

# Joint statistics of natural frequencies corresponding to structural systems with singular random parameter matrices

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## Abstract

An asymptotic approximation methodology for solving standard random eigenvalue problems is generalized herein to account for structural systems with singular random parameter matrices. In this regard, resorting to the concept of Moore-Penrose matrix inverse and generalizing expressions for the rate of change of the eigenvalues, novel closed-form expressions are derived for the joint moments of the system natural frequencies. Two indicative examples pertaining to multi-degree-of-freedom structural systems are considered for demonstrating the reliability of the methodology. Comparisons with pertinent Monte Carlo simulation data are included as well.

**Keywords:** Random eigenvalue problem - Singular matrix - Random vibration - Moore-Penrose inverse

## 1 Introduction

Uncertainty modeling of media properties and of external excitations based on stochastic process theory leads, typically, to stochastic partial differential equations governing the dynamics of engineering systems (e.g., [11, 22]). Naturally, various diverse solution methodologies have been developed over the last few decades for determining the stochastic response of such systems; see, for instance, [28], [23], [36], and [26] for a broad perspective and some recent indicative research efforts.

Further, spatial discretization of the stochastic field modeling the material properties of the engineering system yields stochastic differential equations with random parameter matrices (e.g., [17]). These matrices are related to the formulation of a random eigenvalue problem, which is of paramount importance to the dynamic analysis of stochastic structural systems. In this regard, various solution schemes have been developed for determining statistics of the eigenvalues, or equivalently, of the natural frequencies corresponding to the considered dynamic system. Indicatively, these include methodologies based on Monte Carlo simulation (MCS) (e.g., [49]), on perturbation analysis (e.g., [4, 31, 47]), on polynomial chaos expansions (e.g., [18, 16]), on crossing theory (e.g., [20, 21]),

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on dimensional decomposition approaches (e.g., [43, 44, 45]), and on stochastic order reduction schemes (e.g., [55, 12]).

Note, however, that the aforementioned solution schemes routinely rely on the assumption of parameter matrices with appealing properties, such as symmetry and positive definiteness. Unfortunately, it is not possible to employ these techniques in a straightforward manner for addressing cases of asymmetric and singular random matrices. In fact, the authors and co-workers have generalized recently standard random vibration input-output relationships and related solution techniques to account for systems with singular matrices (e.g., [14, 15, 25, 41, 40, 35, 24, 32]). Indicative cases where singular matrices may appear include multi-body system modeling based on dependent coordinates, hysteresis modeling via auxiliary state equations, and energy harvesters with coupled electro-mechanical equations (e.g., [53, 3, 37, 38, 39]).

In this paper, the asymptotic approximation methodology developed in [1] for solving standard random eigenvalue problems is generalized for determining joint moments of natural frequencies corresponding to systems with singular random parameter matrices. This is done by resorting to the concept of Moore-Penrose matrix inverse and by generalizing expressions for the rate of change of eigenvalues to account for asymmetric matrices. Two indicative examples pertaining to multi-degree-of-freedom (MDOF) structural systems are considered for demonstrating the reliability of the methodology. Comparisons with pertinent MCS data are included as well.

## 2 Random eigenvalue problem formulation to account for dynamic systems with singular parameter matrices

### 2.1 Standard random eigenvalue problem

Following the standard modeling in structural dynamics (e.g., [8, 7]), the governing equation of a linear  $n$ -degree-of-freedom ( $n$ -DOF) undamped system is given by

$$\mathbf{M}(\boldsymbol{\alpha})\ddot{\mathbf{q}} + \mathbf{K}(\boldsymbol{\alpha})\mathbf{q} = \mathbf{0}, \quad (1)$$

where  $\mathbf{q}$  denotes the  $n$ -dimensional displacement vector defined based on generalized coordinates (e.g., [46]). Further, the random parameter matrices  $\mathbf{M}(\boldsymbol{\alpha}), \mathbf{K}(\boldsymbol{\alpha}) : \mathbb{R}^k \rightarrow \mathbb{R}^{n^2}$  are considered to be continuous and at least twice differentiable functions of the random parameter vector  $\boldsymbol{\alpha} \in \mathbb{R}^k$ , which may relate to material or geometric properties (e.g., [1, 9]). In this regard, the statistical properties of the system in Eq. (1) are characterized completely by the joint probability density function (PDF)

$$p_{\boldsymbol{\alpha}}(\boldsymbol{\alpha}) = \exp\{-f(\boldsymbol{\alpha})\}, \quad (2)$$

which is non-Gaussian in general and  $f : \mathbb{R}^k \rightarrow \mathbb{R}$  denotes an arbitrary function of the parameter vector  $\boldsymbol{\alpha}$  (e.g., [1, 2]).

Next, considering a solution of Eq. (1) in the form (e.g., [46])  $\mathbf{q}(t) = \mathbf{q}_0 \exp(i\omega(\boldsymbol{\alpha})t)$ , where  $\omega(\boldsymbol{\alpha})$  denotes the system natural frequency and  $\mathbf{q}_0$  is an  $n$ -dimensional amplitude vector, and substituting into Eq. (1) yields

$$(\mathbf{K}(\boldsymbol{\alpha}) - \omega^2(\boldsymbol{\alpha})\mathbf{M}(\boldsymbol{\alpha}))\mathbf{q}_0 = \mathbf{0}. \quad (3)$$

Clearly, Eq. (3) has a non-trivial solution if, and only if, the determinant of the matrix on the left-hand side is equal to zero, i.e.,

$$\|(\mathbf{K}(\boldsymbol{\alpha}) - \omega^2(\boldsymbol{\alpha})\mathbf{M}(\boldsymbol{\alpha}))\| = 0. \quad (4)$$

It is readily seen that Eq. (4) constitutes an  $n$ -th order polynomial of  $\omega^2(\boldsymbol{\alpha})$ , whose root  $\omega_i^2(\boldsymbol{\alpha})$  is the square of the system  $i$ -th natural frequency  $\omega_i(\boldsymbol{\alpha})$ , for  $i = 1, 2, \dots, n$ . Further, each of the natural frequencies  $\omega_i(\boldsymbol{\alpha})$  corresponds to an eigenvector (or mode shape)  $\boldsymbol{\phi}_i(\boldsymbol{\alpha})$ ; that is, the value of  $\mathbf{q}_0$  in Eq. (3). Further, denoting  $\lambda_i(\boldsymbol{\alpha}) = \omega_i^2(\boldsymbol{\alpha})$  and considering Eq. (3), the random generalized eigenvalue problem takes the form

$$\mathbf{K}(\boldsymbol{\alpha})\boldsymbol{\phi}_i(\boldsymbol{\alpha}) = \lambda_i(\boldsymbol{\alpha})\mathbf{M}(\boldsymbol{\alpha})\boldsymbol{\phi}_i(\boldsymbol{\alpha}), \quad (5)$$

for  $i = 1, 2, \dots, n$ . Specifically, for the pair of random matrices  $\mathbf{M}(\boldsymbol{\alpha})$  and  $\mathbf{K}(\boldsymbol{\alpha})$  the problem in Eq. (5) consists in determining the random eigenvalue  $\lambda_i(\boldsymbol{\alpha}) : \mathbb{R}^k \rightarrow \mathbb{R}$  and its corresponding random eigenvector  $\boldsymbol{\phi}_i(\boldsymbol{\alpha}) : \mathbb{R}^k \rightarrow \mathbb{R}^n$ .

It is worth noting that a large number of researchers have developed various diverse techniques for solving Eq. (5) (e.g., [21, 43, 1]). However, these solution techniques rely on standard modeling of structural systems yielding parameter matrices with appealing properties, such as symmetry and positive definiteness. Unfortunately, it is not possible to employ these techniques in a straightforward manner for addressing cases of asymmetric and singular matrices presented in the following.

## 2.2 Random eigenvalue problem considering singular matrices

As shown in various theoretical analyses and applications (e.g., [53, 54, 14]), it can be argued that adopting a dynamic system modeling based on non-generalized (dependent) coordinates can be advantageous from a computational efficiency perspective (e.g., [29]). Specifically, following [48] and [14], consider the system in Eq. (1) modeled in the form

$$\mathbf{M}_x(\boldsymbol{\alpha})\ddot{\mathbf{x}} + \mathbf{K}_x(\boldsymbol{\alpha})\mathbf{x} = \mathbf{0}, \quad (6)$$

where  $\mathbf{x}$  denotes an  $l$ -dimensional (dependent) coordinates vector ( $l > n$ ). Further, to account for the relationships between the dependent coordinates and other forms of constraints, Eq. (6) is considered in conjunction with

$$\mathbf{A}\ddot{\mathbf{x}} + \mathbf{L}\mathbf{x} = \mathbf{0} \quad (7)$$

where  $\mathbf{A}$  and  $\mathbf{L}$  denote  $m \times l$  matrices. Next, combining Eq. (6) and Eq. (7) yields

$$\bar{\mathbf{M}}_x(\boldsymbol{\alpha})\ddot{\mathbf{x}} + \bar{\mathbf{K}}_x(\boldsymbol{\alpha})\mathbf{x} = \mathbf{0}, \quad (8)$$

where  $\bar{\mathbf{M}}_x(\boldsymbol{\alpha}), \bar{\mathbf{K}}_x(\boldsymbol{\alpha}) : \mathbb{R}^k \rightarrow \mathbb{R}^{(l+m) \times l}$  denote the augmented  $(l+m) \times l$  mass and stiffness random matrices of the system, given by (e.g., [14])

$$\bar{\mathbf{M}}_x(\boldsymbol{\alpha}) = \begin{bmatrix} [(\mathbf{I}_l - \mathbf{A}^+\mathbf{A})\mathbf{M}_x(\boldsymbol{\alpha})]_{l \times l} \\ [\mathbf{A}]_{m \times l} \end{bmatrix}, \quad \bar{\mathbf{K}}_x(\boldsymbol{\alpha}) = \begin{bmatrix} [(\mathbf{I}_l - \mathbf{A}^+\mathbf{A})\mathbf{K}_x(\boldsymbol{\alpha})]_{l \times l} \\ [\mathbf{L}]_{m \times l} \end{bmatrix}. \quad (9)$$

In Eq. (9),  $\mathbf{I}_l$  is the  $l \times l$  identity matrix, and “+” denotes the Moore-Penrose matrix inverse operation (see also Appendix I). A detailed derivation of Eqs. (6)-(9) can be found, indicatively, in [25] and in [52].

Clearly, matrices  $\bar{\mathbf{M}}_x(\boldsymbol{\alpha})$  and  $\bar{\mathbf{K}}_x(\boldsymbol{\alpha})$  in Eq. (9) are rectangular, and thus, a straightforward formulation of the random eigenvalue problem according to Eq. (5), which refers to square matrices, is not possible. In passing, it is worth noting various research efforts for addressing eigenvalue problems associated with rectangular/asymmetric matrices. These include, indicatively, solution frameworks based on perturbation methods (e.g., [50]) and approaches based on the concept of matrix pseudo-spectrum (e.g., [51, 19]).

In the ensuing analysis, the random eigenvalue problem corresponding to Eq. (8) is formulated by exploiting the relationship between the system of Eq. (1) modeled via generalized (independent) coordinates, and the system of Eq. (6) modeled via dependent coordinates. This implies that the system of Eq. (6) has  $n$  non-zero eigenvalues coinciding with the ones obtained by solving Eq. (5), whereas the remaining  $l-n$  eigenvalues are expected to be equal to zero. Further, it was shown in [34], based on the Cauchy-Binet formula of linear algebra, that the same  $n$  non-zero and  $l-n$  zero eigenvalues are obtained by removing  $m$  linearly dependent rows from the  $(l+m) \times l$  matrices  $\bar{\mathbf{M}}_x$  and  $\bar{\mathbf{K}}_x$  in Eq. (8) and by considering the eigenvalue problem corresponding to the resulting  $l \times l$  matrices.

An intuitive explanation of the validity of the above argument is that the  $l-n$  zero eigenvalues correspond to rigid body motions resulting from the  $l-n$  dependent coordinates used to model the system of Eq. (6). In fact, despite the redundant coordinates modeling, the constituent parts of the system under consideration remain the same, and thus, it is anticipated that the non-zero eigenvalues of the system of Eq. (8) are exactly the same as the eigenvalues of the system whose governing equations of motion are modeled by utilizing an  $n$ -dimensional vector of independent coordinates; see also [10].

In this regard, following [34],  $m$  linearly dependent rows of sub-matrices  $[(\mathbf{I}_l - \mathbf{A}^+\mathbf{A})\mathbf{M}_x(\boldsymbol{\alpha})]_{l \times l}$  and  $[(\mathbf{I}_l - \mathbf{A}^+\mathbf{A})\mathbf{K}_x(\boldsymbol{\alpha})]_{l \times l}$  of the block matrices  $\bar{\mathbf{M}}_x(\boldsymbol{\alpha})$  and  $\bar{\mathbf{K}}_x(\boldsymbol{\alpha})$  in Eq. (9) are removed. This leads to the random generalized eigenvalue problem

$$\left( \tilde{\mathbf{K}}(\boldsymbol{\alpha}) - \tilde{\lambda}_i(\boldsymbol{\alpha})\tilde{\mathbf{M}}(\boldsymbol{\alpha}) \right) \tilde{\boldsymbol{\phi}}_i(\boldsymbol{\alpha}) = \mathbf{0}, \quad (10)$$

where  $\tilde{\lambda}_i(\boldsymbol{\alpha})$  and  $\tilde{\boldsymbol{\phi}}_i(\boldsymbol{\alpha})$ ,  $i = 1, 2, \dots, l$ , denote the  $i$ -th random eigenvalue and the corresponding right random eigenvector. Further, the  $l \times l$  random parameter matrices  $\tilde{\mathbf{M}}(\boldsymbol{\alpha}), \tilde{\mathbf{K}}(\boldsymbol{\alpha}) : \mathbb{R}^k \rightarrow \mathbb{R}^{l \times l}$  are defined as

$$\tilde{\mathbf{M}}(\boldsymbol{\alpha}) = \begin{bmatrix} [(\mathbf{I}_l - \mathbf{A}^+\mathbf{A})\mathbf{M}_x(\boldsymbol{\alpha})]_{(l-m) \times l} \\ [\mathbf{A}]_{m \times l} \end{bmatrix}, \quad \tilde{\mathbf{K}}(\boldsymbol{\alpha}) = \begin{bmatrix} [(\mathbf{I}_l - \mathbf{A}^+\mathbf{A})\mathbf{K}_x(\boldsymbol{\alpha})]_{(l-m) \times l} \\ [\mathbf{L}]_{m \times l} \end{bmatrix}. \quad (11)$$

Thus, in the ensuing analysis the random eigenvalue problem defined in Eq. (10) is considered in conjunction with generalizing an asymptotic approximation methodology for determining joint moments of natural frequencies corresponding to systems with singular parameter matrices. Note that although the matrices involved in Eq. (10) are square, they are asymmetric in general. This hinders the straightforward generalization of the technique developed in [1] and poses certain challenges to be addressed in the following.

### 3 Generalization of an asymptotic approximation methodology for determining the joint moments of natural frequencies corresponding to dynamic systems with singular parameter matrices

#### 3.1 Standard formulation

In this section, a concise presentation of the standard approximation methodology proposed in [1] for determining the joint moments of natural frequencies is included for completeness. The methodology pertains to evaluating a multi-dimensional integral based on the saddle point approximation (e.g., [5, 56]).

Specifically, considering that the smooth and twice differentiable function  $g(\boldsymbol{\alpha}) : G \subset \mathbb{R}^k \rightarrow \mathbb{R}$  exhibits its unique global minimum at  $\boldsymbol{\alpha} = \boldsymbol{\alpha}_0 \in G$ , and denoting its positive definite Hessian matrix evaluated at  $\boldsymbol{\alpha}_0$  as  $\mathbf{D}_g(\boldsymbol{\alpha}_0) = \left( \frac{\partial^2 g}{\partial \alpha_i \partial \alpha_j} \right) \Big|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0}$ , the integral

$$\mathcal{I} = \int_G \exp \{-g(\boldsymbol{\alpha})\} d\boldsymbol{\alpha} \quad (12)$$

is approximated by (e.g., [5, 56])

$$\mathcal{I} \approx (2\pi)^{k/2} \exp \{-g(\boldsymbol{\alpha}_0)\} \|\mathbf{D}_g(\boldsymbol{\alpha}_0)\|^{-1/2}. \quad (13)$$

Further, taking into account the system parameter vector  $\boldsymbol{\alpha}$  and its joint PDF defined in Eq. (2), the arbitrary order  $s_i$  ( $i = 1, 2, \dots, n$ ) joint moments of multiple natural frequencies are given by

$$\mu_{i_1, i_2, \dots, i_n}^{(s_1, s_2, \dots, s_n)} = \int_{\mathbb{R}^k} \{\omega_{i_1}^{s_1}(\boldsymbol{\alpha}) \omega_{i_2}^{s_2}(\boldsymbol{\alpha}) \dots \omega_{i_n}^{s_n}(\boldsymbol{\alpha})\} p_{\boldsymbol{\alpha}}(\boldsymbol{\alpha}) d\boldsymbol{\alpha}, \quad (14)$$

where the natural frequencies are determined by solving the standard random generalized eigenvalue problem of Eq. (5). Next, substituting Eq. (2) into Eq. (14) yields

$$\mu_{i_1, i_2, \dots, i_n}^{(s_1, s_2, \dots, s_n)} = \int_{\mathbb{R}^k} \exp \{-(f(\boldsymbol{\alpha}) - s_1 \ln \omega_{i_1}(\boldsymbol{\alpha}) - s_2 \ln \omega_{i_2}(\boldsymbol{\alpha}) - \dots - s_n \ln \omega_{i_n}(\boldsymbol{\alpha}))\} d\boldsymbol{\alpha}. \quad (15)$$

Taking into account the integral form in Eq. (12), setting  $g(\boldsymbol{\alpha}) = f(\boldsymbol{\alpha}) - s_1 \ln \omega_{i_1}(\boldsymbol{\alpha}) - s_2 \ln \omega_{i_2}(\boldsymbol{\alpha}) - \dots - s_n \ln \omega_{i_n}(\boldsymbol{\alpha})$  and employing the approximation of Eq. (13), Eq. (15) becomes

$$\mu_{i_1, i_2, \dots, i_n}^{(s_1, s_2, \dots, s_n)} \approx (2\pi)^{k/2} \{\omega_{i_1}^{s_1}(\boldsymbol{\alpha}_0) \omega_{i_2}^{s_2}(\boldsymbol{\alpha}_0) \dots \omega_{i_n}^{s_n}(\boldsymbol{\alpha}_0)\} \exp \{-f(\boldsymbol{\alpha}_0)\} \|\mathbf{D}_g(\boldsymbol{\alpha}_0)\|^{-1/2}. \quad (16)$$

Clearly, an integral part of the methodology pertains to the efficient calculation of point  $\boldsymbol{\alpha}_0$  to be used in the approximation of Eq. (16). In this regard, considering the exponent of Eq. (15), and applying the condition  $\frac{\partial g(\boldsymbol{\alpha})}{\partial \alpha_k} = 0$ , for all  $k$ , satisfied at point  $\boldsymbol{\alpha} = \boldsymbol{\alpha}_0$  where  $g(\boldsymbol{\alpha})$  exhibits a unique global minimum (see [56]), yields the expression

$$\mathbf{d}_f(\boldsymbol{\alpha}_0) = \frac{s_1}{\omega_{i_1}(\boldsymbol{\alpha}_0)} \mathbf{d}_{\omega_{i_1}}(\boldsymbol{\alpha}_0) + \frac{s_2}{\omega_{i_2}(\boldsymbol{\alpha}_0)} \mathbf{d}_{\omega_{i_2}}(\boldsymbol{\alpha}_0) + \dots + \frac{s_n}{\omega_{i_n}(\boldsymbol{\alpha}_0)} \mathbf{d}_{\omega_{i_n}}(\boldsymbol{\alpha}_0), \quad (17)$$

where  $\mathbf{d}_f(\boldsymbol{\alpha}_0) = \frac{\partial f(\boldsymbol{\alpha})}{\partial \boldsymbol{\alpha}_i} \Big|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0}$  and  $\mathbf{d}_{\omega_{i_\rho}}(\boldsymbol{\alpha}_0) = \frac{\partial \omega_{i_\rho}(\boldsymbol{\alpha})}{\partial \alpha_{i_\rho}} \Big|_{\boldsymbol{\alpha}=\boldsymbol{\alpha}_0}$ , for all  $i_\rho \in \{i_1, i_2, \dots, i_n\}$ . Further, considering the exponent of Eq. (15) and differentiating yields the Hessian matrix evaluated at  $\boldsymbol{\alpha} = \boldsymbol{\alpha}_0$  in the form

$$\mathbf{D}_g(\boldsymbol{\alpha}_0) = \mathbf{D}_f(\boldsymbol{\alpha}_0) + \sum_{i=i_1, j=j_1}^{i_n, j_n} i \omega_i^{-1}(\boldsymbol{\alpha}_0) \left\{ \frac{\mathbf{d}_{\omega_i}(\boldsymbol{\alpha}_0) \mathbf{d}_{\omega_i}^T(\boldsymbol{\alpha}_0)}{\omega_i(\boldsymbol{\alpha}_0)} - \mathbf{D}_{\omega_i}(\boldsymbol{\alpha}_0) \right\}. \quad (18)$$

Obviously, point  $\boldsymbol{\alpha}_0$  can be determined by solving the nonlinear system of equations shown in Eq. (17). This can be done by employing any standard numerical optimization algorithm (e.g., [33]), or by utilizing the simple iterative solution scheme till convergence followed in [1]. It is worth noting that  $\mathbf{d}_f(\boldsymbol{\alpha})$  can be evaluated explicitly based on knowledge of the PDF  $p_{\boldsymbol{\alpha}}(\boldsymbol{\alpha})$ , whereas the computationally cumbersome numerical differentiation of  $\omega_{i_\rho}$  at each step of the solution algorithm is avoided, since closed-form expressions can be derived for  $\mathbf{d}_{\omega_{i_\rho}}$  in terms of the derivative of the mass and stiffness matrices. Specifically, considering the normalization of the eigenvectors  $\boldsymbol{\phi}_j$  of the  $n$ -DOF system in Eq. (1) in the form

$$\boldsymbol{\phi}_i^T(\boldsymbol{\alpha}) \mathbf{M}(\boldsymbol{\alpha}) \boldsymbol{\phi}_j(\boldsymbol{\alpha}) = \delta_{ij}, \quad i, j = 1, 2, \dots, n, \quad (19)$$

where  $\delta_{ij}$  denotes the Kronecker delta function, and differentiating Eq. (5) yields the expressions [13, 42]

$$\frac{\partial \omega_j(\boldsymbol{\alpha})}{\partial \alpha_{j_1}} = \frac{\boldsymbol{\phi}_j^T(\boldsymbol{\alpha}) \mathcal{G}_{j,j_1}(\boldsymbol{\alpha}) \boldsymbol{\phi}_j(\boldsymbol{\alpha})}{2\omega_j(\boldsymbol{\alpha})} \quad (20)$$

and

$$\frac{\partial^2 \omega_j(\boldsymbol{\alpha})}{\partial \alpha_{j_1} \partial \alpha_{j_2}} = \left[ \frac{1}{2\omega_j(\boldsymbol{\alpha})} \frac{\partial^2 (\omega_j^2(\boldsymbol{\alpha}))}{\partial \alpha_{j_1} \partial \alpha_{j_2}} - \frac{1}{\omega_j(\boldsymbol{\alpha})} \frac{\partial \omega_j(\boldsymbol{\alpha})}{\partial \alpha_{j_2}} \frac{\partial \omega_j(\boldsymbol{\alpha})}{\partial \alpha_{j_1}} \right], \quad (21)$$

where

$$\mathcal{G}_{j,j_1}(\boldsymbol{\alpha}) = \left[ \frac{\partial \mathbf{K}(\boldsymbol{\alpha})}{\partial \alpha_{j_1}} - \omega_j^2(\boldsymbol{\alpha}) \frac{\partial \mathbf{M}(\boldsymbol{\alpha})}{\partial \alpha_{j_1}} \right] \quad (22)$$

and

$$\begin{aligned} \frac{\partial^2 (\omega_j^2(\boldsymbol{\alpha}))}{\partial \alpha_{j_1} \partial \alpha_{j_2}} &= \boldsymbol{\phi}_j^T(\boldsymbol{\alpha}) \left[ \frac{\partial^2 \mathbf{K}(\boldsymbol{\alpha})}{\partial \alpha_{j_1} \partial \alpha_{j_2}} - \omega_j^2(\boldsymbol{\alpha}) \frac{\partial^2 \mathbf{M}(\boldsymbol{\alpha})}{\partial \alpha_{j_1} \partial \alpha_{j_2}} \right] \boldsymbol{\phi}_j(\boldsymbol{\alpha}) - \left( \boldsymbol{\phi}_j^T(\boldsymbol{\alpha}) \frac{\partial \mathbf{M}(\boldsymbol{\alpha})}{\partial \alpha_{j_1}} \boldsymbol{\phi}_j(\boldsymbol{\alpha}) \right) \\ &\times \left( \boldsymbol{\phi}_j^T(\boldsymbol{\alpha}) \mathcal{G}_{j,j_2}(\boldsymbol{\alpha}) \boldsymbol{\phi}_j(\boldsymbol{\alpha}) \right) - \left( \boldsymbol{\phi}_j^T(\boldsymbol{\alpha}) \frac{\partial \mathbf{M}(\boldsymbol{\alpha})}{\partial \alpha_{j_2}} \boldsymbol{\phi}_j(\boldsymbol{\alpha}) \right) \left( \boldsymbol{\phi}_j^T(\boldsymbol{\alpha}) \mathcal{G}_{j,j_1}(\boldsymbol{\alpha}) \boldsymbol{\phi}_j(\boldsymbol{\alpha}) \right) \\ &+ 2 \sum_{r=1}^n \frac{\left( \boldsymbol{\phi}_r^T(\boldsymbol{\alpha}) \mathcal{G}_{j,j_1}(\boldsymbol{\alpha}) \boldsymbol{\phi}_j(\boldsymbol{\alpha}) \right) \left( \boldsymbol{\phi}_r^T(\boldsymbol{\alpha}) \mathcal{G}_{j,j_2}(\boldsymbol{\alpha}) \boldsymbol{\phi}_j(\boldsymbol{\alpha}) \right)}{\omega_j^2(\boldsymbol{\alpha}) - \omega_r^2(\boldsymbol{\alpha})}. \end{aligned} \quad (23)$$

It is noted that the application of the methodology developed in [1] and described in this section relies on the assumption of parameter matrices exhibiting symmetry. Specifically, the mass normalization considered in Eq. (19), which holds only for symmetric random eigenvalue problems such as in Eq. (5), is a necessary condition for deriving the expressions in Eqs. (20) and (21) for the rate of change of the eigenvalues. Clearly, Eqs. (20) and (21) are not valid for the herein considered random eigenvalue problem described in Eq. (10), where matrices  $\tilde{\mathbf{M}}(\boldsymbol{\alpha})$  and  $\tilde{\mathbf{K}}(\boldsymbol{\alpha})$  are asymmetric in general. In this regard, new generalized expressions are developed next pertaining to the derivatives of random eigenvalues corresponding to asymmetric matrices. Subsequently, the latter are employed in conjunction with the asymptotic approximation methodology for determining the joint moments of natural frequencies corresponding to dynamic systems with singular parameter matrices.

### 3.2 Proposed generalized formulation

In this section, the random eigenvalue problem of Eq. (10) pertaining to asymmetric matrices is considered. In this regard, novel closed-form expressions for the rate of change of eigenvalues are derived. First, some fundamental definitions are provided for completeness.

**Definition 1.** A matrix pencil for the  $n \times n$  matrices  $\mathbf{A}$  and  $\mathbf{B}$  is defined as  $\mathbf{B} + \lambda \mathbf{A}$ , where  $\lambda \in \mathbb{R}$ .

**Definition 2.** A matrix pencil  $\mathbf{B} + \lambda \mathbf{A}$  is defined as regular if  $\mathbf{A}, \mathbf{B}$  are square matrices and  $\mathbf{A}$  is non-singular.

**Definition 3.** A matrix pencil  $\mathbf{B} + \lambda \mathbf{A}$  is defined as simple of order  $n$ , if it is regular and has  $n$ -linearly independent right eigenvectors.

**Theorem 1.** [27] Consider the  $n \times n$  matrices  $\boldsymbol{\Psi} = [\boldsymbol{\psi}_1 \ \boldsymbol{\psi}_2 \ \dots \ \boldsymbol{\psi}_n]$  and  $\boldsymbol{\Phi} = [\boldsymbol{\phi}_1 \ \boldsymbol{\phi}_2 \ \dots \ \boldsymbol{\phi}_n]$ , where  $\boldsymbol{\psi}_i$  and  $\boldsymbol{\phi}_i$ ,  $i = 1, 2, \dots, n$  denote, respectively, the  $i$ -th left and right eigenvectors of the simple matrix pencil  $\mathbf{B} + \lambda \mathbf{A}$ . Then, the relationships  $\boldsymbol{\Psi}^T \mathbf{A} \boldsymbol{\Phi} = \mathbf{I}_n$  and  $\boldsymbol{\Psi}^T \mathbf{B} \boldsymbol{\Phi} = -\boldsymbol{\Lambda}$  apply, where  $\mathbf{I}_n$  is the  $n \times n$  identity matrix and  $\boldsymbol{\Lambda}$  denotes an  $n \times n$  diagonal matrix, whose diagonal elements are the eigenvalues  $\lambda_i$ ,  $i = 1, 2, \dots, n$ .

Next, attention is directed to the random eigenvalue problem defined in Eq. (10). Note that, in contrast to the standard eigenvalue problem in Eq. (5) pertaining to symmetric and positive definite matrices,  $\tilde{\mathbf{M}}(\boldsymbol{\alpha})$  and  $\tilde{\mathbf{K}}(\boldsymbol{\alpha})$  in Eq. (10) are asymmetric in general. As a result, the standard mass normalization condition of Eq. (19), applicable for symmetric matrices, is not valid anymore. Therefore, Eqs. (20) and (21) for the rate of change of the eigenvalues need to be generalized. In the following, relying on Theorem 1, a generalized solution framework is developed.

In this regard, the left eigenvectors for the eigenvalue problem of Eq. (10) are defined as the  $l$ -dimensional random vectors  $\tilde{\boldsymbol{\psi}}_i(\boldsymbol{\alpha}) : \mathbb{R}^k \rightarrow \mathbb{R}^l$ , such that

$$\tilde{\boldsymbol{\psi}}_i^T(\boldsymbol{\alpha}) \left( \tilde{\mathbf{K}}(\boldsymbol{\alpha}) - \tilde{\lambda}_i(\boldsymbol{\alpha}) \tilde{\mathbf{M}}(\boldsymbol{\alpha}) \right) = \mathbf{0}, \quad (24)$$

for  $i = 1, 2, \dots, l$  (e.g., [30, 27]). Next, considering the matrices  $\tilde{\Psi}(\boldsymbol{\alpha}) = [\tilde{\psi}_1(\boldsymbol{\alpha}) \ \tilde{\psi}_2(\boldsymbol{\alpha}) \ \dots \ \tilde{\psi}_l(\boldsymbol{\alpha})]$  and  $\tilde{\Phi}(\boldsymbol{\alpha}) = [\tilde{\phi}_1(\boldsymbol{\alpha}) \ \tilde{\phi}_2(\boldsymbol{\alpha}) \ \dots \ \tilde{\phi}_l(\boldsymbol{\alpha})]$  of the simple matrix pencil  $\tilde{\mathbf{K}}(\boldsymbol{\alpha}) - \tilde{\lambda}_i(\boldsymbol{\alpha})\tilde{\mathbf{M}}(\boldsymbol{\alpha})$ , and relying on Theorem 1 (see also [27]), the matrix  $\tilde{\Psi}(\boldsymbol{\alpha})$  formulated by the left eigenvectors  $\tilde{\psi}_i(\boldsymbol{\alpha})$ ,  $i = 1, 2, \dots, l$  is given by

$$\tilde{\Psi}(\boldsymbol{\alpha}) = \left( \tilde{\mathbf{M}}(\boldsymbol{\alpha})\tilde{\Phi}(\boldsymbol{\alpha}) \right)^{-1}. \quad (25)$$

Therefore, the orthogonality conditions for the random parameter matrices in Eq. (11) become [27]

$$\tilde{\psi}_i^T(\boldsymbol{\alpha})\tilde{\mathbf{M}}(\boldsymbol{\alpha})\tilde{\phi}_j(\boldsymbol{\alpha}) = \delta_{ij} \quad (26)$$

and

$$\tilde{\psi}_i^T(\boldsymbol{\alpha})\tilde{\mathbf{K}}(\boldsymbol{\alpha})\tilde{\phi}_j(\boldsymbol{\alpha}) = -\delta_{ij}\tilde{\lambda}_i(\boldsymbol{\alpha}), \quad (27)$$

for  $i, j = 1, 2, \dots, l$ . It can be readily seen that Eq. (26) can be construed as a generalization of the mass normalization condition of Eq. (19) to account for asymmetric matrices.

As explained in section "Generalization of the random eigenvalue problem to account for singular parameter matrices", the random eigenvalue problem of Eq. (10) yields the  $n$  distinct eigenvalues corresponding to the original system of Eq. (1) in addition to  $l - n$  zero eigenvalues corresponding to rigid body motions. In the ensuing analysis, the  $n$  non-zero eigenvalues are placed in descending order followed by the  $l - n$  zero eigenvalues. The corresponding left and right eigenvectors  $\tilde{\psi}_i(\boldsymbol{\alpha})$  and  $\tilde{\phi}_i(\boldsymbol{\alpha})$  ( $i = 1, 2, \dots, l$ ) are arranged accordingly, and thus, matrices  $\tilde{\Psi}(\boldsymbol{\alpha})$  and  $\tilde{\Phi}(\boldsymbol{\alpha})$  contain the eigenvectors corresponding to the  $n$  non-zero eigenvalues followed by the eigenvectors corresponding to the  $l - n$  zero eigenvalues.

Further, pre-multiplying Eq. (10) by  $\tilde{\psi}_i^T(\boldsymbol{\alpha})$  and differentiating with respect to the  $j$ -th component of the random parameter  $\boldsymbol{\alpha}^T = [\alpha_1 \ \alpha_2 \ \dots \ \alpha_k] \in \mathbb{R}^k$  leads to

$$\begin{aligned} & \left( \frac{\partial \tilde{\psi}_i(\boldsymbol{\alpha})}{\partial \alpha_j} \right)^T \left( \tilde{\mathbf{K}}(\boldsymbol{\alpha}) - \tilde{\lambda}_i(\boldsymbol{\alpha})\tilde{\mathbf{M}}(\boldsymbol{\alpha}) \right) \tilde{\phi}_i(\boldsymbol{\alpha}) + \tilde{\psi}_i^T(\boldsymbol{\alpha}) \left( \tilde{\mathbf{K}}(\boldsymbol{\alpha}) - \tilde{\lambda}_i(\boldsymbol{\alpha})\tilde{\mathbf{M}}(\boldsymbol{\alpha}) \right) \left( \frac{\partial \tilde{\phi}_i(\boldsymbol{\alpha})}{\partial \alpha_j} \right) \\ & + \tilde{\psi}_i^T(\boldsymbol{\alpha}) \left( \frac{\partial \tilde{\mathbf{K}}(\boldsymbol{\alpha})}{\partial \alpha_j} - \frac{\partial \tilde{\lambda}_i(\boldsymbol{\alpha})}{\partial \alpha_j} \tilde{\mathbf{M}}(\boldsymbol{\alpha}) - \tilde{\lambda}_i(\boldsymbol{\alpha}) \frac{\partial \tilde{\mathbf{M}}(\boldsymbol{\alpha})}{\partial \alpha_j} \right) \tilde{\phi}_i(\boldsymbol{\alpha}) = 0. \end{aligned} \quad (28)$$

Taking into account Eqs. (10), (24) and (26), Eq. (28) yields

$$\frac{\partial \tilde{\lambda}_i(\boldsymbol{\alpha})}{\partial \alpha_j} = \tilde{\psi}_i^T(\boldsymbol{\alpha})\tilde{\mathcal{G}}_{i,j}(\boldsymbol{\alpha})\tilde{\phi}_i(\boldsymbol{\alpha}), \quad (29)$$

for  $i = 1, 2, \dots, l$  and  $j = 1, 2, \dots, k$ , where

$$\tilde{\mathcal{G}}_{i,j}(\boldsymbol{\alpha}) = \frac{\partial \tilde{\mathbf{K}}(\boldsymbol{\alpha})}{\partial \alpha_j} - \tilde{\lambda}_i(\boldsymbol{\alpha}) \frac{\partial \tilde{\mathbf{M}}(\boldsymbol{\alpha})}{\partial \alpha_j}. \quad (30)$$

Finally, considering that for  $i = 1, 2, \dots, l$  the eigenvalue  $\tilde{\lambda}_i$  coincides with the square of the  $i$ -th natural frequency of the system in Eq. (8), i.e.,  $\tilde{\lambda}_i(\boldsymbol{\alpha}) = \tilde{\omega}_i^2(\boldsymbol{\alpha})$ , Eq. (29) is written, equivalently, as

$$\frac{\partial \tilde{\omega}_i(\boldsymbol{\alpha})}{\partial \alpha_j} = \frac{\tilde{\psi}_i^T(\boldsymbol{\alpha})\tilde{\mathcal{G}}_{i,j}(\boldsymbol{\alpha})\tilde{\phi}_i(\boldsymbol{\alpha})}{2\tilde{\omega}_i(\boldsymbol{\alpha})}. \quad (31)$$

Clearly, Eq. (31) can be construed as a generalization of Eq. (20) to account for asymmetric matrices corresponding to the eigenvalue problem of Eq. (10).

Next, attention is directed to generalizing also Eq. (21). First, the derivatives of the left eigenvectors  $\tilde{\psi}_1(\boldsymbol{\alpha}), \tilde{\psi}_2(\boldsymbol{\alpha}), \dots, \tilde{\psi}_l(\boldsymbol{\alpha})$  and right eigenvectors  $\tilde{\phi}_1(\boldsymbol{\alpha}), \tilde{\phi}_2(\boldsymbol{\alpha}), \dots, \tilde{\phi}_l(\boldsymbol{\alpha})$  are expressed in the form [13]

$$\left( \frac{\partial \tilde{\psi}_i(\boldsymbol{\alpha})}{\partial \alpha_j} \right)^T = \sum_{s=1}^l c_{ijs} \tilde{\psi}_s^T(\boldsymbol{\alpha}), \quad \frac{\partial \tilde{\phi}_i(\boldsymbol{\alpha})}{\partial \alpha_j} = \sum_{s=1}^l d_{ijs} \tilde{\phi}_s(\boldsymbol{\alpha}), \quad (32)$$

where  $c_{ijs}$  and  $d_{ijs}$  ( $s = 1, 2, \dots, l$ ) denote real coefficients to be determined. Further, setting  $i = j$  in Eq. (26), differentiating with respect to  $\alpha_j$ , and taking into account the orthogonality conditions in Eqs. (26) and (27) yields

$$c_{iji} + d_{ijs} = -\tilde{\psi}_i^T(\boldsymbol{\alpha}) \frac{\partial \tilde{\mathbf{M}}(\boldsymbol{\alpha})}{\partial \alpha_j} \tilde{\phi}_i(\boldsymbol{\alpha}). \quad (33)$$

Furthermore, considering the left eigenvectors  $\tilde{\Psi}(\boldsymbol{\alpha})$ , Eq. (24) is post-multiplied by  $\tilde{\phi}_\rho(\boldsymbol{\alpha})$  and subsequently differentiated with respect to  $\alpha_j$ . Also, taking into account the orthogonality conditions in Eqs. (26) and (27) leads to

$$c_{ij\rho} = \frac{1}{\tilde{\lambda}_i(\boldsymbol{\alpha}) - \tilde{\lambda}_\rho(\boldsymbol{\alpha})} \tilde{\psi}_i^T(\boldsymbol{\alpha}) \tilde{\mathcal{G}}_{i,j}(\boldsymbol{\alpha}) \tilde{\phi}_\rho(\boldsymbol{\alpha}), \quad (34)$$

for  $\rho \neq i$ . Similarly, pre-multiplying Eq. (10) by  $\tilde{\psi}_\rho^T(\boldsymbol{\alpha})$  and manipulating yields

$$d_{ij\rho} = \frac{1}{\tilde{\lambda}_i(\boldsymbol{\alpha}) - \tilde{\lambda}_\rho(\boldsymbol{\alpha})} \tilde{\psi}_\rho^T(\boldsymbol{\alpha}) \tilde{\mathcal{G}}_{i,j}(\boldsymbol{\alpha}) \tilde{\phi}_i(\boldsymbol{\alpha}), \quad (35)$$

for  $\rho \neq i$ . Next, differentiating Eq. (29) with respect to  $\alpha_r$  and considering Eq. (32) leads to

$$\begin{aligned} \frac{\partial^2 \tilde{\lambda}_i(\boldsymbol{\alpha})}{\partial \alpha_j \partial \alpha_r} &= \tilde{\psi}_i^T(\boldsymbol{\alpha}) \left( \frac{\partial^2 \tilde{\mathbf{K}}(\boldsymbol{\alpha})}{\partial \alpha_j \partial \alpha_r} - \tilde{\lambda}_i(\boldsymbol{\alpha}) \frac{\partial^2 \tilde{\mathbf{M}}(\boldsymbol{\alpha})}{\partial \alpha_j \partial \alpha_r} \right) \tilde{\phi}_i(\boldsymbol{\alpha}) - \left( \tilde{\psi}_i^T(\boldsymbol{\alpha}) \frac{\partial \tilde{\mathbf{M}}(\boldsymbol{\alpha})}{\partial \alpha_j} \tilde{\phi}_i(\boldsymbol{\alpha}) \right) \\ &\quad \times \left( \tilde{\psi}_i^T(\boldsymbol{\alpha}) \tilde{\mathcal{G}}_{i,r}(\boldsymbol{\alpha}) \tilde{\phi}_i(\boldsymbol{\alpha}) \right) - \left( \tilde{\psi}_i^T(\boldsymbol{\alpha}) \frac{\partial \tilde{\mathbf{M}}(\boldsymbol{\alpha})}{\partial \alpha_r} \tilde{\phi}_i(\boldsymbol{\alpha}) \right) \left( \tilde{\psi}_i^T(\boldsymbol{\alpha}) \tilde{\mathcal{G}}_{i,j}(\boldsymbol{\alpha}) \tilde{\phi}_i(\boldsymbol{\alpha}) \right) \\ &\quad + \sum_{\rho=1, \rho \neq i}^l \left\{ \frac{\left( \tilde{\psi}_\rho^T(\boldsymbol{\alpha}) \tilde{\mathcal{G}}_{i,r}(\boldsymbol{\alpha}) \tilde{\phi}_i(\boldsymbol{\alpha}) \right) \left( \tilde{\psi}_i^T(\boldsymbol{\alpha}) \tilde{\mathcal{G}}_{i,j}(\boldsymbol{\alpha}) \tilde{\phi}_\rho(\boldsymbol{\alpha}) \right)}{\tilde{\lambda}_i(\boldsymbol{\alpha}) - \tilde{\lambda}_\rho(\boldsymbol{\alpha})} \right. \\ &\quad \left. + \frac{\left( \tilde{\psi}_i^T(\boldsymbol{\alpha}) \tilde{\mathcal{G}}_{i,r}(\boldsymbol{\alpha}) \tilde{\phi}_\rho(\boldsymbol{\alpha}) \right) \left( \tilde{\psi}_\rho^T(\boldsymbol{\alpha}) \tilde{\mathcal{G}}_{i,j}(\boldsymbol{\alpha}) \tilde{\phi}_i(\boldsymbol{\alpha}) \right)}{\tilde{\lambda}_i(\boldsymbol{\alpha}) - \tilde{\lambda}_\rho(\boldsymbol{\alpha})} \right\}, \end{aligned} \quad (36)$$

whereas differentiating the expression  $\tilde{\lambda}_i(\boldsymbol{\alpha}) = \tilde{\omega}_i^2(\boldsymbol{\alpha})$  with respect to  $\alpha_j$  and  $\alpha_r$  yields

$$\frac{\partial^2 \tilde{\omega}_i(\boldsymbol{\alpha})}{\partial \alpha_j \partial \alpha_r} = \frac{1}{2\tilde{\omega}_i(\boldsymbol{\alpha})} \frac{\partial^2 \tilde{\lambda}_i(\boldsymbol{\alpha})}{\partial \alpha_j \partial \alpha_r} - \frac{1}{\tilde{\omega}_i(\boldsymbol{\alpha})} \frac{\partial \tilde{\omega}_i(\boldsymbol{\alpha})}{\partial \alpha_r} \frac{\partial \tilde{\omega}_i(\boldsymbol{\alpha})}{\partial \alpha_j}, \quad (37)$$

where  $\frac{\partial^2 \tilde{\lambda}_i(\boldsymbol{\alpha})}{\partial \alpha_j \partial \alpha_r}$  is given by Eq. (36).

It is readily seen that Eq. (37) constitutes a generalization of Eq. (21) to account for asymmetric matrices in the eigenvalue problem of Eq. (10). Further, it is worth noting that Eqs. (31) and (37) degenerate to the standard expressions of Eqs. (20) and (21), respectively, for the special case of symmetric matrices  $\mathbf{M}(\boldsymbol{\alpha})$  and  $\mathbf{K}(\boldsymbol{\alpha})$ . Thus, relying on Theorem 1, generalized expressions have been derived herein for the rate of change of the eigenvalues to be used in Eqs. (17) and (18) of the proposed asymptotic approximation methodology.

## 4 Numerical examples

In the following numerical examples, without loss of generality and for facilitating the derivation of compact closed-form expressions, the random vector  $\boldsymbol{\alpha} \in \mathbb{R}^k$  is modeled as a multivariate Gaussian distribution; that is, Eq. (2) takes the form

$$p_{\boldsymbol{\alpha}}(\boldsymbol{\alpha}) = (2\pi)^{-k/2} \|\boldsymbol{\Sigma}\|^{-1/2} \exp \left\{ -\frac{1}{2} (\boldsymbol{\alpha} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{\alpha} - \boldsymbol{\mu}) \right\}, \quad (38)$$

where  $\boldsymbol{\mu} \in \mathbb{R}^k$  and  $\boldsymbol{\Sigma} \in \mathbb{R}^{k \times k}$  denote, respectively, the mean vector and the covariance matrix. Next, utilizing Eq. (38),  $\mathbf{d}_f(\boldsymbol{\alpha})$  and  $\mathbf{D}_f(\boldsymbol{\alpha})$  become

$$\mathbf{d}_f(\boldsymbol{\alpha}) = \boldsymbol{\Sigma}^{-1}(\boldsymbol{\alpha} - \boldsymbol{\mu}), \quad \mathbf{D}_f(\boldsymbol{\alpha}) = \boldsymbol{\Sigma}^{-1}. \quad (39)$$

Further, considering Eq. (39) in conjunction with Eq. (17), and setting  $s_1 = s_2 = 1$ , the equation for determining the optimal point  $\boldsymbol{\alpha}_0$  becomes

$$\boldsymbol{\alpha}_0 = \boldsymbol{\mu} + \boldsymbol{\Sigma} \left( \frac{\mathbf{d}_{\tilde{\omega}_i}(\boldsymbol{\alpha}_0)}{\tilde{\omega}_i(\boldsymbol{\alpha}_0)} + \frac{\mathbf{d}_{\tilde{\omega}_j}(\boldsymbol{\alpha}_0)}{\tilde{\omega}_j(\boldsymbol{\alpha}_0)} \right), \quad (40)$$

which is solved iteratively in the ensuing analysis based on the following scheme proposed in [1]. The scheme is initiated by selecting an initial value for  $\boldsymbol{\alpha}_0 = \boldsymbol{\mu}$  and an error tolerance  $\varepsilon$ . Next, an updated value  $\boldsymbol{\alpha}_0^{upd}$  is computed by resorting to Eq. (40). If the criterion  $|\boldsymbol{\alpha}_0^{upd} - \boldsymbol{\alpha}_0| < \varepsilon$  is satisfied,  $\boldsymbol{\alpha}_0$  is set equal to  $\boldsymbol{\alpha}_0^{upd}$  and the iterative scheme

ends. In a different case,  $\boldsymbol{\alpha}_0 = \boldsymbol{\alpha}_0^{upd}$ , and the iterations continue until convergence. Following determination of the optimal point  $\boldsymbol{\alpha}_0$ , the joint moment of the natural frequencies  $\tilde{\omega}_i(\boldsymbol{\alpha})$ ,  $\tilde{\omega}_j(\boldsymbol{\alpha})$  is obtained from Eq. (16) as

$$\mu_{ij}^{(1,1)} \approx \tilde{\omega}_i(\boldsymbol{\alpha}_0)\tilde{\omega}_j(\boldsymbol{\alpha}_0) \exp \left\{ -\frac{1}{2}(\boldsymbol{\alpha}_0 - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\boldsymbol{\alpha}_0 - \boldsymbol{\mu}) \right\} \|\mathbf{I}_k + \mathbf{D}_g(\boldsymbol{\alpha}_0)\|^{-1/2}, \quad (41)$$

where

$$\mathbf{D}_g(\boldsymbol{\alpha}_0) = \mathbf{I}_k + \boldsymbol{\Sigma}^{-1} \left( \frac{\mathbf{d}_{\tilde{\omega}_i}(\boldsymbol{\alpha}_0)\mathbf{d}_{\tilde{\omega}_i}^T(\boldsymbol{\alpha}_0)}{\tilde{\omega}_i^2(\boldsymbol{\alpha}_0)} - \frac{\mathbf{D}_{\tilde{\omega}_i}(\boldsymbol{\alpha}_0)}{\tilde{\omega}_i(\boldsymbol{\alpha}_0)} + \frac{\mathbf{d}_{\tilde{\omega}_j}(\boldsymbol{\alpha}_0)\mathbf{d}_{\tilde{\omega}_j}^T(\boldsymbol{\alpha}_0)}{\tilde{\omega}_j^2(\boldsymbol{\alpha}_0)} - \frac{\mathbf{D}_{\tilde{\omega}_j}(\boldsymbol{\alpha}_0)}{\tilde{\omega}_j(\boldsymbol{\alpha}_0)} \right). \quad (42)$$

Note that for the evaluation of the mean of the natural frequencies, i.e.,  $s_1 = 1$ , Eq. (40) for the optimal point becomes

$$\boldsymbol{\alpha}_1 = \boldsymbol{\mu} + \boldsymbol{\Sigma} \frac{\mathbf{d}_{\tilde{\omega}_i}(\boldsymbol{\alpha}_1)}{\tilde{\omega}_i(\boldsymbol{\alpha}_1)}, \quad (43)$$

and the mean value is given by

$$E[\tilde{\omega}_i(\boldsymbol{\alpha})] = \tilde{\omega}_i(\boldsymbol{\alpha}_1) \exp \left\{ -\frac{1}{2}(\boldsymbol{\alpha}_1 - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\boldsymbol{\alpha}_1 - \boldsymbol{\mu}) \right\} \|\mathbf{I}_k + \mathbf{D}_g(\boldsymbol{\alpha}_1)\|^{-1/2}, \quad (44)$$

where

$$\mathbf{D}_g(\boldsymbol{\alpha}_1) = \mathbf{I}_k + \boldsymbol{\Sigma}^{-1} \left( \frac{\mathbf{d}_{\tilde{\omega}_i}(\boldsymbol{\alpha}_1)\mathbf{d}_{\tilde{\omega}_i}^T(\boldsymbol{\alpha}_1)}{\tilde{\omega}_i^2(\boldsymbol{\alpha}_1)} - \frac{\mathbf{D}_{\tilde{\omega}_i}(\boldsymbol{\alpha}_1)}{\tilde{\omega}_i(\boldsymbol{\alpha}_1)} \right). \quad (45)$$

#### 4.1 2-DOF system with singular matrices

The 2-DOF system in Fig. 1 is considered next, where mass  $m_1$  is connected to the foundation by a linear spring of stiffness coefficient  $k_1$  and to mass  $m_2$  by a linear spring of stiffness coefficient  $k_2$ . The equation of motion is given by Eq. (1) with

$$\mathbf{M}(\boldsymbol{\alpha}) = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad \mathbf{K}(\boldsymbol{\alpha}) = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}, \quad (46)$$

where the matrix elements  $m_i$ ,  $k_i$  are defined as functions of the random variable  $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{R}^4$ , which follows the Gaussian PDF of Eq. (38) with  $\boldsymbol{\mu} = \mathbf{0}$  and  $\boldsymbol{\Sigma} = \mathbf{I}_4$ . These are given by

$$m_i = \bar{m}_i (1 + \varepsilon_m \alpha_i), \quad k_i = \bar{k}_i (1 + \varepsilon_k \alpha_{2+i}), \quad i = 1, 2. \quad (47)$$

In Eq. (47), the mean parameter values  $\bar{m}_1 = 1.6$ ,  $\bar{m}_2 = 0.1$  and  $\bar{k}_1 = 1$ ,  $\bar{k}_2 = 0.11$  are considered with  $\varepsilon_m = 0.15$  and  $\varepsilon_k = 0.15$ .

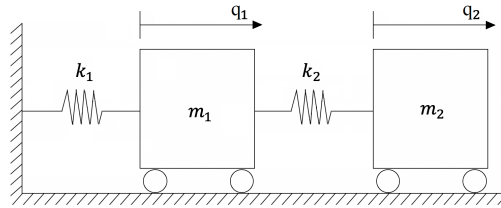


Fig. 1: A 2-DOF linear system with random parameter matrices.

Next, following the standard approach in [1], the mean values of the natural frequencies of the 2-DOF system in Fig. 1(a), as well as their covariance matrix are computed. The obtained results are shown in Table 1.

Further, the system equations of motion are formulated by adopting a redundant DOFs modeling approach (see also [14]). In this regard, considering the dependent coordinates vector  $\mathbf{x}^T = [x_1 \ x_2 \ x_3]$ , the system in Fig. 1 is decomposed into its constituent parts as shown in Fig. 2. The system governing equations of motion are given in the matrix form of Eq. (6), where the random mass and stiffness matrices become

$$\mathbf{M}_x(\boldsymbol{\alpha}) = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & m_2 \\ 0 & m_2 & m_2 \end{bmatrix}, \quad \mathbf{K}_x(\boldsymbol{\alpha}) = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & k_2 \end{bmatrix}. \quad (48)$$



Considering the constraint equations in Eq. (7), i.e.,  $x_2 = x_1 + d$ , where  $d$  denotes the physical length of mass  $m_1$ , leads to  $\mathbf{A} = [1 \ -1 \ 0]$  and  $\mathbf{L} = [0 \ 0 \ 0]$ . Therefore, Eq. (8) is formulated with

$$\bar{\mathbf{M}}_{\mathbf{x}}(\boldsymbol{\alpha}) = \begin{bmatrix} 0.5m_1 & 0.5m_2 & 0.5m_2 \\ 0.5m_1 & 0.5m_2 & 0.5m_2 \\ 0 & m_2 & m_2 \\ 1 & -1 & 0 \end{bmatrix}, \quad \bar{\mathbf{K}}_{\mathbf{x}}(\boldsymbol{\alpha}) = \begin{bmatrix} 0.5k_1 & 0 & 0 \\ 0.5k_1 & 0 & 0 \\ 0 & 0 & k_2 \\ 0 & 0 & 0 \end{bmatrix}. \quad (49)$$

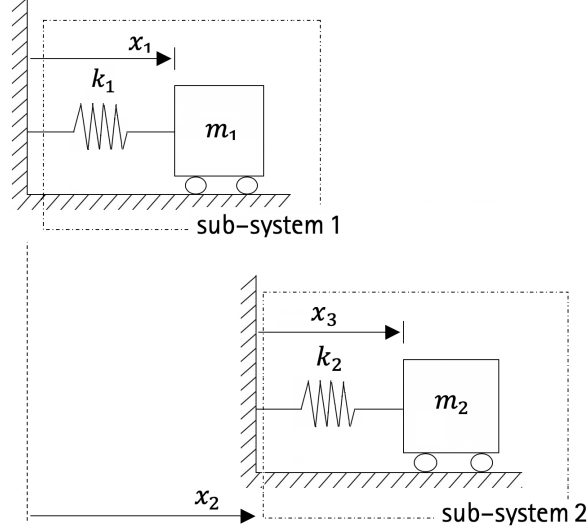


Fig. 2: The system in Fig. 1 modeled by adopting redundant (dependent) DOFs.

Obviously, matrices  $\bar{\mathbf{M}}_{\mathbf{x}}(\boldsymbol{\alpha})$  and  $\bar{\mathbf{K}}_{\mathbf{x}}(\boldsymbol{\alpha})$  in Eq. (49) are rectangular. In this regard, the generalized eigenvalue problem of Eq. (10) is formulated by removing the dependent rows from the matrices in Eq. (49). Also, considering Eq. (47), Eq. (11) yields

$$\tilde{\mathbf{M}}(\boldsymbol{\alpha}) = \begin{bmatrix} 0.5\bar{m}_1(1 + \varepsilon_m\alpha_1) & 0.5\bar{m}_2(1 + \varepsilon_m\alpha_2) & 0.5\bar{m}_2(1 + \varepsilon_m\alpha_2) \\ 0 & \bar{m}_2(1 + \varepsilon_m\alpha_2) & \bar{m}_2(1 + \varepsilon_m\alpha_2) \\ 1 & -1 & 0 \end{bmatrix} \quad (50)$$

and

$$\tilde{\mathbf{K}}(\boldsymbol{\alpha}) = \begin{bmatrix} 0.5\bar{k}_1(1 + \varepsilon_k\alpha_3) & 0 & 0 \\ 0 & 0 & \bar{k}_2(1 + \varepsilon_k\alpha_4) \\ 0 & 0 & 0 \end{bmatrix}. \quad (51)$$

Next, the generalized expressions given by Eqs. (31) and (37) for the rate of change of natural frequencies of systems with singular and asymmetric matrices are used for determining the mean values of the natural frequencies, as well as the corresponding covariance matrix. In this regard, the eigenvalue problem defined by Eq. (10) yields the right eigenvectors  $\tilde{\boldsymbol{\phi}}_i(\boldsymbol{\alpha})$  ( $i = 1, 2, 3$ ) of the system shown in Fig. 2, whereas the corresponding left eigenvectors  $\tilde{\boldsymbol{\psi}}_i(\boldsymbol{\alpha})$  are determined by resorting to the eigenvalue problem in Eq. (24) in conjunction with Eq. (25). The 3-DOF system in Fig. 2 has three distinct eigenvalues, one of which is equal to zero due to employing an additional redundant DOF for modeling the governing equations of motion. The left and right eigenvectors corresponding to the zero eigenvalue are placed last in order when formulating matrices  $\tilde{\boldsymbol{\Phi}}(\boldsymbol{\alpha})$  and  $\tilde{\boldsymbol{\Psi}}(\boldsymbol{\alpha})$ . Further, the derivatives of the random matrices in Eqs. (50) and (51) are required for evaluating Eqs. (31) and (37). Therefore, differentiating Eq. (50) with respect to the random variable  $\boldsymbol{\alpha} \in \mathbb{R}^4$  yields

$$\frac{\partial \tilde{\mathbf{M}}(\boldsymbol{\alpha})}{\partial \alpha_1} = 0.5\varepsilon_m\bar{m}_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \frac{\partial \tilde{\mathbf{M}}(\boldsymbol{\alpha})}{\partial \alpha_2} = 0.5\varepsilon_m\bar{m}_2 \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad (52)$$

and  $\frac{\partial \tilde{\mathbf{M}}(\boldsymbol{\alpha})}{\partial \alpha_i} = \mathbf{0}_{3 \times 3}$ , for  $i = 3, 4$ . Similarly, Eq. (51) yields  $\frac{\partial \tilde{\mathbf{K}}(\boldsymbol{\alpha})}{\partial \alpha_i} = \mathbf{0}_{3 \times 3}$ , for  $i = 1, 2$ , whereas

$$\frac{\partial \tilde{\mathbf{K}}(\boldsymbol{\alpha})}{\partial \alpha_3} = 0.5\varepsilon_k\bar{k}_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \frac{\partial \tilde{\mathbf{K}}(\boldsymbol{\alpha})}{\partial \alpha_4} = 0.5\varepsilon_k\bar{k}_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \quad (53)$$

Resorting to Eqs. (41) and (44), the mean vector and covariance matrix of the non-zero natural frequencies of the system depicted in Fig. 2 are determined and compared in Table 1 with the respective ones obtained based on the standard approach in [1] corresponding to the eigenvalue problem of Eq. (5). It can be readily seen the results from the two approaches are in very good agreement with each other, whereas comparisons with pertinent MCS data (20,000 samples) demonstrate the high accuracy degree of the methodology.

Tab. 1: Mean vector and covariance matrix of the natural frequencies corresponding to systems shown in Figs. 1 and 2. Results obtained by applying both the standard and the generalized approaches are compared with MCS data (20,000 samples).

-	$\mu_{\tilde{\omega}_1}$	$\mu_{\tilde{\omega}_2}$	$\text{COV}_{\tilde{\omega}_1, \tilde{\omega}_1}$	$\text{COV}_{\tilde{\omega}_2, \tilde{\omega}_2}$	$\text{COV}_{\tilde{\omega}_1, \tilde{\omega}_2}$
standard approach	1.1261	0.7425	0.0109	0.0045	0.0029
generalized approach	1.1176	0.7476	0.0111	0.0048	0.0025
MCS	1.1273	0.7411	0.0108	0.0046	0.0028

## 4.2 3-DOF system with singular matrices

In this section, the 3-DOF system in Fig. 3 is considered for demonstrating that the proposed solution framework can be used also for treating classes of systems with repeated zero eigenvalues, as well as cases pertaining to systems with closely spaced eigenvalues.

In this regard, mass  $m_1$  is connected to the foundation by a linear spring of stiffness coefficient  $k_1$  and to masses  $m_2$  and  $m_3$  by linear springs of stiffness coefficients  $k_2$  and  $k_4$ , respectively. Further, mass  $m_2$  is connected to mass  $m_3$  by a linear spring of stiffness coefficient  $k_3$ . The system governing equation of motion is given by Eq. (1) with

$$\mathbf{M}(\boldsymbol{\alpha}) = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}, \quad \mathbf{K}(\boldsymbol{\alpha}) = \begin{bmatrix} k_1 + k_2 + k_4 & -k_2 & -k_4 \\ -k_2 & k_2 + k_3 & -k_3 \\ -k_4 & -k_3 & k_3 + k_4 \end{bmatrix}, \quad (54)$$

where

$$m_i = \bar{m}_i (1 + \varepsilon_m \alpha_i), \quad i = 1, 2, 3, \quad k_j = \bar{k}_j (1 + \varepsilon_k \alpha_{3+j}), \quad j = 1, 2, 3, 4. \quad (55)$$

The matrix elements in Eq. (55) are functions of the random variable  $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_7) \in \mathbb{R}^7$ , which follows the Gaussian PDF of Eq. (38) with  $\boldsymbol{\mu} = \mathbf{0}$  and  $\boldsymbol{\Sigma} = \mathbf{I}_7$ . Moreover, the mean parameter values  $\bar{m}_1 = \bar{m}_3 = 4$ ,  $\bar{m}_2 = 1$ ,  $\bar{k}_1 = \bar{k}_2 = \bar{k}_3 = 1$ ,  $\bar{k}_4 = 0.8$  are considered in conjunction with  $\varepsilon_m = 0.15$  and  $\varepsilon_k = 0.2$ .

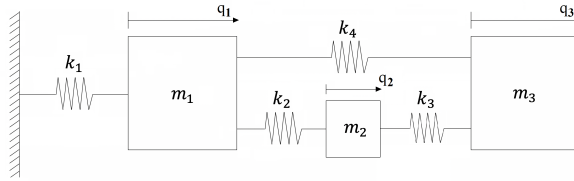


Fig. 3: A 3-DOF linear system with random parameter matrices.

Following the standard approach in [1], the mean values of the system natural frequencies and the corresponding covariance matrix are computed. The obtained results are shown in Tables 2 and 3.

Next, a redundant coordinates modeling for the system in Fig. 3 is considered (e.g., [25]) based on the 5-dimensional (dependent) coordinates vector  $\mathbf{x}^T = [x_1 \ x_2 \ x_3 \ x_4 \ x_5]$  and the system in Fig. 3 is decomposed into its constituent parts as shown in Fig. 4. The system governing equations of motion are written in the matrix form of Eq. (6), where

$$\mathbf{M}_{\mathbf{x}}(\boldsymbol{\alpha}) = \begin{bmatrix} m_1 & 0 & 0 & 0 & 0 \\ 0 & m_2 & m_2 & 0 & 0 \\ 0 & m_2 & m_2 & 0 & 0 \\ 0 & 0 & 0 & m_3 & m_3 \\ 0 & 0 & 0 & m_3 & m_3 \end{bmatrix}, \quad \mathbf{K}_{\mathbf{x}}(\boldsymbol{\alpha}) = \begin{bmatrix} k_1 & 0 & 0 & 0 & 0 \\ 0 & k_4 & 0 & -k_4 & -k_4 \\ 0 & 0 & k_2 & 0 & 0 \\ 0 & -k_4 & 0 & k_4 & k_4 \\ 0 & -k_4 & 0 & k_4 & k_3 + k_4 \end{bmatrix}. \quad (56)$$

Further, considering the constraint equations in Eq. (7), i.e.,  $x_2 = x_1 + d_1$  and  $x_4 = x_2 + x_3 + d_2$ , where  $d_i$  ( $i = 1, 2$ ) denotes the length of mass  $m_i$ , the  $2 \times 5$  matrices  $\mathbf{A}$  and  $\mathbf{L}$  become

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \end{bmatrix}, \quad \mathbf{E} = \mathbf{0}_{2 \times 5}, \quad (57)$$

and thus, the  $7 \times 5$  matrices  $\bar{\mathbf{M}}_{\mathbf{x}}(\boldsymbol{\alpha})$  and  $\bar{\mathbf{K}}_{\mathbf{x}}(\boldsymbol{\alpha})$  in Eq. (8) take the form

$$\bar{\mathbf{M}}_{\mathbf{x}}(\boldsymbol{\alpha}) = \begin{bmatrix} 0.4m_1 & 0.2m_2 & 0.2m_2 & 0.2m_3 & 0.2m_3 \\ 0.4m_1 & 0.2m_2 & 0.2m_2 & 0.2m_3 & 0.2m_3 \\ -0.2m_1 & 0.4m_2 & 0.4m_2 & 0.4m_3 & 0.4m_3 \\ 0.2m_1 & 0.6m_2 & 0.6m_2 & 0.6m_3 & 0.6m_3 \\ 0 & 0 & 0 & m_3 & m_3 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \end{bmatrix} \quad (58)$$

and

$$\bar{\mathbf{K}}_{\mathbf{x}}(\boldsymbol{\alpha}) = \begin{bmatrix} 0.4k_1 & 0.2k_4 & -0.2k_2 & -0.2k_4 & -0.2k_4 \\ 0.4k_1 & 0.2k_4 & -0.2k_2 & -0.2k_4 & -0.2k_4 \\ -0.2k_1 & -0.6k_4 & 0.6k_2 & 0.6k_4 & 0.6k_4 \\ 0.2k_1 & -0.4k_4 & 0.4k_2 & 0.4k_4 & 0.4k_4 \\ 0 & -1.0k_4 & 0 & k_4 & k_3 + k_4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (59)$$

respectively.

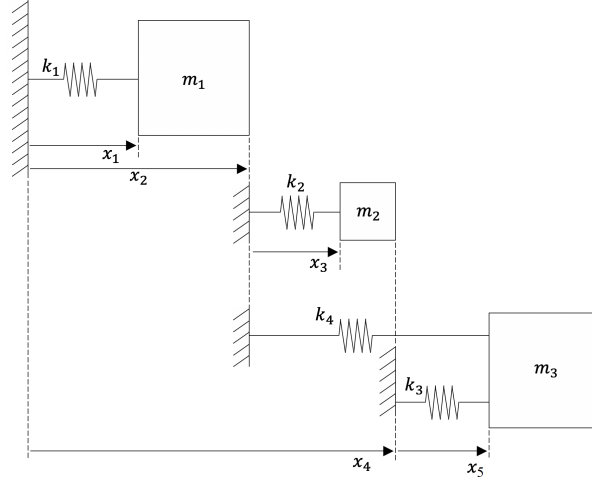


Fig. 4: The system in Fig. 3 modeled by adopting redundant (dependent) DOFs.

Next, directing attention to formulating the generalized eigenvalue problem defined in Eq. (10), the dependent rows in matrices  $\bar{\mathbf{M}}_{\mathbf{x}}(\boldsymbol{\alpha})$  and  $\bar{\mathbf{K}}_{\mathbf{x}}(\boldsymbol{\alpha})$  are removed, and Eq. (11) yields

$$\tilde{\mathbf{M}}(\boldsymbol{\alpha}) = \begin{bmatrix} 0.4m_1 & 0.2m_2 & 0.2m_2 & 0.2m_3 & 0.2m_3 \\ -0.2m_1 & 0.4m_2 & 0.4m_2 & 0.4m_3 & 0.4m_3 \\ 0 & 0 & 0 & m_3 & m_3 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 \end{bmatrix} \quad (60)$$

and

$$\tilde{\mathbf{K}}(\boldsymbol{\alpha}) = \begin{bmatrix} 0.4k_1 & 0.2k_4 & -0.2k_2 & -0.2k_4 & -0.2k_4 \\ -0.2k_1 & -0.6k_4 & 0.6k_2 & 0.6k_4 & 0.6k_4 \\ 0 & -1.0k_4 & 0 & k_4 & k_3 + k_4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (61)$$

Similarly to the analysis pertaining to the 2-DOF example, the 5-DOF system in Fig. 4 has three distinct eigenvalues, coinciding with the ones corresponding to the original system in Fig. 3, and two eigenvalues equal to zero due to redundant coordinates modeling. In this regard, the matrices  $\tilde{\Phi}(\boldsymbol{\alpha})$  and  $\tilde{\Psi}(\boldsymbol{\alpha})$  are formed such that the eigenvectors corresponding to zero eigenvalues are placed last in order. Further, to determine the expressions given by Eqs. (31) and (37) for the rate of change of natural frequencies, first, the random matrix in Eq. (60) is differentiated with respect to the random variable  $\boldsymbol{\alpha} \in \mathbb{R}^7$ , yielding

$$\frac{\partial \tilde{\mathbf{M}}(\boldsymbol{\alpha})}{\partial \alpha_1} = \varepsilon_m \bar{m}_1 \begin{bmatrix} 0.4 & 0 & 0 & 0 & 0 \\ -0.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \frac{\partial \tilde{\mathbf{M}}(\boldsymbol{\alpha})}{\partial \alpha_2} = \varepsilon_m \bar{m}_2 \begin{bmatrix} 0 & 0.2 & 0.2 & 0 & 0 \\ 0 & 0.4 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (62)$$

$$\frac{\partial \tilde{\mathbf{M}}(\boldsymbol{\alpha})}{\partial \alpha_3} = \varepsilon_m \bar{m}_3 \begin{bmatrix} 0 & 0 & 0 & 0.2 & 0.2 \\ 0 & 0 & 0 & 0.4 & 0.4 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad (63)$$

and  $\frac{\partial \tilde{\mathbf{M}}(\boldsymbol{\alpha})}{\partial \alpha_i} = \mathbf{0}_{5 \times 5}$ , for  $i = 4, 5, 6, 7$ . Similarly, taking into account Eq. (61) leads to  $\frac{\partial \tilde{\mathbf{K}}(\boldsymbol{\alpha})}{\partial \alpha_i} = \mathbf{0}_{5 \times 5}$ , for  $i = 1, 2, 3$ ,

$$\frac{\partial \tilde{\mathbf{K}}(\boldsymbol{\alpha})}{\partial \alpha_4} = \varepsilon_k \bar{k}_1 \begin{bmatrix} 0.4 & 0 & 0 & 0 & 0 \\ -0.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \frac{\partial \tilde{\mathbf{K}}(\boldsymbol{\alpha})}{\partial \alpha_5} = \varepsilon_k \bar{k}_2 \begin{bmatrix} 0 & 0 & -0.2 & 0 & 0 \\ 0 & 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (64)$$

and

$$\frac{\partial \tilde{\mathbf{K}}(\boldsymbol{\alpha})}{\partial \alpha_6} = \varepsilon_k \bar{k}_3 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \frac{\partial \tilde{\mathbf{K}}(\boldsymbol{\alpha})}{\partial \alpha_7} = \varepsilon_k \bar{k}_4 \begin{bmatrix} 0 & 0.2 & 0 & -0.2 & -0.2 \\ 0 & -0.6 & 0 & 0.6 & 0.6 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (65)$$

Then, utilizing Eqs. (41) and (44), the mean values of the non-zero natural frequencies of the system in Fig. 4 and the corresponding covariance matrix are computed. The results are shown in Tables 2 and 3, and compared with estimates obtained by the standard approach and by pertinent MCS data (20,000 samples). It is readily seen that the methodology exhibits a high accuracy degree. Furthermore, the correlation coefficients corresponding to the eigenvalues are calculated as

$$\boldsymbol{\rho}_{\bar{\omega}_j} = \begin{bmatrix} 1.0000 & 0.2502 & 0.1416 \\ 0.2502 & 1.0000 & 0.6371 \\ 0.1416 & 0.6371 & 1.0000 \end{bmatrix}. \quad (66)$$

As anticipated, it is seen that the eigenvalues that are closer to each other (e.g.,  $\omega_2$  and  $\omega_3$ ) are more statistically correlated than the ones that are further apart (e.g.,  $\omega_1$  and  $\omega_3$ ).

Finally, the reliability of the proposed methodology with respect to cases of closely spaced eigenvalues is investigated next. In particular, the mean parameter values  $\bar{m}_1 = \bar{m}_3 = 5$ ,  $\bar{m}_2 = 1$ ,  $\bar{k}_1 = \bar{k}_2 = \bar{k}_3 = 1$ ,  $\bar{k}_4 = 3.5$  in conjunction with  $\varepsilon_m = 0.15$  and  $\varepsilon_k = 0.2$  are considered. First, the mean values and the covariance matrix of the natural frequencies of the system in Fig. 3 are computed by applying the standard approach in [1]. The results are shown in Tables 4 and 5, where it is seen that the first and second eigenvalues of the system are relatively close to each other. Next, applying the herein developed methodology, and utilizing Eqs. (41) and (44), leads to the computation of the mean values and covariance matrix of the non-zero natural frequencies of the system in Fig. 4. These results are shown in Tables 4 and 5, respectively, whereas the correlation coefficients are calculated as

$$\boldsymbol{\rho}_{\bar{\omega}_j} = \begin{bmatrix} 1.0000 & 0.0976 & 0.0668 \\ 0.0976 & 1.0000 & 0.3590 \\ 0.0668 & 0.3590 & 1.0000 \end{bmatrix}. \quad (67)$$

In general, it is seen that the natural frequencies are only moderately correlated. Also, the correlation between  $\omega_1$  and  $\omega_2$  is higher than that between  $\omega_1$  and  $\omega_3$ . This is anticipated since  $\omega_1$  and  $\omega_3$  are further apart than  $\omega_1$  and  $\omega_2$ . However, the correlation between  $\omega_2$  and  $\omega_3$  is higher than that between  $\omega_1$  and  $\omega_2$  in spite of  $\omega_1$  being closer to  $\omega_2$  compared to  $\omega_3$ . Comparisons with pertinent MCS data (20,000 samples) are included as well, demonstrating that

the proposed methodology is capable of determining joint statistics of systems with singular parameter matrices and with closely spaced eigenvalues with a reasonable degree of accuracy. Of course, it is noted that, similarly to the standard approach by [1], the exhibited accuracy is unavoidably affected by the saddle point approximation of Eq. (13), and worsens the more closely spaced the eigenvalues are. This can be seen by comparing the accuracy degree of the results in Tables 2 and 3, with the accuracy degree of the estimates in Tables 4 and 5.

Tab. 2: Mean vector of the natural frequencies corresponding to systems shown in Figs. 3 and 4. Results obtained by applying both the standard and the generalized approaches are compared with MCS data (20,000 samples).

-	$\mu_{\tilde{\omega}_1}$	$\mu_{\tilde{\omega}_2}$	$\mu_{\tilde{\omega}_3}$
standard approach	1.5189	0.8940	0.2988
generalized approach	1.5118	0.8936	0.3008
MCS	1.5187	0.8934	0.2985

Tab. 3: Covariance matrix of the natural frequencies corresponding to systems shown in Figs. 3 and 4. Results obtained by applying both the standard and the generalized approaches are compared with MCS data (20,000 samples).

-	$\text{COV}_{\tilde{\omega}_1, \tilde{\omega}_1}$	$\text{COV}_{\tilde{\omega}_2, \tilde{\omega}_2}$	$\text{COV}_{\tilde{\omega}_3, \tilde{\omega}_3}$	$\text{COV}_{\tilde{\omega}_1, \tilde{\omega}_2}$	$\text{COV}_{\tilde{\omega}_1, \tilde{\omega}_3}$	$\text{COV}_{\tilde{\omega}_2, \tilde{\omega}_3}$
standard approach	0.0224	0.0054	0.0009	0.0029	0.0007	0.0013
generalized approach	0.0224	0.0052	0.0008	0.0027	0.0006	0.0013
MCS	0.0228	0.0054	0.0009	0.0030	0.0007	0.0013

Tab. 4: Mean vector of the natural frequencies corresponding to systems shown in Figs. 3 and 4 for the set of mean parameter values  $\bar{m}_1 = \bar{m}_3 = 5$ ,  $\bar{m}_2 = 1$ ,  $\bar{k}_1 = \bar{k}_2 = \bar{k}_3 = 1$ ,  $\bar{k}_4 = 3.5$ . Results obtained by applying both the standard and the generalized approaches are compared with MCS data (20,000 samples).

-	$\mu_{\tilde{\omega}_1}$	$\mu_{\tilde{\omega}_2}$	$\mu_{\tilde{\omega}_3}$
standard approach	1.5061	1.3032	0.2905
generalized approach	1.4940	1.3066	0.2913
MCS	1.5242	1.2865	0.2904

Tab. 5: Covariance matrix of the natural frequencies corresponding to systems shown in Figs. 3 and 4 for the set of mean parameter values  $\bar{m}_1 = \bar{m}_3 = 5$ ,  $\bar{m}_2 = 1$ ,  $\bar{k}_1 = \bar{k}_2 = \bar{k}_3 = 1$ ,  $\bar{k}_4 = 3.5$ . Results obtained by applying both the standard and the generalized approaches are compared with MCS data (20,000 samples).

-	$\text{COV}_{\tilde{\omega}_1, \tilde{\omega}_1}$	$\text{COV}_{\tilde{\omega}_2, \tilde{\omega}_2}$	$\text{COV}_{\tilde{\omega}_3, \tilde{\omega}_3}$	$\text{COV}_{\tilde{\omega}_1, \tilde{\omega}_2}$	$\text{COV}_{\tilde{\omega}_1, \tilde{\omega}_3}$	$\text{COV}_{\tilde{\omega}_2, \tilde{\omega}_3}$
standard approach	0.0208	0.0152	0.0010	0.0040	0.0005	0.0013
generalized approach	0.0224	0.0169	0.0009	0.0019	0.0003	0.0014
MCS	0.0188	0.0138	0.0010	0.0058	0.0007	0.0012

## 5 Concluding remarks

In this paper, the asymptotic approximation methodology developed in [1] for solving standard random eigenvalue problems has been generalized for determining joint moments of natural frequencies corresponding to systems with singular random parameter matrices. This has been done by resorting to the concept of Moore-Penrose matrix inverse and by generalizing expressions for the rate of change of eigenvalues to account for asymmetric matrices. Two indicative examples pertaining to MDOF structural systems have been considered for demonstrating the reliability of the methodology. It has been shown that the eigenvalue statistics obtained by the herein developed methodology, and pertaining to system modeling based on dependent coordinates, agree very well with the respective ones corresponding to the conventionally formulated random eigenvalue problem. In this regard, the proposed framework can be construed as a generalization of the approach in [1] to treat systems with singular random parameter matrices. Also, comparisons with relevant MCS data demonstrate a high degree of accuracy.

## 6 Data Availability Statement

All data, models, or code that support the findings of this study are available from the corresponding author upon reasonable request.

## 7 Acknowledgments

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## 8 APPENDIX I. Moore-Penrose matrix inverse

Consider a linear system of equations in the form  $\mathbf{Ax} = \mathbf{b}$ , where  $\mathbf{A}$  is either a rectangular  $m \times n$ , or a square but singular  $n \times n$  matrix, and  $\mathbf{x}$ ,  $\mathbf{b}$  are  $n$ -dimensional vectors. It is readily seen that solving  $\mathbf{Ax} = \mathbf{b}$  necessitates the generalization of the concept of matrix inverse, which has given birth to the theory of generalized matrix inverses [6]. In particular, the Moore-Penrose generalized matrix inverse is utilized throughout the paper.

**Definition 4.** For any matrix  $\mathbf{A} \in \mathbb{C}^{m \times n}$ , there is a unique matrix  $\mathbf{A}^+ \in \mathbb{C}^{n \times m}$  such that:

(i)  $AA^+A = A$ , (ii)  $A^+AA^+ = A^+$ , (iii)  $(AA^+)^* = AA^+$ , (iv)  $(A^+A)^* = A^+A$ .

The matrix  $\mathbf{A}^+$  in Definition 4 is called the Moore-Penrose inverse of  $\mathbf{A}$ . If  $\mathbf{A}$  is a square, real and non-singular matrix, then  $\mathbf{A}^+$  coincides with the inverse of  $\mathbf{A}$ , i.e.,  $\mathbf{A}^+ = \mathbf{A}^{-1}$ . Using the Moore-Penrose inverse, a closed form solution to the algebraic system  $\mathbf{Ax} = \mathbf{b}$  is attained, which for any matrix  $\mathbf{A} \in \mathbb{R}^{m \times n}$  is given by

$$\mathbf{x} = \mathbf{A}^+\mathbf{b} + (\mathbf{I}_n - \mathbf{A}^+\mathbf{A})\mathbf{y}. \quad (68)$$

In Eq. (68),  $\mathbf{y}$  denotes an arbitrary  $n$ -dimensional vector and  $\mathbf{I}_n$  represents the  $n \times n$  identity matrix. A more detailed presentation of the topic can be found in [6].

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