A CLASS OF QUADRATIC FORWARD-BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. In this work, we study a class of quadratic forward backward stochastic differential equations (QFBSDEs) with measurable drift and continuous generator. We establish some existence and uniqueness results for such QFBSDEs. Our approach is based on a weak decoupling field and an Itô-Krylov formula for BSDE. In the one dimensional case, we derive existence of a unique strong solution. Moreover, assuming that the diffusion of the forward system is the identity matrix, we also obtain the Malliavin differentiability of the solution to the QFBSDE and derive the dependence of the system with respect to the initial parameter.

1. Introduction

Consider the following stochastic basis $(\Omega, \mathfrak{F}, \{\mathfrak{F}_t\}_{t\geqslant 0}, \mathbb{P}, B_t)$, where $\{\mathfrak{F}_t\}_{t\geqslant 0}$ is the standard filtration generated by the d-dimensional Brownian motion B_t, augmented by all $\mathbb{P}$-null sets of $\mathfrak{F}$. In this paper, we aim at studying the solvability of the following fully coupled Markovian-type quadratic forward-backward stochastic differential equation (QFBSDE)

$$\begin{cases} X_s^{t,x} = x + \int_t^s b(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\mathrm{d}r + \int_t^s \sigma(r, X_r^{t,x}, Y_r^{t,x})\mathrm{d}B_r, \\ Y_s^{t,x} = \phi(X_T^{t,x}) + \int_s^T g(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\mathrm{d}r - \int_s^T Z_r^{t,x}\sigma^*(r, X_r^{t,x}, Y_r^{t,x})\mathrm{d}B_r, \\ \forall (s,x) \in [t,T] \times \mathbb{R}^d, \end{cases} \tag{1.1}$$

where the coefficients b, f, σ, ϕ and g are such that for almost every (t,x), the driver g is continuous in (y,z) and satisfies

$$|g(t,x,y,z)| \leq \Lambda(1 + |y| + f(y)|z|^2), \qquad \Lambda > 0. \tag{1.2}$$

The superscript (t,x) in (1.1) denotes the initial condition of the diffusion X whereas σ^* stands for the transpose of the matrix σ.

In the framework of BSDEs, it is known that solving the QFBSDE (1.1) is equivalent to looking for a *"decoupled field"* $u(t,x)$ such that the relation (2.1) (see Section 2.1) holds, with u, solution (in some sense) to the following quasi-linear parabolic PDE

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) + \frac{1}{2}\sum_{i,j}^d a_{ij}(t,x,u(t,x))\frac{\partial^2 u}{\partial x_i \partial x_j}(t,x) + \sum_i^d b_i(t,x,u(t,x),\nabla_x u(t,x))\frac{\partial u}{\partial x_i}(t,x) \\ \qquad + g(t,x,u(t,x),\nabla_x u(t,x)) = 0, \quad (t,x) \in [0,T[\times\mathbb{R}^d, \\ u(T,x) = \phi(x), \quad x \in \mathbb{R}^d. \end{cases} \tag{1.3}$$

Quadratic forward backward SDEs with $b \equiv b(t,x)$ and $\sigma \equiv \sigma(t,x)$ Lipschitz continuous and f constant in (1.2) were first studied in [17]. Viscosity and Sobolev solutions to the associated PDE was studied to justify the relation between the solutions to (1.1) and (1.3) (see (2.1)). When the driver is non-smooth, the authors in [5] introduced the notion of weak solution of BSDE. In the same spirit, the notion of weak solution of the fully-coupled QFBSDEs (1.1) was introduced in [8]. This work extends the work [17] to the setting of FBSDE. In additions, the authors assumed that the diffusion matrix and the final condition are Hölder continuous with respect to space

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whereas the drift and the generator may be discontinuous in second variable. A uniqueness in law result was established by using a decoupling strategy in the *four step scheme*. The authors in [25] assumed that the coefficients are all bounded, continuous in (t, x) and uniformly continuous in (y, z). There has been a lot of work on FBSDEs with the goal of proving existence and uniqueness result under weaker conditions of the coefficients; we refer the reader to [1, 6, 20, 23, 24, 35, 37] and references therein. A new class of FBSDE with distributional drift was also studied in [13].

The notion of FBSDEs is an essential tool in the study of stochastic optimal control problems and stochastic differential games. This is mainly due to the fact that when solving an optimal control problem using the Pontryagin's stochastic maximum principle, the optimal control can be defined in terms of the state process and the adjoint equation which itself is characterised by a BSDE. Other applications include utility maximisation and probabilistic approach to quasilinear parabolic partial differential questions. We refer the reader to [6, 16, 11, 12, 26, 32, 30, 34] and references therein.

In this paper, we aim at studying the solvability of the QFBSDE (1.1) for a class of functions f satisfying some conditions to be made more precise. To the best of our knowledge, such class of FBSDE has not yet been solved in the literature. Our approach to solve this equation relies on decoupled field argument. Firstly we show as in [8] that the QFBSDE (1.1) has a weak solution which is weakly unique. This is done as follows: we derive a-priori bounds to the solution of the regularized version of PDE (1.3), as well as all its derivatives. This allows us to find a solution to the PDE (1.3) via compactness arguments and then prove existence and uniqueness solution to (1.1) by using (2.1).

Let us observe that one of the main ingredients to derive a-priori bounds and to prove the uniqueness in [8], consists in applying the Itô formula to an appropriate $C^2(\mathbb{R})$-function in order to get rid of the quadratic growth on z. We use an analogous approach in this work. Since the function f in (1.2) is not constant the exponential transformation used in [8] is not suitable. We then consider the one-to-one function Φ_f (respectively Ψ_f) defined in Lemma B.1 (respectively Lemma B.2). The function Φ_f (respectively Ψ_f) is not a $C^2(\mathbb{R})$-function, since f is not assumed to be continuous, thus the classical Backward-Itô formula fails. Nevertheless we can apply the Itô-Krylov change of variables for BSDE introduced in [4, Theorem 2.1] to overcome this situation. It is important to mention that the condition on f does not include the class of constant f. This case is under investigation in our companion paper [36] and constitutes an improvement of the existence and uniqueness results in [8].

Secondly, in the one-dimensional setting, assuming that the diffusion is globally Lipschitz continuous in y and Hölder continuous in x with Hölder exponent bigger than $\frac{1}{2}$, we show that pathwise uniqueness holds for the system and hence the existence of a unique strong solution to (1.1) (see [2]). Such result is surprising in the theory of forward-backward SDEs, due to the very mild conditions imposed on the coefficients. The strategy used in this paper is based on an occupation time formula (see [33, 31]).

In the case of multidimensional SDE with diffusion coefficient reduced to identity matrix, we investigate the smoothness of the unique strong solution to the QFBSDE in the Malliavin sense. We show that the solution is Malliavin differentiable in $[0, T-\delta]$ for all $\delta > 0$ such that $\delta < T$ (compare with [22]). There are several works dealing with Malliavin differentiablility of QBSDE (see for example [1]). Our approach in showing the Malliavin differentiability of the QFBSDE is different from the ones in the existing literature on FBSDEs. Since the drift of the forward equation is irregular, we use both a decoupling field and an approximation arguments to show that the solution of the forward equation is Malliavin differentiable. The Malliavin differentiability of the BSDE then follows.

Let us now introduce some notations for later use. For $T > 0$, fixed, $d \in \mathbb{N}\backslash\{0\}$, $p \in [1, \infty)$, we denote by:

- $W^{1,2,d+1}_{\text{loc}}([0, T[\times\mathbb{R}^d, \mathbb{R})$ the Sobolev space of classes of functions u such that u : $[0, T[\times\mathbb{R}^d \to \mathbb{R}$, $|u|, |\partial_t u|, |\nabla_x u|, |\nabla^2_{x,x} u| \in L^{d+1}_{\text{loc}}([0, T[\times\mathbb{R}^d, \times\mathbb{R})$;
- $\mathcal{S}^p(\mathbb{R}^d)$ the space of continuous $\{\mathfrak{F}_s\}_{t\leq s\leq T}$-adapted $\mathbb{R}^d$-valued processes X such that $\|X\|^p_{\mathcal{S}^p(\mathbb{R}^d)} := \mathbb{E}\sup_{s\in[t,T]}|X_s|^p < \infty$,

- $\mathcal{H}^p(\mathbb{R}^d)$ the space of $\{\mathfrak{F}_s\}_{t\leq s\leq T}$-progressively measurable $\mathbb{R}^d$-valued processes Z such that $\|Z\|^p_{\mathcal{H}^p(\mathbb{R}^d)} := \mathbb{E}(\int_t^T |Z_s|^2\mathrm{d}s)^{p/2} < \infty$.
- $\mathcal{S}^\infty(\mathbb{R}^d)$ the space of continuous $\{\mathfrak{F}_s\}_{t\leq s\leq T}$-adapted processes $Y : \Omega\times[t,T]\to\mathbb{R}^d$ such that $\|Y\|_\infty := \mathrm{essup}_{\omega\in\Omega}\sup_{s\in[t,T]}|Y_s| < \infty, \forall t\in[0,T]$.
- $BMO(\mathbb{P})$ the space of continuous square integrable martingales M with $M_0 = 0$ such that $\|M\|_{BMO(\mathbb{P})} = \sup_\tau \|\mathbb{E}[\langle M\rangle_T - \langle M\rangle_\tau/\mathfrak{F}_\tau]\|_\infty^{1/2} < \infty$, where the supremum is taken over all stopping times $\tau\in[0,T]$;
- $\mathcal{H}_{BMO}$ the space of $\mathbb{R}^d$- valued $\mathcal{H}^p$-integrable processes $(Z_s)_{s\in[0,T]}$ for all $p\geq 2$ such that $\int_0 Z_s\mathrm{d}B_s \in BMO(\mathbb{P})$. We define $\|Z\|_{\mathcal{H}_{BMO}} := \|\int Z\mathrm{d}B\|_{BMO(\mathbb{P})}$.

For a function u we define $\mathrm{osc}(u)$ by $\mathrm{osc}(u) := \max(u) - \min(u)$.

The main assumptions of this work are the following

Assumption 1.1. *The coefficients* $b : [0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d \to \times\mathbb{R}^d$, $g : [0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\to\times\mathbb{R}$, $\sigma : [0,T]\times\mathbb{R}^d\times\mathbb{R}\to\times\mathbb{R}^{d\times d}$ *and* $\phi : \mathbb{R}^d\times\mathbb{R}$ *are Borel measurable functions and satisfy: There exist constants* $\Lambda, K\ K_0, \lambda > 0$ *such that*

(H1) $\forall t\in[0,T], \forall(x,y,z)\in\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^{d\times d}$

$$|b(t,x,y,z)| \leqslant \Lambda(1+|y|), \quad |\sigma(t,x,y)| \leqslant \Lambda(1+|y|),$$
$$|g(t,x,y,z)| \leqslant \Lambda(1+|y|+f(y)|z|^2), \quad |\phi(x)|\leq\Lambda,$$

where $f : \mathbb{R}\mapsto\mathbb{R}_+$, *is locally bounded and globally integrable on* $\mathbb{R}$ *and such that* $f(y)\leq f(|y|)$ *for* $y\in\mathbb{R}$.

(H2) $\forall\xi\in\mathbb{R}^d$, $\langle\xi, a(t,x,y)\xi\rangle \geqslant \lambda|\xi|^2, \quad a = \sigma\sigma^*$.

(H3) *For all* $(t,x,y,z),(t,x,y',z')\in[0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^{d\times d}$

$$|g(t,x,y,z) - g(t,x,y',z')| \leq K\Big(1+f(|y-y'|^2)(|z|+|z'|)\Big)\Big(|y-y'|+|z-z'|\Big).$$

(H4) *For all* $(t,x,y,z),(t,x,y',z')\in[0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^{d\times d}$

$$|a(t,x,y) - a(t,x,y')| \leq K|y-y'|$$
$$|b(t,x,y,z) - b(t,x,y',z')| \leq K(|y-y'|+|z-z'|).$$

(H5) *For all* $(t,x,y),(t,x',y)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}$

$$|a(t,x,y) - a(t,x',y)| + |\phi(x)-\phi(x')| \leq K_0|x-x'|^{\alpha_0}, \text{ for } \alpha_0\in(0,1).$$

Let us remark that the condition (H3) will only be used to establish both the uniqueness in law and the pathwise uniqueness to the FBSDE (1.1). This condition is not needed for the solvability of PDE (1.3).

We now give definitions of the notion of solution to the system (1.1). We start with the definition of a strong solution.

Definition 1.2. *For a given standard set-up* $(\Omega,\mathfrak{F},\mathbb{P},\mathbb{F} = \{\mathfrak{F}_s\}_{t\leq s\leq T}, B)$, *a triplet process* $(X^{t,x},Y^{t,x},Z^{t,x})$ *is said to be a strong solution to* (1.1) *if the following are satisfied*

1. $(X^{t,x},Y^{t,x},Z^{t,x})$ *is an* $\mathbb{F}$*-adapted process, and* $X^{t,x},Y^{t,x}$ *are continuous, such that*
$$\mathbb{E}\Big\{\sup_{s\in[t,T]}|X_s^{t,x}|^2 + \sup_{s\in[t,T]}|Y_s^{t,x}|^2 + \int_t^T|Z_s^{t,x}|^2\mathrm{d}s\Big\} < \infty;$$
2. $(X^{t,x},Y^{t,x},Z^{t,x})$ *satisfies* (1.1) $\mathbb{P}$*-almost surely.*

Next we give the definition of a weak solution.

Definition 1.3. *Let* $(\Omega,\mathfrak{F},\mathbb{P},\mathbb{F} = \{\mathfrak{F}_s\}_{t\leq s\leq T}, B)$ *be a standard set-up. The tuple process* $(\Omega,\mathfrak{F},\mathbb{P},\mathbb{F} = \{\mathfrak{F}_s\}_{t\leq s\leq T}, B, X^{t,x},Y^{t,x},Z^{t,x})$ *is called a weak solution to* (1.1) *if*

1. $(X^{t,x},Y^{t,x},Z^{t,x})\in\mathcal{S}^2\times\mathcal{S}^\infty\times\mathcal{H}^2$*;*
2. $\mathbb{P}$*-almost surely* $(X^{t,x},Y^{t,x},Z^{t,x})$ *satisfies* (1.1).

The following definition is the notion of “weak BMO solution”.

Definition 1.4. *Let* $(\Omega, \mathfrak{F}, \mathbb{P}, \mathbb{F} = \{\mathfrak{F}_s\}_{t\leq s\leq T}, B)$ *be a standard set-up. The tuple process* $(\Omega, \mathfrak{F}, \mathbb{P}, \mathbb{F} = \{\mathfrak{F}_s\}_{t\leq s\leq T}, B, X^{t,x}, Y^{t,x}, Z^{t,x})$ *is called a weak BMO solution to* (1.1) *if*

1. $(X^{t,x}, Y^{t,x}, Z^{t,x}) \in \mathcal{S}^2 \times \mathcal{S}^\infty \times \mathcal{H}_{BMO}$*;*
2. $\mathbb{P}$*-almost surely* $(X^{t,x}, Y^{t,x}, Z^{t,x})$ *satisfies* (1.1).

The main results of this work are the following: weak existence and uniqueness in law of the solution of the QFBSDE (1.1), existence of unique strong solution and Malliavin differentiablility of the solution to the QFBSDE (1.1).

Theorem 1.5. *Suppose Assumption 1.1 is satisfied. Then the QFBSDE* (1.1) *has a weak solution.*

Theorem 1.6. *Suppose Assumption 1.1 is satisfied. Then uniqueness in law hold for the solution to the QFBSDE* (1.1).

Theorem 1.7. *Suppose that* $d = 1$*. Suppose Assumption 1.1 is valid with* $\alpha_0 \geq 1/2$*. Then for every* $\delta > 0$ *there exists a unique strong solution to the QFBSDE* (1.1) *on* $[0, T-\delta]$*. Moreover, by the continuity of the solution, and growth of the coefficient pathwise uniqueness holds on* $[0, T]$.

In order to prove Theorem 1.7, we first observe that the QFBSDE (1.1) has a weak solution. Using the Yamada-Watanabe theorem for FBSDE (see for example [2] or [3, Corollary 3.3]), we only need to prove the pathwise uniqueness. This strategy is standard in the theory of SDEs (see for example [33, 31]).

Next we assume that the diffusion is the identity matrix. i.e., $\sigma \equiv I_{d\times d}$, where $I_{d\times d}$ denotes the identity matrix on $\mathbb{R}^d$. As a result, the QFBSDE (1.1) has a unique strong Malliavin differentiable solution (Compare with [22]).

Theorem 1.8. *Suppose Assumption 1.1 are satisfied. Suppose in addition that the diffusion is the identity matrix. i.e.,* $\sigma \equiv I_{d\times d}$*. For every* $\delta > 0$*, the QFBSDE* (1.1) *has a unique strong Malliavin differentiable solution for all* $t \in [0, T-\delta]$ *for* $T > 0$.

Finally we give the dependence with respect to the initial parameter.

Proposition 1.9. *Suppose conditions of Theorem 1.8 are valid. Let* $(X^{s,x}, Y^{s,x}, Z^{s,x})$ *be the unique strong solution to the QFBSDE* (1.1) *and let* $p > 0$ *be an integer. Then there exists a constant* C *only depending on* δ *and the coefficients of the equation such that*

$$\mathbb{E}[|X_{t_1}^{s_1,x_1} - X_{t_2}^{s_2,x_2}|^p] \leq C\Big(|s_1 - s_2|^{p/2} + |t_1 - t_2|^{p/2} + |x_1 - x_2|^p\Big), \tag{1.4}$$

$$\mathbb{E}[|Y_{t_1}^{s_1,x_1} - Y_{t_2}^{s_2,x_2}|^p] \leq C\Big(|s_1 - s_2|^{p/2} + |t_1 - t_2|^{p/2} + |x_1 - x_2|^p\Big), \tag{1.5}$$

$$\mathbb{E}[|Z_{t_1}^{s_1,x_1} - Z_{t_2}^{s_2,x_2}|^p] \leq C\Big(|s_1 - s_2|^{p\alpha_4/4} + |t_1 - t_2|^{p\alpha_4/4} + |x_1 - x_2|^{p\alpha_4}\Big) \tag{1.6}$$

for all $s_1, s_2, t_1, t_2 \in [0, T-\delta]$ $(T > 0)$ *for all* x_1, x_2*, where* α_4 *is given in Theorem 2.5.*

The remainder of the paper is organized as follows: in Section 2, we first provide a-priori estimates of the solution in the regularized framework which will enables us to establish the weak solution to PDE (1.3). Section 3 is devoted to the proof of weak solution to the FBSDE (1.1) whereas Section 4 is concerned with the proof of the preliminary results on the solvability of the PDE 1.3. The paper ends with an appendix in which we give some auxiliary results.

2. Preliminary results

As pointed out earlier, the proof of the main results rely on the decoupling field technique. As in [8], the main idea is to find a solution the PDE (1.3) in a suitable set.

2.1. **A priori estimates.** It is well known that, when the coefficients b, σ, g and ϕ are smooth, i.e. infinitely often differentiable with respect to the variables t, x, y and z, bounded and with bounded derivatives of any order, the associated PDE (1.3) has a unique bounded solution denoted by u and such that $\partial_t u, \nabla_x u, \nabla^2_{x,x} u$ are all bounded and Hölder continuous on $[0, T] \times \mathbb{R}^d$ (see for example [21]). Moreover for an arbitrarily chosen standard set-up $(\Omega, \mathfrak{F}, \mathbb{P}, \mathbb{F} = \{\mathfrak{F}_s\}_{t\leq s\leq T}, B)$, there exists a unique strong solution $(X^{t,x}, Y^{t,x}, Z^{t,x})$ to the FBSDE (1.1). In addition, the following relations hold (see [14])

$$\forall s \in [t,T], \quad u(s, X_s^{t,x}) = Y_s^{t,x} \quad \forall s \in [t,T[, \quad \nabla_x u(s, X_s^{t,x}) = Z_s^{t,x}. \tag{2.1}$$

Let $(b_n)_{n\geq 1}, (\sigma_n)_{n\geq 1}, (g_n)_{n\geq 1}$ and $(\phi_n)_{n\geq 1}$ be sequences obtained by standard mollification of the coefficients b, σ, g and ϕ, respectively. Then b_n, σ_n, g_n and ϕ_n are smooth functions with compact supports. One can show that the sequences b_n, σ_n and ϕ_n satisfy Assumption 1.1, uniformly in n. In addition, $(b_n)_{n\geq 1}, (\sigma_n)_{n\geq 1}$ and $(\phi_n)_{n\geq 1}$ converge respectively to b, σ, and ϕ in the following sense: For a.e. $(t,x) \in [0,T] \times \mathbb{R}^d$, $\forall (y,z) \in \mathbb{R} \times \mathbb{R}^d$ $(b_n, a_n \equiv \sigma_n \sigma_n^*)(t,x,y,z) \to (b,a)(t,x,y,z)$, $\phi_n \to \phi$ uniformly on compact subsets of $\mathbb{R}^d$. Let us now focus on the sequence $(g_n)_{n\geq 1}$. Consider the four mollifiers $(\psi_n)_{n\geq 1}, (\psi_n^1)_{n\geq 1}, (\psi_n^2)_{n\geq 1}, (\psi_n^3)_{n\geq 1}$ defined respectively on $\mathbb{R}, \mathbb{R}^d, \mathbb{R}, \mathbb{R}^d$ by

$$\begin{aligned} \psi_n(\cdot) \quad &= cn\varphi(n|\cdot|), \quad \psi_n^2(\cdot) = c_2 n\varphi(n|\cdot|), \\ \psi_n^1(\cdot) \quad &= c_1 n^d \varphi(n|\cdot|), \quad \psi_n^3(\cdot) = c_3 n^d \varphi(n|\cdot|), \end{aligned}$$

where $\forall x \in \mathbb{R}^d, \varphi(x) = \exp(1/(|x|^2 - 1))1_{[0,1]}(|x|)$ and c, c_1, c_2, c_3 are four constants of normalization. For an extended function g in $\Gamma = \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$, we define g_n by:

$$g_n(t,x,y,z) := \int_\Gamma g(t-s, x-x^1, y-y^1, z-z^1)\psi_n(s)\psi_n^1(x^1)\psi_n^2(y^1)\psi_n^3(z^1)\mathrm{d}s\mathrm{d}x^1\mathrm{d}y^1\mathrm{d}z^1.$$

Lemma 2.1. *There exists a constant $C > 0$ such that the sequence $(g_n)_{n\geq 1}$ satisfies the following inequality:*

$$|g_n(t,x,y,z)| \leq C\Lambda(1 + |y| + f_n(y)(|z|^2 + \frac{1}{n^2})),$$

where the sequence $(f_n)_{n\geq 1}$ stands for the mollifier of the function f. Moreover, the sequence $(g_n)_{n\in\mathbb{N}}$ converges to g in (t,x) a.e. and locally uniformly in $(y,z) \in \mathbb{R} \times \mathbb{R}^d$.

Proof. From the definition of g_n, Assumption (H1) and the Fubini theorem, we obtain

$$\begin{aligned} |g_n(t,x,y,z)| \leq &\int_\Gamma |g(t-s, x-x^1, y-y^1, z-z^1)\psi_n(s)\psi_n^1(x^1)\psi_n^2(y^1)\psi_n^3(z^1)|\mathrm{d}s\mathrm{d}x^1\mathrm{d}y^1\mathrm{d}z^1 \\ \leq &\Lambda \int_\Gamma \left(1 + |y-y^1| + f(y-y^1)|z-z^1|^2\right)\psi_n(s)\psi_n^1(x^1)\psi_n^2(y^1)\psi_n^3(z^1)\mathrm{d}s\mathrm{d}x^1\mathrm{d}y^1\mathrm{d}z^1 \\ \leq &\Lambda(1+|y|) + \Lambda\int_\mathbb{R} |y^1|\psi_n^2(y^1)\mathrm{d}y^1 + 2\Lambda|z|^2\int_\mathbb{R} f(y-y^1)\psi_n^2(y^1)\mathrm{d}y^1 \\ &+ 2\Lambda\int_{\mathbb{R}\times\mathbb{R}^d} f(y-y^1)|z^1|^2\psi_n^2(y^1)\psi_n^3(z^1)\mathrm{d}y^1\mathrm{d}z^1 \\ \leq &\Lambda\Big(1 + (|y| + \frac{1}{n}) + (2f_n(y)|z|^2 + \frac{2}{n^2}f_n(y))\Big) \leq \Lambda\Big(1 + |y| + f_n(y)(2|z|^2 + \frac{2}{n^2})\Big), \end{aligned}$$

which prove the first statement. On the other hand, it is well known that there exists a negligible set $\mathcal{N} \in \mathcal{B}(\mathbb{R}^{d+1})$ i.e. $\mu_{d+1}(\mathcal{N}) = 0$, such that $\forall (t,x) \in \mathcal{N}^c, \forall (y,z) \in \mathbb{R} \times \mathbb{R}^d$,

$$\Big|\int g(s,x^1,y,z)\psi_n(t-s)\psi_n^1(x-x^1)\mathrm{d}s\mathrm{d}x^1 - g(t,x,y,z)\Big| \leq \epsilon/2,$$

where μ_{d+1} stands for the Lebesgue measure on $\mathbb{R}^{d+1}$. Moreover, since $g(t,x,\cdot,\cdot)$ is uniformly continuous on compacts, for all $\epsilon/2 > 0$, $\forall (t,x) \in \mathcal{N}^c$ there is $\eta > 0$ such that for $(y,z), (y',z') \in \mathbb{R} \times \mathbb{R}^d$, satisfying $|y-y'| + |z-z'| \leqslant \eta$ we obtain $|g(t,x,y,z) - g(t,x,y',z')| \leq \epsilon/2$. Therefore, for n large enough

$$\begin{aligned}
&|g_n(t,x,y,z)-g(t,x,y,z)|\\
=&\Big|\int_\Gamma [g(s,x^1,y^1,z^1)-g(t,x,y,z)]\psi_n(t-s)\psi_n^1(x-x^1)\psi_n^2(y-y^1)\psi_n^3(z-z^1)\mathrm{d}s\mathrm{d}x^1\mathrm{d}y^1\mathrm{d}z^1\Big|\\
\leq&\Big|\int_\Gamma [g(s,x^1,y^1,z^1)-g(s,x^1,y,z)]\psi_n(t-s)\psi_n^1(x-x^1)\psi_n^2(y-y^1)\psi_n^3(z-z^1)\mathrm{d}s\mathrm{d}x^1\mathrm{d}y^1\mathrm{d}z^1\Big|\\
&+\Big|\int_{\mathbb{R}^{d+1}} [g(s,x^1,y,z)-g(t,x,y,z)]\psi_n(t-s)\psi_n^1(x-x^1)\mathrm{d}s\mathrm{d}x^1\Big| \leq \epsilon/2+\epsilon/2.
\end{aligned}$$

This ends the proof. □

We are now in position to derive some a priori bounds of the function u solution to PDE (1.3), its derivatives $(\partial_t u, \nabla_x u, \nabla^2_{xx} u)$ in terms of coefficients that appear in Assumption 1.1. These controls will be given as a series of Lemmas and will allow us to introduce a regularization procedure.

Lemma 2.2. *Suppose Assumption 1.1 holds. Then there exists a constant $\Upsilon^{(1)}$ depending only on Λ, λ, T and the norm $L^1(\mathbb{R})$ of the function f, such that*

$$\forall (t,x)\in[0,T]\times\mathbb{R}^d, \quad |u(t,x)|\leqslant \Upsilon^{(1)}.$$

Proof. See Section 4. □

Lemma 2.3. *Suppose conditions of Lemma 2.2 are in force. Then there exist constants $\alpha_1, \Upsilon^{(2)} > 0$, depending only on parameters appearing in Assumption 1.1 such that for all $(t,x),(s,y)\in[0,T]\times\mathbb{R}^d$,*

$$|u(t,x)-u(s,y)|\leq \Upsilon^{(2)}(|t-s|^{\alpha_2/2}+|x-y|^{\alpha_2}), \quad \alpha_2=\alpha_0\wedge\alpha_1.$$

Proof. See Section 4. □

Lemma 2.4. *There exist constants $\alpha_3, \Upsilon^{(3)} > 0$, depending only on parameters appearing in Assumption 1.1 such that, for all $(t,x)\in[0,T[\times\mathbb{R}^d$,*

$$|\nabla_x u(t,x)|\leq \Upsilon^{(3)}(T-t)^{(-1+\alpha_3)/2}.$$

Proof. See Section 4. □

The proofs of the following Lemma can be found in [8, Section 7].

Lemma 2.5. *There exist constants $\alpha_4, \Upsilon^{(4)} > 0$, depending only on parameters appearing in Assumption 1.1 such that for all $(t,x),(s,y)\in[0,T[\times\mathbb{R}^d, t\leq s$,*

$$|\nabla_x u(t,x)-\nabla_x u(s,y)|\leq \Upsilon^{(4)}(T-s)^{(-1+\alpha_4)/2}(|t-s|^{\alpha_4/2}+|x-y|^{\alpha_4}).$$

Lemma 2.6. *Let $p\geq 1$. There exists a constant $\alpha_5\in]0,1]$, depending on parameters appearing in Assumption 1.1 (and not on p) and a constant $\Upsilon^{(5)}(p)$ also depending on the same parameters such that, for all $R\geq 1, \delta\in]0,T], \zeta\in\mathbb{R}^d$,*

$$\int_{T-\delta}^T\int_{B(\zeta,R)}[(T-s)^{1-\alpha_5}(|\partial_t u(s,y)|+|\nabla^2_{x,x}u(s,y)|)]^p\mathrm{d}s\mathrm{d}y\leq \Upsilon^{(5)}(p)\delta R^d.$$

Proof. See Section 4. □

2.2. **Solvability of PDE** (1.3)**.** This subsection is devoted to the solvability of the PDE (1.3). Let us once more stress the fact that, the proof of the existence of solution to QFBSDE (1.1) does not require an a priori uniqueness result of solution to the associated PDE (1.3). This condition is however required in the proof of the uniqueness of both the FBSDE (see the proof of Theorem 1.6 in Section 3) and the PDE (see Theorem 2.7).

We recall that $W^{1,2,d+1}_{\mathrm{Loc}}([0,T[\times\mathbb{R}^d,\mathbb{R})$ stands for the Sobolev space of classes of functions u such that $u:[0,T]\times\mathbb{R}^d\to\mathbb{R}$, $|u|, |\partial_t u|, |\nabla_x u|, |\nabla^2_{x,x}u|\in L^{d+1}_{\mathrm{Loc}}([0,T[\times\mathbb{R}^d,\mathbb{R})$.

Theorem 2.7. *Under Assumption 1.1, the PDE* (1.3) *has a unique solution in the space $\mathcal{L}$ defined by*

$$\mathcal{L} := \Big\{ u \in C^0(R_{T^-}, \mathbb{R}) \cap C^{0,1}(R_T, \mathbb{R}) \cap W^{1,2,d+1}_{Loc}(R_T, \mathbb{R}),$$
$$\exists \gamma > 0, \sup_{(t,x)\in[0,T[\times\mathbb{R}^d} \Big(|u(t,x)| + (T-t)^{1/2-\gamma} |\nabla_x u(t,x)| \Big) < +\infty \Big\},$$

where $R_{T^-} = [0,T[\times\mathbb{R}^d$ and $R_T = [0,T]\times\mathbb{R}^d$.

Proof. Existence of a solution: For every $n \geq 0$, we know from [21] that the PDE (1.3) with coefficients denoted $(b_n, a_n \equiv \sigma_n\sigma_n^T, g_n, \phi_n)$ admits a unique classical solution that we denote by u_n with derivatives by $(\partial_t u_n, \nabla_x u_n, \nabla^2_{xx} u_n)$. In addition, it follows from Lemmas 2.2, 2.3, 2.4 and 2.5 that the sequence $(u_n)_{n\geq 0}$ (respectively $(\nabla_x u_n)_{n\geq 0}$) is uniformly bounded and equi-continuous on every compact subset of $[0,T]\times\mathbb{R}^d$ (respectively $[0,T[\times\mathbb{R}^d$.) Thus by the Arzela-Ascoli theorem, there exists subsequences (still indexed by n) $(u_n)_{n\geq 0}$ (respectively $(\nabla_x u_n)$) converging uniformly on compact in $[0,T]\times\mathbb{R}^d$ (respectively $[0,T[\times\mathbb{R}^d$) to a limit u (respectively $\tilde{u}$). It is readily seen that $\tilde{u} = \nabla_x u$. Furthermore, Lemma 2.6 ensures that the sequence $(\partial_t u_n)_{n\geq 0}$ and $(\nabla^2_{x,x} u_n)_{n\geq 0}$ are bounded in an appropriate L^p space. Thus one can extract subsequences (still indexed by n) $(\partial_t u_n)_{n\geq 0}$ (respectively $(\nabla^2_{x,x} u_n)_{n\geq 0}$) that will converge weakly in L^p. These limits are $\partial_t u$ (respectively $\nabla^2_{xx} u$) in the distribution sense. Thus, it is readily seen that the limit u satisfies almost everywhere the PDE (1.3).
Uniqueness of the solution: To prove this statement, we borrow some results from the next section. Note that, if $u, v \in \mathcal{L}$ denote two solutions to the PDE (1.3), then from (2.1) it is possible to associate them respectively to (weak) solutions $(\Omega, \mathfrak{F}, \mathbb{P}), , \{\mathfrak{F}_s\}_{t\leq s\leq T}, B, X^{t,x}, Y^{t,x}, Z^{t,x})$ and $(\bar{\Omega}, (\bar{\mathfrak{F}}), \bar{\mathbb{P}}, , \{\bar{\mathfrak{F}}_s\}_{t\leq s\leq T}, \bar{B}, U^{t,x}, V^{t,x}, W^{t,x})$ of the QFBSDE (1.1), with initial value $(t,x) \in [0,T]\times\mathbb{R}^d$ (see subsection 3.1). Moreover, from the Markov property of the processes $X^{t,x}$ and $U^{t,x}$ we deduce that: $u(t,x) = \mathbb{E}(Y_t^{t,x})$ and $\bar{\mathbb{E}}(V_t^{t,x}) = v(t,x)$ hold for all $(t,x) \in [0,T]\times\mathbb{R}^d$. Hence, the uniqueness follows from the fact that the processes $Y_t^{t,x}$ and $V_t^{t,x}$ have the same law (see subsection 3.2). □

3. Proof of the main results

Unless otherwise stated, in the sequel we will drop the superscript for the sake of simplicity.

3.1. **Proof of Theorem 1.5.** We start by showing existence of the weak solution to the forward equation. Without loss of generality, we assume that the initial condition is $(0,x)$. To begin with, we set

$$X_t = x + \int_0^t \tilde{b}(s, X_s)\mathrm{d}s + \int_0^t \tilde{\sigma}(s, X_s)\mathrm{d}B_s, \tag{3.1}$$

where, $\tilde{b}(s,X_s) := b(s, X_s, u(s,X_s), \nabla_x u(s,X_s))$, $\quad \tilde{\sigma}(s,X_s) := \sigma(s,X_s,u(s,X_s))$. Our goal is to prove the existence of a standard set-up $(\Omega, \{\mathfrak{F}\}, \mathbb{P}, B)$ and a continuous $\{\mathfrak{F}_t\}_{0\leq t\leq T}$ -adapted process X satisfying (3.1).

Let $(\Omega, \{\mathfrak{F}\}, \mathbb{Q}, \tilde{B})$ be a stochastic basis. Using assumptions (H1), (H2) and the boundedness of u (Lemma 2.2) it is well known that the SDE

$$X_t = x + \int_0^t \tilde{\sigma}(s, X_s)\mathrm{d}\tilde{B}_s \tag{3.2}$$

has a unique weak solution (see for example [15]). Define the function θ by $\theta(t,x) := \tilde{b}(t,x)\tilde{\sigma}^{-1}(t,x)$. Then $\theta(t, X_t)$ is a bounded process and the process

$$\varXi_t := \exp\Big(\int_0^t \theta(s,X_s)\mathrm{d}\tilde{B}_s - \frac{1}{2}\int_0^t |\theta(s,X_s)|^2\mathrm{d}s \Big), \quad t \geq 0, \tag{3.3}$$

is a martingale under $\mathbb{Q}$. Therefore from the Girsanov's theorem, the process

$$B_t := \tilde{B}_t - \int_0^t \theta(s,X_s)\mathrm{d}s, \tag{3.4}$$

defines a Brownian motion under the probability measure $\mathbb{P}$, defined by $\mathrm{d}\mathbb{P} := \Xi_t \mathrm{d}\mathbb{Q}$. Substituting (3.4) into (3.2), one sees that, the triple $(X, B), (\Omega, \mathfrak{F}, \mathbb{P}), \{\mathfrak{F}_t\}$ is a weak solution to (3.1). Moreover, thanks to assumption (H1) and Lemma 2.2, one can show that $\mathbb{E}[\sup_{0\le t\le T}|X_t|^2] < \infty$.

Let us now turn to the proof of a weak solution to the BSDE with parameter (ϕ, g). Recall that $u \in W^{1,2,d+1}_{\text{loc}}([0,T[\times\mathbb{R}^d,\mathbb{R})$ is solution to PDE (1.3) and set $Y_t = u(t, X_t)$ and $Z_t = \nabla_x u(t, X_t)$. For every $R > 0$, we define $\rho(R)$ by $\rho(R) := \inf\{t \ge 0, |X_t| \ge R\} \wedge T(1 - 1/R)$. By applying the Itô-Krylov formula for all $t \in [0, \rho(R)]$ (see [18, Ch 2, Sec 10, Theorem 1]) we deduce that

$$\begin{aligned} Y_t - Y_{\rho(R)} &= u(t, X_t) - u(\rho(R), X_{\rho(R)}) \\ &= \int_t^{\rho(R)} g(s, X_s, u(s, X_s), \nabla_x u(s, X_s))\mathrm{d}s - \int_t^{\rho(R)} \langle \nabla_x u(s, X_s), \sigma(s, X_s, u(s, X_s))\mathrm{d}B_s\rangle. \end{aligned}$$

From the definition of $\rho(R)$, the integrability over $[0, T]$ of the driver is granted $\mathbb{P}$-a.s., and thanks to assumption(H1), Lemmas 2.3 and 2.4, the above stochastic integral is well defined. Thus, by letting $R \to \infty, \rho(R) \to T$ a.s., and due to the regularity properties of u, (continuity and boundedness), the continuity of the process X in T, and the local boundedness of $\nabla_x u$, we deduce that

$$Y_t = \phi(X_T) + \int_t^T g(s, X_s, Y_s, Z_s)\mathrm{d}s - \int_t^T \langle Z_s, \sigma(s, X_s, Y_s)\mathrm{d}B_s\rangle, \tag{3.5}$$

with $Y_s = u(s, X_s)$, $Z_s = \nabla_x u(s, X_s)$. Hence, from Lemma 2.2 we obtain $\mathbb{E}[\sup_{0\le t\le T}|Y_t|] \le \Upsilon^{(1)}$ and from the proof of Lemma 2.2 (see (4.5)), we have $\mathbb{E}\int_0^T |Z_s|^2\mathrm{d}s < \infty$, $\mathbb{P}$-a.s.

On the other hand, for any $\mathfrak{F}_t$-stopping time $\tau \le T$, applying the Itô-Krylov formula for BSDEs to the function Ψ_f in Lemma B.2 and using (H1) and (H2), we deduce that

$$\begin{aligned} \Psi_f(|Y_\tau|) &= \Psi_f(|Y_T|) + \int_\tau^T \Big[\mathrm{sgn}(Y_s)\Psi_f'(|Y_s|)g(s, X_s, Y_s, Z_s) - \frac{1}{2}\Psi_f''(|Y_s|)a(s, X_s, Y_s)|Z_s|^2\Big]\mathrm{d}s \\ &\quad - \int_\tau^T \mathrm{sgn}(Y_s)\Psi_f'(|Y_s|)Z_s\sigma^*(s, X_s, Y_s)\mathrm{d}B_s \\ &\le \Psi_f(|Y_T|) - \frac{\lambda}{2}\int_\tau^T \Big(\Psi_f''(|Y_s|) - \frac{2\Lambda}{\lambda}f(|Y_s|)\Psi_f'(|Y_s|)\Big)|Z_s|^2\mathrm{d}s + \Lambda\int_\tau^T \Psi_f'(|Y_s|)(1 + |Y_s|)\mathrm{d}s \\ &\quad - \int_\tau^T \mathrm{sgn}(Y_s)\Psi_f'(|Y_s|)Z_s\sigma^*(s, X_s, Y_s)\mathrm{d}B_s, \end{aligned}$$

Remark that the above stochastic integral is well defined since $\mathbb{E}[\sup_{0\le t\le T}|Y_t| + \int_0^T |Z_s|^2\mathrm{d}s] < \infty$. In addition, using the properties of the function Ψ_f (see Lemma B.2) and taking the conditional expectation with respect to $\mathfrak{F}_t$ on both sides of the above inequality, we deduce that there exists a constant $\Upsilon^{(6)}$ only depending on parameters in the assumptions such that $\mathbb{E}\Big[\int_\tau^T |Z_s|^2\mathrm{d}s\Big|\mathfrak{F}_\tau\Big] \le \Upsilon^{(6)}$. Thus $\int_0^\cdot Z_s\mathrm{d}B_s$ is a BMO-martingale. The same reasoning is also valid for any initial condition of the form $(t, x) \in [0, T] \times \mathbb{R}^d$.

Therefore, we have shown that the tuple $(\Omega, \{\mathfrak{F}\}, \mathbb{P}, \{\mathfrak{F}_s\}_{t\le s\le T}, B, X, Y, Z)$ is a weak solution to FBSDE (1.1) with initial condition (t, x), in the sense of Definitions 1.3 and 1.4. This ends the proof. □

3.2. **Proof of Theorem 1.6.** The proof of the uniqueness in law of the weak solution to the QFBSDE (1.1) relies on the so call *weak decoupling method* as developed in [8].

The next result gives an L^2_{loc} bound of the second order derivative of the solution to the PDE (1.3). It is similar to [8, Lemma 4.1] and its proof uses Krylov estimates (see [18, Ch.2, Sec.3 Lemma 1]).

Lemma 3.1. *Suppose that conditions of Lemma 2.2 are satisfied. Let* $(\bar{\Omega}, (\bar{\mathfrak{F}}), \bar{\mathbb{P}}, \{\bar{\mathfrak{F}}_t\}_{0\le t\le T}, \bar{B}, U, V, W)$ *be a solution to the FBSDE* (1.1) *with initial condition* $(0, x), x \in \mathbb{R}^d$. *Define* $\tau(t, r) := \inf\{s \ge t, |U_s - U_t| \ge r\} \wedge T$, *with* $r \ge 1$. *Then there exist* *constants $\gamma \in]0,1]$, and $C(p)$, depending only on known parameters appearing in Assumption 1.1 such that for all $p \geq 1, \bar{\mathbb{P}}$-a.s.,*

$$\bar{\mathbb{E}}\Big[\int_t^{\tau(t,r)}[(T-s)^{(1-\gamma)p}|\nabla^2_{xx}u(s,U_s)|^p]\mathrm{d}s\Big|\bar{\mathfrak{F}}_t\Big] \leq C(p)(T-t)^{1/d+1}r^{d/d+1}.$$

Proof of Lemma 3.1. Using Lemma 2.6, we know that for $\beta \leq \alpha_5$, the function $(T-\cdot)^{(1-\beta)p}|\nabla^2_{x,x}u|^p \in L^{d+1}_{\text{loc}}([0,T]\times\mathbb{R}^d,\mathbb{R})$. Then, it is possible to build a sequence $(\psi_n)_{n\geq 1}$ of continuous non-negative functions with compact support such that $\psi_n \to (T-\cdot)^{(1-\beta)p}|\nabla^2_{x,x}u|^p\chi$ in the $L^{d+1}([0,T]\times\mathbb{R}^d)$-norm, where $\chi:\mathbb{R}^d\to[0,1]$ is a smooth cutting indicator function. Recall that under the probability measure $\bar{\mathbb{P}}$, the process U is an Itô process with bounded drift (confer (H1) and Lemma 2.2) and uniformly non-degenerate and bounded diffusion matrix (Confer (H1),(H2)and Lemma 2.2).

It follows from the definition of $\tau(t,r)$ and the Krylov's estimates [18, Ch,2, Sec.3, Theorems 3 and 4] that there exists a constant C only depending on d,λ,Λ and T such that

$$\begin{aligned}
&\Big|\bar{\mathbb{E}}\Big[\int_t^{\tau(t,r)}[(T-s)^{(1-\beta)p}|\nabla^2_{x,x}u(s,U_s)|^p\chi(s,U_s)-\psi_n(s,U_s)]\mathrm{d}s\Big|\bar{\mathfrak{F}}_t\Big]\Big|\\
&\quad\leq \bar{\mathbb{E}}\Big[\int_t^{T}|(T-s)^{(1-\beta)p}|\nabla^2_{x,x}u(s,U_s)|^p\chi(s,U_s)-\psi_n(s,U_s)|\mathrm{d}s\Big|\bar{\mathfrak{F}}_t\Big]\\
&\quad C\Big[\int_t^T\int_{\mathbb{R}^d}|(T-s)^{(1-\beta)p}|\nabla^2_{x,x}u(s,x)|^p\chi(s,x)-\psi_n(s,x)|^{d+1}\mathrm{d}s\mathrm{d}x\Big]^{1/(d+1)}.
\end{aligned}\tag{3.6}$$

Using the definition of $\tau(t,r)$ and the Fatou's Lemma, we deduce for every $n\geq 1$

$$\begin{aligned}
&\bar{\mathbb{E}}\Big[\int_t^{\tau(t,r)}\psi_n(s,U_s)\mathrm{d}s\Big|\bar{\mathfrak{F}}_t\Big]\\
&\quad\leq \mathbb{E}\Big[\int_t^T\psi_n(s,U_s)1_{\{|U_s-U_t|\leq r\}}\mathrm{d}s\Big|\bar{\mathfrak{F}}_t\Big]\\
&\quad= \bar{\mathbb{E}}\Big[\int_t^T\lim_{m\to\infty}\psi_n(s,U_s-U_t+\Sigma^d_m(U_t))1_{\{|U_t-U_s|\leq r\}}\mathrm{d}s\Big|\bar{\mathfrak{F}}_t\Big]\\
&\quad\leq \liminf_{m\to\infty}\bar{\mathbb{E}}\Big[\int_t^T\psi_n(s,U_s-U_t+\Sigma^d_m(U_t))1_{\{|U_t-U_s|\leq r\}}\mathrm{d}s\Big|\bar{\mathfrak{F}}_t\Big]\\
&\quad= \liminf_{m\to\infty}\sum_{x\in 2^{-m}\mathbb{Z}^d}1_{\{\Sigma^d_m(U_t)=x\}}\bar{\mathbb{E}}\Big[\int_t^T\psi_n(s,U_s-U_t+x)1_{\{|U_t-U_s|\leq r\}}\mathrm{d}s\Big|\bar{\mathfrak{F}}_t\Big],
\end{aligned}\tag{3.7}$$

where for $m\in\mathbb{N}$, the functions $\Sigma^1_m : x\in\mathbb{R}\mapsto 2^{-m}(k+1)$ for $x\in]2^{-m}k,2^{-m}(k+1)]$ and $\Sigma^d_m : x\in\mathbb{R}\mapsto(\Sigma^1_{n'}(x_1),\cdots,\Sigma^d_m(x_d))$. (Note that, $\Sigma^d_m(x)\to x$ as $m\to\infty$). Apply again the preceding Krylov's estimates to the process $(U_s-U_t)_{t\leq s\leq T}$ and using Theroem 2.6, we get

$$\begin{aligned}
&\bar{\mathbb{E}}\Big[\int_t^T\psi_n(s,U_s-U_t+x)1_{\{|U_t-U_s|\leq r\}}\mathrm{d}s\Big|\bar{\mathfrak{F}}_t\Big]\leq C\Big[\int_t^T\int_{B(0,r)}|\psi_n(s,y+x)|^{d+1}\mathrm{d}s\mathrm{d}y\Big]^{1/(d+1)}\\
&\leq C\Big[\int_t^T\int_{B(0,r)}|(T-s)^{(1-\beta)p}|\nabla^2_{x,x}u|^p(s,y+x)\chi(s,y+x)-\psi_n(s,y+x)|^{d+1}\mathrm{d}s\mathrm{d}y\Big]^{1/(d+1)}\\
&\quad+C\Big[\int_t^T\int_{B(0,r)}|(T-s)^{(1-\beta)p}|\nabla^2_{x,x}u|^p(s,y+x)\chi(s,y+x)|^{d+1}\mathrm{d}s\mathrm{d}y\Big]^{1/(d+1)}\\
&\leq C\Big[\int_t^T\int_{B(0,r)}|(T-s)^{(1-\beta)p}|\nabla^2_{x,x}u|^p(s,y+x)\chi(s,y+x)-\psi_n(s,y+x)|^{d+1}\mathrm{d}s\mathrm{d}y\Big]^{1/(d+1)}\\
&\quad+C[C_5(p(d+1))(T-t)r^d]^{1/(d+1)}.
\end{aligned}\tag{3.8}$$

First substituting (3.8) into (3.7) then let n goes to infinity. Then using (3.6) we get

$$\bar{\mathbb{E}}\Big[\int_t^{\tau(t,r)}[(T-s)^{(1-\beta)p}|\nabla^2_{x,x}u(s,U_s)|^p\chi(s,U_s)]\mathrm{d}s\Big|\bar{\mathfrak{F}}_t\Big]\leq C[C_5(p(d+1))(T-t)r^d]^{1/(d+1)}.$$

The proof is completed by letting $\chi\to 1$ and using the Beppo-Levi theorem. □

Proof of Theorem 1.6. Let $(\bar{\Omega},(\bar{\mathfrak{F}}),\bar{\mathbb{P}},\{\bar{\mathfrak{F}}_t\}_{0\leq t\leq T},\bar{B},U,V,W)$ be another weak solution to the QF-BSDE (1.1) with initial condition $(0,x),x\in\mathbb{R}^d$. Set

$$\forall t\in[0,T],\bar{V}_t:=u(t,U_t),\quad \bar{W}_t:=\nabla_x u(t,U_t),\forall t\in[0,T[.$$

The strategy is to first establish that $(\bar{V},\bar{W})\equiv(V,W)$, $\bar{\mathbb{P}}$-a.s. and then to identify the forward component of (1.1) with the SDE given by (3.1).

Let us remark that (H1) and Lemma 2.2 imply the boundedness of the drift b. Hence since $u\in W^{1,2,d+1}_{\mathrm{loc}}([0,T[\times\mathbb{R}^d,\mathbb{R})$, one can apply the Itô-Krylov formula to show that the process $\bar{V}$ is still a semimartingale. In order to apply such a formula to the process $u(\cdot,U)$, one needs a localization argument. For that purpose, let $R>0,R_1>0$, and define

$$\bar{\rho}(R):=\inf\{t\geq 0,|U_t|\geq R\}\wedge T(1-1/R)\text{ and }\bar{\rho}_1(R_1):=\inf\{t\geq 0,\int_0^t|W_s|^2\mathrm{d}s\geq R_1\}\wedge T,\tag{3.9}$$

Let $\tau\leq\bar{\rho}(R)\wedge\bar{\rho}_1(R_1)$ be a stopping time. Then using the Itô-Krylov formula, we get for all $t\in[0,\tau]$

$$\begin{aligned}\mathrm{d}\bar{V}_t=\mathrm{d}u(t,U_t)=&\partial_t u(t,U_t)\mathrm{d}t+1/2\sum_{i,j=1}^d a_{ij}(t,U_t,V_t)\partial^2_{x_ix_j}u(t,U_t)\mathrm{d}t\\&+\langle b(t,U_t,V_t,W_t),\nabla_x u(t,U_t)\rangle\mathrm{d}t+\langle\nabla_x u(t,U_t),\sigma(t,U_t,V_t)\mathrm{d}\bar{B}_t\rangle,\end{aligned}\tag{3.10}$$

Recall that since u is the solution of the PDE (1.3) and $\bar{W}_t:=\nabla_x u(t,U_t)$, we have for all $t\in[0,\tau]$

$$\partial_t u(t,U_t)=-\frac{1}{2}\sum_{i,j=1}^d a_{ij}(t,U_t,\bar{V}_t)\partial^2_{x_i,x_j}u(t,U_t)-\langle b(t,U_t,\bar{V}_t,\bar{W}_t),\nabla_x u(t,U_t)\rangle-g(t,\bar{U}_t,\bar{V}_t,\bar{W}_t).$$

Substituting the above into (3.10), we get

$$\begin{aligned}\mathrm{d}\bar{V}_t=&\Big(1/2\sum_{i,j=1}^d\big(a_{ij}(t,U_t,V_t)-a_{ij}(t,U_t,\bar{V}_t)\big)\,\partial^2_{x_ix_j}u(t,U_t)+\langle b(t,U_t,V_t,W_t)-b(t,U_t,\bar{V}_t,\bar{W}_t),\bar{W}_t\rangle\\&-g(t,U_t,\bar{V}_t,\bar{W}_t)\Big)\mathrm{d}t+\langle\bar{W}_t,\sigma(t,U_t,V_t)\mathrm{d}\bar{B}_t\rangle,\ \ \mathbb{P}\text{-a.s. for all }t\in[0,\tau]\end{aligned}\tag{3.11}$$

Using the definition of τ, (H1), Lemmas 2.2 and 2.4, and the Krylov inequalities, the drift term in (3.11) is well defined.

On the other hand, since V is solution to FBSDE (1.1), we have

$$\mathrm{d}V_t=-g(t,U_t,V_t,W_t)\mathrm{d}t+\langle W_t,\sigma(t,U_t,V_t)\mathrm{d}\bar{B}_t\rangle,\ \ \mathbb{P}\text{-a.s. for all }t\in[0,\tau],$$

from which we get for all $t\in[0,\tau]$

$$\begin{aligned}\mathrm{d}(V-\bar{V})_t=&-1/2\sum_{i,j=1}^d\big(a_{ij}(t,U_t,V_t)-a_{ij}(t,U_t,\bar{V}_t)\big)\,\partial^2_{x_ix_j}u(t,U_t)\mathrm{d}t-\big(g(t,U_t,V_t,W_t)-g(t,U_t,\bar{V}_t,\bar{W}_t)\big)\,\mathrm{d}t\\&-\langle b(t,U_t,V_t,W_t)-b(t,U_t,\bar{V}_t,\bar{W}_t),\bar{W}_t\rangle\mathrm{d}t+\langle W_t-\bar{W}_t,\sigma(t,U_t,V_t)\mathrm{d}\bar{B}_t\rangle\end{aligned}\tag{3.12}$$

Let $t\in[0,\tau]$, applying Itô-Krylov change of variable formula for BSDEs (see [4, Theorem 2.1]) to the function Φ_f defined in Lemma B.1, we get

$$
\begin{aligned}
&\Phi_f(|V_t-\bar V_t|^2)\\
=&\Phi_f(|V_\tau-\bar V_\tau|^2)+\int_t^\tau \Phi_f'(|V_s-\bar V_s|^2)(V_s-\bar V_s)\sum_{i,j=1}^d\big(a_{ij}(s,U_s,V_s)-a_{ij}(s,U_s,\bar V_s)\big)\,\partial^2_{x_ix_j}u(s,U_s)\mathrm{d}s\\
&+2\int_t^\tau \Phi_f'(|V_s-\bar V_s|^2)(V_s-\bar V_s)\big(g(s,U_s,V_s,W_s)-g(s,U_s,\bar V_s,\bar W_s)\big)\,\mathrm{d}s\\
&+2\int_t^\tau \Phi_f'(|V_s-\bar V_s|^2)(V_s-\bar V_s)\langle b(s,U_s,V_s,W_s)-b(s,U_s,\bar V_s,\bar W_s),\bar W_s\rangle\mathrm{d}s\\
&-2\int_t^\tau \Phi_f'(|V_s-\bar V_s|^2)(V_s-\bar V_s)\langle W_s-\bar W_s,\sigma(s,U_s,V_s)\mathrm{d}\bar B_s\rangle\\
&-\int_t^\tau \Phi_f'(|V_s-\bar V_s|^2)\langle W_s-\bar W_s,a(s,U_s,V_s)(W_s-\bar W_s)\rangle\mathrm{d}s\\
&-2\int_t^\tau \Phi_f''(|V_s-\bar V_s|^2)|V_s-\bar V_s|^2\langle W_s-\bar W_s,a(s,U_s,V_s)(W_s-\bar W_s)\rangle\mathrm{d}s\\
=&\Phi_f(|V_\tau-\bar V_\tau|^2)+T(1)+T(2)+T(3)+T(4)+T(5)+T(6).
\end{aligned}
\tag{3.13}
$$

Using (H4), we get

$$
T(1)\leqslant K\int_t^T \Phi_f'(|V_s-\bar V_s|^2)|V_s-\bar V_s|^2|\nabla^2_{xx}u(s,U_s)|\mathrm{d}s. \tag{3.14}
$$

Using (H3), we get

$$
\begin{aligned}
T(2)\leq&2K\int_t^\tau \Phi_f'(|V_s-\bar V_s|^2)|V_s-\bar V_s|(1+f(|V_s-\bar V_s|^2)(|W_s|+|\bar W_s|))(|V_s-\bar V_s|+|W_s-\bar W_s|)\mathrm{d}s\\
\leq&2K\int_t^\tau \Phi_f'(|V_s-\bar V_s|^2)|V_s-\bar V_s|(1+f(|V_s-\bar V_s|^2)(|W_s-\bar W_s|+2|\bar W_s|))(|V_s-\bar V_s|+|W_s-\bar W_s|)\mathrm{d}s
\end{aligned}
\tag{3.15}
$$

Using once more (H4), we get

$$
T(3)\leqslant 2K\int_t^\tau \Phi_f'(|V_s-\bar V_s|^2)|V_s-\bar V_s|^2|\bar W_s|\mathrm{d}s+2K\int_t^\tau \Phi_f'(|V_s-\bar V_s|^2)|V_s-\bar V_s||W_s-\bar W_s||\bar W_s|\mathrm{d}s \tag{3.16}
$$

It follows from the properties of $\Phi_f', V, \bar V, W, \bar W$ and σ, that T_4 is a square integrable martingale (see (3.22)) and we write

$$
T(4):=-\int_t^\tau 2\Phi_f'(|V_s-\bar V_s|^2)(V_s-\bar V_s)\langle W_s-\bar W_s,\sigma(s,U_s,V_s)\mathrm{d}\bar B_s\rangle=-\int_t^\tau \mathrm{d}M_s. \tag{3.17}
$$

Using (H2), we get

$$
T(5)\leqslant -\lambda\int_t^\tau \Phi_f'(|V_s-\bar V_s|^2)|W_s-\bar W_s|^2\mathrm{d}s. \tag{3.18}
$$

Using (H2) and Lemma B.1, we have

$$
T(6)\leqslant -\lambda\int_t^\tau 2\kappa|V_s-\bar V_s|^2|W_s-\bar W_s|^2 f(|V_s-\bar V_s|^2)\Phi_f'(|V_s-\bar V_s|^2)\mathrm{d}s. \tag{3.19}
$$

Substituting (3.14)-(3.19) into (3.13) and using the Young inequality repeatedly and the fact that V and $\bar{V}$ are bounded we get

$$
\begin{aligned}
&\Phi_f(|V_t-\bar{V}_t|^2)+\lambda\int_t^\tau\big(1+2\kappa|V_s-\bar{V}_s|^2f(|V_s-\bar{V}_s|^2)\big)\,\Phi_f'(|V_s-\bar{V}_s|^2)|W_s-\bar{W}_s|^2\mathrm{d}s\\
\leq&\Phi_f(|V_\tau-\bar{V}_\tau|^2)+C\int_t^\tau\big(1+|\bar{W}_s|+|\nabla_{xx}^2u(s,U_s)|\big)\,\Phi_f'(|V_s-\bar{V}_s|^2)|V_s-\bar{V}_s|^2\mathrm{d}s-\int_t^\tau\mathrm{d}M_s\\
&+2K\int_t^\tau(1+|\bar{W}_s|)\Phi_f'(|V_s-\bar{V}_s|^2)|V_s-\bar{V}_s||W_s-\bar{W}_s|\mathrm{d}s+2K\int_t^\tau\Phi_f'(|V_s-\bar{V}_s|^2)|V_s-\bar{V}_s|^2f(|V_s-\bar{V}_s|^2)|W_s-\bar{W}_s|\mathrm{d}s\\
&+2K\int_t^\tau\Phi_f'(|V_s-\bar{V}_s|^2)|V_s-\bar{V}_s|f(|V_s-\bar{V}_s|^2)|W_s-\bar{W}_s|^2\mathrm{d}s+4K\int_t^\tau\Phi_f'(|V_s-\bar{V}_s|^2)|V_s-\bar{V}_s|^2f(|V_s-\bar{V}_s|^2)|\bar{W}_s|\mathrm{d}s\\
&+4K\int_t^\tau\Phi_f'(|V_s-\bar{V}_s|^2)|V_s-\bar{V}_s|f(|V_s-\bar{V}_s|^2)|\bar{W}_s||W_s-\bar{W}_s|\mathrm{d}s.
\end{aligned}
$$

Using the boundedness of V and $\bar{V}$, the local boundedness and positivity of f, it holds that there exists $M_1>0$ such that $f(|V_s-\bar{V}_s|^2)\leq M_1$ and the Young inequality yields

$$
\begin{aligned}
&\Phi_f(|V_t-\bar{V}_t|^2)+\lambda\int_t^\tau\big(1+2\kappa|V_s-\bar{V}_s|^2f(|V_s-\bar{V}_s|^2)\big)\,\Phi_f'(|V_s-\bar{V}_s|^2)|W_s-\bar{W}_s|^2\mathrm{d}s\\
\leq&\Phi_f(|V_\tau-\bar{V}_\tau|^2)+C\int_t^\tau\big(1+|\bar{W}_s|+|\nabla_{xx}^2u(s,U_s)|\big)\,\Phi_f'(|V_s-\bar{V}_s|^2)|V_s-\bar{V}_s|^2\mathrm{d}s-\int_t^\tau\mathrm{d}M_s\\
&+2K\int_t^\tau\Phi_f'(|V_s-\bar{V}_s|^2)(\epsilon_1|W_s-\bar{W}_s|^2+\frac{1}{\epsilon_1}|V_s-\bar{V}_s|^2(1+|\bar{W}_s|)^2)\mathrm{d}s\\
&+2K\int_t^\tau\Phi_f'(|V_s-\bar{V}_s|^2)|V_s-\bar{V}_s|^2f(|V_s-\bar{V}_s|^2)(\epsilon_2|W_s-\bar{W}_s|^2+\frac{1}{\epsilon_2})\mathrm{d}s\\
&+2K\int_t^\tau\Phi_f'(|V_s-\bar{V}_s|^2)f(|V_s-\bar{V}_s|^2)|W_s-\bar{W}_s|^2(\epsilon_3+\frac{1}{\epsilon_3}|V_s-\bar{V}_s|^2)\mathrm{d}s\\
&+4KM_1\int_t^\tau\Phi_f'(|V_s-\bar{V}_s|^2)|V_s-\bar{V}_s|^2|\bar{W}_s|\mathrm{d}s\\
&+4KM_1\int_t^\tau\Phi_f'(|V_s-\bar{V}_s|^2)(\frac{1}{\epsilon_4}|\bar{W}_s|^2|V_s-\bar{V}_s|^2+\epsilon_4|W_s-\bar{W}_s|^2)\mathrm{d}s\\
\leq&\Phi_f(|V_\tau-\bar{V}_\tau|^2)+C_{\epsilon_1,\epsilon_4,K,M_1}\int_t^\tau\big(1+|\bar{W}_s|+|\bar{W}_s|^2+|\nabla_{xx}^2u(s,U_s)|\big)\,\Phi_f'(|V_s-\bar{V}_s|^2)|V_s-\bar{V}_s|^2\mathrm{d}s-\int_t^\tau\mathrm{d}M_s\\
&+2K\int_t^\tau(\epsilon_1+2M_1\epsilon_4+\epsilon_3M_1)\Phi_f'(|V_s-\bar{V}_s|^2)|W_s-\bar{W}_s|^2\mathrm{d}s\\
&+2K\int_t^\tau(\epsilon_2+\frac{1}{\epsilon_3})\Phi_f'(|V_s-\bar{V}_s|^2)|V_s-\bar{V}_s|^2f(|V_s-\bar{V}_s|^2)|W_s-\bar{W}_s|^2\mathrm{d}s.
\end{aligned}
$$

We recall (see Lemma 2.4) that for all $t\leq s\leq T$, $\bar{W}_s=\nabla_xu(s,U_s)$ is bounded by $\Upsilon^{(3)}(T-s)^{-1+\alpha_3/2}$ ($\Upsilon^{(3)},\alpha_3>0$). Applying once more the Young inequality, it holds that there exists a constant $C:=C_{\epsilon_1,\epsilon_4,K,M_1}$ such that:

$$
\begin{aligned}
&\Phi_f(|V_t-\bar{V}_t|^2)+2\lambda\int_t^\tau\big(1+2\kappa|V_s-\bar{V}_s|^2f(|V_s-\bar{V}_s|^2)\big)\,\Phi_f'(|V_s-\bar{V}_s|^2)|W_s-\bar{W}_s|^2\mathrm{d}s\\
\leq&\Phi_f(|V_\tau-\bar{V}_\tau|^2)+C\int_t^\tau\big(1+(T-s)^{-1+\alpha_3}+|\nabla_{xx}^2u(s,U_s)|\big)\,\Phi_f'(|V_s-\bar{V}_s|^2)|V_s-\bar{V}_s|^2\mathrm{d}s-\int_t^\tau\mathrm{d}M_s\\
&+2K\int_t^\tau(\epsilon_1+2M_1\epsilon_4+\epsilon_3M_1)\Phi_f'(|V_s-\bar{V}_s|^2)|W_s-\bar{W}_s|^2\mathrm{d}s\\
&+2K\int_t^\tau(\epsilon_2+\frac{1}{\epsilon_3})\Phi_f'(|V_s-\bar{V}_s|^2)|V_s-\bar{V}_s|^2f(|V_s-\bar{V}_s|^2)|W_s-\bar{W}_s|^2\mathrm{d}s.
\end{aligned}
$$

Thus choose $\epsilon_1 = \frac{\lambda}{16K}, \epsilon_3 = \frac{\lambda}{32KM_1}, \epsilon_4 = \frac{\lambda}{64KM_1}$

$$\begin{aligned}
&\Phi_f(|V_t - \bar V_t|^2) + \lambda \int_t^\tau \left(3/4 + 2\kappa |V_s - \bar V_s|^2 f(|V_s - \bar V_s|^2)\right) \Phi_f'(|V_s - \bar V_s|^2)|W_s - \bar W_s|^2 \mathrm{d}s \\
\le &\Phi_f(|V_\tau - \bar V_\tau|^2) + C\int_t^\tau \left(1 + (T-s)^{-1+\alpha_3} + |\nabla^2_{xx} u(s,U_s)|\right) \Phi_f'(|V_s - \bar V_s|^2)|V_s - \bar V_s|^2 \mathrm{d}s - \int_t^\tau \mathrm{d}M_s \\
&+ 2K\int_t^\tau (\epsilon_2 + \frac{1}{\epsilon_3})\Phi_f'(|V_s - \bar V_s|^2)|V_s - \bar V_s|^2 f(|V_s - \bar V_s|^2)|W_s - \bar W_s|^2 \mathrm{d}s.
\end{aligned}$$

Choose now $\kappa = \lambda^{-1}(\epsilon_2 + \frac{1}{\epsilon_3})K$, and use Lemma B.1 to get

$$\begin{aligned}
&\Phi_f(|V_t - \bar V_t|^2) + 3\lambda/4 \int_t^\tau \Phi_f'(|V_s - \bar V_s|^2)|W_s - \bar W_s|^2 \mathrm{d}s \\
\le &\Phi_f(|V_\tau - \bar V_\tau|^2) + C\int_t^\tau \left(1 + (T-s)^{-1+\alpha_3} + |\nabla^2_{xx} u(s,U_s)|\right) \Phi_f(|V_s - \bar V_s|^2) \mathrm{d}s - \int_t^\tau \mathrm{d}M_s. \quad (3.20)
\end{aligned}$$

Fix $t \in [0,T[$ and multiply both sides in (3.20) by $1_{\{t\le\tau\}}$ to get

$$\begin{aligned}
&1_{\{t\le\tau\}}\Phi_f(|V_t - \bar V_t|^2) + 3\lambda/4 1_{\{t\le\tau\}} \int_t^\tau \Phi_f'(|V_s - \bar V_s|^2)|W_s - \bar W_s|^2 \mathrm{d}s \\
\le &1_{\{t\le\tau\}}\Phi_f(|V_\tau - \bar V_\tau|^2) + C 1_{\{t\le\tau\}}\int_t^\tau \left(1 + (T-s)^{-1+\alpha_3} + |\nabla^2_{xx} u(s,U_s)|\right) \Phi_f(|V_s - \bar V_s|^2) \mathrm{d}s \\
&- 1_{\{t\le\tau\}}\int_t^\tau \mathrm{d}M_s. \quad (3.21)
\end{aligned}$$

Let us now focus on the martingale expression in the above inequality. Using the boundedness of V and $\bar V$, the linear growth assumption of the diffusion coefficient, the local boundedness of the gradient and since (U,V,W) is solution to the QFBSDE (1.1) and $\Phi_f'(z) \leqslant \exp(\kappa ||f||_{L^1(\mathbb{R})})$, we obtain the following bound

$$\bar{\mathbb{E}}\int_0^T \mathrm{d}\langle M\rangle_s \leqslant C\bar{\mathbb{E}}\int_0^T |W_s - \tilde W_s|^2 \mathrm{d}s < \infty. \quad (3.22)$$

The above bound ensures that the martingale $(M_t)_{0\le t}$ is square integrable and its conditional expectation with respect to $\bar{\mathfrak{F}}_t$ is zero. Thus taking conditional expectation on both sides of (3.21), we have

$$\begin{aligned}
&1_{\{t\le\tau\}}\Phi_f(|V_t - \bar V_t|^2) + 3\lambda/4 \bar{\mathbb{E}}\Big[1_{\{t\le\tau\}} \int_t^\tau \Phi_f'(|V_s - \bar V_s|^2)|W_s - \bar W_s|^2 \mathrm{d}s \Big| \bar{\mathfrak{F}}_t\Big] \\
&\leqslant \bar{\mathbb{E}}\Big[1_{\{t\le\tau\}}\Phi_f(|V_\tau - \bar V_\tau|^2)\Big|\mathfrak{F}_t\Big] \\
&\quad + C\bar{\mathbb{E}}\Big[1_{\{t\le\tau\}}\int_t^\tau \left(1 + (T-s)^{-1+\alpha_3} + |\nabla^2_{xx} u(s,U_s)|\right) \Phi_f(|V_s - \bar V_s|^2) \mathrm{d}s \Big| \bar{\mathfrak{F}}_t\Big]. \quad (3.23)
\end{aligned}$$

Consider $\tau(t,r)$ as defined in Lemma 3.1 and set $\tau = \tau(t,r) \wedge \bar\rho(R) \wedge \bar{\rho_1}(R_1)$. Assume without loss of generality that $\gamma \leqslant \alpha_3$ and apply the general Young inequality $ab \le a^p/p + b^q/q, 1/p + 1/q = 1$

to obtain

$$
\begin{aligned}
&1_{\{t\leq\tau\}}\Phi_f(|V_t-\bar{V}_t|^2)+3\lambda/41_{\{t\leq\tau\}}\bar{\mathbb{E}}\Big[\int_t^\tau \Phi_f'(|V_s-\bar{V}_s|^2)|W_s-\bar{W}_s|^2\mathrm{d}s\Big|\bar{\mathfrak{F}}_t\Big]\\
\leq&1_{\{t\leq\tau\}}\bar{\mathbb{E}}\Big[\Phi_f(|V_\tau-\bar{V}_\tau|^2)\Big|\bar{\mathfrak{F}}_t\Big]\\
&+C1_{\{t\leq\tau\}}\bar{\mathbb{E}}\Big[\int_t^\tau(1+(T-s)^{-1+\gamma}+(T-s)^{-1+\gamma}(T-s)^{1-\gamma}|\nabla^2_{xx}u(s,U_s)|)\Phi_f(|V_s-\bar{V}_s|^2)\mathrm{d}s\Big|\bar{\mathfrak{F}}_t\Big]\\
\leq&1_{\{t\leq\tau\}}\bar{\mathbb{E}}\Big[\Phi_f(|V_\tau-\bar{V}_\tau|^2)\Big|\bar{\mathfrak{F}}_t\Big]+C1_{\{t\leq\tau\}}\bar{\mathbb{E}}\Big[\int_t^\tau(1+(T-s)^{-1+\gamma}+(T-s)^{(-1+\gamma)(1-\gamma/2)/(1-\gamma)}\\
&+(T-s)^{2(1-\gamma)(1-\gamma/2)/\gamma}|\nabla^2_{x,x}u(s,U_s)|^{2(1-\gamma/2)/\gamma})\Phi_f(|V_s-\bar{V}_s|^2)\mathrm{d}s\Big|\bar{\mathfrak{F}}_t\Big]\\
\leq&1_{\{t\leq\tau\}}\bar{\mathbb{E}}\Big[\Phi_f(|V_\tau-\bar{V}_\tau|^2)\Big|\bar{\mathfrak{F}}_t\Big]\\
&+C1_{\{t\leq\tau\}}\bar{\mathbb{E}}\Big[\int_t^\tau(1+(T-s)^{-1+\gamma/2}+(T-s)^{2(1-\gamma)(1-\gamma/2)/\gamma}|\nabla^2_{x,x}u(s,U_s)|^{2(1-\gamma/2)/\gamma})\Phi_f(|V_s-\bar{V}_s|^2)\mathrm{d}s\Big|\bar{\mathfrak{F}}_t\Big].
\end{aligned}
$$

Applying Lemma 3.1, with $p=2(1-\gamma/2)/\gamma$, we have

$$
\begin{aligned}
&1_{\{t\leq\tau\}}\Phi_f(|V_t-\bar{V}_t|^2)+3\lambda/41_{\{t\leq\tau\}}\bar{\mathbb{E}}\Big[\int_t^\tau \Phi_f'(|V_s-\bar{V}_s|^2)|W_s-\bar{W}_s|^2\mathrm{d}s\Big|\bar{\mathfrak{F}}_t\Big]\\
\leq&1_{\{t\leq\tau\}}\bar{\mathbb{E}}\Big[\Phi_f(|V_\tau-\bar{V}_\tau|^2)\Big|\bar{\mathfrak{F}}_t\Big]\\
&+C[(T-t)^{\gamma/2}+(T-t)^{1/d+1}r^{d/d+1}]\sup_{t\leq s\leq T}\operatorname{essup}_{\omega\in\bar{\Omega}}[\Phi_f(|V_s-\bar{V}_s|^2]. \qquad (3.24)
\end{aligned}
$$

We make the following crucial observation: the probability measure $\bar{\mathbb{P}}$ does not depend on R or R_1. Thus, in (3.23) we can successively let $R\to\infty$ and $R_1\to\infty$ i.e., $\tau\to\tau(t,r)$. Using the continuity of Φ_f(by considering its representative which belongs to $C^1(\mathbb{R})$), V and $\bar{V}$, we obtain

$$
\begin{aligned}
&\Phi_f(|V_t-\bar{V}_t|^2)+3\lambda/4\bar{\mathbb{E}}\Big[\int_t^{\tau(t,r)} \Phi_f'(|V_s-\bar{V}_s|^2)|W_s-\bar{W}_s|^2\mathrm{d}s\Big|\bar{\mathfrak{F}}_t\Big]\\
\leq&\bar{\mathbb{E}}\Big[\Phi_f(|V_{\tau(t,r)}-\bar{V}_{\tau(t,r)}|^2)\Big|\bar{\mathfrak{F}}_t\Big]\\
&+C[(T-t)^{\gamma/2}+(T-t)^{1/d+1}r^{d/d+1}]\sup_{t\leq s\leq T}\operatorname{essup}_{\omega\in\bar{\Omega}}[\Phi_f(|V_s-\bar{V}_s|^2]. \qquad (3.25)
\end{aligned}
$$

Using the boundedness of Φ_f, the first term in the above inequality satisfies:

$$
\begin{aligned}
\bar{\mathbb{E}}\Big[\Phi_f(|V_{\tau(t,r)}-\bar{V}_{\tau(t,r)}|^2)\Big|\bar{\mathfrak{F}}_t\Big]\leq&\bar{\mathbb{E}}\Big[\Phi_f(|V_{\tau(t,r)}-\bar{V}_{\tau(t,r)}|^2)1_{\tau(t,r)<T}\Big|\bar{\mathfrak{F}}_t\Big]+\bar{\mathbb{E}}\Big[\Phi_f(|V_T-\bar{V}_T|^2)\Big|\bar{\mathfrak{F}}_t\Big]\\
\leq& C\bar{\mathbb{P}}\left(\{\tau(t,r)<T\}\big|\bar{\mathfrak{F}}_t\right)+\bar{\mathbb{E}}\Big[\Phi_f(|V_T-\bar{V}_T|^2)\Big|\bar{\mathfrak{F}}_t\Big]. \qquad (3.26)
\end{aligned}
$$

It follows from definition of $\tau(t,r)$ that

$$
\bar{\mathbb{P}}\left(\{\tau(t,r)<T\}\big|\bar{\mathfrak{F}}_t\right)\leq\bar{\mathbb{P}}\Big(\sup_{t\leq s\leq T}|U_s-U_t|\geq r\big|\bar{\mathfrak{F}}_t\Big),
$$

with $U_s=x+\int_0^s b(r,U_r,V_r,W_r)\mathrm{d}r+\int_0^s\sigma(r,U_r,V_r)\mathrm{d}\bar{B}_r$. In order to estimate the above conditional probability, we wish to apply a Bernstein's type of inequality. Since the process $(U_s-U_t)_{t\leq s\leq T}$ is not a continuous local martingale, we use a measure change to remove the drift of $(U_s-U_t)_{t\leq s\leq T}$. In this spirit, define $\theta(\cdot,U_\cdot,V_\cdot):=b(\cdot,U_\cdot,V_\cdot)\sigma^{-1}(\cdot,U_\cdot,V_\cdot)$. Since, (U,V,W) is solution to FBSDE 1.1 and using the condition on b,σ,U and V, we deduce that θ is bounded. Then, the following probability measure

$$
\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\bar{\mathbb{P}}}:=\exp\left(\zeta_0^{T,\bar{B}}(\theta)\right),\quad T\geq 0. \qquad (3.27)
$$

is well defined, with

$$
\zeta_t^{s,\bar{B}}(\theta):=\int_t^s\theta(r,U_r,V_r)\mathrm{d}\bar{B}_r-\frac{1}{2}\int_t^s|\theta(r,U_r,V_r)|^2\mathrm{d}r.
$$

Therefore, the Girsanov's theorem yields

$$\hat{B}_s := \bar{B}_s - \int_0^s \theta(r, U_r, V_r)\mathrm{d}r, \tag{3.28}$$

is an $\bar{\mathfrak{F}}_s$-Brownian motion under $\mathbb{Q}$ for all $t \le s \le T$. Furthermore

$$U_s - U_t = \int_t^s \sigma(r, U_r, V_r)\mathrm{d}\hat{B}_r. \tag{3.29}$$

In addition, $\mathbb{Q}$ and $\bar{\mathbb{P}}$ are equivalent and using the Girsanov's theorem and Hölder's inequality, we obtain

$$\begin{aligned}
&\bar{\mathbb{P}}\Big(\sup_{t\le s\le T}|U_s - U_t| \ge r\big|\bar{\mathfrak{F}}_t\Big)\\
&= \mathbb{E}^{\mathbb{Q}}\Big(\exp\Big(\zeta_t^{T,\hat{B}}(-\theta)\Big)\mathbf{1}_{\{\sup_{t\le s\le T}|U_s-U_t|\ge r\}}\big|\bar{\mathfrak{F}}_t\Big)\\
&\le \mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(\zeta_t^{T,\hat{B}}(-\theta)\Big)^2\big|\bar{\mathfrak{F}}_t\Big]^{1/2} \times \mathbb{Q}\Big(\sup_{t\le s\le T}|U_s - U_t| \ge r\big|\bar{\mathfrak{F}}_t\Big)^{1/2}\\
&\le \mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(\zeta_t^{T,\hat{B}}(-4\theta)\Big)\big|\bar{\mathfrak{F}}_t\Big]^{1/4} \times \mathbb{E}^{\mathbb{Q}}\left[\exp\Big(6\int_t^T|\theta(r, U_r, V_r)|^2\mathrm{d}r\Big)\big|\bar{\mathfrak{F}}_t\right]^{1/4} \times \mathbb{Q}\Big(\sup_{t\le s\le T}|U_s - U_t| \ge r\big|\bar{\mathfrak{F}}_t\Big)^{1/2}.
\end{aligned}$$

It follows from the Novikov's condition that the first term of the above inequality is finite and using the boundedness of the process θ, the second one is bounded. Then, we deduce the existence of a constant $C > 0$ such that

$$\begin{aligned}
\bar{\mathbb{P}}\Big(\sup_{t\le s\le T}|U_s - U_t| \ge r\big|\bar{\mathfrak{F}}_t\Big) \le& C\mathbb{Q}\Big(\sup_{t\le s\le T}|U_s - U_t| \ge r\big|\bar{\mathfrak{F}}_t\Big)^{1/2}\\
=& C\mathbb{Q}\Big(\sup_{t\le s\le T}\Big|\int_t^s \sigma(s, U_s, V_s)\mathrm{d}\hat{B}_s\Big| \ge r\big|\bar{\mathfrak{F}}_t\Big)^{1/2}
\end{aligned}$$

Using the Bernstein inequality, one can get as in [8, (4.19)]

$$\bar{\mathbb{P}}\Big(\sup_{t\le s\le T}|U_s - U_t| \ge r\big|\bar{\mathfrak{F}}_t\Big) \le C\exp(-C^{-1}r^2(T-t)^{-1}). \tag{3.30}$$

Hence from (3.25) and (3.30), there exists a constant C such that

$$\begin{aligned}
\Phi_f(|V_t - \bar{V}_t|^2) \le& \operatorname{essup}_{\omega\in\bar{\Omega}}[\Phi_f(|V_T - \bar{V}_T|^2)] + C\exp\big(-C^{-1}r^2(T-t)^{-1}\big)\\
&+ C[(T-t)^{\gamma/2} + (T-t)^{1/d+1}r^{d/d+1}]\sup_{t\le s\le T}\operatorname{essup}_{\omega\in\bar{\Omega}}[\Phi_f(|V_s - \bar{V}_s|)^2].
\end{aligned} \tag{3.31}$$

The above inequality is similar to [8, Section 4.3.5]. We can also apply the non trivial discrete Gronwall's lemma as developed there. For the sake of completeness, we briefly present it in this paper. The inequality (3.31) is also valid for every $s \in [t, T]$, hence taking supremum on both side, we have

$$\begin{aligned}
\sup_{t\le s\le T}\operatorname{essup}_{\omega\in\bar{\Omega}}\Phi_f(|V_s - \bar{V}_s|^2) \le& \operatorname{essup}_{\omega\in\bar{\Omega}}[\Phi_f(|V_T - \bar{V}_T|^2)] + C\exp\big(-C^{-1}r^2(T-t)^{-1}\big)\\
&+ C[(T-t)^{\gamma/2} + (T-t)^{1/d+1}r^{d/d+1}]\sup_{t\le s\le T}\operatorname{essup}_{\omega\in\bar{\Omega}}[\Phi_f(|V_s - \bar{V}_s|)^2].
\end{aligned} \tag{3.32}$$

Taking $r = (T-t)m$ for a free parameter $m \ge (T-t)^{-1}$, we obtain

$$\begin{aligned}
\sup_{t\le s\le T}\operatorname{essup}_{\omega\in\bar{\Omega}}\Phi_f(|V_s - \bar{V}_s|^2) \le& \operatorname{essup}_{\omega\in\bar{\Omega}}[\Phi_f(|V_T - \bar{V}_T|^2)] + C\exp\big(-C^{-1}m^2(T-t)\big)\\
&+ C[(T-t)^{\gamma/2} + (T-t)^{1/d+1}m^{d/d+1}]\sup_{t\le s\le T}\operatorname{essup}_{\omega\in\bar{\Omega}}[\Phi_f(|V_s - \bar{V}_s|)^2].
\end{aligned} \tag{3.33}$$

Choose $\delta > 0$ such that $C[\delta^{\gamma/2} + \delta m^{d/d+1}] = 1/2$. For m large enough, $\delta m^{d/d+1} \equiv 1/(2C)$, so that $\delta m \geq 1$. Hence (3.33) still valid for $T - t = \delta$

$$\sup_{T-\delta\leq s\leq T} \operatorname{essup}_{\omega\in\bar{\Omega}} \Phi_f(|V_s - \bar{V}_s|^2) \leq 2\operatorname{essup}_{\omega\in\bar{\Omega}}[\Phi_f(|V_T - \bar{V}_T|^2)] + 2C\exp\left(-C^{-1}m^2\delta\right). \quad (3.34)$$

Using the properties of the solution $u$to the PDE (1.3) (boundedness and Hölder continuity) the above inequality (3.34) can be obtained on the following interval of lenght δ: $[T-2\delta, T-\delta], [T-3\delta, T-2\delta], \dots, [T-(i+1)\delta, T-i\delta], \dots [0, T-N\delta], N \equiv \lfloor T\delta^{-1}\rfloor, i+1 \leq N$. In particular (3.34) is valid on each of these intervals, with the same constant C. Set

$$a_0 \equiv \operatorname{essup}_{\omega\in\bar{\Omega}} \Phi_f(|V_T - \bar{V}_T|^2), \quad a_i \equiv \sup_{s\in[T-i\delta, T-(i-1)\delta]} \operatorname{essup}_{\omega\in\bar{\Omega}} \Phi_f(|V_s - \bar{V}_s|^2), \quad 1 \leq i \leq N,$$

$$a_{N+1} \equiv \sup_{s\in[0,T-N\delta]} \operatorname{essup}_{\omega\in\bar{\Omega}} \Phi_f(|V_s - \bar{V}_s|^2).$$

It follows from (3.33)that $a_{i+1} \leq 2a_i + 2C\exp(-C^{-1}m^2\delta)$ for all $i \in \{0, \dots, N\}$.

Applying a version of the discrete Gronwall lemma for all $i \in \{0, \dots, N+1\}$ and the fact that a_0 is reduced to zero, we obtain:

$$\begin{aligned} a_i &\leq 2C(2^i - 1)\exp(-C^{-1}m^2\delta) \leq 2C\exp(i(\ln 2) - C^{-1}m^2\delta) \\ &\leq 2C\exp((N+1)(\ln 2) - C^{-1}m^2\delta) \leq \tilde{C}\exp(\tilde{C}T\delta^{-1} - \tilde{C}^{-1}m^2\delta). \end{aligned}$$

For m large enough, we have $\delta^{-1} \leq 4Cm^{d/d+1}$. In particular, there is a constant $\bar{C}$ such for all $i \in \{0, \dots, N+1\}$

$$a_i \leq \bar{C}\exp(\bar{C}Tm^{d/d+1} - \bar{C}^{-1}m^2 m^{-d/d+1}) = \bar{C}\exp(\bar{C}Tm^{d/d+1} - \bar{C}^{-1}m^{d+2/d+1}).$$

Taking the supremum over all the indices $i \in \{0, \dots, N+1\}$, we obtain

$$\sup_{0\leq s\leq T} \operatorname{essup}_{\omega\in\bar{\Omega}} \Phi_f(|V_s - \bar{V}_s|^2) \leq \bar{C}\exp(\bar{C}Tm^{d/d+1} - \bar{C}^{-1}m^{d+2/d+1}).$$

By letting $m \to +\infty$ in the above inequality, we have

$$\operatorname{essup}_{\omega\in\bar{\Omega}} \Phi_f(|V_t - \bar{V}_t|^2) = 0, \quad \text{for all } t \in [0, T]. \quad (3.35)$$

Using Lemma B.1 and the continuity of V and $\bar{V}$, we deduce from (3.35) that

$$\mathbb{P}\text{-a.s.}, \forall t \in [0, T], \quad V_t = \bar{V}_t = u(t, U_t). \quad (3.36)$$

On the other hand, using (4) in Lemma B.1 and (3.24), it follows from (3.35) that

$$x\forall t \in [0, T[, \quad W_t = \bar{W}_t = \nabla_x u(t, U_t), \ \mathrm{d}t \otimes \mathbb{P}\text{-a.s.}, \quad (3.37)$$

From (3.36) and (3.37), we observe that in order to show $\mathbb{P} \circ (X, Y, Z, B)^{-1} = \bar{\mathbb{P}} \circ (U, V, W, \bar{B})^{-1}$, its enough to prove that (X, B) and $(U, \bar{B})$ have the same law. Under Assumption 1.1, it is readily seen that the martingale problem associated to $b(\cdot, \cdot, u(\cdot, \cdot), \nabla_x u(\cdot, \cdot)), a(\cdot, \cdot, u(\cdot, \cdot))$ is well-posed (see for instance [38]), we deduce that the distribution of $(U, \bar{B})$ under $\bar{\mathbb{P}}$ matches the law of (X, B) under $\mathbb{P}$. This concludes the proof.

□

3.3. **Proof Theorem 1.7.** From the previous result, there exist a standard set-up $(\Omega, \mathfrak{F}, \mathbb{P}, \mathbb{F}, B)$ and a continuous $\{\mathfrak{F}_t\}_{0\leq t\leq T}$ -adapted process X such that for all $t \in [0, T]$,

$$X_t = x + \int_0^t \tilde{b}(s, X_s)\mathrm{d}s + \int_0^t \tilde{\sigma}(s, X_s)\mathrm{d}B_s,$$

where, $\tilde{b}(s, X_s) := b(s, X_s, u(s, X_s), \nabla_x u(s, X_s))$, $\tilde{\sigma}(s, X_s) := \sigma(s, X_s, u(s, X_s))$. Observe that under the assumptions of the Theorem, there exists a constant $C > 0$ such that for all $(t, x, x') \in [0, T-\delta] \times \mathbb{R} \times \mathbb{R}$, $\delta > 0$

$$|\tilde{\sigma}(t, x) - \tilde{\sigma}(t, x')| \leq C|x - x'|^{\alpha_0}. \quad (3.38)$$

Indeed, let $(t,x,x') \in [0,T]\times\mathbb{R}\times\mathbb{R}$, we have

$$\begin{aligned}|\tilde{\sigma}(t,x)-\tilde{\sigma}(t,x')| &= |\sigma(t,x,u(t,x))-\sigma(t,x',u(t,x'))|\\ &\le |\sigma(t,x,u(t,x))-\sigma(t,x',u(t,x))| + |\sigma(t,x',u(t,x))-\sigma(t,x',u(t,x'))|\\ &\le L\big(|x-x'|^{\alpha_0}+|u(t,x)-u(t,x')|\big),\end{aligned}$$

and the bound follows from the Lipschitz continuity of u for $t\in[0,T-\delta]$.

For any continuous semimartingale X, we will denote by $L^a_t(X)$ the right-continuous version of its local time at level a. We have the following result

Lemma 3.2. *Let X^1 and X^2 be two solutions to the SDE* (3.1) *with the same underlying Brownian motion B. Let $t\in[0,T-\delta]$. Then $L^0_t(X^1-X^2)=0$.*

Proof. Assume that, there is $t\in[0,T], \varepsilon>0$ and a set $A\in\mathfrak{F}$, with $\mathbb{P}(A)>0$ such that $L^0_t(X^1-X^2)(\omega)>\varepsilon$ for $\omega\in A$. Since the map $a\mapsto L^a_t(X^1-X^2)$ is right continuous, then there is $\tilde{\delta}>0$ such that for all $a\in[0,\tilde{\delta}]$, $L^a_t(X^1-X^2)\ge\varepsilon/2$, on A. Therefore, using the occupation-times formula and keeping in mind that $2\alpha_0\ge1$, we deduce that:

$$\int_0^t \frac{\mathrm{d}\langle X^1-X^2,X^1-X^2\rangle_s}{|X^1_s-X^2_s|^{2\alpha_0}} = \int_0^{+\infty}\frac{1}{|a|^{2\alpha_0}}L^a_t(X^1-X^2)\mathrm{d}a \ge \frac{\varepsilon}{2}\int_0^{\tilde{\delta}}\frac{1}{a^{2\alpha_0}}\mathrm{d}a=+\infty, \quad \text{on } A. \tag{3.39}$$

On the other hand, using (3.38), we obtain

$$\int_0^t \frac{\mathrm{d}\langle X^1-X^2,X^1-X^2\rangle_s}{|X^1_s-X^2_s|^{2\alpha_0}} = \int_0^t\frac{(\tilde{\sigma}(s,X^1_s)-\tilde{\sigma}(s,X^2_s))^2}{|X^1_s-X^2_s|^{2\alpha_0}}\mathrm{d}s\le Ct, \quad \text{on } A. \tag{3.40}$$

Thus $\mathbb{P}(A)=0$, which contradicts (3.39). Since A was arbitrarily chosen, we conclude that $L^0_t(X^1-X^2)=0$, $\mathbb{P}$-a.s.. □

Lemma 3.3. *The processes $X^1\wedge X^2$ and $X^1\vee X^2$ are also solutions to the SDE* (3.1)*, whenever X^1 and X^2 are solutions to* (3.1)*. Moreover,* (3.1) *has a unique strong solution.*

Proof. Lemma 3.2 and the Tanaka's formula yield

$$\begin{aligned}X^1\vee X^2 &= X^2_t+(X^1_t-X^2_t)^+\\ &= X^2_t+\int_0^t 1_{\{X^1_s>X^2_s\}}\mathrm{d}(X^1_s-X^2_s)+\frac12 L^0_t(X^1-X^2)\\ &= X^2_t+\int_0^t 1_{\{X^1_s>X^2_s\}}\mathrm{d}X^1_s-\int_0^t 1_{\{X^1_s>X^2_s\}}\mathrm{d}X^2_s\\ &= x+\int_0^t 1_{\{X^1_s>X^2_s\}}\mathrm{d}X^1_s+\int_0^t 1_{\{X^1_s\le X^2_s\}}\mathrm{d}X^2_s\\ &= x+\int_0^t\Big(1_{\{X^1_s>X^2_s\}}\tilde{b}(s,X^1_s)+1_{\{X^1_s\le X^2_s\}}\tilde{b}(s,X^2_s)\Big)\mathrm{d}s+\int_0^t\Big(1_{\{X^1_s>X^2_s\}}\tilde{\sigma}(s,X^1_s)+1_{\{X^1_s\le X^2_s\}}\tilde{\sigma}(s,X^2_s)\Big)\mathrm{d}B_s\\ &= x+\int_0^t\tilde{b}(s,X^1_s\vee X^2_s)\mathrm{d}s+\int_0^t\tilde{\sigma}(s,X^1_s\vee X^2_s)\mathrm{d}B_s.\end{aligned}$$

This proves that $X^1\vee X^2$ is also solution to (3.1). Similarly, one can show that $X^1\wedge X^2$ is also a solution to (3.1). Since $X^1\wedge X^2$ and $X^1\vee X^2$ have the same law, we have

$$\mathbb{E}|X^1_t-X^2_t| = \mathbb{E}[X^1\vee X^2-X^1\wedge X^2]=0.$$

This implies that $X^1=X^2$ and hence pathwise uniqueness. Therefore, from the Yamada-Watanabe's theorem, (3.1) has a unique strong solution. □

Proof Theorem 1.7. Let (X,Y,Z) and (X^1,Y^1,Z^1) be two weak solutions to (1.1) under the same underlying stochastic basis $(\Omega,\mathfrak{F},\mathbb{P},(\mathfrak{F}_t)_{0\le t\le T},B)$. From the uniqueness in law (see subsection 3.2) the following connections hold

$$\begin{cases} Y_t = u(t, X_t) \\ Z_t = \nabla_x u(t, X_t) \end{cases} \qquad \begin{cases} Y_t^1 = u(t, X_t^1) \\ Z_t^1 = \nabla_x u(t, X_t^1) \end{cases}$$

from the previous lemma, $X \equiv X^1$ then,

$$\begin{cases} Y_t = u(t, X_t) \\ Z_t = \nabla_x u(t, X_t) \end{cases} \qquad \begin{cases} Y_t^1 = u(t, X_t) \\ Z_t^1 = \nabla_x u(t, X_t). \end{cases}$$

Therefore, $Y \equiv Y^1$ and $Z \equiv Z^1$ □

3.4. **Proof of Theorem 1.8.** Consider the function $\tilde{b} : (t, x) \mapsto b(t, x, u(t, x), \nabla_x u(t, x))$. Therefore the forward equation can be writen as

$$X_t = x + \int_0^t \tilde{b}(s, X_s)\mathrm{d}s + B_t. \tag{3.41}$$

Using (H1) and the boundedness of u, it follows that $\tilde{b}$ is uniformly bounded. Using [27, Theorem 3.3] it holds that (3.41) has a unique strong Malliavin differentiable solution. The relation (2.1) yields that the FBSDE (1.1) has a unique strong solution.

The function $x \mapsto u(t, x)$ is Lipschitz continuous for every $\delta > 0$ and every $t \in [0, T - \delta]$. Therefore, using the chain rule formula for Malliavin calculus (see for example [10] or [29, Proposition 1.2.4]) we get that Y_t is Malliavin differentiable for all $t \in [0, T - \delta]$.

W use both the chain rule formula and [34, Lemma 2.3] to show that Z_t is Malliavin differentiable for all $t \in [0, T - \delta]$. Consider the function $\tilde{g} : (t, x) \mapsto g(t, x, u(t, x), \nabla u(t, x))$. Choose $\delta > 0$ and recall that for $t \in [0, T - \delta]$, the solution to the PDE (1.3) satisfies $\nabla_x u(t, x) \leq C_\delta$. Thus using the boundedness of u, the local boundedness of f and the growth of g, it follows that the functions $\tilde{g}$ is uniformly bounded. Let $\tilde{b}_n$ be a sequence of smooth coefficients with compact support and satisfying (H1) that approximates the coefficient $\tilde{b}$ and let $\tilde{g}_n$ be a similar sequence approximating $\tilde{g}$. Let (X^n, Y^n, Z^n) be the corresponding strong solution to the QFBSDE when $\tilde{b}$ and $\tilde{g}$ are replaced by $\tilde{b}_n$ and $\tilde{g}_n$, respectively. For $\delta > 0$, let A_t^n be the process defined by $A_t^n := \int_t^{T-\delta} Z_r^n \mathrm{d}B_r$. We have the following reprensentation

$$\begin{aligned} A_t^n &= - Y_t^n + Y_{T-\delta}^n + \int_t^{T-\delta} g_n(r, X_r^n, Y_r^n, Z_r^n)\mathrm{d}r \\ &= - Y_t^n + Y_{T-\delta}^n + \int_t^{T-\delta} \tilde{g}_n(s, X_r^n)\mathrm{d}r. \end{aligned} \tag{3.42}$$

Then by applying the boundedness of $\nabla_x u$ in $[0, T - \delta]$, the Girsanov theorem, the boundedness of $\tilde{g}$ and Krylov type estimate, we have that A_t^n converges in L^2 to $A_t := \int_t^{T-\delta} Z_r \mathrm{d}B_r$.

Taking the Malliavin derivative on both sides of the approximating sequence X^n of X and using the chain rule, we get

$$D_s X_t^n = I_{\{s \leq t\}} + \int_s^t D_x \tilde{b}_n(r, X_r^n) D_s X_r^n \mathrm{d}r, \tag{3.43}$$

where D_x is the derivative with respect to the space variable. Iterating the above yields

$$D_s X_t^n = I_{\{s \leq t\}} + \sum_{k=1}^{\infty} \int_{s \leq r_1 \leq \ldots \leq r_k \leq t} D_x \tilde{b}_n(r_1, X_{r_1}^n) : \cdots : D_x \tilde{b}_n(r_k, X_{r_k}^n) \mathrm{d}r_k \ldots \mathrm{d}r_1. \tag{3.44}$$

where the symbol ":" stands for the matrix multiplication.

Taking the Malliavin derivative on both sides of (3.42) and using (3.44), we get

$$\begin{aligned}
D_sA_t^n =& - D_sY_t^n + D_sY_{T-\delta}^n + \int_t^{T-\delta} D_x\tilde{g}_n(r,X_r^n)D_sX_r^n\mathrm{d}r\\
=& - D_sY_t^n + D_sY_{T-\delta}^n + \int_s^{T-\delta} D_x\tilde{g}_n(r,X_r^n)D_sX_r^n\mathrm{d}r - \int_s^t D_x\tilde{g}_n(r,X_r^n)D_sX_r^n\mathrm{d}r\\
=& - D_sY_t^n + D_sY_{T-\delta}^n\\
&+ \int_s^{T-\delta} D_x\tilde{g}_n(r,X_r^n)\Big(I_{\{s\le t\}} + \sum_{k=1}^\infty \int_{s<s_1<\cdots<s_k<r} D_x\tilde{b}_n(s_1,X_{s_1}^n):\cdots:D_x\tilde{b}_n(s_k,X_{s_k}^n)\mathrm{d}s_k\ldots\mathrm{d}s_1\Big)\mathrm{d}r\\
&- \int_s^{t} D_x\tilde{g}_n(r,X_r^n)\Big(I_{\{s\le t\}} + \sum_{k=1}^\infty \int_{s<s_1<\cdots<s_k<r} D_x\tilde{b}_n(s_1,X_{s_1}^n):\cdots:D_x\tilde{b}_n(s_k,X_{s_k}^n)\mathrm{d}s_k\ldots\mathrm{d}s_1\Big)\mathrm{d}r\\
=& - D_sY_t^n + D_sY_{T-\delta}^n + \int_t^{T-\delta} D_x\tilde{g}_n(r,X_r^n)\mathrm{d}r\\
&+ \int_s^{T-\delta} D_x\tilde{g}_n(r,X_r^n)\Big(\sum_{k=1}^\infty \int_{s<s_1<\cdots<s_k<r} D_x\tilde{b}_n(s_1,X_{s_1}^n):\cdots:D_x\tilde{b}_n(s_k,X_{s_k}^n)\mathrm{d}s_k\ldots\mathrm{d}s_1\Big)\mathrm{d}r\\
&- \int_s^{t} D_x\tilde{g}_n(r,X_r^n)\Big(\sum_{k=1}^\infty \int_{s<s_1<\cdots<s_k<r} D_x\tilde{b}_n(s_1,X_{s_1}^n):\cdots:D_x\tilde{b}_n(s_k,X_{s_k}^n)\mathrm{d}s_k\ldots\mathrm{d}s_1\Big)\mathrm{d}r\\
=& - D_sY_t^n + D_sY_{T-\delta}^n + \int_t^{T-\delta} \tilde{D}_xg_n(r,X_r^n)\mathrm{d}r\\
&+ \sum_{k=1}^\infty \int_{s<s_1<\cdots<s_{k+1}<T-\delta} D_x\tilde{g}_n(s_{k+1},X_{s_{k+1}}^n)D_x\tilde{b}_n(s_1,X_{s_1}^n):\cdots:D_x\tilde{b}_n(s_k,X_{s_k}^n)\mathrm{d}s_{k+1}\ldots\mathrm{d}s_1\\
&- \sum_{k=1}^\infty \int_{s<s_1<\cdots<s_{k+1}<t} D_x\tilde{g}_n(s_{k+1},X_{s_{k+1}}^n)D_x\tilde{b}_n(s_1,X_{s_1}^n):\cdots:D_x\tilde{b}_n(s_k,X_{s_k}^n)\mathrm{d}s_{k+1}\ldots\mathrm{d}s_1\\
=& I_1+I_2+I_3+I_4+I_5
\end{aligned}$$

Squaring both sides, using Hölder inequality and taking expectation gives:

$$\mathbb{E}[|D_sA_t^n|^2] \le 2^4\Big(\mathbb{E}[|I_1|^2]+\mathbb{E}[|I_2|^2]+\mathbb{E}[|I_3|^2]+\mathbb{E}[|I_4|^2+\mathbb{E}[|I_5|^2]\Big)$$

We know from [27, Lemma 3.5] that $\mathbb{E}[|D_sX_t^n|^2] < C$, where C is a positive constant only depending on t, d, δ, and $\|\tilde{b}\|_\infty$.

Since $Y_t^n = u_n(t, X_t^n)$ with u_n Lipschitz continuous in the second variable, there exists a constant that we denote again by M depending on the coefficients of the equation and δ such that $\mathbb{E}[|I_i|^2] \le M$, $i = 1, 2$.

Let us now consider the third term. Using the Girsanov transform and the Hölder inequality, we get

$$\begin{aligned}
\mathbb{E}[\|I_3\|^2] =& \mathbb{E}\Big[\Big\|\int_t^{T-\delta} D_x\tilde{g}_n(r,X_r^n)\mathrm{d}r\Big\|^2\Big]\\
=& \mathbb{E}\Big[\Big\|\int_t^{T-\delta} D_x\tilde{g}_n(r,B_r)\mathrm{d}r\Big\|^2\mathcal{E}\Big(\sum_{j=1}^d\int_0^{T-\delta}\tilde{b}_n^j(s,B_s)\mathrm{d}B_s^{(j)}\Big)\Big]\\
\le& \mathbb{E}\Big[\Big\|\int_t^{T-\delta} D_x\tilde{g}_n(r,B_r)\mathrm{d}r\Big\|^4\Big]^{\frac{1}{2}}\mathbb{E}\Big[\mathcal{E}\Big(\sum_{j=1}^d\int_0^{T-\delta}\tilde{b}_n^j(s,B_s)\mathrm{d}B_s^{(j)}\Big)^2\Big]^{\frac{1}{2}} \le M,
\end{aligned}$$

where the last inequality follows from [27, Proposition 3.7] (see also [28, Proposition 7]) and the fact that the Dolean-Dade exponential has finite moments since $\tilde{b}_n^j$ is uniformly bounded.

Next we have the following:

Claim: Let B be a d-dimensional Brownian motion starting from the origin and $\tilde{g}, \tilde{b}_1, \dots, \tilde{b}_n$ be bounded continuous differentiable functions with compact supports. Then there exists a Universal constant C such that

$$
\begin{aligned}
&\Big|\mathbb{E}\Big[\int_{s<s_1<\dots<s_{n+1}<T-\delta} D_x\tilde{g}(s_{n+1}, B_{s_{n+1}})D_x\tilde{b}_1(s_1, B_{s_1}) : \dots : D_x\tilde{b}_n(s_k, B_{s_k})\mathrm{d}s_{k+1}\dots\mathrm{d}s_1\mathrm{d}r\Big]\Big| \\
&\leq \frac{C^{k+1}\|\tilde{g}\|_\infty \prod_{i=1}^k \|\tilde{b}_i\|_\infty (T-\delta-s)^{\frac{n+1}{2}}}{\Gamma(\frac{k+1}{2}+1)} \qquad (3.45)
\end{aligned}
$$

This follows from [27, Proposition 3.7]. For the sake of completeness we provide a bit of details.

proof of the claim. We use arguments analogous to the proof of [27, Proposition 3.7]. Let $z = (z^{(1)}, \dots z^{(d)})$ be a generic element of $\mathbb{R}^d$ and $|\cdot|$ be the Euclidean norm.

Using the join law of the Brownian motion we have

$$
\begin{aligned}
&\Big|\mathbb{E}\Big[\int_{s<s_1<\dots<s_{k+1}<T-\delta} D_x\tilde{g}(s_{k+1}, B_{s_{k+1}}) : D_x b_1(s_1, B_{s_1}) : \dots : D_x b_n(s_k, B_{s_k})\mathrm{d}s_1\dots\mathrm{d}s_{k+1}\Big]\Big| \\
=&\Big|\int_{s<s_1<\dots<s_{k+1}<T-\delta}\int_{\mathbb{R}^{d(k+1)}} D_x\tilde{g}(s_{k+1}, z_{k+1}) : \prod_{i=1}^n D_x b_i(s_i - s_{i-1}, z_i)) \\
&\times P(s_{k+1}-s_n, z_{k+1}-z_k)P(s_i - s_{i-1}, z_i - z_{i-1})\mathrm{d}z_1\dots\mathrm{d}z_{k+1}\mathrm{d}s_1\dots\mathrm{d}s_{k+1}\Big| \\
:=&J^1_{k+1}(s, T, z_0) \qquad (3.46)
\end{aligned}
$$

where $P(t,z) = (2\pi t)^{-d/2}e^{-|z|^2/2t}$ is the Gaussian kernel. Using similar reasoning as in the proof of [27, Proposition 3.7], we have

$$
|J^1_{k+1}(s, T, z_0)| \leq \frac{C^{k+1}(T-\delta-s)^{\frac{k+1}{2}}}{\Gamma(\frac{k+1}{2}+1)}. \qquad (3.47)
$$

□

Now, let us consider the expression I_4. As before, using the Girsanov theorem and the Hölder inequality, we can write

$$
\begin{aligned}
&\mathbb{E}[|I_4|^2]\\
=&\mathbb{E}\Big[\Big|\sum_{k=1}^{\infty}\int_{s<s_1<\cdots<s_{k+1}<T-\delta} D_x\tilde{g}_n(s_{k+1},X^n_{s_{k+1}})D_x\tilde{b}_n(s_1,X^n_{s_1}):\cdots:D_x\tilde{b}_n(s_k,X^n_{s_k})\mathrm{d}s_{k+1}\ldots\mathrm{d}s_1\Big|^2\Big]\\
=&\mathbb{E}\Big[\Big|\sum_{k=1}^{\infty}\int_{s<s_1<\cdots<s_{k+1}<T-\delta} D_x\tilde{g}_n(s_{k+1},B_{s_{k+1}})D_x\tilde{b}_n(s_1,B_{s_1}):\cdots:D_x\tilde{b}_n(s_k,B_{s_k})\mathrm{d}s_{k+1}\ldots\mathrm{d}s_1\Big|^2\\
&\times\mathcal{E}\Big(\sum_{j=1}^{d}\int_0^{T-\delta}\tilde{b}^{n,j}(s,B_s)\mathrm{d}B_s\Big)\Big]\\
\leq&\mathbb{E}\Big[\Big|\sum_{k=1}^{\infty}\int_{s<s_1<\cdots<s_{k+1}<T-\delta} D_x\tilde{g}_n(s_{k+1},B_{s_{k+1}})D_x\tilde{b}_n(s_1,B_{s_1}):\cdots:D_x\tilde{b}_n(s_k,B_{s_k})\mathrm{d}s_{k+1}\ldots\mathrm{d}s_1\Big|^4\Big]^{\frac{1}{2}}\\
&\times\mathbb{E}\Big[\mathcal{E}\Big(\sum_{j=1}^{d}\int_0^{T-\delta}\tilde{b}^{n,j}(s,B_s)\mathrm{d}B_s\Big)^2\Big]^{\frac{1}{2}}\\
\leq&C\mathbb{E}\Big[\Big|\sum_{k=1}^{\infty}\int_{s<s_1<\cdots<s_{k+1}<T-\delta} D_x\tilde{g}_n(s_{k+1},B_{s_{k+1}})D_x\tilde{b}_n(s_1,B_{s_1}):\cdots:D_x\tilde{b}_n(s_k,B_{s_k})\mathrm{d}s_{k+1}\ldots\mathrm{d}s_1\Big|^4\Big]^{\frac{1}{2}}\\
\leq&C\mathbb{E}\Big[\Big|\sum_{k=1}^{\infty}\sum_{i,j=1}^{d}\sum_{\ell_1,\ell_2,\ldots,\ell_k=1}^{d}\int_{s<s_1<\cdots<s_{k+1}<T-\delta}\frac{\partial}{\partial x_{\ell_1}}\tilde{g}_n(s_1,B_{s_1})\frac{\partial}{\partial x_{\ell_2}}\tilde{b}_n^{(i)}(s_2,B_{s_2})\\
&\times\frac{\partial}{\partial x_{\ell_3}}\tilde{b}^{(\ell_2)}(s_3,B_{s_3})\ldots\frac{\partial}{\partial x_{\ell_j}}\tilde{b}^{(\ell_k)}(s_{k+1},B_{s_{k+1}})\mathrm{d}s_1\ldots\mathrm{d}s_{k+1}\Big|^4\Big]^{\frac{1}{2}}. \qquad (3.48)
\end{aligned}
$$

Let us now consider the following term

$$
\mathcal{W}=\int_{s<s_1<\cdots<s_{k+1}<T-\delta}\frac{\partial}{\partial x_{\ell_1}}\tilde{g}_n(s_1,B_{s_1})\frac{\partial}{\partial x_{\ell_2}}\tilde{b}_n^{(i)}(s_2,B_{s_2})\frac{\partial}{\partial x_{\ell_3}}\tilde{b}^{(\ell_2)}(s_3,B_{s_3})\ldots\frac{\partial}{\partial x_{\ell_i}}\tilde{b}^{(\ell_k)}(s_{k+1},B_{s_{k+1}})\mathrm{d}s_1\ldots\mathrm{d}s_{k+1}
$$

Repeated use of the deterministic integration by part allows to show that $\mathcal{W}^2$ can be written as a sum of at most 2^{2k+2} terms of the form

$$
\mathcal{W}=\int_{s<s_1<\cdots<s_{2k+2}<T-\delta}h^1_1(s_1)h^1_2(s_2)h^2_3(s_3)\ldots h^2_{2k+2}(s_{2k+2})\mathrm{d}s_1\ldots\mathrm{d}s_{2k+2}.
$$

with $h^1_q\in\Big\{\frac{\partial}{\partial x_i}\tilde{g}_n(\cdot,B_\cdot),i=1,\ldots,d\Big\}$, $q=1,2$ and $h^2_\ell\in\Big\{\frac{\partial}{\partial x_i}\tilde{b}^{(j)}_n(\cdot,B_\cdot),\ i,j=1,\ldots,d\Big\}$, $\ell=3,4\ldots,2k+2$. Using analogous arguments, $\mathcal{W}^4=\mathcal{W}^2\mathcal{W}^2$ can be written as a sum of at most 2^{8k+8} of similar terms of length $4k+4$. This observation together with the above claim yields

$$
\begin{aligned}
&\mathbb{E}\Big[\Big|\int_{s<s_1<\cdots<s_{k+1}<T-\delta}\frac{\partial}{\partial x_{\ell_1}}\tilde{g}_n(s_1,B_{s_1})\frac{\partial}{\partial x_{\ell_2}}\tilde{b}_n^{(i)}(s_2,B_{s_2})\\
&\times\frac{\partial}{\partial x_{\ell_3}}\tilde{b}^{(\ell_2)}(s_3,B_{s_3})\ldots\frac{\partial}{\partial x_{\ell_i}}\tilde{b}^{(\ell_k)}(s_{k+1},B_{s_{k+1}})\mathrm{d}s_1\ldots\mathrm{d}s_{k+1}\Big|^4\Big]^{\frac{1}{2}}\\
\leq&\Big(\frac{2^{8k+8}C^{4k+4}\|\tilde{g}_n\|^4\|\tilde{b}_n\|^{4k}(T-\delta-s)^{2k+2}}{\Gamma(2k+3)}\Big)^{\frac{1}{2}}\leq\frac{2^{4k+4}C^{2k+2}\|\tilde{g}_n\|^2\|\tilde{b}_n\|^{2k}(T-\delta-s)^{k+1}}{((2k+2)!)^{\frac{1}{2}}}. \qquad (3.49)
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\mathbb{E}[|I_4|^2]\leq&C\sum_{k=1}^{\infty}\frac{d^{k+4}2^{4k+4}C^{2k+2}\|\tilde{g}_n\|^2\|\tilde{b}_n\|^{2k}(T-\delta-s)^{k+1}}{((2k+2)!)^{\frac{1}{2}}}\\
\leq&C_d(\|\tilde{g}_n\|_\infty,\|\tilde{b}_n\|_\infty). \qquad (3.50)
\end{aligned}
$$

The bound for $\mathbb{E}[|I_5|^2]$ follows in a similar way.

It follows from the above computations that A^n is Malliavin differentiable and we have

$$\mathbb{E}[|D_s A_t^n|^2] \leq C_d((\|\tilde{g}_n\|_\infty, \|\tilde{b}_n\|_\infty)),$$

with $\tilde{b}_n$ and $\tilde{g}_n$ uniformly bounded. In addition, A_t^n converges to $\int_t^{T-\delta} Z_s \mathrm{d}B_s$ in $L^2(\Omega)$ for every $t \in [0, T-\delta]$. Thus using [29, Lemma 1.2.3], we deduce that $\int_t^{T-\delta} Z_s \mathrm{d}B_s$ in $L^2(\Omega)$ is Malliavin differentiable with $\mathbb{E}\Big[\Big|D_s \int_t^{T-\delta} Z_r \mathrm{d}B_r\Big|^2\Big] < \infty$.

Now using [29, Lemma 2.3], we conclude that Z_t is Malliavin differentiable for all $t \in [0, T-\delta]$. The proof is completed.

□

3.5. **Proof of Proposition 1.9.** As before we first observe that the function $\tilde{b} : (t,x) \mapsto b(t,x,u(t,x), \nabla_x u(t,x))$ is uniformly bounded for all $t \in [0, T-\delta]$. Then (1.4) follows from [28, Corollary 14]. In addition, we have

$$Y_{t_1}^{s_1,x_1} = u(t_1, X_{t_1}^{s_1,x_1}).$$

Therefore using Lemma 2.3, we havwe

$$\begin{aligned} |Y_{t_1}^{s_1,x_1} - Y_{t_2}^{s_2,x_2}| =& |u(t_1, X_{t_1}^{s_1,x_1}) - u(t_2, X_{t_2}^{s_2,x_2})| \\ \leq& C\Big(|t_1 - t_2|^{1/2} + |X_{t_1}^{s_1,x_1} - X_{t_2}^{s_2,x_2}|\Big), \end{aligned} \tag{3.51}$$

and the result follows. One can show (1.6) in a similar way. □

4. Proof of the preliminary results

This section is devoted to the proof of the auxiliary results.

4.1. **Proof of Lemma 2.2.** Let us consider the $W^{1,2}_{\mathrm{loc}}(\mathbb{R})$ function Φ_f in Lemma B.1. Then applying the Itô-Krylov formula for BSDE, we have

$$\begin{aligned} \Phi_f(|Y_s|) =& \Phi_f(|Y_T|) + \int_s^T \mathrm{sgn}(Y_r)\Phi_f'(|Y_r|) g(r, X_r, Y_r, Z_r)\mathrm{d}r - \frac{1}{2}\int_s^T \Phi_f''(|Y_r|)|Z_r|^2 |a(r, X_r, Y_r)|\mathrm{d}r \\ &- \int_s^T \mathrm{sgn}(Y_r)\Phi_f'(|Y_r|) Z_r \sigma^*(r, X_r, Y_r)\mathrm{d}B_r. \end{aligned}$$

Using assumptions (H1),(H2) and the fact that $\Phi_f, \Phi_f', \Phi_f'' \geq 0$, we deduce that

$$\begin{aligned} \Phi_f(|Y_s|) \leq& \Phi_f(|Y_T|) + \Lambda \int_s^T \Phi_f'(|Y_r|)(1 + |Y_r| + f(|Y_r|)|Z_r|^2)\mathrm{d}r - \frac{\lambda}{2}\int_s^T \Phi_f''(|Y_r|)|Z_r|^2 \mathrm{d}r \\ &- \int_s^T \mathrm{sgn}(Y_r)\Phi_f'(|Y_r|) Z_r \sigma^*(r, X_r, Y_r)\mathrm{d}B_r \\ =& \Phi_f(|Y_T|) + \int_s^T \Big(\Lambda \Phi_f'(|Y_r|) f(|Y_r|) - \frac{\lambda}{2}\Phi_f''(|Y_r|)\Big)|Z_r|^2 \mathrm{d}r + \Lambda \int_s^T \Phi_f'(|Y_r|)(1 + |Y_r|)\mathrm{d}r \\ &- \int_s^T \mathrm{sgn}(Y_r)\Phi_f'(|Y_r|) Z_r \sigma^*(r, X_r, Y_r)\mathrm{d}B_r. \end{aligned}$$

Using Lemma B.1 for $\kappa = 2\Lambda/\lambda$, we obtain

$$\Phi_f(|Y_s|) \leq \Phi_f(|Y_T|) + \Lambda \int_s^T \Phi_f'(|Y_r|)(1 + |Y_r|)\mathrm{d}r - \int_s^T \mathrm{sgn}(Y_r)\Phi_f'(|Y_r|) Z_r \sigma^*(r, X_r, Y_r)\mathrm{d}B_r. \tag{4.1}$$

Again, from Lemma B.1, there exists a positive constant C only depending on Λ and $\exp(\kappa ||f||_{L^1(\mathbb{R})})$, such that

$$\Phi_f(|Y_s|) \leq \Phi_f(|Y_T|) + CT + C\int_s^T \Phi_f(|Y_r|)\mathrm{d}r - \int_s^T \mathrm{sgn}(Y_r)\Phi_f'(|Y_r|) Z_r \sigma^*(r, X_r, Y_r)\mathrm{d}B_r. \tag{4.2}$$

Since (X, Y, Z) is a solution to (1.1), using Lemma B.1 and (H1), we see that the stochastic integral in (4.2) is square integrable, and in particular, its conditional expectation vanishes. Therefore, by taking conditional expectation on both sides of (4.2) and since Y_T is bounded and Φ_f is increasing, we obtain

$$\Phi_f(|Y_s|) \leq \mathbb{E}\Big[\Phi_f(|Y_T|) + CT + C\int_s^T \Phi_f(|Y_r|)\mathrm{d}r\Big|\mathfrak{F}_s\Big]. \tag{4.3}$$

Hence the classical time inverse Gronwall's inequality yields:

$$\Phi_f(|Y_s|) \leq C(T, \Lambda, ||f||_{L^1(\mathbb{R})}), \quad \mathrm{d}\mathbb{P}\text{-a.s.}$$

Therefore, since Φ_f is increasing and invertible with Φ_f^{-1} increasing, locally Lipschitz and with $\Phi_f^{-1}(0) = 0$ (see Lemma B.1) we deduce that there exist a constant $\Upsilon^{(1)}$ only depending on Λ, λ, T and the L^1-norm of f such that

$$|Y_s| \leq \Phi_f^{-1}\Big(C(T, \Lambda, ||f||_{L^1(\mathbb{R})})\Big) \leq \Upsilon^{(1)}, \quad \mathrm{d}\mathbb{P}\text{-a.s.} \tag{4.4}$$

On the other hand, we consider the function Ψ_f in Lemma B.2 and define the following stopping time $\tau_n := \inf\{s > 0, \int_0^s |\Psi_f'(Y_r)|^2|Z_r|^2\mathrm{d}r \geq n\} \wedge T,$ for $n > 0$. Since the map $z \mapsto \Psi_f(|z|)$ belongs to $W^{1,2}_{\text{loc}}(\mathbb{R})$, by applying once more the Itô-Krylov formula for BSDE, we obtain for any $s \in [0, T]$

$$\begin{aligned}\Psi_f(|Y_0|) =&\Psi_f(|Y_{s\wedge\tau_n}|) + \int_0^{s\wedge\tau_n} \mathrm{sgn}(Y_r)\Psi_f'(|Y_r|)g(r, X_r, Y_r, Z_r)\mathrm{d}r - \frac{1}{2}\int_0^{s\wedge\tau_n} \Psi_f''(|Y_r|)a(r, X_r, Y_r)|Z_r|^2\mathrm{d}r\\ &- \int_0^{s\wedge\tau_n} \mathrm{sgn}(Y_r)\Psi_f'(|Y_r|)Z_r\sigma^*(r, X_r, Y_r)\mathrm{d}B_r.\end{aligned}$$

From assumptions (H1) and (H2), we deduce

$$\begin{aligned}\Psi_f(|Y_0|) &\leq \Psi_f(|Y_{s\wedge\tau_n}|) + \Lambda\int_0^{s\wedge\tau_n} \Psi_f'(|Y_r|)(1 + |Y_r| + f(|Y_r|)|Z_r|^2)\mathrm{d}r - \frac{\lambda}{2}\int_0^{s\wedge\tau_n} \Psi_f''(|Y_r|)|Z_r|^2\mathrm{d}r\\ &\quad - \int_0^{s\wedge\tau_n} \mathrm{sgn}(Y_r)\Psi_f'(|Y_r|)Z_r\sigma^*(r, X_r, Y_r)\mathrm{d}B_r.\\ &= \Psi_f(|Y_{s\wedge\tau_n}|) + \int_0^{s\wedge\tau_n} \Big(\Lambda f(|Y_r|)\Psi_f'(|Y_r|) - \frac{\lambda}{2}\Psi_f''(|Y_r|)\Big)|Z_r|^2\mathrm{d}r + \Lambda\int_0^{s\wedge\tau_n} \Psi_f'(|Y_r|)(1 + |Y_r|)\mathrm{d}r\\ &\quad - \int_0^{s\wedge\tau_n} \mathrm{sgn}(Y_r)\Psi_f'(|Y_r|)Z_r\sigma^*(r, X_r, Y_r)\mathrm{d}B_r.\end{aligned}$$

Choosing $\kappa = \frac{2\Lambda}{\lambda}$, in Lemma B.2, we obtain

$$\begin{aligned}\Psi_f(|Y_0|) \leq \Psi_f(|Y_{s\wedge\tau_n}|) &- \frac{\lambda}{2}\int_0^{s\wedge\tau_n} |Z_r|^2\mathrm{d}r + \Lambda\int_0^{s\wedge\tau_n} \Psi_f'(|Y_r|)(1 + |Y_r|)\mathrm{d}r\\ &- \int_0^{s\wedge\tau_n} \mathrm{sgn}(Y_r)\Psi_f'(|Y_r|)Z_r\sigma^*(r, X_r, Y_r)\mathrm{d}B_r.\end{aligned}$$

Therefore, taking expectation on both sides of the above inequality and using the fact that Ψ is positive, we get

$$\frac{\lambda}{2}\mathbb{E}\int_0^{s\wedge\tau_n} |Z_r|^2\mathrm{d}r \leq \mathbb{E}\Psi_f(|Y_{s\wedge\tau_n}|) + \Lambda\mathbb{E}\int_0^T \Psi_f'(|Y_r|)(1 + |Y_r|)\mathrm{d}r.$$

Using the growth of both Ψ_f and Ψ_f' (see Lemma B.2) and we deduce

$$\mathbb{E}\int_0^{s\wedge\tau_n} |Z_r|^2\mathrm{d}r \leq C\mathbb{E}\left(|Y_{s\wedge\tau_n}|^2 + \int_0^T |Y_r|(1 + |Y_r|)\mathrm{d}r\right).$$

Thus, from the boundedness of the process Y obtain in (4.4) and the application the classical Fatou's lemma we deduce that there exist a positive constant that we will be denoted by $\Upsilon^{(1)}$ and depending only on Λ, λ, T and the L^1-norm of f such that

$$\mathbb{E}\int_s^T |Z_r|^2\mathrm{d}r \leq \Upsilon^{(1)}. \tag{4.5}$$

In particular, combining relations (4.4), (4.5) for $s = t$ and relation (2.1) we obtain the desired result. The proof is completed. □

4.2. **Proof of Lemma 2.3.** The proof is based on an extension of the Krylov and Safonov theory of linear parabolic PDE with a non-degenerate, discontinuous diffusion matrix to quasi-linear parabolic PDE. By following the scheme developed in [7, Theorem 1.1] with a slight modification, to prove the function u is uniformly Hölder continuous it is enough to show the subsequent bound

$$\operatorname{osc}_{\mathcal{Q}_{(t,x)}(R)}(u) \leq \Upsilon^{(2)}\left(\left(\frac{R}{R_0(t)}\right)^{\alpha_1}\omega_0(t,x) + RR_0(t)\right), \tag{4.6}$$

with

$$\begin{cases} \mathcal{Q}_{(t,x)}(R) := \left\{(s,y) \in [0,T] \times \mathbb{R}^d, 0 \leq s-t \leq R^2, \max\limits_{i=1,\cdots,d} |y_i - x_i| \leq R\right\}. \\ \omega_0(t,x) := \max\limits_{\varepsilon=\pm 1}\left(\operatorname{osc}_{\mathcal{Q}_{(t,x)}(R_0(t))}(\varepsilon u)\right) \\ R_0(t) := (T-t)^{1/2}. \end{cases}$$

However the bound (4.6) follows by a combination of [9, Lemma 13.5] and the following bound

$$\operatorname{osc}_{\mathcal{Q}_{(t,x)}(R)}(u) \leq \eta \operatorname{osc}_{\mathcal{Q}_{(t,x)}(2R)}(u) + CR^2. \tag{4.7}$$

with $\eta \in (0,1)$ only depending on Λ, λ, d and T. Next we show that (4.7) holds.

Since the coefficients b, σ, ϕ and g are assumed to be smooth thus, the associated PDE (1.3) has a unique solution $u \in C^{1,2}([0,T[\times\mathbb{R}^d, \mathbb{R})$. Using Lemma 2.2 and assumption (H1), the diffusion coefficient σ is uniformly bounded. Furthermore, using assumption (H2), the following SDE

$$X_s = x + \int_t^s \sigma(r, X_r, u(r, X_r))\mathrm{d}B_r, \quad (t,x) \in [0,T] \times \mathbb{R}^d \tag{4.8}$$

has a weak solution (see for instance Theorem 1, Paragraph 6, Chapter II in [18]) that we denote by $(\Omega, \mathfrak{F}, \mathbb{P}, B, \{\mathfrak{F}_s\}_{t\leq s\leq T}, X)$. It follows from the Itô's formula that the couple $(Y_s, Z_s)_{t\leq s\leq \tau}$ (with Y and Z given by (2.1)) satisfies the following BSDE

$$\begin{cases} Y_s = u(\tau, X_\tau) + \displaystyle\int_s^\tau g_1(r, X_r, Y_r, Z_r)\mathrm{d}r - \int_s^\tau Z_r\sigma^*(r, X_r, Y_r)\mathrm{d}B_r, \\ \forall s \in [t,\tau] \text{ and } \quad \mathbb{E}\Big[\displaystyle\int_t^\tau (|X_s|^2 + |Y_s|^2 + |Z_s|^2)\mathrm{d}s\Big] < \infty, \end{cases} \tag{4.9}$$

up to an $\{\mathfrak{F}_s\}_{t\leq s\leq T}$-stopping time τ such that:

$$\exists m^{(0)} \geq 0, \quad \mathbb{P}\{\sup_{t\leq s\leq \tau} |X_s| \leq m^{(0)}\} = 1,$$

and for all $(t,x,y,z) \in [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$, where $g_1(t,x.y.z) = zb(t,x,y,z) + g(t,x,y,z)$.

Fix $\gamma \in \mathbb{R}^*$, and for all $r \in [t,T]$, set $\tilde{Z}_r = \gamma\langle Z_r, \sigma\rangle(r, X_r, Y_r)$. Then

$$\gamma Y_s = \gamma u(\tau, X_\tau) + \gamma\int_s^\tau g(r, X_r, Y_r, Z_r)\mathrm{d}r + \int_s^\tau \tilde{Z}_r(\sigma^{-1}b)(r, X_r, Y_r, Z_r)\mathrm{d}r - \int_s^\tau \tilde{Z}_r\mathrm{d}B_r. \tag{4.10}$$

In particular for all $t \leq s \leq s' \leq \tau$, using (H1) and the boundedness of u i.e. Y (see Lemma 2.2), there exists a constant $M > 0$ only depending on Λ, λ and T such that

$$\begin{aligned} \gamma Y_s &\leq \gamma u(s', X_{s'}) + M\gamma\int_s^{s'} (1 + f(|Y_r|)|Z_r|^2)\mathrm{d}r + M\int_s^{s'} \tilde{Z}_r\mathrm{d}r - \int_s^{s'} \tilde{Z}_r\mathrm{d}B_r. \\ &\leq \gamma u(\tau, X_{s'}) + M(1+\gamma^2)\int_s^{s'} \mathrm{d}r + M\int_s^{s'} \tilde{Z}_r\mathrm{d}r + M\int_s^{s'} f(|Y_r|)|\tilde{Z}_r|^2\mathrm{d}r - \int_s^{s'} \tilde{Z}_r\mathrm{d}B_r. \end{aligned} \tag{4.11}$$

Since f is locally bounded and the process Y is bounded, there exists a positive constant M_0 such that $f(|Y_s|) \leq M_0$. Thus, one can find a constant denoted by M again such that

$$\begin{aligned}\gamma Y_s \leq & \gamma u(s', X_{s'}) + M(1+\gamma^2)\int_s^{s'} \mathrm{d}r + M\int_s^{s'} \tilde{Z}_r \mathrm{d}r + M\int_s^{s'} |\tilde{Z}_r|^2 \mathrm{d}r - \int_s^{s'} \tilde{Z}_r \mathrm{d}B_r \\ = & \gamma u(s', X_{s'}) + M(1+\gamma^2)\int_s^{s'} \mathrm{d}r + M\int_s^{s'} |\tilde{Z}_r|^2 \mathrm{d}r - \int_s^{s'} \tilde{Z}_r \mathrm{d}B_r^{\mathbb{Q}}. \end{aligned} \tag{4.12}$$

Observe that $B_{s'}^{\mathbb{Q}} = -\int_t^{s'} M\mathrm{d}r + B_{s'}$ is a Brownian motion under the measure $\mathbb{Q}$ given by

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\Big|_{\mathfrak{F}_s} = \exp\Big\{\int_t^{s\wedge\tau} M\mathrm{d}B_r - \frac{1}{2}\int_t^{s\wedge\tau} M^2 \mathrm{d}r\Big\}.$$

Let us remark that, the equation on the right side of (4.12) is a quadratic BSDE. Thus, it follows from [17] that there exists a unique progressively measurable process $(\bar{Y}_s, \bar{Z}_s)_{t\leq s\leq\tau}$ satisfying

$$\bar{Y}_s = \gamma u(\tau, X_\tau) + M\int_s^\tau \big((1+\gamma^2) + |\bar{Z}_r|^2\big)\mathrm{d}r - \int_s^\tau \bar{Z}_r \mathrm{d}B_r^{\mathbb{Q}}. \tag{4.13}$$

Moreover, there exists $m^{(1)} \geq 0$, such that :

$$\mathbb{Q}\Big\{\forall t \leq s \leq \tau, |\bar{Y}_s| \leq m^{(1)}\Big\} = 1, \quad \int_0^\cdot \bar{Z}_s \mathrm{d}B_s^{\mathbb{Q}} \in BMO(\mathbb{Q}). \tag{4.14}$$

Hence the comparison principle for quadratic BSDE (see [17, Theorem2.6]) yields

$$\gamma u(t,x) \leq \bar{Y}_t. \tag{4.15}$$

Thus for all $t \leq s \leq T$, the following probability measure $\bar{\mathbb{P}}$ on $(\mathfrak{F}_s)_{t\leq s\leq T}$

$$\frac{\mathrm{d}\bar{\mathbb{P}}}{\mathrm{d}\mathbb{Q}}\Big|_{\mathfrak{F}_s} := \exp\Big(M\int_t^{s\wedge\tau} \bar{Z}_r \mathrm{d}B_r^{\mathbb{Q}} - \frac{M^2}{2}\int_t^{s\wedge\tau} |\bar{Z}_r|^2 \mathrm{d}r\Big) \tag{4.16}$$

is well defined and thanks to Girsanov theorem, the process

$$\forall t \leq s \leq T, \quad \bar{B}_s = B_s^{\mathbb{Q}} - M\int_t^{s\wedge\tau} \bar{Z}_r \mathrm{d}r \tag{4.17}$$

is an $(\mathfrak{F}_s)_{t\leq s\leq T}$-Brownian motion. Substituting (4.17) into (4.13) and noting that $\{\int_t^{s\wedge\tau} \bar{Z}_r \mathrm{d}\bar{B}_r\}_{t\leq s\leq T}$ is a bounded martingale (see for example [17, Proposition 2.1]), we deduce that

$$\bar{Y}_t = \bar{\mathbb{E}}[\gamma u(\tau, X_\tau) + M(1+\gamma^2)(\tau - t)]. \tag{4.18}$$

Hence, (4.15) yields

$$u(t,x) \leq \bar{\mathbb{E}}[u(\tau, X_\tau) + \frac{M(1+\gamma^2)}{\gamma}(\tau - t)], \tag{4.19}$$

where $\bar{\mathbb{E}}$ stands for the expectation under the probability $\bar{\mathbb{P}}$. Choose now $\gamma = 10\Upsilon^{(1)}$, where $\Upsilon^{(1)}$ is given in Theorem 2.2. In the sequel for $(t_0, x_0) \in [0,T] \times \mathbb{R}^d$, we define $\mathcal{Q}(r) := \mathcal{Q}_{(t_0,x_0)}(r)$, for all $0 \leq r \leq R_0(t_0)$.

Moreover for a fixed $R > 0$ such that $t_0 + 4R^2 \leq T$, we set for every $(t,x) \in \mathcal{Q}(2R)$

$$m^+ = \max_{\mathcal{Q}(2R)}(u), \quad m^- = \max_{\mathcal{Q}(2R)}(-u).$$

It is readily seen that

$$\mu_{d+1}(B^+) \geq \mu_{d+1}(\mathcal{Q}(2R)) \quad \text{or} \quad \mu_{d+1}(B^-) \geq \mu_{d+1}(\mathcal{Q}(2R)),$$

where

$$B^+ = \Big\{(s,y) : m^+ - u(s,y) \geq \frac{1}{2}\operatorname{osc}_{\mathcal{Q}(2R)}(u)\Big\}, \quad B^- = \Big\{(s,y) : m^- + u(s,y) \geq \frac{1}{2}\operatorname{osc}_{\mathcal{Q}(2R)}(u)\Big\},$$

and μ_{d+1} stands for the Lebesgue measure on $\mathbb{R}^{d+1}$. Let us assume that $\mu_{d+1}(B^+) \geq \mu_{d+1}(\mathcal{Q}(2R))$, and define

$$\begin{cases} \tau_1 := \inf\left\{s \geq t, (s, X_s) \in B^+\right\}, \\ \tau_2 := \inf\left\{s \geq t, (s, X_s) \in \partial\mathcal{Q}(2R)\right\}. \end{cases}$$

Define τ by

$$\tau := \tau_1 \wedge \tau_2. \tag{4.20}$$

Using (4.19) and τ as given in (4.20), there exists a constant $m^{(2)}$ only depending on Λ, λ and T such that

$$\begin{aligned} u(t,x) &\leq \bar{\mathbb{E}}[u(\tau, X_\tau)1_{\{\tau_2<\tau_1\}}] + \bar{\mathbb{E}}[u(\tau, X_\tau)1_{\{\tau_2\geq\tau_1\}}] + m^{(2)}R^2 \\ &\leq m^+\bar{\mathbb{P}}\{\tau_2 < \tau_1\} + \left(m^+ - \frac{1}{2}\operatorname{osc}_{\mathcal{Q}(2R)}(u)\right)\bar{\mathbb{P}}\{\tau_2 \geq \tau_1\} + m^{(2)}R^2 \\ &= m^+\left(1 - \bar{\mathbb{P}}\{\tau_2 \geq \tau_1\}\right) + \left(m^+ - \frac{1}{2}\operatorname{osc}_{\mathcal{Q}(2R)}(u)\right)\bar{\mathbb{P}}\{\tau_2 \geq \tau_1\} + m^{(2)}R^2 \\ &= m^+ - \frac{1}{2}\operatorname{osc}_{\mathcal{Q}(2R)}(u)\bar{\mathbb{P}}\{\tau_2 \geq \tau_1\} + m^{(2)}R^2. \end{aligned} \tag{4.21}$$

If follows from [19] (compare with [7, (1.33)]) that there exists a constant $\eta^{(1)}$ which depends only on Λ, λ and d such that

$$\mathbb{P}\{\tau_2 \geq \tau_1\} \geq \eta^{(1)}. \tag{4.22}$$

Using the Girsanov transform and the Hölder inequality, we get

$$\begin{aligned} \eta^{(1)} \leq& \mathbb{P}\{\tau_2 \geq \tau_1\} \\ =& \mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-M\int_t^\tau \mathrm{d}B_r^{\mathbb{Q}} - \frac{M^2}{2}\int_t^\tau \mathrm{d}r\Big)1_{\{\tau_2\geq\tau_1\}}\Big] \\ \leq& \mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(-2M(B_\tau^{\mathbb{Q}} - B_t^{\mathbb{Q}}) - 2M^2(\tau - t)\Big)\Big]^{1/2}\mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(2M^2(\tau - t)\Big)\Big]^{1/4}\mathbb{Q}\{\tau_2 \geq \tau_1\}^{1/4} \\ \leq& C\mathbb{Q}\{\tau_2 \geq \tau_1\}^{1/4}. \end{aligned} \tag{4.23}$$

From [17, Proposition 2.1], there exists a constant c only depending on Λ, λ, d and T such that

$$\Big|\int_t^\tau \bar{Z}_r \mathrm{d}\bar{B}_r\Big| < c.$$

Thus, using once more the Hölder inequality, the Girsanov theorem and the above inequality, we have

$$\begin{aligned} \mathbb{Q}\{\tau_2 \geq \tau_1\} &= \bar{\mathbb{E}}\Big[\exp\Big(-M\int_t^\tau \bar{Z}_r\mathrm{d}\bar{B}_r - \frac{M^2}{2}\int_t^\tau |\bar{Z}_r|^2\mathrm{d}r\Big)1_{\{\tau_2\geq\tau_1\}}\Big] \\ &\leq \bar{\mathbb{E}}\Big[\exp\Big(-4M\int_t^\tau \bar{Z}_r\mathrm{d}\bar{B}_r\Big)\Big]^{1/4}\bar{\mathbb{E}}\Big[\exp\Big(-2M^2\int_t^\tau |\bar{Z}_r|^2\mathrm{d}r\Big)\Big]^{1/4}\bar{\mathbb{P}}\{\tau_2 \geq \tau_1\}^{1/2} \\ &\leq C\bar{\mathbb{P}}\{\tau_2 \geq \tau_1\}^{1/2}. \end{aligned} \tag{4.24}$$

Therefore there exists a constant $\eta^{(2)}$ depending on Λ, λ and d such that

$$\bar{\mathbb{P}}\{\tau_2 \geq \tau_1\} \geq \eta^{(2)}. \tag{4.25}$$

Hence from (4.21), we deduce that

$$u(t,x) \leq m^+ - \frac{\eta^{(2)}}{2}\operatorname{osc}_{\mathcal{Q}(2R)}(u) + m^{(2)}R^2. \tag{4.26}$$

This implies that

$$0 \leq (1 - \frac{\eta^{(2)}}{2})\operatorname{osc}_{\mathcal{Q}(2R)}(u) + m^{(2)}R^2. \tag{4.27}$$

Therefore, we can find $0 < \eta^{(3)} < 1$ only depending on Λ, λ, d and T such that

$$\operatorname{osc}_{\mathcal{Q}(R)}(u) \leq (1 - \eta^{(3)})\operatorname{osc}_{\mathcal{Q}(2R)}(u) + m^{(2)}R^2. \tag{4.28}$$

This completes the proof.

□

4.3. **Proof of Lemma 2.6.** In order to prove Lemma 2.6, we need the following Gagliardo-Nirenberg inequality (see [8, Lemma 6.3])

Lemma 4.1. *Let* $q_1.q_2 \in [1,+\infty]$ *and* $r > 0$. *Assume that* $\frac{1}{p} = \frac{1}{2q_1} + \frac{1}{2q_2}$. *Then there exists a constant* $C(p,q_1,q_2)$ *such that for every smooth function* ϱ *from* $B(0,r)$ *into* $\mathbb{R}$*:*

$$\begin{aligned} r^p \int_{B(0,r)} |\nabla_x \varrho(t,x)|^p \mathrm{d}x \leq & C(p,q_1,q_2)\Big[\int_{B(0,r)} |\varrho(t,x)|^{q_2}\mathrm{d}x\Big]^{p/2q_2} \\ & \times \Big[\int_{B(0,r)} \big(|\varrho(t,x)|^{q_1} + r^{q_1}|\nabla_x\varrho(t,x)|^{q_1} + r^{2q_1}|\nabla^2_{xx}\varrho(t,x)|^{q_1}\big)\,\mathrm{d}x\Big]^{p/2q_1}. \end{aligned}$$

We also need the subsequent result

Lemma 4.2. *The linear PDE*

$$\begin{cases} \dfrac{\partial w}{\partial t}(t,x) + \dfrac{1}{2}\displaystyle\sum_{i,j}^{d} a_{i,j}(t,x,u(t,x))\dfrac{\partial^2 w}{\partial x_i \partial x_j}(t,x) = 0, \quad (t,x) \in [0,T[\times\mathbb{R}^d, \\ w(T,x) = \phi(x), \quad x\in\mathbb{R}^d, \end{cases} \tag{4.29}$$

has a unique bounded solution $w \in C^{1,2}([0,T]\times\mathbb{R}^d,\mathbb{R})$. *Moreover, there exist two constants* C_w *and* β_w*, depending only* $\alpha_0, K_0, d, \lambda, \Lambda$ *and* T *such that for all* $(t,x)\in[0,T]\times\mathbb{R}^d$,

$$|\nabla^2_{xx} w(t,x)| \leq C_w (T-t)^{-1+\beta_w}. \tag{4.30}$$

Using the above results, we can now give a hint to the prove of Lemma 2.6.

Proof of Lemma 2.6. We assume without loss generality that $\zeta = 0$ and recall that the coefficients in this section are assumed to be smooth i.e. $a(t,x,u(t,x))$ and $\ell(t,x)$ defined below, are Hölder continuous. Clearly the following linear PDE

$$\begin{cases} \dfrac{\partial w^{(0)}}{\partial t}(t,x) + \dfrac{1}{2}\displaystyle\sum_{i,j}^{d} a_{i,j}(t,x,u(t,x))\dfrac{\partial^2 w^{(0)}}{\partial x_i \partial x_j}(t,x) = -\ell(t,x), \quad (t,x) \in [0,T[\times\mathbb{R}^d, \\ w^{(0)}(T,x) = 0, \quad x\in\mathbb{R}^d, \end{cases} \tag{4.31}$$

with $\ell(t,x) = g(t,x,u(t,x),\nabla_x u(t,x)) + \langle b(t,x,u(t,x),\nabla_x u(t,x)), \nabla_x u(t,x)\rangle$, has a unique bounded solution $w^0 \in C^{1,2}([0,T]\times\mathbb{R}^d,\mathbb{R})$ with bounded and uniformly Hölder continuous partial derivatives of order one in t and order two in x. Set $w^{(1)} = u - w^{(0)}$, it is readily seen that $w^{(1)}$ matches the solution of the linear PDE(4.29), then from (4.30) there exist $\gamma, C > 0$ only depending on parameters appearing in Lemma 2.6 such that the following estimate holds

$$\int_{T-\delta}^{T}\int_{B(0,R)} \Big[(T-s)^{1-\beta}|\nabla^2_{x,x} w^{(1)}(s,y)|\Big]^p \mathrm{d}s\mathrm{d}y \leq C\delta R^d. \tag{4.32}$$

Moreover, using once more Lemma 4.1, it can be shown that for all $(t,x),(t',x') \in [0,T[\times\mathbb{R}^d$

$$\begin{cases} |w^{(1)}(t,x)| \leq C \\ |w^{(1)}(t,x) - w^{(1)}(t',x')| \leq C(|x-x'|^\beta + |t-t'|^{\beta/2}). \end{cases} \tag{4.33}$$

Let us now define the following operators

$$\mathcal{L}_t := \frac{1}{2}\sum_{i,j}^{d} a_{i,j}(t,x,u(t,x))\frac{\partial^2}{\partial x_i \partial x_j}, \quad \mathcal{L}^0_t := \frac{1}{2}\sum_{i,j}^{d} a_{i,j}(t,0,u(t,0))\frac{\partial^2}{\partial x_i \partial x_j}, \tag{4.34}$$

then, $(T-t)^{1-\gamma}w^{(0)}$ satisfies the following PDE

$$\begin{aligned} &\Big[\frac{\partial}{\partial t} + \mathcal{L}^0_t\Big][(T-t)^{1-\gamma}w^{(0)}(t,x)] \\ &= -(T-t)^{1-\gamma}\ell(t,x) + \big[\mathcal{L}^0_t - \mathcal{L}_t\big][(T-t)^{1-\gamma}w^{(0)}(t,x)] + (1-\gamma)(T-t)^{1-\gamma}w^{(0)}(t,x). \end{aligned} \tag{4.35}$$

Note that, the nonlinear function ℓ, the partial derivatives of order two of w^0 and $(T-\cdot)^{1-\gamma}w^{(0)}$ are Hölder continuous. Hence, we can apply Theorem A.1 iv) on the interval $]T-\delta,T[$ to $v\equiv(T-\cdot)^{1-\gamma}w^{(0)}$ and obtain

$$\begin{aligned}
&(1-\theta)^{2p}\rho^{2p}\int_{T-\delta}^{T}\int_{B(0,\theta\rho)}(T-t)^{(1-\gamma)p}|\nabla^2_{x,x}w^{(0)}(t,x)|^p\mathrm{d}t\mathrm{d}x\\
&\quad\leq C(1-\theta')^{2p}\rho^{2p}\int_{T-\delta}^{T}\int_{B(0,R)}(T-t)^{(1-\gamma)p}|\ell(t,x)|^p\mathrm{d}t\mathrm{d}x\\
&\quad+C(1-\theta')^{2p}\rho^{2p}\sum_{i,j}^{d}\int_{T-\delta}^{T}\int_{B(0,\theta'\rho)}(T-t)^{(1-\gamma)p}|a_{i,j}(t,x,u(t,x))-a_{i,j}(t,0,u(t,0))|^p|\nabla^2_{x,x}w^{(0)}(t,x)|^p\mathrm{d}t\mathrm{d}x\\
&\quad+C(1-\theta')^{2p}\rho^{2p}\int_{T-\delta}^{T}\int_{B(0,\theta'\rho)}(T-t)^{-\gamma p}|w^{(0)}(t,x)|^p\mathrm{d}t\mathrm{d}x\\
&\quad+C\int_{T-\delta}^{T}\int_{B(0,\theta'\rho)}(T-t)^{(1-\gamma)p}|w^{(0)}(t,x)|^p\mathrm{d}t\mathrm{d}x\\
&\quad+\frac{1}{2}(1-\theta')^{2p}\rho^{2p}\int_{T-\delta}^{T}\int_{B(0,\theta'\rho)}(T-t)^{(1-\gamma)p}|\nabla^2_{x,x}w^{(0)}(t,x)|^p\mathrm{d}t\mathrm{d}x. \qquad (4.36)
\end{aligned}$$

Let us focus on the first term of the right side of (4.36). From the growth of g,b (Assumption (H1)) and the boundedness of the function u (Lemma 2.2) , we deduce that

$$|\ell(t,x)|^p\leq C(p,\Lambda)\Big(1+|\nabla_x u(t,x)|^p+f^p(|u(t,x)|)|\nabla_x u(t,x)|^{2p}\Big). \qquad (4.37)$$

Therefore, it follows from the local boundedness of the function f and application of Lemma 4.1 to the triple $(2p,p,+\infty)$ that

$$\begin{aligned}
&(1-\theta')^{2p}\rho^{2p}\int_{T-\delta}^{T}\int_{B(0,R)}(T-t)^{(1-\gamma)p}|\ell(t,x)|^p\mathrm{d}t\mathrm{d}x\\
&\quad\leq C(p,\Lambda)(1-\theta')^{2p}\rho^{2p}\int_{T-\delta}^{T}\int_{B(0,R)}(T-t)^{(1-\gamma)p}\Big(1+|\nabla_x u(t,x)|^{2p}\Big)\mathrm{d}t\mathrm{d}x\\
&\quad\leq C(p,\Lambda)(1-\theta')^{2p}\rho^{(2+\gamma)p}\Big[\int_{T-\delta}^{T}\int_{B(0,R)}(T-t)^{(1-\gamma)p}|\nabla^2_{xx}u(t,x)|^p\mathrm{d}t\mathrm{d}x+\delta\rho^{-2p}\Big]. \qquad (4.38)
\end{aligned}$$

Note that from here, we can simply follows similar steps as in the proof of [8, Therom 3.5]. The proof is completed.

□

4.4. **Proof of Lemma 2.4.** Let $\pi\in C^\infty(\mathbb{R}^d,[0,1])$ be a smooth function taking value 1 on the ball $B(0,1)$ and zero outside the ball $B(0,2)$. For all (t,x), set $\tilde{u}(t,,x):=\pi(x)u(t,x)$,$\tilde{\varphi}(x):=\pi(x)\varphi(x)$ and

$$\tilde{\ell}(t,x):=(\mathcal{L}_t-\mathcal{L}^0_t)\tilde{u}(t,x)+\pi(x)\ell(t,x)-\langle\nabla_x\pi(x),a(t,x,u(t,x))\nabla_x u(t,x)\rangle-\frac{1}{2}u(t,x)\mathcal{L}_t(\pi)(x)$$

Using Assumption 1.1, Lemma 2.2 and (4.36), we have

$$|\tilde{\ell}(t,x)|\leq C\left(1+|\nabla_x u(t,x)|+f(|u(t,x)|)|\nabla_x u(t,x)|^2+|\nabla^2_{x,x}u(t,x)|\right)1_{\{|x|\leq 2\}}$$

for any (t,x). Since f is locally bounded, we deduce that

$$|\tilde{\ell}(t,x)|\leq C\left(1+|\nabla_x u(t,x)|^2+|\nabla^2_{x,x}u(t,x)|\right)1_{\{|x|\leq 2\}}.$$

The proof is completed by applying Theorem A.1 (ii). □

Appendix A. Some PDE estimates

In this section, the following PDE

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) + \dfrac{1}{2}\displaystyle\sum_{i,j}^{d} c_{i,j}(t)\dfrac{\partial^2 v}{\partial x_i \partial x_j}(t,x) + \varphi(t,x) = 0, \\ v(T,x) = h(x), \quad x \in \mathbb{R}^d, \end{cases} \tag{A.1}$$

plays an eminent role in deriving the desired a priori estimates to the partial derivatives of the function u. We assume that the function $c : [0,T] \mapsto \mathcal{S}_d(\mathbb{R})$ is bounded and measurable in the sense that there exist $0 < \lambda_0 < \Lambda_0 < \infty$, such that $\forall t \in [0,T], \forall \theta \in \mathbb{R}^d\ \lambda_0|\theta|^2 \leqslant \langle \theta, c(t)\theta\rangle \leqslant \Lambda_0|\theta|^2$. Moreover, we define $\Gamma(t,s) := \int_t^s c(s_1)\mathrm{d}s_1$ for all $0 \leqslant t \leqslant s \leqslant T$ and $x, y \in \mathbb{R}^d$. Aslo, we define

$$\Psi^{(c)}(t,x;s,y) :=(2\pi)^{-d/2}\left(\det[\Gamma(t,s)]\right)^{-1/2}\exp\Big(-\frac{1}{2}\langle x-y, \Gamma^{-1}(t,s)(x-y)\rangle\Big) \tag{A.2}$$

The following results on the solution to the PDE (A.1) can be found in [8, Sections 5, 6 and 7].

Theorem A.1. *Assume $\varphi \in C^{\beta/2,\beta}([0,T]\times\mathbb{R}^d,\mathbb{R})$ is bounded, uniformly Hölder continuous, $\beta > 0$, and h is a bounded function in $C^{2+\beta}(\mathbb{R}^d,\mathbb{R})$ with bounded and uniformly Hölder continuous derivatives of order one and two. Then*

(i) *The PDE (A.1), has a unique bounded solution $v \in C^{1,2}([0,T]\times\mathbb{R}^d,\mathbb{R})$, such that, the solution $v(t,x)$ has the following representation*

$$\forall (t,x) \in [0,T]\times\mathbb{R}^d \quad v(t,x) = \mathcal{T}_{t,T}h(x) + \int_t^T \mathcal{T}_{t,s}\varphi(s,x)\mathrm{d}s; \tag{A.3}$$

where $\mathcal{T}_{t,T}\psi(x) = \int_{\mathbb{R}^d}\psi(y)\Psi^{(c)}(t,x,s,y)\mathrm{d}y$.

(ii) *There exists a constant C_v, only depending on λ_0, Λ_0 and d such that for all $(t,x) \in [0,T)\times\mathbb{R}^d$*

$$\begin{aligned} |\nabla_x v(t,x)| \leqslant & C_v\Big[(T-t)^{-1/2}\int_{\mathbb{R}^d}\Big|h(x+\Gamma^{1/2}(t,T)z) - h(x)\Big||z|\exp(-|z|^2/2)\mathrm{d}z \\ & + \int_t^T\int_{\mathbb{R}^d}(s-t)^{-1/2}\Big|\varphi(s,x+\Gamma^{1/2}(t,s)z)\Big||z|\exp(-|z|^2/2)\mathrm{d}z\mathrm{d}s\Big]. \end{aligned} \tag{A.4}$$

(iii) *In addition, if $\varphi(t,0)$ vanishes for every $t \in [0,T]$, then for any $t \in [0,T)$,*

$$\begin{aligned} |\nabla^2_{xx} v(t,0)| \leqslant & C_v\Big[(T-t)^{-1}\int_{\mathbb{R}^d}\Big|h(x+\Gamma^{1/2}(t,T)z) - h(0)\Big|(1+|z|^2)\exp(-|z|^2/2)\mathrm{d}z \\ & + \int_t^T\int_{\mathbb{R}^d}(s-t)^{-1}\Big|\varphi(s,x+\Gamma^{1/2}(t,s)z)\Big|(1+|z|^2)\exp(-|z|^2/2)\mathrm{d}z\mathrm{d}s\Big]. \end{aligned} \tag{A.5}$$

(iv) *Furthermore, if h is zero and the support of the function φ is bounded, then for given $\rho > 0$ and $\theta \in (0,1)$, set $\theta' = (1+\theta)/2$, there exists a constant C_v^p depending only on λ_0, Λ_0, d and p such that*

$$\begin{aligned} & (1-\theta)^{2p}\rho^{2p}\int_0^T\int_{B(z,\theta\rho)}|\nabla^2_{xx}v(t,x)|^p\mathrm{d}x\mathrm{d}t \\ \leqslant & C_v^p\Big[(1-\theta')^{2p}\rho^{2p}\int_0^T\int_{B(z,\theta'\rho)}|\varphi(t,x)|^p(1+|z|^2)\mathrm{d}x\mathrm{d}t + \int_0^T\int_{B(z,\theta'\rho)}|v(t,x)|^p(1+|z|^2)\mathrm{d}x\mathrm{d}t\Big] \\ & + \frac{1}{2}(1-\theta')^{2p}\rho^{2p}\int_0^T\int_{B(z,\theta'\rho)}|\nabla^2_{xx}v(t,x)|^p\mathrm{d}x\mathrm{d}t; \end{aligned} \tag{A.6}$$

Appendix B. Auxiliary results

The first part of the following Lemma can be found in [3, Lemma 5.1] (see also [4])

Lemma B.1. *Let $f \in L^1(\mathbb{R})$ be a positive function but not necessarily continuous. Then the function*

$$\Phi_f(z) := \int_0^z \exp\left(\kappa \int_0^y f(t)\mathrm{d}t\right)\mathrm{d}y,$$

with κ stands for a free nonnegative parameter, satisfies the differential equation:

$$\Phi_f''(z) - \kappa f(z)\Phi_f'(z) = 0 \text{ a.e. on } \mathbb{R},$$

and has the following properties

(1) *Φ_f is a one to one function from $\mathbb{R}$ onto $\mathbb{R}$. Both Φ_f and its inverse Φ_f^{-1} are locally Lipschitz.*
(2) *Both Φ_f and its inverse Φ_f^{-1} belong to $W^{1,2}_{loc}(\mathbb{R})$. If f is continuous, then both Φ_f and its inverse Φ_f^{-1} belong to $C^2(\mathbb{R})$.*
(3) *Φ_f and Φ_f' are positive on $\mathbb{R}_+$. Moreover, $\Phi_f(z) = 0$ iff $z = 0$.*
(4) *There exists $c > 0$, such that for all $z \in \mathbb{R}_+$,: $c \leqslant \Phi_f'(z) \leqslant \exp(\kappa||f||_{L^1(\mathbb{R})})$.*
(5) *For all $z \in \mathbb{R}_+$, $0 \leqslant z\Phi_f'(z) \leqslant \exp(\kappa||f||_{L^1(\mathbb{R})})\Phi_f(z)$.*

Proof. For the proof of (1) and (2), see [3, Lemma 5.1].

Condition (3) follows from the definition and the fact that $f \geq 0$. Clearly, the function Φ_f and its inverse Φ_f^{-1} are continuous, one to one, strictly increasing functions. From direct computations, $\Phi_f'(z) = \exp(\kappa \int_0^z f(t)\mathrm{d}t)$ and $\Phi_f''(z) = \kappa f(z)\Phi_f'(z)$. Then, the differential equation is satisfied. Since, $\forall z \in \mathbb{R}, |\Phi_f'(z)| \leq \exp(\kappa||f||_{L^1(\mathbb{R})})$, condition (4) follows. We shall prove assertion *iii*). Since $1 \leq \exp(\kappa \int_0^y f(t)\mathrm{d}t)$. Then $z \leq \int_0^z \exp(\kappa \int_0^y f(t)\mathrm{d}t)\mathrm{d}y$. We deduce from (5) that $z\Phi_f'(z) \leq \exp(\kappa||f||_{L^1(\mathbb{R})})\Phi_f(z)$. □

Lemma B.2 (see [4], Lemma A.1)**.** *We consider the following function*

$$K_f(y) := \int_0^y \exp\left(-\kappa \int_0^x f(t)\mathrm{d}t\right)\mathrm{d}x,$$

where $\kappa > 0$ stands for a free non negative parameter. Then the function

$$\Psi_f(z) := \int_0^z K_f(y)\exp\left(\kappa \int_0^y f(t)\mathrm{d}t\right)\mathrm{d}y,$$

satisfies the differential equations $\Psi_f''(z) - \kappa f(z)\Psi_f'(z) = 1$ a.e. on $\mathbb{R}$ and has the following properties

i) *Ψ_f and Ψ_f' are positive on $\mathbb{R}_+$, and Ψ_f belongs to $W^{1,2}_{loc}(\mathbb{R})$.*
ii) *The map $z \mapsto \Psi_f(|z|) \in W^{1,2}_{loc}(\mathbb{R})$.*
iii) *There exist $c_1, c_2 > 0$, such that for every $z \in \mathbb{R}$, $\Psi_f(|z|) \leq c_1|z|^2$ and $\Psi_f'(|z|) \leq c_2|z| \quad \forall z \in \mathbb{R}$.*

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