STRONG CONVERGENCE OF THE EULER-MARUYAMA APPROXIMATION FOR SDES WITH UNBOUNDED DRIFT.

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ABSTRACT. In this work, we prove strong convergence on small time interval of order $1/2-\epsilon$ for arbitrarily small $\epsilon > 0$ of the Euler-Maruyama approximation for additive Brownian motion with Hölder continuous drift satisfying a linear growth condition. The proof is based on direct estimations of functional of the Euler-Maruyama approximation. The order of convergence does not depend on the Hölder index of the drift, thus generalising the results obtained in [10] to both Linear growth and to an optimal convergence order.

1. INTRODUCTION

Let $B = \{B_t, t \geq 0\}$ denote a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. We denote by $(\mathcal{F}_t)_{t\in[0,T]}$ the augmented filtration generated by B. Consider the following stochastic differential equation

$$X_t = X_0 + \int_0^t f(X_s)\mathrm{d}s + B_t, \quad t \in [0,T], \tag{1.1}$$

where $T \in (0, \infty)$, the drift coefficient $f : \mathbb{R}^d \to \mathbb{R}^d$ is Hölder continuous, of linear growth and the initial condition X_0 is independent of B. It is known that (see for example [8, 4]) the above equation has a unique strong solution. Due to the nonlinearity and complexity of such equations, finding the explicit solutions is a virtually impossible task. Thus developing approximation methods for the solution becomes a natural alternative. Among other approximations, the Euler-Maruyama approximation is a simple and efficient numerical approximation to simulate solutions of SDEs.

The purpose of this paper is to study the strong rate of convergence of the Euler-Maruyama approximation for (1.1) given by

$$X_t^n = X_0 + \int_0^t f\left(X_{k_n(s)}^n\right)\mathrm{d}s + B_t, \tag{1.2}$$

where $k_n(t) := [nt]/n$ with $[tn]$ denotes the integer part of nt and $t \in [0,T], n \geq 1$ is the number of steps. We aim at analysing the L^2-approximation $\sup_{t\in[0,T]} \mathbf{E}|X_t^n - X_t|^2$ which depends on n (the rate of convergence).

Stochastic differential equations with irregular coefficients often appear in real world problems (statistical mechanics, physics, etc) and thus it is important to study numerical approximations of such equation. Analysis of the Euler-Maruyama approximation for additive Brownian motion with singular drift coefficients has received much attention because of their importance in applications. In the pioneering work [6], the author proves the almost sure convergence of the Euler-Maruyama approximation when the drift satisfies a monotonicity condition. In [7], the authors show that the Euler-Maruyama approximation converges strongly for SDEs with a discontinuous monotone drift coefficient. Recently, the authors in [10] used the regularity of the solution of the Kolmogorov equation associated to the SDE (1.1) to prove the strong rate convergence for the Euler-Maruyama approximation of SDEs with Hölder continuous drift coefficient assuming that the driving noise is either a Brownian motion or truncated α-stable process. The rate of convergence depends on

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Hölder exponent. The work of [10] is extended in [1] to the class of continuous SDEs with Hölder Dini-continuous, possibly unbounded drift.

Most recently the authors in [3], studied the strong convergence of a d-dimensional SDEs when the drift coefficient is bounded and Dini continuous. Surprisingly, they showed that the convergence in L^2 is of order $1/2-\epsilon$ for arbitrarily small $\epsilon>0$. In addition, they proved that in the one-dimensional case one has order $1/2-\epsilon$ for drift coefficients which are bounded and integrable over $\mathbb{R}$. The results obtained in [3] constitute a real improvement of the previous work since the order of convergence does not depend on the Hölder power of the drift f. It is also worth mentioning the work [9] in which the author assume that $d=1$ and the drift f is the sum of two bounded functions a and b, with a continuous, differentiable and having bounded derivatives, and b integrable in the Sobolev-Slobodeckij spaces of order $\kappa\in(0,1)$,. They proved a convergence in L^2 of order $(\frac{3}{4}\vee\frac{1+\kappa}{2})-\epsilon$ for the Euler-Maruyama scheme (1.2).

In this work, using both arguments in [3] and [10], we show strong convergences of the Euler-Maruyama approximation of SDEs when the drift is possibly unbounded and is Hölder of order $1/2-\epsilon$. In order to show this result, we consider a function g of spatial linear growth and show that the process $u^n=(u^n_t)_{t\geq 0}$ defined by:

$$u^n_t=\int_0^t g(s,B_s)\mathrm{d}s-\int_0^t g(s,B_{k_n(s)})\mathrm{d}s,$$

is bounded in $L^2(\Omega)$ by $C\{n^{-1}\log(n)\}^{1/2}$. C is a constant depending on d, T and the linear growth constant (see Lemma 2.1). We then use the solution of the Kolmogorov equation associated to the SDE (1.2) along with Girsanov theorem to prove that the L^2-convergence of the Euler scheme converges with the order independent of the of the Hölder power of the drift f of the SDE.

The next result which gives the order of convergence of the Euler-Maruyama scheme of an SDE in the case of time homogeneous Hölder continuous and unbounded drift is the main result of this paper:

Theorem 1.1. *Let $\epsilon,\delta\in(0,1)$. Assume that $b\in\mathbf{C}^\delta(\mathbb{R}^d,\mathbb{R}^d)$. Then for all $n\geq 3$ we have the bound*

$$\sup_{t\in[0,T_\epsilon]}\mathbf{E}|X^n_t-X_t|^2\leq Mn^{-1+\epsilon},\tag{1.3}$$

where $M=M(d,M_0,\epsilon,\delta,T,\|f\|_\delta)$ and T small enough depending on ϵ.

The remainder of the paper is organised as follows. In Section 2 we give the framework for the analysis of the approximation of (1.2) for the additive Brownian motion (1.1). In Section 3, we prove the main result whereas the appendix provides some result on existence of solution Kolmogorov backward equation.

2. Preliminary results

In this section we present and prove the preliminary results we will use for the analysis of the approximation stated in the introduction. From now on, we set $Q_T:=(0,T)\times\mathbb{R}^d$ and X_0 independent of the $\mathbf{P}$-augmented filtration generated by the Brownian motion B. The following results generalise [3] to the case of unbounded drift.

Lemma 2.1. *Let g be a Borel measurable function on $[0,T]\times\mathbb{R}^d$. Suppose that $g(s,z)\leq k(1+|z|)$ (that is $\tilde g\in L_\infty(Q_T)$ where $\tilde g(t,z)=\dfrac{g(t,z)}{1+|z|}$). Then there exists a constant $C=C(T)$ such that*

$$\mathbf{E}\Big|\int_0^r(g(s,B_s)-g(s,B_{k_n(s)}))\mathrm{d}s\Big|^2\leq C\|\tilde g\|^2_\infty n^{-1}\log(n)\tag{2.1}$$

for all $n\geq 3$, $r\in[0,T]$.

Remark 2.2. *In order to prove this estimate we use an averaging argument as opposed to [3] in which the authors use the independence of the increment of the Brownian and a conditioning argument.*

Proof of Lemma 2.1. We first assume without loss of generality that the function g in the lemma is for example C^1 and compactly supported. Let $M = M(d,T)$ be a constant that may change from one line to the other. For n big enough so that $1/n \in [0,T]$, we distinguish two cases:

Case 1: $r \in [0, 1/n]$.

Using Hölder inequality, Fubini theorem and the linear growth condition, we have

$$\begin{aligned}\mathbf{E}\Big|\int_0^r (g(s,B_s)-g(s,B_{k_n(s)}))\mathrm{d}s\Big|^2 \leq & r\mathbf{E}\int_0^r \big|g(s,B_s)-g(s,B_{k_n(s)})\big|^2\,\mathrm{d}s\\ \leq & r\|\tilde{g}\|^2_{L_\infty(Q_T)}\mathbf{E}\int_0^r (1+|B_s|+1+|B_{k_n(s)}|)^2\mathrm{d}s\\ \leq & r^2\|\tilde{g}\|^2_\infty \mathbf{E}(2+2\sup_{s\in[0,r]}|B_s|)^2\\ \leq & M(d)\|\tilde{g}\|^2_\infty r^2 \leq M(d)\|\tilde{g}\|^2_\infty n^{-2}\\ \leq & M\|\tilde{g}\|^2_\infty n^{-1}\log(n).\end{aligned}$$

Case 2: $r \in [1/n, T]$.

By the averaging argument, it suffices to prove the inequality (2.1) for g_ℓ, the ℓ-th component of g. For simplicity, we assume g takes values in $\mathbb{R}$. Let us first observe that, for $x, y \in \mathbb{R}^d$, we have

$$\begin{aligned}g(t,x_1,\dots,x_d)-g(t,y_1,\dots,y_d) =& g(t,x_1,\dots,x_d)-g(t,y_1,x_2\dots,x_d)\\ &+g(t,y_1,x_2\dots,x_d)-g(t,y_1,y_2,x_3,\dots,x_d)\\ &+g(t,y_1,y_2,\dots,x_d)-g(t,y_1,y_2,y_3,\dots,x_d)\\ &+g(t,y_1,y_2,y_3,\dots,x_d)-g(t,y_1,y_2,y_3,y_4\dots,x_d)\\ &+\dots+g(t,y_1,y_2,\dots,y_{d-1},x_d)-g(t,y_1,\dots,y_d).\end{aligned}$$

Set $x_i = B^i_t$ and $y_i = B^i_{k_n(t)}$ for $i = 1,\dots,d$ and define

$$Z^i(t,z) := (B^1_{k_n(t)},\dots,B^{i-1}_{k_n(t)}, z, B^{i+1}_t,\dots,B^d_t)$$

for $z \in \mathbb{R}$. Using the above telescoping sum, taking the square on both side and taking the expectation, if follows from the triangle inequality that

$$\mathbf{E}\Big|\int_0^r (g(s,B_s)-g(s,B_{k_n(s)}))\mathrm{d}s\Big|^2 \leq M_d\sum_{i=1}^d \mathbf{E}\Big|\int_0^r \Big\{g(s,Z^i(s,B^i_s))-g(s,Z^i(s,B^i_{k_n(s)}))\Big\}\mathrm{d}s\Big|^2. \tag{2.2}$$

We wish show that each term on the right of (2.2) satisfies the bound in the Lemma. To this end, define $\mathbf{J}_n$ by:

$$\mathbf{J}_n := \frac{1}{2}\mathbf{E}\Big|\int_{1/n}^r (g(s,Z^i(s,B^i_s))-g(s,Z^i(s,B^i_{k_n(s)})))\mathrm{d}s\Big|^2.$$

We will prove

$$2\mathbf{J}_n := \mathbf{E}\Big|\int_{1/n}^r (g(s,Z^i(s,B^i_s))-g(s,Z^i(s,B^i_{k_n(s)})))\mathrm{d}s\Big|^2 \leq M\|\tilde{g}\|^2_{L_\infty(Q_T)}n^{-1}\log(n), \tag{2.3}$$

for every $i = 1,\dots,d$. Using the mean-value theorem, we get

$$2\mathbf{J}_n = \mathbf{E}\Big|\int_{1/n}^r\int_0^1 (B^i_s - B^i_{k_n(s)})\frac{\partial_i g}{\partial x_i}(s,Z^i(s,B^i_s-\theta(B^i_s-B^i_{k_n(s)})))\mathrm{d}\theta\mathrm{d}s\Big|^2.$$

Using the deterministic integration by part formula, we can rewrite $\mathbf{J}_n$ as follows

$$
\begin{aligned}
\mathbf{J}_n =&\mathbf{E}\int_0^1\int_0^1\int_{1/n}^r\int_{1/n}^t (B_s^i - B_{k_n(s)}^i)\frac{\partial_i g}{\partial x_i}(s, Z^i(s, B_s^i-\theta(B_s^i-B_{k_n(s)}^i)))\\
&\times (B_t^i - B_{k_n(t)}^i)\frac{\partial_i g}{\partial x_i}(t, Z^i(t, B_t^i-\lambda(B_t^i-B_{k_n(t)}^i)))\mathrm{d}s\mathrm{d}t\mathrm{d}\theta\mathrm{d}\lambda\\
=&\mathbf{E}\int_0^1\int_0^1\int_{2/n}^r\int_{1/n}^{k_n(t)-1/n}\cdots\mathrm{d}s\mathrm{d}t\mathrm{d}\theta\mathrm{d}\lambda\\
&+\mathbf{E}\int_0^1\int_0^1\int_{2/n}^r\int_{k_n(t)-1/n}^{t}\cdots\mathrm{d}s\mathrm{d}t\mathrm{d}\theta\mathrm{d}\lambda\\
&+\mathbf{E}\int_0^1\int_0^1\int_{1/n}^{2/n}\int_{1/n}^{t}\cdots\mathrm{d}s\mathrm{d}t\mathrm{d}\theta\mathrm{d}\lambda\\
=:&\mathbf{J}_n^1+\mathbf{J}_n^2+\mathbf{J}_n^3
\end{aligned}
\tag{2.4}
$$

where the integrands are the same in all three expressions. Remark that for $s < k_n(t)$, we have $k_n(s) \leq s < k_n(t) \leq t$. A simple rearrangement gives (see for example [3]),

$$
\begin{aligned}
\frac{\partial_i g}{\partial x_i}(s, Z^i(s, B_s^i-\theta(B_s^i-B_{k_n(s)}^i))) =&\frac{\partial_i g}{\partial x_i}(s, Z^i(s, (1-\theta)(B_s^i-B_{k_n(s)}^i)+B_{k_n(s)}^i)),\\
\frac{\partial_i g}{\partial x_i}(t, Z^i(t, B_t^i-\lambda(B_t^i-B_{k_n(t)}^i))) =&\frac{\partial_i g}{\partial x_i}(t, Z^i(t, (1-\lambda)(B_t^i-B_{k_n(t)}^i)+(B_{k_n(t)}^i-B_s^i)\\
&+(B_s^i-B_{k_n(s)}^i)+B_{k_n(s)}^i)).
\end{aligned}
$$

For $i = 1,\ldots,d$ and fixed s,t such that $s < k_n(t)$, define the following random variables

$$
\begin{aligned}
Y_1^i(s) :=&B_{k_n(s)}^i,\\
Y_2^i(s) :=&B_s^i - B_{k_n(s)}^i,\\
Y_3^i(s,t) :=&B_{k_n(t)}^i - B_s^i,\\
Y_4^i(t) :=&B_t^i - B_{k_n(t)}^i.
\end{aligned}
$$

Using the properties of the increment of the Brownian motion, if follows that the random variables $Y_1^i(s), Y_2^i(s), Y_3^i(s,t), Y_4^i(t)$ are independent and Gaussian with mean 0 and variance $(\sigma_j^i)^2 = (\sigma_j^i)^2(s,t)$ given respectively by

$$
\begin{aligned}
(\sigma_{y_1}^i)^2(s,t) = k_n(s), &\qquad (\sigma_{y_2}^i)^2(s,t) = s - k_n(s),\\
(\sigma_{y_3}^i)^2(s,t) = k_n(t) - s, &\qquad (\sigma_{y_4}^i)^2(s,t) = t - k_n(t).
\end{aligned}
$$

Next we consider the following $(d+3)$-dimensional Gaussian random vector given by

$$
W_i = W_i(s,t) = (B_{k_n(s)}^1,\ldots,B_{k_n(s)}^{i-1}, Y_1^i(s), Y_2^i(s), Y_3^i(s,t), Y_4^i(t), B_s^{i+1},\ldots,B_s^d)
$$

Let ${W_i}^j$ be the j-th component of the above Gaussian random vector W_i. Note that for $j = 1,\ldots,d{+}3, \alpha\in\mathbb{N}$, we have $\mathbf{E}|\left({W_i}^j\right)^\alpha| \leq c_\alpha(\sigma_j^i)^\alpha$ where $c_\alpha > 0$ and σ_j^i denotes the variance of ${W_i}^j$. Using the fact that for fixed s,t the variance of ${W_i}^j, j = 1,\ldots,i-1$ (respectively $j = i+1,\ldots,d$) is $k_n(s)$(respectively s), we deduce that the joint density of W_i is given by

$$
\begin{aligned}
P(s,t,u,y,v) =&\Big((2\pi)^{(d+3)}(k_n(s))^{i-1}s^{d-i}\Pi_{j=1}^4(\sigma_{y_j}^i)^2(s,t)\Big)^{-1/2}\exp\Big(-\sum_{j=1}^{i-1}\frac{u_j^2}{2k_n(s)}\Big)\\
&\times\exp\Big(-\sum_{j=1}^4\frac{y_j^2}{2(\sigma_{y_j}^i)^2(s,t)}\Big)\exp\Big(-\sum_{j=1}^{d-i}\frac{v_j^2}{2s}\Big)
\end{aligned}
$$

for $(u,y,v)\in\mathbb{R}^{i-1}\times\mathbb{R}^4\times\mathbb{R}^{d-i}$.

Let us now define the following functions

$$G_1(t,u,y,v) := g(t,u,(1-\theta)y_2 + y_1, v), \tag{2.5}$$
$$G_2(t,u,y,v) := g(t,u,(1-\lambda)y_4 + y_3 + y_2 + y_1, v), \tag{2.6}$$

where $(t,u,y,v) \in [0,T] \times \mathbb{R}^{i-1} \times \mathbb{R}^4 \times \mathbb{R}^{d-i}$.

Claim 1: We claim that

$$|\mathbf{J}_n^1| \le M\|\tilde{g}\|_{L_\infty}^2 n^{-1}\log n. \tag{2.7}$$

Proof of the Claim 1. $\mathbf{J}_n^1$ can be written in terms of G_1 and G_2 as:

$$\mathbf{J}_n^1 = \int_0^1\int_0^1\int_{2/n}^r\int_{1/n}^{k_n(t)-1/n}\int_{\mathbb{R}^{d+3}} y_4y_2\frac{\partial G_1}{\partial y_1}(s,u,y,v)\frac{\partial G_2}{\partial y_3}(t,u,y,v)P(s,t,u,y,v)\mathrm{d}u\mathrm{d}y\mathrm{d}v\mathrm{d}s\mathrm{d}t\mathrm{d}\theta\mathrm{d}\lambda.$$

Using the deterministic integration by parts with respect to y_1 we have

$$\begin{aligned}
\mathbf{J}_n^1 &= \int_0^1\int_0^1\int_{2/n}^r\int_{1/n}^{k_n(t)-1/n}\int_{\mathbb{R}^{d+3}} y_4y_2\frac{\partial G_1}{\partial y_1}(s,u,y,v)\frac{\partial G_2}{\partial y_3}(t,u,y,v)P(s,t,u,y,v)\mathrm{d}u\mathrm{d}y\mathrm{d}v\mathrm{d}s\mathrm{d}t\mathrm{d}\theta\mathrm{d}\lambda \\
&= \int_0^1\int_0^1\int_{2/n}^r\int_{1/n}^{k_n(t)-1/n}\int_{\mathbb{R}^{d+2}}\left[\int_{\mathbb{R}} y_4y_2\frac{\partial G_1}{\partial y_1}(s,u,y,v)\frac{\partial G_2}{\partial y_3}(t,u,y,v)P(s,t,u,y,v)\mathrm{d}y_1\right]\mathrm{d}u\mathrm{d}y_2\mathrm{d}y_3\mathrm{d}y_4\mathrm{d}v\mathrm{d}s\mathrm{d}t\mathrm{d}\theta\mathrm{d}\lambda \\
&= \int_0^1\int_0^1\int_{2/n}^r\int_{1/n}^{k_n(t)-1/n}\int_{\mathbb{R}^{d+2}}\left[\int_{\mathbb{R}} G_1(s,u,y,v)\frac{\partial}{\partial y_1}\left\{y_4y_2\frac{\partial G_2}{\partial y_3}(t,u,y,v)P(s,t,u,y,v)\right\}\mathrm{d}y_1\right. \\
&\qquad\left. - \left[y_4y_2\frac{\partial G_1}{\partial y_1}(s,u,y,v)\frac{\partial G_2}{\partial y_3}(t,u,y,v)P(s,t,u,y,v)\right]_{y_1=-\infty}^{y_1=+\infty}\right]\mathrm{d}u\mathrm{d}y_2\mathrm{d}y_3\mathrm{d}y_4\mathrm{d}v\mathrm{d}s\mathrm{d}t\mathrm{d}\theta\mathrm{d}\lambda \\
&= \int_0^1\int_0^1\int_{2/n}^r\int_{1/n}^{k_n(t)-1/n}\int_{\mathbb{R}^{d+2}}\Bigg(\left[\int_{\mathbb{R}} G_1(s,u,y,v)\frac{\partial}{\partial y_1}\left\{y_4y_2\frac{\partial G_2}{\partial y_3}(t,u,y,v)P(s,t,u,y,v)\right\}\mathrm{d}y_1\right] \\
&\qquad - 0\Bigg)\times\mathrm{d}u\mathrm{d}y_2\mathrm{d}y_3\mathrm{d}y_4\mathrm{d}v\mathrm{d}s\mathrm{d}t\mathrm{d}\theta\mathrm{d}\lambda \\
&= \int_0^1\int_0^1\int_{2/n}^r\int_{1/n}^{k_n(t)-1/n}\int_{\mathbb{R}^{d+2}}\left[\int_{\mathbb{R}}\left\{G_1(s,u,y,v)y_4y_2\frac{\partial^2 G_2}{\partial y_1\partial y_3}(t,u,y,v)P(s,t,u,y,v)\right.\right. \\
&\qquad\left.\left. + G_1(s,u,y,v)y_4y_2\frac{\partial G_2}{\partial y_3}(t,u,y,v)\frac{\partial P}{\partial y_1}(s,t,u,y,v)\right\}\mathrm{d}y_1\right]\mathrm{d}u\mathrm{d}y_2\mathrm{d}y_3\mathrm{d}y_4\mathrm{d}v\mathrm{d}s\mathrm{d}t\mathrm{d}\theta\mathrm{d}\lambda \\
&= \int_0^1\int_0^1\int_{2/n}^r\int_{1/n}^{k_n(t)-1/n}\int_{\mathbb{R}^{d+3}} G_1(s,u,y,v)y_4y_2\frac{\partial^2 G_2}{\partial y_1\partial y_3}(t,u,y,v)P(s,t,u,y,v)\mathrm{d}u\mathrm{d}y\mathrm{d}v\mathrm{d}s\mathrm{d}t\mathrm{d}\theta\mathrm{d}\lambda \\
&\quad + \int_0^1\int_0^1\int_{2/n}^r\int_{1/n}^{k_n(t)-1/n}\int_{\mathbb{R}^{d+3}} G_1(s,u,y,v)y_4y_2\frac{\partial G_2}{\partial y_3}(t,u,y,v)\frac{\partial P}{\partial y_1}(s,t,u,y,v)\mathrm{d}u\mathrm{d}y\mathrm{d}v\mathrm{d}s\mathrm{d}t\mathrm{d}\theta\mathrm{d}\lambda
\end{aligned}$$

where we have used the fact that g is smooth and compactly supported.
Applying once more an integration by parts with respect to y_3, we obtain that

$$\begin{aligned}
\mathbf{J}_n^1 &= \int_0^1\int_0^1\int_{2/n}^r\int_{1/n}^{k_n(t)-1/n}\int_{\mathbb{R}^{d+3}} \frac{\partial G_1}{\partial y_3}(s,u,y,v)y_4y_2\frac{\partial G_2}{\partial y_1}(t,u,y,v)P(s,t,u,y,v)\mathrm{d}u\mathrm{d}y\mathrm{d}v\mathrm{d}s\mathrm{d}t\mathrm{d}\theta\mathrm{d}\lambda \\
&\quad + \int_0^1\int_0^1\int_{2/n}^r\int_{1/n}^{k_n(t)-1/n}\int_{\mathbb{R}^{d+3}} G_1(s,u,y,v)y_4y_2\frac{\partial G_2}{\partial y_1}(t,u,y,v)\frac{\partial P}{\partial y_3}(s,t,u,y,v)\mathrm{d}u\mathrm{d}y\mathrm{d}v\mathrm{d}s\mathrm{d}t\mathrm{d}\theta\mathrm{d}\lambda \\
&\quad + \int_0^1\int_0^1\int_{2/n}^r\int_{1/n}^{k_n(t)-1/n}\int_{\mathbb{R}^{d+3}} \frac{\partial G_1}{\partial y_3}(t,u,y,v)y_4y_2G_2(s,u,y,v)P(s,t,u,y,v)\mathrm{d}u\mathrm{d}y\mathrm{d}v\mathrm{d}s\mathrm{d}t\mathrm{d}\theta\mathrm{d}\lambda \\
&\quad + \int_0^1\int_0^1\int_{2/n}^r\int_{1/n}^{k_n(t)-1/n}\int_{\mathbb{R}^{d+3}} G_1(s,u,y,v)y_4y_2G_2(t,u,y,v)\frac{\partial^2 P}{\partial y_1\partial y_3}(s,t,u,y,v)\mathrm{d}u\mathrm{d}y\mathrm{d}v\mathrm{d}s\mathrm{d}t\mathrm{d}\theta\mathrm{d}\lambda
\end{aligned}$$

Observe that by its definition, G_1 does not depend on y_3, thus $\frac{\partial G_1}{\partial y_3} = 0$, we get that

$$\begin{aligned}\mathbf{J}_n^1 &= \int_0^1 \int_0^1 \int_{2/n}^r \int_{1/n}^{k_n(t)-1/n} \int_{\mathbb{R}^{d+3}} y_4 y_2 G_1(s,u,y,v) G_2(t,u,y,v) \frac{\partial^2 P}{\partial y_1 \partial y_3}(s,t,u,y,v) \mathrm{d}u\mathrm{d}y\mathrm{d}v\mathrm{d}s\mathrm{d}t\mathrm{d}\theta\mathrm{d}\lambda \\ &+ \int_0^1 \int_0^1 \int_{2/n}^r \int_{1/n}^{k_n(t)-1/n} \int_{\mathbb{R}^{d+3}} y_4 y_2 G_1(s,u,y,v) \frac{\partial G_2}{\partial y_1}(t,y) \frac{\partial P}{\partial y_3}(s,t,u,y,v) \mathrm{d}u\mathrm{d}y\mathrm{d}v\mathrm{d}s\mathrm{d}t\mathrm{d}\theta\mathrm{d}\lambda \\ &=: \mathbf{J}_n^1(1) + \mathbf{J}_n^1(2).\end{aligned}$$

It follows from the smoothness of the Gaussian density P that

$$\begin{aligned}|\mathbf{J}_n^1(1)| \leq \Big| \int_0^1 \int_0^1 \int_{2/n}^r \int_{1/n}^{k_n(t)-1/n} \int_{\mathbb{R}^{d+3}} & y_1 y_2 y_3 y_4 G_1(s,u,y,v) G_2(t,u,y,v) P(s,t,u,y,v) \\ & \times (\sigma_{y_2}^i)^{-2}(s,t)(\sigma_{y_4}^i)^{-2}(s,t) \mathrm{d}u\mathrm{d}y\mathrm{d}v\mathrm{d}s\mathrm{d}t\mathrm{d}\theta\mathrm{d}\lambda \Big|\end{aligned}$$

The linear growth of g and the fact that $\lambda, \theta \in [0,1]$ yield

$$\begin{aligned}|\mathbf{J}_n^1(1)| \leq & \|\tilde{g}\|_{L_\infty}^2 \int_0^1 \int_0^1 \int_{2/n}^r \int_{1/n}^{k_n(t)-1/n} \int_{\mathbb{R}^{d+3}} |y_1 y_2 y_3 y_4| P(s,t,u,y,v) (\sigma_{y_1}^i)^{-2}(s,t)(\sigma_{y_3}^i)^{-2}(s,t) \\ & \times (1 + |[u,(1-\theta)y_2 + y_1, v]|)(1 + |[u,(1-\lambda)y_4 + y_3 + y_2 + y_1, v]|) \mathrm{d}s\mathrm{d}t\mathrm{d}u\mathrm{d}y\mathrm{d}v\mathrm{d}\theta\mathrm{d}\lambda \\ \leq & M(d)\|\tilde{g}\|_{L_\infty}^2 \int_{2/n}^r \int_{1/n}^{k_n(t)-1/n} \int_{\mathbb{R}^{d+3}} \sum_{q_j=0,1,2,3, j=1,\dots,d+3, q_{i+1}>0, q_{i+3}>0} \\ & |(u_1)^{q_1} \dots (u_{i-1})^{q_{i-1}} (y_1)^{q_i} (y_2)^{q_{i+1}} (y_3)^{q_{i+2}} (y_4)^{q_{i+3}} (v_1)^{q_{i+4}} \dots (v_{d-i})^{q_d}| \times \\ & \times P(s,t,u,y,v)(\sigma_{y_2}^i)^{-2}(s,t)(\sigma_{y_4}^i)^{-2}(s,t) \mathrm{d}s\mathrm{d}t\mathrm{d}u\mathrm{d}y\mathrm{d}v,\end{aligned}$$

where the finite sum under the integrals is a polynomial that maximises the expression: $|y_1 y_2 y_3 y_4|(1 + |[u,(1-\theta)y_2 + y_1, v]|)(1 + |[u,(1-\lambda)y_4 + y_3 + y_2 + y_1, v]|)$. Using the moments of a Gaussian process, we have

$$\begin{aligned}|\mathbf{J}_n^1(1)| \leq & M(d)\|\tilde{g}\|_{L_\infty}^2 \int_{2/n}^r \int_{1/n}^{k_n(t)-1/n} \sum_{q_j=0,1,2,3, j=1,\dots,d+3, q_{i+1}>0, q_{i+2}>0} \Big(\Pi_{\ell=1}^{d+3} \mathbf{E}|(W_i^\ell(s,t))^{q_j}| \Big) \\ & \times (\sigma_{y_2}^i)^{-2}(s,t)(\sigma_{y_4}^i)^{-2}(s,t) \mathrm{d}s\mathrm{d}t \\ \leq & M(d)\|\tilde{g}\|_{L_\infty}^2 \int_{2/n}^r \int_{1/n}^{k_n(t)-1/n} \Big(\sum_{q_j=0,1,2,3, j=1,\dots,d+3, q_{i+1}>0, q_{i+3}>0} \Pi_{\ell=1}^{d+3} c_{q_j} (\sigma_\ell^i)^{q_j}(s,t) \Big) \times \\ & \times (\sigma_{y_2}^i)^{-2}(s,t)(\sigma_{y_4}^i)^{-2}(s,t) \mathrm{d}s\mathrm{d}t \\ \leq & M(T,d) n^{-1} \|\tilde{g}\|_{L_\infty}^2 \int_{2/n}^r \int_{1/n}^{k_n(t)-1/n} (k_n(s)(k_n(t)-s))^{-1/2} \mathrm{d}s\mathrm{d}t, \end{aligned} \tag{2.8}$$

where we have used the fact that $\sigma_{y_2}^i(s,t), \sigma_{y_4}^i(s,t) \leq n^{-1/2}$ and $\sigma_\ell^i, \sigma_{y_1}^i(s,t), \sigma_{y_3}^i(s,t) \leq T^{1/2}$ for all $\ell = 1, \dots, i-1, i+4, \dots, d+3$.

In what follows, $M(T,d) = M$ denotes a constant depending on T and d and that may change from one line to the other. Note that one has $k_n(s) \geq s - 1/n$ and by a change of variables one sees that

$$\int_{1/n}^{k_n(t)} ((s-n^{-1})(k_n(t)-s))^{-1/2} \mathrm{d}s = \int_0^1 (s(1-s))^{-1/2} \mathrm{d}s = \pi \tag{2.9}$$

for all $t \in [2/n, T]$. Consequently,

$$|\mathbf{J}_n^1(1)| \leq M n^{-1} \|\tilde{g}\|_{L_\infty(Q_T)}^2. \tag{2.10}$$

For $\mathbf{J}_n^1(2)$, observe that $\frac{\partial G_2}{\partial y_1}=\frac{\partial G_2}{\partial y_3}$, thus integrating by parts with respect to y_3 gives

$$\begin{aligned}
&|\mathbf{J}_n^1(2)|\\
=&\Big|\int_0^1\int_0^1\int_{2/n}^r\int_{1/n}^{k_n(t)-1/n}\int_{\mathbb{R}^{d+3}} y_4y_2G_1(s,u,y,v)G_2(t,u,y,v)\frac{\partial^2 P}{\partial y_3^2}(s,t,u,y,v)\mathrm{d}u\mathrm{d}y\mathrm{d}v\mathrm{d}s\mathrm{d}t\mathrm{d}\theta\mathrm{d}\lambda\Big|\\
\leq&\Big|\int_0^1\int_0^1\int_{2/n}^r\int_{1/n}^{k_n(t)-1/n}\int_{\mathbb{R}^{d+3}} y_4y_2G_1(s,u,y,v)G_2(t,u,y,v)y_3^2(\sigma^i_{y_3})^{-4}(s,t)P(s,t,u,y,v)\mathrm{d}u\mathrm{d}y\mathrm{d}v\mathrm{d}s\mathrm{d}t\mathrm{d}\theta\mathrm{d}\lambda\Big|\\
&+\Big|\int_0^1\int_0^1\int_{2/n}^r\int_{1/n}^{k_n(t)-1/n}\int_{\mathbb{R}^{d+3}} y_4y_2G_1(s,u,y,v)G_2(t,u,y,v)y_3^2(\sigma^i_{y_3})^{-2}P(s,t,u,y,v)\mathrm{d}u\mathrm{d}y\mathrm{d}v\mathrm{d}s\mathrm{d}t\mathrm{d}\theta\mathrm{d}\lambda\Big|.
\end{aligned}$$

As before, thanks to the linear growth of the function g, using once more the polynomial maximising the product among y_3^2, y_2, y_1 and the definition of G_1 and G_2 (see (2.5)-(2.6)), we have

$$\begin{aligned}
&|\mathbf{J}_n^1(2)|\\
\leq&M(d)\|\tilde g\|^2_{L_\infty}\int_{2/n}^r\int_{1/n}^{k_n(t)-1/n}\int_{\mathbb{R}^{d+3}}\sum_{q_j=0,1,2,3,j=1,\ldots,d+3,q_{i+1}>0,q_{i+2}>1,q_{i+3}>0}\\
&|(u_1)^{q_1}\ldots(u_{i-1})^{q_{i-1}}(y_1)^{q_i}(y_2)^{q_{i+1}}(y_3)^{q_{i+2}}(y_4)^{q_{i+3}}(v_1)^{q_{i+41}}\ldots(v_{d-i})^{q_d}|\\
&\times(\sigma^i_{y_3})^{-4}P(s,t,u,y,v)\mathrm{d}u\mathrm{d}y\mathrm{d}v\mathrm{d}s\mathrm{d}t\\
&+M(d)\|\tilde g\|^2_{L_\infty}\int_{2/n}^r\int_{1/n}^{k_n(t)-1/n}\int_{\mathbb{R}^{d+3}}\sum_{q_j=0,1,2,3,j=1,\ldots,d+3,q_{i+1}>0,q_{i+3}>0}\\
&|(u_1)^{q_1}\ldots(u_{i-1})^{q_{i-1}}(y_1)^{q_i}(y_2)^{q_{i+1}}(y_3)^{q_{i+2}}(y_4)^{q_{i+3}}(v_1)^{q_{i+41}}\ldots(v_{d-i})^{q_d}|\\
&\times(\sigma^i_{y_3})^{-2}P(s,t,u,y,v)\mathrm{d}u\mathrm{d}y\mathrm{d}v\mathrm{d}s\mathrm{d}t\\
\leq&M(d)\|\tilde g\|^2_{L_\infty}\int_{2/n}^r\int_{1/n}^{k_n(t)-1/n}\int_{\mathbb{R}^{d+3}}\sum_{q_j=0,1,2,3,j=1,\ldots,d+3,q_{i+1}>0,q_{i+2}>1,q_{i+3}>0}\Big(\Pi_{\ell=1}^{d+3}\mathbf{E}|(W_i^\ell)^{q_j}(s,t)|\Big)\times\\
&\times(\sigma^i_{y_3})^{-4}\mathrm{d}s\mathrm{d}t\\
&+M(d)\|\tilde g\|^2_{L_\infty}\int_{2/n}^r\int_{1/n}^{k_n(t)-1/n}\int_{\mathbb{R}^{d+3}}\sum_{q_j=0,1,2,3,j=1,\ldots,d+3,q_{i+1}>0,q_{i+3}>0}\Big(\Pi_{\ell=1}^{d+3}\mathbf{E}|(W_i^\ell)^{q_j}(s,t)|\Big)\\
&\times(\sigma^i_{y_3})^{-2}\mathrm{d}s\mathrm{d}t.
\end{aligned}$$

Using similar argument as in (2.8), we obtain

$$\begin{aligned}
&|\mathbf{J}_n^1(2)|\\
\leq&M(d)\|\tilde g\|^2_{L_\infty}\int_{2/n}^r\int_{1/n}^{k_n(t)-1/n}\int_{\mathbb{R}^{d+3}}\sum_{q_j=0,1,2,3,j=1,\ldots,d+3,q_{i+1}>0,q_{i+3}>0}\Big(\Pi_{\ell=1}^{d+3}c_{q_j}(\sigma^i_\ell)^{q_j}(s,t)\Big)(\sigma^i_{y_3})^{-2}\mathrm{d}s\mathrm{d}t\\
\leq&M(T,d)\|\tilde g\|^2_{L_\infty}n^{-1}\int_{2/n}^r\int_{1/n}^{k_n(t)-1/n}(\sigma^i_{y_3})^{-2}\mathrm{d}s\mathrm{d}t=M\|\tilde g\|^2_{L_\infty}n^{-1}\int_{2/n}^r\int_{1/n}^{k_n(t)-1/n}(k_n(t)-s)^{-1}\mathrm{d}s\mathrm{d}t\\
\leq&M\|\tilde g\|^2_{L_\infty}n^{-1}\int_{2/n}^r-\log(n^{-1})+\log(k_n(t)-n^{-1})\mathrm{d}t\\
=&M\|\tilde g\|^2_{L_\infty}n^{-1}\int_{2/n}^r\log(E(nt)-1)\mathrm{d}t\\
\leq&M\|\tilde g\|^2_{L_\infty}n^{-1}\int_{2/n}^r\log(nT)\mathrm{d}t\leq M\|\tilde g\|^2_{L_\infty}n^{-1}T\log(nT)\\
\leq&M\|\tilde g\|^2_{L_\infty}n^{-1}T(\log n+\log(T))\leq M\|\tilde g\|^2_{L_\infty}n^{-1}\log n,
\end{aligned}$$

where we have used the same arguments and techniques as in (2.8). Therefore,

$$|\mathbf{J}_n^1(2)| \leq M\|\tilde{g}\|_{L_\infty}^2 n^{-1}\log n. \tag{2.11}$$

□

Claim 2: We claim that

$$|\mathbf{J}_n^2| \leq M\|\tilde{g}\|_{L_\infty}^2 n^{-1}. \tag{2.12}$$

Proof of the Claim 2. For $\mathbf{J}_n^2$ we have

$$\mathbf{J}_n^2 = \mathbf{E}\int_{2/n}^r\int_{k_n(t)-1/n}^t (g(s,Z^i(s,B_s)) - g(s,Z^i(s,B_{k_n(s)})))(g(t,Z^i(t,B_t)) - g(t,Z^i(t,B_{k_n(t)})))\mathrm{d}s\mathrm{d}t$$

Then,

$$\begin{aligned}
|\mathbf{J}_n^2| =& \Big|\mathbf{E}\int_{2/n}^r\int_{k_n(t)-1/n}^t (g(s,Z^i(s,B_s)) - g(s,Z^i(s,B_{k_n(s)})))(g(t,Z^i(t,B_t)) - g(t,Z^i(t,B_{k_n(t)})))\mathrm{d}s\mathrm{d}t\Big| \\
\leq& \|\tilde{g}\|_{L_\infty}^2 \int_{2/n}^r\int_{k_n(t)-1/n}^t 3\mathbf{E}\Big(8 + |Z^i(s,B_s)|^2 + |Z^i(s,B_{k_n(s)})|^2 + |Z^i(t,B_t)|^2 + |Z^i(t,B_{k_n(t)})|^2\Big)\mathrm{d}s\mathrm{d}t \\
\leq& \|\tilde{g}\|_{L_\infty}^2 \int_{2/n}^r\int_{k_n(t)-1/n}^t 3\mathbf{E}\Big(8 + M(d)\sum_{j=1}^{i-1} 2\Big[|B^j_{k_n(s)}|^2 + |B^j_{k_n(t)}|^2\Big] + M(d)|B^i_t|^2 + M(d)|B^i_{k_n(t)}|^2 \\
&+ M(d)\sum_{j=i+1}^{d}\Big[|B^j_s|^2 + |B^j_t|^2\Big] + M(d)|B^i_s|^2 + M(d)|B^i_{k_n(s)}|^2\Big)\mathrm{d}s\mathrm{d}t \\
\leq& \|\tilde{g}\|_{L_\infty}^2 \int_{2/n}^r\int_{k_n(t)-1/n}^t 3\Big(8 + M(d)\sum_{j=1}^{i-1} 2k_n(s) + k_n(t) + t + k_n(t) + M(d)\sum_{j=i+1}^{d} s + t + s + k_n(s)\Big)\mathrm{d}s\mathrm{d}t \\
\leq& M(d,T)\|\tilde{g}\|_{L_\infty}^2 \int_{2/n}^r\int_{k_n(t)-1/n}^t \mathrm{d}s\mathrm{d}t \\
\leq& M\|\tilde{g}\|_{L_\infty}^2 \int_{2/n}^r t - k_n(t) + 1/n\mathrm{d}t \\
\leq& M\|\tilde{g}\|_{L_\infty}^2 \int_{2/n}^r 2/n\mathrm{d}t \leq M/n\|\tilde{g}\|_{L_\infty}^2
\end{aligned}$$

Therefore,

$$|\mathbf{J}_n^2| \leq Mn^{-1}\|\tilde{g}\|_{L_\infty(Q_T)}^2. \tag{2.13}$$

The claim is proved. □

In a similar one can prove that

$$|\mathbf{J}_n^3| \leq Mn^{-1}\|\tilde{g}\|_{L_\infty(Q_T)}^2 \tag{2.14}$$

Putting the estimates (2.7), (2.13) and (2.14) together, we obtain (2.3). In addition, (2.3) does not depend on i, it follows that

$$\mathbf{E}\Big|\int_0^r (g(s,B_s) - g(s,B_{k_n(s)}))\mathrm{d}s\Big|^2 \leq M\|\tilde{g}\|_{L_\infty}^2 n^{-1}\log n, \tag{2.15}$$

where g is chosen smooth and of linear growth. Finally, let g be a given Borel on $[0,T]\times\mathbb{R}^d$ satisfying the condition of the theorem, one can find a sequence of compactly supported smooth functions $\tilde{g}_m$ with $\tilde{g}_m \in L_\infty$, converging to $\tilde{g}$ *a.e.* on $[0,T]\times\mathbb{R}^d$, where $\tilde{g}(z) = g(z)/(1+|z|)$. Setting $g_m(z) := (1+|z|)\tilde{g}_m(z)$, we have that $g_m(t,W(t))$ converges to $\tilde{g}(t,W(t))$ *a.s.* for *a.a.* $t \in [0,T]$ when m goes to infinity, and the same for $\tilde{g}_m(t,W(t)+x)$, so by Fatou's lemma the required estimation follows. □

Lemma 2.3. *Let g and f be $\mathbb{R}^d$ valued Borel functions on $[0,T]\times\mathbb{R}^d$ and $\mathbb{R}^d$, respectively. Assume that $(\tilde{g},\tilde{f}) \in L_\infty(Q_T)\times L_\infty(\mathbb{R}^d)$, where $\tilde{g}(t,z) = g(t,z)/(1+|z|)$ and $\tilde{f}(z) = f(z)/(1+|z|)$. Let X be the solution the SDE (1.1). Then for $p \geq 2$ there exists a constant M depending on $\|\tilde{g}\|_{L_\infty}, \|\tilde{f}\|_\infty, \mathbf{E}|X_0|^p, p$ and T such that*

$$\mathbf{E}\left|\int_0^t \Big(g(s,X_s^n) - g(s,X^n_{k_n(s)})\Big)\,\mathrm{d}s\right|^p \leq M \tag{2.16}$$

for all $t\in[0,T]$ and $n\geq 3$.

Proof. The result follows by applying successfully: Hölder inequality, the fact that $(a+b)^p \leq 2^{p-1}(a^p+b^p)$ for $a,b\geq 0$ and $p>1$, the linear growth property of the coefficient and Gronwall's inequality. □

Lemma 2.4. *Let f and g be as in Lemma 2.3. Choose X^n as in (1.2). Then there exists $r\in[0,T]$ small enough such that for all $\epsilon\in(0,1)$, we have*

$$\mathbf{E}\Big|\int_0^r \Big(g(s,X_s^n) - g(s,X^n_{k_n(s)})\Big)\mathrm{d}s\Big|^2 \leq Mn^{-1+\epsilon} \tag{2.17}$$

for all $n\geq 3, r\in[0,T]$, where $M = M(\epsilon, T, \|\tilde{f}\|_\infty, \|\tilde{g}\|_{L_\infty})$

Proof. We start by defining

$$(\gamma^n)_t := \exp\Big(\int_0^t -\sum_{i=1}^d f^i(X^n_{k_n(s)})dB_s^i - (1/2)\int_0^t \sum_{i=1}^d f^i(X^n_{k_n(s)})^2\mathrm{d}s\Big).$$

Using Equation (1.2), the linear growth condition and the Cauchy-Schwartz inequality, we have with probability one

$$\begin{aligned}
|X^n_{k_n(t)}|^2 &= \Big|X_0 + \int_0^{k_n(t)} f\left(X^n_{k_n(s)}\right)\mathrm{d}s + B_{k_n(t)}\Big|^2 \\
&\leq 3|X_0|^2 + 3k_n(t)\int_0^{k_n(t)} \Big|f\Big(X^n_{k_n(s)}\Big)\Big|^2\mathrm{d}s + 3|B_{k_n(t)}|^2 \\
&\leq 3|X_0|^2 + 6k_n(t)\|\tilde{f}\|^2_\infty \int_0^{k_n(t)} \Big(1+\Big|X^n_{k_n(s)}\Big|^2\Big)\mathrm{d}s + 3|B_{k_n(t)}|^2 \\
&\leq 3|X_0|^2 + 6[k_n(t)]^2\|\tilde{f}\|^2_\infty + 3|B_{k_n(t)}|^2 + 6k_n(t)\|\tilde{f}\|^2_\infty \int_0^{k_n(t)} \Big|X^n_{k_n(s)}\Big|^2\mathrm{d}s.
\end{aligned} \tag{2.18}$$

By Gronwall inequality, we get that

$$|X^n_{k_n(t)}|^2 \leq \Big[3|X_0|^2 + 6[k_n(t)]^2\|\tilde{f}\|^2_\infty + 3|B_{k_n(t)}|^2\Big]\exp\Big(6\|\tilde{f}\|^2_\infty k_n(t)^2\Big) \tag{2.19}$$

uniformly on $n\geq 3$. In addition, using (2.18), once more, we have with probability one

$$\begin{aligned}
&(1/2)\int_0^t \Big|f(X^n_{k_n(s)})\Big|^2\mathrm{d}s \\
&\leq 2\|\tilde{f}\|^2_\infty \int_0^t \Big(1+\Big|X^n_{k_n(s)}\Big|^2\Big)\mathrm{d}s \\
&\leq 2\|\tilde{f}\|^2_\infty \int_0^t \Big(1+\Big[3|X_0|^2 + 6[k_n(s)]^2\|\tilde{f}\|^2_\infty + 3|B_{k_n(s)}|^2\Big]\exp\Big(6\|\tilde{f}\|^2_\infty k_n(s)^2\Big)\Big)\mathrm{d}s \\
&\leq 2\|\tilde{f}\|^2_\infty\Big(t + \Big[3|X_0|^2 t + 6t^3\|\tilde{f}\|^2_\infty + 3\int_0^t |B_{k_n(s)}|^2\mathrm{d}s\Big]\exp\Big(6\|\tilde{f}\|^2_\infty t^2\Big)\Big) \\
&\leq 2\|\tilde{f}\|^2_\infty\Big(T + \Big[3|X_0|^2 T + 6T^3\|\tilde{f}\|^2_\infty\Big]\exp\Big(6\|\tilde{f}\|^2_\infty T^2\Big)\Big) + 6\|\tilde{f}\|^2_\infty \exp\Big(6\|\tilde{f}\|^2_\infty T^2\Big)\int_0^T |B_{k_n(s)}|^2\mathrm{d}s \\
&:= U(T) + V(T)\int_0^1 |B_{k_n(Ts)}|^2\mathrm{d}s,
\end{aligned} \tag{2.20}$$

where

$$U(T) = 2\|\tilde{f}\|_\infty^2\Big(T + \Big[3|X_0|^2T + 6T^3\|\tilde{f}\|_\infty^2\Big]\exp\Big(6\|\tilde{f}\|_\infty^2T^2\Big)\Big)$$
$$V(T) = 6T\|\tilde{f}\|_\infty^2\exp\Big(6\|\tilde{f}\|_\infty^2T^2\Big).$$

Taking exponential on both sides, taking the expectation on both sides on both sides of (2.20) and using the fact that X_0 is independent of $\{B_t\}_{t\in[0,T]}$, applying Jensen inequality and Fubini theorem, we get

$$\begin{aligned}
\mathbf{E}\exp\Big[(1/2)\int_0^t\Big|f(X^n_{k_n(s)})\Big|^2\mathrm{d}s\Big] \leq& \mathbf{E}\exp\Big\{U(T)+V(T)\int_0^T|B_{k_n(s)}|^2\mathrm{d}s\Big\}\\
\leq& \mathbf{E}\exp\Big\{U(T)\Big\}\times\int_0^T\mathbf{E}\exp\Big\{V(T)|B_{k_n(s)}|^2\Big\}\mathrm{d}s\\
\leq& \mathbf{E}\exp\Big\{U(T)\Big\}T\sup_{s\in[0,T]}\mathbf{E}\exp\Big\{V(T)|B_{k_n(s)}|^2\Big\}\\
\leq& \mathbf{E}\exp\Big\{U(T)\Big\}T\sup_{s\in[0,T]}\mathbf{E}\exp\Big\{V(T)|B_s|^2\Big\}. \qquad (2.21)
\end{aligned}$$

Since $V(t)$ goes to zero when t tends to zero, then for T small enough and depending only on $\|\tilde{f}\|_\infty$ and independent of X_0, we have by Fernique theorem that

$$\mathbf{E}\exp\Big\{V(t)|B_s|^2\Big\} = \int_{\mathbb{R}^d}\frac{1}{(2\pi)^{d/2}}\exp(V(t)|z|^2)\exp\Big(-\frac{|z|^2}{2s}\mathrm{d}z\Big) < \infty. \qquad (2.22)$$

Then for such small enough time T, the process $(\gamma^n_s)_{s\in[0,T]}$ is a martingale uniformly in n.

Now by Hölder's inequality we have for $r\in[0,T]$

$$\begin{aligned}
&\mathbf{E}\Big|\int_0^r g(s,X^n_s)-g(s,X^n_{k_n(s)})\mathrm{d}s\Big|^2\\
\leq& \mathbf{E}\Big[\Big|\int_0^r g(s,X^n_s)-g(s,X^n_{k_n(s)})\mathrm{d}s\Big|^{2-\epsilon}(\gamma^n_r)^{(2-\epsilon)/2}\Big|\int_0^r g(s,X^n_s)-g(s,X^n_{k_n(s)})\mathrm{d}s\Big|^{\epsilon}(\gamma^n_r)^{(\epsilon-2)/2}\Big]\\
\leq& \Big[\mathbf{E}\Big|\int_0^r g(s,X^n_s)-g(s,X^n_{k_n(s)})\mathrm{d}s\Big|^2(\gamma^n_r)\Big]^{(2-\epsilon)/2}\\
&\times\Big[\mathbf{E}\Big|\int_0^r g(s,X^n_s)-g(s,X^n_{k_n(s)})\mathrm{d}s\Big|^2(\gamma^n_r)^{(\epsilon-2)/\epsilon}\Big]^{\epsilon/2}. \qquad (2.23)
\end{aligned}$$

Thanks to Girsanov's theorem under Bĕnes conditions, we have that $Y^n := X^n - X_0$ is an $(\mathbb{P}^n,(\mathcal{F}_t)_{t\in[0,r]})$-adapted Brownian motion on $[0,r]$, where $\mathbb{P}^n$ is defined by the measure $\mathrm{d}\mathbb{P}^n = (\gamma^n_r)\mathrm{d}\mathbf{P}$. Consequently by Lemma 2.1, we have

$$\begin{aligned}
&\Big(\mathbf{E}\Big|\int_0^r g(s,X_0+Y^n_s)-g(s,X_0+Y^n_{k_n(s)})\mathrm{d}s\Big|^2(\gamma^n_r)\Big)^{(2-\epsilon)/2}\\
\leq& \Big(\mathbf{E}_{\mathbb{P}^n}\Big|\int_0^r g(s,X_0+Y^n_s)-g(s,X_0+Y^n_{k_n(s)})\mathrm{d}s\Big|^2\Big)^{(2-\epsilon)/2}\\
\leq& \Big(Mn^{-1}\log(n)\Big)^{(2-\epsilon)/2} \leq Mn^{-1+\epsilon}.
\end{aligned}$$

In addition using Lemma 2.3, we have

$$\begin{aligned}
&\Big(\mathbf{E}\Big|\int_0^r g(s,X^n_s)-g(s,X^n_{k_n(s)})\mathrm{d}s\Big|^2(\gamma^n_r)^{(\epsilon-2)/\epsilon}\Big)^{\epsilon/2}\\
\leq& \Big(\mathbf{E}\Big|\int_0^r g(s,X^n_s)-g(s,X^n_{k_n(s)})\mathrm{d}s\Big|^4\mathbf{E}(\gamma^n_r)^{2(\epsilon-2)/\epsilon}\Big)^{\epsilon/4}\\
\leq& \Big(M\mathbf{E}(\gamma^n_r)^{2(\epsilon-2)/\epsilon}\Big)^{\epsilon/4}. \qquad (2.24)
\end{aligned}$$

The process $(\gamma^n_s)_{s\in[0,r]}$ is an exponential martingale with $\mathbf{E}((\gamma^n_r)) = 1$. Take $\alpha = 2(\epsilon-2)/\epsilon$ and set $b_1 = 2\alpha(-f)$. Then the exponential martingale Γ^n with b_1 instead of f is such that $\mathbf{E}(\Gamma^n_r) = 1$

where $r = r(\epsilon)$ is small enough (Novikov condition (2.21)) depending on ϵ and $\|\tilde{f}\|_\infty$. Therefore, for β such that $\beta = 2\alpha^2$,

$$
\begin{aligned}
&\mathbf{E}((\gamma_r^n)^\alpha) \\
=&\mathbf{E}\Big[\exp\Big(\alpha\int_0^r -f(X^n_{k_n(s)})\mathrm{d}B_s - \frac{\beta}{2}\int_0^r f(X^n_{k_n(s)})^2\mathrm{d}s\Big)\exp\Big(\frac{(\beta-\alpha)}{2}\int_0^r f(X^n_{k_n(s)})^2\mathrm{d}s\Big)\Big] \\
\leq&\Big[\mathbf{E}\exp\Big(2\alpha\int_0^r -f(X^n_{k_n(s)})\mathrm{d}B_s - \beta\int_0^r f(X^n_{k_n(s)})^2\mathrm{d}s\Big)\Big]^{1/2}\Big[\mathbf{E}\exp\Big((\beta-\alpha)\int_0^r f(X^n_{k_n(s)})^2\mathrm{d}s\Big)\Big]^{1/2} \\
\leq&\Big[\mathbf{E}\exp\Big(\int_0^r 2\alpha(-b)(X^n_{k_n(s)})\mathrm{d}B_s - \frac{1}{2}\int_0^r 4\alpha^2 f(X^n_{k_n(s)})^2\mathrm{d}s\Big)\Big]^{1/2}\Big[\mathbf{E}\exp\Big((\beta-\alpha)\int_0^r f(X^n_{k_n(s)})^2\mathrm{d}s\Big)\Big]^{1/2} \\
=&\Big[\mathbf{E}(\Gamma_r^n)\Big]^{1/2}\Big[\mathbf{E}\exp\Big(\alpha(\beta-1)\int_0^r f(X^n_{k_n(s)})^2\mathrm{d}s\Big)\Big]^{1/2} \\
\leq&\Big[\mathbf{E}\exp\Big(\alpha(\beta-1)\int_0^r f(X^n_{k_n(s)})^2\mathrm{d}s\Big)\Big]^{1/2} \leq M(\epsilon,\|\tilde{f}\|_\infty) < \infty, \qquad (2.25)
\end{aligned}
$$

where the last inequality follows using arguments similar to those used to obtain the above Běnes condition, since the Borel function $\alpha(\beta-1)f$ satisfies the linear growth property. The desired result follows for r small enough depending on $\epsilon > 0$. □

3. Proof of the main result

In this section, we wish to use the above results along with some results in partial differential equation and namely the regularity of the associated Kolmogorov equation (see Appendix A) to prove Theorem 1.3.

Proof of Theorem 1.3. Let $M(d)$ denotes a positive constant that may change from one line to the other. We denote by $(X_t)_{t\in[0,T]}$ the strong solution of (1.1) when $f(t,x) \equiv f(x)$. By Lemma A.1, there exists a classical solution $u \in \mathbf{C}^{2+\delta'}(\mathbb{R}^d,\mathbb{R}^d)$ of

$$(1/2)\Delta u^i + f\cdot\nabla u^i - \lambda u^i = -f^i, \quad i = 1,\dots,d$$

which satisfies the bounds

$$\|\nabla u^i\|_\infty + \|\Delta u^i\|_\infty \leq M(\lambda)\|f^i\|_\delta. \qquad (3.1)$$

Applying Itô's formula for $u(X_t)$ we have

$$\int_0^t f^i(X_s)\mathrm{d}s = u^i(X_0) - u^i(X_t) + \int_0^t \lambda u^i(X_s)\mathrm{d}s + \int_0^t \nabla u^i(X_s)\mathrm{d}B_s. \qquad (3.2)$$

Similarly, we have

$$
\begin{aligned}
\int_0^t f^i(X_s^n)\mathrm{d}s =& u^i(X_0) - u^i(X_t^n) + \int_0^t \lambda u^i(X_s^n)\mathrm{d}s \\
&- \int_0^t \nabla u^i(X_s^n).\Big(f^i(X_s^n) - f\Big(X^n_{k_n(s)}\Big)\Big)\mathrm{d}s + \int_0^t \nabla u^i(X_s^n)\mathrm{d}B_s. \qquad (3.3)
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
\mathbf{E}|X_t - X_t^n|^2 \leq& 2\mathbf{E}\Big|\int_0^t (f(X_t) - f(X_t^n))\mathrm{d}s\Big|^2 + 2\mathbf{E}\Big|\int_0^t \Big(f(X_t^n) - f(X^n_{k_n(s)})\Big)\mathrm{d}s\Big|^2 \\
=:& \mathbf{J}_n^1 + \mathbf{J}_n^2.
\end{aligned}
$$

For $\mathbf{J}_n^1$ we have by (3.2) and (3.3).

$$
\begin{aligned}
\mathbf{J}_n^1 \leq & M(d)\sum_{i=1}^d \mathbf{E}\Big|\int_0^t (f^i(X_t)-f^i(X_t^n))\mathrm{d}s\Big|^2 \\
\leq & M(d)\sum_{i=1}^d \mathbf{E}|u^i(X_t)-u^i(X_t^n)|^2 \\
& + M(d)\sum_{i=1}^d \mathbf{E}\Big|\int_0^t \Big(\nabla u^i(X_s)-\nabla u^i(X_s^n)\Big)\mathrm{d}B_s\Big|^2 + M(d)\sum_{i=1}^d \lambda^2\mathbf{E}\Big|\int_0^t \Big(u^i(X_s)-u^i(X_s^n)\Big)\mathrm{d}s\Big|^2 \\
& + M(d)\sum_{i=1}^d \mathbf{E}\Big|\int_0^t \nabla u^i(X_s^n)\cdot\Big(f(X_s^n)-f(X_{k_n(s)}^n)\Big)\mathrm{d}s\Big|^2 \\
=: & \sum_{i=1}^4 \mathbf{J}_n^1(i).
\end{aligned} \tag{3.4}
$$

Then by (3.1) we have the following estimates:

$$
\mathbf{J}_n^1(1) \leq M(d)\|\nabla u\|_\infty^2\mathbf{E}|X_t-X_t^n|^2 \leq M(d)M(\lambda)^2\|f\|_\delta^2\mathbf{E}|X_t-X_t^n|^2.
$$

By Itô's isometry, (3.1) and Hölder inequality we have

$$
\begin{aligned}
\mathbf{J}_n^1(2)+\mathbf{J}_n^1(3) &\leq M(d)\int_0^t \mathbf{E}|\nabla u(X_s)-\nabla u(X_s^n)|^2\mathrm{d}s + M(d)\lambda^2 T\int_0^t \mathbf{E}|u(X_s)-u(X_s^n)|^2\mathrm{d}s \\
&\leq M(d)\sum_{i=1}^d \|\Delta u^i\|_\infty^2\int_0^t \mathbf{E}|X_s-X_s^n|^2\mathrm{d}s + M(d)\lambda^2\sum_{i=1}^d \|\nabla u^i\|_\infty^2 T\int_0^t \mathbf{E}|X_s-X_s^n|^2\mathrm{d}s \\
&\leq K\int_0^t \mathbf{E}|X_s-X_s^n|^2\mathrm{d}s;
\end{aligned}
$$

where $K = M(d)(1+\lambda^2 T)M(\lambda)^2\|f\|_\delta^2$. Consider the terms $\mathbf{J}_n^1(4)$, we have

$$
\begin{aligned}
\mathbf{J}_n^1(4) &= M(d)\sum_{i=1}^d \mathbf{E}\left|\int_0^t \nabla u^i(X_s^n).f(X_s^n)-\nabla u^i(X_s^n).f\left(X_{k_n(s)}^n\right)\mathrm{d}s\right|^2 \\
&\leq M(d)\sum_{i=1}^d \mathbf{E}\left|\int_0^t \left[\nabla u^i(X_s^n).f(X_s^n)-\nabla u^i\left(X_{k_n(s)}^n\right).f\left(X_{k_n(s)}^n\right)\right.\right. \\
&\quad \left.\left. +\ \nabla u^i\left(X_{k_n(s)}^n\right).f\left(X_{k_n(s)}^n\right)-\nabla u^i(X_s^n).f\left(X_{k_n(s)}^n\right)\right]\mathrm{d}s\right|^2 \\
&\leq M(d)\sum_{i=1}^d \mathbf{E}\left|\int_0^t \left[\nabla u^i(X_s^n).f(X_s^n)-\nabla u^i\left(X_{k_n(s)}^n\right).f\left(X_{k_n(s)}^n\right)\right]\mathrm{d}s\right|^2 \\
&+ M(d)\sum_{i=1}^d \mathbf{E}\left|\int_0^t \left[\nabla u^i\left(X_{k_n(s)}^n\right)-\nabla u^i(X_s^n)\right].f\left(X_{k_n(s)}^n\right)\mathrm{d}s\right|^2 \\
&\quad =: F_n^1+F_n^2.
\end{aligned} \tag{3.5}
$$

We estimate the last term of the above, F_n^2. By Cauchy-Schwartz and Hölder inequality, we have

$$
\begin{aligned}
F_n^2 \leq& M(d)\sum_{i=1}^d \mathbf{E}\int_0^t \left(\nabla u^i\left(X^n_{k_n(s)}\right)-\nabla u^i(X^n_s)\right)^2\left|f(X^n_{k_n(s)})\right|^2 \mathrm{d}s\\
\leq& \|\tilde f\|^2_{L_\infty} M(d)\sum_{i=1}^d \|\Delta u^i\|^2_\infty \int_0^t \mathbf{E}\left[\left|X^n_{k_n(s)}-X^n_s\right|^2\left(1+\left|X^n_{k_n(s)}\right|^2\right)\right]\mathrm{d}s\\
\leq& \|\tilde f\|^2_{L_\infty} M(d)\sum_{i=1}^d \|\Delta u^i\|^2_\infty \int_0^t \sqrt{\mathbf{E}\left[\left|X^n_{k_n(s)}-X^n_s\right|^4\right]\mathbf{E}\left[\left(1+\left|X^n_{k_n(s)}\right|^2\right)^2\right]}\mathrm{d}s\\
\leq& M(d)\left(\|\tilde f\|_{L_\infty}\|f\|_\delta\right)^2\int_0^t \sqrt{\mathbf{E}\left[\left|X^n_{k_n(s)}-X^n_s\right|^4(\gamma^n_r)^{1/2}(\gamma^n_r)^{-1/2}\right]}\mathrm{d}s\\
\leq& M(d)\left(\|\tilde f\|_{L_\infty}\|f\|_\delta\right)^2\int_0^t \sqrt[4]{\mathbf{E}\left[\left|X^n_{k_n(s)}-X^n_s\right|^8\gamma^n_r\right]\mathbf{E}\left[(\gamma^n_r)^{-1}\right]}\mathrm{d}s\\
\leq& M(d)\left(\|\tilde f\|_{L_\infty}\|f\|_\delta\right)^2\sqrt[4]{\mathbf{E}\left[(\gamma^n_r)^{-1}\right]}\int_0^t \sqrt[4]{\mathbf{E}_{\mathbb{P}^n}\left[\left|Y^n_{k_n(s)}-Y^n_s\right|^8\right]}\mathrm{d}s\\
\leq& M(d)\left(\|\tilde f\|_{L_\infty}\|f\|_\delta\right)^2\sqrt[4]{\mathbf{E}\left[(\gamma^n_r)^{-1}\right]}\int_0^t \sqrt[4]{M_8\left[|s-k_n(s)|^8\right]}\mathrm{d}s\\
\leq& M(d)\left(\|\tilde f\|_{L_\infty}\|f\|_\delta\right)^2\sqrt[4]{\mathbf{E}\left[(\gamma^n_r)^{-1}\right]}t(1/n)^2\\
\leq& M(d)\left(\|\tilde f\|_{L_\infty}\|f\|_\delta\right)^2 M^{2.25}M(\lambda)^2 n^{-1+\epsilon}\\
=:& K'M(\lambda)^2n^{-1+\epsilon}.
\end{aligned}
\tag{3.6}
$$

Both the terms F_n^1 and $\mathbf{J}_n^2$ can be estimated using Lemma 2.4, with $g(t,x)=f(x)$ and $g(t,x)=\nabla u(x)f(x)$), respectively. Thus we get

$$F_n^1+\mathbf{J}_n^2\leq Mn^{-1+\epsilon}.$$

Putting the above bounds together we get

$$\mathbf{E}|X_t-X_t^n|^2\leq\left[M+K'M(\lambda)^2\right]n^{-1+\epsilon}+M(d)M(\lambda)^2\|f\|^2_\delta T\mathbf{E}|X_t-X^n_t|^2+K\int_0^t\mathbf{E}|X_s-X_s^n|^2\mathrm{d}s.$$

For λ big enough (say $M(d)M(\lambda)^2\|f\|^2_\delta T<1/2$), we have

$$\mathbf{E}|X_t-X_t^n|^2\leq 2\left[M+K'M(\lambda)^2\right]n^{-1+\epsilon}+K\int_0^t\mathbf{E}|X_s-X_s^n|^2\mathrm{d}s.$$

Hence the desired result follows by Gronwall's Lemma. □

Appendix A. Results on Kolmogorov backward PDE

In this section, we provide a result on the regularity of the Kolmogorov equation (see for example [4, Theorem 5]) under Hölder continuity assumptions of the drift coefficient. Let us recall that for $\delta\in(0,1)$, $\mathbf{C}^\delta(\mathbb{R}^d)$ denotes the set of locally uniformly δ-Hölder continuous functions i.e $g\in\mathbf{C}^\delta(\mathbb{R}^d)$ if

$$[g]_\delta:=\sup_{x\neq y\in\mathbb{R}^d,|x-y|\leq 1}\frac{|g(x)-g(y)|}{|x-y|^\delta}<\infty.$$

We assume that the functions in $\mathbf{C}^\delta$ have at most linear growth and define the norm $\|\cdot\|_\delta$ of a function g in $\mathbf{C}^\delta(\mathbb{R}^d)$ by $\|g\|_\delta:=\|\tilde g\|_\infty+[g]_\delta$, where $\tilde g(x):=g(x)/(1+|x|)$. A function $g\in\mathbf{C}^{2,\delta}(\mathbb{R}^d)$ if $g\in\mathbf{C}^\delta(\mathbb{R}^d)$ and moreover, the first and second order derivatives D^1g and D^2g are bounded and δ-Hölder continuous. The following result corresponds to [4, Theorem 5].

Lemma A.1. *Let $\delta \in (0,1)$. For any $\delta' \in (0,\delta)$, there exists $\lambda_0 > 0$ (depending on $\delta, \delta', [f]_\delta$, $f \in \mathbf{C}^\delta(\mathbb{R}^d, \mathbb{R}^d)$ fixed) such that, for $\lambda \geq \lambda_0$, for any $g \in \mathbf{C}^\delta(\mathbb{R}^d)$, the following elliptic partial differential equation*

$$\lambda u - (1/2)\Delta u - \nabla u \cdot f = g$$

admits a unique classical solution $u = u_\lambda \in \mathbf{C}^{2+\delta}(\mathbb{R}^d)$ for which

$$\|\tilde{u}\|_\infty + \|\nabla u\|_\infty + \|\Delta u\|_\infty \leq M(\lambda)\|g\|_\delta,$$

with $M(\lambda)$ (independent on u and g) such that $M(\lambda)$ decays to zero when λ goes to infinity.

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