Fundamental and Phenomenological Aspects of Anti-D-Brane Supersymmetry Breaking

Ph.D. Thesis

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Abstract

This dissertation focusses on the investigation of both the fundamental and the phenomenological features of non-supersymmetric string theories based on supersymmetry breaking by anti-D-branes. The study of non-supersymmetric string theories is shedding light on an important corner of the string landscape that might ultimately explain the reason why, so far, supersymmetry has not been detected in the observed universe.

The first line of research aims at enriching the understanding of misaligned supersymmetry in String Theory. Misaligned supersymmetry consists in cancellations between bosonic and fermionic contributions at different energy levels in the whole string spectrum. This is interpreted as a physical mechanism that helps visualising the origin of the finiteness of string constructions, otherwise motivated based on the behaviour of the theory under modular transformations. A review is presented of how misaligned supersymmetry in closed-string theories leads to a cancellation between bosons and fermions even in non-supersymmetric scenarios. Then, it is shown that an entirely analogous cancellation can take place in non-supersymmetric open-string theories, too, by studying anti-D-branes placed on top of O-planes. These ideas are then developed via a systematic analysis of the net physical degeneracies at each energy level, studying their non-trivial cancellations and relating them to the modular properties of the partition function. Eventually, the whole concept of misaligned supersymmetry in String Theory is analysed in a mathematically rigorous way, showing the details of how the boson-fermion cancellations can take place in physical quantities, and the role of misalignment in all known 10-dimensional tachyon-free non-supersymmetric string constructions is finally discussed.

The second line of research, instead, is devoted to a phenomenological investigation and description of a class of quasi-realistic non-supersymmetric vacua including anti-D-branes. In particular, it discusses model-building scenarios featuring intersecting anti-D3- and D7-branes. Effectively, supersymmetry is broken spontaneously, despite having no scale at which sparticles appear and standard supersymmetry is restored. If the branes are placed on singularities at the tip of warped throats in Calabi-Yau orientifold flux compactifications, they may give rise to quasi-realistic particle spectra, closed- and open-string moduli stabilisation with a Minkowski/de Sitter uplift, together with a geometrical origin for the scale hierarchies. A derivation is given of the low-energy effective field theory description for such scenarios, i.e. a non-linear supergravity theory for standard and constrained supermultiplets, including soft supersymmetry-breaking matter couplings. The effect of closed-string moduli stabilisation on the open-string matter sector is worked out, incorporating non-perturbative and perturbative effects, and the mass and coupling hierarchies are computed with a view towards phenomenology.

To my family.

Declaration

I hereby declare that the material presented in this doctoral thesis is the result of my own research activity together with my collaborators. All references to other people's work are cited explicitly. All my work has been carried out in the String Phenomenology Group in the Department of Mathematical Sciences at the University of Liverpool during my doctoral studies.

The results presented in this thesis are based on the following publications:

– ref. [1]: Non-supersymmetric String Models from Anti-D3-/D7-branes in Strongly Warped Throats, published in the *Journal of High Energy Physics* 12 (2020) 174 (e-print: hep-th/2007.11333), in collaboration with Susha Parameswaran;

– ref. [2]: Misaligned Supersymmetry and Open Strings, published in the *Journal of High Energy Physics* 04 (2021) 099 (e-print: hep-th/2012.04677), in collaboration with Niccolò Cribiori, Susha Parameswaran and Timm Wrase;

– ref. [3]: Modular invariance, misalignment and finiteness in non-supersymmetric strings, published in the *Journal of High Energy Physics* 01 (2022) 127 (e-print: hep-th/2110.11973), in collaboration with Niccolò Cribiori, Susha Parameswaran and Timm Wrase.

During my doctoral studies, I have also contributed to an article that is not discussed in this thesis:

– ref. [4]: Branes, fermions, and superspace dualities, published in the *Journal of High Energy Physics* 10 (2021) 243 (e-print: hep-th/2106.02090), in collaboration with Ander Retolaza, Jamie Rogers and Radu Tatar.

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CONTENTS

1 INTRODUCTION

A search for a fundamental theory that is capable of describing nature is intrinsic to the defining features of Physics. The observation that several phenomena can be described and explained by relying on the understanding of only a few essential principles leads one to always wonder what the fundamental structure of the physical world is and to try to lead as many physical laws as possible back to a more fundamental level.

On the one hand, based on the basic principles of Quantum Mechanics and Special Relativity, Quantum Field Theory unifies the description of three of the four known fundamental forces of nature. These take place among the elementary particles that constitute the world and include atomic and subatomic interactions. In fact, the Standard Model of Particle Physics has undergone an extraordinary series of experimental confirmations, culminating in the observation of the Higgs boson. On the other hand, General Relativity is the theory of gravitation, the other fundamental observed force. Along with providing a description of the spacetime itself and how things move through it, compatibly with astronomical observations too, this theory also describes phenomena such as black holes and gravitational waves, and it lays the basis for the study of the cosmological evolution of the Universe. However, despite an overwhelming experimental evidence of success in the description of the fundamental interactions of the observed world, it is still unknown how to reconcile these two essential cornerstones. In technical terms, the core issue is that a quantum theory of gravity is not renormalisable. Further open issues include the explanation of the various scale hierarchies appearing in nature, coherent descriptions of inflation in the early Universe as well as of the current-time accelerated expansion of the latter, and a satisfactory modelling of dark matter. A proposed way-out that in principle is believed to be able to explain all these open questions is String Theory.

String Theory assumes the fundamental constituents of the world to be 1-dimensional strings, rather than point-like particles. Astonishingly, this is sufficient to define a theory whose intrinsic excitations represent a graviton as well as a huge set of particles that may well include the observed ones. Moreover, for its own consistency, it determines the number of spacetime dimensions, ten, and, in principle, it fixes dynamically in the vacuum of the theory all the numerical values of the particle coupling constants, which instead are input parameters in the Standard Model and in General Relativity. Despite being extremely promising and appealing, though, String Theory works based on a very complex set of physical and mathematical tools that make it hard to understand its reach and its limitations. In particular, the choice of which string construction to consider and of the details of the compactification to four dimensions, in the lack of reasons to select them based on reasons of internal coherence, gives a virtually unlimited array of possibilities that should be considered to reconcile the theory with the observed world. This means that there is room for all sorts of analyses, but also that it is not extremely clear which is the direction to be pursued.

The aim of String Phenomenology is to investigate over the connection between the core string-based theory presumed to be underlying the universe and the observations of the world. This requires both control over the internal consistency of the theory and the capability of the theory to generate realistic interactions in the effective four-dimensional description. The problem of internal consistency of such constructions involves keeping all the necessary approximations under control. On the other hand, adherence to the observed world includes more specifically the need for a particle spectrum that resembles the standard-model one, possibly accommodating for exotic particles, such as those that could constitute the inflaton and dark matter, and for instance the attainability of a vacuum energy that is compatible with the tiny observed cosmological constant, requiring a theory in which all the fields are stabilised in a stable or metastable vacuum. The feasibility of a vacuum in which tree-level and quantum corrections from all particles result in tiny scales compared to the scale governing Quantum Gravity is a particularly tough issue to solve in ordinary Quantum Field Theory, since an intermediate cutoff scale is expected to be involved, which tends to drive all the standard-model scales up to its value, but String Theory is presumed to contain the solution of this hierarchy problem, too. Traditional approaches to solve the hierarchy problem involve the concept of supersymmetry, i.e. an extension of the Poincaré algebra that also includes a symmetry between the bosons and the fermions of the theory. In fact, however, mere supersymmetric extensions of the Standard Model are still hardly compatible with for instance a tiny cosmological constant, since the physical cancellations between bosonic and fermionic terms that could make sense of the hierarchies stop working exactly once supersymmetry is broken, which nevertheless is necessary to reconcile with real-world observations. Supersymmetry can be postulated within String Theory, too, and it is believable that this may be a key towards understanding this puzzle. More generally, as the observed world contains both bosons and fermions, a way to generate them in the string-derived particle spectrum is by postulating supersymmetry.

Of course, supersymmetry is not observed in present-day experiments. This means that, if it exists at all, it is a symmetry that is broken in the vacuum, and ultimately a realistic theory must reproduce a non-supersymmetric vacuum. As follows from this introduction, the fundamental motivation that has driven the development of the research presented in this work is the exploration of non-supersymmetric string theories. In these theories, loosely speaking, supersymmetry is present at the worldsheet level, i.e. at the level where the 1-dimensional string is defined, but, in the effective theory that comes out of them, they lack the traditional features of supersymmetric constructions, i.e. in particular a one-to-one correspondence between all the bosonic and all the fermionic particles of the theory. An aspect that renders such theories particularly interesting consists in the fact that the lack of space-time supersymmetry bypasses the problem of understanding supersymmetry breaking at the field-theory level, which is a typical source of tension between hierarchies and supersymmetry breaking, and yet their defining features are understandable via the rich underlying superstring-derived structure that characterises them, which leaves room for consistently explaining the presence different characteristic energy scales. Another noteworthy common aspect of non-supersymmetric string theories lies in the fact that they typically tend to result in positive-energy vacua. Although this is a desirable characteristic of a string-based theory aimed at describing the observed universe – since a de Sitter vacuum is an explanation to cosmological observations of a universe undergoing an accelerated expansion –, it is a difficult feature to achieve in string-based constructions. Understanding the extent to which such a vacuum can be reached consistently within a purely stringy framework is therefore advisable, and non-supersymmetric string theories are good candidates for this.

There is only a small number of known 10-dimensional superstring models with no tachyons and they can be distinguished according to the number of supersymmetries they possess [5]. The supersymmetric string theories are type IIA and type IIB theories [6,7], i.e. closed-string theories with no gauge groups, the type I theory [8], i.e. a theory with closed and open strings and the gauge group SO(32), and the heterotic $\mathrm{E}_8 \times \mathrm{E}_8$- and SO(32)-theories [9–11], i.e. closed-string theories with the eponimous gauge groups. In type II theories, gauge groups can still arise in the presence of branes and the associated open strings. In all these theories, some mechanism to break all the supersymmetries must be accounted for, in such a way that the corresponding low-energy four-dimensional theory is not effectively supersymmetric. The inherently non-supersymmetric string theories are the heterotic $\mathrm{SO}(16)\times\mathrm{SO}(16)$-theory [12, 13], with the eponymous gauge group, the Sugimoto model [14], with a gauge group USp(32), and the type 0'B theory [15–17], with a gauge group SU(32). In these cases, too, some scenario for a compactification to a realistic four-dimensional theory should eventually be constructed. A key difference of the Sugimoto model with the other non-supersymmetric models is the presence of a massless gravitino in its spectrum. This is related to the fact that supersymmetry is only non-linearly realised, and it can be interpreted as a symmetry broken at the string scale. This is the simplest instance of a scenario that goes by the name of 'brane supersymmetry breaking' [14, 18–26]. The breaking of supersymmetry in these models originates in the fact that, although individually their constituents are supersymmetric, they preserve different supersymmetries with respect to each other. The other non-supersymmetric theories, instead, are more deeply not supersymmetric, not possessing a massless gravitino.

In loose terms, this work focusses on an enrichment of the understanding of superstring theories where the reason for the lack of supersymmetry lies in the presence of specific brane configurations, similarly to the case of brane supersymmetry breaking hinted above. In particular, Dp-branes and anti-Dp-branes are intrinsic objects of type I and type II theories, and they can be imagined as $(p+1)$-dimensional spacetime hypersurfaces to which the endpoints of open strings are attached. These degrees of freedom happen to partially break the supersymmetries generated by the closed strings due to their boundary conditions, i.e. in other words they preserve only certain supersymmetries. At the same time, note that closed strings, among other fields, generate the graviton, and they are always presents. Even for the latter, symmetries of the worldsheet or of the internal space can reduce the number of supersymmetries that are being preserved. Inherently with these facts, constructions exist where the supersymmetry preserved by the closed-string sector is not realised by the

D-brane setup. This results in a theory which is effectively non-supersymmetric. Another aspect that renders anti-D-branes remarkable for model-building consists in the fact that they provide a positive contribution to the vacuum energy.

An outlook on the organisation of this thesis is below.

- In an attempt to provide a relatively self-contained contextualisation of the work presented later on, chapter 2 reviews a few of the basic ideas of String Theory and String Phenomenology. In particular, the core aspects that are reviewed are the properties of string partition functions, typical compactification scenarios and the algebraic structure of anti-D-brane supersymmetry breaking.

- Based on refs. [2, 3], chapter 3 interprets the breaking of supersymmetry by anti-D-branes on O-planes in terms of an asymptotic supersymmetry characterising the whole spectrum of string states. This idea of a 'misaligned supersymmetry' characterising non-supersymmetric string theories [27, 28], present in the literature with reference to closed-string models, is reviewed and extended to open-string models. Taking the heterotic SO(16) $\times$ SO(16)- and the anti-Dp-brane/Op-plane theory as notable examples, by a detailed analysis of the Hardy-Ramanujan-Rademacher expansion for the partition-function coefficients, it is shown that systematic cancellations – similar to the standard supersymmetry-like boson-fermion cancellations – take place involving the whole towers of bosonic and fermionic states, allowing for a physically intuitive understanding of the finiteness of for instance the one-loop cosmological constant.

- Based on ref. [1], chapter 4 analyses theories where supersymmetry is broken by anti-D-branes in a purely low-energy-based point of view. This has been motivated by the interest in formulating a description of non-supersymmetric anti-D3-/D7-brane constructions at orbifold singularities [29, 30] using the standard supergravity tools by embedding the degrees of freedom of the supersymmetry-breaking sectors in constrained superfields [31, 32]. The result is an instance of a phenomenological construction where the standard-like model fields belong to constrained supermultiplets and in a characterisation of the orders of magnitude that emerge in this construction for a KKLT-like moduli stabilisation.

- To conclude, chapter 5 presents a recap of the main results that have been found and overviews future related prospects.

Both lines of research address the problem of enriching the understanding of anti-D-brane supersymmetry breaking. The first one does so in quite a formal way, inspecting the fundamental and mathematical nature of anti-D-brane supersymmetry breaking and, actually, of non-supersymmetric string theories more in general. The second one, instead, takes on quite the opposite direction, exploring in a practical way, as required by the number of approximations needed to carry on calculations mimicking a presumedly quasi-realistic construction, how such theories can be formulated in purely phenomenological terms. Their common feature, i.e. anti-D-brane supersymmetry breaking, is at the core of several present-day studies and this thesis is aimed at adding new insights on both its fundamental nature and on its impact on real-world modelling proposals.

2 BASICS IN STRINGS AND D-BRANES

This chapter overviews a few core aspects of string theory, with the aim of introducing in a relatively self-contained way most of the topics that are going to be discussed in the core part of the thesis. This will help the fundamental ideas to lie on coherently well-defined grounds and the notation to be always clear.

Starting from the superstring worldsheet action, the chapter introduces the general mode expansions and partition functions for both closed and open type IIB strings in section 2.1. Further, the basic features of string compactifications are overviewed, with particular reference Calabi-Yau orientifolds in the presence of warping, in section 2.2. After this, focussing on the open-string sector, it reviews further basic facts about D-branes in section 2.3 and, finally, it motivates the use of constrained multiplets for brane-induced supersymmetry breaking in section 2.4.

2.1 RNS-superstrings and Partition Functions

This section provides a short introduction to supersymmetric strings, needed to introduce the notation and set the stage for later discussions, and it is mostly based on refs. [33, 34]; excellent introductions are also refs. [35–37]. Being a core subject of string theory, only a brief (but essentially self-contained) overview is presented here.

2.1.1 Classical Superstrings

String theory can be introduced as a self-consistent theory unifying gravity and quantum mechanics in a simple framework, starting from the only assumption that the fundamental constituents of the universe are relativistic 1-dimensional strings to be quantised and not point-like particles. As such, though, bosonic string theory has two crucial issues: it is tachyonic and it does not posses fermions. Therefore, it is not suitable for phenomenological applications. A way to reach a more realistic scenario consists in formulating a theory which is supersymmetric at the worldsheet level. Notice that supersymmetry in target spacetime is not guaranteed to be there. This is the Ramond-Neveu-Schwarz superstring.

The Polyakov action describes a 2-dimensional string worldsheet Σ embedded in the D-dimensional flat spacetime $\mathbb{R}^{1,D-1}$, with the $(-,+^{D-1})$-signature metric η_{MN}. Such an embedding is defined by the coordinates $X^M = X^M(\xi^\alpha)$, where $\xi^\alpha = \tau, \sigma$. The action is a generalisation of the Nambu-Goto action, which, generalising the wordline action of a particle, represents the volume swept by the worldsheet, and is obtained by defining an auxiliary worldsheet metric $h_{\alpha\beta} = h_{\alpha\beta}(\xi)$ with $(-,+)$-signature, in order to make calculations manageable. In order to have supersymmetry on the worldsheet, one has to introduce a 2-dimensional Majorana spinor ψ^M (in components, this is $\psi^M_{\dot{\alpha}} = (\psi^M_+, \psi^M_-)^T$), which is the superpartner of the coordinate X^M, and a gravitino χ_α, which is the superpartner of the graviton $h_{\alpha\beta}$. The supersymmetric generalisation of the bosonic Polyakov action is the Ramond-Neveu-Schwarz action

$$S_{\mathrm{RNS}} = -\frac{T}{2}\int_\Sigma \mathrm{d}^2\xi\,\sqrt{-h}\,\Big[h^{\alpha\beta}\partial_\alpha X^M\partial_\beta X^N\eta_{MN} + \mathrm{i}\overline{\psi}^M\rho^\alpha\partial_\alpha\psi^N\eta_{MN} \\ +\mathrm{i}\overline{\chi}_\alpha\rho^\beta\rho^\alpha\psi^M\Big(\partial_\beta X^N - \frac{\mathrm{i}}{4}\overline{\chi}_\beta\psi^N\Big)\eta_{MN}\Big], \tag{2.1.1}$$

where T is the string tension, defined in terms of the so-called Regge slope α' as

$$T = \frac{1}{2\pi\alpha'}. \tag{2.1.2}$$

The worldsheet coordinates have a domain $(\tau/l, \sigma/l) \in \mathbb{R}\times[0,\pi[$, where l is some characteristic length of the spacelike extension of the string. The first line contains the kinetic terms and the second line contains couplings which are necessary to ensure supersymmetry, with the 2-dimensional γ-matrices representing the Clifford algebra $\{\rho_\alpha, \rho_\beta\} = 2\,h_{\alpha\beta}$. The RNS-action possesses the following symmetries: local supersymmetry, Weyl and super-Weyl invariance, 2-dimensional local Lorentz transformations, worldsheet reparametrisations and D-dimensional spacetime Poincaré transformations.

One can find the field equations via the principle of least action. Crucially, it is possible to make use of the action symmetries to reduce the number of fields involved. In particular, local supersymmetry, local Lorentz transformations and 2-dimensional reparametrisations allow one to choose a gauge where the metric and the gravitino read

$$h_{\alpha\beta}(\xi) = \mathrm{e}^{2\omega(\xi)}\eta_{\alpha\beta}, \tag{2.1.3a}$$
$$\chi_\alpha(\xi) = \rho_\alpha\chi(\xi). \tag{2.1.3b}$$

In this superconformal gauge, a Weyl and a super-Weyl transformations always allow one to gauge away all the metric and gravitino degrees of freedom. Therefore, the RNS-action in the superconformal gauge reads

$$S_{\mathrm{RNS}} = -\frac{T}{2}\int_\Sigma \mathrm{d}^2\xi\,\Big[\eta^{\alpha\beta}\partial_\alpha X^M\partial_\beta X^N\eta_{MN} + \mathrm{i}\overline{\psi}^M\rho^\alpha\partial_\alpha\psi^N\eta_{MN}\Big]. \tag{2.1.4}$$

In the superconformal gauge, the dynamical field equations take the form

$$\eta^{\alpha\beta}\partial_\alpha\partial_\beta X^M = 0, \tag{2.1.5a}$$
$$\rho^\alpha\partial_\alpha\psi^M = 0. \tag{2.1.5b}$$

Before gauge-fixing, given the energy-momentum tensor $T_{\alpha\beta} = -(2/\sqrt{-h})\,\delta S_{\mathrm{RNS}}/\delta h^{\alpha\beta}$ and its fermionic counterpart $t_\alpha = -(2/\sqrt{-h})\,\delta S_{\mathrm{RNS}}/\mathrm{i}\delta\overline{\chi}^\alpha$, setting them to zero corresponds to the field equations for the auxiliary fields. In the superconformal gauge, these read

$$T_{\alpha\beta} = T\Big[\partial_\alpha X^M\partial_\beta X_M - \frac{1}{2}\eta_{\alpha\beta}\big(\partial^\gamma X^M\partial_\gamma X_M\big) + \frac{\mathrm{i}}{4}\overline{\psi}^M\rho_\beta\partial_\alpha\psi_M + \frac{\mathrm{i}}{4}\overline{\psi}^M\rho_\alpha\partial_\beta\psi_M\Big] = 0, \tag{2.1.6a}$$

$$t_\alpha = T\big[\rho^\beta \rho_\alpha \psi^M \partial_\beta X_M\big] = 0. \tag{2.1.6b}$$

Notice that, imposing the spinor field equation (but not the scalar field equation), both energy-momentum and its fermionic counterpart (in γ-matrix terms) are traceless, being

$$\eta^{\alpha\beta} T_{\alpha\beta} = 0, \tag{2.1.7a}$$
$$\rho^\alpha t_\alpha = 0. \tag{2.1.7b}$$

These are consequences of Weyl and super-Weyl invariance. One can also verify that these fields also satisfy an on-shell continuity equation, i.e.

$$\partial^\alpha T_{\alpha\beta} = 0, \tag{2.1.8a}$$
$$\partial^\alpha t_\alpha = 0. \tag{2.1.8b}$$

In the superconformal gauge, a general solution to the field equations (2.1.5a, 2.1.5b) can be found in terms of the lightcone coordinates

$$\sigma_\pm = \tau \pm \sigma. \tag{2.1.9}$$

In fact, they read

$$\partial_+ \partial_- X^M = 0, \tag{2.1.10a}$$
$$\partial_\pm \psi^M_\mp = 0. \tag{2.1.10b}$$

Before setting them to zero as a way to impose the constraints of eqs. (2.1.6a, 2.1.6b), it is convenient to express the conserved currents in the new coordinates, i.e.

$$T_{\pm\pm} = T\left[\partial_\pm X^M \partial_\pm X_M + \frac{\mathrm{i}}{2} \psi^M_\pm \partial_\pm \psi_{\pm M}\right], \tag{2.1.11a}$$
$$t_\pm = 2T\big[\psi^M_\pm \partial_\pm X_M\big]. \tag{2.1.11b}$$

For the fermionic term only the non-zero components are shown (i.e. $(t_\pm)_\pm = t_\pm$, with $(t_\pm)_\mp = 0$); on the other hand, the component $T_{+-} = \mathrm{i}T[\psi^M_+ \partial_- \psi_{+M} + \psi^M_- \partial_+ \psi_{-M}]/4$ vanishes on-shell. In these coordinates, it is also elementary to find the continutity equations (which can be inferred from eqs. (2.1.8a, 2.1.8b) as well)

$$\partial_\mp T_{\pm\pm} = 0, \tag{2.1.12a}$$
$$\partial_\mp t_\pm = 0. \tag{2.1.12b}$$

These indicate that the currents are functions of only one lightcone coordinate, i.e. $T_{\pm\pm} = T_{\pm\pm}(\sigma^\pm)$ and $t_\pm = t_\pm(\sigma^\pm)$.

In order to proceed to the quantisation of the problem, one needs to find the classical brackets of the theory. For the superconformal-gauge action in eq. (2.1.4), one can define the canonical momenta $P_M = \delta\mathcal{L}/\delta\dot{X}^M = T\dot{X}_M$ and $\Pi^{\dot\alpha}_M = \delta\mathcal{L}/\delta\dot\psi^M_{\dot\alpha} = (\mathrm{i}/2)\, T\psi_{M\dot\beta}(C\rho^0)^{\dot\beta\dot\alpha}$. A little inspection shows that this system has the second-class constraint

$$\Sigma^{\dot\alpha}_M = \Pi^{\dot\alpha}_M - \frac{\mathrm{i}}{2}\, T\psi_{M\dot\beta}(C\rho^0)^{\dot\beta\dot\alpha},$$

being its Poisson bracket with itself nonzero, but rather

$$M^{M}{}_{N}{}^{\dot\alpha\dot\beta}(\sigma'\sigma'') = \{\Sigma^{M\dot\alpha}(\tau,\sigma'), \Sigma^{\dot\beta}_{N}(\tau,\sigma'')\}_{\mathrm{P}} = \mathrm{i}T\delta^M_N \left(C\rho^0\right)^{\dot\alpha\dot\beta}\delta(\sigma'-\sigma'').$$

One thus needs to substitute the Poisson brackets with the Dirac brackets.[2.1] For the lightcone coordinates in the lightcone gauge, these read

$$\{X^M(\tau,\sigma'), P_N(\tau,\sigma'')\}_{\mathrm{D}} = \delta^M_N\delta(\sigma'-\sigma''), \tag{2.1.13a}$$
$$\{\psi^M_\pm(\tau,\sigma'), \psi_{N\pm}(\tau,\sigma'')\}_{\mathrm{D}} = -\mathrm{i}\,T^{-1}\delta^M_N\delta(\sigma'-\sigma''). \tag{2.1.13b}$$

A crucial point that should be discussed in detail concerns boundary conditions. In order to derive the field equations written above, boundary terms from the action have to be zero. In particular, assuming the variations to be zero at infinite distance in time, i.e. $\delta X^M(\tau,\sigma)|_{\tau=\tau',\tau''} = 0$ and $\delta\psi^M(\tau,\sigma)|_{\tau=\tau',\tau''} = 0$, the variation of thr superconformal-gauge action of eq. (2.1.4) gives a boundary term

$$\delta S^{\mathrm{B}}_{\mathrm{RNS}} = -T\int_{\tau'}^{\tau''}\mathrm{d}\tau\ \left[(\delta X^M)\partial_\sigma X_M + \frac{\mathrm{i}}{2}\left(\psi^M_+(\delta\psi_{M+}) - \psi^M_-(\delta\psi_{M-})\right)\right]\Big|_{\sigma=0}^{\sigma=l}.$$

To impose the condition $\delta S^{\mathrm{B}}_{\mathrm{RNS}} = 0$, one encounters two different possibilities.

- The string is closed, and therefore the coordinates have some periodicity conditions. In the bosonic sector, the boundary conditions must be periodic, i.e.

  $$X^M(\tau,\sigma+l) = X^M(\tau,\sigma). \tag{2.1.14}$$

  In the fermionic sector, one can either have periodic or anti-periodic boundary conditions, i.e.

  $$\psi^M_\pm(\tau,\sigma+l) = \mathrm{e}^{2\pi\mathrm{i}\phi_\pm}\psi^M_\pm(\tau,\sigma), \tag{2.1.15}$$

  where periodic conditions, for $\phi_\pm = 0$, are called Ramond boundary conditions, while anti-periodic conditions, for $\phi_\pm = 1/2$, are called Neveu-Schwarz boundary conditions.
  Poincaré invariance requires all the M-directions to have the same boundary conditions. Nevertheless, the boundary conditions for the two spinors can be chosen independently, therefore RR-, RNS-, NSR- and NSNS-boundaries are all possible.

- The string is open, and therefore the boundary terms cancel off independently at each string end. In the bosonic sector, one has two possibilities, called Neumann and Dirichlet boundary conditions, respectively, i.e.

  $$\partial_\sigma X^M(\tau,\sigma)|_{\sigma=0,l} = 0, \tag{2.1.16a}$$
  $$\delta X^M(\tau,\sigma)|_{\sigma=0,l} = 0. \tag{2.1.16b}$$

[2.1] For a theory with a complete set of second-class constraints σ_i with Poisson brackets $\{\phi_i,\phi_j\}_{\mathrm{P}} = M_{ij}$, Dirac brackets are defined as $\{A,B\}_{\mathrm{D}} = \{A,B\}_{\mathrm{P}} - \{A,\phi_i\}_{\mathrm{P}}(M^{-1})^{ij}\{\phi_j,B\}_{\mathrm{P}}$.

Dirichlet boundaries require the string endpoints to be fixed and the locus of the points where, instead, the endpoints can freely move – those with Neumann conditions – is a Dirichlet-brane, or D-brane for short. If Dirichlet boundaries are imposed on a $(p+1)$-dimensional spacetime, that defines a Dp-brane. This breaks Poincaré invariance down to this hypersurface.
In the fermionic sector, one can choose among the boundary conditions $\psi_+^M(\tau,0) = \pm\psi_-^M(\tau,0)$ and $\psi_+^M(\tau,l) = \pm\psi_-^M(\tau,l)$, where only the relative sign is relevant. One calls Neumann-Neumann directions those where[2.2]

$$\psi_+^\alpha(\tau,0) = \psi_-^\alpha(\tau,0), \tag{2.1.17a}$$
$$\psi_+^\alpha(\tau,l) = \eta\psi_-^\alpha(\tau,l). \tag{2.1.17b}$$

On the other hand, Dirichlet-Dirichlet directions are such that

$$\psi_+^{\dot m}(\tau,0) = -\psi_-^{\dot m}(\tau,0), \tag{2.1.18a}$$
$$\psi_+^{\dot m}(\tau,l) = -\eta\psi_-^{\dot m}(\tau,l). \tag{2.1.18b}$$

Finally, Neumann-Dirichlet and Dirichlet-Neumann directions, respectively, are defined in such a way that

$$\psi_+^\iota(\tau,0) = \psi_-^\iota(\tau,0), \tag{2.1.19a}$$
$$\psi_+^\iota(\tau,l) = -\eta\psi_-^\iota(\tau,l); \tag{2.1.19b}$$

$$\psi_+^\iota(\tau,0) = -\psi_-^\iota(\tau,0), \tag{2.1.20a}$$
$$\psi_+^\iota(\tau,l) = \eta\psi_-^\iota(\tau,l). \tag{2.1.20b}$$

In all cases, the values $\eta = +1, -1$ originate the Ramond and Neveu-Schwarz sectors, respectively.

The final step to perform consists in writing the general solutions to the field equations (2.1.5a, 2.1.5b), i.e. eqs. (2.1.10a, 2.1.10b). Closed and open strings must be discussed separately.

- For closed strings, in view of eq. (2.1.10a), the bosonic coordinate must be expressed as $X^M(\sigma^+,\sigma^-) = X_-^M(\sigma^-) + X_+^M(\sigma^+)$, and, to satisfy the periodic boundary conditions of eq. (2.1.14), the right- and left-moving parts can be expanded, respectively, as

$$X_-^M(\sigma^-) = x_-^M + \frac{p^M}{2T}\frac{\sigma^-}{l} + \frac{\mathrm{i}}{2}\frac{1}{\sqrt{\pi T}}\sum_{n\in\mathbb{Z}_*}\frac{1}{n}\,\alpha_n^M\,\mathrm{e}^{-2\pi\mathrm{i}n\sigma^-/l}, \tag{2.1.21a}$$
$$X_+^M(\sigma^+) = x_+^M + \frac{p^M}{2T}\frac{\sigma^+}{l} + \frac{\mathrm{i}}{2}\frac{1}{\sqrt{\pi T}}\sum_{n\in\mathbb{Z}_*}\frac{1}{n}\,\overline{\alpha}_n^M\,\mathrm{e}^{-2\pi\mathrm{i}n\sigma^+/l}, \tag{2.1.21b}$$

[2.2] These α-indices should not be confused with the generic worldsheet coordinates ξ^α, which will never be used again in the following.

where $x^M = x_-^M + x_+^M$ is the center of mass position at time $\tau = 0$, p^M represents the centre-of-mass momentum, and α_n^M and $\overline{\alpha}_n^M$ are arbitrary Fourier coefficients; moreover, $\mathbb{Z}_* = \mathbb{Z} \setminus \{0\}$. Because the coordinates are real, x^M and p^M are also real and the Fourier modes are such that

$$(\alpha_n^M)^* = \alpha_{-n}^M, \tag{2.1.22a}$$
$$(\overline{\alpha}_n^M)^* = \overline{\alpha}_{-n}^M. \tag{2.1.22b}$$

For future use, it is convenient to define the 0-modes as $\alpha_0^M = \overline{\alpha}_0^M = p^M / \sqrt{4\pi T}$.[2.3] To satisfy eq. (2.1.10b), the fermionic coordinates must be such that $\psi_-^M = \psi_-^M(\sigma^-)$ and $\psi_+^M = \psi_+^M(\sigma^+)$. In both cases, to satisfy the boundary conditions of eq. (2.1.15), the functions can be expanded as

$$\psi_-^M(\sigma^-) = \frac{1}{\sqrt{lT}} \sum_{r \in \mathbb{Z}+\phi} b_r^M \, \mathrm{e}^{-2\pi \mathrm{i} r \sigma^- / l}, \tag{2.1.23a}$$
$$\psi_+^M(\sigma^+) = \frac{1}{\sqrt{lT}} \sum_{r \in \mathbb{Z}+\phi} \overline{b}_r^M \, \mathrm{e}^{-2\pi \mathrm{i} r \sigma^+ / l}, \tag{2.1.23b}$$

where b_r^M and $\overline{b}_r^M$ are Fourier modes. Due the Majorana condition on the spinor, these modes must be such that

$$(b_r^M)^* = b_{-r}^M, \tag{2.1.24a}$$
$$(\overline{b}_r^M)^* = \overline{b}_{-r}^M. \tag{2.1.24b}$$

Because the energy-momentum tensor and the fermionic current satisfy the continuity equations (2.1.12a, 2.1.12b), one can define conserved charges. Given an arbitrary smooth periodic function $f = f(\sigma^-)$ such that $f(\sigma^-) = f(\sigma^- + l)$, the integral $L[f] = \int_0^l \mathrm{d}\sigma \; f(\tau - \sigma) \, T_{--}(\tau - \sigma)$ is a conserved charge, i.e. $\mathrm{d}L[f]/\mathrm{d}\tau = 0$, and similarly for the fermionic current. Therefore, one can define the sets of conserved charges $L_m = (l/2\pi) \int_0^l \mathrm{d}\sigma \; \mathrm{e}^{-2\pi \mathrm{i} m \sigma / l} \, T_{--}(0, \sigma)$ and $G_r = \sqrt{l/4\pi} \int_0^l \mathrm{d}\sigma \; \mathrm{e}^{-2\pi \mathrm{i} r \sigma / l} \, t_-(0, \sigma)$, where the time $\tau = 0$ has been considered for simplicity, being them constant. Plugging in the mode expansions in the expressions of eq. (2.1.11a, 2.1.11b), one finds

$$L_m = \frac{1}{2} \sum_{n \in \mathbb{Z}} \alpha_{m-n}^M \alpha_{n M} + \frac{1}{2} \sum_{r \in \mathbb{Z}+\phi} r \, b_{m-r}^M b_{r M}, \tag{2.1.25a}$$
$$G_r = \sum_{n \in \mathbb{Z}} \alpha_n^M b_{r-n\, M}, \tag{2.1.25b}$$

As a matter of fact, the physical constraints of eqs. (2.1.6a, 2.1.6b) translate into the requirements

$$L_m = 0, \tag{2.1.26a}$$
$$G_r = 0. \tag{2.1.26b}$$

An analogous analysis applies for the left-moving terms too.

[2.3] This identification works if all directions are non-compact. For compact directions with periodicity conditions, one has to consider different definitions which eventually lead to T-duality (see e.g. ref. [38]).

- For open strings, there are different solutions depending on the combination of the boundary conditions at the two endpoints. As far as worldsheet bosons are concerned, one can have the mode expansions

$$\text{NN:} \qquad X^\alpha = x^\alpha + \frac{p^\alpha}{T}\frac{\tau}{l} + \frac{\mathrm{i}}{\sqrt{\pi T}} \sum_{n\in\mathbb{Z}_*} \frac{1}{n}\, \alpha_n^\alpha\, \mathrm{e}^{-\mathrm{i}\pi n\tau/l} \cos\Big(\frac{\pi n\sigma}{l}\Big), \tag{2.1.27a}$$

$$\text{DD:} \qquad X^{\dot m} = x_0^{\dot m} + (x_1^{\dot m} - x_0^{\dot m})\frac{\sigma}{l} + \frac{1}{\sqrt{\pi T}} \sum_{n\in\mathbb{Z}_*} \frac{1}{n}\, \alpha_n^{\dot m}\, \mathrm{e}^{-\mathrm{i}\pi n\tau/l} \sin\Big(\frac{\pi n\sigma}{l}\Big), \tag{2.1.27b}$$

$$\text{ND:} \qquad X^\iota = x^\iota + \frac{\mathrm{i}}{\sqrt{\pi T}} \sum_{s\in\mathbb{Z}+1/2} \frac{1}{s}\, \alpha_s^\iota\, \mathrm{e}^{-\mathrm{i}\pi s\tau/l} \cos\Big(\frac{\pi s\sigma}{l}\Big), \tag{2.1.27c}$$

$$\text{DN:} \qquad X^\iota = x^\iota + \frac{1}{\sqrt{\pi T}} \sum_{s\in\mathbb{Z}+1/2} \frac{1}{s}\, \alpha_s^\iota\, \mathrm{e}^{-\mathrm{i}\pi s\tau/l} \sin\Big(\frac{\pi s\sigma}{l}\Big). \tag{2.1.27d}$$

In this case, it is convenient to define $\alpha_0^\alpha = p^\alpha/\sqrt{\pi T}$ and $\alpha_0^{\dot m} = \sqrt{\pi T}(x_1^{\dot m} - x_0^{\dot m})/\pi$. On the other hand, as regards worldsheet fermions, the mode expansions read

$$\text{NN:} \qquad \psi_\pm^\alpha = \frac{1}{\sqrt{2lT}} \sum_{r\in\mathbb{Z}+\phi} b_r^\alpha\, \mathrm{e}^{-\mathrm{i}\pi r\sigma^\pm/l}, \tag{2.1.28a}$$

$$\text{DD:} \qquad \psi_\pm^{\dot m} = \pm\frac{1}{\sqrt{2lT}} \sum_{r\in\mathbb{Z}+\phi} b_r^{\dot m}\, \mathrm{e}^{-\mathrm{i}\pi r\sigma^\pm/l}, \tag{2.1.28b}$$

$$\text{ND:} \qquad \psi_\pm^\iota = \frac{1}{\sqrt{2lT}} \sum_{r\in\mathbb{Z}+\phi'} b_r^\iota\, \mathrm{e}^{-\mathrm{i}\pi r\sigma^\pm/l}, \tag{2.1.28c}$$

$$\text{DN:} \qquad \psi_\pm^\iota = \pm\frac{1}{\sqrt{2lT}} \sum_{r\in\mathbb{Z}+\phi'} b_r^\iota\, \mathrm{e}^{-\mathrm{i}\pi r\sigma^\pm/l}, \tag{2.1.28d}$$

where $\phi = 0$ and $\phi' = 1/2$ for the Ramond sector, while $\phi = 1/2$ and $\phi' = 0$ for the Neveu-Schwarz sector. Again, as a consequence of the continuity equations (2.1.12a, 2.1.12b) there exist conserved charges. Given two smooth functions $f = f(\sigma^-)$ and $g = g(\sigma^-)$ such that $f = g$ at the endpoints $\sigma = 0, l$, the integral $L[f] = \int_0^l \mathrm{d}\sigma\, \big[f(\tau-\sigma)\, T_{--}(\tau-\sigma) + g(\tau+\sigma)\, T_{++}(\tau+\sigma)\big]$ is a conserved charge, and similarly for the fermionic current. Therefore, setting again $\tau = 0$, one can define the conserved charges $L_m = (l/\pi) \int_0^l \mathrm{d}\sigma\, \big[\mathrm{e}^{-\pi\mathrm{i}m\sigma/l}\, T_{--}(0,\sigma) + \mathrm{e}^{\pi\mathrm{i}m\sigma/l}\, T_{++}(0,\sigma)\big]$ and $G_r = \sqrt{l/2\pi} \int_0^l \mathrm{d}\sigma\, \big[\mathrm{e}^{-\pi\mathrm{i}r\sigma/l}\, t_-(0,\sigma) + \mathrm{e}^{\pi\mathrm{i}r\sigma/l}\, t_+(0,\sigma)\big]$. Plugging in the mode expansions in the expressions of eq. (2.1.11a, 2.1.11b), one finds[2.4]

$$L_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} \alpha_{m-n}^M \alpha_{nM} + \frac{1}{2}\sum_{r\in\mathbb{Z}+\phi} r\, b_{m-r}^M b_{rM}, \tag{2.1.29a}$$

$$G_r = \sum_{n\in\mathbb{Z}} \alpha_n^M b_{r-n\,M}, \tag{2.1.29b}$$

[2.4] The expressions below are written in a sloppy notation in order not to overcomplicate them: they are correct if there are only NN- and DD-directions, while in the presence of ND- and/or DN-directions too one has to consider, for each M-direction, the values that the modes can take. This will be taken into account explicitly when discussing masses, as it will be relevant.

Again, the physical constraints of eqs. (2.1.6a, 2.1.6b) imply the requirements

$$L_m = 0, \tag{2.1.30a}$$
$$G_r = 0. \tag{2.1.30b}$$

2.1.2 RNS-Superstring Quantisation

In order to quantise the RNS-superstring, one needs to promote all the fields to operators and to promote the classical brackets to commutators and anticommutators according to the rule

$$AB + (-1)^{f_A f_B + 1} BA = i\{A, B\}_{\mathrm{D}},$$

where anticommutators appear if both A and B are fermionic operators and commutators are instead defined in all the other cases.

So, one has to consider the Dirac brackets of eqs. (2.1.13a, 2.1.13b) and to express the fields as in the mode expansions of eqs. (2.1.21a, 2.1.21b) and (2.1.23a, 2.1.23b). To reproduce the standard equal-time commutators and anticommutators $[X^M(\tau,\sigma'), P^N(\tau,\sigma'')] = \mathrm{i}\eta^{MN}\delta(\sigma' - \sigma'')$ and $\{\psi^M_\pm(\tau,\sigma'), \psi^N_\pm(\tau,\sigma'')\} = T^{-1}\eta^{MN}\delta(\sigma' - \sigma'')$, the algebra has to be

$$[\alpha^M_m, \alpha^N_n] = m\,\delta_{m,-n}\,\eta^{MN}, \tag{2.1.31a}$$
$$\{b^M_r, b^N_r\} = \delta_{r,-s}\,\eta^{MN}, \tag{2.1.31b}$$

with $[x^M, p^N] = \mathrm{i}\,\eta^{MN}$, if spacetime momentum is defined, and similarly in the left-moving sector for closed-strings, with all other mutual commutators and anticommutators being zero. The reality conditions are turned into the hermiticity conditions

$$(\alpha^M_n)^\dagger = \alpha^M_{-n}, \tag{2.1.32a}$$
$$(b^M_r)^\dagger = b^M_{-r}. \tag{2.1.32b}$$

One can see that eqs. (2.1.31a, 2.1.31b) define the algebra of creation and annihilation operators. It is possible to define the vacuum of the theory in the NS-sector as the state $|\mathrm{NS}\rangle$ such that

$$\alpha^M_n|\mathrm{NS}\rangle = 0, \qquad \forall\, m \in \mathbb{N}; \tag{2.1.33a}$$
$$b^M_r|\mathrm{NS}\rangle = 0, \qquad \forall\, r \in \mathbb{N}_0 + \frac{1}{2}. \tag{2.1.33b}$$

In the R-sector one defines an R-vacuum as a state $|\mathrm{R}\rangle$ such that

$$\alpha^M_n|\mathrm{R}\rangle = 0, \qquad \forall\, m \in \mathbb{N}; \tag{2.1.34a}$$
$$b^M_r|\mathrm{R}\rangle = 0, \qquad \forall\, r \in \mathbb{N}. \tag{2.1.34b}$$

Actually, the R-vacuum is degenerate due to the freedom to act with the operators b^M_0: because $[M^2, b^N_0] = 0$, the state $|\mathrm{R}\rangle$ has the same mass independently of how many 0-operators act on it. The 0-operators satisfy the Clifford algebra $\{b^M_0, b^N_0\} = \eta^{MN}$. If D is even, one can define the operators $b_\pm(0) = \mathrm{i}(b^0_0 \pm b^1_0)/\sqrt{2}$ and $b_\pm(i) = (b^{2i}_0 \pm \mathrm{i}b^{2i+1}_0)/\sqrt{2}$ for $i = 1, \dots, D/2 - 1$. One can thus verify the Clifford algebra $\{b_+(\alpha), b_-(\beta)\} = \delta_{\alpha\beta}$ for

$\alpha, \beta = 0, \ldots, D/2 - 1$. So, if one defines the highest-weight state $|\mathrm{R}\rangle$ as the state such that $b_+(\alpha)|\mathrm{R}\rangle = 0$, then any state $b_-(\alpha)|\mathrm{R}\rangle$ is such that $b_+(\alpha)b_-(\beta)|\mathrm{R}\rangle = \delta_{\alpha\beta}|\mathrm{R}\rangle$, and similarly for states with multiple $b_-(\alpha)$-operators acting. This means that there is a total of $n(D) = 2^{D/2}$ different fermionic states.

In the Hilbert space associated to the oscillator algebra, the quantum theory can be described in terms of the operator-valued observables found in the classical formulation. The operators need to be defined to be normal-ordered to avoid ambiguities in their definition. This framework leads to covariant quantisation, but it brings non-trivial technical challenges. In particular, for the Hilbert subspace of physical (i.e. satisfying the constraints) and gauge-inequivalent (all states differing by a null state are identified) states, the absence of negative-norm states and the unitarity of the S-matrix are not trivial to attain. One needs the tools of conformal field theory to address these challenges in an efficient way. This eventually fixes the number of spacetime dimensions and the normal-ordering constants. An alternative but not manifestly-covariant method to quantise the theory is lightcone quantisation. Because it is less mathematically demanding, this is the approach discussed below.

Lightcone quantisation is based on taking advantage of classical symmetries to fix a gauge and to work only with physical states, solving the classical constraints explicitly. In the superconformal gauge, one can take advantage of the conformal and residual local supersymmetry transformations to define the lightcone gauge. Given the definitions

$$
\begin{aligned}
X(\pm) &= \frac{1}{\sqrt{2}}(X^0 \pm X^1), \\
\psi(\pm) &= \frac{1}{\sqrt{2}}(\psi^0 \pm \psi^1).
\end{aligned}
$$

the lightcone gauge is the gauge where, defining $p^\pm = (p^0 \pm p^1)/\sqrt{2}$, one fixes

$$
\begin{aligned}
X(+) &= \frac{p^+\tau}{lT}, \\
\psi(+) &= 0.
\end{aligned}
$$

In fact, in this way, it is possible to to express the on-shell constraints for the metric and gravitino constraints of eqs. (2.1.11a, 2.1.11b), namely $T_{\pm\pm} = 0$ and $t_\pm = 0$, as

$$
\frac{p^+}{lT}\partial_\pm X(-) = \partial_\pm X^I \partial_\pm X_I + \frac{\mathrm{i}}{2}\psi^I_\pm \partial_\pm \psi_{\pm I}, \tag{2.1.35a}
$$

$$
\frac{p^+}{2lT}\psi_\pm(-) = \psi^I_\pm \partial_\pm X_I, \tag{2.1.35b}
$$

where $I = 2, \ldots, D-1$ denotes all the directions except the two gauge-fixed ones. The key point of lightcone quantisation resides in the fact that it allows one to solve explicitly the classical constraints.

- For closed strings, in the right-moving sector related to σ^-, one can express the constraints reported in eqs. (2.1.35a, 2.1.35b) as

$$
\frac{p^+}{\sqrt{\pi T}}\alpha_n(-) = \sum_{m\in\mathbb{Z}} \alpha^I_{n-m}\alpha_{mI} + \sum_{r\in\mathbb{Z}+\phi} r b^I_{n-r} b_{rI}, \tag{2.1.36a}
$$

$$\frac{p^+}{\sqrt{4\pi T}} b_r(-) = \sum_{m\in\mathbb{Z}} b^I_{r-m}\alpha_{mI}, \tag{2.1.36b}$$

respectively, where $\alpha_n(-) = (\alpha^0_n - \alpha^1_n)/\sqrt{2}$ and $b_r(-) = (b^0_r - b^1_r)/\sqrt{2}$. One can perform a similar analysis for the left-moving sector related to σ^+, thus constraining the coefficients $\overline{\alpha}_n(-)$ and $\overline{b}_r(-)$. According to the definition of the 0-mode operators, one finds the identity $\alpha_0(-) = \overline{\alpha}_0(-)$. This means that the metric constraint relates the right- and left-moving oscillators, which is the so-called level-matching condition. In a relativistic theory, the squared mass operator is defined as $M^2 = -p^M p_M$. In view of the level-matching condition, in lightcone coordinates and taking into account the definition of the 0-mode operators, one can write

$$M^2 = 2p^+p^- - p^I p_I = \sqrt{4\pi T}\big(p^+\alpha_0(-) + p^+\overline{\alpha}_0(-)\big) - p^I p_I = m^2 + \overline{m}^2,$$

where m^2 and $\overline{m}^2$ are the right- and left-moving sector masses, respectively, with the level-matching condition reading $m^2 = \overline{m}^2$. More explicitly, for the right-moving sector, in view of the metric constraints, one can write

$$m^2 = \sqrt{4\pi T}\, p^+\alpha_0(-) - \frac{1}{2}p^I p_I = 2\pi T\left[\sum_{m\in\mathbb{Z}_*} \alpha^I_{-m}\alpha_{mI} + \sum_{r\in\mathbb{Z}+\phi} r b^I_{-r} b_{rI}\right]$$

and similarly for the left-moving sector term. In the quantum theory, the ordering of the oscillator operators in such a way that all creators are on the left of all the destructors results in constant terms proportional to the regularised sums $\sum_{n\in\mathbb{N}} n = -1/12$ and $\sum_{r\in\mathbb{N}+\frac{1}{2}} r = 1/24$. Defining the right-moving transverse number operators

$$\tilde{N}_{\mathrm{b}} = \sum_{n\in\mathbb{N}} \alpha^I_{-n}\alpha_{nI}, \tag{2.1.37a}$$

$$\tilde{N}_{\phi} = \sum_{r\in\mathbb{N}-\phi} r b^I_{-r} b_{rI}, \tag{2.1.37b}$$

and analogously in the left-moving sector, the closed-string mass operator reads

$$M^2 = 4\pi T\left[\tilde{N}_{\mathrm{b}} + \tilde{N}_{\phi} + \tilde{\overline{N}}_{\mathrm{b}} + \tilde{\overline{N}}_{\phi} - \frac{d}{12} - \frac{d}{12}a_\phi\right], \tag{2.1.38}$$

where the constant is defined as

$$\frac{1}{12}a_\phi = \sum_{r\in\mathbb{N}-\phi} r = \begin{cases} \dfrac{1}{24}, & \phi = \dfrac{1}{2}; \\[2ex] -\dfrac{1}{12}, & \phi = 0. \end{cases}$$

- For open strings, assuming the gauge-fixed directions to have NN-boundaries, one can express the constraints reported in eqs. (2.1.35a, 2.1.35b) respectively as[2.5]

$$\frac{p^+}{\sqrt{\pi T}}\alpha_n(-) = \frac{1}{2}\sum_{m\in\mathbb{Z}+\varphi(I)} \alpha^I_{n-m}\alpha_{mI} + \frac{1}{2}\sum_{r\in\mathbb{Z}+\phi(I)} r b^I_{n-r} b_{rI}, \tag{2.1.39a}$$

[2.5]The constant $\varphi(I)$ is needed to write the bosonic open-string expansion in a consistent way. It takes the values $\varphi = 0$ for NN- and DD-directions and $\varphi = 1/2$ for ND- and DN-directions.

$$\frac{p^+}{\sqrt{\pi T}} b_r(-) = \sum_{s\in\mathbb{Z}+\phi(I)} \alpha^I_{r-s} b_{sI}, \tag{2.1.39b}$$

where $\alpha_n(-) = (\alpha^0_n - \alpha^1_n)/\sqrt{2}$ and $b_r(-) = (b^0_r - b^1_r)/\sqrt{2}$. In each I-direction, the function ϕ must be fixed according to the associated boundary conditions.[2.6]

In lightcone coordinates, and in view of the definition of the 0-mode operators, one can write the squared mass operator as $M^2 = -p^\alpha p_\alpha$, where only the NN-directions contribute to the physical mass (and indeed have nonzero spacetime momentum). Therefore, labelling as i the remaining NN-directions aside from the gauge-fixed ones, one finds a mass

$$M^2 = 2p^+p^- - p^i p_i = \sqrt{4\pi T}\, p^+ \alpha_0(-) - p^i p_i.$$

This, thanks to the metric constraint, can be expressed as

$$M^2 = \pi T \left[\sum_{m\in\mathbb{Z}_*+\varphi(I)} \alpha^I_{-m}\alpha_{mI} + \sum_{r\in\mathbb{Z}+\phi(I)} r b^I_{-r} b_{rI} \right] + T^2 \delta_{\dot m \dot n} \Delta x^{\dot m} \Delta x^{\dot n},$$

where the spacetime momentum terms cancel off for the NN-directions and the remaining non-zero 0-modes, coming only from the DD-directions, have been written explicitly in terms of the separation $\Delta x^{\dot m} = x^{\dot m}_1 - x^{\dot m}_0$; the latter will be ignored below for brevity, considering $\Delta x^{\dot m} = 0$. After quantisation, by ordering of the oscillator operators with all creators on the left of the destructors, one finds different contributions for each direction, depending on its boundary conditions. For a Dp-brane, one has $p-1$ non-gauge-fixed NN-directions, and then there can be q directions with DD-boundaries, with $s = p+q+1$ directions with NN- and DD-boundaries in total, and $l = D - s$ directions with either ND- or DN-boundaries. Taking all possible boundaries into account, for open strings the total mass takes the form

$$M^2_\phi = 2\pi T \left[\tilde N_\mathrm{b} + \tilde N_\phi - \frac{c_\phi(s)}{16} \right], \tag{2.1.40}$$

where the bosonic and fermionic number operators are defined as in eqs. (2.1.37a, 2.1.37b), respectively, with the modes taking their values as in the open-string mode expansions in each direction, depending on the boundary conditions, and the normal-ordering constant is defined as

$$\frac{1}{16} c_\phi(s) = \begin{cases} \dfrac{2(s-1)-D}{16}, & \phi = \dfrac{1}{2}; \\ 0, & \phi = 0. \end{cases}$$

Because the masses in the tower of string states are multiples of $M^2_s = 1/\alpha'$, this is set as the reference string mass scale. In the quantum theory, it can be shown that spacetime Lorentz invariance is preserved if and only if the number of spacetime dimensions is $D = 10$.

[2.6] All these expressions are valid independently of the kind of boundary conditions in each of the I-directions. In fact, although the individual mode expansions can have overall $\pm$-signs and/or different domains for the labels n and r, depending on the specific boundary conditions, these can be seen to always cancel out in the product direction by direction.

2.1.2.1 Superstring Perturbative Series

In order to perform a covariant analysis, one should eventually pursue a path-integral quantisation of the RNS-superstring worldsheet action of eq. (2.1.1) in the Euclidean formulation of the theory. To do this, one has to consider the fact the RNS-action depends on the embedding coordinates X^M, their superpartners ψ^M, the auxiliary metric h and the gravitino χ^M. Along with all the possible configurations of these fields, one should also sum over all the possible worldsheet topologies τ_Σ. In order to quantify the weight of each topology, one can introduce in the worldsheet action a 2-dimensional spacetime Einstein-Hilbert term. Moreover, one can also add a boundary term, which arises for open strings, whose worldsheet has a boundary $\partial\Sigma$. This total term happens to be a topological invariant, i.e.

$$\frac{1}{4\pi}\int_\Sigma \mathrm{d}^2\xi\,\sqrt{-h}\;R[h] + \frac{1}{2\pi}\int_{\partial\Sigma}\mathrm{d}s\,k = \chi(\Sigma), \tag{2.1.41}$$

where $\chi(\Sigma)$ is the worldsheet Euler character. In this action, $R[h]$ is the Riemann tensor corresponding to the metric $h_{\alpha\beta}$ and k is the trace of the extrinsic curvature on the boundary. It turns out that these are the only two terms that one can add to the RNS-action without changing the field equations. By the Gauss-Bonnet theorem, the Euler character of a surface Σ with h handles, b boundaries and c crosscaps is[2.7]

$$\chi = 2 - 2h - b - c. \tag{2.1.42}$$

Schematically, the path integral for a superstring theory may then be formulated as

$$Z = \frac{1}{\mathrm{vol}\,\gamma}\sum_{\tau_\Sigma}\int DX\int D\psi\int Dh\int D\chi\;\mathrm{e}^{-S_{\mathrm{RNS}}[X,\psi,h,\chi]-\lambda\chi}, \tag{2.1.43}$$

where λ is some coupling parameters and $\mathrm{vol}\,\gamma$ represents the redundant gauge-equivalent configurations that are being overcounted. In fact, it turns out that the topological term arises naturally in the presence of a non-zero constant background dilaton Φ, with $\lambda = \Phi$. Therefore, one finds a perturbative series in the string coupling

$$g_s = \mathrm{e}^{\Phi}, \tag{2.1.44}$$

where the order in the perturbation is dictated by the Euler character corresponding to the worldsheet topology being considered. Now, tree-level scattering amplitudes, after a conformal map, correspond to the topology of a sphere and of a disk for closed and open strings, respectively, so they are weighted by powers g_s^{-2} and g_s^{-1}, respectively. The one-loop corrections correspond to the torus and the annulus for closed and open strings, respectively, both with powers $g^0 = 1$, and therefore a relative suppression with respect to the tree-level results of g_s^2 and g_s, respectively.

[2.7] This includes both orientable and non-orientable surfaces for generality, but in this introductory section only orientable surfaces are being considered.

2.1.3 Superstring Partition Functions

This section contains a brief (yet self-contained) overview on the determination of partition functions in string theory. First, bosonic strings are reviewed for pedagogical reasons, then superstrings are analysed. The guidance for this section is the material in refs. [33, 34, 39].

2.1.3.1 Bosonic Strings

For bosonic strings, given the toroidal parameter τ and the squared nome $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$, the one-loop partition function is defined as the trace

$$Z_{\mathrm{b}}(\tau,\overline{\tau}) = \mathrm{tr}_{\mathcal{H}_{\mathrm{b}}}\, q^{\tilde{L}_0^{\mathrm{b}}}\overline{q}^{\overline{\tilde{L}}_0^{\mathrm{b}}}, \tag{2.1.45}$$

where $\tilde{L}_0^{\mathrm{b}}$ and $\overline{\tilde{L}}_0^{\mathrm{b}}$ are the right- and left-moving transverse Virasoro 0-operators, respectively. The total Hilbert space $\mathcal{H}_{\mathrm{b}}$ contains spacetime momenta and oscillators, and the latter part is factorised in right- and left-moving parts, $\mathcal{H}_{\mathrm{bo}}$ and $\overline{\mathcal{H}}_{\mathrm{bo}}$. To start, spacetime momenta can be ignored for brevity.

As concerns oscillatory excitations, without loss of generality, one can focus on the right-moving sector. In the framework of lightcone quantisation, the Virasoro operator reads

$$\tilde{L}_0^{\mathrm{b}} = \frac{1}{2}\sum_{n\in\mathbb{Z}} \alpha_{-n}^{I}\alpha_{nI} = \frac{1}{8\pi T}\tilde{p}^2 + \tilde{N}_{\mathrm{b}} - \frac{d}{24}. \tag{2.1.46}$$

where the transverse squared momentum is $\tilde{p}^2 = p^I p_I$ and the transverse number operator is defined as in eq. (2.1.37a). It is convenient to focus on one direction at a time, knowing that each of the directions $I = 2, \ldots, D-1$ contributes identically. Dealing with only one direction, one can observe the commutation relation

$$[\tilde{N}_{\mathrm{b}}, \alpha_m] = -m\alpha_m. \tag{2.1.47}$$

Ignoring any overall normalisation factors $f = f(\{n_k\})$ for brevity, all the elements of an orthonormal basis of the Fock space can be written as

$$|n_1, n_2, \ldots, n_M\rangle \equiv (\alpha_{-1})^{n_1}(\alpha_{-2})^{n_2}\ldots(\alpha_{-M})^{n_M}|0\rangle, \qquad \{n_j \in \mathbb{N}_0\}_{j=1}^{M}.$$

By direct inspection, one finds the eigenvalue equation

$$\tilde{N}_{\mathrm{b}}|n_1, n_2, \ldots, n_M\rangle = \left[\sum_{j=1}^{M} j n_j\right]|n_1, n_2, \ldots, n_M\rangle. \tag{2.1.48}$$

Therefore, following the definition, one finds the partition function

$$\mathrm{tr}_{\mathcal{H}_{\mathrm{bo}}}\big(q^{\tilde{L}_0^{\mathrm{b}}}\big) = q^{\frac{1}{8\pi T}\tilde{p}^2 - \frac{1}{24}} \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\ldots\langle n_1, n_2, \ldots|\, q^{\tilde{N}_{\mathrm{b}}}\, |n_1, n_2, \ldots\rangle.$$

Using the eigenvalue equation for the number operator, a few elementary manipulations with sums and products and the geometric series allow one to conclude with the relation,

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\ldots\langle n_1, n_2, \ldots|\, q^{\tilde{N}_{\mathrm{b}}}\, |n_1, n_2, \ldots\rangle = \prod_{k=1}^{\infty}\sum_{n_k=0}^{\infty} q^{k n_k} = \prod_{k=1}^{\infty}\frac{1}{1-q^k} = q^{\frac{1}{24}}\eta(\tau)^{-1},$$

where $\eta = \eta(\tau)$ is the Dedekind η-function. Therefore one can write

$$\mathrm{tr}_{\mathcal{H}_{\mathrm{bo}}}\big(q^{\tilde{L}_0^{\mathrm{b}}}\big) = q^{\frac{1}{8\pi T}\tilde{p}^2}\eta(\tau)^{-1}.$$

To take into account spacetime momentum in the proper way, one needs to include the left-moving sector too. Doing this, one finds the extra trace

$$\sqrt{\frac{2\pi}{T}}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\langle k|q^{\frac{1}{8\pi T}\tilde{p}^2}\overline{q}^{\frac{1}{8\pi T}\tilde{p}^2}|k\rangle = \sqrt{\frac{2\pi}{T}}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}q^{\frac{1}{8\pi T}k^2}\overline{q}^{\frac{1}{8\pi T}k^2} = \sqrt{\frac{1}{\tau_2}}.$$

In bosonic string theory, the number of spacetime dimensions is $D = 26$ and therefore, being $d = 24$, the full partition function reads

$$Z_{\mathrm{b}}(\tau,\overline{\tau}) = \frac{1}{\tau_2^{12}}\frac{1}{\eta^{24}(\tau)\overline{\eta}^{24}(\overline{\tau})}. \tag{2.1.49}$$

One can make use of the results above to determine the partition function of open bosonic strings. For the open strings associated to a Dp-brane, the partition function is defined as

$$A_{\mathrm{b}}(\tau_2;p) = \mathrm{tr}_{\mathcal{H}_{\mathrm{b}}^p}q[\mathrm{i}\tau_2]^{\frac{1}{2}\tilde{L}_0^{\mathrm{b}}}. \tag{2.1.50}$$

where the 1/2-factor is motivated by the different Regge trajectory in the spectrum as compared to the closed-string case and the argument in $q[\mathrm{i}\tau_2] = \mathrm{e}^{-2\pi\tau_2}$ is restricted to $\tau = \mathrm{i}\tau_2$ as a consequence of the fact that the open string is not periodic along the worldsheet spacelike direction. Therefore, the bosonic open-string partition function is

$$A_{\mathrm{b}}(\tau_2;p) = \frac{1}{\tau_2^{\frac{1}{2}(p+1)}}\frac{1}{\eta^{24}(\mathrm{i}\tau_2/2)}. \tag{2.1.51}$$

The dependence on p comes from the fact that the string propagates through spacetime only in the $(p+1)$-dimensional region with NN-boundaries.[2.8]

2.1.3.2 Superstrings

For superstrings, the L_0-operator involves both the bosonic and the fermionic oscillation operators, being

$$\tilde{L}_0 = \tilde{L}_0^{\mathrm{b}} + \tilde{L}_0^{\phi},$$

in which $\tilde{L}_0^{\mathrm{b}}$ is the bosonic operator and $\tilde{L}_0^{\phi}$ represents the fermionic counterparts in the NS- and in the R-sectors, where $\phi = 1/2, 0$, respectively. The latter reads

$$\tilde{L}_0^{\phi} = \sum_{r\in\mathbb{N}-\phi} r b_{-r}^I b_{rI} - \frac{d}{24}a_{\phi},$$

[2.8] It is conventional to write this amplitude counting all the $p+1$ directions. This facilitates the discussion of the one-loop cosmological constant. Up to overall factors, in a D-dimensional quantum field theory, this is the integral over a Schwinger time t of the power $t^{-1-D/2}$ multiplied by $a_n\,\mathrm{e}^{-2\pi t M_n^2/\mu^2}$, where a_n are the physical degeneracies at the mass levels M_n^2, for some scale μ^2. In string-theory constructions, the exponentials come from the variable q. For a closed-string theory, it happens that $1 + D/2 = 2 + (D-2)/2$, with τ_2^{-2} being needed to define the modular-invariant measure and $\tau_2^{-(D-2)/2}$ corresponding to the power found in the partition function. For open strings, with $D = p+1$, instead, it is customary to split the power as τ_2^{-1} and $\tau_2^{-(p+1)/2}$, considering the latter as part of the partition function.

Therefore, given the fermion-number operator in eq. (2.1.37b), one can write

$$\tilde{L}_0 = \frac{1}{8\pi T}\tilde{p}^2 + \tilde{N}_{\mathrm{b}} + \tilde{N}_\phi - \frac{d}{24} - \frac{d}{24}a_\phi. \tag{2.1.52}$$

Again, one can focus on a single direction. One can observe the commutation relation

$$[\tilde{N}_\phi, b_r] = -r b_r. \tag{2.1.53}$$

For a state in the Fock space, i.e.[2.9]

$$|n_1, n_2, \ldots, n_M\rangle_\phi \equiv (b_{-1+\phi})^{n_1}(b_{-2+\phi})^{n_2}\ldots(b_{-M+\phi})^{n_M}|\phi\rangle, \qquad \{n_j = 0,1\}_{j=1}^{M},$$

a similar algebra applies as in the bosonic case and therefore one can write

$$\tilde{N}_\phi|n_1, n_2, \ldots, n_M\rangle_\phi = \left[\sum_{j=1}^{M} n_j(j-\phi)\right]|n_1, n_2, \ldots, n_M\rangle_\phi. \tag{2.1.54}$$

It is now possible to compute the trace over the fermionic Hilbert space. In this case, the calculation is simplified by the fact that the occupation numbers at each level can only take the values $n_j = 0, 1$. One finds

$$\mathrm{tr}_{\mathcal{H}_\phi}\big(q^{\tilde{L}_0^\phi}\big) = \sum_{n_1=0,1}\sum_{n_2=0,1}\ldots{}_\phi\langle n_1, n_2, \ldots|q^{\tilde{L}_0^\phi}|n_1, n_2, \ldots\rangle_\phi = q^{-\frac{1}{24}a_\phi}\prod_{r\in\mathbb{N}-\phi}(1+q^r).$$

Now one can consider the full right-moving Hilbert space in order to find all the right terms. The results turn out to be combinations of the Dedekind η- and of the Jacobi ϑ-functions. In the NS-sector, one finds

$$\mathrm{tr}_{\mathcal{H}_{\mathrm{bo}}\otimes\mathcal{H}_{\mathrm{NS}}}\big(q^{\tilde{N}_{\mathrm{b}}+\tilde{N}_{\mathrm{NS}}-\frac{1}{24}-\frac{1}{48}}\big) = q^{-\frac{1}{16}}\frac{\prod_{n=1}^{\infty}(1+q^{n-\frac{1}{2}})}{\prod_{m=1}^{\infty}(1-q^m)} = \frac{\vartheta_3(\tau)^{\frac{1}{2}}}{\eta(\tau)^{\frac{3}{2}}}. \tag{2.1.55}$$

In the R-sector, one finds

$$\mathrm{tr}_{\mathcal{H}_{\mathrm{bo}}\otimes\mathcal{H}_{\mathrm{R}}}\big(q^{\tilde{N}_{\mathrm{b}}+\tilde{N}_{\mathrm{R}}-\frac{1}{24}+\frac{1}{24}}\big) = 2\frac{\prod_{n=1}^{\infty}(1+q^{n})}{\prod_{m=1}^{\infty}(1-q^m)} = \frac{\vartheta_2(\tau)^{\frac{1}{2}}}{\eta(\tau)^{\frac{3}{2}}}. \tag{2.1.56}$$

where the overall factor is due to the action of the b_0-operator on the vacuum, which has not been written explicitly above.

For a physical spectrum to be built, one needs to impose the GSO-projection on the spectrum. This can be achieved by defining the worldsheet fermion-number operator $(-1)^F$ and defining a projector $P_\pm = (1 \pm (-1)^F)/2$, which can remove tachyons while making the spacetime spectrum supersymmetric. An analysis of this operator can be performed quite straightforwardly.

[2.9] In the R-sector, one has the further freedom to act on the vacuum with the operator b_0, thus actually identifying different vacua.

- In the NS-sector, the worldsheet fermion-number is $F = \sum_{r\in\mathbb{N}-1/2} b_{-r}b_r - 1$, and it is thus defined in such a way that its action is

$$(-1)^F b^{I_1}_{-r_1}\dots b^{I_M}_{-r_M}|\mathrm{NS}\rangle = (-1)^{M+1} b^{I_1}_{-r_1}\dots b^{I_M}_{-r_M}|\mathrm{NS}\rangle,$$

which means that all states with even occupation numbers are removed out by the projector $P_+ = (1+(-1)^F)/2$. Dealing with just one direction, one can write

$$\mathrm{tr}_{\mathcal{H}_{\mathrm{NS}}}\big((-1)^F q^{\tilde{L}_0^{\mathrm{NS}}}\big) = q^{-\frac{1}{48}} \sum_{n_1=0,1}\sum_{n_2=0,1}\dots {}_{\mathrm{NS}}\langle n_1,n_2,\dots|(-1)^F q^{\tilde{N}_{\mathrm{NS}}}|n_1,n_2,\dots\rangle_{\mathrm{NS}}.$$

Now, since it is possible to write

$${}_{\mathrm{NS}}\langle n_1,n_2,\dots|(-1)^F q^{\tilde{N}_{\mathrm{NS}}}|n_1,n_2,\dots\rangle_{\mathrm{NS}} = -(-1)^{n_1+n_2+\cdots} q^{\frac{1}{2}n_1+\frac{3}{2}n_2+\cdots},$$

one finds

$$\mathrm{tr}_{\mathcal{H}_{\mathrm{NS}}}\big((-1)^F q^{\tilde{L}_0^{\mathrm{NS}}}\big) = -q^{-\frac{1}{48}} \prod_{r\in\mathbb{N}-\phi}\sum_{n_r=0,1}(-1)^{n_r} q^{rn_r} = -q^{-\frac{1}{48}}\prod_{r\in\mathbb{N}-\phi}(1-q^r).$$

- In the R-sector, the operator to be considered is the generalised chirality operator $(-1)^F = 2^{D/2-1}\big(\prod_{I=2}^{D-1} b_0^I\big)(-1)^{\sum_{r\in\mathbb{N}} b^I_{-r}b_{rI}}$, whose action is

$$\begin{aligned}(-1)^F b^{I_1}_{-r_1}\dots b^{I_M}_{-r_M}|\mathrm{R}_+\rangle &= (-1)^M(-1)^{\sum_i \delta_{r_i,0}} b^{I_1}_{-r_1}\dots b^{I_M}_{-r_M}|\mathrm{R}_+\rangle,\\ (-1)^F b^{I_1}_{-r_1}\dots b^{I_M}_{-r_M}|\mathrm{R}_-\rangle &= -(-1)^M(-1)^{\sum_i \delta_{r_i,0}} b^{I_1}_{-r_1}\dots b^{I_M}_{-r_M}|\mathrm{R}_-\rangle,\end{aligned}$$

where $|\mathrm{R}_\pm\rangle$ are two R-vacua, with chiralities ±1. In fact, in lightcone quantisation vacua in the R-sector can be labelled as $|0\rangle_{\mathrm{R}} = |s_1,s_2,s_3,s_4\rangle$, with $s_i = 1/2$ or $s_i = -1/2$: defining the highest-weight state as the state such that $b_+(i)|\mathrm{R}\rangle = 0$ for all $i = 1,2,3,4$, this can be denoted as $|+1/2,+1/2,+1/2,+1/2\rangle$, with each operator $b_-(i)$ lowering s_i by one unit. The definitions are such that $\sum_{i=1}^4 s_i \in 2\mathbb{Z}$ and $\sum_{i=1}^4 s_i \in 2\mathbb{Z}+1$ for the states $|\mathrm{R}_+\rangle$ and $|\mathrm{R}_-\rangle$, respectively. One can write

$$\mathrm{tr}_{\mathcal{H}_{\mathrm{R}}}\big((-1)^F q^{\tilde{L}_0^{\mathrm{R}}}\big) = q^{\frac{1}{24}} \sum_{n_1=0,1}\sum_{n_2=0,1}\dots {}_{\mathrm{R}}\langle n_1,n_2,\dots|(-1)^F q^{\tilde{N}_{\mathrm{R}}}|n_1,n_2,\dots\rangle_{\mathrm{R}}.$$

Because the action of worldsheet parity is exactly specular in the two sectors, the result of the trace is clearly zero, i.e.

$${}_{\mathrm{R}}\langle n_1,n_2,\dots|(-1)^F q^{\tilde{N}_{\mathrm{R}}}|n_1,n_2,\dots\rangle_{\mathrm{R}} = {}_{\mathrm{R}}\langle n_1,n_2,\dots|(-1)^F q^{n_1+n_2+\cdots}|n_1,n_2,\dots\rangle_{\mathrm{R}} = 0.$$

Therefore, one can write

$$\mathrm{tr}_{\mathcal{H}_{\mathrm{R}}}\big((-1)^F q^{\tilde{L}_0^{\mathrm{R}}}\big) = 0.$$

To sum up, including worldsheet parity, the NS-sector is the only one having a non-zero trace and this reads

$$\mathrm{tr}_{\mathcal{H}_{\mathrm{bo}}\otimes\mathcal{H}_{\mathrm{NS}}}\big((-1)^F q^{\tilde{N}_{\mathrm{b}}+\tilde{N}_{\mathrm{NS}}-\frac{1}{24}-\frac{1}{48}}\big) = -q^{-\frac{1}{16}}\frac{\prod_{n=1}^\infty\big(1-q^{n-\frac{1}{2}}\big)}{\prod_{m=1}^\infty(1-q^m)} = -\frac{\vartheta_4(\tau)^{\frac{1}{2}}}{\eta(\tau)^{\frac{3}{2}}}.$$

To find the actual partition function, one needs to insert the spacetime momentum term, obtaining an extra factor $f(\tau_2) = \tau_2^{-1/2}$ which comes from spacetime momentum integration when both the right- and left-moving sectors are properly taken into account.

The partition function of type IIB string theory can be written by requiring that, in the GSO-projection, all states are $+1$-eigenvalues of the chirality operator $(-1)^F$ both in the right- and left-moving sectors, in a 10-dimensional spacetime with $d = 8$. Splitting the calculation into the NS- and R-sectors, with a relative phase ω to be determined by requiring modular invariance, one can write

$$
\begin{aligned}
Z_{\mathrm{IIB}}(\tau,\overline{\tau}) &= \mathrm{tr}_{\mathcal{H}}\left[\frac{1}{2}(1+(-1)^F)\, q^{\tilde{L}_0}\right]\left[\frac{1}{2}(1+(-1)^{\overline{F}})\, \overline{q}^{\tilde{\overline{L}}_0}\right] \\
&= \frac{1}{4\tau_2^4}\left[\frac{\vartheta_3^4(\tau)}{\eta^{12}(\tau)} - \frac{\vartheta_4^4(\tau)}{\eta^{12}(\tau)} + \omega\frac{\vartheta_2^4(\tau)}{\eta^{12}(\tau)}\right]\left[\frac{\overline{\vartheta}_3^4(\overline{\tau})}{\overline{\eta}^{12}(\overline{\tau})} - \frac{\overline{\vartheta}_4^4(\overline{\tau})}{\overline{\eta}^{12}(\overline{\tau})} + \overline{\omega}\frac{\overline{\vartheta}_2^4(\overline{\tau})}{\overline{\eta}^{12}(\overline{\tau})}\right].
\end{aligned}
$$

An S-generating modular transformation maps the function to itself if and only if it is $\omega = -1$. In terms of the so(8)-characters, the partition function for type IIB string theory reads

$$
Z_{\mathrm{IIB}}(\tau,\overline{\tau}) = \frac{1}{\tau_2^4}\frac{\left[V_8(\tau) - S_8(\tau)\right]\left[\overline{V}_8(\overline{\tau}) - \overline{S}_8(\overline{\tau})\right]}{\eta^8(\tau)\overline{\eta}^8(\overline{\tau})}. \tag{2.1.57}
$$

Obviously, the spectrum is supersymmetric and in fact the partition function vanishes, as one can verify as a consequence of the Jacobi equation $\vartheta_3^4(\tau) - \vartheta_4^4(\tau) - \vartheta_2^4(\tau) = 0$, or $V_8 = S_8$, which implies that actually it is

$$
Z_{\mathrm{IIB}}(\tau,\overline{\tau}) = 0.
$$

One can nonetheless recognise immediately how the sectors in the type IIB spectrum are represented in the terms of the partition function. In fact, the numerical coefficients of the terms $q^n\overline{q}^n$ in the Laurent expansion of the partition function represent the number of states with a mass $M^2 = 8\pi n T$. As also discussed in subsection 2.2.1, at the massless level:

- the NSNS-states come from the product $V_8\overline{V}_8$ and represent the graviton, the dilaton, and the Kalb-Ramond field;
- the RR-states come from the product $S_8\overline{S}_8$ and represent the 0-, 2- and self-dual 4-form fields;
- the NSR- and RNS-states come from the products $V_8\overline{S}_8$ and $S_8\overline{V}_8$ and represent two gravitini and two dilatini.

The number of states can be counted easily by extracting the diagonal elements in the series expansion of Z_{IIB}. This confirms the absence of tachyons and at the massless level it confirms the numbers of degrees of freedom $n_{\mathrm{NSNS}} = n_{\mathrm{RR}} = n_{\mathrm{NSR}} = n_{\mathrm{RNS}} = 64$. The interpretation of the states stems from the Fock-space construction, as in subsection 2.2.1. From the modular-invariant partition function, one defines the one-loop torus scattering amplitude as

$$
\mathcal{T} = \int_{\mathbb{F}} \frac{\mathrm{d}^2\tau}{(\mathrm{Im}\,\tau)^6}\frac{\left[V_8(\tau) - S_8(\tau)\right]\left[\overline{V}_8(\overline{\tau}) - \overline{S}_8(\overline{\tau})\right]}{\eta^8(\tau)\overline{\eta}^8(\overline{\tau})}, \tag{2.1.58}
$$

where $\mathbb{F} = \{\tau \in \mathbb{C} : \operatorname{Re}\tau \in [-1/2, 1/2] \wedge \operatorname{Im}\tau \in [0, +\infty[\wedge |\tau| \in [1, +\infty[\}$ is the fundamental domain and $\mathrm{d}^2\tau/(\operatorname{Im}\tau)^2$ is the invariant measure of the modular group (for more details, see appendix A.1.1).

One can make use of the results above to determine the partition function of open fermionic strings. For the open strings associated to a Dp-brane, the partition function is defined as

$$A(\tau_2; p) = \operatorname{tr}_{\mathcal{H}_p}\left[\frac{1}{2}(1 + (-1)^F)\, q[\mathrm{i}\tau_2]^{\frac{1}{2}\tilde{L}_0}\right]. \tag{2.1.59}$$

In a straightforward analogy with the previous calculations, it thus follows that the supersymmetric open-string partition function reads

$$A(\tau_2; p) = \frac{1}{\tau_2^{\frac{1}{2}(p+1)}} \frac{V_8(\mathrm{i}\tau_2/2) - S_8(\mathrm{i}\tau_2/2)}{\eta^8(\mathrm{i}\tau_2/2)}. \tag{2.1.60}$$

Again, one can see that the spectrum is supersymmetric due to the Jacobi identity $V_8 = S_8$. In this case, the numerical coefficients of the terms q^n in the Laurent expansion of the partition function represent the number of states with a mass $M^2 = 2\pi n T$, and, as also discussed in subsection 2.3.1, the spectrum can be described as follows:

- the NS-states come from the term V_8 and the represent the scalar and/or vector fields on the Dp-brane worldvolume;
- the R-states come from the term S_8 and represent the spinor fields on the Dp-brane worldvolume.

The number of degrees of freedom at each mass level corresponds to the coefficients in the series expansion of the partition function and, at the massless level, it can be checked to be $n_{\mathrm{NS}} = n_{\mathrm{R}} = 8$. The interpretation of the states stems from the Fock-space construction, as in subsection 2.3.1. This partition function is not modular invariant and the so-called one-loop annulus amplitude is defined, selecting $p = 9$, as (the factor $1/2$ is conventional, for future use)

$$\mathcal{A} = \frac{n^2}{2} \int_0^\infty \frac{\mathrm{d}\tau_2}{\tau_2^6} \frac{V_8 - S_8}{\eta^8}\left[\frac{\mathrm{i}\tau_2}{2}\right]. \tag{2.1.61}$$

In this expression, an extra factor n^2 has been inserted to account for the situation where n branes are present. One has to add Chan-Paton labels to each state, as explained in subsection 2.3.1, so the trace receives a contribution $\sum_{a=1}^n \sum_{b=1}^n \langle a, b | a, b\rangle = n^2$. In fact, the number of degrees of freedom corresponds precisely to an U(n)-gauge theory with scalars and spinors in the adjoint representation.

The choice of the GSO-projection is not unique. In fact, one can define another tachyon-free theory by choosing the same GSO-projection as above in the NS-sectors but requiring an opposite parity in the right- and left-moving R-sectors. This defines the type IIA closed-string theory. Because the trace involving the $(-1)^F$-operator in the R-sector is zero, the partition function is the same as above, so

$$Z_{\mathrm{IIA}}(\tau, \overline{\tau}) = \frac{1}{\tau_2^4} \frac{\left[V_8(\tau) - S_8(\tau)\right]\left[\overline{V}_8(\overline{\tau}) - \overline{C}_8(\overline{\tau})\right]}{\eta^8(\tau)\overline{\eta}^8(\overline{\tau})}. \tag{2.1.62}$$

In the left-moving sector the function C_8 appears, instead of S_8. Of course numerically the functions are the same, i.e. $C_8 = S_8$, but the notation stems from the fact that the trace in the R-sector is proportional to $\vartheta_1 = 0$, so changing its signs formally changes the function S_8 into the function C_8. This reflects a physical fact: the type IIA spectrum is analogous to the type IIB one, but it is non-chiral. In the perturbative spectrum, this is precisely the consequence of the different GSO-projection.

Combining the functions $O_8/(\tau_2^4\eta^8)$, $V_8/(\tau_2^4\eta^8)$, $S_8/(\tau_2^4\eta^8)$ and $C_8/(\tau_2^4\eta^8)$, i.e. the elements that have appeared above from the traces and the $(-1)^F$-projectors, it is possible to identify further modular-invariant theories. Requiring to have a single graviton and to always have bosons and fermions to contribute with opposite signs, one finds that, along with the type IIA and type IIB theories, two more exist. These are the so-called type 0A and type 0B theories [13, 17] and their partition functions read

$$Z_{0\mathrm{A}}(\tau,\overline{\tau}) = \frac{1}{\tau_2^4}\frac{O_8\overline{O}_8 + V_8\overline{V}_8 + S_8\overline{C}_8 + C_8\overline{S}_8}{\eta^8\overline{\eta}^8}[\tau,\overline{\tau}], \tag{2.1.63}$$

$$Z_{0\mathrm{B}}(\tau,\overline{\tau}) = \frac{1}{\tau_2^4}\frac{O_8\overline{O}_8 + V_8\overline{V}_8 + S_8\overline{S}_8 + C_8\overline{C}_8}{\eta^8\overline{\eta}^8}[\tau,\overline{\tau}]. \tag{2.1.64}$$

These theories do not have any spacetime fermions and therefore they are not supersymmetric. Furthermore, they also both contain a tachyon, as apparent due to the presence of the term $O_8\overline{O}_8$.

The Hilbert space of closed superstrings involves two separate spaces, corresponding to the right- and the left-moving sectors. One can define a theory that is invariant under the exchange of right- and left-moving oscillators. This can be done by defining the worldsheet parity operator Ω_P. This is an operator acting on any state $|r\rangle|l\rangle$, where $|r\rangle$ and $|l\rangle$ are states constructed in the right- and left-moving sectors, respectively, by exchanging the right- and left-moving terms with each other, i.e. acting as $\Omega_P|r\rangle|l\rangle = |l\rangle|r\rangle$. Schematically, one can understand the presence of this operator in the Hilbert-space traces as

$$\begin{aligned}
\mathrm{tr}_{\mathcal{H}}\, q^{\tilde{L}_0}\overline{q}^{\tilde{\overline{L}}_0}\Omega &= \sum_r\sum_l \langle r|\langle l|q^{\tilde{L}_0}\overline{q}^{\tilde{\overline{L}}_0}\Omega_P|r\rangle|l\rangle \\
&= \sum_r\sum_l \langle r|\langle l|q^{\tilde{L}_0}\overline{q}^{\tilde{\overline{L}}_0}|l\rangle|r\rangle \\
&= \sum_r \langle r|\langle r|(q\overline{q})^{\tilde{L}_0}|r\rangle|r\rangle,
\end{aligned}$$

as a consequence of the fact that considering the same states $|r\rangle$ effectively identifies the L_0-operators. Adapting the previous calculations, and adding a factor $1/2$ as this contribution enters the partition function via a projector $P = (1+\Omega_P)/2$, one finds the so-called Klein-bottle partition function

$$Z_K = \frac{1}{2}\frac{1}{\tau_2^4}\frac{V_8(2\mathrm{i}\tau_2) - S_8(2\mathrm{i}\tau_2)}{\eta^8(2\mathrm{i}\tau_2)}.$$

To obtain this, the NSNS-states provide the V_8-term, whereas the RR-term give the S_8-term, with the opposite sign due to exchaning fermion states. This function is not modular-invariant, and it does not even involve a real part for the complex variable τ. This impacts the choice for the domain of integration in the determination of the scattering amplitudes. The Klein-bottle amplitude therefore reads

$$\mathcal{K} = \frac{1}{2}\int_0^\infty \frac{\mathrm{d}\tau_2}{\tau_2^6}\frac{V_8 - S_8}{\eta^8}[2\mathrm{i}\tau_2]. \tag{2.1.65}$$

To conclude, the type IIB closed-string sector modded out by the projector $P = (1+\Omega_P)/2$ has a partition function that corresponds to half the contribution of the torus amplitude in eq. (2.1.58) plus the Klein-bottle term in eq. (2.1.65). The counting of states reveals an equal number of bosons and fermions: in the fermionic NSR- and RNS-sectors, worldsheet parity only leaves half of states from the halved-torus, with $n_{\mathrm{NSR}}+n_{\mathrm{RNS}} = (64+64)/2 = 64$, whereas in the NSNS- and RR-sectors, respectively, it leaves only symmetric and antisymmetric combinations, with $n_{\mathrm{NSNS}} = (64+8)/2 = 36$ and $n_{\mathrm{RR}} = (64-8)/2 = 28$.

For the open-string sector, the situation is similar. In this case worldsheet parity acts by mapping the worldsheet spacetime coordinate as $\sigma \xrightarrow{\Omega_P} \pi - \sigma$ and exchanging the spinor components, rendering this an unoriented theory as well. In this case the calculations are slightly more involved,[2.10] but eventually one finds the so-called Möbius-strip amplitude[2.11]

$$Z_M = -\frac{\epsilon n}{2}\frac{1}{\tau_2^{\frac{1}{2}(p+1)}}\frac{V_8(\mathrm{i}\tau_2/2+1/2) - S_8(\mathrm{i}\tau_2/2+1/2)}{\eta^8(\mathrm{i}\tau_2/2+1/2)},$$

where the factor ϵn accounts for the trace over the gauge group. In fact, the group $\mathrm{U}(n)$ has to be restricted in such a way that worldsheet parity is still a projector. One needs to have $\Omega_P^2|a,b\rangle = \gamma_{bd}\Omega_P|d,c\rangle\gamma_{ca}^{-1} = \gamma_{bd}\gamma_{ce}|e,f\rangle\gamma_{fd}^{-1}\gamma_{ca}^{-1} = |a,b\rangle$, where γ is the matrix representing the orientifold action on the gauge indices, which requires $\gamma^T = \epsilon\gamma$, with $\epsilon = \pm 1$. So the trace gives a factor $\sum_{a=1}^n\sum_{b=1}^n\langle a,b|\Omega|a,b\rangle = \mathrm{tr}\,\gamma^T\gamma^{-1} = \epsilon n$. The scattering amplitude is

$$\mathcal{M} = -\frac{\epsilon n}{2}\int_0^\infty \frac{\mathrm{d}\tau_2}{\tau_2^6}\frac{V_8 - S_8}{\eta^8}\left[\frac{\mathrm{i}\tau_2}{2}+\frac{1}{2}\right]. \tag{2.1.66}$$

Adding up the partition functions in eqs. (2.1.61, 2.1.66), which represents the theory modded out by the operator $P = (1+\Omega_P)/2$, the total number of degrees of freedom for the open string is then to be multiplied by a factor $m_g = n(n-\epsilon)/2$. So the case $\epsilon = 1$ defines a theory with gauge group $\mathrm{SO}(n)$, and it defines O9$^-$-planes, whereas the case $\epsilon = -1$ is associated to the group $\mathrm{USp}(n)$ and to O9$^+$-planes.

The annulus and Möbius-strip amplitudes of eqs. (2.1.61, 2.1.66) can be generalised to account for the presence of n_+ D9-branes and n_- anti-D9-branes. Anti-D-branes are analogous objects to D-branes, with the only difference being their opposite RR-charge (see sections 2.3 and 2.4 for more details). In this case, including NS- and R-factors $\epsilon_{\mathrm{NS}}, \epsilon_{\mathrm{R}} = \pm 1$,

[2.10] In fact, all the essential mathematics is exactly the one discussed in subsubsections 3.4.1.3 and 3.4.1.3, leading to the partition function in eq. (3.4.7). Although it would have been possible to report such results in this introductory chapter, the actual calculation is left in a later chapter for pedagogical reasons.

[2.11] Notice that, with respect to the notation in ref. [39], here the hat on the characters with shifted argument is understood for simplicity, since for the simple cases in consideration it does not imply any actual change (see also comments in subsection 3.4.2).

the amplitudes are

$$\mathcal{A} = \frac{1}{2}\int_0^\infty \frac{\mathrm{d}\tau_2}{\tau_2^6}\frac{(n_+^2+n_-^2)(V_8-S_8)+2n_+n_-(O_8-C_8)}{\eta^8}\left[\frac{\mathrm{i}\tau_2}{2}\right], \tag{2.1.67}$$

$$\mathcal{M} = -\frac{1}{2}\int_0^\infty \frac{\mathrm{d}\tau_2}{\tau_2^6}\frac{\epsilon_{\mathrm{NS}}(n_++n_-)V_8-\epsilon_{\mathrm{R}}(n_+-n_-)S_8}{\eta^8}\left[\frac{\mathrm{i}\tau_2}{2}+\frac{1}{2}\right]. \tag{2.1.68}$$

Along with the characters V_8 and S_8, the characters O_8 and C_8 appear too, following the presence of possible tachyons and an opposite GSO-projection for brane/antibrane states [40–43]. The Klein-bottle, annulus and Möbius-strip amplitudes in eqs. (2.1.65, 2.1.67, 2.1.68) are 'direct-channel amplitudes', and they are computed straightforwardly from the string spectrum. Taking advantage of the closed/open string duality, one can exploit them to define the 'transverse-channel amplitudes' and gain information about the tadpole cancellation conditions. Via the transformation $\ell = 1/(2\tau_2)$, $\ell = 2/\tau_2$ and $\ell = 1/(2\tau_2)$, respectively, the three transverse-channel amplitudes read

$$\tilde{\mathcal{K}} = \frac{1}{2}2^5\int_0^\infty \mathrm{d}\ell\,\frac{V_8-S_8}{\eta^8}[\mathrm{i}\ell], \tag{2.1.69}$$

$$\tilde{\mathcal{A}} = \frac{1}{2}2^{-5}\int_0^\infty \mathrm{d}\ell\,\frac{(n_++n_-)^2V_8-(n_+-n_-)^2S_8}{\eta^8}[\mathrm{i}\ell], \tag{2.1.70}$$

$$\tilde{\mathcal{M}} = -\frac{1}{2}2\int_0^\infty \mathrm{d}\ell\,\frac{\epsilon_{\mathrm{NS}}(n_++n_-)V_8-\epsilon_{\mathrm{R}}(n_+-n_-)S_8}{\eta^8}\left[\mathrm{i}\ell+\frac{1}{2}\right]. \tag{2.1.71}$$

The lack of a cancellation among the constant terms proportional to V_8 and S_8 signals the presence of NSNS- and RR-tadpoles, respectively. An NSNS-tadpole signals the presence of a dilaton potential in the effective action, with a term proportional to $f(\phi) = \mathrm{e}^{\gamma_0\phi}$, for some constant γ_0. This, in itself, is believed not to be a fundamental inconsistency of the theory.[2.12] On the other hand, an RR-tadpole would indicate the violation of a field equation for an RR-form field, and therefore is unacceptable. The absence of tadpoles is guaranteed by the conditions

$$2^5 - \epsilon_{\mathrm{NS}}(n_++n_-) = 0, \tag{2.1.72a}$$

$$2^5 - \epsilon_{\mathrm{R}}(n_+-n_-) = 0. \tag{2.1.72b}$$

A simple solution to both constraints is given by $n_- = 0$ and $n_+ = 32$, with $\epsilon_{\mathrm{NS}} = \epsilon_{\mathrm{R}} = 1$. This is type I string theory, whose closed-string sector corresponds to the orientifold-invariant type IIB one and whose open-string sector contains a stack of D9-branes generating the gauge group SO(32). Such a theory has $N_{10} = 1$ supersymmetry. Solutions with both $n_+, n_- \neq 0$ suffer tachyonic instabilities. A consistent solution with no D9-branes, i.e. $n_+ = 0$, is represented by the Sugimoto model, which has $\epsilon_{\mathrm{NS}} = \epsilon_{\mathrm{R}} = -1$ and $n_- = 32$. This theory contains anti-D9-branes generating the gauge group USp(32) and it has an NSNS-tadpole, but no RR-tadpole. Due to the presence of anti-D9-branes, this theory is non-supersymmetric.

[2.12] See refs. [44, 45] for seminal work in this direction.

It should be mentioned that it is also possible to define an orientifold of the type 0B theory that generates a chiral spectrum, with fermions in the open-string spectrum and an SU(32)-gauge group, and also without tachyons. This is the type 0'B theory. Such a theory is also non-supersymmetric.

To conclude, it is necessary to specify that two more fundamental supersymmetric theories exist, i.e. the heterotic $\mathrm{E}_8 \times \mathrm{E}_8$- and SO(32)-theories. These correspond to theories with $N_{10} = 1$ supersymmetry. Their construction is not going to be discussed here for brevity. An orbifold projection of the former generates the $\mathrm{SO}(16) \times \mathrm{SO}(16)$-theory, which is a consistent non-tachyonic and non-supersymmetric construction.

2.2 Elements of String Compactifications

This section introduces some essential elements of string compactifications, focussing on the closed-string sector of type IIB theories. Because an analysis of the geometrical details of the dimensional reduction of a theory are not at the core of the topics discussed in later chapters, this section is less complete and formal than the others. It is only meant to outline the essential basic ideas that underlie the subject of string compactifications.

2.2.1 Type IIB Low-Energy Effective Action

According to the closed-string mass formula of eq. (2.1.38), the lightest states in the type IIB GSO-invariant closed-string theory are massless states from the NSNS-, RR-, NSR- and RNS-sectors.

- In the NSNS-sector, the massless states are the graviton G_{MN}, the dilaton Φ and the Kalb-Ramond 2-form B_2 corresponding to the symmetric, trace and anti-symmetric part, respectively, of the state $b^M_{-1/2}|\mathrm{NS}\rangle \otimes \bar{b}^N_{-1/2}|\overline{\mathrm{NS}}\rangle$. In the RR-sector, the GSO-invariant massless states are the scalar C_0, the 2-form C_2 and the self-dual 4-form C_4 corresponding to the decomposition of the state $|\mathrm{R}_+\rangle \otimes |\overline{\mathrm{R}}_+\rangle$.
- In the NSR- and RNS-sectors, the massless states are two dilatini λ^i and two gravitini ψ^i_M, where each dilatino-gravitino pair corresponds to the decomposition of the states $|\mathrm{R}_+\rangle \otimes \bar{b}^M|\overline{\mathrm{NS}}\rangle$ and $b^M|\mathrm{NS}\rangle \otimes |\overline{\mathrm{R}}_+\rangle$, respectively. Both gravitini and both dilatini have the same chirality, with the gravitino chirality being opposite to that of the dilatini.

The particle content is supersymmetric since it contains equal numbers of bosons and fermions. Because there are two gravitini with the same chirality, this corresponds to a chiral $N_{10} = 2$ supergravity theory.

In string theory, one way to determine the low-energy action corresponding to a string model is to reconstruct the action that generates the same scattering amplitudes for the fields as the corresponding amplitudes computed in the complete string-theoretic framework. This low-energy effective theory is formulated in terms of the string-frame metric $ds^2_{10} = G_{MN}\,\mathrm{d}x^M \mathrm{d}x^N$. Given the gravitational coupling $2\kappa^2_{10} = l_s^8/2\pi$, where the string length is

$l_s = 2\pi\sqrt{\alpha'}$, the type IIB 10-dimensional massless bosonic action reads [35]

$$\begin{aligned} S_{\text{IIB}}^{\text{boson}} = \frac{1}{2\kappa_{10}^2} \int_{X_{1,9}} &\left[\mathrm{e}^{-2\Phi} \left(R_{10} \star 1 + 4\, \mathrm{d}\Phi \wedge \star\, \mathrm{d}\Phi - \frac{1}{2}\, H_3 \wedge \star\, H_3 \right) \right] \\ + \frac{1}{2\kappa_{10}^2} \int_{X_{1,9}} &\left[-\frac{1}{2}\, F_1^s \wedge \star\, F_1^s - \frac{1}{2}\, \tilde{F}_3^s \wedge \star\, \tilde{F}_3^s - \frac{1}{4}\, \tilde{F}_5^s \wedge \star\, \tilde{F}_5^s \right] \\ &- \frac{1}{4\kappa_{10}^2} \int_{X_{1,9}} C_4^s \wedge H_3 \wedge\, \tilde{F}_3^s, \end{aligned}$$

where R_{10} is the Ricci scalar for string-frame metric, with Hodge dual '$\star$', and the NSNS- and RR-sector field-strength tensors are respectively defined as $H_3 = \mathrm{d}B_2$ and $\tilde{F}^s = \mathrm{d}C^s - H_3 \wedge C^s$. The string coupling is defined as $g_s = \mathrm{e}^{\langle\Phi\rangle}$, where Φ is the dilaton. It is not necessary to write the fermionic action since it can be reconstructed via supersymmetry.

In order for the Einstein-Hilbert term to be expressed in the canonical normalisation, a metric redefinition is necessary, moving to the so-called Einstein frame. The 10-dimensional Einstein-frame metric is defined as

$$\hat{g}_{MN} = \mathrm{e}^{-(\Phi - \langle\Phi\rangle)/2} G_{MN}, \tag{2.2.1}$$

and it can be expressed even more easily in terms of the shifted dilaton $\phi = \Phi - \langle\Phi\rangle$. In this way, in a more compact notation, the Einstein-frame type IIB low-energy effective action can eventually be written as [46]

$$\begin{aligned} S_{\text{IIB}}^{\text{boson}} = \frac{1}{2\hat{\kappa}_{10}^2} \int_{X_{1,9}} &\left[\hat{R}_{10} \hat{\ast} 1 - \frac{\mathrm{d}\tau \wedge \hat{\ast}\, \mathrm{d}\overline{\tau}}{2\, (\operatorname{Im}\tau)^2} - \frac{G_3 \wedge \hat{\ast}\, \overline{G}_3}{2 \operatorname{Im}\tau} - \frac{1}{4}\, \tilde{F}_5 \wedge \hat{\ast}\, \tilde{F}_5 \right] \\ &- \frac{\mathrm{i}}{8\hat{\kappa}_{10}^2} \int_{X_{1,9}} \frac{1}{\operatorname{Im}\tau}\, C_4 \wedge G_3 \wedge \overline{G}_3, \end{aligned} \tag{2.2.2}$$

where the physical 10-dimensional gravitational coupling is $2\hat{\kappa}_{10}^2 = g_s^2 l_s^8 / 2\pi$ and the RR-fields have been rescaled as $C = g_s C^s$. Further, the axio-dilaton and the complexified 3-form flux have been defined as $\tau = C_0 + \mathrm{i}\,\mathrm{e}^{-\phi}$ and $G_3 = \tilde{F}_3 - \mathrm{i}\,\mathrm{e}^{-\phi} H_3 = F_3 - \tau H_3$, respectively. This action is a fundamental tool in the dimensional reduction of type IIB string theory. Understanding how it is dimensionally reduced provides knowledge on many of the fundamental properties of the closed-string sector.

2.2.2 Dimensional Reductions and Kaluza-Klein States

A string-derived field theory is typically formulated in higher dimensions compared to the observed 4-dimensional world. A typical interpretation of this fact is that the phenomenological 4-dimensional theory emerges from a compactification of the original theory. The basic idea is that the complete 10-dimensional space $X_{1,9}$ is factorised as $X_{1,9} = X_{1,3} \times Y_6$, where $X_{1,3}$ is the observed 4-dimensional geometry and Y_6 is an internal space that below a certain energy scale is compactified.

Given the field content of a 10-dimensional theory, a key part in the compactification is the dimensional reduction of all the fields. This typically generates an infinite number of so-called Kaluza-Klein 4-dimensional fields for each 10-dimensional field, with an infinite tower of different masses. This section reports an overview of the dimensional reduction for the different kinds of fields that are typically encountered in string compactifications, i.e. scalars, spinors, vectors, gravitinos, gravitons and p-forms. This is a vast topic and this section only outlines the basic features. For complete reviews, see refs. [47, 48].

In what follows, the discussion is referred to compactifications on spacetimes of the kind $X_{1,9} = \mathbb{M}^{1,3} \times Y_6$, with no assumptions on specific features of the internal manifold unless explicitly stated. The notation is as follows: 10-dimensional directions are denoted as x^M, with $M = 0, 1, \ldots, 9$, 4-dimensional non-compact directions are denoted as x^μ, with $\mu = 0, 1, 2, 3$,and internal 6-dimensional directions are denoted as y^m, with $m = 4, \ldots, 9$.

2.2.2.1 Lorentz Group Representations

In a D-dimensional quantum field theory, the spin of a field $u = u(x)$ is specified according to the representation of the Lorentz group $\mathrm{SO}(1, D-1)$ under which it transforms. If the spacetime is factorised in the form $X_{1,9} = \mathbb{M}^{1,3} \times Y_6$, then one must restrict to representations of the tangent space group $\mathrm{SO}(1,3) \times \mathrm{SO}(6)$, which is a subgroup $\mathrm{SO}(1,3) \times \mathrm{SO}(6) \subset \mathrm{SO}(1,9)$ of the original group. The basic properties of the dimensional reductions of interest are outlined below.

- Scalar fields $\phi = \phi(x)$ are in the trivial representation of both the groups $\mathrm{SO}(1,9)$ and $\mathrm{SO}(1,3) \times \mathrm{SO}(6)$, therefore their reduction does not present any particular issue.

- For spinor fields $\psi^A = \psi^A(x)$, the reduction is based on Dirac matrices. In 10-dimensional spacetime Dirac matrices are ten matrices of dimension $2^5 = 32$. They can be labelled as Γ_M^{AB}, with $A, B = 1, 2, \ldots, 32$, and they satisfy the Clifford algebra $\{\Gamma_M, \Gamma_N\} = 2g_{MN}$. The generators of the group $\mathrm{SO}(1,9)$ can then be seen to be $\Sigma_{MN} = [\Gamma_M, \Gamma_N]/4$.

  In a spacetime of the form $X_{1,9} = \mathbb{M}^{1,3} \times Y_6$, one can take the first four matrices $\Gamma^\mu = \gamma^\mu$ as the Dirac matrices for the space $\mathbb{M}^{1,3}$ and the final six matrices $\gamma^m = \Gamma^m$ as the Dirac matrices for the space Y_6. In fact, they satisfy the correct Clifford algebra and they allow one to construct the generators of the groups $\mathrm{SO}(1,3)$ and $\mathrm{SO}(6)$ as $\sigma_{\mu\nu} = [\gamma_\mu, \gamma_\nu]/4$ and $s_{mn} = [\gamma_m, \gamma_n]/4$, respectively. Therefore, a spinor field of $\mathrm{SO}(1,9)$ transforms as a spinor field of $\mathrm{SO}(1,3)$ as well as a spinor of $\mathrm{SO}(6)$ too. Ultimately this means that one can write the spinor as $\psi^{\alpha a}$, with $\alpha = 1, \ldots, 4$ and $a = 1, \ldots, 8$, in such a way that the indices α and a account for $\mathrm{SO}(1,3)$- and $\mathrm{SO}(6)$ transformations, respectively.

- Gauge fields $A_M = A_M(x^M)$ transform under the vector representation $\mathbf{10}$ of the group $\mathrm{SO}(1,9)$, which decomposes in terms of $\mathrm{SO}(1,3) \times \mathrm{SO}(6)$ as the representation $(\mathbf{4}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{6})$, namely a vector of $\mathrm{SO}(1,3)$ plus a vector of $\mathrm{SO}(6)$. Such a decomposition of A_M into subrepresentation simply means that the components A_μ transform as a 4-dimensional gauge field while the components A_m as a 6-dimensional vector.

For higher-spin fields, the concepts are generalisations of the results for spin-1/2 and spin-1 fields which result from product representations. Only a few basic ideas are reported below.

- Gravitinos are spin-3/2 particles which we can describe by a field $\psi_M^A = \psi_M^A(x)$, where M is a vector index and A is a spinor index. Evidently, one can implement the same decompositions as before to the SO(1,3)- and SO(6)-vectors that are obtained by splitting the Lorentz-index. One thus gets the SO(1,3) vector-spinor $\chi^{\mu}_{\alpha a}$ and the SO(6) vector-spinor $\chi^{m}_{\alpha a}$.
- The graviton is described by the metric tensor $g_{MN} = g_{MN}(x)$. Intuitively, the components $g_{\mu\nu}$ would form the 4-dimensional metric tensor, the components $g_{\mu m}$ would form a 4-dimensional vector and the components g_{mn} would form several 4-dimensional scalars.
- As far as p-form fields $A_p = A_p(x)$ are concerned, the field $A_{M_1 \dots M_p}$ decomposes in terms of $\mathrm{SO}(1,3) \times \mathrm{SO}(6)$ quite easily: components with r indices in $\mathbb{M}^{1,3}$ and $p-r$ indices in Y_6 just correspond to an r-form in $\mathbb{M}^{1,3}$ and a $(p-r)$-form in Y_6.

Given these basic decompositions, the actual study of the field dimensional reduction is quite complicated due to the requirement of masslessness, as is going to be shown below. In fact, requiring the 4-dimensional fields to be massless implies precise characteristics on their 6-dimensional counterparts.

2.2.2.2 Massles Modes

This subsection outlines the effects of dimensional reduction in spacetimes structured as $X_{1,9} = \mathbb{M}^{1,3} \times Y_6$ on the different fields that a theory does contain.

In principle, the core ideas are the same for each spin, and can thus be understood thanks to the case of scalars. Nevertheless, the analysis of fields with more advanced Lorentz index structure turns out to be more involved due to further non-trivial technical issues.

Scalar Fields

A 10-dimensional massless scalar field $\phi = \phi(x^M)$ satisfies the wave equation $\nabla^M \nabla_M \, \phi = 0$, where ∇_M is the 10-dimensional covariant derivative. If the spacetime is of the form $X_{1,9} = \mathbb{M}^{1,3} \times Y_6$, then the wave operator splits naturally in the form $\nabla^M \nabla_M = \partial^\mu \partial_\mu + \nabla^m \nabla_m$. The field can be expanded in the Fourier expansion

$$\phi(x^M) = \sum_\omega \phi^\omega(x^\mu) \, \tilde{\phi}^\omega(x^m)$$

where ω is some label that parametrises the expansion in such a way that the 6-dimensional fields $\tilde{\phi}^\omega$ are the eigenfunctions of the 6-dimensional Laplacian, i.e.

$$\nabla^m \nabla_m \tilde{\phi}^\omega = -m_\omega^2 \, \tilde{\phi}^\omega.$$

Such a decomposition is always admissible because the eigenfunctions of the Laplacian form a basis of functions over the space Y_6. Furthermore, for a compact space Y_6 such eigenvalues are non-positive, i.e. the values m_ω^2 are non-negative. Following this decomposition, the 4-dimensional fields ϕ_i do satisfy the field equations

$$(\partial^\mu \partial_\mu - m_\omega^2) \phi^\omega = 0,$$

which means that each field ϕ^ω is a scalar field of mass m_ω^2. Roughly, one expects the order of magnitude of the compact space Laplacian eigenvalues to be of the same order of magnitude as $1/R^2$, R being the size of the manifold Y_6 that scales the internal derivatives. Therefore this the reference Kaluza-Klein mass scale M_{KK}^2 can be defined as

$$M_{KK}^2 = 1/R^2.$$

Because the masses are then approximately of the same order of magnitude as the Kaluza-Klein mass scale, i.e. $m_\omega^2 \sim M_{KK}^2$, generally speaking, the phenomenological interest is only in the massless modes, since the massive Kaluza-Klein tower is above the 4-dimensional cutoff energy. So, the number of fields present below the cutoff is the number of zero-modes solving the equation $\nabla^m \nabla_m \tilde{\phi} = 0$: in a compact space, this has only one solution.

Spinor Fields

A massless 10-dimensional spinor $\psi = \psi(x^M)$ satisfies the field equation $\mathrm{i}\Gamma^M \nabla_M \psi = 0$. Such an equation can be immediately rewritten as $\mathrm{i}(\gamma^\mu \nabla_\mu + \gamma^m \nabla_m)\psi = 0$. The operators $\not{\nabla}_4 = \gamma^\mu \nabla_\mu$ and $\not{\nabla}_6 = \gamma^m \nabla_m$ do not commute, so a further operation is required before a Fourier expansion. One can introduce the 4-dimensional chirality matrix $\gamma_{(4)} = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$ and define the operators $\tilde{\not{\nabla}}_4 = \gamma_{(4)}\gamma^\mu \nabla_\mu$ and $\tilde{\not{\nabla}}_6 = \gamma_{(4)}\gamma^m \nabla_m$. The wave equation can be expressed as $\mathrm{i}\,(\tilde{\not{\nabla}}_4 + \tilde{\not{\nabla}}_6)\psi = 0$ and, because $\tilde{\not{\nabla}}_4$ and $\tilde{\not{\nabla}}_6$ commute, they can be diagonalised simultaneously. If one determines the complete set of normalised solutions of the differential equation

$$\mathrm{i}\tilde{\not{\nabla}}_6 \tilde{\psi}_\omega(x^m) = m_\omega\, \tilde{\psi}_\omega(x^m),$$

then the complete wave equation can be solved by expanding the solution ψ in terms of such solutions, i.e.

$$\psi(x^M) = \sum_\omega \psi_\omega(x^\mu)\tilde{\psi}_\omega(x^m).$$

In fact, the the wave equation for the 4-dimensional field is now

$$(\mathrm{i}\tilde{\not{\nabla}}_4 + m_\omega)\psi_\omega(x^\mu) = 0,$$

and describes a different 4-dimensional spinor ψ_ω of mass m_ω for each value of ω. The equations in terms of $\tilde{\not{\nabla}}_6$ and $\tilde{\not{\nabla}}_4$ are equivalent to the equations with the usual slashed operators $\not{\nabla}_6$ and $\not{\nabla}_4$ because both the matrices γ^m and γ^μ and the matrices $\gamma_{(4)}\gamma^m$ and $\gamma_{(4)}\gamma^\mu$ satisfy the same Clifford algebra over Y_6 and $\mathbb{M}^{1,3}$.

The covariant derivative incorporates the effects of both the spacetime curvature and the coupling to gauge fields. If no gauge fields are present, the equation $\not{\nabla}_6\tilde{\psi} = 0$ can be studied easily in the context of Calabi-Yau compactifications. In fact, a Calabi-Yau threefold (see subsection 2.2.3) has a single solution to the Killing equation $\nabla_m \xi(x^m) = 0$. More generally, in the presence of additional interactions such as the gauge ones, the analysis is more complicated and one needs the so-called index-theorems counting the solutions to the the Killing equation for each chirality. This is not going to be explored further here.

Gravitinos

The study of gravitinos is quite similar to that of spinors. For a spin-3/2 field ψ_M^A, with M and A a vector and a spinor index, respectively, one encounters the following scenario.

- Massless terms from the SO(1, 3)-group vector-spinor $\psi_\mu^{\alpha a}$ provide 4-dimensional gravitinos, therefore their number is equal to the number of supersymmetries preserved by the compactification. For a Calabi-Yau threefold compactification, there exists a single Killing spinor which satisfies the equation $\nabla_m \xi(x^m) = 0$, which means for each 10-dimensional gravitino there is one only 4-dimensional gravitino.

- In general, the components $\psi_m^{\alpha a}$ originate 4-dimensional spinors which are also in the representation **6** of the group SO(6). Therefore, the zero-mode equation is generalised non-trivially to an equation of the form $\not\nabla \xi_m = 0$. For Calabi-Yau compactifications, it is possible to show that such massless fields modes provide the superpartners of Kähler and complex-structure moduli (to be introduced later).

Differential Forms

The dimensional reduction of p-forms can be performed by studying their field equation. In general, p-form fields A_p are invariant under the gauge transformations $A_p \to A_p + \mathrm{d} f_{p-1}$, for an arbitrary $(p-1)$-form f_{p-1} and their action in an n-dimensional spacetime is in the form

$$S = -\frac{1}{2}\int_X \mathrm{d}A_p \wedge * \mathrm{d}A_p = -\frac{1}{2}\int_X A_p \wedge * \mathrm{d}_n^\dagger \mathrm{d}_n A_p,$$

where $F_{p+1} = \mathrm{d}A_p$ is the $(p+1)$-form field-strength tensor. It is then immediate to infer the field equations $\mathrm{d}_n^\dagger \mathrm{d}_n A_p = 0$. However, it is convenient to write them in terms of the Hodge-de Rham operator $\Delta = \mathrm{d}^\dagger \mathrm{d} + \mathrm{d}\mathrm{d}^\dagger$ as

$$\begin{cases} \Delta A_p = 0, \\ \mathrm{d}^\dagger A_p = 0, \end{cases}$$

where the gauge condition $\mathrm{d}^\dagger A_p = 0$ has been imposed.

Before moving on, it is worthwhile to consider in more detail what happens to p-forms in a space of the kind $X_{1,9} = \mathbb{M}^{1,3} \times Y_6$. In the following, $(r, p-r)$-forms denote the components of a p-form C_p with r and $s = p-r$ indices in the tangent spaces of $\mathbb{M}^{1,3}$ and Y_6, respectively. It turns out that the Hodge-de Rham operator Δ_{10} splits in terms of the Hodge-De Rham operators Δ_4 and Δ_6 on the manifolds $\mathbb{M}^{1,3}$ and Y_6, respectively, as

$$\Delta_{10} = \Delta_4 + \Delta_6,$$

In view of the decomposition of the Hodge-de Rham operator, if it acts on wedge products of 4- and 6-dimensional differential forms, it is convenient to split the 10-dimensional p-forms A_p into $(r, p-r)$-form fields $A_{r,p-r}$ that are expanded as

$$A_{r,p-r}(x^M) = \sum_\omega A_r^\omega(x^\mu) \wedge \tilde{A}_{p-r}^\omega(y^m),$$

where A_r^ω and $\tilde{A}_{p-r}^\omega$ are $(r,0)$- and $(0,p-r)$-forms defined on $\mathbb{M}^{1,3}$ and Y_6, respectively. Therefore, the field equations can easily be seen to read

$$\Delta_{10} A_{r,p-r} = \sum_\omega \left[\Delta_4 A_r^\omega \wedge \tilde{A}_{p-r}^\omega + A_r^\omega \wedge \Delta_6 \tilde{A}_{p-r}^\omega\right] = 0.$$

In particular, one can immediately observe the field equation of massless r-forms on the space $\mathbb{M}^{1,3}$, i.e.

$$\Delta_4 A_r^\omega = 0,$$

for each of the solutions $\tilde{A}_{p-r}^\omega$ of the equation

$$\Delta_6 \tilde{A}_{p-r}^\omega = 0.$$

Equivalently, $\tilde{A}_{p-r}^i$ must be a harmonic form in the internal space Y_6. The number of $(0,p-r)$-forms which are zero eigenvalues of Δ_6 is the Betti number $b_{p-r} = \dim H^{p-r}(Y_6)$, i.e. the dimension of the cohomology class $H^{p-r}(Y_6)$ over the compact space Y_6.

The gauge condition undergoes an analogous decomposition. Indeed, because it reads $\mathrm{d}_{10}^\dagger A_{r,p-r} = \sum_\omega \left[\mathrm{d}_4^\dagger A_r^\omega \wedge \tilde{A}_{p-r}^\omega + (-1)^r A_r^\omega \wedge \mathrm{d}_6^\dagger \tilde{A}_{p-r}^\omega\right] = 0$, since $\tilde{A}_{p-r}^\omega$ is harmonic by assumption for massless 4-dimensional r-forms A_r^ω, it also implies the condition

$$\mathrm{d}_4^\dagger A_r^\omega = 0.$$

Graviton

The analysis of the metric g_{MN} can be performed qualitatively by studying the index-structure of the field.

- The purely non-compact term $g_{\mu\nu}$ corresponds to a scalar field in the compact space Y_6, therefore the zero-mode equation is the same as for scalar fields and we thus have a single 4-dimensional graviton field $g_{\mu\nu} = g_{\mu\nu}(x^\mu)$.

- The mixed terms $g_{\mu m}$ are 4-dimensional vectors and also vectors of SO(6), thus the massless 4-dimensional vectors would correspond to covariantly-constant vectors under the group SO(6). However, Calabi-Yau threefolds do not admit such vectors.

- The compact terms g_{mn} are 4-dimensional scalars and in fact, for a Calabi-Yau threefold, they correspond exactly to the scalar degrees of freedom that determine the metric, namely Kähler moduli and complex-structure moduli (introduced later).

2.2.3 Calabi-Yau Orientifold Compactifications

This subsection outlines the essential elements of Calabi-Yau orientifold compactifications, which constitute a common framework to generate 4-dimensional chiral theories from type IIB string theory.

2.2.3.1 Supersymmetry and Holonomy

Type IIB string theory is a 10-dimensional theory with $N_{10} = 2$ supersymmetries. A compactification on a trivial space Y_6 comes with too many 4-dimensional supersymmetries. In fact, a 4-dimensional compactified type IIB theory has $N_4 = 8$ supersymmetries unless the compactification plays a non-trivial role in the dimensional reduction of the supersymmetry charges. Evidently, such a spectrum is not compatible with real-world observations of chiral interactions. At the same time, a certain amount of supersymmetry is often assumed to be important in order to simplify the solution of the hierarchy problem.

A compactification on a manifold Y_6 impacts the 4-dimensional supersymmetry content due to the reduction undergone by the supersymmetry charges. Ultimately, this is related to the holonomy group of such a manifold Y_6. In short, the holonomy group H_M of a manifold M is the group of the rotations a vector v^m undergoes under parallel transport on a closed path γ. Following refs. [48, 49], this can be explained naively by just thinking of Kaluza-Klein compactifications. The 4-dimensional fields that are visible at low energies are the zero-modes that are constant in the extra-dimensional internal space. Similarly, supersymmetries that are unbroken in the 4-dimensional spacetime of a higher-dimensional theory are those corresponding to supersymmetry parameters that are covariantly constant over the internal space. Such supersymmetry parameters are spinors η that in general can be decomposed in terms of 4- and 6-dimensional spinors as $\eta = \epsilon(x^\mu)\xi(y^m)$. The requirement of covariantly-constant spinors over the compact manifold Y_6 amounts then to selecting Killing spinors, i.e. spinors such that $\nabla_m \xi = 0$. Equivalently, covariantly-constant spinors correspond to singlets under the compact-space holonomy group. Indeed, such spinors do not change under parallel transport over closed loops. In other words, a supercharge that gets rotated by parallel transport over a closed path in the internal space cannot determine a well-defined 4-dimensional supersymmetry.

Therefore, on the one hand, if the manifold Y_6 admits some covariantly-constant spinors, then some supersymmetries may survive in the compactified 4-dimensional theory. On the other hand, if it admits none, no supersymmetries are present, while if all spinors are covariantly constant all the supersymmetries are preserved. From a phenomenological point of view, a great deal of interest is in the situation where only certain classes of covariantly-constant spinors are allowed. Indeed, a desirable aim is to rule out the situations with either no or too many supersymmetries. The holonomy group of a generic manifold of dimension $d = 2n$ is $\mathrm{SO}(2n)$ and thus does not admit singlets, i.e. covariantly-constant spinors. So, a generic internal space Y_6 with holonomy group $\mathrm{SO}(6)$ is not acceptable. In more detail, because the spin group of the group $\mathrm{SO}(6)$ is $\mathrm{Spin}(6) \simeq \mathrm{SU}(4)$, chiral spinors over the compact space Y_6 in principle transform under its representation $\mathbf{4}$ or $\overline{\mathbf{4}}$. In order to have some covariantly-constant spinors, it is necessary to consider an internal space Y_6 whose holonomy group is at most $\mathrm{SU}(3)$, in which case spinors decompose into the representations $\mathbf{3}+\mathbf{1}$ or $\overline{\mathbf{3}}+\mathbf{1}$ of the holonomy group $\mathrm{SU}(3)$. In fact, the decomposition of a 10-dimensional Weyl spinor in the representation $\mathbf{16}$ of the group $\mathrm{SO}(1,9)$, is as summarised in the table

Spin(1, 9)		Spin(1, 3) × Spin(6)		Spin(1, 3) × SU(3)
$\mathbf{16}$	$\rightarrow$	$(\mathbf{2},\mathbf{4})+(\mathbf{2'},\overline{\mathbf{4}})$	$\rightarrow$	$(\mathbf{2},\mathbf{3})+(\mathbf{2'},\overline{\mathbf{3}})+(\mathbf{2},\mathbf{1})+(\mathbf{2'},\overline{\mathbf{1}})$

and the representations $\mathbf{2}$ and $\mathbf{2'}$ denote the two possible chiralities under the 4-dimensional Lorentz group. The components that lead to 4-dimensional supersymmetries are those in SU(3)-singlets, namely $(\mathbf{2},\mathbf{1})$ and $(\mathbf{2'},\overline{\mathbf{1}})$, that eventually consist in the 4-dimensional supercharges Q_α and $\overline{Q}_{\dot\alpha}$ once the Majorana condition is imposed. This discussion shows that the easiest supersymmetric 4-dimensional theory obtainable from a type IIB compactification is a theory with $N_4 = 2$ supersymmetries generated by compactifying on 6-dimensional spaces with holonomy group SU(3). Supersymmetries can further be reduced by postulating other symmetries in the theory, such as orientifold invariance. Choosing to focus on manifolds of complex dimension 3, it turns out that Calabi-Yau 3-folds CY_3 are in fact the complex 3-dimensional spaces with holonomy group SU(3). This motivates their use in string compactifications.

Calabi-Yau manifolds are a specific class of complex manifolds. A generic geometrical description of these manifolds is an advanced topic that involves several tools from differential complex geometry, including an analysis of almost-complex, complex, Hermitean and Kähler manifolds. Since giving an account of these topics is somewhat beyond the scope of this thesis, only a few results are outlined below. An excellent review is in refs. [50, 51] and a short list of essential results is summarised in appendix A.3. For the purposes of this thesis, a Calabi-Yau n-fold CY_n can be defined as a complex n-dimensional manifold with holonomy group $\mathrm{H} = \mathrm{SU}(n)$. A great number of relationships between Hodge numbers exist for complex, Kähler and Calabi-Yau manifolds and these eventually allow one to characterise n-dimensional Calabi-Yau manifolds in terms of such Hodge numbers. For instance 3-dimensional Calabi-Yau manifolds and can be classified in terms of two Hodge numbers as shown by the so-called Hodge diamond depicted in fig. 2.1.

$$
\begin{array}{ccccccc}
 & & & h^{0,0} & & & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & h^{2,0} & & h^{1,1} & & h^{0,2} & \\
h^{3,0} & & h^{1,2} & & h^{2,1} & & h^{0,3} \\
 & h^{3,1} & & h^{2,2} & & h^{1,3} & \\
 & & h^{3,2} & & h^{2,3} & & \\
 & & & h^{3,3} & & &
\end{array}
\quad = \quad
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{2,1} & & h^{2,1} & & 1 \\
 & 0 & & h^{1,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

Figure 2.1: Hodge diamond for 3-dimensional Calabi-Yau manifolds.

2.2.3.2 Calabi-Yau Orientifold Compactifications

As type IIB compactifications on Calabi-Yau threefolds lead to $N_4 = 2$ theories, they cannot provide reliable extensions of the Standard Model. However, postulating the presence of another symmetry such as worldsheet parity is among the easiest ways to achieve a chiral spectrum. This subsection describes the compactification of type IIB theories on Calabi-Yau orientifolds and the 4-dimensional supersymmetric $N_4 = 1$ effective field theory it leads to. Again, the analysis is going to be essential. It is based on refs. [52, 53]; helpful material is also in refs. [33, 49].

2.2.3.2.1 Particle Spectrum

Let type IIB string theory be defined on a 10-dimensional spacetime of the form $X_{1,9} = \mathbb{M}^{1,3} \times \mathrm{CY}_3$, where CY_3 is a Calabi-Yau threefold. In addition, invariance under worldsheet parity and a geometric action on the internal space are also postulated. In more detail, generalising the notion of orientifold for non-compactified theories, the so-called orientifold operator O is of the form

$$O = (-1)^{F_L} \Omega_P \sigma^*, \tag{2.2.3}$$

where Ω_P is the worldsheet parity operator, σ^* is the pull-back of σ, i.e. a holomorphic isometric involution on the internal manifold, and F_L is the 4-dimensional fermion-number operator in the left-moving sector. A $(p+1)$-dimensional fixed point of the involution is called an Op-plane, in analogy with O9-planes in type I theories (see subsubsection 2.1.3.2), and, unless stated explicitly, this means Op^--planes. The involution acts trivially on the Kähler form, i.e. $\sigma^* \omega_{1,1} = \omega$, and it is assumed to act on the holomorphic 3-form as $\sigma^* \Omega = -\Omega$. This leads to the possibility to have O3- and O7-planes as the fixed points of the involution. Moreover, it requires the presence of F_L in order to have $O^2 = 1$ on all states. The opposite choice $\sigma^* \Omega = \Omega$ is possible and it leads to O5- and O9-planes.

In detail, for arbitrary states $|\overline{\alpha}\rangle_{\mathrm{NS}}$ and $|\overline{\beta}\rangle_{\mathrm{R}}$ in the left-moving NS- and R-sectors, the left-moving fermion-number operator F_L is such that

$$\begin{aligned} (-1)^{F_L} |\overline{\alpha}\rangle_{\mathrm{NS}} &= +|\overline{\alpha}\rangle_{\mathrm{NS}}, \\ (-1)^{F_L} |\overline{\beta}\rangle_{\mathrm{R}} &= -|\overline{\beta}\rangle_{\mathrm{R}}, \end{aligned}$$

whereas the worldsheet parity operator Ω_P is defined to act as

$$\begin{aligned} \Omega_P \big[|\overline{\mathrm{NS}}\rangle \times |\mathrm{NS}\rangle \big] &= +|\overline{\mathrm{NS}}\rangle \times |\mathrm{NS}\rangle, \\ \Omega_P \big[|\overline{\mathrm{NS}}\rangle \times |\mathrm{R}_+\rangle \big] &= +|\overline{\mathrm{R}}_+\rangle \times |\mathrm{NS}\rangle, \\ \Omega_P \big[|\overline{\mathrm{R}}_+\rangle \times |\mathrm{NS}\rangle \big] &= +|\overline{\mathrm{NS}}\rangle \times |\mathrm{R}_+\rangle \\ \Omega_P \big[|\overline{\mathrm{R}}_+\rangle \times |\mathrm{R}_+\rangle \big] &= -|\overline{\mathrm{R}}_+\rangle \times |\mathrm{R}_+\rangle. \end{aligned}$$

As can be inferred by expanding on the defining condition $\Omega_P\, \psi_\pm^M(\tau,\sigma)\, \Omega_P^{-1} = \psi_\mp^M(\tau, \pi - \sigma)$, the action of worldsheet parity on closed-string fermionic creation operators is

$$\begin{aligned} \Omega_P\, b_r^M\, \Omega_P^{-1} &= \mathrm{e}^{-2\pi \mathrm{i} r}\, \overline{b}_r^M, \\ \Omega_P\, \overline{b}_r^M\, \Omega_P^{-1} &= \mathrm{e}^{-2\pi \mathrm{i} r}\, b_r^M, \end{aligned}$$

The type IIB massless spectrum on Calabi-Yau orientifolds is the result of the dimensional reduction of the type IIB massless spectrum of the states which are invariant under the action of the orientifold projection operator O. Thanks to supersymmetry, it is enough to consider the bosonic sector.

- In the NSNS-sector, the left-moving spacetime fermion-number operator acts as the identity and worldsheet parity symmetrises spacetime indices, so schematically one finds $O\phi = \sigma^*\phi$, $O\hat{g}_{MN} = \sigma^*\hat{g}_{MN}$ and $OB_2 = -\sigma^* B_2$.
- In the RR sector, the left-moving spacetime fermion-number flips the sign of the state, while worldsheet parity antisymmetrises indices, so one has $OC_0 = \sigma^* C_0$, $OC_2 = -\sigma^* C_2$ and $OC_4 = \sigma^* C_4$.

Because the operator σ is a holomorphic involution, each of the cohomology groups $H^{p,q}$ splits into a direct sum of cohomology groups $H^{p,q}_+$ and $H^{p,q}_-$ as

$$H^{p,q} = H^{p,q}_+ \oplus H^{p,q}_-,$$

with $H^{p,q}_+$ and $H^{p,q}_-$ being the $(+1)$- and (-1)-eigenspaces of σ with dimensions $h^{p,q}_+$ and $h^{p,q}_-$, respectively. Furthermore, starting from the Hodge diamond, one can deduce the following properties:

- since the Hodge operator $*$ commutes with σ^*, as σ preserves the Calabi-Yau orientation and metric, one has $h^{1,1}_+ = h^{2,2}_+$ and $h^{1,1}_- = h^{2,2}_-$;
- since σ is holomorphic, one has $h^{1,2}_+ = h^{2,1}_+$ and $h^{1,2}_- = h^{2,1}_-$;
- since $\sigma^*\Omega = -\Omega$, one has $h^{3,0}_+ = h^{0,3}_+ = 0$ and $h^{3,0}_- = h^{0,3}_- = 1$;
- since the volume-form must be proportional to $\Omega \wedge \overline{\Omega}$, which is invariant under σ^*, one has $h^{0,0}_+ = h^{3,3}_+ = 1$ and $h^{0,0}_- = h^{3,3}_- = 0$.

All the non-trivial cohomology group and their basis are summarised in table 2.1.

cohomology group	dimension	basis elements
$H^{1,1}_+$, $H^{1,1}_-$	$h^{1,1}_+$, $h^{1,1}_-$	ω_i, ω_ι
$H^{2,1}_+$, $H^{2,1}_-$	$h^{2,1}_+$, $h^{2,1}_-$	χ_a, χ_α
$H^{2,2}_+, H^{2,2}_-$	$h^{1,1}_+$, $h^{1,1}_-$	Ω^i, Ω^ι
H^3_+, H^3_-	$2\,h^{1,2}_+$, $2 + 2\,h^{1,2}_-$	$(\alpha_k, \beta^l), (\alpha_\kappa, \beta^\lambda)$

Table 2.1: Cohomology groups and cohomology group basis for a Calabi-Yau orientifold.

Under these premises, it is now possible to expand the 10-dimensional massless spectrum in terms of harmonic forms over the Calabi-Yau orientifold.

- For the metric $\hat{g}_{MN}$, because the fundamental form $\omega_{1,1}$ is such that $\sigma^*\omega_{1,1} = \omega_{1,1}$, the deformations of $\omega_{1,1} = \mathrm{i} g_{a\bar{b}}\, \mathrm{d}z^a \wedge \mathrm{d}\bar{z}^{\bar{b}}$, with $\hat{g}_{mn} = g_{mn}$, can be expanded as

$$[\omega_{1,1}(x)] = t^i(x)\,\omega_i, \qquad\qquad i = 1, \ldots, h^{1,1}_+,$$

which defines $h^{1,1}_+$ real scalars t^i, called Kähler moduli. Further deformations of the complex structure can be parametrised via the contraction with harmonic $(3,0)$- and $(2,1)$-forms, defining $h^{1,2}_-$ complex scalars u^α, called complex-structure moduli, as

$$\Omega_{abd} J_{\bar{c}}{}^d(x) = -\mathrm{i} \sum_{\alpha=1}^{h^{1,2}_-} u^\alpha(x) \chi_{\alpha\, ab\bar{c}}.$$

- Scalars, i.e. the dilaton ϕ and the 0-form C_0, are automatically two scalars in the 4-dimensional as well as in the 10-dimensional spacetime.
- As concerns the differential p-form fields, on the one hand the 2-forms B_2 and C_2 can only be expanded on (-1)-eigenspaces, i.e.

  $$\begin{aligned} [B_2(x)] &= b^\iota(x)\,\omega_\iota, && \iota = 1,\dots,h^{1,1}_-, \\ [C_2(x)] &= c^\iota(x)\,\omega_\iota, && \iota = 1,\dots,h^{1,1}_-, \end{aligned}$$

  where the basis coefficients b^ι and c^ι correspond to 4-dimensional real scalars. On the other hand, the 4-form C_4 can be expanded only on $(+1)$-eigenspaces, i.e.

  $$[C_4(x)] = \theta_2^i(x) \wedge \omega_i + V_1^k(x) \wedge \alpha_k - U_{k1}(x) \wedge \beta^k + \theta_i(x)\Omega^i, \qquad \begin{cases} i = 1,\dots,h^{1,1}_+, \\ k = 1,\dots,h^{1,2}_+. \end{cases}$$

  Because of the self-duality condition, only half of these degrees of freedom are independent, namely $h^{1,1}_+$ real scalars θ_i and $h^{1,2}_+$ real vectors V^k_μ.

In the fermionic sector, only a linear combination of the dilatinos and a linear combination of the gravitinos is invariant under the projection, which in fact hints at a particle content with one supersymmetry.

To conclude, given the particle content above, one can eventually individuate a unique general way to make up 4-dimensional $N_4 = 1$ multiplets. This is summarised in table 2.2.

multiplet	number	bos. field
gravity	1	$g_{\mu\nu}$
chiral	1	(C_0, ϕ)
chiral	$h^{1,2}_-$	u^α

multiplet	number	bos. field
chiral	$h^{1,1}_+$	(t^i, θ_i)
chiral	$h^{1,1}_-$	(b^ι, c^ι)
vector	$h^{1,2}_+$	V^k_μ

Table 2.2: $N_4 = 1$ massless spectrum from type IIB Calabi-Yau orientifold compactifications.

The volume of the extra-dimensions can be written as

$$\mathrm{vol}\,\mathrm{CY}_3 = \frac{\mathrm{i} l_s^6}{6} \int_{\mathrm{CY}_3} \omega_{1,1} \wedge \omega_{1,1} \wedge \omega_{1,1} = \frac{l_s^6}{6}\, t^i t^j t^l\, k_{ijk} = l_s^6 \ell_{(0)} \mathcal{V}, \tag{2.2.4}$$

where $\mathcal{V}$ captures the field-dependence given by the Kähler moduli t^i, the string scale l_s is the internal-space scale and $\ell_{(0)}$ is a constant depending on the extremes of integration.

2.2.3.2.2 Effective Supergravity Formulation

In view of the dimensional reductions of the fields in paragraph 2.2.3.2.1, it is possible to dimensionally reduce the Einstein-frame type IIB action of eq. (2.2.2). This can be done by allowing for the presence of non-trivial background 3-form fluxes H_3 and F_3. This is compatible with the generation of a warp factor, but the latter is not going to be considered here for simplicity. For a discussion of warping, see subsection 2.2.4. The 4-dimensional Einstein-frame metric is parametrised, with respect to the 10-dimensional Einstein-frame metric $ds_{10}^2 = \hat{g}_{MN}\mathrm{d}x^M\mathrm{d}x^N$, as

$$ds_{10}^2 = \mathrm{e}^{-6(u-\langle u\rangle)}g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + \mathrm{e}^{2u}g_{mn}\mathrm{d}y^m\mathrm{d}y^n,$$

where u is a field parametrising the internal volume as $\mathrm{vol}\,\mathrm{CY}_3 = \mathrm{e}^{6u}l_s^6\ell_{(0)}$, where $\ell_{(0)}$ is some numerical constant. With these choices, one finds a Einstein-Hilbert term defining a 4-dimensional Planck mass $m_P^2 = 4\pi\,\mathrm{e}^{\langle 6u\rangle}\ell_{(0)}/g_s^2l_s^2$.

The calculations for the dimensional reduction are quite lengthy,[2.13] and they can be found in ref. [52]. Remarkably, it turns out that the dimensionally-reduced action can be reproduced exactly and elegantly in the formalism of $N_4 = 1$ supergravity, as expected. A review of the latter can be found in appendix C. Although a general analysis is possible, it is easier to restrict the discussion to the case where $h_-^{1,1} = 0$, which is enough to capture all the essential elements of the theory. Along with the axio-dilaton $\tau = C_0 + \mathrm{i}\mathrm{e}^{-\phi}$ and the complex-structure moduli u^α, the Kähler coordinates must be defined as

$$T_i = \frac{3\mathrm{i}}{2}\,\theta_i + \frac{3}{4}\,k_{ijk}t^jt^k.$$

In this context, the superfields T_i are also called Kähler moduli. Then, apart from the gauge sector, the complete supergravity action is reproduced by the Kähler and super-potentials

$$\kappa_4^2\hat{K} = -\ln\left[-\mathrm{i}(\tau-\overline{\tau})\right] - \ln\left[-\mathrm{i}\int_{\mathrm{Y}_6}\Omega(u)\wedge\overline{\Omega}(\overline{u})\right] - 2\ln\mathcal{V} - \ln(4\pi[\ell_{(0)}]^2) \tag{2.2.5a}$$

$$\kappa_4^3\hat{W} = \frac{g_s}{l_s^2}\int_{\mathrm{CY}_3}G_3(\tau)\wedge\Omega(u), \tag{2.2.5b}$$

where $G_3 = F_3 - \tau H_3$ is the complexified 3-form flux. The volume-term $\mathcal{V}$ provides an implicit dependence on the moduli T_i by inverting its definition in terms of the moduli t^i. Unfortunately this is an impossible task in general; for compactifications with $h^{1,1} = 1$, however, one can define $T = 3\mathrm{i}\theta/2 + 3k_{111}t^2/4 = (243\ell_{(0)}^2/16k_{111})^{1/3}(-\mathrm{i}\rho)$, with $\rho = \chi + \mathrm{i}\mathrm{e}^{4u}$ being the single Kähler modulus controlling the internal volume.[2.14] The gauge sector is described entirely by means of the gauge kinetic functions $f_{kl} = (\mathrm{i}/2)\,\partial Y_k/\partial X^l$, with $X^k = \int_{\mathrm{CY}_3}\Omega\wedge\beta^k$ and $Y_k = \int_{\mathrm{CY}_3}\Omega\wedge\alpha_k$.

[2.13] The dimensional reduction of the axio-dilaton and of the volume modulus, in a compactification with $h_+^{1,1} = 1$, are discussed in detail, in a more general context, in appendix B.

[2.14] The factors can be worked out by knowing that the superfield ρ must be proportional to the superfield T and by taking advantage of the identities $k_{111}t^3/6 = \ell_{(0)}\mathcal{V} = \ell_{(0)}\mathrm{e}^{6u}$.

2.2.3.2.3 Vacuum Solutions and Imaginary Self-Dual Flux Background

The $N_4 = 1$ supergravity formulation in terms of a Kähler potential and a superpotential allows one way address the study of two crucial issues: the characteristics of the vacuum state of the system, i.e. the moduli space of the theory, and, more in detail, the corresponding supersymmetry properties.

A general analysis is possible, but for pedagogical reasons it is sufficient to consider the case where $h^{1,1} = 1$, working in terms of the field ρ, for which $-2\ln\mathcal{V} = -3\ln[-\mathrm{i}(\rho-\overline{\rho})]+\ln 8$, and knowing that $\nabla_{u^\alpha}\Omega = \nabla_\alpha\Omega = \mathrm{i}\chi_\alpha$, the Kähler-covariant derivatives read

$$\hat{\nabla}_\alpha \hat{W} = -\frac{\mathrm{i}g_s}{\kappa_4^3 l_s^2}\int_{\mathrm{CY}_3} G_3 \wedge \chi_\alpha, \tag{2.2.6a}$$

$$\hat{\nabla}_\tau \hat{W} = \frac{\mathrm{i}g_s}{\kappa_4^3 l_s^2}\frac{1}{[-\mathrm{i}(\tau-\overline{\tau})]}\int_{\mathrm{CY}_3} \hat{\overline{G}}_3 \wedge \Omega, \tag{2.2.6b}$$

$$\hat{\nabla}_\rho \hat{W} = \frac{-3\mathrm{i}\hat{W}}{[-\mathrm{i}(\rho-\overline{\rho})]}. \tag{2.2.6c}$$

In fact, because the Kähler metric for the Kähler modulus reads $\kappa_4^2\hat{K}_{\rho\overline{\rho}} = 3/[-\mathrm{i}(\rho-\overline{\rho})]^2$, it is easy to see that the F-term scalar potential undergoes a net cancellation of the negative-definite term proportional to the gravitino mass. This is the well-known no-scale structure. In fact, one finds

$$\begin{aligned}\hat{V}_F &= \mathrm{e}^{\kappa_4^2\hat{K}}\big(\hat{K}^{\alpha\overline{\beta}}\hat{\nabla}_\alpha\hat{W}\hat{\overline{\nabla}}_{\overline{\beta}}\hat{\overline{W}} + \hat{K}^{\tau\overline{\tau}}\hat{\nabla}_\tau\hat{W}\hat{\overline{\nabla}}_{\tau}\hat{\overline{W}} + \hat{K}^{\rho\overline{\rho}}\hat{\nabla}_\rho\hat{W}\hat{\overline{\nabla}}_{\rho}\hat{\overline{W}} - 3\kappa_4^2\hat{W}\hat{\overline{W}}\big)\\ &= \mathrm{e}^{\kappa_4^2\hat{K}}\big(\hat{K}^{\alpha\overline{\beta}}\hat{\nabla}_\alpha\hat{W}\hat{\overline{\nabla}}_{\overline{\beta}}\hat{\overline{W}} + \hat{K}^{\tau\overline{\tau}}\hat{\nabla}_\tau\hat{W}\hat{\overline{\nabla}}_{\tau}\hat{\overline{W}}\big),\end{aligned} \tag{2.2.7}$$

which eventually means the Kähler modulus ρ is not trivially stabilised. This is a crucial issue of type IIB compactifications, since this no-scale structure is an intrinsic obstacle and there are only a few (and still currently debated) scenarios, involving non-perturbative and perturbative effects [54,55], which are argued to break it in such a way that ρ is consistently stabilised.[2.15] The compactification volume should always be large, in string units, in order to consistently rely on the effective-theory approximation neglecting higher-derivative interactions. On the other hand, the leftover potential can be minimised straightforwardly at the value $\langle\hat{V}_F\rangle = 0$, which corresponds to the conditions

$$\int_{\mathrm{CY}_3}\langle G_3\rangle \wedge \langle\chi_\alpha\rangle = 0, \tag{2.2.8a}$$

$$\int_{\mathrm{CY}_3}\langle\hat{\overline{G}}_3\rangle \wedge \langle\Omega\rangle = 0. \tag{2.2.8b}$$

In fact, the Kähler metric are symmetric and positive-definite, therefore the minimum of the potential is fixed at zero when the Kähler-covariant derivatives for the complex-structure and axio-dilaton fields are zero. Therefore, the vacuum expectation values for the complex-structure moduli $\langle u^\alpha\rangle$ and for the axio-dilaton $\langle\tau\rangle$ are fixed as the solutions of eqs. (2.2.8a,

[2.15] An instance capturing all the essential elements of a KKLT-like [54] compactification is going to be discussed in subsection 4.4.3.

2.2.8b). For the same reason, it is immediate to conclude that the F-terms for these fields are also zero, i.e. $\langle F^\alpha \rangle = \langle \mathrm{e}^{\kappa_4^2 \hat{K}/2} \hat{K}^{\alpha\overline{\beta}} \hat{\overline{\nabla}}_{\overline{\beta}} \hat{\overline{W}} \rangle = 0$ and $\langle F^\tau \rangle = \langle \mathrm{e}^{\kappa_4^2 \hat{K}/2} \hat{K}^{\tau\overline{\tau}} \hat{\overline{\nabla}}_{\overline{\tau}} \hat{\overline{W}} \rangle = 0$, which means that these fields do not break supersymmetry. Whether or not supersymmetry is actually broken spontaneously or not in the vacuum depends on the details of how the Kähler modulus is stabilised and on the features of the other quantities that are present in the full theory, such as branes and anti-branes.

The 3-form flux G_3 on a Calabi-Yau threefold modded by an orientifold belongs to the cohomology class H^3_-, according to the definition $G_3 = F_3 - \tau H_3$. The fact they are harmonic stems from Bianchi identities and their field equations, and their σ-parity is dictated by the one of its components. For a given 3-form flux G_3, it is convenient to define its imaginary self-dual and imaginary anti-self-dual components G_3^- and G_3^+ as

$$G_3^\pm \equiv \frac{1}{2}\,(G_3 \pm \mathrm{i} *_6 G_3), \tag{2.2.9}$$

which satisfy the property $*_6 G_3^\pm = \mp \mathrm{i}\, G_3^\pm$, and so to decompose the 3-form flux G_3 as $G_3 = G_3^- + G_3^+$. In a Calabi-Yau threefold CY_3 it is possible to show that the holomorphic harmonic 3-form Ω and the harmonic $(2,1)$-forms χ_a are such that

$$*_6 \Omega = -\mathrm{i}\Omega, \tag{2.2.10a}$$

$$*_6 \chi_\alpha = +\mathrm{i}\chi_\alpha. \tag{2.2.10b}$$

This means that the imaginary self-dual component G_3^- of the 3-form flux must belong to the cohomology group $H^{0,3}_- \oplus H^{2,1}_-$, whereas the imaginary anti-self-dual component G_3^+ of the 3-form flux must belong to the cohomology group $H^{3,0}_- \oplus H^{1,2}_-$. More general compactifications require this point to be deal with less ease.[2.16] The vacuum conditions of eqs. (2.2.8a, 2.2.8b) require the 3-form flux to have only $(2,1)$- and $(0,3)$-components. This means that the 3-form flux must be imaginary self-dual.

2.2.4 Giddings-Kachru-Polchinski Compactifications

Based on ref. [56], this subsection describes the scenario arising within flux compactifications with a warped-metric Ansatz that is compatible with a 4-dimensional flat spacetime. In particular, it discusses how the warp factor is related to background fluxes in type IIB string theory. The 10-dimensional Einstein-frame line element $ds^2_{10} = \hat{g}_{MN} \mathrm{d}x^M \mathrm{d}x^N$ is parametrised to be

$$ds^2_{10} = \mathrm{e}^{2A(y)}\, \breve{g}_{\mu\nu}(x)\, \mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{e}^{-2A(y)}\, \breve{g}_{mn}(y)\, \mathrm{d}y^m \mathrm{d}y^n, \tag{2.2.11}$$

where the term e^{2A} is the so-called warp factor. This can generate a hierarchy between the 4-dimensional spacetime at a certain position y^m in the internal space and the generic bulk reference Planck scale, in a string-theory realisation of the Randall-Sundrum mechanism

[2.16]In a Calabi-Yau threefold, there cannot be a non-primitive harmonic $(2,1)$-form, so all harmonic $(2,1)$-forms are primitive and hence imaginary self-dual (and viceversa), whereas all $(3,0)$-forms are instead always primitive [56].

[57]. Basically, the idea is to study the field equations for type IIB string theory assuming such a warped-metric Ansatz in order to discuss its reliability.

The field content and the corresponding interactions are determined by the action of eq. (2.2.2). The dynamics of the volume-controlling modulus $u = u(x)$ is going to be ignored here, imagining to work with its expectation value; for details, see e.g. ref. [58]. In detail, the setup under consideration can be described as follows.

- The metric is assumed to be of the standard warped product kind with a conformally flat 4-dimensional spacetime, as introduced in eq. (2.2.11).
- As scalars cannot develop a field-strength in the non-compact directions due to Lorentz invariance, the axio-dilaton τ is allowed to vary only over the internal manifold, i.e.

$$\tau = \tau(y). \tag{2.2.12}$$

- Due to Lorentz invariance, the complex 3-form flux G_3 is assumed to have only internal indices, which means that

$$G_3 = \frac{1}{3!}\, G_{mnl}\, \mathrm{d}y^m \wedge \mathrm{d}y^n \wedge \mathrm{d}y^l. \tag{2.2.13}$$

- Due to Lorentz invariance, the self-dual 5-form flux $\tilde{F}_5$ is postulated to be of the form

$$\tilde{F}_5 = (1 + \hat{*}_{10})\, \mathrm{d}\alpha \wedge \check{\mathrm{vol}}_4, \tag{2.2.14}$$

  for some real scalar $\alpha = \alpha(y)$.

Further, the theory is assumed to include a set of localised sources described by an action S_σ. Such sources are a combination of D- and anti-D-branes as well as O- and anti-O-planes, compatibly with the Ansatz metric, as will be shown. In fact, branes are natural sources for closed-string sector background fields and, moreover, they happen to be crucial tools in type IIB model building, and similarly for orientifold planes. So, the total action is of the form

$$S = S_{\mathrm{IIB}} + S_\sigma. \tag{2.2.15}$$

2.2.4.1 4-dimensional Einstein-Equation Constraint

One can analyse the trace-reversed Einstein's equations to determine whether the warped-metric Ansatz is admissible. The trace-reversed 10-dimensional Einstein equations read

$$\hat{R}_{MN} = \hat{\kappa}_{10}^2 \left(\hat{T}_{MN} - \frac{1}{8}\, \hat{g}_{MN}\, \hat{T} \right), \tag{2.2.16}$$

In general, given an action $S = S[g, \phi^a]$, not including the Einstein-Hilbert term and depending on some fields ϕ^a, along with the 10-dimensional metric tensor g_{MN}, the energy-momentum tensor T_{MN} of the theory is defined as

$$\hat{T}_{MN}[\phi^a] \equiv -\frac{2}{\sqrt{-\hat{g}_{10}}} \frac{\delta S[\hat{g}, \phi^a]}{\delta \hat{g}^{MN}},$$

It is possible to relate the warp factor $A = A(y)$ to the total energy-momentum tensor of the theory. In particular, focussing on the purely 4-dimensional term, one can equate the Ricci-tensor components $\hat{R}_{\mu\nu}$ computed dynamically with those determined geometrically from the warped metric employing the standard definition.

- One can start with the Ricci tensor evaluated via Einstein equations. For the axio-dilaton τ, one can write in general

$$\hat{T}_{MN}[\tau] = \frac{1}{2\hat{\kappa}_{10}^2\,(\mathrm{Im}\,\tau)^2}\left[\partial_M\tau\,\partial_N\overline{\tau} - \frac{1}{2}\,\hat{g}_{MN}\,\partial_P\tau\,\partial^{\hat{P}}\overline{\tau}\right],$$

  and the contribution to the Ricci tensor turns out to be null, being $\Delta\hat{R}_{\mu\nu}[\tau] = 0$, as can be checked with the assumption that $\tau = \tau(y)$. For the 3-form flux G_3, one finds

$$\hat{T}_{MN}[G_3] = \frac{1}{4\hat{\kappa}_{10}^2\,\mathrm{Im}\,\tau}\left[G_{MM_2M_3}\overline{G}_N{}^{\hat{M}_2\hat{M}_3} - \frac{1}{6}\,\hat{g}_{MN}\,G_{M_1M_2M_3}\overline{G}^{\hat{M}_1\hat{M}_2\hat{M}_3}\right],$$

  and, considering the fact that this form has only internal components, the contribution of the flux to the 4-dimensional Ricci tensor is readily seen to be

$$\Delta\hat{R}_{\mu\nu}[G_3] = -\frac{1}{48\,\mathrm{Im}\,\tau}\,\hat{g}_{\mu\nu}\,G_{mnp}\overline{G}^{\hat{m}\hat{n}\hat{p}}.$$

  For the self-dual 5-form the generic energy-momentum tensor components are

$$\hat{T}_{MN}[\tilde{F}_5] = \frac{1}{96\hat{\kappa}_{10}^2}\left[\tilde{F}_{MPQRS}\tilde{F}_N{}^{\hat{P}\hat{Q}\hat{R}\hat{S}} - \frac{1}{10}\,\hat{g}_{MN}\,\tilde{F}_{NPQRS}\tilde{F}^{\hat{N}\hat{P}\hat{Q}\hat{R}\hat{S}}\right].$$

  One can express the flux as $\tilde{F}_5 = \hat{\mathrm{vol}}_4 \wedge \mathrm{d}\varphi - \hat{*}_6\mathrm{d}\varphi$, defining $\mathrm{d}\varphi = \mathrm{e}^{-4A}\mathrm{d}\alpha$ for simplicity, and its only-nonzero components can be written as $\tilde{F}_{\mu\nu\rho\sigma m} = \hat{\epsilon}_{\mu\nu\rho\sigma}\,\partial_m\varphi$ and $\tilde{F}_{klpqr} = -\hat{g}^{mn}\,\hat{\epsilon}_{mklpqr}\,\partial_n\varphi$. The combinations $\tilde{F}_{\mu PQRS}\tilde{F}_\nu{}^{\hat{P}\hat{Q}\hat{R}\hat{S}} = -4!\,\hat{g}_{\mu\nu}\hat{g}^{mn}\,\partial_m\varphi\,\partial_n\varphi$ and $\tilde{F}_{NPQRS}\tilde{F}^{\hat{N}\hat{P}\hat{Q}\hat{R}\hat{S}} = 0$ indicate that the 4-dimensional components of the energy-momentum tensor are $\hat{T}_{\mu\nu} = 1/(96\hat{\kappa}_{10}^2)\,[-24\,\hat{g}_{\mu\nu}\hat{g}^{mn}\,\partial_m\varphi\,\partial_n\varphi]$. Also, one can easily observe that the trace of the full 10-dimensional energy-momentum tensor vanishes, i.e. $\hat{T}[\tilde{F}_5] = 0$. To conclude, the 4-dimensional Ricci tensor contribution is

$$\Delta R_{\mu\nu}[\tilde{F}_5] = -\frac{1}{4}\,\hat{g}_{\mu\nu}\hat{g}^{mn}\,\partial_m\varphi\,\partial_n\varphi = -\frac{1}{4}\,\mathrm{e}^{-8A}\,\hat{g}_{\mu\nu}\hat{g}^{mn}\,\partial_m\alpha\,\partial_n\alpha.$$

  One must also consider the localised source. In the end, summing up, the total 4-dimensional Ricci tensor as evaluated through Einstein equations can be written as

$$\begin{aligned}\hat{R}_{\mu\nu} = &-\frac{1}{4}\,\hat{g}_{\mu\nu}\left[\frac{1}{12\,\mathrm{Im}\,\tau}\,G_{mnp}\overline{G}^{\hat{m}\hat{n}\hat{p}} + \mathrm{e}^{-8A}\,\hat{g}^{mn}\,\partial_m\alpha\,\partial_n\alpha\right]\\ &+\hat{\kappa}_{10}^2\left(\hat{T}_{\mu\nu} - \frac{1}{8}\,\hat{g}_{\mu\nu}\,\hat{T}\right)_\sigma.\end{aligned} \tag{2.2.17}$$

- On the other hand, one has to determine the Ricci tensor for 10-dimensional metrics of the warped form in eq. (2.2.11). For a metric of the form $ds_{10}^2 = \hat{g}_{MN}\mathrm{d}x^M\mathrm{d}x^N =$

$\mathrm{e}^{2A(y)}\,\breve{g}_{\mu\nu}(x)\,\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{e}^{-2A(y)}\,\breve{g}_{mn}(y)\,\mathrm{d}y^m \mathrm{d}y^n$, the 4-dimensional components of the 10-dimensional Ricci tensor, computed with the metric $\hat{g}_{MN}$, can be written as

$$\begin{aligned} \hat{R}_{\mu\nu} &= \breve{R}_{\mu\nu} - \mathrm{e}^{4A}\,\breve{g}_{\mu\nu}\breve{\nabla}^{\breve{m}}\breve{\nabla}_m A \\ &= \breve{R}_{\mu\nu} - \frac{1}{4}\,\breve{g}_{\mu\nu}\big(\breve{\nabla}^{\breve{m}}\breve{\nabla}_m \mathrm{e}^{4A} - \mathrm{e}^{-4A}\,\breve{\nabla}^{\breve{m}}\mathrm{e}^{4A}\,\breve{\nabla}_m \mathrm{e}^{4A}\big), \end{aligned} \tag{2.2.18}$$

where $\breve{R}_{\mu\nu}$ is the Ricci tensor of the unwarped 4-dimensional metric $\breve{g}_{\mu\nu}$ and the covariant derivatives $\breve{\nabla}_m$ are with respect to the unwarped 6-dimensional internal metric $\breve{g}_{mn}$. Depending on the context, both expressions may be more or less useful.

Finally, the different definitions of the Ricci tensor can be compared in order to get a defining equation for the warp factor. By equating the dynamical calculation of eq. (2.2.17) and the geometric expression of eq. (2.2.18), one gets

$$\begin{aligned} -\frac{1}{4}\,\breve{g}_{\mu\nu}\,\breve{\nabla}^{\breve{m}}\breve{\nabla}_m \mathrm{e}^{4A} = -\breve{R}_{\mu\nu} &- \frac{1}{4}\,\breve{g}_{\mu\nu}\,\mathrm{e}^{-6A}\,\hat{\nabla}^{\hat{m}}\mathrm{e}^{4A}\,\hat{\nabla}_m \mathrm{e}^{4A} \\ &- \frac{1}{4}\,\mathrm{e}^{2A}\breve{g}_{\mu\nu}\left[\frac{1}{12\,\mathrm{Im}\,\tau}\,G_{mnp}\overline{G}^{\hat{m}\hat{n}\hat{p}} + \mathrm{e}^{-8A}\,\hat{g}^{mn}\,\partial_m\alpha\,\partial_n\alpha\right] \\ &+ \hat{\kappa}_{10}^2\left(\hat{T}_{\mu\nu} - \frac{1}{8}\,\mathrm{e}^{2A}\,\breve{g}_{\mu\nu}\,\hat{T}\right)_\sigma, \end{aligned}$$

where quantities are written in terms of the warped or unwarped internal metric for future convenience. In the end, the trace of this expression allows one to write

$$\begin{aligned} \breve{\nabla}^{\breve{m}}\breve{\nabla}_m \mathrm{e}^{4A} = \breve{R}_4 + \frac{1}{12\,\mathrm{Im}\,\tau}\,\mathrm{e}^{2A}\,G_{mnp}\overline{G}^{\hat{m}\hat{n}\hat{p}} + \mathrm{e}^{-6A}\left(\hat{\nabla}^{\hat{m}}\mathrm{e}^{4A}\,\hat{\nabla}_m \mathrm{e}^{4A} + \hat{\nabla}^{\hat{m}}\alpha\hat{\nabla}_m\alpha\right) \\ - \frac{\hat{\kappa}_{10}^2}{2}\,\mathrm{e}^{2A}\left(\hat{g}^{\mu\nu}\hat{T}_{\mu\nu} - \hat{g}^{mn}\hat{T}_{mn}\right)_\sigma. \end{aligned} \tag{2.2.19}$$

This equation is a necessary condition which the warp factor e^{4A} has to satisfy and it constrains the kind of flux and brane configurations which are compatible with a warped metric in a compact manifold. In particular, for flat 4-dimensional solutions, i.e. for $\breve{g}_{\mu\nu} = k\,\eta_{\mu\nu}$ for some constant k, with $\tilde{R} = 0$, one can observe that on a compact internal manifold the left-handside integral is null whereas, in the absence of localised sources, the right-handside is positive semidefinite. This implies that the 3-form fluxes must vanish and that the potential α and the warp factor e^{4A} must be constant. This is an alternative version of the Maldacena-Nuñez no-go theorem [59]. Interestingly, for simple warped flat solutions the source trace term must be negative so that the other terms can be positive and not vanishing. In particular, without such localised sources there can be no non-trivial warping.

2.2.4.1.1 Example: p-Brane Source Terms

As an example, one can consider a p-brane wrapping the 4-dimensional spacetime $X_{1,3}$ and a $(p-3)$-cycle Σ_{p-3} of the internal space Y_6. It can be shown (see also eqs. (2.3.2, 2.3.3)) that the worldvolume action of a p-brane at leading α'-order is of the form

$$S_{p\text{-brane}} = -\tau_p \int_{X_{1,3}\times\Sigma_{p-3}} \mathrm{d}^{p+1}\xi\,\mathrm{e}^{(p-3)\phi/4}\,\sqrt{-\det\hat{g}_{\alpha\beta}} + \mu_p \int_{X_{1,3}\times\Sigma_{p-3}} C_{p+1},$$

with $g_{\alpha\beta}$ and ϕ the pull-backs of the metric and of the dilaton on the worldvolume, for $\alpha = 0, 1, \dots, p$. Here, τ_p and μ_p are the tension and the RR-charge of the p-brane, respectively. For Dp- and anti-Dp-branes one has $\tau_p = \tau_{\mathrm{D}p}$, whereas for O$p$-planes one effectively finds $\tau_p = -2^{p-4}\tau_{\mathrm{D}p}$, with $\tau_{\mathrm{D}p} = 2\pi/g_s l_s^{p+1}$. For D$p$- and anti-D$p$-branes, the charge is $\mu_p = q\tau_{\mathrm{D}p}$, with $q = \pm 1$, whereas for Op-planes it is $\mu_p = -2^{p-4}\tau_{\mathrm{D}p}$. Here the notation is such that Op- indicates Op^--planes; Op^+-planes have opposite tension and charge and one may also define anti-O$p^\pm$-planes, with opposite charges.

Only the DBI-term depends on the metric and therefore it is the only relevant term for the energy-momentum tensor. In the static gauge $X^{\dot\mu} = \xi^\alpha \delta_\alpha^{\dot\mu}$ and $Y^{\dot m} = 0$, for $\dot\mu = \mu, m'$, with $\mu = 0, 1, 2, 3$ and $m' = 4, \dots, p$, and $\dot m = p+1, \dots, 9$, it can be written as

$$S_{\mathrm{DBI}}^{p\text{-brane}} = -\tau_p \int_{X_{1,3}\times Y_6} \mathrm{d}^{10}x\, \mathrm{e}^{(p-3)\phi/4} \sqrt{-\det \hat g_{\alpha\beta}}\, \sqrt{\det \hat g_{m'n'}}\, \frac{\delta(\Sigma_{p-3})}{l_s^{9-p}},$$

with $\hat g_{\dot m \dot n}$ representing the metric over the transverse space. This gives

$$\hat T_{MN} = -\frac{2}{\sqrt{-\hat g_{10}}}\,\frac{\delta S}{\delta \hat g^{MN}} = -\frac{2}{\sqrt{-\hat g_{10}}}\,\frac{\delta S}{\delta \hat g^{\alpha\beta}}\,\frac{\delta \hat g^{\alpha\beta}}{\delta \hat g^{MN}}.$$

In the static gauge, the metric pullback is $\hat g_{\alpha\beta} = \partial_\alpha X^M \partial_\beta X^N \hat g_{MN} = \hat g_{\mu\nu}\, \delta_\alpha^\mu \delta_\beta^\nu + \hat g_{m'n'}\, \delta_\alpha^{m'} \delta_\beta^{n'}$ and one immediately finds

$$\begin{aligned}\hat T_{\mu\nu} &= -\tau_p\, \mathrm{e}^{(p-3)\phi/4}\, \hat g_{\mu\nu}\, \delta(\Sigma_{p-3})/l_s^{9-p},\\ \hat T_{mn} &= -\tau_p\, \mathrm{e}^{(p-3)\phi/4}\, \hat g_{m'n'} \delta_m^{m'} \delta_n^{n'}\, \delta(\Sigma_{p-3})/l_s^{9-p}.\end{aligned}$$

In particular, one finds the combination

$$-\frac{1}{2}\big(\hat g^{\mu\nu}\hat T_{\mu\nu} - \hat g^{mn}\hat T_{mn}\big) = \frac{7-p}{2}\,\tau_p\, \mathrm{e}^{(p-3)\phi/4}\, \delta(\Sigma_{p-3})/l_s^{9-p}. \tag{2.2.20}$$

So, for positive-tension objects like Dp- and anti-Dp-branes of spatial dimension $p \le 7$, warped-metric solutions are not compatible with non-anti-de Sitter spacetimes. However, negative-tension objects like Op- and anti-Op-planes also exist and thus turn out to be natural sources for warped metrics even in scenarios with Minkowski or de Sitter vacuum.

2.2.4.2 5-form Field-Equation Constraint

One can now consider another constraint which arises from the low-energy effective action, i.e. the 5-form flux field equations. The action involving the 4-form C_4 is

$$S\,[C_4] = -\frac{1}{8\hat\kappa_{10}^2}\int_{X_{1,9}} \tilde F_5 \wedge \hat{*}_{10} \tilde F_5 - \frac{\mathrm{i}}{8\hat\kappa_{10}^2}\int_{X_{1,9}} \frac{1}{\operatorname{Im}\tau}\, C_4^+ \wedge G_3 \wedge \overline{G}_3 + S_\sigma[C_4]$$

where the presence of a generic localised source, which generally couples to the 4-form field C_4 through its pull-back in the Chern-Simons action, has been taken into account. In what follows, this will be considered as a p-brane, without specifying its details. The action variation with respect to the field C_4 is (the source electric coupling is halved because of the self-duality of the 4-form field-strength tensor [56])

$$\frac{\delta S\,[C_4]}{\delta C_4} = -\frac{1}{4\hat\kappa_{10}^2}\,\hat{*}_{10}\mathrm{d}^\dagger \tilde F_5 - \frac{\mathrm{i}}{8\hat\kappa_{10}^2}\,\frac{1}{\operatorname{Im}\tau}\, G_3 \wedge \overline{G}_3 + \frac{1}{2}\,\tau_{\mathrm{D3}}\, \hat{*}_{10} j_4^\sigma = 0,$$

with $\hat{*}_{10} j_4^\sigma$ accounting for the 6-form outcome of the functional derivation by the 4-form of the localised source action in terms of the D3-brane tension τ_{D3}, being j_4^σ the current density. The generic RR-coupling to the local source term reads (see eqs. (2.3.2, 2.3.3))

$$S_\sigma[C_4] = \tau_{\mathrm{D3}} \int_{X_{1,9}} \varphi_* C_4 \wedge \hat{*}_{10} j_4^\sigma,$$

with the tension τ_{D3} written explicitly for future convenience and the generic electric coupling μ_p being absorbed in the definition of the current density. In the case of 3-branes, this coupling is especially simple to express as a worldvolume integral. Because $\hat{*}_{10} \mathrm{d}^\dagger \tilde{F}_5 = -\mathrm{d}\tilde{F}_5$, one can write

$$\mathrm{d}\tilde{F}_5 = \frac{\mathrm{i}}{2\,\mathrm{Im}\,\tau} G_3 \wedge \overline{G}_3 - 2\hat{\kappa}_{10}^2 \tau_{\mathrm{D3}}\, \hat{*}_{10} j_4^\sigma. \tag{2.2.21}$$

In view of the identity $G_3 \wedge \overline{G}_3 = -2\mathrm{i}\,\mathrm{Im}\,\tau\, H_3 \wedge \tilde{F}_3$, which can be seen to correspond to the 5-form flux Bianchi identity. A few manipulations show the left handside to be $\mathrm{d}\tilde{F}_5 = \mathrm{e}^{-8A}\mathrm{d}\mathrm{e}^{4A} \wedge \hat{*}_6 \mathrm{d}\alpha + \mathrm{e}^{-4A} \hat{\Delta}\alpha\, \hat{\mathrm{vol}}_6$, i.e. $\mathrm{d}\tilde{F}_5 = \left(\mathrm{e}^{-8A} \hat{\nabla}^{\hat{m}} \mathrm{e}^{4A} \hat{\nabla}_m \alpha - \mathrm{e}^{-4A} \hat{\nabla}^{\hat{m}} \hat{\nabla}_m \alpha\right) \hat{\mathrm{vol}}_6$. It will be useful to make use of the identity $\hat{\nabla}^{\hat{m}} \hat{\nabla}_m \alpha = \mathrm{e}^{2A}\, \breve{\nabla}^{\breve{m}} \breve{\nabla}_m \alpha - \mathrm{e}^{-4A}\, \hat{\nabla}^{\hat{m}} \mathrm{e}^{4A} \hat{\nabla}_m \alpha$. In the right handside, the 3-form flux product can be recast into the form $G_3 \wedge \overline{G}_3 = -G_{mnp} (\hat{*}_6 \overline{G})^{\hat{m}\hat{n}\hat{p}}\, \hat{\mathrm{vol}}_6/3!$. For the source term, instead, it is assumed that the localised p-brane sources are wrapping the 4-dimensional spacetime, along with an internal $(p-3)$-dimensional cycles, and therefore they are such that their contribution can be generally arranged in the suitable form $\hat{*}_{10} j_4^\sigma = \rho_{(3)}^\sigma\, \hat{\mathrm{vol}}_6$. Summing up, eq. (2.2.21) gives

$$\breve{\nabla}^{\breve{m}} \breve{\nabla}_m \alpha = \frac{\mathrm{i}}{12\,\mathrm{Im}\,\tau}\, \mathrm{e}^{2A}\, G_{mnp} (\hat{*}_6 \overline{G})^{\hat{m}\hat{n}\hat{p}} + 2\,\mathrm{e}^{-6A}\, \hat{\nabla}^{\hat{m}} \mathrm{e}^{4A}\, \hat{\nabla}_m \alpha + 2\hat{\kappa}_{10}^2 \tau_{\mathrm{D3}}\, \mathrm{e}^{2A}\, \rho_{(3)}^\sigma. \tag{2.2.22}$$

2.2.4.2.1 Example: 3-Brane Source Terms

It is instructive to analyse the term $\hat{*}_{10} j_4^\sigma$ for the easiest case, i.e. that of 3-branes, such as D3- and anti-D3-branes as well as O3- and anti-O3-planes. The Chern-Simons coupling of the 4-form to the 3-brane reads

$$S_{\text{3-brane}}[C_4] = rq\tau_{\mathrm{D3}} \int_{W_{1,3}} (\varphi_* C_4) = rq\tau_{\mathrm{D3}} \int_{X_{1,9}} (\varphi_* C_4) \wedge a_6^{W_{1,3}},$$

where $W_{1,3}$ represents the 3-brane worldvolume, with $a_6^{W_{1,3}} = \delta(y) \hat{\mathrm{vol}}_6 / l_s^6$ being its Poincaré. Here, $q = \pm 1$ is the brane RR-charge and r accounts for the possible nature of the 3-brane, i.e. $r = 1$ for D3-/anti-D3-branes and $r = -1/4$ for O3-/anti-O3-planes [33]. In the static gauge, with $X^\mu = \xi^\alpha \delta_\alpha^\mu$ and $Y^m = 0$, the 4-form pull-back is $(\varphi_* C)_{\alpha\beta\gamma\delta} = \delta_\alpha^\mu \delta_\beta^\nu \delta_\gamma^\rho \delta_\sigma^\delta\, C_{\mu\nu\rho\sigma}$, so the functional derivative for the source term is

$$\frac{\delta}{\delta C_4} S_{\text{3-brane}}[C_4] = rq\tau_{\mathrm{D3}}\, \delta(y)\, \frac{\hat{\mathrm{vol}}_6}{l_s^6} = \tau_{\mathrm{D3}}\, j_6^{(3)} = \tau_{\mathrm{D3}}\, \rho_{(3)}^{\text{3-brane}} \frac{\hat{\mathrm{vol}}_6}{l_s^6}.$$

This means that D3- and anti-D3-branes give the source terms $\rho_{(3)} = q\, \delta(y)/l_s^6$, while O3- and anti-O3-planes give $\rho_{(3)} = -(q/4)\, \delta(y)/l_s^6$.

2.2.4.2.2 Tadpole Conditions

To conclude the discussion, it is worthwhile to briefly introduce some terminology and to discuss the emergence of tadpole conditions.

- The D3-brane charge from localised sources is the integral of the localised source term $\rho_{(3)}$ over the internal space, which is not wrapped, with a normalisation with respect to the D3-brane tension τ_{D3}. As the electric charge of a p-brane is the integral over a $(9-p)$-dimensional ball B_{9-p} that surrounds it, this is

$$Q_3^\sigma = \int_{Y_6} \hat{*}_{10} j_4^\sigma = \int_{Y_6} \rho_{(3)}^\sigma \, \hat{\mathrm{vol}}_6.$$

  For instance, D3- and anti-D3-branes have charges $Q_3 = q$, and similarly for O3-anti-O3-planes, with $Q_3 = -q/4$.

- The total D3-brane charge follows from the general definition of the electric charge of a p-brane as the integral over the border of the $(9-p)$-dimensional ball B_{9-p}, i.e. the $(8-p)$-dimensional sphere $\mathrm{S}^{8-p} = \partial \mathrm{B}_{9-p}$, of the Hodge-dual of the field-strength tensor F_{p+2}. As follows from the Bianchi identity of eq. (2.2.21), this is

$$Q_3 = -\frac{1}{2\hat{\kappa}_{10}^2 \tau_{\mathrm{D3}}} \oint_{\mathrm{S}^5} \hat{*}_{10} \tilde{F}_5 = -\frac{1}{2\hat{\kappa}_{10}^2 \tau_{\mathrm{D3}}} \int_{Y_6} \mathrm{d}\hat{*}_{10} \tilde{F}_5 = Q_3^\sigma - \frac{1}{2\hat{\kappa}_{10}^2 \tau_{\mathrm{D3}}} \int_{Y_6} H_3 \wedge \tilde{F}_3.$$

The integrated Bianchi identity indicates that the total D3-brane charge vanishes, i.e.

$$Q_3 = -\frac{1}{2\hat{\kappa}_{10}^2 \tau_{\mathrm{D3}}} \int_{Y_6} H_3 \wedge \tilde{F}_3 + Q_3^\sigma = 0.$$

Because the string-frame integrals of background fluxes over cycles are quantised, one can expand the harmonic 3-form fluxes in the cohomology class H^3 in terms of integer numbers, and the vanishing of the total D3-brane charge results in a restrictive condition on such integers [60]. This is an instance of an RR-tadpole condition. If the unwarped internal space is a Calabi-Yau orientifold, for some integers m^κ, e_κ, m^κ_{RR} and $e_{\mathrm{RR}\,\kappa}$, one can write

$$\begin{aligned} H_3 &= 4\pi^2 \alpha' (m^\kappa \alpha_\kappa - e_\kappa \beta^\kappa), \\ F_3 &= 4\pi^2 \alpha' g_s (m^\kappa_{\mathrm{RR}} \alpha_\kappa - e_{\mathrm{RR}\,\kappa} \beta^\kappa). \end{aligned}$$

In a model with a total number of N_{D3}, $N_{\overline{\mathrm{D3}}}$, N_{O3} and $N_{\overline{\mathrm{O3}}}$ of D3-branes, anti-D3-branes, O3-planes and anti-O3-planes, respectively, the requirement of total vanishing D3-brane charge reads

$$N_{\mathrm{D3}} - N_{\overline{\mathrm{D3}}} - \frac{1}{4} N_{\mathrm{O3}} + \frac{1}{4} N_{\overline{\mathrm{O3}}} = m^\kappa_{\mathrm{RR}}\, e_\kappa - e_{\mathrm{RR}\,\kappa}\, m^\kappa.$$

2.2.4.3 Giddings-Kachru-Polchinski Solutions

Given the conditions of eqs. (2.2.19) and (2.2.22) from the purely 4-dimensional Einstein's equations and the 5-form flux field equation can be combined in an enlightening way, after a simple subtraction one finds [61]

$$\begin{aligned}\breve{\nabla}^{\breve{m}}\breve{\nabla}_m(\mathrm{e}^{4A}-\alpha) = \breve{R}_4 &+ \frac{\mathrm{e}^{2A}}{24\,\mathrm{Im}\,\tau}\left[\mathrm{i}G_{mnp}-(\hat{*}_6 G)_{mnp}\right]\left[-\mathrm{i}\overline{G}^{\hat{m}\hat{n}\hat{p}}-(\hat{*}_6\overline{G})^{\hat{m}\hat{n}\hat{p}}\right]\\ &+\mathrm{e}^{-6A}\left[\hat{\nabla}_m(\mathrm{e}^{4A}-\alpha)\right]\left[\hat{\nabla}^{\hat{m}}(\mathrm{e}^{4A}-\alpha)\right]\\ &-2\hat{\kappa}_{10}^2\,\mathrm{e}^{2A}\left[\frac{1}{4}\left(\hat{g}^{\mu\nu}\hat{T}_{\mu\nu}-\hat{g}^{mn}\hat{T}_{mn}\right)_\sigma+\tau_{\mathrm{D3}}\,\rho^\sigma_{(3)}\right].\end{aligned}\tag{2.2.23}$$

This is a crucial equation for the study of warped compactifications. In fact, the left handside integrates to zero over a compact internal manifold, so the same must happen with the right-handside too. In a 4-dimensional Minkowski spacetime, with $\breve{R}_4=0$, in the right handside the flux terms are positive semidefinite, so in order to allow for warped metric solutions it is necessary that, globally, the localised terms must be such that the total source term is vanishing too. Such a condition, i.e. the identity

$$\frac{1}{4}\left(\hat{g}^{\mu\nu}\hat{T}_{\mu\nu}-\hat{g}^{mn}\hat{T}_{mn}\right)_\sigma+\tau_{\mathrm{D3}}\,\rho^\sigma_{(3)}=0 \tag{2.2.24}$$

is in principle always achievable if the theory includes the correct number of localised sources. If this is the case, then eq. (2.2.23) has two fundamental implications: the 3-form flux must be imaginary self-dual and the warp factor e^{4A} must be identified with the 5-form potential α, i.e.

$$\hat{*}_6 G_3 = \mathrm{i}G_3, \tag{2.2.25}$$

$$\mathrm{e}^{4A}=\alpha. \tag{2.2.26}$$

Note that for 3-forms on 6-dimensional spaces, the Hogde operator acts exactly in the same way for conformally equivalent metrics, so $\breve{*}_6 G_3=\hat{*}_6 G_3$. Importantly, as seen in subsubsection 2.2.3.2 an imaginary self-dual 3-form flux G_3 is the standard solution which guarantees the minimum of the potential energy in type IIB supergravity.

A couple of possible scenarios is discussed below.

- Let the sources be D3-branes and O3-planes. For D3-brane sources, following the discussions in paragraphs (2.2.4.1.1, 2.2.4.2.1), one finds

$$\frac{1}{4}\left(\hat{g}^{\mu\nu}\hat{T}_{\mu\nu}-\hat{g}^{mn}\hat{T}_{mn}\right)_{\mathrm{D3}}+\tau_{\mathrm{D3}}\,\rho^{\mathrm{D3}}_{(3)}=-\frac{7-3}{4}\,\tau_{\mathrm{D3}}\,\delta(\Sigma)+\tau_{\mathrm{D3}}\,\delta(\Sigma)=0.$$

  The same calculation holds for O3-planes, with just an irrelevant overall $-1/4$-factor.

- Let there be anti-D3-branes as localised sources. Then, the source term reads

$$\frac{1}{4}\left(\hat{g}^{\mu\nu}\hat{T}_{\mu\nu}-\hat{g}^{mn}\hat{T}_{mn}\right)_{\overline{\mathrm{D3}}}+\tau_{\mathrm{D3}}\,\rho^{\overline{\mathrm{D3}}}_{(3)}=-\frac{7-3}{4}\,\tau_{\mathrm{D3}}\,\delta(\Sigma)-\tau_{\mathrm{D3}}\,\delta(\Sigma)=-2\tau_{\mathrm{D3}}\,\delta(\Sigma),$$

  which contributes positively to the right-handside integral. So anti-D3-branes alone would not be adequate sources for allowing warped metrics in imaginary self-dual backgrounds. However, for instance, the source term for anti-O3-planes reads

$$\frac{1}{4}\left(\hat{g}^{\mu\nu}\hat{T}_{\mu\nu}-\hat{g}^{mn}\hat{T}_{mn}\right)_{\overline{\mathrm{O3}}}+\tau_{\mathrm{D3}}\,\rho^{\overline{\mathrm{O3}}}_{(3)}=\frac{1}{2}\tau_{\mathrm{D3}}\,\delta(\Sigma).$$

2.2.4.4 Other Field Equation Constraints

For completeness, this section reports the remaining field equations of the type IIB theory, showing they are compatible with the Giddings-Kachru-Polchinski solution. For simplicity, a flat 4-dimensional spacetime is assumed and localised sources are ignored.

One can show the general 10-dimensional type IIB field equations to be [46]

$$\hat{*}_{10}\mathrm{d}^\dagger\mathrm{d}\phi + \mathrm{e}^{2\phi}\, F_1 \wedge \hat{*}_{10}F_1 + \frac{1}{2}\,\mathrm{e}^{\phi}(\tilde{F}_3 \wedge \hat{*}_{10}\tilde{F}_3 - \mathrm{e}^{-2\phi} H_3 \wedge \hat{*}_{10}H_3) = 0, \tag{2.2.27}$$

$$\hat{*}_{10}\mathrm{d}^\dagger(\mathrm{e}^{2\phi}F_1) - \mathrm{e}^{\phi}H_3 \wedge \hat{*}_{10}\tilde{F}_3 = 0, \tag{2.2.28}$$

$$\mathrm{d}\hat{*}_{10}(\mathrm{e}^{-\phi}H_3 - \mathrm{e}^{\phi}C_0\,\tilde{F}_3) - \tilde{F}_3 \wedge \tilde{F}_5 - C_0 H_3 \wedge \tilde{F}_5 = 0, \tag{2.2.29}$$

$$\mathrm{d}\hat{*}_{10}(\mathrm{e}^{\phi}\tilde{F}_3) + H_3 \wedge \tilde{F}_5 = 0 \tag{2.2.30}$$

$$\mathrm{d}\hat{*}_{10}\tilde{F}_5 = H_3 \wedge \tilde{F}_3, \tag{2.2.31}$$

together with the Einstein equations

$$\begin{aligned}\hat{R}_{MN} = \frac{1}{2}\,\partial_M\phi\partial_N\phi + \frac{1}{2}\,\mathrm{e}^{2\phi}\partial_M C_0\partial_N C_0 + \frac{1}{4}\,\mathrm{e}^{-\phi}H_{MPQ}H_N{}^{\hat{P}\hat{Q}} + \frac{1}{4}\,\mathrm{e}^{\phi}\tilde{F}_{MPQ}\tilde{F}_N{}^{\hat{P}\hat{Q}} \\ -\frac{1}{48}\,g_{MN}\left(\mathrm{e}^{-\phi}H_{PQR}H^{\hat{P}\hat{Q}\hat{R}} + \mathrm{e}^{\phi}\tilde{F}_{PQR}\tilde{F}^{\hat{P}\hat{Q}\hat{R}}\right) + \frac{1}{96}\,\tilde{F}_{PQRS}\tilde{F}_N{}^{\hat{P}\hat{Q}\hat{R}\hat{S}}.\end{aligned} \tag{2.2.32}$$

Also, Bianchi identities (following directly from the definitions of the fields) are to be imposed too. All these equations undergo simplifications under the fundamental GKP-assumptions of eqs. (2.2.11-2.2.14).

- According to subsubsection 2.2.4.3, the 4-dimensional components of the Einstein equations and the 5-form flux field equation, in eqs. (2.2.32) and (2.2.31), respectively, can be combined together into the defining GKP-condition of eq. (2.2.23).

- The dynamical conditions for the scalars and ϕ and C_0 are in eqs. (2.2.27) and (2.2.28), respectively. These can be combined to give an equation for the axio-dilaton τ, of course. A few maniupluations, under the GKP-assumptions of eqs. (2.2.11-2.2.14), show the axio-dilaton field equation to be

$$\hat{\nabla}^{\hat{m}}\hat{\nabla}_m\tau + \frac{\mathrm{i}}{\mathrm{Im}\tau}\,(\hat{\nabla}^{\hat{m}}\tau)(\hat{\nabla}_m\tau) + \frac{\mathrm{i}}{6}\,(G_3^-)_{mnp}(G_3^+)^{\hat{m}\hat{n}\hat{p}} = 0. \tag{2.2.33}$$

  This fixes the axio-dilaton profile $\tau = \tau(y)$. In this case, however, it would be important to insert the terms from the D7-brane sources.

- To discuss the field equations for the 2-forms B_2 and C_2, i.e. eqs. (2.2.29) and (2.2.30), respectively, it is customary to define a 3-form $\Lambda_3 \equiv \mathrm{e}^{4A}\hat{*}_6 G_3 - \mathrm{i}\alpha G_3$. In this way, the field equations read

$$\mathrm{d}\Lambda_3 + \frac{\mathrm{i}}{2\,\mathrm{Im}\,\tau}\,\mathrm{d}\tau \wedge (\Lambda_3 + \overline{\Lambda}_3) = 0. \tag{2.2.34}$$

  This equation is trivially satisfied in a GKP-background.

- As far as the Einstein equations are concerned, it is necessary to analyse those with mixed 4-dimensional/internal and purely internal indices. The former can be easily seen to give $R_{\mu m} = 0$, both in terms of the energy-momentum tensor, as given by eq. (2.2.32), and in purely geometrical terms with a warped metric of the form in eq. (2.2.11). On the other hand, Einstein equations for purely internal directions are not trivial. A few lengthy manipulations eventually give the condition on the unwarped-metric Ricci tensor [62]

$$\begin{aligned}\breve{R}_{mn} = \frac{1}{2(\mathrm{Im}\,\tau)^2}\,\partial_m\tau\partial_n\overline{\tau} + \frac{1}{2}\,\mathrm{e}^{-8A}\,\partial_m\mathrm{e}^{4A}\partial_n\mathrm{e}^{4A} - \frac{1}{2}\,\mathrm{e}^{-8A}\partial_m\alpha\partial_n\alpha \\ + \frac{1}{4\mathrm{Im}\,\tau}\left(G_{mpq}\overline{G}_n{}^{\breve{p}\breve{q}} - \frac{1}{6}\,\breve{g}_{mn}\,\mathrm{e}^{-2A}\,G_{pqr}\overline{G}^{\breve{p}\breve{q}\breve{r}}\right).\end{aligned} \tag{2.2.35}$$

  Again, one should also insert terms from the D7-brane sources.

2.2.4.5 Warped Solutions at Deformed Conifolds

There exist flux configurations that can be shown to generate warped throats, i.e. local deformations of the bulk space where the warping is significative. Therefore, the tip of such warped throats is an ideal location for some D-branes on which to construct standard-like models.

A useful starting point to understand such a construction is the analysis of D3-brane configuations of ref. [63]. In the absence of fluxes, in the vicinity of N coincident D3-branes, the warp factor is approximately $\mathrm{e}^{-4A(\breve{r})} \simeq 4\pi g_{\mathrm{s}}N/\breve{r}^4$, where $\breve{r}$ is the distance from the stack of D3-branes measured by the unwarped metric $\breve{g}_{mn}$. In the vicinity of the D3-branes the geometry is of the form $\mathrm{AdS}_5 \times \mathrm{S}^5$. To achieve a setup with a large but finite hierarchy in the supposedly observable sector, the D3-branes should be slightly separated from the others, giving a tiny but still finite hierarchy, however in the present model there is no mechanism to generate and keep stable this kind of separation.

For a more realistic construction, one can introduce the presence of background fluxes. In particular, following the original construction of the Klebanov-Strassler throat in ref. [64], which is then specialised to compact setups in ref. [56], one can consider the internal Calabi-Yau threefold to locally host a deformed 3-dimensional conifold. As discussed in ref. [65], a conifold is a singular space which is smooth everywhere but for a number of isolated conical singularities and with vanishing first Chern class (after deleting the singular points). In a neighbourhood of the singular point, the conifold can be expressed as the subspace of $\mathbb{C}^4$ such that $w_1^2+w_2^2+w_3^2+w_4^2 = 0$. Such a 3-dimensional conifold singularity is a cone with a 5-dimensional basis given by the product of a 3- and a 2-sphere, i.e. $\mathrm{T}_{1,1} = \mathrm{S}^3 \times \mathrm{S}^2$. The so-called deformed conifold is one of the two possible Calabi-Yau manifolds which result from smoothing the conifold singularity. The deformed conifold is described in a neighbourhood of the used-to-be singularity as the submanifold of $\mathbb{C}^4$ such that

$$w_1^2 + w_2^2 + w_3^2 + w_4^2 = z$$

for some complex parameter z. In the deformed conifold the 3-sphere S^3 is shrunk to a finite size whose volume is controlled by the parameter z. It can be shown that z is a complex-structure modulus. In a compact Calabi-Yau orientifold with the Hodge numbers $h^{1,1} = h^{1,1}_{+} = 1$ and $h^{1,2} = h^{1,2}_{-} = 1$, i.e. with a single Kähler modulus and a single complex-structure modulus, where the geometry is that of the deformed conifold with the complex-structure modulus being z, there exists a number $2 + 2\,h^{1,2} = 4$ of 3-cycles: the 3-sphere $A = \mathrm{S}^3$ and its B-dual, with one single intersection, plus two more cycles A' and B', with one single intersection too. Formally, the cycle B would be non-compact in the local conifold model, but is actually compact when the throat is embedded in the full Calabi-Yau threefold model, with a cutoff radial distance from the tip of the cone would-be singularity. Let there be M units of flux F_3 on the 3-sphere A and K units of flux H_3 on the cycle B, in such a way that

$$\int_A F_3 = \int_{\mathrm{CY}_3} F_3 \wedge \beta = 4\pi^2\alpha' g_s M,$$
$$\int_B H_3 = \int_{\mathrm{CY}_3} H_3 \wedge \alpha = 4\pi^2\alpha' K.$$

In other words, these fluxes may be expanded as $F_3 = 4\pi^2\alpha' g_s M\alpha$ and $H_3 = -4\pi^2\alpha' K\beta$. It can be proven that the complex structure of the conifold z is the period on the collapsing 3-sphere A, i.e.

$$z = \int_A \Omega = \int_{\mathrm{CY}_3} \Omega \wedge \beta,$$

and that the dual period is

$$\int_B \Omega = \int_{\mathrm{CY}_3} \Omega \wedge \alpha = G(z) = -\frac{\mathrm{i}z}{2\pi} \ln z + h(z),$$

where $h(z)$ is a holomorphic function. Therefore, the supergravity formulation of this theory is determined by the Kähler and super-potentials [66]

$$\kappa_4^2 \hat{K} = -\ln\left[-\mathrm{i}\int \Omega \wedge \overline{\Omega}\right] - \ln\left[-\mathrm{i}(\tau - \overline{\tau})\right] - 3\ln\left[-\mathrm{i}(\rho - \overline{\rho})\right] + C,$$
$$\kappa_4^3 \hat{W} = \frac{g_s}{l_s^2}\int \Omega \wedge G_3 = -g_s[K\tau z - g_s M G(z)],$$

for some constant C. The vacuum of this system is fixed by the requirements that the covariant derivatives $\hat{\nabla}_z \hat{W}$ and $\hat{\nabla}_\tau \hat{W}$ be zero.

- On the one hand, from the complex-structure modulus z we find in the first place the minimisation condition

$$\hat{\nabla}_z \hat{W} = -\frac{g_s^2}{\kappa_4^3}\left[\frac{K\tau}{g_s} - M\left(-\frac{\mathrm{i}\ln z}{2\pi} - \frac{\mathrm{i}}{2\pi} + \partial_z h\right) + \kappa_4^2 \partial_z \hat{K}\left[\frac{K\tau z}{g_s} - M\left(-\frac{\mathrm{i}z}{2\pi}\ln z + h\right)\right]\right] = 0.$$

  Taking z to be real, positive and small, and assuming the ratio K/g_s to be large, the vacuum condition can be written as

$$-\frac{\mathrm{i}M}{2\pi}\ln\langle z\rangle - \frac{\mathrm{i}K}{g_s} = 0.$$

  This means that the vacuum expectation value of the complex-structure modulus is exponentially small, being given by

$$\langle z\rangle = \mathrm{e}^{-2\pi K/M g_\mathrm{s}}.$$

- On the other hand, the axio-dilaton minimisation condition

$$\hat{\nabla}_\tau \hat{W} = \frac{g_s^2}{\kappa_4^3} \frac{\mathrm{i}}{[-\mathrm{i}(\tau - \overline{\tau})]} \left[\frac{Kz}{g_s} \overline{\tau} - M\left(-\frac{\mathrm{i}z}{2\pi} \ln z + h \right) \right] = 0$$

  cannot be solved consistently as for an exponentially small value $\langle z \rangle$. A natural generalisation of the theory consists in a setup with fluxes on both the cycles A, B and A', B'. This can be shown to render the minimisation of τ possible.

As an important remark, it should be noticed that one can argue that this mechanism can be destabilised by the presence of anti-D3-branes, which are usually implemented in semi-realistic models in order to uplift the otherwise anti-de Sitter solutions that are typical of many string vacua. This can lead to the re-emergence of the conifold singularity. Specific hierarchies among the flux integers may help against the destabilisation, but, due to a tension with the tadpole cancellation conditions, such a solution tends to reduce considerably the hierarchy that would be created by a tiny complex-structure modulus [67, 68].

In principle, the analytic behaviour of the warp factor can be found by solving eq. (2.2.19). However, some arguments can be made in order to assess the warp factor in the vicinity of a stack of D3-branes at the tip of the throat, as explained by ref. [56]. Useful are also the reviews in refs. [36, 48]. On general grounds any Dp-brane interacts with closed strings via the interplay between its open and closed strings. This creates an effective supergravity background. It is possible to prove that, in type IIB string theory, the supergravity solution for N coincident Dp-branes in flat spacetime, with $p = 1, 3, 5$, is

$$ds_{10}^2 = Z(\breve{r})^{-\frac{1}{2}}\, \eta_{\dot{\mu}\dot{\nu}}\, \mathrm{d}x^{\dot{\mu}} \mathrm{d}x^{\dot{\nu}} + Z(\breve{r})^{\frac{1}{2}}\, (\mathrm{d}\breve{r}^2 + \breve{r}^2 \mathrm{d}\Omega_{8-p}^2),$$

where the index $\dot{\mu}$ spans the worldvolume directions, whereas $\mathrm{d}l_{\mathbb{R}^6}^2 = \mathrm{d}\breve{r}^2 + \breve{r}^2 \mathrm{d}\Omega_{8-p}^2$ is just the Euclidean metric in the transverse directions. The warp factor is $Z(\breve{r}) = 1 + (\rho/\breve{r})^{7-p}$, with the characteristic size $\rho^{7-p} = (4\pi)^{(5-p)/2}\, N g_{\mathrm{s}}\, \alpha'^{(7-p)/2}\, \Gamma[(7-p)/2]$, and $\breve{r}$ is the radial coordinate in the transverse space to the worldvolume of the Dp-brane. In the case of a stack of D3-branes, the metric takes the form

$$ds_{10}^2 = Z(\breve{r})^{-\frac{1}{2}}\, \eta_{\mu\nu}\, \mathrm{d}x^\mu \mathrm{d}x^\nu + Z(\breve{r})^{\frac{1}{2}}\, (\mathrm{d}\breve{r}^2 + \breve{r}^2 \mathrm{d}l_{\mathrm{S}^5}^2),$$

where the warp factor is

$$Z(\breve{r}) = 1 + \frac{\rho^4}{\breve{r}^4},$$

with the characteristic size $\rho = (4\pi g_{\mathrm{s}}\, N \alpha'^2)^{1/4}$. On the one hand, the metric tends asymptotically to flat 10-dimensional spacetime $\mathbb{R}^{1,9}$ as the radial coordinate increases, i.e. as $\breve{r} \sim \infty$. On the other hand, close to the D3-brane, i.e. in the so-called near-horizon limit $\breve{r} \sim 0$, one gets the special structure $\mathrm{AdS}_5 \times \mathrm{S}^5$, being

$$ds_{10}^2 \overset{r \sim 0}{\simeq} \frac{\breve{r}^2}{\rho^2}\, \eta_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu + \frac{\rho^2}{\breve{r}^2} \mathrm{d}\breve{r}^2 + \rho^2 ds_{\mathrm{S}^5}^2.$$

In fact, the 4-dimensional directions and the radial coordinate combine to form a 5-dimensional anti-de Sitter spacetime AdS_5, whereas the remaining coordinates correspond to a 5-sphere

S^5. A sketch is reported in fig. 2.2 (see e.g. refs. [48, 69]).

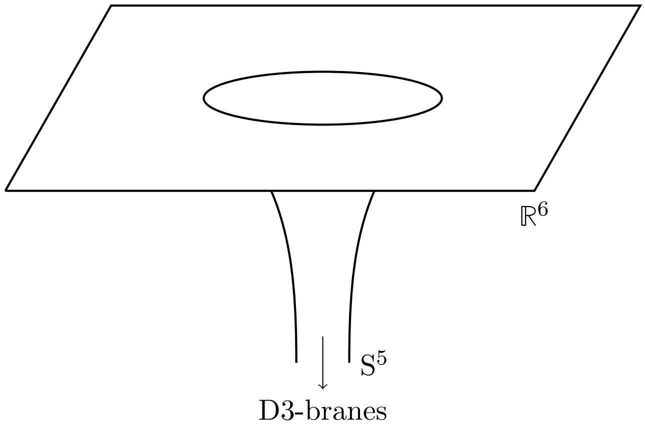

Figure 2.2: A scheme of the throat geometry in the internal dimensions.

In the case of the deformed conifold, in the minimal case one considers the presence of M units of F_3-flux on the 3-sphere $A = S^3$ and K units of H_3-flux on the dual cycle B, and then the 5-form integrated Bianchi identity gives the identification $N_{D3} = MK$. The unwarped metric is the deformed conifold metric $dl_6^2 = \mathrm{d}\breve{r}^2 + \breve{r}^2 ds^2_{T_{1,1}}$ and the actual supergravity solution can be argued to be

$$ds^2_{10} = Z(\breve{r})^{-\frac{1}{2}}\, \eta_{\mu\nu}\, \mathrm{d}x^\mu \mathrm{d}x^\nu + Z(\breve{r})^{\frac{1}{2}}\, (\mathrm{d}\breve{r}^2 + \breve{r}^2 \mathrm{d}l^2_{T_{1,1}}),$$

with $\rho = (4\pi g_s\, MK\alpha'^2)^{1/4}$. This approximation holds as long as the radial distance $\breve{r}$ from the singularity is bigger than the 3-sphere characteristic size $\langle z\rangle^{1/3}$: closer to the would-be singularity, the throat is cut off. In other words, moving towards the position $\breve{r} = 0$, the minimal would-be distance from the D3-branes must be of the same order of magnitude as the only physical size into play, i.e. the size $\langle z\rangle^{1/3}$ of the 3-sphere S^3. A sketch of the situation is reported in fig. 2.3 (see e.g. ref. [70]).

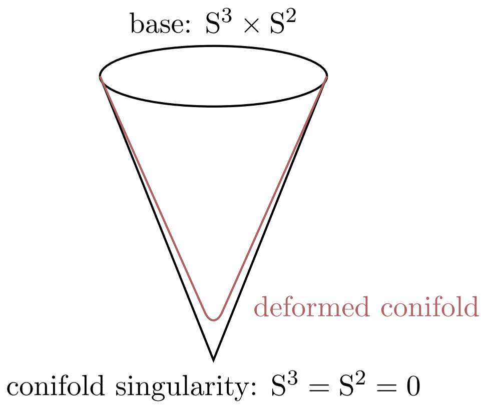

Figure 2.3: A scheme of the deformed conifold geometry.

So, in such a scenario the minimum value that the warp factor reaches is approximately

$$\mathrm{e}^{A_0} \sim \langle z \rangle^{\frac{1}{3}} \sim \mathrm{e}^{-2\pi K/3Mg_s}. \tag{2.2.36}$$

In other words, around the tip of the throat, all mass hierarchies turn out to be exponentially redshifted down to small values via a Randall-Sundrum-like mechanism.

2.3 D-Branes

Ever since the fundamental discovery that D-branes are intrinsic elements of type II string theories [71], they have been essential in a huge variety of studies. This section overviews the basic properties of D-branes as seen from the perturbative spectrum of open strings. For definiteness, the discussion is referred to type IIB theories.

2.3.1 D-Brane Massless Spectra

The Fock-space description of open strings in section 2.1 allows for an easy interpretation of the massless spectrum of a Dp-brane. An open string with NN-boundaries along the directions $\alpha = 0, \dots, p$ and DD-boundaries along the transverse $\dot{m} = p+1, \dots, 9$ defines a Dp-brane. Assuming the lightcone gauge-fixed directions to be NN-ones, the mass formula of eq. (2.1.40) identifies the following massless states.

- In the NS-sector, the massless states are $b^{\alpha}_{-1/2}|\mathrm{NS}\rangle$ and $b^{\dot{m}}_{-1/2}|\mathrm{NS}\rangle$, which correspond to a vector field A_α, with $p-1$ dynamical degrees of freedom, and to $9-p$ real scalars $\varphi^{\dot{m}}$, respectively. All massless bosons are preserved by the GSO-projection, unlike the tachyon $|\mathrm{NS}\rangle$.

- In the R-sector, the GSO-invariant massless state is $|\mathrm{R}_+\rangle$, which corresponds to 8 physical fermionic degrees of freedom. In view of a 4-dimensional formulation in terms of Weyl spinors, these can be arranged as four fields ψ^ω, with $\omega = 0, 1, 2, 3$.

This spectrum is manifestly supersymmetric, since it contains an equal number of bosonic and fermionic degrees of freedom of the same mass.

In the presence of stack of n coincident Dp-branes, the massless spectrum is enhanced to a $\mathrm{U}(n)$-group gauge theory. This can be explained intuitively as follows. For an oriented open string, each state carries a couple of indices (a, b) that label the specific Dp-brane where the string starts and ends, with $a, b = 1, \dots, n$. These are the Chan-Paton indices. In particular, the ground states can be expressed as $|a, b; \mathrm{NS}\rangle = t^I_{ab}|I; \mathrm{NS}\rangle$ and $|a, b; \mathrm{R}_+\rangle = t^I_{ab}|I; \mathrm{R}_+\rangle$, where t^I_{ab} are n-dimensional matrices labelled by an index $I = 0, 1, \dots, n^2-1$. Defining Hermitian conjugation as $(|a, b\rangle)^\dagger = \langle a, b| = \langle I|t^I_{ba}$, for Hermitian matrices $(t^I)^\dagger = t^I$ normalised as $\mathrm{tr}\, t^I t^J = \delta^{IJ}$, one can choose the condition $t^I_{ba} t^I_{cd} = \delta_{ac}\delta_{bd}$ to find the orthonormality condition $\langle a, b|c, f\rangle = \delta_{ac}\delta_{bd}$. The massless spectrum now consists of the gauge vectors $A^\alpha_{ab} = t^I_{ab} A^\alpha_I$ corresponding to the states $b^\alpha_{-1/2}|a, b; \mathrm{NS}\rangle$, of the scalars $\varphi^{\dot{m}}_{ab} = t^I_{ab}\varphi^{\dot{m}}_I$ associated to the states $b^{\dot{m}}_{-1/2}|a, b; \mathrm{NS}\rangle$ and, similarly, of the spinors $\psi^\omega_{ab} = t^I_{ab}\psi^\omega_I$ corresponding to the states $|a, b; \mathrm{R}_+\rangle$. This particle content corresponds to a supersymmetric non-Abelian

$\mathrm{U}(n)$-gauge theory with scalars and spinors transforming in the adjoint representation. This is equivalent to saying that the matrices t^I are the elements of the $\mathfrak{u}(n)$-algebra defining the $\mathrm{U}(n)$-group in the adjoint representation.

2.3.1.1 Example: D3-, Anti-D3- and D7-Branes

An instructive example is represented by theories with D3-, anti-D3- and D7-branes. One can see easily the connection between the straightforward Fock-space construction of subsection 2.1.2 and the formalism of partion functions in subsection 2.1.3. The setup discussed below corresponds to a model with a stack of n D3- or anti-D3-branes, labelled by an index $a = 1, \dots, n$, and an intersecting stack of w D7-branes, labelled by an index $a' = 1, \dots, w$.

- The 33-states correspond to strings stretching between two D3-branes a and b. At the massless bosonic level, there is a $\mathrm{U}(n)$-gauge vector $A^\mu_{ab} = t^I_{ab} A^\mu_I$ for the states $b^\mu_{-1/2}|a, b; \mathrm{NS}\rangle$, and six real scalars $\varphi^m_{ab} = t^I_{ab}\varphi^m_I$ for the states $b^m_{-1/2}|a, b; \mathrm{NS}\rangle$, preserved by the GSO-projection. At the massless fermionic level, there are four spinors $\psi^\omega_{ab} = t^I_{ab}\psi^\omega_I$, with $\omega = 0, \dot\omega$, and $\dot\omega = 1, 2, 3$, corresponding to the states $|a, b; \mathrm{R}_+\rangle$, preserved by the GSO-projection. It is convenient to label the R-vacua as $|\mathrm{R}_+\rangle = |s_0, s_1, s_2, s_3\rangle$, with $\sum_{r=0}^3 s_r \in 2\mathbb{Z}$, after redefining $s_0 = s_4$.

  In 4-dimensional terms, this particle content corresponds to the vector multiplet $V_4 = (A_\mu, \varphi^m, \psi^\omega)$ of a $N_4 = 4$ non-Abelian $\mathrm{U}(n)$-gauge theory. This reflects the fact that branes break half of the closed-string sector supersymmetries. Such a spectrum can be decomposed into one vector $V_1 = (A_\mu, \psi^0)$ and three adjoint chiral multiplets $\varphi^{\dot m}_1 = (\varphi^{\dot\omega}, \psi^{\dot\omega})$ of a $N_4 = 1$ non-Abelian $\mathrm{U}(n)$-gauge theory, with the scalars suitably complexified.

  In an analogous description, $\overline{33}$-states correspond to the same particle content.

- The 77-states correspond to strings stretching between two D7-branes a' and b'. At the massless bosonic level, they give the $\mathrm{U}(w)$-gauge vector $B^\alpha_{a'b'} = \tau^K_{a'b'} B^\alpha_K$ for the states $b^\alpha_{-1/2}|a', b'; \mathrm{NS}\rangle$, for $\alpha = 0, \dots, 7$, and two real scalars $\sigma^{\dot m}_{a'b'} = \tau^K_{a'b'}\sigma^{\dot m}_K$ for the states $b^{\dot m}_{-1/2}|a', b'; \mathrm{NS}\rangle$, preserved by the GSO-projection, with $\dot m = 8, 9$. At the massless fermionic level, there are the degrees of freedom associated with the states $|a', b'; \mathrm{R}_+\rangle$ preserved by the GSO-projection.

  This is an 8-dimensional theory, and the details of a reduction to four dimensions depends on the details of the compactification. Naively, in the language of $N_4 = 1$ supersymmetry, the 4-dimensional components of the gauge vector and one spinor make up a vector multiplet $U_1 = (B_\mu, \eta^0)$, whereas the complexified internal components of the gauge vector and the complexified scalar plus the three remaining spinors make up three adjoint chiral multiplets $\sigma^{\dot\omega}_1 = (\sigma^{\dot\omega}, \eta^{\dot\omega})$.

- The 37-states correspond to strings starting from the D3-brane a and ending on the D7-brane b'. In this case, at the massless level bosons are provided by the R-sector and fermions are provided by the NS-sector, as follows from the mode expansions in eqs. (2.1.27, 2.1.28) and the mass formula of eq. (2.1.40). In the NS-sector, the vacuum is massless and it corresponds to the states $|a, b'; s_1, s_2\rangle$: not having fermionic degeneracies in the non-compact directions, since the b-operators with half integer indices are in the internal ND- and DN-directions, these are four real scalar degrees of freedom, reduced to two by the GSO-projection, i.e. to one complex scalar φ. In the R-sector, the vacuum is massless and it corresponds to the states $|a, b'; s_0, s_3\rangle$, with $s_0 = s_4$, which are spinorial degrees of freedom, halved by a subsequent GSO-projection. This corresponds to one 4-dimensional Weyl spinor ψ. Generalising the Chan-Paton index-structure, all these fields are in the fundamental representation of the group $\mathrm{U}(n)$ and in the antifundamental representation of the group $\mathrm{U}(w)$. The 73-states correspond to strings starting from the D7-brane a' and ending on the D3-brane b and are entirely analogous, so they are a complex scalar φ_* and a Weyl spinor ψ_*, with the only difference that they are in the conjugate representation.

In the language of 4-dimensional supersymmetry, this field content makes up one $N_4 = 2$ hypermultiplet $H_2 = (\varphi, \varphi_*; \psi, \psi_*)$ in a bifundamental representation of the group $\mathrm{U}(n) \times \mathrm{U}(w)$. This evidences the fact that the intersecting states break half of the supersymmetries of the same-brane states. Equivalently, this spectrum can be arranged into two $N_4 = 1$ chiral multiplets $\varphi_1 = (\varphi, \psi)$ and $\varphi_{*1} = (\varphi_*, \psi_*)$.

For $\overline{3}$7- and 7$\overline{3}$-states, the construction is completely analogous with the only but crucial difference that the GSO-projection is opposite.[2.17]

2.3.2 General D-Brane Action

Let the 10-dimensional spacetime $X_{1,9} = X_{1,3} \times X_6$ be spanned by the index M, with $M = 0, \dots, 9$, with the 4-dimensional subspace spanned by the index μ, with $\mu = 0, 1, 2, 3$, and the internal coordinates varying as $m = 4, \dots, 9$. The Dp- or anti-Dp-brane worldvolume $W_{1,p}$ is defined via the embedding

$$\varphi : \; W_{1,p} \hookrightarrow \mathbb{M}^{1,9}$$

in a notation such that:

- indices $\dot{\mu}$ span both the 4-dimensional spacetime and the $p-3$ internal directions wrapped by the Dp-brane, i.e. $\dot{\mu} = \mu, m'$, with $m' = 4, \dots, p$;
- indices $\dot{m}$ span the internal directions which are not wrapped, i.e. $\dot{m} = p+1, \dots, 9$.

The $(p+1)$-dimensional worldvolume theory of a stack of n coincident Dp- or anti-Dp-branes consists of the following massless degrees of freedom:

- from the NS-sector, a vector A_α which gauges the non-Abelian gauge group $\mathrm{U}(n)$ and $9-p$ scalars $\phi^{\dot{m}}$ in the adjoint representation of the group $\mathrm{U}(n)$, with the indices α and $\dot{m}$ respectively running over the worldvolume longitudinal and transverse directions, meaning $\alpha = 0, \dots, p$ and $\dot{m} = p+1, \dots, 9$;

[2.17] This impacts more elaborated constructions, such as setups with branes at orbifold singularities, where the details of the orbifold projection depend on the specific values of the terms s_r in the states $|\{s_r\}\rangle$. For more details, see subsection 4.5.3.

- from the R-sector, some spinors ψ^A in the adjoint representation of the group $\mathrm{U}(n)$, where the family index A counts the number of $(p+1)$-dimensional spinors descending from a single 10-dimensional Majorana-Weyl spinor.

The defining difference between D-branes and anti-D-branes is their charge under the RR-fields, which is $q = 1, -1$, respectively. The effective action describing the massless degrees of freedom of coincident D-branes is a non-Abelian generalisation of the effective action describing a single D-brane [72]. In detail, it is the summation of a Dirac-Born-Infeld and a Chern-Simons action, i.e.

$$S^{\mathrm{D}p} = S^{\mathrm{D}p}_{\mathrm{DBI}} + S^{\mathrm{D}p}_{\mathrm{CS}}. \tag{2.3.1}$$

For brevity, only the bosonic action is discussed below. An analysis of the general Dp-brane fermionic action can be found in refs. [73–75] (see also ref. [76]).

2.3.2.1 Dirac-Born-Infeld Action

In the string frame, the Dirac-Born-Infeld term for a stack of Dp-branes at a generic smooth point in the internal manifold takes the form[2.18]

$$S^{\mathrm{D}p}_{\mathrm{DBI}} = -T_p \int_{W_{1,p}} \mathrm{d}^{p+1}\xi \ \mathrm{str}\left[\,\mathrm{e}^{-\Phi}\sqrt{-\det\left[\Gamma_{\alpha\beta}\right]\cdot\det\left[Q^{\dot n}{}_{\dot m}\right]}\,\right],$$

where $T_p = 2\pi/l_s^{p+1}$ is the Dp-brane tension. Also, one has the rank-2 tensor

$$\Gamma_{\alpha\beta} \equiv E_{\alpha\beta} + E_{\alpha\dot m}\left(Q^{-1}-1\right)^{\dot m}{}_{\dot l}\, E^{\dot l\dot n}\, E_{\dot n\beta} + 2\pi\alpha' F_{\alpha\beta},$$

along with the combination of the string-frame metric tensor and the 2-form NSNS-field, $E_{MN} = G_{MN} + B_{MN}$, with $E_{\alpha\beta}$ being its pull-back on the worldvolume, as well as the purely non-Abelian rank-$(1,1)$ tensor

$$Q^{\dot n}{}_{\dot m} = \delta^{\dot n}_{\dot m} + 2\pi\mathrm{i}\alpha'\left[\phi^{\dot n},\phi^{\dot k}\right] E_{\dot k\dot m}.$$

The determinant 'det' is with respect to spacetime indices, while the trace 'str' is the symmetrised trace over the gauge group indices such that the Lie matrix-valued terms $F_{\alpha\beta}$, $D_\alpha\phi^{\dot m}$ and $[\phi^{\dot m},\phi^{\dot n}]$ are treated as commuting (no other terms are treated as commuting).

One can write the action in the Einstein frame by redefining the metric and NSNS-field combination as $\hat e_{MN} = \mathrm{e}^{-\phi/2}\, E_{MN} = \hat g_{MN} + \mathrm{e}^{-\phi/2}\, B_{MN}$. Elementary operations then reveal the action to take the form

$$S^{\mathrm{D}p}_{\mathrm{DBI}} = -\tau_{\mathrm{D}p} \int_{W_{1,p}} \mathrm{d}^{p+1}\xi \ \mathrm{str}\left[\mathrm{e}^{(p-3)\,\phi/4}\sqrt{-\det\left[\hat\gamma_{\alpha\beta}\right]\cdot\det\left[Q^{\dot n}{}_{\dot m}\right]}\,\right], \tag{2.3.2}$$

where the physical Dp-brane tension turns out to be $\tau_{\mathrm{D}p} = 2\pi/g_s l_s^{p+1}$. Also, one redefines the rank-2 tensor as

$$\hat\gamma_{\alpha\beta} \equiv \hat e_{\alpha\beta} + \hat e_{\alpha\dot m}\left(Q^{-1}-1\right)^{\dot m}{}_{\dot l}\, \hat e^{\dot l\dot n}\, \hat e_{\dot n\beta} + 2\pi\alpha'\,\mathrm{e}^{-\phi/2}\, F_{\alpha\beta},$$

whilst the rank-$(1,1)$ tensor is still

$$Q^{\dot n}{}_{\dot m} = \delta^{\dot n}_{\dot m} + 2\pi\mathrm{i}\alpha'\,\mathrm{e}^{\phi/2}\left[\phi^{\dot n},\phi^{\dot k}\right]\hat e_{\dot k\dot m}.$$

[2.18] Notice that one can define the brane tension as $\tau_{\mathrm{D}p} = T_p/g_s$ as a direct consequence of writing the dilaton factor in the action in terms of the shifted dilaton field $\phi = \Phi - \langle\Phi\rangle$.

2.3.2.2 Chern-Simons Action

The Chern-Simons action is the same both in the string and the 10-dimensional Einstein frame up to the rescaling of the RR-fields and it takes the form

$$S_{\mathrm{CS}}^{\mathrm{D}p} = q\,\tau_{\mathrm{D}p} \int\limits_{W_{1,p}} \mathrm{str} \left\{ \left[\varphi_* \left(\mathrm{e}^{2\pi i \alpha' \mathrm{i}_\phi \mathrm{i}_\phi} \left(\sum_{l=0}^{4} C_{2l} \wedge \mathrm{e}^{B_2} \right) \right) \right] \wedge \mathrm{e}^{2\pi\alpha' F_2} \right\}, \tag{2.3.3}$$

where i_ϕ denotes the interior product with the vector field $\phi^{\dot{m}}$, i.e. for a general n-form

$$\mathrm{i}_\phi A_n = \frac{1}{(p-1)!} \phi^{\dot{m}}\, A_{\dot{m}\, M_1 \dots M_{n-1}}\, \mathrm{d}x^{M_1} \wedge \dots \mathrm{d}x^{M_{n-1}}.$$

2.3.2.3 Further Remarks

One typically chooses to work in the so-called static gauge, in which, given the expansion parameter $\sigma_s = l_s^2/2\pi$ for ease of notation, the brane position is parametrised as

$$X^{\dot{\mu}}(\xi) = \delta_\alpha^{\dot{\mu}}\, \xi^\alpha, \tag{2.3.4a}$$

$$Y^{\dot{m}}(\xi) = y_0^{\dot{m}} + \sigma_s \phi^{\dot{m}}(\xi), \tag{2.3.4b}$$

where $y_0^{\dot{m}}$ are the background brane positions in the Dirichlet directions while the terms $\delta Y^{\dot{m}} = \sigma_s \phi^{\dot{m}}$ represent fluctuations thereof. The following remarks are useful.

- The DBI- and CS-actions involve pull-backs of 10-dimensional fields onto the brane worldvolume: these are a generalised version of the standard pull-back as they involve non-Abelian fields. For instance the non-Abelian pull-back on the worldvolume of a 1-form $v = v_M\, \mathrm{d}x^M$ is

$$\varphi_* v = v_{\dot{\mu}}\, \delta_\alpha^{\dot{\mu}}\, \mathrm{d}\xi^\alpha + \sigma_s\, \nabla_\alpha \phi^{\dot{m}}\, v_{\dot{m}}\, \mathrm{d}\xi^\alpha,$$

  where ∇_α is the standard gauge covariant derivative, as a generalisation of the standard pull-back expression involving $\partial_\alpha y^m$. Generalisations to n-forms are immediate.

- Fields on the brane worldvolume must be expressed as functions of the coordinates ξ^α. A generic 10-dimensional function $f = f(x^M)$ can be written as a non-Abelian Taylor expansion on the worldvolume, i.e.

$$f(x^{\dot{\mu}}, y^{\dot{m}}) = \sum_{k=0}^{\infty} \frac{\sigma_s^k}{k!}\, \phi^{\dot{m}_1} \phi^{\dot{m}_2} \dots \phi^{\dot{m}_k}\, \partial_{\dot{m}_1} \partial_{\dot{m}_2} \dots \partial_{\dot{m}_k}\, f(x^{\dot{\mu}}, y_0^{\dot{m}}),$$

  which accounts for the fluctuations of the Dp-brane in terms of the non-Abelian displacements from the original position $y_0^{\dot{m}}$.

2.4 D-Branes and Non-Linear Supersymmetry

This section discusses and summarises the main results of ref. [77] about the general form of the constrained $N_4 = 1$ supermultiplets that can be used to package the field content of an anti-D3-brane in a flat 10-dimensional background in view of a compactification to a 4-dimensional spacetime. Particularly useful is also the discussion of ref. [78] on the general form of the D-brane action in superspace. The final part on the form of constrained supermultiplets is thoroughly based on ref. [31].

2.4.1 D-Branes in Flat Superspace

This subsection reports the general form of the Dp-brane low-energy effective action in superspace. In particular, it considers a single Dp- or anti-Dp-brane of a type IIB theory in a flat 10-dimensional spacetime. The guideline is the summary in ref. [78].

2.4.1.1 General D-Brane Action in Flat Spacetime

Let the background geometry be the 10-dimensional flat spacetime $\mathbb{M}^{1,9} = \mathbb{M}^{1,3} \times \mathbb{R}^6$, in the same notation as in subsection 2.3. The action for the massless states of a Dp- or anti-Dp-brane can be described in a manifestly supersymmetric notation at the cost of dealing with 10-dimensional quantities. For the purpose of studying supersymmetry transformations, this is the most convenient formulation. Given the worldvolume coordinate ξ^α, in general the degrees of freedom are encoded in:

- a gauge vector $A_\alpha = A_\alpha(\xi)$ and the Dp-brane embedding coordinates $X^M = X^M(\xi)$, before fixing the static gauge;
- the doublet of 10-dimensional positive-chirality Majorana-Weyl spinors

$$\Theta = \begin{pmatrix} \theta \\ \theta' \end{pmatrix}.$$

Some gauge symmetries reduce the number of the proper physical degrees of freedom, obtaining a boson-fermion match. According to ref. [79], the effective Dp-superbrane effective action in a flat background geometry consists of the summation of the Dirac-Born-Infeld and Wess-Zumino actions

$$S_{\mathrm{DBI}}^{\mathrm{D}p} = -T_{\mathrm{D}p} \int_{W_{1,p}} \mathrm{d}^{p+1}\xi \, \sqrt{-\det\left(\mathrm{G}_{\alpha\beta} + \sigma_s \mathrm{F}_{\alpha\beta}\right)} \tag{2.4.1}$$

and

$$S_{\mathrm{WZ}}^{\mathrm{D}p} = q T_{\mathrm{D}p} \int_{W_{1,p}} \Omega_{p+1}, \tag{2.4.2}$$

where the area $\sigma_s = 2\pi\alpha'$ has been defined for brevity. Each term requires to be explained carefully. First of all, the generalised metric pull-back on the Dp-brane, $\mathrm{G}_{\alpha\beta}$, is defined as

$$\mathrm{G}_{\alpha\beta} \equiv \eta_{MN} \, \Pi_\alpha{}^M \Pi_\beta{}^N,$$

with the generalised pull-back derivative[2.19] $\Pi_\alpha{}^M \equiv \partial_\alpha X^M - \sigma_s^2\,\bar{\theta}\,\Gamma^M\partial_\alpha\theta$, where $\bar{\theta} = (\bar{\theta}, \bar{\theta}')$. Second, the field-strength tensor $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\beta$ is generalised to

$$\mathrm{F}_{\alpha\beta} = F_{\alpha\beta} - b_{\alpha\beta} = F_{\alpha\beta} - 2\sigma_s\,\bar{\theta}\,\sigma^3\Gamma_M\partial_{[\alpha}\theta\left[\partial_{\beta]}X^M - \frac{\sigma_s^2}{2}\,\bar{\theta}\,\Gamma^M\partial_{\beta]}\theta\right].$$

Third, Ω_{p+1} is a $(p+1)$-form which can be defined through its differential, i.e. the $(p+2)$-form [79, 81, 82]

$$I_{p+2} = -\sigma_s^2\,\mathrm{d}\bar{\theta}\,t_p\,\mathrm{d}\theta,$$

where the p-form t_p can be read off the formal summation

$$T = \sum_{p=2n+1} t_p = \mathrm{e}^{\mathrm{F}}\,S(\Upsilon)\,\sigma^1,$$

with the 1-form $\Upsilon_\alpha = \Pi_\alpha{}^M\,\Gamma_M$ and the formal sum $S(\Upsilon) \equiv \sum_{n=0}^\infty (\sigma^3)^n\,\Upsilon^{2n+1}/(2n+1)!$. Of course, Γ_M are flat spacetime 10-dimensional γ-matrices.

Action Symmetries

The super-Dp-brane effective action is invariant under several symmetries: global supersymmetry, local κ-symmetry, local coordinate transformations and U(1)-gauge transformations. In type IIB theories, the space-like directions spanned by the brane are always an odd number $p = 2n+1$.

- For any given 10-dimensional Majorana-Weyl constant spinor ϵ, the supersymmetry field transformations read

$$\sigma_s\,\delta_\epsilon\theta = \epsilon, \tag{2.4.3a}$$

$$\delta_\epsilon X^M = -\sigma_s\,(\bar{\theta}\,\Gamma^M\epsilon), \tag{2.4.3b}$$

$$\delta_\epsilon A_\alpha = -(\bar{\theta}\Gamma_M\sigma^3\epsilon)\partial_\alpha X^M + \frac{\sigma_s^2}{6}\left[(\bar{\theta}\Gamma_M\sigma^3\epsilon)(\bar{\theta}\Gamma^M\partial_\alpha\theta) + (\bar{\theta}\Gamma_M\epsilon)(\bar{\theta}\Gamma^M\sigma^3\partial_\alpha\theta)\right]. \tag{2.4.3c}$$

  Notice that terms like $\partial_\alpha\theta$, $\Pi_\alpha{}^M$, $\mathrm{G}_{\alpha\beta}$ and $\mathrm{F}_{\alpha\beta}$ are identically supersymmetric.

- Given an arbitrary double 10-dimensional Majorana-Weyl spinor $\kappa = \kappa(\xi)$ of definite chirality, the κ-symmetry variations are

$$\sigma_s\,\delta_\kappa\theta = (1 + q\Gamma^{(p)})\kappa, \tag{2.4.4a}$$

$$\delta_\kappa X^M = \sigma_s\left[\bar{\theta}\,\Gamma^M(1 + q\Gamma^{(p)})\kappa\right], \tag{2.4.4b}$$

$$\delta_\kappa A_\alpha = -\sigma_s(\bar{\theta}\Gamma_M\sigma^3\delta_\kappa\theta)\,\partial_\alpha X^M - \frac{\sigma_s^3}{2}\left[(\bar{\theta}\Gamma_M\sigma^3\delta_\kappa\theta)(\bar{\theta}\Gamma^M\partial_\alpha\theta) + (\bar{\theta}\Gamma_M\delta_\kappa\theta)(\bar{\theta}\Gamma^M\sigma^3\partial_\alpha\theta)\right], \tag{2.4.4c}$$

[2.19] With respect to ref. [78], the double Majorana-Weyl spinor θ has been shifted to $\theta \to \sigma_s\,\theta$ in such a way as to be fully consistent with the notation of ref. [80].

where the matrix $\Gamma^{(p)}$ reads

$$\Gamma^{(p)} = \begin{pmatrix} 0 & \beta_- \\ (-1)^n \beta_+ & 0 \end{pmatrix},$$

with $\beta_\pm \beta_\mp = (-1)^n$, the matrices β_+ and β_- being

$$\beta_\pm = \frac{\sqrt{-\det \mathrm{G}}}{\sqrt{-\det\left(\mathrm{G} + \sigma_s \mathrm{F}\right)}} \sum_{k=0}^{n+1} \frac{(\pm\sigma_s)^k}{2^k k!}\, \hat{\Gamma}^{\alpha_1 \dots \alpha_{2k}}\, \mathrm{F}_{\alpha_1\alpha_2} \dots \mathrm{F}_{\alpha_{2k-1}\alpha_{2k}}\, \Gamma^{\mathrm{D}p}_{(0)}.$$

In particular, the matrices $\hat{\Gamma}_\alpha$ are defined as the generalised γ-matrix pull-back

$$\hat{\Gamma}_\alpha = \Pi_\alpha{}^M \, \Gamma_M,$$

with indices being raised by the inverse $G^{\alpha\beta}$ of the generalised metric pull-back, whereas the matrix $\Gamma^{\mathrm{D}p}_{(0)}$ is defined as

$$\Gamma^{\mathrm{D}p}_{(0)} = \frac{1}{(p+1)!} \frac{1}{\sqrt{-\det \mathrm{G}}}\, \varepsilon^{\alpha_1 \dots \alpha_{p+1}}\, \hat{\Gamma}_{\alpha_1 \dots \alpha_{p+1}}.$$

- The action is also invariant under local general coordinate transformations on the worldvolume, i.e. $\xi^\alpha \to \xi^\alpha - \eta(\xi)$, with variations

$$\delta_\eta \theta = \eta^\alpha \partial_\alpha \theta, \tag{2.4.5a}$$

$$\delta_\eta X^M = \eta^\alpha \partial_\alpha X^M, \tag{2.4.5b}$$

$$\delta_\eta A_\alpha = \eta^\beta \partial_\beta A_\alpha, \tag{2.4.5c}$$

  and under U(1)-gauge transformations, with a variation $\delta_{\mathrm{U}(1)} A_\alpha = \partial_\alpha \phi$, for an arbitrary function ϕ.

2.4.1.2 Gauge-Fixed Super-D-Brane Action

If κ-symmetry is gauge-fixed, then only the degrees of freedom of one 10-dimensional Majorana-Weyl spinor remain and the redundant fermionic directions are removed. Also, worldvolume general coordinate transformations can be fixed too in order to remove non-physical scalar degrees of freedom. The κ-symmetry gauge fixing may vary depending on the most convenient choice for the case at hand. For future use, a convenient κ-symmetry gauge is

$$\frac{1}{2}\,(1_2 + \sigma^3)\theta = 0, \tag{2.4.6}$$

which implies that $\theta = 0$. In particular, the κ-symmetry fixing allows to see that the Wess-Zumino term is constant since I_{p+2} is zero as results from multiplications of the form $\bar{\theta}\,(\sigma^3)^k \sigma^1 \theta$, i.e.

$$I_{p+2} = \mathrm{d}\Omega_{p+1} = 0.$$

Further, the static gauge condition reads

$$X^{\dot{\mu}}(\xi) = \delta^{\dot{\mu}}_{\alpha}\, \xi^\alpha, \qquad X^{\dot{m}}(\xi) = y_0^{\dot{m}} + \sigma_s\, \phi^{\dot{m}}(\xi). \tag{2.4.7}$$

In this way, the physical degrees of freedom are encoded in one vector A_α, $9-p$ scalars $\phi^{\dot{m}}$ and one 10-dimensional Majorana-Weyl spinor θ'.

In the selected gauge, the dynamical part of the action is the DBI-term, so the total action reads

$$S^{\mathrm{D}p} = -T_{\mathrm{D}p} \int_{W_{1,p}} \mathrm{d}^{p+1}\xi \left[\sqrt{-\det\left(\mathrm{G}_{\alpha\beta} + \sigma_s \mathrm{F}_{\alpha\beta}\right)} - qV_0^{\mathrm{WZ}}\right]. \tag{2.4.8}$$

Now, the generalised metric pull-back on the Dp-brane $\mathrm{G}_{\alpha\beta}$ reads

$$\mathrm{G}_{\alpha\beta} = \eta_{\dot{\mu}\dot{\nu}}\, \Pi_\alpha{}^{\dot{\mu}}\Pi_\beta{}^{\dot{\nu}} + \delta_{\dot{m}\dot{n}}\, \Pi_\alpha{}^{\dot{m}}\Pi_\beta{}^{\dot{n}}.$$

with $\Pi_\alpha{}^{\dot{\mu}} = \delta_\alpha^{\dot{\mu}} - \sigma_s^2\, \overline{\theta}'\Gamma^{\dot{\mu}}\partial_\alpha\theta'$ and $\Pi_\alpha{}^{\dot{m}} = \sigma_s\, \partial_\alpha\phi^{\dot{m}} - \sigma_s^2\, \overline{\theta}'\Gamma^{\dot{m}}\partial_\alpha\theta'$, and the generalised Born-Infeld field-strength tensor $F_{\alpha\beta}$ is

$$\begin{aligned}\mathrm{F}_{\alpha\beta} = F_{\alpha\beta} - b_{\alpha\beta} = F_{\alpha\beta} &+ 2\sigma_s\, \overline{\theta}'\, \sigma^3\Gamma_{\dot{\mu}}\partial_{[\alpha}\theta' \left[\delta^{\dot{\mu}}_{\beta]} - \frac{\sigma_s^2}{2}\, \overline{\theta}'\, \Gamma^{\dot{\mu}}\partial_{\beta]}\theta'\right] \\ &+ 2\sigma_s\, \overline{\theta}'\, \sigma^3\Gamma_{\dot{m}}\partial_{[\alpha}\theta' \left[\sigma_s\partial_{\beta]}\phi^{\dot{m}} - \frac{\sigma_s^2}{2}\, \overline{\theta}'\, \Gamma^{\dot{m}}\partial_{\beta]}\theta'\right].\end{aligned}$$

In order to preserve the gauge-fixing conditions, each supersymmetry transformation (i.e. the non-gauge-fixed transformation) must be accompanied by a κ-symmetry and a local coordinate transformation such that the total transformation does not change the gauge choice, i.e. such that

$$\begin{cases} \delta\,\theta = 0, \\ \delta X^{\dot{\mu}} = 0. \end{cases}$$

It is possible to observe that, to preserve this gauge, each supersymmetry transformation with infinitesimal parameter ϵ must be accompanied by a κ-symmetry and a local coordinate transformation with parameters κ and η^α such as to satisfy the constraints

$$\epsilon_1 + \kappa_1 + q\beta_-\kappa_2 = 0, \tag{2.4.9a}$$

$$\eta^\alpha = \sigma_s\, \overline{\theta}'\delta^\alpha_{\dot{\mu}}\Gamma^{\dot{\mu}}\epsilon_2 + (-1)^n\, q\sigma_s\, \overline{\theta}'\delta^\alpha_{\dot{\mu}}\Gamma^{\dot{\mu}}\beta_+\epsilon_1. \tag{2.4.9b}$$

Given these constraints, one can work out the total variations generated by an arbitrary supersymmetry parameter ϵ. With some work, the general gauge-fixed supersymmetry transformations generated by the spinor doublet ϵ on a Dp-brane can be seen to take the form

$$\sigma_s\, \delta\theta' = \epsilon_2 + (-1)^{n+1} q\beta_+\epsilon_1 + \sigma_s\, \eta^\alpha\partial_\alpha\theta', \tag{2.4.10a}$$

$$\delta\phi^{\dot{m}} = -\,\overline{\theta}'\Gamma^{\dot{m}}\epsilon_2 + (-1)^{n+1}\, \overline{\theta}'\Gamma^{\dot{m}} q\beta_+\epsilon_1 + \eta^\alpha\partial_\alpha\phi^{\dot{m}}, \tag{2.4.10b}$$

$$\begin{aligned}\delta A_\alpha = &-\eta^\beta\, F_{\alpha\beta} + [\overline{\theta}'(\delta^{\dot{\mu}}_\alpha\Gamma_{\dot{\mu}} + \sigma_s\, \Gamma_{\dot{m}}\partial_\alpha\phi^{\dot{m}})(\epsilon_2 + (-1)^n\, q\beta_+\epsilon_1)] \\ &- \sigma_s^2\Big[\overline{\theta}'\Gamma_{\dot{\mu}}\Big(\frac{\epsilon_2}{3} + (-1)^n\, q\beta_+\epsilon_1\Big)\Big](\overline{\theta}'\Gamma^{\dot{\mu}}\partial_\alpha\theta') \\ &- \sigma_s^2\Big[\overline{\theta}'\Gamma_{\dot{m}}\Big(\frac{\epsilon_2}{3} + (-1)^n\, q\beta_+\epsilon_1\Big)\Big](\overline{\theta}'\Gamma^{\dot{m}}\partial_\alpha\theta').\end{aligned} \tag{2.4.10c}$$

Such supersymmetry transformations are generated by two independent 10-dimensional Majorana-Weyl spinors ϵ_1 and ϵ_2, which means that each of them generates an independent set of supersymmetry transformations.

To single out the two distinct groups of transformations, a parameter redefinition is necessary. One can change the basis of spinors from ϵ_1 and ϵ_2 to χ and ζ by defining

$$\epsilon_1 \equiv -\frac{\mathrm{i}^n}{2}\,\Gamma_*\,\chi, \tag{2.4.11a}$$

$$\epsilon_2 \equiv -\frac{1}{2}\,\chi + \zeta, \tag{2.4.11b}$$

where the definition has been made $\Gamma_* = (-\mathrm{i})^n \Gamma^0 \dots \Gamma^{2n+1}$. In this way, the total variations induced by supersymmetry in the basis (χ, ζ) acquire quite an easy form. Defining $\beta \equiv (-\mathrm{i})^n\,\beta_+\,\Gamma_*$, the supersymmetry transformations generated by χ are

$$\sigma_s\,\delta_\chi\theta' = -\frac{1}{2}(1-q\beta)\chi - \frac{\sigma_s^2}{2}\,[\bar{\theta}'\delta^\alpha_{\dot\mu}\Gamma^{\dot\mu}\big(1+q\beta\big)\chi]\,\partial_\alpha\theta' \tag{2.4.12a}$$

$$\delta_\chi\phi^{\dot m} = \frac{1}{2}[\bar{\theta}'\Gamma^{\dot m}(1+q\beta)\chi] - \frac{\sigma_s}{2}\,[\bar{\theta}'\delta^\alpha_{\dot\mu}\Gamma^{\dot\mu}(1+q\beta)\chi]\,\partial_\alpha\phi^{\dot m}, \tag{2.4.12b}$$

$$\begin{aligned}\delta_\chi A_\alpha = &-\frac{1}{2}[\bar{\theta}'(\delta^{\dot\mu}_\alpha\Gamma_{\dot\mu} + \sigma_s\,\Gamma_{\dot m}\partial_\alpha\phi^{\dot m})(1+q\beta)\chi] + \frac{\sigma_s}{2}[\bar{\theta}'\delta^\beta_{\dot\mu}\Gamma^{\dot\mu}(1+q\beta)\chi]\,F_{\alpha\beta} \\ &+\frac{\sigma_s^2}{2}\Big[\bar{\theta}'\Gamma_{\dot\mu}\Big(\frac{1}{3}+q\beta\Big)\chi\Big](\bar{\theta}'\Gamma^{\dot\mu}\partial_\alpha\theta') + \frac{\sigma_s^2}{2}\Big[\bar{\theta}'\Gamma_{\dot m}\Big(\frac{1}{3}+q\beta\Big)\chi\Big](\bar{\theta}'\Gamma^{\dot m}\partial_\alpha\theta'),\end{aligned} \tag{2.4.12c}$$

whereas the supersymmetry transformations generated by ζ are

$$\sigma_s\,\delta_\zeta\theta' = \zeta + \sigma_s^2\,[\bar{\theta}'\delta^\alpha_{\dot\mu}\Gamma^{\dot\mu}\zeta]\,\partial_\alpha\theta' \tag{2.4.13a}$$

$$\delta_\zeta\phi^{\dot m} = -(\bar{\theta}'\Gamma^{\dot m}\zeta) + \sigma_s\,[\bar{\theta}'\delta^\alpha_{\dot\mu}\Gamma^{\dot\mu}\zeta]\,\partial_\alpha\phi^{\dot m}, \tag{2.4.13b}$$

$$\begin{aligned}\delta_\zeta A_\alpha = &[\bar{\theta}'(\delta^{\dot\mu}_\alpha\Gamma_{\dot\mu} + \sigma_s\,\Gamma_{\dot m}\partial_\alpha\phi^{\dot m})\zeta] - \sigma_s\,[\bar{\theta}'\delta^\beta_{\dot\mu}\Gamma^{\dot\mu}\zeta]\,F_{\alpha\beta} \\ &-\frac{\sigma_s^2}{3}\,(\bar{\theta}'\Gamma_{\dot\mu}\zeta)(\bar{\theta}'\Gamma^{\dot\mu}\partial_\alpha\theta') - \frac{\sigma_s^2}{3}\,(\bar{\theta}'\Gamma_{\dot m}\zeta)(\bar{\theta}'\Gamma^{\dot m}\partial_\alpha\theta').\end{aligned} \tag{2.4.13c}$$

Out of the two supersymmetry transformations, it is apparent that there is a distinction: for Dp-branes with odd n, the supersymmetry variation generated by χ is linearly realised since, as $\beta = 1$ in the asbence of worldvolume flux, one has $\delta_\chi\langle\theta'\rangle = 0$, whereas the supersymmetry variation generated by ζ is non-linearly realised as one necessarily has $\delta_\zeta\langle\theta'\rangle = \sigma_s^{-1}\zeta$. For anti-D$p$-branes, or D$p$-branes with even n, in an appropriate basis, one would obtain analogous results.

2.4.2 D3-Branes and Orientifolds

Thanks to the results of subsection 2.4.1, it is possible to identify the half of supersymmetries that are preserved by a D3-brane in the presence of an orientifold projection. The orientifold symmetry is assumed to preserve O3- and O7-planes.

In the pre-orientifold theory, one starts with two supersymmetry parameters ϵ_1 and ϵ_2. An O3-/O7-plane orientifold symmetry projects out of the spectrum half of the bulk supersymmetries and the preserved supersymmetry is generated by a combination of the two original supersymmetries such that

$$\epsilon_1 = -\mathrm{i}\Gamma_{(4)}\epsilon_2 = \mathrm{i}\Gamma_{(6)}\epsilon_2, \tag{2.4.14}$$

where $\Gamma_{(4)} = \mathrm{i}\Gamma^{0123}$ is the 4-dimensional chirality matrix in the directions parallel to the observed universe and $\Gamma_{(6)} = -\mathrm{i}\Gamma^{456789}$ is the chirality matrix in the internal space.

In the basis of the spinor parameters χ and ζ, which for a D3-brane can be defined as via the indentifications

$$\epsilon_1 \equiv \frac{\mathrm{i}}{2}\,\Gamma_{(4)}\,\chi, \tag{2.4.15a}$$

$$\epsilon_2 \equiv -\frac{1}{2}\,\chi + \zeta, \tag{2.4.15b}$$

the supersymmetry preserved by the orientifold is especially simple to identify. In fact, the condition of eq. (2.4.14) in the basis of spinors defined in eqs. (2.4.15a, 2.4.15b) requires the condition $\Gamma_{(4)}\zeta = 0$, i.e. $\zeta = 0$. In other words, in the (χ, ζ)-basis, the spinor χ generates the bulk supersymmetry transformations that are also preserved by the D3-brane. These can be immediately read in eqs. (2.4.12a, 2.4.12b, 2.4.12c).

The half of the supersymmetries that is not preserved by the orientifold projection satisfies the condition $\epsilon_1 = \mathrm{i}\Gamma_{(4)}\epsilon_2$. In the (χ, ζ)-basis, this means $\Gamma_{(4)}\chi = \Gamma_{(4)}\zeta$, so one can set $\chi = \zeta$. With this condition, combining eqs. (2.4.12a, 2.4.12b, 2.4.12c) and eqs. (2.4.13a, 2.4.13b, 2.4.13c), one finds the variations

$$\sigma_s\,\delta_\eta\theta' = \frac{1}{2}(1+\beta)\,\eta + \frac{\sigma_s^2}{2}\,[\overline{\theta}'\delta^\alpha_\mu\Gamma^\mu(1-\beta)\eta]\,\partial_\alpha\theta', \tag{2.4.16a}$$

$$\delta_\eta\phi^m = -\frac{1}{2}[\overline{\theta}'\Gamma^m(1-\beta)\,\eta] + \frac{\sigma_s}{2}\,[\overline{\theta}'\delta^\alpha_\mu\Gamma^\mu(1-\beta)\eta]\,\partial_\alpha\phi^m, \tag{2.4.16b}$$

$$\begin{aligned}\delta_\eta A_\alpha = {}& \frac{1}{2}[\overline{\theta}'(\delta^\mu_\alpha\Gamma_\mu + \sigma_s\,\Gamma_m\partial_\alpha\phi^m)(1-\beta)\eta] - \frac{\sigma_s}{2}[\overline{\theta}'\delta^\beta_\mu\Gamma^\mu(1-\beta)\,\eta]\,F_{\alpha\beta} \\ & - \frac{\sigma_s^2}{2}\left[\overline{\theta}'\Gamma_\mu\Big(\frac{1}{3}-\beta\Big)\eta\right](\overline{\theta}'\Gamma^\mu\partial_\alpha\theta') - \frac{\sigma_s^2}{2}\left[\overline{\theta}'\Gamma_m\Big(\frac{1}{3}-\beta\Big)\eta\right](\overline{\theta}'\Gamma^m\partial_\alpha\theta').\end{aligned} \tag{2.4.16c}$$

These are the supersymmetry transformations $\delta_\eta = \delta_\chi + \delta_\zeta$ generated by a spinor $\eta = \chi = \zeta$ in the (χ, ζ)-basis. All the preliminary steps are now completed and it is possible to write the non-linear supersymmetry transformations in a manageable way to then relate them to the 4-dimensional supersymmetry formalism. In what follows, only a finite number of terms is going to be kept, and this will be set by the power of the area σ_s appearing, which counts the number of fields involved. Higher orders are subleading corrections and they can be neglected for the purposes of this work.

In order to deal efficiently with supersymmetry transformations, it is necessary to expand the β-matrix in terms of the physical degrees of freedom A_α, ϕ^m and θ'. A lengthy calculations shows that it reads

$$\begin{aligned}\beta = 1 &+ \frac{\sigma_s}{2}\,\Gamma^{\alpha\beta}F_{\alpha\beta} + \sigma_s\,\partial^\alpha\phi^m\,\Gamma_{m\alpha} + \frac{\sigma_s^2}{2}\,F^{\alpha\beta}\partial^\gamma\phi^m\Gamma_{m\alpha\beta\gamma} - \frac{\sigma_s^2}{4}\,F^{\alpha\beta}F_{\alpha\beta} \\ &+ \frac{\sigma_s^2}{8}\,\Gamma^{\alpha\beta\gamma\delta}F_{\alpha\beta}F_{\gamma\delta} - \sigma_s^2(\overline{\theta}'\Gamma^\alpha\partial^\beta\theta')\,\Gamma_{\alpha\beta} - \sigma_s^2\,(\overline{\theta}'\Gamma^m\partial^\alpha\theta')\,\Gamma_{m\alpha} \\ &- \frac{\sigma_s^2}{2}\,\delta_{mn}\,\partial^\alpha\phi^m\partial_\alpha\phi^n - \frac{\sigma_s^2}{2}\,\partial^\alpha\phi^m\,\partial^\beta\phi^n\Gamma_{mn\alpha\beta} + O(\sigma_s)^3,\end{aligned}$$

where all indices lowered and raised by the flat Minkowski metric η_α and its inverse.

Now the method to follow is quite clear, and one can simply make use of the expansion of the β-matrix in the non-linear supersymmetry variations in eqs. (2.4.16a, 2.4.16b, 2.4.16c). At this stage the calculations are quite involved, but by means of γ-matrix properties and Fierz identitites the results can be substantially simplified. Eventually, it is possible to redefine the spinor, scalar and vector fields in such a way as to get the standard form of non-linearly realised supersymmetry transformations, which is also easier to deal with and shorter to dimensionally reduce. Defining the new fields ϑ', φ^m and V_α as

$$\begin{aligned}\vartheta' \equiv\ & \theta' - \frac{\sigma_s}{4} F_{\alpha\beta}\Gamma^{\alpha\beta}\theta' - \frac{\sigma_s}{2}\delta_{mn}\partial_\alpha\phi^n\Gamma^{m\alpha}\theta' + \frac{\sigma_s^2}{4}[\overline{\theta}'\Gamma_\alpha\partial_\beta\theta']\Gamma^{\alpha\beta}\theta' + \frac{\sigma_s^2}{8}F^{\alpha\beta}F_{\alpha\beta}\theta' \\ &+ \frac{\sigma_s^2}{4}[\overline{\theta}'\Gamma_m\partial_\alpha\theta']\Gamma^{m\alpha}\theta' - \frac{\sigma_s^2}{16}F_{\alpha\beta}F_{\gamma\delta}\Gamma^{\alpha\beta\gamma\delta}\theta' + \frac{\sigma_s^2}{4}\delta_{mn}\partial_\alpha\phi^m\partial^\alpha\phi^n\theta' \\ &+ \frac{\sigma_s^2}{4}\delta_{ml}\delta_{nk}\partial_\alpha\phi^l\partial_\beta\phi^k\Gamma^{mn\alpha\beta}\theta' - \frac{\sigma_s^2}{4}\delta_{mn}F_{\alpha\beta}\partial_\gamma\phi^n\Gamma^{m\alpha\beta\gamma}\theta' + O(\sigma_s)^3,\end{aligned} \tag{2.4.17a}$$

$$\varphi^m \equiv \phi^m - \frac{\sigma_s^2}{8}F_{\alpha\beta}\left[\overline{\theta}'\Gamma^{m\alpha\beta}\theta'\right] - \frac{\sigma_s^2}{4}\delta_{nl}\partial_\alpha\phi^l\left[\overline{\theta}'\Gamma^{mn\alpha}\theta'\right] + O(\sigma_s)^3, \tag{2.4.17b}$$

$$V_\alpha \equiv A_\alpha - \frac{\sigma_s^2}{4}\left[\overline{\theta}'\Gamma_{m\alpha\beta}\theta'\right]\partial^\beta\phi^m + \frac{\sigma_s^2}{8}F^{\beta\gamma}\left[\overline{\theta}'\Gamma_{\alpha\beta\gamma}\theta'\right] + O(\sigma_s)^3, \tag{2.4.17c}$$

and further shifting the spinor exploiting the zilch symmetry [78] variation $\delta\theta' = -\sigma_s[\overline{\eta}\,\Gamma^\alpha\theta' - \overline{\theta}'\,\Gamma^\alpha\eta + (\overline{\eta}\,\Gamma^M\theta')\Gamma_M\Gamma^\alpha]\,\partial_\alpha\theta'/4$, one obtains the supersymmetry variations

$$\sigma_s\delta'_\eta\vartheta' = \eta + \frac{\sigma_s^2}{2}\,\delta^\alpha_\mu\,\partial_\alpha\vartheta'\,[\overline{\vartheta}'\Gamma^\mu\eta] + O(\sigma_s)^3, \tag{2.4.18a}$$

$$\sigma_s\delta'_\eta\varphi^m = \frac{\sigma_s^2}{2}\,\delta^\alpha_\mu\,\partial_\alpha\varphi^m\,[\overline{\vartheta}'\Gamma^\mu\eta] + O(\sigma_s)^3, \tag{2.4.18b}$$

$$\sigma_s\delta'_\eta V_\alpha = \frac{\sigma_s^2}{2}\,\delta_{mn}\,\partial_\alpha\varphi^m[\overline{\vartheta}'\Gamma^n\eta] - \frac{\sigma_s^2}{2}\,U_{\alpha\beta}\,\delta^\beta_\mu[\overline{\vartheta}'\Gamma^\mu\eta] + O(\sigma_s)^3, \tag{2.4.18c}$$

being $U_{\alpha\beta}$ the field-strength of V_α, where for simplicity the supersymmetry variations generated by including the effect of the zilch symmetry transformation have been denoted as 'δ'_η'. Notice that no transformations may be further simplified by field redefinitions because of the identity $\overline{\vartheta}'\Gamma^M\vartheta' = 0$.

Now that the general non-linear supersymmetry transformations are known in the 10-dimensional formulation, it is possible to dimensionally-reduce them in view of a 4-dimensional theory. In particular, spinors and γ-matrices can be reduced in such a way as to get the general non-linear supersymmetry transformations in terms of 4-dimensional Weyl spinors. As locally the full space has the form $\mathbb{M}^{1,9} = \mathbb{M}^{1,3} \times \mathbb{R}^6$, one can focus on the internal tangent-space subgroup SU(3), which is sufficient to visualise the one supersymmetry preserved in an $N_4 = 1$ Calabi-Yau orientifold compactification, as done e.g. in ref. [80]. A 10-dimensional Majorana-Weyl spinor can be decomposed as the summation of one 4-dimensional Weyl spinor ρ_α tensored with a 6-dimensional Weyl spinor belonging to an SU(3)-singlet plus three 4-dimensional Weyl spinors χ^a, with $a = 1, 2, 3$, combined with an SU(3)-triplet in the internal space. For the supersymmetry parameter η, only the term containing just the SU(3)-singlet is considered as it is the associated supersymmetry parameter ϵ that generates the 4-dimensional $N_4 = 1$ supersymmetry transformations. As far as bosons are concerned, the vector field can be shifted into a field a_μ to get rid of the scalar derivative in the supersymmetry transformation, while the scalars can be complexified into three complex scalars φ^a, for $a = 1, 2, 3$. So, decomposing eqs. (2.4.18a, 2.4.18b, 2.4.18c), one eventually finds

$$\delta_\varepsilon \rho = \sigma_s^{-1} \varepsilon + \mathrm{i}\sigma_s \, \delta_\mu^\alpha \, \partial_\alpha \rho \left[\overline{\rho}\,\overline{\sigma}^\mu \,\varepsilon + \rho\,\sigma^\mu\,\overline{\varepsilon}\right] + O(\sigma_s)^2, \tag{2.4.19a}$$

$$\delta_\varepsilon \chi^m = \mathrm{i}\sigma_s \, \delta_\mu^\alpha \, \partial_\alpha \chi^m \left[\overline{\rho}\,\overline{\sigma}^\mu \,\varepsilon + \rho\,\sigma^\mu\,\overline{\varepsilon}\right] + O(\sigma_s)^2, \tag{2.4.19b}$$

$$\delta_\varepsilon \varphi^m = \mathrm{i}\sigma_s \, \delta_\mu^\alpha \, \partial_\alpha \varphi^m \left[\overline{\rho}\,\overline{\sigma}^\mu \,\varepsilon + \rho\,\sigma^\mu\,\overline{\varepsilon}\right] + O(\sigma_s)^2, \tag{2.4.19c}$$

$$\delta_\varepsilon a_\alpha = -\mathrm{i}\sigma_s \, f_{\alpha\beta} \, \delta_\mu^\beta \left[\overline{\rho}\,\overline{\sigma}^\mu \,\varepsilon + \rho\,\sigma^\mu\,\overline{\varepsilon}\right] + O(\sigma_s)^2. \tag{2.4.19d}$$

Because the supersymmetry variation of the Weyl spinor ρ is never vanishing in the vacuum, such a field is the general goldstino living on the D3-brane worldvolume. In the end, the conclusion is that, along with the standard linearly realised supersymmetry, a D3- or anti-D3-brane at a generic smooth point in the internal manifold of a superstring compactification is expected to be invariant under 4-dimensional non-linearly realised supersymmetry as well. Notice that the supersymmetry variation of the goldstino sets an F-breaking term at the string scale $F \sim \sigma_s^{-1} \sim M_s^2$.

2.4.3 Anti-D3-Branes and Orientifolds

The discussion of anti-D3-branes in the presence of an O3-/O7-plane orientifold is analogous to the D3-brane one. In fact, one can simply observe that, on an anti-D3-brane, the supersymmetry variations preserved by the bulk, generated by $(\chi, \zeta) = (\chi, 0)$, in eqs. (2.4.12a, 2.4.12b, 2.4.12c), are exactly the same as the supersymmetry variations that are projected out for a D3-brane, in eqs. (2.4.16a, 2.4.16b, 2.4.16c), up to an overall sign. Therefore, one can follow exactly the same steps as above. This simply means that the supersymmetry transformations preserved in the bulk theory by O3-/O7-plane orientifold are non-linearly realised by the anti-D3-brane. This is a fundamental conclusion of ref. [83].

2.4.4 Constrained Supermultiplets

A theory with the non-linear supersymmetry transformations of eqs. (2.4.19a, 2.4.19b, 2.4.19c, 2.4.19d) breaks supersymmetry. To describe such a theory in an effective 4-dimensional approach, one can use constrained superfields [31]. This subsection is devoted to explaining this and to associating the degrees of freedom of an anti-D3-brane to a specific constrained supermultiplet [77]. Here, all expressions are written ignoring higher-order terms, i.e. involving more spinor contractions – which is controlled by the σ_s-order – as this is enough to understand the logic behind the identifications. A fully-fledged analysis is in refs. [77, 84].

Notation and conventions for the $N_4 = 1$ supersymmetry transformations are as in ref. [85] and they are reviewed in appendix A.4. It should also be noted that there are instances of fields denoted differently here and elsewhere, but the context is always such that the field being meant is unambiguous.

2.4.4.1 Nilpotent Chiral Superfield

The fundamental constrained superfield which is ubiquitous in non-linear supersymmetry descriptions is a nilpotent chiral superfield X, i.e. a chiral superfield satisfying the constraint

$$X^2 = 0. \tag{2.4.20}$$

Such a constraint removes the scalar degree of freedom and, via a field redefinition, contains a fermion degree of freedom which transforms non-linearly.

Nilpotency Condition on Chiral Superfields

Given a chiral superfield $X = (\varphi, \psi, F)$, let the auxiliary field be non-zero, i.e. $F \neq 0$, and let the nilpotency condition $X^2 = 0$ in eq. (2.4.20) be imposed. Expanding the superfield in superspace coordinates, this constraint is solved by the condition

$$\varphi = \frac{1}{2F}\,(\psi\psi). \tag{2.4.21}$$

This means that the scalar degrees of freedom are dependent on those of the spinor ψ. Although all the dynamical scalar degrees of freedom are encoded in those of the spinor, the supersymmetry transformation of the spinor ψ is still of the standard linear form.

Anti-D3-Brane Goldstino

Following ref. [84], one may perform a field redefinition which determines a non-linearly transforming spinor. A closed-form expression can be written but in the following a perturbative analysis will be performed for brevity. Although not suitable for a rigorous proof, this will give quite a clear feel of the logic behind the discussion. The expansion will be in orders of a supersymmetry breaking scale which is set to be $M^2_{\text{SUSY}} = \sigma_s^{-1}$.

As a matter of fact, given the X-multiplet spinor ψ, one can define a new spinor ρ as

$$\rho_\alpha = \rho^{(0)}_\alpha - \mathrm{i}\sigma_s^2\,[\rho^{(0)}\,\sigma^\mu\,\overline{\rho}^{(0)}]\,\partial_\mu\rho^{(0)}_\alpha + O(\sigma_s)^3, \tag{2.4.22}$$

where the 'zeroth-order' spinor is $\rho^{(0)}_\alpha = \psi_\alpha/[\sqrt{2}\sigma_s F]$. This can be inferred from the closed recursive definition of the non-linearly transforming field in ref. [84] by simply expanding it out. By employing the equality $\overline{\xi}\,\overline{\sigma}^\mu\chi = -\chi\sigma^\mu\overline{\xi}$ and the Fierz identity $\chi_\alpha(\xi\eta) = -\xi_\alpha(\eta\chi) - \eta_\alpha(\chi\xi)$, the usual chiral supersymmetry transformations imply the variation

$$\delta_\epsilon\rho^{(0)}_\alpha = \sigma_s^{-1}\epsilon_\alpha + 2\mathrm{i}\sigma_s\partial_\mu\rho^{(0)}_\alpha(\rho^{(0)}\sigma^\mu\overline{\epsilon}),$$

which in turn gives

$$\delta_\epsilon\rho_\alpha = \sigma_s^{-1}\epsilon_\alpha + \mathrm{i}\sigma_s\big[(\rho\,\sigma^\mu\overline{\epsilon}) + (\overline{\rho}\,\overline{\sigma}^\mu\epsilon)\big]\partial_\mu\rho_\alpha + O(\sigma_s)^2. \tag{2.4.23}$$

This should be compared with the supersymmetry variation in eq. (2.4.19a), and it is apparent that they are the same transformation. This means that the anti-D3-brane goldstino can be identified with the spinor field of a nilpotent chiral superfield. To conclude, notice that the scalar field φ can be written as $\varphi = \sigma_s^2\,(\rho\rho)F + O(\sigma_s)^3$.

2.4.4.2 Orthogonal Chiral Superfield

In order to describe spinors which transform non-linearly in the presence of a different goldstino, one can introduce an orthogonal chiral superfield Y such that

$$XY = 0, \tag{2.4.24}$$

where X is the nilpotent chiral superfield. Such a constraint removes the scalar degree of freedom from the multiplet Y.

Orthogonality Condition on Chiral Superfields

Given two chiral superfields $X = (\varphi, \psi, F)$ and $Y = (\phi, \zeta, G)$, where X is a nilpotent supermultiplet satisfying $X^2 = 0$, the solution to the constraint $XY = 0$ can be written as

$$\phi = \frac{(\psi\zeta)}{F} - \frac{(\psi\psi)}{2F^2}G. \tag{2.4.25}$$

This means that the degrees of freedom of the scalar ϕ are dependent on those of its spinor superpartner ζ_α and of the goldstino ψ_α.

Anti-D3-Brane Modulini

In terms of the goldstino, the constrained scalar reads $\phi = \sqrt{2}\sigma_s\,(\zeta\rho) - \sigma_s^2(\rho\rho)G + O(\sigma_s)^3$. Following ref. [84], the proper non-linear spinor degree of freedom can be defined as

$$\chi_\alpha = \zeta_\alpha - \sqrt{2}\sigma_s\, G\rho_\alpha + 2\mathrm{i}\,\sigma_s^2\,(\zeta\,\partial_\mu\rho)(\sigma^\mu\overline{\rho})_\alpha - \mathrm{i}\,\sigma_s^2\,(\rho\,\sigma^\mu\overline{\rho})\,\partial_\mu\zeta_\alpha + O(\sigma_s)^3. \tag{2.4.26}$$

Focussing on the leading orders in σ_s, which can be done by studying $\sigma_s\,\delta_\epsilon\chi_\alpha$, one finds

$$\delta_\epsilon\chi_\alpha = \mathrm{i}\sigma_s\left[(\rho\,\sigma^\mu\overline{\epsilon}) + (\overline{\rho}\,\overline{\sigma}^\mu\epsilon)\right]\partial_\mu\chi_\alpha + O(\sigma_s)^2. \tag{2.4.27}$$

Again, this corresponds to one of the anti-D3-brane supersymmetry variations, i.e. eq. (2.4.19b). This means thar anti-D3-brane modulini can be packaged into chiral superfields orthogonal to the nilpotent superfield.

2.4.4.3 Chiral-Product Superfield

Another class of constrained chiral superfields which keeps only the scalar component is that of chiral superfields H such that, given a nilpotent superfield X fulfilling the relation $X^2 = 0$, the condition holds

$$X\overline{D}_{\dot{\alpha}}\overline{H} = 0, \tag{2.4.28}$$

where D_α represents the supersymmetric covariant derivative. In other words, the product superfield $X\overline{H}$ is chiral. Such a constraint removes the spinor and auxiliary degrees of freedom from the multiplet H.

Chiral-Product Superfield Condition

Given two chiral superfields $X = (\varphi, \psi, F)$ and $H = (\phi_H, \psi_H, F_H)$, where X is a nilpotent supermultiplet satisfying $X^2 = 0$, let the product superfield $K(X, \overline{H}) = X\overline{H}$ be chiral.

In this case, solving the constraint is less easy. It is convenient to expand the superfield H around the shifted coordinate $\overline{y}^\mu = x^\mu - \mathrm{i}\,(\overline{\theta}\overline{\sigma}^\mu\theta)$ and to require the product $\overline{K} = \overline{X}H$ to be antichiral, i.e. requiring involving at least one direction θ to be zero. Notably, doing this requires two inequivalent conditions, namely

$$\psi_{H\alpha} = -\frac{\mathrm{i}}{\overline{F}}\,(\sigma^\mu\overline{\psi})_\alpha\,\partial_\mu\phi_H, \tag{2.4.29a}$$

$$F_H = \frac{1}{\overline{F}^2}\,(\overline{\psi}\,\overline{\sigma}^\mu\sigma^\nu\overline{\psi})\,\partial_\mu\partial_\nu\phi_H - \frac{1}{\overline{F}}\left[\overline{\psi}\,\overline{\sigma}^\mu\sigma^\nu\partial_\mu\left(\frac{\overline{\psi}}{\overline{F}}\right)\right]\partial_\nu\phi_H. \tag{2.4.29b}$$

Unlike the solutions to previous constraints, in this case both a physical field and the auxiliary field are forced to depend directly on other physical fields. This means that superpotential terms $W = \mu H^2/2$ do not generate mass terms but more fermionic interactions.

Anti-D3-Brane Scalars

In this case, the analysis of a full field redefinition which generates the expected non-linear transformations is even more involved than above. However, up to the lowest order in σ_s the analysis is elementary. In physical units, the constrained spinor and auxiliary scalar fields read $\psi_{H\alpha} = -\mathrm{i}\sigma_s\sqrt{2}\,(\sigma^\mu\overline{\rho})_\alpha\,\partial_\mu\phi_H + O(\sigma_s)^3$ and $F_H = -2\sigma_s^2\,\eta^{\mu\nu}(\overline{\rho}\overline{\rho})\,\partial_\mu\partial_\nu\phi_H + 4\sigma_s^2\,\eta^{\mu\nu}(\overline{\rho}\partial_\mu\overline{\rho})\,\partial_\nu\phi_H + O(\sigma_s)^4$. Again inspired by ref. [84], the appropriate non-linearly transforming scalar φ_H can be simply defined as

$$\varphi_H \equiv \phi_H + \mathrm{i}\sigma_s^2\,(\rho\,\sigma^\mu\overline{\rho})\,\partial_\mu\phi_H + O(\sigma_s)^3. \tag{2.4.30}$$

Indeed, as a matter of fact, studying $\sigma_s\,\delta_\epsilon\varphi_H$, one finds the supersymmetry transformation

$$\sigma_s\,\delta_\epsilon\varphi_H = \mathrm{i}\sigma_s^2\left[(\overline{\rho}\,\overline{\sigma}^\mu\epsilon) + (\rho\,\sigma^\mu\overline{\epsilon})\right]\partial_\mu\varphi_H + O(\sigma_s)^3. \tag{2.4.31}$$

Because this corresponds to the supersymmetry variation in eq. (2.4.19c), the anti-D3-brane scalars can be described by a constrained superfield of the form discussed here.

2.4.4.4 Orthogonal Field-Strength Vector Superfields

As far as vector superfields are concerned, it is possible to impose a constraint that removes the gaugino. Such a constraint can be expressed as

$$XW_\alpha = 0, \tag{2.4.32}$$

where X is a nilpotent chiral superfield and W_α is the chiral superfield describing the supersymmetric field-strength tensor of a vector multiplet $V = (\lambda, A_\mu, D)$.

Orthogonality Condition on Supersymmetric Field-Strengths

A way to remove the spinor degree of freedom from a vector multiplet in the presence of a nilpotent superfield consists in imposing an orthogonality condition on its supersymmetric field-strength tensor. However, for gauge fields things are generally slightly more complicated due to gauge transformations. On the other hand, the analysis of the constraint itself is not really involved.

The issue of supergauge transformations and the constraints on superfields can be addressed by analysing an explicit example. In a general theory with a Goldstone boson a and a gauge field A_μ, gauge transformations read

$$\begin{aligned} a &\to a + f, \\ A_\mu &\to A_\mu + \partial_\mu f, \end{aligned}$$

for an arbitrary real function f. As discussed in ref. [31], the Goldstone boson a can be described by a chiral superfield $A = (a + ib, \psi_A, F_A)$ satisfying the constraint

$$X(A - \overline{A}) = 0,$$

where X is the nilpotent chiral superfield. On the other hand, the vector field A_μ is encoded in a vector superfield V. So one can define the supergauge transformations

$$\begin{aligned} A &\to A + \Omega/2, \\ V &\to V - \mathrm{i}(\Omega - \overline{\Omega}), \end{aligned}$$

which means that the theory can be supergauge invariant only if the chiral superfield Ω is such as to satisfy the condition

$$X(\Omega - \overline{\Omega}) = 0.$$

This fact has the important consequence that the WZ-gauge cannot be selected for the vector superfield V as the field Ω is subject to some constraints and is thus not completely arbitrary. A good gauge choice is then the one in which

$$XV = 0,$$

since it is gauge-invariant with the above constraint on Ω. In this gauge, the vector superfield V no longer has the especially simple form with only the goldstino, the vector and the auxiliary scalar fields, but rather has three more scalars and one more spinor which depend on the non-redundant degrees of freedom. This gauge choice, although technically different from the WZ-gauge, sets the components that are normally zero to be proportional to powers of the goldstino.

The easiest way to remove the spinor degree of freedom from a vector multiplet consists in imposing an orthogonality condition between the nilpotent chiral superfield X and the supersymmetric field-strength tensor W_α associated to the vector supermultiplet V. It is convenient to express the supersymmetric field-strength superfield as

$$W_\alpha = \lambda_\alpha + (L_\alpha{}^\beta \theta_\beta) + \mathrm{i}(\theta\theta)(\sigma^\mu \partial_\mu \overline{\lambda})_\alpha.$$

where the spinor-valued term ${L_\alpha}^\beta$ is defined as

$$ {L_\alpha}^\beta = \delta_\alpha^\beta \, D + \frac{\mathrm{i}}{2} \, {(\sigma^\mu \overline{\sigma}^\nu)_\alpha}^\beta \, F_{\mu\nu}. $$

By a direct comparison with the expression of the chiral superfield Y and of the orthogonality condition $XY = 0$, the solution to this constraint can be derived straightforwardly and it reads

$$ \lambda_\alpha(x) = \frac{1}{\sqrt{2}F(x)} \, {L_\alpha}^\beta \psi_\beta - \frac{\psi^\beta \psi_\beta}{2F^2(x)} \, \mathrm{i}(\sigma^\mu \, \partial_\mu \overline{\lambda})_\alpha. \tag{2.4.33} $$

Such an equation could be solved explicitly by substituting recursively the definition of the gaugino inside it, ending with a finite number of terms due to the Grassmannian nature of the former. However, this is not particularly useful for the following description.

Anti-D3-Brane Gauge Field

The constraint on the spinor component can be expressed in physical units in terms of the goldstino as $\lambda_\alpha = \sigma_s {L_\alpha}^\beta \rho_\beta - \mathrm{i}\sigma_s^2 (\rho\rho) \, (\sigma^\mu \partial_\mu \overline{\lambda})_\alpha + O(\sigma_s)^3$. Note that in the alternative gauge, the supersymmetry variation of the gauge vector would also involve the derivative of the spinor field that is usually set to zero in the WZ-gauge but that in this case is $\xi^\alpha = \sigma_s (\overline{\rho}\, \overline{\sigma}^\mu)^\alpha A_\mu + O(\sigma_s)^2$, as shown by ref. [31]. If the WZ-gauge can be imposed, however, following ref. [86], the vector field which transforms non-linearly can be expressed simply as

$$ a_\mu = A_\mu - \sigma_s^2 \, (\rho \, \sigma^\nu \overline{\rho}) \left(\eta_{\mu\nu} D - \frac{1}{2} F^{\lambda\kappa} \varepsilon_{\mu\nu\lambda\kappa} \right) + O(\sigma_s)^3. \tag{2.4.34} $$

Indeed, in this case, a few manipulations, together with the Pauli-matrix relationship $\overline{\sigma}^\mu \sigma^\nu \overline{\sigma}^\lambda = -\eta^{\mu\nu} \overline{\sigma}^\lambda - \eta^{\nu\lambda} \overline{\sigma}^\mu + \eta^{\mu\lambda} \overline{\sigma}^\nu + \mathrm{i} \, \varepsilon^{\mu\nu\lambda\kappa} \overline{\sigma}_\kappa$, indicate that the supersymmetry variation turns out to be

$$ \sigma_s \, \delta_\epsilon a_\mu = -\mathrm{i}\sigma_s^2 \left[(\rho \, \sigma^\nu \overline{\epsilon}) + (\overline{\rho} \, \overline{\sigma}^\nu \epsilon) \right] f_{\mu\nu} + O(\sigma_s)^3, \tag{2.4.35} $$

which matches the variation of eq. (2.4.19d). This means that the gauge field of an anti-D3-brane can be described in terms of a constrained multiplet of the form discussed here.

3 MISALIGNED SUPERSYMMETRY IN STRING THEORY

This chapter discusses misaligned supersymmetry in string-theoretic realisations, going over the material presented in the articles [2,3].

The contents are organised as follows. To start, section 3.1 contextualises the research about misaligned supersymmetry in string theory. More ideas to motivate how misaligned supersymmetry is a potential way-out to address the hierarchy problem are then outlined in section 3.2. After this, a review of the core ideas underlying misaligned supersymmetry in closed-string theories is presented in section 3.3. As an example, the non-supersymmetric heterotic SO(16) $\times$ SO(16)-theory is discussed in detail, showing that misaligned supersymmetry is present at leading order. Then, section 3.4 shows that the structural features of misaligned supersymmetry can also be present for open strings. In particular, it is argued that the spectrum of an anti-Dp-brane on top of an Op-plane shows that supersymmetry is not just broken but also misaligned. This serves as a prototypical example. Moving on, in sections 3.5 and 3.6, a proof that the cancellations characterising misaligned supersymmetry take place at any order in the Hardy-Ramanujan-Rademacher expansion of the state degeneracies is provided for a vast and generic class of models whose partition functions are Dedekind η-quotients. The heterotic SO(16) $\times$ SO(16)-theory and the anti-Dp-brane/Op-plane system are studied in full detail. After this, in section 3.7, misaligned supersymmetry is analysed in a more formal perspective, showing the precise connection between a misaligned particle spectrum and a finite one-loop cosmological constant. Then, section 3.8 comments on string-based supertraces. To conclude, in section 3.9, the known 10-dimensional tachyon-free non-supersymmetric theories are discussed in terms of misaligned supersymmetry. Finally, section 3.10 offers a recap with the main conclusions.

3.1 Context

Understanding the presence of supersymmetry in string-theory constructions is of the utmost importance, if one aims to describe the real world from a microscopic point of view in this framework. In this respect, it is essential to distinguish models in which the spectrum is not supersymmetric only below a certain energy scale from those in which supersymmetry cannot be restored in an effective field theory since either it is not present at all or it is broken at the string scale. Such models, i.e. those that can experience only the non-supersymmetric phase, are going to be the subject of the present analysis.

Among many non-supersymmetric string constructions, two notable examples, which are going to be investigated in detail as prototypical instances in what follows, are the non-supersymmetric heterotic $SO(16) \times SO(16)$-theory [12, 13] and the open-string system constituted by an anti-Dp-brane on top of an Op-plane in type II string theories. This second class of models also provides a realisation of the general phenomenon of brane supersymmetry breaking, discussed in refs. [5,14,18–24,87], and studied also in refs. [25,26]. Further notable examples that are going to be studied are the Sugimoto USp(32)-model [14], indeed a prototypical instance for brane supersymmetry breaking, and the type 0'B SU(32)-theory [15–17].

A crucial point to discuss is the structure that explains how string-theory constructions can be capable of maintaining finiteness and stability, even in the absence of supersymmetry in the target spacetime. Ultimately, this has to be related to modular invariance: for closed strings, this is apparent in the expression of the one-loop cosmological constant; for open strings, only a subgroup is preserved, but nonetheless this is sufficient for a constraint. A proposal to explain such a finiteness in physically-intuitive terms, introduced in refs. [27,28] and further developed in refs. [28, 88–92], is the idea of misaligned supersymmetry. In this perspective, the string-theory finiteness is explained as a consequence of exponentially-growing oscillations between the net number of bosons and fermions, at each mass level, that lead to cancellations when considering the entire infinite tower of string states as a whole, thanks to the underlying string-based structure of the theory. Instead, standard supersymmetric spectra would only lead to finite quantities as a consequence of cancellations taking place at each individual mass level, due to an exact matching in the number of fermionic and bosonic degrees of freedom. The identification of an almost exact cancellation between closed-string spacetime bosons and fermions in the asymptotic density of states in the worldsheet theory dates back to the work of ref. [93]. The fundamental intuition of refs. [27, 28] is that such a cancellation does not need to be necessarily level-by-level, as for standard supersymmetry, hence the name 'misaligned supersymmetry', or, sometimes, 'asymptotic supersymmetry'. A first extension to open-string models is in ref. [94], where it is argued that misaligned supersymmetry is needed to decouple the open-string sector from a closed-string tachyon.

A variety of aspects renders non-supersymmetric constructions and misaligned supersymmetry remarkable. To start, a peculiar trait of the brane supersymmetry-breaking scenario, compared to the other non-supersymmetric models, is the presence of a gravitino in the spectra of the former. In this case, the absence of a mass term for the gravitino is still compatible with supersymmetry breaking at the string scale, which leads to a non-linear realisation of supersymmetry in the spacetime effective theory [23] – see also refs. [24, 83, 95, 96]. This in itself represents a notable fact since in more standard scenarios supersymmetry is broken at lower energies by some F- or D-term dynamically-fixed non-zero value. Yet, it is going to be argued that misaligned supersymmetry manifests more generally in non-supersymmetric theories. It is also worth mentioning that, while the heterotic $SO(16) \times SO(16)$-theory and the Sugimoto model have their entire spectrum in a standardly or misalignedly supersymmetric phase, and therefore do exhibit misaligned supersymmetry, this is not the case for the type 0'B theory. The latter is somewhat special since it presents misaligned supersymmetry only in the open-string sector (annulus and Möbius strip), whereas its closed-string sector does not present any sort of supersymmetry whatsoever, containing only bosons. It turns nonetheless out that the structure ensuring the absence of UV-divergences in the latter can also be described in terms of a misaligned action of the orientifold symmetry (in the Klein bottle).

From a different perspective, the interplay between anti-Dp-branes and Op-planes plays an important role in phenomenologically important type II string-theory constructions with broken supersymmetry. In particular, an anti-D3-brane in a Calabi-Yau orientifold background with an O3-plane is at the core of the KKLT- and LVS-proposals for the string-theory realisation of de Sitter vacua [54, 55]. The anti-D3-brane can be placed on top of an O3-plane that is located at the bottom of a warped throat [95, 97]. As a simplified version of this scenario, the analysis presented here studies an analogous flat-space model and discovers an enlightening connection to misaligned supersymmetry. More generically, non-supersymmetric branes lead to 4-dimensional low-energy effective theories with broken supersymmetry, which constitute a much wider class of models than supersymmetric ones [96] and could lead to phenomenologically interesting non-supersymmetric realisations of the Standard Model [1, 30, 98]. These are phenomenological motivations for which it is desirable to revisit and investigate further anti-Dp-branes and Op-planes in string theory, uncovering their connection to misaligned supersymmetry. In string-theory model building, heterotic string models exhibiting misaligned supersymmetry have been previously analysed in refs. [90, 91, 99–103], while for open strings relevant developments are for instance in refs. [26, 87, 104–107]. More on the mathematical side, in ref. [89] an intriguing connection to the Riemann hypothesis has been proposed, relating the zeros of the ζ-function to the oscillation of the physical degeneracies, thus pointing towards the presence of a rich and interesting structure behind non-supersymmetric models.

Sufficient conditions for misaligned supersymmetry in closed-string theories have been found to be modular invariance and the absence of physical tachyons [27]. Indeed, it is well known that modular invariance is inherent for closed strings and, in particular, it dictates how left- and right- moving sectors are coupled. In closed-string theories, the cancellations implied by misaligned supersymmetry occur precisely in accordance with the way modular invariance fixes the couplings among sectors: changing these couplings would in general spoil modular invariance and prevent such cancellations from occurring. In this work, the original literature is reviewed, discussing then in detail the non-supersymmetric heterotic SO(16)×SO(16)-theory and showing how the predicted cancellations occur. Then, the attention is turned towards open-string models and, in particular, the paradigmatic example under consideration is the system in which an anti-Dp-brane is placed on top of an Op-plane in type II string theories. Here, the orientifold involution generically breaks modular invariance down to a subgroup. Nevertheless, misaligned supersymmetry can still be argued to be at work. It is such a result that points out that brane supersymmetry breaking and misaligned supersymmetry are related. This hints at a deeper connection that would substantially improve the understanding of non-supersymmetric type II string theory compactifications.

To identify misaligned supersymmetry, it is necessary to study the net boson-fermion degeneracies of the physical states, which are encapsulated in the partition function of the theory. Because of the underlying modular properties of such partition functions, these net physical degeneracies can be obtained from a Hardy-Ramanujan-Rademacher sum. Prior to this work, misaligned supersymmetry has been demonstrated only by looking at the leading exponentials in such an expansion, but here a method is introduced that allows one to study all terms. In order to do that, the discussion is specialised to the case in which the partition function can be written entirely in terms of quotients of Dedekind η-functions and, for such a subclass of theories, it is shown analytically how misaligned supersymmetry is at work at any order of the Hardy-Ramanujan-Rademacher expansion. In particular, a systematic procedure is developed to show the occurrence of the required cancellations at all orders in the expansions defining the number of states. This very same procedure can be applied for both open- and closed-string theories. Such a formalism allows one to prove that the state degeneracies vanish in an averaged sense, where to take the average one needs to introduce the envelope functions interpolating the net degeneracies. Once more, the heterotic SO(16)$\times$SO(16)-theory and the type II models with an anti-Dp-brane on top of an Op-plane are going to serve as two explicit examples. This can be an important step in understanding misaligned supersymmetry as a general property of String Theory and of String Phenomenology in particular.

As a further expansion of the discussion, this work contains a detailed analysis of the mathematical structure of misaligned supersymmetry. In particular, it relates in a definite way the machinery for the cancellation of the envelope functions for the net state degeneracies to the fermion-boson cancellations that are interpreted as the origin of finite physical observables in theories exhibiting misaligned supersymmetry. In fact, although intuitively immediate, the actual mathematical meaning of such envelope functions is not apparent in their original formulation [27, 88], as they only serve as a conceptual tool to visualise the cancellations that are reflected in the amplitude finiteness. So, whilst the original literature has interpreted the finiteness of non-supersymmetric theories exhibiting misaligned supersymmetry as a consequence of this misalignment, here it is shown how to observe such cancellations at work, in an exact way, in the fully-fledged mathematical expression of the one-loop cosmological constant. This represents a physically intuitive explanation that should parallel the usual arguments based on modular invariance, and in fact the discussion is going to emphasise the close relationship in the two descriptions.

In detail, it is possible to show that the cancellations that take place in the heuristic sector-average net degeneracy also appear manifestly in the one-loop cosmological constant. Indeed it is going to be shown that misalignment leads to a cancellation of all exponential divergences in the latter, and a similar structure is expected to emerge in the other quantum-corrected observables too. The modular properties of the partition functions further lead leftover power-law divergences to cancel out, leading to an overall UV-finite result. This is going to be discussed for both open and closed strings. Although the role of modular invariance in ensuring finiteness is well-known for closed-string theories, and therefore these results are simply a parallel explanation, it is going to be proven that a similar modularly-constrained structure also exists for open strings, as anticipated. In principle, this might be considered surprising, since modular invariance is explicitly broken by the worldsheet boundary: in this respect, it is nonetheless going to be argued that a remnant of the original modular group is enough to explain finiteness in the open-string models under scrutiny. All in all, the crucial aspect is simply the well-behaved behaviour under S-transformations, although not necessarily covariant.

3.2 Hierarchy Problem and Supersymmetry

One of the core issues of the Standard Model of Particle Physics is the hierarchy problem. Because of quantum loop corrections, this model has to be considered as an effective field theory valid below a cutoff scale m_{c}. The cutoff scale must be much larger than the typical scales characterising the physics of the Standard Model, a reference for which can be taken to be the electroweak scale m_{EW}, and actually larger than the current experimentally accessible scales m_{exp} at particle colliders, since no new particles have been observed up to that scale yet. At the same time, this cutoff scale should not exceed the Planck scale m_P, at which quantum-gravity effects enter into play. This represents a problem since the standard-model physics takes place at scales near the scale m_{EW} but the quantum corrections, in the absence of fine tuning, tend to bring this up to scales of order m_{c}. This is not just a naturalness problem about the explanation of the hierarchy $m_{\mathrm{EW}}/m_P \sim 10^{-16} \ll 1$, but an issue itself about the mechanisms that keep the electroweak scale much smaller than the Planck scale even after quantum corrections [85, 108].

3.2.1 Supersymmetry

To show more concretely how the hierarchy problem presents itself and how one can attempt to solve it, as an example, following ref. [85], one can consider the complex Higgs field H, with the usual scalar potential $V = -\mu_H^2\, H\overline{H} + \lambda_H (H\overline{H})^2$. The mass-like parameter roughly sets the electroweak scale m_{EW}, so its input value should be of order $\mu_H^2 \sim m_{\mathrm{EW}}^2$, while the self-interaction quartic coupling is some perturbative parameter $\lambda_H < 1$. If this Higgs field is coupled to a Dirac spinor ψ of mass m_ψ via a Yukawa coupling $\mathcal{L}_y = -yH\overline{\psi}\psi$, then the one-loop correction to the Higgs mass parameter is

$$\Delta\mu_H^2 = \frac{|y|^2}{8\pi^2} m_{\mathrm{c}}^2 - f|y|^2 m_\psi^2 \ln\frac{m_{\mathrm{c}}}{m_\psi} + \delta_y\mu_H^2,$$

where f is a factor that is actually different for the real and imaginary components of H and $\delta_y\mu_H^2$ is a cutoff-independent term. This correction makes it apparent that the Higgs mass parameter tends to be driven up to scales of order m_{c}. At the same time, if the Higgs is also coupled to a complex scalar ϕ of mass m_ϕ with the quartic coupling $\mathcal{L}_\lambda = -\lambda H\overline{H}\phi\overline{\phi}$, then the one-loop correction to the Higgs mass parameter is

$$\Delta\mu_H^2 = -\frac{\lambda}{16\pi^2}\left[m_{\mathrm{c}}^2 - 2\, m_\phi^2 \ln\frac{m_{\mathrm{c}}}{m_\phi}\right] + \delta_\lambda\mu_H^2,$$

with $\delta_\lambda\mu_H^2$ being a cutoff-independent term. This correction has a similar impact as above. However, a crucial observation to make is the following: if the presence of each Dirac spinor is accompanied by a pair of complex scalars with coupings such that $|y|^2 = \lambda$, then the quadratic cutoff corrections are exactly such as to cancel out. Imposing the condition that all loop corrections actually cancel out is equivalent to requiring the spectrum to be supersymmetric. Supersymmetry is the most general symmetry relating bosons and fermions consistently with relativistic quantum field theory and it emerges as the element that extends the standard Poincaré group to its graded boson-fermion generalisation, i.e. the super-Poincaré group [109, 110]. Supersymmetry constrains all bosons of integer spin $s_b \in \mathbb{N}_0$ to come in pair with fermions of semi-integer spin $s_f \in \mathbb{N}_0 + 1/2$, and viceversa. In particular, supersymmetry requires the masses and the charges of these pairs of spinors and bosons to be the same, which is the reason for the cancellations.

After formulating a supersymmetric quantum field theory with particles p_i of mass M_i^2 and spin s_i, for some particle label $i \in \mathrm{P}$, one can generically write the one-loop corrections to the Higgs mass and to the cosmological constant as [88]

$$\Delta\mu_H^2 = \epsilon_0 \big(\mathrm{str}\, 1\big)\, m_\mathrm{c}^2 + \epsilon_2 \big(\mathrm{str}\, M^2\big) \ln\left(\frac{m_\mathrm{c}}{m_0}\right) + \delta\mu_H^2,$$
$$\Lambda = \xi_0 \big(\mathrm{str}\, 1\big)\, m_\mathrm{c}^4 + \xi_2 \big(\mathrm{str}\, M^2\big)\, m_\mathrm{c}^2 + \xi_4 \big(\mathrm{str}\, M^4\big) \ln\left(\frac{m_\mathrm{c}}{m_0}\right) + \delta\Lambda,$$

where m_0 is some reference mass scale of the order of the mass of the particles in the model, $\epsilon_{2\beta}$ and $\xi_{2\beta}$ are some parameters depending on the interaction couplings, and $\delta\mu_H^2$ and $\delta\Lambda$ are further cutoff-independent terms. These expressions depend on the values of the supertraces, which are defined as

$$\mathrm{str}\, M^{2\beta} = \sum_{i \in \mathrm{P}} (-1)^{2s_i} M_i^{2\beta}.$$

In a supersymmetric theory, the supertraces are all vanishing, with a net cancellation of all the particle contributions against the contribution of their superpartner. Whilst in principle this is an elegant solution to the hierarchy problem, it is hard to reconcile with observations since the observed universe does not show supersymmetry at the experimentally accessible scales. In fact, even if supersymmetry is just spontaneously broken, to account for the current experimental lack of superpartners, the scale m_SUSY of this breaking must be above the highest experimental scale m_exp. As soon as the boson-fermion matching is perturbed, though, the supertraces are no longer zero, but rather they generically are of the same order of magnitude as the mass splittings. Although in principle this may still leave some room for explaining the scale of the Higgs mass, it is very hardly compatible with the tiny observed cosmological constant $\Lambda_0^{1/4} \sim 10^{-15} m_\mathrm{EW} \ll m_\mathrm{c}$. The difference in the Higgs-mass and cosmological-constant scales is an inherent problem for the broken-supersymmetry solution. Among many possibilities to explain this conundrum, it is compelling to wonder whether misaligned supersymmetry and non-supersymmetric string theories may offer a satisfactory and elegant solution [88].

3.2.2 Misaligned Supersymmetry

The fundamental feature defining a theory that exhibits misaligned supersymmetry consists in a particle spectrum such that all the quantum-corrected physical observables are finite as an effect of an overall net boson-fermion cancellation. This represents a generalisation of the supersymmetric pairwise boson-fermion cancellations.

In more detail, the finiteness of a theory with misaligned supersymmetry is a consequence of the overall summation of the quantum effects of all the bosons and the fermions in the theory, even in the lack of a supersymmetric boson-fermion pairing. In view of the quantum field theory results overviewed so far, in order for such a scenario to be possible, one is led to consider the presence of an infinite number of particles and to admit the absence of symmetry-breaking scales. In fact, in order to have a systematic cancellation of the quantum divergences, an infinite number of finely distributed states is expected to be needed, without considering a cutoff scale. This is the reason why 'misaligned supersymmetry' may as well be referred to as 'asymptotic supersymmetry'. Furthermore, misaligned supersymmetry does not necessarily need to be broken in the vacuum, so the electroweak and cosmological-constant scales do not need to be related by any symmetry-breaking scale. In the presence of an infinite number of states and without supersymmetry, the standard definition of supertrace obviously diverges. One can introduce a regularised version of the supertraces by defining them as

$$\mathrm{Str}\, M^{2\beta} = \lim_{y \to 0^+} \left[\sum_{i \in \mathrm{P}} (-1)^{2s_i} M_i^{2\beta}\, \mathrm{e}^{-y M_i^2/\mu^2} \right],$$

where μ^2 is some mass scale. Obviously this definition at this point is arbitrary, and not related to any physical observable, but for a moment it can be assumed to be a generalisation of the standard supertrace definition for a theory with an infinite number of states. It contains an exponential regulator and it reduces to the standard definition for a finite number of states. If, for instance, one considers a toy model with net boson-fermion degeneracies $N_b(n) - N_f(n) = (-1)^n n^2$ and mass levels $M_n^2 = \mu^2 n$, then the regularised supertraces read $\mathrm{Str}\, 1 = 0$, $\mathrm{Str}\, M^2 = \mu^2/8$ and $\mathrm{Str}\, M^4 = 0$. Of course, just this by itself is not enough to explain the validity or the meaning of the new supertraces, but it should give a feeling that infinitely many states may combine together in such a way as to give finite results. It is conceivable that this may then help in addressing the hierarchy problem, if for instance cancellations within supertraces could ensure not simply finite but actually highly-suppressed loop corrections.

This whole situation shares analogies with non-supersymmetric string theories. These are string-theory constructions that lack spacetime supersymmetry but that nonetheless still give finite one-loop quantum corrections to the tree-level results. This, in fact, can be interpreted as a manifestation of misaligned supersymmetry. The key aspect to bear in mind is that string theory is not just a theory that in a low-energy limit produces an effective quantum field theory, but rather it is a more complex theory that possesses additional features. In particular, its one-loop corrections are invariant under the modular group $\mathrm{PSL}_2(\mathbb{Z})$, in the case of closed strings, or, at least, are constrained by some dualities related to their behaviour under the modular group, in the case of open strings. This requires these models to have an infinite number of degrees of freedom and, in particular, it relates the low-energy states to the infinitely many extremely heavy ones in specific ways. In the easiest realisations, such properties ensure for instance one-loop cosmological constants that, besides the string-coupling powers parametrising the perturbative loop orders, are set at orders of magnitude around the string scale. More elaborated constructions may be studied to understand whether configurations exist in which such finite corrections are also small.

The following sections in this chapter review these ideas and illustrate developments that make them more precise in a vast class of non-supersymmetric string theories, including both closed and open strings.

3.3 Misaligned Supersymmetry in Closed Strings

This section is devoted to a review of the concept of misaligned supersymmetry in closed-string theories, following mainly the discussion in refs. [27, 28, 88]. Then, the heterotic SO(16)×SO(16)-theory is analysed in detail as an instructive example in which misaligned supersymmetry is at work. Unless differently specified, in this chapter the term ‘closed strings’ refers to torus amplitudes: while, generally speaking, this is unambiguous in heterotic theories, in type II theories the presence of orientifolds requires the introduction of Klein-bottle, annulus and Möbius-strip amplitudes. The analysis of ‘open strings’ in this chapter covers also the Klein-bottle amplitudes, since technically their structure is analogous to that of annuli and Möbius strips, i.e. the actual open-string terms. In all cases where these distinctions are relevant, they will be pointed out explicitly.

3.3.1 Misaligned Supersymmetry Review

The presence of misaligned supersymmetry in String Theory can be understood from several perspectives. It can be formulated as the occurrence of exponentially-growing oscillations in the net number of bosonic minus fermionic physical states at each energy level. Equivalently, it can be related to the presence of unphysical tachyons in the partition function, namely virtual excitations with negative squared mass, which are not dangerous as long as they remain off-shell. Alternatively, it can be deduced by looking at the asymptotic behaviour of an appropriately defined sector-averaged number of states that is argued to grow more slowly than the various state degeneracies as the energy increases. In all descriptions, these features should be manifested in finite quantum corrections in the physical observables. Below, a discussion of how all of these concepts are intertwined is outlined.

The natural starting point for the discussion of misaligned supersymmetry is the string one-loop torus partition function.[3.1] Given the squared nome $q = \mathrm{e}^{2\pi \mathrm{i}\tau}$, in a conformal field theory with sectors labelled by an index $i = 0, \ldots, N-1$, let $\chi_i(q)$ be the characters of an irreducible representation $|h_i\rangle$ of highest weight h_i, with $h_i \geq 0$ in unitary theories, i.e.

$$\chi_i(\tau) = \mathrm{tr}_{\mathcal{H}_i}\, q^{L_0 - c/24}, \tag{3.3.1}$$

[3.1] Despite the approach based on lightcone quantisation presented in subsection 2.1.3, in this subsection the more general approach based on conformal field theory is considered; for reviews, see refs. [33, 34].

where L_0 is the zeroth Virasoro generator, c is the central charge of the worldsheet conformal field theory, and $\mathcal{H}_i$ is the Hilbert space built via the application of the raising Virasoro operators L_{-n} on the highest-weight state $|h_i\rangle$, with $n > 0$. The characters constitute an N-dimensional representation of the modular group with weight $k \in \mathbb{Z}/2$, meaning that, for a modular $\mathrm{PSL}_2(\mathbb{Z})$-transformation $M\tau = (a\tau + b)/(c\tau + d)$, with τ being defined in the fundamental $\mathrm{PSL}_2(\mathbb{Z})$-domain $\mathbb{F} = \{\tau \in \mathbb{C} :\ \mathrm{Re}\,\tau \in [-1/2, 1/2] \wedge \mathrm{Im}\,\tau \in [0, +\infty[\wedge |\tau| \in [1, +\infty[\}$, they transform as

$$\chi_i(M\tau) = (c\tau + d)^k \sum_{j=1}^{N} M_i{}^j \chi_j(\tau),$$

where $M_i{}^j$ is the matrix representing the action of the modular-group element in the basis of the characters χ_i. It is assumed that the characters can be expanded in terms of non-negative coefficients $a_{i,n}$ in a Laurent series of the form

$$\chi_i(\tau) = q^{H_i} \sum_{n=0}^{\infty} a_{i,n} q^n, \tag{3.3.2}$$

where $H_i = h_i - c/24$ is the vacuum energy of the sector i. The coefficients $a_{i,n}$ count the degeneracy of states of the sector i at the excited level n.

In a closed-string theory formulated in a D-dimensional non-compact spacetime, one has to consider the mixing of the characters of both the right- and left-moving highest-weight sectors χ_i and $\overline{\chi}_{\overline{\jmath}}$, respectively. In fact, the one-loop torus partition function can be written in general as

$$Z(\tau, \overline{\tau}) = (\mathrm{Im}\,\tau)^{1-\frac{D}{2}} \sum_{i=0}^{N-1} \sum_{\overline{\jmath}=0}^{N-1} A^{i\overline{\jmath}} \chi_i(q) \overline{\chi}_{\overline{\jmath}}(\overline{q}), \tag{3.3.3}$$

where $A^{i\overline{\jmath}}$ is a character-mixing matrix. In consistent physical models, the partition function must be modular-invariant, i.e. it must be unchanged under the action of the modular group, transforming as $Z(\tau, \overline{\tau}) = Z(M\tau, \overline{M\tau})$. This requires the modular weights to be $k = 1 - D/2$ and it constrains the shape of the character-mixing matrix to be such that $A^{i\overline{\jmath}} = \sum_p \sum_{\overline{q}} M_p{}^i A^{p\overline{q}} \overline{M}_{\overline{q}}{}^{\overline{\jmath}}$. Also, a T-transformation can be seen to require the condition $H_i - \overline{H}_{\overline{\jmath}} \in \mathbb{Z}$ for all $(i, \overline{\jmath})$-pairs such that $A^{i\overline{\jmath}} \neq 0$. Moreover, all the entries of the mixing matrix must be non-negative as they correspond to the coefficients defining the total Hilbert space, with $\mathcal{H} = \bigoplus_{i,\overline{\jmath}} A^{i\overline{\jmath}} \mathcal{H}_i \otimes \overline{\mathcal{H}}_{\overline{\jmath}}$. In practice, however, one often deals with pseudocharacters, i.e. linear combinations of characters. In the following, physical tachyons are assumed to be absent from the spectrum.

3.3.1.1 HRR-Expansions and Net Boson-Fermion Cancellations

For simplicty, it is assumed that all the characters are normalised in such a way that $a_{i,0} = 1$. As long as one can define a set of characters that is closed under the modular $\mathrm{PSL}_2(\mathbb{Z})$-group, one can write the general expression of the Laurent coefficients $a_{i,n}$ of the characters

$\chi_i(q)$ by means of a Hardy-Ramanujan-Rademacher series as [27, 111, 112]

$$a_{i,n} = \sum_{\alpha=1}^{\alpha_{\max}} \frac{2\pi}{\alpha} \sum_{j=0}^{N-1} Q(\alpha;n)_i^{\ j} f_j(\alpha;n+H_i) + \delta a_{i,n}(\alpha_{\max}), \tag{3.3.4}$$

where the definitions have been made

$$Q(\alpha;n)_i^{\ j} = \mathrm{e}^{-\frac{\mathrm{i}\pi k}{2}} \sum_{\substack{0\le\beta<\alpha: \\ \gcd(\alpha,\beta)=1}} (M^{-1}_{\alpha\beta,\,\beta'})_i^{\ j}\, \mathrm{e}^{2\pi\mathrm{i}\left[H_j\frac{\beta'}{\alpha} - H_i\frac{\beta}{\alpha}\right]}\, \mathrm{e}^{-2\pi\mathrm{i}n\frac{\beta}{\alpha}}, \tag{3.3.5a}$$

$$f_j(\alpha;n+H_i) = \left[\frac{H_j}{n+H_i}\right]^{\frac{1}{2}-\frac{k}{2}} J_{k-1}\left[\frac{4\pi}{\alpha}\left[H_j(n+H_i)\right]^{\frac{1}{2}}\right], \tag{3.3.5b}$$

with $f_j(\alpha;n+H_i) = [2\pi(n+H_i)/\alpha]^{k-1}$ for $H_j = 0$. In these expressions, $J_\nu = J_\nu(x)$ represents the Bessel functions of the first kind (see appendix A.2.1 for details) and the matrix $(M_{\alpha\beta,\,\beta'})_i^{\ j}$ is the representation of the modular group acting on the characters $\chi_i(q)$ corresponding to the $\mathrm{PSL}_2(\mathbb{Z})$-group element

$$M_{\alpha\beta,\,\beta'} = \begin{pmatrix} -\beta' & (1+\beta\beta')/\alpha \\ -\alpha & \beta \end{pmatrix}, \tag{3.3.6}$$

where β' is an arbitrary integer parametrising the freedom of acting on $M_{\alpha\beta,\,\beta'}$ on the left with a phase-chaning T-transfomation. The matrix $Q(\alpha;n)_i^{\ j}$ is real and independent of the choice of β'. In the generic expression, the term $\delta a_{i,n}$ is an error that depends on the value chosen for $\alpha_{\max}$: the minimum error is obtained for $\alpha_{\max} \sim n^{1/2}$ and it is of order $\delta a_{i,n} = O(n^{k/2}; +\infty)$, if $k \le 0$.

At large values of n, the asymptotic behaviour of the function $f_j(\alpha;n+H_i)$ depends critically on the sign of H_j, since the growth of Bessel function changes in a drastic way. Expanding the Bessel function asymptotically, depending on the sign of H_j, one finds the asymptotic behaviours

$$f_j(\alpha;n+H_i) \overset{n\sim\infty}{\simeq} \frac{1}{2\pi}\left(\frac{\alpha}{2}\right)^{\frac{1}{2}} |H_j|^{\frac{1}{4}-\frac{k}{2}} (n+H_i)^{\frac{k}{2}-\frac{3}{4}}\, \mathrm{e}^{\frac{4\pi}{\alpha}[|H_j|(n+H_i)]^{\frac{1}{2}}}, \qquad H_j < 0; \tag{3.3.7a}$$

$$f_j(\alpha;n+H_i) \overset{n\sim\infty}{\simeq} \left[\frac{2\pi(n+H_i)}{\alpha}\right]^{k-1}, \qquad H_j = 0; \tag{3.3.7b}$$

$$f_j(\alpha;n+H_i) \overset{n\sim\infty}{\simeq} \frac{(2\alpha)^{\frac{1}{2}}}{2\pi} H_j^{\frac{1}{4}-\frac{k}{2}} (n+H_i)^{\frac{k}{2}-\frac{3}{4}} \cos\theta_j(\alpha;n+H_i), \qquad H_j > 0. \tag{3.3.7c}$$

Here, the angle has been defined $\theta_j(\alpha;n+H_i) = 4\pi\left[H_j(n+H_i)\right]^{\frac{1}{2}}/\alpha - k\pi/2 + \pi/4$. It is apparent that, in the summation over α that defines the terms $a_{i,n}$, the leading-order results are given by the term $\alpha = 1$ in sectors with negative energy $H_j < 0$. For the leading-order term $\alpha = 1$, the choice $\beta' = 0$ gives $M_{10} = S^{-1}$, so that $Q(1;n)_i^{\ j} = \mathrm{e}^{-\mathrm{i}\pi k/2} S_i^{\ j}$. In particular, the dominant contribution comes from the identity sector, i.e. the sector with vanishing weight $h = 0$ and therefore the minimum energy $H = -c/24$; without loss of generality, this is taken to correspond to the sector $i = 0$. To conclude, the asymptotic behaviour for the coefficients $a_{i,n}$ dictated by eq. (3.3.4) is

$$a_{i,n} \overset{n\sim\infty}{\simeq} A_i\,(n+H_i)^{-B} \mathrm{e}^{C\,(n+H_i)^{\frac{1}{2}}}, \tag{3.3.8}$$

where the coefficients have been defined

$$A_i = \frac{1}{\sqrt{2}}\, \mathrm{e}^{-\frac{\mathrm{i}\pi k}{2}}\, S_i{}^0 \left[\frac{c}{24}\right]^{\frac{1}{4}-\frac{k}{2}}, \tag{3.3.9a}$$

$$B = \frac{3}{4} - \frac{k}{2}, \tag{3.3.9b}$$

$$C = 4\pi \left[\frac{c}{24}\right]^{\frac{1}{2}}. \tag{3.3.9c}$$

In particular, the exponential coefficient $C = 1/T_{\mathrm{H}}$ is the inverse Hagedorn temperature of the theory. An important and well-known result is that the entries $S_i{}^0$ are always non-vanishing [113]. As a consequence, the coefficients A_i are non-vanishing as well and, within each sector, the degeneracy of states grows exponentially with the energy. All sectors experience the same kind of exponentially-growing behaviour, since they are all coupled to the identity sector by the non-zero coefficients $S_i{}^0$.

To understand how misaligned supersymmetry can work, one needs to look more closely at the partition function, in order to make use of the knowledge on the characters. To this purpose, one can insert the character expansion of eq. (3.3.2) into the generic partition function in eq. (3.3.3) and, after the index shifts $m_i = m + H_i$ and $n_{\bar{\jmath}} = n + \overline{H}_{\bar{\jmath}}$, express the latter as

$$Z(\tau,\overline{\tau}) = (\operatorname{Im}\tau)^{1-\frac{D}{2}} \sum_{i=0}^{N-1} \sum_{\bar{\jmath}=0}^{N-1} A^{i\bar{\jmath}} \sum_{m_i\in\mathbb{N}_0+H_i}\ \sum_{n_{\bar{\jmath}}\in\mathbb{N}_0+\overline{H}_{\bar{\jmath}}} a_{i,\,m_i-H_i}\, \overline{a}_{\bar{\jmath},\,n_{\bar{\jmath}}-\overline{H}_{\bar{\jmath}}}\, q^{m_i}\overline{q}^{n_{\bar{\jmath}}}. \tag{3.3.10}$$

The physical states correspond to the level-matched products with $m_i = n_{\bar{\jmath}}$, while all the other terms represent unphysical states. In general, because of the condition $m_i - n_{\bar{\jmath}} \in \mathbb{Z}$, which is inherent due to modular invariance, the partition function restricted to physical states can be defined via a τ_1-integration as

$$\begin{aligned} g(\tau_2) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}\tau_1\, Z(\tau_1,\tau_2) \\ &= \tau_2^{1-\frac{D}{2}} \sum_{i=0}^{N-1} \sum_{\bar{\jmath}=0}^{N-1} \sum_{n_{i\bar{\jmath}}\in\mathbb{N}_0+H_{i\bar{\jmath}}} A^{i\bar{\jmath}}\, a_{i,\,n_{i\bar{\jmath}}-H_i} \overline{a}_{\bar{\jmath},\,n_{i\bar{\jmath}}-\overline{H}_{\bar{\jmath}}}\, \mathrm{e}^{-4\pi\tau_2 n_{i\bar{\jmath}}}, \end{aligned} \tag{3.3.11}$$

with the definition $H_{i\bar{\jmath}} = \max(H_i, \overline{H}_{\bar{\jmath}})$. In particular, it is therefore possible to indicate the net physical degeneracies of the theory as

$$a_{nn} = \sum_{i=0}^{N-1} \sum_{\bar{\jmath}=0}^{N-1} A^{i\bar{\jmath}}\, a_{i,\,n_{i\bar{\jmath}}-H_i} \overline{a}_{\bar{\jmath},\,n_{i\bar{\jmath}}-\overline{H}_{\bar{\jmath}}} = \sum_{i=0}^{N-1} \sum_{\bar{\jmath}=0}^{N-1} A^{i\bar{\jmath}}\, a_{i\bar{\jmath},\,nn}. \tag{3.3.12}$$

Here, it should be stressed that the right- and left-moving state degeneracies that are being considered, i.e. the terms $a_{i,\,n_{i\bar{\jmath}}-H_i}$ and $\overline{a}_{\bar{\jmath},\,n_{i\bar{\jmath}}-\overline{H}_{\bar{\jmath}}}$, are generally defined only for certain values of $n = n_{i\bar{\jmath}}$, for each $(i,\bar{\jmath})$-pair. In fact, to write the partition function in terms of these coefficients a_{nn} just as a sum over n, the sums over the sectors of the theory should be swapped with the sum over n, but the latter actually depends on the $(i,\bar{\jmath})$-pair.

From the HRR-formula in eq. (3.3.4), thanks to the asymptotic expansions (3.3.7a, 3.3.7b, 3.3.7c), one can see that tachyonic states with $H_j < 0$ have an exponentially-growing behaviour, while states with $H_j \geq 0$ are power-law suppressed, since $k = 1 - D/2 < 0$, for the common case $D > 2$. Therefore, the fastest-growing contribution in the net physical degeneracies of eq. (3.3.12) is provided by the terms $i = \bar{\jmath} = 0$ for all the sectors, which have $H_0 = -c/24$ and $\overline{H}_{\bar{0}} = -\bar{c}/24$. It is in this sense that the unphysical tachyons (i.e. not level-matched tachyonic states) play a fundamental role in determining the asymptotic behaviours of the physical state degeneracies. Thanks to the expansion of eq. (3.3.8), each pair of sectors then contributes with a term

$$a_{i\bar{\jmath},nn} \overset{n\sim\infty}{\simeq} A_i \overline{A}_{\bar{\jmath}}\, n^{-2B}\, \mathrm{e}^{(C+\overline{C})\, n^{\frac{1}{2}}} = \frac{1}{2}\, S_i{}^0 \overline{S}_{\bar{\jmath}}{}^{\bar{0}}\, |H_0 \overline{H}_{\bar{0}}|^{\frac{1}{4}-\frac{k}{2}}\, n^{k-\frac{3}{2}}\, \mathrm{e}^{C_{\mathrm{tot}}\, n^{\frac{1}{2}}}. \tag{3.3.13}$$

In this expression, the total inverse Hagedorn temperature has been defined as $C_{\mathrm{tot}} = C + \overline{C}$ and it can be expressed as

$$C_{\mathrm{tot}} = 4\pi \big[|H_0|^{\frac{1}{2}} + |\overline{H}_{\bar{0}}|^{\frac{1}{2}} \big] = \lim_{n\to\infty} \frac{\ln |a_{i\bar{\jmath},nn}|}{n^{\frac{1}{2}}}. \tag{3.3.14}$$

As anticipated before, the definition of C_{tot} does not make a distinction in the $(i,\bar{\jmath})$-indices, since all sectors grow with the same exponential behaviour. Notice that, generally, the degeneracies a_{nn} in eq. (3.3.12) have a similar asymptotic behaviour to the terms $a_{i\bar{\jmath},nn}$ in eq. (3.3.13), since they are made out of their products. Misaligned supersymmetry predicts then that these asymptotic exponential behaviours cancel out when summing over all sectors in a specific way, leading to an effective exponential growth governed by some effective inverse Hagedorn temperature $C_{\mathrm{eff}} < C_{\mathrm{tot}}$, as is going to be explained now.

In general, the presence of exponentially-growing numbers of states can lead to divergences in physical quantities. Misaligned supersymmetry avoids this by modelling the occurrence of specific cancellations taking place in a generalisation of the degeneracies a_{nn} in eq. (3.3.12), namely a sector-averaged number of states that grows more slowly than the degeneracy contributions $a_{i\bar{\jmath},nn}$ in eq. (3.3.13) themselves. One should notice that, although for each index n the net physical degeneracies a_{nn} are written as a sum of the contributions $a_{i\bar{\jmath},nn}$ over all the sectors $(i,\bar{\jmath})$, in general not all sectors contribute to the total degeneracy for all indices. In fact, for any specific sector $(i,\bar{\jmath})$, the coefficients $a_{i\bar{\jmath},nn}$ are defined for indices in a specific discrete set of values, so at a given level n not all sectors necessarily appear in the actual sum that defines the degeneracy a_{nn} (e.g. only certain subsectors may be defined for semi-integer n, with other sectors being defined instead for integer n). To define an appropriate sector average, one formally replaces the asymptotic state degeneracies $a_{i\bar{\jmath},nn}$ with their interpolating functions $\Phi_{i\bar{\jmath}}(n)$, i.e.

$$a_{i\bar{\jmath},nn} \mapsto \Phi_{i\bar{\jmath}}(n),$$

where n is no longer assumed to take discrete values only, but is a real positive variable $n \in \mathbb{R}^+$. The interpolating functions $\Phi_{i\bar{\jmath}}(n)$ are usually called ‘envelope functions’. The crucial point here is that, while the index n in $a_{i\bar{\jmath},nn}$ also depends on the sectors $(i,\bar{\jmath})$, as is clear from eq. (3.3.12), the functional forms $\Phi_{i\bar{\jmath}}(n)$ are defined for all positive real values of n. In other words, the argument n in $\Phi_{i\bar{j}}(n)$ is independent from the sectors $(i,\bar{j})$, so the envelope functions can be used to define a generalisation of the degeneracies a_{nn} that actually includes an average from all the sectors in the theory. In fact, once the envelope functions are introduced, it is possible define the sector-averaged number of states $\langle a_{nn}\rangle$ as the sum of these functions over all sectors in the theory, i.e.

$$\langle a_{nn}\rangle = \sum_{i=0}^{N-1}\sum_{\bar{j}=0}^{N-1} A^{i\bar{j}}\Phi_{i\bar{j}}(n). \tag{3.3.15}$$

Then, in analogy with the definition in eq. (3.3.14), the effective inverse Hagedorn temperature C_{eff} is defined as

$$C_{\text{eff}} = \lim_{n\to\infty}\frac{\ln|\langle a_{nn}\rangle|}{n^{\frac{1}{2}}}. \tag{3.3.16}$$

The fundamental result of ref. [27] can now be explained as follows. Let $Z = Z(\tau,\overline{\tau})$ be a modular-invariant partition function in the form of eq. (3.3.3), with an oscillatory term $f(\tau,\overline{\tau}) = (\operatorname{Im}\tau)^{-k}Z(\tau,\overline{\tau})$. Then, let the mixing matrix be such that for all non-zero entries $A^{i\bar{j}} \neq 0$ at least one of the two characters χ_i and $\overline{\chi}_{\bar{j}}$ has a non-negative vacuum energy, i.e. at least one condition between $H_i \geq 0$ and $\overline{H}_{\bar{j}} \geq 0$ is satisfied. This means that the physical theory has no physical tachyons, since level-matched states require the equality $H_i + m = \overline{H}_{\bar{j}} + n$, so if e.g. $\overline{H}_{\bar{j}} < 0$ but $H_i \geq 0$, then $\overline{H}_{\bar{j}} + n \geq 0$, and therefore the total mass units are $H_i + m + \overline{H}_{\bar{j}} + n \geq 0$. However, unphysical tachyons are allowed. If these conditions hold, then it can be shown that the sector-averaged degeneracies $\langle a_{nn}\rangle$ grow asymptotically with an effective inverse Hagedorn temperature such that

$$C_{\text{eff}} < C_{\text{tot}}. \tag{3.3.17}$$

This can be thought of as a generalisation of a result in the theory of modular forms, i.e. the fact that a one-variable modular form $f(\tau) = q^H\sum_{n=0}^{\infty} a_n q^n$ of modular weight $k < 12H$ must be identically zero, i.e. its coefficients are $a_n = 0$ for all $n \in \mathbb{N}_0$. The two-variable case $f = f(\tau,\overline{\tau})$ is fundamentally different, since modular invariance only restricts the product of two different modular forms. In the absence of physical tachyons, ignoring the case in which for any non-zero entries $A^{i\bar{j}}$ one has both $H_i \geq 0$ and $\overline{H}_{\bar{j}} \geq 0$, which can be shown to correspond to spacetime supersymmetry with $a_{nn} = 0$, it is not possible to constrain the degeneracies a_{nn} but only the sector-averaged degeneracies $\langle a_{nn}\rangle$ as in eq. (3.3.17). This works by considering cancellations in the average $\langle a_{nn}\rangle$ that occur among sectors that are misaligned, since for any given n, the functions $\Phi_{i\bar{j}}(n)$ participating in the average come from terms $a_{i\bar{j},nn}$ that are not necessarily defined for that n.

For completeness, a brief overview of the proof is outlined below. Thanks to eq. (3.3.4), in view of eqs. (3.3.12, 3.3.15), and neglecting the power-law remainders, the sector-averaged net degeneracies $\langle a_{nn}\rangle$ can be written as

$$\begin{aligned}\langle a_{nn}\rangle = \sum_i\sum_{\bar{j}} A^{i\bar{j}}&\left[\sum_{\alpha}\frac{2\pi}{\alpha}\sum_k Q(\alpha;n-H_i)_i{}^k f_k(\alpha;n)\right]\\ &\cdot\left[\sum_{\beta}\frac{2\pi}{\beta}\sum_{\bar{l}}\overline{Q}(\beta;n-\overline{H}_{\bar{j}})_{\bar{j}}{}^{\bar{l}}\overline{f}_{\bar{l}}(\beta;n)\right],\end{aligned}$$

where the shifts imply that in the summations the dependence on the $(i, \bar{j})$-sectors is only in the mixing matrix and in the Q and $\overline{Q}$-functions. This means that one can single out the two $(i, \bar{j})$-summations, writing

$$\begin{aligned} P(\alpha, \beta; n)^{k\bar{l}} &= \sum_i \sum_{\bar{j}} A^{i\bar{j}} Q(\alpha; n - H_i)_i{}^k \overline{Q}(\beta; n - \overline{H}_{\bar{j}})_{\bar{j}}{}^{\bar{l}} \\ &= \sum_\beta \sum_\gamma \mathrm{e}^{2\pi\mathrm{i}\left[H_k \frac{\gamma'}{\alpha} - n\frac{\gamma}{\alpha}\right]} \, \mathrm{e}^{-2\pi\mathrm{i}\left[\overline{H}_{\bar{l}} \frac{\delta'}{\beta} - n\frac{\delta}{\beta}\right]} \sum_i \sum_{\bar{j}} A^{i\bar{j}} (M^{-1}_{\alpha\gamma,\,\gamma'})_i{}^k (\overline{M}^{-1}_{\beta\delta,\,\delta'})_{\bar{j}}{}^{\bar{l}} \\ &= \sum_\beta \sum_{\bar{\beta}} \mathrm{e}^{2\pi\mathrm{i}\left[H_k \frac{\gamma'}{\alpha} - n\frac{\gamma}{\alpha}\right]} \, \mathrm{e}^{-2\pi\mathrm{i}\left[\overline{H}_{\bar{l}} \frac{\delta'}{\beta} - n\frac{\delta}{\beta}\right]} \left[(M^{-1}_{\alpha\gamma,\,\gamma'})^T (M_{\beta\delta,\,\delta'})^T A\right]^{k\bar{l}}, \end{aligned}$$

where, in matrix notation, advantage has been taken of the modular-invariance requirement on the mixing matrix $A = M^T A \overline{M}$. At leading order $\alpha = \beta = 1$, it is possible to determine that $P(\alpha, \beta; n)^{k\bar{l}} = A^{k\bar{l}}$, which implies the equation

$$\langle a_{nn} \rangle \overset{n\sim\infty}{\simeq} \sum_k \sum_{\bar{l}} A^{k\bar{l}} f_k(1; n) \overline{f}_{\bar{l}}(1; n).$$

Under the assumption that either $H_i \geq 0$ or $H_{\bar{j}} \geq 0$ for each non-zero entry $A^{i\bar{j}}$, none of the terms contributing to $\langle a_{nn} \rangle$ experience the maximal growth given by the identity sectors for both right- and left-movers. This is enough to prove eq. (3.3.17).

The most immediate physical consequence of eq. (3.3.17) consists in the fact that the spectrum of physical degeneracies a_{nn} must oscillate. In fact, an oscillation in the net number of states among different levels is necessary to motivate the existence of a net cancellation. This means that the modular-invariant theory, with no physical tachyons, does not need to be supersymmetric, but it is enough for it to show a 'misaligned supersymmetry'. The heterotic SO(16)×SO(16)-theory analysed in detail in subsection 3.3.2 will serve as an instance to show the main features of misaligned supersymmetry in concrete.

In addition to all this, ref. [27] further conjectures that the effective exponential coefficient is actually zero, i.e.

$$C_{\text{eff}} = 0, \tag{3.3.18}$$

which implies that the sector-averaged number of states does not grow exponentially, but at most polynomially. The conditions in eqs. (3.3.17, 3.3.18) will be referred to as the 'weak' and the 'strong' forms of misaligned supersymmetry, respectively. The strong condition in eq. (3.3.18) implies the occurrence of cancellations also at all subleading orders, since the weak condition is a result of leading-order cancellations only.

To conclude, it must be emphasised that the study of subleading corrections in the general HRR-sum is extremely intricate to perform, as should be apparent from the expansion in eq. (3.3.4). In fact, this requires a general method to select the integers β' to define a consistent $\mathrm{PSL}_2(\mathbb{Z})$-matrix $M_{\alpha\beta,\beta'}$ and a general way to express the corresponding matrix representation in the space of the characters, possibly in terms of the modular T- and S-transformations, which is quite arduous. Furthermore, the functions $Q(\alpha; n)_i{}^j$ depend on n for $\alpha > 1$, complicating both the calculations and the meaning of the operation of sector-averaging itself. Besides this, the asymptotic expansions in eqs. (3.3.7a, 3.3.7b, 3.3.7c) evidence the presence of power-law corrections in all the exponentially-growing sectors and of further power-law terms in all the other sectors, coming from the asymptotic expansions of the Bessel functions, as well as the presence of infinite series of similar form coming from all terms with $\alpha > 1$. All these terms should eventually be combined together. Among the issues above, perhaps the most critical of all lies in the fact that all subleading orders with $\alpha > 1$ go beyond the reach of the methods leading to the result of eq. (3.3.17), since this is based on the leading-order identity $\sum_{i=0}^{N-1}\sum_{\bar{\jmath}=0}^{N-1} A^{i\bar{\jmath}} Q(1; n-H_i)_i{}^k \overline{Q}(1; n-\overline{H}_{\bar{\jmath}})_{\bar{\jmath}}{}^{\bar{l}} = A^{k\bar{l}}$, which is only valid for $\alpha = \beta = 1$. Indeed the fact that the functions $Q(\alpha; n)_i{}^j$ depend on n for all subleading orders $\alpha > 1$ and the M-matrix shortcomings render the generalisation of this identity untreatable. The validity of eq. (3.3.17) is still guaranteed, but these technical complications prevent any easy refinement of this result. To conclude, it is also worthwhile to emphasise that, for practical purposes, when introducing the functions $\Phi_{i\bar{\jmath}}(n)$, the variable n is taken to be continuous, since this helps in visualising the cancellations at leading order, but the cancellations could be seen by just extending the indices n in $a_{i\bar{\jmath},nn}$ to take all of the possible discrete levels that can appear in any sector, even if they are not defined for the specific $(i, \bar{\jmath})$-sector. However, the extension of the index n beyond its original domain is hard to generalise beyond leading order: whilst taking $n \in \mathbb{R}^+$ works perfectly for $\alpha = 1$, the functions $\Phi_{i\bar{\jmath}}(n)$ are in general not real when subleading orders are considered, and in fact the functions $Q(\alpha; n)_i{}^j$ in eq. (3.3.5a) can be seen to be real for $\alpha > 1$ only for $n \in \mathbb{Z}$ [27]. This fact is ambiguous since the envelope functions are directly associated to counting physical degrees of freedom.

Unnormalised Characters

It is often convenient to work with characters that are not normalised, i.e. which are such that $a_{\iota,0} \neq 1$, for $\iota > 0$, and similarly for the anti-holomorphic sector. In the identity sector, instead, the normalisation $a_{0,0} = 1$ is always going to be preserved.

An asymptotic description can can be easily worked out for the unnormalised characters in terms of the analysis of normalised characters. Let χ_i be a basis of unnormalised characters and let $\breve{\chi}_\iota = \chi_i / a_{i,0}$ be the corresponding basis of normalised characters, whose discussion has been performed up to this point. Working with the unnormalised characters simply requires the mixing and the S-matrix terms to be rescaled as $\breve{A}^{i\bar{\jmath}} = a_{i,0}\bar{a}_{\bar{\jmath},0}A^{i\bar{\jmath}}$ and $\breve{S}_i{}^j = (a_{j,0}/a_{i,0}) S_i{}^j$, where unaccented and accented terms represent terms referred to unnormalised and normalised basis, respectively. So, for instance, adapting eq. (3.3.13), the coefficients for the unnormalised theory, i.e. $a_{i\bar{\jmath},nn} = (a_{i,0}\bar{a}_{\bar{\jmath},0})\,\breve{a}_{i\bar{\jmath},nn}$, can be written as

$$a_{i\bar{\jmath},nn} \overset{n\sim\infty}{\simeq} (a_{i,0}\bar{a}_{\bar{\jmath},0}) \frac{1}{2} \breve{S}_i{}^0 \breve{\overline{S}}_{\bar{\jmath}}{}^{\bar{0}} \frac{|H_0 \overline{H}_{\bar{0}}|^{\frac{1}{4}-\frac{k}{2}}}{n^{\frac{3}{2}-k}} \, \mathrm{e}^{C_{\mathrm{tot}}\, n^{\frac{1}{2}}} = \frac{1}{2} S_i{}^0 \overline{S}_{\bar{\jmath}}{}^{\bar{0}} \frac{|H_0 \overline{H}_{\bar{0}}|^{\frac{1}{4}-\frac{k}{2}}}{n^{\frac{3}{2}-k}} \, \mathrm{e}^{C_{\mathrm{tot}}\, n^{\frac{1}{2}}},$$

and, adapting eq. (3.3.15), the unnormalised sector-averaged degeneracies are

$$\langle a_{nn} \rangle = \sum_{i=0}^{N-1}\sum_{\bar{\jmath}=0}^{N-1} \breve{A}^{i\bar{\jmath}} \breve{\Phi}_{i\bar{\jmath}}(n) = \sum_{i=0}^{N-1}\sum_{\bar{\jmath}=0}^{N-1} a_{i,0}\bar{a}_{\bar{\jmath},0} A^{i\bar{\jmath}} \frac{1}{a_{i,0}\bar{a}_{\bar{\jmath},0}} \Phi_{i\bar{\jmath}}(n) = \sum_{i=0}^{N-1}\sum_{\bar{\jmath}=0}^{N-1} A^{i\bar{\jmath}} \Phi_{i\bar{\jmath}}(n).$$

All results up to this point had been presented for normalised characters in order to avoid unnecessary notational complications. From now on, there should be no ambiguities in working with unnormalised characters.

3.3.1.2 Cosmological Constant in Closed-String Theories

A simple one-loop amplitude is the one-loop cosmological constant. In this subsubsection, arguments are provided to disentangle the relationship between misaligned supersymmetry and finite quantum-corrected physical observables.[3.2] In a string-theory model with a D-dimensional target spacetime and a modular-invariant partition function $Z = Z(\tau, \overline{\tau})$, it can be shown that the one-loop cosmological constant can be defined as

$$\tilde{\Lambda}_D = -\frac{1}{2\kappa_D^2}\frac{1}{4\pi l_s^2}\int_{\mathbb{F}} \frac{\mathrm{d}^2\tau}{(\operatorname{Im}\tau)^2}\, Z(\tau, \overline{\tau}), \tag{3.3.19}$$

where the D-dimensional gravitational coupling constant is $2\kappa_D^2 = l_s^{D-2}/2\pi$. The region of integration $\mathbb{F}$ is the $\mathrm{PSL}_2(\mathbb{Z})$-group fundamental domain. In the absence of physical tachyons, this one-loop cosmological constant is finite, even in the absence of supersymmetry. Both physical and unphysical states take part in determining the final result. In fact, although in the region $(\tau_1, \tau_2) \in [-1/2, 1/2] \times [1, \infty[$ only the level-matched states appear, the unphysical ones being projected out by the integration over τ_1, in the area enclosed between the fundamental domain and the square $(\tau_1, \tau_2) \in [-1/2, 1/2] \times [0, 1]$ unphysical states appear too. In particular, the partition function typically has unphysical tachyons, which therefore play a role in fixing the cosmological constant.

Taking advantage of the modular invariance of the partition function, given the partition function restricted to the sum over physical states $g(\tau_2) = \int_{-1/2}^{1/2} \mathrm{d}\tau_1\, Z(\tau, \overline{\tau})$, it is possible to express the one-loop cosmological constant purely in terms of physical states. In fact, for a modular-invariant partition function $Z = Z(\tau, \overline{\tau})$ that is also free of physical tachyons, the Kutasov-Seiberg identity [93] states that

$$\tilde{\Lambda}_D = -\frac{1}{2\kappa_D^2}\frac{1}{12 l_s^2}\lim_{\sigma\to\infty} g(\sigma^{-1}). \tag{3.3.20}$$

A key difference among the two expressions of the one-loop cosmological constant is that in the Kutasov-Seiberg formula only physical states appear. Modular invariance constrains the physical number of states in such a way that their effect in the region $\tau_2 \sim 0^+$ also accounts for what would be the effect of unphysical states, too, in the integral expression over the fundamental domain $\mathbb{F}$. This stems from the fact that a generic modular transformation of the partition function relates physical and unphysical states to each other.

Compared to the $\mathbb{F}$-domain integration, the Kutasov-Seiberg identity allows one to draw general conclusions more easily as it only involves level-matched states, making it more direct to interpret the cosmological-constant finiteness in view of effective cancellations interpretable in terms of misaligned supersymmetry. Moreover, matching an integral definition

[3.2] The details of the relationship between the string-theory one-loop cosmological constant and misaligned supersymmetry constitute the main topic of section 3.7. Nevertheless, to conclude the overview on closed-string misaligned supersymmetry, it is useful to also review the essential closed-string features here.

with the result of a limit, it makes it easier to see how oscillations can actually take place in a physical observable. In fact, expanding the function $g(\tau_2)$ in terms of the net degeneracies a_{nn}, it is possible to infer the small-τ_2 behaviour

$$\sum_{n\in\frac{1}{2}\mathbb{N}_0} a_{nn}\,\mathrm{e}^{-4\pi\tau_2 n} \overset{\tau_2\sim 0^+}{\sim} -24\,\kappa_D^2 l_s^2\,\tilde{\Lambda}_D\,\tau_2^{D/2-1}. \tag{3.3.21}$$

Now, the relationship between a finite one-loop cosmological constant and misaligned supersymmetry should be intuitively clear. The fundamental aspect of String Theory that guarantees the possibility to have a finite one-loop cosmological constant is the absence of physical tachyons, with high-energy divergences being cut off via the modular invariance of the partition function. Moreover, the Kutasov-Seiberg identity, building on these features, shows how the finiteness may be seen from the point of view of a specific behaviour of the series of all the physical degeneracies, in which bosonic and fermionic contributions must eventually cancel out. At the same time, misaligned supersymmetry emerges in non-supersymmetric tachyon-free modular-invariant closed-string models, exhibiting systematic cancellations in appropriately-defined sector-averaged net physical degeneracies. Therefore, it is natural to interpret the finiteness of the one-loop cosmological constant in terms of physical cancellations, all over the string perturbative spectrum, that can be visualised as an effect of misaligned supersymmetry.

To conclude, it is worthwhile to mention supertraces. Because the coefficients a_{nn} correspond to the physical degeneracies at some mass level $M_n^2 = \mu^2 n$, where $\mu = \ell/\sqrt{\alpha'}$, for some model-dependent constant ℓ, one can define the generalised supertraces as

$$\mathrm{Str}\, M^{2\beta} = \lim_{t\to 0^+}\left[\sum_{n\in\frac{1}{2}\mathbb{N}_0} a_{nn} M_n^{2\beta}\mathrm{e}^{-4\pi t M_n^2/\mu^2}\right] = \mu^{2\beta}\lim_{t\to 0^+}\left[\left(-\frac{1}{4\pi}\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\beta}\sum_{n\in\frac{1}{2}\mathbb{N}_0} a_{nn}\,\mathrm{e}^{-4\pi\tau_2 n}\right].$$

In particular, fixing $\mu = 2/\sqrt{\alpha'}$ for typical closed-string theories, as a consequence of the Kutasov-Seiberg identity one immediately verifies that the first non-zero supertrace corresponds to the power $2\beta = D-2$, with

$$\begin{aligned}
&\mathrm{Str}\, M^0 = \mathrm{Str}\, M^2 = \cdots = \mathrm{Str}\, M^{D-4} = 0,\\
&\mathrm{Str}\, M^{D-2} = \frac{96\pi(D/2-1)!}{(-4\pi)^{D/2}}\left(\frac{4}{\alpha'}\right)^{\frac{D}{2}-1}\kappa_D^2 l_s^2\tilde{\Lambda}_D.
\end{aligned}$$

Ultimately, the regularised supertraces emerge as another conceptual tool to visualise the cancellations coming from a misaligned spectrum. In fact, for them to be vanishing or finite, and more generally for the series $g(\tau_2) = \tau_2^{1-D/2}\sum_{n\in\mathbb{N}_0/2} a_{nn}\,\mathrm{e}^{-4\pi\tau_2 n}$ to be finite as $\tau_2 \sim 0^+$, it is apparent that specific cancellations need to take place, entailing the whole towers of both bosonic and fermionic states.

3.3.1.3 New Perspectives

A general comment is in order. In the discussion overviewed up to this point, it is explained in quite some detail how, once the envelope functions are defined, it is in fact possible to show that they undergo cancellations. This is generally the case for the discussion of closed- and open-string misaligned supersymmetry in the whole of sections 3.3 and 3.4, and also 3.6. However, the details of how the machinery of misaligned supersymmetry works at subleading order and of how it mathematically connects to the finiteness of physical observables in concrete is not spelled out in definite terms yet. In particular, it is not fully clear how to physically interpret the extension of the net state degeneracies to levels exceeding their original domains of definition, especially when subleading orders are concerned, and how to explain the one-loop cosmological constant finiteness in view of misaligned supersymmetry in precise mathematical terms, with an explicit observation of the cancellations. These are essentially unexplored corners of the literature about misaligned supersymmetry and the main purpose of this chapter is precisely to shed light on both these topics. More precisely, sections 3.5 and 3.6 are aimed at showing the manifestation of misaligned supersymmetry at all subleading orders in terms of the envelope functions, while section 3.7 eventually motivates and explains the physical meaning of the envelope-function cancellations at all orders in precise and definite terms, showing the way in which such cancellations explain a finite one-loop cosmological constant.

3.3.2 Example: Heterotic SO(16)×SO(16)-Theory

The non-supersymmetric heterotic SO(16)×SO(16)-theory, originally constructed in refs. [12, 13], is a prototypical example of a non-supersymmetric and yet tachyonic-free closed-string theory. This subsection shows the way in which this model presents the main features of misaligned supersymmetry reviewed in subsection 3.3.1.

It is instructive to start by recalling how the heterotic SO(16)×SO(16)-theory can be obtained from an orbifold of the heterotic-$E_8\times E_8$ superstring theory [9]. The one-loop torus partition function of the heterotic $E_8\times E_8$-theory is given by

$$Z_{E_8\times E_8} = \frac{(\operatorname{Im}\tau)^{-4}}{\eta^8\overline{\eta}^8}\left[V_8 - S_8\right]\left[\overline{O}_{16}+\overline{S}_{16}\right]^2. \tag{3.3.22}$$

Here and in the following, the so($2n$)-characters O_{2n}, V_{2n}, S_{2n} and C_{2n} are defined as in ref. [39] (see appendix A.1.1 for a review). The partition function in eq. (3.3.22) is vanishing due to the well known Jacobi's *aequatio identica satis abstrusa*, i.e. $V_8 = S_8$. Physically, this is a consequence of spacetime supersymmetry, namely the fact that the number of bosons and fermions is the same at each energy level. The heterotic SO(16)×SO(16)-theory is obtained via an orbifold of the heterotic $E_8\times E_8$-theory. The orbifold projector acting on the partition function in eq. (3.3.22) is $P_g = (1+g)/2$. In this projector, the orbifold generator is $g = (-1)^{F+F_1+F_2}$, where F is the spacetime fermion number, while F_1 and F_2 are the fermion numbers of the first and second E_8-factors, respectively. The insertion of the projector gives

$$P_g(Z_{E_8\times E_8}) = \frac{1}{2}\left[Z_{E_8\times E_8} + g(Z_{E_8\times E_8})\right],$$

where the action of the orbifold reads

$$g(Z_{E_8\times E_8}) = \frac{(\operatorname{Im}\tau)^{-4}}{\eta^8\overline{\eta}^8}(V_8+S_8)(\overline{O}_{16}-\overline{S}_{16})^2,$$

and it is obtained by flipping the signs of the S_8- and S_{16}-sectors in eq. (3.3.22). However, as it stands, the projected partition function is not modular-invariant. To obtain a modular-invariant expression, one acts repeatedly with modular T- and S-transformations, adding at each step the new terms thus generated, until the final result is modular-invariant. Eventually, one finds

$$\begin{aligned} Z_{\mathrm{SO}(16)\times\mathrm{SO}(16)} = \frac{(\mathrm{Im}\,\tau)^{-4}}{\eta^8\overline{\eta}^8}\Big[& V_8(\overline{O}_{16}\overline{O}_{16}+\overline{S}_{16}\overline{S}_{16}) - S_8(\overline{O}_{16}\overline{S}_{16}+\overline{S}_{16}\overline{O}_{16}) \\ & + O_8(\overline{V}_{16}\overline{C}_{16}+\overline{C}_{16}\overline{V}_{16}) - C_8(\overline{V}_{16}\overline{V}_{16}+\overline{C}_{16}\overline{C}_{16})\Big], \end{aligned} \tag{3.3.23}$$

which is the partition function of the heterotic SO(16) × SO(16)-theory. This is related to the supersymmetric partition function $Z_{\mathrm{E}_8\times\mathrm{E}_8}$ of eq. (3.3.22) as

$$Z_{\mathrm{SO}(16)\times\mathrm{SO}(16)} = \frac{1}{2}\big[Z_{\mathrm{E}_8\times\mathrm{E}_8} + Z_{g(\mathrm{E}_8\times\mathrm{E}_8)}\big], \tag{3.3.24}$$

where $Z_{g(\mathrm{E}_8\times\mathrm{E}_8)}$ is the modular-invariant expression

$$\begin{aligned} Z_{g(\mathrm{E}_8\times\mathrm{E}_8)} = \frac{(\mathrm{Im}\,\tau)^{-4}}{\eta^8\overline{\eta}^8}\big[& (V_8+S_8)(\overline{O}_{16}-\overline{S}_{16})^2 + (O_8-C_8)(\overline{V}_{16}+\overline{C}_{16})^2 \\ & -(O_8+C_8)(\overline{V}_{16}-\overline{C}_{16})^2\big]. \end{aligned} \tag{3.3.25}$$

As will be shown in the rest of this subsection, the partition function in eq. (3.3.23) exhibits the characteristic features of misaligned supersymmetry. For many practical purposes, since $Z_{\mathrm{E}_8\times\mathrm{E}_8}=0$, one may as well just focus on the partition function in eq. (3.3.25). The heterotic SO(16)×SO(16)-theory is an instance of a non-supersymmetric string construction. In fact, the partition function in eq. (3.3.23) is non-zero. The lack of supersymmetry can be observed even at the massless level, whose particle content is summarised in table 3.1 (for a description of the degrees of freedom of this theory, see e.g. refs. [114, 115]). In particular, it is worth to emphasise that there is no gravitino in the theory.

bosons	fermions
graviton g_{MN}, dilaton Φ, Kalb-Ramond 2-form B_{MN}	
gauge fields A_M: $\boldsymbol{r_A} = \mathbf{(120,1)\oplus(1,120)}$	
	spinors ψ: $\boldsymbol{r_\psi} = \mathbf{(128,1)\oplus(1,128)}$
	co-spinors ξ: $\boldsymbol{r_\xi} = \mathbf{(16,16)}$

Table 3.1: Massless particle content in the heterotic SO(16) × SO(16)-theory. Each entry lists the name of the particle and the SO(16)×SO(16)-group representation it transforms in. No superpartners appear; in particular, the net state degeneracy is $N_b(0) - N_f(0) = [(28+1+35)+(8\cdot 120\cdot 2)] - [(8\cdot 128\cdot 2)+(8\cdot 16\cdot 16)] = -2112$.

Expanding the partition function in eq. (3.3.23) in powers of q and $\overline{q}$, one can observe the defining properties of misaligned supersymmetry. To start, one can single out the partition function restricted to physical states, i.e. those with equal powers of q and $\overline{q}$. This can be defined as $g_{\mathrm{SO}(16)\times\mathrm{SO}(16)}(\tau_2) = \int_{-1/2}^{1/2} \mathrm{d}\tau_1\, Z_{\mathrm{SO}(16)\times\mathrm{SO}(16)}(\tau_1, \tau_2)$ and, noticing that $q\overline{q} = \mathrm{e}^{-4\pi\tau_2}$, it has an expansion that reads

$$g_{\mathrm{SO}(16)\times\mathrm{SO}(16)}(\tau_2) = \tau_2^{-4}\Big[-2112 + 147456\,(\mathrm{e}^{-4\pi\tau_2})^{\frac{1}{2}} - 4713984\,(\mathrm{e}^{-4\pi\tau_2}) + O((\mathrm{e}^{-4\pi\tau_2})^{\frac{3}{2}};0)\Big].$$

So, one can easily recognise one of the features of misaligned supersymmetry, i.e. an exponentially-growing oscillation in the net number of bosons minus fermions at each physical energy level. This is shown clearly in fig. 3.1. Moreover, looking at terms with different powers of q and $\overline{q}$ in the complete partition function, one would see that some of them have in fact negative powers of q and/or $\overline{q}$. These correspond to unphysical tachyons and, in the absence of a physical tachyon, they are yet another signal of misaligned supersymmetry.

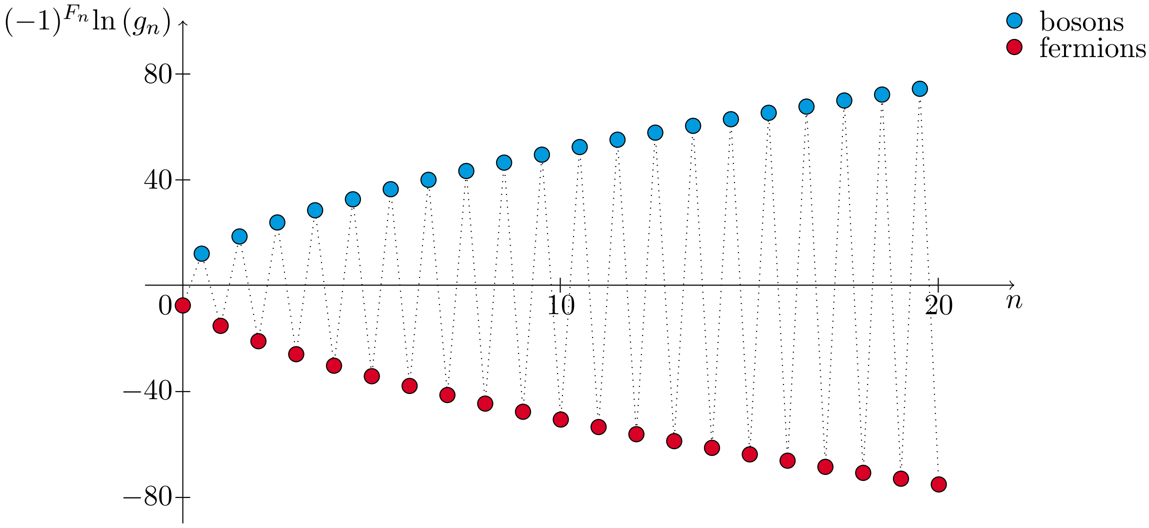

Figure 3.1: The net number of physical degrees of freedom for the lightest energy levels of the heterotic SO(16)×SO(16)-theory, defined as $(-1)^{F_n} g_n = N_b(n) - N_f(n)$. Each point corresponds to string states with mass $M_n^2 = 4n/\alpha'$, for $n = 0, 1/2, 1, \ldots, 20$. A positive value indicates a surplus of bosonic states compared to the fermionic ones, and vice versa for negative values. The presence of two misaligned sectors is clearly visible: fermions are associated to integer values of n, while bosons are associated to half-integer values. As predicted by misaligned supersymmetry, one can observe an exponentially-growing oscillation between the net number of bosons and fermions.

A more quantitative way to discuss the presence of misaligned supersymmetry is by employing the formalism presented in subsection 3.3.1. Looking at the partition function of the heterotic SO(16) × SO(16)-theory in eq. (3.3.23) and comparing it with the general partition function in eq. (3.3.3), a convenient basis for the right- and left-moving characters χ_i and $\overline{\chi}_{\overline{j}}$ is

$$\chi_i = \frac{1}{\eta^8}\begin{pmatrix} O_8 \\ V_8 \\ S_8 \\ C_8 \end{pmatrix} = \begin{pmatrix} q^{-\frac{1}{2}}[1 + 36q + O(q^2;0)] \\ 8 + 128q + O(q^2;0) \\ 8 + 128q + O(q^2;0) \\ 8 + 128q + O(q^2;0) \end{pmatrix}, \tag{3.3.26}$$

$$\overline{\chi}_{\bar{j}} = \frac{1}{\overline{\eta}^8}\begin{pmatrix} \overline{O}_{16}\overline{O}_{16} + \overline{S}_{16}\overline{S}_{16} \\ \overline{V}_{16}\overline{C}_{16} + \overline{C}_{16}\overline{V}_{16} \\ \overline{V}_{16}\overline{V}_{16} + \overline{C}_{16}\overline{C}_{16} \\ \overline{O}_{16}\overline{S}_{16} + \overline{S}_{16}\overline{O}_{16} \end{pmatrix} = \begin{pmatrix} \overline{q}^{-1}[1 + 248\overline{q} + O(\overline{q}^2;0)] \\ \overline{q}^{\frac{1}{2}}[4096 + 245760\overline{q} + O(\overline{q}^2;0)] \\ 256 + 36864\overline{q} + O(\overline{q}^2;0) \\ 256 + 36864\overline{q} + O(\overline{q}^2;0) \end{pmatrix}. \tag{3.3.27}$$

This basis is chosen in such a way that the characters are eigenfunctions of T, they are covariant under S, and they have a series expansion with positive coefficients. In this basis, the matrix $A^{i\bar{j}}$ is given by

$$A^{i\bar{j}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{3.3.28}$$

The modular transformations T and S act on the right-moving sector χ_i as

$$T_i^{\ j} = \mathrm{diag}(-1,1,1,1), \qquad S_i^{\ j} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \tag{3.3.29}$$

and on the left-moving sector $\overline{\chi}_{\bar{j}}$ as

$$\overline{T}_{\bar{i}}^{\ \bar{j}} = \mathrm{diag}(1,-1,1,1), \qquad \overline{S}_{\bar{i}}^{\ \bar{j}} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}. \tag{3.3.30}$$

Notice that the elements in the basis are ordered in such a way that the identity sector in both the right- and the left-moving sectors resides in the first component of the characters χ_i and $\overline{\chi}_{\bar{j}}$.

Since $S_i^{\ 0} \equiv 1/2$ and $\overline{S}_{\bar{j}}^{\ \bar{0}} \equiv 1/2$, i.e. they are both non-zero, each sector $a_{i\bar{j},nn}$ grows with an exponential behaviour dictated by eq. (3.3.13). In particular, the heterotic string theory has central charges $c = 12$ and $\bar{c} = 24$, which means vacuum energies $H_0 = -1/2$ and $\overline{H}_{\bar{0}} = -1$, and it is formulated in a 10-dimensional spacetime, with $k = -4$. Therefore, the coefficients asymptotically grow as

$$a_{i\bar{j},nn} \overset{n\sim\infty}{\simeq} \frac{1}{32\cdot 2^{\frac{1}{4}}}\,\frac{1}{n^{\frac{11}{2}}}\,\mathrm{e}^{4\pi[1+(\frac{1}{2})^{\frac{1}{2}}]\,n^{\frac{1}{2}}}. \tag{3.3.31}$$

In particular, the inverse Hagedorn temperature of the theory is

$$C_{\mathrm{tot}} = 4\pi\Big[1 + \Big(\frac{1}{2}\Big)^{\frac{1}{2}}\Big]. \tag{3.3.32}$$

One can also verify the counting of the number of states in eq. (3.3.12), as follows. The massless level has a net degeneracy $a_{00} = a_{1,0}\overline{a}_{\bar{0},1} - a_{2,0}\overline{a}_{\bar{3},0} - a_{3,0}\overline{a}_{\bar{2},0} = -2112$, with the $(0,\bar{1})$-sector not being defined for $n = 0$, and the first excited level has a degeneracy $a_{1/2\,1/2} = a_{0,1}\overline{a}_{\bar{1},0} = 147456$, with the $(1,\bar{0})$-, $(2,\bar{3})$- and $(3,\bar{2})$-sectors not being defined. This procedure can be followed iteratively at any mass level, in principle.

When considering the sum over all sectors entering the partition function, if misaligned supersymmetry is present, as the oscillations in fig. 3.1 hint, then cancellations are expected in the sector-averaged number of states. To verify that this is indeed the case for the system under investigation, one introduces the functional forms $\Phi_{i\bar{j}}(n)$ associated to the degeneracies $a_{i\bar{j},nn}$. These can be written as

$$\Phi_{i\bar{j}}(n) = \frac{1}{8 \cdot 2^{\frac{1}{4}}} \frac{1}{n^{\frac{11}{2}}} {S_i}^0 {\overline{S}_{\bar{j}}}^{\bar{0}} \, e^{4\pi[1+(\frac{1}{2})^{\frac{1}{2}}]\, n^{\frac{1}{2}}} + \phi_{i\bar{j}}(n), \tag{3.3.33}$$

where the functions $\phi_{i\bar{j}}(n)$ stand for all the subleading terms. Then, using the explicit form of S-transformation matrices in eqs. (3.3.29, 3.3.30), one can check that all the leading exponentials cancel when summing over all sectors, being

$$\sum_{i=0}^{3} \sum_{\bar{j}=\bar{0}}^{\bar{3}} A^{i\bar{j}} {S_i}^0 {\overline{S}_{\bar{j}}}^{\bar{0}} = {S_1}^0 {\overline{S}_{\bar{0}}}^{\bar{0}} + {S_0}^0 {\overline{S}_{\bar{1}}}^{\bar{0}} - {S_3}^0 {\overline{S}_{\bar{2}}}^{\bar{0}} - {S_2}^0 {\overline{S}_{\bar{3}}}^{\bar{0}} = 0. \tag{3.3.34}$$

This means that the sector-averaged number of states defined in eq. (3.3.15) is determined by the subleading terms $\phi^{i\bar{j}}(n)$ as the leading-order terms cancel out, i.e.

$$\langle a_{nn} \rangle = \sum_{i=0}^{3} \sum_{\bar{j}=\bar{0}}^{\bar{3}} A^{i\bar{j}} \Phi_{i\bar{j}}(n) = \sum_{i=0}^{3} \sum_{\bar{j}=\bar{0}}^{\bar{3}} A^{i\bar{j}} \phi_{i\bar{j}}(n). \tag{3.3.35}$$

Such subleading terms can have different asymptotic behaviours in different sectors, but their exponential growth is fixed by a coefficient C_{eff} which is intrinsically smaller than C_{tot}. This result shows that, in the heterotic SO(16)×SO(16)-theory, misaligned supersymmetry is present at least in its weak form, leading to a sector-averaged number of physical states growing at a rate $C_{\text{eff}} < C_{\text{tot}}$.

The next step would be to check whether in fact misaligned supersymmetry is present in its strong from, namely whether it is $C_{\text{eff}} = 0$. Proving this conjecture requires a careful analysis of the subleading contributions to each sector. As explained previously, this task cannot be performed immediately with the functional forms $\Phi_{i\bar{j}}(n)$, as the latter are substantially more difficult to deal with for $\alpha > 1$.

It is possible to evaluate numerically the one-loop cosmological constant and verify that it is indeed finite, as expected. In particular, it is amounts to

$$\tilde{\Lambda} = \frac{1}{2\kappa_{10}^2} \frac{1}{l_s^2} I,$$

with $I = -(1/4\pi) \int_{\mathbb{F}} \mathrm{d}^2\tau Z_{\mathrm{SO}(16)\times\mathrm{SO}(16)}(\tau, \overline{\tau})/(\mathrm{Im}\,\tau)^2 \simeq 57.8 > 0$. The fact that it is positive can be understood in terms of the abundance of fermions at the massless level, which is the least suppressed one in the fundamental-domain integration. This corresponds to the one-loop correction to vanishing tree-level potential. So, the string-frame constant scalar potential generates a term

$$S_\Lambda = -\frac{1}{2\kappa_{10}^2} \int \mathrm{d}^{10}x \sqrt{-G}\, \frac{1}{l_s^2} I = -\frac{1}{2\hat{\kappa}_{10}^2} \int \mathrm{d}^{10}x \sqrt{-\hat{g}}\, \frac{1}{l_s^2} g_s^2 I \, e^{5\phi/2}.$$

It is also possible to notice the g_s^2-suppression in the one-loop term, as expected due to the toroidal origin of the quantum correction. For recent reviews and discussions about the role of dilaton potential terms of this kind, see e.g. refs. [5, 116–120].

3.4 Misaligned Supersymmetry in Open Strings

This section discusses the presence of misaligned supersymmetry in open-string theories. The usual discussion of misaligned supersymmetry focuses on closed strings and a major role is played by the modular invariance of the torus partition function. In the following, as a prototypical example of the emergence of the features of misaligned supersymmetry for open strings, a particular class of models is going to be examined: these are the theories obtained by placing an anti-Dp-brane on top of an Op-plane, in type II string theories, which also provide examples of brane supersymmetry breaking. A key difference compared to the closed-string analysis is the lack of modular invariance in the partition function.

3.4.1 D-Branes and Orientifolds

As a motivation for its choice as a case-study example for open-string misaligned supersymmetry, this subsection shows that the mass levels of the perturbative spectrum of an anti-Dp-brane sitting on top of an Op-plane in fact present an increasing oscillation in the net number of bosons and fermions. This strongly supports the idea that misaligned supersymmetry is underlying it, as is going to be shown later in subsection 3.4.2.

In particular, this subsection describes the effects of an orientifold projection on the theory of Dp- and anti-Dp-branes. To start, it analyses the perturbative mass spectrum associated to the theory with one of these branes sitting on top of an orientifold Op-plane. A heuristic method to compute the number of the degrees of freedom at each mass level is presented as well. Eventually, the form of the partition functions that the theories are described by is also motivated and commented.

3.4.1.1 Orientifold-Invariant D-Brane Spectra

The starting point of the analysis is the perturbative spectrum of a Dp-brane sitting on top of an Op-plane. As overviewed in chapter 2, in lightcone quantization, one has the NS- and R-vacua $|\mathrm{NS}\rangle$ and $|\mathrm{R}_\pm\rangle$, as well as the bosonic and fermionic creation operators α^I_{-n} and b^I_{-r}, for $n \in \mathbb{N}$ and $r \in \mathbb{N}_0 + \phi$, with $\phi = 1/2, 0$ in the NS- and R-sectors, respectively; as usual, the index I denotes all directions but the gauge-fixed ones. Along with the GSO-projection, a requirement for states to be physical in the presence of an orientifold symmetry is that the states be invariant under the action of an orientifold operator O, i.e. an eigenstate of the operator $P_O = (1 + O)/2$ with unit eigenvalue.

This analysis can be performed by studying the action of the orientifold symmetry on the GSO-invariant states, based on an analogous discussion in ref. [33]. If the orientifold symmetry acts on a compact space, it is not only composed of the action of worldsheet parity Ω_P, but also of a geometric $\mathbb{Z}_2$-action. Moreover, the orientifold breaks half of the closed-string supersymmetry, distinguishing the RR-charge of the branes. In fact, it can be argued that the orientifold operator acts on the NS- and R-vacua of a Dp-brane as

$$O|\mathrm{NS}\rangle = \mathrm{e}^{-\frac{\mathrm{i}\pi}{2}}|\mathrm{NS}\rangle, \tag{3.4.1a}$$

$$O|\mathrm{R}\rangle = -|\mathrm{R}\rangle, \tag{3.4.1b}$$

and on the creation operators as

$$O\alpha_m^I O^{-1} = (-1)^m \alpha_m^I, \tag{3.4.2a}$$
$$Ob_r^I O^{-1} = \mathrm{e}^{\mathrm{i}\pi r} b_r^I. \tag{3.4.2b}$$

These relations are formally the same as those for worldsheet parity alone, when acting on a D9-brane/O9-plane system: this can be seen as a consequence of the fact that, for NN-directions, the calculations are effectively unchanged, whereas for DD-directions, that are in the internal space, the opposite sign that worldsheet parity requires on the creation operators is cancelled by the geometric $\mathbb{Z}_2$-action. Under these premises, it is then possible to determine the partition function of Dp- and anti-Dp-branes sitting on an orientifold Op-plane. This requires two further observations.

- The generic NS-state of mass $\alpha' M^2 = n$, with $n \in \mathbb{N}_0$, requires a total number $n + 1/2$ of oscillatory excitations and, suppressing spacetime indices for brevity, it can be written as
$$|\mathrm{NS}_n\rangle = \alpha_{-n_1}^{I_1} \dots \alpha_{-n_k}^{I_k} b_{-r_1}^{J_1} \dots b_{-r_l}^{J_l} \, |\mathrm{NS}\rangle,$$
with $\sum_{i=1}^k n_i + \sum_{j=1}^l r_j = n + 1/2$. In this way, the orientifold action reads
$$O|\mathrm{NS}_n\rangle = (-1)^{n+1} |\mathrm{NS}_n\rangle. \tag{3.4.3}$$

- The generic R-state of mass $\alpha' M^2 = n$, with $n \in \mathbb{N}_0$, requires a total number of excitations n and, again suppressing spacetime indices, it can be written as
$$|\mathrm{R}_n\rangle = \alpha_{-n_1}^{I_1} \dots \alpha_{-n_k}^{I_k} b_{-r_1}^{J_1} \dots b_{-r_l}^{J_l} \, |\mathrm{R}\rangle,$$
with $\sum_{i=1}^k n_i + \sum_{j=1}^l r_j = n$, where $|\mathrm{R}\rangle$ is either of the R-vacua. Therefore, for a Dp-brane one can observe the action
$$O|\mathrm{R}_n\rangle = (-1)^{n+1} |\mathrm{R}_n\rangle. \tag{3.4.4}$$

This discussion shows that the spectrum of a Dp-brane sitting on top of an orientifold Op-plane is supersymmetric. In particular, compared to the spectrum of a Dp-brane away from the orientifold singularity, all even-mass levels $\alpha' M^2 = 2n$ are projected out and all odd-mass levels $\alpha' M^2 = 2n + 1$ are preserved by the symmetry.

The analysis of the spectrum of an anti-Dp-brane on top of an Op-plane is identical except for the fact that the orientifold operator acts with the opposite sign on the R-vacuum, which implies that the projection on the fermions is opposite. This means that the spectrum has a pure fermion/boson alternance: in contrast with the locally-supersymmetric spectrum of a Dp- or anti-Dp-brane at a smooth internal point, levels with mass $\alpha' M^2 = 2n$ contain all the fermions of the spectrum but no bosons, whereas levels with mass $\alpha' M^2 = 2n+1$ contain all the bosons of the spectrum but no fermions. As shown by ref. [121], the configuration of a single anti-Dp-brane on top of an Op-plane is stable. For multiple coincident anti-Dp-branes, only Op^+-planes give a stable theory.

Below, figs. 3.2 and 3.3 show the number of physical states for a Dp- and an anti-Dp-brane on top of an Op-plane, respectively, which one can calculate directly from the

Hilbert-space construction. Notice that, whilst the pattern is clear from the discussion above, it is in the following subsections that it will be explained how to compute the exact degeneracies at each level in an efficient way.

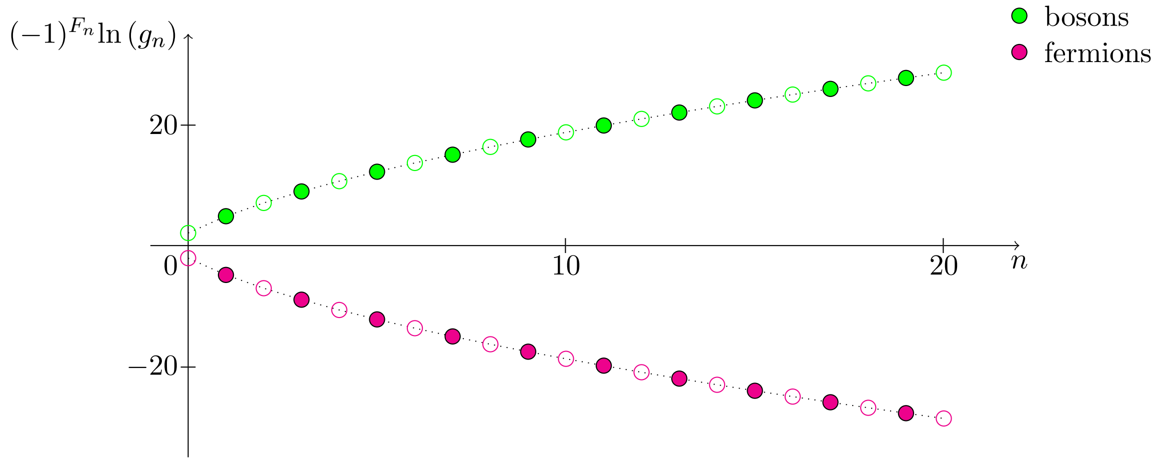

Figure 3.2: The number of bosonic and fermionic physical degrees of freedom for the lightest energy levels of a Dp-brane on top of an Op-plane, defined as $(-1)^{F_n}g_n = N_b(n) - N_f(n)$. Each point corresponds to states with mass $M_n^2 = n/\alpha'$, with $n = 0, 1, \ldots, 20$. Filled points correspond to states that are invariant under the orientifold projection, whereas empty dots represent states that would be there if the Dp-brane was at a smooth point but that are projected out by the orientifold. The number of bosonic and fermionic states is the same at each mass level and the partition function vanishes as required by supersymmetry.

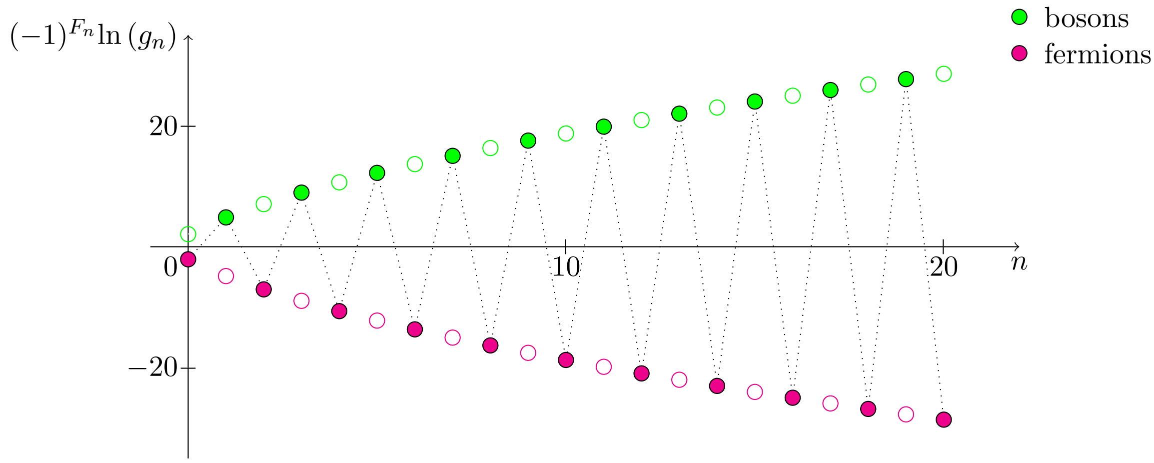

Figure 3.3: The number of bosonic and fermionic physical degrees of freedom for the lightest energy levels of an anti-Dp-brane on top of an Op-plane, defined as $(-1)^{F_n}g_n = N_b(n) - N_f(n)$. Each point corresponds to states with mass $M_n^2 = n/\alpha'$, with $n = 0, 1, \ldots, 20$. Filled points correspond to states that are invariant under the orientifold projection, whereas empty dots represent states that would be there if the anti-Dp-brane was at a smooth point but that are projected out by the orientifold. One clearly sees the presence of a misalignment.

3.4.1.2 Number of Degrees of Freedom by Level

Given the general form of the states in the NS- and R-sectors, a combinatoric analysis allows one to determine the number of the fermionic and bosonic degrees of freedom, $N_f(m)$ and $N_b(m)$ respectively, at each level in a relatively straightforward way.

For each level $\alpha' M^2 = m$, one needs to account for all the possible ways in which it is possible to excite the vacuum giving that mass and to keep in consideration all the symmetrisation and antisymmetrisation factors that are implied by the creation operators. A careful analysis indicates that the number of fermionic degrees of freedom in the R-sector at the level m is expressible as

$$\frac{1}{8}\,N(m) = \sum_{r=0}^{m}\left\{\sum_{\lambda\in P(m-r)}\ \sum_{\mu\in P(r)}\left[\prod_{j=1}^{m-r}\prod_{l=1}^{r}\binom{8+n_j^{(m-r)}-1}{n_j^{(m-r)}}\binom{8}{n_l^{(r)}}\right]\right\}, \tag{3.4.5}$$

where $P(k)$ denotes the set of all the partitions of the integer number k, with $n_j^{(k)}$ representing the coefficients in the partition λ written as $k(\lambda) = \sum_{j=1}^{k} j n_j^{(k)}$. This is a formal way to summarise the results of a counting which, if needed, can be performed explicitly at the desired level. Due to the boson-fermion alternance, the net degeneracy at level m is

$$(-1)^{F_m} g_m = N_b(m) - N_f(m) = (-1)^{m+1} N(m). \tag{3.4.6}$$

From an explicit analysis of the formulae above up to the fourth mass level, one can observe that the spectrum gives the degeneracies $g_0 = 8$, $g_1 = 128$, $g_2 = 1152$ and $g_3 = 7680$. So, as expected, one finds oscillations in the net numbers of bosons and fermions that increase as the mass level of interest grows. This is a necessary condition for misaligned supersymmetry.

A final comment is in order. This analysis of the Hilbert space generated by the fermionic and bosonic creation and annihilation operators acting on the NS- and R-vacua shows that the net number of degrees of freedom at a given mass level is actually the number of the degrees of freedom that are present. In other words, each level has just either bosons or fermions. The particle content at the massless level is shown in table 3.2.

bosons	fermions
	spinors ψ^ω

Table 3.2: Massless particle content for an anti-Dp-brane on top of an Op-plane. No superpartners appear, and the net state degeneracy is $N_b(0) - N_f(0) = 0 - 8 = -8$. The $N_f(0) = 8$ degrees of freedom represent four 4-dimensional Weyl spinors: comparing with the flat-spacetime U(1)-theory with $N_4 = 4$ supersymmetries, these are the would-be gaugino and the three would-be modulini.

3.4.1.3 Partition Functions

One can make use of the results in subsubsection 3.4.1.1 to determine the partition function associated to Dp- and anti-Dp-branes sitting on top of an orientifold Op-plane. For the open strings associated to a Dp-brane, the partition function is defined as

$$M(\tau_2; p) = \mathrm{tr}_{\mathcal{H}_p}\left[\frac{1}{4}(1+O)(1+(-1)^F)\, q[\mathrm{i}\tau_2]^{\frac{1}{2}\tilde{L}_0}\right].$$

In all the basic calculations of partition functions, the traces over the Fock space eventually involve the product $\langle n_1, n_2, \ldots | n_1, n_2, \ldots \rangle$, multiplied by terms of the form $q^{\sum_j j n_j + h}$, with $\sum_j j n_j = n - h$ schematically counting the appropriate number of excitations provided for each level-n mass, with vacuum energies $h = -1/2$ and $h = 0$ in the NS- and R-sectors, respectively. If this product contains the orientifold operator too, following the eigenvalue eqs. (3.4.3, 3.4.4), it gives $\langle n_1, n_2, \ldots | O | n_1, n_2, \ldots \rangle = -(-1)^{\sum_j j n_j - h}$, which effectively corresponds to shifting the modular parameter $\tau = \mathrm{i}\tau_2/2$ to $\tau = \mathrm{i}\tau_2/2 + 1/2$ and an overall sign change. Therefore, the full partition function reads [121, 122]

$$M(\tau_2; p) = \frac{1}{2} \frac{1}{\tau_2^{\frac{1}{2}(p+1)}} \left[\frac{V_8(\mathrm{i}\tau_2/2) - S_8(\mathrm{i}\tau_2/2)}{\eta^8(\mathrm{i}\tau_2/2)} - \frac{V_8(\mathrm{i}\tau_2/2 + 1/2) - S_8(\mathrm{i}\tau_2/2 + 1/2)}{\eta^8(\mathrm{i}\tau_2/2 + 1/2)} \right]. \quad (3.4.7)$$

Again, one can see that the spectrum is supersymmetric due to the Jacobi identity $V_8 = S_8$.

The analysis of the spectrum of an anti-Dp-brane on top of an Op-plane is identical except for the fact that the orientifold operator acts with an opposite sign on the R-vacuum. Therefore, leaving intact the NS-sector but changing sign in the R-sector, one finds the partition function

$$M(\tau_2; \overline{p}) = \frac{1}{2} \frac{1}{\tau_2^{\frac{1}{2}(p+1)}} \left[\frac{V_8(\mathrm{i}\tau_2/2) - S_8(\mathrm{i}\tau_2/2)}{\eta^8(\mathrm{i}\tau_2/2)} - \frac{V_8(\mathrm{i}\tau_2/2 + 1/2) + S_8(\mathrm{i}\tau_2/2 + 1/2)}{\eta^8(\mathrm{i}\tau_2/2 + 1/2)} \right]. \quad (3.4.8)$$

In this case, supersymmetry is broken by the anti-Dp-brane on the orientifold plane and indeed there is always a non-zero contribution to the partition function. These calculations represent an explicit derivation of well-known results that however one can infer quite directly from the analysis of the perturbative string state spectrum. In fact, this will be helpful in both making contact and interpreting such results in subsection 3.4.2 and in discussing more formally the emergence of misaligned supersymmetry in subsection 3.4.3.

3.4.2 Anti-D-Branes on Top of O-Planes

The one-loop amplitude of oriented closed strings is associated to a torus, i.e. the sole closed orientable Riemann surface of vanishing Euler character, having $h = 1$ handles. For open strings, one finds the amplitude associated to the annulus, i.e. another orientable surface, with $b = 2$ boundaries, of vanishing Euler character. With unoriented strings, the situation is more involved. There are indeed two additional Riemann surfaces, with the corresponding amplitude, of vanishing Euler character: the Klein bottle, with $c = 2$ crosscaps, and the Möbius strip, with $b = 1$ boundary and $c = 1$ crosscap. For instance, the partition function of a Dp-brane or anti-Dp-brane in flat space is encoded in the annulus amplitude, while that of a Dp-brane or anti-Dp-brane on top of an Op-plane also involves the Möbius-strip amplitude. In this subsection, the first of these setups is briefly reviewed and then the second one is discussed in detail, making contact with subsections 2.1.3 and 3.4.1.

In type IIB theories, the one-loop partition function of a Dp-brane in flat space is encoded in the annulus amplitude $2A_p(t)$, where the overall factor is instrumental for a definition of $A_p(t)$ that is useful to discuss orientifolds later on. This annulus amplitude reads [121–123]

$$A_p(t) = \frac{1}{2} \frac{1}{(2t)^{\frac{p+1}{2}}} \frac{1}{\eta^8} \big(V_{p-1} O_{9-p} + O_{p-1} V_{9-p} - S_{p-1} S_{9-p} - C_{p-1} C_{9-p} \big) [\mathrm{i}t],$$

Here and in the following, in squared brackets is indicated the argument of the Dedekind η- and Jacobi ϑ-functions and of the so($2n$)-characters. It should be stressed that, for both the annulus and the Möbius strip, here the amplitude parameter t is normalised as in ref. [33], which is half the parameter τ_2 used in ref. [39] and in subsections 2.1.3 and 3.4.1, i.e. $t = \tau_2/2$. A decomposition of the characters simplifies the function $A_p(t)$ to

$$A_p(t) = \frac{1}{2}\frac{1}{(2t)^{\frac{p+1}{2}}}\frac{(V_8 - S_8)}{\eta^8}[\mathrm{i}t]. \tag{3.4.9}$$

The one-loop amplitude is obtained by integrating over the whole spectrum

$$\mathcal{A}_p = \int_0^\infty \frac{\mathrm{d}t}{2t}\,(2t)^{-\frac{p+1}{2}}\frac{(V_8 - S_8)}{\eta^8}[\mathrm{i}t],$$

This partition function and the associated amplitude are vanishing since $V_8 = S_8$. This is a manifestation of the well-known fact that a Dp-brane preserves supersymmetry and thus the net number of bosons minus fermions is vanishing at each energy level. In flat space, there is no real distinction between a Dp-brane and an anti-Dp-brane, indeed the partition function is the same. Things are different after the inclusion of Op-planes.

When an O$p^\pm$-plane is introduced, the 2-dimensional surface of interest is not just the annulus anymore, but also the Möbius strip. Notice that for an odd number of branes, only Op^--planes are allowed, but the discussion here is being kept as generic as possible. The partition function of a Dp-brane on a O$p^\pm$-plane is [121, 122]

$$M_{p\pm}(t) = \pm\frac{1}{2}\frac{1}{(2t)^{\frac{p+1}{2}}}\frac{(\hat{V}_8 - \hat{S}_8)}{\hat{\eta}^8}[\mathrm{i}t]. \tag{3.4.10}$$

As is customary, the Möbius-strip theory is formulated in terms of hatted characters. Given the term $q[\mathrm{i}t] = \mathrm{e}^{-2\pi t}$, these are defined to be manifestly real as

$$\hat{\chi}_i(\mathrm{i}t) = \mathrm{e}^{-\mathrm{i}\pi H_i}\chi_i\left(\mathrm{i}t + \frac{1}{2}\right) = q[\mathrm{i}t]^{H_i}\sum_{n=0}^{\infty}(-1)^n a_{i,n} q[\mathrm{i}t]^n.$$

Indeed, the Möbius strip has $\tau = \mathrm{i}t + 1/2$, with a non-vanishing constant real part. On the one hand, this real part is crucial for misaligned supersymmetry, since it introduces relative signs for the number of states at each mass level. On the other hand, as a consequence of the fixed real part of the argument, $\chi(\mathrm{i}t+1/2)$ acquires a phase, $\mathrm{e}^{\mathrm{i}\pi H}$, which can be conveniently eliminated by defining the manifestly real quantity $\hat{\chi}(\mathrm{i}t)$. Anyway, one can see that for the combinations V_8/η^8 and S_8/η^8, the phase is trivial, i.e. $\mathrm{e}^{\mathrm{i}\pi H} = 1$, therefore in the following the hatted notation is not going to be used, not playing a role in the mathematical analysis of the function. Similarly to the annulus partition function in eq. (3.4.9), the Möbius-strip partition function in eq. (3.4.10) vanishes due to supersymmetry, since $\hat{V}_8 = \hat{S}_8$.

When considering an anti-Dp-brane, the orientifold-projection operator in the partition function does not alter the Neveu-Schwarz sector but it gives the opposite sign in the Ramond sector, with respect to the Dp-brane. This is similar to what happens in the heterotic SO(16)×SO(16)-theory, discussed in subsection 3.3.2, where the orbifold projection introduces additional minus signs into a supersymmetric (and hence vanishing) partition function. So, the partition function of an anti-Dp-brane on an O$p^{\pm}$-plane is

$$M_{\overline{p}\pm}(t) = \pm\frac{1}{2}\frac{1}{(2t)^{\frac{p+1}{2}}}\frac{(V_8+S_8)}{\eta^8}\left[\mathrm{i}t+\frac{1}{2}\right]. \tag{3.4.11}$$

Crucially, due to the fermionic sign flip, this is not vanishing anymore, evidencing the breaking of supersymmetry. The associated amplitude is

$$\mathcal{M}_{\overline{p}\pm} = \pm\int_0^\infty \frac{\mathrm{d}t}{2t}(2t)^{-\frac{p+1}{2}}\frac{(V_8+S_8)}{\eta^8}\left[\mathrm{i}t+\frac{1}{2}\right].$$

It is worth to observe that the interpretation of these results follows straightforwardly from the premises of subsection 3.4.1.3. In fact, the amplitudes associated to the annulus and the Möbius strip are exactly the terms that, when combined, appear in the explicit calculation of the partition functions for Dp- and anti-Dp-branes in the presence of an orientifold symmetry. One can easily verify that for a Dp- and an anti-Dp-brane on top of an Op^--plane, the partition functions in eqs. (3.4.7, 3.4.8), respectively, can be written as

$$M(\tau_2;p) = A_p(\tau_2/2) + M_{p-}(\tau_2/2), \tag{3.4.12}$$

$$M(\tau_2;\overline{p}) = A_p(\tau_2/2) + M_{\overline{p}-}(\tau_2/2). \tag{3.4.13}$$

These simple observations will prove helpful in the following in order to interpret and manipulate the Laurent expansions of the partition functions. In particular, the Möbius-strip amplitude can be used to count the net physical degeneracies of the theory, since the annulus amplitude is identically zero. It is in this sense that the so(8)-characters V_8 and S_8 count both bosons and fermions, depending on the situation. As eqs. (3.4.12, 3.4.13) show, this is because the actual boson-fermion counting should include the interplay of the annulus and Möbius-strip contributions when discussing bosons and fermions separately.

Now, a first indication of the presence of misaligned supersymmetry in the anti-Dp-brane/Op-plane system can be obtained pretty easily by expanding the integrand in powers of q. One just needs the p-independent factor

$$M(\mathrm{i}t) = -\frac{1}{2}\frac{V_8+S_8}{\eta^8}\left[\mathrm{i}t+\frac{1}{2}\right]. \tag{3.4.14}$$

In fact, in terms of the variable $q[\mathrm{i}t] = \mathrm{e}^{-2\pi t}$, a simple Laurent expansion reveals the lightest-level degeneracies as the q-coefficients appearing in the series

$$-\frac{1}{2}\frac{(V_8+S_8)}{\eta^8}\left[\mathrm{i}t+\frac{1}{2}\right] = -8+128\,(\mathrm{e}^{-2\pi t})-1152\,(\mathrm{e}^{-2\pi t})^2+7680\,(\mathrm{e}^{-2\pi t})^3+O((\mathrm{e}^{-2\pi t})^4;0).$$

One notices an increasing oscillation in the number of bosons and fermions at each energy levels, as shown in fig. 3.3, and the degeneracies correspond to the ones found in eq. (3.4.6). As anticipated, the alternating signs giving rise to the oscillation are precisely due to the fixed real part in the argument of the characters. Indeed one can check that setting it to zero would result in the same expansion in powers of $q[\mathrm{i}t]$, but without any sign flip, i.e.

$$\frac{1}{2}\frac{(V_8+S_8)}{\eta^8}[\mathrm{i}t] = 8+128\,(\mathrm{e}^{-2\pi t})+1152\,(\mathrm{e}^{-2\pi t})^2+7680\,(\mathrm{e}^{-2\pi t})^3+O((\mathrm{e}^{-2\pi t})^4;0).$$

These expansions are meaningful for the Op^--plane case. The Op^+-plane case has just an overall sign difference, but one would also need to include the Chan-Paton degeneracies (which would also modify the Op^--plane case, of course).

As done for closed strings and the example of the heterotic SO×SO(16)-theory, one can make use of the formalism of subsection 3.3.1 to perform a quantitative study of the asymptotic growths of the state degeneracies and show the presence of misaligned supersymmetry.

3.4.3 Asymptotic Number of States

In the general discussion about the counting of states and misaligned supersymmetry for closed strings in subsection 3.3.1, and in the example in subsection 3.3.2, a major role is played by modular properties. Indeed, the determination of the asymptotic degeneracies of the characters χ_i and $\overline{\chi}_{\bar{j}}$ in eq. (3.3.8) is based on the fact that they are a representation of the modular group. This is then used to obtain the physical degeneracies in eq. (3.3.13) to be combined by means of the matrix $A^{i\bar{j}}$. In addition, the very proof that misaligned supersymmetry is present in its weak form, i.e. eq. (3.3.17), relies on the particular structure of the matrix $A^{i\bar{j}}$, which is constrained by the partition-function modular invariance. In the heterotic SO(16)×SO(16)-theory, this can be seen in eqs. (3.3.28, 3.3.34).

Open strings are not necessarily $\mathrm{PSL}_2(\mathbb{Z})$-covariant, but this is not necessarily a problem for computing their asymptotic net degeneracies and for discussing the presence of misaligned supersymmetry, as long as the theory can be formulated in terms of a basis of characters that is closed under the modular group and that has non-negative expansion coefficients $a_{i,n}$. In fact, this is enough to determine the asymptotic net degeneracies as in eq. (3.3.8). In a given theory, it is then possible to check explicitly whether the sector-averaged degeneracies cancel out. This is discussed below for the anti-Dp-brane/Op-plane theory.

For the anti-Dp-brane/Op-plane system, it is possible to check that the partition function $M(\tau) = -(1/2)(V_8 + S_8)/\eta^8[\tau + 1/2]$ does not transform covariantly under modular S-transformations. On the other hand, the function $A(\tau) = (1/2)(V_8 - S_8)/\eta^8[\tau]$ is closed and in fact it is a modular form of weight $k = -4$. This has implications on the basis of the characters that one can choose in order to study the presence of misaligned supersymmetry, since a direct generalisation of the closed-string-like analysis of the theory of an anti-Dp-brane sitting on an Op-plane is not feasible. In fact, a simple observation allows one to describe the degeneracies of the function $M(\tau)$ in terms of modular forms. Indeed, the only difference generated by the shift $\tau \to \tau + 1/2$ is in the transformation $q[\tau] = \mathrm{e}^{2\pi\mathrm{i}\tau} \to q[\tau + 1/2] = -q[\tau]$, which in a Laurent series of the form

$$M(\tau) = \sum_{n=0}^{\infty} a_n q^n[\tau]$$

just reflects in the change

$$\tilde{M}(\tau) = M(\tau + 1/2) = \sum_{n=0}^{\infty} (-1)^n a_n q^n[\tau] = \sum_{n=0}^{\infty} \tilde{a}_n q^n[\tau].$$

For the function $\tilde{M}(\tau) = -(1/2)(V_8 + S_8)/\eta^8[\tau + 1] = -(1/2)(V_8 + S_8)/\eta^8[\tau]$, one can simply notice that a closed basis of characters involving $V_8/\eta^8[\tau]$ and $S_8/\eta^8[\tau]$ is in fact known. Indeed one can take the basis in eq. (3.3.26), also used for the right-moving sector of the SO(16)×SO(16)-theory, with the modular T- and S-transformations acting as in eqs. (3.3.29). The basis elements $\chi_1 = V_8/\eta^8 = \sum_{n=0}^{\infty} a_{1,n} q^n$ and $\chi_2 = S_8/\eta^8 = \sum_{n=0}^{\infty} a_{2,n} q^n$ are coupled to the identity sector $i = 0$ via the element $S_i{}^0 = 1/2$ and their leading exponential contribution to the asymptotic number of states is

$$\begin{aligned} a_{1,n} &\overset{n\sim\infty}{\simeq} \frac{1}{2\cdot 2^{\frac{11}{4}}} \frac{1}{n^{\frac{11}{4}}} \mathrm{e}^{4\pi(\frac{1}{2})^{\frac{1}{2}} n^{\frac{1}{2}}}, \\ a_{2,n} &\overset{n\sim\infty}{\simeq} \frac{1}{2\cdot 2^{\frac{11}{4}}} \frac{1}{n^{\frac{11}{4}}} \mathrm{e}^{4\pi(\frac{1}{2})^{\frac{1}{2}} n^{\frac{1}{2}}}, \end{aligned}$$

where use has been made of the general result of eq. (3.3.8), with $H_0 = -1/2$ and $k = -4$. In particular, the inverse Hagedorn temperature can be read off to be

$$C_{\text{tot}} = 4\pi \left(\frac{1}{2}\right)^{\frac{1}{2}}. \tag{3.4.15}$$

Although $M(\tau)$ itself is not a modular form, its close relationship with the modular form $A(\tau)$ allows one to find its asymptotic coefficients. As a matter of fact, the coefficients of $\tilde{M}(\tau)$ are the opposite of those of $\chi_1(\tau) = \chi_2(\tau)$, i.e. $\tilde{a}_n = -a_{1,n} = -a_{2,n}$, and related to those of $M(\tau)$ as $\tilde{a}_n = (-1)^n a_n$, so the coefficients a_n have an asymptotic expansion

$$a_n \overset{n\sim\infty}{\simeq} \frac{(-1)^{n+1}}{2\cdot 2^{\frac{11}{4}}} \frac{1}{n^{\frac{11}{4}}} \mathrm{e}^{4\pi(\frac{1}{2})^{\frac{1}{2}} n^{\frac{1}{2}}}. \tag{3.4.16}$$

Trying to mimic the formalism adopted for closed strings, one can write

$$M(\mathrm{i}t) = \chi_b[\mathrm{i}t] - \chi_f[\mathrm{i}t],$$

where, for $q[\mathrm{i}t] = \mathrm{e}^{-2\pi t}$, the bosonic and fermionic contributions have been arranged into the two contributions

$$\begin{aligned} \chi_b[\mathrm{i}t] &= \frac{1}{2}\left(\frac{V_8}{\eta^8}[\mathrm{i}t] - \frac{V_8}{\eta^8}\left[\mathrm{i}t + \frac{1}{2}\right]\right) = 128\,(\mathrm{e}^{-2\pi t}) + 7680\,(\mathrm{e}^{-2\pi t})^3 + O((\mathrm{e}^{-2\pi t})^5; 0), \\ \chi_f[\mathrm{i}t] &= \frac{1}{2}\left(\frac{S_8}{\eta^8}[\mathrm{i}t] + \frac{S_8}{\eta^8}\left[\mathrm{i}t + \frac{1}{2}\right]\right) = 8 + 1152\,(\mathrm{e}^{-2\pi t})^2 + O((\mathrm{e}^{-2\pi t})^4; 0). \end{aligned}$$

It is then possible to define two envelope functions, for the bosonic and fermionic sectors separately, i.e.

$$\Phi_{b,f}(n) = \frac{(-1)^{2s_{b,f}}}{2\cdot 2^{\frac{11}{4}}} \frac{1}{n^{\frac{11}{4}}} \mathrm{e}^{4\pi(\frac{1}{2})^{\frac{1}{2}} n^{\frac{1}{2}}} + \phi_{b,f}(n),$$

where the sign is obviously given by $(-1)^{2s_b} = 1$ and $(-1)^{2s_f} = -1$ and $\phi_{b,f}(n)$ represents all subleading contributions. This is simply a formal way to be able to express the idea that the leading exponentials in $\Phi_{b,f}(n)$ cancel out when summing over all sectors. Indeed it is possible to write the sector-averaged net degeneracies as

$$\langle a_n \rangle = \Phi_b(n) - \Phi_f(n) = \phi_b(n) - \phi_f(n). \tag{3.4.17}$$

This suggests that misaligned supersymmetry is present for the anti-Dp-brane on top of an Op-plane in type II string theory and that the effective inverse Hagedorn temperature C_{eff} of this theory is such that

$$C_{\text{eff}} < C_{\text{tot}}. \tag{3.4.18}$$

It should be noted that this whole discussion essentially provides a heuristic argument for a deeper idea, i.e. the fact that the anti-Dp-brane/Op-brane theory has a vanishing effective inverse Hagedorn temperature $C_{\text{eff}} = 0$, once all subleading corrections are taken into account. Of course, the point is that the coefficients of the theory of an anti-Dp-brane on top of an Op-plane match precisely with those of a Dp-brane, except for the fact that half of them are zero. In an interpolation of the Dp-brane bosonic and fermionic degeneracies, as figs. 3.2 and 3.3 suggest, one expects to find an all-order cancellations due to supersymmetry, therefore the same conclusion is to be expected for the envelope functions of the anti-Dp-brane/Op-brane theory.

It is worth reminding that up to this point, similarly to the case of closed strings, the cancellation of the envelope functions for open strings is not immediately related to actual physical cancellations, but rather it should serve as a tool to visualise how the theory may be capable of maintaining finiteness. In section 3.6, it is going to be shown that cancellations do indeed occur also at subleading orders, in a variety of models including an anti-Dp-brane on top of an Op-plane and the heterotic SO(16)$\times$SO(16)-theory as well. In particular, it will be shown fairly generically that the condition $C_{\text{eff}} = 0$ holds true in the presence of misaligned supersymmetry. The role of such cancellations in physical observables is then going to be discussed in section 3.7.

3.5 HRR-Expansions Beyond Leading Order

In sections 3.3 and 3.4, misaligned supersymmetry has been discussed just by looking at the leading exponentials in the asymptotic expansion of the net string-state degeneracies of a given model. This is enough to conclude that there exists a notion of an effective inverse Hagedorn temperature that is smaller than the one characterising the individual sectors of the theory, i.e. $C_{\text{eff}} < C_{\text{tot}}$. A natural further step is the discussion of subleading orders and the verification of whether the conjecture $C_{\text{eff}} = 0$ is realised.

As explained in subsection 3.3.1, the formalism of the functional forms $\Phi_{i\bar{j}}(n)$ is not well-suited when going beyond leading order in the HRR-expansion for the sector-averaging procedure of the state degeneracies $a_{i\bar{j},nn}$. The core of the problem is that, for subleading orders $\alpha > 1$, it is hard to deal with the terms $Q(\alpha; n)_i^{\ j}$ appearing in the general term of eq. (3.3.4). A general prescription for analysing an arbitrary order in α is indeed complicated to implement practically due to the intricacies in the definition of the function $Q(\alpha; n)_i^{\ j}$. Also, to extend the variable n beyond its original domain, i.e. the key part of sector-averaging, such functions become typically complex, while any function $\Phi_{i\bar{j}}(n)$ should be real. This is of course a problem of definition and not any intrinsic issue of the theory. State degeneracies are real and the envelope functions are just visualising tools to interpret the spectrum of the theory. Quite simply, one just needs to find a general way to define these envelope functions at all orders and such that it makes the calculations manageable.

It turns out that, for a specific class of Dedekind η-quotients, a simple HRR-formula can be derived for the Laurent coefficients at all orders, as shown by ref. [124]. One can then use this knowledge to determine the state degeneracies of a given string model and to also define envelope functions at all orders. Such a class of functions is going to be the focus of the attention just because it happens to be associated with a simple HRR-series, and it is sufficiently general that all the prototypical models of interest happen to be amenable to an analysis via this simplified expansion. There is no deeper meaning in this choice.

The mathematical results are reviewed in subsection 3.5.1 and then, in subsections 3.5.2 and 3.5.3, the partition functions of the heterotic SO(16) × SO(16)-theory and of the anti-Dp-brane/Op-plane theory, respectively, are recast in terms that are amenable for such an analysis and further discussed. This framework is going to be sufficient to formulate a general procedure to study cancellations at all HRR-orders in the envelope functions in section 3.6 and also in the one-loop cosmological constant in section 3.7.

3.5.1 Rademacher Series for η-quotients

Let $\tau \in \mathbb{H}$ be a variable defined in the complex upper-half plane, with the squared nome being $q = \mathrm{e}^{2\pi \mathrm{i}\tau}$. Let $\delta = \{\delta_m \in \mathbb{Z}\}_{m=1}^{\infty}$ be a sequence of integers such that only a finite number of them is non-zero and let $Z : \mathbb{H} \to \mathbb{C}$ be a function of the variable τ that can be written as

$$Z(\tau) = \prod_{m=1}^{\infty} \left[\eta(m\tau)\right]^{\delta_m}. \tag{3.5.1}$$

This is a Dedekind η-quotient and its properties are entirely encoded in the sequence δ. This function can be expressed in a Laurent series in terms of the squared nome as

$$Z(\tau) = q^{-n_0} \sum_{n=0}^{\infty} a_n q^n, \tag{3.5.2}$$

where a_n are the coefficients to be determined and the constant n_0 can be written as

$$n_0 = -\frac{1}{24} \sum_{m=1}^{\infty} m\, \delta_m. \tag{3.5.3}$$

In order to express the coefficients a_n in a general form just based on the information contained in the sequence δ, a few more definitions are in order.

- Let the constant c_1 and the functions $c_2 = c_2(\alpha)$, $c_3 = c_3(\alpha)$ be defined as

$$c_1 = -\frac{1}{2} \sum_{m=1}^{\infty} \delta_m, \tag{3.5.4}$$

$$c_2(\alpha) = \prod_{m=1}^{\infty} \left[\frac{\gcd(m,\alpha)}{m}\right]^{\frac{\delta_m}{2}}, \tag{3.5.5}$$

$$c_3(\alpha) = -\sum_{m=1}^{\infty} \delta_m \frac{[\gcd(m,\alpha)]^2}{m}. \tag{3.5.6}$$

  Under the full modular group, the function $Z = Z(\tau)$ does not necessarily transform covariantly. Indeed, it is a covariant form of weight $k = -c_1$ just under the congruence $\mathrm{PSL}_2(\mathbb{Z})$-subgroup

$$\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) : \, c = 0 \bmod n \right\},$$

where $n = \mathrm{lcm}\,\{m \in \mathbb{N} : \delta_m \neq 0\}$. Although the S-transformation is not necessarily part of $\Gamma_0(n)$, however, the Dedekind quotient $Z(\tau) = \prod_m [\eta(m\tau)]^{\delta_m}$ still transforms in such a way that $-c_1$ may be loosely interpreted as a weight, as can be verified by noticing the transformation rule $Z(-1/\tau) = (-\mathrm{i}\tau)^{-c_1} (\prod_m m^{-\delta_m/2}) \prod_m [\eta(\tau/m)]^{\delta_m}$.

- Then, given the function

$$\varphi(\beta, \alpha) = \mathrm{e}^{-\mathrm{i}\pi \sum_{m=1}^{\infty} \delta_m \, s\left(\frac{m\beta}{\gcd(m,\alpha)}, \frac{\alpha}{\gcd(m,\alpha)}\right)},$$

where $s(\beta, \alpha)$ represents the Dedekind sum

$$s(\beta, \alpha) = \sum_{n=1}^{\alpha-1} \frac{n}{\alpha} \left(\frac{\beta n}{\alpha} - \left\lfloor \frac{\beta n}{\alpha} \right\rfloor - \frac{1}{2} \right),$$

let the function $P_\alpha = P_\alpha(n)$ be

$$P_\alpha(n) = \sum_{\substack{0 \leq \beta < \alpha, \\ \gcd(\beta,\alpha)=1}} \mathrm{e}^{-2\pi \mathrm{i} n \frac{\beta}{\alpha}} \varphi(\beta, \alpha). \tag{3.5.7}$$

In the following, these functions $P_\alpha = P_\alpha(n)$ are indicated as Kloosterman-like sums. These can be verified to be real for all values of $\alpha \in \Gamma$ and $n \in \mathbb{N}_0$.

- To conclude, let the function $G = G(\alpha)$ be

$$G(\alpha) = \min_{m \in \mathbb{N}:\, \delta_m \neq 0} \left\{ \frac{[\gcd(m, \alpha)]^2}{m} \right\} - \frac{c_3(\alpha)}{24}. \tag{3.5.8}$$

This function essentially controls the number of Bessel functions appearing in the final HRR-expansion: if it is non-negative, there is just one Bessel function appearing in the HRR-expansion.

In this setup, the main result of ref. [124] is the following theorem.

Theorem (Sussman). If the constant c_1 is positive the function $G = G(\alpha)$ is non-negative, i.e. if $c_1 > 0$ and $G(\alpha) \geq 0$, then, for an arbitrary integer $n > n_0$, the coefficients a_n in the series expansion of the function $Z = Z(\tau)$ in eq. (3.5.2) can be written as

$$a_n = \frac{2\pi}{[24(n-n_0)]^{\frac{c_1+1}{2}}} \sum_{\alpha \in \Gamma} c_2(\alpha) \, [c_3(\alpha)]^{\frac{c_1+1}{2}} \, \frac{P_\alpha(n)}{\alpha} \, I_{c_1+1}\left[\left(\frac{2\pi^2}{3\alpha^2} c_3(\alpha)(n-n_0) \right)^{\frac{1}{2}} \right], \tag{3.5.9}$$

where Γ is the set $\Gamma = \{\alpha \in \mathbb{N} : c_3(\alpha) > 0\}$ and $I_\nu = I_\nu(z)$ represents the modified Bessel function of the first kind.

Although the result in eq. (3.5.9) is a consequence of the HRR-expansion, for practical reasons, in what follows it will also be indicated as a 'Sussman HRR-expansion'. Compared to the general HRR-formula in eq. (3.3.4), one can see that in the case of eq. (3.5.9) there is no mixing between different sectors and that all contributions are in the form of the modified Bessel functions of the first kind. Moreover, the Kloosterman-like sums are expressed in an essentially simpler form.

This formula allows one to have control over each of the various contributions, i.e. the leading and all of the subleading terms, to a given state degeneracy a_n. Because of the asymptotic expansion $I_\nu(x) \overset{x\sim\infty}{\simeq} \mathrm{e}^x/(2\pi x)^{\frac{1}{2}}$, each descreasing value $c_3(\alpha)/\alpha^2$, for $\alpha \in \Gamma$, represents a successively subleading exponential correction to the coefficient a_n. If α_0 is the integer maximising $c_3(\alpha)/\alpha^2$, defining $c_0 = c_3(\alpha_0)/\alpha_0^2$, the asymptotic expression of a_n is

$$a_n \overset{n\sim\infty}{\simeq} \frac{1}{8^{\frac{1}{2}}} \frac{1}{(n-n_0)^{\frac{2c_1+3}{4}}} \left[\frac{2c_0}{3}\right]^{\frac{2c_1+1}{4}} c_2(\alpha_0)\, P_{\alpha_0}(n) \left[\frac{\alpha_0}{4}\right]^{c_1} \mathrm{e}^{\left[\frac{2\pi^2 c_0}{3}(n-n_0)\right]^{\frac{1}{2}}}. \tag{3.5.10}$$

3.5.1.1 Properties of the Function $P_\alpha(n)$

The series coefficients a_n in eq. (3.5.9) involve the n- and α-dependent functions $P_\alpha(n)$, defined in eq. (3.5.7), that are sums over phases. Being invariant under the shift $n \to n + m\alpha$ for any $m \in \mathbb{Z}$, namely $P_\alpha(n) = P_\alpha(n + m\alpha)$, these terms $P_\alpha(n)$ can take only up to α different values, at a fixed order α. These values can be denoted as $P_\alpha(\beta)$, with $\beta = 1, \ldots, \alpha$. An important property for the forthcoming discussion is that specific sums of such terms $P_\alpha(\beta)$ are vanishing. This is a consequence of the following lemma.

Lemma. Given the natural numbers $m, \alpha, n \in \mathbb{N}$, and defining $q_\alpha(m) = \gcd(\alpha, m)$, if $\alpha > 1$ and if $\nexists\, p \in \mathbb{N}:\ m = p\alpha$, i.e. if m is not a multiple of α, then the identity holds

$$\sum_{\beta=0}^{\alpha/q_\alpha(m)-1} P_\alpha(n + m\beta) = 0. \tag{3.5.11}$$

The proof of this result is straightforward. Indeed one can easily observe the validity of the series of identities

$$\begin{aligned}
\sum_{\beta=0}^{\alpha/q_\alpha(m)-1} P_\alpha(n + m\beta) &= \sum_{\beta=0}^{\alpha/q_\alpha(m)-1} \sum_{\substack{0\le\gamma<\alpha,\\ \gcd(\gamma,\alpha)=1}} \mathrm{e}^{-2\pi\mathrm{i}(n+m\beta)\frac{\gamma}{\alpha}} \varphi(\gamma,\alpha) \\
&= \sum_{\substack{0\le\gamma<\alpha,\\ \gcd(\gamma,\alpha)=1}} \mathrm{e}^{-2\pi\mathrm{i}n\frac{\gamma}{\alpha}} \varphi(\gamma,\alpha) \sum_{\beta=0}^{\alpha/q_\alpha(m)-1} \mathrm{e}^{-2\pi\mathrm{i}m\beta\frac{\gamma}{\alpha}} \\
&= \sum_{\substack{0\le\gamma<\alpha,\\ \gcd(\gamma,\alpha)=1}} \mathrm{e}^{-2\pi\mathrm{i}n\frac{\gamma}{\alpha}} \varphi(\gamma,\alpha) \left[\frac{1-\mathrm{e}^{-2\pi\mathrm{i}m\gamma/q_\alpha(m)}}{1-\mathrm{e}^{-2\pi\mathrm{i}m\frac{\gamma}{\alpha}}}\right] = 0,
\end{aligned}$$

where use has been made of the geometric sum $\sum_{n=0}^{s-1} r^n = (1-r^s)/(1-r)$, with $r = \mathrm{e}^{-2\pi\mathrm{i}m\frac{\gamma}{\alpha}}$ and $s = \alpha/q_\alpha(m)$, and of the fact that $\alpha/q_\alpha(m), m\gamma/q_\alpha(m) \in \mathbb{N}$. An important subcase of this result is for $n = 0$ and $m = 1$, which gives

$$\sum_{\beta=0}^{\alpha-1} P_\alpha(\beta) = 0. \tag{3.5.12}$$

This is a key result to prove the cancellations among the various sectors beyond leading order. A similar result also applies for the functions $Q(\alpha;n)_i^{\,j}$ in eq. (3.3.5a), but working with Dedekind η-quotients amenable to the Sussman HRR-expansion of eq. (3.5.9) proves to be an easier and yet sufficiently general task. Hence, the interest in this formulation.

3.5.2 Heterotic SO(16)×SO(16)-Theory in Terms of Dedekind η-quotients

This subsection shows in detail how to recast the partition function of the heterotic SO(16)× SO(16)-theory in a form that is suitable for determining the net state degeneracies in terms of the results in eqs. (3.5.9, 3.5.10). To this purpose, one has to employ standard identities for modular functions in order to tackle some subtleties concerning the applicability of the Sussman HRR-expansion, as is going to be explained.

To analyse the partition function of the heterotic SO(16)×SO(16)-theory, it is convenient to ignore the factor coming from spacetime momentum integration. So, defining the function $Z(\tau,\overline{\tau}) = (\operatorname{Im}\tau)^4\, Z_{\mathrm{SO}(16)\times\mathrm{SO}(16)}(\tau,\overline{\tau})$, starting from the partition function in eq. (3.3.23) and with the help of eqs. (3.3.24, 3.3.25), in terms of Dedekind η- and Jacobi ϑ-functions one can write

$$Z(\tau,\overline{\tau}) = \frac{1}{2}\,\frac{1}{\eta^{12}\overline{\eta}^{24}}\left[\vartheta_2^4\overline{\vartheta}_3^8\overline{\vartheta}_4^8 - \vartheta_3^4\overline{\vartheta}_2^8\overline{\vartheta}_4^8 + \vartheta_4^4\overline{\vartheta}_2^8\overline{\vartheta}_3^8\right][\tau,\overline{\tau}]. \tag{3.5.13}$$

It is convenient to separate the three terms in the sum and to factorise the contributions from right- and left-movers, which will be denoted as $R_\sigma(\tau)$ and $L_\sigma(\tau)$, respectively, with $\sigma = 1,2,3$, by writing the partition function as

$$Z(\tau,\overline{\tau}) = \sum_{\sigma=1}^{3} Z_\sigma(\tau,\overline{\tau}) = \frac{1}{2}\sum_{\sigma=1}^{3} R_\sigma(\tau)\overline{L}_\sigma(\overline{\tau}). \tag{3.5.14}$$

Each Jacobi ϑ-function can be expressed in terms of a combination of Dedekind η-functions with different powers and arguments, as reviewed in appendix A.1.1. Below, taking advantage of this, the three terms in eq. (3.5.14) are recast in terms of Dedekind η-quotients amenable to the expansion of eq. (3.5.9) and analysed separately. It should be highlighted that the individual terms $Z_\sigma(\tau,\overline{\tau})$ are not modular-invariant, but their combination is.

In each term, with $q[\tau] = \mathrm{e}^{2\pi\mathrm{i}\tau}$, the partition function takes the form

$$Z_\sigma(\tau,\overline{\tau}) = \frac{1}{2}R_\sigma(\tau)\overline{L}_\sigma(\overline{\tau}) = \frac{1}{2}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} a^{R_\sigma}_{d_\sigma k}\overline{a}^{L_\sigma}_{d_\sigma l}\, q[\tau]^{d_\sigma(k-n_0^{R_\sigma})}\overline{q}[\overline{\tau}]^{d_\sigma(l-n_0^{L_\sigma})},$$

in a self-explaining notation, where the possibility of fractional arguments has been taken into account (see eqs. (3.5.27)), with either $d_\sigma = 1$ or $d_\sigma = 1/2$. Level-matched physical states can be seen to correspond to levels such that $k - n_0^{R_\sigma} = l - n_0^{L_\sigma}$. To deal with an expansion with only integer indices, one can perform a rescaling $\tau' = \tau/d_\sigma$, which gives

$$\begin{aligned} Z'_\sigma(\tau,\overline{\tau}) = Z_\sigma\big(\frac{\tau}{d_\sigma},\frac{\overline{\tau}}{d_\sigma}\big) &= \frac{1}{2}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} a^{R_\sigma}_{d_\sigma k}\overline{a}^{L_\sigma}_{d_\sigma l}\, q[\tau]^{k-n_0^{R_\sigma}}\,\overline{q}[\overline{\tau}]^{l-n_0^{L_\sigma}} \\ &= \frac{1}{2}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} a^{R'_\sigma}_{k}\overline{a}^{L'_\sigma}_{l}\, q[\tau]^{k-n_0^{R_\sigma}}\,\overline{q}[\overline{\tau}]^{l-n_0^{L_\sigma}}. \end{aligned}$$

This means that, for $n \in d_\sigma \mathbb{N}_0$, the level-matched degeneracies read

$$a^\sigma_{nn} = \frac{1}{2}\, a^{R_\sigma}_{n+d_\sigma n_0^{R_\sigma}}\, \overline{a}^{L_\sigma}_{n+d_\sigma n_0^{L_\sigma}} = \frac{1}{2}\, a^{R'_\sigma}_{n/d_\sigma + n_0^{R_\sigma}}\, \overline{a}^{L'_\sigma}_{n/d_\sigma + n_0^{L_\sigma}}\,.$$

This sets the notation for situations where, in order to make use of eq. (3.5.9) for functions with fractional expansion indices, one performs a variable rescaling, making Z' a suitable function for the Sussman HRR-expansion.

- In the first product, the factors are

$$R_1(\tau) = \frac{\vartheta_2^4(\tau)}{\eta^{12}(\tau)} = \frac{16\,\eta^8(2\tau)}{\eta^{16}(\tau)}, \tag{3.5.15a}$$

$$L_1(\tau) = \frac{\vartheta_3^8(\tau)\vartheta_4^8(\tau)}{\eta^{24}(\tau)} = \frac{\eta^8(\tau)}{\eta^{16}(2\tau)}. \tag{3.5.15b}$$

  In both cases, one finds $c_1 = 4$ and $G(\alpha) \geq 0$, therefore the expansion in eq. (3.5.9) applies and provides complete knowledge over all of the subleading contributions. In particular, for R_1 one finds $n_0 = 0$, $c_2(2\omega+1) = 1/16$ and $c_3(2\omega+1) = 12$, with $c_3(2\omega+2) = 0$, while for L_1 one finds $n_0 = 1$, $c_2(2\omega+2) = 1$ and $c_3(2\omega+2) = 24$, with $c_3(2\omega+1) = 0$. The variable ω is a number $\omega \in \mathbb{N}_0$ used to emphasise whether α is even or odd, with $\alpha = 2\omega+2$ or $\alpha = 2\omega+1$.

  Further, one can easily evaluate the asymptotic forms, thanks to eq. (3.5.10). For R_1, one has $c_0 = 12$ for $\alpha_0 = 1$, with $c_2(1) = 1/16$ and $P_1(n) = 1$ (with an overall factor 16), while for L_1 one finds $c_0 = 6$ for $\alpha_0 = 2$, with $c_2(2) = 1$ and $P_2(n) = (-1)^n$, so

$$a^{R_1}_n \overset{n\sim\infty}{\simeq} \frac{1}{4\cdot 8^{\frac{1}{4}}}\,\frac{1}{n^{\frac{11}{4}}}\,\mathrm{e}^{(8\pi^2 n)^{\frac{1}{2}}}, \tag{3.5.16a}$$

$$a^{L_1}_n \overset{n\sim\infty}{\simeq} \frac{1}{2}\,\frac{(-1)^n}{(n-1)^{\frac{11}{4}}}\,\mathrm{e}^{(4\pi^2(n-1))^{\frac{1}{2}}}. \tag{3.5.16b}$$

  Therefore, the asymptotic behaviour of the coefficients of the first term in the partition function follows the pattern

$$a^1_{nn} = \frac{1}{2}\, a^{R_1}_n\, \overline{a}^{L_1}_{n+1} \overset{n\sim\infty}{\simeq} \frac{1}{16\cdot 8^{\frac{1}{4}}}\,\frac{(-1)^{n+1}}{n^{\frac{11}{2}}}\,\mathrm{e}^{[(8\pi^2)^{\frac{1}{2}}+(4\pi^2)^{\frac{1}{2}}]\,n^{\frac{1}{2}}}. \tag{3.5.17}$$

- In the second product, the two factors are

$$R_2(\tau) = -\frac{\vartheta_3^4(\tau)}{\eta^{12}(\tau)} = -\frac{\eta^8(\tau)}{\eta^8(\tau/2)\eta^8(2\tau)}, \tag{3.5.18a}$$

$$L_2(\tau) = \frac{\vartheta_2^8(\tau)\vartheta_4^8(\tau)}{\eta^{24}(\tau)} = \frac{256\,\eta^{16}(\tau/2)\eta^{16}(2\tau)}{\eta^{40}(\tau)}. \tag{3.5.18b}$$

  Clearly, one cannot employ directly eq. (3.5.9) for these expressions because a fractional argument appears. Since it just amounts to an index relabelling, however, one can simply just consider the argument $\tau' = 2\tau$, focussing on the functions

$$R_2'(\tau) = R_2(2\tau) = -\frac{\eta^8(2\tau)}{\eta^8(\tau)\eta^8(4\tau)}, \tag{3.5.19a}$$

$$L_2'(\tau) = L_2(2\tau) = \frac{256\,\eta^{16}(\tau)\eta^{16}(4\tau)}{\eta^{40}(2\tau)}. \tag{3.5.19b}$$

While $c_1 = 4$ and $G(\alpha) \geq 0$ for the factor $-R_2'(\tau)$, the factor $L_2'(\tau)$ does not have a non-negative function $G(\alpha)$. Nevertheless, it turns out that a further shift $\tilde{\tau} = \tau + 1/2$, which amounts to flipping half of the signs in the series expansion (something one can keep track of), even this term happens to have a positive semidefinite function $G(\alpha)$, along with $c_1 = 4$. Appendix A.1.1 reviews the shift $\tau \to \tau + 1/2$ in the Dedekind η-function. So one has to consider the function

$$\tilde{L}_2'(\tau) = L_2'(\tau + 1/2) = \frac{256\,\eta^{8}(2\tau)}{\eta^{16}(\tau)}. \tag{3.5.20}$$

Now these functions are suitable for the expansion in eq. (3.5.9) and it is possible to obtain again a complete understanding of all of the subleading contributions. For $-R_2'$ one finds $n_0 = 1$, $c_2(2\omega+1) = 16$, $c_2(4\omega+4) = 1$, $c_3(2\omega+1) = 6$ and $c_3(4\omega+4) = 24$, with $c_3(4\omega+2) = 0$, while for $\tilde{L}_2'$ one finds $n_0 = 0$, $c_2(2\omega+1) = 1/16$ and $c_3(2\omega+1) = 12$, with $c_3(2\omega+2) = 0$.

One can easily evaluate the asymptotic forms. For $-R_2'$, one has $c_0 = 6$ for $\alpha_0 = 1$, with $c_2(1) = 16$ and $P_1(n) = 1$, while for L_2' (the result for $L_2'(\tau)$ can be obtained by studying $L_2'(\tau+1/2)$ and inserting a factor $(-1)^n$ in the coefficients) one finds $c_0 = 12$ for $\alpha_0 = 1$, $c_2(1) = 1/16$ and $P_1(n) = (-1)^n$ (with an overall factor 256), therefore

$$a_n^{R_2'} \overset{n\sim\infty}{\simeq} -\frac{1}{2}\,\frac{1}{(n-1)^{\frac{11}{4}}}\,\mathrm{e}^{(4\pi^2(n-1))^{\frac{1}{2}}}, \tag{3.5.21a}$$

$$a_n^{L_2'} \overset{n\sim\infty}{\simeq} \frac{2\cdot 2^{\frac{1}{2}}}{2^{\frac{1}{4}}}\,\frac{(-1)^n}{n^{\frac{11}{4}}}\,\mathrm{e}^{(8\pi^2 n)^{\frac{1}{2}}}. \tag{3.5.21b}$$

The coefficients appearing in the second term of the partition function are actually $a_{n+1/2}^{R_2} = a_{2n+1}^{R_2'}$ and $a_n^{L_2} = a_{2n}^{L_2'}$, and they provide the asymptotic behaviour

$$a_{nn}^2 = \frac{1}{2}\,a_{n+\frac{1}{2}}^{R_2}\,\overline{a}_n^{L_2} \overset{n\sim\infty}{\simeq} \frac{1}{64\cdot 2^{\frac{1}{4}}}\,\frac{(-1)^{2n+1}}{n^{\frac{11}{2}}}\,\mathrm{e}^{[(16\pi^2)^{\frac{1}{2}}+(8\pi^2)^{\frac{1}{2}}]\,n^{\frac{1}{2}}}. \tag{3.5.22}$$

- In the third product, the factors are

$$R_3(\tau) = \frac{\vartheta_4^4(\tau)}{\eta^{12}(\tau)} = \frac{\eta^8(\tau/2)}{\eta^{16}(\tau)}, \tag{3.5.23a}$$

$$L_3(\tau) = \frac{\vartheta_2^8(\tau)\vartheta_3^8(\tau)}{\eta^{24}(\tau)} = \frac{256\,\eta^8(\tau)}{\eta^{16}(\tau/2)}. \tag{3.5.23b}$$

Again, in order to employ the expansion of eq. (3.5.9), one can consider the functions

$$R_3'(\tau) = R_3(2\tau) = \frac{\eta^8(\tau)}{\eta^{16}(2\tau)}, \tag{3.5.24a}$$

$$L_3'(\tau) = L_3(2\tau) = \frac{256\,\eta^8(2\tau)}{\eta^{16}(\tau)}. \tag{3.5.24b}$$

These have $c_1 = 4$ and $G(\alpha) \geq 0$, as required. Once more, eq. (3.5.9) provides the complete information on the leading and subleading contributions of this sector. For R_3', one has $n_0 = 1$, $c_2(2\omega+2) = 1$ and $c_3(2\omega+2) = 24$, with $c_3(2\omega+1) = 0$, while for L_3' one finds $n_0 = 0$, $c_2(2\omega+1) = 1/16$ and $c_3(2\omega+1) = 12$, with $c_3(2\omega+2) = 0$.

One can easily evaluate the asymptotic forms. For R_3', one has $c_0 = 6$ for $\alpha_0 = 2$, with $c_2(2) = 1$ and $P_2(n) = (-1)^n$, while for L_3' one finds $c_0 = 12$ for $\alpha_0 = 1$, with $c_2(1) = 1/16$ and $P_1(n) = 1$ (with an overall factor 256), therefore

$$a_n^{R_3'} \overset{n\sim\infty}{\simeq} \frac{1}{2}\,\frac{(-1)^n}{(n-1)^{\frac{11}{4}}}\mathrm{e}^{(4\pi^2(n-1))^{\frac{1}{2}}}, \tag{3.5.25a}$$

$$a_n^{L_3'} \overset{n\sim\infty}{\simeq} \frac{2\cdot 2^{\frac{1}{2}}}{2^{\frac{1}{4}}}\,\frac{1}{n^{\frac{11}{4}}}\mathrm{e}^{(8\pi^2 n)^{\frac{1}{2}}}. \tag{3.5.25b}$$

As above, the coefficients appearing in the third term of the partition function are $a_{n+1/2}^{R_3} = a_{2n+1}^{R_3'}$ and $a_n^{L_3} = a_{2n}^{L_3'}$. The corresponding asymptotic behaviour reads

$$a_{nn}^3 = \frac{1}{2}\,a_{n+\frac{1}{2}}^{R_3}\,a_n^{L_3} \overset{n\sim\infty}{\simeq} \frac{1}{64\cdot 2^{\frac{1}{4}}}\,\frac{(-1)^{2n+1}}{n^{\frac{11}{2}}}\mathrm{e}^{[(16\pi^2)^{\frac{1}{2}}+(8\pi^2)^{\frac{1}{2}}]\,n^{\frac{1}{2}}}. \tag{3.5.26}$$

In terms of the variable $q = \mathrm{e}^{2\pi i\tau}$, the six factors have the Laurent expansions

$$R_1(\tau) = 16 + 256q + 2304q^2 + 15360q^3 + 84224q^4 + O(q^5;0), \tag{3.5.27a}$$
$$L_1(\tau) = q^{-1}\big[1 - 8q + 36q^2 - 128q^3 + 402q^4 + O(q^5;0)\big], \tag{3.5.27b}$$
$$R_2(\tau) = -q^{-\frac{1}{2}}\big[1 + 8q^{\frac{1}{2}} + 36q + 128q^{\frac{3}{2}} + 402q^2 + O(q^{\frac{5}{2}};0)\big], \tag{3.5.27c}$$
$$L_2(\tau) = 256 - 4096q^{\frac{1}{2}} + 36864q - 245760q^{\frac{3}{2}} + 1347584q^2 + O(q^{\frac{5}{2}};0), \tag{3.5.27d}$$
$$R_3(\tau) = q^{-\frac{1}{2}}\big[1 - 8q^{\frac{1}{2}} + 36q - 128q^{\frac{3}{2}} + 402q^2 + O(q^{\frac{5}{2}};0)\big], \tag{3.5.27e}$$
$$L_3(\tau) = 256 + 4096q^{\frac{1}{2}} + 36864q + 245760q^{\frac{3}{2}} + 1347584q^2 + O(q^{\frac{5}{2}};0). \tag{3.5.27f}$$

From the leading-order results from each contribution a_{nn}^{σ}, in eqs. (3.5.17, 3.5.22, 3.5.26), it is evident that, extending the asymptotic forms to pairs of functional forms $\pm\Phi^{\sigma}(n)$, one can confirm that such envelope functions cancel out, even in this formulation. Below, fig. 3.4 reports a plot of the three different sectors that one individuates when writing the partition function in terms of Dedekind η-quotients.

It is worth highlighting the fact that, up to the scaling $\tau \to \tau' = 2\tau$ and to the shift $\tau \to \tilde{\tau} = \tau + 1/2$, the heterotic SO(16)×SO(16)-theory can be written entirely in terms of two functions. Indeed, one has the identities

$$R_3'(\tau) = L_1(\tau) = R_2'(\tau + 1/2), \tag{3.5.28a}$$
$$L_3'(\tau) = 16\,R_1(\tau) = L_2'(\tau + 1/2). \tag{3.5.28b}$$

As a final remark, notice that a quotient of Dedekind η-functions does not necessarily satisfy the requirements of applicability of the HRR-expansion of eq. (3.5.9). In the specific case at hand, nevertheless, simple manipulations on τ allow one to bypass the problem of a non-positive semi-definite function $G(\alpha)$ in eq. (3.5.8).

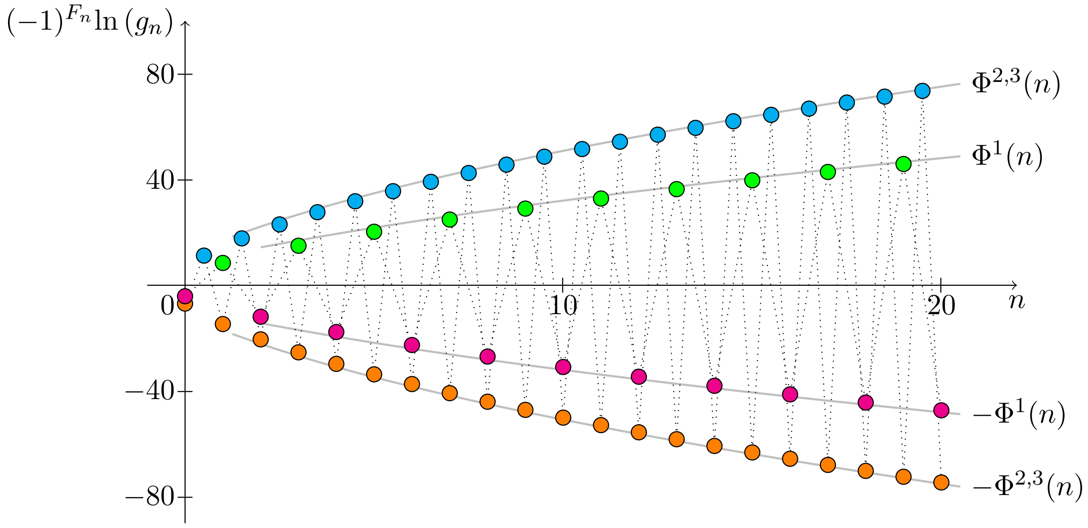

Figure 3.4: The lightest string states in the heterotic SO(16) × SO(16)-theory. The interpolating functions $\pm\Phi^\sigma(n)$, for $\sigma = 1, 2, 3$, correspond to the three terms Z_1, Z_2 and Z_3 that combine into the total partition function, and in particular they are simply the degeneracies a_n^σ at leading order plotted for a continuous variable n. Notice that, although $\Phi^2(n) = \Phi^3(n)$, i.e. Z_2 and Z_3 contribute equally to physical states, the associated off-shell coefficients are different.

3.5.3 Anti-D-Branes on O-Planes in Terms of Dedekind η-quotients

This subsection shows how to formulate the theory of an anti-Dp-brane on top of an Op-plane as an η-quotient, in order to then employ the expansions in eq. (3.5.9, 3.5.10) to study the asymptotic expansion of and the subleading contributions to the state degeneracies.

To discuss the anti-Dp-brane/Op-plane theory, it is convenient to just focus on the p-independent part of the partition function in eq. (3.4.14). This can be simplified by exploiting the properties of the so($2n$)-characters. First of all, it is convenient to remove the dependence on the constant real part in the argument by means of the identities [33]

$$(S_{2n} + C_{2n})\left[\mathrm{i}t + \frac{1}{2}\right] = (S_{2n} + C_{2n})(O_{2n} - V_{2n})[2\mathrm{i}t],$$
$$\eta^{2n}\left[\mathrm{i}t + \frac{1}{2}\right] = \eta^{2n}(O_{2n} + V_{2n})[2\mathrm{i}t].$$

Second, the Jacobi triple-product identity can be recast as

$$(S_{2n} + C_{2n})(O_{2n} + V_{2n})(O_{2n} - V_{2n}) = 2^n.$$

Both expressions can be simplified by noticinng that $S_{2n} = C_{2n}$. Further, in the particular case $n = 4$, a further simplification arises as a consequence of the Jacobi identity $V_8 = S_8$. Summing up, one has

$$V_8\left[\mathrm{i}t + \frac{1}{2}\right] = V_8(O_8 - V_8)[2\mathrm{i}t],$$

$$\eta^8\left[\mathrm{i}t+\frac{1}{2}\right]=\eta^8(O_8+V_8)[2\mathrm{i}t],$$
$$V_8(O_8+V_8)(O_8-V_8)=8.$$

Using all these relationships, one finds the identity

$$\frac{V_8}{\eta^8}\left[\mathrm{i}t+\frac{1}{2}\right]=\frac{2^3}{\eta^8}(O_8+V_8)^{-2}[2\mathrm{i}t]=8\vartheta_3[2\mathrm{i}t]^{-8}. \tag{3.5.29}$$

Therefore, the p-independent Möbius strip amplitude associated to the anti-Dp-brane/Op-plane theory, from eq. (3.4.14), can be written as

$$M(\mathrm{i}t)=-\frac{V_8}{\eta^8}\left[\mathrm{i}t+\frac{1}{2}\right]=-8\vartheta_3[2\mathrm{i}t]^{-8} \tag{3.5.30}$$

In order to take advantage of the Sussman HRR-analysis, it is necessary to consider a function with a generic complex argument, namely $M=M(\tau)$, knowing that the physical properties are actually contained at the locus $\tau=\mathrm{i}t$. Following the discussion above, the generic function to be analysed is

$$M(\tau)=-8\,\vartheta_3^{-8}(2\tau)=-\frac{8\,\eta^{16}(\tau)\,\eta^{16}(4\tau)}{\eta^{40}(2\tau)}, \tag{3.5.31}$$

where the Jacobi ϑ-function has been written as a Dedekind quotient. Appendix A.1.1 contains details about the relevant identities. Of course, one could obtain the same result by simply starting from the function $M=M(\tau)$ as given in eq. (3.4.14) and then directly expanding the Jacobi ϑ-functions appearing in it in terms of Dedekind η-function factors, via other well-known identities in appendix A.1.1. However, it is sometimes convenient to deal with the function $\vartheta_3(\tau)$ alone, hence the method above, which singles it out immediately.

The function in eq. (3.5.31) has $c_1=4$, but it turns out not to be amenable to the expansion of eq. (3.5.9) since the condition $G(\alpha)\geq 0$ is not satisfied. Nevertheless, as for the heterotic SO(16)×SO(16)-string example, this problem can be easily overcome by shifting the variable as $\tau\to\tau+1/2$, which amounts to dealing with an expansion in powers of q without alternating signs. This leads to a 'dual' function

$$\tilde{M}(\tau)=M(\tau+1/2)=-\frac{8\,\eta^8(2\tau)}{\eta^{16}(\tau)}. \tag{3.5.32}$$

For the function $-\tilde{M}(\tau)$, the condition $G(\alpha)\geq 0$ is satisfied and therefore the expansion of eq. (3.5.9) is applicable and allows one to work out all leading and subleading contributions. In particular, one finds $n_0=0$, $c_1=4$, $c_2(2\omega+1)=1/16$, $c_3(2\omega+2)=0$ and $c_3(2\omega+1)=12$. One can easily evaluate the asymptotic form. Indeed, one has $c_0=12$ for $\alpha_0=1$ and $c_2(1)=1/16$, with an overall factor 8. Therefore, inserting a factor $(-1)^{n+1}$ to trace back to the coefficients of $M(\tau)=\sum_{n=0}^{\infty}a_n q[\tau]^n$, the asymptotic form of a_n is

$$a_n\overset{n\sim\infty}{\simeq}\frac{1}{8\cdot 8^{\frac{1}{4}}}\,\frac{(-1)^{n+1}}{n^{\frac{11}{4}}}\mathrm{e}^{(8\pi^2 n)^{\frac{1}{2}}}. \tag{3.5.33}$$

As expected, this result confirms exactly the asymptotic behaviour found in eq. (3.4.16).

To conclude, it is worthwhile to emphasise that the function in eq. (3.5.32) is related to the heterotic SO(16)×SO(16)-theory by the series of identities

$$-M(\tau+1/2)=\frac{1}{2}R_1(\tau)=\frac{1}{32}L_2'(\tau+1/2)=\frac{1}{32}L_3'(\tau). \tag{3.5.34}$$

3.6 Systematic Cancellations at All Orders

This section finally shows that, in the class of models for which the tools presented in section 3.5 can be employed, the cancellations implied by misaligned supersymmetry occur at all orders in the HRR-expansions and the conjecture $C_{\text{eff}} = 0$ holds. This outcome relies crucially on the result of eq. (3.5.11).

To start, subsection 3.6.1 provides a general prescription to construct the envelope functions at any HRR-order and to verify their net cancellation. This result is based on the easy structure of the Sussman HRR-expansion and it is also such that these functions are real when n is continuos even beyond leading order, thus overcoming the problems mentioned in subsection 3.3.1. Then, the discussion is specialised to the two models analysed explicitly in the present work: open strings and the anti-Dp-brane/Op-plane system, in subsection 3.6.2, and closed strings and the heterotic SO(16)×SO(16)-theory, in subsection 3.6.3. Their order of presentation is reverted compared to previous sections for clarity of exposition.

3.6.1 General Procedure

This subsection outlines the general features of the method that is going to be used in subsections 3.6.2 and 3.6.3 to discuss all-order cancellations in misaligned supersymmetry.

3.6.1.1 All-Order Envelope Functions

The starting point for the discussion is the framework presented in subsection 3.5.1. In the same notation therein introduced, for any function $Z(\tau) = q^{-n_0} \sum_{n=0}^{\infty} a_n q^n$ amenable to the Sussman HRR-expansion in eq. (3.5.9), the Laurent coefficients in the q-expansion for $n > n_0$ can be written in the form

$$a_n = \sum_{\alpha \in \Gamma} a_n(\alpha) = \sum_{\alpha \in \Gamma} P_\alpha(n) f_n(\alpha), \tag{3.6.1}$$

where the terms $P_\alpha(n)$ are the Kloosterman-like sums and the functions $f_n = f_n(\alpha)$ are defined as

$$f_n(\alpha) = \frac{2\pi\, c_2(\alpha)\, [c_3(\alpha)]^{\frac{c_1+1}{2}}}{\alpha[24(n-n_0)]^{\frac{c_1+1}{2}}}\, I_{c_1+1}\left[\left(\frac{2\pi^2}{3\alpha^2}\, c_3(\alpha)(n-n_0)\right)^{\frac{1}{2}}\right].$$

Each term $a_n(\alpha)$ represents a different contribution to the coefficient a_n, whose relevance in the limit $n \sim \infty$ is weighed by the ratio $c_3(\alpha)/\alpha^2$. However, the analysis in this subsection is going to be referred to all possible values of n.

In order to go over the usual analysis of misaligned supersymmetry presented in sections 3.3 and 3.4, and extend it beyond leading order, it is necessary to define a notion of 'envelope functions'. The general prescription presented there amounts to individuating the characters of the theory and, for each of them, i.e. for all the degeneracies a_n that belong to each of them, to letting n be a continuous variable, thus promoting the discretised terms a_n to their functional forms with $n \in \mathbb{R}^+$. Here, for any term appearing in the partition function, 'sectors' may be identified by the leading-order results.[3.3] In the case of open strings, these can be typically identified with the bosonic and fermionic states (cfr. subsection 3.5.3). For closed strings, this is more complicated since each term comes from the interplay of right- and left-moving oscillators, but one can just consider the leading-order terms given by each of the products of Dedekind η-quotients in which one can arrange the full partition function (cfr. subsection 3.5.2). Anyway, for the time being, one can focus on open strings; closed strings will be discussed later, in subsection 3.6.3.

One can extend the idea of interpolating functions to all the contributions $a_n(\alpha)$, thus getting some envelope functions for each admitted value of α. Now, whilst the terms with $\alpha = 1$ are manifestly real, being $P_1(n) \equiv 1$, the terms with $\alpha > 1$ can be complex, due to the fact that $P_\alpha(n)$ is a complex number, in general, for $n \in \mathbb{R}^+$. To overcome this problem and to construct a real envelope function, one should notice two facts.

(i) In general, the leading-order contribution in the expression of the degeneracies a_n can underestimate or overestimate the correct value. This means that subleading corrections may come with either positive or negative signs.

(ii) As noticed in section 3.5, for each fixed α, there are only α independent real values of the function $P_\alpha(n)$ as $n \in \mathbb{N}$ varies. To stress the fact that they are being employed with this fact in mind, it is convenient to introduce a notation such that, for all values $\beta = 1, \dots, \alpha$, the Kloosterman-like terms are identified as

$$P_\alpha(n) = P_\alpha(\beta) \equiv p_\alpha(\beta) \in \mathbb{R}, \qquad \forall n \in \mathbb{N}_\alpha(\beta), \tag{3.6.2}$$

where $\mathbb{N}_\alpha(\beta) = \{n \in \mathbb{N} : \; n = \beta \bmod \alpha\}$ are the α subsets of $\mathbb{N}$ in which the function $P_\alpha(n)$ takes each of its specific values. The result in eq. (3.5.11) indicates that the sum of $p_\alpha(\beta)$ over $\beta = 1, \dots, \alpha$ is zero, i.e. $\sum_{\beta=1}^{\alpha} p_\alpha(\beta) = 0$.

Because the aim of this discussion is to define functional forms that interpolate between the physical degeneracies at discrete n, one can define α different subsectors, depending on the value taken by the function $P_\alpha(n)$. In particular, for each of these α different values, one can define functional forms $\Phi_\alpha(n;\beta)$, with $\beta = 1, \dots, \alpha$, such that

$$\Phi_\alpha(n;\beta) = p_\alpha(\beta)\, f_n(\alpha). \tag{3.6.3}$$

The crucial step here is that the terms $P_\alpha(n)$, that are generically complex if $n \in \mathbb{R}$, are being replaced by the discrete and manifestly real terms $p_\alpha(\beta)$, which instead are independent of n. Therefore, the functions in eq. (3.6.3) are now real even if n is a continuous variable. So,

[3.3] It might be helpful to think of the terms $\alpha = 1$ and $\alpha > 1$ as leading and subleading orders, respectively. However, this identification could be misleading. Indeed, what governs the exponential growth in eq. (3.5.9) is the quantity $c_3(\alpha)/\alpha^2$ and it is not guaranteed to be maximized at $\alpha = 1$. Moreover, the condition $c_3(1) \leq 0$ may also happen, and therefore the corresponding term with $\alpha = 1$ would not appear in the sum. In fact, such cases indeed appear for instance in subsection 3.5.2. Nevertheless, when present, the contribution with $\alpha = 1$ has $P_1(n) = 1$ and there is only one subsector $\beta = 0$, thus making this case somehow special. These subtleties will be commented on when they come up in the examples of subsections 3.6.2 and 3.6.3, but for the time being the discussion is going to be kept as plain as possible.

taking into account the fact that the theory has sectors σ – recalling that, for open strings, one can simply think of bosonic and fermionic states –, at each $\alpha > 1$ this procedure is in fact introducing α different subsectors within each sector σ, and the envelope functions are introduced via the enhancements

$$a_n^\sigma(\alpha) \mapsto \Phi_\alpha(n;\beta), \tag{3.6.4}$$

which are defined for each $\beta = 1, \ldots, \alpha$. Indeed, in a given sector σ, for a given integer n and a given order α, the term $a_n^\sigma(\alpha)$ is expressed in terms of a specific Kloosterman-like term $P_\alpha(n) = p_\alpha(\beta)$, for some β, but the presence of the other Kloosterman-like terms at different values of n in the same sector means that, when defining the interpolating functions of a given sector, one is associating all the possible α terms $p_\alpha(\beta)$ to each value of n.

The total number of subsectors increases with α in the HRR-expansion, up to an infinite number of them. As explained above, in general these subsectors are populated by positive and negative contributions, therefore one should expect that cancellations can occur among them. That this is indeed the case is a consequence of the lemma on the functions $P_\alpha(\beta)$, in eq. (3.5.11). To see this explicitly, one just needs to average the envelope functions $\Phi_\alpha(n;\beta)$ over all the subsectors at the fixed value of α, labelled by $\beta = 1, \ldots, \alpha$. In fact, because the functions $f_n(\alpha)$ do not depend on the label β, at any fixed order $\alpha > 1$ one can immediately conclude that such an average is vanishing, i.e.

$$\sum_{\beta=1}^{\alpha} \Phi_\alpha(n;\beta) = \left[\sum_{\beta=1}^{\alpha} p_\alpha(\beta)\, f_n(\alpha)\right] = \left[\sum_{\beta=1}^{\alpha} p_\alpha(\beta)\right] f_n(\alpha) = 0, \tag{3.6.5}$$

where use has been made of the identity $\sum_{\beta=1}^{\alpha} p_\alpha(\beta) = 0$. As shown in the proof of eq. (3.5.11), this result holds for every integer $\alpha > 1$, including the limit $\alpha \to \infty$. Observing these cancellations for every order in the HRR-expansions, one is left at most with $\alpha = 1$, if it is present in the original expansion in eq. (3.5.9). This value is special in some sense, since there are no subsectors associated to it and therefore the mechanisms outlined above cannot work. However, here comes to rescue the presence of the other sectors. Indeed, cancellations among terms with $\alpha = 1$ have to occur among different sectors, analogous to the original formulation of misaligned supersymmetry reviewed in subsection 3.3. Therefore, due to the cancellations of all sectors against each other for $\alpha = 1$ and the cancellations among subsectors $\beta = 1, \ldots, \alpha$ for $\alpha > 1$, a net cancellation of the envelope functions follows and hence the vanishing of the effective inverse Hagedorn temperature, i.e. $C_{\text{eff}} = 0$.

As an example, one can consider for instance the function $\tilde{M}(\tau)$, defined in eq. (3.5.32), that is associated to the description of an anti-Dp-brane on top of an Op-plane. The sectors are the bosonic and fermionic sectors, which at leading order $\alpha = 1$ are interpolated by the same functions, up to a sign. Therefore, the leading-order envelope functions cancel out against each other, as observed in a different formalism in subsection 3.4.3. Now, as discussed in subsection 3.5.3, this function is such that the even values $\alpha = 2\omega + 2$ do not take part in the HRR-expansion, the only contributions to the Laurent coefficient being from odd values $\alpha = 2\omega + 1$. For the first correction beyond leading order, i.e. $\alpha = 3$, one has the complex-valued function $\tilde{P}_3(n) = \mathrm{e}^{-\frac{2\pi \mathrm{i}}{3}(n-2)} + \mathrm{e}^{-\frac{4\pi \mathrm{i}}{3}(n+1)}$. Restricting to integer values of n, one finds the three possible real values $\tilde{P}_3(1) = -1$, $\tilde{P}_3(2) = +2$ and $\tilde{P}_3(3) = -1$, and these add up to zero, as expected. All values of α behave in a similar way. One can therefore follow the argument above to claim that $C_{\text{eff}} = 0$ in this case. In subsection 3.6.2, the cancellations in the anti-Dp-brane/Op-plane system are discussed in full detail.

3.6.1.2 Remarks

There are a few subtle points in the previous reasoning that have been omitted for convenience of presentation and that are going to be addressed below.

To start, an important remark is due on the definition of the envelope functions, which is going to impact the discussion of cancellations in subsections 3.6.2 and 3.6.3. It is necessary to specify that, technically, instead of eq. (3.6.4), the proper enhancement of the state degeneracies $a_n^\sigma(\alpha)$ to define envelope functions should be of the kind

$$a_n^\sigma(\alpha) \mapsto \Phi_\alpha^\sigma(n; \beta^\sigma), \tag{3.6.6}$$

with the superscript in β^σ indicating the fact that, for the values of n appearing in a given sector σ, as determined by the physical degeneracies a_n^σ belonging to that sector, not necessarily all the terms $p_\alpha(\beta)$ appear, but only a subset, with elements denoted as $p_\alpha(\beta^\sigma)$. In fact, only the values $n \in \mathbb{N}^\sigma$ appear among the net degeneracies a_n^σ in a given sector σ, for some subset $\mathbb{N}^\sigma \subset \mathbb{N}$, and not all terms $p_\alpha(\beta)$ appear in that subsector if the subsets $\mathbb{N}^\sigma$ and $\mathbb{N}_\alpha(\beta)$ do not intersect, i.e. if $\mathbb{N}^\sigma \cap \mathbb{N}_\alpha(\beta) = \emptyset$.[3.4] The envelope functions Φ_α^σ are also labelled following this reasoning. This is a technical subtlety that has been be ignored in the general introduction for the sake of clarity, as the replacement of eq. (3.6.4) has been enough to explain the fundamental ideas. However a definition as in eq. (3.6.6) may be necessary if the coefficients in each sector are computed separately, as in subsection 3.6.2.

Second, in sections 3.3 and 3.4, a defining feature of misaligned supersymmetry has been identified in the presence of a boson-fermion oscillation at leading order in the HRR-expansion. Therefore, in general it is going to be assumed that the partition functions that are being dealt with have this property.

For simplicity, let the partition function be given by a single term which is also a Dedekind η-quotient. In this case, the Laurent series must have oscillating-sign coefficients. In particular, this case also corresponds to open-string theories and the anti-Dp-brane/Op-plane theory that is being used as an example is no exception. If one is considering a closed-string theory, instead, then typically one has to deal with a product of right- and left-movers, which individually are Dedekind η-quotients. Some slight differences may occur, for instance just either the right- or left-moving sector contributions may oscillate, with the other one having coefficients of a given sign. The heterotic SO(16) $\times$ SO(16)-theory is indeed of this latter type. Moreover, if the Dedekind η-quotient describes an open string, it is also assumed to be tachyon-free, which instead is not necessary for the closed-string case scenario, but this does not play a direct role in the following discussion. All these

[3.4] A prototypical instance of this kind is as follows. Let the sector a_n^σ be defined only for even values $n \in 2\mathbb{N}$ (as is a typical case if the sectors are bosons and fermions): for a given even order $\alpha = 2\omega + 2$ appearing in eq. (3.5.9), then a subset of the possible terms $p_\alpha(n)$ would be sufficient to interpolate all the values $a_n^\sigma(2\omega + 2)$, namely those corresponding to an even argument $p_\alpha(2), p_\alpha(4), \ldots, p_\alpha(2\omega + 2)$.

complications are not fundamental for the following arguments and anyway they will be taken into account in the specific examples later on. If several Dedekind η-quotients are present, one can just repeat the analysis for each of them separately. Shortcomings related to fractional indices are not going to be considered since they merely represent a need for a suitable index labelling, when computing the degeneracies. As a final remark, it should also be noted that the terminology so far has implicitly assumed the discussion of single Dedekind η-quotients to refer to open strings and of products thereof to closed strings. The annulus and the Möbius-strip amplitudes, which involve single Dedekind η-quotients, are intrinsically open-string terms, but Klein-bottle terms technically refer to closed strings. Of course, their analysis is analogous.

Now, let the function $Z = Z(\tau)$ be a single η-quotient $Z(\tau) = q^{-n_0} \sum_{n=0}^{\infty} a_n q^n$ with oscillating-sign coefficients $a_n \lesseqgtr 0$. It is convenient to distinguish the following situations.

1. The conditions of applicability of eq. (3.5.9), i.e. a positive constant $c_1 > 0$ and a non-negative function $G(\alpha) \geq 0$, are met either by the function $Z(\tau)$ or by the function $\tilde{Z}(\tau) = Z(\tau + 1/2)$, up to overall factors. Then, two subcases must be considered.
   (a) If the conditions of applicability of eq. (3.5.9) are met by $\tilde{Z}(\tau) = Z(\tau + 1/2)$, which corresponds to a Laurent series with definite-sign coefficients, as the original $Z(\tau)$ has oscillating coefficients by assumption, one can work out the alternating coefficients a_n of the original series from the definite-sign coefficients $\tilde{a}_n$ of the new series by just noticing that they are trivially related as $a_n = (-1)^n \tilde{a}_n$. One will then simply have to keep track of which states have positive/negative coefficients before the shift of τ is performed. A concrete example is the partition function $M(\tau)$ of an anti-Dp-brane on top of an Op-plane, as in subsection 3.5.3.
   (b) If the conditions of applicability of eq. (3.5.9) are met by $Z(\tau)$, which has oscillating coefficients a_n, then the latter are given immediately by the Sussman HRR-expansion. A concrete example are the functions $L_1(\tau)$ and $R'_3(\tau)$ in subsection 3.5.2.

   Both the functions $Z(\tau)$ and $\tilde{Z}(\tau)$ may be amenable to the expansion of eq. (3.5.9). Of course, in such a situation one can choose which subcase to adopt for the discussion.
2. The conditions of applicability of eq. (3.5.9) are met neither by the function $Z(\tau)$ nor by the function $\tilde{Z}(\tau) = Z(\tau + 1/2)$.

In the following, case 2 is not going to be considered. In fact, none of the examples taken into account presents an incompatibility with the Sussman HRR-expansion. On the other hand, for a partition function $Z(\tau)$, case 1b[3.5] happens to be trivially described as in the general treatment above, since all the degeneracies a_n can be extended to envelope functions

[3.5] Notice that, in case 1b, the leading-order sector should be expected to be for $\alpha > 1$, since, with $\alpha = 1$, the function $P_1(n) = 1$ cannot give sign oscillations in the series coefficients. In such a circumstance, if the term $\alpha = 1$ is absent from the series, the reasoning on the cancellation of the envelope functions at all orders proceeds straightforwardly. However this should always be checked for the specific function under consideration: in case the term $\alpha = 1$ appeared, one should reconsider its whole cancellation mechanism.

that cancel out against each other, without the need to make any distinction among 'sectors' to compute the degeneracies. Instead, case 1a is intrinsically more subtle: in fact, in this case, in order to compute the degeneracies, one has to deal explicitly with two different sectors, typically the bosonic and the fermionic states, due to the distinction made when using the function $\tilde{Z}(\tau)$ instead of the function $Z(\tau)$. This is going to be the central focus of subsection 3.6.2. Such a scenario also corresponds to the open-string system of interest with anti-Dp-branes on top of Op-planes.

As anticipated, the situation is more involved for closed strings, where a product of two factors has to be considered and the array of possibilities is much larger. Here, the partition functions may involve a combination of factors falling both in cases 1a and 1b, but also of factors with all their coefficients being of a definite sign. For the latter, the Laurent coefficients may still be worked out by means of the Sussman HRR-expansion. So, the methods to discuss the presence of misaligned supersymmetry for closed strings are intrinsically different and more complicated compared to those for open strings. Extensions of the open-string reasoning to the closed-string scenario are going to be discussed in subsection 3.6.3, focussing again on the heterotic SO(16)×SO(16)-model as an example.

3.6.2 All-Order Cancellations for Open Strings

This subsection discusses in detail open-string partition functions falling into the scenario presented in case 1a and then specialises it to the case of an anti-Dp-brane sitting on top of an Op-plane. Instead, case 1b is not treated explicitly since it is immediate.

3.6.2.1 General Discussion

In the function $\tilde{Z}(\tau) = Z(\tau + 1/2)$, the shift $\tau + 1/2$ flips the signs of the state degeneracies and leads to coefficients $\tilde{a}_n$ that are all of the same sign, e.g. positive for definiteness. Therefore, one cannot distinguish anymore which states are bosons and which are fermions, in $\tilde{Z} = \tilde{Z}(\tau)$. In order to discuss cancellations among bosons and fermions for the actual model $Z = Z(\tau)$, therefore, one must treat separately the values of n that correspond to original bosonic degeneracies, namely $a_n = \tilde{a}_n$, and those that correspond to original fermionic degeneracies, namely $a_n = -\tilde{a}_n$. Using a tilde-notation to stress that all the quantities refer to the function $\tilde{Z} = \tilde{Z}(\tau)$, the degeneracies can be written as

$$a_n = \tilde{a}_n = \sum_{\alpha \in \tilde{\Gamma}} \tilde{a}_n(\alpha) = \sum_{\alpha \in \tilde{\Gamma}} \tilde{P}_\alpha(n) \tilde{f}_n(\alpha), \qquad \forall n \in \mathbb{N}_b, \tag{3.6.7a}$$

$$-a_n = \tilde{a}_n = \sum_{\alpha \in \tilde{\Gamma}} \tilde{a}_n(\alpha) = \sum_{\alpha \in \tilde{\Gamma}} \tilde{P}_\alpha(n) \tilde{f}_n(\alpha), \qquad \forall n \in \mathbb{N}_f, \tag{3.6.7b}$$

where $\mathbb{N}_b \subset \mathbb{N}$ and $\mathbb{N}_f \subset \mathbb{N}$ represent the subsets of values of n with a bosonic degeneracy $a_n = \tilde{a}_n$ and a fermionic degeneracy $a_n = -\tilde{a}_n$, respectively. Although this formally looks the same as the generic case above, the fact that n only takes values in a subset of $\mathbb{N}$ is crucial. Indeed, since the periodicity of the function $\tilde{P}_\alpha(n)$ is such that $\tilde{P}_\alpha(n) = \tilde{P}_\alpha(n + \alpha)$, one is no longer guaranteed that the values of n at disposal in $\mathbb{N}_b$ are sufficient for $\tilde{P}_\alpha(n)$ to take all the distinct α values $\tilde{p}_\alpha(\beta)$ that it would assume if its domain was $\mathbb{N}$, and similarly for $\mathbb{N}_f$. In particular, one may write

$$a_n(\alpha) = \tilde{a}_n(\alpha) = \tilde{p}_\alpha(\beta)\tilde{f}_n(\alpha), \qquad \forall n \in \mathbb{N}_\alpha(\beta) \cap \mathbb{N}_b; \tag{3.6.8a}$$

$$-a_n(\alpha) = \tilde{a}_n(\alpha) = \tilde{p}_\alpha(\beta)\tilde{f}_n(\alpha), \qquad \forall n \in \mathbb{N}_\alpha(\beta) \cap \mathbb{N}_f. \tag{3.6.8b}$$

It is therefore manifest that, e.g. in the bosonic sector, the value $\tilde{P}_\alpha(n) = \tilde{p}_\alpha(\beta)$ is only found if $\mathbb{N}_\alpha(\beta) \cap \mathbb{N}_b \neq \emptyset$. Of course, the missing values of $\tilde{p}_\alpha(\beta)$ would be found in the fermionic sector, and vice versa, but this means that they would contribute with an extra (-1)-factor, invalidating the cancellation as presented in eq. (3.5.11). For definiteness, let the bosonic sector be such that $\mathbb{N}_b = 2\mathbb{N}_0 + 1$. In this case, the contributions that appear in the corrections to the bosonic degeneracies in eq. (3.6.7a) are

$$\ldots,\ \tilde{a}_{2n-1}(\alpha),\ \overbrace{\tilde{a}_{2n+1}(\alpha)}^{\tilde{P}_\alpha(2n+1)},\ \overbrace{\tilde{a}_{2n+3}(\alpha)}^{\tilde{P}_\alpha(2n+3)},\ \ldots,\ \overbrace{\tilde{a}_{2n+2\alpha-1}(\alpha)}^{\tilde{P}_\alpha(2n+2\alpha-1)},\ \underbrace{\tilde{a}_{2n+2\alpha+1}(\alpha)}_{\tilde{P}_\alpha(2n+1)},\ \ldots$$

and it is possible to observe how the periodicity mod 2α in the functions $\tilde{P}_\alpha(n)$ allows one to recognise the sectors $\tilde{P}_\alpha(2n+1+2l)$, with $l = 0, \ldots, \alpha - 1$. Now one needs to understand whether the values $\tilde{P}_\alpha(2n+1)$, $\tilde{P}_\alpha(2n+3)$, $\ldots$, $\tilde{P}_\alpha(2n+2\alpha-1)$ suffice to individuate all the α terms $\tilde{p}_\alpha(\beta)$ that add up to zero. The answer is affirmative if α is odd, as one can see by direct inspection. The situation is similar for the fermionic terms in eq. (3.6.7b). A more explicit treatment is below. See also figs. 3.5 and 3.6 for an explicit example.

- In the bosonic sector, one has the sequence

  $$\underbrace{\tilde{a}_1(\alpha)}_{\tilde{p}_\alpha(1)},\ \tilde{a}_3(\alpha),\ \tilde{a}_5(\alpha),\ \ldots,\ \tilde{a}_{2\alpha-5}(\alpha),\ \underbrace{\tilde{a}_{2\alpha-3}(\alpha)}_{\tilde{p}_\alpha(\alpha-3)},\ \underbrace{\tilde{a}_{2\alpha-1}(\alpha)}_{\tilde{p}_\alpha(\alpha-1)},\ \overbrace{\tilde{a}_{2\alpha+1}(\alpha)}^{\tilde{p}_\alpha(1)},\ \ldots$$

  and, therefore, for odd α each sequence of α consecutive terms contains all the α different terms $\tilde{p}_\alpha(\beta)$, which ultimately happens because the difference of two odd numbers is even, whereas for even α half of the terms $\tilde{p}_\alpha(\beta)$ are never hit by the degeneracies because $\alpha - (2l+1)$, for any l, is never even if α is even.

- In the fermionic sector, one has the sequence

  $$\underbrace{\tilde{a}_0(\alpha)}_{\tilde{p}_\alpha(0)},\ \tilde{a}_2(\alpha),\ \tilde{a}_4(\alpha),\ \ldots,\ \tilde{a}_{2\alpha-6}(\alpha),\ \underbrace{\tilde{a}_{2\alpha-4}(\alpha)}_{\tilde{p}_\alpha(\alpha-4)},\ \underbrace{\tilde{a}_{2\alpha-2}(\alpha)}_{\tilde{p}_\alpha(\alpha-2)},\ \overbrace{\tilde{a}_{2\alpha}(\alpha)}^{\tilde{p}_\alpha(0)},\ \ldots$$

  and, therefore, for odd α every sequence of α consecutive terms contains all the α different terms $\tilde{p}_\alpha(\beta)$, whereas for even α half of the terms $\tilde{p}_\alpha(\beta)$ are never hit by the degeneracies.

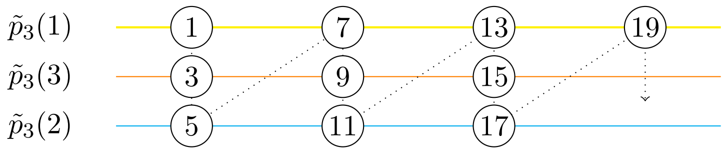

Figure 3.5: Periodicity of the function $\tilde{P}_\alpha(n)$ for $\alpha = 3$, with odd argument $n \in 2\mathbb{N}_0 + 1$. Each circle contains increasing odd integers $n = 2l+1$, while the horizontal lines represent the associated term $\tilde{P}_3(n)$, expressed as $\tilde{p}_3(\beta)$ for the appropriate β. The periodicity $\tilde{P}_3(n) = \tilde{P}_3(n \bmod 3)$ permits to group all odd numbers n in $\alpha = 3$ different groups. All different values of $\tilde{p}_\alpha(\beta)$ can be populated by $\tilde{P}_\alpha(n)$ for odd values of α. Even arguments $n \in 2\mathbb{N}_0$ behave in the same way for odd values of α.

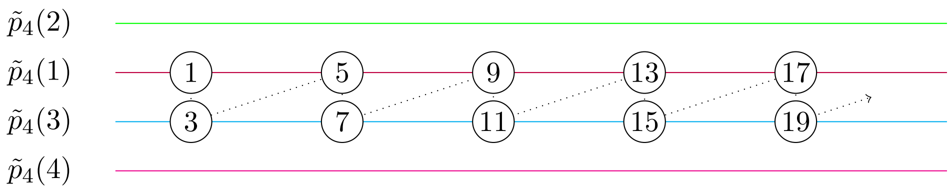

Figure 3.6: Periodicity of the function $\tilde{P}_\alpha(n)$ for $\alpha = 4$, with odd argument $n \in 2\mathbb{N}_0+1$. Each circle contains increasing odd integers $n = 2l+1$, while the horizontal lines represent the corresponding value $\tilde{P}_4(n)$, expressed as $\tilde{p}_4(\beta)$ for the appropriate β. The periodicity $\tilde{P}_4(n) = \tilde{P}_4(n \bmod 4)$ necessarily leaves out half of the possible values $\tilde{p}_\alpha(\beta)$ in the first column. Even arguments $n \in 2\mathbb{N}_0$ behave in the same way for odd values of α.

In all cases in which the subsectors of a given sector are enough to individuate all the Kloosterman-like terms $\tilde{p}_\alpha(\beta)$ at an order α, one can define envelope functions $\pm\Phi_\alpha(n;\beta)$ via the replacement in eq. (3.6.6), based on eqs. (3.6.8a, 3.6.8b), and therefore conclude that their average is zero, according to the property in eq. (3.5.11), for which $\sum_{\beta=1}^{\alpha} \tilde{p}_\alpha(\beta) = 0$ (similarly as with eqs. (3.6.4, 3.6.5)). Therefore, the outcome of this analysis is that models exhibiting an oscillatory degeneracy pattern, i.e. misaligned supersymmetry, experience a net cancellation, taking place among the pure bosonic and pure fermionic sectors individually, at all odd subleading orders $\alpha = 2\omega + 1$. If these odd orders are the only ones appearing in the HRR-expansion of eq. (3.5.9), and if the leading-order terms are also such as to cancel out their envelope functions, then one can conclude that the effective inverse Hagedorn temperature of the theory is zero, i.e.

$$C_{\text{eff}} = 0. \tag{3.6.9}$$

On the other hand, at all even subleading orders $\alpha = 2\omega+2$, the pure bosonic and fermionic sectors generally do still combine together among themselves into nonzero values. In short, this suggests that the even orders $\alpha = 2\omega+2$ may be problematic within the case 1a under consideration and it is not possible to draw any general conclusion for them at this stage. However, this situation does not appear in the open-string case example of interest, the anti-Dp-brane/Op-plane theory. Notice that, instead, these cases are easily tractable for the closed-string cases of interest below.

3.6.2.2 All-Order Cancellations for Anti-D-Branes on O-Planes

As anticipated, for the partition function $M = M(\tau)$ of an anti-Dp-brane sitting on top of an Op-plane, given in eq. (3.5.31), the shifted function $\tilde{M}(\tau)$ in eq. (3.5.32) that one needs to consider is such that all the even orders in the HRR-expansions are vanishing, the only contributions being from $\alpha = 2\omega + 1$. This is discussed in subsection 3.5.3. Therefore, one does not have to deal with additional complications and the machinery presented above can be applied directly. This shows that the interpolating functions cancel at all subleading orders, with bosonic and fermionic corrections averaging out to zero independently from each other. On the other hand, the leading-order contributions cancel out straightforwardly, as discussed in subsection 3.4.3. Ultimately, this reasoning proves that the effective inverse Hagedorn temperature for an anti-Dp-brane sitting on top of an Op-plane is

$$C_{\text{eff}}^{\overline{\text{D}p}/\text{O}p} = 0. \tag{3.6.10}$$

This is a formal derivation of the result that has been anticipated in subsection 3.4.3. This method, nevertheless, allows for a detailed analysis even in cases in which the effective cancellations may not be so apparent. Below, fig. 3.7 reports a schematic representation of the cancellations taking place at leading and next-to-leading order in the anti-Dp-brane/Op-plane theory.

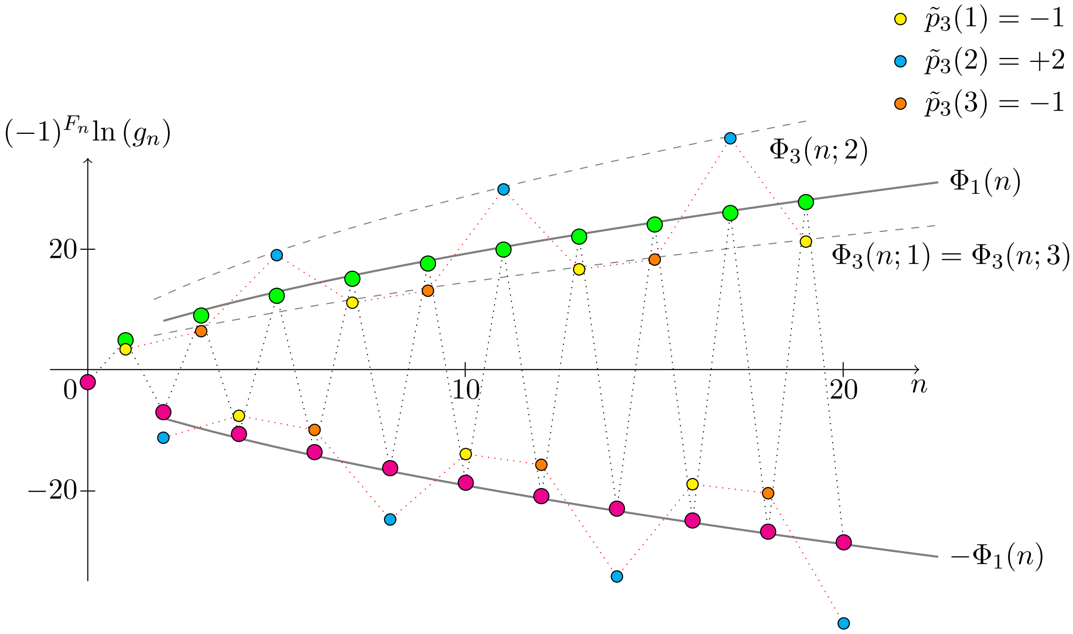

Figure 3.7: A schematic plot representing the spectrum of an anti-Dp-brane on top of an Op-plane, including the terms at leading order, for $\alpha = 1$, and the (magnified) corrections at next-to-leading order, for $\alpha = 3$. One has to consider bosons (odd n) and fermions (even n) separately as the corrections to the coefficients of the partition function $M(\tau)$ are computed with the dual function $\tilde{M}(\tau)$. Then, levels $n = 1 \bmod 3$, $n = 2 \bmod 3$ and $n = 3 \bmod 3$ have corrections multiplied by the values $\tilde{p}_3(1) = -1$, $\tilde{p}_3(2) = +2$ and $\tilde{p}_3(3) = -1$, respectively. For each different value the function $\tilde{p}_\alpha(n)$ can take, one can individuate a different interpolating function, both for bosons and for fermions. Evidently, the average of such interpolating functions vanishes, independently from each other, both in the bosonic and in the fermionic sector.

3.6.3 All-Order Cancellations for Closed Strings

In dealing with closed strings, one typically has to consider products of two functions, one for the right- and one for the left-moving sector. Usually, there are several such products that need to be added to make up the full partition function of the theory. However, the tools of section 3.5 allow for a separate treatment of each of these products, so in an actual partition function one just needs to combine the results obtained from each of these terms.

3.6.3.1 General Discussion

The generic function that is considered in this subsubsection is a product of the form $Z(\tau,\overline{\tau}) = R(\tau)\overline{L}(\overline{\tau})$, where $R = R(\tau)$ and $L = L(\tau)$ are the functions of $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$ coming from the string oscillations in the right- and left-moving sectors, respectively. In general, the level-matched net degeneracies have an expression of the kind

$$a_{nn} = a^R_{n+n_0^R}\overline{a}^L_{n+n_0^L} = \left[\sum_\alpha a^R_{n+n_0^R}(\alpha)\right]\left[\sum_\beta \overline{a}^L_{n+n_0^L}(\beta)\right],$$

where $a^R_{n+n_0^R}$ and $a^L_{n+n_0^L}$ are the Laurent coefficients of the series $R(\tau)$ and $L(\tau)$, respectively, and the summations over α and β represent their HRR-expansions. In this case, addressing an all-encompassing analysis is quite arduous, so the discussion is going to be centred on the instance where both $R(\tau)$ and $L(\tau)$ are immediately amenable to eq. (3.5.9), which is enough for example to discuss the heterotic $\mathrm{SO}(16)\times\mathrm{SO}(16)$-theory, without extra complications such as the shift $\tau + 1/2$. This should suffice to provide all the basic elements that one may need to take care of in the investigation of a given theory. In detail, the full state degeneracies can be expressed as

$$a_{nn} = \sum_{\alpha\in\Gamma_R}\sum_{\beta\in\Gamma_L} a_{nn}(\alpha,\beta),$$

with the indices 'R' and 'L' referring of course to the functions $R(\tau)$ and $L(\tau)$, respectively, and their related quantities, where, emphasising the Kloosterman-like terms, the definition has been made

$$a_{nn}(\alpha,\beta) = P^R_\alpha(n+n_0^R)\overline{P}^L_\beta(n+n_0^L)f^R_{n+n_0^R}(\alpha)\overline{f}^L_{n+n_0^L}(\beta). \tag{3.6.11}$$

Then, for any given couple of values $\alpha\in\Gamma_R$ and $\beta\in\Gamma_L$, the series of terms $a_{nn}(\alpha,\beta)$ can be associated to some envelope functions $\Phi_{\ell_{\alpha\beta}}(n;\alpha,\beta)$, for an index $\ell_{\alpha\beta} = 1,\ldots,\mathrm{lcm}(\alpha,\beta)$, with the least common multiple being $\mathrm{lcm}(\alpha,\beta) = \alpha\beta/\gcd(\alpha,\beta)$. Indeed, the series of contributions $a_{nn}(\alpha,\beta)$ allows one to define a number $\mathrm{lcm}(\alpha,\beta)$ of continuous functions of $n\in\mathbb{R}^+$ as

$$\Phi_{\ell_{\alpha\beta}}(n;\alpha,\beta) = P^R_\alpha(\ell_{\alpha\beta}+n_0^R)\overline{P}^L_\beta(\ell_{\alpha\beta}+n_0^L)f^R_{n+n_0^R}(\alpha)\overline{f}^L_{n+n_0^L}(\beta). \tag{3.6.12}$$

For any given (α,β)-pair, observing the two different periodicities $P^R_\alpha(m) = P^R_\alpha(m \bmod \alpha)$ and $\overline{P}^L_\beta(m) = \overline{P}^L_\beta(m \bmod \beta)$, this defines an envelope function for each of the possible outcomes of the product $P^R_\alpha(n+n_0^R)\overline{P}^L_\beta(n+n_0^L)$. Such envelope functions are parametrised by the counter $\ell_{\alpha\beta}$, which is defined in such a way that, if $n = \ell_{\alpha\beta} \bmod \operatorname{lcm}(\alpha, \beta)$, then the condition holds $P^R_\alpha(n + n^R_0)\overline{P}^L_\beta(n + n^L_0) = P^R_\alpha(\ell_{\alpha\beta} + n^R_0)\overline{P}^L_\beta(\ell_{\alpha\beta} + n^L_0)$. In fact, at all the values of $n \in \mathbb{N}$ such that $n = \ell_{\alpha\beta} \bmod \operatorname{lcm}(\alpha, \beta)$, one has the identification $a_{nn} = \Phi_{\ell_{\alpha\beta}}(n; \alpha, \beta)$, with the envelope functions then being extended to any continuous value $n \in \mathbb{R}^+$.

In order to show that misaligned supersymmetry takes place at any order in the Sussman HRR-series, one needs to show that these envelope functions average out to zero, i.e. that $\sum_{\ell_{\alpha\beta}=1}^{\operatorname{lcm}(\alpha,\beta)} \Phi_{\ell_{\alpha\beta}}(n; \alpha, \beta) = 0$. To do that, it is sufficient to rearrange the sum over $\ell_{\alpha\beta}$ in terms of a simple double sum. In fact, defining the term $\xi_{\alpha\beta} = \beta/\gcd(\alpha, \beta)$, one can write

$$
\begin{aligned}
\sum_{\ell_{\alpha\beta}=1}^{\operatorname{lcm}(\alpha,\beta)} \Phi_{\ell_{\alpha\beta}}(n; \alpha, \beta) &= \sum_{k_\alpha=1}^{\alpha} \sum_{m=0}^{\xi_{\alpha\beta}-1} \Phi_{k_\alpha+m\alpha}(n; \alpha, \beta) \\
&= \sum_{k_\alpha=1}^{\alpha} \sum_{m=0}^{\xi_{\alpha\beta}-1} P^R_\alpha(k_\alpha + m\alpha + n^R_0)\overline{P}^L_\beta(k_\alpha + m\alpha + n^L_0) f^R_{n+n^R_0}(\alpha)\overline{f}^L_{n+n^L_0}(\beta) \\
&= \sum_{k_\alpha=1}^{\alpha} P^R_\alpha(k_\alpha + n^R_0) f^R_{n+n^R_0}(\alpha)\overline{f}^L_{n+n^L_0}(\beta) \left[\sum_{m=0}^{\xi_{\alpha\beta}-1} \overline{P}^L_\beta(k_\alpha + m\alpha + n^L_0)\right] \\
&= 0,
\end{aligned}
$$

where it has been taken advantage of the periodicity of the right-moving P-function and, in the last line, use has been made of eq. (3.5.11) for the left-moving one. In order to exploit this result, the condition $\xi_{\alpha\beta} = \beta/\gcd(\alpha, \beta) > 1$ must be fulfilled. This is the case if $\beta > \alpha$, and, reverting the roles, an analogous discussion holds if $\alpha > \beta$. However, the reasoning does not generally hold if $\alpha = \beta$, in which case showing the cancellation is not possible in these terms. Therefore, the sector-averaging mechanism cannot be argued to work so generally. Nevertheless, a systematic cancellation takes place for instance in all situations where the function R has only odd αs and the function $\overline{L}$ has only even βs, or vice versa.[3.6] If so, in the calculation above, notice that one could not draw the same conclusion by splitting the sum over $l_{\alpha\beta}$ as a sum over $k_\beta = 1, \ldots, \beta$ and $m = 0, \ldots, \alpha/\gcd(\alpha, \beta) - 1$ since, in the case where $\beta = 2r\alpha$, for $r \in \mathbb{N}$, one would have $\alpha/\gcd(\alpha, \beta) = 1$. To conclude, in such a scenario, the cancellation thus shown is a general proof of the vanishing of the average of the envelope function at any order $(\alpha, \beta) \in \Gamma_R \times \Gamma_L$, i.e.

$$
\sum_{\ell_{\alpha\beta}=1}^{\operatorname{lcm}(\alpha,\beta)} \Phi_{\ell_{\alpha\beta}}(n; \alpha, \beta) = 0. \tag{3.6.13}
$$

Therefore, in theories that exhibit misaligned supersymmetry and that are amenable to the discussion above, one can claim that the effective inverse Hagedorn temperature is zero, i.e.

$$
C_{\text{eff}} = 0. \tag{3.6.14}
$$

[3.6] This is a situation that should somehow be expected to be common. In fact, typically one should have one sector with definite signs and another one with oscillations, with the former expected to have odd αs and the latter expected to have even βs. Nevertheless, claiming this to be general would need a dedicated analysis.

The key to this result is just eq. (3.5.11), which is a general property of the Dedekind η-quotients for which eq. (3.5.9) holds. Here the discussion has been centred around a specific closed-string scenario, but the machinery that has been described is easy to adapt to a large variety of closed-string models. Cancellations beyond leading order depend on the specific details of the right- and left-moving factors that define their partition function.

3.6.3.2 All-Order Cancellations for the Heterotic SO(16)×SO(16)-Theory

As discussed in subsection 3.5.2, the partition function of the heterotic SO(16)×SO(16)-theory can be written as a sum of three terms that are separate products of right- and left-moving Dedekind η-quotients. In particular, given eq. (3.5.14), the simplest contribution to study is the product $Z_1(\tau,\overline{\tau}) = R_1(\tau)\overline{L}_1(\overline{\tau})/2$. Both functions $R_1(\tau)$ and $L_1(\tau)$, defined in eqs. (3.5.15a, 3.5.15b), are Dedekind η-quotients whose Laurent coefficients are given by eq. (3.5.9). Moreover, the right-moving term R_1 receives contributions only from odd αs, and vice versa the left-moving term L_1 is only determined by even βs. Therefore, the analysis above applies immediately and one concludes that the average of the corresponding envelope functions vanishes, i.e.

$$\sum_{\ell_{\alpha\beta}=1}^{\mathrm{lcm}(\alpha,\beta)} \Phi^1_{\ell_{\alpha\beta}}(n;\alpha,\beta) = 0$$

For the remaining two contributions, one should consider the functions R_2, L_2, R_3 and L_3, defined in eqs. (3.5.18a, 3.5.18b) and (3.5.23a, 3.5.23b). Actually, though, thanks to the list of dualities in eqs. (3.5.28a, 3.5.28b), it is apparent that the functions $Z_2 = R_2\overline{L}_2/2$ and $Z_3 = R_3\overline{L}_3/2$ contribute to physical states, which are what defines the envelope functions, in an identical way, just with a different index labelling and with different overall factors. Therefore, the total cancellation shown for Z_1 holds for Z_2 and Z_3, too, i.e.

$$\sum_{\ell_{\alpha\beta}=1}^{\mathrm{lcm}(\alpha,\beta)} \Phi^2_{\ell_{\alpha\beta}}(n;\alpha,\beta) = 0,$$
$$\sum_{\ell_{\alpha\beta}=1}^{\mathrm{lcm}(\alpha,\beta)} \Phi^3_{\ell_{\alpha\beta}}(n;\alpha,\beta) = 0.$$

To conclude, even for the heterotic SO(16)×SO(16)-theory it has been proved that

$$C_{\mathrm{eff}}^{\text{het-SO(16)}\times\text{SO(16)}} = 0. \tag{3.6.15}$$

3.6.3.3 Alternative Method for Open Strings

The open-string case discussed in subsection 3.6.2 can also be discussed with the tools presented in subsubsection 3.6.3.1. Given the partition function $M = M(\tau)$ of an anti-Dp-brane on top of an Op-plane, one can imagine a closed-string theory where the right-moving sector is $R(\tau) = -\tilde{M}(\tau) = -M(\tau+1/2)$ and the left-moving sector has coefficients

$l_n = (-1)^{n+1}$ for $\beta = 2$ and vanishing for all other βs. In this case, the right-moving sector is defined only for odd values of α, so the number of envelope functions is $\mathrm{lcm}(\alpha, 2) = 2\alpha$. For $\ell_\alpha = 1, \ldots, 2\alpha$, the envelope functions can be defined as

$$\Phi_{\ell_\alpha}(n;\alpha) = (-1)^{\ell_\alpha+1}\tilde{P}_\alpha(\ell_\alpha + n_0)\tilde{f}_{n+n_0}(\alpha).$$

Therefore, one simply has

$$\sum_{\ell_\alpha=1}^{2\alpha} \Phi_{\ell_\alpha}(n;\alpha) = \sum_{k=1}^{2}\sum_{m=0}^{\alpha-1} \Phi_{k+2m}(n;\alpha) = \sum_{k=1}^{2}(-1)^{k+1}\tilde{f}_{n+n_0}(\alpha)\sum_{m=0}^{\alpha-1}\tilde{P}_\alpha(k+2m+n_0) = 0.$$

3.7 Formal Interpretation of Misaligned Supersymmetry

In the literature, the vanishing of the sector-averaged envelope functions in misaligned spectra is interpreted as the manifestation of the structure for which one-loop observables can be finite, despite the infinitely-growing mismatch in bosonic and fermionic degrees of freedom. However, the details of how the necessary cancellations take place have never been elucidated. A description of this is the main purpose of this section.

In subsection 3.7.1, the relationship between the one-loop cosmological constant and the string-theory partition function is reviewed. Then, subsection 3.7.2 describes and comments on the mathematical details of how the presence of misaligned supersymmetry is tied to a finite one-loop cosmological constant for open strings. Finally, subsection 3.7.3 discusses this for closed strings. It is going to be apparent that the mathematics of the physical cancellations of exponential divergences is analogous to the envelope-function averaging.

3.7.1 String-Theory One-loop Cosmological Constant

As the cosmological constant is the observable on which the focus is going to be put, this subsection reviews its definition in perturbative String Theory at one-loop level and points out the most relevant aspects for the future analysis.

For a given string-theory construction, one can consider the D-dimensional quantum field theory consisting of the associated tower of perturbative string states. In general, these states can be labelled by a discrete index $n \in \mathrm{S}(\mathbb{N}_0)$ indicating the state mass level M_n^2, for some index set $\mathrm{S}(\mathbb{N}_0)$. At each of these mass levels, the net physical degeneracy, i.e. the difference of the number of bosonic and fermionic states $N_b(n)$ and $N_f(n)$, respectively, is denoted as $(-1)^{F_n} g_n = N_b(n) - N_f(n)$, with $g_n \geq 0$ counting its absolute value and F_n representing its fermionic parity. Given an arbitrary mass scale μ^2, in terms of a Schwinger proper-time parameter t, the one-loop cosmological constant then reads [28, 88]

$$\Lambda = -\frac{1}{2}\left(\frac{\mu^2}{8\pi^2}\right)^{D/2} \sum_{n\in \mathrm{S}(\mathbb{N}_0)} (-1)^{F_n} g_n \int_0^\infty \frac{\mathrm{d}t}{t^{1+D/2}}\, \mathrm{e}^{-2\pi M_n^2 t/\mu^2}. \tag{3.7.1}$$

In this expression, the region where $t \sim \infty$ leads to divergences only in the presence of tachyons, whereas the region where $t \sim 0^+$ is instead generally singular, unless cancellations occur due to the structure of the net physical degeneracies or it is cut off via a physical principle. For ease of presentation, this expression is going to be discussed for open and closed strings below, separately, in subsubsections 3.7.1.1 and 3.7.1.2. In what follows, open strings are always going to represent Dp- and anti-Dp-brane states.

3.7.1.1 One-loop Cosmological Constant for Open Strings

For tachyon-free open strings, the mass spectrum in both the NS- and R-sectors follows the pattern $M_n^2 = n/\alpha'$ for each mass level $n \in \mathbb{N}_0$, so it is convenient to set $\mu = 1/\sqrt{\alpha'}$. Moreover, for the field theory of a Dp- or anti-Dp-brane, one must consider a spacetime of dimension $D = p+1$. Therefore, eq. (3.7.1) can be rearranged as

$$\Lambda_{\mathrm{D}p} = -\frac{1}{2\pi}\, T_{\mathrm{D}p} \int_0^\infty \frac{\mathrm{d}t}{2t}\, M_{\mathrm{D}p}(t), \tag{3.7.2}$$

where the tension of the Dp-brane is $T_{\mathrm{D}p} = 2\pi/l_s^{p+1}$, with the string length being $l_s = 2\pi\sqrt{\alpha'}$, and where the partition function has been singled out

$$M_{\mathrm{D}p}(t) = \frac{1}{(2t)^{\frac{1}{2}(p+1)}} \sum_{n\in\mathbb{N}_0} (-1)^{F_n} g_n\, \mathrm{e}^{-2\pi t n}. \tag{3.7.3}$$

In eq. (3.7.2), the cosmological constant is UV-divergent unless cancellations occur such that the partition function in eq. (3.7.3) approaches the origin $t = 0$ at least as a power t^ϵ, with $\epsilon > 0$. This is the case for supersymmetric theories, where the partition function is identically zero, by definition, due to the level-by-level exact matching in the number of fermions and bosons, i.e. $g_n \equiv 0$ for all $n \in \mathbb{N}_0$. In theories lacking level-by-level supersymmetry, with $g_n \geq 0$, this condition is far from being guaranteed. In subsection 3.7.2, nevertheless, this is going to be shown the case for theories with misaligned supersymmetry. Of course, IR-divergences are absent in the lack of tachyons.

A comment on the counting of states is in order. According to eqs. (2.1.40, 2.1.52), the open-string mass operator and the Virasoro 0-generator are related as $\tilde{L}_0 = \alpha'\tilde{p}^2/4 + \alpha' M^2$. Here, the mass term takes values $M^2 = n/\alpha'$, with $n \in \mathbb{N}_0$ denoting the number-operator eigenvalue up to the vacuum-energy shift, following the GSO-projection. Therefore, in accordance with eq. (2.1.50), ignoring spacetime momentum, open-string partition functions are of the form $Z \sim \operatorname{tr} q[\mathrm{i}\tau_2]^{\alpha' M^2/2} \sim \sum_{n\in\mathbb{N}_0} (-1)^{F_n} g_n\, q[\mathrm{i}\tau_2]^{n/2}$. Defining a t-parameter as $t = \tau_2/2$, one finds $Z \sim \sum_{n\in\mathbb{N}_0} (-1)^{F_n} g_n\, q[\mathrm{i}t]^n$. In fact, this is precisely the rescaling performed in section 3.4, eventually considering partition functions $M(\tau)$ with $\operatorname{Im}\tau = t$. This confirms that the coefficients of the term $q[\tau]^n$ in these expansions effectively count the net degeneracies at the n-level mass.

3.7.1.2 One-loop Cosmological Constant for Closed Strings

For tachyon-free closed strings, the mass spectrum typically follows the pattern $M_n^2 = 4n/\alpha'$ for each mass level $n \in \mathbb{N}_0/2$. This is provided by two identical contributions from the right- and left-moving sectors $m_n^2 = \overline{m}_n^2 = 2n/\alpha'$. Therefore, it is convenient to set $\mu = 2/\sqrt{\alpha'}$. Defining a complex variable $\tau = \tau_1 + \mathrm{i}\tau_2$, with $\tau_2 = t/2$, the right-handside in eq. (3.7.1) can accommodate a further integration $\int_{-1/2}^{1/2} \mathrm{d}\tau_1 = 1$. More generally, any term $a_{mn}\, \mathrm{e}^{2\pi \mathrm{i}\tau_1(m-n)} \mathrm{e}^{-2\pi\tau_2(m+n)}$ can be added, with $m \neq n$, leaving the result invariant, since the τ_1-integration trivially means $\int_{-1/2}^{1/2} \mathrm{d}\tau_1\, \mathrm{e}^{2\pi\mathrm{i}\tau_1 k} = \delta_{k0}$. Note that this always works since invariance under T-transformations requires $m - n \in \mathbb{Z}$ in string-theory constructions. In fact, for $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$, the cosmological constant can be expressed as

$$\Lambda_D = -\frac{1}{8\pi} \frac{1}{\kappa_D^2 l_s^2} \int_{\mathbb{S}} \frac{\mathrm{d}^2\tau}{\tau_2^2}\, Z(\tau, \overline{\tau}), \tag{3.7.4}$$

with the D-dimensional gravitational coupling constant being $2\kappa_D^2 = l_s^{D-2}/2\pi$, where the integrand is identified with the string-theory partition function, written as

$$Z(\tau, \overline{\tau}) = \tau_2^{1-D/2} \sum_{m \in \frac{1}{2}\mathbb{N}_0} \sum_{n \in \frac{1}{2}\mathbb{N}_0} a_{mn}\, q^m \overline{q}^n, \tag{3.7.5}$$

together with the $\mathrm{PSL}_2(\mathbb{Z})$-invariant measure $\mathrm{d}^2\tau/\tau_2^2$. This is integrated over the domain $\mathbb{S} = \{\tau \in \mathbb{C} :\ \mathrm{Re}\,\tau \in [-1/2, 1/2] \wedge \tau_2 \in [0, +\infty[\}$. Here, one should note the identification $a_{nn} \equiv (-1)^{F_n} g_n$. In general, the one-loop cosmological constant in eq. (3.7.4) is free of IR-divergences in the region $\tau_2 \sim \infty$ if the theory is free of physical tachyons, which has been assumed. On the other hand, it is UV-divergent in the region $\tau_2 \sim 0^+$. Thanks to modular invariance, this can nonetheless be fixed, restricting the domain of integration to non-redundant configurations.

In detail, because $Z = Z(\tau, \overline{\tau})$ represents the one-loop partition function of a closed-string theory, it is invariant under the modular group $\mathrm{PSL}_2(\mathbb{Z})$, and the divergence can be interpreted as a gauge divergence. In fact, a manifestly finite result can be obtained by factorising out the redundant volume, restricting the integration to the fundamental domain

$$\mathbb{F} = \{\tau \in \mathbb{C} :\ \tau_1 \in [-1/2, 1/2] \wedge \tau_2 \in [0, +\infty[\wedge |\tau| \in [1, +\infty[\}.$$

Explicitly, therefore, the regularised version of the cosmological constant of eq. (3.7.4) reads

$$\tilde{\Lambda}_D = -\frac{1}{8\pi} \frac{1}{\kappa_D^2 l_s^2} \int_{\mathbb{F}} \frac{\mathrm{d}^2\tau}{\tau_2^2}\, Z(\tau, \overline{\tau}). \tag{3.7.6}$$

This is an integral definition. Because the singular region corresponding to $\tau_2 = 0$ has been removed, the UV-divergence is absent. In the absence of physical tachyons, this one-loop cosmological constant is finite.

One may also express the regularised one-loop cosmological constant in a different way, by means of the so-called Kutasov-Seiberg identity [93]. A heuristic argument to motivate it is helpful, before commenting it. One can account for the gauge divergence in the volume of integration by defining a regulated domain $\mathbb{S}_\sigma = \{\tau \in \mathbb{C} :\ \mathrm{Re}\,\tau \in [-1/2, 1/2] \wedge \mathrm{Im}\,\tau \in [\sigma^{-1}, +\infty[\}$, with $\sigma \gg 1$, and establishing the relationship

$$\frac{1}{\mathrm{vol}_{\mathrm{PSL}_2(\mathbb{Z})}\, \mathbb{F}} \int_{\mathbb{F}} \frac{\mathrm{d}^2\tau}{\tau_2^2}\, Z(\tau, \overline{\tau}) \overset{\sigma \sim \infty}{\simeq} \frac{1}{\mathrm{vol}_{\mathrm{PSL}_2(\mathbb{Z})}\, \mathbb{S}_\sigma} \int_{\mathbb{S}_\sigma} \frac{\mathrm{d}^2\tau}{\tau_2^2}\, Z(\tau, \overline{\tau}),$$

where the volumes of $\mathbb{S}_\sigma$ and $\mathbb{F}$ with respect to the modular-invariant measure are

$$\mathrm{vol}_{\mathrm{PSL}_2(\mathbb{Z})}\,\mathbb{S}_\sigma \equiv \int_{\mathbb{S}_\sigma} \frac{\mathrm{d}^2\tau}{\tau_2^2} = \int_{-1/2}^{1/2} \mathrm{d}\tau_1 \int_{\sigma^{-1}}^{\infty} \frac{\mathrm{d}\tau_2}{\tau_2^2} = \sigma,$$
$$\mathrm{vol}_{\mathrm{PSL}_2(\mathbb{Z})}\,\mathbb{F} \equiv \int_{\mathbb{F}} \frac{\mathrm{d}^2\tau}{\tau_2^2} = \int_{\sqrt{3}/2}^{\infty} \frac{\mathrm{d}\tau_2}{\tau_2^2} - 2\int_{\sqrt{3}/2}^{1} \frac{\mathrm{d}\tau_2}{\tau_2^2}\sqrt{1-\tau_2^2} = \frac{\pi}{3}.$$

In the $\mathbb{S}_\sigma$-integration, the partition function effectively receives contributions only from the physical states. In terms of the function

$$g(\tau_2) = \int_{-1/2}^{1/2} \mathrm{d}\tau_1\, Z(\tau_1, \tau_2), \tag{3.7.7}$$

one can write

$$\lim_{\sigma\to\infty}\left[\frac{1}{\mathrm{vol}_{\mathrm{PSL}_2(\mathbb{Z})}\,\mathbb{S}_\sigma}\int_{\mathbb{S}_\sigma}\frac{\mathrm{d}^2\tau}{\tau_2^2}\,Z(\tau,\overline{\tau})\right] = \lim_{\sigma\to\infty}\frac{1}{\sigma}\int_{\sigma^{-1}}^{\infty}\frac{\mathrm{d}\tau_2}{\tau_2^2}\,g(\tau_2) = \lim_{\sigma\to\infty} g(\sigma^{-1}),$$

assuming the integral to be dominated by the region around $\tau_2 \sim \sigma^{-1} \sim 0^+$ and ignoring the τ_2-dependence of $g(\tau_2)$. This represents a heuristic derivation of the Kutasov-Seiberg identity

$$\tilde{\Lambda}_D = -\frac{1}{24}\frac{1}{\kappa_D^2 l_s^2}\lim_{\sigma\to\infty} g(\sigma^{-1}). \tag{3.7.8}$$

This equivalence matches an integral with a limit definition. It is proven in physical terms in ref. [93], and it assumes a theory that is free of physical tachyons. In the mathematical literature, this identity can be shown via a generalisation of the Rankin-Selberg-Zagier technique that lies in unfolding the $\mathbb{F}$-domain integration into an $\mathbb{S}$-domain integration by taking advantage of the modular invariance of the partition function [89, 125–128], as recently reviewed by ref. [129]. Similarly to the case of open strings, it might be feared that the cosmological constant in eq. (3.7.8) could be divergent, when approaching the UV-region $\sigma \sim \infty$. However, such a divergence must be absent due to modular invariance. Therefore, the finiteness of eq. (3.7.8) can be interpreted as the result of some sort of boson-fermion cancellation. This is a manifestation of misaligned supersymmetry.

In particular, expanding $g(\tau_2)$ in terms of g_n, it is possible to infer the small-τ_2 behaviour

$$\sum_{n\in\frac{1}{2}\mathbb{N}_0} (-1)^{F_n} g_n\, \mathrm{e}^{-4\pi\tau_2 n} \overset{\tau_2\sim 0^+}{\sim} -24\,\kappa_D^2 l_s^2\,\tilde{\Lambda}_D\,\tau_2^{D/2-1}. \tag{3.7.9}$$

This expression is important for the definition of the regularised supertraces (as also mentioned in section 3.3). Some of these are finite as a consequence of the identity in eq. (3.7.9) and of the finiteness of the regularised cosmological constant [28]. Note that for open strings one can formally define supertraces, but they are not manifestly related to the cosmological constant in an obvious way. Along with further comments on closed-string supertraces, an interpretation is proposed in section 3.8.

Again, it is worthwhile to conclude with a comment on the states counting. According to eqs. (2.1.38, 2.1.52), the closed-string mass operators and the Virasoro 0-generators are related as $\tilde{L}_0 = \alpha'\tilde{p}^2/4 + \alpha' m^2/2$ and $\tilde{\overline{L}}_0 = \alpha'\tilde{p}^2/4 + \alpha'\overline{m}^2/2$, with $m^2 = \overline{m}^2 = 2n/\alpha'$, with

$n \in \mathbb{N}_0/2$ denoting the number-operator eigenvalue up to the vacuum-energy shift. Therefore, in accordance with eq. (2.1.45), ignoring spacetime momentum, closed-string partition functions are of the form $Z \sim \operatorname{tr} q[\tau]^{\alpha' m^2/2} \overline{q}[\overline{\tau}]^{\alpha' \overline{m}^2/2} \sim \sum_{k\in\mathbb{N}_0/2} \sum_{l\in\mathbb{N}_0/2} a_{kl}\, q[\tau]^k \overline{q}[\overline{\tau}]^l$. This confirms that the coefficients of the term $q^n\overline{q}^n$ in these expansions effectively count the net degeneracies at the mass level $\alpha' M^2 = 4n$, as in section 3.3.

3.7.2 Open-String Misaligned Supersymmetry

For simplicity, it is convenient to start by considering open strings since their partition functions just involve a single term. According to eq. (3.7.2), the key fact to make sure there are no UV-divergences is that the partition function

$$M_{\mathrm{D}p}(t) = \frac{1}{(2t)^{\frac{p+1}{2}}} M(\mathrm{i}t) \tag{3.7.10}$$

approaches the region $t \sim 0^+$ like t^ϵ, with $\epsilon > 0$, which guarantees a finite cosmological constant in the absence of tachyons. The main topic of this section is the motivation of the conditions under which the one-loop open-string comsological constant is finite. The focus is going to be on the p-independent term $M(\mathrm{i}t) = M(\tau = \mathrm{i}t)$, where the function $M = M(\tau)$ is typically expressed as a pure Dedekind η-quotient, with the power-law prefactor being the only difference between branes of different spacetime dimensions. It is going to be shown that the misaligned symmetry in the associated state degeneracies leads to cancellation of exponential divergences in the one-loop partition function. A remnant modular symmetry moreover ensures that all power-law divergences cancel, leading to a finite result.

3.7.2.1 Setup

For definiteness, the focus is going to be on the class of tachyon-free open-string theories in which the partition function $M = M(\tau)$ is not amenable to the special Sussman HRR-expansion of eq. (3.5.9), but the negative of the shifted-argument function $\tilde{M}(\tau) = M(\tau + 1/2)$ is. This reduces in fact to case 1a in the discussion of subsection 3.6.2, but case 1b could be discussed analogously. Note that generically there can also be an overall numerical positive prefactor that leads to trivial modifications of the equations below, but it is immediate to include this rescaling in the results.

For $q[\tau] = \mathrm{e}^{2\pi\mathrm{i}\tau}$, if the partition function $M(\tau)$ has the Laurent expansion

$$M(\tau) = \sum_{n\in\mathbb{N}_0} (-1)^{F_n} g_n\, q[\tau]^n, \tag{3.7.11}$$

then the negative of the shifted-argument function $\tilde{M}(\tau)$ reads

$$-\tilde{M}(\tau) = \sum_{n\in\mathbb{N}_0} (-1)^{n+1} (-1)^{F_n} g_n\, q[\tau]^n. \tag{3.7.12}$$

An instance of this scenario is that of an anti-Dp-brane on top of an Op-plane as discussed in great detail in subsections 3.5.3 and 3.6.2. Extensions to other more complicated scenarios are immediate. Employing the Sussman HRR-expansion of eq. (3.5.9) for the function $f(\tau) = -\tilde{M}(\tau)$, the coefficients are found to be

$$(-1)^{n+1}(-1)^{F_n} g_n = \sum_{\alpha\in\Gamma} \frac{2\pi\, c_2(\alpha)\,[c_3(\alpha)]^{\frac{c_1+1}{2}}}{[24n]^{\frac{c_1+1}{2}}} \,\frac{P_\alpha(n)}{\alpha}\, I_{c_1+1}\left[\left(\frac{2\pi^2}{3\alpha^2}\, c_3(\alpha)\, n\right)^{\frac{1}{2}}\right]. \qquad (3.7.13)$$

The terms in eq. (3.7.13) are only valid for $n > 0$, since the general expression of the coefficients does not cover the case corresponding to $n = 0$. Taking all this into account, one can now restrict the attention to the case $\tau = \mathrm{i}\tau_2$, and the function $M(\mathrm{i}\tau_2)$ can be expressed in the form

$$g(\tau_2) \equiv M(\mathrm{i}\tau_2) = (-1)^{F_0} g_0 + \sum_{\alpha\in\Gamma}\sum_{\beta=1}^{\alpha} P_\alpha(\beta) g_\alpha(\tau_2;\beta). \qquad (3.7.14)$$

In this expression, the terms $P_\alpha(\beta)$, with $\beta = 1,\ldots,\alpha$, are the α different values that the periodic function $P_\alpha(n)$ can assume. Moreover, the sets $\mathbb{N}_\alpha(\beta) = \{n \in \mathbb{N} : n = \beta \bmod \alpha\}$ denote the sets of integers which satisfy $P_\alpha(n) = P_\alpha(\beta)$ for all $n \in \mathbb{N}_\alpha(\beta)$. Furthermore, the functions $g_\alpha(\tau_2;\beta)$ are defined as

$$g_\alpha(\tau_2;\beta) = \sum_{n\in\mathbb{N}_\alpha(\beta)} (-1)^{n+1} \frac{2\pi c_2(\alpha)[c_3(\alpha)]^{\frac{c_1+1}{2}}}{\alpha[24\,n]^{\frac{c_1+1}{2}}} I_{c_1+1}\left[\left(\frac{2\pi^2}{3\alpha^2} c_3(\alpha)\, n\right)^{\frac{1}{2}}\right] \mathrm{e}^{-2\pi\tau_2 n}. \qquad (3.7.15)$$

Up to this point, for each $\alpha \in \Gamma$, the sum over $n \in \mathbb{N}$ has been reorganised into α sums over $n \in \mathbb{N}_\alpha(\beta)$, for $\beta = 1,\ldots,\alpha$. For each of these sums, the quantity $P_\alpha(\beta)$ factorizes out, due to its α-periodicity.

It is actually convenient to make a further distinction, namely to distinguish the contributions for which $(-1)^{n+1}$ is positive from those for which it is negative. As in subsection 3.6.2, it is assumed that only odd terms $\alpha = 2\omega + 1 \in 2\mathbb{N}_0 + 1$ can appear in the HRR-sum. Then, one can introduce the two sets $\mathbb{N}^{\pm}_\alpha(\beta) = \{n \in \mathbb{N}_\alpha(\beta) : (-1)^{n+1} = \pm 1\}$ and express the full function $g(\tau_2)$ as

$$g(\tau_2) = (-1)^{F_0} g_0 + \sum_{\alpha\in\Gamma}\sum_{\beta=1}^{\alpha} P_\alpha(\beta)\left[g^{+}_\alpha(\tau_2;\beta) - g^{-}_\alpha(\tau_2;\beta)\right], \qquad (3.7.16)$$

where the two definite-sign functions $g^{\pm}_\alpha(\tau_2;\beta)$ have been defined as

$$g^{\pm}_\alpha(\tau_2;\beta) = \sum_{n\in\mathbb{N}^{\pm}_\alpha(\beta)} \frac{2\pi c_2(\alpha)[c_3(\alpha)]^{\frac{c_1+1}{2}}}{\alpha[24\,n]^{\frac{c_1+1}{2}}} I_{c_1+1}\left[\left(\frac{2\pi^2}{3\alpha^2} c_3(\alpha)\, n\right)^{\frac{1}{2}}\right] \mathrm{e}^{-2\pi\tau_2 n}. \qquad (3.7.17)$$

Notice that for the functions $g^{\pm}_\alpha(\tau_2;\beta)$ the superscript sign does not relate to their effective contribution to $g(\tau_2)$ being positive or negative: this also depends on the sign of the overall term $P_\alpha(\beta)$ they are multiplied with. In the rest of this section, eqs. (3.7.16, 3.7.17) will constitute the fundamental tool to discuss misaligned supersymmetry.

3.7.2.2 Cancellation of Exponential Divergences

In order to discuss the behaviour of the function $g(\tau_2)$ in eq. (3.7.16), one can take advantage of the series expansion of the modified Bessel function of the first kind, reviewed in appendix A.2.1, i.e.

$$I_\delta(z) = \left(\frac{z}{2}\right)^\delta \sum_{k=0}^{\infty} \frac{\left(\frac{z^2}{4}\right)^k}{k!\,(\delta+k)!}, \tag{3.7.18}$$

where it is understood that δ is a positive integer. Thanks to this, setting $\delta = c_1 + 1$, the functions in eq. (3.7.17) can be expressed as

$$g^{\pm}_\alpha(\tau_2;\beta) = \frac{2\pi c_2(\alpha)}{\alpha^{1-\delta}} \sum_{n\in\mathbb{N}^{\pm}_\alpha(\beta)} \sum_{k=0}^{\infty} \left[\frac{\pi}{12}\frac{c_3(\alpha)}{\alpha^2}\right]^{k+\delta} \frac{(2\pi n)^k}{k!\,(k+\delta)!}\, \mathrm{e}^{-2\pi\tau_2 n}. \tag{3.7.19}$$

This expression makes it possible to study the region $\tau_2 \sim 0^+$ in quite a fruitful way. In what follows, a finite $\tau_2 > 0$ will be considered in order to carry out the calculations with the infinite sums, then the behaviour of the functions of interest will be assessed in the limit $\tau_2 \to 0^+$.

Because the elements in the infinite summations over k and n are positive-definite, the order of the two summations in eq. (3.7.19) can be interchanged. The sum for $n \in \mathbb{N}^{\pm}_\alpha(\beta)$ can be rearranged by observing that its elements can be written as $n = m^{\pm}_\alpha(\beta) \bmod \gamma_\alpha$, where $m^{\pm}_\alpha(\beta)$ is an integer depending on α and β and $\gamma_\alpha = \mathrm{lcm}\,(2,\alpha) = 2\alpha$, with α assumed to be odd. For the time being, it is convenient to leave the notation γ_α, with the explicit α-dependence not written for brevity, to facilitate the analogous closed-string discussion below. Note that $m^{\pm}_\alpha(\beta)$ is by definition the smallest element in the set $\mathbb{N}^{\pm}_\alpha(\beta)$, and it is generally not corresponding to β. For instance, $m^{+}_\alpha(\beta)$ is the smallest positive odd natural equal to $\beta \bmod \alpha$. As α is assumed to be odd, if β is odd too one has $m^{+}_\alpha(\beta) = \beta$, while if β is even one has $m^{+}_\alpha(\beta) = \beta + \alpha$, which is odd. An analogous reasoning applies to $m^{-}_\alpha(\beta)$. In general, it is possible to write

$$m^{\pm}_\alpha(\beta) = \beta + \frac{(1 \pm (-1)^\beta)}{2}\alpha. \tag{3.7.20}$$

This relationship will be helpful later on, but for now the terms $m^{\pm}_\alpha(\beta)$ can be left unexpanded. With this parametrisation, the summation over n can be performed in terms of the geometric series, resulting in

$$\begin{aligned} \sum_{n\in\mathbb{N}^{\pm}_\alpha(\beta)} (2\pi n)^k\, \mathrm{e}^{-2\pi\tau_2 n} &= \sum_{l=0}^{\infty} \left[2\pi(m^{\pm}_\alpha(\beta) + l\gamma_\alpha)\right]^k \mathrm{e}^{-2\pi\tau_2\left[m^{\pm}_\alpha(\beta)+l\gamma_\alpha\right]} \\ &= (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}\tau_2^k} \sum_{l=0}^{\infty} \mathrm{e}^{-2\pi\tau_2\left[m^{\pm}_\alpha(\beta)+l\gamma_\alpha\right]} \\ &= (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}\tau_2^k} \left[\frac{\mathrm{e}^{2\pi\left[\gamma_\alpha - m^{\pm}_\alpha(\beta)\right]\tau_2}}{\mathrm{e}^{2\pi\gamma_\alpha\tau_2} - 1}\right]. \end{aligned} \tag{3.7.21}$$

In this way, to finally explore the region where $\tau_2 \sim 0^+$,[3.7] it is sufficient to Taylor-expand the leftover order-k derivative. In the notation reviewed in appendix A.1.2, from the expansion

$$\frac{\mathrm{e}^{2\pi\left[\gamma_\alpha - m_\alpha^\pm(\beta)\right]\tau_2}}{\mathrm{e}^{2\pi\gamma_\alpha\tau_2}-1} = \frac{1}{2\pi\gamma_\alpha}\frac{1}{\tau_2} + \frac{\gamma_\alpha - 2m_\alpha^\pm(\beta)}{2\gamma_\alpha} + O(\tau_2;0),$$

one learns that the function to be differentiated k times at leading order is $1/(2\pi\gamma_\alpha\tau_2)$. It should be noted that this is the only β- and $(\pm)$-independent term, since the leftover power-law term depends on β and the $(\pm)$-sign via the terms $m_\alpha^\pm(\beta)$. In more detail, one obtains

$$(-1)^k\frac{\mathrm{d}^k}{\mathrm{d}\tau_2^k}\left[\frac{\mathrm{e}^{2\pi\left[\gamma_\alpha - m_\alpha^\pm(\beta)\right]\tau_2}}{\mathrm{e}^{2\pi\gamma_\alpha\tau_2}-1}\right] = \frac{1}{2\pi\gamma_\alpha}\frac{k!}{\tau_2^{1+k}} + \sum_{l=0}^{\infty} f_l(k, m_\alpha^\pm(\beta))\tau_2^l, \tag{3.7.22}$$

where $f_l(k, m_\alpha^\pm(\beta))$ are constants not depending on τ_2 that will be discussed later on (see eq. (3.7.32)) to keep the discussion here as plain as possible. Therefore, performing the sum over n, eq. (3.7.22) provides the β- and $(\pm)$-independent divergent term in the function $g_\alpha^\pm(\tau_2;\beta)$ defined in eq. (3.7.17). In particular, thanks to the expansion of eq. (3.7.22), the original function $g_\alpha^\pm(\tau_2;\beta)$ appearing in eq. (3.7.17), and rearranged into a different form in eq. (3.7.19), can now be written as

$$g_\alpha^\pm(\tau_2;\beta) = \frac{1}{\tau_2}\frac{c_2(\alpha)}{\alpha^{1-\delta}\gamma_\alpha}\left[\frac{\pi}{12}\frac{c_3(\alpha)}{\alpha^2}\right]^\delta \sum_{k=0}^{\infty}\frac{\left[\frac{\pi}{12}\frac{c_3(\alpha)}{\alpha^2}\frac{1}{\tau_2}\right]^k}{(k+\delta)!} + \Delta g_\alpha^\pm(\tau_2;\beta), \tag{3.7.23}$$

where, according to eq. (3.7.22), the remainder is

$$\Delta g_\alpha^\pm(\tau_2;\beta) = \frac{2\pi c_2(\alpha)}{\alpha^{1-\delta}}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\left[\frac{\pi}{12}\frac{c_3(\alpha)}{\alpha^2}\right]^{k+\delta}}{k!\,(k+\delta)!} f_l(k, m_\alpha^\pm(\beta))\,\tau_2^l. \tag{3.7.24}$$

So, according to eq. (3.7.23), the function $g_\alpha^\pm(\tau_2;\beta)$ contains a β- and $(\pm)$-independent singular part as $\tau_2 \sim 0^+$ and a β- and $(\pm)$-dependent power-series remainder. As anticipated above, the key difference among these two terms consists in the fact that only the series has a dependence on β and the $(\pm)$-sign. In the singular part, one can recognise the leftover sum to be

$$\sum_{k=0}^{\infty}\frac{\left[\frac{\pi}{12}\frac{c_3(\alpha)}{\alpha^2}\frac{1}{\tau_2}\right]^k}{(k+\delta)!} = \frac{\mathrm{e}^{\frac{\pi}{12}\frac{c_3(\alpha)}{\alpha^2}\frac{1}{\tau_2}}}{\left[\frac{\pi}{12}\frac{c_3(\alpha)}{\alpha^2}\frac{1}{\tau_2}\right]^\delta}\left[1 - \frac{1}{(\delta-1)!}\Gamma\left[\delta, \frac{\pi}{12}\frac{c_3(\alpha)}{\alpha^2}\frac{1}{\tau_2}\right]\right], \tag{3.7.25}$$

[3.7] To evaluate the series by writing $(2\pi n)^k \mathrm{e}^{-2\pi\tau_2 n} = (-1)^k (\mathrm{d}/\mathrm{d}\tau_2)^k\, \mathrm{e}^{-2\pi\tau_2 n}$, one must invert the order of the differentiation with respect to k and of the summation over n. For a series of functions $f_n(x)$, if their series $f(x) = \sum_{n\in\mathbb{N}} f_n(x)$ is convergent and if the series of their derivatives $\sum_{n\in\mathbb{N}} f_n'(x)$ is uniformly convergent, then the identity holds $f'(x) = \sum_{n\in\mathbb{N}} f_n'(x)$ (see eq. (0.307, [130])). For the case at hand, the series is not convergent in the region $\tau_2 \sim 0^+$, as shown by the term $1/\tau_2$, so one should remove this and consider the leftover sum. This is the working assumption being adopted. An alternative way to compute the required series rigorously is to make use of the results for the arithmetico-geometric sum (see eq. (0.113, [130])).

where $\Gamma(\nu, z)$ is the incomplete Γ-function. In the region $\tau_2 \sim 0^+$, the incomplete Γ-function can also be expanded to write

$$\sum_{k=0}^{\infty} \frac{\left[\frac{\pi}{12}\frac{c_3(\alpha)}{\alpha^2}\frac{1}{\tau_2}\right]^k}{(k+\delta)!} = \frac{\mathrm{e}^{\frac{\pi}{12}\frac{c_3(\alpha)}{\alpha^2}\frac{1}{\tau_2}}}{\left[\frac{\pi}{12}\frac{c_3(\alpha)}{\alpha^2}\frac{1}{\tau_2}\right]^{\delta}} - \frac{1}{(\delta-1)!}\frac{\tau_2}{\left[\frac{\pi}{12}\frac{c_3(\alpha)}{\alpha^2}\right]} + O(\tau_2;0)^2. \tag{3.7.26}$$

One can eventually conclude that the function $g_\alpha^{\pm}(\tau_2;\beta)$ around the point $\tau_2 \sim 0^+$ reads

$$g_\alpha^{\pm}(\tau_2;\beta) \overset{\tau_2\sim 0^+}{\simeq} \frac{c_2(\alpha)}{\gamma_\alpha}\,\alpha^{\delta-1}\,\tau_2^{\delta-1}\,\mathrm{e}^{\frac{\pi}{12}\frac{c_3(\alpha)}{\alpha^2}\frac{1}{\tau_2}} + r(\alpha,\tau_2) + \Delta g_\alpha^{\pm}(\tau_2;\beta), \tag{3.7.27}$$

where the exponential term comes from the leading divergent term in eq. (3.7.26), with an associated finite remainder

$$r(\alpha,\tau_2) = -\frac{1}{(\delta-1)!}\frac{c_2(\alpha)}{\gamma_\alpha}\,\alpha^{\delta-1}\left(\frac{\pi}{12}\frac{c_3(\alpha)}{\alpha^2}\right)^{\delta-1} + O(\tau_2;0), \tag{3.7.28}$$

and $\Delta g_\alpha^{\pm}(\tau_2;\beta)$ is the power-series remainder defined in eq. (3.7.24). In the limit $\tau_2 \to 0^+$, the functions in eq. (3.7.27) obviously diverge sector-by-sector due to the exponential of $1/\tau_2$. However, the complete physical information relating to the one-loop cosmological constant is contained in the complete $(\pm)$- and β-averaged function $g(\tau_2)$ defined in eq. (3.7.16), and in this one the singular part is automatically cancelled out by the fermion-boson oscillation appearing therein at order $\alpha = 1$ and by the HRR-expansion property $\sum_{\beta=1}^{\alpha} P_\alpha(\beta) = 0$ in eq. (3.5.11) for higher orders $\alpha > 1$. This is true not only for the leading exponentially-divergent term in eq. (3.7.27), but also for the remainder terms in $r(\alpha,\tau_2)$, since they all are independent of the $(\pm)$-sign and of β. In other words, this structure carries over to all the subleading terms that descend from the term scaling as $1/\tau_2^{k+1}$ in the expansion of eq. (3.7.22), i.e. those contained in eq. (3.7.25). Due to the $(\pm)$-independence, one may also still explain the subleading-order cancellations as more standard boson-fermion cancellations, but, noticeably, this is not necessary (moreover, the perspective in terms of the $P_\alpha(\beta)$-cancellation is instrumental for the closed-string analysis in section 3.7.3). All cancellations find an intuitive interpretation in the anti-Dp-brane/Op-plane example represented in fig. 3.7. In fact, since all the β- and $(\pm)$-independent terms in eq. (3.7.23) cancel out, the function $g = g(\tau_2)$ appearing in eq. (3.7.16) can be simply written as

$$g(\tau_2) = (-1)^{F_0} g_0 + \sum_{\alpha\in\Gamma}\sum_{\beta=1}^{\alpha} P_\alpha(\beta)\big[\Delta g_\alpha^{+}(\tau_2;\beta) - \Delta g_\alpha^{-}(\tau_2;\beta)\big]. \tag{3.7.29}$$

Remarkably, this is just a constant term plus a power-series difference. Therefore, this is a proof that all of the exponentially-divergent contributions to the one-loop cosmological constant coming from the β- and $(\pm)$-independent part of eq. (3.7.23) cancel out when summing over all of the sectors of the theory, leaving at most a power-law dependence on τ_2. The only thing that matters to reach this conclusion is that all of these singular contributions to $g_\alpha^{\pm}(\tau_2;\beta)$ are identical for a given α (i.e. they are independent of β and of the $(\pm)$-sign) and therefore cancel out when averaging over the sectors labelled by β and/or when taking into account the difference between positive and negative terms.

All this is a formalisation of the cancellations taking place for open strings among the so-called envelope functions. These are the functions $\Phi^{\pm}_{\alpha}(n;\beta)$ defined for a continuous variable $n \in \mathbb{R}^+$ in subsection 3.6.2 as the functions that interpolate between the contributions to the degeneracy numbers $(-1)^{F_n} g_n$ in the Hardy-Ramanujan-Rademacher-expansion from the sector β, at a given order α. In fact, the mathematics underlying the cancellations is exactly the same. In the case of the envelope functions, one reaches an all-order cancellation as the envelope functions, by definition, depend on β only through the terms $P_\alpha(\beta)$, but their actual meaning in physical quantities that depend on sums over discrete integers is not apparent. In the calculation above, instead, it has been shown explicitly that the exponential divergences appearing in the one-loop cosmological constant are indeed dependent on β just via the terms $P_\alpha(\beta)$ and therefore their cancellation is automatic if the cancellation takes place among envelope functions. In other words, the vanishing of the average of the envelope functions discussed in subsection 3.6.2 is a sufficient condition to claim that the exponential divergence in the function $g(\tau_2)$ cancels out.

3.7.2.3 Power-Series Terms

In order to claim the finiteness of the one-loop cosmological constant, the leftover power-law terms in eq. (3.7.29) need to be studied carefully as $\tau_2 \sim 0^+$. Indeed, although the exponential divergences are proven to be absent, the integral defining the cosmological constant may still be singular as a power-law. In general one can write

$$g(\tau_2) = (-1)^{F_0} g_0 + \Delta g^+(\tau_2) - \Delta g^-(\tau_2) = (-1)^{F_0} g_0 + \sum_{l=0}^{\infty} b_l \tau_2^l, \tag{3.7.30}$$

where the τ_2-dependence comes from the difference of the two power-series expansions

$$\Delta g^{\pm}(\tau_2) = \sum_{\alpha\in\Gamma}\sum_{\beta=1}^{\alpha} P_\alpha(\beta)\Delta g^{\pm}_{\alpha}(\tau_2;\beta) = \sum_{l=0}^{\infty} b^{\pm}_l \tau_2^l, \tag{3.7.31}$$

with the definition $b_l = b^+_l - b^-_l$. On the other hand, the constant term is $g(0) = (-1)^{F_0} g_0 + b_0$. It turns out that a few manipulations allow one to determine an analytic expression for the coefficients of these power-series expansions $\Delta g^{\pm}(\tau_2)$.

To start, it is possible to characterise the power series $\Delta g^{\pm}_{\alpha}(\tau_2;\beta)$ appearing in eq. (3.7.24) in quite an explicit way. This requires knowledge of the coefficients $f_l(k, m^{\pm}_{\alpha}(\beta))$, which can be gained by going back to their original introduction. As reviewed in appendix A.2.2, from the definition of the Bernoulli polynomials one finds

$$\begin{aligned}\frac{\mathrm{e}^{2\pi\left[\gamma_\alpha - m^{\pm}_{\alpha}(\beta)\right]\tau_2}}{\mathrm{e}^{2\pi\gamma_\alpha\tau_2}-1} &= \frac{1}{2\pi\gamma_\alpha\tau_2}\,\frac{2\pi\gamma_\alpha\tau_2\,\mathrm{e}^{2\pi\gamma\tau_2\left[1-\frac{m^{\pm}_{\alpha}(\beta)}{\gamma_\alpha}\right]}}{\mathrm{e}^{2\pi\gamma_\alpha\tau_2}-1}\\ &= \sum_{n=0}^{\infty}(-1)^n B_n\left[\frac{m^{\pm}_{\alpha}(\beta)}{\gamma_\alpha}\right]\frac{(2\pi\gamma_\alpha\tau_2)^{n-1}}{n!},\end{aligned}$$

where $B_n(x)$ are the Bernoulli polynomials, with $B_n(1-x) = (-1)^n B_n(x)$. So, in the expansion of eq. (3.7.22) one finds that the series coefficients read

$$f_l(k, m^{\pm}_{\alpha}(\beta)) = \frac{(-1)^{l+1}}{l!}\,\frac{(2\pi\gamma_\alpha)^{l+k}}{k+l+1} B_{k+l+1}\left[\frac{m^{\pm}_{\alpha}(\beta)}{\gamma_\alpha}\right]. \tag{3.7.32}$$

Therefore, the coefficients $b_l^\pm$ for the functions $\Delta g^\pm(\tau_2)$ written in eq. (3.7.31) can be determined by inserting eq. (3.7.32) in eq. (3.7.24) and read

$$b_l^\pm = \frac{(-1)^{l+1}}{l!} \sum_{\alpha\in\Gamma} \frac{2\pi c_2(\alpha)}{\alpha^{1-\delta}} \sum_{\beta=1}^{\alpha} P_\alpha(\beta) \sum_{k=0}^{\infty} \frac{\left[\frac{\pi}{12}\frac{c_3(\alpha)}{\alpha^2}\right]^{k+\delta}}{k!\,(k+\delta)!} \frac{(2\pi\gamma_\alpha)^{k+l}}{k+l+1} B_{k+l+1}\left[\frac{m_\alpha^\pm(\beta)}{\gamma_\alpha}\right].$$

Thanks to this result, one can write the total coefficient $b_l = b_l^+ - b_l^-$ as

$$\begin{aligned} b_l = \frac{(-1)^{l+1}}{l!} \sum_{\alpha\in\Gamma} \pi c_2(\alpha) \frac{\alpha^{l-1}}{(2\pi)^{\delta-l}} \sum_{\beta=1}^{\alpha}\sum_{k=0}^{\infty} \frac{\left[\frac{\pi^2}{6}\frac{c_3(\alpha)}{\alpha}\right]^{k+\delta}}{k!\,(k+\delta)!} P_\alpha(\beta) \\ \cdot \frac{2^{k+l+1}}{k+l+1}\left[B_{k+l+1}\left[\frac{m_\alpha^+(\beta)}{2\alpha}\right] - B_{k+l+1}\left[\frac{m_\alpha^-(\beta)}{2\alpha}\right]\right], \end{aligned}$$

where the identity $\gamma_\alpha = 2\alpha$ has also been enforced. By plugging in the definition of $m_\alpha^\pm(\beta)$ in eq. (3.7.20), one can see that the difference of Bernoulli polynomials can be written as

$$\begin{aligned} B_{k+l+1}\left[\frac{m_\alpha^+(\beta)}{2\alpha}\right] - B_{k+l+1}\left[\frac{m_\alpha^-(\beta)}{2\alpha}\right] &= (-1)^\beta\left[B_{k+l+1}\left(\frac{\beta}{2\alpha}+\frac{1}{2}\right) - B_{k+l+1}\left(\frac{\beta}{2\alpha}\right)\right] \\ &= \frac{k+l+1}{2^{k+l+1}}(-1)^\beta E_{k+l}\left(\frac{\beta}{\alpha}\right), \end{aligned}$$

where $E_n(x)$ are the Euler polynomials, whose definition and relationship with Bernoulli polynomials is also reviewed in appendix A.2.2. In view of this, the power-series coefficients can be written as

$$\begin{aligned} b_l &= \frac{(-1)^{l+1}}{l!} \sum_{\alpha\in\Gamma} \pi c_2(\alpha) \frac{\alpha^{l-1}}{(2\pi)^{\delta-l}} \sum_{\beta=1}^{\alpha}\sum_{k=0}^{\infty} \frac{\left[\frac{\pi^2}{6}\frac{c_3(\alpha)}{\alpha}\right]^{k+\delta}}{k!\,(k+\delta)!} (-1)^\beta P_\alpha(\beta) E_{k+l}\left(\frac{\beta}{\alpha}\right) \\ &= \frac{(-1)^{l}}{l!} \sum_{\alpha\in\Gamma} \pi c_2(\alpha) \frac{\alpha^{l-1}}{(2\pi)^{\delta-l}} \sum_{r=0}^{\alpha-1}\sum_{k=0}^{\infty} \frac{\left[\frac{\pi^2}{6}\frac{c_3(\alpha)}{\alpha}\right]^{k+\delta}}{k!\,(k+\delta)!} (-1)^r P_\alpha(-r) E_{k+l}\left(1-\frac{r}{\alpha}\right), \end{aligned}$$

where the change of variable $\beta = \alpha - r$ has been employed, knowing that $\alpha = 2\omega+1$ is odd by assumption, and it has been made use of the periodicity condition $P_\alpha(\alpha - r) = P_\alpha(-r)$. Because the Euler polynomials are such that $E_n(1-x) = (-1)^n E_n(x)$, one can conclude that the power-series coefficients read

$$b_l = \frac{\pi}{l!} \sum_{\alpha\in\Gamma} \frac{c_2(\alpha)\alpha^{l-1}}{(2\pi)^{\delta-l}} \sum_{k=0}^{\infty} \frac{\left[\frac{\pi^2}{6}\frac{c_3(\alpha)}{\alpha}\right]^{k+\delta}}{k!\,(k+\delta)!} \sum_{r=0}^{\alpha-1}(-1)^{k+r} P_\alpha(-r) E_{k+l}\left(\frac{r}{\alpha}\right). \tag{3.7.33}$$

This represents the coefficient of the order-l term in the power series $\Delta g(\tau_2)$ for a generic open-string model where only odd values of α appear in eq. (3.5.9). Unfortunately, this expression is too complicated, in general, to simplify it further in an analytic way. The fundamental complication in eq. (3.7.33) lies in the form of the Kloosterman-like term $P_\alpha(-r)$ defined in eq. (3.5.7): this is very hard to be dealt with analytically, which renders the remaining sums untreatable too.

Although an explicit derivation of the characteristics of the power-series terms in a similar way as for the exponential divergences is beyond reach, it is nonetheless possible to determine the form of such power-law terms by relying on the mathematical properties of the Dedekind η-function. Let the partition function be a Dedekind η-quotient

$$M(\tau) = \xi \prod_{m=1}^{\infty} \left[\eta(m\tau)\right]^{\delta_m}, \tag{3.7.34}$$

where ξ is some numerical constant. By exploiting the modular properties of the Dedekind η-function, it is possible to determine the behaviour of this function on the imaginary axis $\tau_1 = 0$ as $\tau_2 \sim 0^+$. In fact, under the generating S-transformation $S(\tau) = -1/\tau$ of the modular group $\mathrm{PSL}_2(\mathbb{Z})$, the Dedekind η-function transforms as $\eta(-1/\tau) = \sqrt{-\mathrm{i}\tau}\,\eta(\tau)$, so, restricting to the imaginary axis $\tau = \mathrm{i}\tau_2$, one can write

$$\eta\left(\frac{\mathrm{i}}{\tau_2}\right) = \sqrt{\tau_2}\,\eta(\mathrm{i}\tau_2). \tag{3.7.35}$$

The definition of the Dedekind η-function allows one to write $\eta(\mathrm{i}t) = \mathrm{e}^{-\frac{\pi t}{12}} \prod_{m=1}^{\infty}(1-\mathrm{e}^{-2\pi m t})$, which in turn gives

$$\ln \eta(\mathrm{i}t) = -\frac{\pi t}{12} + \sum_{m=1}^{\infty} \ln\left(1 - \mathrm{e}^{-2\pi m t}\right) = -\frac{\pi t}{12} + O(\mathrm{e}^{-2\pi t}; \infty). \tag{3.7.36}$$

So, combining the S-transformation relation and the limit as $1/\tau_2 \sim \infty$, one concludes that, in the region where $\tau_2 \sim 0^+$, the Dedekind η-function behaves as (see appendix A.1.2 for more details)

$$\eta(\mathrm{i}\tau_2) \overset{\tau_2 \sim 0^+}{\simeq} \tau_2^{-\frac{1}{2}}\, \mathrm{e}^{-\frac{\pi}{12\tau_2}}. \tag{3.7.37}$$

Therefore, recalling the definition of the constant $c_1 = -\sum_{m=1}^{\infty} \delta_m/2$ and introducing the coefficients

$$s = \prod_{m=1}^{\infty} m^{\delta_m}, \tag{3.7.38a}$$

$$c_4 = -\sum_{m=1}^{\infty} \frac{\delta_m}{m}, \tag{3.7.38b}$$

one can simply write the asymptotic behaviour of the open-string partition function as

$$M(\mathrm{i}\tau_2) \overset{\tau_2 \sim 0^+}{\simeq} \xi\, s^{-\frac{1}{2}}\, \tau_2^{c_1}\, \mathrm{e}^{\frac{\pi c_4}{12\tau_2}}. \tag{3.7.39}$$

In the absence of an exponential divergence, i.e. for $c_4 = 0$, which one can assume and indeed verify in all the explicit examples, including the situations where the cancellations of subsubsection 3.7.2.2 take place, this provides a direct way to compute the series coefficients appearing in the expansion of eq. (3.7.30). Assuming c_1 to be integer, which is also verified in the string-derived examples considered here, eq. (3.7.39) indicates that the constant term and the first $c_1 - 1$ coefficients are zero and that the first non-zero one is b_{c_1}, i.e.

$$(-1)^{F_0} g_0 + b_0 = b_1 = \cdots = b_{c_1-1} = 0, \tag{3.7.40a}$$

$$b_{c_1} = \xi\, s^{-\frac{1}{2}}. \tag{3.7.40b}$$

Note that it is not just possible to easily find this leading power-law term, but that it is actually possible to also show that all the coefficients b_l except b_{c_1} are vanishing. Indeed, by plugging eq. (3.7.36) into eq. (3.7.37), one finds

$$\eta(\mathrm{i}\tau_2) = \tau_2^{-\frac{1}{2}}\, \mathrm{e}^{-\frac{\pi}{12\tau_2}} \Big[1 + O\Big(\mathrm{e}^{-\frac{2\pi}{\tau_2}}, 0\Big)\Big]. \tag{3.7.41}$$

This means that eq. (3.7.39) is only corrected by terms that are exponentially-suppressed compared to the leading term. Therefore, if $c_4 = 0$, the conclusion is that $b_{c_1}\tau_2^{c_1}$ is the only non-zero power-law term in the region $\tau_2 \sim 0^+$. From the discussion in subsubsection 3.7.1.1, one can see then that the cosmological constant of a Dp-brane theory is not divergent if $c_1 > (p+1)/2$.

Although neat, the result of eq. (3.7.39) hides its origin in terms of oscillations in the spectrum degeneracies as following from the HRR-expansion. Even though it has not been possible to show it directly, modular properties must constrain the state degeneracies in such a way as to ensure such an asymptotic behaviour. In fact, in purely technical terms, the discussion of the cancellations leading to eq. (3.7.29) may have been bypassed, relying on the modular properties of open-string partition functions, but of course the reason for showing those cancellations has been the necessity to give a physical interpretation to the mechanisms underlying the finiteness of the one-loop cosmological constant. This is also the reason for which it has been sensible to gather as much information as possible on the power-series remainder, in the same spirit that has led to showing the cancellation of the exponential divergences, despite the eventual unmanageability of eq. (3.7.33).

To conclude, it is worthwhile to notice that the expansion of eq. (3.7.39) has been explained as a consequence of the $\mathrm{PSL}_2(\mathbb{Z})$-properties of the Dedekind η-function. However, this can also be inferred from simpler considerations in mathematical analysis [131]. Details about both methods are in appendix A.1.2.

3.7.2.4 Example: Anti-Dp-Brane/Op-Plane Theory

It may be enlightening to discuss an explicit example. As usual, the prototypical model consists of the theory of an anti-Dp-brane sitting on top of an Op-plane.

The cancellation of the exponential divergence in the region $\tau_2 \sim 0^+$ has been demonstrated in subsection 3.7.2.2. In fact, it has been shown that the shifted-argument function $\tilde{M}(\tau) = -8\,\eta^8(2\tau)/\eta^{16}(\tau)$ by which it can be described, defined in eq. (3.5.32), only receives odd-α contributions in the Sussman HRR-expansion, so the cancellations therein described take place and one is left at most with a power series in the form of eq. (3.7.29). To discuss the details of the latter, the tools of subsection 3.7.2.3, and in particular the expansion in eq. (3.7.39), are necessary.

For an anti-Dp-brane on top of an Op-plane, according to eq. (3.5.31), the p-independent partition function is

$$M(\tau) = -8\,\frac{\eta^{16}(\tau)\,\eta^{16}(4\tau)}{\eta^{40}(2\tau)}. \tag{3.7.42}$$

For this, besides $\xi = -8$, one finds $s = 1/256$, $c_1 = 4$, and the exponential disappears as $c_4 = 0$, which means

$$g(\tau_2) = M(\mathrm{i}\tau_2) \overset{\tau_2 \sim 0^+}{\simeq} -128\tau_2^4. \tag{3.7.43}$$

So, in this case it is apparent that the expected cancellation of divergent terms take place, and it is also possible to determine explicitly the full power-law dependence. In fact, there is only one non-zero term. Unfortunately, for the open string there is no analogue of the Kutasov-Seiberg formula and one has to calculate the integral in eq. (3.7.2) in order to determine the cosmological constant $\Lambda_{\overline{p}-}$. The integral receives contributions from small and large τ_2 so that the above expansion only ensures its finiteness for small τ_2, but it does not allow one to determine analytically its explicit value.

3.7.3 Closed-String Misaligned Supersymmetry

To describe misaligned supersymmetry for closed strings, following eq. (3.7.8), the fundamental tool to discuss is the function $g(\tau_2)$ defined in eq. (3.7.7). The key for a finite one-loop cosmological constant is a function $g(\tau_2)$ approaching the region $\tau_2 \sim 0^+$ as a constant, according to the Kutasov-Seiberg identity in eq. (3.7.8). The discussion is inherently more complicated compared to the case of open strings since the partition function is the product of right- and left-moving sectors, but most of the analysis follows the same pattern.

3.7.3.1 Setup

Let the closed-string partition function be of the form $Z(\tau,\overline{\tau}) = \tau_2^{1-D/2} R(\tau)\overline{L}(\overline{\tau})$, where the terms $R(\tau) = q^{-n_0^R}\sum_{n=0}^{\infty} a_n^R q^n$ and $L(\tau) = q^{-n_0^L}\sum_{n=0}^{\infty} a_n^L q^n$ are the right- and left-moving contributions, respectively, with $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$. More generally, the closed-string partition function can be the sum of several such terms, i.e. $Z(\tau,\overline{\tau}) = \tau_2^{1-D/2}\sum_\sigma Z_\sigma(\tau,\overline{\tau})$, with $Z_\sigma(\tau,\overline{\tau}) = R_\sigma(\tau)\overline{L}_\sigma(\overline{\tau})$, in which case the discussion of exponential divergences below may be applied to each term $Z_\sigma(\tau,\overline{\tau})$ individually. This is the case for example for the heterotic SO(16)$\times$SO(16)-theory discussed in subsections 3.5.2 and 3.6.3. It should be pointed out that it is conceivable that there also may be models in which the cancellations happen between different terms, and this would require an adaptation of the procedure discussed below. Also notice that for simplicity here the analysis is referred to the case where $n \in \mathbb{N}_0$: terms with $n \in \mathbb{N}_0/2$ can be studied similarly after a variable rescaling $\tau' = 2\tau$. The constant terms n_0^R and n_0^L are also assumed to be integer, which can also follow from a variable rescaling. Then, one can write

$$g(\tau_2) = \tau_2^{1-D/2}\sum_{n=-n_0}^{\infty} (-1)^{F_n} g_n\,\mathrm{e}^{-4\pi\tau_2 n}, \tag{3.7.44}$$

where, defining $n_0 = \min(n_0^R, n_0^L)$, the net physical degeneracies read

$$(-1)^{F_n} g_n = a^R_{n+n_0^R}\, \overline{a}^L_{n+n_0^L}. \tag{3.7.45}$$

If both the functions $R(\tau)$ and $L(\tau)$ are Dedekind η-quotients that are amenable to the Sussman HRR-expansion of eq. (3.5.9), then it is possible to express the Laurent coefficients $a^R_{n+n_0^R}$ and $\overline{a}^L_{n+n_0^L}$ as simplified HRR-sums, for $n > 0$. In fact, it is possible to write

$$g(\tau_2) = \tau_2^{1-D/2}\big[h_0(\tau_2) + h(\tau_2)\big], \tag{3.7.46}$$

where $h_0(\tau_2)$ represents the sum restricted to coefficients not amenable to the simplified Sussman HRR-expansion and $h(\tau_2)$ stands for the remaining infinite series, i.e.

$$h_0(\tau_2) = \sum_{n=0}^{n_0} (-1)^{F_{-n}} g_{-n}\, \mathrm{e}^{4\pi\tau_2 n}, \tag{3.7.47a}$$

$$h(\tau_2) = \sum_{n\in\mathbb{N}} \sum_{\alpha\in\Gamma_R} \sum_{\beta\in\Gamma_L} P^R_\alpha(n+n_0^R)\overline{P}^L_\beta(n+n_0^L) f^R_{n+n_0^R}(\alpha)\overline{f}^L_{n+n_0^L}(\beta)\, \mathrm{e}^{-4\pi\tau_2 n}. \tag{3.7.47b}$$

Here, as usual, the two sets containing the contributions to the coefficients are defined as $\Gamma_R = \{\alpha \in \mathbb{N} : c_3^R(\alpha) > 0\}$ and $\Gamma_L = \{\beta \in \mathbb{N} : c_3^L(\beta) > 0\}$, whilst the functions $f_n^R(\alpha)$ and $f_n^L(\beta)$ represent the rest of the HRR-expansion factors aside from the P-functions. One can see from eq. (3.7.6) that the cosmological constant is divergent due to the term $h_0(\tau_2)$ if and only if $n_0 \neq 0$. In this case there are physical tachyons in the spectrum and therefore there is no stable vacuum around which one can study the theory. For such cases the Kutasov-Seiberg identity in eq. (3.7.8) is not applicable and therefore the analysis will be restricted to theories with $n_0 = 0$, which implies $h_0(\tau_2) = (-1)^{F_0} g_0$.

Because of the periodicity of the functions $P^R_\alpha(n)$ and $P^L_\beta(n)$, given the index $\ell = 1, \ldots, \mathrm{lcm}(\alpha, \beta)$, with the dependence on α and β within its range being left implicit for brevity, one can rearrange the infinite sum over n in $h(\tau_2)$ by writing[3.8]

$$h(\tau_2) = \sum_{\alpha\in\Gamma_R} \sum_{\beta\in\Gamma_L} \sum_{\ell=1}^{\mathrm{lcm}(\alpha,\beta)} P^R_\alpha(\ell+n_0^R)\overline{P}^L_\beta(\ell+n_0^L)\, h_{\alpha\beta}(\tau_2;\ell), \tag{3.7.48}$$

where the functions have been defined

$$h_{\alpha\beta}(\tau_2;\ell) = \sum_{n\in\mathbb{N}_{\alpha\beta}(\ell)} f^R_{n+n_0^R}(\alpha)\overline{f}^L_{n+n_0^L}(\beta)\, \mathrm{e}^{-4\pi\tau_2 n}, \tag{3.7.49}$$

with the sets $\mathbb{N}_{\alpha\beta}(\ell) = \{n \in \mathbb{N} : n = \ell \bmod \mathrm{lcm}(\alpha, \beta)\}$ being defined in such a way that the condition $P^R_\alpha(n+n_0^R)\overline{P}^L_\beta(n+n_0^L) = P^R_\alpha(\ell+n_0^R)\overline{P}^L_\beta(\ell+n_0^L)$ holds for all $n \in \mathbb{N}_{\alpha\beta}(\ell)$. In a

[3.8] When summing over n, one has to be careful in exploiting properly the periodicity of the P-functions. The correct strategy is the same as the one explained in subsection 3.6.3. One splits the sum over n into $\ell = 1, \ldots, \mathrm{lcm}(\alpha, \beta)$ contributions, in front of which the P-functions factorise. Then, within each of these contributions one has to sum the f-functions over all of the possible values of n associated to the fixed ℓ, namely those for which $n = \ell \bmod \mathrm{lcm}(\alpha, \beta)$. This is needed since one wants to sum over all n such that $P^R_\alpha(n+n_0^R)\bar{P}^L_\beta(n+n_0^L) = P^R_\alpha(\ell+n_0^R)\bar{P}^L_\beta(\ell+n_0^L)$, for a fixed ℓ, to be able to ignore in this step the details of the P-functions. In fact, the product $P^R_\alpha(n+n_0^R)\bar{P}^L_\beta(n+n_0^L)$ is unchanged by an $\mathrm{lcm}(\alpha, \beta)$-step.

straightforward calculation, analogous to the open-string one discussed above, one can show that if the functions $h_{\alpha\beta}(\tau_2;\ell)$ have a divergent exponential term which is independent of ℓ, then the vanishing of the pure P-function combinations in one sector is enough to conclude that such exponential divergences cancel out. This is discussed below.

3.7.3.2 Cancellation of Exponential Divergences

Defining $\delta_R = c_1^R + 1$ and $\delta_L = c_1^L + 1$ for brevity, by making use of the explicit form of the functions $f_n^R(\alpha)$ and $f_n^L(\beta)$, and thanks to the Taylor expansion of the Bessel function, one can write the functions in eq. (3.7.49) as

$$h_{\alpha\beta}(\tau_2;\ell) = \\ = \frac{4\pi^2 c_2^R(\alpha) c_2^L(\beta)}{\alpha^{1-\delta_R}\beta^{1-\delta_L}} \sum_{a=0}^{\infty}\sum_{b=0}^{\infty} \left[\frac{\pi}{12}\frac{c_3^R(\alpha)}{\alpha^2}\right]^{\delta_R+a} \left[\frac{\pi}{12}\frac{c_3^L(\beta)}{\beta^2}\right]^{\delta_L+b} \sum_{n\in\mathbb{N}_{\alpha\beta}(\ell)} \frac{(2\pi n)^{a+b}\,\mathrm{e}^{-4\pi\tau_2 n}}{a!b!(\delta_R+a)!(\delta_L+b)!}. \tag{3.7.50}$$

Defining the step $\gamma_{\alpha\beta} = \mathrm{lcm}\,(\alpha,\beta)$, according to the definition of the sets $\mathbb{N}_{\alpha\beta}(\ell)$ above, it is possible to write

$$\sum_{n\in\mathbb{N}_{\alpha\beta}(\ell)} (2\pi n)^{a+b}\,\mathrm{e}^{-4\pi\tau_2 n} = \sum_{k=0}^{\infty}[2\pi(\ell+k\gamma_{\alpha\beta})]^{a+b}\,\mathrm{e}^{-4\pi\tau_2(\ell+k\gamma_{\alpha\beta})} = \left(-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau_2}\right)^{a+b} \frac{\mathrm{e}^{4\pi(\gamma_{\alpha\beta}-\ell)\tau_2}}{\mathrm{e}^{4\pi\gamma_{\alpha\beta}\tau_2}-1}.$$

So the Bernoulli polynomials appear again, enabling one to write in general

$$\left(-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau_2}\right)^{r} \frac{\mathrm{e}^{4\pi(\gamma_{\alpha\beta}-\ell)\tau_2}}{\mathrm{e}^{4\pi\gamma_{\alpha\beta}\tau_2}-1} = \frac{1}{4\pi\gamma_{\alpha\beta}\tau_2}\frac{r!}{(2\tau_2)^r} + \sum_{m=0}^{\infty} B_{m+r+1}\left(\frac{\ell}{\gamma_{\alpha\beta}}\right)\frac{(-1)^{m+1}(2\pi\gamma_{\alpha\beta})^{m+r}(2\tau_2)^m}{(m+r+1)\,m!},$$

which eventually means

$$\sum_{n\in\mathbb{N}_{\alpha\beta}(\ell)} (2\pi n)^{a+b}\,\mathrm{e}^{-4\pi\tau_2 n} = \\ = \frac{1}{4\pi\gamma_{\alpha\beta}\tau_2}\frac{(a+b)!}{(2\tau_2)^{a+b}} + \sum_{m=0}^{\infty} B_{m+a+b+1}\left(\frac{\ell}{\gamma_{\alpha\beta}}\right)\frac{(-1)^{m+1}(2\pi\gamma_{\alpha\beta})^{a+b+m}(2\tau_2)^m}{(m+a+b+1)\,m!}. \tag{3.7.51}$$

This formally looks the same as for open strings, as expected. In particular, the first term could again give rise to exponential divergences, once it is resummed over a and b. However, as in the open-string case, such a divergent term is manifestly independent of ℓ and therefore, in the full expression of $h(\tau_2)$ in eq. (3.7.48), one can immediately perform the sum over such ℓ. Defining $\xi_{\alpha\beta} = \beta/\gcd(\alpha,\beta)$, for instance, for $\beta > \alpha$, this sum gives

$$\begin{aligned}\sum_{\ell=1}^{\mathrm{lcm}(\alpha,\beta)} P_\alpha^R(\ell+n_0^R)\overline{P}_\beta^L(\ell+n_0^L) &= \sum_{k_\alpha=1}^{\alpha}\sum_{m=0}^{\xi_{\alpha\beta}-1} P_\alpha^R(k_\alpha+m\alpha+n_0^R)\overline{P}_\beta^L(k_\alpha+m\alpha+n_0^L) \\ &= \sum_{k_\alpha=1}^{\alpha} P_\alpha^R(k_\alpha+n_0^R)\left[\sum_{m=0}^{\xi_{\alpha\beta}-1}\overline{P}_\beta^L(k_\alpha+m\alpha+n_0^L)\right] \\ &= 0,\end{aligned} \tag{3.7.52}$$

implying the absence of exponential divergences, as for the open-string case. To achieve this result, the periodicity of the P^R_α-function in the right-moving sector has been used and it has been observed that the sum over the P^L_β-function in the left-moving sector vanishes due to the identity of eq. (3.5.11). For $\alpha > \beta$, of course, one can simply exchange them above. So, for an exhaustive analysis, one is left with $\alpha = \beta$, in which case though one cannot generically show a cancellation. This means that in such cases divergences can arise. However, if for example only odd αs and even βs appear in the right- and left-moving sectors, respectively, or viceversa, then such a condition is simply not encountered. In such cases, all the exponential divergences cancel out. For other theories, the cancellations may also happen to take place between different terms in the partition function (i.e. different right-left products cancelling their contributions against each other). This may be the case for theories where one also has to deal with any possible set of terms α and β. The formalism developed up to this point is expected to be applicable to all such cases, but for the time being a more general analysis is deferred for a case-by-case study.

As expected, this is a proof of the fact that the cancellation of the envelope functions at all orders in the HRR-expansion implies a cancellation of the exponential divergence in the cosmological constant of closed-string theories, if the condition $\alpha \neq \beta$ holds for all right-left products of Kloosterman-like sums. In fact, it is immediate to observe that the mathematics underlying the envelope-function cancellation discussed in subsection 3.6.3 is precisely the same as the one appearing in eq. (3.7.52). Moreover, it should also be apparent that the mathematical structure is exactly the same as for open strings, with minor technical complications only induced by the product of right- and left-moving sectors, again not unexpectedly. As mentioned above, in terms of envelope functions, an explicit example is given by the heterotic SO(16)×SO(16)-theory as discussed in detail in subsection 3.6.3.

3.7.3.3 Power-Series Terms

As the exponential divergences have been shown to be absent, the remaining contributions to the one-loop cosmological constant are encoded in the function $g(\tau_2) = \tau_2^{1-D/2}\big[(-1)^{F_0} g_0 + h(\tau_2)\big]$, as in eq. (3.7.46), where $h = h(\tau_2)$ is just a power series. Indeed, following the separation of the divergent part from the power series in the expansion of eq. (3.7.51), the remaining uncancelled ℓ-dependent part means that the function $h(\tau_2)$ takes the form

$$h(\tau_2) = \sum_{m=0}^{\infty} b_m \tau_2^m, \tag{3.7.53}$$

with coefficients

$$\begin{aligned} b_m = & \sum_{\alpha\in\Gamma_R}\sum_{\beta\in\Gamma_L}\sum_{\ell=1}^{\mathrm{lcm}(\alpha,\beta)} P^R_\alpha(\ell+n^R_0)\overline{P}^L_\beta(\ell+n^L_0) \\ & \cdot \frac{4\pi^2 c^R_2(\alpha)c^L_2(\beta)}{\alpha^{1-\delta_R}\beta^{1-\delta_L}} \sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\left[\frac{\pi}{12}\frac{c^R_3(\alpha)}{\alpha^2}\right]^{\delta_R+a}\left[\frac{\pi}{12}\frac{c^L_3(\beta)}{\beta^2}\right]^{\delta_L+b} \\ & \cdot \frac{(2\pi\gamma_{\alpha\beta})^{a+b}}{a!\,b!\,(\delta_R+a)!\,(\delta_L+b)!}\, B_{m+a+b+1}\left(\frac{\ell}{\gamma_{\alpha\beta}}\right)\frac{(-1)^{m+1}(4\pi\gamma_{\alpha\beta})^m}{(m+a+b+1)\,m!}. \end{aligned} \tag{3.7.54}$$

The term $(-1)^{F_0}g_0$ can be computed straightforwardly, and its treatment does not present technical obstructions. On the other hand, the coefficients b_m cannot be dealt with explicitly since they consists of several infinite sums that, in particular due to the presence of the Kloosterman-like terms, are very difficult to study analytically.

All in all, this is again reminiscent of the open-string result, where only a power series in τ_2 with coefficients given by an expression involving Bernoulli polynomials appears. As anticipated, though, in the case of closed strings the expressions are even more involved due to the mixing of the different right- and left-moving sectors. Moreover, a simple argument giving the specific power-law behaviour is prevented, even if the closed-string partition function can be written in terms of Dedekind η-quotients, due to the different structure of the theory. Indeed, for closed strings, results similar to the one in eq. (3.7.39) for open strings are harder to implement because, in order to discuss the function $g(\tau_2)$ explicitly without relying on the HRR-expansion, one has to integrate over the variable τ_1, and the resulting small-τ_2 behaviour cannot be written as simply. So, for the closed-string case one cannot obtain a trivial expression for the power series that arises when expanding $g(\tau_2)$ for $\tau_2 \sim 0^+$ based off the properties of the Dedekind η-function asymptotics.

To make progress, one has to be careful about the fact that the partition function may be composed of several terms $Z_\sigma(\tau,\overline{\tau}) = R_\sigma(\tau)\overline{L}_\sigma(\overline{\tau})$, each modular non-invariant but combining into a modular-invariant sum $Z(\tau,\overline{\tau}) = \tau_2^{1-D/2}\sum_\sigma Z_\sigma(\tau,\overline{\tau})$. In this case, the function to be eventually considered is of the form $g(\tau_2) = \tau_2^{1-D/2}\big[(-1)^{F_0}g_0 + \sum_\sigma h^\sigma(\tau_2)\big]$, where each function $h^\sigma(\tau_2) = \sum_{m=0}^{\infty} b_m^\sigma \tau_2^m$ is a power series of the form outlined in eqs. (3.7.53, 3.7.54). Having restored modular invariance, the one-loop cosmological constant is of course finite and the limit of $g(\tau_2)$ is finite too, according to the Kutasov-Seiberg identity in eq. (3.7.8). In fact, in the derivation of the latter, an asymptotic behaviour analogous to the open-string one in eq. (3.7.39) can be established by considering the Mellin transform $I(s)$ of the function $g(\tau_2)/\tau_2$. Following refs. [89, 129, 131], one can show the relationship

$$I\left[\frac{g(\tau_2)}{\tau_2}\right](s) \equiv \int_0^\infty \mathrm{d}\tau_2\, \tau_2^{s-1}\frac{g(\tau_2)}{\tau_2} = \int_{\mathbb{F}} \frac{\mathrm{d}^2\tau}{\tau_2^2}\, E(\tau,\overline{\tau};s)Z(\tau,\overline{\tau}), \tag{3.7.55}$$

where $E(\tau,\overline{\tau};s)$ is the non-holomorphic Eisenstein series. Denoting as $s_i \in \{s_0, s_a\} = \Pi_I$ the poles of the Mellin transform and as $r(s_i)$ the corresponding residues, one can then invert the Mellin transform to write

$$\frac{g(\tau_2)}{\tau_2} \overset{\tau_2\sim 0^+}{\sim} \sum_{s_i\in\Pi_I} r(s_i)\tau_2^{-s_i}, \tag{3.7.56}$$

On the real axis, $s_0 = 1$ is the only pole. In the rest of the complex plane, these poles can be seen to be related to the non-trivial zeros of the Riemann ζ-function as $s_a = \rho_a/2$, where $\rho_a = 1/2 \pm \mathrm{i}\gamma_a$, for $\gamma_a \in \mathbb{R}$, assuming the Riemann hypothesis to be correct. In the function $g(\tau_2)$, the leading term for $\tau_2 \sim 0^+$ is clearly given by the real pole $s_0 = 1$, implying the finite limit $\lim_{\tau_2\to 0^+} g(\tau_2) = r(s_0)$. The associated residue can be seen to be $r(s_0) = 3I/\pi$, where $I = \int_{\mathbb{F}} \mathrm{d}^2\tau\, Z(\tau,\overline{\tau})/\tau_2^2$. It should be noted that this is in fact the essence of the proof of the Kutasov-Seiberg identity in eq. (3.7.8). To conclude, in analogy with the open-string result in eqs. (3.7.40a, 3.7.40b), the result is that the coefficients of the power series can be written as

$$(-1)^{F_0} g_0 + \sum_\sigma b_0^\sigma = \sum_\sigma b_1^\sigma = \cdots = \sum_\sigma b_{D/2-2}^\sigma = 0, \tag{3.7.57a}$$

$$\sum_\sigma b_{D/2-1}^\sigma = r(s_0). \tag{3.7.57b}$$

3.8 Open-String Supertraces

This section discusses supertraces in String Theory. After a review of their definition in closed-string theories, it provides an interpretation for the meaning of supertraces in open-string theories.

3.8.1 Recap: Closed-String Supertraces

In string-theoretic models, there is an infinite number of degrees of freedom, and therefore any deviation from supersymmetry generally implies that the standard supertraces $\operatorname{str} M^{2\beta} = \sum_{n\in\mathbb{N}_0/2} (-1)^{F_n} g_n\, M_n^{2\beta}$ are infinite. In order to a provide a meaningful notion of string-based supertraces, ref. [28] proposes the definition of supertraces of the form

$$\operatorname{Str} M^{2\beta} = \lim_{t\to 0} \left[\sum_{n\in\frac{1}{2}\mathbb{N}_0} (-1)^{F_n} g_n\, M_n^{2\beta}\, \mathrm{e}^{-2\pi t M_n^2/\mu^2} \right], \tag{3.8.1}$$

for an arbitrary mass parameter μ. These reduce to the standard supertraces for a finite number of degrees of freedom, but they are also well-defined quantities for theories with an infinite number of fields, with the exponential of the mass operator playing the role of a natural cut-off. In fact, in terms of the supertraces of eq. (3.8.1), the one-loop cosmological constant of a closed-string theory in even D non-compact dimensions can be expressed as [28]

$$\tilde{\Lambda}_D = \frac{1}{\kappa_D^2 l_s^2} \frac{(-4\pi)^{\frac{D}{2}}}{96\pi (D/2-1)!} \operatorname{Str} \left(\frac{\alpha' M^2}{4} \right)^{\frac{D}{2}-1}, \tag{3.8.2}$$

with all the supertraces of smaller powers of M^2 being zero, i.e. $\operatorname{Str} M^0 = \operatorname{Str} M^2 = \cdots = \operatorname{Str} M^{D-4} = 0$. This can be inferred by expressing $\tilde{\Lambda}_D$ in eq. (3.7.8) in view of the expansion of eq. (3.7.9), after fixing $\mu = 2/\sqrt{\alpha'}$. In particular, eq. (3.8.2) can be interpreted as a generalisation of the QFT-expression for the one-loop cosmological constant, which is the sum of a series of terms depending on the usual supertraces (see e.g. refs. [85, 132]). The work of ref. [129] generalises these ideas to the scalar-mass corrections too.

As the one-loop cosmological constant $\tilde{\Lambda}_D$ is finite in theories exhibiting misaligned supersymmetry, in the cases where eq. (3.8.2) holds, misaligned supersymmetry motivates the finiteness of the supertraces. However, for open strings there is no analogue of a relationship such as eq. (3.8.2). This dates back to the lack of modular invariance and thus of a Kutasov-Seiberg-like identity in this case, which prevents the expression of the integral defining the one-loop cosmological constant in terms of a simple limit. Below it is shown that an expression in the spirit of eq. (3.8.1) also makes sense for open strings and how to interpret it.

3.8.2 Supertraces for Open Strings

In accordance with the definitions of eqs. (3.7.2, 3.7.3, 3.7.14), the one-loop cosmological constant for the theory of a Dp- or anti-Dp-brane can be written as

$$\Lambda_{\mathrm{D}p} = -\frac{T_{\mathrm{D}p}}{2\pi} \int_0^\infty \frac{\mathrm{d}t}{(2t)^{\frac{p+3}{2}}}\, g(t),$$

with

$$g(t) = \sum_{n=0}^{\infty} (-1)^{F_n} g_n\, \mathrm{e}^{-2\pi t n}.$$

For tachyon-free theories, the integral can diverge at $t = 0$, whereas the limit $t \sim \infty$ is finite thanks to the exponential suppression factor $\mathrm{e}^{-2\pi t n}$, for $n > 0$, and the power-law damping $t^{-(p+3)/2}$, for $n = 0$. For masses $M_n^2 = n/\alpha'$, setting $\mu^2 = 1/\alpha'$, the supertraces defined in eq. (3.8.1) read

$$\operatorname{Str} M^{2\beta} = \lim_{t\to 0} \sum_{n=0}^{\infty} (-1)^{F_n} g_n \left(\frac{n}{\alpha'}\right)^{\beta} \mathrm{e}^{-2\pi t n} = \lim_{t\to 0} \left[\left(-\frac{1}{2\pi\alpha'}\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\beta} g(t)\right]. \tag{3.8.3}$$

So far, supertraces for open strings are simple quantities that one can compute explicitly, but their interpretation as physical observables is not apparent. A physical interpretation is provided below.

Because it has been shown that the exponential divergences of the form $\mathrm{e}^{1/t}$ cancel out and the function $g(t)$ is just a series of non-negative powers of t, as shown in the derivation of eq. (3.7.29), the function $g(t)$ can be Taylor-expanded around the point $t = 0$ as

$$g(t) = \sum_{\beta=0}^{\infty} \frac{t^\beta}{\beta!} \left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\beta} g(t)\right]_{t=0} = \sum_{\beta=0}^{\infty} \frac{t^\beta}{\beta!} (-2\pi\alpha')^{\beta} \operatorname{Str} M^{2\beta}.$$

Since the integration over $t \in [\epsilon, \infty[$ gives a finite result for an arbitrary $\epsilon \in \mathbb{R}^+$, the potentially divergent term in the cosmological constant corresponds to the part integrated over $t \in [0, \epsilon[$. The latter can be written as

$$\delta\Lambda_{\mathrm{D}p} = -\frac{T_{\mathrm{D}p}}{2\pi} \int_0^\epsilon \frac{\mathrm{d}t}{(2t)^{\frac{p+3}{2}}}\, g(t) = -\frac{T_{\mathrm{D}p}}{2\pi} \int_0^\epsilon \frac{\mathrm{d}t}{(2t)^{\frac{p+3}{2}}} \sum_{\beta=0}^{\infty} \frac{t^\beta}{\beta!} (-2\pi\alpha')^{\beta} \operatorname{Str} M^{2\beta}.$$

For any given β, such an integral is convergent if $\beta > (p+1)/2$, which means that for the one-loop cosmological constant to be finite one needs to have

$$\operatorname{Str} M^{2\beta} = 0, \qquad \beta = 0, 1, \ldots, \frac{p+1}{2}. \tag{3.8.4}$$

This resembles the closed-string result, where the first non-zero supertrace likewise has to be $\operatorname{Str} M^{D-2}$. Now, comparing the Taylor expansion of $g(t)$ with the power series defined in eq. (3.7.30), one can see the identification

$$b_l + (-1)^{F_0} g_0\, \delta_{l0} = \frac{(-1)^l}{l!} (2\pi\alpha')^l \operatorname{Str} M^{2l}. \tag{3.8.5}$$

This is an alternative way to interpret the need for vanishing series coefficients: the fact that the first few of them are zero can be interpreted via the vanishing of the first few supertraces. Note that eq. (3.7.39) does not receive any power-law corrections. This also allows one to actually conclude that all supertraces except one vanish in open-string models. If the non-zero coefficient is b_{c_1}, with $c_1 > 0$, the non-zero supertrace is

$$\mathrm{Str}\, M^{2c_1} = (-1)^{c_1} \frac{c_1!}{(2\pi\alpha')^{c_1}}\, b_{c_1}. \tag{3.8.6}$$

It is interesting to wonder whether the region near $t \sim \infty$ can provide additional information about the supertraces. This is not necessarily the case due to the peculiar properties of string-theory one-loop partition functions under modular transformations. In fact, well-behaved changes under S-transformations typically relate the regions around $t \sim 0$ and $t \sim \infty$. For instance, let the function $M(\tau)$ transform as $M(\mathrm{i}/\tau_2) = s^{-1/2}\tau_2^{-c_1} M(\mathrm{i}\tau_2/k)$, for some positive constant k. Then the leftover integration over $t \in [\epsilon, \infty[$ reads

$$\Lambda_{\mathrm{D}p} - \delta\Lambda_{\mathrm{D}p} = -\frac{T_{\mathrm{D}p}}{2\pi}\int_\epsilon^\infty \frac{\mathrm{d}t}{(2t)^{\frac{p+3}{2}}}\, g(t) = -\frac{T_{\mathrm{D}p}}{2\pi}\frac{s^{-\frac{1}{2}}}{2^{c_1}}\left(\frac{k}{4}\right)^{\frac{p+1}{2}-c_1}\int_0^{1/k\epsilon} \frac{\mathrm{d}y\; g(y)}{(2y)^{2+c_1-\frac{p+3}{2}}},$$

thanks to the change of variable $t = 1/ky$. The potential divergence comes from the region near $y \sim 0^+$. Taking again advantage of the expansion of the function $g(t)$, one infers that the integral is finite so long as $\beta > (2c_1 - p - 1)/2$. This condition expresses an IR-UV duality that relates Dp-branes to D$(2c_1 - 2 - p)$-branes.

3.8.2.1 Example: Anti-Dp-Brane/Op-Plane Theory

For an anti-Dp-brane on top of an Op-plane, in view of eq. (3.4.11), the one-loop cosmological constant in eq. (3.7.2) can be expressed as

$$\Lambda_{\overline{p}-} = -\frac{g_s}{2\pi}\,\tau_{\mathrm{D}p}\int_0^\infty \frac{\mathrm{d}t}{2t}\, M_{\overline{p}-}(t) = -\frac{g_s}{2\pi}\,\tau_{\mathrm{D}p}\int_0^\infty \frac{\mathrm{d}t\, M(\mathrm{i}t)}{(2t)^{p+3}}, \tag{3.8.7}$$

where $M(\mathrm{i}t)$ is the function defined in eq. (3.5.30) and the Einstein-frame anti-Dp-brane tension reads $\tau_{\mathrm{D}p} = T_{\mathrm{D}p}/g_s$. It should be noted that the oscillatory part of the integrand transforms under S-transformations as

$$M\left(\frac{\mathrm{i}}{t}\right) = \left(\frac{t}{2}\right)^{-4} M\left(\frac{\mathrm{i}t}{4}\right), \tag{3.8.8}$$

which is manifested as a duality relating anti-Dp- and anti-D$(6-p)$-branes, for $p < 7$, i.e.

$$-\frac{g_s\tau_{\mathrm{D}p}}{2\pi}\int_0^\infty \frac{\mathrm{d}t\; M(\mathrm{i}t)}{(2t)^{\frac{p+3}{2}}} = -\frac{g_s\tau_{\mathrm{D}p}}{2\pi}\int_0^\infty \frac{\mathrm{d}y\; M(\mathrm{i}y)}{(2y)^{\frac{(6-p)+3}{2}}},$$

obtained by a simple change of variable $t = 1/(4y)$, which means

$$l_s^p \Lambda_{\overline{p}-} = l_s^{6-p}\Lambda_{\overline{(6-p)}-}. \tag{3.8.9}$$

Such a condition confirms the fact that the information available in the region near $t \sim \infty$ is equivalent to the information available around $t \sim 0^+$.

The integral is easy to calculate explicitly. It is convenient to define the integral I_p as

$$I_p = -\int_0^\infty \frac{\mathrm{d}t}{2t}\, M_{\overline{p}-}(t) = \int_0^\infty \frac{\mathrm{d}t}{(2t)^{\frac{1}{2}(p+3)}}\, \frac{8}{\vartheta_3^8[2\mathrm{i}t]} = 4\int_0^\infty \frac{\mathrm{d}t}{t^{\frac{1}{2}(p+3)}}\, \frac{1}{\vartheta_3^8[\mathrm{i}t]}, \tag{3.8.10}$$

so that

$$\Lambda_{\overline{p}-} = \frac{g_s}{2\pi}\, \tau_{\mathrm{D}p}\, I_p. \tag{3.8.11}$$

The value I_p is finite as long as $p = 0, 1, 2, 3, 4, 5, 6$, as can be seen immediately thanks to the small-t expansion

$$\vartheta_3^{-8}[\mathrm{i}t] \overset{t \sim 0^+}{\simeq} t^4. \tag{3.8.12}$$

One can evaluate the integral numerically and find

$$\begin{aligned} I_0 &= I_6 \simeq 8.32542, \\ I_1 &= I_5 \simeq 4.54293, \\ I_2 &= I_4 \simeq 3.4919, \\ I_3 &\simeq 3.2305. \end{aligned}$$

As anticipated, the equality for S-dual anti-Dp-/anti-D$(6-p)$-branes can be easily seen to follow from the modular transformation

$$\vartheta_3[\mathrm{i}t^{-1}] = t^{\frac{1}{2}}\vartheta_3[\mathrm{i}t]. \tag{3.8.13}$$

For $p > 6$, the one-loop cosmological constant diverges and the flat-spacetime calculation does not give a sensible answer. This can be attributed to the fact that the corresponding anti-Dp-brane on top of an Op-plane strongly backreacts and no asymptotic flat-spacetime solution exists.

The full vacuum energy consists of the tree-level potential, which prior to the compactification corresponds to the constant part of the DBI-term, plus the one-loop correction. Let the shifted dilaton be $\phi = \Phi - \langle \Phi \rangle$. Then, in the string frame, the contribution to the action reads

$$S_\Lambda^{\overline{p}-} = -\tau_{\mathrm{D}p} \int_{W_{1,p}} \mathrm{d}^{p+1}\xi\, \sqrt{-G_{p+1}}\, \mathrm{e}^{-\phi} \left[1 + \frac{g_s}{2\pi}\, I_p\, \mathrm{e}^{\phi}\right], \tag{3.8.14}$$

where G_{p+1} is the determinant of the pulled-back metric, with $\varphi : W_{1,p} \hookrightarrow X_{1,9}$ being the embedding function of the anti-Dp-brane worldvolume $W_{1,p}$ into the 10-dimensional spacetime $X_{1,9}$. As expected, the one-loop correction to the tree-level vacuum energy is suppressed by a factor g_s, which is the open-string coupling. The 10-dimensional Einstein frame is defined by the metric $\hat{g}_{MN} = \mathrm{e}^{-\frac{\phi}{2}} G_{MN}$, which gives

$$S_\Lambda^{\overline{p}-} = -\tau_{\mathrm{D}p} \int_{W_{1,p}} \mathrm{d}^{p+1}\xi\, \sqrt{-\hat{g}_{p+1}}\, \mathrm{e}^{\frac{(p-3)}{4}\phi} \left[1 + \frac{g_s}{2\pi}\, I_p\, \mathrm{e}^{\phi}\right]. \tag{3.8.15}$$

To conclude, one can eventually compute the supertraces. From the previous analysis, it appears that the function $g(t)$ is

$$g(t) = -8\vartheta_3^{-8}[2\mathrm{i}t], \tag{3.8.16}$$

which means that the supertraces can be computed as

$$\mathrm{Str}\, M^{2\beta} = -8 \lim_{t\to 0}\left[\left(-\frac{1}{2\pi\alpha'}\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\beta} \vartheta_3^{-8}[2it]\right]. \tag{3.8.17}$$

From the small-t expansion in eq. (3.8.12), one can see immediately that the only non-zero supertrace arises for $\beta = 4$, with $\mathrm{Str}\, M^8 = -128\cdot 4!/(2\pi\alpha')^4$. This is consistent with the fact that in the anti-Dp-brane/Op-plane theory one has $b_l + (-1)^{F_0} g_0\, \delta_{l0} = 0$ for $l = 0, 1, 2, 3$, with the first non-zero coefficient being $b_4 = (2\pi\alpha')^4\, \mathrm{Str}\, M^8/4! = -128$ and a one-loop cosmological constant finite up to $p = 6$.

3.9 Non-Supersymmetric Strings and Misaligned Supersymmetry

In this section, a review is provided of the known consistent tachyon-free 10-dimensional non-supersymmetric models, i.e. the heterotic SO(16)×SO(16)-theory, the Sugimoto USp(32)-model and the type 0'B SU(32)-theory. It is going to be argued that they all exhibit the defining features of misaligned supersymmetry in parts of their spectra.[3.9] This supports the idea that misaligned supersymmetry is a generic feature appearing in the non-supersymmetric string landscape.

3.9.1 Heterotic SO(16)×SO(16)-Theory

As discussed thoroughly in sections 3.3, 3.6 and 3.7, the heterotic SO(16)×SO(16)-theory is a prototypical 10-dimensional non-supersymmetric closed-string theory exhibiting misaligned supersymmetry. Its misalignment has to be studied in relation to a closed-string partition function involving the product of right- and left-moving sectors.

As signalled by the dualities of eqs. (3.5.28a, 3.5.28b), the partition function can be described in terms of the two functions defined in eqs. (3.5.15a, 3.5.15b), i.e.

$$R_1(\tau) = \frac{2S_8}{\eta^8}(\tau) = \frac{\vartheta_2^4(\tau)}{\eta^{12}(\tau)} = \frac{16\,\eta^8(2\tau)}{\eta^{16}(\tau)}, \tag{3.9.1}$$

$$L_1(\tau) = \frac{\vartheta_3^8(\tau)\vartheta_4^8(\tau)}{\eta^{24}(\tau)} = \frac{\eta^8(\tau)}{\eta^{16}(2\tau)}. \tag{3.9.2}$$

For both R_1 and L_1, the Sussman HRR-expansion of eq. (3.5.9) applies, with only odd αs and even βs appearing, respectively. Not all terms in the partition function are of the form of eq. (3.9.1) or eq. (3.9.2), but, apart from the rescaling $\tau' = 2\tau$ that only amounts to index labelling, they differ at most due to 1/2-shifts as

$$\tilde{R}_1(\tau) = R_1(\tau + 1/2) = \frac{16\,\eta^{16}(\tau)\eta^{16}(4\tau)}{\eta^{40}(2\tau)}, \tag{3.9.3}$$

[3.9] It is worth pointing out ref. [118] for a recent review on non-supersymmetric strings and ref. [133] for an analysis of the interactions between branes in them.

$$\tilde{L}_1(\tau) = L_1(\tau + 1/2) = -\frac{\eta^8(2\tau)}{\eta^8(\tau)\eta^8(4\tau)}. \tag{3.9.4}$$

The function $\tilde{L}_1$ can be studied via eq. (3.5.9). On the other hand, the function $\tilde{R}_1$ is not amenable to this Sussman HRR-expansion, but this does not constitute a problem since for counting the state degeneracies one can just work with R_1 and keep track of the signs produced by the shift. A plot of the lightest energy states in the spectrum is in fig. 3.1.

3.9.2 Sugimoto USp(32)-Theory

The Sugimoto USp(32)-theory, introduced in section 2.1.3, is a non-supersymmetric open-string theory and, in fact, its partition function presents the same structure as the anti-Dp-brane/Op-plane theory. The latter has been considered in detail in sections 3.4, 3.6 and 3.7 as a prototypical case-study of open-string misaligned supersymmetry.

Effectively, implementing the Jacobi identity $V_8 = S_8$, the Sugimoto model is described by the Möbius-strip amplitude in eq. (2.1.68), which can be written as

$$\mathcal{S} = \frac{1}{2}\int_0^\infty \frac{\mathrm{d}t}{t^6}\frac{V_8 + S_8}{\eta^8}\left[\mathrm{i}t + \frac{1}{2}\right]. \tag{3.9.5}$$

Ignoring the power-law term, the integrand can be analysed by considering it as the restriction to imaginary arguments of the function

$$S(\tau) = \frac{1}{2}\frac{V_8 + S_8}{\eta^8}\left[\tau + \frac{1}{2}\right]. \tag{3.9.6}$$

Not unexpectedly, this has exactly the same structure as the open-string theory shown to exhibit misaligned supersymmetry, i.e. an anti-Dp-brane sitting on top of an Op-plane, whose p-independent partition function in eq. (3.5.31) is in fact $M = -S$. This is not amenable to the Sussman HRR-expansion, but the shifted-argument function is. This means that the exponential UV-divergences cancel automatically as in eq. (3.7.29). In the classification of section 3.6, this is case 1a. One should also rememeber that in fact, up to a constant factor, this is the function $\tilde{R}_1$, according to the dualities in eq. (3.5.34).

Importantly, one should note that there is an IR-divergence. Since the function above is $\tilde{R}_1$ as defined in eq. (3.9.3), its small-τ_2 expansion can be obtained from eq. (3.7.43) and starts with a power τ_2^4. So, the integral in eq. (3.9.5) is IR-divergent. This is due to the uncancelled NSNS-tadpole. For these codimension-zero sources, such a tadpole leads to a runaway potential for the dilaton and could be cancelled by a non-trivial dilaton profile, see for example refs. [5, 22].

It is interesting to interpret the physical content of the Sugimoto USp(32)-model. The closed-string sector is the same as the one of the type I theory, and it is supersymmetric. The open-string sector presents misaligned supersymmetry, and this is reflected in the fact that the gauge representations of bosons and fermions follow an alternating misaligned pattern: even-mass level bosons are in symmetric representations and even-mass level fermions are in antisymmetric representations of USp(32), and vice versa at odd mass levels. This can be seen easily by counting the degrees of freedom stemming from the combination of the

V_8-terms in the annulus and in the Möbius strip to count the bosons, and the S_8-terms to count fermions [5]. A plot of the open-string sector states can be reconstructed from the one in fig. 3.3.

3.9.3 Type 0'B SU(32)-Theory

The type 0'B SU(32)-theory represents an especially instructive instance of a 10-dimensional non-supersymmetric theory, and therefore it is going to be discussed in detail here.

Modular invariance allows one to define type 0A and type 0B theories, as discussed in subsection 2.1.3. Unlike the case of the type 0A theory, where chirality cannot be achieved, an orientifold projection of the type 0B theory reveals the existence of a theory with a chiral spectrum hosting both bosons and fermions. Actually, there exist three possible such projections with chiral spectra [15, 16, 26, 134], and only one of them, remarkably, removes the tachyon. For this theory, referred to as type 0'B theory [15], the open-descendant direct-channel amplitudes are

$$
\begin{aligned}
\mathcal{K} &= -\frac{1}{2}\int_0^\infty \frac{\mathrm{d}\tau_2}{\tau_2^6}\,\frac{O_8 - V_8 - S_8 + C_8}{\eta^8}[2\mathrm{i}\tau_2],\\
\mathcal{A} &= -\frac{1}{2}\int_0^\infty \frac{\mathrm{d}\tau_2}{\tau_2^6}\,\frac{1}{\eta^8}\big[-2\,(n_v n_c + n_o n_s)O_8 - 2(n_v n_s + n_o n_c)V_8\\
&\qquad\qquad + 2(n_o n_v + n_s n_c)S_8 + (n_o^2 + n_v^2 + n_s^2 + n_c^2)C_8\big]\left[\frac{\mathrm{i}\tau_2}{2}\right],\\
\mathcal{M} &= \frac{1}{2}\int_0^\infty \frac{\mathrm{d}\tau_2}{\tau_2^6}\,\frac{(n_o - n_v - n_s + n_c)C_8}{\eta^8}\left[\frac{\mathrm{i}\tau_2}{2}+\frac{1}{2}\right],
\end{aligned}
$$

where n_o, n_v, n_s and n_c are non-negative integers fixed by consistency conditions that are shortly going to be discussed. In the transverse channel, these amplitudes read

$$
\begin{aligned}
\tilde{\mathcal{K}} &= -\frac{1}{2}\,2^6\int_0^\infty \mathrm{d}\ell\,\frac{C_8}{\eta^8}[\mathrm{i}\ell],\\
\tilde{\mathcal{A}} &= \frac{1}{2}\,2^{-6}\int_0^\infty \mathrm{d}\ell\,\frac{1}{\eta^8}\big[-(n_o + n_v - n_s - n_c)^2 O_8 + (n_o + n_v + n_s + n_c)^2 V_8\\
&\qquad\qquad + (n_o - n_v + n_s - n_c)^2 S_8 - (n_o - n_v - n_s + n_c)^2 C_8\big][\mathrm{i}\ell],\\
\tilde{\mathcal{M}} &= \frac{1}{2}\,2\int_0^\infty \mathrm{d}\ell\,\frac{(n_o - n_v - n_s + n_c)C_8}{\eta^8}\left[\mathrm{i}\ell+\frac{1}{2}\right].
\end{aligned}
$$

Focussing on the consistency conditions stemming from these, one should set the coefficients of the O_8- and S_8-terms to zero, since they describe bosonic and fermionic contributions with the wrong sign: this pair of conditions reduces to $n_o = n_c$ and $n_v = n_s$. Further, tadpole cancellation requires that the ubiquitous C_8-contributions vanish, thus fixing $n_o = 32 + n_v$. Next, one should notice that the closed-string tachyon in the halved torus amplitude is removed by the Klein-bottle term, in the direct channel. To additionally remove the open-string tachyon from the annulus term, too, in view of the tadpole constraints, one has to fix $n_v = n_s = 0$, which means $n_o = n_c = 32$. Note that these conditions still leave a dilaton tadpole [15] from the V_8-term in the transverse-channel annulus.

To recap, the total type 0'B one-loop amplitude, proportional to the one-loop cosmological constant, is

$$\begin{aligned}
&\mathcal{T}/2+\mathcal{K}+\mathcal{A}+\mathcal{M} = \\
&= \frac{1}{2}\int_{\mathbb{F}} \frac{\mathrm{d}^2\tau}{\tau_2^6} \frac{|O_8|^2+|V_8|^2+|S_8|^2+|C_8|^2}{|\eta|^{16}}[\tau,\bar{\tau}] - \frac{1}{2}\int_0^\infty \frac{\mathrm{d}\tau_2}{\tau_2^6} \frac{O_8-V_8-S_8+C_8}{\eta^8}[2\mathrm{i}\tau_2] \\
&-\frac{1}{2}\int_0^\infty \frac{\mathrm{d}\tau_2}{\tau_2^6} \frac{\left[-2\cdot 32^2\, V_8 + 2\cdot 32^2\, C_8\right]}{\eta^8}\left[\frac{\mathrm{i}\tau_2}{2}\right] + \frac{1}{2}\int_0^\infty \frac{\mathrm{d}\tau_2}{\tau_2^6} \frac{2\cdot 32\, C_8}{\eta^8}\left[\frac{\mathrm{i}\tau_2}{2}+\frac{1}{2}\right].
\end{aligned} \tag{3.9.7}$$

Along with the functions R_1 and $\tilde{R}_1$ of eqs. (3.9.1, 3.9.3) – recalling that $S_8 = C_8$ –, this amplitude also involves the functions already defined in eqs. (3.5.18a, 3.5.23a), i.e.

$$-R_2(\tau) = \frac{O_8+V_8}{\eta^8}(\tau) = \frac{\vartheta_3^4(\tau)}{\eta^{12}(\tau)} = \frac{\eta^8(\tau)}{\eta^8(\tau/2)\eta^8(2\tau)}, \tag{3.9.8}$$

$$R_3(\tau) = \frac{O_8-V_8}{\eta^8}(\tau) = \frac{\vartheta_4^4(\tau)}{\eta^{12}(\tau)} = \frac{\eta^8(\tau/2)}{\eta^{16}(\tau)}. \tag{3.9.9}$$

a convenient τ-rescaling of which gives the functions in eqs. (3.5.19a, 3.5.24a), i.e.

$$-R_2'(\tau) = \frac{\eta^8(2\tau)}{\eta^8(\tau)\eta^8(4\tau)}, \tag{3.9.10}$$

$$R_3'(\tau) = \frac{\eta^8(\tau)}{\eta^{16}(2\tau)}. \tag{3.9.11}$$

In accordance with the dualities in eq. (3.5.28a), one should notice the identities $L_1(\tau) = R_2'(\tau+1/2) = R_3'(\tau)$.

In the amplitude of eq. (3.9.7), one has to study term by term, but luckily this is a relatively easy task for most contributions. The open-string sector is analogous to the Sugimoto USp(32)-theory one. On the other hand, a plot representing the total number of closed-string states for the type 0'B theory is in fig. 3.8.

- The open-string sector exhibits misaligned supersymmetry. The annulus amplitude happens to vanish by the Jacobi identity, so it represents a supersymmetric term. On the other hand, the Möbius-strip term is proportional to $\tilde{R}_1(\tau)$, and therefore its exponential divergences cancel out in the same way as for anti-Dp-branes/Op-planes and the Sugimoto model. This is a manifestation of misaligned supersymmetry, and it refers to the so-called case 1a. In fact, this open-string sector follows exactly the same pattern as the Sugimoto USp(32)-model.

- In the closed-string sector, the spectrum is purely bosonic. Yet, the interpretation that one may give of it can be seen in the perspective of a misalignment. To start, one has to observe that the torus amplitude has a tachyonic term which is only cancelled by the combination with the Klein bottle. This eliminates IR-divergences. Then, UV-divergences can be seen to be absent from the spectrum since the Klein bottle, described by the function R_3', undergoes the cancellations discussed in sections 3.6 and 3.7. In particular, this corresponds to case 1b. Although the physical interpretation of this fact cannot be phrased in terms of bosonic and fermionic oscillations, the mathematics is the same and in fact one can observe the cancellation of the divergences of the form e^{1/τ_2} as $\tau_2 \sim 0^+$ coming from O_8 and V_8. The correct physical interpretation regards the projection undergone by the bosons of the closed-string sector after the interplay of the halved torus with the Klein bottle. The oscillation given by the Klein-bottle function $-R_3'(\tau) = -q^{-1}\left[1 - 8q + 36q^2 - 128q^3 + O(q,0)^4\right]$, with $q = e^{2\pi i\tau}$, implies an alternating pattern in the spectrum when combined with the halved torus [5].

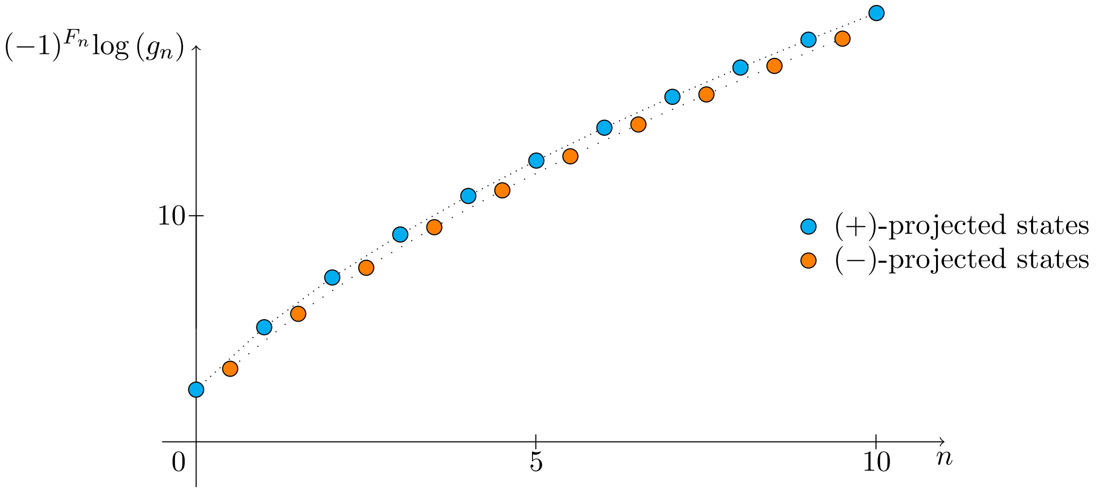

Figure 3.8: The net number of physical degrees of freedom for the lightest energy levels in the closed-string sector of the type 0'B theory, defined as g_n, at the n-th mass level. All states are bosonic, and each point corresponds to states with mass $M_n^2 = 4n/\alpha'$, with $n = 0, 1/2, 1, \ldots, 10$. There is a clear alternance between states receiving a positive contribution from both the torus and the Klein bottle, i.e. undergoing the '(+)-projection', and states receiving a positive contribution from the torus and a negative contribution from the Klein bottle, i.e. undergoing the '(−)-projection'.

The type 0'B theory illustrates several important points in the closed-string sector. Bearing in mind that the tachyon in the halved torus is removed by the tachyonic term in the Klein bottle, the remaining integration of the torus amplitude is finite since the UV-region is cut off from the domain thanks to modular invariance. This specific result does not require misaligned supersymmetry, but also it does not violate the claim that all non-tachyonic modular-invariant theories are either supersymmetric or misalignedly-supersymmetric [27], since in fact this specific amplitude technically contains a tachyon. Moreover, the presence of the tachyon also prevents one from making use of the Kutasov-Seiberg identity, bypassing any possible physical interpretation based on the torus physical states near the region $\tau_2 \sim 0^+$. Of course, the tachyon is actually removed due to an orientifold projection, which brings in a Klein-bottle amplitude, whose mathematical structure is indeed of a misaligned-form – though the physical interpretation of this closed-string term is just a peculiar orientifold-induced modding-out of the closed-string states –, as well as misalignedly supersymmetric open-string sectors. These observations also appear in ref. [135]. An interesting analysis of the open strings appearing in the type 0'B theory is also in ref. [104].

3.10 Conclusions

In this chapter, misaligned supersymmetry has been shown to be a feature that can characterise non-supersymmetric theories for both closed and open strings and it has been shown to be a useful conceptual tool to explain the finiteness of quantum-corrected observables. An analysis has been performed of the mechanisms by which string theory is capable of giving finite results in the absence of spacetime supersymmetry. Working at one-loop level in perturbation theory, it has been reviewed how this is possible due to modular invariance, which plays a role even when broken by the worldsheet boundaries. Then, such a finiteness has been interpreted as a consequence of cancellations between bosonic and fermionic terms in the full infinite tower of string states. Since the spectrum is not supersymmetric, such cancellations have been named 'misaligned' (or 'asymptotic') supersymmetry in the literature [27, 88], and in fact here it has been shown that the analogy with standard supersymmetric scenarios is indeed accurate.

To start, previous results on misaligned supersymmetry for closed strings, here epitomised by the heterotic SO(16)×SO(16)-theory, have been extended to the open-string case for models in which an anti-Dp-brane is placed on top of an Op-plane. In all cases, misaligned supersymmetry leads to cancellations between bosons and fermions at all different energy levels. Such cancellations are usually visualised by proving that the sector-averaged state degeneracies grow at an exponential rate governed by a coefficient C_{eff} that is smaller than the inverse Hagedorn temperature, i.e. $C_{\text{eff}} < C_{\text{tot}}$. Here, it has been shown that in a large class of theories such a coefficient is actually zero, i.e. $C_{\text{eff}} = 0$. This proves a total cancellation, in the envelope functions necessary to define the sector-averaged degeneracies, that previously has only been conjectured.

Then, given the fact that the formula for the net physical degeneracies is exact, taking advantage of the results for the exact cancellation of the envelope functions, the way in which non-zero results actually arise in physical quantities, such as the one-loop cosmological constant, has been analysed in full mathematical detail. The finite results do arise when performing discrete sums over the states instead of using the interpolating envelope functions. In fact, the improved understanding of all the subleading corrections has been implemented in a mathematically rigorous way to show explicitly how discrete sums over the number of states do indeed lead to a finite non-zero result. In particular, it has been shown that the cancellation of the envelope functions is a sufficient condition to conclude that the exponential high-energy divergences do cancel out. Eventually, the modular properties of string-theory partition functions have been taken advantage of to confirm these results and to describe the leftover power-law terms.

As a complement, an interpretation has been given of supertraces for open strings, relating them to the series coefficients of the function whose integral gives the one-loop cosmological constant. This is reminiscent of the closed-string results of ref. [28]. It would be interesting to examine the formulation of the light-fermion conjecture proposed in ref. [136], which makes use of standard supertraces, in terms of these string-based supertraces for non-supersymmetric models.

Finally, the presence of misaligned supersymmetry in all known 10-dimensional non-supersymmetric tachyon-free string constructions has been discussed. While the heterotic $SO(16)\times SO(16)$-theory and the Sugimoto USp(32)-model, along with the single anti-Dp-brane/Op-plane theory, clearly exhibit misaligned supersymmetry, the type 0'B theory is more interesting. Its closed spectrum is purely bosonic and thus it cannot realise misaligned supersymmetry, strictly speaking. However, the Klein bottle, needed to remove the closed-string tachyon, does exhibit a misalignment. Likewise the open-string annulus and Möbius-strip amplitudes do realise misaligned supersymmetry, as conjectured in ref. [135].

It has been observed that misaligned supersymmetry is present in systems which in principle do not share any common feature. For instance, the heterotic $SO(16)\times SO(16)$-theory is a non-supersymmmetric closed-string model, while anti-Dp-branes on Op-planes, including the Sugimoto model, involve open-string states that spontaneously break the supersymmetry preserved by the type-II closed-string sector they are coupled to. Moreover, while the presence of misaligned supersymmetry in closed strings can be interpreted as a consequence of the underlying modular invariance [27], this reasoning cannot be directly applied to open strings. In the reference open-string theory example, the partition function turns out to be invariant under a congruence subgroup of $PSL_2(\mathbb{Z})$, which is crucial for using a specific version of the Hardy-Ramanujan-Rademacher sum. Certainly, misaligned supersymmetry seems to be a general phenomenon that is capable of explaining in physical terms the reason for which String Theory gives finite answers, even in the absence of supersymmetry. This is an intriguing feature that might help interpreting the hierarchy of scales in the observed world if, formulated in these terms, one were to discover phenomenologically viable string constructions resulting in not just finite but also highly-suppressed loop corrections.

A great deal of attention has been devoted to models with anti-Dp-branes on top of Op-planes, which are examples of brane supersymmetry breaking. The results thus point towards a relation between this scenario and misaligned supersymmetry. Some models of brane supersymmetry breaking can give finite answers thanks to misaligned supersymmetry. Here, it has been shown that, in flat spacetime, the one-loop cosmological constant of these models is finite and such a finiteness can be explained thanks to the presence of misaligned supersymmetry. In addition, it is known that the worldvolume field theory living on an anti-D3-brane on top of an O3-plane is described by non-linear supersymmetry [83, 137]. In this sense, this work also suggests that low-energy effective theories with non-linear supersymmetry of this kind are completed in the high-energy regime into string theories with misaligned supersymmetry. A key observation for this is that the mass scale of the non-linear realisation of supersymmetry is the anti-D3-brane tension, $m \sim \tau_{\mathrm{D3}}^{1/4}$, and similarly this is the scale that characterises the infinite tower of string states that define the realisation of misaligned supersymmetry. It would be nice to relate this to the results of ref. [96]. Note that the anti-D3-brane/O3-plane theory is of particular interest due to its relation to the KKLT- and LVS-constructions.

There are various directions in which this work can be extended. First, it would be compelling to extend the analysis to more realistic setups, including quasi-realistic particle spectra and the compactification to a 4-dimensional space. Here, the main working example has consisted of only a single anti-Dp-brane in a flat 10-dimensional background modded out by an orientifold projection. It would be interesting to analyse deviations from this, including multiple coincident and/or intersecting branes and considering compactified 4-dimensional theories. In these cases, a central question would be the stability of the resulting construction, as discussed for example in ref. [121], and the role of the gauge degrees of freedom in the counting of states. The role played by the Kaluza-Klein towers of states should also be investigated further, to check their misalignment. Similar considerations hold for the heterotic SO(16) × SO(16)-theory and for all other theories exhibiting misaligned supersymmetry, the Sugimoto model and type 0'B theory. In the literature, heterotic string models exhibiting misaligned supersymmetry have appeared for instance in refs. [90, 91, 99–103]; see also refs. [138–140] and refs. [141–143]. Noteworthy constructions involving open strings are for instance refs. [26, 87, 104–107]. Such constructions have been attracting significant attention of late, as evidence for supersymmetry in nature remains elusive. Second, at a more formal level, the analysis of the partition function has been performed only at the one-loop level. Therefore, a natural development would be to understand whether or not higher-order loops introduce significant alterations to the scenario pictured here. For the two-loop level, one can see for example ref. [91]. On the mathematical side, in principle there may exist models that require extensions of the analysis presented here: the proofs of misaligned boson-fermion cancellations may need to be extended by trying to relax the assumptions on the form of the partition functions, such as the parity of the values α denoting successively subleading terms in the HRR-sums, for both open and closed strings, and the fact that cancellations take place individually within each term for open strings and within each right-left product for closed strings, as one may expect cancellations of different terms against each other. Furthermore, recently ref. [129] has discussed the calculation of the one-loop scalar masses in string-theoretic constructions. It would be interesting to analyse the expression of such masses with the tools presented here, in order to see how misaligned supersymmetry acts concretely in observables other than the cosmological constant. Ultimately, one would like to understand the extent to which modular invariance, misaligned supersymmetry and the infinite towers of string states can help with the long-standing hierarchy problems in the cosmological constant and Higgs mass. To conclude, recently a connection between misaligned supersymmetry and swampland conjectures has been pointed out in ref. [144]. It would be interesting to pursue along all these lines of investigation.

4 ANTI-D3-/D7-BRANES IN WARPED THROATS

This chapter presents an instance of a quasi-realistic standard-like model construction employing anti-D3- and D7-branes, going over the material presented in the article [1].

The contents are organised as follows. To start, section 4.1 contextualises the research for standard-like model realisations with anti-D3- and D7-branes. Then, section 4.2 reviews strongly-warped scenarios in type IIB string theory, highlighting the hierarchies among the string, Kaluza-Klein and flux-induced energy scales as well as the conditions for a low-energy supergravity formulation to be valid, with focus on the role of anti-D3-branes. As a helpful example, section 4.3 discusses the supergravity description of models with intersecting D3-/D7-branes in strongly-warped regimes, including possible supersymmetry breaking by fluxes. Then, section 4.4 extends to intersecting anti-D3-/D7-branes models, making use of the tools of constrained superfields, and embeds them into scenarios where the closed-string sector moduli are stabilised. Section 4.5 discusses the supergravity description of quasi-realistic standard-like models on anti-D3-/D7-brane models at orbifold singularities. Finally, a summary of possible mass scales in these setups is provided in section 4.6 and section 4.7 outlines the main conclusions.

4.1 Context

In the string-theory literature, supersymmetry has been a key ingredient of model building and a leading candidate for a solution to the long-standing gauge hierarchy problem [108, 145]. However, the present-day absence of supersymmetric partners in all experimental observations [146], together with the failure of supersymmetry to explain the even bigger cosmological-constant problem, suggests that the nature of supersymmetry breaking has not yet been understood. In view of this, it is worthwhile to study the viability of string-theoretic non-supersymmetric constructions also from a phenomenological standpoint.

The fact that anti-D-branes in type II Calabi-Yau orientifold compactifications [56, 60] spontaneously break supersymmetry has received a great deal of attention [67, 68, 77, 83, 84, 95–98, 147–167] (for earlier analyses, see refs. [18–21, 23, 24, 121]). Together with fluxes, non-perturbative and perturbative effects, whose interplay can address the moduli-stabilisation problem, the positive-definite energy density of anti-D-branes may then also help to obtain a (quasi-)de Sitter vacuum corresponding to the observed universe [54, 55]. Whilst the consistency of these de Sitter constructions is still under debate (for an incomplete list, see refs. [58, 168–218]), the spontaneous breaking of supersymmetry by anti-D-branes means that these objects can be used in string model building whilst maintaining the powerful machinery of supersymmetry.

In more detail, there is a precise identification between the anti-D3-brane action in flat space placed on an orientifold plane and the Volkov-Akulov theory of non-linearly realised supersymmetry [83, 137]. Moreover, all the degrees of freedom on an anti-D3-brane can be described using the tools familiar from linear supergravity by placing the low-energy fields in constrained supermultiplets [77, 84, 96], where the constraints ensure that either only the bosonic or only the fermionic component is an independent dynamical degree of freedom [31,32]. In particular, the anti-D3-brane gaugino plays the role of the goldstino and falls in a nilpotent superfield x. Here, the constraint $x^2 = 0$ fixes the scalar φ^x in terms of the spinor ψ^x and of the auxiliary field F^x, with the F-term being non-zero by assumption. Moreover, the standard non-linear supersymmetry transformation for the goldstino, i.e. $\sqrt{2}\,\delta_\epsilon\lambda \sim \epsilon/l^2$, can be seen after the field redefinition $\lambda \sim \psi^x/(2l^2F^x)$, where l is the scale at which the massive string states come into play.

This progress has made it possible to describe how the anti-D3-brane couples to bulk fields in type IIB Calabi-Yau orientifold flux compactifications, including the closed-string moduli, and to determine the mutual interplay between the closed- and the open-string sectors [67,68,95,97,98,147,149,156,163]. The low-energy effective field theory corresponds to a non-linear supergravity theory, including standard and constrained superfields, with the anti-D3-brane uplift corresponding to an F^x-term contribution to the scalar potential. In particular, ref. [163] has derived the complete action for an anti-D3-brane in the KKLT-scenario by means of constrained superfields, and ref. [68] has considered the coupling of the anti-D3-brane goldstino to the complex-structure modulus controlling the warp factor in a Klebanov-Strassler throat [64]. Non-linear supersymmetry strongly constrains the theory; for example, the well-known non-renormalisation theorems fulfilled by low-energy effective linearly realised supergravities descending from string theory extend to the non-linear supergravity theories [156].

Given the null results thus far in searches for superpartners, the recent insights into anti-D-brane supersymmetry breaking, and the potential importance of the latter in cosmological model building, the material presented in this chapter develops the idea that quasi-realistic particle physics models, with non-standard realisations of supersymmetry, may be obtained using anti-D3-branes. Anti-D3-/D7-brane systems placed at orbifold singularities are known to lead to interesting low-energy particle spectra, comprising non-Abelian gauge groups, adjoint fermions, bifundamental scalars and bifundamental fermions [29, 30, 219–224] (for reviews, see e.g. refs. [225, 226]). Intriguingly, as a consequence of the orbifold projection, the scalars and the fermions from the intersecting $\overline{3}7$- and $7\overline{3}$-sectors fall into distinct bifundamental representations of the gauge groups, and so the low-energy spectrum does not fulfil the usual superpartner pairing. It is therefore interesting to consider such systems at the tip of a strongly-warped throat, which may be dynamically obtained since anti-D3-branes minimize their energy there. Depending on the warping, the volume and the mass-sourcing fluxes, both closed- and open-string sectors may localise either in the highly-redshifted region or in the bulk, and hierarchical mass scales may be explained via geometrical warping [53, 56, 57, 227, 228]. For definiteness, the focus is going to be put on strongly-warped scenarios in which most of the degrees of freedom, from both the closed- and the open-string sectors, tend to localise in the highly-redshifted region of the internal space [228], but all the results could easily be adapted to any model with intersecting anti-D3-/D7-branes. In fact, the strong-warping regime is chosen just because it also allows one to explore further a class of 4-dimensional effective theories that may be phenomenologically relevant in the landscape of string-theory solutions. Eventually, interesting bottom-up particle-physics models may then plausibly be embedded into complete string compactifications [29, 30], with in principle all closed- and open-string moduli stabilised via fluxes, perturbative and non-perturbative effects.

Towards this objective, this work computes the low-energy effective field theory describing an anti-D3-/D7-brane system at an orbifold singularity at the tip of a strongly-warped throat, within a supersymmetric type IIB Calabi-Yau orientifold flux compactification [53, 56, 227–229]. Whilst the closed-string and 77-sector degrees of freedom fulfil a linear supersymmetry, and fall into standard supermultiplets [52, 56, 230–237], the $\overline{3}\overline{3}$- and $\overline{3}7$-/$7\overline{3}$-sector degrees of freedom have non-linear supersymmetry transformations, and fall into constrained supermultiplets [77, 84, 96, 156, 163, 235]. By a dimensional reduction of the bulk and worldvolume actions, and by exploiting how the internal-spacetime symmetries transform the intersecting states (for which no action is known), one can infer the non-linear supergravity action, encapsulated as usual in a Kähler potential, a superpotential, the gauge kinetic functions and the Fayet-Iliopoulos terms. This non-linear supergravity theory allows one to infer the interactions related by supersymmetry, both linear and non-linear, and to work out the consequences of closed-string moduli stabilisation, including perturbative and non-perturbative effects, on the open-string sectors. Previous studies on the supersymmetry-breaking effects of anti-D3-branes in the KKLT-setup have considered the possibility in which the matter sector originates from D3- and D7-branes [98, 238–241], while in this construction the anti-D3-brane sectors provide both the uplift energy and matter.

It is interesting to compare the effective field-theory description of anti-D3-brane supersymmetry breaking with the standard hidden-sector supersymmetry breaking via some non-zero closed-string field F-term. For this purpose, pure anti-D3-brane breaking may be assumed, though in the main text setups with both open- and closed-string breaking active will also be considered. Similarly to the standard procedure, one considers a vacuum that spontaneously breaks supersymmetry via a non-zero auxiliary field F^x and expands the action around this F-term, to obtain a set of soft-breaking terms in the Lagrangian. The anti-D3-/D7-brane systems give rise to several further low-energy fields – beyond the goldstino – which also lie in constrained superfields without physical superpartners and which can acquire soft-breaking terms. As in standard gravity-mediated hidden-sector supersymmetry-breaking scenarios, the scale of the soft-breaking masses is $m_{\text{soft}} \sim f_x/m_P$, where f_x sets the uplift energy of the anti-D3-brane provided by the F^x-term. Whereas, in a standard supersymmetry-breaking scenario, the light fields would fall in constrained superfields below the scale m_{soft}, for the anti-D3-brane, constrained superfields are necessary even above m_{soft}, and there is no scale at which superpartners appear. Instead, the structure that gives the remarkable finiteness properties of string theory is expected to involve the entire spectrum of string states, which appear at the warped string scale for anti-D3-branes at the tip of strongly-warped throats. To contextualise these ideas for anti-D3-/D7-brane constructions in the presence of moduli stabilisation, this work discusses the scales that emerge in theories with anti-D3-/D7-brane systems embedded in a KKLT-like scenario after the interplay between open- and closed-string F-terms.

4.2 Warped IIB Closed-String Sector

Focussing on strongly-warped type IIB compactifications, this section introduces the appropriate 10-dimensional metric, shows the hierarchies between the mass scales and discusses the conditions for well-defined 4-dimensional supergravity formulations.

4.2.1 Warped Type IIB Closed-String Supergravity

In warped type IIB compactifications, the 10-dimensional metric takes the form [227, 229]

$$ds_{10}^2 = \frac{\gamma^{\frac{3}{2}}\, \mathrm{e}^{2\Omega[c]}}{[\mathrm{e}^{-4A}+c]^{\frac{1}{2}}} \left[g_{\mu\nu}\, \mathrm{d}x^\mu \mathrm{d}x^\nu + 2\partial_\mu c\, \partial_m b\, \mathrm{d}x^\mu \mathrm{d}y^m\right] + [\mathrm{e}^{-4A}+c]^{\frac{1}{2}}\, g_{mn}\, \mathrm{d}y^m \mathrm{d}y^n, \quad (4.2.1)$$

where the coordinates x^μ and y^m describe the non-compact 4-dimensional spacetime $X_{1,3}$ and the compact 6-dimensional space Y_6, respectively, $\mathrm{e}^{2\Omega} = \mathrm{e}^{2\Omega[c(x)]}$ is a Weyl-rescaling factor to the 4-dimensional Einstein frame, defined as

$$\mathrm{e}^{2\Omega[c]} = \frac{\displaystyle\int_{Y_6} \mathrm{d}^6 y \sqrt{g_6}}{\displaystyle\int_{Y_6} \mathrm{d}^6 y \sqrt{g_6}\, \left[\mathrm{e}^{-4A}+c\right]}, \quad (4.2.2)$$

γ is an extra arbitrary constant, and $b = b(y)$ is a compensator field needed to solve the Einstein equations [229] but ignored in the following as it is sources only derivative couplings with the open-string excitations. The warp factor $\mathrm{e}^{-4A} = \mathrm{e}^{-4A(y)}$ and the volume-controlling real Kähler modulus $c = c(x)$ combine together into the generalised warp factor

$$\mathrm{e}^{-4A[c(x),y]} = \mathrm{e}^{-4A(y)} + c(x). \quad (4.2.3)$$

From the metric above, the physical internal volume in the Einstein frame is

$$\mathrm{vol}_6 = \int_{Y_6} \mathrm{d}^6 y \sqrt{g_6}\, [\mathrm{e}^{-4A}+c]^{\frac{3}{2}},$$

whilst the dimensionless unwarped and warped internal volumes are defined respectively as

$$l_s^6 \ell_{(0)} = \int_{Y_6} \mathrm{d}^6 y \sqrt{g_6},$$
$$l_s^6 \ell_w = \int_{Y_6} \mathrm{d}^6 y \sqrt{g_6}\, \mathrm{e}^{-4A}.$$

Moreover, the dimensionless physical internal volume is defined as $\mathcal{V} = \mathrm{vol}_6/\ell_{(0)} l_s^6$, in units of the unwarped volume. Given the 10-dimensional gravitational coupling $2\hat{\kappa}_{10}^2 = g_s^2 l_s^8/2\pi$, with the string coupling g_s and the string length l_s, the 4-dimensional reduced Planck length κ_4 turns out to be

$$2\kappa_4^2 = \frac{2\hat{\kappa}_{10}^2}{\gamma^{\frac{3}{2}} l_s^6 \ell_{(0)}} = \frac{g_s^2 l_s^2}{2\pi \gamma^{\frac{3}{2}} \ell_{(0)}}, \tag{4.2.4}$$

with the reduced Planck mass m_P being defined as the inverse $m_P = 1/\kappa_4$. In the large-volume limit, where warping becomes negligible, one can identify the field c as $c = \mathrm{e}^{4u} = \mathcal{V}^{2/3}$ and the Weyl factor as $\mathrm{e}^{2\Omega} = 1/c = \mathrm{e}^{-4u}$, and fixing the constant $\gamma = \langle c \rangle$ ensures that the string and Planck scales are related by the physical internal volume [55, 56].

In a type IIB Calabi-Yau orientifold compactification with Hodge number $h_+^{1,1} = 1$, the 4-dimensional closed-string effective action, involving the axio-dilaton $\tau = C_0 + \mathrm{i}\,\mathrm{e}^{-\phi}$, the complex-structure moduli u^α, with $\alpha = 1, \dots, h_-^{2,1}$, and the Kähler modulus $\rho = \chi + \mathrm{i}c$, can be reproduced by means of a Kähler potential $\hat{K}$ and a superpotential $\hat{W}$ of the form [53, 229, 242]

$$\kappa_4^2 \hat{K} = -\ln\left[-\mathrm{i}(\tau - \overline{\tau})\right] - \ln\left[-\mathrm{i}\int_{Y_6} \mathrm{e}^{-4A}\, \Omega \wedge \overline{\Omega}\right] - 3\ln\left[2\,\mathrm{e}^{-2\Omega}\right] + \ln\left[\frac{2}{\pi}\frac{\ell_w}{[\ell_{(0)}]^3}\right], \tag{4.2.5a}$$

$$\kappa_4^3 \hat{W} = \frac{g_s}{l_s^2}\int_{Y_6} G_3 \wedge \Omega. \tag{4.2.5b}$$

Note that $\mathrm{e}^{-2\Omega} = \mathrm{Im}\,\rho + c_0$, with $c_0 = \ell_w/\ell_{(0)}$, gives a Kähler potential for the volume modulus of the usual no-scale form. In fact, it is immediate to show the identity $\partial_\rho \mathrm{e}^{2\Omega} = \mathrm{i}\,\mathrm{e}^{4\Omega}/2$, thanks to which one finds the derivatives

$$\kappa_4^2 \hat{K}_\rho = \frac{3\mathrm{i}}{2}\,\mathrm{e}^{2\Omega}, \qquad \kappa_4^2 \hat{K}_{\rho\overline{\rho}} = \frac{3}{4}\,\mathrm{e}^{4\Omega},$$

and the Kähler-covariant derivative

$$\kappa_4^3 \nabla_\rho \hat{W} = \frac{3\mathrm{i} g_s}{2 l_s^2}\,\mathrm{e}^{2\Omega}\int_{Y_6} G_3 \wedge \Omega = \frac{3\mathrm{i}}{2}\,\mathrm{e}^{2\Omega}\kappa_4^3 \hat{W}.$$

The no-scale structure is a consequence of the identity $\kappa_4^2\, \hat{K}^{\rho\overline{\rho}} \hat{K}_\rho \hat{K}_{\overline{\rho}} = 3$. The F-term for the Kähler modulus is

$$\hat{F}^\rho = -2\mathrm{i}\,\mathrm{e}^{\kappa_4^2 \hat{K}/2} \mathrm{e}^{-2\Omega}\,\kappa_4^2 \overline{\hat{W}}.$$

Some more details of these results are reviewed in appendix B, including the notation employed later on for approximations. The focus in the current work is on local configurations of intersecting anti-D3-/D7-branes within such warped geometries, and it will be assumed throughout that the global configuration of fluxes, branes and O-planes within the Calabi-Yau orientifold compactifications considered satisfy the RR-tadpole cancellation conditions necessary for overall consistency.

4.2.2 Field Localisation and 4-dimensional Supergravity Conditions

In the presence of a highly-warped throat, there can be non-trivial localisation effects for the closed-string sector fields; further, there are interesting hierarchies between mass scales in the bulk and in the redshifted region. These scenarios are studied in detail by ref. [228] and, because they are relevant in the model-building setups considered in this work, a review of their main features is provided below. For brevity, the normalisation $\ell_{(0)} = 1$ is assumed in the rest of the subsection.

4.2.2.1 Closed-String Sector Field Localisation

As a guiding example for the closed-string sector fields with a flux-induced mass, one can study the behaviour of the axio-dilaton τ in a strongly-warped compactification. The linearised field equation for the axio-dilaton wavefunction, labelled as $\tau_w = \tau_w(y)$, takes the form [227, 243]

$$-\frac{\mathrm{e}^{2\Omega}}{[\mathrm{e}^{-4A}+c]}\, K_6\, \tau_w(y) + \frac{m_4^2}{\gamma^{\frac{3}{2}}}\, \tau_w(y) = \frac{1}{12\,\mathrm{Im}\,\tau}\, \frac{\mathrm{e}^{2\Omega}}{[\mathrm{e}^{-4A}+c]^2}\, G_{mnp}\overline{G}^{mnp}\, \tau_w(y).$$

In this expression, m_4^2 represents the 4-dimensional mass of the axio-dilaton $\tau = \tau(x)$. The term involving the 3-form product $G_3 \cdot \overline{G}_3$ constitutes the flux-induced mass, whereas the differential operator $K_6 = -[\partial_m(\sqrt{g}\, g^{mn}\partial_n)]/\sqrt{g} + [\mathrm{e}^{-4A}+c]^{1/2} m_{10}^2$ sources the Kaluza-Klein tower of states and accounts for further reduced 10-dimensional mass contributions m_{10}^2. By estimating these terms, one can qualitatively understand non-trivial localisation effects. Without loss of generality, it will be assumed that integrals are dominated by the bulk; this only changes the behaviour of the Weyl factor $\mathrm{e}^{2\Omega}$, but since the estimates are just going to be used for relative comparisons, this is actually irrelevant.

- In the bulk, the unwarped metric g_{mn} is of order one and the 3-form flux is of the order of its quantisation integer n_f, that is $G_{mnp} \sim n_f/l_s$. The background warp factor is negligible compared to the volume modulus, that is

  $$\mathrm{e}^{-4A} \ll c \simeq \mathcal{V}^{\frac{2}{3}}.$$

  Following these estimates, the order of magnitude of the flux-induced moduli masses in the bulk is

  $$m_{\text{flux}}^2 = \frac{\gamma^{\frac{3}{2}}}{12}\, \frac{\mathrm{e}^{2\Omega}}{[\mathrm{e}^{-4A}+c]^2}\, G_{mnp}\overline{G}^{mnp} \sim \frac{n_f^2}{\mathcal{V}^2}\, \frac{\gamma^{\frac{3}{2}}}{l_s^2} \sim \frac{g_s^2 n_f^2}{\mathcal{V}^2}\, \frac{1}{\kappa_4^2}. \tag{4.2.6}$$

  Also, given the characteristic length scale of the bulk λ as measured in terms of the unwarped metric g_{mn} (with $\lambda^6 \sim \ell_{(0)}$ in general), the bulk Kaluza-Klein scale is

  $$m_{\text{KK}}^2 \sim \frac{\mathrm{e}^{2\Omega}}{[\mathrm{e}^{-4A}+c]}\, \frac{\gamma^{\frac{3}{2}}}{\lambda^2 l_s^2} \sim \frac{1}{\lambda^2 \mathcal{V}^{\frac{4}{3}}}\, \frac{\gamma^{\frac{3}{2}}}{l_s^2} \sim \frac{g_s^2}{\lambda^2 \mathcal{V}^{\frac{4}{3}}}\, \frac{1}{\kappa_4^2}. \tag{4.2.7}$$

  Finally, one can observe that the (reduced) string mass is $m_s^2 = g_s^2 m_P^2/4\pi\gamma^{3/2} = 1/l_s^2$, so the bulk string scale must be defined as

  $$m_s^2 = \frac{g_s^2}{4\pi\mathcal{V}}\, \frac{1}{\kappa_4^2}. \tag{4.2.8}$$

- At the tip of a highly-warped throat, where $\mathrm{e}^{-4A} \gg c$, the scenario changes drastically. Let n_f^0 be the order of magnitude the 3-form flux quanta therein, such as with a Klebanov-Strassler throat threaded by M units of F_3-flux on the 3-sphere and K units of H_3-flux on the dual 3-cycle of the deformed conifold. In the vicinity of the would-be conifold singularity, the 10-dimensional Einstein-frame metric takes the form [56, 64]

$$ds_{10}^2 = \mathrm{e}^{2A_0}\, \breve{g}_{\mu\nu}\, \mathrm{d}x^\mu \mathrm{d}x^\nu + r_0^2 \left[\frac{1}{2}\,\mathrm{d}\xi^2 + \mathrm{d}\Omega_3^2 + \frac{1}{4}\,\xi^2\, \mathrm{d}\Omega_2^2\right],$$

where ξ is the radial coordinate of the deformed conifold, the tip being located at $\xi = 0$, while the other line elements describe the 3- and 2-sphere of the conifold base, and r_0 is the radius of the 3-sphere at the tip of the throat, such that $r_0^2 \sim n_f^0$.[4.1] This indicates that the internal metric at the tip of the throat has the behaviour

$$g_{mn}^0 \sim n_f^0\, \mathrm{e}^{2A_0}, \tag{4.2.9}$$

where A_0 is the warp factor at the tip of the throat, with the 3-form flux scaling as $G_{mnp}^0 \sim n_f^0/l_s$. In this way, the characteristic scale of the closed-string sector flux-induced mass evaluated at the tip of the throat is

$$(m_{\text{flux}}^w)^2 = \frac{\gamma^{\frac{3}{2}}}{12}\, \mathrm{e}^{2\Omega+8A_0}\, G_{mnp}^0 \overline{G}_0^{mnp} \sim \frac{\mathrm{e}^{2A_0}}{n_f^0 \mathcal{V}^{\frac{2}{3}}} \frac{\gamma^{\frac{3}{2}}}{l_s^2} \sim \frac{g_s^2}{n_f^0 \mathcal{V}^{\frac{2}{3}}} \frac{1}{\kappa_4^2}\, \mathrm{e}^{2A_0}. \tag{4.2.10}$$

On the other hand, according to the definition of the metric, the generic throat Kaluza-Klein scale is

$$(m_{\text{KK}}^w)^2 \sim \frac{\mathrm{e}^{2\Omega+4A_0}}{\lambda_0^2} \frac{\gamma^{\frac{3}{2}}}{l_s^2} \sim \frac{\mathrm{e}^{2A_0}}{n_f^0 \chi^2 \mathcal{V}^{\frac{2}{3}}} \frac{\gamma^{\frac{3}{2}}}{l_s^2} \sim \frac{g_s^2}{n_f^0 \chi^2 \mathcal{V}^{\frac{2}{3}}} \frac{1}{\kappa_4^2}\, \mathrm{e}^{2A_0}, \tag{4.2.11}$$

where the length scale of a cycle at the tip of the throat, measured by g_{mn}^0, has been written as $\lambda_0^2 \sim n_f^0\, \mathrm{e}^{2A_0}\, \chi^2$, with χ a parameter independent of the warp factor. To conclude, one may also infer that the warped string scale can be defined as

$$(m_s^w)^2 = \frac{g_s^2}{4\pi \mathcal{V}^{\frac{2}{3}}} \frac{1}{\kappa_4^2}\, \mathrm{e}^{2A_0}. \tag{4.2.12}$$

The factor controlling the size of the throat is preferably taken to be $\chi > 1$, so that the warped Kaluza-Klein scale is smaller than the warped string scale.

In particular, if the warped mass of eq. (4.2.10) is smaller than the bulk mass of eq. (4.2.6), then it is energetically favourable for the closed-string sector fields to be mostly localised at the tip of the throat. Roughly, the condition for this to happen is therefore

$$\frac{\mathcal{V}^{\frac{2}{3}}}{n_f (n_f^0)^{\frac{1}{2}}} \lesssim \mathrm{e}^{-A_0}. \tag{4.2.13}$$

[4.1] Adapting from ref. [64], it is being assumed that the scalings are, roughly, of the form $g_{mn}^0 \sim e^{2A_0} g_s M$ and $G_{mnp} \sim (g_s M + K)/l_s$. One can thus set $g_s M \sim K \sim n_f^0$. Any more refined condition can be implemented easily.

Notably, the warped flux-induced and warped Kaluza-Klein scales m^w_{flux} and m^w_{KK} are comparable. Because the cutoff for the 4-dimensional effective theory has to be at most the warped Kaluza-Klein scale, most of the degrees of freedom from the closed-string sector fall above the 4-dimensional threshold. Fields surviving the cutoff include the Kähler volume modulus, which does not have a flux-induced mass, and potentially some complex-structure moduli associated to the geometry at the infrared end of the throat.

4.2.2.2 Conditions for a 4-dimensional Supergravity Formulation

Whilst below the warped Kaluza-Klein scale the effective theory is 4-dimensional, an $N_4 = 1$ supergravity formulation is not always possible. In fact, in the presence of supersymmetry breaking, the gravitino gauging the broken supersymmetry becomes massive and may happen to be localised by warping in the infrared end of the throat. In this case, it would have stronger couplings than those of a simple 4-dimensional supergravity description, since they would be suppressed by the warped Planck scale $m_P^w = \mathrm{e}^{A_0} m_P$ rather than by the Planck scale m_P governing the graviton interactions [228]. This will now be discussed in more detail, beginning with supersymmetry breaking by fluxes, and followed by comments on supersymmetry breaking with an anti-D3-brane.

The 4-dimensional gravitino corresponding to the least broken supersymmetry (i.e. broken at the smallest scale) is identified with the lightest Kaluza-Klein mode, which becomes massless as the supersymmetry-breaking parameter is taken to zero. Taking this 4-dimensional gravitino ψ_μ to be embedded in the 10-dimensional gravitino as $\Psi_\mu(x,y) = \psi_\mu(x) \otimes \eta(y)$, the qualitative behaviour of the gravitino wavefunction η in the extra dimensions can be determined from the 10-dimensional gravitino field equation, which implies a flux-induced mass for ψ_μ that is of order

$$\frac{m_{3/2}}{\gamma^{\frac{3}{4}}} \sim \frac{\mathrm{e}^{\Omega}}{[\mathrm{e}^{-4A} + c]} \, G_{mnp} \gamma^{mnp},$$

where γ^m are the Dirac matrices representing the Clifford algebra $\{\gamma^m, \gamma^n\} = 2g^{mn}$ and G_{mnp} is the supersymmetry-breaking 3-from flux. Similarly to the case of the axio-dilaton described above, this mass gives rise to two possible scales across the internal manifold:

(i) a 3-form flux of order $G_{mnp} \sim n_f \theta / l_s$ in the bulk generates a gravitino mass

$$m_{3/2} \sim \frac{\mathrm{e}^{\Omega} \gamma^{\frac{3}{4}}}{[\mathrm{e}^{-4A} + c]} \, G_{mnp} \gamma^{mnp} \sim \frac{n_f \theta}{\mathcal{V}} \, \frac{\gamma^{\frac{3}{4}}}{l_s} \sim \frac{g_s n_f \theta}{\mathcal{V}} \, \frac{1}{\kappa_4}; \tag{4.2.14}$$

(ii) a 3-form flux of order $G_{mnp} \sim n_f^0 \theta_0 / l_s$ in the throat generates a gravitino mass

$$m^w_{3/2} \sim \mathrm{e}^{\Omega + 4A_0} \, G^0_{mnp} \gamma_0^{mnp} \gamma^{\frac{3}{4}} \sim \frac{\theta_0 \, \mathrm{e}^{A_0}}{(n_f^0)^{\frac{1}{2}} \mathcal{V}^{\frac{1}{3}}} \, \frac{\gamma^{\frac{3}{4}}}{l_s} \sim \frac{g_s \theta_0}{(n_f^0)^{\frac{1}{2}} \mathcal{V}^{\frac{1}{3}}} \, \frac{1}{\kappa_4} \, \mathrm{e}^{A_0}. \tag{4.2.15}$$

The numbers θ and θ_0 represent the fact that the fluxes do not necessarily break supersymmetry, so the magnitude of the breaking can vary. The scales in eqs. (4.2.14, 4.2.15) are also the expected orders of magnitude of the mass splittings among the fields of any supermultiplet, depending on where the fields are localised. For supersymmetry-breaking flux parameters such that $m^w_{3/2} \ll m_{3/2}$, which is the naive expectation from eq. (4.2.13), it is energetically favourable for the lightest gravitino to localise at the infrared end of the throat. Its interactions are then suppressed in terms of warped scales, in contrast to the Planck-suppressed graviton interactions, making a standard supergravity description difficult. However, when the flux parameters satisfy

$$\frac{\theta}{\theta_0} \ll \frac{\mathrm{e}^{A_0}\mathcal{V}^{\frac{2}{3}}}{n_f (n_f^0)^{\frac{1}{2}}}, \tag{4.2.16}$$

which is fulfilled in particular as $\theta \to 0$, the gravitino mass scales in eqs. (4.2.14, 4.2.15) are such that $m_{3/2} \ll m^w_{3/2}$. In this case, the 4-dimensional gravitino does not localise in the throat, allowing it to have standard m_P-suppressed interactions. Nevertheless, the actual gravitino mass is still warped-down, that is $\hat{m}^w_{3/2} = \mathrm{e}^{A_0} m_{3/2}$, as the warp factor governs the physical scales at the tip of the throat, including the scale of supersymmetry breaking [228].

This is the framework considered in this work and, in it, it is thus sensible to formulate an $N_4 = 1$ supergravity theory below a cutoff scale set as the warped Kaluza-Klein scale m^w_{KK} if the supergravity condition in eq. (4.2.16) holds, in the regime set by the localisation condition in eq. (4.2.13). In particular, one can reproduce the supergravity description of a highly-warped theory by means of a Kähler potential with the structure

$$\kappa_4^2 \mathcal{K} = 2A_0 + \kappa_4^2 K, \tag{4.2.17}$$

where K is the Kähler potential that one would define in the absence of the extremely strong warping effects discussed above and A_0 is the warp factor at the tip of the throat, with the superpotential W (and the gauge kinetic functions f_{AB}) unchanged.[4.2] Indeed, such a formulation manifestly provides redshifted energy scales and, in particular, all the 4-dimensional masses are warped down. This includes the warped-down gravitino mass, $\hat{m}^w_{3/2} = \mathrm{e}^{A_0} m_{3/2}$, where the redshift is induced by the $2A_0$-shift and the unwarped mass is $m_{3/2} = \mathrm{e}^{\kappa_4^2 K/2} W$, as given by eq. (4.2.14). From now on, $\mathcal{F}^M = \mathrm{e}^{A_0} F^M$ and $\mathcal{V}_F = \mathrm{e}^{2A_0} V_F$ also denote the F-terms and the F-term potentials associated to a highly-warped scenario, respectively.

To summarise, some fields are localised in the bulk region, like the graviton and the gravitino, while others are localised at the tip of the warped throat, like the Kähler modulus (see discussion below) and possible open-string states. These provide the degrees of freedom for the standard-like model realisations of interest in this work. In more detail, one can distinguish:

- fields that are localised at the tip of the throat, which can have redshifted mass scales and be part of the low-energy effective theory, including the Kähler modulus and local open-string states, or be heavier than the low-energy theory cutoff scale, like the axio-dilaton and possible complex-structure moduli;

[4.2] In general, purely closed-string contributions to K, W and f_{AB} are then independent of A_0, but note that the open-string terms (or local geometric closed-string moduli terms) may have a dependence on A_0 if they are located in a region of strong warping.

- fields that are localised in the bulk, which typically have masses above the cutoff scale (like bulk complex-structure moduli) and/or highly suppressed couplings with the throat-localised degrees of freedom (like bulk branes, which could provide massless degrees of freedom), and therefore can be neglected.

In ref. [228], this discussion refers to the spontaneous supersymmetry breaking by fluxes. In this work, supersymmetry breaking by anti-D3-branes at the tip of a throat is also considered. Although the way in which anti-D3-branes break supersymmetry is conceptually different to flux supersymmetry breaking, the arguments on the localisation of the gravitino in the bulk follow through in the same way, for small enough bulk gravitino mass-sourcing fluxes. Hence, the following sections show how to incorporate open-string degrees of freedom in a description with the $2A_0$-shift in the Kähler potential as in eq. (4.2.17).

Kähler Modulus Localisation

In KKLT-like constructions, in which the Kähler modulus is stabilised by non-perturbative effects such as D7-brane gaugino condensates [244–247] or Euclidean D3-brane instantons [248], the Kähler potential shift in eq. (4.2.17) implies that the scalar potential sourced by non-perturbative effects is redshifted by the warp factor, even though the non-perturbative effects are not necessarily localised near the throat.

To understand this redshifting, one should consider the localisation of the Kähler modulus ρ. The field ρ is massless before the compactification, so naively one expects it to be not localised. However, an explicit analysis is performed in ref. [229] and reveals that:

(i) the wavefunction of the 4-dimensional graviton $g_{\mu\nu}$ is strongly peaked in the bulk region, both in the presence and in the absence of strong warping;

(ii) the wavefunction of the Kähler modulus ρ tends to be more and more peaked in the throat as the warping becomes stronger.

Notice that even with non-perturbative effects, the Kähler modulus is very light and well below the warped KK-scale cutoff, suggesting that its wavefunction is perturbed only slightly and in particular that it is still peaked in the throat. Then, ρ should feel any non-perturbative effects localised in the bulk via a redshifted mediation to the tip of the throat. Consistently with this picture, one can observe that with a warped-down non-perturbative contribution to the scalar potential, the stronger the warping is – i.e. the longer the throat is – the less efficient the stabilisation becomes. Another challenge is that the supersymmetry-breaking $(0,3)$-flux tends to localise around the gaugino condensate [61, 249], which in some cases could result in the gravitino localising at the throat tip, making a supergravity description difficult.

4.2.3 Note on Coupling Estimates

For definiteness, when providing estimates of the orders of magnitude of the field couplings, all the results are always going to be referred to the case where integrations over the internal space are dominated by the bulk region. Numerical order-1 constants are also going to be ignored for the estimates, unless explicitly stated. In particular, unless explicitly re-inserted, the flux integers n_f and n_f^0 are going to be dropped in the remaining sections because, in the qualitative analysis of the volume- and warp-factor-dependences of the mass scales, they are irrelevant in fixing the reference orders of magnitude.

4.3 Warped D3- and D7-Branes

This section considers D3- and D7-branes in strongly-warped Calabi-Yau orientifold compactifications, as a warm-up before the anti-D3-/D7-brane constructions. As D3-/D7-brane systems preserve the same $N_4 = 1$ supersymmetry as the closed-string sector, the only sources of supersymmetry breaking considered here are $(0,3)$-fluxes. An $N_4 = 1$ supergravity description can be derived by matching with the operators that are obtained from the dimensional reduction. As discussed in section 2.3, the low-energy degrees of freedom from the branes, here composing the matter sector, are the following.

- D3-branes contain three complex scalars φ^a parametrising the position of the brane in the internal space and three spinors ψ^a in an SU(3)-triplet with respect to the internal tangent space group, which form three chiral multiplets, as well as one Abelian gauge vector A_μ and a spinor λ in an SU(3)-singlet, which form a vector multiplet.

- D7-branes wrapping a 4-cycle in the internal space contain one complex scalar σ^3 parametrising the position of the brane in the internal space and a spinor η^3, which together form a chiral multiplet, as well as one Abelian gauge vector B_μ and a spinor ζ, which form a vector multiplet. Extra degrees of freedom associated to the Wilson lines are absent if the wrapped cycle has no non-contractible 1-cycles.

- When D3- and D7-branes overlap, the intersecting 37- and 73-states correspond to two complex scalars φ and φ_* and two spinors ψ and ψ_*, which form two chiral multiplets in conjugate representations of the gauge groups. Specifically, the chiral multiplets φ and φ_* have charges $q_{\mathrm{D3}} = +1, -1$ and $q_{\mathrm{D7}} = -1, +1$, respectively, under the D3- and D7-brane U(1)-gauge groups.

A summary of the supergravity expansions for models with matter and supersymmetry-breaking hidden sectors, the latter including bulk moduli, is given in appendix C.2. In the following subsections, the specific form of these interactions is going be derived from the dimensional reduction of D3-/D7-brane actions in warped flux compactifications and the intersecting states are also going to be discussed. The total Kähler potential and the total superpotential are going to be found to take the form

$$\begin{aligned}\mathcal{K} = \frac{2A_0}{\kappa_4^2} + \hat{K} + Z_{\sigma^3\overline{\sigma}^3}\sigma^3\overline{\sigma}^3 + \frac{1}{2}\left[H_{\sigma^3\sigma^3}\sigma^3\sigma^3 + \text{c.c.}\right] \\ + Z_{\varphi^a\overline{\varphi}^b}\varphi^a\overline{\varphi}^b + Z_{\varphi\overline{\varphi}}\varphi\overline{\varphi} + Z_{\varphi_*\overline{\varphi}_*}\varphi\overline{\varphi}_*,\end{aligned} \tag{4.3.1a}$$

$$W = \hat{W} + \frac{1}{2}\tilde{\mu}_{\sigma^3\sigma^3}\sigma^3\sigma^3 + \tilde{y}(\beta\sigma^3 - \varphi^3)\,\varphi\varphi_*, \tag{4.3.1b}$$

where $\hat{K}$ and $\hat{W}$ are the pure closed-string potentials of eqs. (4.2.5a, 4.2.5b) and all the other terms represent the open-string couplings. The gauge kinetic functions, the D-term potentials and – in the case of supersymmetry-breaking fluxes – the soft-breaking terms are also going to be worked out. The $2A_0$-shift will be inserted if working under the conditions of eqs. (4.2.13, 4.2.16), with all masses being redshifted by the warp factor.

The details of the open-string sector terms depend on the brane configuration, with two main constructions considered. The D3-brane is going to be placed at the tip of a highly-warped throat, whereas the D7-brane will wrap a 4-cycle either located at the tip of the throat or extending from the tip into the bulk. Detailed global constructions are deferred for future work and, when explicit, the wrapped 4-cycle will be assumed to be a toroidal orbifold for simplicity (throats with such cycles have been constructed e.g. in ref. [30]). Unless otherwise stated, only a pure $(2,1)$-flux is going to be assumed to exist at the tip of the throat. The dimensional reduction is not going to capture the complex-structure moduli couplings, but the supergravity formulation will correctly account for them. Stabilisation of the volume modulus ρ is going to be considered in subsection 4.4.3 for the main case of interest, which is the presence of KKLT-like non-perturbative corrections and uplifting anti-D3-branes. Further, notice that worldvolume fluxes will not be considered in this work.

4.3.1 Pure D3- and D7-Brane States

This subsection overviews the analysis of D3- and D7-branes in type IIB Calabi-Yau orientifold compactifications, adapting it to the strongly-warped metric of eq. (4.2.1). In the following, superscripts and subscripts '0' denote quantities evaluated at the tip of the throat.

4.3.1.1 Warped D3-Branes

As discussed in appendix B.2, it is possible to express the action of the D3-brane degrees of freedom by adapting the results of the dimensional reductions from refs. [80, 149, 231, 232].

4.3.1.1.1 D3-Brane Chiral Superfields

In the 4-dimensional Einstein frame, the pure kinetic action for the D3-brane scalars takes the form (see also refs. [228, 250, 251], which work directly in the regime of strong warping)

$$S_{\mathrm{kin}}^{\text{D3-scalars}} = -\frac{1}{2\pi g_s}\int_{X_{1,3}} \mathrm{d}^4x\,\sqrt{-g_4}\,\mathrm{e}^{2\Omega}\,g^0_{a\bar{b}}\,g^{\mu\nu}\,\nabla_\mu\varphi^a\nabla_\nu\overline{\varphi}^b.$$

Therefore, one can include this term within the Kähler potential of the Kähler modulus as

$$\kappa_4^2 K = -3\ln\left[2\,\mathrm{e}^{-2\Omega} - \frac{\kappa_4^2}{3\pi g_s}\,g^0_{a\bar{b}}\,\varphi^a\overline{\varphi}^b\right].$$

This logarithmic no-scale structure, with K of the form $K = -3\log\left[f_{\mathrm{hid}}(\rho,\overline{\rho}) + f_{\mathrm{vis}}(\varphi,\overline{\varphi})\right]$, is a common feature of D-brane supergravity and suggests the possibility of sequestering [53, 252] (see also ref. [232]). It implies that the brane scalars do not feel hidden-sector supersymmetry breaking at tree-level, and it turns out that brane fermions also stay massless at tree-level. From the expression above, it follows that the Kähler matter metric is

$$Z_{\varphi^a\overline{\varphi}^b} = \frac{1}{2\pi g_s}\, \mathrm{e}^{2\Omega}\, g^0_{a\bar{b}}. \tag{4.3.2}$$

Due to supersymmetry, the D3-brane modulini are also captured by these couplings. Since the chiral multiplet φ^a is massless in an imaginary self-dual flux background, this Kähler potential is enough to account for the D3-brane chiral field couplings.

As discussed in subsection 4.2.2, for a low-energy effective field theory describing fields at the tip of a highly-warped throat, the Kähler potential is shifted by the constant $2A_0$. This clearly does not change the Kähler matter metric for the D3-brane fields.

4.3.1.1.2 D3-Brane Gauge Sector

The Weyl scaling from the 10- to the 4-dimensional Einstein frame does not affect the D3-brane gauge kinetic terms in the action, so one has

$$S^{\text{D3-vector}}_{\text{kin}} = -\frac{1}{4\pi g_s}\int_{X_{1,3}} \mathrm{e}^{-\phi}\, F_2 \wedge *F_2 + \frac{1}{4\pi g_s}\int_{X_{1,3}} C_0\, F_2 \wedge F_2$$

and the gauge kinetic function is as usual

$$f_{\text{D3}} = -\frac{\mathrm{i}\tau}{2\pi g_s}. \tag{4.3.3}$$

This does not depend on the warp factor due to the cancellation happening in the metric-dependent factors. The dimensional reduction of the gaugino is not performed as the action can be reproduced by supersymmetry arguments.

4.3.1.2 D7-Branes Extending from the Tip of a Warped Throat into the Bulk

This subsubsection describes a D7-brane wrapping a 4-cycle Σ_4 that extends from the tip of a warped throat up into the bulk region. Details of the dimensional reduction of the D7-brane worldvolume action can be found in refs. [235, 236, 253] (see also refs. [233, 234, 237]) and are briefly overviewed in appendix B.2. A toy model is described below, including the geometric configuration and the corresponding dimensional reduction. This is not a fully realistic construction, but it gives a simple way to capture the relevant orders of magnitude in the mass scales that are sought after. In particular, the warp factor is assumed to be only a function of the directions parallel to the 4-cycle.

4.3.1.2.1 D7-Brane Configuration and Field Localisation Conditions

It is assumed that the internal space, locally in the neighbourhood of the wrapped D7-brane, takes the form $\Sigma_4 \ltimes \Sigma_2$. Let the coordinates $y^{m'}$ span the 4-space Σ_4, for $m' = 4, \dots, 7$, with z^1, z^2 the corresponding complexified directions, and let the coordinates $y^{\dot{m}}$, for $\dot{m} = 8, 9$, parametrise the transverse 2-space Σ_2, with z^3 the associated complex coordinate. Given some convenient coordinates $r^{m'} = r^{m'}(y^{n'})$ and $\theta^{\dot{m}} = \theta^{\dot{m}}(y^{\dot{n}})$, the metric is of the form

$$ds_6^2 = \mathrm{e}^{-2A}\, g_{mn} \mathrm{d}y^m \mathrm{d}y^n = \mathrm{e}^{-2A(r)} \left(g_{m'n'}(r)\, \mathrm{d}y^{m'} \mathrm{d}y^{n'} + g_{3\bar{3}}(r,\theta)\, \mathrm{d}z^3 \mathrm{d}\bar{z}^3 \right).$$

At some $r^2 = r_{m'} r^{m'} = r_{\mathrm{UV}}^2$, the bulk is glued to a warped throat, which ends at its tip with a tiny warp factor $\mathrm{e}^{2A}(r=0) = \mathrm{e}^{2A_0}$. The D7-brane wraps the slice corresponding to the coordinates $\theta = 0$. See fig. 4.1 for a schematic represention of the geometry under consideration.

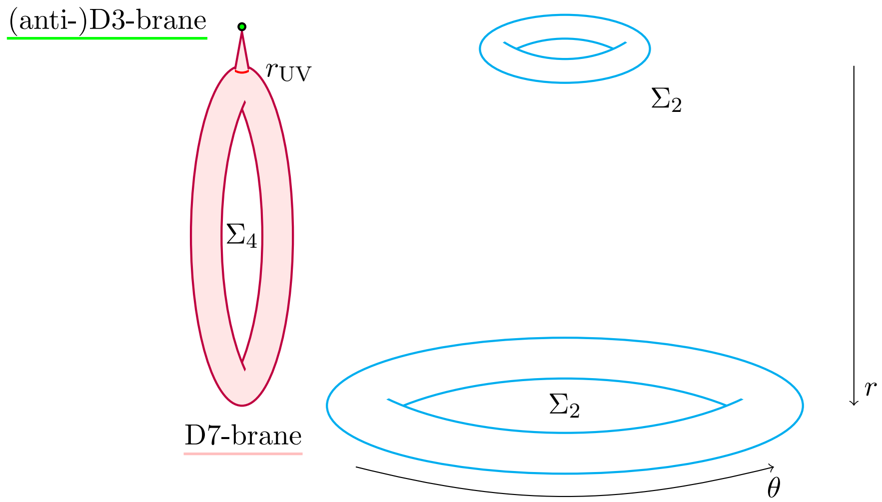

Figure 4.1: A sketch of the toy configuration under consideration, with the D7-brane wrapping the 4-space at $\theta = 0$ and some throat being glued to the bulk at $r = r_{\mathrm{UV}}$. The D3- or anti-D3-brane provides extra open-string states, as discussed in sections 4.3 and 4.4, respectively.

In order to be able to perform explicit calculations, the warp factor is assumed to be a function of only the 4-space coordinates. Further, the 4-cycle is assumed to be the orbifold $\Sigma_4 = \mathrm{T}^4/\mathbb{Z}_2$ and locally the orthogonal directions correspond to the 2-torus T^2, i.e. the metric is such that

$$g_{m'n'}(r \in \Sigma_4) = g_{m'n'}^{\mathrm{T}^4/\mathbb{Z}_2}, \qquad g_{3\bar{3}}(r \in \Sigma_4, \theta = 0) = g_{3\bar{3}}^{\mathrm{T}^2}. \tag{4.3.4}$$

Finally, in analogy with the Klebanov-Strassler throat, it is assumed that at the throat tip the metric scales with the constant e^{2A_0}, as in eq. (4.2.9), that is

$$g_{m'n'}(r < r_{\mathrm{UV}}) \overset{r\sim 0}{\simeq} \mathrm{e}^{2A_0}, \qquad g_{3\bar{3}}(r < r_{\mathrm{UV}}, \theta = 0) \overset{r\sim 0}{\simeq} \mathrm{e}^{2A_0}. \tag{4.3.5}$$

Localisation Scenarios

In analogy with what happens for the closed-string sector, one might guess that the open-string moduli of the wrapped D7-brane can become localised at the tip of the throat too. The conditions under which this occurs will now be worked out.

One can analyse the internal wavefunction of the D7-brane scalar fields by dimensionally reducing the real fields $\sigma^{\dot{m}} = \sigma^{\dot{m}}(x, y)$, with $\dot{m} = 8, 9$, in a similar way to refs. [253, 254].

The D7-brane 8-dimensional scalar action can be written in terms of the 4-dimensional Einstein frame metric as

$$S_{\mathrm{D7}}^{\mathrm{scalar}} = -\tau_{\mathrm{D7}}\sigma_s^2 \int_{X_{1,3}} \mathrm{d}^4x\,\sqrt{-g_4}\int_{\Sigma_4}\mathrm{d}^4y\,\sqrt{g_{\Sigma_4}}\,\Big[\mathrm{e}^{2\Omega+\phi}\,[\mathrm{e}^{-4A}+c]\,g_{\dot r\dot s}\,g^{\mu\nu}\,\nabla_\mu\sigma^{\dot r}\nabla_\nu\sigma^{\dot s}$$
$$+\gamma^{\frac{3}{2}}\,\mathrm{e}^{4\Omega+\phi}\,g_{\dot r\dot s}\,g^{m'n'}\nabla_{m'}\sigma^{\dot r}\nabla_{n'}\sigma^{\dot s}+\frac{1}{2}\,\frac{\gamma^{\frac{3}{2}}\,\mathrm{e}^{4\Omega+2\phi}}{\mathrm{e}^{-4A}+c}\,G_{m'n'\dot r}\overline{G}^{m'n'}{}_{\dot s}\,\sigma^{\dot r}\sigma^{\dot s}\Big],$$

where it is understood that only some of the 3-form fluxes contribute, as determined by the interference of the DBI- and CS-actions [235]. For constant Kähler-modulus and axio-dilaton backgrounds, one finds the field equation

$$\gamma^{-\frac{3}{2}}\,\Delta_4\sigma^{\dot r}+\frac{\mathrm{e}^{2\Omega}}{[\mathrm{e}^{-4A}+c]}\Delta_{\Sigma_4}\sigma^{\dot r}+\frac{1}{2\,\mathrm{Im}\,\tau}\,\frac{\mathrm{e}^{2\Omega}}{[\mathrm{e}^{-4A}+c]^2}\,G^{m'n'\dot r}\overline{G}_{m'n'\dot s}\,\sigma^{\dot s}=0.$$

Then, defining the Kaluza-Klein decomposition of the field as

$$\sigma^{\dot r}(x,y)=\sum_\omega \sigma_\omega^{\dot r}(x)\,\zeta_\omega^{\dot r}(y)$$

and imposing the Klein-Gordon equations $\Delta_4\sigma_\omega^{\dot r} = -m_\omega^2\sigma_\omega^{\dot r}$, one eventually obtains the internal wavefunction field equation

$$-\frac{\mathrm{e}^{2\Omega}}{[\mathrm{e}^{-4A}+c]}\Delta_{\Sigma_4}\zeta_\omega^{\dot r}+\frac{m_\omega^2}{\gamma^{\frac{3}{2}}}\,\zeta_\omega^{\dot r}=\frac{1}{2\,\mathrm{Im}\,\tau}\,\frac{\mathrm{e}^{2\Omega}}{[\mathrm{e}^{-4A}+c]^2}\,G^{m'n'\dot r}\overline{G}_{m'n'\dot s}\,\zeta_\omega^{\dot s}.$$

This internal wave equation gives the same 4-dimensional mass contributions as the equation defining the axio-dilaton wavefunction, with the only difference that here the wavefunction is 4- rather than 6-dimensional. Notice that the axio-dilaton equation also includes generic 10-dimensional mass terms that here have been neglected for simplicity.

Following subsection 4.2.2, the compactification volume can be sufficiently large that warped-down masses are still greater than bulk masses (see eq. (4.2.13)), and fields tend to localise in the bulk. In any case, the D7-brane chiral superfield is localised near the tip of the throat whenever the warped-down mass m_{D7}^w is smaller than the unwarped bulk mass m_{D7}, that is if

$$\frac{\mathrm{e}^{A_0}\mathcal{V}^{\frac{2}{3}}}{n_f\,(n_f^0)^{\frac{1}{2}}}\lesssim\frac{\theta'}{\theta_0'}, \tag{4.3.6}$$

where the fluxes sourcing the D7-brane field masses have been taken to be $G_{mnp}\sim\theta' n_f/l_s$ in the bulk and $G_{mnp}\sim\theta_0' n_f^0/l_s$ near the tip. For generic flux parameters, θ' and θ_0', the warped mass is of the same order as the warped flux-induced axio-dilaton mass m_{flux}^w of eq. (4.2.10) and the warped Kaluza-Klein scale m_{KK}^w of eq. (4.2.11), i.e.

$$(m_{\mathrm{D7}}^w)^2\sim\frac{g_s^2{\theta_0'}^2}{n_f^0\mathcal{V}^{\frac{2}{3}}}\,\frac{1}{\kappa_4^2}\,\mathrm{e}^{2A_0},$$

so that these fields are too heavy to stay in the low-energy theory. However, if θ_0' is small enough, it may be that fluxes sourcing the D7-brane masses allow both $m_{\mathrm{D7}}^w\lesssim m_{\mathrm{D7}}$, so fields are localised, and also $m^w_{\mathrm{D7}} \ll m^w_{\mathrm{KK}}$, so fields stay in the low-energy theory. It may also happen that θ' is small enough that the hierarchy is $m^w_{\mathrm{D7}} \gtrsim m_{\mathrm{D7}}$, so it is energetically favourable for the D7-brane fields to be localised in the bulk, and yet the mass is warped-down in the effective field theory, analogously to what happens to the gravitino. Such possible scenarios are now going to be discussed in detail.

4.3.1.2.2 D7-Brane Chiral Superfield in the Bulk

For large enough internal volumes that do not satisfy the localisation condition of eq. (4.3.6), $m_{\mathrm{D7}} \lesssim m^w_{\mathrm{D7}}$ and D7-brane fields generally extend along the throat from the tip into the bulk. Before the compactification over the wrapped 4-cycle, the kinetic term for the D7-brane transverse complexified scalar σ^3 reads

$$S^{\text{D7-scalar}}_{\text{kin}} = -\frac{1}{2\pi g_s l_s^4} \int_{X_{1,3}} \mathrm{d}^4 x \sqrt{-g_4} \int_{\Sigma_4} \mathrm{d}^4 y \sqrt{g_{\Sigma_4}}\, [\mathrm{e}^{-4A} + c]\, \mathrm{e}^{2\Omega+\phi}\, g_{3\overline{3}}\, g^{\mu\nu}\, \nabla_\mu \sigma^3 \nabla_\nu \overline{\sigma}^3.$$

Since the warp factor varies only longitudinally with respect to the brane, one can respectively define the dimensionless unwarped and warped 4-dimensional volumes as

$$\begin{aligned} l_s^4 \ell^{\Sigma_4}_{(0)} &= \int_{\Sigma_4} \mathrm{d}^4 y \sqrt{g_{\Sigma_4}}, \\ l_s^4 \ell^{\Sigma_4}_{w} &= \int_{\Sigma_4} \mathrm{d}^4 y \sqrt{g_{\Sigma_4}}\, \mathrm{e}^{-4A}. \end{aligned}$$

In particular, the internal metric, being that of a torus, is independent of the 4-cycle coordinates and, following the definition of the Weyl factor in eq. (4.2.2), it is apparent that the kinetic term becomes

$$S^{\text{D7-scalar}}_{\text{kin}} = -\frac{\ell^{\Sigma_4}_{(0)}}{2\pi g_s} \int_{X_{1,3}} \mathrm{d}^4 x \sqrt{-g_4}\, \mathrm{e}^{\phi}\, g_{3\overline{3}}\, g^{\mu\nu}\, \nabla_\mu \sigma^3 \nabla_\nu \overline{\sigma}^3.$$

One can reproduce this within the type IIB supergravity action by modifying the axio-dilaton Kähler potential as

$$\kappa_4^2 K = -\ln \left[-\mathrm{i}(\tau - \overline{\tau}) - \frac{\kappa_4^2}{\pi g_s} \ell^{\Sigma_4}_{(0)} g_{3\overline{3}}\, \sigma^3 \overline{\sigma}^3 \right],$$

or equivalently by defining the Kähler matter metric

$$Z_{\sigma^3\overline{\sigma}^3} = \frac{\ell^{\Sigma_4}_{(0)}}{\pi g_s} \frac{g_{3\overline{3}}}{[-\mathrm{i}(\tau - \overline{\tau})]}. \tag{4.3.7}$$

As far as the mass term is concerned, one can again proceed by engineering a way to reproduce the 4-dimensional mass obtained via dimensional reduction in supergravity. In real notation, one finds an action of the form[4.3]

$$S^{\text{D7-scalar}}_{\text{mass}} = -\frac{1}{2\pi g_s l_s^4} \int_{X_{1,3}} \mathrm{d}^4 x \sqrt{-g_4} \int_{\Sigma_4} \mathrm{d}^4 y \sqrt{g_{\Sigma_4}}\, \frac{1}{8\pi \ell_{(0)}} \frac{g_s^2}{\kappa_4^2} \frac{\mathrm{e}^{4\Omega+2\phi}}{\mathrm{e}^{-4A} + c}\, l_s^2\, G_{m'n'\dot{r}}\, \overline{G}^{m'n'}{}_{\dot{s}}\, \sigma^{\dot{r}} \sigma^{\dot{s}}.$$

[4.3] Notice that the overall volume factor is not related to any non-locality. It appears only by inserting a factor $1 = l_s^2/l_s^2$ in the action to have the appropriate scalings for 4- and 6-dimensional operators and expressing one string-length factor l_s in terms of the Planck length κ_4. In fact, in these dimensional reductions one finds a volume factor $\ell_{(0)}$ for each factor κ_4^2.

D7-branes have a supersymmetric mass sourced by a $(2,1)$-flux. In the toy model under consideration, in the vicinity of the brane it is possible to decompose forms in the 6-dimensional space into products of forms in the 4- and 2-dimensional spaces $\Sigma_4 = \mathrm{T}^4/\mathbb{Z}_2$ and T^2, respectively. In particular, the specific mass-sourcing $(2,1)$-flux can be written as [235] (the hat denotes the specific component)

$$\hat{G}_3(r,0) = f(r,0)\,\chi_\vartheta,$$

where the $(2,1)$-form $\chi_\vartheta = \eta \wedge \mathrm{d}\overline{w}^3$ is defined in terms of the $(2,0)$-form $\eta = \mathrm{d}w^1 \wedge \mathrm{d}w^2$ of the 4-cycle and $\mathrm{d}\overline{w}^3$, with $w^a = z^a/l_s$ a dimensionless coordinate, and $f = f(r,\theta)$ is a function representing the near-brane dependences. For definiteness, let the integrals be dominated by the throat region, where $\mathrm{e}^{-4A} \gg \langle c \rangle$. As $\mathrm{e}^{4A}\hat{G}_3$ is a harmonic form, one can express the 2-form component $g_2 = f(r,\theta=0)\,\eta$ in terms of the harmonic $(2,0)$-form η as

$$\mathrm{e}^{4A} g_2 = \frac{1}{\omega_w^{\Sigma_4}}\,\eta \int_{\Sigma_4} g_2 \wedge \overline{\eta},$$

with $\omega_w^{\Sigma_4} = \int_{\Sigma_4} \mathrm{e}^{-4A}\eta \wedge \overline{\eta}$. Appendix B.3 provides useful supplementary details. Now, starting from the general action above, the supersymmetric mass term can be expressed as

$$S^{\text{D7-scalar}}_{\text{mass}} = -\frac{1}{2\pi g_s}\int_{X_{1,3}} \mathrm{d}^4x\sqrt{-g_4}\int_{\Sigma_4} \mathrm{d}^2w\,\mathrm{d}^2\overline{w}\,\sqrt{g_{\Sigma_4}}\,\frac{1}{8\pi\ell_{(0)}}\frac{g_s^2}{\kappa_4^2}\,\mathrm{e}^{4\Omega+4A+2\phi}\,l_s^2(g_2\cdot\overline{g}_2)\,\sigma^3\overline{\sigma}^3.$$

Notice that, to respect the dimensionality of G_3, i.e. $G_3 \sim l_s^2$, with $G_{mnp} \sim l_s^{-1}$, the 2-form is defined as $g_2 = l_s\,g_{a'b'}\,\mathrm{d}z^{a'} \wedge \mathrm{d}z^{b'}$, with $g_{a'b'} \sim l_s^{-1}$. Of course the l_s-factors just come from considering dimensionless coordinates. Now, because g_2 is automatically self-dual, i.e. $*_4 g_2 = g_2$, the 4-cycle integral is

$$\int_{\Sigma_4} \mathrm{d}^2w\,\mathrm{d}^2\overline{w}\sqrt{g_{\Sigma_4}}\,\mathrm{e}^{4A}\,g_2\cdot\overline{g}_2 = \frac{1}{l_s^6}\int_{\Sigma_4} \mathrm{e}^{4A}\,g_2\wedge\overline{g}_2 = \frac{1}{(\omega_w^{\Sigma_4})^2}\frac{1}{l_s^6}\int_{\Sigma_4} \mathrm{e}^{-4A}\,\eta\wedge\overline{\eta}\int_{\Sigma_4} g_2\wedge\overline{\eta}\int_{\Sigma_4}\overline{g}_2\wedge\eta.$$

The first integral factor can be written as

$$\lambda_{\Sigma_4} = \int_{\Sigma_4} \mathrm{e}^{-4A}\,\eta\wedge\overline{\eta} = \omega_w^{\Sigma_4} \simeq \omega_w^{\Sigma_4}\frac{\ell_w^{\Sigma_4}}{\ell_{(0)}^{\Sigma_4}}\mathrm{e}^{2\Omega},$$

where an approximate unit factor has been introduced in the final relation for convenience in the comparison of the dimensionally reduced action with the supergravity. In the end the scalar mass term becomes

$$S^{\text{D7-scalar}}_{\text{mass}} = -\frac{1}{2\pi g_s}\int_{X_{1,3}} \mathrm{d}^4x\sqrt{-g_4}\,\frac{1}{8\pi\ell_{(0)}}\frac{g_s^2}{\kappa_4^2}\frac{\mathrm{e}^{6\Omega+2\phi}}{\omega_w^{\Sigma_4}}\frac{\ell_w^{\Sigma_4}}{\ell_{(0)}^{\Sigma_4}}\frac{1}{l_s^4}\int_{\Sigma_4}\overline{g}_2\wedge\overline{\eta}\int_{\Sigma_4} g_2\wedge\eta\,\sigma^3\overline{\sigma}^3.$$

The opposite approximation to that used above, where integrals are dominated by the bulk region, can be obtained easily by taking formally $\mathrm{e}^{4A} = 1$ everywhere, and $\mathrm{e}^{2\Omega} = 1/c$.

In view of ref. [237], to generate the $(2,1)$-flux-induced mass one introduces the holomorphic superpotential bilinear coupling

$$\begin{aligned}\tilde{\mu}_{\sigma^3\sigma^3} &= -\frac{\ell_{(0)}}{\pi}\,\frac{1}{\kappa_4 l_s^2}\partial_\tau\partial_{u^\vartheta}\int_{Y_6}\Big[G_3\wedge\Omega\Big]\,\delta^{(2)}(\theta)\\ &= \Bigg[\frac{\ell_{(0)}}{\pi[-\mathrm{i}(\tau-\overline{\tau})]\kappa_4 l_s^2}\int_{Y_6}(G_3-\overline{G}_3)\wedge\left(\frac{\mathrm{i}}{\omega_w}\left(\partial_{u^\vartheta}\omega_w\right)\Omega-\chi_{u^\vartheta}\right)\delta^{(2)}(\theta)\Bigg],\end{aligned} \tag{4.3.8}$$

where use has been made of the identity $\partial_{u^\alpha}\Omega = [\partial_{u^\alpha}\ln\omega_w]\,\Omega + \mathrm{i}\chi_\alpha$. Dirac δ-functions on a compact space I are normalised as $\int_I \mathrm{d}x\,\delta(x) = l$, where l is the length dimension of x. Indeed, in the specific case in which the background is pure $(2,1)$-flux, this is (for the calculation of the mass, notice that $g_{3\overline{3}} = 1$)

$$\left[\tilde{\mu}_{\sigma^3\sigma^3}\right]_{(2,1)} = \left[\frac{\ell_{(0)}}{\pi\,[-\mathrm{i}(\tau-\overline{\tau})]\kappa_4 l_s^2}\int_{\Sigma_4}\overline{g}_2\wedge\eta\right]\delta_{33}.$$

As required, the effective coupling $\mu_{\sigma^3\sigma^3} = \mathrm{e}^{\kappa_4^2\hat{K}/2}[\tilde{\mu}_{\sigma^3\sigma^3}]_{(2,1)}$, reproduces a supersymmetric mass $m^2_{\sigma^3\overline{\sigma}^3} = Z^{\sigma^3\overline{\sigma}^3}\mu_{\sigma^3\sigma^3}\overline{\mu}_{\overline{\sigma}^3\overline{\sigma}^3}$ that corresponds precisely to the one inferred from the dimensional reduction. The identification takes place if $\mathrm{e}^{\kappa_4^2\hat{K}_{\mathrm{cs}}}\ell_w = \ell_w^{\Sigma_4}/\omega_w^{\Sigma_4}$, otherwise the bilinear coupling $\tilde{\mu}_{\sigma^3\sigma^3}$ should be rescaled by an order-1 factor $(\ell_w/\omega_w)^{-1/2}\left(\ell_w^{\Sigma_4}/\omega_w^{\Sigma_4}\right)^{1/2}$, in which the apparent non-holomorphicity is expected to cancel. For the canonically normalised field, one recognises the expected scale

$$m^2_{\mathrm{D7}} \sim \frac{g_s^2}{\mathcal{V}^2}\frac{1}{\kappa_4^2}.$$

As will be seen from all the dimensional reductions, all the couplings of the theory have 4-dimensional scales which are defined in terms of the reduced Planck length with, depending on the interactions, various suppressions from the string coupling, the volume and/or the warp factor, while the string length factors precisely account for the integrations over the compact space.

Comment on Generic Flux Backgrounds

For a generic flux background, one can again take advantage of the results of refs. [235, 237] and a similar dimensional reduction follows as above: one obtains the same supersymmetric mass just found, plus some soft-breaking scalar mass terms.

In detail, ref. [237] considers unwarped toroidal orbifold compactifications and shows that all these terms can be generated by the holomorphic coupling $\tilde{\mu}_{\sigma^3\sigma^3}$ of eq. (4.3.8) and a non-vanishing Kähler-potential H-term, which, together with the axio-dilaton and complex-structure moduli F-terms, give the same effective μ-coupling as above, generated only by (2,1)-fluxes, along with the soft-breaking terms (see eqs. (C.2.2a, C.2.5a, C.2.5b)).

For less isotropic scenarios, where for instance only the wrapped cycle is a toroidal orbifold $O_4 = \mathrm{T}^4/\mathbb{Z}_2$, some difficulties may arise. The complex-structure moduli Kähler potential should include $\kappa_4^2\hat{K}(u,\overline{u}) = -\ln\omega_w$, with $\omega_w = \mathrm{i}\,\ell_w^{\Sigma_4}\ell_{(0)}^{\mathrm{T}^2}\prod_{a=1}^{3}[-\mathrm{i}(u^a-\overline{u}^a)]$, where $u^3 = u^\vartheta$ is the modulus associated to the $(2,1)$-form χ^ϑ, and the H-coupling should be

$$H_{\sigma^3\sigma^3} = -\frac{1}{\pi g_s}\,\frac{\ell_{(0)}^{\Sigma_4}}{[-\mathrm{i}(\tau-\overline{\tau})][-\mathrm{i}(u^3-\overline{u}^3)]}\,\delta_{33}. \tag{4.3.9}$$

The interplay between the various terms in eq. (C.2.2a) can take place here only if the closed-string sector terms are all properly factorised, singling out integrations over the 4-cycle. This is true only if the 3-form flux is constant over the whole transverse space. Similar considerations hold for the soft-breaking masses of eq. (C.2.5a). The B-term also follows from eq. (C.2.5b). This reasoning has only considered purely flux-induced F-terms; the effects of moduli-stabilisation effects are going to be discussed in KKLT-like stabilisation scenarios with anti-D3-brane supersymmetry breaking in section 4.4.

4.3.1.2.3 Strongly-Warped Throats with D7-Brane Chiral Superfield at Tip

If the internal volume is sufficiently small as to satisfy the condition of eq. (4.2.13) and in particular the D7-brane mass flux parameters satisfy eq. (4.3.6), i.e. $m^w_{\mathrm{D7}} \lesssim m_{\mathrm{D7}}$, the D7-brane chiral superfield field localises at the tip of the throat.

One can impose the localisation of the D7-brane scalar at the level of the dimensional reduction by means of a delta-function that accompanies the superfield σ^3, meaning the substitution $\sigma^3\overline{\sigma}^3 \to \delta^{(4)}(y-y_0)\,\sigma^3\overline{\sigma}^3$. Adapting the previous results (in particular, notice that the integration over the 4-cycle gives a factor e^{2A_0} originating from the metric, which at the tip scales as $g_{m'n'} \sim \mathrm{e}^{2A_0}$), one finds the action

$$\begin{aligned} S_{\text{D7-scalar}} = &- \frac{1}{2\pi g_s}\int_{X_{1,3}} \mathrm{d}^4x\,\sqrt{-g_4}\;\mathrm{e}^{2\Omega+2A_0+\phi}\,g^{\mu\nu}\,\nabla_\mu\sigma^3\nabla_\nu\overline{\sigma}^3 \\ &- \frac{1}{2\pi g_s}\int_{X_{1,3}} \mathrm{d}^4x\,\sqrt{-g_4}\;\frac{1}{8\pi\ell_{(0)}}\,\frac{g_s^2}{\kappa_4^2}\,\mathrm{e}^{4\Omega+8A_0+2\phi}\,l_s^2(g_2^0\cdot\overline{g}_2^0)\,\sigma^3\overline{\sigma}^3. \end{aligned}$$

Again, this analysis manifestly neglects the dependence on the complex-structure moduli. The 2-form g_2^0 is the component of the mass-sourcing flux precisely at the tip of the throat, with $G_3^0 = g_2^0\wedge\mathrm{d}w^3$. It is convenient to absorb the warp factors into the scalar $\dot{\sigma}^3 = \mathrm{e}^{A_0}\sigma^3$, for which the kinetic action becomes

$$\begin{aligned} S_{\text{D7-scalar}} = -\int_{X_{1,3}} \mathrm{d}^4x\,\sqrt{-g_4}\,\Bigg[&\frac{1}{\pi g_s}\,\frac{\mathrm{e}^{2\Omega}}{[-\mathrm{i}(\tau-\overline{\tau})]}\,g^{\mu\nu}\,\nabla_\mu\dot{\sigma}^3\nabla_\nu\dot{\overline{\sigma}}^3 \\ &+\frac{g_s}{4\pi^2\ell_{(0)}}\,\frac{\mathrm{e}^{4\Omega+6A_0}}{[-\mathrm{i}(\tau-\overline{\tau})]^2}\,l_s^2(g_2^0\cdot\overline{g}_2^0)\,\frac{1}{\kappa_4^2}\,\dot{\sigma}^3\dot{\overline{\sigma}}^3\Bigg]. \end{aligned}$$

The action can be reproduced by means of the Kähler matter metric

$$Z_{\dot{\sigma}^3\dot{\overline{\sigma}}^3} = \frac{1}{\pi g_s}\,\frac{\mathrm{e}^{2\Omega}}{[-\mathrm{i}(\tau-\overline{\tau})]} \tag{4.3.10}$$

and, in the presence of only $(2,1)$-flux at the tip, the superpotential bilinear coupling

$$\big[\tilde{\mu}_{\dot{\sigma}^3\dot{\sigma}^3}\big]_{(2,1)} = \frac{\ell_{(0)}\,l_s\overline{g}_{12}^0}{\pi[-\mathrm{i}(\tau-\overline{\tau})]\kappa_4}. \tag{4.3.11}$$

Notice that the bilinear coupling is holomorphic since it can be seen to arise from the GVW-superpotential deformation

$$\delta W = -\frac{\ell_{(0)}}{2\pi}\,\frac{1}{\kappa_4 l_s^2}\,\partial_\tau\partial_{u^\vartheta}\int_{Y_6}\Big[G_3\wedge\Omega\Big]\,\delta^{(4)}(r)\,\delta^{(2)}(\theta)\,\dot{\sigma}^3\dot{\sigma}^3 \equiv \frac{1}{2}\,\tilde{\mu}_{\sigma^3\sigma^3}\dot{\sigma}^3\dot{\sigma}^3.$$

This reproduces the mass term when the total Kähler potential contains the $2A_0$-shift, namely when the theory is formulated as in eq. (4.2.17). Note that the supergravity calculation does provides an extra factor ℓ_w/ω_w, which accounts for the ignorance on the complex-structure moduli above. As expected, the canonically normalised mass reads

$$(m^w_{\mathrm{D7}})^2 \sim \frac{g_s^2}{\mathcal{V}^{\frac{2}{3}}}\,\frac{1}{\kappa_4^2}\,\mathrm{e}^{2A_0}.$$

The structure in the Kähler- and super-potential couplings for the D7-brane chiral superfields here is identical to the case in which the D7-brane wraps a 4-cycle localised at the tip of the throat, as is going to be discussed in subsubsection 4.3.1.3, after replacing the flux evaluated at the warped end of the 4-cycle with the integral of the flux in the 4-cycle at the tip. Therefore, the case discussed above will not be treated separately in the following.

4.3.1.2.4 Strongly-Warped Scenarios with D7-Brane fields in the Bulk

An interesting scenario arises in the presence of fluxes at the tip of the throat that would give a warped-down mass for the D7-brane fields, m_{D7}^{w}, that is still heavier than flux-induced masses in the bulk, m_{D7}. In this case, the D7-brane fields minimise their energy by localising in the bulk, so the D7-brane couplings are those in eqs. (4.3.7, 4.3.8). However, as discussed above, strongly-warped scenarios fulfilling eq. (4.2.13), which allow a supergravity description thanks to eq. (4.2.16), have a Kähler potential with the structure in eq. (4.2.17). So, similarly to what happens with the gravitino when eq. (4.2.16) is satisfied, in the 4-dimensional effective theory the canonically normalised D7-brane scalar mass scales as

$$\mathrm{e}^{A_0}\, m_{\mathrm{D7}} \sim \frac{\theta' g_s}{\mathcal{V}}\, \frac{1}{\kappa_4} \mathrm{e}^{A_0}.$$

4.3.1.2.5 D7-Brane Gauge Sector

From the DBI-action of a stack of D7-branes one can observe the kinetic action for the 4-dimensional gauge field to be

$$S_{\mathrm{kin}}^{\text{D7-vector}} = -\frac{1}{8\pi g_s l_s^4} \int_{X_{1,3}} \mathrm{d}^4x\, \sqrt{-g_4} \int_{\Sigma_4} \mathrm{d}^4y\, \sqrt{g_{\Sigma_4}}\, [\mathrm{e}^{-4A} + c]\, g^{\mu\rho} g^{\nu\sigma}\, F_{\mu\nu} F_{\rho\sigma}.$$

It is thus possible to recognise the inverse of the Weyl factor and write

$$S_{\mathrm{kin}}^{\text{D7-vector}} = -\frac{\ell_{(0)}^{\Sigma_4}}{8\pi g_s} \int_{X_{1,3}} \mathrm{d}^4x\, \sqrt{-g_4}\, \mathrm{e}^{-2\Omega}\, g^{\mu\rho} g^{\nu\sigma}\, F_{\mu\nu} F_{\rho\sigma},$$

so that from the Yang-Mills coupling condition

$$\frac{4\pi}{g_{\mathrm{YM}}^2} = \operatorname{Im} \tau_{\mathrm{YM}} = \frac{1}{g_s}\, \mathrm{e}^{-2\Omega} \ell_{(0)}^{\Sigma_4} = \frac{1}{g_s} \left[-\frac{\mathrm{i}}{2} \left(\rho - \overline{\rho}\right) + \frac{\ell_w^{\Sigma_4}}{\ell_{(0)}^{\Sigma_4}} \right] \ell_{(0)}^{\Sigma_4},$$

together with holomorphicity, one concludes that the gauge kinetic function has to be

$$f_{\mathrm{D7}} = -\frac{\mathrm{i}\ell_{(0)}^{\Sigma_4}}{2\pi g_s} \left[\rho + \mathrm{i} c_0'\right], \tag{4.3.12}$$

with the constant $c_0' = \ell_w^{\Sigma_4}/\ell_{(0)}^{\Sigma_4}$. With a volume facorisation such that $c_0 = c_0'$, this preserves exactly the usual structure of D7-brane gauge couplings, provided the inclusion of the shift suggested by ref. [229]. In the limit where integrals are dominated by the bulk region, the gauge-kinetic function becomes $f_{\mathrm{D7}} = -\mathrm{i}\ell_{(0)}^{\Sigma_4}\rho/2\pi g_s$.

It would be interesting to study localisation effects such as those that can take place in the chiral sector. The gaugino soft-breaking mass is provided by $(0,3)$-fluxes, following eq. (C.2.7). Meanwhile, similar mechanisms seem to be prevented for the gauge field, since the vectors do not have flux-induced masses.

4.3.1.3 D7-Branes at the Tip of Warped Throats

This subsubsection describes the dimensional reduction and the supergravity formulation of a D7-brane wrapping a 4-cycle Σ_4 at the tip of a warped throat, assuming that the warp factor varies only transversally with respect to the brane. A toy model is described below, including the geometric configuration and the corresponding dimensional reduction.

4.3.1.3.1 D7-Brane Configuration

Let the internal 6-dimensional space in the vicinity of the D7-brane wrapped at the tip of the warped throat take the form $\Sigma_4 \rtimes \Sigma_2$. Let the coordinates $y^{m'}$ span a 4-space, for $m' = 4, \ldots, 7$, with z^1, z^2 their complexified version, and let $y^{\dot{m}}$ parametrise the transverse 2-space, for $\dot{m} = 8, 9$, with z^3 the corresponding complex direction. Given some convenient coordinates $\psi^{m'} = \psi^{m'}(y^{n'})$ and $r^{\dot{m}} = r^{\dot{m}}(y^{\dot{n}})$ for the 4- and 2-dimensional spaces, respectively, the internal metric near the throat tip is

$$ds_6^2 = \mathrm{e}^{-2A}\, g_{mn} \mathrm{d}y^m \mathrm{d}y^n = \mathrm{e}^{-2A(r)} \left(g_{m'n'}(\psi, r)\, \mathrm{d}y^{m'} \mathrm{d}y^{n'} + g_{3\overline{3}}(r)\, \mathrm{d}z^3 \mathrm{d}\overline{z}^3 \right).$$

The D7-brane is assumed to wrap the 4-dimensional slice corresponding to the position $r = 0$ at the tip and this 4-space is assumed to see a warp factor which ends up at the tiny value $\mathrm{e}^{2A}(r = 0) = \mathrm{e}^{2A_0}$. The warped throat is glued to some conformal Calabi-Yau orientifold representing the bulk at $r^2 = r_{\dot{m}} r^{\dot{m}} = r_{\mathrm{UV}}^2$, for some r_{UV}. See fig. 4.2 for a schematic represention of the geometry under consideration.

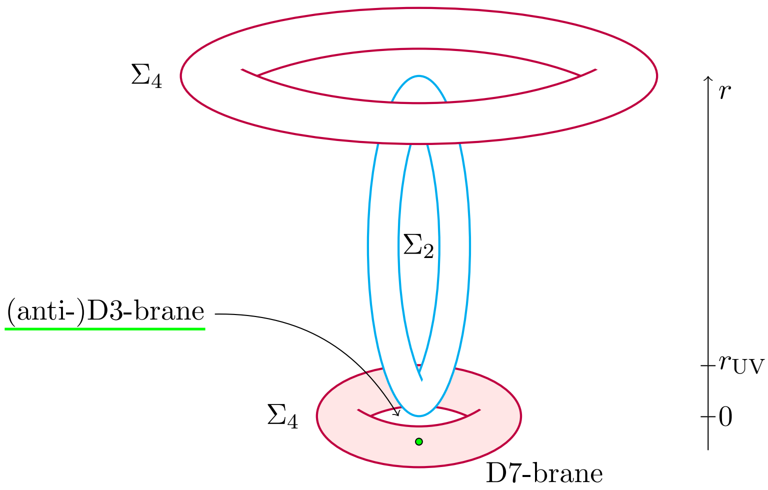

Figure 4.2: A sketch of the toy configuration under consideration, with the D7-brane wrapping the 4-space at $r = 0$. The D3- or anti-D3-brane provides extra open-string states, as discussed in sections 4.3 and 4.4, respectively.

To make calculations explicit, it will be assumed that the metric at the tip of the throat corresponds to the geometry $(\mathrm{T}^4/\mathbb{Z}_2) \times \mathrm{T}^2$. Moreover, in analogy with the KS-metric at the throat tip in eq. (4.2.9), an overall scaling with the constant e^{2A_0} is assumed, giving

$$g_{m'n'}(\psi, r < r_{\mathrm{UV}}) \overset{r \sim 0}{\simeq} g_{m'n'}^{(\mathrm{T}^4/\mathbb{Z}_2)}\, \mathrm{e}^{2A_0}, \qquad g_{3\overline{3}}(r < r_{\mathrm{UV}}) \overset{r \sim 0}{\simeq} g_{3\overline{3}}^{(\mathrm{T}^2)}\, \mathrm{e}^{2A_0}. \tag{4.3.13}$$

4.3.1.3.2 D7-Brane Chiral Superfield

If the D7-brane wraps a 4-cycle which is entirely localised at the tip of the warped throat, then the metric of the 4-cycle needs to be evaluated at that point in the transverse space. In view of the strong-warping condition $e^{-4A_0} \gg c$, the kinetic term for the D7-brane scalar field takes the form

$$S_{\text{kin}}^{\text{D7-scalar}} = -\frac{1}{2\pi g_s l_s^4} \int_{X_{1,3}} d^4x \sqrt{-g_4} \int_{\Sigma_4} d^4y \sqrt{g_{\Sigma_4}^0}\, e^{2\Omega-4A_0+\phi}\, g_{3\overline{3}}^0\, g^{\mu\nu}\, \nabla_\mu \sigma^3 \nabla_\nu \overline{\sigma}^3.$$

Because in the current setup neither the warp factor nor the internal metric depend on the 4-cycle coordinates, one can easily observe that such an action reads

$$S_{\text{kin}}^{\text{D7-scalar}} = -\frac{\ell_4^0}{2\pi g_s} \int_{X_{1,3}} d^4x \sqrt{-g_4}\, e^{2\Omega-4A_0+\phi}\, g_{3\overline{3}}^0\, g^{\mu\nu}\, \nabla_\mu \sigma^3 \nabla_\nu \overline{\sigma}^3,$$

where the 4-cycle dimensionless unwarped volume at the tip of the throat is defined as

$$\ell_4^0 = \frac{1}{l_s^4} \int_{\Sigma_4} d^4y \sqrt{g_{\Sigma_4}^0} \sim e^{4A_0}.$$

In the end, the Kähler matter metric has to be

$$Z_{\sigma^3\overline{\sigma}^3} = \frac{1}{\pi g_s} \frac{e^{2\Omega-4A_0}}{[-i(\tau-\overline{\tau})]}\, \ell_4^0\, g_{3\overline{3}}^0.$$

Notably, the D7-brane scalar Kähler matter metric shows two distinct features now that the D7-brane lies at the strongly-warped throat-tip rather than extending along the throat:

- a dependence on the warp factor, which is reasonable because the whole D7-brane is localised at strong warping;
- a dependence on the Kähler modulus, which means the D7-brane fields are sequestered and effectively very similar to a D3-brane localised at the tip of the throat.

Also notice that the matter metric has the effective volume and warp factor scaling $Z_{\sigma^3\overline{\sigma}^3} \sim e^{2\Omega+2A_0}/g_s$, in accord with the result of ref. [228], following the scaling of the metric $g_{3\overline{3}}^0$ and, correspondingly, of the volume of the 4-cycle at the tip of the throat ℓ_4^0.

Again, the total mass term emerges from the interference of the DBI- and CS-actions, but for the purposes of determining the suppression factors one can simply focus on e.g. the DBI-action, which, in real notation, is of the form

$$S_{\text{mass}}^{\text{D7-scalar}} = -\frac{1}{2\pi g_s} \int_{X_{1,3}} d^4x \sqrt{-g_4} \int_{\Sigma_4} d^4y \sqrt{g_{\Sigma_4}^0}\, \frac{g_s^2\, e^{4\Omega+4A_0+2\phi}}{8\pi \ell_{(0)} \kappa_4^2 l_s^2}\, G_{m'n'\dot{r}}^0\, \overline{G}_{p'q'\dot{s}}^0\, g_0^{m'p'} g_0^{n'q'}\, \sigma^{\dot{r}} \sigma^{\dot{s}}.$$

As the theory at the tip of the throat sees a constant warp factor, one can expand the harmonic mass-sourcing $(2,1)$-flux easily. The supersymmetric mass-sourcing $(2,1)$-flux is still proportional to the harmonic form $\chi_\vartheta = \eta \wedge d\overline{w}^3$, with η the holomorphic $(2,0)$-form of the space $T^4/\mathbb{Z}_2$, and can be written as [235]

$$\hat{G}_3^0 = f(0)\chi_\vartheta,$$

where $f = f(r)$ is a function of the transverse direction (again, the hat denotes the component of the flux that sources the mass term). In terms of the 2-form component, which can be identified as $g_2^0 = f\,\eta$, the expansion thus reads

$$g_2^0 = \frac{1}{\omega_{(0)}^{\Sigma_4}}\,\eta \int_{\Sigma_4} g_2^0 \wedge \overline{\eta},$$

where $\omega_{(0)}^{\Sigma_4} = \int_{\Sigma_4} \eta \wedge \overline{\eta}$. The mass term can thus be expressed as

$$S_{\text{mass}}^{\text{D7-scalar}} = -\frac{1}{2\pi g_s} \int\limits_{X_{1,3}} \mathrm{d}^4x\sqrt{-g_4} \int\limits_{\Sigma_4} \mathrm{d}^4y \sqrt{g_{\Sigma_4}^0}\, \frac{1}{8\pi\ell_{(0)}} \frac{g_s^2}{\kappa_4^2 l_s^2}\, \mathrm{e}^{4\Omega+4A_0+2\phi}\, g_2^0 \cdot \overline{g}_2^0\, \sigma^3\overline{\sigma}^3.$$

It turns out that the 4-cycle integral can be performed straightforwardly and reads

$$\int_{\Sigma_4} \mathrm{d}^4y \sqrt{g_{\Sigma_4}^0}\, g_2^0 \cdot \overline{g}_2^0 = \frac{1}{l_s^2} \int_{\Sigma_4} g_2 \wedge \overline{g}_2 = \frac{1}{\omega_{(0)}^{\Sigma_4}} \frac{1}{l_s^2} \int_{\Sigma_4} g_2^0 \wedge \overline{\eta} \int_{\Sigma_4} \overline{g}_2^0 \wedge \eta,$$

so the scalar mass term is simply

$$S_{\text{mass}}^{\text{D7-scalar}} = -\frac{1}{2\pi g_s} \int_{X_{1,3}} \mathrm{d}^4x\sqrt{-g_4}\, \frac{g_s^2}{8\pi\ell_{(0)}} \frac{\mathrm{e}^{4\Omega+4A_0+2\phi}}{\omega_{(0)}^{\Sigma_4}\kappa_4^2 l_s^4} \int_{\Sigma_4} g_2^0 \wedge \overline{\eta} \int_{\Sigma_4} \overline{g}_2^0 \wedge \eta\, \sigma^3\overline{\sigma}^3.$$

With a pure $(2,1)$-flux background at the tip, such a mass can be generated by means of the superpotential bilinear coupling

$$\begin{aligned} \left[\tilde{\mu}_{\sigma^3\sigma^3}\right]_{(2,1)} &= -\frac{\mathrm{e}^{-A_0}\ell_{(0)}}{\pi\kappa_4 l_s^2} \left[g_{3\overline{3}}^0 \frac{\ell_4^0}{\ell_{(0)}^{\Sigma_4}}\right]^{\frac{1}{2}} \left[\partial_\tau \partial_{u^\vartheta} \int_{Y_6} \left[G_3 \wedge \Omega\right] \delta^{(2)}(r)\right]_{(2,1)} \\ &= \frac{\mathrm{e}^{-A_0}\ell_{(0)}}{\pi\left[-\mathrm{i}(\tau-\overline{\tau})\right]\kappa_4 l_s^2} \left[g_{3\overline{3}}^0 \frac{\ell_4^0}{\ell_{(0)}^{\Sigma_4}}\right]^{\frac{1}{2}} \int_{\Sigma_4} \overline{g}_2^0 \wedge \eta, \end{aligned} \tag{4.3.14}$$

which works in the presence of the $2A_0$-shift in eq. (4.2.17). Similarly to the case of eq. (4.3.8), the identification is made assuming the relationship $\mathrm{e}^{\kappa_4^2\hat{K}_{\text{cs}}}\ell_w = \ell_{(0)}^{\Sigma_4}/\omega_{(0)}^{\Sigma_4}$ to hold. This is not necessarily true in every compactification, in which case an additional factor $[(\ell_{(0)}^{\Sigma_4}\omega_{(0)})/(\omega_{(0)}^{\Sigma_4}\ell_{(0)}))]^{1/2}$ can be inserted in $\tilde{\mu}_{\sigma^3\sigma^3}$.

Comment on Generic Flux Backgrounds

For generic flux-backgrounds, similar challenges arise as in paragraph 4.3.1.2.2. However, for ISD-fluxes, if the Kähler modulus is stabilised by non-perturbative effects in the bulk, the $(0,3)$-flux is localised away from the tip [61, 249]. So, the $(0,3)$-flux does not contribute to the integral in $\tilde{\mu}_{\sigma_3\sigma_3}$, and the Kähler potential coupling $H_{\sigma^3\sigma^3}$ can also be set to zero.

Following eqs. (C.2.5a, C.2.5b), even if $(0,3)$-flux is present in the bulk, and so there is a non-zero F-term for the volume modulus, for such a pure flux-induced F-term cancellations hold such that if $H_{\sigma^3\sigma^3} = 0$ then it follows that $B_{\sigma^3\sigma^3} = 0$ and $m^2_{\sigma^3\sigma^3,\,\text{soft}} = 0$, consistently with the fact that the tip of the throat only sees $(2,1)$-fluxes [235, 237].[4.4] The effects of non-perturbative corrections besides flux localisation are going to be discussed in KKLT-like stabilisation scenarios with anti-D3-brane supersymmetry breaking in section 4.4.

[4.4] As an aside, notice the discrepancy between eqs. (3.25, [235]) and (6.24, [237]).

Warp Factors and Field Redefinitions

The superpotential bilinear coupling $\tilde{\mu}_{\sigma^3\sigma^3}$ in eq. (4.3.14) depends on the warp factor through e^{-A_0}, $g^0_{3\overline{3}}$ and ℓ^0_4. It is convenient to make the warp-factor dependences explicit. Two possible approaches are now discussed.

In order to match the D7-brane chiral multiplet kinetic and mass terms with such a structure, first of all one has to redefine the D7-brane scalar field as

$$\check{\sigma}^3\check{\overline{\sigma}}^3 = \mathrm{e}^{-4A_0}\,\ell^0_4\, g^0_{3\overline{3}}\,\sigma^3\overline{\sigma}^3, \tag{4.3.15}$$

and analgously its superpartner too. In this way, the kinetic and mass terms read

$$\begin{aligned} S^{\text{D7-scalar}} =& -\frac{1}{2\pi g_s}\int_{X_{1,3}} \mathrm{d}^4x\,\sqrt{-g_4}\;\mathrm{e}^{2\Omega+\phi}\,g^{\mu\nu}\,\nabla_\mu\check{\sigma}^3\nabla_\nu\check{\overline{\sigma}}^3 \\ & -\frac{1}{2\pi g_s}\int_{X_{1,3}} \mathrm{d}^4x\sqrt{-g_4}\,\frac{g_s^2}{8\pi\ell_{(0)}}\,\frac{\mathrm{e}^{4\Omega+2A_0+2\phi}}{\omega^{\Sigma_4}_{(0)}\kappa_4^2 l_s^4}\,\frac{\mathrm{e}^{6A_0}}{\ell^0_4\, g^0_{3\overline{3}}}\int_{\Sigma_4} g^0_2\wedge\overline{\eta}\int_{\Sigma_4}\overline{g}^0_2\wedge\eta\;\check{\sigma}^3\check{\overline{\sigma}}^3. \end{aligned}$$

By relabelling the fields as $\check{\sigma}^3 \to \sigma^3$ for simplicity, one obtains the final action via the Kähler matter metric

$$Z_{\sigma^3\overline{\sigma}^3} = \frac{1}{\pi g_s}\,\frac{\mathrm{e}^{2\Omega}}{[-\mathrm{i}(\tau-\overline{\tau})]} \tag{4.3.16}$$

and the superpotential bilinear coupling

$$\left[\tilde{\mu}_{\sigma^3\sigma^3}\right]_{(2,1)} = \left[\frac{\mathrm{e}^{6A_0}}{g^0_{3\overline{3}}}\,\frac{1}{\ell^0_4\ell^{\Sigma_4}_{(0)}}\right]^{\frac{1}{2}}\left[\frac{\ell_{(0)}}{\pi[-\mathrm{i}(\tau-\overline{\tau})]\kappa_4 l_s^2}\int_{\Sigma_4}\overline{g}^0_2\wedge\eta\right]. \tag{4.3.17}$$

Therefore, thanks to the field redefinition and the Kähler-potential shift, the bilinear potential is now effectively independent of the warp factor.

A second possibility is to replace the original e^{A_0}-dependence in the bilinear coupling $\tilde{\mu}_{\sigma^3\sigma^3}$ with a trilinear term coupling $z^{\frac{1}{3}}$ to the product $\sigma^3\sigma^3$ [67,68], where z is the complex-structure modulus fixing the warp factor at the tip as $\langle z\rangle^{1/3} \sim \mathrm{e}^{A_0}$, assuming for concreteness a Klebanov-Strassler throat. This will be further commented on below.

4.3.1.3.3 D7-Brane Gauge Sector

From the D7-brane DBI-action, one can observe the kinetic action for the 4-dimensional gauge field to be

$$S^{\text{D7-vector}}_{\text{kin}} = -\frac{1}{8\pi g_s}\int_{X_{1,3}} \mathrm{d}^4x\,\sqrt{-g_4}\,[\mathrm{e}^{-4A_0}+c]\,\ell^0_4\,g^{\mu\rho}g^{\nu\sigma}\,F_{\mu\nu}F_{\rho\sigma}$$

and therefore, following the condition $\mathrm{e}^{-4A_0} \gg c$, the Yang-Mills coupling is

$$\frac{4\pi}{g^2_{\text{YM}}} = \operatorname{Im}\tau_{\text{YM}} = [\mathrm{e}^{-4A_0}+c]\,\frac{\ell^0_4}{g_s} = \left[\mathrm{e}^{-4A_0}-\frac{\mathrm{i}}{2}\,(\rho-\overline{\rho})\right]\frac{\ell^0_4}{g_s} \simeq \mathrm{e}^{-4A_0}\,\frac{\ell^0_4}{g_s}.$$

One can thus conclude that the gauge kinetic function has to be

$$f_{\text{D7}} = \frac{\ell^0_4}{2\pi g_s}\left[\mathrm{e}^{-4A_0}-\mathrm{i}\rho\right] \simeq \frac{\ell^0_4}{2\pi g_s}\,\mathrm{e}^{-4A_0}. \tag{4.3.18}$$

Notice that, as the volume of the wrapped 4-cycle depends on the warp factor due to the behaviour of the metric in eq. (4.3.13), the term $\ell_4^0\,\mathrm{e}^{-4A_0}$ is actually independent of the warp factor. The subleading term in f_{D7} instead depends on the warp factor, and, as for the $\tilde{\mu}$-term above, it may be written as a holomorphic contribution in the complex-structure modulus $z^{4/3}$ [67, 68]. Also, although the subleading term in f_{D7} contributes a soft gaugino mass, due to the e^{4A_0}-redshift factor it is always suppressed with respect to the anomaly-mediated mass contributions discussed below.

4.3.2 D3-/D7-Brane Intersecting States

Interactions in the low-energy effective action involving D3-/D7-brane intersecting states are now going to be worked out. Tools other than dimensional reduction need to be used since a higher-dimensional effective theory for such states is unknown.

4.3.2.1 D3-Brane and D7-Brane Extending from the Throat Tip into the Bulk

For intersecting D3-/D7-branes, where the D3-brane is at the tip of a warped throat and the D7-brane wraps a 4-cycle extending from the tip into the bulk with the configuration described in paragraph 4.3.1.2.4, the couplings for the intersecting states in the Kähler- and super-potential of eqs. (4.3.1a, 4.3.1b) are as follows.

- Following the studies of scattering amplitudes of the intersecting D3-/D7-brane states, refs. [230, 233, 234] find the structure $Z_{\varphi\overline{\varphi}} = Z_{\varphi_*\overline{\varphi}_*} = 1/[-\mathrm{i}(\rho-\overline{\rho})]$ in an unwarped compactification, so, generalising this, for a warped compactification one can define the Kähler matter metrics for the intersecting states as

$$Z_{\varphi\overline{\varphi}} = Z_{\varphi_*\overline{\varphi}_*} = \frac{1}{2\pi g_s}\,\mathrm{e}^{2\Omega}. \tag{4.3.19}$$

  Further, symmetry arguments reveal that the fields φ and φ_* do not have flux-induced masses [235]. In fact the resulting no-scale structure implies they can be included within the logarithmic Kähler potential (together with the other chiral superfields) by defining the ρ-term as

$$\kappa_4^2 K = -3\ln\left[2\,\mathrm{e}^{-2\Omega} - \frac{\kappa_4^2}{3\pi g_s}\,\varphi\overline{\varphi}\right].$$

- As they need to be massless, the intersecting states do not have any bilinear H- or $\tilde{\mu}$-coupling. However, one needs to account for a would-be mass term in the case in which the D3- and D7-brane are separated, as explained by ref. [235]. As will also be argued in subsubsection 4.3.2.3, the superpotential term which accounts for this interaction is generated by the Yukawa couplings

$$\tilde{Y}_{\sigma^3\varphi\varphi_*} = -\tilde{Y}_{\varphi^3\varphi\varphi_*} = \frac{1}{g_s}\left[\frac{2}{\pi}[\ell_{(0)}]^3\right]^{\frac{1}{2}} = \tilde{\xi}. \tag{4.3.20}$$

It will be shown below that such terms are fundamental in order to generate the leading-order flux-mediated couplings between the D7-brane and the intersecting states. Notice that the canonically normalised physical Yukawa couplings involving σ^3 are suppressed by the warp factor, while those involving φ^3 are not, consistently with their different localisations with respect to φ and φ_*.

The corresponding low-energy supergravity action has a D-term potential, an F-term potential, and some soft supersymmetry-breaking couplings.

- The D-term potential emerges because the intersecting states are charged under the D3- and the D7-brane gauge fields, with couplings

$$\begin{aligned} g_{\mathrm{D3}}^{-2} &= -\frac{\mathrm{i}}{4\pi g_s}\,(\tau-\overline{\tau}), \\ g_{\mathrm{D7}}^{-2} &= -\frac{\mathrm{i}\ell_{(0)}^{\Sigma_4}}{4\pi g_s}\,(\rho-\overline{\rho}+2\mathrm{i}c_0) = \frac{\ell_{(0)}^{\Sigma_4}}{2\pi g_s}\,\mathrm{e}^{-2\Omega}, \end{aligned}$$

where for simplicity it is being implied that $c_0 = c_0'$. It is now easy to infer that the D-term potential for the field φ is

$$\begin{aligned} V_D^{(\mathrm{susy})} &= \frac{1}{2}\,g_{\mathrm{D3}}^2\,(Z_{\varphi\overline{\varphi}}\varphi\overline{\varphi})^2 + \frac{1}{2}\,g_{\mathrm{D7}}^2\,(Z_{\varphi\overline{\varphi}}\varphi\overline{\varphi})^2 \\ &= \frac{\mathrm{e}^{4\Omega}}{2\pi g_s\,[-\mathrm{i}(\tau-\overline{\tau})]}\,(\varphi\overline{\varphi})^2 + \frac{\mathrm{e}^{6\Omega}}{4\pi g_s \ell_{(0)}^{\Sigma_4}}\,(\varphi\overline{\varphi})^2, \end{aligned} \tag{4.3.21}$$

and similarly for the field φ_*. It is worthwhile to observe that the specific value of the redshift factor at the tip of the throat does not appear.

- On the other hand, in an ISD-background the F-term potential comes from the effective superpotential

$$W_{\mathrm{susy}} = \frac{1}{2}\mu_{\sigma^3\sigma^3}\sigma^3\sigma^3 + \xi\,(\sigma^3-\varphi^3)\,\varphi\varphi_*,$$

where for the sake of simplicity the trilinear term $\xi = \mathrm{e}^{\kappa_4^2\mathcal{K}/2}\tilde{\xi}$ has been defined, and reads $V_F^{\mathrm{susy}} = Z^{i\overline{j}}\left[\partial_i W_{\mathrm{susy}}\right]\left[\partial_{\overline{j}}\overline{W}_{\mathrm{susy}}\right]$. This potential gives the redshifted D7-brane supersymmetric mass, but also the couplings between the pure and the intersecting-brane states. First of all, one has the cubic interaction

$$\begin{aligned} V_{\mathrm{cubic}}^{(\sigma^3\varphi\varphi_*)} &= Z^{\sigma^3\overline{\sigma}^3}\mu_{\sigma^3\sigma^3}\overline{\xi}\sigma^3\,\overline{\varphi}\,\overline{\varphi}_* + Z^{\sigma^3\overline{\sigma}^3}\overline{\mu}_{\overline{\sigma}^3\overline{\sigma}^3}\xi\overline{\sigma}^3\,\varphi\,\varphi_* \\ &= \frac{1}{4\pi\kappa_4}\frac{\mathrm{e}^{6\Omega+2A_0}}{[-\mathrm{i}(\tau-\overline{\tau})][-\mathrm{i}\omega_w^{\Sigma_4}]}\left[\frac{2}{\pi\ell_{(0)}}\right]^{\frac{1}{2}}\frac{\ell_w^{\Sigma_4}}{\ell_{(0)}^{\Sigma_4}}\left[\left[\frac{1}{l_s^2}\int_{\Sigma_4}\overline{g}_2\wedge\eta\right]\sigma^3\,\overline{\varphi}\,\overline{\varphi}_* + \mathrm{c.c.}\right]. \end{aligned} \tag{4.3.22}$$

Additionally, one can observe two distinct quartic interactions which involve only the intersecting states. First of all, there is the standard quartic potential

$$\begin{aligned} V_{\mathrm{quartic}}^{(\varphi\overline{\varphi})} &= Z^{\sigma^3\overline{\sigma}^3}\xi\overline{\xi}\,\varphi\varphi_*\,\overline{\varphi}\,\overline{\varphi}_* + Z^{\varphi^3\overline{\varphi}^3}\xi\overline{\xi}\,\varphi\varphi_*\,\overline{\varphi}\,\overline{\varphi}_* \\ &= \frac{1}{2\pi g_s}\frac{\mathrm{e}^{6\Omega+2A_0}}{[-\mathrm{i}\omega_w^{\Sigma_4}]}\frac{\ell_w^{\Sigma_4}}{\ell_{(0)}^{\Sigma_4}}\,\varphi\varphi_*\,\overline{\varphi}\,\overline{\varphi}_* + \frac{\ell_w^{\Sigma_4}}{\pi g_s}\frac{\mathrm{e}^{4\Omega+2A_0}\,g_0^{3\overline{3}}}{[-\mathrm{i}(\tau-\overline{\tau})]\,[-\mathrm{i}\omega_w^{\Sigma_4}]}\,\varphi\varphi_*\,\overline{\varphi}\,\overline{\varphi}_*, \end{aligned} \tag{4.3.23}$$

in which the warp factor redshifts the D7-brane term, but not the D3-brane one due to the cancellation induced by the inverse metric $g_0^{3\overline{3}} \sim \mathrm{e}^{-2A_0}$. This does not happen for the D7-brane because its matter metric is determined by the bulk metric $g_{3\overline{3}} = 1$. Second, there are the quartic interactions that represent the would-be mass terms, i.e.

$$\begin{aligned} V_{\text{quartic}}^{(\sigma^3\varphi\overline{\varphi})} &= \xi\overline{\xi}\, Z^{\varphi_*\overline{\varphi}_*}\, (\sigma^3 - \varphi^3)(\overline{\sigma}^3 - \overline{\varphi}^3)\, \varphi\overline{\varphi} \\ &= \frac{1}{\pi g_s}\, \frac{\mathrm{e}^{4\Omega+2A_0}\, \ell_w^{\Sigma_4}}{[-\mathrm{i}(\tau-\overline{\tau})]\,[-\mathrm{i}\omega_w^{\Sigma_4}]}\, (\sigma^3 - \varphi^3)(\overline{\sigma}^3 - \overline{\varphi}^3)\, \varphi\overline{\varphi} \end{aligned} \tag{4.3.24}$$

and the equivalent term for the field φ_*, which are redshifted by the warp factor as must be due to the location of the intersection at the tip of the throat.

- In order to determine the supersymmetry-breaking terms for the states φ and φ_*, instead, it is necessary to determine the Riemann tensor associated to the Kähler matter metrics. In order to show the general structure of the couplings, in this discussion the possibility of having both $(2,1)$- and $(0,3)$-fluxes is considered.[4.5] One finds the Levi-Civita connection $\Gamma^{\varphi}_{\rho\varphi} = \mathrm{i}\,\mathrm{e}^{2\Omega}/2$, which implies that the only non-vanishing component of the Riemann tensor is

$$R_{\rho\overline{\rho}\varphi\overline{\varphi}} = \frac{1}{2\pi g_s}\frac{1}{4}\,\mathrm{e}^{6\Omega}.$$

So, as a manifestation of sequestering, in an ISD-background the identity still holds

$$m^2_{\varphi\overline{\varphi},\,\text{soft}} = \hat{m}^w_{3/2}\hat{\overline{m}}^w_{3/2}\, Z_{\varphi\overline{\varphi}} - \hat{\mathcal{F}}^{\rho}\hat{\overline{\mathcal{F}}}^{\rho}\, R_{\rho\overline{\rho}\varphi\overline{\varphi}} = 0,$$

and the fields φ and φ_* stay massless even when supersymmetry is broken by ρ. Due to the lack of an H- or a $\tilde{\mu}$-term for these fields, there is no B-coupling either.

Finally, one has to consider the supersymmetry-breaking scalar trilinear couplings, which must be studied with some care. For the couplings to the D7-brane scalar σ^3, one finds

$$\nabla_{\rho} Y_{\sigma^3\varphi\varphi_*} = \partial_{\rho} Y_{\sigma^3\varphi\varphi_*} + \frac{1}{2}\,\kappa_4^2 \hat{K}_{\rho} Y_{\sigma^3\varphi\varphi_*} - 3\Gamma^{l}_{\rho\sigma^3} Y_{l\varphi\varphi_*} = \frac{3\mathrm{i}}{2}\,\mathrm{e}^{2\Omega} Y_{\sigma^3\varphi\varphi_*}$$

as a consequence of the fact that, because of the form of the D7-brane matter metric, its associated Levi-Civita connection vanishes, i.e. $\Gamma^{\sigma^3}_{\rho\sigma^3} = 0$. One also finds

$$\begin{aligned} \nabla_{\rho} Y_{\varphi\varphi_*\sigma^3} &= \partial_{\rho} Y_{\varphi\varphi_*\sigma^3} + \frac{1}{2}\,\kappa_4^2 \hat{K}_{\rho}\, Y_{\varphi\varphi_*\sigma^3} - 3\Gamma^{l}_{\rho\varphi}\, Y_{l\varphi_*\sigma^3} = 0, \\ \nabla_{\rho} Y_{\varphi_*\sigma^3\varphi} &= \partial_{\rho} Y_{\varphi_*\sigma^3\varphi} + \frac{1}{2}\,\kappa_4^2 \hat{K}_{\rho}\, Y_{\varphi_*\sigma^3\varphi} - 3\Gamma^{l}_{\rho\varphi_*}\, Y_{l\sigma^3\varphi} = 0, \end{aligned}$$

because in this case the connection is exactly such as to cancel the first two terms. For the couplings with the D3-brane scalar φ^3, one finds that all the covariant derivatives

[4.5] Notice that a $(0,3)$-flux does not affect the supersymmetric couplings: the D3-brane does not have supersymmetric couplings depending on ISD-fluxes, while the D7-brane effective μ-term is correct so long as the conditions around eq. (4.3.9) are fulfilled.

vanish too as a consequence of the form of the Kähler matter metric. Therefore, the only supersymmetry-breaking trilinear coupling is $A_{\sigma^3\varphi\varphi_*}$ (see eq. (C.2.5c)). If one writes the $(0,3)$-flux as $G_3' = g_2'(w^3, \overline{w}^3) \wedge \mathrm{d}\overline{w}^3$, with a suitable $(0,2)$-form $g_2' = g_2'(w^3, \overline{w}^3)$ on the 4-cycle, then this becomes[4.6]

$$A_{\sigma^3\varphi\varphi_*} = \frac{3}{4\pi\kappa_4} \, \frac{\mathrm{e}^{6\Omega+2A_0}}{[-\mathrm{i}(\tau-\overline{\tau})]\,[-\mathrm{i}\omega_w^{\Sigma_4}]} \, \frac{\ell_w^{\Sigma_4}}{\ell_{(0)}^{\Sigma_4}} \left[\frac{2r^2}{\pi\ell_{(0)}}\right]^{\frac{1}{2}} \frac{1}{l_s^2} \int_{\Sigma_4} \overline{g}_2' \wedge \overline{\eta}, \tag{4.3.25}$$

where the ratio has been defined $r = \ell_{(0)}^{\Sigma_4} \ell_{(0)}^{\mathrm{T}^2} / \ell_{(0)}$, with $\ell_{(0)}^{\mathrm{T}^2}$ coming from the tranverse integration of the superpotential term. Of course, in general one can always set $r = 1$.

Evidently, in the presence of supersymmetry-breaking imaginary anti-self-dual fluxes, one would obtain mass corrections for the scalars φ and φ_* sourced by both the axio-dilaton and the complex-structure modulus. Also, one would obtain new trilinear terms coupling these fields to the D3-brane scalar φ^3 too.

Notice that in an ISD-background the intersecting D3-/D7-brane states couple to the background fluxes only via the mediation of the D7-brane fields as the interactions with the D3-brane fields are protected by the no-scale structure of the latter.

4.3.2.2 D3-Brane and D7-Brane at the Tip of the Throat

For a system of intersecting D3-/D7-branes where the D7-brane wraps a 4-cycle that is localised at the tip of a warped throat, as in subsubsection 4.3.1.3, or where the D7-brane wraps a 4-cycle extending through the throat with fields localised at the tip, as in paragraph 4.3.1.2.3, the intersecting-state parameters of the Kähler- and super-potentials of eqs. (4.3.1a, 4.3.1b) are as follows:

- the Kähler matter metric is

$$Z_{\varphi\overline{\varphi}} = Z_{\varphi_*\overline{\varphi}_*} = \frac{1}{2\pi g_s}\,\mathrm{e}^{2\Omega}; \tag{4.3.26}$$

- setting $\beta = \mathrm{e}^{-A_0}$, as argued in subsubsection 4.3.2.3, the Yukawa couplings are

$$\tilde{Y}_{\sigma^3\varphi\varphi_*} = \frac{1}{g_s}\left[\frac{2}{\pi}\,[\ell_{(0)}]^3\right]^{\frac{1}{2}} \beta = \tilde{\xi}\beta, \tag{4.3.27a}$$

$$\tilde{Y}_{\varphi^3\varphi\varphi_*} = -\tilde{\xi}. \tag{4.3.27b}$$

In this case the canonically normalised physical Yukawa couplings are not redshifted.

These account for the sequestered nature of the fields as well as for the presence of the would-be mass term due to any brane separation.

For the intersecting-state contributions to the D-term potential, F-term potential and soft supersymmetry-breaking terms, the fact that the D7-brane is localised and therefore has a no-scale-like matter metric (cfr. eqs. (4.3.7, 4.3.16)) gives rise to particular features.

[4.6] In this calculation the coupling involving the intersecting states is present only if there is a $(0,3)$-flux at the tip of the throat. This is not necessarily what happens in a fully stabilised model, where the non-perturbative corrections that stabilise the volume modulus localise the $(0,3)$-flux in the bulk.

- The D3- and the D7-brane gauge couplings are (neglecting the ρ-dependent term for the D7-brane)

$$g_{\mathrm{D3}}^{-2} = -\frac{\mathrm{i}}{4\pi g_s}\,(\tau - \overline{\tau}),$$
$$g_{\mathrm{D7}}^{-2} = \frac{\ell_4^0}{2\pi g_s}\,\mathrm{e}^{-4A_0},$$

so the D-term potential for the field φ reads

$$V_D^{(\mathrm{susy})} = \frac{1}{2\pi g_s\,[-\mathrm{i}(\tau - \overline{\tau})]}\,\mathrm{e}^{4\Omega}\,(\varphi\overline{\varphi})^2 + \frac{1}{4\pi g_s\,\ell_4^0}\,\mathrm{e}^{4\Omega+4A_0}\,(\varphi\overline{\varphi})^2. \tag{4.3.28}$$

The volume dependence is now different for the D7-brane-induced potential. However, the warp factor at the tip of the throat is still effectively missing.

- As usual, the F-term potential comes from the effective superpotential and, in addition to the D7-brane supersymmetric mass, there are couplings between the pure and the intersecting-brane states. One finds the cubic interaction

$$V_{\mathrm{cubic}}^{(\sigma^3\varphi\varphi_*)} = \frac{1}{4\pi\kappa_4}\,\frac{\mathrm{e}^{4\Omega+A_0}}{[-\mathrm{i}(\tau-\overline{\tau})]\,[-\mathrm{i}\omega_{(0)}^{\Sigma_4}]}\left[\frac{2\varsigma}{\pi}\,\frac{\ell_{(0)}^{\Sigma_4}}{\ell_{(0)}}\right]^{\frac{1}{2}}\left[\left[\frac{1}{l_s^2}\int_{\Sigma_4}\overline{g}_2^0\wedge\eta\right]\sigma^3\,\overline{\varphi}\,\overline{\varphi}_* + \mathrm{c.c.}\right]. \tag{4.3.29}$$

where the ratio $\varsigma = \mathrm{e}^{6A_0}/(g^0_{3\overline{3}}\ell_4^0)$ has been defined for brevity. Compared to the potential of eq. (4.3.22), this potential is less warped-down due to the term $\beta = \mathrm{e}^{-A_0}$. The quartic interactions for pure intersecting states are

$$V_{\mathrm{quartic}}^{(\varphi\overline{\varphi})} = \frac{\mathrm{e}^{4\Omega}}{2\pi g_s}\,\frac{\ell_{(0)}^{\Sigma_4}}{[-\mathrm{i}\omega_{(0)}^{\Sigma_4}]}\varphi\varphi_*\,\overline{\varphi}\,\overline{\varphi}_* + \frac{1}{\pi g_s}\,\frac{\mathrm{e}^{4\Omega+2A_0}\,g_0^{3\overline{3}}}{[-\mathrm{i}(\tau-\overline{\tau})]}\,\frac{\ell_{(0)}^{\Sigma_4}}{[-\mathrm{i}\omega_{(0)}^{\Sigma_4}]}\,\varphi\varphi_*\,\overline{\varphi}\,\overline{\varphi}_*, \tag{4.3.30}$$

where for the D3-brane induced term, the redshift effect is again cancelled by the metric, while for the D7-brane the cancellation arises due to the specific setup with the wrapped 4-cycle at the tip of the throat and the field redefinition of eq. (4.3.15) (see subsubsection 4.3.2.3). There is also the quartic would-be separation mass interaction

$$V_{\mathrm{quartic}}^{(\sigma^3\varphi\overline{\varphi})} = \frac{1}{\pi g_s}\,\frac{\mathrm{e}^{4\Omega+2A_0}}{[-\mathrm{i}(\tau-\overline{\tau})]}\,\frac{\ell_{(0)}^{\Sigma_4}}{[-\mathrm{i}\omega_{(0)}^{\Sigma_4}]}\,(\sigma^3\mathrm{e}^{-A_0} - \varphi^3)(\overline{\sigma}^3\mathrm{e}^{-A_0} - \overline{\varphi}^3)\,\varphi\overline{\varphi}. \tag{4.3.31}$$

- For the supersymmetry-breaking terms, it is obvious that in a pure $(2,1)$-flux there cannot be any. In particular, one finds no flux-dependent A-couplings for the intersecting D3-/D7-brane states. In fact, even if there were a $(0,3)$-flux, the trilinear scalar coupling $A_{\sigma^3\varphi\varphi_*}$ would vanish due to the no-scale structure of the modulus ρ.

4.3.2.3 A 6-dimensional Picture of the Intersecting States

One can heuristically motivate the form of the Kähler- and super-potential for the D3-/D7-brane intersecting states by a qualitative analysis of their would-be effective field theory.

One can consider the setup in which the branes are separated by a non-zero string-frame coordinate distance $\delta z^3 = (\langle\pi^3\rangle - \langle\phi^3\rangle)\, l_s^2/(2\pi)$, in the static gauge, where π^3 and ϕ^3 are the D7- and D3-brane positions in the D7-brane transverse direction, respectively, with $\sigma^3 = \gamma^{3/4}\pi^3$ and $\varphi^3 = \gamma^{3/4}\phi^3$ (see appendix B.2). A displacement of the D3-brane in the D7-brane longitudinal directions does not induce mass terms, so the intersecting states can be pictured as 6-dimensional fields living in the non-compact 4-dimensional spacetime plus the 2-dimensional compact space Y_2, in the z^3-direction, along which the D3- and D7-branes can be separated. In the string frame, the supersymmetric mass term for the 6-dimensional intersecting states θ and θ_* is (see subsection 2.1.2)

$$M^2_{\theta\bar\theta} = M^2_{\theta_*\bar\theta_*} = \frac{4\pi^2}{l_s^4} G_{3\bar 3}\delta z^3 \delta\bar z^3,$$

with G_{MN} the string-frame metric, and θ and θ_* are soon going to be related to the 4-dimensional fields φ and φ_*. The kinetic action must be of the form

$$S_{\mathrm{D3/D7}} = -\frac{1}{2\pi l_s^2}\int_{X_{1,3}\times Y_2} \mathrm{d}^6x\, \sqrt{-G_6}\; \mathrm{e}^{-n\Phi}\left[G^{\mu\nu}\partial_\mu\theta\partial_\nu\bar\theta + \frac{4\pi^2}{l_s^4}G_{3\bar 3}\delta z^3\delta\bar z^3\,\theta\bar\theta\right],$$

with n a constant representing the fact that usually actions in the string frame are normalised with overall dilaton factors. Then, in the 4-dimensional Einstein frame one obtains

$$S_{\mathrm{D3/D7}} = -\frac{\ell^{Y_2}_{(0)}}{2\pi g_s^n}\int_{X_{1,3}} \mathrm{d}^4x\,\sqrt{-g_4}\;\mathrm{e}^{2\Omega+(1-n)\phi}\left[g^{\mu\nu}\partial_\mu\varphi\partial_\nu\overline{\varphi} + \mathrm{e}^{2\Omega+\phi}\left(g_{3\bar 3}\delta\zeta^3\delta\overline{\zeta}^3\right)\varphi\overline{\varphi}\right], \quad (4.3.32)$$

where the brane position moduli have been rescaled as explained in appendix B.2, leading to $\delta\zeta^3 = \gamma^{3/4}\delta z^3 = (\beta\langle\sigma^3\rangle - \langle\varphi^3\rangle)\, l_s^2/(2\pi)$, and the same scaling has been performed on the intersecting states, i.e. $\varphi = \gamma^{3/4}\theta$. The extra factor is $\beta = 1$ for a D7-brane extended from tip to bulk and $\beta = \mathrm{e}^{-A_0}$ for a D7-brane localised at tip, following the extra field redefinition in eq. (4.3.15). Such a construction is compatible with a simple supersymmetric description, i.e. by means of a $\tilde\mu$-tilde coupling, only if the dilaton power takes the value $n = 1$ as a different choice cannot reproduce in supergravity the action of eq. (4.3.32).

So far, this action applies to any intersecting D3-/D7-brane setup, but it is convenient to specialise to the case in which the D3-brane is located at the tip of a warped throat. As the intersection takes place at the tip of the throat, the action has the form

$$S_{\mathrm{D3/D7}} = -\frac{1}{2\pi g_s}\int_{X_{1,3}} \mathrm{d}^4x\,\sqrt{-g_4}\left[\mathrm{e}^{2\Omega}\, g^{\mu\nu}\partial_\mu\varphi\partial_\nu\overline{\varphi} + \mathrm{e}^{4\Omega+2A_0+\phi}\,\delta\zeta^3\delta\overline{\zeta}^3\,\varphi\overline{\varphi}\right],$$

where advantage has been taken of the fact that the internal metric scales as $g^0_{3\bar 3} \sim \mathrm{e}^{2A_0}$ and the 2-space volume factor has been absorbed into the field φ. Assuming the formulation with the Kähler potential $2A_0$-shift, the action above can be reproduced in a supersymmetric way by means of the Kähler- and super-potential terms

$$Z_{\varphi\overline{\varphi}} = \frac{1}{2\pi g_s}\,\mathrm{e}^{2\Omega}, \quad (4.3.33a)$$

$$\tilde\mu_{\varphi\varphi} = \frac{1}{g_s}\left[\frac{2}{\pi}\left[\ell_{(0)}\right]^3\right]^{\frac12}\delta\zeta^3. \quad (4.3.33b)$$

This generates an additional factor $\ell_w/[-\mathrm{i}\omega_w]$ that has not been captured by the previous discussion picturing a dimensional reduction. In a theory in which the D3- and D7-brane scalars are dynamical, the terms $\tilde{\mu}_{\varphi\varphi}$ can be used to fix the trilinear couplings as $\tilde{Y}_{\sigma^3\varphi\varphi} = \tilde{\mu}_{\varphi\varphi}|_{\langle\varphi^3\rangle=0}/\langle\sigma\rangle^3$, $\tilde{Y}_{\varphi^3\varphi\varphi} = \tilde{\mu}_{\varphi\varphi}|_{\langle\sigma\rangle^3=0}/\langle\varphi\rangle^3$. This is a simplified example since it contains only one intersecting field, while in reality there are both the 37- and the 73-states. However, provided a diagonalisation of the states, the structure of the Yukawa couplings is correct. In this way, from the bilinear couplings in eq. (4.3.33b) one obtains the trilinear couplings in eqs. (4.3.20) and (4.3.27).

As commented on in subsubsection 4.3.1.3, if a complex-structure modulus z associated to the tip of the throat controls the warp factor, then one might choose to not use the redefinition eq. (4.3.15) of the D7-brane scalars at the tip of the warped throat, instead obtaining couplings to z^p, with $p \geq 0$.

4.4 Warped Anti-D3- and D7-Branes

This section discusses the supergravity description of intersecting anti-D3- and D7-branes in strongly-warped Calabi-Yau orientifold compactifications.

First, the description of anti-D3-branes in terms of constrained superfields is overviewed, adapting the results of ref. [163], which refer to a different metric Ansatz to eq. (4.2.1) and outside the regime of field localisation of eqs. (4.2.13, 4.2.16) (see also ref. [156]). Second, it is shown how to extend these results to anti-D3-/D7-brane constructions, including in particular the intersecting states, building on the results for D3-/D7-branes. Finally, considering how these local models may eventually be embedded in global compactifications, the effects of moduli stabilisation and anomaly mediation on the open-string degrees of freedom are worked out, referring to the KKLT-scenario for definiteness. Along with the dimensional reductions in appendix B.2, use is made of appendix C.3, which reports the supergravity expansions that are suitable in the presence of non-linear supersymmetry.

4.4.1 Pure Anti-D3-Brane

The particle content of D3- and anti-D3-branes is the same, but the couplings with the bulk and other sources are different due to their opposite RR-charge, with implications on their supersymmetry transformations too. This subsection begins with a brief general discussion on anti-D3-brane supersymmetry breaking, then the anti-D3-brane field content and its low-energy effective action in terms of constrained superfields are described in detail.

4.4.1.1 Anti-D3-Brane Supersymmetry Breaking

In type IIB Calabi-Yau orientifolds, anti-D3-branes do not preserve the same supersymmetry as the closed-string sector since the orientifold-invariant supersymmetry charge realises supersymmetry only non-linearly on their worldvolume, whereas the supersymmetry charge that would be linearly realised on the brane is projected out. In particular, the gaugino transformation under the surviving supersymmetry takes the form $\sqrt{2}\delta_\epsilon\lambda \sim \epsilon/l^2$, where the factor $l \sim 1/\tilde{m}_s$ is the relevant string scale at the brane location, i.e. either the string scale

m_s or the warped string scale m_s^w. This can be seen by adapting the analysis of subsections 2.3.2 and 2.4.2 in view of the warped compactification introduced in section 4.2.

Because the scale $1/l$ never vanishes, there is no scale at which supersymmetry becomes linearly realised, and the 4-dimensional effective theory does not involve the usual F- or D-terms whose vacuum expectation value may become zero to restore linear supersymmetry. Nevertheless, because the worldvolume action remains supersymmetric, whilst there is no vacuum in which the anti-D3-brane goldstino has a non-zero supersymmetry transformation, this is effectively a spontaneous supersymmetry breaking. As a consequence of non-linearity, the anti-D3-brane degrees of freedom cannot be encoded in standard $N_4 = 1$ multiplets; instead, all the massless degrees of freedom of the anti-D3-brane must be packaged into constrained superfields. Once the tool of constrained supermultiplets is introduced, there is no technical difference with respect to the low-energy effective theory describing standard F-term spontaneous supersymmetry breaking below the supersymmetry-breaking scale.

Constrained superfields in global supersymmetry are thoroughly discussed in ref. [31] as a tool to describe effective theories with broken supersymmetry when the superpartners that become heavy due to the mass-splitting are integrated out. The simplest example is the nilpotent chiral superfield, whose only physical degree of freedom is its fermion playing the role of Volkov-Akulov goldstino for broken supersymmetry [137]. A generic treatment of constrained superfields in both global and local supersymmetry can be found in ref. [32]. As discussed in ref. [96], it should be noted that, although the massless degrees of freedom realise non-linear supersymmetry as if their superpartners had been integrated out, above the supersymmetry-breaking scale the full infinite tower of string states is necessary for a consistent supersymmetric theory, and there is no energy scale above which supersymmetry in the usual sense is restored. In fact, in view of the discussion in chapter 3, this is not surprising since the anti-D3-brane spectrum exhibits a form of misaligned supersymmetry.

4.4.1.2 Anti-D3-Brane Constrained Multiplets

To place the anti-D3-brane fields in constrained supermultiplets, one matches the non-linear supersymmetry transformations for the brane fields with those of a specific constrained superfield [77, 78, 84], as reviewed in subsection 2.4.4. For notational convenience, the notation from now on is different, and constrained superfields are denoted with lower-case letters.

- The anti-D3-brane gaugino λ, which plays the role of the goldstino, is described in terms of the fermion component ψ^x of a chiral superfield x that satisfies the nilpotency condition [255–258]
$$x^2 = 0. \tag{4.4.1}$$
This effectively removes its scalar φ^x in favour of the spinor ψ^x, indeed implying the identification $\varphi^x = \psi^x\psi^x/(2F^x)$, with the auxiliary field F^x being non-vanishing by assumption. At leading order in l, i.e. the scale at which the tower of string states enters into play, the gaugino λ and the goldstino ψ^x are then related as
$$\lambda \sim \frac{1}{2l^2}\frac{\psi^x}{F^x},$$

with the non-linear supersymmetry variation $\sqrt{2}\delta_\epsilon\lambda \sim \epsilon/l^2$. If the anti-D3-brane sits at the tip of a warped throat, then this supersymmetry-breaking scale is the warped string scale $l \sim 1/m_s^w$.

As the goldstino is contained in a chiral multiplet, the would-be gaugino D-term breaking is actually described as an F-term breaking. Eventually the gaugino is fixed as $\lambda = 0$ in the unitary gauge. The supergravity generalisation of this construction is discussed by refs. [259–263].

- The anti-D3-brane Abelian gauge field A_μ is contained in the vector degrees of freedom of a field-strength chiral multiplet W_α satisfying the constraint [31, 86]

$$xW_\alpha = 0, \tag{4.4.2}$$

which removes the gaugino ζ^W by making it proportional to the goldstino ψ^x.

- The anti-D3-brane modulini ψ^a are described by the fermionic degrees of freedom of three chiral superfields y^a satisfying the constraints [264, 265]

$$xy^a = 0, \tag{4.4.3}$$

which remove the scalars φ^{y^a} by making them proportional to the goldstino ψ^x.

- The anti-D3-brane scalars φ^a describing position fluctuations are encoded in the scalar degrees of freedom of three chiral superfields h^a satisfying the constraints [31, 32]

$$\overline{x}\mathrm{D}_\alpha h^a = 0, \tag{4.4.4}$$

with D_α the supersymmetry-covariant derivative, which makes both the spinors ψ^{h^a} and the auxiliary fields F^{h^a} proportional to the goldstino ψ^x. As it is constrained, the solution to the F-term field equation is not the usual $F^{h^a} = \mathrm{e}^{\kappa_4^2 K/2} K^{h^a \overline{I}} \nabla_{\overline{I}} \overline{W}$, but rather a goldstino-dependent expression which vanishes in the unitary gauge.

4.4.1.3 Anti-D3-Brane Supergravity

The supergravity formulation of a single anti-D3-brane at the tip of a warped throat in an orientifold compactification with Hodge number $h_+^{1,1} = 1$ is reported below. One can follow the dimensional reductions of refs. [80, 149, 163, 231, 232] and adapt them to the metric of eq. (4.2.1).

4.4.1.3.1 Anti-D3-Brane Uplift Energy

Anti-D3-branes provide a positive energy uplift to the vacuum energy at the classical level. Given the warp factor A_0 at the anti-D3-brane location, in the 4-dimensional Einstein frame it reads

$$S_\Lambda^{\overline{\mathrm{D3}}} = -\frac{1}{\kappa_4^4} \int \mathrm{d}^4 x \sqrt{-g_4}\, \frac{g_s^3}{4\pi[\ell_{(0)}]^2}\, \frac{\mathrm{e}^{4\Omega}}{\mathrm{e}^{-4A_0} + c}.$$

In the setup with the anti-D3-brane at the tip of the throat, the warp factor dominates over the volume modulus, so that the effective form of the term above is

$$S_{\Lambda}^{\overline{\mathrm{D3}}} = -\frac{1}{\kappa_4^4} \int \mathrm{d}^4 x \sqrt{-g_4}\, \frac{g_s^3}{4\pi[\ell_{(0)}]^2}\, \mathrm{e}^{4\Omega+4A_0}.$$

Such a vacuum energy can be reproduced in supergravity as the F-term potential contribution of the nilpotent superfield x introduced in eq. (4.4.1) by defining the Kähler- and super-potential

$$\begin{aligned} \kappa_4^2 \hat{K} =&\ -\ln[-\mathrm{i}(\tau-\overline{\tau})] - \ln[-\mathrm{i}\omega_w] + \ln\left[\frac{2}{\pi}\frac{\ell_w}{[\ell_{(0)}]^3}\right] \\ &- 3\ln\left[2\,\mathrm{e}^{-2\Omega} - \frac{4\kappa_4^2}{3g_s}\frac{\ell_w}{\ell_{(0)}}\frac{\mathrm{e}^{-2A_0}\, x\overline{x}}{[-\mathrm{i}(\tau-\overline{\tau})][-\mathrm{i}\omega_w]}\right], \end{aligned} \tag{4.4.5a}$$

$$\kappa_4^3 \hat{W} = \frac{g_s}{l_s^2}\int_{Y_6} G_3\wedge\Omega + \sqrt{2}\, g_s\kappa_4 x, \tag{4.4.5b}$$

with the actual total Kähler potential being $\kappa_4^2\mathcal{K} = 2A_0 + \kappa_4^2\hat{K}$. In fact, in the unitary gauge, the only change to the closed-string sector theory induced by the nilpotent superfield is the anti-D3-brane uplift contribution to the F-term potential, as long as the goldstino is aligned completely with the x-spinor [147], since the gauge fixing sets $\varphi^x = \psi^x = 0$. Explicitly, the correction to the F-term potential is $\delta\mathcal{V}_F = \mathrm{e}^{2A_0+\kappa_4^2\hat{K}}\, \hat{K}^{x\overline{x}} \nabla_x\hat{W}\nabla_{\overline{x}}\overline{\hat{W}}$, with the terms

$$\begin{aligned} \hat{K}_{x\overline{x}} &= \frac{2}{g_s}\frac{\mathrm{e}^{2\Omega-2A_0}}{[-\mathrm{i}(\tau-\overline{\tau})][-\mathrm{i}\omega_w]}\frac{\ell_w}{\ell_{(0)}}, \\ \nabla_x\hat{W} &= \frac{\sqrt{2}g_s}{\kappa_4^2}. \end{aligned}$$

Notice that ref. [163] does not work with the $2A_0$-shift in the Kähler potential, as is appropriate in regimes not fulfilling eq. (4.2.13). In fact, in ref. [163] the warp factor in the Kähler potential depends on the brane scalars, i.e. $A = A(h^a, \overline{h}^a)$, which would imply a kinetic-term correction for the scalars due to the $2A_0$-shift in the Kähler potential. In the formulation presented here, instead, the term A_0 is independent of the brane scalars.

Complex-Structure Moduli in Warped Throats

In type IIB $N_4 = 1$ compactifications, the axio-dilaton and the complex-structure moduli are typically stabilised at high energy scales. However, in a KS-throat, the complex-structure modulus z controlling the size of 3-sphere at the throat tip stays in the low-energy effective theory [67]. For a dimensionless field z, its vacuum expectation value fixes the warp factor at the tip of the throat as [56]

$$\langle z\overline{z}\rangle^{\frac{1}{3}} = \mathrm{e}^{2A_0} = \mathrm{e}^{-4\pi K/3g_s M}, \tag{4.4.6}$$

where M and K are the quantised F_3- and H_3-fluxes through the conifold 3-sphere and its dual 3-cycle, respectively.

The Kähler metric for the complex-structure modulus z is computed in ref. [266]. Moreover, ref. [68] shows the way to include such a field within the supergravity formulation together with an uplifting anti-D3-brane. Including the Kähler-modulus shift used here, one can write the Kähler- and super-potential as

$$\kappa_4^2 \hat{K} = -3 \ln\left[2\,\mathrm{e}^{-2\Omega} - \frac{4\kappa_4^2}{3g_s}\frac{\ell_w}{\ell_{(0)}}\frac{x\overline{x}}{[-\mathrm{i}(\tau-\overline{\tau})][-\mathrm{i}\omega_w]}\right] + Z_{z\overline{z}}(z\overline{z})z\overline{z},$$
$$\kappa_4^3 \hat{W} = \frac{g_s}{l_s^2}\int_{Y_6} G_3 \wedge \Omega + W(z) + \sqrt{2}\,g_s\kappa_4 z^{\frac{1}{3}} x,$$

where the Kähler metric $Z_{z\overline{z}}$ and the superpotential $W(z)$ determine the vacuum expectation value of the field z to be that in equation (4.4.6); for brevity, the constant term and the axio-dilaton and other complex-structure moduli have been dropped. Also, one may include the Kähler potential shift as the extra Kähler potential coupling

$$\kappa_4^2 \delta\hat{K} = \frac{1}{3}\ln z\overline{z} = 2A_0.$$

Such a term does not participate in the Kähler metric but only in the overall scaling of the energy scales, as it needs to do, and to some scalar and fermionic couplings.

In the KS-throat, the unwarped metric at the tip of the throat scales as $g^0_{mn} \sim \mathrm{e}^{2A_0}$, which is crucial as it sets the Kähler matter metric of the open-string degrees of freedom sitting at the tip of the throat. Therefore, writing the warp factor at the tip in terms of the complex-structure modulus leads, for example, to a coupling from the would-be kinetic term of the form

$$\delta\mathcal{K} = \frac{1}{2\pi g_s}\,\mathrm{e}^{2\Omega} g^0_{a\overline{b}} h^a \overline{h}^b \sim \frac{1}{2\pi g_s}\,(z\overline{z})^{\frac{1}{3}}\,\mathrm{e}^{2\Omega}\delta_{a\overline{b}} h^a \overline{h}^b.$$

It would be interesting to incorporate all such interactions between z and the open-string fields in a complete supergravity description.

Obviously, if the throat is not of the Klebanov-Strassler type, the details of the potentials are different, but by analogy one should expect qualitatively similar results.

4.4.1.3.2 Anti-D3-Brane Modulini

For the modulini of an anti-D3-brane, the pure kinetic term reads

$$S^{\overline{\mathrm{D3}}\text{-modulini}}_{\mathrm{kin}} = -\frac{\mathrm{i}}{2\pi g_s}\int_{X_{1,3}} \mathrm{d}^4x\,\sqrt{-g_4}\,\mathrm{e}^{2\Omega}\,g^0_{a\overline{b}}\,\overline{\psi}^b \overline{\sigma}^\mu \nabla_\mu \psi^a.$$

This can be matched with a supergravity formulation by encoding the spinors ψ^a in the constrained multiplets y^a defined in eq. (4.4.3) and using the Kähler potential

$$\kappa_4^2 \hat{K} = -3\ln\left[2\,\mathrm{e}^{-2\Omega} - \frac{4\kappa_4^2}{3g_s}\frac{\mathrm{e}^{-2A_0}}{[-\mathrm{i}(\tau-\overline{\tau})][-\mathrm{i}\omega_w]}\frac{\ell_w}{\ell_{(0)}}x\overline{x} - \frac{\kappa_4^2}{3\pi g_s}g^0_{a\overline{b}}\,y^a\overline{y}^b\right],$$

or alternatively, after an easy logarithmic expansion, with the Kähler matter metric

$$Z_{y^a\overline{y}^b} = \frac{1}{2\pi g_s}\,\mathrm{e}^{2\Omega}\,g^0_{a\overline{b}}. \tag{4.4.7}$$

For the mass term, from the dimensional reduction one finds

$$S^{\overline{\mathrm{D3}}\text{-modulini}}_{\mathrm{mass}} = -\frac{1}{2\pi g_s}\int_{X_{1,3}} \mathrm{d}^4x\,\sqrt{-g_4}\,\left[m_{\psi^a\psi^b}\psi^a\psi^b + \mathrm{c.c.}\right],$$

with the mass[4.7]

$$m_{\psi^a\psi^b} = -\frac{1}{\left[4\pi\ell_{(0)}\right]^{\frac{1}{2}}}\,\frac{g_s}{4\kappa_4}\,\mathrm{e}^{3\Omega+4A_0+\phi/2}\,l_s^4 g^0_{\bar{c}(a}\,\Omega^0_{b)de}\,(\overline{G}{}_3^{0-})^{\bar{c}de}.$$

Following the method of ref. [163], this mass term can be generated via a Kähler-potential bilinear coupling

$$H_{y^a y^b} = \frac{1}{4\pi g_s^2}\,\frac{\ell_w^{\frac{1}{2}}}{[-\mathrm{i}(\tau-\overline{\tau})][-\mathrm{i}\omega_w]^{\frac{1}{2}}}\,\mathrm{e}^{2\Omega+A_0}\,l_s^4\,g^0_{\bar{c}(a}\,\Omega^0_{b)de}\,(\overline{G}{}_3^{0-})^{\bar{c}de}\,\kappa_4\overline{x}. \tag{4.4.8}$$

Indeed, as required, in an imaginary self-dual background one obtains the effective μ-term

$$\mu_{y^a y^b} = -\hat{\overline{\mathcal{F}}}{}^{x}\partial_{\overline{x}} H_{y^a y^b} = \frac{1}{2\pi g_s}\,m_{\psi^a\psi^b}.$$

It is possible to observe that the scale of the canonically normalised mass is [228]

$$m^w_{\overline{\mathrm{D3}}} \sim \frac{g_s}{\mathcal{V}^{\frac{1}{3}}}\,\frac{1}{\kappa_4}\,\mathrm{e}^{A_0}.$$

4.4.1.3.3 Anti-D3-Brane Scalars

The pure kinetic action for the anti-D3-brane scalars takes the form

$$S^{\overline{\mathrm{D3}}\text{-scalars}}_{\mathrm{kin}} = -\frac{1}{2\pi g_s}\int_{X_{1,3}} \mathrm{d}^4x\,\sqrt{-g_4}\,\mathrm{e}^{2\Omega}\,g^0_{a\bar{b}}\,g^{\mu\nu}\,\nabla_\mu\varphi^a\nabla_\nu\overline{\varphi}^b.$$

In order to correctly account for the expected no-scale structure (also see paragraph 4.3.1.1.1), one needs to generalise the full Kähler potential for the Kähler modulus as

$$\kappa_4^2\hat{K} = -3\ln\!\left[2\,\mathrm{e}^{-2\Omega} - \frac{4\kappa_4^2}{3g_s}\,\frac{\mathrm{e}^{-2A_0}}{[-\mathrm{i}(\tau-\overline{\tau})][-\mathrm{i}\omega_w]}\,\frac{\ell_w}{\ell_{(0)}}\,x\overline{x} - \frac{\kappa_4^2}{3\pi g_s}\,g^0_{a\bar{b}}\,y^a\overline{y}^b - \frac{\kappa_4^2}{3\pi g_s}\,g^0_{a\bar{b}}\,h^a\overline{h}^b\right],$$

where h^a are the constrained chiral multiplets defined in eq. (4.4.4) and containing the scalars φ^a. An expansion of the logarithm shows that the Kähler matter metric is

$$Z_{h^a\overline{h}^b} = \frac{1}{2\pi g_s}\,\mathrm{e}^{2\Omega}\,g^0_{a\bar{b}} + \frac{\kappa_4^2}{3\pi g_s^2}\,\frac{\mathrm{e}^{4\Omega-2A_0}}{[-\mathrm{i}(\tau-\overline{\tau})][-\mathrm{i}\omega_w]}\,\frac{\ell_w}{\ell_{(0)}}\,x\overline{x}\,g^0_{a\bar{b}}. \tag{4.4.9}$$

As far as scalar masses are concerned, from the combination of the relevant parts of the DBI- and CS-actions one finds the term

$$S^{\overline{\mathrm{D3}}\text{-scalars}}_{\mathrm{mass}} = -\frac{1}{2\pi g_s}\int_{X_{1,3}} \mathrm{d}^4x\,\sqrt{-g_4}\,\frac{\mathrm{e}^{4\Omega}}{4\pi\ell_{(0)}}\,\frac{g_s^2}{\kappa_4^2}\,[l_s^2\nabla_a\nabla_{\bar{b}}(\mathrm{e}^{4A}+\alpha)]_0\,\varphi^a\overline{\varphi}^b.$$

[4.7] In ref. [80] the holomorphic 3-form is defined in terms of the γ-matrices that are suitable for the geometry at the tip of the throat. Given the internal Dirac matrices γ_m and the internal spinor η_+ of positive chirality and norm $\eta_+^\dagger\eta_+ = 1$ which defines the SU(3)-structure of the space, with η_- its conjugate, the holomorphic 3-form and the Kähler form are defined as

$$l_s^3\Omega_{mnp} = \eta_-^\dagger\gamma_{mnp}\eta_+, \qquad \tilde{\omega}_{mn} = \mathrm{i}\,\eta_+^\dagger\gamma_{mn}\eta_+.$$

To make estimates in terms of the warp-factor scaling, then one needs to consider the qualitative behaviour $l_s^3\Omega^0 \sim \mathrm{e}^{3A_0}\,(n_f^0)^{3/2}$, consistently with the metric behaviour. This observation is useful for section 4.6.

If only $(2,1)$-flux is present at the tip of the throat, the anti-D3-brane scalar mass-squared trace can be evaluated at leading order thanks to the GKP-equations, which, at a position in the internal space with pure $(2,1)$-flux background, for some order-1 constant k, imply the relation [56, 231, 267][4.8]

$$g^{a\bar{b}}\,\nabla_a\nabla_{\bar{b}}\mathrm{e}^{4A} = \frac{k}{12}\,\mathrm{e}^{8A+\phi}\,G^-_{2,1}\cdot\overline{G}^-_{2,1}.$$

In fact, in a pure $(2,1)$-flux background, the anti-D3-brane mass supertrace vanishes [231], which reflects the fact that in the absence of the orientifold the theory would be supersymmetric. So, the scalar masses are provided by a μ-term equivalent to the modulini one. It is then natural to try to generate the μ-term by using the same H-coupling as for the modulini, in eq. (4.4.8). However, for an H-coupling of the form $H_{h^a h^b}$, because the constrained superfield h^a does not have an independent F-term, the supergravity expansions are different to the standard case, as shown in appendix C.3. It turns out that the coupling $H_{h^a h^b} = H_{y^a y^b}$, is still able to generate a mass

$$m^2_{\varphi^a\overline{\varphi}^b} = 2Z^{h^c\overline{h}^d}\,\hat{\mathcal{F}}^M\hat{\overline{\mathcal{F}}}^N\,H_{h^a h^c,\overline{N}}\overline{H}_{\overline{h}^b\overline{h}^d,M},$$

but this also originates unwanted bilinear couplings, as is going to be seen around the derivation of eq. (4.4.13). An alternative way to describe the mass term is to use a mixed $y^a h^b$-coupling $H_{y^a h^b}$, with $H_{y^a h^b} = H_{y^a y^b}$. One now obtains a scalar mass

$$m^2_{\varphi^a\overline{\varphi}^b} = Z^{y^c\overline{y}^d}\,\hat{\mathcal{F}}^x\hat{\overline{\mathcal{F}}}^x\,H_{h^a y^c,\overline{x}}\overline{H}_{\overline{h}^b\overline{y}^d,x}$$

and it also turns out that the unwanted bilinear interactions are avoided. Such an H-term also contributes a coupling $m^2_{y^a\overline{y}^b}y^a\overline{y}^b$, but this is actually a fermionic term that vanishes in the unitary gauge. The key for this structure is the presence, in the expression of the mass, of the inverse of the diagonal Kähler matter metric combined with two off-diagonal H-terms. All this can be checked explicitly by working out the details of the F-term potential expansion, as explained in appendix C.4. In conclusion, the would-be supersymmetric scalar mass can be reproduced by means of the Kähler potential bilinear coupling

$$H_{h^a y^b} = \frac{1}{4\pi g_s^2}\,\frac{\ell_w^{\frac{1}{2}}}{[-\mathrm{i}(\tau-\overline{\tau})][-\mathrm{i}\omega_w]^{\frac{1}{2}}}\,\mathrm{e}^{2\Omega+A_0}\,l_s^4\,g^0_{\overline{c}(a}\,\Omega^0_{b)de}\,(\overline{G}_3^{0-})^{\overline{c}de}\,\kappa_4\overline{x}. \tag{4.4.10}$$

[4.8] In the GKP-setup [56], by rearranging the 4-dimensional components of the Einstein equations and the field equation of the 4-form potential, one can show the condition (see subsection 2.2.4)

$$\begin{aligned}\nabla^m\nabla_m\big(\mathrm{e}^{4A}+\alpha\big) = \frac{\mathrm{e}^{2A}}{24\,\mathrm{Im}\,\tau}&\big[\mathrm{i}G_{mnp}+(\hat{*}_6G)_{mnp}\big]\big[-\mathrm{i}\overline{G}^{\hat{m}\hat{n}\hat{p}}+(\hat{*}_6\overline{G})^{\hat{m}\hat{n}\hat{p}}\big]\\ &+\mathrm{e}^{-6A}\,\big[\hat{\nabla}_m\big(\mathrm{e}^{4A}+\alpha\big)\big]\big[\hat{\nabla}^{\hat{m}}\big(\mathrm{e}^{4A}+\alpha\big)\big]\\ &-2\hat{\kappa}_{10}^2\,\mathrm{e}^{2A}\left[\frac{1}{4}\left(\hat{g}^{\mu\nu}\hat{T}_{\mu\nu}-\hat{g}^{mn}\hat{T}_{mn}\right)_\sigma - T_3\,\rho^\sigma_{(3)}\right].\end{aligned}$$

In a background with ISD-fluxes $G_3 = -\mathrm{i}*_6 G_3$ and the condition $\mathrm{e}^{4A}=\alpha$, one can observe that:

- the source term vanishes for an anti-D3-brane (and is subleading in the string length for a D7-brane);
- all the flux contributions are expected to have the same functional dependence as the 3-form term.

Therefore, in a pure $(2,1)$-flux, one finds the equation in the main text. Obviously a similar result holds for a generic imaginary self-dual $(2,1)$- and $(0,3)$-flux background.

The analysis of subsection C.3 shows that in general there is also a would-be soft supersymmetry-breaking coupling mass of the form

$$m^2_{\varphi^a\overline{\varphi}^b,\,\mathrm{soft}} = \kappa_4^2\hat{\mathcal{V}}_F Z_{h^a\overline{h}^b} - \hat{\mathcal{F}}^M\hat{\overline{\mathcal{F}}}^N\big[Z_{h^a\overline{h}^b,M\overline{N}} - 2\Gamma^{h^c}_{Mh^a}\,Z_{h^c\overline{h}^d}\,\Gamma^{\overline{h}^d}_{\overline{N}\overline{h}^b}\big] \\ +\big[\hat{m}^w_{3/2}\hat{\mathcal{F}}^M Z_{h^a\overline{h}^b,M} + \hat{\overline{m}}^w_{3/2}\hat{\overline{\mathcal{F}}}^N Z_{h^a\overline{h}^b,N}\big].$$

In a pure $(2,1)$-flux background, the only contribution is from the x-field F-term, which gives the would-be soft-breaking mass

$$m^2_{\varphi^a\overline{\varphi}^b,\,\mathrm{soft}} = \frac{2}{3}\,\kappa_4^2 V_{\overline{\mathrm{D3}}}\,Z_{h^a\overline{h}^b}. \tag{4.4.11}$$

This term can be seen to emerge in the dimensional reduction as follows. In the presence of the anti-D3-brane scalars, the volume is shifted and the total Weyl factor should be such that [227, 268]

$$\mathrm{e}^{-2\Omega'} = \mathrm{e}^{-2\Omega} - \frac{\kappa_4^2}{6\pi g_s}\,g^0_{a\overline{b}}h^a\overline{h}^b, \tag{4.4.12}$$

with the actual uplift energy $V'_{\overline{\mathrm{D3}}} = 4\pi\gamma^3\,\mathrm{e}^{4\Omega'+4A_0}/g_s l_s^4$. If one expands this energy in h^a, then what is obtained is exactly the sought-after factor, being

$$V'_{\overline{\mathrm{D3}}}(\mathrm{e}^{2\Omega'}) = V_{\overline{\mathrm{D3}}}(\mathrm{e}^{2\Omega})\left[1 + \frac{2}{3}\,\kappa_4^2 Z_{h^a\overline{h}^b}h^a\overline{h}^b\right].$$

If a non-zero $(0,3)$-flux were present at the tip of the throat too, the scalar masses would receive extra contributions in the dimensional reduction. This flux instead would not modify the modulini masses, but it would provide a mass to the gaugino. The scalar mass correction cannot be added as a would-be supersymmetric μ-term, since an F^x-induced extra contribution gives cross-terms between $(2,1)$- and $(0,3)$-fluxes in the scalar mass-squared trace, which are not seen in the dimensional reduction [231], and an F^ρ-induced would-be μ-coupling cannot work either because it is impossible to find a scaling $H_{ab} \propto \mathrm{e}^{n\Omega}$ giving a mass $m^2_{a\overline{b}} \propto \mathrm{e}^{4\Omega}$. Instead, the matching can be achieved via a would-be soft-breaking term, by adding an extra $x\overline{x}$-term in the Kähler metric in eq. (4.4.9). Notice that, even in the presence of a non-vanishing F^ρ-term, the scalar masses are still partially protected by a no-scale cancellation

$$Z_{h^a\overline{h}^b,\,\rho\overline{\rho}} - 2\Gamma^{h^c}_{\rho h^a}\,Z_{h^c\overline{h}^d}\,\Gamma^{\overline{h}^d}_{\overline{\rho}\overline{h}^b} = 0.$$

This is a specific feature of the constrained-superfield would-be supersymmetry-breaking mass expression, since the usual soft supersymmetry-breaking mass vanishes in the presence of a logarithmic structure but due to a different cancellation involving the gravitino mass. However, there is an extra F^ρ-induced term giving an unwanted mass $m^2_{a\overline{b}} \propto \mathrm{e}^{6\Omega}$: this is a common issue for highly-warped setups if working with just one Kähler modulus, but, if needed, it can be removed by an extra $x\overline{x}$-term. In the main scenario considered, only a $(2,1)$-flux is present at the tip of the throat, so the $(0,3)$-flux-induced mass must vanish.

From the dimensional reduction one also obtains bilinear and trilinear couplings. For an Abelian anti-D3-brane, the bilinear coupling is

$$S^{\overline{\mathrm{D3}}\text{-scalars}}_{\text{bilinear}} = -\frac{1}{2\pi g_s}\int_{X_{1,3}}\mathrm{d}^4x\,\sqrt{-g_4}\,\frac{\mathrm{e}^{4\Omega}}{8\pi\ell_{(0)}}\,\frac{g_s^2}{\kappa_4^2}\left([l_s^2\nabla_a\nabla_b(\mathrm{e}^{4A}+\alpha)]_0\,\varphi^a\varphi^b + \text{c.c.}\right),$$

whilst there are no trilinear couplings. The description within supergravity follows from the discussions in subsections C.2 and C.3. As there are no bilinear $\tilde{\mu}$-couplings, for a term $H_{h^a h^b}$ the generic B-coupling would be

$$\begin{aligned} B_{\varphi^a\varphi^b} = \kappa_4^2 \hat{\mathcal{V}}_F H_{h^a h^b} + \hat{\overline{m}}{}^w_{3/2} \hat{\overline{\mathcal{F}}}{}^M \partial_{\overline{M}} H_{h^a h^b} + \hat{m}^w_{3/2} \hat{\mathcal{F}}^M \hat{\nabla}_M H_{h^a h^b} \\ - \hat{\mathcal{F}}^M \hat{\overline{\mathcal{F}}}{}^N \left(H_{h^a h^b, M\overline{N}} - 4\Gamma^l_{Mi} H_{h^a h^b, \overline{N}} \right). \end{aligned}$$

One can now observe that if a term $H_{h^a h^b} \propto \overline{x}\, \mathrm{e}^{2\Omega}$ were used to generate the mass term $m_{\varphi^a \overline{\varphi}^b}$, it would also give a B-term scaling as $B_{\varphi^a\varphi^b} \propto \mathrm{e}^{6\Omega + 4A_0}$, which is not present in the dimensional reduction. Although this may be cancelled by a suitable counter-term $H'_{h^a h^b} \propto x\overline{x}\, \mathrm{e}^{4\Omega - 2A_0}$, it is simpler to instead obtain the mass term via the coupling $H_{h^a y^b}$, as chosen in eq. (4.4.10); this only generates a bilinear term $B_{h^a y^b}$, which is not a scalar coupling and vanishes in the unitary gauge. The required coupling $B_{\varphi^a\varphi^b}$ above can be obtained by defining an extra H-term

$$H'_{h^a h^b} = \frac{1}{2\pi g_s} \frac{\mathrm{e}^{4\Omega}}{8\pi \ell_{(0)}} \frac{g_s^2}{\kappa_4^2} [l_s^2 \nabla_a \nabla_b (\mathrm{e}^{4A} + \alpha)]_0 \frac{x\overline{x}}{\hat{\mathcal{F}}^x \hat{\overline{\mathcal{F}}}{}^{x}}. \tag{4.4.13}$$

This generates only the required coupling since it just affects the B-term because this is the only operator with a term scaling as a second x-derivative of the H-term.

4.4.1.3.4 Anti-D3-Brane Gauge Field

Compared to the D3-brane gauge field, the anti-D3-brane gauge field is described by the same DBI-term but by an opposite CS-term, which results in the 4-dimensional action

$$S_{\text{kin}}^{\overline{\text{D3}}\text{-vector}} = -\frac{1}{4\pi g_s} \int_{X_{1,3}} \mathrm{e}^{-\phi}\, F_2 \wedge *F_2 - \frac{1}{4\pi g_s} \int_{X_{1,3}} C_0\, F_2 \wedge F_2.$$

Of course, the gauge kinetic function cannot be simply $f_{\overline{\text{D3}}} = \mathrm{i}\overline{\tau}/2\pi g_s$ as it is not holomorphic in the axio-dilaton. A solution to this issue is given in ref. [163], which finds

$$f_{\overline{\text{D3}}} = (\overline{\mathrm{D}}^2 - 8\mathrm{R}) \left(\frac{\overline{x} \overline{f}_{\text{D3}}(\overline{\tau})}{\overline{\mathrm{D}}^2 \overline{x}} \right),$$

with D_α the supergravity fermionic derivative and R the gravity multiplet. This function is holomorphic thanks to the projectors but at the same time has a superspace expansion

$$f_{\overline{\text{D3}}} = \frac{\mathrm{i}\overline{\tau}}{2\pi g_s} + O(x). \tag{4.4.14}$$

Because x is the nilpotent superfield, all the extra terms are proportional to the goldstino and therefore vanish in the unitary gauge.

4.4.2 Anti-D3-/D7-Brane Intersecting States

For intersecting anti-D3-/D7-branes systems, the pure anti-D3- and pure D7-states have been described in the previous subsections. It is also possible to provide a supergravity formulation of anti-D3-/D7-brane intersecting states:

- on the one hand, one can infer the scaling factors for the kinetic and interaction terms of anti-D3-/D7-brane intersecting states using the D3-/D7-brane system discussed in subsection 4.3.2;
- on the other hand, the tools of constrained superfields allow one to formulate the low-energy theory in the language of supergravity.

4.4.2.1 Anti-D3-/D7-Brane Constrained Superfields and Couplings

The strings stretching between the anti-D3- and the D7-brane give two scalar fields φ and φ_* as well as two Weyl spinors ψ and ψ_*; in particular, the fields (φ, ψ) and (φ_*, ψ_*) are in conjugate representations of the gauge groups.

Similarly to the pure anti-D3-brane states, because the anti-D3-/D7-brane intersecting states do not preserve the supersymmetry of the Calabi-Yau orientifold bulk, the natural tool to describe them consists in constrained superfields. It is impossible to identify the constrained superfields for the intersecting states by comparing with supersymmetry variations because the latter are unknown as they cannot be inferred directly from a dimensional reduction. However, one can postulate the following ones:

(i) the scalar fields φ and φ_* belong to the chiral superfields h and h_* satisfying the spinor-removing constraints

$$x\overline{x}\,\mathrm{D}_\alpha h = 0, \tag{4.4.15a}$$

$$x\overline{x}\,\mathrm{D}_\alpha h_* = 0; \tag{4.4.15b}$$

(ii) the Weyl spinors ψ and ψ_* belong to the chiral superfields y and y_* satisfying the scalar-removing constraints

$$xy = 0, \tag{4.4.16a}$$

$$xy_* = 0. \tag{4.4.16b}$$

These constraints have been chosen because they are the easiest way [32] to remove the undesired degrees of freedom from the effective theory below the anti-D3-brane supersymmetry-breaking scale. In particular, notice that the constraint for the scalar fields is such as to leave an independent F-term [269].

In the strongly-warped regime set by eqs. (4.2.13, 4.2.16), the Kähler potential contains the $2A_0$-shift as in eq. (4.2.17). Given the closed-string and anti-D3-brane goldstino potentials $\hat{K}$ and $\hat{W}$ of eqs. (4.4.5a, 4.4.5b), one can argue that the total Kähler potential and superpotential are

$$\begin{aligned} K = \hat{K} &+ Z_{y^a\overline{y}^b} y^a \overline{y}^b + \frac{1}{2}\big[H_{h^a h^b} y^a y^b + \text{c.c.}\big] \\ &+ Z_{h^a\overline{h}^b} h^a \overline{h}^b + \frac{1}{2}\big[H_{y^a h^b} y^a h^b + \text{c.c.}\big] \\ &+ Z_{\sigma^3\overline{\sigma}^3} \sigma^3 \overline{\sigma}^3 + \frac{1}{2}\big[H_{\sigma^3\sigma^3} \sigma^3 \sigma^3 + \text{c.c.}\big] \\ &+ Z_{h\overline{h}} h\overline{h} + Z_{y\overline{y}} y\overline{y} + Z_{h_*\overline{h}_*} h_*\overline{h}_* + Z_{y_*\overline{y}_*} y_*\overline{y}_*, \end{aligned} \tag{4.4.17a}$$

$$W = \hat{W} + \frac{1}{2}\,\tilde{\mu}_{\sigma^3\sigma^3}\sigma^3\sigma^3 + \tilde{\xi}(\beta\sigma^3 - y^3 - h^3)yy_* + \tilde{\xi}(\beta\sigma^3 - y^3 - h^3)hy_* + \tilde{\xi}(\beta\sigma^3 - y^3 - h^3)yh_*, \tag{4.4.17b}$$

The pure anti-D3- and D7-brane terms follow from those discussed in subsubsections 4.3.1.2, 4.3.1.3, 4.4.1.3, and their theory is the same except for the anti-D3-brane uplift effect on the D7-brane theory to be discussed. The other terms represent the intersecting states and they are going to be discussed below.

The fields belonging to the supermultiplets h, y, and h_*, y_* have charges $q_{\overline{\mathrm{D3}}} = 1, -1$ and $q_{\mathrm{D7}} = -1, 1$, respectively, under the anti-D3- and D7-brane gauge groups.

4.4.2.2 Anti-D3-Brane with D7-Brane from the Throat Tip into the Bulk

In the setup in which the anti-D3-brane sits at the tip of the warped throat and the D7-brane wraps a 4-cycle extending from the throat tip into the bulk, the couplings for the intersecting states in eqs. (4.4.17a, 4.4.17b) are as follows.

- Because the kinetic terms are not affected by the flux-induced supersymmetry breaking, for anti-D3-/D7-brane intersecting states one can make use of the same Kähler matter metric terms as for the D3-/D7-brane case. The logarithmic structure that is equivalent to eq. (4.3.19) for D3-/D7-branes is generaralised to

$$\kappa_4^2 K = -3\ln\left[2\,\mathrm{e}^{-2\Omega} - \frac{4\kappa_4^2}{3g_s}\frac{\ell_w}{\ell_{(0)}}\frac{\mathrm{e}^{-2A_0}\,x\overline{x}}{[-\mathrm{i}(\tau-\overline{\tau})][-\mathrm{i}\omega_w]} - \frac{\kappa_4^2}{3\pi g_s}\,\varphi\overline{\varphi}\right],$$

  so the matter metrics for anti-D3-/D7-branes are defined to be

$$Z_{h\overline{h}} = \frac{1}{2\pi g_s}\,\mathrm{e}^{2\Omega} + \frac{\kappa_4^2}{3\pi g_s^2}\,\frac{\mathrm{e}^{4\Omega-2A_0}}{[-\mathrm{i}(\tau-\overline{\tau})][-\mathrm{i}\omega_w]}\,\frac{\ell_w}{\ell_{(0)}}\,x\overline{x}, \tag{4.4.18a}$$

$$Z_{y\overline{y}} = \frac{1}{2\pi g_s}\,\mathrm{e}^{2\Omega}. \tag{4.4.18b}$$

  This is consistent with the intersecting states not acquiring flux-induced masses [235] due to similar cancellations to those discussed for the intersecting D3-/D7-brane scalars. This is also going to be discussed explicitly in subsection 4.4.3.

- For the trilinear couplings in the superpotential, further explanations are required, as two related but distinct features from the higher-dimensional setup need considering.

  (i) Using the internal-space symmetries of the flux-dependent couplings, ref. [235] shows that the anti-D3-/D7-brane intersecting states couple only to the pure anti-D3-brane states and not to the pure D7-brane states. The coupling 3-form flux can be written as $G_3'' = g_2'' \wedge \mathrm{d}w^3$, where $g_2'' = g_2''(w^3, \overline{w}^3)$ is a combination of $(1,1)$-forms on the 4-cycle, and the scalar trilinear couplings are of the kind

$$t_{\alpha\beta\gamma} = \frac{1}{\kappa_4}\,u(\mathrm{e}^{2\Omega}, \mathrm{e}^{2A_0})\,c_{\alpha\beta\gamma},$$

where (see appendix B.3 for the explicit expressions of the $(1,1)$-forms ζ_i)

$$c_{h^3\overline{h}\overline{h}_*} = \frac{1}{l_s^2}\int_{\Sigma_4} g_2'' \wedge \zeta_1, \tag{4.4.19a}$$

$$c_{h^3hh_*} = \frac{1}{l_s^2}\int_{\Sigma_4} g_2'' \wedge \zeta_2, \tag{4.4.19b}$$

$$c_{h^3h_*\overline{h}_*} = \frac{1}{l_s^2}\int_{\Sigma_4} g_2'' \wedge (\zeta_3+\zeta_4) = c_{h^3h\overline{h}}, \tag{4.4.19c}$$

with the overall factor

$$u(\mathrm{e}^{2\Omega}, \mathrm{e}^{2A_0}) = \frac{1}{4\pi}\,\frac{\mathrm{e}^{7\Omega+3A_0}}{[-\mathrm{i}(\tau-\overline{\tau})]^{\frac{1}{2}}\,[-\mathrm{i}\omega_w^{\Sigma_4}]}\left[\frac{1}{\pi\ell_{(0)}}\right]^{\frac{1}{2}}\frac{\ell_w^{\Sigma_4}}{(\ell_{(0)}^{\Sigma_4})^{\frac{1}{2}}}.$$

A $(2,1)$-flux sources the coupling, but it is not the same flux that sources the D7-brane mass. Indeed ref. [235] identifies the flux components that the couplings depend on, while the overall scaling u has been inferred from the D3-/D7-brane case by matching the canonically normalised couplings (see eqs. (4.3.22, 4.3.25), and notice the ratio $(Z_{\varphi^3\overline{\varphi}^3}/Z_{\sigma^3\overline{\sigma}^3})^{1/2} = (g^0_{3\overline{3}}/g_{3\overline{3}})^{1/2}\mathrm{e}^{\Omega}[-\mathrm{i}(\tau-\overline{\tau})]^{1/2}/(2\ell^{\Sigma_4}_{(0)})^{1/2})$.

(ii) Also, one needs to account for the mass due to the brane separation in a supersymmetric way since both the scalars and the spinors acquire the same separation mass. A way to do that is via a trilinear coupling in the superpotential.

A natural guess to implement both these facts in the 4-dimensional effective theory is a generalisation of the trilinear coupling in eq. (4.3.20), with all the permutations accounting for the fact that now scalars and spinors are in different multiplets. Because for ISD-fluxes both the anti-D3- and the D7-brane have an effective superpotential bilinear coupling, though, such a term would again generate a coupling of the D7-brane state σ^3 with the intersecting states. A way to avoid it is to exclude the coupling[4.9]

$$\delta W = \tilde{\xi}(\sigma^3 - h^3 - y^3)hh_*.$$

As a matter of fact the trilinear couplings of the proposed superpotential in eq. (4.4.17b), namely

$$\tilde{Y}_{\sigma^3yy_*} = \tilde{Y}_{\sigma^3hy_*} = \tilde{Y}_{\sigma^3yh_*} = \tilde{\xi}, \tag{4.4.20a}$$

$$\tilde{Y}_{y^3yy_*} = \tilde{Y}_{y^3hy_*} = \tilde{Y}_{y^3yh_*} = -\tilde{\xi}, \tag{4.4.20b}$$

$$\tilde{Y}_{h^3yy_*} = \tilde{Y}_{h^3hy_*} = \tilde{Y}_{h^3yh_*} = -\tilde{\xi}, \tag{4.4.20c}$$

are enough to generate the desired couplings apart from a couple, which however will be dealt with in paragraph 4.4.2.2.2.

[4.9] The removal of the term $\delta_1 W = \tilde{\xi}\sigma^3hh_*$ prevents the couplings with the D7-brane, the absence of the term $\delta_2 W = -\tilde{\xi}h^3hh_*$ prevents the generation of quartic couplings of the anti-D3-brane with the intersecting states already generated by the other terms – which however generate the would-be separation mass terms in an easy way including the D7-brane scalar too – and the absence of the term $\delta_3 W = -\tilde{\xi}y^3hh_*$ prevents the cubic scalar coupling $\delta V_F = \overline{\xi}\mu_{33}\varphi^3\overline{\varphi\varphi}_* + \text{c.c.}$, which is also forbidden by the symmetry arguments of ref. [235]. All these facts can be checked in the F-term potential in eq. (4.4.27).

4.4.2.2.1 Standard Supergravity Terms

One now needs to determine the effective D- and F-term potentials as well as the soft would-be supersymmetry-breaking couplings. Most of the terms have already been worked out in the earlier discussions on anti-D3- and D7-brane states, so one can focus on the interplay between the branes and on the new terms from intersecting states.

- For the D7-brane, most of the calculations hold as in the analysis of the pure D7-brane in subsubsection 4.3.1.2 and others are similar to the case of intersecting D3-/D7-branes in subsubsection 4.3.2.1, as now summarised.

  For the supersymmetric terms, the effective μ-coupling and the corresponding supersymmetric mass is exactly the same as for the pure D7-brane construction. On the other hand, the effective superpotential couplings follow straightforwardly from the superpotential and are

  $$Y_{\sigma^3 yy_*} = Y_{\sigma^3 hy_*} = Y_{\sigma^3 yh_*} = \xi. \tag{4.4.21}$$

  Notice that the superpotential gives exactly the same (and no extra) Yukawa couplings as the D3-/D7-brane construction, since only the terms with one scalar and two spinors generate proper Yukawa terms.

  For the supersymmetry-breaking terms, assuming that the Kähler metric and the H-term do not depend on x since they come from a deformation of the axio-dilaton Kähler potential, from the general expression one can observe the soft-breaking mass

  $$\begin{aligned} m^2_{\sigma^3\overline{\sigma}^3,\,\mathrm{soft}} &= \left(\hat{m}^w_{3/2}\hat{\overline{m}}^w_{3/2} + \kappa_4^2 \mathcal{V}_F\right) Z_{\sigma^3\overline{\sigma}^3} - \mathcal{F}^M \overline{\mathcal{F}}^N R_{M\overline{N}\sigma^3\overline{\sigma}^3} \\ &= (m^{\mathrm{flux}}_{\sigma^3\overline{\sigma}^3,\,\mathrm{soft}})^2 + \delta m^2_{\sigma^3\overline{\sigma}^3,\,\mathrm{soft}}, \end{aligned}$$

  where $m^{\mathrm{flux}}_{\sigma^3\overline{\sigma}^3,\,\mathrm{soft}}$ represents the flux-induced soft-breaking terms, which then get an extra uplifting contribution due to the supersymmetry breaking by the anti-D3-brane, with

  $$\delta m^2_{\sigma^3\overline{\sigma}^3,\,\mathrm{soft}} = \kappa_4^2 V_{\overline{\mathrm{D3}}}\, Z_{\sigma^3\overline{\sigma}^3} = \left[\frac{g_s}{2\pi\ell_{(0)}}\right]^2 \frac{\mathrm{e}^{4\Omega+4A_0}\ell^{\Sigma_4}_{(0)}}{\kappa_4^2[-\mathrm{i}(\tau-\overline{\tau})]}. \tag{4.4.22}$$

  The effective B-term follows a similar destiny since it can be seen to read

  $$B_{\sigma^3\sigma^3} = B^{\mathrm{flux}}_{\sigma^3\sigma^3} + \kappa_4^2 V_{\overline{\mathrm{D3}}} H_{\sigma^3\sigma^3}. \tag{4.4.23}$$

  Finally, the trilinear A-terms do not generate any scalar trilinear coupling as the trilinear terms of eq. (4.4.21) never involve three scalars due to the constraints, which means that the would-be scalar trilinear coupling is actually a fermionic interaction.

- For the anti-D3-brane, there is no substantial difference with respect to the analysis of subsubsection 4.4.1.3 since there are no new bilinear couplings in the Kähler potential or in the superpotential. One also has the superpotential trilinear couplings

  $$Y_{y^3 yy_*} = Y_{y^3 hy_*} = Y_{y^3 yh_*} = -\xi, \tag{4.4.24a}$$
  $$Y_{h^3 yy_*} = Y_{h^3 hy_*} = Y_{h^3 yh_*} = -\xi. \tag{4.4.24b}$$

Evidently, these terms just add couplings between the anti-D3-brane and the intersecting states, but do not cause any particular modification to the pure anti-D3-brane action. Again, the superpotential also gives exactly the same Yukawa couplings as in the D3-/D7-brane construction.

- For the anti-D3-/D7-brane intersecting states, because their Kähler potential and superpotential expansion terms do not involve bilinear terms apart from the Kähler matter metric, one simply has the trilinear superpotential couplings discussed above and the soft-breaking masses

$$m^2_{\varphi\overline{\varphi},\,\mathrm{soft}} = \big(\hat{m}^w_{3/2}\hat{\overline{m}}^w_{3/2}\, Z_{h\overline{h}} - \mathcal{F}^\rho\overline{\mathcal{F}}^\rho\, R_{\rho\overline{\rho}h\overline{h}}\big) + \big(\kappa_4^2 V_{\overline{\mathrm{D3}}}\, Z_{h\overline{h}} - \mathcal{F}^x\overline{\mathcal{F}}^x\, R_{x\overline{x}h\overline{h}}\big),$$

and similarly for the counterpart φ_*. This is referred to an ISD-background for definiteness. The first contribution always vanishes before non-perturbative corrections kick in, but the second one does not and reads

$$\delta m^2_{\varphi\overline{\varphi},\,\mathrm{soft}} = \frac{2}{3}\,\kappa_4^2 V_{\overline{\mathrm{D3}}}\, Z_{h\overline{h}} = \left[\frac{g_s}{2\pi\ell_{(0)}}\right]^2 \frac{\mathrm{e}^{6\Omega+4A_0}}{3\kappa_4^2}. \tag{4.4.25}$$

Because these fields have no pure bilinear and trilinear couplings in the Kähler- and super-potential, they do not have further couplings among themselves alone.

To conclude, one must consider the complete effective D- and F-term potentials. First of all, for the D-term potential, one has again the positive semi-definite quartic self-interaction terms (and similarly for the corresponding field φ_*)

$$\begin{aligned} V_D^{(\mathrm{susy})} &= \frac{1}{2}\, g^2_{\overline{\mathrm{D3}}}\, (Z_{h\overline{h}}\varphi\overline{\varphi})^2 + \frac{1}{2}\, g^2_{\mathrm{D7}}\, (Z_{h\overline{h}}\varphi\overline{\varphi})^2 \\ &= \frac{\mathrm{e}^{4\Omega}}{2\pi g_s\,[-\mathrm{i}(\tau-\overline{\tau})]}\, (\varphi\overline{\varphi})^2 + \frac{\mathrm{e}^{6\Omega}}{4\pi g_s \ell_{(0)}^{\Sigma_4}}\, (\varphi\overline{\varphi})^2. \end{aligned} \tag{4.4.26}$$

Second, for the F-term potential, most of the terms that are generated are actually fermionic interactions and not scalar couplings. Taking into account the effective bilinear terms from the pure D7- and anti-D3-branes as well as the Yukawa couplings in eqs. (4.4.21, 4.4.24), one obtains the effective superpotential

$$\begin{aligned} W_{\mathrm{susy}} = \frac{1}{2}\,\mu_{\sigma^3\sigma^3}\sigma^3\sigma^3 + \frac{1}{2}\,\mu_{y^a y^b} y^a y^b + \mu_{y^a h^b} y^a h^b + \xi\,(\sigma^3 - y^3 - h^3) y y_* \\ + \xi\,(\sigma^3 - y^3 - h^3) h y_* + \xi\,(\sigma^3 - y^3 - h^3) y h_*. \end{aligned}$$

Therefore, the effective F-term potential takes the form

$$\begin{aligned} V_F^{(\mathrm{susy})} = Z^{\sigma^3\overline{\sigma}^3}\, \mu_{\sigma^3\sigma^3}\overline{\mu}_{\overline{\sigma}^3\overline{\sigma}^3}\sigma^3\overline{\sigma}^3 + Z^{y^a\overline{y}^b}\, \mu_{y^a h^c}\overline{\mu}_{\overline{y}^b\overline{h}^d}\varphi^c\overline{\varphi}^d \\ + Z^{y\overline{y}}\big[\xi\big(\sigma^3 - \varphi^3\big)\varphi_*\big]\big[\overline{\xi}\big(\overline{\sigma}^3 - \overline{\varphi}^3\big)\overline{\varphi}_*\big] \\ + Z^{y_*\overline{y}_*}\big[\xi\big(\sigma^3 - \varphi^3\big)\varphi\big]\big[\overline{\xi}\big(\overline{\sigma}^3 - \overline{\varphi}^3\big)\overline{\varphi}\big]. \end{aligned} \tag{4.4.27}$$

One immediately recognises the D7-brane supersymmetric mass, the anti-D3-brane scalar mass and the would-be separation mass for the anti-D3-/D7-brane intersecting states, with the same volume scaling as for the D3-/D7-brane case.[4.10]

The constrained multiplets h^a have constrained F-terms, but they always appear in mixed $h^a y^b$-, $h^a y y_*$-, $h^a h y_*$- and $h^a y h_*$-couplings. Therefore they both contribute the non-standard couplings discussed in appendix C.3, which turn out be fermionic and vanishing in the unitary gauge, and standard couplings via the effect of y^b, y and y_*, which have unconstrained F-terms and end up providing bosonic terms in the action. This is exactly the same mechanism discussed in paragraph 4.4.1.3.3.

4.4.2.2.2 $x\overline{x}$-Dependent Interaction Terms

The supergravity formulation described so far incorporates all the expected couplings, except the trilinear flux couplings in eqs. (4.4.19a, 4.4.19b, 4.4.19c) and an anti-D3-/D7-brane version of the D3-/D7-brane quartic potential in eq. (4.3.23).

These couplings can be obtained by considering a specific class of supersymmetric terms, introduced in refs. [157] (for further developments and applications, see e.g. refs. [161, 163, 270, 271]). This involves the nilpotent goldstino field in such a way as to only contribute bosonic terms to the component action, with the fermionic terms vanishing in the unitary gauge. Indeed, the coupling in eqs. (4.4.19a, 4.4.19b, 4.4.19c) can be described by adding to the Kähler potential in eq. (4.4.17a) the deformation

$$\delta\mathcal{K} = \frac{2[\ell_w^{\Sigma_4}]^2}{g_s^4} \left[\frac{1}{\pi}\,\frac{\ell_{(0)}}{\ell_{(0)}^{\Sigma_4}}\right]^{\frac{1}{2}} \frac{\kappa_4^2 x\overline{x}\,\mathrm{e}^{5\Omega-3A_0}}{[-\mathrm{i}(\tau-\overline{\tau})]^{\frac{3}{2}}\,[-\mathrm{i}\omega_w^{\Sigma_4}]^2} \left[\kappa_4\, c_{\alpha\beta\gamma} h^\alpha h^\beta h^\gamma + \text{c.c.}\right]. \tag{4.4.28}$$

The only modification that this induces in the bosonic action comes from the second derivative with respect to x and $\overline{x}$, namely $\delta V_F = \delta\mathcal{K}_{x\overline{x}}\mathcal{F}^x\overline{\mathcal{F}}^x$ as all the other terms contain the scalar component of x, which is proportional to the goldstino. One can similarly include the coupling of eq. (4.3.23).

4.4.2.3 Anti-D3-Brane and D7-Brane at the Tip of the Throat

If the anti-D3-brane and the D7-brane are localised at the tip of the warped throat, the supergravity couplings for the intersecting states in eqs. (4.4.17a, 4.4.17b) are given explicitly as follows.

[4.10] For ease of notation, only the non-fermionic terms have been reported. Denoting the fermionic terms that one would have in a generic way as r_x, the actual expression one finds is

$$\begin{aligned} V_F^{(\text{susy})} = {} & Z^{\sigma^3\overline{\sigma}^3}\big(\mu_{\sigma^3\sigma^3}\sigma^3 + r_x\big)\big(\overline{\mu}_{\overline{\sigma}^3\overline{\sigma}^3}\overline{\sigma}^3 + \overline{r}_{\overline{x}}\big) + Z^{y^a\overline{y}^b}\big(\mu_{y^a h^c}\varphi^c + r_x\big)\big(\overline{\mu}_{\overline{y}^b\overline{h}^d}\overline{\varphi}^d + \overline{r}_{\overline{x}}\big) \\ & + Z^{y\overline{y}}\big[\xi(\sigma^3-\varphi^3)\varphi_* + r_x\big]\big[\overline{\xi}(\overline{\sigma}^3-\overline{\varphi}^3)\overline{\varphi}_* + \overline{r}_{\overline{x}}\big] \\ & + Z^{y_*\overline{y}_*}\big[\xi(\sigma^3-\varphi^3)\varphi + r_x\big]\big[\overline{\xi}(\overline{\sigma}^3-\overline{\varphi}^3)\overline{\varphi} + \overline{r}_{\overline{x}}\big]. \end{aligned}$$

- As in subsubsection 4.4.2.2, the Kähler matter metric terms for the anti-D3-/D7-brane intersecting states read

$$Z_{h\bar{h}} = \frac{1}{2\pi g_s}\,\mathrm{e}^{2\Omega} + \frac{\kappa_4^2}{3\pi g_s^2}\,\frac{\mathrm{e}^{4\Omega-2A_0}}{[-\mathrm{i}(\tau-\overline{\tau})][-\mathrm{i}\omega_w]}\,\frac{\ell_w}{\ell_{(0)}}\,x\overline{x}, \tag{4.4.29a}$$

$$Z_{y\bar{y}} = \frac{1}{2\pi g_s}\,\mathrm{e}^{2\Omega}. \tag{4.4.29b}$$

- For the cubic superpotential term, one can again follow subsubsection 4.4.2.2. For a localised D7-brane there is no $(0,3)$-flux coupling for the intersecting D3-/D7-brane states, so, following the tangent-space symmetry arguments of ref. [235] and the scaling factors determined therein, the trilinear scalar couplings are still of the form

$$t_{\alpha\beta\gamma} = \frac{1}{\kappa_4}\,u(\mathrm{e}^{2\Omega},\mathrm{e}^{2A_0})\,c_{\alpha\beta\gamma},$$

where the flux and index structure is

$$c_{h^3\overline{h}\overline{h}_*} = \frac{1}{l_s^2}\int_{\Sigma_4} g_2'' \wedge \zeta_1, \tag{4.4.30a}$$

$$c_{h^3h_*\overline{h}_*} = \frac{1}{l_s^2}\int_{\Sigma_4} g_2'' \wedge (\zeta_2+\zeta_3) = c_{h^3h\overline{h}}, \tag{4.4.30b}$$

but with an overall factor

$$u(\mathrm{e}^{2\Omega},\mathrm{e}^{2A_0}) = \frac{1}{4\pi}\,\frac{\mathrm{e}^{4\Omega+2A_0}}{[-\mathrm{i}(\tau-\overline{\tau})]^{\frac{1}{2}}\,[-\mathrm{i}\omega_{(0)}^{\Sigma_4}]}\left[\frac{2\varsigma}{\pi}\,\frac{\ell_{(0)}^{\Sigma_4}}{\ell_{(0)}}\right]^{\frac{1}{2}}.$$

The matching with the scaling for the D3-/D7-brane coupling in eq. (4.3.29) is done in terms of the canonically normalised fields. Anyway, as in subsubsection 4.4.2.2, the Yukawa couplings are still simply

$$\tilde{Y}_{\sigma^3yy_*} = \tilde{Y}_{\sigma^3hy_*} = \tilde{Y}_{\sigma^3yh_*} = \tilde{\xi}\beta, \tag{4.4.31a}$$

$$\tilde{Y}_{y^3yy_*} = \tilde{Y}_{y^3hy_*} = \tilde{Y}_{y^3yh_*} = -\tilde{\xi}, \tag{4.4.31b}$$

$$\tilde{Y}_{h^3yy_*} = \tilde{Y}_{h^3hy_*} = \tilde{Y}_{h^3yh_*} = -\tilde{\xi}, \tag{4.4.31c}$$

with $\beta = \mathrm{e}^{-A_0}$, from the discussion of subsubsection 4.3.2.3.

4.4.2.3.1 Standard Supergravity Terms

Again, one can study the interactions term by term.

- For the D7-brane, the results of subsubsection 4.3.1.3 still hold with the further anti-D3-brane contribution to the soft-breaking mass[4.11]

$$\delta m^2_{\sigma^3\overline{\sigma}^3,\,\mathrm{soft}} = \kappa_4^2 V_{\overline{\mathrm{D3}}}\,Z_{\sigma^3\overline{\sigma}^3} = \left[\frac{g_s}{2\pi\ell_{(0)}}\right]^2 \frac{\mathrm{e}^{6\Omega+4A_0}}{\kappa_4^2[-\mathrm{i}(\tau-\overline{\tau})]} \tag{4.4.32}$$

[4.11] Since the Kähler metric now contains a factor $\mathrm{e}^{2\Omega}/\mathrm{Im}\,\tau$, it is ambiguous whether it comes from a shift in the axio-dilaton Kähler potential or the Kähler-modulus one. In the latter case, the D7-brane Kähler metric acquires an $x\overline{x}$-dependence, and there is an additional contribution to $\delta m^2_{\sigma^3\overline{\sigma}^3,\,\mathrm{soft}}$, which results in an overall factor $f = 2/3$ in the total expression.

and the B-term

$$B_{\sigma^3\sigma^3} = \kappa_4^2 V_{\overline{\mathrm{D3}}} H_{\sigma^3\sigma^3}. \tag{4.4.33}$$

Further, now one has the effective superpotential couplings

$$Y_{\sigma^3 yy_*} = Y_{\sigma^3 hy_*} = Y_{\sigma^3 yh_*} = \xi\, \mathrm{e}^{-A_0}. \tag{4.4.34}$$

Finally, the trilinear A-terms do not generate any scalar trilinear coupling since in fact they correspond to fermionic interactions.

- For the anti-D3-brane, the same results as in subsubsection 4.4.1.3 hold identically. Further, there are the superpotential trilinear couplings

$$Y_{y^3 yy_*} = Y_{y^3 hy_*} = Y_{y^3 yh_*} = -\xi, \tag{4.4.35a}$$
$$Y_{h^3 yy_*} = Y_{h^3 hy_*} = Y_{h^3 yh_*} = -\xi. \tag{4.4.35b}$$

- For the anti-D3-/D7-brane intersecting states, once again the only thing to add is the soft-breaking mass

$$\delta m^2_{\varphi\overline{\varphi},\,\mathrm{soft}} = \frac{2}{3}\kappa_4^2 V_{\overline{\mathrm{D3}}}\, Z_{h\overline{h}} = \left[\frac{g_s}{2\pi\ell_{(0)}}\right]^2 \frac{\mathrm{e}^{6\Omega+4A_0}}{3\kappa_4^2}. \tag{4.4.36}$$

To conclude, one must discuss the effective D- and F-term potentials. For the D-term potential, one has again

$$V_D^{(\mathrm{susy})} = \frac{\mathrm{e}^{4\Omega}}{2\pi g_s\,[-\mathrm{i}(\tau-\overline{\tau})]}\,(\varphi\overline{\varphi})^2 + \frac{\mathrm{e}^{4\Omega+4A_0}}{4\pi g_s\,\ell_4^0}\,(\varphi\overline{\varphi})^2. \tag{4.4.37}$$

For the F-term potential, from the effective superpotential

$$\begin{aligned} W_{\mathrm{susy}} = \frac{1}{2}\,\mu_{\sigma^3\sigma^3}\sigma^3\sigma^3 + \frac{1}{2}\,\mu_{y^a y^b} y^a y^b + \mu_{y^a h^b} y^a h^b + \xi\,(\sigma^3\mathrm{e}^{-A_0} - y^3 - h^3) yy_* \\ +\xi\,(\sigma^3\mathrm{e}^{-A_0} - y^3 - h^3) hy_* + \xi\,(\sigma^3\mathrm{e}^{-A_0} - y^3 - h^3) yh_*, \end{aligned}$$

one can see that the effective F-term potential reads as usual

$$\begin{aligned} V_F^{(\mathrm{susy})} = Z^{\sigma^3\overline{\sigma}^3}\, \mu_{\sigma^3\sigma^3}\overline{\mu}_{\overline{\sigma}^3\overline{\sigma}^3}\sigma^3\overline{\sigma}^3 + Z^{y^a\overline{y}^b}\, \mu_{y^a h^c}\overline{\mu}_{\overline{y}^b\overline{h}^d}\varphi^c\overline{\varphi}^d \\ +Z^{y\overline{y}}\big[\xi(\sigma^3\mathrm{e}^{-A_0} - \varphi^3)\varphi_*\big]\big[\overline{\xi}(\overline{\sigma}^3\mathrm{e}^{-A_0} - \overline{\varphi}^3)\overline{\varphi}_*\big] \\ +Z^{y_*\overline{y}_*}\big[\xi(\sigma^3\mathrm{e}^{-A_0} - \varphi^3)\varphi\big]\big[\overline{\xi}(\overline{\sigma}^3\mathrm{e}^{-A_0} - \overline{\varphi}^3)\overline{\varphi}\big]. \end{aligned} \tag{4.4.38}$$

4.4.2.3.2 $x\overline{x}$-Dependent Interaction Terms

For completeness, one has to include in the theory the flux-dependent trilinear couplings between the anti-D3-brane and the intersecting states in eqs. (4.4.30a, 4.4.30b). Again, one can do so by means of the Kähler potential

$$\delta\mathcal{K} = \frac{2\ell_{(0)}^{\Sigma_4}}{g_s^4}\,\frac{\kappa_4^2 x\overline{x}\,\mathrm{e}^{2\Omega-4A_0}}{[-\mathrm{i}(\tau-\overline{\tau})]^{\frac{3}{2}}\,[-\mathrm{i}\omega_{(0)}^{\Sigma_4}]^2}\left[\frac{2\varsigma}{\pi}\ell_{(0)}\ell_{(0)}^{\Sigma_4}\right]^{\frac{1}{2}}\Big[\kappa_4\, c_{\alpha\beta\gamma}h^\alpha h^\beta h^\gamma + \mathrm{c.c.}\Big]. \tag{4.4.39}$$

One can do the same for the quartic coupling in eq. (4.3.30).

4.4.3 Moduli Stabilisation and Anomaly Mediation

The scenario presented so far provides a toy model picturing anti-D3-/D7-brane constructions where supersymmetry is non-linearly realised. However, the volume modulus is a runaway direction due to the anti-D3-brane uplift and its stabilisation affects the other fields of the theory. Moreover, as will also be discussed, some fields receive non-negligible mass contributions from anomaly-mediation effects.

4.4.3.1 Perturbative and Non-Perturbative Corrections

Due to the no-scale structure of the theory, tree-level type IIB flux compactifications lack the stabilisation of the Kähler modulus controlling the internal volume; nevertheless, this can be fixed once non-perturbative and α'-corrections are included.

For concreteness, the KKLT-scenario [54] for the Kähler-modulus stabilisation is going to be considered here, but analogous computations could be performed for the Large-Volume Scenario [55]. To start, the two classes of modifications occurring to the closed-string potentials $\hat{K}$ and $\hat{W}$ are introduced, and then these are analysed for the strongly-warped compactifications of interest.

(i) In KKLT-like constructions, the Kähler-modulus potential receives non-perturbative corrections from effects such as D7-brane gaugino condensation[4.12] or Euclidean D3-brane instantons. Both effects can be described in the low-energy supergravity theory by means of a superpotential of the form

$$\delta\hat{W}_{\mathrm{np}} = \frac{1}{\kappa_4^3}\, A\, \mathrm{e}^{a\mathrm{i}\rho},$$

where A and a are parameters whose details depend on the string origin of the non-perturbative effects. This correction against a non-zero flux superpotential stabilises the volume modulus and, together with the anti-D3-brane uplift, it can be argued to give a 4-dimensional non-supersymmetric de Sitter vacuum.

(ii) The perturbative α'-corrections modify the Kähler potential for the volume modulus as [272, 273]

$$\kappa_4^2\hat{K} = -2\ln\left[\left(2\,\mathrm{e}^{-2\Omega}\right)^{\frac{3}{2}} + \frac{1}{2}\,\xi'\right],$$

where, given the parameter $\xi = -\zeta(3)\chi/16\pi^3$, with $\zeta = \zeta(s)$ the Riemann ζ-function and the Euler number $\chi = 2\,(h^{1,1} - h^{2,1})$ taken to be positive, the deformation is

$$\xi' = \xi'(\tau,\overline{\tau}) = [-\mathrm{i}(\tau-\overline{\tau})]^{\frac{3}{2}}\,\xi.$$

Although α'-corrections to a KKLT-setup with anti-D3-brane uplift do not qualitatively modify the stabilisation of ρ, as they are subleading in the volume suppression,

[4.12] This mechanism and its stability after the anti-D3-brane uplift have been scrutinised carefully in the literature and, despite the criticisms, there is no clear proof for it to be inconsistent. For recent discussions, see for instance refs. [58, 201–203, 206, 210–213, 215–217].

they turn out to provide leading-order contributions to some open-string masses, including the intersecting anti-D3-/D7-brane scalars.

Note that, as discussed in subsubsection 4.2.2.2, in strongly-warped scenarios, the effects of supergravity corrections are warped down in the scalar potential due to localisation effects, leading to a modification of the usual scales. This stems from the $2A_0$-shift in the Kähler potential (see eq. (4.2.17)).

Typically the axio-dilaton and complex-structure moduli are fixed at higher energy scales than the Kähler-modulus and the open-string degrees of freedom, determining the flux background to be imaginary self-dual. This happens also in highly-warped compactifications, as discussed in subsection 4.2.2, so in the low-energy effective field theory they can be regarded as constant terms. An exception may be the complex-structure moduli associated to the throat base at the strongly-warped end [67, 68]. For the open-string sector, the anti-D3-brane scalars receive leading-order flux-induced mass contributions, so non-perturbative and α'-corrections give at most subleading corrections. A similar reasoning applies to the D7-brane scalars. Spinors are less affected than the scalars since they do not get soft-breaking contributions. On the other hand, the intersecting states do not have flux-induced masses, so such corrections play a relevant role.

Including perturbative and non-perturbative corrections, the relevant terms in the supergravity theory for the volume modulus ρ and for the anti-D3-/D7-brane intersecting-state scalars φ are

$$\kappa_4^2 K = -2\ln\left[\left([-\mathrm{i}(\rho-\overline{\rho})+2c_0] - \frac{\kappa_4^2}{3}\gamma x\overline{x} - \frac{\kappa_4^2}{3}\,\omega h\overline{h}\right)^{\frac{3}{2}} + \frac{\xi'}{2}\right], \tag{4.4.40a}$$

$$\kappa_4^3 W = W_0 + A\,\mathrm{e}^{a\mathrm{i}\rho} + \kappa_4 s x, \tag{4.4.40b}$$

where $\mathrm{e}^{-2\Omega} = \operatorname{Im}\rho + c_0$ with $c_0 = \ell_w/\ell_{(0)}$. Here, from eqs. (4.4.18, 4.4.29), the constant ω can be seen to be

$$\omega = \frac{1}{\pi g_s},$$

while the definitions of the constant GVW-term and of the anti-D3-brane parameters W_0, γ and s, respectively, can be extracted from eqs. (4.4.5a, 4.4.5b) and read

$$\gamma = \frac{4}{g_s}\frac{\mathrm{e}^{-2A_0}}{\langle -\mathrm{i}(\tau-\overline{\tau})\rangle\langle -\mathrm{i}\omega_w\rangle}\frac{\ell_w}{\ell_{(0)}},$$

$$W_0 = \frac{g_s}{l_s^2}\left\langle \int_{Y_6} G_3\wedge\Omega\right\rangle,$$

$$s = \sqrt{2}g_s.$$

For brevity, the contributions from the vacuum expectation values of the axio-dilaton and of the complex-structure moduli as well as the constant terms, including the constants in eqs. (4.4.5a, 4.4.5b) and the $2A_0$-shift in eq. (4.2.17), have not been reported in the Kähler potential, but they are going to be reinserted when discussing physical scales.

Although the underlying string construction is different, as far as the scalar fields are concerned, up to constant terms, the supergravity theory of eqs. (4.4.40a, 4.4.40b) is structurally equivalent to the one studied in detail in ref. [98],[4.13] so this subsubsection simply summarises the main results of the calculations. It turns out that the F-term potential of this model can be written as

$$V_F = \hat{V}_F(\rho,\overline{\rho}) + \Delta V_F(\rho,\overline{\rho};\varphi,\overline{\varphi}),$$

where $\hat{V}_F$ is the pure Kähler-modulus potential, as a consequence of the breaking of the no-scale structure by the corrections and uplift term, while ΔV_F is the scalar potential for the scalar field φ, generating a mass term among other interactions.

4.4.3.1.1 Kähler-Modulus Stabilisation and Minkowski Vacuum

On the one hand, one can show that the leading-order hidden-sector supersymmetry-breaking F-term potential reads

$$\hat{V}_F = V_F^{\mathrm{KKLT}+\alpha'} + V_F^{\overline{\mathrm{D3}}+\alpha'}, \tag{4.4.41}$$

where the α'-corrected KKLT-potential and uplift energy respectively read

$$\begin{aligned} V_F^{\mathrm{KKLT}+\alpha'} &= \frac{1}{\kappa_4^4}\left[\frac{a^2 A\overline{A}\,\mathrm{e}^{\mathrm{i}a(\rho-\overline{\rho})}}{3\,[-\mathrm{i}(\rho-\overline{\rho})+2c_0]} + \frac{a(W_0\overline{A}\,\mathrm{e}^{-\mathrm{i}a\overline{\rho}} + \overline{W}_0 A\,\mathrm{e}^{\mathrm{i}a\rho})}{[-\mathrm{i}(\rho-\overline{\rho})+2c_0]^2}\right] + \delta_{\alpha'}V_F^{\mathrm{KKLT}}, \\ V_F^{\overline{\mathrm{D3}}+\alpha'} &= \frac{s^2}{\gamma[-\mathrm{i}(\rho-\overline{\rho})+2c_0]^2\kappa_4^4} + \delta_{\alpha'}V_F^{\overline{\mathrm{D3}}}, \end{aligned}$$

with the α'-corrections being

$$\begin{aligned} \delta_{\alpha'}V_F^{\mathrm{KKLT}} &= \frac{\xi'}{2\kappa_4^4}\left[\frac{1}{6}\frac{a^2 A\overline{A}\,\mathrm{e}^{\mathrm{i}a(\rho-\overline{\rho})}}{[-\mathrm{i}(\rho-\overline{\rho})+2c_0]^{\frac{5}{2}}} - \frac{a[W_0\overline{A}\,\mathrm{e}^{-\mathrm{i}a\overline{\rho}} + \overline{W}_0 A\,\mathrm{e}^{\mathrm{i}a\rho}]}{2[-\mathrm{i}(\rho-\overline{\rho})+2c_0]^{\frac{7}{2}}} + \frac{3\,W_0\overline{W}_0}{2[-\mathrm{i}(\rho-\overline{\rho})+2c_0]^{\frac{9}{2}}}\right], \\ \delta_{\alpha'}V_F^{\overline{\mathrm{D3}}} &= -\frac{\xi'}{2\kappa_4^4}\,\frac{s^2}{\gamma[-\mathrm{i}(\rho-\overline{\rho})+2c_0]^{\frac{7}{2}}}. \end{aligned}$$

By parametrising the superpotential constants as $W_0 = |W_0|\,\mathrm{e}^{\mathrm{i}\theta}$ and $A = |A|\,\mathrm{e}^{\mathrm{i}\alpha}$, given the definition of the Kähler modulus

$$\rho = \chi + \mathrm{i}\,c,$$

one finds that at leading order the axion χ is minimised as $a\langle\chi\rangle = \theta - \alpha + n\pi$. Without loss of generality, one can set $\theta = 0$. Then, the leading-order c-dependent scalar potential is

$$\hat{V}(c) = \frac{1}{\kappa_4^4}\,\frac{a|A|}{2}\left[\frac{1}{3}\,\frac{a|A|\,\mathrm{e}^{-2ac}}{[c+c_0]} - \frac{|W_0|\,\mathrm{e}^{-ac}}{[c+c_0]^2}\right] + \frac{1}{\kappa_4^4}\,\frac{s^2}{4\gamma[c+c_0]^2}. \tag{4.4.42}$$

Defining the shifted variables $c' = c + c_0$ and $|B| = |A|\,\mathrm{e}^{ac_0}$ [229], one obtains results that are formally equivalent to those of ref. [98]. In the large-volume regime, in which $c \gg 1$, the stationary condition $\partial V/\partial c = 0$ gives the solution

$$|W_0| = \frac{2}{3}\,\langle a[c+c_0]\rangle|A|\mathrm{e}^{-\langle ac\rangle} + \frac{1}{a\gamma}\frac{s^2}{\langle a[c+c_0]\rangle|A|}\mathrm{e}^{\langle ac\rangle}. \tag{4.4.43}$$

[4.13] In ref. [98], the matter sector is realised on a D3-brane, with supersymmetry being broken by a distant anti-D3-brane. For the scalar fields, this turns out to have an analogous supergravity formulation, the only differences being the $2A_0$-shift to the Kähler potential and the c_0-shift to the Kähler modulus.

Further, a Minkowski vacuum $\langle \hat{V}_F \rangle = 0$ can be obtained at leading order in the volume if the anti-D3-brane uplift parameter s is such as to fulfil the equality

$$s^2 = \frac{2}{3}\, a\gamma \, \langle a[c + c_0] \rangle |A|^2 \mathrm{e}^{-2\langle ac \rangle}. \tag{4.4.44}$$

This could be tuned even more precisely at subleading orders in the volume. Of course, one may as well want to impose a de Sitter vacuum, but in any case the vacuum energy has to be small. The ignored α'-corrections would only modify the vacuum conditions at subleading order in the volume.

One can write the vacuum expectation value in eq. (4.4.43) in view of the volume leading-order Minkowksi vacuum condition of eq. (4.4.44) as $|W_0| = (2/3)\,(\langle a[c+c_0] \rangle + 1)\,|A|\,\mathrm{e}^{-\langle ac \rangle}$, or, more conveniently and simply expressing the leading-order term in the volume, as

$$|W_0|^2 = \frac{2s^2}{3\gamma}\, \langle [c + c_0] \rangle.$$

By taking this into account, the gravitino mass, namely $\hat{m}_{3/2}^2 = \langle \mathrm{e}^{\kappa_4^2 \hat{K}} \kappa_4^4 \hat{W} \hat{\overline{W}} \rangle$, at leading order in the volume is

$$\hat{m}_{3/2}^2 = \frac{1}{\kappa_4^2}\, \frac{s^2}{12\gamma \langle [c + c_0] \rangle^2}.$$

Similarly, one can see that the not-yet canonically normalised Kähler-modulus mass is

$$\hat{m}_{cc}^2 = \frac{1}{2} \frac{\partial^2 \hat{V}}{\partial c^2}\bigg|_{c = \langle c \rangle} = \frac{1}{\kappa_4^4}\, \frac{a^2 s^2}{4\gamma \langle [c + c_0] \rangle^2}.$$

Finally, the combination of fluxes, non-perturbative corrections and anti-D3-brane uplift induces a non-zero F-term for the field ρ, along with the one for x. In the Minkowski vacuum of eqs. (4.4.43, 4.4.44), at leading order in the volume one finds[4.14]

$$\hat{F}^x = \left[\frac{6}{\gamma}\right]^{\frac{1}{2}} (\langle c \rangle + c_0)^{\frac{1}{2}}\, \frac{\hat{\overline{m}}_{3/2}}{\kappa_4},$$

$$\hat{F}^\rho = \frac{\mathrm{i}}{a}\, \hat{\overline{m}}_{3/2}.$$

This means that the goldstino ψ_{g} is now a linear combination of the anti-D3-brane gaugino and of the Kähler modulino (see e.g. refs. [262, 274] for progress in the couplings between the gravitino and ψ^x). The unitary gauge does not exactly set to zero the spinor component ψ^x of the nilpotent superfield, but rather the goldstino. This means that the anti-D3-brane models in this section have a plethora of interactions between the fields coupled to x and/or ρ and the linear combination of ψ^x and ψ^ρ that is orthogonal to the goldstino. This spinor ψ_{g}' is massive, with a mass of at least the same order as the Kähler-modulus mass. However, from the scalar potential, one can see that the scales characterising each F-term have a different volume suppression, being [68]

$$f_x = \left[\frac{1}{3} \hat{K}_{x\overline{x}} \hat{F}^x \hat{\overline{F}}^x\right]^{\frac{1}{2}} \simeq \frac{\hat{m}_{3/2}}{\kappa_4}, \tag{4.4.45a}$$

[4.14] In the presence of perturbative and non-perturbative corrections (and an anti-D3-brane), the axio-dilaton F-term becomes non-zero too. However, it is small compared to the F-terms for x and ρ [98].

$$f_\rho = \left[\frac{1}{3}\hat{K}_{\rho\overline{\rho}}\hat{F}^{\rho}\hat{\overline{F}}^{\rho}\right]^{\frac{1}{2}} \simeq \frac{1}{2a\mathcal{V}^{\frac{2}{3}}}\frac{\hat{m}_{3/2}}{\kappa_4}. \tag{4.4.45b}$$

This suggests that, due to the hierarchically smaller volume suppression, the anti-D3-brane still provides the dominant contribution to the goldstino ψ_{g}, thus not changing drastically the scenario compared to the case where the goldstino is provided by the anti-D3-brane alone.

4.4.3.1.2 Open-String Mass Terms

In order to write the open-string scalar potential in a convenient way it is helpful to consider the complete canonical normalisation of the scalar field, including the α'-corrections. At the end of the day, one finds the φ-field scalar potential

$$\Delta V_F = \left[\frac{2\kappa_4^2}{3}\left(V_F^{\mathrm{KKLT}+\alpha'} + V_F^{\overline{\mathrm{D3}}+\alpha'}\right) + \Theta_F\right]\frac{\left(1+\delta_{Z_{\varphi\overline{\varphi}}}\right)\omega\varphi\overline{\varphi}}{[-\mathrm{i}(\rho-\overline{\rho})+2c_0]}. \tag{4.4.46}$$

where the correction to the field normalisation is $\delta_{Z_{\varphi\overline{\varphi}}} = -\xi'/2[-\mathrm{i}(\rho-\overline{\rho})+2c_0]^{3/2}$. In this form, it is easy to impose the vacuum solutions. The Θ_F-term reads

$$\Theta_F = \Theta_F^{\mathrm{KKLT}+\alpha'} + \Theta_F^{\overline{\mathrm{D3}}+\alpha'},$$

with the KKLT- and uplift-like terms

$$\begin{aligned}
\Theta_F^{\mathrm{KKLT}+\alpha'} &= \frac{5\xi'}{72\kappa_4^2}\left[\frac{a^2 A\overline{A}\,\mathrm{e}^{a\mathrm{i}(\rho-\overline{\rho})}}{[-\mathrm{i}(\rho-\overline{\rho})+2c_0]^{\frac{5}{2}}} + \frac{3a(\overline{A}W_0\mathrm{e}^{-a\mathrm{i}\overline{\rho}} + A\overline{W}_0\mathrm{e}^{a\mathrm{i}\rho})}{[-\mathrm{i}(\rho-\overline{\rho})+2c_0]^{\frac{7}{2}}} + \frac{9W_0\overline{W}_0}{[-\mathrm{i}(\rho-\overline{\rho})+2c_0]^{\frac{9}{2}}}\right],\\
\Theta_F^{\overline{\mathrm{D3}}+\alpha'} &= \frac{\xi'}{12\kappa_4^2}\,\frac{s^2}{\gamma[-\mathrm{i}(\rho-\overline{\rho})+2c_0]^{\frac{7}{2}}}.
\end{aligned}$$

In the Minkowski vacuum of eqs. (4.4.43, 4.4.44), only Θ_F contributes to the scalar masses. At leading order, its KKLT-like term happens to vanish, so the potential is fixed by its uplift-like term and it is positive definite. In particular, one finds the mass term

$$\Delta V_F|_{\rho=\langle\rho\rangle} = \frac{s^2\omega}{12\gamma[-\mathrm{i}\langle\rho-\overline{\rho}\rangle+2c_0]^{\frac{9}{2}}}\,\frac{\xi'}{\kappa_4^2}\,\varphi\overline{\varphi}. \tag{4.4.47}$$

4.4.3.1.3 Complete Scalar Potential and Mass Terms

For a fully-fledged calculation, one must insert the axio-dilaton and complex-structure modulus Kähler potentials and the constant term, as in eqs. (4.4.5a, 4.4.5b). Further, the $2A_0$-shift in K also needs to be included, as in eq. (4.2.17), and the consequent redshift will be indicated by the superscript 'w', in line with the notation in the rest of the chapter. Finally, one should bear in mind that hatted quantities mean they are purely determined by the supersymmetry-breaking hidden-sector potentials.

Developing the observations made at the end of subsubsection 4.2.2.2 on the redshifting of non-perturbative contributions to the scalar potential in strongly-warped scenarios, notice that the $2A_0$-shift in the Kähler potential does not change qualitatively the shape of the scalar potential in the presence of KKLT-like non-perturbative corrections and anti-D3-brane uplift, but it affects it quantitatively. Indeed, the uplift term from the anti-D3-brane is scaled by the usual factor e^{4A_0}, but the pure closed-string sector term, which is usually unwarped, is now also scaled down by a factor e^{2A_0}. The moduli stabilisation is thus somewhat more delicate, as the uplift from the anti-D3-brane should not be too large with respect to the close string stabilisation so as to cause a runaway. Also, all the masses are now redshifted by an extra factor e^{2A_0}.

In detail, in the closed-string sector, the gravitino mass and the canonically normalised Kähler-modulus mass respectively read

$$(\hat{m}^w_{3/2})^2 = \frac{1}{\kappa_4^2} \frac{g_s^3}{12\pi[\ell_{(0)}]^2}\, \mathrm{e}^{4\Omega+4A_0} \sim \frac{g_s^3}{\mathcal{V}^{\frac{4}{3}}} \frac{1}{\kappa_4^2} \mathrm{e}^{4A_0}, \tag{4.4.48}$$

$$(\hat{m}^w_c)^2 = 4a^2 \mathrm{e}^{-4\Omega} (\hat{m}^w_{3/2})^2 \sim a^2 g_s^3 \frac{1}{\kappa_4^2} \mathrm{e}^{4A_0}. \tag{4.4.49}$$

Notice that two factors contribute to make the gravitino mass highly suppressed, i.e. the e^{2A_0}-redshift and the small bulk $(0,3)$-flux, which in the tuning towards a de Sitter/Minkowski vacuum ends up providing a lower volume- but enhanced warp factor-suppression.

Moreover, because the open-string scalars are the intersecting-state fields φ and φ_*, in terms of the gravitino mass their canonically normalised mass is

$$m_{3\overline{7}}^2 = m_{\overline{7}3}^2 = \frac{\xi}{4}\, \mathrm{e}^{3\Omega-3\phi/2}\, (\hat{m}^w_{3/2})^2 \sim \frac{\xi g_s^3}{\mathcal{V}^{\frac{7}{3}}} \frac{1}{\kappa_4^2} \mathrm{e}^{4A_0}. \tag{4.4.50}$$

Such a mass is quite small due to a large volume suppression and the effect of warping, but it is necessarily positive definite. Notice that it vanishes in the absence of the α'-corrections, namely if one sets $\xi = 0$.

Further, for D7-branes extending from the bulk to the throat the gauge kinetic function is determined by the volume modulus (see eq. (4.3.12)) and one finds the F^ρ-induced gaugino mass

$$m_{1/2}^{\mathrm{D7}} = \frac{\mathrm{e}^{2\Omega}}{2a}\, \hat{m}^w_{3/2} \sim \frac{1}{a} \frac{g_s^3}{\mathcal{V}^2} \frac{1}{\kappa_4^2} \mathrm{e}^{4A_0}. \tag{4.4.51}$$

For D7-branes at the tip of the throat, there is a dependence on the volume modulus but it is highly redshifted (see eq. (4.3.18)).

4.4.3.1.4 Corrections to Pure Anti-D3- and D7-Brane Couplings

The effect of the Kähler-modulus stabilisation on the masses and the couplings of the pure anti-D3- and D7-brane states can also be worked out using supergravity, as is now going to be summarised. It is useful to note that the F-term for the volume modulus ρ has an extra volume-power suppression in the presence of non-perturbative corrections, while the F-term for the goldstino x is unchanged. A key observation will be that the non-perturbative corrections induce scales that are never bigger than the flux-induced ones discussed before, so in the end the orders of magnitude for masses and couplings are unchanged.

For the pure D7- and anti-D3-brane chiral multiplets, the canonically normalised would-be supersymmetric masses are $\mathrm{e}^{A_0} m_{\mathrm{D7}}$ or m^w_{D7} for the D7-brane fields localised in the bulk and at the tip, respectively (see subsubsection 4.3.1.2 and 4.3.1.3), and $m^w_{\overline{\mathrm{D3}}}$ for the anti-D3-brane (see subsubsection 4.4.1.3). On the one hand, for such fields, the ρ-field F-term does not participate in the effective μ-terms, leaving these would-be supersymmetric masses unchanged. On the other hand, the would-be soft-breaking masses can be seen to be never bigger than these flux-induced terms, being at most of the order of the gravitino mass. Indeed, after canonically normalising, the would-be soft-breaking masses scale as

$$m_{\mathrm{soft}} \sim \hat{m}^w_{3/2}. \tag{4.4.52}$$

For both bulk- and throat-localised D7-branes, one finds canonically normalised soft-breaking masses $m^{77}_{\mathrm{soft}} \sim \hat{m}^w_{3/2}$, with $e^{A_0} m_{\mathrm{D7}} \sim \hat{m}^w_{3/2}$, assuming $\theta \sim \theta'$, and $m^w_{\mathrm{D7}} \gg \hat{m}^w_{3/2}$. For anti-D3-branes, one similarly finds $(m^{\overline{33}}_{\mathrm{soft}})^2 \sim -(\hat{m}^w_{3/2})^2$, where, at leading order in the volume, the key role is played by the F-term of the goldstino x. The B-terms are unaffected for the anti-D3-branes, coming from an $x\overline{x}$-term, while they receive normalised contributions for the D7-branes of order $B_i \sim (e^{A_0} m_{\mathrm{D7}} + \hat{m}^w_{3/2})\hat{m}^w_{3/2}$ or $B_i \sim m^w_{\mathrm{D7}}\hat{m}^w_{3/2}$, for bulk or tip localisation, respectively. In the former case, the soft-breaking corrections compete with the flux-induced ones, but do not dominate, while in the latter the corrections are irrelevant for the mass eigenvalues. Finally, notice that the trilinear would-be soft-breaking couplings with the intersecting states are inserted via the $x\overline{x}$-coupling and are thus unaffected.

In this construction, the non-perturbative effects do not directly affect the open-string sector $x\overline{x}$-couplings. However, one may expect corrections for all the couplings, with a scale set by $\hat{m}^w_{3/2}$. For the pure anti-D3-brane, such corrections would be irrelevant, as $m^w_{\overline{\mathrm{D3}}} \gg \hat{m}^w_{3/2}$. On the other hand, considering the counter-part D3-/D7-branes, the soft-breaking trilinear coupling depends on the ρ-field F-term and is thus suppressed in the presence of non-perturbative corrections. Such changes can be implemented by hand, modifying the scalings in the $x\overline{x}$-terms.

In all these couplings, the α'-corrections may only contribute at most with volume-suppressed terms and are thus irrelevant for fixing the orders of magnitude. An intuitive explanation for this can be seen in the fact that they do not participate in the stabilisation of the Kähler modulus and they are subleading in the F-terms.

4.4.3.2 Anomaly Mediation

In supersymmetric theories with a hidden sector, anomaly mediation provides a one-loop contribution to gaugino masses and trilinear scalar couplings, and a two-loop contribution to charged scalar masses [252,275]. Again, this is discussed in a setup similar to the current one in ref. [98], so only an essential review is reported below.

In the case of a diagonalisable Kähler matter metric, given the corresponding canonically normalised fields f^i, with indices lowered and raised by Kronecker-deltas $\delta_{i\bar{j}}$ and $\delta^{i\bar{j}}$, one can show that the anomaly-mediated gaugino masses, the scalar masses and the trilinear couplings read [252,275–279]

$$\dot{m}^a_{1/2} = \frac{\beta_{g_a}}{g_a} \left[\hat{m}^w_{3/2} - \frac{\kappa_4^2}{3}\,\hat{\mathcal{F}}^M \hat{K}_M\right], \tag{4.4.53a}$$

$$\dot{m}_i^2 = \frac{1}{2}\,\beta_v \frac{\partial\gamma^i}{\partial v}\left[\hat{m}^w_{3/2} - \frac{\kappa_4^2}{3}\,\hat{\mathcal{F}}^M\hat{K}_M\right]\left[\hat{\overline{m}}^w_{3/2} - \frac{\kappa_4^2}{3}\,\hat{\overline{\mathcal{F}}}^M\hat{K}_{\overline{M}}\right], \tag{4.4.53b}$$

$$\dot{a}_{ijk} = \frac{1}{2}\,y_{ijk}\,(\gamma^i+\gamma^j+\gamma^k)\left[\hat{m}^w_{3/2} - \frac{\kappa_4^2}{3}\,\hat{\mathcal{F}}^M\hat{K}_M\right]. \tag{4.4.53c}$$

Here, y_{ijk} are the canonically normalised Yukawa couplings, v represents any running coupling and β_v is the corresponding beta-function, with γ^i the f^i-field anomalous dimension. More details on the corrections generated by anomaly mediation are provided below.

- Given the quadratic Casimir invariant in the adjoint representation $C_2(G)$ and the generator normalisation factor $C(r_G)$ for the representation r_G, respectively, the beta-functions for the gauge couplings g read

  $$\beta_g = -\frac{g^3}{16\pi^2}\,b,$$

  where b is the coefficient

  $$b = \frac{11}{3}\,C_2(G) - \frac{2}{3}\,n_f\,C(r_G^f) - \frac{1}{3}\,n_s\,C(r_G^s),$$

  with n_f and n_s being the spinors and scalars in the representations r_G^f and r_G^s of the gauge group G, respectively. For the special unitary group SU(n), with $n>1$, one has the set of values

particle representation	C	C_2
$\boldsymbol{n}$	$\frac{1}{2}$	$\frac{n^2-1}{2n}$
$(\boldsymbol{n},\overline{\boldsymbol{n}})$	n	n

  and for an Abelian group U(1) one finds $C(q)=q^2$ and $C_2(q)=0$, where q is the particle charge.

- One can write schematically the beta-functions for the Yukawa couplings y_{ijk} as

  $$\beta_{y_{ijk}} = f^l{}_i\,(g,y)\,y_{ljk} + f^l{}_j\,(g,y)\,y_{ilk} + f^l{}_k\,(g,y)\,y_{ijl},$$

  where $f^i{}_j\,(g,y)$ are functions generally scaling as $f(g,y)\sim b'g^2+b''y\overline{y}$, for some model-dependent coefficients b' and b'', where the details of the index structure can be ignored for simplicity for the present purposes of determining just parametric dependences [252].

- Finally, the anomalous dimension γ^i can be written as

  $$\gamma^i = \frac{1}{16\pi^2}\left(\frac{1}{2}\sum_{j,k} y_{ijk}\overline{y}_{\overline{ijk}} - 2\sum_a g_a^2 C_2(r^i_{G_a})\right).$$

The relevant mass scales are worked out below for intersecting anti-D3-/D7-branes. For single branes, the only non-neutral fields of the model are in the intersecting sector, which are thus the only one receiving corrections. More realistic non-Abelian models with multiple branes have a larger non-neutral spectrum, but the mass scales, being fixed by the gauge couplings, are analogous. In particular, the b-coefficients are typically negative due to the large number of degrees of freedom.

- For a D7-brane wrapping a 4-cycle extending along the throat, the anomaly-mediated gaugino mass is slightly more suppressed than the volume-modulus F-term contribution, being

$$\dot{m}_{1/2}^{\mathrm{D7}} \simeq -\frac{g_{\mathrm{D7}}^2}{16\pi^2}\, b_{\mathrm{D7}}\, \hat{m}_{3/2}^w = -\frac{g_s b_{\mathrm{D7}}}{8\pi \ell_{(0)}^{\Sigma_4}}\, \mathrm{e}^{2\Omega}\, \hat{m}_{3/2}^w, \tag{4.4.54}$$

  Instead, if the D7-brane wraps a 4-cycle that is localised at the infrared end of the throat, the anomaly-mediated mass is

$$\dot{m}_{1/2}^{\mathrm{D7}} \simeq -\frac{g_{\mathrm{D7}}^2}{16\pi^2}\, b_{\mathrm{D7}}\, \hat{m}_{3/2}^w = -\frac{g_s b_{\mathrm{D7}}}{8\pi (\mathrm{e}^{-4A_0} \ell_4^0)}\, \hat{m}_{3/2}^w. \tag{4.4.55}$$

  In the presence of non-Abelian anti-D3-branes, there are extra would-be gaugini apart from the goldstino and their anomaly-mediated mass is[4.15]

$$\dot{m}_{1/2}^{\overline{\mathrm{D3}}} \simeq -\frac{g_{\overline{\mathrm{D3}}}^2}{16\pi^2}\, b_{\overline{\mathrm{D3}}}\, \hat{m}_{3/2}^w = -\frac{g_s b_{\overline{\mathrm{D3}}}}{4\pi[-\mathrm{i}(\tau-\overline{\tau})]}\, \hat{m}_{3/2}^w. \tag{4.4.56}$$

- For the intersecting-state scalars, which classically are massless, the full anomaly-mediated mass term is

$$\dot{m}_\varphi^2 \simeq \frac{1}{2}\left[g_{\mathrm{D7}}^4 b_{\mathrm{D7}} C_2(r_{\mathrm{D7}}^\varphi) + g_{\overline{\mathrm{D3}}}^4 b_{\overline{\mathrm{D3}}} C_2(r_{\overline{\mathrm{D3}}}^\varphi)\right]\left(\frac{\hat{m}_{3/2}^w}{8\pi^2}\right)^2 + \delta_y \dot{m}_\varphi^2,$$

  where $\delta_y \dot{m}_\varphi^2$ represents the Yukawa coupling-dependent contribution. This scales as $\delta_y \dot{m}_\varphi^2 \sim y\overline{y}(b' g^2 + b'' y\overline{y})$. For anti-D3-branes and localised D7-branes, one finds the scalings $g^2 \sim g_s$ and $y \sim g_s^{1/2}$, while for extended D7-branes the anti-D3-brane terms, unchanged, are the dominating ones. So the leading gauge coupling- and Yukawa coupling-dependent corrections have the same parametric dependence, and in the following the focus is going to be on the former for simplicity. In particular, notice that these tend to be negative-definite in quasi-realistic constructions with $b < 0$, and they therefore compete with the positive-definite α'-induced correction. So, for a D7-brane wrapping a 4-cycle extending along the throat, the leading-order anomaly-mediated scalar mass has a scale set by the anti-D3-brane contribution and reads

$$\dot{m}_\varphi^2 \simeq \frac{g_s^2 b_{\overline{\mathrm{D3}}} C_2(r_{\overline{\mathrm{D3}}}^\varphi)}{8\pi^2[-\mathrm{i}(\tau-\overline{\tau})]^2}(\hat{m}_{3/2}^w)^2 + \delta_y \dot{m}_\varphi^2. \tag{4.4.57}$$

[4.15] If one considers the effects of a non-zero axio-dilaton F-term, the gaugino mass contribution is at most of order $m_{1/2}^{\overline{\mathrm{D3}}} \sim \hat{m}_{3/2}^w/(a\mathcal{V}^{2/3})$ [98], so it is usually subleading with respect to the anomaly-mediated one.

On the other hand, for a D7-brane wrapping a 4-cycle localised at the tip of the throat, the leading-order term is

$$\dot{m}_\varphi^2 \simeq \left[\frac{g_s^2 b_{\overline{\mathrm{D3}}} C_2(r^\varphi_{\overline{\mathrm{D3}}})}{8\pi^2[-\mathrm{i}(\tau-\overline{\tau})]^2} + \frac{g_s^2 b_{\mathrm{D7}} C_2(r^\varphi_{\mathrm{D7}})}{32\pi^2(\mathrm{e}^{-4A_0}\ell_4^0)^2}\right](\hat{m}^w_{3/2})^2 + \delta_y \dot{m}_\varphi^2. \tag{4.4.58}$$

As long as the b-coefficients are negative, which happens typically in Standard-like Model extensions, the gauge coupling-induced scalar mass corrections are negative-definite. These contributions are in a very close competition with the α'-induced terms and the tachyonic terms might dominate, leading to an instability.

- The contributions to the trilinear couplings are again determined in view of the gauge-coupling terms and read

$$\dot{a}_{ijk} \simeq y_{ijk} \sum_a \frac{1}{b_a} \sum_{l=i,j,k} C_2(r^l_{G_a})\, \dot{m}^a_{1/2} + \delta_y \dot{a}_{ijk},$$

where the Yukawa-coupling contribution is of order $\delta_y \dot{a}_{ijk} \sim y_{ijk}\, y\overline{y}\, \hat{m}^w_{3/2}$. This means that such trilinear terms are of up to order $\dot{a} \sim g_s\, y\, \hat{m}^w_{3/2}$. Compared to the pure flux-induced terms, one can see that these tend to be leading for D7-branes extending along the throat and subleading for D7-branes wrapping 4-cycles at the tip of the throat.

4.5 Extension to Non-Abelian Theories

This section outlines a way to extend the previous results on single anti-D3- and D7-branes to multiple coincident branes at orbifold singularities, which can provide quasi-realistic particle sepctra with non-Abelian gauge groups and matter fields in bifundamental representations. The identification of the non-Abelian sectors with appropriate constrained superfields is worked out, and the new supergravity interactions are found, first for anti-D3-brane stacks, then for anti-D3-/D7-brane systems. Finally, the low-energy effective field theory corresponding to anomaly-free combinations of anti-D3-/D7-branes on orbifold-like singularities within flux compactifications is spelled out in some detail.

Note that, although the explicit realisation of Calabi-Yau orientifolds with orbifold-like singularities would be a successive step in this analysis, the core results in sections 4.3 and 4.4 hold in any such construction. In particular, the consequences of the orbifolding are in the richer array of gauge group representations particles may fall into, but the orders of magnitude of gauge couplings and masses are generally unchanged. At the same time, there is a very interesting interplay between the orbifolding and supersymmetry breaking by anti-D3-branes, whereby, after the orbifolding, the bifundamental matter stretching between anti-D3-branes and D7-branes has scalars and fermions in different gauge representations. Other minor differences, due for instance to orbifold symmetries projecting out certain background fluxes, are commented on explicitly. In a complete construction, RR-tadpole cancellation conditions would restrict the combinations of fluxes, anti-D3-branes and wrapped D7-branes appearing at each fixed point of the geometry.

4.5.1 Non-Abelian Anti-D3-Branes

First of all, it is necessary to describe a stack of coincident anti-D3-branes in the language of $N_4 = 1$ supergravity by extending its constrained superfields to the non-Abelian framework and adding a few new couplings which are non-zero only in the non-Abelian case.

4.5.1.1 Particle Content

The gauge group of a stack of n coincident anti-D3-branes at a smooth point in the internal space is the non-Abelian group $\mathrm{U}(n)$. The group $\mathrm{U}(n)$ fulfils the isomorphism

$$\mathrm{U}(n) \simeq \mathrm{SU}(n) \times \mathrm{U}(1)/\mathbb{Z}_n,$$

so its generators t^I, with $I = 0, i$, consist of the n-dimensional identity $t^0 = 1_n$ and of the n-dimensional Hermitean generators t^i of the group $\mathrm{SU}(n)$, with $i = 1, \ldots, n^2 - 1$. As explained in subsection 2.3.1,[4.16] the particle content consists of the following degrees of freedom:

- a non-Abelian gauge vector, i.e.
$$\hat{A}^\mu = \hat{A}^\mu_I t^I = A^\mu 1_n + A^\mu_i t^i;$$
- a gaugino in the adjoint representation, i.e.
$$\hat{\lambda} = \hat{\lambda}_I t^I = \lambda 1_n + \lambda_i t^i;$$
- three complex scalars in the adjoint representation, i.e.
$$\hat{\varphi}^a = \hat{\varphi}^a_I t^I = \varphi^a 1_n + \varphi^a_i t^i;$$
- three modulini in the adjoint representation, i.e.
$$\hat{\psi}^a = \hat{\psi}^a_I t^I = \psi^a 1_n + \psi^a_i t^i.$$

The field A_μ gauges the $\mathrm{U}(1)$-component and the fields A^μ_i gauge the non-Abelian $\mathrm{SU}(n)$-component. Also, the fields λ, φ^a and ψ^a are netural under the Abelian group and singlets of the $\mathrm{SU}(n)$-component, whereas the fields λ_i, φ^a_i and ψ^a_i are neutral under the Abelian group and in the adjoint representation of the $\mathrm{SU}(n)$-component.

Because it is a singlet under all the gauge groups, the gaugino λ can be identified as the goldstino of the theory. Therefore, it can be placed in a nilpotent chiral superfield x just as in eq. (4.4.1), with

$$x^2 = 0. \tag{4.5.1}$$

Being a singlet, the nilpotent superfield is sufficient to define the other constraints in a similar fashion as for a single anti-D3-brane, thanks to the linearity of such constraints [31], as shown in subsection 2.4.4.

[4.16] For clarity, compared to subsection 2.3.1, here matrix-valued fields carry hats or tildes in order to distinguish them from the matrix components.

- The non-Abelian gaugini λ^i can be packaged in the chiral superfield

$$\tilde{x} = x_i t^i,$$

which is neutral under the U(1)- and in the adjoint of the SU(n)-component of the gauge group, with the scalars removed by a constraint like the one in eq. (4.4.3), i.e.

$$x\tilde{x} = 0. \tag{4.5.2}$$

- Similarly, the full gauge vector can be described by the field-strength chiral superfield

$$\hat{W}_\alpha = W_\alpha + \tilde{W}_\alpha,$$

with $W_\alpha = W_\alpha 1_n$ and $\tilde{W}_\alpha = W_{i\alpha}\, t^i$, where the spinor components are removed by the constraints[4.17] (generalising that of eq. (4.4.2))

$$xW_\alpha = 0, \tag{4.5.3a}$$
$$x\tilde{W}_\alpha = 0. \tag{4.5.3b}$$

As the nilpotent superfield x is a singlet, these constraints are gauge-invariant.[4.18] Also notice that the condition $x\hat{W}_\alpha = 0$ is equivalent to the two constraints written above.

- For the modulini, one can define the chiral superfields

$$\hat{y}^a = y^a + \tilde{y}^a,$$

with $y^a = y^a 1_n$ and $\tilde{y}^a = y^a_i t^i$, and remove the scalar components by means of the constraints (generalising the ones in eq. (4.4.3))

$$xy^a = 0, \tag{4.5.4a}$$
$$x\tilde{y}^a = 0. \tag{4.5.4b}$$

Again, gauge invariance is preserved and an equivalent condition is $x\hat{y}^a = 0$.

- Finally, the scalars can again be encoded in the chiral superfields

$$\hat{h}^a = h^a + \tilde{h}^a,$$

with $h^a = h^a 1_n$ and $\tilde{h}^a = h^a_i t^i$, with the spinor and auxilary-field components removed by the constraints (generalising those of eq. (4.4.4))

$$\overline{x}\mathrm{D}_\alpha h^a = 0; \tag{4.5.5a}$$
$$\overline{x}\mathrm{D}_\alpha \tilde{h}^a = 0. \tag{4.5.5b}$$

These are gauge-invariant for gauge transformations with a chiral superfield Ω such that $\overline{x}(\Omega - \overline{\Omega}) = 0$, which implies the constraint $\overline{x}\mathrm{D}_\alpha\Omega = 0$ and is consistent with the gauge-fixing choice $xV = 0$ (see ref. [31] for more details). Again, one can simply write the condition as $x\mathrm{D}_\alpha\hat{h}^a = 0$.

[4.17] In addition to the constraint, there may be a modified Wess-Zumino gauge condition, as discussed in the Abelian case by ref. [31], which easily extends to the non-Abelian case.

[4.18] Notice that, if the constraint reads $x\tilde{W} = 0$, then, given the gauge transformation induced by the chiral superfield Λ, the constraint $x[\mathrm{e}^{\mathrm{i}\Lambda}\tilde{W}\mathrm{e}^{-\mathrm{i}\Lambda}] = \mathrm{e}^{\mathrm{i}\Lambda}x\tilde{W}\mathrm{e}^{-\mathrm{i}\Lambda} = 0$ holds too.

4.5.1.2 Supergravity Formulation

Given the superfield spectrum above, one needs to extend the $N_4 = 1$ description of subsubsection 4.4.1.3 to a non-Abelian theory. Adapting the existing Abelian couplings to their non-Abelian version is straightforward. Moreover, to match the dimensionally-reduced effective action of refs. [80, 232], one needs to generate further cubic and quartic scalar interactions as well as some Yukawa couplings.

Quite remarkably, one can verify that the only extra terms which need to be included in the supergravity theory are those in the trilinear superpotential

$$\delta\hat{W} = \frac{v}{4\pi g_s}\, l_s^3 \Omega^0_{abc}\, \mathrm{tr}\, \hat{y}^a \hat{y}^b \hat{h}^c + \frac{v}{4\pi g_s}\, l_s^3 \Omega^0_{abc}\, \mathrm{tr}\, \hat{y}^a \hat{h}^b \hat{h}^c, \tag{4.5.6}$$

where the normalisation constant is $v^2 = 4\pi\, \mathrm{e}^{-2A_0}\, [\ell_{(0)}]^3$. One could account for the warp factor by considering the throat complex-structure modulus [67, 68].

- Since it contains two spinors and one scalar, the first term in the superpotential only represents a Yukawa coupling between the modulini $\hat{\psi}^a$ and the scalars $\hat{\varphi}^a$ of the form

$$y_{\hat{\psi}^a \hat{\psi}^b \hat{\varphi}^c} = y_{\hat{y}^a \hat{y}^b \hat{h}^c} = \frac{\mathrm{e}^{3\Omega}}{2\pi g_s [-\mathrm{i}(\tau - \overline{\tau})]^{\frac{1}{2}}}\, \frac{\ell_w^{\frac{1}{2}}}{[-\mathrm{i}\omega_w]^{\frac{1}{2}}}\, l_s^3 \Omega^0_{abc},$$

which corresponds to the couplings in refs. [80, 232], provided the insertion of the complex-structure moduli in ω_w (not captured explicitly in the dimensional reduction).

- In a similar way as for D3-branes, the Yukawa terms also generate the quartic scalar potential and part of the cubic potential [232]. Indeed, now one has the effective anti-D3-brane superpotential

$$W^{\overline{\mathrm{D3}}}_{\mathrm{susy}} = \frac{1}{2}\, \mu_{\hat{y}^a \hat{y}^b} \mathrm{tr}\, \hat{y}^a \hat{y}^b + \frac{1}{2}\, \mu_{\hat{y}^a \hat{h}^b} \mathrm{tr}\, \hat{y}^a \hat{h}^b + \frac{1}{2}\, y_{\hat{y}^a \hat{y}^b \hat{h}^c} \mathrm{tr}\, \hat{y}^a \hat{y}^b \hat{h}^c + \frac{1}{2}\, y_{\hat{y}^a \hat{h}^b \hat{h}^c} \mathrm{tr}\, \hat{y}^a \hat{h}^b \hat{h}^c,$$

which in the unitary gauge generates the F-term scalar potential

$$V_F^{(\mathrm{susy})} = Z^{\hat{y}^a \hat{\overline{y}}^b}\, \mathrm{tr}\, \big(\mu_{\hat{y}^a \hat{h}^c} \hat{\varphi}^c + y_{\hat{y}^a \hat{h}^c \hat{h}^d} \hat{\varphi}^c \hat{\varphi}^d\big) \big(\overline{\mu}_{\hat{\overline{y}}^b \hat{\overline{h}}^e} \hat{\overline{\varphi}}^e + \overline{y}_{\hat{\overline{y}}^b \hat{\overline{h}}^e \hat{\overline{h}}^f} \hat{\overline{\varphi}}^e \hat{\overline{\varphi}}^f\big).$$

Further, the D-term potential now reads

$$V_D = \frac{1}{2}\, g^2_{\overline{\mathrm{D3}}}\, \mathrm{tr}\, \big(Z_{\hat{h}^a \hat{\overline{h}}^b} \hat{\varphi}^a \hat{\overline{\varphi}}^b\big) \big(Z_{\hat{h}^c \hat{\overline{h}}^d} \hat{\varphi}^c \hat{\overline{\varphi}}^d\big).$$

Obviously, the quadratic term in the F-term potential is the usual anti-D3-brane mass term. Then, in accordance with the results of refs. [80, 232], the cubic term reads[4.19]

$$V^{\overline{\mathrm{D3}}}_{\mathrm{cubic}} = \frac{\mathrm{e}^{4\Omega + 4A_0}}{8\pi[-i(\tau - \overline{\tau})]\kappa_4} \left[\frac{1}{2\pi[-\mathrm{i}\omega_w]}\, \frac{\ell_w}{\ell_{(0)}}\right]^{\frac{1}{2}} \cdot$$
$$l_s \big[-g_0^{d\overline{e}} g^0_{\overline{f}(d} \Omega^0_{a)gh} \overline{\Omega}^0_{\overline{e}\overline{b}\overline{c}} l_s^6 (\overline{G}_3^{0-})^{\overline{f}gh}\big]\, \mathrm{tr}\, \hat{\varphi}^a \hat{\overline{\varphi}}^b \hat{\overline{\varphi}}^c + \mathrm{c.c.},$$

[4.19] In the presence of $(0, 3)$-flux at the tip of the throat, there would be a further soft-breaking contribution to the trilinear scalar potential.

while a combination the D-term potential and the quartic term of the F-term potential is consistent with the usual would-be $N_4 = 4$ scalar potential

$$V^{\overline{\mathrm{D3}}}_{\text{quartic}} = \frac{e^{4\Omega}}{2\pi g_s[-i(\tau-\overline{\tau})]}\, g^0_{a\bar{b}} g^0_{c\bar{d}}\, \mathrm{tr}\left[[\hat{\varphi}^a, \hat{\varphi}^c][\hat{\overline{\varphi}}^b, \hat{\overline{\varphi}}^d] + [\hat{\varphi}^a, \hat{\overline{\varphi}}^d][\hat{\overline{\varphi}}^b, \hat{\varphi}^c]\right].$$

4.5.2 Non-Abelian Anti-D3-/D7-Brane Systems

As a further step towards quasi-realistic constructions, one can add a stack of w intersecting D7-branes to the system with n anti-D3-branes. New states appear in the theory along with those listed in subsubsection 4.5.1.1.

The D7-brane worldvolume is enhanced to a non-Abelian $\mathrm{U}(w)$-theory, where the gauge group is factorisable as $\mathrm{U}(w) \simeq \mathrm{SU}(w) \times \mathrm{U}(1)/\mathbb{Z}_w$, with generators τ^K, for $K = 0, k$, and $k = 1, \ldots, w^2 - 1$. The degrees of freedom are then:

- a non-Abelian gauge vector and a spinor in the adjoint representation, i.e.

$$\hat{B}^\mu = B^\mu 1_w + B^\mu_k \tau^k,$$
$$\hat{\zeta} = \zeta 1_w + \zeta_k \tau^k;$$

- a scalar and another spinor in the adjoint representation, i.e.

$$\hat{\sigma}^3 = \sigma^3 1_w + \sigma^3_k \tau^k,$$
$$\hat{\eta}^3 = \eta^3 1_w + \eta^3_k \tau^k.$$

As D7-branes do not break supersymmetry, these fields make up standard multiplets. In particular, there are an Abelian vector superfield W'_α, containing B^μ and ζ, a non-Abelian $\mathrm{SU}(w)$-group vector superfield $\tilde{W}'_\alpha$, containing B^μ_k and ζ_k, a neutral chiral multiplet σ^3, containing σ^3 and η^3, and a chiral multiplet $\tilde{\sigma}^3$, containing σ^3_k and η^3_k.

Along with the pure anti-D3- and D7-brane states, new particles arise from strings stretching between these branes. For such anti-D3-/D7-brane intersecting states, the situation does not differ too much from the setup with single branes. The degrees of freedom are:

- two scalar fields $\hat{\varphi}$ and $\hat{\varphi}_*$ from the $\overline{3}7$- and $7\overline{3}$-sectors, respectively, with the former in the fundamental representation of the group $\mathrm{U}(n)$ and in the antifundamental representation of the group $\mathrm{U}(w)$, and the latter in the conjugate representation;
- two spinor fields $\hat{\psi}$ and $\hat{\psi}_*$ from the $\overline{3}7$- and $7\overline{3}$-sectors, respectively, with the former in the fundamental representation of the group $\mathrm{U}(n)$ and in the antifundamental representation of the group $\mathrm{U}(w)$, and the latter in the conjugate representation.

As usual, these fields cannot be packaged in standard supermultiplets with respect to the closed-string sector supersymmetry, but rather they fall into constrained superfields.

- The scalars can be encoded in the chiral superfields $\hat{h}$ and $\hat{h}_*$ such as to remove their spinor components, generalising eqs. (4.4.15), i.e.

$$x\overline{x}\, \mathrm{D}_\alpha \hat{h} = 0, \tag{4.5.7a}$$
$$x\overline{x}\, \mathrm{D}_\alpha \hat{h}_* = 0. \tag{4.5.7b}$$

- The spinors can be encoded in the chiral superfields $\hat{y}$ and $\hat{y}_*$ such as to remove their scalar components, generalising eqs. (4.4.16), i.e.

$$x\hat{y} = 0, \tag{4.5.8a}$$

$$x\hat{y}_* = 0. \tag{4.5.8b}$$

Again, thanks to the linearity of the constraints, their solutions are simple generalisations of the Abelian ones. Notice that a superfield in the fundamental representation of a group $\mathrm{U}(p)$ has a charge $q = +1$ under the corresponding Abelian subgroup and is in the fundamental representation of the $\mathrm{SU}(p)$-subgroup, and correspondingly for the antifundamental representation.

4.5.3 Anti-D3-/D7-Branes at Orbifold Singularities

An interesting class of model-building setups is the one with anti-D3-branes and D7-branes at orbifold singularities, introduced by ref. [29] and implemented by ref. [30] in a more complete quasi-realistic geometrical setup (see also ref. [222]). The fact that the branes sit at an orbifold singularity breaks each gauge group $\mathrm{U}(m)$ into several subgroups $\mathrm{U}(m_i)$. Interestingly, the anti-D3-/D7-intersecting scalars and spinors now transform in different representations of the unbroken gauge groups, and so have no semblance to being superpartners.

4.5.3.1 Gauge Group Breaking and Massless Spectrum

In a given 10-dimensional spacetime of the kind $X_{1,9} = \mathbb{M}^{1,3} \times Y_6$, the internal 6-dimensional space can be assumed to host locally an orbifold singularity of the type appearing in the quotient space $Q_6 = \mathbb{C}^3/\mathbb{Z}_N$, for some integer N. The general principle is to build a quasi-realistic matter spectrum locally at this singular point and then to embed this into a global model at a later stage, in a purely bottom-up approach. The action θ of the $\mathbb{Z}_N$-twist on the complex internal coordinates z^a, for $a = 1, 2, 3$, is

$$\mathbb{Z}_N : \quad z^a \overset{\theta}{\mapsto} \alpha^{l_a} z^a,$$

with the definition $\alpha = \mathrm{e}^{2\pi \mathrm{i}/N}$. Under the condition $\sum_{a=1}^{3} l_a = 0 \bmod N$, an $N_4 = 2$ supersymmetry is preserved in the bulk. Moreover, for simplicity, only the case where l_3 is even is discussed. The action of the $\mathbb{Z}_N$-twist on the massless degrees of freedom of a stack of n anti-D3-branes is then as follows (see refs. [29, 221]).

- Gauge vector fields $\hat{A}^\mu$ correspond to states $b^\mu_{-1/2}|\mathrm{NS}\rangle$ that are orthogonal to the orbifolded directions, therefore the $\mathbb{Z}_N$-twist only affects the Chan-Paton degrees of freedom. This means that the action of the orbifold on the non-Abelian gauge vector can be written as

$$\mathbb{Z}_N : \quad \hat{A}_\mu \overset{\theta}{\mapsto} \Gamma_{\theta,\overline{3}}\, \hat{A}_\mu\, \Gamma^{-1}_{\theta,\overline{3}},$$

  where, given N arbitrary integers n_i, with $i = 0, 1, \ldots, N-1$, such that $\sum_{i=0}^{N-1} n_i = n$, the representation of the orbifold matrix is chosen to be

$$\Gamma_{\theta,\overline{3}} = \mathrm{diag}\left(1_{n_0}, \alpha 1_{n_1}, \ldots, \alpha^{N-1} 1_{n_{N-1}}\right).$$

Expressing the matrix-valued gauge vector in terms of $n_i \times n_j$-dimensional blocks as $\hat{A}^\mu = (\hat{A}^\mu_{n_i n_j})$, the twist-invariant condition reads $(\hat{A}^\mu_{n_i n_j}) = \alpha^{i-j} (\hat{A}^\mu_{n_i n_j})$, so only the blocks on the diagonal survive the projection. Therefore, the twist-invariant generators generate the subgroup

$$G(N) = \bigotimes_{i=0}^{N-1} \mathrm{U}(n_i).$$

- The three complex scalars $\hat{\varphi}^a$ correspond to states $b^a_{-1/2}|\mathrm{NS}\rangle$ that carry an internal index, so under the orbifold twist they undergo a transformation that affects them both as twisted directions, in the static gauge, and due to the action on the gauge degrees of freedom. In detail, their transformation is of the form

$$\mathbb{Z}_N : \quad \hat{\varphi}^a \overset{\theta}{\mapsto} \alpha^{l_a}\, \Gamma_{\theta,\overline{3}}\, \hat{\varphi}^a\, \Gamma^{-1}_{\theta,\overline{3}}.$$

In terms of $n_i \times n_j$-dimensional defined by $\hat{\varphi}^a = (\hat{\varphi}^a_{n_i n_j})$, the twist-invariant condition reads $(\hat{\varphi}^a_{n_i n_j}) = \alpha^{i-j+l_a} (\hat{\varphi}^a_{n_i n_j})$. This implies that the orbifold-invariant scalar fields fall into the representations

$$\sum_{a=1}^{3} \sum_{i=0}^{N-1} (\boldsymbol{n}_i, \overline{\boldsymbol{n}}_{i+l_a}).$$

- The four Weyl spinors are associated to the states $|\{s_m\}\rangle_{m=0}^3$, where the half-integers $s_m = \pm 1/2$ define their chirality, and, compatibly with the GSO-projection, they can be labelled as $\hat{\psi}^\omega$, with $\omega = 0$ corresponding to the would-be gaugino $\hat{\lambda}$ and $\omega = a = 1, 2, 3$ corresponding to the three would-be modulini $\hat{\psi}^a$. The orbifold twist takes the form

$$\mathbb{Z}_N : \quad \hat{\psi}^\omega \overset{\theta}{\mapsto} \alpha^{s_m k_m}\, \Gamma_{\theta,\overline{3}}\, \hat{\psi}^\omega\, \Gamma^{-1}_{\theta,\overline{3}},$$

where k_m are integers defining the orbifold action on the fermions, with $\sum_{m=0}^3 k_m = 0 \bmod N$, and $l_1 = k_2 + k_3$, $l_2 = k_1 + k_3$ and $l_3 = k_1 + k_2$. A supersymmetric singularity with $\sum_{a=1}^3 l_a = 0 \bmod N$ requires $k_0 = 0$, which fixes $l_a = -k_a$. The calculations show that the orbifold-invariant subset of the spinor $\hat{\lambda}$ transforms in the representation

$$\sum_{i=0}^{N-1} (\boldsymbol{n}_i, \overline{\boldsymbol{n}}_i),$$

while from the would-be modulini $\hat{\psi}^a$ one obtains the representations

$$\sum_{a=1}^{3} \sum_{i=0}^{N-1} (\boldsymbol{n}_i, \overline{\boldsymbol{n}}_{i+l_a}).$$

In the presence of D7-branes, the reasoning is analogous. Just as for the action of the orbifold twist on the anti-D3-brane gauge degrees of freedom, one defines the matrix

$$\Gamma_{\theta,7} = \mathrm{diag}\left(1_{w_0}, \alpha 1_{w_1}, \dots, \alpha^{N-1} 1_{w_{N-1}}\right),$$

and then essentially the same reasoning as above can be followed, projecting the gauge field $\hat{B}^\mu$, the gaugino $\hat{\zeta}$, the scalar field $\hat{\sigma}^3$ and the modulino $\hat{\eta}^3$.

The description of the orbifold action on the anti-D3-/D7-brane intersecting states can also be worked out in a similar way. A crucial difference with respect to all the other states is that, in this case, the twist-invariant scalars and spinors transform under different representations of the gauge group, and therefore could never form supersymmetric multiplets with respect to any supersymmetry generator. This is a consequence of the fact that the GSO-projection for brane/antibrane states needs to be opposite compared to brane/brane and antibrane/antibrane states [20, 40–43]. Therefore, the projection is different for the scalar $\hat{\varphi}$ and the spinor $\hat{\psi}$, and similarly it is different for the scalar $\hat{\varphi}_*$ and the spinor $\hat{\psi}_*$. For D3-/D7-brane intersecting states, instead, scalars and spinors would still combine into chiral multiplets in conjugate representations.

To sum up, all the orbifold-invariant open-string states living on the the $\mathbb{C}^3/\mathbb{Z}_N$-singularity can be worked out explicitly as explained above. One can directly implement the tools presented above with the states discussed in detail in subsubsection 2.3.1.1. The full spectrum is summarised below.

- The $\overline{3}3$-sector provides a simple would-be supersymmetric massless spectrum.
  - (i) The vector fields and adjoint Weyl spinors transform in identical representations of the group $\bigotimes_{i=0}^{N-1} \mathrm{U}(n_i)$, i.e. in particular:

$$\overline{3}3\text{-sector vectors:} \qquad \boldsymbol{r}_{\boldsymbol{v}}^{(\overline{\boldsymbol{3}}\boldsymbol{3})} = \sum_{i=0}^{N-1} (\boldsymbol{n}_i, \overline{\boldsymbol{n}}_i); \tag{4.5.9a}$$

$$\overline{3}3\text{-sector Weyl spinors:} \qquad \boldsymbol{r}_{\boldsymbol{W_0}}^{(\overline{\boldsymbol{3}}\boldsymbol{3})} = \sum_{i=0}^{N-1} (\boldsymbol{n}_i, \overline{\boldsymbol{n}}_i). \tag{4.5.9b}$$

  - (ii) The $3N$ complex scalar fields and the remaining $3N$ Weyl spinors transform in identical bi-fundamental representations of the group $\bigotimes_{i=0}^{N-1} \mathrm{U}(n_i)$, namely:

$$\overline{3}3\text{-sector scalars:} \qquad \boldsymbol{r}_{\boldsymbol{s}}^{(\overline{\boldsymbol{3}}\boldsymbol{3})} = \sum_{a=1}^{3} \sum_{i=0}^{N-1} (\boldsymbol{n}_i, \overline{\boldsymbol{n}}_{i+l_a}); \tag{4.5.10a}$$

$$\overline{3}3\text{-sector Weyl spinors:} \qquad \boldsymbol{r}_{\boldsymbol{W}}^{(\overline{\boldsymbol{3}}\boldsymbol{3})} = \sum_{a=1}^{3} \sum_{i=0}^{N-1} (\boldsymbol{n}_i, \overline{\boldsymbol{n}}_{i+l_a}). \tag{4.5.10b}$$

- The $7\overline{3}$- and $\overline{3}7$-sectors provide the following non-supersymmetric massless spectrum, transforming in distinct bifundamental representations:
  - (i) two sets of N scalar fields:

$$7\overline{3}\text{-sector scalars:} \qquad \boldsymbol{r}_{\boldsymbol{s}}^{(\boldsymbol{7}\overline{\boldsymbol{3}})} = \sum_{i=0}^{N-1} (\overline{\boldsymbol{n}}_i, \boldsymbol{w}_{i+(l_1-l_2)/2}), \tag{4.5.11a}$$

$$\overline{3}7\text{-sector scalars:} \qquad \boldsymbol{r}_{\boldsymbol{s}}^{(\overline{\boldsymbol{3}}\boldsymbol{7})} = \sum_{i=0}^{N-1} (\boldsymbol{n}_{i+(l_1-l_2)/2}, \overline{\boldsymbol{w}}_i); \tag{4.5.11b}$$

(ii) two sets of N Weyl spinors:

$$\text{7}\overline{3}\text{-sector Weyl spinors:} \qquad \boldsymbol{r}_{\boldsymbol{W}}^{(\mathbf{7\overline{3}})} = \sum_{i=0}^{N-1} (\overline{\boldsymbol{n}}_i, \boldsymbol{w}_{i-l_3/2}), \tag{4.5.12a}$$

$$\overline{3}\text{7-sector Weyl spinors:} \qquad \boldsymbol{r}_{\boldsymbol{W}}^{(\mathbf{\overline{3}7})} = \sum_{i=0}^{N-1} (\boldsymbol{n}_{i-l_3/2}, \overline{\boldsymbol{w}}_i). \tag{4.5.12b}$$

- Finally, in the 77-sector, one has a supersymmetric spectrum, as follows:

  (i) the vector fields and a class of Weyl spinors form a number N of $N_4 = 1$ vector multiplets:

$$\text{77-sector vector multiplets:} \qquad \boldsymbol{r}_{\boldsymbol{V}}^{(\mathbf{77})} = \sum_{i=0}^{N-1} (\boldsymbol{w}_i, \overline{\boldsymbol{w}}_i); \tag{4.5.13}$$

  (ii) the scalars fields and the Weyl spinors form a number N of $N_4 = 1$ chiral multiplets:

$$\text{77-sector chiral multiplets:} \qquad \boldsymbol{r}_{\boldsymbol{C}}^{(\mathbf{77})} = \sum_{i=0}^{N-1} (\boldsymbol{w}_i, \overline{\boldsymbol{w}}_{i+l_3}). \tag{4.5.14}$$

Such representations factorise according to the factorisation of the groups $\mathrm{U}(p_i)$. For instance, if a field is in the representation $\boldsymbol{p}_i$ with respect to the group $\mathrm{U}(p_i)$, it has charge $q_i = 1$ under its $\mathrm{U}(1)$-component, denoted as $\mathrm{U}(1)_i$ from now on, and it is in the representation $\boldsymbol{p}_i$ of its $\mathrm{SU}(p_i)$-component.

As models with anti-D3- and D7-branes at orbifold singularities contain chiral fermions in fundamental representations of the gauge groups, the theory is anomalous unless special cancellations occur, which is usually guaranteed by the cancellation of RR-tadpoles [221]. The specific configurations which make the theory anomaly-free are spelled out below and amount to the combinations of the sets of integers $\{n_i\}_{i=0}^{N-1}$ and $\{w_i\}_{i=0}^{N-1}$ that happen to give a theory in which all the anomalous Feynman diagrams add up to zero.

1. It can be shown that the condition whose fulfilment guarantees the cancellation of all the cubic non-Abelian anomalies arising from the $\mathrm{SU}(n_i)$- and $\mathrm{SU}(w_i)$-subgroups is [29, 30, 221]

$$4\left[\prod_{a=1}^{3} \sin\left(\frac{\pi k l_a}{N}\right)\right] \operatorname{tr} \Gamma_{\theta^k,\overline{3}} - \sin\left(\frac{\pi k l_3}{N}\right) \operatorname{tr} \Gamma_{\theta^k,7} = 0, \tag{4.5.15}$$

   for all values $k = 0, 1, \ldots, N-1$. This condition also gaurantees the absence of mixed Abelian-gravitational anomalies.

2. Under the condition above, the mixed Abelian/non-Abelian diagrams as well as the cubic Abelian diagrams are pseudo-anomalous, which implies that the Abelian factors

actually acquire a mass via the Green-Schwarz mechanism, apart from the linear combination[4.20] [29, 30, 280]

$$Q = \sum_{i=0}^{N-1} \frac{Q_i}{n_i}, \tag{4.5.16}$$

with Q_i being the generators of the $\mathrm{U}(1)_i$-factors. Depending on the model, there may be additional non-anomalous combinations.

In principle, the gauge fields in the multiplets from the spectrum reported in eq. (4.5.9a) are the vectors[4.21] $\hat{A}^\mu_i = A^\mu_i + \tilde{A}^\mu_i$, one for each different $\mathrm{U}(n_i)$-subgroup, and similarly for the $\mathrm{U}(w_i)$-subgroups. However:

i. the non-Abelian gauge fields $\tilde{A}^\mu_i$ of the $\mathrm{SU}(n_i)$-components are non-anomalous if the condition in eq. (4.5.15) is satisfied, and similarly for the $\mathrm{SU}(w_i)$-components;

ii. all the Abelian gauge vectors A^μ_i are anomalous and hence disappear from the low-energy effective theory, apart from the linear combination given in eq. (4.5.16), i.e.

$$V^\mu = \sum_{i=0}^{N-1} \frac{1}{n_i} A^\mu_i .$$

Additional anti-D3-branes at other fixed points are also included in order to cancel the D7-brane anomaly induced there. Even though the corresponding new U(1)-factors are anomaly-free, they still acquire a mass via the Stückelberg coupling [30, 280].

4.5.3.2 Supergravity Formulation

Given the massless spectrum of anti-D3-/D7-branes at orbifold singularities, one can now describe the effective theory in the language of $N_4 = 1$ supergravity. In particular, one needs to identify the goldstino and understand how to encode the remaining degrees of freedom in supermultiplets.

If the anti-D3-brane sits at an orbifold singularity, the goldstino survives and the same supersymmetry breaking takes place as if it is at a smooth point. A similar breaking also happens for anti-D3-branes sitting at an orientifold singularity, as in ref. [83]. With multiple anti-D3-branes, the goldstino undergoes the projection in eq. (4.5.9b) and the following reasoning holds.

(i) At a smooth point, the anti-D3-brane goldstino would be the neutral singlet contained in the would-be $\mathrm{U}(n)$-gaugino $\hat{\lambda}$. At an orbifold singularity, the original spinor $\hat{\lambda}$ suffers the orbifold projection

$$\hat{\lambda} = \Gamma_{\theta,\overline{3}}\, \hat{\lambda}\, \Gamma^{-1}_{\theta,\overline{3}},$$

[4.20] Actually, this combination exists as long as all the integers n_i are non-zero. Moreover, some $\mathbb{Z}_N$-orbifolds might have further anomaly-free linear combinations. An explanation to this is in ref. [29], ss. 2.3.

[4.21] One must be careful with the notation, if hatted or tilded fields carry an i-index: for instance, $\tilde{A}^\mu_i$ denotes the vector field gauging the $\mathrm{SU}(n_i)$-subgroup, and it can be expanded as $\tilde{A}^\mu_i = (\tilde{A}^\mu_{n_i n_i}) = \tilde{A}^\mu_{ik} t^k_i$, with t^k_i being the Hermitian generators of $\mathrm{SU}(n_i)$, for $k = 1, \dots, n_i^2 - 1$.

which singles out several diagonal components as several gaugini $\hat{\lambda}_i$ for each of the subgroups $\mathrm{U}(n_i)$. For each of these, one extracts a neutral singlet λ_i under the $\mathrm{U}(1)_i$ and $\mathrm{SU}(n_i)$ subgroups.

(ii) Only one linear combination of the gaugini and their would-be vector superpartners is actually massless, with orthogonal combinations acquiring a mass via the Green-Schwarz mechanism [30]. In accordance with eq. (4.5.16), the goldstino of the theory is thus the linear combination

$$\psi_{\mathrm{g}} = \sum_{i=0}^{N-1} \frac{1}{n_i} \lambda_i,$$

since it is the only massless gauge-neutral spinor on the anti-D3-brane worldvolume.

The goldstino can be encoded as usual in a nilpotent superfield x. After the identification of the goldstino, one can easily infer the main characteristics of the supergravity effective field theory of the remaining fields in the massless spectrum. Details are below.

- In the $\overline{3}3$-sector, the situation is as follows.
  (i) The vectors and the Weyl spinors that transform in the adjoint representations $\boldsymbol{r}_{\boldsymbol{v}}^{(\overline{\mathbf{3}}\mathbf{3})} = \boldsymbol{r}_{\boldsymbol{W_0}}^{(\overline{\mathbf{3}}\mathbf{3})}$ of eqs. (4.5.9a, 4.5.9b) correspond to the orbifold-invariant blocks of the fields $\tilde{A}^{\mu}$ and $\tilde{\lambda}$, plus the non-anomalous Abelian component V^{μ} and the goldstino ψ_{g}. Therefore, they belong to the orbifold-invariant blocks from the constrained superfields $\tilde{W}_{\alpha}$ and $\tilde{x}$, to the constrained superfield W_{α} and to the fundamental nilpotent chiral mulitplet x, respectively.

  The vectors are massless and provide the standard-like model gauge fields, with the goldstino being set to zero in the unitary gauge. On the other hand, the would-be non-Abelian gaugini are extra massless degrees of freedom that are made massive by non-trivial effects such as anomaly mediation.

  (ii) The complex scalars and the Weyl spinors transforming in the bifundamental representations $\boldsymbol{r}_{\boldsymbol{s}}^{(\overline{\mathbf{3}}\mathbf{3})} = \boldsymbol{r}_{\boldsymbol{W}}^{(\overline{\mathbf{3}}\mathbf{3})}$ of eqs. (4.5.10a, 4.5.10b) are the orbifold-invariant blocks of the fields φ^a, $\tilde{\varphi}^a$, ψ^a and $\tilde{\psi}^a$, and therefore belong to the orbifold-invariant blocks from the constrained superfields h^a, $\tilde{h}^a$, y^a and $\tilde{y}^a$, respectively.

  All these fields are massive in the presence of $(2,1)$-flux at the anti-D3-brane location. Scalars receive further subleading contributions originating from perturbative and non-perturbative corrections to the theory. Notice that not all the orbifold singularities allow for $(2,1)$-fluxes, in which case the corrections become leading for the scalars, with the spinors staying massless.[4.22]

- In the $7\overline{3}$- and $\overline{3}7$-sectors, the situation is as follows.

[4.22] For a supersymmetric $\mathbb{Z}_N$-twist, a necessary condition for the $(2,1)$-flux to survive the orbifold projection is that at least one of the l_a-coefficients be $l_a = N/2$, which is not satisfied e.g. by a $\mathbb{C}^3/\mathbb{Z}_3$-singularity, but it is for instance by $\mathbb{C}^3/\mathbb{Z}_4$; the flux can also be preserved for singularities of the form $(\mathbb{C}^2/\mathbb{Z}_N) \times \mathbb{C}$, $\mathbb{C}^3/[\mathbb{Z}_M \times \mathbb{Z}_N]$ and $\mathbb{C}^3/[\mathbb{Z}_M \times \mathbb{Z}_N \times \mathbb{Z}_K]$ [29,231,235]. Moreover, depending on the orbifold action, the specific flux components that render the modulini massive [231] might be projected out. The trace condition permits this situation while keeping the scalars massive.

(i) The scalars transforming in the bifundamental representations $\boldsymbol{r_s^{(\bar{3}7)}}$ and $\boldsymbol{r_s^{(7\bar{3})}}$ of eqs. (4.5.11a, 4.5.11b) are the orbifold-invariant blocks of the fields $\hat{\varphi}$ and $\hat{\varphi}_*$, and therefore belong to the corresponding blocks from the constrained superfields $\hat{h}$ and $\hat{h}_*$, respectively.

In theories stabilised by non-perturbative and including perturbative effects, such fields are massive and, moreover, they also receive contributions from anomaly mediation. Anomaly-mediated mass contributions can be negative and lead to tachyonic instabilities, but they may be balanced by other effects such as the α'-corrected uplift contribution.

(ii) The Weyl spinors belonging to the bifundamental representations $\boldsymbol{r_W^{(\bar{3}7)}}$ and $\boldsymbol{r_W^{(7\bar{3})}}$ of eqs. (4.5.12a, 4.5.12b) are the orbifold-invariant blocks of the fields $\hat{\psi}$ and $\hat{\psi}_*$, and therefore belong to the corresponding blocks from the constrained superfields $\hat{y}$ and $\hat{y}_*$, respectively.

Such fields are always massless and therefore they always contribute to the massless matter content of the standard-like model extension built at the orbifold-like singularity.

- In the 77-sector, the situation is the following.

(i) The fields in the vector multiplets in the adjoint representations $\boldsymbol{r_V^{(77)}}$ are the invariant blocks from the fields B_μ (if anomaly-free), $\tilde{B}_\mu$, ζ and $\tilde{\zeta}$, and therefore belong to the corresponding blocks of the vector multiplets W'_α and $\tilde{W}'_\alpha$.

Such gauge fields are modelled to correspond to interactions in a hidden sector. In a pure-flux background, the gaugini are massive only in the presence of $(0,3)$-flux, which is not present at the tip of the throat. However, for bulk-extended D7-branes they acquire masses from a non-zero volume-modulus F-term and even for throat-localised D7-branes they acquire a mass by anomaly mediation.

(ii) The fields in the chiral multiplets in the bifundamental representations $\boldsymbol{r_C^{(77)}}$ are the invariant blocks of the fields σ^3, $\tilde{\sigma}^3$, η^3 and $\tilde{\eta}^3$, and therefore belong to the corresponding blocks of the chiral superfields σ^3 and $\tilde{\sigma}^3$.

All these fields are massive in the presence of $(2,1)$-flux, with further contributions from perturbative and non-perturbative corrections to the theory.

Now that the supermultiplets have been identified, given the $N_4 = 1$ supergravity formulation of a system with intersecting anti-D3- and D7-branes at a smooth point in the internal space, in order to describe the theory of intersecting anti-D3- and D7-branes at an orbifold singularity one can simply reduce the original superfields to the subset that is invariant under the orbifold twist. An extensive work in the construction of quasi-realistic particle spectra on anti-D3-/D7-branes at orbifold singualities can be found in refs. [29,30,231,235].

All the analysis carried out so far can be adapted to more complicated orbifolds than the one hosting just $\mathbb{C}^3/\mathbb{Z}_N$-singularities. Such orbifolds induce additional structure in the matter spectrum, but they do not bring in fundamental differences in terms of the tools that one has to use to set up a supergravity description in terms of constrained superfields. The most prominent difference is that there exist singularities with further massless would-be vector superfields. In particular, this is a feature of orbifolds which leave invariant at least one of the complex directions [29]. In this case this gives rise to extra massless Abelian gauge fields and neutral spinor fields.

4.6 Analysis of the Mass Hierarchies

Together, sections 4.3, 4.4 and 4.5 provide the tools to formulate the supergravity description of chiral gauge theories from intersecting anti-D3-/D7-branes on warped orbifold singularities in type IIB Calabi-Yau orientifold flux compactifications. For completness, the physical mass scales that emerge in such constructions are now discussed, with a view towards quasi-realistic standard-like models. In the scenario considered:

- the localisation condition of eq. (4.2.13) is assumed, implying that closed-string sector fields, apart from the gravitino, tend to localise near the redshifted end of the throat;
- the hierarchy of eq. (4.2.16) between the gravitino mass-sourcing fluxes is assumed, implying that the gravitino is localised in the bulk and a low-energy supergravity description is consistent.

For definiteness, it is also assumed that $(2,1)$-fluxes are present at the tip of the throat, with no other fluxes. For ease of notation, the normalisation $\ell_{(0)} = 1$ is considered throughout the rest of this section, and similarly for the other numerical constants. The visible sector is constituted by particles that are charged under the anti-D3-brane gauge groups; all the rest represents hidden sectors. This section should also serve as a recap of the mechanisms and effects that have been discussed in this chapter.

4.6.1 Pure D7- and Anti-D3-Brane States

Pure D7- and anti-D3-brane states are discussed first, as their masses are essentially determined by the dimensional reduction of the worldvolume actions. In particular, except for some of the gaugini, the 77- and $\overline{33}$-states are not critically dependent on the interplay between each other and neither on the way in which the Kähler modulus is stabilised nor on anomaly-mediation effects.

- For D7-branes that wrap 4-cycles extending from the tip of a warped throat into the bulk, the fate of the hidden matter chiral multiplets can be one of two possibilities, in accord with subsubsection 4.3.1.2.
  - If the mass-sourcing fluxes do not have specific hierarchies, then the D7-brane chiral superfield is localised near the tip of the throat with a mass of the order of the flux-induced axio-dilaton one, that is, from the normalisation induced by the matter metric and the μ-coupling in eqs. (4.3.10, 4.3.11), a canonical supersymmetric mass

$$m_{77}^2 \sim (m_{\mathrm{D7}}^w)^2 \sim \frac{g_s^2}{\mathcal{V}^{\frac{2}{3}}} \frac{1}{\kappa_4^2} \, \mathrm{e}^{2A_0}, \tag{4.6.1}$$

which is of the same order of magnitude as the warped Kaluza-Klein scale $m^w_{\rm KK}$ of eq. (4.2.11), above the cutoff scale of the theory.

– If the fluxes are such that the D7-brane chiral multiplet does not localise near the tip, then, from the matter metric and the μ-coupling in eqs. (4.3.7, 4.3.8), the canonically normalised supersymmetric mass is

$$m^2_{77} \sim {\rm e}^{2A_0} m^2_{\rm D7} \sim \frac{\theta'^2 g_s^2}{\mathcal{V}^2} \frac{1}{\kappa_4^2} {\rm e}^{2A_0}, \tag{4.6.2}$$

where θ' is a small number representing the small bulk flux. In this case, the chiral multiplet survives the warped Kaluza-Klein cutoff.

Again following subsubsection 4.3.1.2, given the gauge kinetic function of eq. (4.3.12), the hidden-sector gauge couplings are of order

$$g^2_{\rm D7} \simeq \frac{2\pi g_s}{\mathcal{V}^{\frac{2}{3}}}. \tag{4.6.3}$$

In the absence of $(0,3)$-flux, if there are no supersymmetry-breaking or anomaly-mediation effects, the D7-brane gaugino is massless.

- For D7-branes that wrap 4-cycles localised at the tip of a warped throat, from the discussion in subsubsection 4.3.1.3 with the matter metric and the μ-coupling of eqs. (4.3.16, 4.3.17), the hidden chiral matter multiplets acquire the canonical mass [228]

$$m^2_{77} \sim (m^w_{\rm D7})^2 \sim \frac{g_s^2}{\mathcal{V}^{\frac{2}{3}}} \frac{1}{\kappa_4^2} {\rm e}^{2A_0}. \tag{4.6.4}$$

This means that the fields do not survive the cutoff unless the mass-sourcing $(2,1)$-flux is parametrically smaller than other fluxes in the throat that generate the warped Kaluza-Klein scale. Also, subsubsection 4.3.1.3, thanks to the gauge kinetic function of eq. (4.3.18), indicates that the hidden gauge couplings scale as

$$g^2_{\rm D7} \simeq 2\pi g_s. \tag{4.6.5}$$

Again, the gaugino is massless in the absence of supersymmetry-breaking or anomaly-mediation effects.

- For anti-D3-branes, the modulini and scalar exotics have masses of the same order of magnitude, as discussed in subsubsection 4.4.1.3. From the matter metrics of eqs. (4.4.7, 4.4.9) and the H-couplings of eqs. (4.4.8, 4.4.10), one finds once again that a $(2,1)$-flux sources a canonical mass [228]

$$m^2_{33} \sim (m^w_{\rm D3})^2 \sim \frac{g_s^2}{\mathcal{V}^{\frac{2}{3}}} \frac{1}{\kappa_4^2} {\rm e}^{2A_0}. \tag{4.6.6}$$

Further, given the gauge kinetic function in eq. (4.4.14), the gauge coupling scales as

$$g^2_{\rm D3} \simeq 2\pi g_s. \tag{4.6.7}$$

As the anti-D3-brane gaugino is the goldstino of the theory, it is always massless. For non-Abelian branes, there can be anomaly-mediation effects, otherwise the gaugini are always massless in the models under consideration.

4.6.2 Anti-D3-/D7-Brane Intersecting states and Stable Kähler Modulus

In the presence of intersecting anti-D3- and D7-branes, following subsection 4.4.2, the spinors from the $\overline{3}7$- and $7\overline{3}$-sectors are massless, with the scalars just receiving a small uplift-induced mass correction. However, the string perturbative and non-perturbative effects are crucial for both stabilising the Kähler modulus and for making the intersecting scalars massive, as discussed in subsection 4.4.3. In contrast, for the pure anti-D3- and D7-brane states, these only induce suppressed extra contributions that are only significant for some of the gaugini. A relevant role can also be played by anomaly mediation.

As discussed in subsection 4.4.3, following eq. (4.4.48), the interplay between non-perturbative corrections and the anti-D3-brane uplift implies that the gravitino acquires a mass of order

$$(\hat{m}^w_{3/2})^2 \sim \frac{g_s^3}{\mathcal{V}^{\frac{4}{3}}}\frac{1}{\kappa_4^2}\,\mathrm{e}^{4A_0}. \tag{4.6.8}$$

Roughly, this can be written in terms of the warped Kaluza-Klein scale and the condition of eq. (4.2.13) shows that this mass is bounded above as

$$(\hat{m}^w_{3/2})^2 \lesssim \frac{g_s}{\mathcal{V}^2}\,(m^w_{\mathrm{KK}})^2.$$

This means that the gravitino is well within the cutoff of the theory. Also, the Kähler modulus is stabilised and, from eq. (4.4.49), its canonically normalised[4.23] mass is of order

$$(\hat{m}^w_{\mathcal{V}})^2 \sim a^2\mathcal{V}^{\frac{4}{3}}\,(\hat{m}^w_{3/2})^2, \tag{4.6.9}$$

with the upper bound

$$(\hat{m}^w_{\mathcal{V}})^2 \lesssim \frac{a^2 g_s}{\mathcal{V}^{\frac{2}{3}}}\,(m^w_{\mathrm{KK}})^2,$$

leaving it well within the warped Kaluza-Klein cutoff too. Finally, in accordance with eq. (4.4.50), the canonical masses for the $\overline{3}7$-/$7\overline{3}$-sector visible scalars are of order

$$m^2_{\overline{3}7} \sim m^2_{7\overline{3}} \sim \frac{\xi}{\mathcal{V}}\,(\hat{m}^w_{3/2})^2. \tag{4.6.10}$$

Again, one can easily verify that these fields survive the 4-dimensional cutoff, being

$$m^2_{\overline{3}7} \sim m^2_{7\overline{3}} \lesssim \frac{g_s\xi}{\mathcal{V}^3}\,(m^w_{\mathrm{KK}})^2.$$

As discussed in subsection 4.4.3, moduli stabilisation has effects on the gaugini, and anomaly mediation affects both the gaugini and the intersecting states.

- For D7-branes wrapping a 4-cycle extended from the throat tip into the bulk, from eq. (4.4.51), the non-zero volume F-term induces D7-brane hidden gaugini masses of order

$$m^{\mathrm{D7}}_{1/2} \sim \frac{1}{a\mathcal{V}^{\frac{2}{3}}}\,\hat{m}^w_{3/2}, \tag{4.6.11}$$

[4.23] For ease of notation, although this is the mass of the canonically normalised modulus, the symbol $\mathcal{V}$ is maintained from now on since the volume is what is controlled by the field c.

An anomaly-mediated contribution is also there but it has an extra string-coupling suppression, as can be seen in eq. (4.4.54). Also, from eq. (4.4.57), the anomaly-mediated mass contribution for the anti-D3-/D7-brane visible-sector intersecting scalars is of order

$$\delta m_{37}^2 \sim \delta m_{7\overline{3}}^2 \sim -g_s^2 (\hat{m}_{3/2}^w)^2, \tag{4.6.12}$$

which competes closely with the α'-induced volume-suppressed contribution. One generally also has a Yukawa coupling-induced term of the same order of magnitude.

- If the D7-brane wraps a 4-cycle which is localised at the tip of the warped throat, then eq. (4.4.55) indicates that the D7-brane hidden gaugino acquires an anomaly-mediated mass of order

  $$m_{1/2}^{\mathrm{D7}} \sim g_s \, \hat{m}_{3/2}^w. \tag{4.6.13}$$

  Further, from eq. (4.4.58), the anomaly-mediated contribution for the anti-D3-/D7-brane visible-sector intersecting scalars is of order

  $$\delta m_{\overline{3}7}^2 \sim \delta m_{7\overline{3}}^2 \sim -g_s^2 \, (\hat{m}_{3/2}^w)^2. \tag{4.6.14}$$

  This can dominate the term originated by the α'-induced contribution, generating an instability, depending on how the volume and the string coupling are tuned. A Yukawa coupling-induced term of the same order of magnitude is also generally there.

- Anomaly mediation also generates masses for the anti-D3-brane visible-sector would-be gaugini apart from the goldstino, which are present for non-Abelian anti-D3-branes. In this case, thanks to eq. (4.4.56), the order of magnitude is

  $$m_{1/2}^{\overline{\mathrm{D3}}} \sim g_s \, \hat{m}_{3/2}^w. \tag{4.6.15}$$

An interesting scenario is the one in which the mass-sourcing $(2,1)$-flux is such that the pure anti-D3- and D7-brane chiral multiplets are heavier than the cutoff scale. Since their positions are stabilised at the expectation values $\langle \varphi^a \rangle = 0$ and $\langle \sigma^3 \rangle = 0$, the trilinear couplings disappear. One is left with an effective theory in which the 4-dimensional degrees of freedom are:

- the non-anomalous visible and hidden gauge vectors from anti-D3- and D7-branes, which are massless, and the corresponding gaugini, which have masses of the order of magnitude in eqs. (4.6.11)/(4.6.13) and (4.6.15);

- the intersecting anti-D3-/D7-brane states, i.e. some standard-like model spinors and exotic scalars in fundamental representations of the gauge groups, where the spinors are massless and the scalars have masses of the order of magnitude in eq. (4.6.10);

- the graviton, which is obviously massless, and a gravitino with a mass of the order of magnitude in eq. (4.6.8), after the combination with the anti-D3-brane goldstino;

- the Kähler modulus and its superpartner, with masses of the order in eq. (4.6.9), which constitute the lightest closed-string hidden-sector particles after the gravitino.

In models at orbifold singularities, the intersecting states are generally such that the scalars and the spinors are in different representations of the gauge groups, meaning that they do not even have would-be superpartners, but rather represent just a bunch of different charged spin-0 and spin-1/2 particles.

4.6.3 Sample Mass Scales

A qualitative spectrum reproducing the typical mass scales in models with intersecting anti-D3- and D7-branes for strongly-warped compactifications, i.e. satisfying the condition in eq. (4.2.13), and in the limit where the bulk $(0, 3)$-flux is sufficiently small that a 4-dimensional supergravity formulation is allowed, i.e. satisfying eq. (4.2.16), is reported below.

In detail, fig. 4.3 reports a qualitative sample spectrum, in units of the reduced Planck mass $m_P = 1/\kappa_4$, in the case where the anti-D3-brane sits at the tip of the warped throat and the D7-brane wraps a 4-cycle extending from the throat tip into the bulk, with its chiral multiplet localised at the tip (see paragraph 4.3.1.2.3 and subsubsections 4.4.1.3, 4.4.2.3). A similar spectrum emerges if the D7-brane wraps a 4-cycle localised at the throat tip (see subsubsections 4.3.1.3, 4.4.1.3, 4.4.2.3), with only minor changes in the gauge sector. Instead, if the D7-brane wraps a 4-cycle extending from the bulk into the tip, with the chiral multiplet localised in the bulk, the difference is also in the smaller mass of the latter (see paragraph 4.3.1.2.4 and subsubsections 4.4.1.3, 4.4.2.2).

The volume-controlling modulus is stabilised by KKLT-like non-perturbative corrections and α'-corrections are inserted too, as in subsection 4.4.3. The sample values are $g_s = 5{\cdot}10^{-2}$ and $\mathrm{e}^{2A_0} = 10^{-8}$ as well as $a = 0.1$, $|A| = 1$ and $|W_0| = 10^{-5}$, with $\langle \mathrm{Im}\,\tau \rangle = 1$, $\langle -\mathrm{i}\omega_w \rangle = 1$, $\ell_w/\ell_{(0)} = 1$ and $c_0 = 1$, which, for the scalar potential in eq. (4.4.42), give a volume vacuum expectation value $\langle \mathcal{V} \rangle \simeq 1.6 \cdot 10^3$ and a minimum energy $\Lambda \simeq 2.2 \cdot 10^{-26} m_P^4$ (which can of course be adjusted with a more precise fine-tuning). As usual, these parameters have been tuned to ensure the volume-modulus stabilisation (for recent progress towards a top-down understanding of the parameter space for the KKLT-setup see e.g. refs. [67, 204, 206, 281]). In particular, the sample values chosen here are close to the original ref. [54] and roughly satisfy the assumptions of the current setup, but are only one example in a vast parameter space. Along with the Minkowski vacuum condition of eqs. (4.4.43, 4.4.44), the most stringent bounds are:

- the localisation condition in eq. (4.2.13), which requires a small enough volume, compared to the warp factor, such that $\langle \mathcal{V} \rangle^{2/3} \lesssim \mathrm{e}^{-A_0}$;
- a small GVW-superpotential $|W_0|$, which is necessary for the KKLT-vacuum but also to accomplish the supergravity condition in eq. (4.2.16);
- a string coupling that is large enough to be a reasonable gauge coupling in the visible sector, being $g_{\text{vis}}^2 \simeq 2\pi g_s$, but also sufficiently small, as to prevent tachyons in the intersecting sector.[4.24]

[4.24] This is a highly model-dependent problem: it is quite delicate and it requires taking into account the

Roughly, in order to have reasonable gauge couplings and to avoid open-string tachyons, the string coupling has to be of order $g_s \sim 10^{-2}$ and the volume is thus forced to be roughly at most of the order of magnitude $\langle \mathcal{V} \rangle \sim 10^3$. Therefore, the gravitino mass in eq. (4.6.8) - to which all the other 4-dimensional effective masses are proportional - indicates that what really suppresses the masses is the redshift factor e^{A_0}. In particular, the parameters chosen here place the gravitino mass and scalar exotics just above the current observational bounds. However, by stretching the parameters of the non-perturbative superpotential correction, one may achieve scenarios where the redshift e^{A_0} is small enough to make the gravitino – and consequently all the other low-energy fields – arbitrarily light. On the other hand, bigger values of the redshift e^{A_0} are also possible and provide masses that can be a few orders of magnitude larger.

Although a detailed exploration of the phenomenological implications of such scenarios is not the main aim of this analysis, a few comments are due. Notice that in the mass scales all the numerical factors have been dropped and only the parametric dependences on e^{A_0}, g_s and $\langle \mathcal{V} \rangle$ have been taken into account, i.e. numerical values are computed with the estimates summarised in this section.

- From the cosmological perspective, the models do not present the cosmological moduli problem [282–285] since all the hidden moduli are heavier than the visible scalars. Whether or not there is a gravitino problem depends on the decay channels and abundances, but, in any case, the gravitino, with a mass of order $\hat{m}^w_{3/2} \simeq 8 \cdot 10^{-13} m_P$, is sufficiently heavy to decay soon enough as not to spoil the BBN-physics, with a lower bound at roughly $m^{\text{min}}_{3/2} \sim 10^{-13} m_P$ [285–287]. The models also contain some massless hidden U(1)-gaugini and some heavy non-Abelian gaugini from the 77-sector, with masses $m^{\text{D7}}_{1/2} \simeq 6 \cdot 10^{-14} m_P$ for a wrapped 4-cycle extending into the bulk, with a very small gauge coupling of order $g^2_{\text{hid}} \simeq 2 \cdot 10^{-3}$, or $m^{\text{D7}}_{1/2} \simeq 4 \cdot 10^{-14} m_P$ for a wrapped 4-cycle at the throat tip, with coupling $g^2_{\text{hid}} \simeq 0.3$. If the D7-brane chiral multiplet localises near the tip, its mass scale is above the cutoff, while if its mass-sourcing bulk flux is small enough and it stays in the bulk, then its mass is comparable to the gravitino one, i.e. $m_{77} \sim \hat{m}^w_{3/2}$.

- From the particle-physics point of view, the visible sector consists of one Abelian and a few non-Abelian gauge groups plus some charged massless spinors in bifundamental representations as well as some heavy charged bifundamental scalars and a few slightly heavier non-Abelian gaugini. All the gauge couplings are of order $g^2_{\text{vis}} \simeq 0.3$. For a gravitino with a mass of order $\hat{m}^w_{3/2} \simeq 8 \cdot 10^{-13} m_P$, these scalar masses are of order $m^{\text{scalar}}_{37} \sim m^{\text{scalar}}_{7\overline{3}} \simeq 7 \cdot 10^{-15} m_P$ (including the factor $\xi/4 \simeq 1/8$), while the gaugino masses are $m^{\text{D3}}_{1/2} \simeq 4 \cdot 10^{-14} m_P$. Such values are not inconsistent with the observational bounds [146].

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details of the mass terms. As an example, taking a sample value $\xi \simeq 0.5$, one finds a soft-breaking α'-induced mass of order $m^2_{3\overline{7}}/(\hat{m}^w_{3/2})^2 = \xi/(4\langle \mathcal{V} \rangle) \simeq 8 \cdot 10^{-5}$, while, taking a sample value $\sum_i b_i C_2(r^i) = -1$ (often the groups $\mathrm{SU}(n_i)$ with the largest n_i tend to have $b > 0$, with larger C_2, and the opposite happens for smaller values n_i; for instance this is the case in the MSSM and in the models of ref. [30]), the anomaly-mediated corrections are $\delta m^2_{3\overline{7}}/(\hat{m}^w_{3/2})^2 = g_s^2 b C_2/(32\pi^2) \simeq -8 \cdot 10^{-6}$.

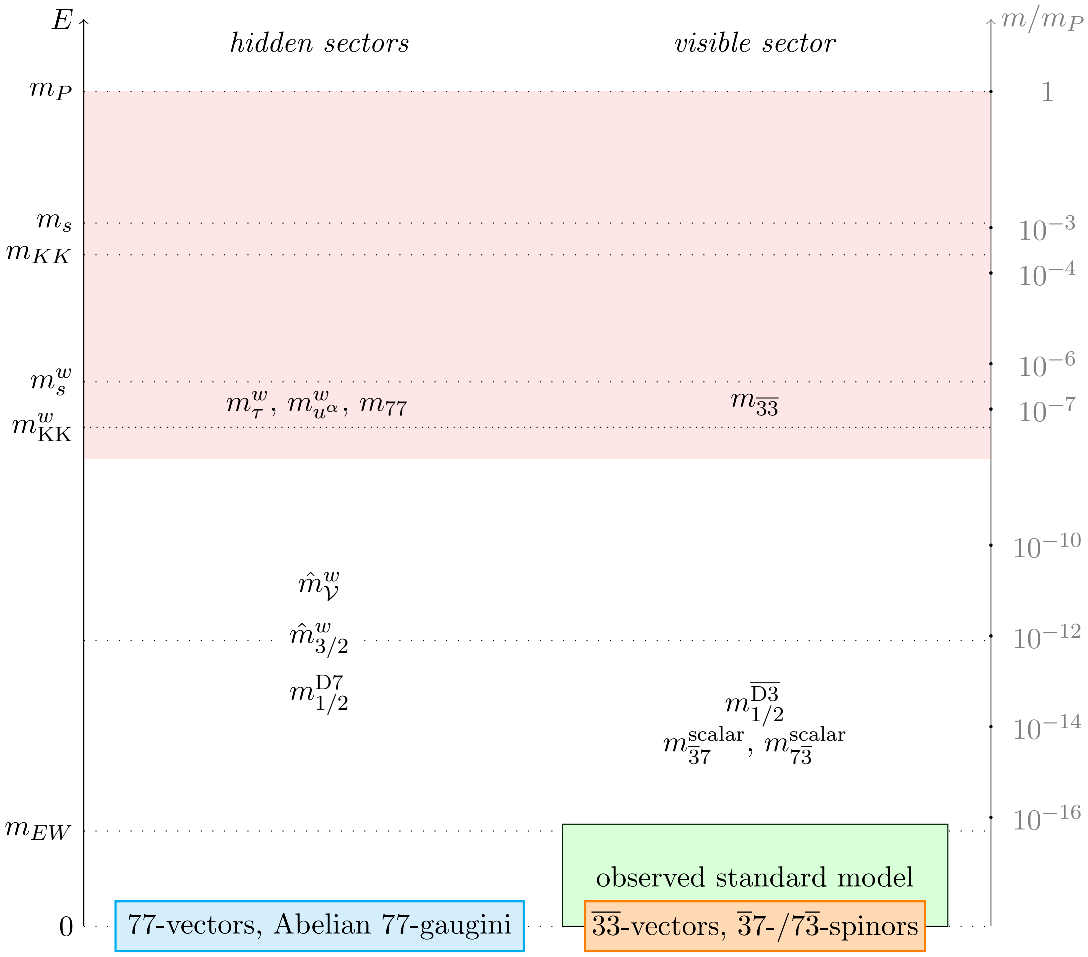

Figure 4.3: A qualitative sample of the mass scales in models with intersecting anti-D3- and D7-branes in highly-warped compactifications, i.e. such that $\langle\mathcal{V}\rangle^{2/3} \leq \mathrm{e}^{-A_0}$, with KKLT-like non-perturbative corrections and α'-corrections, and a small bulk $(0,3)$-flux such that the gravitino localises in the bulk. Where the spin is not indicated, the masses refer to the supermultiplet as the soft-breaking corrections do not dominate. The observed standard-model energy range and the relevant scales above the cutoff are shown explicitly. The graph refers to an anti-D3-brane sitting at the throat tip and a D7-brane wrapping a 4-cycle extending from the tip into the bulk, with the D7-brane chiral multiplet localised at the tip, where the gauge couplings are $g_{\overline{\mathrm{D3}}}^2 \simeq 0.3$ and $g_{\mathrm{D7}}^2 \simeq 2 \cdot 10^{-3}$. A similar spectrum emerges if the D7-brane wraps a 4-cycle localised at the tip, with then the D7-brane scales similar to the anti-D3-brane scales, so $g_{\mathrm{D7}}^2 \simeq g_{\overline{\mathrm{D3}}}^2$ and $m_{1/2}^{\mathrm{D7}} \sim m_{1/2}^{\overline{\mathrm{D3}}}$. If the D7-brane wraps a 4-cycle extending into the bulk and the mass-sourcing $(2,1)$-fluxes are such that the D7-brane chiral multiplet localises in the bulk, then the latter approaches the gravitino mass scale, $m_{77} \sim \hat{m}_{3/2}^w$.

It is important to discuss the scale at which the supersymmetry-breaking mass splittings come into play. Indeed, whilst there is no scale at which superpartners emerge for the $\overline{3}\overline{3}$- and $\overline{3}7$-/$7\overline{3}$-states, closed-string and 77-multiplets do have supersymmetry-breaking mass splittings, and $\overline{3}\overline{3}$-states and $\overline{3}7$-/$7\overline{3}$-scalars also acquire soft mass contributions from supersymmetry-breaking effects. The breaking of supersymmetry by the anti-D3-branes takes place at the warped-string scale m_s^w, where the full tower of string states comes into play [2, 96]. However, the relevant mass scale for supersymmetry breaking in the low-energy theory is instead controlled by the gravitino mass scale $\hat{m}_{3/2}^w$, as will now be commented on. In a near-Minkowski vacuum, the orders of magnitude of the contributions to the F-term scalar potential are fixed by the scales [68, 98]

$$f_x^w = \left[\frac{1}{3}\mathcal{K}_{x\bar{x}}\mathcal{F}^x\overline{\mathcal{F}}^x\right]^{\frac{1}{2}} \simeq \hat{m}_{3/2}^w m_P, \tag{4.6.16a}$$

$$f_\rho^w = \left[\frac{1}{3}\mathcal{K}_{\rho\bar{\rho}}\mathcal{F}^\rho\overline{\mathcal{F}}^\rho\right]^{\frac{1}{2}} \simeq \frac{1}{2a\mathcal{V}^{\frac{2}{3}}}\hat{m}_{3/2}^w m_P, \tag{4.6.16b}$$

although the anti-D3-brane uplift energy and the KKLT-like Kähler-modulus potentials combine non-trivially with the gravitino mass-dependent contribution to give a near-zero cosmological constant, being the leading-order contributions in the scalar potential such that $\langle \mathcal{V}_F/3\rangle \simeq (f_x^w)^2 - (\hat{m}_{3/2}^w m_P)^2 \simeq 0$. One may then define a supersymmetry-breaking scale in the low-energy theory as $m_{\text{SUSY}} \sim (f_x^w)^{1/2}$. Nevertheless, for both the Kähler modulus and the open-string sector, the orders of magnitude of the mass splittings read

$$\begin{aligned} \hat{m}_{\mathcal{V}}^w &\sim a\mathcal{V}^{\frac{2}{3}}\,\hat{m}_{3/2}^w, \\ m_{\text{soft}}^{\text{open}} &\sim \hat{m}_{3/2}^w. \end{aligned}$$

So, even though there is no order parameter able to restore supersymmetry for the anti-D3-brane, the mass-splittings are not at the scale m_s^w or m_{SUSY}, but rather they are fixed by the gravitino mass $\hat{m}_{3/2}^w$ in the stabilised model: as usual, the canonical normalisation in physical units sets the volume-modulus mass at a slightly volume-enhanced gravitino scale, whereas for the open-string contributions the order of magnitude is immediately set at the scale $m_{\text{soft}} \sim m_{\text{SUSY}}^2/m_P \sim \hat{m}_{3/2}^w$ by the mediation of gravity. Moreover, for the low-energy bifundamental scalars, this scale is further reduced by cancellations at leading order and they are the lightest (exotic) visible particles.

To end, it is worthwhile to stress that the particle spectra discussed here represent the generic low-energy effective theory corresponding to intersecting anti-D3-/D7-branes at an orbifold-like singularity, located at the tip of a strongly-warped throat in a Calabi-Yau orientifold flux compactification, with the Kähler modulus stabilised in a KKLT-like framework. An explicit and globally consistent realisation of such constructions is left for future work.

4.7 Conclusions

This chapter has developed the supergravity description for the low-energy effective field theory of intersecting anti-D3-/D7-brane systems on orbifold singularities at the tip of warped throats, in KKLT-stabilised type IIB Calabi-Yau orientifold flux compactifications. Such string configurations could plausibly provide a realisation of the gauge and matter sectors of the Standard Model of Particle Physics, along with a rich hidden sector, with a geometric origin for large hierarchies of scales and a non-standard realisation of supersymmetry breaking. The anti-D3-brane degrees of freedom realise the bulk $N_4 = 1$ supersymmetry only non-linearly, and thus break supersymmetry spontaneously, with the goldstino corresponding to the neutral massless gaugino that is always present. When the branes are placed on orbifold singularities, moreover, the anti-D3-/D7-brane intersecting fermions and bosons transform in different bifundamental representations of the gauge groups, thus they in no way resemble superpartners. A new description is therefore necessary, namely non-linear supergravity using constrained superfields. The focus has been on the main distinctive features of these novel non-supersymmetric scenarios and their low-energy descriptions, while the realisation of globally consistent concrete models is left for future studies.

The analysis has begun by reviewing the properties of warped flux compactifications in section 4.2. In particular, for strongly-warped throats and bulk volumes that are not too large, i.e. satisfying eq. (4.2.13), bulk fields tend to dynamically localise near the tip of the throat, where energy scales are suppressed due to a gravitational redshift. In order to have a 4-dimensional gravitino localised in the bulk, with Planck-suppressed couplings to match those of the graviton, as expected in supergravity, special fluxes satisfying eq. (4.2.16) have also been assumed. The strong warping can eventually be captured in the low-energy supergravity theory describing degrees of freedom at the bottom of the throat via a constant shift of the Kähler potential by the redshift logarithm $\ln \mathrm{e}^{2A_0} = 2A_0$ [228].

Taking this highly-warped flux background, the low-energy effective theory for a supersymmetric D3-/D7-brane system has been reviewed in section 4.3. Two qualitatively different scenarios have been considered: first with the D7-brane wrapping a 4-cycle extending from the tip along the throat into the bulk, second with the wrapped 4-cycle localised at the tip. Moreover, in the first case, the D7-brane chiral supermultiplet may be localised in the bulk or at the tip, depending on its mass-sourcing fluxes. The possibility of internal integrals being dominated by the warped throat or the bulk has also been considered. For the 33- and 77-states, the effective field theory for the light degrees of freedom can be found by simply matching the 4-dimensional interactions found via dimensional reduction with those obtained in linear supergravity (including soft-breaking terms in the presence of supersymmetry-breaking fluxes). For the 37- and 73-states, further tools are necessary, and in particular the allowed interactions can be inferred using the internal-space symmetries [235]. The power of linear supergravity is that, having identified the Kähler potential, superpotential, gauge kinetic functions and Fayet-Iliopoulos terms by matching with a few dimensionally-reduced interactions, the complete action necessary for supersymmetry can be inferred, including couplings to bulk moduli.

With these preparations, the low-energy description of anti-D3-/D7-branes at the bottom of warped throats in supersymmetric warped flux compactifications has been worked out, first for Abelian setups in section 4.4 and then for non-Abelian stacks of branes on orbifold singularities in section 4.5. Despite supersymmetry breaking, the non-linear supergravity construction provides a useful framework for the low-energy theory, including the couplings with bulk fields. After identifying the appropriate constrained superfields to encapsulate the low-energy fields, their interactions have been worked out, building on both the single anti-D3-brane case [163] and the supersymmetric D3-/D7-brane cases above. Most of the interactions can be described within standard supergravity expansions with hidden-sector supersymmetry breaking and soft-breaking terms. However, in the presence of constrained superfields where the constraint also fixes the auxiliary field in the multiplet in terms of the goldstino, the supergravity expansions are non-standard, and are computed in appendix C.3. Another consequence of the anti-D3-brane supersymmetry breaking is in a few couplings involving intersecting states, which would follow from analogy with the supersymmetric D3-/D7-brane case, but do not appear to fit in to the non-linear supergravity expansions. These can instead be realised via a new interaction proportional to the nilpotent goldstino superfield, i.e. an $x\overline{x}$-term [157], which provides each coupling term by term, plus further interactions proportional to the goldstino and vanishing in the unitary gauge. Although this somewhat weakens the power of the supergravity formulation, at least in the current understanding, the latter allows for an embedding of bottom-up open-string scenarios with brane supersymmetry breaking into fully stabilised compactifications, including perturbative and non-perturbative effects. This is essential to understand their phenomenology and cosmology.

To this end, the D-brane setups were embedded in the KKLT-scenario, with the anti-D3-branes providing both gauge and matter sectors as well as the anti-D3-brane uplift to a Minkowski/de Sitter vacuum energy. Attractively, the small bulk $(0,3)$-flux backgrounds, necessary to balance against non-perturbative effects and stabilise the Kähler modulus, also help satisfy condition (4.2.16) allowing for a supergravity description [228]. The technology developed can easily be applied to other moduli-stabilisation scenarios, and less warped scenarios, outside the validity of eqs. (4.2.13, 4.2.16).

The low-energy effective actions thus found have several interesting features. The complex-structure, axio-dilaton, 77-, and $\overline{3}\overline{3}$-sector chiral multiplets acquire would-be supersymmetric mass terms from $(2,1)$-fluxes, at a scale that can be above the cut-off (as well as subleading soft-breaking masses from the anti-D3-brane supersymmetry breaking). Physically, this means that the open-string moduli corresponding to brane positions are stabilised at the tip of the throat. Instead, fermionic $\overline{3}7$- and $7\overline{3}$-states remain massless and could provide the standard-like model light visible sector, whilst scalar visible-sector exotic $\overline{3}7$- and $7\overline{3}$-states – in distinct bifundamental representations – always receive would-be soft-breaking mass contributions, due to the anti-D3-brane and volume-modulus supersymmetry breaking. Because the latter are suppressed by no-scale-like cancellations, α'-corrections (positive-definite) and anomaly mediation (tachyonic) actually compete in setting the scale of the exotic scalar masses [98], and which contribution wins depends on the parameter choices. Moduli stabilisation and anomaly mediation also provide mass terms for the 77- and $\overline{3}\overline{3}$-sector gaugini. As well as the mass scales, the leading supersymmetric and soft-breaking bilinear and trilinear couplings have all been computed. The visible $\overline{3}\overline{3}$-sector gauge couplings are fixed by the string coupling, while for the hidden 77-sector the volume can also play a role. All this is spelled out in section 4.6.

As well as the lighter part of the visible sector (standard-like model gauge fields and fermions, and some scalar exotics and gaugini), and a light hidden gauge sector plus matter, when embedding in KKLT-like scenarios for the Kähler modulus stabilisation, the volume modulus and gravitino remain in the effective field theory, whereby cosmological bounds on the gravitino constrain the parameter space. Notice that the KKLT-like small GVW-parameter $|W_0|$ implies a small gravitino mass, which is then further reduced by warping. Although the precise mass scales are model-dependent, the pattern of masses and their parametric dependence on the warp factor, volume and string-coupling are fairly universal within the KKLT-scenario. Whilst a thorough phenomenological study, including renormalisation-group flows of the scales, has been beyond the reach of this work, if the warping is too strong, the gravitino mass $\hat{m}^w_{3/2} \sim (g_s^{3/2} e^{A_0}/\mathcal{V}^{2/3})\, m_P\, e^{A_0}$ may be so suppressed as to be ruled out by the observational bounds that confirm the BBN-physics, while the exotic scalar masses $m_{\bar{3}7} \sim \hat{m}^w_{3/2}/\mathcal{V}^{1/2}$ may be ruled out by observation in accelerators. Conversely, weaker warping allows scales to be pushed far beyond current experimental bounds.

This work leads to several interesting and important open questions. First and foremost is a rigorous understanding of the extent to which non-linearly realised supersymmetry and strong warping can help resolve hierarchy problems like the gauge hierarchy. The presence of spontaneous supersymmetry breaking, and yet no scale at which the usual superpartners appear, is an intriguing feature of these scenarios. Recently there has been a great deal of interest towards non-supersymmetric constructions in string theory (see e.g. refs. [90, 91, 99–103, 115, 288–293]) and it is very compelling to understand the relation between the D-brane supersymmetry breaking considered here and other approaches in the literature. The work of refs. [2, 3] is also aligned in this direction. It is also worth pointing out that recently ref. [294] has argued for the fact that, in pure 4-dimensional supergravity terms, constructions based on constrained superfields can be stable against loop corrections coming from the integrated-out heavy non-supersymmetric particles.

From a model-building point of view, it would be essential to build warped throats that allow for viable singularities at their tip, and the presence of simple 4-cycles (like for instance the K3-surface or the 4-torus T^4) at their tip or along their length would then allow for easy explicit dimensional reductions. Geometric constructions with warped throats hosting a 4-torus T^4 at the tip and $\mathbb{Z}_3$-singularities are built in ref. [30]. It would be fruitful to extend the present work to anti-D3/D7-brane systems on more general toric singularities, such as in refs. [223, 295–298], at the tip of warped throats. Related work on the construction of throats with branes at singularities can already be found e.g. in refs. [95, 97, 223, 297–301] and on throats with wrapped D7-branes in refs. [302–304]. Ultimately, the results presented here could be applied for globally consistent compactifications, with appropriate singularities, cycles and sources that fulfil RR-tadpole cancellation conditions, in a standard bottom-up approach.

Various possible instabilities arising from anti-D3-branes in flux backgrounds should also be explored, since this work has completely neglected the brane backreaction and the details of the complex-structure modulus that governs the warp factor at the throat tip. In particular, as shown by ref. [168], p anti-D3-branes in the flux background of the KS-throat with M units of RR-flux are metastable and long-lived for sufficiently small ratio p/M, with brane-flux decay occurring non-perturbatively via brane polarisation à la Myers [72] (for an overview of past debates on this picture, see ref. [205]). Recently, ref. [212] has provided an effective interpretation of this KPV-like decay in 4-dimensional supergravity in terms of the evaporation of the nilpotent goldstino superfield (see also refs. [154, 160]). Also, ref. [67] has shown that for a KS-throat the anti-D3-branes may induce a complex-structure instability, depending on the amount of flux relative to the branes. It would be interesting to investigate these dynamics in other relevant throats and in the presence of orbifold singularities. Additionally, so far, world-volume fluxes on the D7-branes have been neglected for simplicity, though they can contribute interesting D-terms and F-terms.

Once globally-consistent realistic constructions approaching the standard model of particle physics have been identified, detailed phenomenological and cosmological studies would be possible.

5 CONCLUSIONS

This work has been centred around aspects of the breaking of supersymmetry by anti-D-branes in String Theory. This has been done in two different perspectives. On the one hand, the nature of anti-D-brane supersymmetry breaking has been considered in its fundamental aspects, evidencing the emergence of patterns that are characteristic of misaligned supersymmetry. On the other hand, the study of a purely phenomenological scenario has been discussed, with the description of a standard-like model realised on intersecting anti-D3-/D7-branes at orbifold-like singularities in the formalism of constrained multiplets.

The work has started, in chapter 2, with a detailed contextualisation of the basic conceptual tools that constitute the foundational common ground for the material presented later on. Starting from the quantisation of the worldsheet superstring theory, it has overviewed core topics of String Theory and String Phenomenology such as the role and the interpretation of partition functions, string compactifications and the conceptual nature of D-brane supersymmetry breaking.

The key concept of the formal analysis of anti-D-brane supersymmetry breaking has been misaligned supersymmetry, in chapter 3. In fact, it has been shown that the open-string theory corresponding to an anti-D-brane sitting on top of an orientifold plane exhibits the defining features of misaligned supersymmetry, i.e. an oscillating degeneracy between fermionic and bosonic abundances at each mass level and yet a finite one-loop cosmological constant. The oscillating and exponentially-growing net degeneracies are defining features of other closed-string non-supersymmetric models. Focussing on the paradigmatic examples of the heterotic $\mathrm{SO}(16)\times\mathrm{SO}(16)$-theory and of the anti-Dp-brane/Op-plane theory, it has been shown that, in a vast class of cases in which the partition function can be written in terms of a family of Dedekind η-quotients, these theories experience a net cancellation in the interpolating functions of the state degeneracies. This has been proven to take place at all orders in the Hardy-Ramanujan-Rademacher series that describes them, extending pre-existing results in the literature, which only concern the leading-order terms in closed-string theories. Furthermore, a mathematical analysis of the one-loop cosmological constant has been carried out. For both open and closed strings, this evidences that the cancellation of the interpolating functions is sufficient to conclude that the exponential divergences appearing in the calculation do indeed cancel out, at all orders. The results are valid for a generic class of non-supersymmetric string-theory constructions, and the Sugimoto USp(32)- and type 0'B SU(32)-theories have also been argued to show misaligned elements in their spectra. Further generalisations are possible, both in physical and mathematical terms. An achievement to pursue is the description of more realistic constructions, in compactified theories and with non-Abelian gauge sectors, in terms of misaligned supersymmetry, in order to understand more about non-supersymmetric theories and their fate in hierarchy naturnalness problems. On the other hand, the proofs of misaligned-like boson-fermion cancellations can be extended by trying to relax the assumptions on the form of the partition functions.

As a parallel line of investigation, an inherently phenomenological analysis has been performed of quasi-realistic standard-like model realisations revolving around anti-D3-branes, in chapter 4. It is known that anti-D3-branes in type IIB theories come with the effect of breaking the supersymmetry that is preserved by Calabi-Yau orientifolds and provide a positive contribution to the vacuum energy. At the same time, when they sit at orbifold singularities, intersecting with D7-branes as well, the corresponding low-energy chiral spectrum can accommodate for non-Abelian gauge theories with scalars and spinors in different representations. This has motivated the interest in the analysis of a model in which the anti-D3-branes do the double job of breaking supersymmetry, uplifting the vacuum energy, and of providing the particle content of a quasi-realistic standard model extension, in their interplay with the D7-branes. The technical tool that has been used for this task consists in constrained superfields. These allow one to make use of many of the well-known supergravity results while working in a model where supersymmetry is intrinsically broken by the theory. For definiteness, the work has focussed on highly-warped scenarios, working out the relevant mass scales and arguing for their reliability as noteworthy toy models, with the perspective of more refined future constructions.

These analyses provide an enrichment of the current understanding of supersymmetry breaking in String Theory. Since supersymmetry, if present at all, must be broken in the vacuum, and, at the same time, since its conceptual and mathematical features provide one with sharp computational tools, this is a topic of great relevance. In line with notable results in the literature, this work has provided further evidence for the idea that the mathematical and conceptual tools inherent to standard supersymmetric theories can be adapted, to a specific extent, in order to also describe string theories that lack spacetime supersymmetry. Such a conclusion is expected to help addressing fundamental puzzles of Theoretical Physics such as the hierarchy problem and the potential description of a consistent string-theoretic de Sitter vacuum. A compelling open question whose answer this work lays some grounds for is the relation between non-linear realisations of supersymmetry, intrinsically non-supersymmetric theories, and misaligned supersymmetry. In fact, for instance, anti-D-branes can both realise the supersymmetry of the closed-string sector non-linearly and exhibit a misaligned spectrum. On the other hand, the heterotic $\mathrm{SO}(16) \times \mathrm{SO}(16)$-theory has a misaligned spectrum, but such a spectrum does not even have a gravitino at the massless level, differently from the anti-D-brane case. This leads one to wonder what this means, at a fundamental level, for non-supersymmetric string theories. In fact, misaligned supersymmetry seems to be a common feature of most theories in this class. A feature that can often be overlooked consists in the fact that String Theory, having an infinite number of degrees of freedom, has a deeper structure than just the effective low-energy theory that it gives rise to. Therefore even mechanisms such as supersymmetry breaking may take place in an intrinsically stringy way, different from any possible pure field-theoretic realisation. A refinement of the description of both the cases of non-linear supersymmetry and of intrinsically non-supersymmetric string-theoretic realisations represents a fundamental advancement in the study of String Theory. Knowledge of non-supersymmetric string theories in a unified picture is of crucial importance in order to understand their effective role in model-building and solving the hierarchy naturalness issues of present-day Theoretical Physics. Any further understanding in this direction is worthwhile to be pursued.

This work has focussed on idealistic setups, leaving aside a variety of complications that at some point have to emerge if looking for a theory of the universe based on String Theory. In particular, the fundamental aspects of non-supersymmetric string theories in relation to misaligned supersymmetry have been discussed prior to compactification and without aiming for realistic particle spectra. The phenomenological modelling of quasi-realistic theories based on anti-D3-branes has been performed in highly simplified setups, in a bottom-up approach and ignoring features such as the brane backreaction and the details of globally well-defined compactifications, thus avoiding certain technical complications. Similarly, it has also been assumed that to keep control of all the approximations is possible. All these restrictions have been dictated by the necessity to select the essential features of the matter in order to make the initial step towards enriching its complete understanding, and not by conceptual obstacles. Of course, this means that developments in this direction are not only desirable but actually doable.

A NOTATION AND CONVENTIONS

This appendix is a collection of the recurrent symbols and functions that appear throughout the main text, with relevant references also provided.

A.1 Modular Forms

A modular form of weight k is a holomorphic function $f = f(\tau)$ defined on the complex upper-half plane $\mathbb{H} = \{\tau \in \mathbb{C} : \operatorname{Im} z > 0\}$ that under the action of the modular group transforms as

$$f(M\tau) = (c\tau + d)^k f(\tau), \tag{A.1.1}$$

and that is also holomorphic at the cusp. The modular group is the group $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\mathbb{Z}_2$ that can be identified with the elements of the matrices

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \overset{\mathbb{Z}_2}{\approx} -M, \qquad a, b, c, d \in \mathbb{Z} : \ \det M = ad - bc = 1.$$

The modular group appears often because it maps a 2-dimensional torus into another torus, when $\tau = \tau_1 + \mathrm{i}\tau_2$ represents the Teichmüller parameter of the torus. The fundamental domain of the modular group is the region in the upper-half complex plane that represents parameters τ that cannot be mapped into one another by a modular transformation and it corresponds to

$$\mathbb{F} = \big\{\tau \in \mathbb{C} : \ \operatorname{Re}\tau \in [-1/2, 1/2] \wedge \operatorname{Im}\tau \in [0, +\infty[\wedge |\tau| \in [1, +\infty[\big\}.$$

Any transformation in the modular group can be generated by a repeated application of the generating T- and S-transformations, which act on the modular parameter as $T(\tau) = \tau + 1$ and $S(\tau) = -1/\tau$. By a slight abuse of notation, sometimes the group $\mathrm{SL}_2(\mathbb{Z})$ itself is called the modular group, ignoring the equivalence $M \sim -M$. Ubiquitous modular forms in string theory are the Dedekind η-function and the Jacobi ϑ-functions.

A.1.1 Dedekind η- and Jacobi ϑ-functions

For the Dedekind η-function and the Jacobi ϑ-functions, the notation follows the definitions of refs. [33, 39]; see also ref. [305]. Here the main relationships employed throughout the main text are reported explicitly for clarity. In terms of the variable $q = \mathrm{e}^{2\pi \mathrm{i}\tau}$, the Dedekind

η-function is defined as

$$\eta(\tau) = q^{\frac{1}{24}} \sum_{n=-\infty}^{\infty} (-1)^n q^{n\frac{(3n-1)}{2}} = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n), \tag{A.1.2}$$

whereas the Jacobi ϑ-functions are defined as

$$\begin{aligned} \vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z|\tau) &= \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}(n+a)^2} \mathrm{e}^{2\pi \mathrm{i}(n+a)(z+b)} \\ &= \mathrm{e}^{2\pi \mathrm{i} a(z+b)} q^{\frac{a^2}{2}} \prod_{n=1}^{\infty} (1-q^n) \Big[1 + q^{n+a-\frac{1}{2}} \mathrm{e}^{2\pi \mathrm{i}(z+b)}\Big] \Big[1 + q^{n-a-\frac{1}{2}} \mathrm{e}^{-2\pi \mathrm{i}(z+b)}\Big]. \end{aligned} \tag{A.1.3}$$

Four specific functions are particularly common, i.e.

$$\vartheta_1(q) \equiv \vartheta\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(0|\tau) = \sum_{n=-\infty}^{\infty} (-1)^{n+\frac{1}{2}} q^{\frac{1}{2}(n+\frac{1}{2})^2} = \mathrm{i} q^{\frac{1}{8}} \prod_{n=1}^{\infty} (1-q^n)^2 (1-q^{n-1}), \tag{A.1.4a}$$

$$\vartheta_2(q) \equiv \vartheta\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}(0|\tau) = \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}(n+\frac{1}{2})^2} = 2q^{\frac{1}{8}} \prod_{n=1}^{\infty} \big(1-q^n\big)\big(1+q^n\big)^2, \tag{A.1.4b}$$

$$\vartheta_3(q) \equiv \vartheta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(0|\tau) = \sum_{n=-\infty}^{\infty} q^{\frac{1}{2}n^2} = \prod_{n=1}^{\infty} \big(1-q^n\big)\big(1+q^{n-\frac{1}{2}}\big)^2, \tag{A.1.4c}$$

$$\vartheta_4(q) \equiv \vartheta\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}(0|\tau) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{1}{2}n^2} = \prod_{n=1}^{\infty} \big(1-q^n\big)\big(1-q^{n-\frac{1}{2}}\big)^2. \tag{A.1.4d}$$

Useful relations related to these functions are

$$\vartheta_1 = 0, \tag{A.1.5}$$

$$\vartheta_3^4 - \vartheta_4^4 - \vartheta_2^4 = 0, \tag{A.1.6}$$

$$\vartheta_2 \vartheta_3 \vartheta_4 = 2\eta^3. \tag{A.1.7}$$

The second and third equalities are the Jacobi identity and the Jacobi triple-product identity. Under the action of the generating transformations T and S of the modular group, the Dedekind η-function transforms as

$$\mathrm{T}: \qquad \eta(\tau+1) = \mathrm{e}^{\frac{\mathrm{i}\pi}{12}} \eta(\tau), \tag{A.1.8}$$

$$\mathrm{S}: \qquad \eta(-1/\tau) = \sqrt{-\mathrm{i}\tau}\, \eta(\tau), \tag{A.1.9}$$

whereas the general form for the transformation of the Jacobi ϑ-functions is

$$\mathrm{T}: \qquad \vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z|\tau+1) = \mathrm{e}^{-\mathrm{i}\pi a(a-1)}\, \vartheta\begin{bmatrix} a \\ a+b-\frac{1}{2} \end{bmatrix}(z|\tau) \tag{A.1.10}$$

$$\mathrm{S}: \qquad \vartheta\begin{bmatrix} a \\ b \end{bmatrix}(z|-1/\tau) = \sqrt{-\mathrm{i}\tau}\, \mathrm{e}^{2\pi \mathrm{i} ab + \mathrm{i}\pi\frac{z^2}{\tau}}\, \vartheta\begin{bmatrix} b \\ -a \end{bmatrix}(z|\tau). \tag{A.1.11}$$

For the functions of interest, one can write

$$\vartheta_2(\tau+1) = \mathrm{e}^{\frac{\mathrm{i}\pi}{4}} \vartheta_2(\tau), \tag{A.1.12a}$$

$$\text{T}: \qquad \vartheta_3(\tau+1) = \vartheta_4(\tau), \tag{A.1.12b}$$

$$\vartheta_4(\tau+1) = \vartheta_3(\tau), \tag{A.1.12c}$$

$$\vartheta_2(-1/\tau) = \sqrt{-\mathrm{i}\tau}\,\vartheta_4(\tau), \tag{A.1.13a}$$

$$\text{S}: \qquad \vartheta_3(-1/\tau) = \sqrt{-\mathrm{i}\tau}\,\vartheta_3(\tau), \tag{A.1.13b}$$

$$\vartheta_4(-1/\tau) = \sqrt{-\mathrm{i}\tau}\,\vartheta_2(\tau). \tag{A.1.13c}$$

By use of the infinite-product definitions, it is possible to express the Jacobi ϑ-functions in terms of the Dedekind η-function and vice versa via the identities

$$\vartheta_2(\tau) = \frac{2\eta^2(2\tau)}{\eta(\tau)}, \tag{A.1.14a}$$

$$\vartheta_3(\tau) = \frac{\eta^5(\tau)}{\eta^2(\tau/2)\eta^2(2\tau)}, \tag{A.1.14b}$$

$$\vartheta_4(\tau) = \frac{\eta^2(\tau/2)}{\eta(\tau)}. \tag{A.1.14c}$$

An application of the modular T-transformation also shows the relationship

$$\eta(\tau+1/2) = \mathrm{e}^{\frac{\mathrm{i}\pi}{24}}\,\frac{\eta^3(2\tau)}{\eta(\tau)\eta(4\tau)}. \tag{A.1.15}$$

For practical purposes, it is also useful to introduce the characters of the so($2n$)-algebras, which are defined as

$$O_{2n} = \frac{\vartheta_3^n + \vartheta_4^n}{2\eta^n}, \tag{A.1.16a}$$

$$V_{2n} = \frac{\vartheta_3^n - \vartheta_4^n}{2\eta^n}, \tag{A.1.16b}$$

$$S_{2n} = \frac{\vartheta_2^n + \mathrm{i}^{-n}\vartheta_1^n}{2\eta^n}, \tag{A.1.16c}$$

$$C_{2n} = \frac{\vartheta_2^n - \mathrm{i}^{-n}\vartheta_1^n}{2\eta^n}. \tag{A.1.16d}$$

Their modular properties can be determined straightforwardly from their definitions.

A.1.2 Asymptotic Expansion of the Dedekind η-function

It is instructive to discuss in some detail the derivation of the asymptotic expansion of the Dedekind η-function. There are two ways for doing this: one relies on the modular properties of the function, whilst another is just a result of mathematical analysis.

Based on the definitions in ref. [131], the notation and the terminology is as follows.

- The expression $f(x) = O(g(x); x_0)$ means that there exists a value $M \in \mathbb{R}^+$ such that $|f(x)| \leq M g(x)$ for any x in a sufficiently small neighbourhood I_{x_0}. The expression $f(x) = o(g(x); x_0)$ means that $\lim_{x\to x_0}(f(x)/g(x)) = 0$.

- The expression $f(x) \overset{x\sim x_0}{\simeq} g(x)$ means that $\lim_{x\to x_0} f(x)/g(x) = 1$. The expression $f(x) \overset{x\sim 0}{\sim} \sum_n f_n x^n$ means that $f(x) - \sum_{n=0}^{m} f_n x^n = o(x^m; 0)$ for any natural number $m \in \mathbb{N}$.
- The function $f : \mathbb{R}^+ \to \mathbb{R}$ is a function of rapid decay at the point x_0 if $f(x) \overset{x\to x_0}{\to} 0$ faster than any power $(x - x_0)^m$, i.e. $f(x) = o((x - x_0)^m; x_0)$ for any natural number $m \in \mathbb{N}$.

The two derivations of the asymptotic behaviour of the function $\eta(\mathrm{i}\tau_2)$ are discussed below. They are reviews of the discussion in ref. [131].

1. One can make use of the behaviour of the Dedekind η-function under the modular group $\mathrm{PSL}_2(\mathbb{Z})$. Under the generating S-transformation $S(\tau) = -1/\tau$, the Dedekind η-function restricted to an imaginary argument $\tau = \mathrm{i}\tau_2$ transforms as

$$\eta\Big(\frac{\mathrm{i}}{\tau_2}\Big) = \sqrt{\tau_2}\,\eta(\mathrm{i}\tau_2). \tag{A.1.17}$$

At the same time, for a pure imaginary argument the Dedekind η-function can be written as $\eta(\mathrm{i}t) = \mathrm{e}^{-\frac{\pi t}{12}} \prod_{n=1}^{\infty}(1 - \mathrm{e}^{-2\pi n t})$, so one finds that

$$\ln \eta(\mathrm{i}t) = -\frac{\pi t}{12} + \sum_{n=1}^{\infty} \ln\left(1 - \mathrm{e}^{-2\pi n t}\right) = -\frac{\pi t}{12} + O(\mathrm{e}^{-2\pi t}; +\infty). \tag{A.1.18}$$

The magnitude of the subleading terms stems from the Taylor-Maclaurin expansion $\ln(1 + x) = O(x; 0)$: one finds $\ln(1 - \mathrm{e}^{-2\pi t}) = O(\mathrm{e}^{-2\pi t}; +\infty)$. So, combining the S-transformation relation of eq. (A.1.17) and the expansion of eq. (A.1.18) for $t = 1/\tau_2$, one finds

$$\ln \eta(\mathrm{i}\tau_2) = -\frac{1}{2}\ln \tau_2 - \frac{\pi}{12\tau_2} + O(\mathrm{e}^{-2\pi/\tau_2}; 0). \tag{A.1.19}$$

This represents a proof for the τ_2-expansion of the Dedekind η-function with imaginary argument as $\eta(\mathrm{i}\tau_2) \overset{\tau_2\sim 0^+}{\simeq} \tau_2^{-1/2}\mathrm{e}^{-\pi/12\tau_2}$, and it also quantifies the magnitude of the subleading terms. Such an asymptotic behaviour is a direct consequence of the modular properties of the Dedekind η-function.

2. One can make use of tools from mathematical analysis. One can prove the following theorem (details of the proof can be found in ref. [131]).

Theorem. Let the function $g = g(x)$ be defined as

$$g(x) = \sum_{m=1}^{\infty} f(mx), \tag{A.1.20}$$

where f is a smooth function on the positive real line with the following properties:

- at the origin, f has the asymptotic development

$$f(x) \overset{x\sim 0}{\sim} b \ln\frac{1}{x} + \sum_{n=0}^{\infty} f_n x^n; \tag{A.1.21}$$

- at infinity, f and all of its derivatives are of rapid decay.

Further, let the definite integral of f be

$$I_f = \int_0^\infty \mathrm{d}x\, f(x). \tag{A.1.22}$$

Then, the function $g = g(x)$ at the origin has the asymptotic development

$$g(x) \overset{x\sim 0}{\sim} \frac{I_f}{x} - \frac{b}{2}\ln\frac{2\pi}{x} + \sum_{n=0}^{\infty}(-1)^n \frac{f_n B_{n+1}}{n+1}x^n. \tag{A.1.23}$$

This theorem is enough to determine the asymptotic behaviour of the Dedekind η-function. Let the function f be

$$f(x) = \ln\left(1 - \mathrm{e}^{-x}\right).$$

This has the asymptotic expansion[A.1] and the definite integral

$$f(x) \overset{x\sim 0}{\sim} \ln x + \sum_{n=1}^{\infty} \frac{B_n}{n\cdot n!}x^n,$$
$$I_f = -\frac{\pi^2}{6}.$$

So, one can apply the theorem with $b = -1$, $f_0 = 0$, $f_n = \frac{B_n}{n\cdot n!}$ for $n \geq 1$, and write

$$g(x) = \sum_{m=1}^{\infty} f(mx) \overset{x\sim 0}{\sim} -\frac{\pi^2}{6x} + \frac{1}{2}\ln\frac{2\pi}{x} + \sum_{n=1}^{\infty}(-1)^n \frac{B_n B_{n+1}}{n\cdot(n+1)!}x^n = -\frac{\pi^2}{6x} - \frac{1}{2}\ln\frac{x}{2\pi} + \frac{x}{24}.$$

In particular, there are no powers beyond x^1 since all the even Bernoulli numbers vanish beyond $B_2 = 1/6$, with moreover $B_0 = 1$ and $B_1 = -1/2$. This can be used to write

$$\ln\eta(\mathrm{i}\tau_2) = -\frac{\pi\tau_2}{12} + \sum_{m=1}^{\infty}\ln\left(1 - \mathrm{e}^{-2\pi m\tau_2}\right) \overset{\tau_2\sim 0}{\sim} -\frac{\pi}{12\tau_2} - \frac{1}{2}\ln\tau_2,$$

in agreement with the expansion of eq. (A.1.19).

[A.1] To see this, one can expand the derivative as

$$\frac{\mathrm{d}f}{\mathrm{d}x}(x) = \frac{1}{x}\frac{x}{\mathrm{e}^x - 1} = \frac{1}{x} + \sum_{n=0}^{\infty}\frac{B_{n+1}}{(n+1)!}x^n$$

and integrate it to

$$f(x;c) = \ln x + \sum_{n=0}^{\infty}\frac{B_{n+1}}{(n+1)(n+1)!}x^{n+1} + c.$$

By requiring that $f(1) = f(1;c)$, for instance, one finds $c = 0$.

Jacobi θ_3-function

Since the elliptic function $\vartheta_3(z,\tau)$ has an important role in the discussion of anti-D-branes, below are recored further useful properties. The Jacobi elliptic function $\vartheta_3 = \vartheta_3(z,\tau)$ can be defined as the infinite sum

$$\vartheta_3(z,\tau) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\pi n^2 \tau} \mathrm{e}^{2n\mathrm{i}z} \tag{A.1.24}$$

and it is a solution of the heat equation

$$\frac{1}{4}\mathrm{i}\pi \frac{\partial^2 \vartheta_3}{\partial z^2}(z,\tau) + \frac{\partial \vartheta_3}{\partial \tau}(z,\tau) = 0. \tag{A.1.25}$$

In this work, many calculations involve the Jacobi ϑ_3-constant, defined as

$$\vartheta_3(\tau) \equiv \vartheta_3(z=0,\tau). \tag{A.1.26}$$

It is often needed to restrict the attention to the case in which the argument is purely imaginary, namely $\tau = \mathrm{i}t$, with $t > 0$. In this case, $\vartheta_3(\mathrm{i}t)$ satisfies the functional equation

$$\vartheta_3(\mathrm{i}t^{-1}) = t^{\frac{1}{2}} \vartheta_3(\mathrm{i}t), \tag{A.1.27}$$

which can be interpreted as a modular S-transformation. It is also possible to show the asymptotic behaviours

$$\vartheta_3(\mathrm{i}t) \overset{t \sim 0^+}{\simeq} t^{-\frac{1}{2}}, \tag{A.1.28}$$

$$\vartheta_3(\mathrm{i}t) \overset{t \sim \infty}{\simeq} 1. \tag{A.1.29}$$

These can be seen to follow from the relationship of the Dedekind η-function with the Jacobi θ_3-function, in eq. (A.1.14b), in view of the expansions of eqs. (A.1.18, A.1.19).

A.2 Useful Special Functions

This appendix reviews basic properties of the special functions appearing in the discussion of misaligned supersymmetry, i.e. Bessel functions and Bernoulli and Euler polynomials.

A.2.1 Bessel Functions

The notation and the conventions for the Bessel functions are mutuated from ref. [306]. Here a short summary is reported in order to facilitate the reading.

The Bessel function of the first kind $J_\nu = J_\nu(z)$ is defined as a solution, along with the Bessel functions of the second and third kind, of the differential equation

$$z^2 \frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + z\frac{\mathrm{d}w}{\mathrm{d}z} + (z^2 - \nu^2) = 0.$$

The function $J_\nu(z)$ can be Taylor-expanded as

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{z^2}{4}\right)^k}{k!\,\Gamma(k+\nu+1)}, \tag{A.2.1}$$

whereas its asymptotic expansion as $|z| \sim \infty$, if $|\arg(z)| < \pi$, reads

$$J_\nu(z) \overset{|z|\sim\infty}{\simeq} \left[\frac{2}{\pi z}\right]^{\frac{1}{2}} \left[\sum_{k=0}^{\infty} (-1)^k \frac{(\nu, 2k)}{(2z)^{2k}} \cos\theta_\nu(z) - \sum_{k=0}^{\infty} (-1)^k \frac{(\nu, 2k+1)}{(2z)^{2k+1}} \sin\theta_\nu(z)\right], \tag{A.2.2}$$

where the angle has been defined $\theta_\nu(z) = z - \pi\nu/2 - \pi/4$ and the Hankel's symbol has been used $(m,n) = \Gamma(m+n+1/2)/[n!\,\Gamma(m-n+1/2)]$.

The modified Bessel function of the first kind $I_\nu = I_\nu(z)$ is defined as a solution, along with the modified Bessel function of the second kind, of the differential equation

$$z^2 \frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + z\frac{\mathrm{d}w}{\mathrm{d}z} - (z^2 + \nu^2) = 0.$$

The function $I_\nu(z)$ can be Taylor-expanded as

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{\left(\frac{z^2}{4}\right)^k}{k!\,\Gamma(k+\nu+1)}, \tag{A.2.3}$$

whereas its asymptotic expansion as $|z| \sim \infty$, if $|\arg(z)| < \pi/2$, reads

$$I_\nu(z) \overset{|z|\sim\infty}{\simeq} \frac{\mathrm{e}^z}{(2\pi z)^{\frac{1}{2}}} \left[\sum_{k=0}^{\infty} \frac{(\nu, 2k)}{(2z)^{2k}} - \sum_{k=0}^{\infty} \frac{(\nu, 2k+1)}{(2z)^{2k+1}}\right], \tag{A.2.4}$$

where the definition has been made $\mu_\nu = 4\nu^2$.

The Bessel functions $J_\nu(z)$ and $I_\nu(z)$ are related to each other, if $-\pi < \arg(z) \leq \pi/2$, by the transformation

$$I_\nu(z) = \mathrm{e}^{-\mathrm{i}\pi\nu/2} J_\nu(\mathrm{i}z). \tag{A.2.5}$$

If $\pi/2 < \arg(z) \leq \pi$, this is replaced by $I_\nu(z) = \mathrm{e}^{3\mathrm{i}\pi\nu/2} J_\nu(\mathrm{i}z)$.

A.2.2 Bernoulli and Euler Polynomials

It is useful to collect the relevant expressions used in the main text about Bernoulli and Euler polynomials. The main guidance is ref. [306].

For a given real number $x \in \mathbb{R}$, Bernoulli and Euler polynomials $B_n = B_n(x)$ and $E_n = E_n(x)$, respectively, are defined as the coefficients appearing in the Taylor expansions

$$\frac{t\,\mathrm{e}^{xt}}{\mathrm{e}^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \tag{A.2.6}$$

$$\frac{2\,\mathrm{e}^{xt}}{\mathrm{e}^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \tag{A.2.7}$$

For the variable $1-x$, one finds

$$B_n(1-x) = (-1)^n B_n(x), \tag{A.2.8}$$

$$E_n(1-x) = (-1)^n E_n(x). \tag{A.2.9}$$

A simple equation relates them to each other for $n>0$, i.e.

$$E_{n-1}(x) = \frac{2^n}{n}\left[B_n\left(\frac{x+1}{2}\right) - B_n\left(\frac{x}{2}\right)\right]. \tag{A.2.10}$$

Bernoulli numbers are defined as $B_n = B_n(0)$, whilst Euler numbers are $E_n = 2^n E_n(1/2)$, for all $n \in \mathbb{N}$

A.3 Differential Forms and Cohomology

This section provides a brief account of the relevant operations on and between p-forms of a real smooth manifold. The main references are ref. [307] for the theoretical background and refs. [50, 308] for the more advanced formulae.

A.3.1 Basic Operations

A rank-(p,q) tensor T on a real smooth manifold of dimension n in the coordinate basis is defined as the expression

$$T = T^{\mu_1\dots\mu_p}{}_{\nu_1\dots\nu_q}\, \partial_{\mu_1} \otimes \cdots \otimes \partial_{\mu_p} \otimes \mathrm{d}x^{\nu^1} \otimes \cdots \otimes \mathrm{d}x^{\nu_q}.$$

A differential p-form A_p is a completely antisymmetric rank-$(0,p)$ tensor defined as

$$A_p = \frac{1}{p!}\, A_{\mu_1\dots\mu_p}\, \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p},$$

where the basis elements are $\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p} = \mathrm{d}x^{[\mu_1} \otimes \cdots \otimes \mathrm{d}x^{\mu_p]}$, the antisymmetrisation being defined including the division by $p!$.

Within this general framework, some standard and very common operations can easily be defined on spacetimes of arbitrary curvature.

- Given a p-form A_p and a q-form B_q, the wedge product $A_p \wedge B_q$ of A_p and B_q is the $(p+q)$-form

  $$A_p \wedge B_q = \frac{1}{p!\,q!}\, A_{\mu_1\dots\mu_p} B_{\mu_{p+1}\dots\mu_{p+q}}\, \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p} \wedge \mathrm{d}x^{\nu_1} \wedge \cdots \wedge \mathrm{d}x^{\nu_q},$$

  which means its components are defined as

  $$(A \wedge B)_{\mu_1\dots\mu_{p+q}} = \frac{(p+q)!}{p!\,q!}\, A_{[\mu_1\dots\mu_p} B_{\mu_{p+1}\dots\mu_{p+q}]}. \tag{A.3.1}$$

  The wedge product turns out to satisfy the condition

  $$A_p \wedge B_q = (-1)^{pq}\, B_q \wedge A_p.$$

- Given a p-form A_p, its exterior derivative $\mathrm{d}A_p$ is a $(p+1)$-form defined as

$$\mathrm{d}A_p = \frac{1}{p!}\,\partial_{\mu_1} A_{\mu_2\ldots\mu_{p+1}}\,\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_{p+1}},$$

i.e. whose components are

$$(\mathrm{d}A)_{\mu_1\ldots\mu_{p+1}} = (p+1)\,\partial_{[\mu_1} A_{\mu_2\ldots\mu_{p+1}]}. \tag{A.3.2}$$

Notably, even if the manifold is curved, the Riemann-tensor factors turn out to cancel out and thus we consider only partial derivatives. Moreover, for any p-form A_p one has

$$\mathrm{d}(\mathrm{d}A_p) = 0$$

and, given a q-form B_q, the property holds

$$\mathrm{d}(A_p\wedge B_q) = \mathrm{d}A_p\wedge B_q + (-1)^p\,A_p\wedge\mathrm{d}B_q.$$

- The Hodge duality operator $*$ is a map of p-forms into $(n-p)$-forms which acts on the basis as

$$*(\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_p}) = \frac{1}{(n-p)!}\,(\det g)^{\frac{1}{2}}\,g^{\mu_1\nu_1}\ldots g^{\mu_p\nu_p}\,\varepsilon_{\nu_1\ldots\nu_p\nu_{p+1}\ldots\nu_n}\,\mathrm{d}x^{\nu_{p+1}}\wedge\cdots\wedge\mathrm{d}x^{\nu_n}.$$

where $\det g$ is the metric determinant and $\varepsilon_{\mu_1\ldots\mu_n}$ is the Levi-Civita symbol. Therefore, given a p-form A_p, its Hodge dual $*A_p$ is an $(n-p)$-form

$$*A_p = \frac{1}{p!(n-p)!}\,(\det g)^{\frac{1}{2}}\,g^{\mu_1\nu_1}\ldots g^{\mu_p\nu_p}\,\varepsilon_{\nu_1\ldots\nu_p\nu_{p+1}\ldots\nu_n}\,A_{\mu_1\ldots\mu_p}\,\mathrm{d}x^{\nu_{p+1}}\wedge\cdots\wedge\mathrm{d}x^{\nu_n}$$

whose components read[A.2]

$$(*A)_{\mu_1\ldots\mu_{n-p}} = \frac{1}{p!}\,(\det g)^{\frac{1}{2}}\,g^{\nu_1\rho_1}\ldots g^{\nu_p\rho_p}\,\varepsilon_{\rho_1\ldots\rho_p\mu_1\ldots\mu_{n-p}}\,A_{\nu_1\ldots\nu_p}. \tag{A.3.3}$$

A double application of the Hodge star operator gives

$$**A_p = (-1)^{s+p(n-p)}A_p,$$

where s is the number of negative eigenvalues in the metric.

Stokes' theorem states that, given a manifold M with boundary ∂M, the volume integral of an $(n-1)$-form is such that

$$\int_M \mathrm{d}A_{n-1} = \int_{\partial M} A_{n-1}.$$

[A.2] In the string theory literature, it is common to find the definition of Hodge dual

$$(*'A)_{\mu_1\ldots\mu_{n-p}} = \frac{1}{p!}\,(\det g)^{\frac{1}{2}}\,g^{\nu_1\rho_1}\ldots g^{\nu_p\rho_p}\,\varepsilon_{\mu_1\ldots\mu_{n-p}\rho_1\ldots\rho_p}\,A_{\nu_1\ldots\nu_p} = (-1)^{p(n-p)}\,(*A)_{\mu_1\ldots\mu_p}.$$

The Levi-Civita symbol $\varepsilon_{\mu_1\ldots\mu_n}$, normalised as $\varepsilon_{1\ldots n} = +1$, is a tensor density of weight $+1$ and the Levi-Civita tensor is defined as

$$\epsilon_{\mu_1\ldots\mu_p} = (\det g)^{\frac{1}{2}}\, \varepsilon_{\mu_1\ldots\mu_n}.$$

Similarly, one can define the symbol $\varepsilon^{\mu_1\cdots\mu_n} \equiv (-1)^s\, \varepsilon_{\mu_1\ldots\mu_n}$ and

$$\epsilon^{\mu_1\cdots\mu_p} = (\det g)^{-\frac{1}{2}}\, \varepsilon^{\mu_1\cdots\mu_n}.$$

Importantly, the identity turns out to hold true

$$\epsilon^{\mu_1\cdots\mu_p\rho_{p+1}\cdots\rho_n}\, \epsilon_{\nu_1\ldots\nu_p\rho_{p+1}\ldots\rho_n} = (-1)^s\, p!\, (n-p)!\, \delta^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_p},$$

where the definition has been made

$$\delta^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_p} = \delta^{[\mu_1}_{[\nu_1} \ldots \delta^{\mu_p]}_{\nu_p]}.$$

Remarkably, for any p-form A_p one has the identity

$$A_{\mu_1\ldots\mu_p} = \delta^{\nu_1\ldots\nu_p}_{\mu_1\ldots\mu_p} A_{\nu_1\ldots\nu_p}.$$

Also, it is possible to write the volume element vol_n as

$$\mathrm{vol}_n = \frac{1}{n!}\, (\det g)^{\frac{1}{2}}\, \varepsilon_{\mu_1\ldots\mu_n}\, \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_n} = \frac{1}{n!}\, \epsilon_{\mu_1\ldots\mu_n}\, \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_n}$$

as one notice that we may write $\mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^n = \varepsilon_{\mu_1\ldots\mu_n}\, \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_n}/n!$.

The Hodge operator $*$ allows one to define an inner product on the space of real forms.

- Given two p-forms A_p and B_p, it is possibe to define the inner product (A_p, B_p) as

  $$(A_p, B_p) = \int A_p \wedge *B_p,$$

  which is symmetric, i.e. $(A_p, B_p) = (B_p, A_p)$, and which can be written explicitly as

  $$(A_p, B_p) = \int A_p \wedge *B_p = \frac{1}{p!} \int \mathrm{d}^n x\, (\det g)^{\frac{1}{2}}\, A_{\mu_1\ldots\mu_p} B^{\mu_1\cdots\mu_p}. \tag{A.3.4}$$

There are two more operators defined on differential forms that are used quite often.

- The adjoint $\mathrm{d}^\dagger$ of the exterior derivative operator is defined as the operator such that, given a p-form A_p and a $(p-1)$-form B_{p-1}, satisfies the condition $(A_p, \mathrm{d}B_{p-1}) = (\mathrm{d}^\dagger A_p, B_{p-1})$. Given a p-form A_p, one can show that

  $$\mathrm{d}^\dagger = (-1)^{n(p-1)+s+1} * \mathrm{d}\, *$$

  and that

  $$\mathrm{d}^\dagger A_p = -\frac{1}{(p-1)!}\, \nabla^\mu A_{\mu\mu_1\ldots\mu_{p-1}}\, \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_{p-1}},$$

  i.e., in terms of components,

  $$(\mathrm{d}^\dagger A)_{\mu_1\ldots\mu_{p-1}} = -\nabla^\mu A_{\mu\mu_1\ldots\mu_{p-1}}.$$

- The Hodge-de Rham operator is defined as

  $$\Delta = \mathrm{d}\mathrm{d}^\dagger + \mathrm{d}^\dagger\mathrm{d}$$

  and, given a p-form A_p, in terms of components one has

  $$(\Delta A)_{\mu_1\ldots\mu_p} = -\nabla^\mu \nabla_\mu A_{\mu_1\ldots\mu_p} - p\, R_{\mu[\mu_1} A^\mu{}_{\mu_2\ldots\mu_p]} - \frac{1}{2}\, p(p-1)\, R_{[\mu_1\nu_1|\mu\nu} A^{\mu\nu}{}_{|\mu_3\ldots\mu_p]}.$$

A.3.2 Cohomology on Smooth Manifolds

Given a smooth manifold M, let $\Omega^p(M)$ denote the space of p-forms on the manifold. A p-form ω_p is closed if its exterior derivative vanishes, i.e. if $\mathrm{d}\omega_p = 0$, and Z^p is defined as the space of closed p-forms, i.e.

$$Z^p = \{\omega_p \in \Omega^p : \mathrm{d}\omega_p = 0\}.$$

A p-form χ_p is exact if there exists a $(p-1)$-form β_{p-1} such that $\chi_p = \mathrm{d}\beta_{p-1}$ and B^p is defined as the space of exact forms, i.e.

$$B^p = \{\chi_p \in \Omega^p : \chi_p = \mathrm{d}\beta_{p-1}, \beta_{p-1} \in \Omega^{p-1}(M)\}.$$

The p-th de Rham cohomology group Γ^p is defined as the quotient space

$$\Gamma^p = Z^p / B^p.$$

Given a closed p-form ω_p, the cohomology class $[\omega_p]$ is defined as

$$[\omega_p] \in \Gamma^p$$

via the equivalence relation $\omega_p = \omega_p + \mathrm{d}\beta_{p-1}$ and ω_p is called a representative of the cohomology class. Equivalent definitions and relations can be estabilished with respect to the adjoint exterior derivative $\mathrm{d}^\dagger$ instead of the exterior derivative d. This gives co-exact forms.

A p-form ω_p is harmonic if it is annihilated by the Hodge-de Rham operator, i.e. if

$$\Delta\omega_p = 0.$$

Because the inner product is positive definite, the p-form ω_p is harmonic if and only if it is both closed and co-closed, i.e. if it is such that $\mathrm{d}\omega_p = 0$ as well as $\mathrm{d}^\dagger\omega_p = 0$. The space of harmonic p-forms is denoted as H^p.

Let M be a closed compact Riemannian manifold. Then each p-form ω_p admits a unique decomposition into the sum of an harmonic p-form α_p, an exact p-form $\mathrm{d}\beta_{p-1}$ and a co-exact p-form $\mathrm{d}^\dagger\gamma_{p+1}$, i.e.

$$\omega_p = \alpha_p + \mathrm{d}\beta_{p-1} + \mathrm{d}^\dagger\gamma_{p+1}.$$

Such an expression is called the Hodge decomposition. As a corollary, a closed p-form ω_p can always be written in terms of an harmonic p-form α_p and an exact p-form $\mathrm{d}\beta_{p-1}$ as

$$\omega_p = \alpha_p + \mathrm{d}\beta_{p-1}$$

and similarly for co-closed p-forms. In fact, the cohomology group Γ^p and the space of harmonic p-forms H^p are isomorphic, i.e.

$$\Gamma^p \simeq H^p.$$

In other words, each cohomology class Γ^p contains a unique harmonic representative α_p in the space H^p.

A p-form ω_p is harmonic if and only if its Hodge dual $*\omega_p$ is harmonic. This relationship is known as Poincaré duality. In fact, on an n-dimensional manifold, the cohomology classes Γ^p and Γ^{n-p} are isomorphic. The Betti numbers b^p are the dimensions of the p-th cohomology group, i.e.

$$b^p = \dim \Gamma^p .$$

So, on a closed compact oriented n-dimensional manifold, the Betti numbers satisfy the conditions

$$b^p = b^{n-p} .$$

The Euler characteristic of a manifold is the number χ defined as

$$\chi = \sum_{p=0}^{n} (-1)^p \, b^p .$$

A.3.3 Complex, Kähler and Calabi-Yau Manifolds

This purely mathematical subsection is just a summary of the results of refs. [50, 51] and it just reports the fundamental definitions and theorems which allow to study complex, Hermitian, Kähler and finally Calabi-Yau manifolds.

As concerns notation, from now on real coordinates are denoted by x^m, with $m = 1, 2, \ldots, 2n$, and complex coordinates are written as z^α, with $\alpha = 1, 2, \ldots, n$. An (r, s)-form $A_{r,s}$ in the complex-manifold formalism is defined as

$$A_{r,s} = \frac{1}{r!s!} A_{\alpha_1 \ldots \alpha_r \overline{\beta}_1 \ldots \overline{\beta}_s} \, \mathrm{d}z^{\alpha_1} \wedge \ldots z^{\alpha_r} \wedge \mathrm{d}z^{\overline{\beta}_1} \wedge \cdots \wedge \mathrm{d}z^{\overline{\beta}_s} .$$

Such a convention emerges naturally by expressing the p-form A_p in complex notation in terms of the (r, s)-form $A_{r,s}$, with $p = r + s$.

A.3.3.1 Complex Manifolds

An m-dimensional almost-complex manifold is a manifold which admits a globally defined rank-$(1, 1)$ tensor field J, called an almost-complex structure, such that

$$J_m{}^k J_k{}^n = -\delta_m{}^n .$$

No mention of complex conjugation has been made, and the tensor J is necessarily real. It can be shown that an almost-complex manifold must have even dimension $m = 2n$.

An n-dimensional complex manifold is a $2n$-dimensional differentiable manifold with an atlas of charts to open subsets U of $\mathbb{C}^n$ such that the transition functions are holomorphic. It turns out that in a complex manifold, for any open neighbourhood with complex coordinates z^α one can define a rank-$(1, 1)$ tensor $J_\alpha{}^\beta$ such that

$$J_\alpha{}^\beta \frac{\partial}{\partial z^\beta} = +\mathrm{i} \, \frac{\partial}{\partial z^\alpha} , \qquad\qquad J_{\overline{\alpha}}{}^{\overline{\beta}} \frac{\partial}{\partial \overline{z}^\beta} = -\mathrm{i} \, \frac{\partial}{\partial \overline{z}^\alpha} .$$

In terms of components, this tensor reads $J_\alpha{}^\beta = \mathrm{i} \, \delta^\beta_\alpha$, $J_{\overline{\alpha}}{}^{\overline{\beta}} = -\mathrm{i}\delta^{\overline{\beta}}_{\overline{\alpha}}$ and $J_\alpha{}^{\overline{\beta}} = J_{\overline{\alpha}}{}^\beta = 0$. In other words, this tensor $J_\alpha{}^\beta$ is a complex structure.

It can be shown that complex manifolds are almost-complex manifolds. Because of the definition of the complex structure $J_\alpha{}^\beta$ in complex manifolds, it is possible to have globally well-defined complex coordinates. This is not the case for almost-complex manifolds because the almost-complex structure must be expressed at each single point.

A.3.3.2 Hermitian Manifolds

Let M be an n-dimensional complex manifold with complex structure $J_m{}^n$, and let g_{mn} be a Riemannian metric on M. The metric g_{mn} is a Hermitian metric if it is of the form

$$ds^2 = g_{\alpha\overline{\beta}}\,\mathrm{d}z^\alpha\,\mathrm{d}\overline{z}^{\overline{\beta}},$$

i.e. if its diagonal components vanish, namely $g_{\alpha\beta} = g_{\overline{\alpha}\overline{\beta}} = 0$. It can be shown that any complex manifold admits a Hermitian metric.

Given a Hermitian manifold with metric g_{mn}, the fundamental or Hermitian 2-form ω_{mn} is defined as

$$\omega_{mn} = J_m{}^l g_{ln}.$$

Equivalently, it is possible to write $\omega_2 = \omega_{mn}\,\mathrm{d}x^m \wedge \mathrm{d}x^n/2$. Importantly, the components can be written as $\omega_{\alpha\overline{\beta}} = \mathrm{i}g_{\alpha\overline{\beta}}$.

On a Hermitian manifold, a connection that is compatible with the metric g_{mn} and the complex structure $J_m{}^k$, i.e. such that $\nabla_k g_{mn} = 0$ and $\nabla_m J_k{}^l = 0$, is called a Hermitian metric. Hermitian metrics are not unique, and only a further constraint can identify a unique metric. It can be shown that on a Hermitian manifold there exists a unique Hermitian connection, called the Chern connection, such that, for any holomorphic and antiholomorphic vectors V^α and $W^{\overline{\alpha}}$, it satisfies the properties

$$\begin{aligned}\nabla_{\overline{\alpha}} V^\beta &= \partial_{\overline{\alpha}} V^\beta,\\ \nabla_\alpha W^{\overline{\beta}} &= \partial_\alpha W^{\overline{\beta}}.\end{aligned}$$

It turns out that the only non-vanishing components in the Chern connection are those with pure indices, with $\Gamma_{\alpha\beta}{}^\gamma = g^{\gamma\overline{\delta}}\,\partial_\alpha g_{\beta\overline{\delta}}$ and $\Gamma_{\overline{\alpha}\overline{\beta}}{}^{\overline{\gamma}} = g^{\overline{\gamma}\delta}\,\partial_\alpha g_{\overline{\beta}\delta}$, while the only non-vanishing components in the Riemann tensor are

$$\begin{aligned}R^\alpha{}_{\beta\gamma\overline{\delta}} &= -\partial_{\overline{\delta}}\Gamma_{\gamma\beta}{}^\alpha,\\ R^{\overline{\alpha}}{}_{\overline{\beta}\overline{\gamma}\delta} &= -\partial_\delta\Gamma_{\overline{\gamma}\overline{\beta}}{}^{\overline{\alpha}},\end{aligned}$$

with the convention $R^k{}_{lmn} = \partial_m\Gamma_{nl}{}^k - \partial_n\Gamma_{ml}{}^k + \Gamma_{mr}{}^k\Gamma_{nl}{}^r - \Gamma_{nr}{}^k\Gamma_{ml}{}^r$. Of course, the Ricci tensor is defined as $R_{mn} = R^k{}_{mkn}$.

Let M be a Hermitian manifold with complex structure $J_m{}^k$ and metric g_{mn}, with a Chern connection. The Ricci form is defined as

$$r_2 = \frac{1}{4}\,R^m{}_{nkl}\,J_m{}^n\,\mathrm{d}x^k \wedge \mathrm{d}x^l.$$

Then, it can be shown that the Ricci form is

$$r_2 = -\mathrm{i}\partial\overline{\partial}\ln\left(\det g\right)^{\frac{1}{2}},$$

where ∂ and $\overline{\partial}$ are exterior derivatives which act only on holomorphic and antiholomorphic coordinates, respectively. In particular, ∂ and $\overline{\partial}$ are called Dolbeault operators and satisfy the condition $\mathrm{d} = \partial + \overline{\partial}$. Further, they are such that $\partial\overline{\partial} = -\mathrm{d}(\partial - \overline{\partial})/2$. It can be shown that the Ricci form r_2 is not exact. However, it is closed.

The first Chern class c_1 is the cohomology class of the Ricci form, namely

$$c_1 = [r_2/2\pi].$$

A.3.3.3 Kähler Manifolds

A Kähler manifold is a Hermitian manifold whose Hermitian form ω_{mn} is closed, namely

$$\mathrm{d}\omega_{1,1} = 0.$$

In a Kähler manifold, the metric g_{mn} is called the Kähler metric and the fundamental form ω_{mn} is called the Kähler form.

It can be shown that, for some $(p-1, q-1)$-form χ, any closed (p, q)-form can be written locally as

$$\omega_{p,q} = \partial\overline{\partial}\chi_{p-1,q-1}.$$

In a Kähler manifold, it can be shown that, for some scalar function $K = K(z, \overline{z})$ called the Kähler potential, the Kähler metric can be written locally as

$$g_{\alpha\overline{\beta}} = \partial_\alpha \partial_{\overline{\beta}} K.$$

On a Kähler manifold, it can be shown that the Chern connection is the Levi-Civita connection. Moreover, it can be shown that, in a Kähler manifold, the Ricci tensor R_{mn} and the Ricci form r_{mn} components are such that (with all the other components being vanishing)

$$R_{\alpha\overline{\beta}} = -\mathrm{i} r_{\alpha\overline{\beta}}.$$

Curvature of Kähler Manifolds and Parallel Transport

Essentially, the holonomy group H_M of a manifold M is the group of the rotations a vector V^m undergoes under parallel transport on a closed path γ. In practice, one considers a closed path $\gamma : [t_0, t'] \to M$ on the $2n$-dimensional manifold M, such that $x^m(t_0) = x^m(t') = p$. A vector $V^m = V^m(x)$ is parallel-transported along the path γ if it satisfies the equation, with the corresponding initial condition,

$$\frac{\mathrm{d}x^l}{\mathrm{d}s} \nabla_l V^m = 0, \qquad\qquad V^m(t_0) = V^m,$$

where s is the proper length parameter. Because it is a linear first-order differential equation, the solution $V'^m = V^m(t')$ at the final point can be written as as

$$V'^m = S^m{}_n V^n.$$

Because parallel transport preserves vector lengths, in an orthonormal basis $e_a = e_a^m \partial_m$ at the point p, i.e. a basis with metric $g_{ab} = \delta_{ab}$, the tensor $S^a{}_b$ is just an orthogonal matrix of the group $\mathrm{O}(2n)$. The set of all such tensors is the holonomy group H_M, which is therefore a subgroup of $\mathrm{O}(2n)$, i.e.

$$\mathrm{H}_M \subseteq \mathrm{O}(2n).$$

Notably, the holonomy group H_K of an n-dimensional Kähler manifold K is a subgroup of the group $\mathrm{U}(n) \subset \mathrm{O}(2n)$, i.e.

$$H_K \subseteq \mathrm{U}(n).$$

This is a consequence of the form of the connection: because the connection does not have mixed complex indices, e.g. a vector with a holomorphic index v^α is parallel transported into a vector v'^α with a holomorphic index too. For instance, the orthonormal basis element $e_A = e^\alpha_A \partial_\alpha$ is rotated to $e'_A = S_A{}^B e_B$, therefore the element $S_A{}^B$ is a unitary matrix.

Further, one can see that a Ricci-flat Kähler manifold $\tilde{K}$ admits a holonomy group $H_{\tilde{K}}$ which is contained in $\mathrm{SU}(n) \subset \mathrm{U}(n)$, i.e.

$$\mathrm{H}_{\tilde{K}} \subseteq \mathrm{SU}(n).$$

To see this, one can consider the parallel transport of a vector V^m along an infinitesimal closed path whose edges are parallel to the directions ∂_m and ∂_n. The definition of the Riemann tensor itself allows one to write the transformed vector as

$$V'^k = V^k + \delta a^{mn}\, R^k{}_{lmn} V^l.$$

Of course, the elements $T^k{}_l = \delta^k{}_l + \delta a^{mn}\, R^k{}_{lmn}$ are the elements of the holonomy group $\mathrm{U}(n)$ which are infinitesimally close to the identity. In a Kähler manifold, an holonomy tranformation maps e.g. a tensor with holomorphic components into a tensor with holomorphic components, and, moreover, the Riemann tensor $R^k{}_{lmn}$ is pure in the indices (k, l). Because of the isomorphism $\mathrm{U}(n) \simeq \mathrm{SU}(n) \times \mathrm{U}(1)/\mathbb{Z}_n$, one can see that the transformation $\mathrm{U}(1)$, that which does not mix vector indices, is proportional to the trace $\delta a^{mn}\, R^k{}_{kmn}$, i.e. $\delta a^{mn}\, R^\gamma{}_{\gamma mn} = -4\delta a^{\alpha\overline{\beta}} R_{\alpha\overline{\beta}}$. This means that a Ricci-flat metric forbids the $\mathrm{U}(1)$-part of the holonomy transformations, thus reducing the holonomy group to $\mathrm{SU}(n)$.

Cohomology on Kähler Manifolds

For a complex manifold M, let $\Omega^{p,q}(M)$ be the space of (p, q)-forms on the manifold. The relevant operators to define cohomology groups and harmonic forms are the Dolbeault operators ∂ and $\overline{\partial}$ and it is possible to define the same spaces as above. In a complex as well as in a Hermitian manifold the Dolbeault and de Rham cohomology classes turn out to be different, while they do coincide in a Kähler manifold.

It can be shown that in a Kähler manifold the exterior derivative d and the Dolbeault operators ∂ and $\overline{\partial}$ are such that

$$\partial\partial^\dagger + \partial^\dagger\partial = \overline{\partial}\overline{\partial}^\dagger + \overline{\partial}^\dagger\overline{\partial} = \frac{1}{2}\,(\mathrm{d}\mathrm{d}^\dagger + \mathrm{d}^\dagger\mathrm{d}).$$

This means that in a Kähler manifold the Dolbeault and de Rham cohomology classes are equivalent.

In a Kähler manifold, denoting as H^r the space of harmonic r-forms with respect to d and as $H^{r,s}_{\overline{\partial}}$ the spaces of harmonic (r,s)-forms with respect to $\overline{\partial}$, the relationship can be shown to hold

$$H^r = \bigoplus_{p=0}^{r} H^{p,r-p}_{\overline{\partial}}.$$

The Hodge numbers $h^{p,q}$ are defined as the dimensions of the (p,q)-th cohomology groups $H^{r,s}_{\overline{\partial}}$, i.e.

$$h^{r,s} = \dim H^{r,s}_{\overline{\partial}}.$$

One can see that in a Kähler manifold, Betti and Hodge numbers satisfy the conditions

$$b^r = \sum_{p=0}^{r} h^{p,r-p}.$$

It can be shown that in an n-dimensional complex manifold the Hodge numbers satisfy the identities

$$\begin{aligned} h^{p,q} &= h^{q,p}, \\ h^{p,q} &= h^{n-p,n-q}. \end{aligned}$$

One can shown that in a Kähler manifold the Kähler form is harmonic. This means that in a Kähler manifold, one has $h^{1,1} > 0$ and hence $b^2 > 0$. More generally, it can be shown that, in an n-dimensional compact closed Kähler manifold, the Hodge number satisfy the condition $h^{p,p} > 0$ for $p \leq m$, where $n = 2m$, and hence the Betti numbers satisfy the conditions $b^{2p} > 0$.

A.3.3.4 Calabi-Yau Manifolds

Calabi-Yau manifolds can be defined in a number of equivalent ways, all of which relying on Yau's theorem.

To start, it is possible to show that in a complex manifold, under a smooth variation of the metric g_{mn} as $g'_{mn} = g_{mn} + \delta g_{mn}$, the first Chern class is invariant. Further, it is possible to prove that, if a Kähler manifold admits a Ricci-flat metric, then its first Chern class vanishes, i.e. $c_1 = 0$.

Now, a necessary condition for a Kähler manifold to admit a Ricci-flat metric is that it has a vanishing first Chern class c_1. Calabi's conjecture states that, in turn, if the first Chern class of a Kähler manifold is vanishing, then it admits a Ricci-flat metric. Basically, the proof of this conjecture is the main result of Yau's theorem.

More precisely, Yau's theorem states the following: given a Kähler manifold M with metric g_{mn} and Kähler form ω_{mn}, let r_{mn} be any representative of the first Chern class; then, there exists a unique metric g'_{mn} on the manifold M with Kähler form ω'_{mn} in the same cohomology class as ω_{mn} whose Ricci form is r_{mn}. A corollary is as follows: let M be a Kähler manifold with metric g_{mn} and Kähler form ω_{mn} with vanishing first Chern class $c_1 = 0$; then there exists a unique Ricci-flat metric g'_{mn} with a Kähler form ω'_{mn} in the same cohomology class as ω_{mn}.

A Calabi-Yau manifold is defined as a Kähler manifold with vanishing first Chern class. As a consequence, it can be proven that an n-dimensional Calabi-Yau manifold is a Kähler manifold with the following equivalent properties:

- its first Chern class is vanishing, i.e. $c_1 = 0$;
- it admits a Ricci-flat metric;
- it admits a metric whose holonomy group is $\mathrm{SU}(n)$;
- it admits a unique holomorphic harmonic $(n, 0)$-form $\Omega_{n,0}$;
- it admits a pair of globally well-defined covariantly constant spinors.

Further interesting properties characterising a Calabi-Yau manifold are as follows.

- A Calabi-Yau manifold with nonzero Euler number χ has $h^{1,0} = 0$.
- An n-dimensional Calabi-Yau manifold has $h^{n,0} = 1$.
- An n-dimensional Calabi-Yau manifold has $h^{p,0} = h^{n-p,0}$.

A.4 Supersymmetry Conventions

This subsection summarises the conventions employed in the main text to discuss $N_4 = 1$ supersymmetry, based on ref. [85].

A.4.1 Essential Notions

Supersymmetry is conveniently described in superspace, i.e. an extension of the usual 4-dimensional spacetime, with coordinates x^μ, with $\mu = 0, 1, 2, 3$, spanned by two more anti-commuting coordinates θ^α and $\overline{\theta}_{\dot\alpha}$, with $\alpha, \dot\alpha = 1, 2$ representing chiral spinor indices. Left- and right-handed spinors are denoted as ξ_α and $\overline{\chi}^{\dot\alpha}$, respectively, with conjugation such that $(\xi_\alpha)^\dagger = \overline{\xi}_{\dot\alpha}$; spinor indices are raised and lowered by the Levi-Civita symbol $\varepsilon_{\alpha\beta}$ and $\varepsilon^{\alpha\beta}$, normalised as $\varepsilon_{21} = \varepsilon^{12} = 1$, as $\xi_\alpha = \varepsilon_{\alpha\beta}\xi^\beta$ and $\xi^\alpha = \varepsilon^{\alpha\beta}\xi_\beta$. Spinor contractions are defined as $\xi\chi = \xi^\alpha\chi_\alpha$ and $\overline{\xi}\overline{\chi} = \overline{\xi}_{\dot\alpha}\overline{\chi}^{\dot\alpha}$. The Pauli-matrix 4-vectors are defined as $\sigma^\mu_{\alpha\dot\alpha} = (1_2, \sigma^i)$ and $\overline{\sigma}^{\mu\,\dot\alpha\alpha} = (1_2, -\sigma^i)$. Superfields are fields defined in the superspace.

A.4.2 Chiral Superfields

A chiral superfield Φ is defined as a superfield solving the equation $\overline{D}_{\dot\alpha}\Phi = 0$, where $D_\alpha = \partial_\alpha - \mathrm{i}(\sigma^\mu\overline{\theta})_\alpha\partial_\mu$ is the supersymmetric chiral-covariant derivative. It contains a complex scalar φ, a Weyl spinor ψ and an auxiliary scalar F and can be expanded in the superspace coordinates $\{x^\mu, \theta^\alpha, \overline{\theta}_{\dot\alpha}\}$ as

$$\Phi = \varphi(y) + \sqrt{2}\,(\theta\psi(y)) + (\theta\theta)\,F(y), \tag{A.4.1}$$

with shifted coordinate $y^\mu = x^\mu + \mathrm{i}\,(\overline{\theta}\overline{\sigma}^\mu\theta)$. The supersymmetry variations are

$$\delta_\epsilon\varphi = \sqrt{2}(\epsilon\psi), \tag{A.4.2a}$$
$$\delta_\epsilon\psi_\alpha = -\mathrm{i}\sqrt{2}\,(\sigma^\mu\overline{\epsilon})_\alpha\,\partial_\mu\varphi + \sqrt{2}\,\epsilon_\alpha F, \tag{A.4.2b}$$
$$\delta_\epsilon F = -\mathrm{i}\sqrt{2}\,(\overline{\epsilon}\,\overline{\sigma}^\mu\partial_\mu\psi). \tag{A.4.2c}$$

Due to Lorentz invariance, only scalars can acquire a non-zero vacuum expectation value. Therefore, the only possibility to spontaneously break supersymmetry is if the auxiliary field is non-zero, giving $\delta_\epsilon\langle\psi_\alpha\rangle = \sqrt{2}\,\epsilon_\alpha\langle F\rangle$.

A.4.3 Vector Superfields

A vector superfield V is defined as a general real scalar superfield V, i.e. a scalar superfield such that $V = \overline{V}$. Expanding in the superspace, such a field can be generally written as

$$\begin{aligned} V(x,\theta,\overline{\theta}) = a &+ (\theta\xi) + (\overline{\theta\xi}) + (\theta\theta)\,b + (\overline{\theta\theta})\,\overline{b} + (\overline{\theta}\overline{\sigma}^\mu\theta)\,A_\mu \\ &+ (\theta\theta)\,\big[\overline{\theta}\big(\overline{\lambda} - \frac{\mathrm{i}}{2}\,\overline{\sigma}^\mu\partial_\mu\xi\big)\big] + (\overline{\theta\theta})\,\big[\theta\big(\lambda - \frac{\mathrm{i}}{2}\,\sigma^\mu\partial_\mu\overline{\xi}\big)\big] \\ &+ \frac{1}{2}\,(\theta\theta)\,(\overline{\theta\theta})\,\big[D + \frac{1}{2}\,\eta^{\mu\nu}\partial_\mu\partial_\nu a\big], \end{aligned}$$

where a is a real scalar, b is a complex scalar, λ and χ are Weyl spinors and A_μ is a vector. Component by component, the supersymmetry variations read

$$\delta_\epsilon a = (\epsilon\xi) + (\overline{\epsilon}\overline{\xi}), \tag{A.4.3a}$$
$$\delta_\epsilon\xi_\alpha = 2\epsilon_\alpha b - (\sigma^\mu\overline{\epsilon})_\alpha(A_\mu + \mathrm{i}\partial_\mu a), \tag{A.4.3b}$$
$$\delta_\epsilon b = (\overline{\epsilon}\overline{\lambda}) - \mathrm{i}(\overline{\epsilon}\,\overline{\sigma}^\mu\partial_\mu\xi), \tag{A.4.3c}$$
$$\delta_\epsilon A_\mu = \mathrm{i}(\epsilon\partial_\mu\xi) - \mathrm{i}(\overline{\epsilon}\partial_\mu\overline{\xi}) + (\epsilon\sigma^\mu\overline{\lambda}) - (\overline{\epsilon}\,\overline{\sigma}^\mu\lambda), \tag{A.4.3d}$$
$$\delta_\epsilon\lambda = \epsilon_\alpha D + \mathrm{i}(\sigma^\mu\overline{\sigma}^\nu\epsilon)_\alpha\partial_{[\mu}A_{\nu]}, \tag{A.4.3e}$$
$$\delta_\epsilon D = -\mathrm{i}(\epsilon\sigma^\mu\partial_\mu\overline{\lambda}) - \mathrm{i}(\overline{\epsilon}\,\overline{\sigma}^\mu\partial_\mu\lambda). \tag{A.4.3f}$$

Given an arbitrary chiral superfield $\Omega = (\phi_\Omega, \psi_\Omega, F_\Omega)$, then the generalised gauge transformation

$$V \to V - \mathrm{i}(\Omega - \overline{\Omega})$$

gives a new vector superfield. In terms of components, the transformations read

$$\begin{aligned} a &\to a - \mathrm{i}(\phi_\Omega - \overline{\phi}_{\overline{\Omega}}), \\ \xi &\to \xi - \mathrm{i}\sqrt{2}\,\psi_\Omega, \\ b &\to b - \mathrm{i}F_\Omega, \\ A_\mu &\to A_\mu + \partial_\mu(\phi_\Omega + \overline{\phi}_{\overline{\Omega}}), \\ \lambda &\to \lambda, \\ D &\to D. \end{aligned}$$

So the only non-redundant degrees of freedom are the spinor λ, the vector A_μ, that undergoes a general gauge transformation, and the scalar D. It is therefore possible to fix the so-called WZ-gauge

$$V(x,\theta,\overline{\theta}) = (\overline{\theta}\overline{\sigma}^\mu\theta)\,A_\mu + (\theta\theta)(\overline{\theta\lambda}) + (\overline{\theta\theta})(\theta\lambda) + \frac{1}{2}\,(\theta\theta)(\overline{\theta\theta})D.$$

A supersymmetry transformation can be shown to change the gauge, but any vector superfield can be brought back to WZ-gauge by a subsequent supergauge transformation.

It is possible to show that the physical degrees of freedom of a vector superfield can be described by means of a generalised field-strength tensor, namely the chiral superfield

$$W_\alpha = -\frac{1}{4}\,\overline{D}_{\dot\alpha}\overline{D}^{\dot\alpha}D_\alpha V,$$

which can be expanded in superspace as

$$W_\alpha = \lambda_\alpha(y) + \left[\delta_\alpha^\beta\,D(y) + \frac{\mathrm{i}}{2}\,(\sigma^\mu\overline{\sigma}^\nu)_\alpha{}^\beta\,F_{\mu\nu}(y)\right]\theta_\beta + \mathrm{i}(\theta\theta)\,[\sigma^\mu\partial_\mu\overline{\lambda}(y)]_\alpha.$$

B DIMENSIONAL REDUCTIONS IN WARPED SPACETIMES

This section outlines the dimensional reduction of closed- and open-string sectors in warped compactifications.[B.1] It is meant to set the notation for the main text and to provide a review of how the scaling factors are obtained in warped dimensional reductions.

It is useful to specify that a relationship $a \sim b$ means that the two terms a and b are roughly of the same order of magnitude, ignoring constant order-1 factors, whereas the relationship $a \simeq b$ means that the two terms are roughly the same. One should notice that the notation $f(x) \overset{x \sim x_0}{\simeq} g(x)$ indicates the fact that the two functions f and g are similar for $x \simeq x_0$ (this lightens the notation, rather than $f(x) \overset{x \simeq x_0}{\simeq} g(x)$, and it is usually employed at $x \sim 0$ or $x \sim \infty$, avoiding ambiguities).

B.1 Type IIB Closed-String Sector

The generic bosonic action for the type IIB closed-string sector in the 10-dimensional Einstein frame is discussed in subection 2.2.1, and can be read in eq. (2.2.2). This is the starting point for the following discussion on the dimensional reduction of the closed-string sector action in warped Calabi-Yau orientifold compactifications.

In a Calabi-Yau orientifold compactification with non-zero background fluxes, the field equations imply a non-trivial warp factor [53, 56]. Following refs. [227, 229], the volume-controlling real Kähler modulus $c = c(x)$ appears as a shift in the warp factor $\mathrm{e}^{A} = \mathrm{e}^{A(y)}$, leading to the definition of the generalised warp factor

$$\mathrm{e}^{-4A[c(x),y]} = \mathrm{e}^{-4A(y)} + c(x),$$

with the 10-dimensional Einstein-frame metric taking the form

$$ds_{10}^2 = \mathrm{e}^{2A[c]}\, \breve{g}_{\mu\nu}\, \mathrm{d}x^{\mu}\mathrm{d}x^{\nu} + \mathrm{e}^{-2A[c]}\, \breve{g}_{mn}\, \mathrm{d}y^{m}\mathrm{d}y^{n}.$$

As discussed by ref. [229], one can Weyl-rescale this to the 4-dimensional Einstein frame, while also introducing a compensator field $b = b(y)$ that is necessary to solve the Einstein equations, with the full metric reading

$$ds_{10}^2 = \gamma^{\frac{3}{2}}\, \mathrm{e}^{2\Omega}\, \mathrm{e}^{2A[c]}\, (g_{\mu\nu}\, \mathrm{d}x^{\mu}\mathrm{d}x^{\nu} + 2\partial_{\mu}c\, \partial_{m}b\, \mathrm{d}x^{\mu}\mathrm{d}y^{m}) + \mathrm{e}^{-2A[c]}\, g_{mn}\, \mathrm{d}y^{m}\mathrm{d}y^{n}. \tag{B.1.1}$$

[B.1] See also ref. [309] for a recent discussion of the scaling properties of the closed- and open-string effective theories in string compactifications.

In particular, in the Weyl rescaling one has the Kähler modulus-dependent factor

$$e^{2\Omega} = \frac{\displaystyle\int_{Y_6} d^6y\,\sqrt{g_6}}{\displaystyle\int_{Y_6} d^6y\,\sqrt{g_6}\,[e^{-4A}+c]}$$

and for generality also an arbitrary constant $\gamma^{3/2}$ has been introduced, which in this case will be chosen as $\gamma = \langle c\rangle$ [55].[B.2] The warp factor has the following behaviours:

- in the infrared region of the throat τ_6, the background warp factor is much larger than the volume modulus, that is $e^{-4A}(y \in \tau_6) \gg \langle c\rangle \gg 1$ so that

$$e^{-\langle 4A[c]\rangle} \simeq e^{-4A}, \qquad y \in \tau_6;$$

- in the bulk region of the compact space, the background warp factor is negligible, that is $e^{-4A}(y \in Y_6\backslash\tau_6) \ll c$, so

$$e^{-\langle 4A[c]\rangle} \simeq \langle c\rangle, \qquad y \in Y_6\backslash\tau_6.$$

The dimensional reduction of the closed-string sector action, to find the 4-dimensional low-energy effective theory corresponding to the flux compactification, is now reviewed for the most relevant degrees of freedom. Following the very definition of the 4-dimensional Einstein frame, the type IIB Einstein-Hilbert action contained in eq. (2.2.2) becomes

$$S_{\text{EH}}^{\text{IIB}} = \frac{1}{2\hat{\kappa}_{10}^2}\int_{X_{1,9}} d^{10}x\,\sqrt{-\hat{g}_{10}}\,\hat{R}_{10} = \frac{1}{2\kappa_4^2}\int_{X_{1,3}} d^4x\,\sqrt{-g_4}\,R_4 + \delta S_{\text{EH}}^{\text{IIB}},$$

with the 4-dimensional gravitational coupling defined as

$$2\kappa_4^2 = \frac{2\hat{\kappa}_{10}^2}{\gamma^{\frac{3}{2}} l_s^6 \ell_{(0)}} = \frac{g_s^2 l_s^2}{2\pi\gamma^{\frac{3}{2}}\ell_{(0)}} \tag{B.1.2}$$

and the term $\delta S_{\text{EH}}^{\text{IIB}}$ standing for the internal curvature and other derivative terms, emerging from the remainder of the Ricci scalar, which provide contributions to the kinetic terms and the scalar potential for the geometric moduli. In particular, the Kähler-modulus kinetic term is reproduced by means of the Kähler potential [229]

$$\kappa_4^2 \hat{K}(\rho,\overline{\rho}) = -3\ln\left[-i(\rho-\overline{\rho}) + 2c_0\right],$$

with $c_0 = \ell_w/\ell_{(0)}$, where the complexified Kähler modulus ρ is defined as

$$\rho(x) = \chi(x) + i\,c(x),$$

with χ being the 4-form axion. The description of the other closed-string sector fields follows with specific features determined by warping effects [53, 266].

[B.2] Notice that the canonically normalised masses in Planck units are independent of constant Weyl rescalings and most references work with $\gamma = 1$.

- For the axio-dilaton τ, it is immediate to check that the kinetic term contained in eq. (2.2.2) is

$$
\begin{aligned}
S_{\text{axio-dilaton}} &= \frac{1}{2\hat{\kappa}_{10}^2} \int_{X_{1,9}} \mathrm{d}^{10}x \, \sqrt{-\hat{g}_{10}} \left[-\frac{1}{2\,(\mathrm{Im}\,\tau)^2}\, \hat{g}^{MN} \partial_M \tau \partial_N \overline{\tau} \right] \\
&= \frac{1}{2\kappa_4^2} \int_{X_{1,3}} \mathrm{d}^4x \, \sqrt{-g_4} \Big[-\frac{1}{2\,(\mathrm{Im}\,\tau)^2}\, g^{\mu\nu}\, \partial_\mu \tau \partial_\nu \overline{\tau} \Big],
\end{aligned}
$$

  which is reproduced by the usual Kähler potential $\kappa_4^2 \hat{K}(\tau,\overline{\tau}) = -\ln\left[-i(\tau - \overline{\tau})\right]$.

- For the complex-structure moduli u^α, with $\alpha = 1, \dots, h^{2,1}_-$, the dimensional reduction is more involved. In particular, one needs the quantities

$$
\begin{aligned}
\omega_w &= \int_{Y_6} \mathrm{e}^{-4A} \Omega \wedge \overline{\Omega}, \\
\hat{K}_{\alpha\overline{\beta}} &= -\frac{1}{\omega_w} \int_{Y_6} \mathrm{e}^{-4A}\, \chi_\alpha \wedge \overline{\chi}_\beta,
\end{aligned}
$$

  which provide the warped version of the complex-structure moduli Kähler potential, $\kappa_4^2 \hat{K}(u, \overline{u}) = -\ln\left[-\mathrm{i}\omega_w\right]$, and the explicit Kähler metric [53, 310], where Ω and χ_α are the unwarped harmonic 3-form and $(2,1)$-form basis, respectively.

To have a complete supergravity formulation, one must also match the scalar potential that arises from the dimensional reduction. The following calculation only captures the axio-dilaton and complex-structure moduli potential as it neglects the details of the coupling with the warp factor, the volume modulus and the compensator field. It is just meant to argue the emergence of the GVW-superpotential [311] and to fix the overall constants. The functional dependence of the scalar potential is set by the 3-form term as the remaining terms from the Einstein-Hilbert and 5-form actions can be combined with the 3-form action, cancelling the contribution from imaginary self-dual fluxes G_3^- and leaving pure imaginary anti-self-dual fluxes G_3^+ [53, 56], with

$$
G_3^\pm = \frac{1}{2}\,(1 \pm \mathrm{i}*_6) G_3.
$$

Now, refs. [227, 229] show that if the warp factor e^{-4A} solves the field equations, so does the shifted warp factor $\mathrm{e}^{-4A} + c$. Assuming then for simplicity the background value for the volume $\langle c \rangle$, one can express this 10-dimensional potential in terms of the 4-dimensional Einstein-frame metric, i.e.

$$
\begin{aligned}
S_{\text{3-form}} &= \frac{1}{2\hat{\kappa}_{10}^2} \int_{X_{1,9}} \mathrm{d}^{10}x \, \sqrt{-\hat{g}_{10}} \left[-\frac{1}{12\,\mathrm{Im}\,\tau}\, G_3^+ \,\hat{\cdot}\, \overline{G}_3^+ \right] \\
&= \frac{\gamma^3}{2\hat{\kappa}_{10}^2} \int_{X_{1,3}} \mathrm{d}^4x \, \sqrt{-g_4} \int_{Y_6} \mathrm{d}^6y \, \sqrt{g_6} \left[-\frac{\mathrm{e}^{\langle 4\Omega \rangle + \langle 4A[c] \rangle}}{12\,\mathrm{Im}\,\tau}\, G_3^+ \cdot \overline{G}_3^+ \right].
\end{aligned}
$$

The most interesting case to consider is the one where integrations are dominated by the throat region τ_6, in which $\mathrm{e}^{-\langle 4A[c] \rangle} \simeq \mathrm{e}^{-4A}$. Because the GKP field equations require the imaginary anti-self-dual 3-forms $\mathrm{e}^{4A}G_3^+$ to be harmonic [53, 56], without loss of generality one can focus on the $(3,0)$-component and expand it as

$$\mathrm{e}^{4A}\, G_{(3,0)} = \frac{1}{\omega_w}\, \Omega \int_{\tau_6} G_3 \wedge \overline{\Omega},$$

so that the action can be written as

$$S_{3\text{-form}} = \frac{\gamma^3}{2\hat{\kappa}_{10}^2} \int_{X_{1,3}} \mathrm{d}^4x \sqrt{-g_4} \int_{\tau_6} \left[-\frac{\mathrm{i}}{2} \frac{\mathrm{e}^{\langle 4\Omega\rangle - 4A}}{\operatorname{Im}\tau\, \omega_w^2}\, \Omega \wedge \overline{\Omega} \left[\int_{\tau_6} G_3 \wedge \overline{\Omega} \right] \left[\int_{\tau_6} \overline{G}_3 \wedge \Omega \right] \right].$$

The integral over the internal space is now easily seen to be

$$\lambda \simeq \int_{Y_6} \mathrm{e}^{-4A}\, \Omega \wedge \overline{\Omega} = \omega_w \simeq \omega_w\, \frac{\ell_w}{\ell_{(0)}}\, \mathrm{e}^{\langle 2\Omega\rangle}, \tag{B.1.3}$$

where an approximate unit factor has been introduced in the final relation, for convenience in the comparison with the supergravity action below. At the end of the day, the 3-form action is (the numerical factor can be determined by properly taking into account the axio-dilaton and 5-form contributions to the scalar potential [53])

$$\begin{aligned} S_{3\text{-form}} &= \frac{\gamma^3}{2\hat{\kappa}_{10}^2} \int_{X_{1,3}} \mathrm{d}^4x\, \sqrt{-g_4} \left[-\frac{\mathrm{i}\,\mathrm{e}^{\langle 6\Omega\rangle}}{\operatorname{Im}\tau\, \omega_w} \frac{\ell_w}{\ell_{(0)}} \left[\int_{Y_6} G_3 \wedge \overline{\Omega} \right] \left[\int_{Y_6} \overline{G}_3 \wedge \Omega \right] \right] \\ &= \frac{1}{2\kappa_4^4} \frac{g_s^2}{4\pi} \int_{X_{1,3}} \mathrm{d}^4x\, \sqrt{-g_4} \left[-\frac{\mathrm{i}\,\mathrm{e}^{\langle 6\Omega\rangle}}{\operatorname{Im}\tau\, \omega_w} \frac{\ell_w}{[\ell_{(0)}]^3} \frac{1}{l_s^4} \left[\int_{Y_6} G_3 \wedge \overline{\Omega} \right] \left[\int_{Y_6} \overline{G}_3 \wedge \Omega \right] \right]. \end{aligned}$$

The last step takes into account the definition of the 4-dimensional Planck units while keeping the bulk integrals scaled with the appropriate string length factors (recalling the scalings $G_3 \sim l_s^2$ and $\Omega \sim 1$). This result gives a way to understand how to insert the volume and warped-volume factors in the effective supergravity formulation whereby the axio-dilaton scalar potential from the Kähler and superpotential of eqs. (4.2.5a, 4.2.5b) reproduce it exactly. The complex-structure moduli scalar potential is found in the dimensional reduction by considering the $(1,2)$-components in the 3-form flux. On the other hand, the Kähler modulus ρ is a flat direction since its F-term potential contribution cancels against the negative-definite potential term corresponding to the squared gravitino mass.

A similar analysis can be done with the opposite approximation that bulk integrals dominate over throat integrals, which leads to the unwarped limit. The calculation follows analogously but it is easier since the warping in the integrations is irrelevant, i.e. $\ell_w \simeq \ell_{(0)}$ and $\omega_w \simeq \omega_{(0)} = \int_{Y_6} \Omega \wedge \overline{\Omega}$. In more detail, one may start from the 10-dimensional potential written above noticing the identities $\mathrm{e}^{4A[c]} = 1/c = \mathrm{e}^{-4u}$ and $\mathrm{e}^{2\Omega} = 1/c = \mathrm{e}^{-4u}$, and reduce it along the same lines, with the 3-form flux G_3^+ being harmonic. Alternatively, formally this limit can be found by setting $\mathrm{e}^{4A} = 1$ in all the final integrated expressions, so that $\ell_w = \ell_{(0)}$ and $\omega_w = \omega_{(0)}$. One obtains the famous results of refs. [56, 236]. The warped expressions are always kept in the main text for the sake of generality.

B.2 D3-, Anti-D3- and D7-Branes

The generic bosonic action for Dp-branes in the 10-dimensional Einstein frame is discussed in subection 2.3.2, and can be read in the DBI- and CS-actions in eqs. (2.3.2) and (2.3.3), respectively. This is the starting point for the following discussion on the dimensional reduction of D3, anti-D3- and D7-brane massless actions in warped Calabi-Yau orientifold compactifications.

In the probe approximation, an explicit dimensional reduction of the D3- and anti-D3-brane action has been performed in refs. [80, 149, 163, 231, 232], while the study of the D7-brane action can be found in refs. [235, 236, 253]. Most references work with a metric of the form

$$ds^2_{10} = \mathrm{e}^{2A[c]}\,\breve{g}_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{e}^{-2A[c]}\,\breve{g}_{mn}\,\mathrm{d}y^m \mathrm{d}y^n .$$

In this subsubsection the results are adapted directly from such references. For a 4-dimensional theory, the worldvolume degrees of freedom must be reduced, and they are sensitive to the details of the wrapped $(p-3)$-cycle. It is also convenient to combine pairs of real scalars into single complex scalars as $\phi^a = \phi^{\dot{m}=2a+2} + \mathrm{i}\,\phi^{\dot{m}=2a+3}$, and similarly for the modulini.

- For D3- and anti-D3-branes, the dimensional reduction proceeds in the same way except for the different interference between the DBI- and CS-actions due to the different RR-charge. All the terms evaluated at the brane location carry a symbol '0'.

  First of all one finds the cosmological constant contribution

  $$S^{\mathrm{D3}_q}_{\Lambda} = -(1-q)\tau_{\mathrm{D3}} \int \mathrm{d}^4x\,\sqrt{-\breve{g}_4}\,\mathrm{e}^{4A_0[c]},$$

  which explains the anti-D3-brane uplift energy.

  Further, the pure scalar kinetic and mass terms turn out to be (there are also bilinear $\phi^a\phi^b$-couplings with the same scaling as the mass terms)

  $$S^{\mathrm{D3}_q}_{\text{scalars}} = -\tau_{\mathrm{D3}}\sigma_s^2 \int_{X_{1,3}} \mathrm{d}^4x\,\sqrt{-\breve{g}_4}\left[\breve{g}^0_{a\bar{b}}\,\breve{g}^{\mu\nu}\,\breve{\nabla}_\mu\phi^a\breve{\nabla}_\nu\overline{\phi}^b + [\nabla_a\nabla_{\bar{b}}(\mathrm{e}^{4A[c]} - q\alpha)]_0\,\phi^a\overline{\phi}^b\right].$$

  Following the GKP-equations [53, 56, 227], the anti-D3-brane scalars are massive for imaginary self-dual $(2,1)$- and $(0,3)$-fluxes, whereas for D3-branes they are massless.

  For the modulini, one finds the kinetic and mass action[B.3]

  $$S^{\mathrm{D3}_q}_{\text{modulini}} = -\tau_{\mathrm{D3}}\sigma_s^2 \int_{X_{1,3}} \mathrm{d}^4x\,\sqrt{-\breve{g}_4}\left[\mathrm{i}\breve{g}^0_{a\bar{b}}\,\overline{\breve{\psi}}^b\,\overline{\breve{\sigma}}^\mu\breve{\nabla}_\mu\breve{\psi}^a + \left(m^{(q)}_{\breve{\psi}^a\breve{\psi}^b}\,\breve{\psi}^a\breve{\psi}^b + \text{c.c.}\right)\right].$$

  For anti-D3-branes, the modulini masses are purely sourced by $(2,1)$-fluxes and read

  $$m^{(q=-1)}_{\breve{\psi}^a\breve{\psi}^b} = -\frac{1}{4}\,\mathrm{e}^{4A_0[c]+\phi/2}\,\breve{g}^0_{\bar{c}(a}\,l_s^3\breve{\Omega}^0_{b)de}\,(\overline{G_3^-})_0^{\breve{\bar{c}}\breve{d}\breve{e}},$$

[B.3] The dimensional reduction of the 10-dimensional Majorana-Weyl spinor to the 4-dimensional Weyl spinors is the same as in ref. [80] since $\mathrm{e}^{-4A_0[c]} \simeq \mathrm{e}^{-4A_0}$ for branes at the tip of the throat.

while for D3-branes they are sourced by imaginary anti-self-dual $(1,2)$-fluxes.

One also finds the gauge vector action

$$S^{\mathrm{D3}_q}_{\text{gauge}} = -\frac{\tau_{\mathrm{D3}}\sigma_s^2}{2}\int_{X_{1,3}} \mathrm{e}^{-\phi}\, F_2 \wedge \breve{*} F_2 + \frac{q\tau_{\mathrm{D3}}\sigma_s^2}{2}\int_{X_{1,3}} C_0\, F_2 \wedge F_2 .$$

The gaugino mass is sourced by $(0,3)$- and $(3,0)$-fluxes for anti-D3- and D3-branes, respectively.

- For D7-branes, the reduction to a 4-dimensional action depends on the wrapped internal 4-cycle, so only the general features of bosons will be discussed. Let the 4-cycle be spanned by the coordinates (z^1, z^2) and let z^3 be transverse direction.

  For the transverse scalar $\pi^3 = \phi^3$, the pure kinetic action is

$$S^{\text{D7-scalar}}_{\text{kin}} = -\tau_{\mathrm{D7}}\sigma_s^2 \int_{X_{1,3}} \mathrm{d}^4x\, \sqrt{-\breve{g}_4}\int_{\Sigma_4} \mathrm{d}^4y\, \sqrt{\breve{g}_{\Sigma_4}}\, \mathrm{e}^{\phi}\, \mathrm{e}^{-4A[c]}\, \breve{g}_{3\overline{3}}\, \breve{g}^{\mu\nu}\, \breve{\nabla}_\mu \pi^3 \breve{\nabla}_\nu \overline{\pi}^3 .$$

  The total mass term emerges from the interference of the DBI- and CS-actions, with the terms adding up or cancelling out. The full expression is complicated, but the scalings can be read from the DBI-term and the mass action has the form

$$S^{\text{D7-scalar}}_{\text{mass}} = -\frac{\tau_{\mathrm{D7}}\sigma_s^2}{2}\int_{X_{1,3}} \mathrm{d}^4x\sqrt{-\breve{g}_4}\int_{\Sigma_4} \mathrm{d}^4y\sqrt{\breve{g}_{\Sigma_4}}\, \mathrm{e}^{2\phi}\, \mathrm{e}^{4A[c]}\, G_{m'n'\dot{r}}\, \overline{G}^{\breve{m}'\breve{n}'}{}_{\dot{s}}\, \pi^{\dot{r}}\pi^{\dot{s}},$$

  in real notation. As D7-branes preserve the same supersymmetry as the orientifold, the supersymmetric mass is sourced by a $(2,1)$-flux (but IASD-fluxes source supersymmetry-breaking masses as well). For the theory to have no Freed-Witten anomalies [312], the 2-form B_2 must be constant over the 4-cycle and in this case the supersymmetric mass is sourced specifially by the flux $G_{12\overline{3}}$.

  One also finds the gauge vector kinetic action

$$S^{\text{D7-vector}}_{\text{kin}} = -\frac{\tau_{\mathrm{D7}}\sigma_s^2}{4}\int_{X_{1,3}} \mathrm{d}^4x\, \sqrt{-\breve{g}_4}\int_{\Sigma_4} \mathrm{d}^4y\, \sqrt{\breve{g}_{\Sigma_4}}\, \mathrm{e}^{-4A[c]}\, \breve{g}^{\mu\rho}\breve{g}^{\nu\sigma}\, F_{\mu\nu}F_{\rho\sigma},$$

  with gaugino masses sourced by $(0,3)$-fluxes.

In order to switch to the 4-dimensional Einstein frame defined in eq. (4.2.1), which is necessary to single out the leading order Kähler-modulus couplings, one can make the identifications

$$\breve{g}_{\mu\nu} = \mathrm{e}^{2\Omega}\,\gamma^{\frac{3}{2}}\, g_{\mu\nu}, \qquad \breve{g}_{mn} = g_{mn}.$$

Notice that one also needs to transform the Pauli matrix 4-vector as $\breve{\sigma}^\mu = \mathrm{e}^{-\Omega}\,\gamma^{-3/4}\,\sigma^\mu$ and to rescale the spinors as $\breve{\psi} = \mathrm{e}^{-\Omega/2}\gamma^{-3/8}\tilde{\psi}$ (for similar calculations, see e.g. refs. [232,313]).

It is also convenient to renormalise the fields in such a way as to remove the γ-factors, which turns out to be very helpful in order to obtain 4-dimensional quantities expressed in the appropriate (string coupling, volume and/or warp factor suppressed) Planck units. So for the D3- and anti-D3-branes one has

$$\varphi^a = \gamma^{\frac{3}{4}}\phi^a, \qquad \psi^a = \gamma^{\frac{3}{4}}\tilde{\psi}^a,$$

while for D7-branes one has

$$\sigma^3 = \gamma^{\frac{3}{4}}\pi^3.$$

Further couplings that arise from the redefinition of the volume modulus are given in the main text (see eq. (4.4.12)). A complete analysis including the compensator field (see eq. (B.1.1) is beyond the scope of this appendix but for progresses in that direction see ref. [251], where it is shown that cancellations occur such that the D3-brane kinetic term is unaffected. Worldvolume fluxes are also not considered.

B.3 Geometry of Warped 4-cycles

This appendix contains a few observations about the geometry of a 4-cycle wrapped by a D7-brane in the two setups discussed in the main text.

B.3.1 Products of 2- and 4-cycles

In the main text, whenever it is necessary to consider the cycles wrapped by the D7-branes explicitly, as in e.g. subsubsections 4.3.1.2 and 4.3.1.3, they are assumed to be (conformally) a 4-dimensional orbifold $O_4 = \mathrm{T}^4/\mathbb{Z}_2$, and the 6-dimensional space is locally assumed to be (conformally) the product of the orbifold O_4 and the 2-torus T^2.

To be concrete, following refs. [234, 234, 237], one considers the 4-dimensional orbifold O_4 spanned by the coordinates (z^1, z^2) and the 2-torus T^2 spanned by z^3, with $w^a = z^a/l_s$ the dimensionless coordinates. Then:

- on the 4-cycle $O_4 = \mathrm{T}^4/\mathbb{Z}_2$, the untwisted $(2,0)$- and $(1,1)$-forms are

  $$\eta = \mathrm{d}w^1 \wedge \mathrm{d}w^2,$$

  and

  $$\zeta_1 = \mathrm{d}w^1 \wedge \mathrm{d}\overline{w}^2, \qquad \zeta_2 = \mathrm{d}\overline{w}^1 \wedge \mathrm{d}w^2,$$
  $$\zeta_3 = \mathrm{d}w^1 \wedge \mathrm{d}\overline{w}^1, \qquad \zeta_4 = \mathrm{d}w^2 \wedge \mathrm{d}\overline{w}^2;$$

- the untwisted harmonic 3-forms on the 6-dimensional space $(\mathrm{T}^4/\mathbb{Z}_2) \times \mathrm{T}^2$ are then the holomorphic 3-form

  $$\Omega = \eta \wedge \mathrm{d}w^3 = \mathrm{d}w^1 \wedge \mathrm{d}w^2 \wedge \mathrm{d}w^3,$$

  and the $(2,1)$-forms

  $$\chi_1 = \mathrm{d}w^1 \wedge \mathrm{d}\overline{w}^2 \wedge \mathrm{d}w^3, \qquad \chi_2 = \mathrm{d}\overline{w}^1 \wedge \mathrm{d}w^2 \wedge \mathrm{d}w^3, \qquad \chi_\vartheta = \mathrm{d}w^1 \wedge \mathrm{d}w^2 \wedge \mathrm{d}\overline{w}^3,$$

  as well as

  $$\chi_3 = \mathrm{d}w^1 \wedge \mathrm{d}\overline{w}^1 \wedge \mathrm{d}w^3, \qquad \chi_4 = \mathrm{d}w^2 \wedge \mathrm{d}\overline{w}^2 \wedge \mathrm{d}w^3.$$

  One could insert the the complex-structure moduli u^a into the relevant elements of the basis by the definition $\mathrm{d}z^a = \mathrm{d}y^a + u^a \mathrm{d}y^{a+3}$, for $a = 1, 2, 3$.

Also, there are extra moduli corresponding to blown-up singularities which are ignored. One can show that the unwarped complex-structure Kähler potential reads

$$\hat{K}_{\text{cs}}^{(0)} = -\ln\left[-i\int_{Y_6}\Omega\wedge\overline{\Omega}\right] = -\ln\Big([-i(u^1-\overline{u}^1)][-i(u^2-\overline{u}^2)][-i(u^3-\overline{u}^3)]\Big) - \ln\ell_{(0)}.$$

In warped scenarios, if the identification of the bulk complex-structure moduli still holds, one finds analogus results with the substitution of the unwarped volume with ℓ_w.

B.3.2 Complex-Structure Kähler Metrics

It is convenient to collectively label the basis of the harmonic $(1,1)$-forms on the orbifold $O_4 = \Sigma_4$ as ζ_i, with $i = 1, \ldots, 4$, and the basis of harmonic $(2,1)$-forms on the 6-dimensional product $O_4 \times \mathrm{T}^2$ as χ_α, with $\alpha = 1, \ldots, 4, \vartheta$. Further there are the harmonic $(2,0)$-form η and the harmonic $(3,0)$-form Ω. The explicit complex-structure moduli factors $[-i(u^a - \overline{u}^a)]$ will be ignored for brevity. It is then possible to observe the following equivalences.

- If the wrapped 4-cycle is extended in the bulk and the warp factor does not vary over the transverse space, then one can observe the identities

$$\omega_w = \int_{Y_6} e^{-4A}\Omega\wedge\overline{\Omega} = \ell_{(0)}^{\mathrm{T}^2}\int_{\Sigma_4} e^{-4A}\eta\wedge\overline{\eta}$$

  and

$$\int_{Y_6} e^{-4A}\chi_\alpha\wedge\overline{\chi}_\beta = \ell_{(0)}^{\mathrm{T}^2}\left[\delta_\alpha^i\delta_\beta^j\int_{\Sigma_4} e^{-4A}\zeta_i\wedge\overline{\zeta}_j - \delta_\alpha^\vartheta\delta_\beta^\vartheta\int_{\Sigma_4} e^{-4A}\eta\wedge\overline{\eta}\right].$$

  This implies that the complex-structure moduli metric can be written as

$$\hat{K}_{\alpha\overline{\beta}} = -\frac{1}{\omega_w}\int_{Y_6} e^{-4A}\chi_\alpha\wedge\overline{\chi}_\beta = \delta_\alpha^i\delta_\beta^j\,\hat{K}_{i\overline{j}} + \delta_\alpha^\vartheta\delta_{\overline{\beta}}^{\overline{\vartheta}},$$

  with the definitions

$$\hat{K}_{i\overline{j}} = -\frac{1}{\omega_w^{\Sigma_4}}\int_{\Sigma_4} e^{-4A}\zeta_i\wedge\overline{\zeta}_j, \qquad \omega_w^{\Sigma_4} = \int_{\Sigma_4} e^{-4A}\eta\wedge\overline{\eta}.$$

- In a setup with the wrapped 4-cycle being localised at the tip of a warped throat, i.e. with the warp factor varying only along the 2-torus, the analysis of the complex-structure moduli is also easy. Then, one can observe the identities

$$\omega_w = \int_{Y_6} e^{-4A}\Omega\wedge\overline{\Omega} = \ell_w^{\mathrm{T}^2}\int_{\Sigma_4}\eta\wedge\overline{\eta}$$

  and

$$\int_{Y_6} e^{-4A}\chi_\alpha\wedge\overline{\chi}_\beta = \ell_w^{\mathrm{T}^2}\left[\delta_\alpha^i\delta_\beta^j\int_{\Sigma_4}\zeta_i\wedge\overline{\zeta}_j - \delta_\alpha^\vartheta\delta_\beta^\vartheta\int_{\Sigma_4}\eta\wedge\overline{\eta}\right]$$

  so that the warped version of the complex-structure moduli metric is the same as the unwarped one, i.e.

$$\hat{K}_{\alpha\overline{\beta}} = -\frac{1}{\omega_w}\int_{Y_6} e^{-4A}\chi_\alpha\wedge\overline{\chi}_\beta = \delta_\alpha^i\delta_\beta^j\,\hat{K}_{i\overline{j}}^{(0)} + \delta_\alpha^\vartheta\delta_{\overline{\beta}}^{\overline{\vartheta}},$$

with the definitions

$$\hat{K}^{(0)}_{i\bar{j}} = -\frac{1}{\omega^{\Sigma_4}_{(0)}} \int_{\Sigma_4} \zeta_i \wedge \overline{\zeta}_j, \qquad \omega^{\Sigma_4}_{(0)} = \int_{\Sigma_4} \eta \wedge \overline{\eta}.$$

As an example to see this at work, one can expand a $(2,1)$-flux for a warp factor not varying over the transverse space. By defining the $(2,0)$-form g_2 via the identification $G_3 = g_2 \wedge \mathrm{d}\overline{w}^3$, given the 3-form expansion $e^{4A} G_3 = -\hat{K}^{\vartheta\overline{\vartheta}} \chi_\vartheta \int_{Y_6} G_3 \wedge \overline{\chi}_\vartheta / \omega_w$, one finds the same expansion that is used in the main text, i.e.

$$e^{4A} g_2 = \frac{1}{\omega^{\Sigma_4}_w} \eta \int_{\Sigma_4} g_2 \wedge \overline{\eta}.$$

C SOFT TERMS FOR LINEAR AND NON-LINEAR SUPERSYMMETRY

This section outlines the structure of the $N_4 = 1$ low-energy effective theories of type IIB compactifications with hidden-sector supersymmetry breaking: first it reviews the well-known results for standard multiplets, then it discusses the modifications that occur in the presence of constrained superfields.

C.1 Classification of Superfields in Type IIB Low-Energy Supergravity

A convenient way to study the low-energy effective $N_4 = 1$ theory of type IIB Calabi-Yau orientifold compactifications starts from observing that the degrees of freedom of the model are divided in three groups.

- Chiral superfields ϕ^M that are gauge-neutral and may acquire a non-zero expectation value and/or a non-zero F-term. These constitute the hidden sector responsible for the breaking of supersymmetry and typically correspond to the closed-string moduli but may also include open-string fields.
- Chiral superfields φ^i that, in order to preserve the gauge symmetries, necessarily have vanishing vacuum expectation values and F-terms, meaning they do not directly break supersymmetry either. These are typically open-string degrees of freedom and constitute the matter sector.
- Vector multiplets W^A which come from both the closed- and the open-string sectors and provide both hidden and observable gauge sectors.

In the main text, the breaking of supersymmetry is described as an F-term breaking, so the vector superfields play quite a marginal role. Also, the terms in the action with a number n of φ^i-fields correspond to order-n couplings as these have zero vacuum expectation values, which motivates the expansion of their theory around the vacuum defined by the fields ϕ^M.

From the expansion of the F-term potential, one can compute the couplings of the theory for all the chiral multiplets in the theory. To start, it is convenient to express the total Kähler potential K and the total superpotential W of the theory in the form

$$K = \hat{K}(\phi, \overline{\phi}) + Z_{i\overline{j}}(\phi, \overline{\phi})\, \varphi^i \overline{\varphi}^j + \frac{1}{2} \left[H_{ij}(\phi, \overline{\phi}) \varphi^i \varphi^j + \overline{H}_{\overline{i}\overline{j}}(\phi, \overline{\phi}) \overline{\varphi}^i \overline{\varphi}^j \right], \tag{C.1.1a}$$

$$W = \hat{W}(\phi) + \frac{1}{2}\,\tilde{\mu}_{ij}(\phi)\,\varphi^i\varphi^j + \frac{1}{3}\,\tilde{Y}_{ijk}(\phi)\,\varphi^i\varphi^j\varphi^k, \tag{C.1.1b}$$

along with the gauge kinetic functions

$$f_{AB} = f_{AB}(\phi), \tag{C.1.2}$$

where the Kähler potential $\hat{K}$ and the superpotential $\hat{W}$ describe the pure supersymmetry-breaking hidden sector, while the gauge kinetic functions f_{AB} and the expansion parameters $Z_{i\bar{j}}$, H_{ij}, $\tilde{\mu}_{ij}$ and $\tilde{Y}_{ijk}$ describe their couplings to the fluctuations φ^i. All matter couplings are assumed to be symmetric in their φ^i-indices and the gauge kinetic functions are always assumed to be block-diagonal. Notice that the Kähler potential is expanded including only the relevant renormalisable terms, whereas the superpotential only contains terms that lead up to quartic scalar interactions.

Then, from an analysis of the general $N_4 = 1$ supergravity action [314] for the theory (C.1.1a, C.1.1b) and (C.1.2), one finds the standard low-energy effective component action for the supersymmetry-breaking hidden sector ϕ^M and just a few relevant couplings involving the matter sector φ^i. In detail, denoting all the chiral multiplets of the theory with the indices $I = M, i$, one can simply insert the potentials in eqs. (C.1.1a, C.1.1b) into the F-term scalar potential

$$V_F = K_{I\overline{J}} F^I \overline{F}^J - 3\kappa_4^2\, \mathrm{e}^{\kappa_4^2 K}\, W\overline{W},$$

where the F-terms are fixed by their algebraic field equations to be $F^I = \mathrm{e}^{\kappa_4^2 K/2}\, K^{I\overline{J}} \nabla_{\overline{J}}\overline{W}$, with $\nabla_I W = \partial_I W + (\kappa_4^2 \partial_I K) W$.[C.1] Fermionic interactions can be discussed in a similar way, and a similar analysis applies for the gauge sectors in eq. (C.1.2). A spontaneous breaking of supersymmetry taking place in the hidden sector is also transmitted to the matter sector with the emergence of mass splittings and certain softly non-supersymmetric couplings.

C.2 Theories with Linear Supersymmetry

If all the fields realise supersymmetry linearly, then all the degrees of freedom are encoded within standard chiral and vector superfields and the expansions are lengthy but straightforward. This subsection summarises the results of refs. [232, 315, 316].

- All the hatted quantities represent the pure ϕ^M-field terms generated by the Kähler and superpotential $\hat{K}$ and $\hat{W}$, namely the F-term scalar potential

  $$\hat{V}_F = \mathrm{e}^{\kappa_4^2 \hat{K}}\, (\hat{K}^{M\overline{N}} \hat{\nabla}_M \hat{W} \hat{\overline{\nabla}}_{\overline{N}} \hat{\overline{W}} - 3\kappa_4^2\, \hat{W}\hat{\overline{W}}),$$

  with the Kähler-covariant derivative $\hat{\nabla}_M \hat{W} = \partial_M \hat{W} + (\kappa_4^2 \partial_M \hat{K})\hat{W}$, the auxilary fields

  $$\hat{F}^M = \mathrm{e}^{\kappa_4^2 \hat{K}/2}\, \hat{K}^{M\overline{N}}\, \hat{\overline{\nabla}}_{\overline{N}} \hat{\overline{W}}$$

[C.1] The matter sector is always expressed in standard mass units, i.e. $[\varphi^i] = \mathrm{M}$, whereas the hidden sector may be in standard units or dimensionless, with $[\phi^M] = 1$. As usual, the Kähler and superpotential units are $[K] = \mathrm{M}^2$ and $[W] = \mathrm{M}^3$. If all the fields are in standard units, the remaining mass dimensions are $[K_{I\overline{J}}] = 1$ and $[F^I] = \mathrm{M}^2$. For dimensionless hidden sector fields, units are $[K_{M\overline{N}}] = \mathrm{M}^2$ and $[F^M] = \mathrm{M}$.

and the gravitino mass

$$\hat{m}_{3/2} = \mathrm{e}^{\kappa_4^2 \hat{K}/2} \kappa_4^2 \hat{W}.$$

As explained above, the pure supersymmetry-breaking hidden-sector effective theory is the same independently of the matter sector. In particular, the F-term scalar potential $\hat{V}_F$ generically sets the supersymmetry-breaking scale at the order of magnitude $m_{\mathrm{SUSY}} \sim [\hat{K}_{M\overline{N}} \hat{F}^M \hat{\overline{F}}^N]^{1/4} \sim [\hat{m}_{3/2} m_P]^{1/2}$.

- As far as the bosonic interactions are concerned, one can see that the theory generates a low-energy theory described by the Lagrangian

$$\mathcal{L}_{\varphi\text{-bosons}} = -Z_{i\overline{j}}\, \partial_\mu \varphi^i \partial^\mu \overline{\varphi}^j - V_{\mathrm{susy}} - V_{\mathrm{soft}},$$

where V_{susy} and V_{soft} are the φ^i-sector supersymmetric and soft supersymmetry-breaking potentials, respectively, given by

$$V_{\mathrm{susy}} = \frac{1}{2}\, D^2 + Z^{i\overline{j}}\, \partial_i W_{\mathrm{susy}} \partial_{\overline{j}} \overline{W}_{\mathrm{susy}}, \tag{C.2.1a}$$

$$V_{\mathrm{soft}} = m^2_{i\overline{j},\,\mathrm{soft}}\, \varphi^i \overline{\varphi}^j + \left(\frac{1}{2}\, B_{ij}\, \varphi^i \varphi^j + \frac{1}{3}\, A_{ijk}\, \varphi^i \varphi^j \varphi^k + \text{c.c.} \right). \tag{C.2.1b}$$

In detail, one can conveniently define the effective superpotential as

$$W_{\mathrm{susy}} = \frac{1}{2}\, \mu_{ij}\, \varphi^i \varphi^j + \frac{1}{3}\, Y_{ijk}\, \varphi^i \varphi^j \varphi^k,$$

where the effective supersymmetric couplings read

$$\mu_{ij} = \mathrm{e}^{\kappa_4^2 \hat{K}/2}\, \tilde{\mu}_{ij} + \hat{m}_{3/2} H_{ij} - \hat{\overline{F}}^N \partial_{\overline{N}} H_{ij}, \tag{C.2.2a}$$

$$Y_{ijk} = \mathrm{e}^{\kappa_4^2 \hat{K}/2}\, \tilde{Y}_{ijk}. \tag{C.2.2b}$$

In particular this generates the supersymmetric masses

$$m^2_{i\overline{j}} = Z^{k\overline{l}}\, \mu_{ik} \overline{\mu}_{\overline{j}\overline{l}} \tag{C.2.3}$$

as well as supersymmetric trilinear and supersymmetric quartic couplings. Another supersymmetric term is the D-term potential determined by

$$D = -g\, Z_{i\overline{j}}\, \varphi^i \overline{\varphi}^j,$$

with the gauge coupling being

$$g^{-2} = \frac{1}{2}\, (f + \overline{f}). \tag{C.2.4}$$

Second, one finds the soft supersymmetry-breaking terms

$$m^2_{i\overline{j},\,\mathrm{soft}} = (\hat{m}_{3/2} \hat{\overline{m}}_{3/2} + \kappa_4^2 \hat{V}_F)\, Z_{i\overline{j}} - \hat{F}^M \hat{\overline{F}}^N\, R_{M\overline{N}i\overline{j}}, \tag{C.2.5a}$$

$$\begin{aligned} B_{ij} = {} & (2\, \hat{m}_{3/2} \hat{\overline{m}}_{3/2} + \kappa_4^2 \hat{V}_F)\, H_{ij} - \hat{\overline{m}}_{3/2}\, \hat{\overline{F}}^M\, \partial_{\overline{M}} H_{ij} + \hat{m}_{3/2}\, \hat{F}^M \hat{\nabla}_M H_{ij} \\ & - \hat{F}^M \hat{\overline{F}}^N\, \hat{\nabla}_M \partial_{\overline{N}} H_{ij} - \mathrm{e}^{\kappa_4^2 \hat{K}/2}\, \tilde{\mu}_{ij}\, \hat{\overline{m}}_{3/2} + \hat{F}^M \hat{\nabla}_M (\mathrm{e}^{\kappa_4^2 \hat{K}/2} \tilde{\mu}_{ij}), \end{aligned} \tag{C.2.5b}$$

$$A_{ijk} = \hat{F}^M \hat{\nabla}_M Y_{ijk}, \tag{C.2.5c}$$

where, given the Levi-Civita connection of the Kähler metric $Z_{i\bar{j}}$, i.e. $\Gamma^j_{Mi} = Z^{j\bar{k}} \partial_M Z_{i\bar{k}}$, the Riemann tensor reads

$$R_{M\overline{N}i\bar{j}} = \partial_M \partial_{\overline{N}} Z_{i\bar{j}} - \Gamma^k_{Mi}\, Z_{k\bar{l}}\, \overline{\Gamma}^{\bar{l}}_{\overline{N}\bar{j}},$$

while the Kähler-covariant derivatives are

$$\hat{\nabla}_M (\mathrm{e}^{\kappa_4^2 \hat{K}/2} \tilde{\mu}_{ij}) = \partial_M (\mathrm{e}^{\kappa_4^2 \hat{K}/2} \tilde{\mu}_{ij}) + \frac{1}{2}\, \kappa_4^2 \hat{K}_M\, \mathrm{e}^{\kappa_4^2 \hat{K}/2} \tilde{\mu}_{ij} - 2\, \Gamma^k_{Mi}\, \mathrm{e}^{\kappa_4^2 \hat{K}/2} \tilde{\mu}_{kj},$$
$$\hat{\nabla}_M Y_{ijk} = \partial_M Y_{ijk} + \frac{1}{2}\, \kappa_4^2 \hat{K}_M\, Y_{ijk} - 3\, \Gamma^l_{Mi}\, Y_{ljk},$$

along with $\hat{\nabla}_M H_{ij} = \partial_M H_{ij} - 2\, \Gamma^k_{Mi}\, H_{kj}$ and $\hat{\nabla}_M H_{ij,\overline{N}} = \partial_M H_{ij,\overline{N}} - 2\, \Gamma^k_{Mi}\, H_{kj,\overline{N}}$. Unless there are further suppressions due to a systematic cancellation, the order of magnitude of the canonically normalised matter soft-breaking terms is set by the scale $m_{\mathrm{soft}} \sim m^2_{\mathrm{SUSY}}/m_P \sim \hat{m}_{3/2}$.

- As far as fermionic interactions are concerned, the relevant terms are the ψ^i-field fermionic masses m^{f}_{ij} and Yukawa couplings y_{ijk} from the supersymmetric Lagrangian

$$\mathcal{L}_{\psi\text{-fermions}} = -Z_{i\bar{j}}\, \overline{\psi}^j \overline{\sigma}^\mu \partial_\mu \psi^i - \left(\frac{1}{2}\, m^{\mathrm{f}}_{ij}\, \psi^i \psi^k + \frac{1}{3}\, y_{ijk}\, \varphi^i \psi^j \psi^k + \text{c.c.} \right),$$

which turn out to be

$$m^{\mathrm{f}}_{ij} = \mu_{ij}, \tag{C.2.6a}$$
$$y_{ijk} = Y_{ijk}. \tag{C.2.6b}$$

Also, the supersymmetry-breaking gaugino masses read

$$m_{1/2} = \hat{F}^M \partial_M \ln(f + \overline{f}). \tag{C.2.7}$$

C.3 Theories with Both Linear and Non-Linear Supersymmetry

If the theory also contains fields that realise supersymmetry non-linearly, then it is necessary to describe such degrees of freedom using constrained supermultiplets. This is the case for instance of type IIB orientifold models with anti-D3-branes.

Non-linearly realised supersymmetry comes in by means of a nilpotent chiral superfield X, whose scalar is constrained to be $\varphi^X = \psi^X \psi^X / 2F^X$ by the nilpotency condition $X^2 = 0$: such a multiplet has a non-zero F-term and therefore must be included in the supersymmetry-breaking hidden sector. Other fields may realise supersymmetry non-linearly due to similar constraints with similar solutions, but usually they do not have non-zero F-terms and thus are not in this sector. Anyway, for all such constrained multiplets, there are two distinct scenarios.

- If the constraint does not fix the F-term of the multiplet, the usual supergravity expansions of subsection C.2 still hold and the constraint only fixes either its bosonic or fermionic dynamical degrees of freedom in the final action. In the unitary gauge the fixed components vanish.

- If the constraint also fixes the F-term, then the expansions of subsection C.2 do not hold anymore since they are derived by expanding the F-term too. If the fields φ^i correspond to chiral multiplets without independent spinor and auxiliary fields, then the calculation proceeds as follows:

  - in principle, the full F-term potential is $V_F = K_{I\overline{J}} F^I \overline{F}^J - 3\kappa_4^2\, \mathrm{e}^{\kappa_4^2 K}\, W\overline{W}$, with the auxiliary fields given by the well-known solutions to their algebraic field equations $\overline{F}^J = \mathrm{e}^{\kappa_4^2 K/2} K^{I\overline{J}} \nabla_I W$;
  - however, the constraints on the φ^i-multiplet auxiliary fields make them purely fermionic terms before algebraically fixing them, so that the actual F-term potential is just $V_F = K_{M\overline{N}} \dot{F}^M \dot{\overline{F}}^N - 3\kappa_4^2\, \mathrm{e}^{\kappa_4^2 K}\, W\overline{W}$, with $\dot{\overline{F}}^N = \mathrm{e}^{\kappa_4^2 K/2} K^{M\overline{N}} \nabla_M W$.

By performing an expansion as in equations (C.1.1a, C.1.1b), one can show that the scalar potential for the fields φ^i is of the form

$$V = m^2_{i\overline{j}} \varphi^i \overline{\varphi}^j + m^2_{i\overline{j},\,\mathrm{soft}}\, \varphi^i \overline{\varphi}^j + \left[\frac{1}{2}\, B_{ij}\, \varphi^i \varphi^j + \frac{1}{3}\, A_{ijk}\, \varphi^i \varphi^j \varphi^k + \mathrm{c.c.}\right].$$

Obviously there is no distinction between supersymmetric and supersymmetry-breaking terms, but the notation is meant to emphasise the differences with respect to the standard case. In particular, the two mass contributions read

$$m^2_{i\overline{j}} = 2 Z^{l\overline{k}}\, \hat{\overline{F}}^N \hat{F}^M H_{il,\overline{N}} \overline{H}_{\overline{j}\overline{k},M}, \tag{C.3.1a}$$

$$\begin{aligned} m^2_{i\overline{j},\,\mathrm{soft}} = \kappa_4^2 \hat{V}_F Z_{i\overline{j}} - \hat{F}^M \hat{\overline{F}}^N \big[Z_{i\overline{j},M\overline{N}} - 2\,\Gamma^k_{Mi}\, Z_{k\overline{l}}\, \overline{\Gamma}^{\overline{l}}_{\overline{N}\overline{j}}\big] \\ + \big[\hat{m}_{3/2} \hat{F}^M Z_{i\overline{j},M} + \hat{\overline{m}}_{3/2} \hat{\overline{F}}^N Z_{i\overline{j},\overline{N}}\big]. \end{aligned} \tag{C.3.1b}$$

Instead the bilinear B-coupling reads

$$\begin{aligned} B_{ij} = \kappa_4^2 \hat{V}_F H_{ij} + \mathrm{e}^{\kappa_4^2 \hat{K}/2} \hat{F}^M \hat{\nabla}_M \tilde{\mu}_{ij} + \hat{\overline{m}}_{3/2} \hat{\overline{F}}^N H_{ij,\overline{N}} + \hat{m}_{3/2} \hat{F}^M H_{ij,M} \\ - \hat{F}^M \hat{\overline{F}}^N \left(H_{ij,M\overline{N}} - 4\,\Gamma^l_{Mi} H_{lj,\overline{N}}\right) - 3\,\mathrm{e}^{\kappa_4^2 \hat{K}/2}\, \hat{\overline{m}}_{3/2} \tilde{\mu}_{ij}. \end{aligned} \tag{C.3.2}$$

As for the trilinear terms, one only has the would-be supersymmetry-breaking coupling, namely

$$A_{ijk} = \mathrm{e}^{\kappa_4^2 \hat{K}/2} \big[\hat{F}^M \hat{\nabla}_M \tilde{Y}_{ijk} - 3\, \hat{\overline{m}}_{3/2} \tilde{Y}_{ijk}\big]. \tag{C.3.3}$$

The covariant derivatives are defined as below. Notably, although the structure of all the coupling terms is different, one can observe that the theory is still invariant under the usual Kähler transformations as all the terms are individually covariant. The case where the scalar and the F-term components of a multiplet are constrained may be discussed in a similar fashion. It is not encountered in the main text and thus left for future study.

C.4 Computational Details

This subsection explains the details of how to compute the soft-breaking terms up to the cubic scalar potential; quartic interactions would be possible to determine analogously but the calculations become extremely unwieldy.

As outlined above, the method to determine the form of the soft-breaking scalar terms simply consists in computing the F-term scalar potential around the vacuum determined by the pure hidden sector. This means that one should express all the relevant operators stemming from eqs. (C.1.1a, C.1.1b) as φ-expansions of the hidden operators. In the following, this will be denoted indicating as 'φ' any contribution of the form φ^i and/or $\overline{\varphi}^i$. The Kähler potential in eq. (C.1.1a) defines a Kähler metric $K_{I\overline{J}} = \partial_I \partial_{\overline{J}} K$ that can be visualised as a block matrix, together with its inverse such that $K_{I\overline{P}} K^{J\overline{P}} = \delta_I^J$, as

$$K_{I\overline{J}} = \begin{pmatrix} K_{M\overline{N}} & K_{M\overline{j}} \\ K_{i\overline{N}} & K_{i\overline{j}} \end{pmatrix}, \qquad K^{I\overline{J}} = \begin{pmatrix} K^{M\overline{N}} & K^{i\overline{N}} \\ K^{M\overline{j}} & K^{i\overline{j}} \end{pmatrix},$$

where the blocks are

$$K_{M\overline{N}} = \hat{K}_{M\overline{N}} + Z_{i\overline{j},M\overline{N}} \varphi^i \overline{\varphi}^j + \frac{1}{2}\big(H_{ij,M\overline{N}} \varphi^i \varphi^j + \overline{H}_{\overline{i}\overline{j},M\overline{N}} \overline{\varphi}^i \overline{\varphi}^j\big), \tag{C.4.1a}$$

$$K_{M\overline{j}} = Z_{i\overline{j},M} \varphi^i + \overline{H}_{\overline{i}\overline{j},M} \overline{\varphi}^i, \tag{C.4.1b}$$

$$K_{i\overline{N}} = Z_{i\overline{j},\overline{N}} \overline{\varphi}^j + H_{ij,\overline{N}} \varphi^j, \tag{C.4.1c}$$

$$K_{i\overline{j}} = Z_{i\overline{j}}, \tag{C.4.1d}$$

and[C.2]

$$\begin{aligned} K^{M\overline{N}} = {} & \hat{K}^{M\overline{N}} - \hat{K}^{M\overline{Q}} \Big[Z_{i\overline{j},P\overline{Q}} \varphi^i \overline{\varphi}^j + \frac{1}{2}\big(H_{ij,P\overline{Q}} \varphi^i \varphi^j + \overline{H}_{\overline{i}\overline{j},P\overline{Q}} \overline{\varphi}^i \overline{\varphi}^j\big)\Big] \hat{K}^{P\overline{N}} \\ & + \hat{K}^{M\overline{Q}} \big(Z_{l\overline{j},P} \varphi^l + \overline{H}_{\overline{l}\overline{j},P} \overline{\varphi}^l\big) Z^{i\overline{j}} \big(Z_{i\overline{k},\overline{Q}} \overline{\varphi}^k + H_{ik,\overline{Q}} \varphi^k\big) \hat{K}^{P\overline{N}} + O(\varphi)^4, \end{aligned} \tag{C.4.2a}$$

$$K^{M\overline{j}} = -Z^{l\overline{j}} \big[Z_{l\overline{k},\overline{Q}} \overline{\varphi}^k + H_{lk,\overline{Q}} \varphi^k \big] \hat{K}^{M\overline{Q}} + O(\varphi)^3, \tag{C.4.2b}$$

$$K^{i\overline{N}} = -\hat{K}^{P\overline{N}} \big[Z_{k\overline{l},P} \varphi^k + \overline{H}_{\overline{k}\overline{l},P} \overline{\varphi}^k \big] Z^{i\overline{l}} + O(\varphi)^3, \tag{C.4.2c}$$

$$K^{i\overline{j}} = Z^{i\overline{j}} + O(\varphi)^2. \tag{C.4.2d}$$

In these expressions, only the orders in φ that are going to be necessary later on have been written explicitly. If needed, they can be determined straightforwardly by inverting the block Kähler metric up to the desired order. The superpotential in eq. (C.1.1b) induces the Kähler-covariant derivatives

$$\begin{aligned} \nabla_M W = {} & \hat{\nabla}_M \hat{W} + \frac{1}{2} \hat{\nabla}_M \tilde{\mu}_{ij} \varphi^i \varphi^j + \frac{1}{3} \hat{\nabla}_M \tilde{Y}_{ijk} \varphi^i \varphi^j \varphi^k \\ & + \Big[Z_{i\overline{j},M} \varphi^i \overline{\varphi}^j + \frac{1}{2}\big(H_{ij,M} \varphi^i \varphi^j + \overline{H}_{\overline{i}\overline{j},M} \overline{\varphi}^i \overline{\varphi}^j\big)\Big] \kappa_4^2 \hat{W} + O(\varphi)^4, \end{aligned} \tag{C.4.3a}$$

[C.2] Notice that the term $K^{M\overline{N}}$ does not receive $O(\varphi)^3$-corrections since these are also ignored in the original reference Kähler potential K and therefore in the Kähler metric $K_{M\overline{N}}$.

$$\nabla_i W = \tilde{\mu}_{ij}\varphi^j + \tilde{Y}_{ijk}\varphi^j\varphi^k + \big[Z_{i\bar{j}}\overline{\varphi}^j + H_{ij}\varphi^j\big]\kappa_4^2\hat{W} + O(\varphi)^3, \tag{C.4.3b}$$

where the Kähler-covariant derivatives are defined as $\hat{\nabla}_M\tilde{\mu}_{ij} = \partial_M\tilde{\mu}_{ij} + (\kappa_4^2\hat{K}_M)\tilde{\mu}_{ij}$ and $\hat{\nabla}_M\tilde{Y}_{ijk} = \partial_M\tilde{Y}_{ijk} + (\kappa_4^2\hat{K}_M)\tilde{Y}_{ijk}$. Auxiliary fields are then determined by employing the definitions above. One finds

$$\begin{aligned}
\overline{F}^N = \hat{\overline{F}}^N&\Big[1 + \frac{\kappa_4^2}{2}Z_{i\bar{j}}\varphi^i\overline{\varphi}^j + \frac{\kappa_4^2}{4}\big(H_{ij}\varphi^i\varphi^j + \overline{H}_{\bar{i}\bar{j}}\overline{\varphi}^i\overline{\varphi}^j\big)\Big]\\
&+ \frac{1}{2}\mathrm{e}^{\kappa_4^2\hat{K}/2}\hat{K}^{P\overline{N}}\hat{\nabla}_P\tilde{\mu}_{ij}\varphi^i\varphi^j + \frac{1}{3}\mathrm{e}^{\kappa_4^2\hat{K}/2}\hat{K}^{P\overline{N}}\hat{\nabla}_P\tilde{Y}_{ijk}\varphi^i\varphi^j\varphi^k\\
&+ \mathrm{e}^{\kappa_4^2\hat{K}/2}\hat{K}^{P\overline{N}}\kappa_4^2\hat{W}\Big[Z_{i\bar{j},P}\varphi^i\overline{\varphi}^j + \frac{1}{2}\big(H_{ij,P}\varphi^i\varphi^j + \overline{H}_{\bar{i}\bar{j},P}\overline{\varphi}^i\overline{\varphi}^j\big)\Big]\\
&- \hat{K}^{P\overline{N}}\hat{\overline{F}}^Q\Big[Z_{i\bar{j},P\overline{Q}}\varphi^i\overline{\varphi}^j + \frac{1}{2}\big(H_{ij,P\overline{Q}}\varphi^i\varphi^j + \overline{H}_{\bar{i}\bar{j},P\overline{Q}}\overline{\varphi}^i\overline{\varphi}^j\big)\Big]\\
&+ \hat{K}^{P\overline{N}}\hat{\overline{F}}^Q\big[Z_{l\bar{j},P}\varphi^l + \overline{H}_{\bar{l}\bar{j},P}\overline{\varphi}^l\big]Z^{i\bar{j}}\big[Z_{i\bar{k},\overline{Q}}\overline{\varphi}^k + H_{ik,\overline{Q}}\varphi^k\big]\\
&- \mathrm{e}^{\kappa_4^2\hat{K}/2}\hat{K}^{P\overline{N}}\big[Z_{k\bar{l},P}\varphi^k + \overline{H}_{\bar{k}\bar{l},P}\overline{\varphi}^k\big]Z^{i\bar{l}}\tilde{\mu}_{ij}\varphi^j\\
&- \mathrm{e}^{\kappa_4^2\hat{K}/2}\hat{K}^{P\overline{N}}\big[Z_{k\bar{l},P}\varphi^k + \overline{H}_{\bar{k}\bar{l},P}\overline{\varphi}^k\big]Z^{i\bar{l}}\tilde{Y}_{ipq}\varphi^p\varphi^q\\
&- \mathrm{e}^{\kappa_4^2\hat{K}/2}\hat{K}^{P\overline{N}}\kappa_4^2\hat{W}\big[Z_{k\bar{l},P}\varphi^k + \overline{H}_{\bar{k}\bar{l},P}\overline{\varphi}^k\big]Z^{i\bar{l}}\big[Z_{i\bar{j}}\overline{\varphi}^j + H_{ij}\varphi^j\big] + O(\varphi)^4,
\end{aligned} \tag{C.4.4a}$$

$$\begin{aligned}
\overline{F}^j = &- \hat{\overline{F}}^P Z^{l\bar{j}}\big[Z_{l\bar{k},\overline{P}}\overline{\varphi}^k + H_{lk,\overline{P}}\varphi^k\big]\\
&+ \mathrm{e}^{\kappa_4^2\hat{K}/2}Z^{i\bar{j}}\tilde{\mu}_{ik}\varphi^k + \mathrm{e}^{\kappa_4^2\hat{K}/2}Z^{i\bar{j}}\tilde{Y}_{ikl}\varphi^k\varphi^l\\
&+ \mathrm{e}^{\kappa_4^2\hat{K}/2}\kappa_4^2\hat{W}Z^{i\bar{j}}\big[Z_{i\bar{k}}\overline{\varphi}^k + H_{ik}\varphi^k\big] + O(\varphi)^3.
\end{aligned} \tag{C.4.4b}$$

Moreover, one has

$$\begin{aligned}
K_{M\overline{N}}\overline{F}^N + K_{M\bar{j}}\overline{F}^j = \hat{\overline{F}}_M&\Big[1 + \frac{\kappa_4^2}{2}Z_{i\bar{j}}\varphi^i\overline{\varphi}^j + \frac{\kappa_4^2}{4}\big(H_{ij}\varphi^i\varphi^j + \overline{H}_{\bar{i}\bar{j}}\overline{\varphi}^i\overline{\varphi}^j\big)\Big]\\
&+ \frac{1}{2}\mathrm{e}^{\kappa_4^2\hat{K}/2}\hat{\nabla}_M\tilde{\mu}_{ij}\varphi^i\varphi^j + \frac{1}{3}\mathrm{e}^{\kappa_4^2\hat{K}/2}\hat{\nabla}_M\tilde{Y}_{ijk}\varphi^i\varphi^j\varphi^k\\
&+ \mathrm{e}^{\kappa_4^2\hat{K}/2}\kappa_4^2\hat{W}\Big[Z_{i\bar{j},M}\varphi^i\overline{\varphi}^j + \frac{1}{2}\big(H_{ij,M}\varphi^i\varphi^j + \overline{H}_{\bar{i}\bar{j},M}\overline{\varphi}^i\overline{\varphi}^j\big)\Big] + O(\varphi)^4,
\end{aligned} \tag{C.4.5a}$$

$$\begin{aligned}
K_{i\overline{N}}\overline{F}^N + K_{i\bar{j}}\overline{F}^j = &\,\mathrm{e}^{\kappa_4^2\hat{K}/2}\tilde{\mu}_{ij}\varphi^j + \mathrm{e}^{\kappa_4^2\hat{K}/2}\tilde{Y}_{ijk}\varphi^j\varphi^k\\
&+ \mathrm{e}^{\kappa_4^2\hat{K}/2}\kappa_4^2\hat{W}\big[Z_{i\bar{j}}\overline{\varphi}^j + H_{ij}\varphi^j\big] + O(\varphi)^3.
\end{aligned} \tag{C.4.5b}$$

These expressions eventually determine the scalar potential expansion up to cubic scalar interactions in the matter sector for theories with only linearly-realised supersymmetry. In fact, working out the φ-expansion of the F-term scalar potential

$$V_F = K_{M\overline{N}}F^M\overline{F}^N + K_{M\bar{j}}F^M\overline{F}^j + K_{i\overline{N}}F^i\overline{F}^N + K_{i\bar{j}}F^i\overline{F}^j - 3\kappa_4^2\,\mathrm{e}^{\kappa_4^2 K}W\overline{W},$$

one finds exactly the scalar matter couplings that have been spelled out in subsection C.2. In the case in which the matter-sector F-terms are constrained to be fermionic, one needs to use

$$\begin{aligned}
\dot{\overline{F}}^N &= \hat{\overline{F}}^N\Big[1 + \frac{\kappa_4^2}{2} Z_{i\bar{j}}\varphi^i\overline{\varphi}^j + \frac{\kappa_4^2}{4}\big(H_{ij}\varphi^i\varphi^j + \overline{H}_{\overline{i}\overline{j}}\overline{\varphi}^i\overline{\varphi}^j\big)\Big] \\
&\quad + \frac{1}{2}\mathrm{e}^{\kappa_4^2\hat{K}/2}\hat{K}^{P\overline{N}}\hat{\nabla}_P\tilde{\mu}_{ij}\varphi^i\varphi^j + \frac{1}{3}\mathrm{e}^{\kappa_4^2\hat{K}/2}\hat{K}^{P\overline{N}}\hat{\nabla}_P\tilde{Y}_{ijk}\varphi^i\varphi^j\varphi^k \\
&\quad + \mathrm{e}^{\kappa_4^2\hat{K}/2}\hat{K}^{P\overline{N}}\kappa_4^2\hat{W}\Big[Z_{i\bar{j},P}\varphi^i\overline{\varphi}^j + \frac{1}{2}\big(H_{ij,P}\varphi^i\varphi^j + \overline{H}_{\overline{i}\overline{j},P}\overline{\varphi}^i\overline{\varphi}^j\big)\Big] \\
&\quad - \hat{K}^{P\overline{N}}\hat{\overline{F}}^Q\Big[Z_{i\bar{j},P\overline{Q}}\varphi^i\overline{\varphi}^j + \frac{1}{2}\big(H_{ij,P\overline{Q}}\varphi^i\varphi^j + \overline{H}_{\overline{i}\overline{j},P\overline{Q}}\overline{\varphi}^i\overline{\varphi}^j\big)\Big] \\
&\quad + \hat{K}^{P\overline{N}}\hat{\overline{F}}^Q\big[Z_{l\bar{j},P}\varphi^l + \overline{H}_{\overline{l}\overline{j},P}\overline{\varphi}^l\big]Z^{i\bar{j}}\big[Z_{i\bar{k},\overline{Q}}\overline{\varphi}^k + H_{ik,\overline{Q}}\varphi^k\big] + O(\varphi)^4,
\end{aligned} \tag{C.4.6}$$

the scalar F-term potential being

$$V_F = K_{M\overline{N}}\dot{F}^M\dot{\overline{F}}^N - 3\kappa_4^2\,\mathrm{e}^{\kappa_4^2 K}W\overline{W}.$$

In this case, expanding the theory in the fluctuations represented by the matter sector, one finds the scalar matter couplings discussed in subsection C.3.

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