The Quarterly Journal of Mathematics
Quart. J. Math. 00 (2021), 1–31; doi:https://doi.org/10.1093/qmath/haab058

CONTACT WITH CIRCLES AND EUCLIDEAN INVARIANTS OF SMOOTH SURFACES IN $\mathbb{R}^3$

by PETER GIBLIN[†]

(*Department of Mathematical Sciences, The University of Liverpool, Liverpool L69 7ZL, UK*)

GRAHAM REEVE[‡]

(*Department of Mathematics and Computer Science, Liverpool Hope University, Liverpool L16 9JD, UK*)

and RICARDO URIBE-VARGAS[§]

(*Institut de Mathématiques de Bourgogne, UMR 5584, CNRS, Université Bourgogne Franche-Comté, Dijon F-21000, France*)

[Received 27 June 2020]

Abstract

We investigate the vertex curve, that is the set of points in the hyperbolic region of a smooth surface in real 3-space at which there is a circle in the tangent plane having at least 5-point contact with the surface. The vertex curve is related to the differential geometry of planar sections of the surface parallel to and close to the tangent planes, and to the symmetry sets of isophote curves, that is level sets of intensity in a 2-dimensional image. We investigate also the relationship of the vertex curve with the parabolic and flecnodal curves, and the evolution of the vertex curve in a generic 1-parameter family of smooth surfaces.

1. Introduction

This article is a contribution to the study of Euclidean invariants of surfaces, and generic families of surfaces, in Euclidean space $\mathbb{R}^3$. There have been many previous such studies, involving among others contact of surfaces with spheres (ridge curves, see for example [**9**, **13**, **20**]), right circular cylinders [**12**], and, as in the present article, circles. In [**4**] Bruce, following on from earlier work of [**18**], considers the contact of circles with surfaces, but the problems studied are different from ours. Another approach is given in Porteous's book [**20**, Chapter 15].

In [**10**] Diatta and the first author studied vertices and inflexions of sections of a smooth surface M in $\mathbb{R}^3$ by planes parallel to, and close to, the tangent plane T_pM at a point p. This was in the context of families of curves which have a singular member (namely the section of M by the tangent plane itself) and the behaviour of the symmetry sets of the curves in such a family. (The corresponding evolution of symmetry sets of a 1-parameter family of *smooth* plane curves was classified in [**6**].) This in turn was motivated by the fact that isophotes (lines of equal intensity) in a camera image can

[†]Corresponding author. E-mail: pjgiblin@liv.ac.uk
[‡]E-mail: reeveg@hope.ac.uk
[§]E-mail: r.uribe-vargas@u-bourgogne.fr

be regarded as level sets of a function of position in the image, namely the intensity function. The evolution of vertices on planar sections parallel to T_pM changes when p crosses a certain curve on M, first studied in [**10**], and which we call the vertex curve (V-curve) in this article. *Thus the V-curve is a Euclidean invariant bifurcation set on M.*

We first recall that the sign of the Gauss curvature K distinguishes three types of points of a smooth surface : *elliptic region* ($K>0$), *hyperbolic region* ($K<0$) and *parabolic set* ($K=0$); and that a *vertex* of a plane curve γ is a point where γ has higher order of contact than usual with its osculating circle (at least 4-point contact). *At a vertex the radius of curvature of γ is critical.*

Vertex curve. Let M be a smooth surface in $\mathbb{R}^3$. The *vertex curve*, or *V-curve*, on M is the closure of the set of points p in the hyperbolic region of M for which there exists a circle, lying in the tangent plane T_pM to M at p, having (at least) 5-point contact with M at p. Such a point p is also called a *vertex point*, or *V-point*, of M. (The V-curves in this article were called 'VT-sets', for 'vertex transition sets', in [**10**].)

At a hyperbolic point $p \in M$ the tangent plane T_pM cuts the surface along two smooth transverse branches. Thus, for a circle lying in the tangent plane T_pM, 5-point contact with M at p can be expected to mean that the circle meets one branch transversely (one of the five contacts) and the other branch at a vertex of that branch (the other four contacts); compare [**10**, Section 3.4(1)]. Whence the name V-curve.

We study the structure of the V-curve and its interactions with the parabolic and flecnodal curves for a generic smooth surface in $\mathbb{R}^3$ and investigate the changes which occur on V-curves during a generic 1-parameter deformation of the underlying surface.

The article is organized as follows. In Section 2 we give two complementary methods for measuring the contact between a surface M at $p \in M$ and a circle lying in the tangent plane to M at p. In Section 3 we show that the V-curve is smooth on the hyperbolic region of M, and in Section 3.2 we show how to distinguish between vertices which are maxima or minima of the absolute radius of curvature. In Section 4 we study the V-curve near a special parabolic point of M, namely a 'cusp of Gauss' or 'godron' (defined below), showing in Proposition 4.5 that at any 'hyperbolic cusp of Gauss' the V-curve has two smooth branches tangent to the parabolic curve (it is empty near an 'elliptic cusp of Gauss'). In Section 4.1.1 we introduce a Euclidean invariant of a cusp of Gauss, defined in two geometric ways. In Section 5 we find the interactions between the V-curve and the flecnodal curve of M; the various possibilities are illustrated in Figure 4. In Section 6 we investigate, partly experimentally, the evolution of the V-curve in a generic 1-parameter family of surfaces and finally in Section 7 we mention some further ongoing work.

1.1. *A note on genericity*

The word 'generic' occurs frequently in this article, and we pause here to give a brief explanation and some references. Although we use the term 'generic surface' it is, strictly speaking, *properties* of (smooth, and usually closed) surfaces which are generic, not surfaces themselves. A generic property P is one such that, if the surface M satisfies P, then so do all surfaces obtained from M by a sufficiently small perturbation (openness), and, if M does not satisfy P, then a suitable arbitrarily small perturbation of M does satisfy P (density). Proofs that a given property is generic use one of the standard 'transversality' results, the most basic being the transversality theorem of Thom (or Sard and Thom).

Transversality. A smooth map $f: A \to B$ between smooth manifolds is said to be *transverse to a submanifold $C \subset B$ at $a \in A$* if either $f(a) \notin C$ or the image of the tangent space to A at a under the derivative f_{*a} together with the tangent space to C span the whole tangent space to B at $f(a)$:

$$f_{*a}T_aA + T_{f(a)}C = T_{f(a)}B.$$

The map f is said to be *transverse to* C if it is transverse to C at every point of A.

In practice transversality is used to show that two submanifolds *do not meet* because they are transverse and the sum of their dimensions is less than that of the ambient space. The following is easy to prove from the implicit function theorem:

If $f : A \to B$ is transverse to C then $f^{-1}(C)$ is a smooth submanifold of A whose codimension in A is the codimension of C in B.

k**-jet extension**. The *k-jet extension* j^kf of a smooth map $f : A \to B$ is the map $j^kf : A \to J^k(A,B)$, $x \mapsto j^k_xf$, which associates to each $x \in A$ the k-jet of f at x.

Thom Transversality Theorem. *If A is a closed manifold and C is a closed submanifold of $J^k(A,B)$, then the set of maps $f : A \to B$, whose k-jet extension is transverse to C, is an open everywhere dense set in the space of all smooth maps from A to B.*

To quote [**1**, p.38] (this book is a general reference for transversality results and applications): 'This theorem means that by a small shift of a smooth map it is possible to put it into general position not only with respect to an arbitrary submanifold in the target space but also with respect to an arbitrary condition placed on the derivatives up to some finite order.'

In practice f is a local embedding of a parameterized surface M in $\mathbb{R}^3$ and C represents either (i) some geometrical structure which we wish to avoid as being 'non-generic' (C is of codimension >2) or (ii) a geometrical structure such that $f^{-1}(C)$ is a set (of the surface) which we wish to study (C is of codimension 1 or 2).

In our case, C can be a hypersurface given by a polynomial of low degree in the coefficients of some finite jet space, and then $f^{-1}(C)$ is a curve of our surface. The k-jet extension of f is a 2-dimensional surface which, for almost any f, is transverse to C. Since transversality is always stable, the above curve is a stable feature of the surface.

Naturally when global questions about surfaces are involved, such as statements about the evolution of parabolic curves in a 'generic 1-parameter family of surfaces' then the transversality arguments become more technical. Full details of this case, which we use in Section 6, are given in [**8**]. Our concern in this article is chiefly with local configurations, and in Section 6 the results are largely experimental in nature.

Other references to generic geometry proofs are [**14**, **5**] (an elementary exposition of the method) and [**3**], particularly Section 8; the main technical result is proved in [**19**].

2. Contact function and contact map

Here, we describe two alternative ways to calculate the contact between a circle and the surface M. For the majority of this article we adopt the 'standard calculation' below, in which we parameterize the circle and use a local equation $z = f(x,y)$ for M. But for some purposes in Section 6 we have found another useful approach in Section 2.2: we parameterize the surface and use two equations for the circle. Two equations are also used in [**18**], defining the circle as an intersection of two spheres, whereas we use a plane and a sphere centred on the plane. This has the advantage of uniqueness. The general theory of contact between submanifolds is contained in [17].

2.1. *Computing the contact by the contact map*

We assume M to be locally given in Monge form $z=f(x,y)$, where f and its partial derivatives f_x,f_y vanish at the origin $(0,0)$, so that the tangent plane to M at the origin is the coordinate plane $z=0$. In this method, we calculate the contact by composing a parameterization of the circle with the equation $z=f(x,y)$. Consider a circle or line through the origin in, say, the (x_1,y_1)-plane, given by

$$r(x_1^2+y_1^2)+sx_1+y_1=0, \text{ with curvature } \frac{2r}{\sqrt{1+s^2}} \text{ and centre } \left(-\frac{s}{2r},-\frac{1}{2r}\right), r\neq 0. \tag{1}$$

The fact that this can represent a line ($r=0$) will be useful later.

We shall map this circle isometrically to a circle in the tangent plane T_p at $p\in M$ by choosing an orthonormal basis for $\mathbb{R}^3$ as follows. Write $p=(x_0,y_0,f(x_0,y_0))$ and let f_x,f_y stand for the partial derivatives of f at $x=x_0, y=y_0$.

$$\mathbf{e}_1=\frac{(1+f_y^2,-f_xf_y,f_x)}{||(1+f_y^2,-f_xf_y,f_x)||},\ \mathbf{e}_2=\frac{(0,1,f_y)}{||(0,1,f_y)||},\ \mathbf{e}_3=\frac{(-f_x,-f_y,1)}{||(-f_x,-f_y,1)||}. \tag{2}$$

Thus $\mathbf{e}_1,\mathbf{e}_2$ span the tangent plane at p and $\mathbf{e}_3$ is a unit normal to M at p. For $p=(0,0)$ the three vectors form the standard basis for $\mathbb{R}^3$. We map a point (x_1,y_1) of the circle (1) to

$$(X,Y,Z)=(x_0,y_0,f(x_0,y_0))+x_1\mathbf{e}_1+y_1\mathbf{e}_2,$$

which lies on an arbitrary circle through p, lying in the tangent plane to M at p. When $x_0=y_0=0$ the map takes the circle in the (x_1,y_1)-plane identically to the same circle in the (x,y)-plane which is the tangent plane to M at the origin.

We shall parameterize the circle (1) by x_1 close to the origin in the (x_1,y_1) plane; then the *contact function* between the corresponding circle in T_p and the surface M is

$$G(x_1,x_0,y_0,r,s)=Z-f(X,Y), \tag{3}$$

where on the right-hand side y_1 is written as a function of x_1. The vertex curve is the locus of points $(x_0,y_0,f(x_0,y_0))$ for which the contact is at least five, and we shall need to find this curve close to the origin $(x_0,y_0)=(0,0)$. The contact is at least five provided the first four derivatives of G with respect to x_1 vanish at $(0,x_0,y_0,r,s)$.

An important observation is that x_1^2 is a *factor* of the function G, that is $G(0,x_0,y_0,r,s)\equiv 0$, $G_{x_1}(0,x_0,y_0,r,s)\equiv 0$. This is because the circle (1) *always* has at least 2-point contact with the surface for $x_1=0$, at the point $(x_0,y_0,f(x_0,y_0))$, since it passes through the intersection of the two curves in which the surface is met by its tangent plane (or, at a parabolic point, through the corresponding singularity of the intersection).

DEFINITION 2.1 The smooth function H determined by the equality

$$G(x_1, x_0, y_0, r, s) = x_1^2 H(x_1, x_0, y_0, r, s)$$

will be called the reduced contact function.

We can now re-interpret the conditions that the first four derivatives of G with respect to x_1 vanish at $x_1 = 0$ in terms of the function H, as follows.

$$\begin{aligned}
G_{x_1} &= 2x_1 H + x_1^2 H_{x_1}, \\
G_{x_1x_1} &= 2H + 4x_1 H_{x_1} + x_1^2 H_{x_1x_1}, \\
G_{3x_1} &= 6H_{x_1} + 6x_1 H_{x_1x_1} + x_1^2 H_{3x_1}, \\
G_{4x_1} &= 12H_{x_1x_1} + 8x_1 H_{3x_1} + x_1^2 H_{4x_1}.
\end{aligned}$$

Thus we now require $H = H_{x_1} = H_{x_1x_1} = 0$ at $x_1 = 0$, that is we consider the map

$$\begin{aligned}
\widetilde{H} \quad &: \quad (\mathbb{R}^4, 0) \to (\mathbb{R}^3, 0), \\
(x_0, y_0, r, s) \quad &\mapsto \quad (H(0, x_0, y_0, r, s), H_{x_1}(0, x_0, y_0, r, s), H_{x_1x_1}(0, x_0, y_0, r, s)).
\end{aligned} \tag{4}$$

The projection to the (x_0, y_0)-plane of $\widetilde{H}^{-1}(0,0,0)$ is the set of points on M, near the origin, at which there is a circle in the tangent plane having 5-point contact or higher with the surface.

2.2. *An alternative approach*

Instead of parameterizing the circle and using an equation $z = f(x, y)$ for the surface M we can parameterize the surface by $(x, y) \mapsto (x, y, f(x, y))$ and write down two equations for the circle. We can specify a plane, namely the tangent plane to M at a given point $P_0 = (x_0, y_0, f(x_0, y_0))$, and a sphere centred at a point of this plane and passing through P_0. This gives a contact map $\mathbb{R}^2 \to \mathbb{R}^2$, which we can reduce using contact equivalence ($\mathcal{K}$-equivalence). We shall use this method in Section 6 as it makes the direct computations much easier.

With the notation above, let (u, v, w) be a point in the tangent plane to M at P_0. The equation of this tangent plane is $G_1(x_0, y_0; x, y, z) = 0$, given by the inner product

$$G_1 = \langle (x - x_0, y - y_0, z - f(x_0, y_0)), (-f_x, -f_y, 1) \rangle = 0,$$

and the partial derivatives are evaluated at P_0. Thus $w = (u - x_0)f_x + (v - y_0)f_y + f(x_0, y_0)$. The equation of the sphere centred at (u, v, w) and passing through P_0 is $G_2(x_0, y_0, u, v; x, y, z) = 0$ where $G_2 = (x-u)^2 + (y-v)^2 + (z-w)^2 - (x_0-u)^2 - (y_0-v)^2 - (z_0-w)^2$, w being substituted as above. The intersection of this sphere with the tangent plane is the circle whose contact with M at P_0 we wish to calculate.

To calculate the contact we must parameterize M close to P_0. Thus let $(x_0 + p, y_0 + q)$ be parameters for M, where p and q are small. The contact map, with variables p, q and for fixed x_0, y_0, u, v, is then the composite of the parameterization

$$(x_0, y_0; p, q) \mapsto (x, y, z) = (x_0 + p, y_0 + q, f(x_0 + p, y_0 + q))$$

with the map

$$(x_0, y_0, u, v; x, y, z) \mapsto (G_1(x_0, y_0; x, y, z),\ G_2(x_0, y_0, u, v; x, y, z)).$$

We shall call the components of this composite map $(H_1(x_0, y_0, p, q), H_2(x_0, y_0, u, v, p, q))$. Note that when we use a polynomial approximation to f both H_1 and H_2 are *polynomial* functions.

For fixed x_0, y_0, u, v it is a map (germ) $H : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ and its $\mathcal{K}$-class is an alternative way of measuring the contact between a circle in the tangent plane to M and the surface M. The parameterization of the V-curve consists of those x_0, y_0 for which, for some u, v, this map has the contact type A_4 or higher at $p = q = 0$.

3. Vertex curve properties in the hyperbolic domain

We start with some basic background. A generic smooth surface M in $\mathbb{R}^3$ has three (possibly empty) parts: (H) an open *domain of hyperbolic points*: at such points there are two tangent lines having greater than 2-point contact with M, called *asymptotic lines*; (E) an open *domain of elliptic points*: at such points there is no such line; and (P) a smooth *curve of parabolic points*: at such points the two asymptotic lines of H have coincided to give a unique such line.

If M is generic and locally given in Monge form $z = f(x, y)$ around p, then p is hyperbolic, elliptic or parabolic if and only if the quadratic part of f, called *second fundamental form of M at p*, is respectively indefinite, definite or degenerate. The zeros of this quadratic form (for p hyperbolic or parabolic) are the asymptotic tangent lines at p.

The integral curves of the fields of asymptotic tangent lines are called *asymptotic curves*.

Left and right. Fix an orientation in $\mathbb{R}^3$. A regularly parameterized smooth space curve is said to be a *left* (*right*) *curve* on an interval if its first three derivatives at each point form a negative (resp. positive) frame. Thus a left (right) curve has negative (resp. positive) torsion and twists like a left (resp. right) screw.

FACT *At each hyperbolic point p one asymptotic curve is left and the other is right* (cf. [21]).

Proof. A point of a curve is right, left or flattening if the torsion at that point satisfies $\tau > 0, \tau < 0$ or $\tau = 0$, respectively. The Gaussian curvature K of a smooth surface is negative on the hyperbolic domain. The Beltrami–Enneper Theorem states that the torsion of the two asymptotic curves passing through a hyperbolic point with Gaussian curvature K has the values $\tau = \pm\sqrt{-K}$. So one is left and the other is right. □

The respective tangents L_ℓ, L_r, called *left* and *right asymptotic lines*, are tangent to the smooth branches of the section $M \cap T_pM$. We call them *left* and *right* branches, respectively.

This left-right distinction depends only on the orientation of $\mathbb{R}^3$, *but not of the surface.*

Left and right vertex curve. The *left* (*right*) vertex curve V_ℓ (resp. V_r) of a surface M consists of the points p for which the left (resp. right) branch of $M \cap T_pM$ has a vertex. We must clarify that

the curve formed by the 'left' (resp. 'right') vertices is not a left (resp. right) curve. We use this terminology just to associate V_ℓ and V_r to the left and right *asymptotic directions*.
We study the behaviour of the V-curve close to the parabolic curve in Section 4. For more information on the behaviour of asymptotic curves close to the parabolic curve see for example [**2**, Chapter 3], [**14**, Chapter 6].

At an elliptic point $p \in M$ the intersection with the tangent plane consists of two complex conjugate curves. In principle one can ask whether a complex circle in that tangent plane could have 5-point contact with M at p. A calculation shows that this imposes two conditions on the point p, which implies that it is only possible at isolated points of a generic surface M. The two conditions imposed on the (real) coefficients in the Monge form of the surface do not appear to have any other geometrical meaning.

3.1. *Smoothness of the vertex curve at a hyperbolic point*

We shall take a surface in local Monge form at a hyperbolic point p so that one asymptotic tangent line at p is the x-axis $y=0$ and the other one is the line $x=ay$:

$$f(x,y) = xy - ay^2 + b_0x^3 + b_1x^2y + b_2xy^2 + b_3y^3 + c_0x^4 \\ + c_1x^3y + c_2x^2y^2 + c_3xy^3 + c_4y^4 + d_0x^5 + \dots, \tag{5}$$

where, if they are needed, the degree 5 terms will have coefficients $d_0, \dots, d_5$, and so on.
In what follows, we shall consider the asymptotic direction along the x-axis.

Proposition 3.1 *In a generic smooth surface M, each branch (the left V_ℓ and the right V_r) of the V-curve is nonsingular on the hyperbolic domain.*

Proof. Applying Section 2.1 to $x_0 = y_0 = 0$, we clearly need $s=0$ for the circle to be tangent to the branch of $f=0$ tangent to the x-axis (for f given in (5)). The contact function then becomes

$$(r-b_0)x_1^3 + (ar^2 + b_1r - c_0)x_1^4 + (r^3 - b_2r^2 + c_1r - d_0)x_1^5 + \text{ higher terms},$$

so that for 4-point contact we need $r = b_0$ and for exactly 5-point contact we add

$$ab_0^2 + b_1b_0 - c_0 = 0, \text{ and } A \neq 0 \text{ with } A := b_0^3 - b_2b_0^2 + c_1b_0 - d_0. \tag{6}$$

The condition $r = b_0$ ensures that the circle osculates the branch of $f=0$ tangent to the x-axis, that is $b_0(x^2+y^2)+y=0$ is the equation, in (x,y)-coordinates in the plane $z=0$, of the osculating circle of this branch, with centre $(0, -\frac{1}{2b_0})$ and curvature $2b_0$. The additional condition $ab_0^2 + b_1b_0 - c_0 = 0$ ensures that the origin is a V-point, while the condition $A \neq 0$, with A as given in (6), ensures that the corresponding circle has exactly 5-point contact with the surface.
Referring to (4), we need to study $\widetilde{H}^{-1}(0,0,0)$ and its Jacobian matrix at $(0,0,r_0,s_0)$ for suitable values of r_0 and s_0, that is for values which correspond to those for a circle which *does* have 5-point contact with the surface at the origin. Of course this requires the origin on the surface M to be a vertex point. As we shall see, for a hyperbolic point on the surface this gives a single condition on

the point, meaning that vertex points generically lie on a curve on the surface. Thus for a smooth vertex curve—the locus of vertex points—at p we require that

(1) $\widetilde{H}(0,0,r_0,s_0)=(0,0,0)$ for some r_0,s_0; this is the same as (6), that is $r_0=b_0, s_0=0$ and $ab_0^2+b_1b_0-c_0=0$,
(2) the 3×4 Jacobian matrix of $\widetilde{H}$ at $(0,0,r_0,s_0)$ has rank 3, and
(3) the third and fourth columns of the Jacobian matrix are independent.

The second condition ensures that $\widetilde{H}^{-1}(0,0,0)$ is smooth at $(0,0,r_0,s_0)$ and the third condition ensures that the projection of this set to the (x_0,y_0)-plane is also smooth at $p=(0,0)$.

From now on in this section we assume condition (6) on c_0. The Jacobian matrix J of $\widetilde{H}$ at $(0,0,b_0,0)$ takes the form (from a direct calculation)

$$J=\begin{pmatrix} -3b_0 & -b_1 & 0 & 1 \\ -4ab_0^2-2b_0b_1 & 2b_0b_2-c_1 & 1 & 2ab_0+b_1 \\ -2b_0^2b_2+6b_0c_1-10d_0 & -6b_0^2b_3+4b_0c_2-2d_1 & 4ab_0+2b_1 & 4b_0^2-4b_0b_2+2c_1 \end{pmatrix}.$$

The last two columns of J are always independent, so that provided one of the minors consisting of columns 1,3,4 or 2,3,4 is nonzero, the whole matrix has rank 3 and the vertex curve is smooth in a neighbourhood of our point p. Putting both these minors equal to zero gives formulas for d_0 and d_1 in terms of b_0,b_1,b_2,c_1,c_2, bearing in mind that $c_0=ab_0^2+b_0b_1$. This imposes two additional conditions on the point, and hence does not occur on a generic surface. □

A generic surface may have isolated points at which the circle has higher contact :

Bi-vertex. A point of the surface where a circle in the tangent plane has 6-point contact with M, that is where one branch of the curve $M\cap T_pM$ has a degenerate vertex, is called a *bi-vertex*.

REMARK 3.2

(1) On a generic surface, the condition $A\neq 0$ holds along the V-curve, except at the bi-vertices. Since the equality $A=0$ (implying 6-point contact) does not affect the proof for the V-curve to be smooth (Proposition 3.1), the V-curve is still smooth at a bi-vertex.
(2) The above proof shows that the tangent vector to the vertex curve at p depends on the terms b_0,b_1,b_2,c_1,c_2,d_0 and d_1. This tangent vector comes to

$$\begin{aligned}&(4a^2b_0^2b_1+4ab_0^2b_2+4ab_0b_1^2-2ab_0c_1-2b_0^2b_1+3b_0^2b_3+4b_0b_1b_2+b_1^3-2b_0c_2-2b_1c_1+d_1,\\ &-4a^2b_0^3-4ab_0^2b_1+6b_0^3-7b_0^2b_2+b_0b_1^2+6b_0c_1-5d_0).\end{aligned} \tag{7}$$

(3) If the second component of (7) is 0 and the first is nonzero then the V-curve is tangent to the corresponding branch of the intersection of M with its tangent plane at the origin.

A generic surface may have also the following isolated points :

Vertex-crossing or **V-crossing**. A point of transverse intersection of V_ℓ and V_r, the left and right (smooth) branches of the V-curve, is called *vertex-crossing* or *V-crossing*.

Thus, at a vertex-crossing each of the two smooth curves comprising $M\cap T_pM$ has a vertex.

Assuming $c_0 = ab_0^2 + b_0b_1$ for M locally given in Monge form (5) so that the branch of $M \cap T_pM$ tangent to the x-axis has a vertex, the additional condition for the branch tangent to $x = ay$ to have a vertex is the following, which can be regarded as a condition on c_4 :

$$b_2b_3 - c_4 + (2b_1b_3 + b_2^2 - 2b_3^2 - c_3)a + (3b_0b_3 + 3b_1b_2 - 3b_2b_3 - c_2 - c_4)a^2 + (4b_0b_2 + 2b_1^2 - 2b_1b_3 - b_2^2 - c_1 - c_3)a^3 + (4b_0b_1 - b_0b_3 - b_1b_2 - c_2)a^4 + (2b_0^2 - c_1)a^5 = 0.$$

The osculating circle of this branch at the origin is of the form (1) with

$$r = -\frac{a^3b_0 + a^2b_1 + ab_2 + b_3}{a(a^2+1)} \text{ and } s = -\frac{1}{a},$$

provided $a \neq 0$. (The form (1) is not adapted to circles whose centre is on the x-axis.)

3.2. *Maximum and minimum points*

We now seek to distinguish between *maximum* and *minimum* points. This means: consider the intersection $X = M \cap T_pM$ at a hyperbolic point, where p belongs to the V-curve. Then one branch of X, say X_ℓ, has a vertex at p. Does this vertex correspond to a maximum or a minimum of the (absolute) radius of the osculating circle at points of X_ℓ?

PROPOSITION 3.3 *Let $p \in M$ be a hyperbolic point of the V-curve, and take A as in (6).*

(a) *The absolute radius of curvature (that is the reciprocal $|\kappa^{-1}|$ of the absolute curvature) of the corresponding branch of $M \cap T_pM$ has a minimum (maximum) if and only if $rA > 0$(resp. $rA < 0$).*
(b) *The corresponding branch of $M \cap T_pM$ has a degenerate vertex (having greater than 4-point contact with the osculating circle) if and only if $A = 0$. In this case, p is a bi-vertex and locally separates the V-curve into a half-branch of maxima and a half-branch of minima* (See Figure 1, left).

Proof.

(a) The curvature of the local branch of X which is tangent to the x-axis comes to

$$\kappa = -2b_0 + 12Ax^2 + \dots \text{ where } A = b_0^3 - b_0^2b_2 + b_0c_1 - d_0, \text{ as in (6)}. \tag{8}$$

This implies the absolute radius of curvature $|\kappa^{-1}|$ has a minimum (maximum) at $x = 0$ if and only if $b_0A > 0$ (resp. $b_0A < 0$). Furthermore, as (6), b_0 is the value of r at p. To find A we consider the next derivative of the reduced contact function H (Definition 2.1). We get

$$\frac{\partial^3 H}{\partial x_1^3}(0,0,0,b_0,0) = 6A,$$

which is zero if and only if the branch of the intersection $M \cap T_pM$ tangent to the x-axis has a degenerate vertex : a circle in the tangent plane has 6-point contact with the surface (a bi-vertex; see Remarks 3.2(1)).

(b) In a generic surface, the function A has only simple zeros on the V-curve (at the bi-vertices). Thus a bi-vertex p locally separates the V-curve into two half-branches: in one branch $A>0$ and in the other $A<0$. Item (b) follows from item (a) because r does not change sign at p. □

3.3. *Flecnodal curve and biflecnodes*

Flecnodal curve. In the closure of the hyperbolic domain of M there is a smooth immersed *flecnodal curve* F formed by the points satisfying any of the equivalent conditions (F1)–(F4):

(F1) An asymptotic line (left or right) exceeds 3-point contact with M at p.
(F2) An asymptotic curve through $p \in M$ (left or right) has an inflexion—that is, for a regular parameterization the first two derivatives are dependent (proportional) vectors.
(F3) A smooth branch (left or right) of the tangent section $M \cap T_pM$ at p has an inflexion.
(F4) In terms of the Monge form (5), $b_0 = 0$ and $c_0 \neq 0$.

To see why (F4) is equivalent to both (F2) and (F3), note that, in (5), the asymptotic curve through p, whose asymptotic direction is $y=0$, has degree 3 expansion (as a space curve)

$$x \mapsto \left(x, -\tfrac{3}{2}b_0x^2 + \tfrac{1}{2}(6ab_0^2 + 5b_0b_1 - 4c_0)x^3 + \ldots, 0\right)$$

and the corresponding branch of the plane curve $M \cap T_pM$ has expansion

$$x \mapsto \left(x, -b_0x^2 + (ab_0^2 + b_0b_1 - c_0)x^3 + \ldots\right).$$

Left and Right Flecnodal Curve. The *left* (*right*) *flecnodal curve* F_ℓ (resp. F_r) of M consists of the points of F at which the over-osculating *asymptotic line* is of left (resp. right) type.

A generic surface may have isolated points of transverse intersection of the left and right branches of the flecnodal curve, called *hyperbonodes*. The presence of hyperbonodes is necessary for the metamorphosis of the parabolic curve in generic 1-parameter families of surfaces [**22**]. A detailed study on the geometry of hyperbonodes was done in [**23**, **15**]. We can also find isolated points of the flecnodal curve at which the asymptotic line exceeds 4-point contact :

Biflecnode. A point at which a line has 5-point contact with the surface is called *biflecnode*.

Hence a biflecnode is a V-point with $r=0$ (a circle of infinite radius). Therefore *a biflecnode is a point of transverse intersection of the left (or right) branches of the flecnodal and vertex curves*. At a *biflecnode* both the asymptotic curve and the intersection curve $M \cap T_pM$ have a *second order* inflexion.

REMARK 3.4 We obviously get a biflecnode from (5) by taking $b_0 = c_0 = 0,\ d_0 \neq 0$.

PROPOSITION 3.5 *A left (right) biflecnode locally separates the left (resp. right) V-curve into a half-branch of maxima and a half-branch of minima.*

Proof. The statement follows from Proposition 3.3 (a) because at a biflecnode p of a generic surface we have $A \neq 0$ and the value of r (that is, of b_0) changes sign at p (cf. Remark 3.4). □

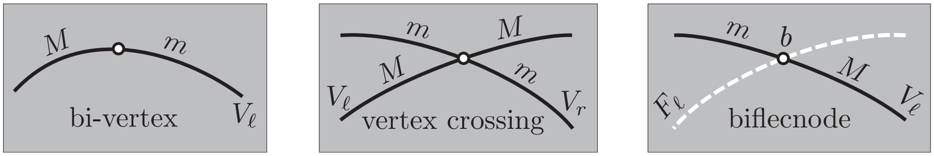


Figure 1. A left bi-vertex, a vertex crossing (with V_ℓ-Max, V_r-min) and a left biflecnode.

3.4. *Stable isolated vertex points in the hyperbolic domain*

Some of the different possibilities for the above isolated vertex points are shown in Figure 1 (see Proposition 3.3 and 3.5). A bi-vertex may be left or right; there are four types of V-crossings (the branches V_ℓ and V_r may consist of maxima or of minima); a biflecnode may be left or right.

A corollary of Proposition 3.3 (*b*) and Proposition 3.5 is

THEOREM 3.6 *On each parametric closed component of the V-curve of a compact generic surface M in $\mathbb{R}^3$ the number of bi-vertices plus the number of biflecnodes is even.*

If, moreover, M is orientable, then on each parametric closed connected component of the V-curve there is an even number (*possibly* 0) *of bi-vertices and an even number* (*possibly* 0) *of biflecnodes.*

4. Vertex curves at a cusp of Gauss

One of the most remarkable points of a generic surface M is a

Cusp of Gauss. Assume that the parabolic curve of M is smooth. A *cusp of Gauss* is a parabolic point at which the unique (but double) asymptotic line is tangent to the parabolic curve.

Note on terminology. Two other common names for a cusp of Gauss (that is, a cusp of the Gauss map) are 'godron', favoured by René Thom, and 'ruffle', used in J.Koenderink's well-known book [**16**]. These names have the advantage that they do not suggest a Euclidean setting, and indeed the cusp of Gauss is actually a *projectively* invariant concept; see Section 4.1.1. This article is about Euclidean concepts so we shall stick to 'cusp of Gauss', except in circumstances where this would prove unwieldy, as in 'flecgodron' (Section 6.5).

4.1. *Some basic properties of cusps of Gauss*

Cusps of Gauss have lots of interesting properties (see for example [**2**, **16**]). Let us mention two of them :

All curves on M tangent to the parabolic curve at a cusp of Gauss g have torsion zero at g, [**21**].

Therefore the space of 2-jets of such tangent curves, $J^2_g := \{(t, \frac{1}{2}ct^2, 0) : c \in \mathbb{R}\} \approx \mathbb{R}$, is identified (up to a factor $\frac{1}{2}$) with the set of their curvatures $\{c \in \mathbb{R}\}$.

Separating 2-jet Lemma ([**21**]). *Given a cusp of Gauss $g \in M$, there exists a unique 2-jet* (*curvature*) *σ in J^2_g (called separating 2-jet at g) satisfying the following properties* :

(a) *The images, by the Gauss map $\Gamma : M \to \mathbb{S}^2 \subset \mathbb{R}^3$, of all curves of M tangent to the asymptotic line at g and whose curvature at g is different from σ are semi-cubic cusps of $\mathbb{S}^2$ sharing the same tangent line at $\Gamma(g)$.*

(b) *Separating property: The images under Γ of any two curves tangent to the asymptotic line at g, whose 2-jets (curvatures) are separated by σ, are cusps pointing in opposite directions.*

Separating invariant. The number σ given in the above lemma is a Euclidean invariant of the cusp of Gauss $g \in M$ that we call the *separating invariant*.

Monge form. Let $p \in M$ be a parabolic point. We shall take p as the origin and the asymptotic line at p as the x-axis. Then the (degenerate) quadratic part of the Monge form is y^2 :

$$z = y^2 + b_0x^3 + b_1x^2y + b_2xy^2 + b_3y^3 + c_0x^4 + c_1x^3y + c_2x^2y^2 + c_3xy^3 + c_4xy^4 + d_0x^5 + \dots . \quad (9)$$

LEMMA 4.1 *Assume as before that the parabolic curve is smooth. A parabolic point of a surface in Monge form* (9) *is a cusp of Gauss if and only if $b_0 = 0$ and $b_1 \neq 0$.* (We will see below that $-b_1$ is the separating invariant of the given cusp of Gauss).

Proof. The local equation of the parabolic curve P, $f_{xx}f_{yy} - f_{xy}^2 = 0$, starts with the terms

$$3b_0x + b_1y + \ldots = 0.$$

Thus the asymptotic line at p $(y = 0)$ is tangent to the parabolic curve at p if and only if $b_0 = 0$ and $b_1 \neq 0$ (because the parabolic curve is smooth). □

4.2 Simple and special cusps of Gauss

(a) The condition for the image of the Gauss map at the origin to be an ordinary (semi-cubical) cusp, using the above form (9), is $b_1^2 - 4c_0 \neq 0$. When this holds, we say the cusp of Gauss is *simple* (sometimes called *nondegenerate*). *On a generic surface all cusps of Gauss are simple.*
(b) The condition $b_1^2 - 4c_0 \neq 0$ is also the condition for the height function $z(x, y)$ in the normal direction $(0, 0, 1)$ at the origin, that is the contact function between M and its tangent plane at the origin, to have type exactly A_3.
(c) The height function can degenerate in two ways: to type A_4 or to D_4. Both these are non-generic for a single surface but occur generically in 1-parameter families; we explore such families in Section 6.
In the case of A_4, also called a double cusp of Gauss, or *bigodron*, the parabolic curve remains smooth ($b_1 \neq 0$), and $b_1^2 - 4c_0 = 0$, $b_1^2b_2 - 2b_1c_1 + 4d_0 \neq 0$. This can be regarded as the collapse of two simple cusps of Gauss, one elliptic and one hyperbolic. See Section 6.3. (This is also sometimes called a *degenerate* cusp of Gauss but the term is ambiguous and 'double' is a more descriptive term.)
In the case of D_4, also called a *flat umbilic*, the parabolic curve becomes singular. See Section 6.2.
(d) There is also the possibility that the parabolic curve undergoes a 'Morse transition', becoming singular at the moment of transition. See Section 6.1.

We now show the following. *The only common points of the vertex curve and the parabolic curve are cusps of Gauss*:

PROPOSITION 4.2 *If a parabolic point p of a generic surface is a vertex point, then p is a cusp of Gauss.*

Proof. Let p be a parabolic point of M, for M is locally given in Monge form (9). If p is also a vertex point, it is easy to check that the contact function (3) at $p = (0,0)$ takes the form

$$b_0 x_1^3 + (r^2 - b_1 r + c_0) x_1^4 + \dots.$$

Referring to the circle given by (1), we must have $s = 0$ to ensure that the 5-point contact circle is tangent to the intersection curve $M \cap T_p M$. This implies

LEMMA 4.3 *There is 5-point contact at $(x_0, y_0) = (0,0)$ if and only if $b_0 = 0$, r is a real solution of $r^2 - b_1 r + c_0 = 0$, and $s = 0$. The curvature of this circle is $2r$.*

Lemma 4.1 and Lemma 4.3 imply that p is a cusp of Gauss for which $b_1^2 - 4c_0 > 0$. □

DEFINITION 4.4 A cusp of Gauss is said to be *hyperbolic* if the intersection with the tangent plane is two tangential curves, that is $b_1^2 - 4c_0 > 0$. A cusp of Gauss is said to be *elliptic* if the intersection with the tangent plane is an isolated point, that is $b_1^2 - 4c_0 < 0$.

(In [**21**] there are five other geometric characterizations of elliptic and hyperbolic cusps of Gauss.)

A cusp of Gauss belongs to the vertex curve if and only if it is hyperbolic. (By Lemma 4.3.)

REMARK 4.5 At a hyperbolic cusp of Gauss neither of the two tangential curves comprising $M \cap T_p M$ has a vertex at p, but their respective osculating circles have 5-point contact with the surface (3-point contact with the osculating branch and 2-point contact with the other tangent branch).

Projective and Euclidean invariants of cusps of Gauss

Cusps of Gauss are projectively invariant. Platonova's (projective) normal form of the 4-jet of a surface at a cusp of Gauss g is $z = \frac{1}{2}y^2 - x^2 y + \frac{1}{2}\rho x^4$, where ρ is a projective invariant defined in [**21**] as a cross ratio. A cusp of Gauss g is hyperbolic (resp. elliptic) if and only if $\rho < 1$ (resp. $\rho > 1$), and simple if and only if $\rho \neq 1$. Computing the cross-ratio invariant ρ in Monge form (9) (with $b_0 = 0$), we get $\rho = 4c_0/b_1^2$.

In our Euclidean case, other coefficients of (9) will also play a role. For example,

(Uribe-Vargas, unpublished) *At a cusp of Gauss, the curvature of the line of (zero) principal curvature is equal to the separating invariant σ. In Monge form (9), with $b_0 = 0$, this curvature is equal to $-b_1$.*

Then the coefficient $-b_1$ represents the geometric and purely Euclidean invariant σ (the above separating invariant). Thus we shall write

$$b_1 = -\sigma, \quad c_0 = \tfrac{1}{4}\sigma^2 \rho. \tag{10}$$

4.2. *Tangency of the parabolic and vertex curves at a cusp of Gauss*

'Naturally' oriented coordinates. At each elliptic point p the surface lies locally on one of the two half-spaces determined by its tangent plane at p, called the *positive half-space at p*. By continuity, the positive half-space is well defined at parabolic points. At a cusp of Gauss g, direct the positive z-axis to the positive half-space at g, the positive y-axis towards the hyperbolic domain, and the positive x-axis in such way that any basis (e_x, e_y, e_z) of x, y, z forms a positive frame of the oriented $\mathbb{R}^3$.

Using the local Monge form of M at a cusp of Gauss (see (10))

$$z = f(x,y) = y^2 - \sigma x^2 y + b_2 xy^2 + b_3 y^3 + \tfrac{1}{4}\sigma^2 \rho x^4 + c_1 x^3 y + c_2 x^2 y^2 + c_3 x y^3 + c_4 y^4 + \ldots \qquad (11)$$

we find that, when $b_1 < 0$, the elliptic domain is on the side $y<0$ of the tangent line $y = z = 0$ to the parabolic curve at the origin and the positive z-axis is the limit of normals to M directed into the positive half-space supporting M at these elliptic points. Therefore the x, y, z axes are naturally oriented as above.

We therefore assume $b_1 = -\sigma < 0$ from now on.

PROPOSITION 4.5 *Let g be a simple (Section 4.2) hyperbolic cusp of Gauss of a generic smooth surface M. In a neighbourhood of g, the V-curve consists of two smooth local branches, tangent to the parabolic curve at g, and having at least 3-point contact with each other. For M locally given in Monge form (11) the condition for exactly 3-point contact is $c_1 + \sigma b_2 \neq 0$.*

Proof. At a hyperbolic cusp of Gauss g there are two distinct circles having 5-point contact with the surface at g (Lemma 4.3). Thus there are two branches of the vertex curve through the cusp of Gauss g. We shall prove that these branches are smooth and tangential there.[1]

Following the method of Section 2, we evaluate the Jacobian matrix J of the map $\widetilde{H}$ at $(0,0,r_0,0)$ (see (4)), where $r_0 = \frac{1}{2}(-\sigma + \sqrt{\sigma^2 - 4c_0}) = -\frac{1}{2}\sigma(1 - \sqrt{1-\rho})$ is one of the two values of r, we obtain a matrix whose third and fourth columns are

$$\left(0, 0, -2\sigma\sqrt{1-\rho}\right)^\top \text{ and } \left(0, -\sigma\sqrt{1-\rho}, 2c_1 - 4b_2 r_0\right)^\top,$$

which are independent since $\rho \neq 1$ for a simple cusp of Gauss. The 3×3 minors formed by columns 1,3,4 and 2,3,4 are respectively 0 and $-2\sigma^3(1-\rho)$, therefore the branch of $\widetilde{H}^{-1}(0)$ and the corresponding branch of the vertex curve of the surface M at the origin are smooth and can both be parameterized locally by x_0, provided the cusp of Gauss is simple.

The first row of the Jacobian matrix is $(0, \sigma, 0, 0)$ and this implies that (given $\sigma \neq 0$) a kernel vector of this matrix has the form $(\xi_1, 0, \xi_3, \xi_4)$ for some ξ_1, ξ_3, ξ_4 where $\xi_1 \neq 0$ since the projection of the tangent vector to the first two coordinates is not zero. Hence the tangent to this local branch of the vertex curve at g is $(1,0)$ in the (x,y)-plane, or $(1,0,0)$ in the ambient 3-space.

The same applies to the other local branch of the vertex curve, and therefore both local branches are tangent to the parabolic curve at g.

Applying the same method as Section 3 to (11), we find the initial terms of the parameterization of the two local branches of the vertex curve

$$V^1 : y = \tfrac{1}{2}\sigma\rho x^2 + B_1 x^3 + \ldots, \qquad V^2 : y = \tfrac{1}{2}\sigma\rho x^2 + B_2 x^3 + \ldots, \qquad (12)$$

where $B_1 - B_2 = 8\sqrt{1-\rho}(\sigma b_2 + c_1)$. Thus provided $c_1 + \sigma b_2 \neq 0$, the two local branches have exactly 3-point contact, and therefore will cross tangentially at g. □

[1] In [10, p.86] it is stated that a V-curve does not always exist in a neighbourhood of a hyperbolic cusp of Gauss. This is incorrect.

REMARK 4.6 It is well known that at every cusp of Gauss of a generic smooth surface the flecnodal curve F is also tangent to the parabolic curve P. Moreover, *cusps of Gauss locally separate the flecnodal curve into left and right half-branches, and the local right-to-left orientation of F, at a hyperbolic cusp of Gauss, coincides with the negative-to-positive orientation of the x-axis in our oriented coordinates* [**21**]. For the V-curve we have a similar statement :

PROPOSITION 4.7 *At a hyperbolic cusp of Gauss g, each tangential local branch V^1, V^2 of the* V*-curve is locally separated by g into left and right half-branches. The local right-to-left orientation of the local branch V^2 coincides with the negative-to-positive orientation of the x-axis* (like the flecnodal curve F) *and is opposite to that of V^1.*

Proof. For a point p of the V-curve close to g we shall write the asymptotic directions on M at p, projected to the plane $z=0$, as $(1,P)$. One can easily verify that the two asymptotic directions (at the hyperbolic points near g), projected to the (x,y)-plane, satisfy that the slope of the left asymptotic line is < the slope of the right one.

For a point $(x_0, y_0(x_0), f(x_0, y_0))$ of V^1 close to g we find that to first order in x_0 the asymptotic direction tangent to the branch of $M \cap T_pM$, having a vertex, is $(1, \sigma(1+\sqrt{1-\rho})x_0, 0)$, and the respective asymptotic direction for $(x_0, y_0(x_0), f(x_0, y_0))$ of V^2 is $(1, \sigma(1-\sqrt{1-\rho})x_0, 0)$.

For a parabolic point near g the unique asymptotic direction, to first order in x_0, is $(1, \sigma x_0, 0)$.

Then for the hyperbolic points, with fixed $x = x_0$, near g the slope P_ℓ of their left asymptotic line must satisfy $P_\ell < \sigma x_0$. The condition for the right asymptotic lines is $P_r > \sigma x_0$.

Thus for points $p = (x_0, y_0, f(x_0, y_0))$ of the tangential local branch V^1 or V^2 of the V-curve close to g we have

$$V^1 \text{ is right at } p \iff \sigma(1+\sqrt{1-\rho})x_0 > \sigma x_0 \iff x_0 > 0,$$
$$V^2 \text{ is left at } p \iff \sigma(1-\sqrt{1-\rho})x_0 < \sigma x_0 \iff x_0 > 0.$$

Therefore Proposition 4.7 is proved. □

5. Further interactions at cusps of Gauss

5.1. *Configurations of geometrically defined curves at cusps of Gauss*

Write T_- and T_+ for the two branches of the tangent section $M \cap T_gM$ at g. We shall determine the relative positions (near g) of the local branches V^1, V^2 of the vertex curve, the flecnodal curve F, the parabolic curve P, the branches $T_\pm$ of $M \cap T_gM$, and the line C of (zero) principal curvature through g. Since all these curves are tangent to the asymptotic line at p, their 2-jet is a curve in the tangent plane of the form $y = \frac{1}{2}cx^2 + \ldots$. Therefore the local configurations of F, P, V, $T_\pm$, C and the asymptotic line at g are determined by the relative positions of their respective curvatures c_F, c_P, c_V, c_{T_-}, c_{T_+} and $c_C = \sigma$ on the real line.

THEOREM 5.1 *Given a simple hyperbolic cusp of Gauss g of M, there are seven possible configurations of the curves F, P, V, $T_\pm$, C and the asymptotic tangent line at g* (Figure 2). *The actual configuration depends on which of the intervals defined by the exceptional values* $\cos\frac{5\pi}{6}$, $\cos\frac{4\pi}{6}$,

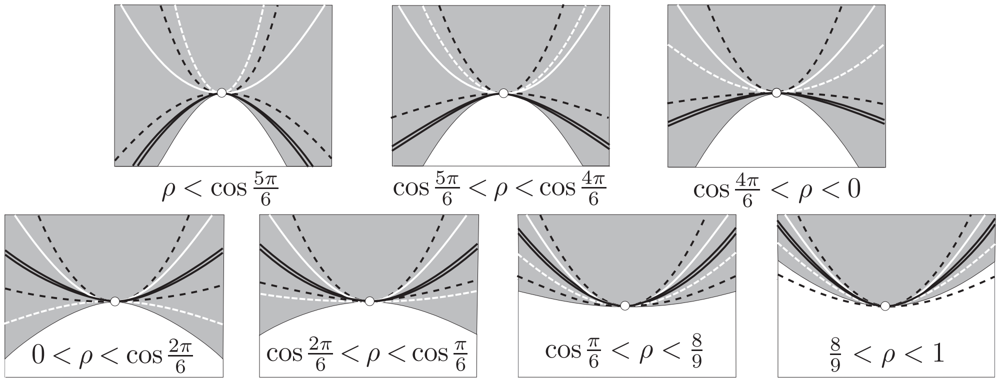


Figure 2. The seven generic configurations, at a hyperbolic cusp of Gauss, of the curves: flecnodal F (white broken), parabolic P (boundary between white and grey domains), V-curve V (black curves which are very close together), tangent section $T_\pm$ (black broken curves) and line of principal curvature C (white).

$\cos\frac{3\pi}{6} = 0$, $\cos\frac{2\pi}{6}$, $\cos\frac{\pi}{6}$, $8/9$, *the invariant* ρ *belongs to, respectively*

$$\begin{array}{lcl}
\rho \in (-\infty, \cos\frac{5\pi}{6}) & \Longleftrightarrow & c_P < c_V < c_{T_-} < \sigma < c_{T_+} < c_F\,; \\
\rho \in (\cos\frac{5\pi}{6}, \cos\frac{4\pi}{6})) & \Longleftrightarrow & c_P < c_V < c_{T_-} < \sigma < c_F < c_{T_+}\,; \\
\rho \in (\cos\frac{4\pi}{6}, \cos\frac{3\pi}{6}) & \Longleftrightarrow & c_P < c_V < c_{T_-} < c_F < \sigma < c_{T_+}\,; \\
\rho \in (\cos\frac{3\pi}{6}, \cos\frac{2\pi}{6}) & \Longleftrightarrow & c_P < c_F < c_{T_-} < c_V < \sigma < c_{T_+}\,; \\
\rho \in (\cos\frac{2\pi}{6}, \cos\frac{\pi}{6}) & \Longleftrightarrow & c_P < c_F < c_{T_-} < c_V < \sigma < c_{T_+}\,; \\
\rho \in (\cos\frac{\pi}{6}, \frac{8}{9}) & \Longleftrightarrow & c_P < c_{T_-} < c_F < c_V < \sigma < c_{T_+}\,; \\
\rho \in (\frac{8}{9}, 1) & \Longleftrightarrow & c_{T_-} < c_P < c_F < c_V < \sigma < c_{T_+}\,.
\end{array}$$

Proof. To determine the flecnodal curve F near g we consider tangent lines to M at points $(x, y, f(x, y))$ and impose the condition that the line should have at least 4-point contact with M. It is then straightforward to calculate the local equation:

$$\text{flecnodal curve } F: \quad y = \tfrac{1}{2}\sigma\rho(2\rho - 1)x^2 + \dots. \tag{13}$$

The local equation of P, $y = \frac{1}{2}\sigma(3\rho - 2)x^2 + \dots$, is given by the Hessian: $f_{xx}f_{yy} - f_{xy}^2 = 0$. We get the local equations of $T_\pm$ from (11) by solving $f(x, y) = 0$: $y = \frac{1}{2}\sigma(1 \pm \sqrt{1-\rho}) + \dots$.

If in addition we use (13), (12) and Proposition 4.1.1, we find that the 2-jets of the curves F, P, V, T_-, T_+ and C on M, are curves in the tangent plane written as $y = h(x)$, where h is given by the following respective functions:

$$\tfrac{1}{2}\sigma\rho(2\rho - 1)x^2, \quad \tfrac{1}{2}\sigma(3\rho - 2)x^2, \quad \tfrac{1}{2}\sigma\rho x^2, \quad \tfrac{1}{2}\sigma(1 + \sqrt{1-\rho})x^2, \quad \tfrac{1}{2}\sigma(1 - \sqrt{1-\rho})x^2, \quad \tfrac{1}{2}\sigma x^2.$$

Thus the respective curvatures are $c_F = \sigma\rho(2\rho - 1)$, $c_P = \sigma(3\rho - 2)$, $c_V = \sigma\rho$, $c_{T_-} = \sigma(1 + \sqrt{1-\rho})$, $c_{T_+} = \sigma(1 - \sqrt{1-\rho})$ and $c_C = \sigma$. Since all these curvatures have σ as factor, their relative positions in the real line are determined by ρ. Thus in Figure 3 the curvatures are divided by σ.

The expressions for the curvatures determine the exceptional values of ρ. □

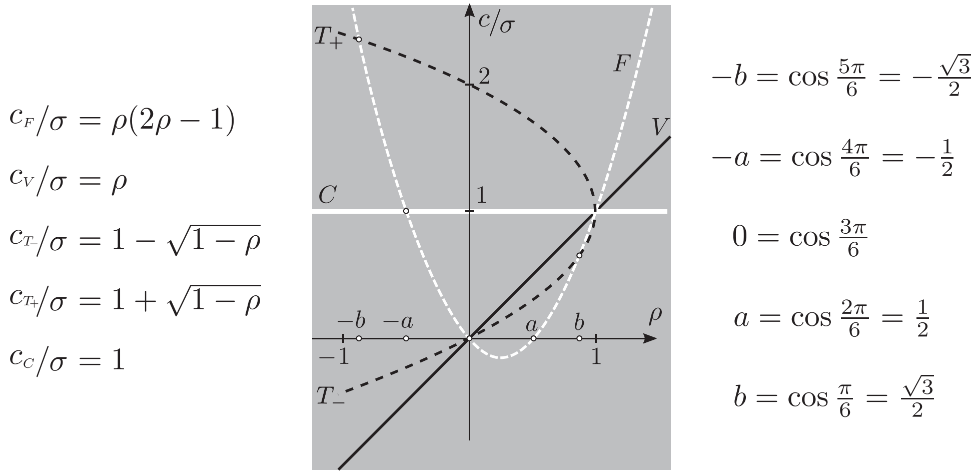


Figure 3. The curvatures c_F (white broken), c_V (black), $c_{T_\pm}$ (broken) and c_C (white), all divided by σ.

NOTE For simplicity we omit from Figure 3 the graph of $c_P/\sigma = (3\rho - 2)$, which is a line. It cuts c_{T_-} at $\rho = \frac{8}{9}$ and the ρ-axis at $\rho = \frac{2}{3}$ (the value of ρ where P changes its convexity).

5.2. *Relative positions of the flecnodal and vertex curves considering left, right branches and minimum, maximum types*

Consider a hyperbolic cusp of Gauss $g \in M$ with given cr-invariant ρ and separating invariant σ, and write $r_1 = -\frac{1}{2}\sigma(1 + \sqrt{1-\rho}),\ r_2 = -\frac{1}{2}\sigma(1 - \sqrt{1-\rho})$.

Take M in Monge form (11).

PROPOSITION 5.2 *Near g, at points of the tangential local branches V^1, V^2 of the V-curve the absolute radius function has*

$$\begin{aligned} &\textit{a maximum on } V^1 &&\iff && G_1 > 0; \\ &\textit{a maximum on } V^2 &&\iff && \rho G_2 > 0 \quad (G_2,\ \rho \textit{ have equal signs}), \end{aligned}$$

where $G_1 := -r_1^2 b_2 + r_1 c_1 - d_0$ *and* $G_2 := -r_2^2 b_2 + r_2 c_1 - d_0$.

Note that this is a different use of the notation G_1, G_2 from Section 2.2.

Proof. We use Proposition 3.3 to find the conditions for the points of these two branches to represent maximum/minimum of the absolute radius function. The function r on the two tangential local branches V^1, V^2 takes the respective forms $\widehat{r}_1 = r_1 + \ldots$ and $\widehat{r}_2 = r_2 + \ldots$.

The function $\partial^3 H/\partial x^3$, on V^1 and V^2, has the respective signs of G_1 and G_2.

According to Proposition 3.3 a point on V^i has a maximum if and only if $\widehat{r}_i G_i < 0$, that is if and only if $r_i G_i < 0$. Clearly $r_1 < 0$, so we get the last inequality if and only if $G_1 > 0$. On the other hand, it is easy to check that $r_2 \rho < 0$; hence $r_2 G_2 < 0$ if and only if $\rho G_2 > 0$. $\square$

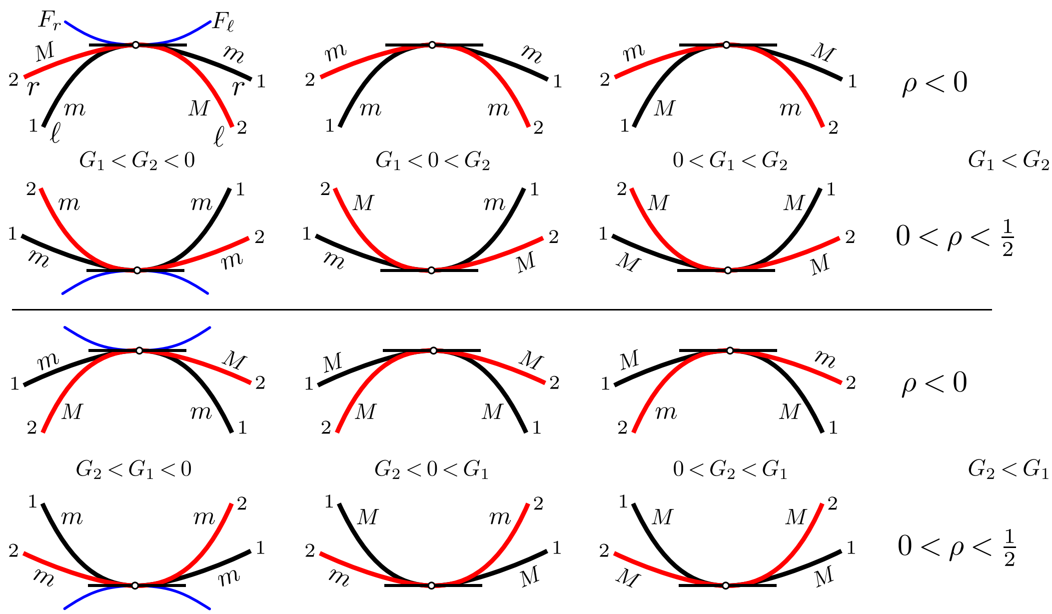


Figure 4. The 12 generic local configurations of the flecnodal curve F and of the local branches V^1, V^2 (denoted 1 and 2) counting their 'left' (ℓ), 'right' (r) half-branches, and their maximum (M), minimum (m) types at a hyperbolic cusp of Gauss. In all cases, the parabolic curve is 'below' the other curves but is not drawn.

REMARK 5.3 The quantities G_1, G_2 arise elsewhere. Consider the branches of the intersection $M \cap T_pM$ of M with its tangent plane at p, that is the plane curve $f(x,y) = 0$. The local equations are

$$y = \tfrac{1}{2}\sigma\left(1 + \sqrt{1-\rho}\right)x^2 + \tfrac{1}{\sigma\sqrt{1-\rho}}G_1x^3 + \ldots, \quad y = \tfrac{1}{2}\sigma\left(1 - \sqrt{1-\rho}\right)x^2 - \tfrac{1}{\sigma\sqrt{1-\rho}}G_2x^3 + \ldots.$$

PROPOSITION 5.4 *Close to g the tangential local branch V^1 is 'below' V^2 for $x < 0$* (that is, has lower y values: $y_1 < y_2$) *if and only if $G_1 < G_2$.*

Proof. The relative size of B_1 and B_2 determines the relative position of V^1 and V^2. But we have: $G_1 - G_2 = -\sigma\sqrt{1-\rho}\,(\sigma b_2 + c_1)$. Thus by (12), $G_1 < G_2$ if and only if $B_2 < B_1$. □

THEOREM 5.5 *There are 12 generic types of hyperbolic cusps of Gauss, according to the relative positions of the half-branches F_ℓ, F_r of the flecnodal curve and the half-branches V_r^1, V_ℓ^1, V_r^2, V_ℓ^2 of the V-curve, counting their maximum and minimum types. These types are listed in Figure 4.*

Proof. The relative position 'below/above' between the tangential local branches V^1, V^2 of the V-curve are given by the inequalities $G_1 < G_2$ and $G_2 < G_1$ (Proposition 5.3); each one has three realizations, for example: $G_1 < G_2 < 0$, $G_1 < 0 < G_2$, $0 < G_1 < G_2$. The two relative positions 'below/above' between the V-curve and the flecnodal curve are given by the inequalities $\rho < 0$ and $\rho > 0$ (Theorem 5.1). We get the type maximum or minimum for V^1 and V^2 from Proposition 5.2 applied to all these inequalities. Then we obtain the 12 generic types shown in Figure 4. □

In Figure 4, the half-branches (F_r, F_ℓ) of the flecnodal curve are shown only in the left-hand diagrams. In all cases, the right and left half-branches of the two local branches of the V-curve (denoted 1, 2) and of the flecnodal curve correspond to those indicated in the first diagram. Observe that for every position of G_1 and G_2, passing from $\rho<0$ to $\rho>0$ only the local branch V^2 changes from maximum to minimum or vice versa. It is explained because if we pass from $\rho<0$ to $\rho>0$ continuously, at $\rho=0$ there is a flecgodron transition (Figure 13) where only the component V^2 changes its type.

6. Some codimension 1 transitions on the V-curve

In this section we shall investigate some transitions on the V-curve, and other curves, which occur in generic 1-parameter families of surfaces, $\{M_t\}$, where t is in some open interval of real numbers containing $t=0$. We consider mainly transitions in which the parabolic curve undergoes a transition. The results of this section are obtained by exact calculation for M_0 and are largely experimental for nearby members of the family.

We consider the following cases:

6.1 The parabolic set of the family M_t is undergoing a 'Morse transition'
 6.1a: the parabolic set of M_0 has an isolated point;
 6.1b: the parabolic set of M_0 has a self-intersection consisting of two transverse smooth branches.
 These two cases are referred to as 'non-transversal A_3 transitions' in [7, 8], the A_3 referring to contact between the tangent plane at the origin and M_0. Case 6.1a is A_3^+ and 6.1b is A_3^-.

6.2 The parabolic set is undergoing a 'D_4 transition' (through a flat umbilic) as in [7, 8]. The symbol D_4 refers to the contact between the surface and its tangent plane at the origin and means that the quadratic terms in the Monge form for M vanish identically. The cubic terms can have one real root (D_4^+) or three (D_4^-).

6.3 M_0 has a degenerate cusp of Gauss, that is in the notation of Definition 4.4 and (9), $b_0=0,\ c_0=\frac{1}{4}\sigma^2$. This means that the contact of M with its tangent plane at the origin is of type at least A_4. To ensure that the contact is no higher than A_4 we require $\sigma^2 b_2+2\sigma c_1+4d_0\neq 0$. Such a point of M is called a 'bigodron' in [21] and an 'A_4 transition' in [7, Section 3.2]; it occurs when an elliptic and a hyperbolic cusp of Gauss come into coincidence and disappear and is generic in a 1-parameter family of surfaces. The parabolic curve remains nonsingular throughout.

6.4 The V-curve is *singular* because both components of the vector (7) are zero; this amounts to saying that d_0 and d_1 are expressible in terms of coefficients a, b_i, c_j in the hyperbolic case.

6.5 In Section 6.5 we describe a different kind of transition, the 'flecgodron' which is the coincidence of a biflecnode and a cusp of Gauss.

The generic transitions of the parabolic set of a surface in 3-space are enumerated in [7, 8], the second of these articles providing full mathematical details of results summarized in the first. Since the V-curve does not intersect the parabolic set except at hyperbolic cusps of Gauss, the cases 6.1–2 are restricted to those where hyperbolic cusps of Gauss are created or destroyed. For 6.1a this means that an 'elliptic island' appears in a hyperbolic region of M_t as t passes through 0, in which case two hyperbolic cusps of Gauss are created on a newly created closed curve of the parabolic set. See Figure 5. For 6.2 only one of the two cusps of Gauss which are created or destroyed in the transition can intersect the V-curve.

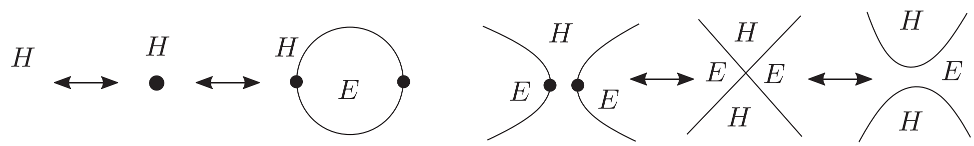


Figure 5. A diagram of local transitions on the parabolic curve P in a generic 1-parameter family of surfaces as in 6.1a,b (from [7, Figure 2]), where H, E denote the hyperbolic and elliptic regions respectively. (left): from empty to a closed loop of P which has two cusps of Gauss (the dots); (right): through a crossing of smooth branches in which two cusps of Gauss are created or destroyed. These are the cases in which *hyperbolic* cusps of Gauss are involved, so that the V-curve is involved too.

6.1. *'Morse* (A_3) *transition' on the parabolic curve*

(Figure 5.)

THEOREM 6.1 *Assume the parabolic set of a generic family of smooth surfaces M_t has an A_3 (Morse) transition (at $t=0$) in a point p of the surface M_0. Then*

(a) *If the parabolic set of M_0 locally consists of p, then the V-curve also consists of p.*
(b) *If the parabolic set of M_0 has two transverse smooth branches at p, then the V-curve is composed of two singular local components each of which consists of two transverse smooth branches at p.*
(c) *The two local components of case (b) share a common pair of tangent lines and these two common tangent lines are distinct from the tangents to the parabolic curve.*

Proof. For a surface M_0 in Monge form (9) the parabolic curve has local equation

$$3b_0x + b_1y + (3b_0b_2 - b_1^2 + 6c_0)x^2 + (9b_0b_3 - b_1b_2 + 3c_1)xy + (3b_1b_3 - b_2^2 + c_2)y^2 = 0, \qquad (14)$$

up to order 2 in x, y. It is singular provided $b_0 = b_1 = 0$ so that the Monge form becomes

$$z = y^2 + b_2xy^2 + b_3y^3 + c_0x^4 + c_1x^3y + c_2x^2y^2 + c_3xy^3 + c_4y^4 + d_0x^5 + \dots .$$

The parabolic curve has a Morse (that is, nondegenerate) singularity provided the discriminant $\Delta := 8b_2^2c_0 - 8c_0c_2 + 3c_1^2$ of the quadratic terms in (14) is nonzero. So we get

$$\text{a crossing for } \Delta > 0, \qquad \text{an isolated point for } \Delta < 0. \qquad (15)$$

To examine the V-curve of M_0, we use the reduced contact function H (in Definition 2.1) and find that the values of r for 5-point contact at the origin are given by $r^2 + c_0 = 0$. Thus we require $c_0 < 0$ for real 5-point contact circles ($c_0 = 0$ is of higher codimension). In this case, write $C_0 = \sqrt{-c_0}$ so that $r = \pm C_0$ and $\Delta = -8b_2^2C_0^2 + 8C_0^2c_2 + 3c_1^2$.

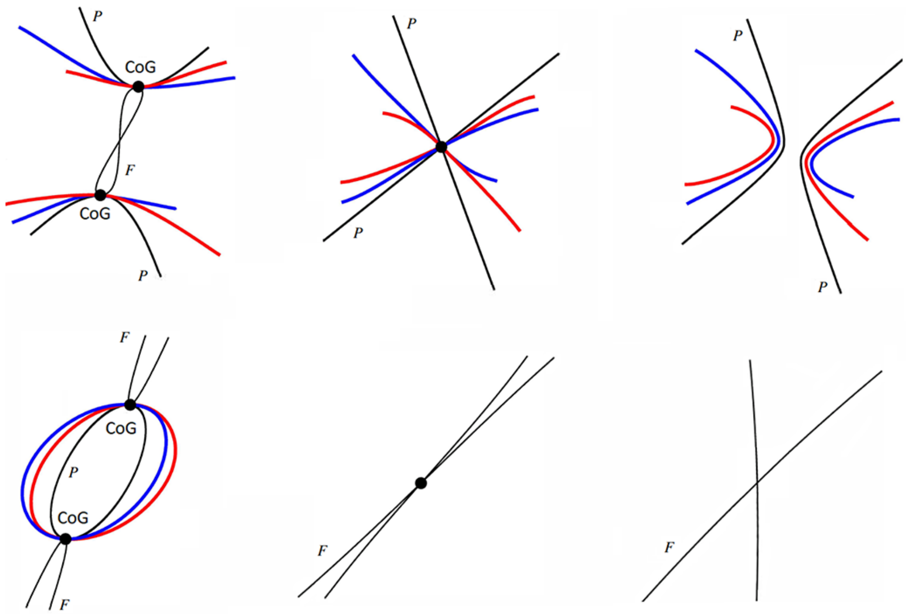


Figure 6. Local (Morse) transitions on the parabolic set P and their effect on the V-curve and the flecnodal curve F in a generic family of surfaces M_t. CoG stands for (hyperbolic) cusp of Gauss. Above centre: a crossing on the parabolic set, case 6.1b. The figure-of-eight in the top left diagram is the flecnodal curve. Below centre: an isolated point on the parabolic set, case 6.1a. The central oval in the lower left diagram is the parabolic curve and the other curves labelled F represent the flecnodal curve. The family of surfaces used in the upper diagram is $z = y^2 - x^4 + x^3y + x^2y^2 + tx^2$ for small t with $t = 0$ in the middle, and $z = y^2 - x^4 + x^3y - x^2y^2 + tx^2$ for the lower diagram.

Then the V-curve has two local components which correspond to the values $r = \pm C_0$, and their respective local equations have the following common 2-jet:

$$8C_0^4x_0^2 - 4C_0^2c_1x_0y_0 + (4C_0^2b_0^2 - 4C_0^2c_2 - c_1^2)y_0^2 = 0. \tag{16}$$

Its discriminant, $\widehat{\Delta} := 16C_0^4(-8b_2^2C_0^2 + 8C_0^2c_2 + 3c_1^2)$, is a positive multiple of the discriminant Δ of the 2-jet of the local equation of the parabolic curve, $\widehat{\Delta} = 16\,C_0^4\Delta$.

Thus using this last equality together with (15) we prove items a and b.

Proof of item c: The first statement follows because the pairs of tangents of the two local components of the V-curve are defined by a common quadratic form (16). One can check that the coincidence of these tangents with the tangents to the parabolic curve would lead to $\Delta = 0$. □

Note that $c_0 < 0$ is also the condition for the intersection between M_0 and its tangent plane T_pM_0 at p to consist of two tangential curves—a tacnode—rather than an isolated point.

The local transitions occurring on the V-curves in a generic family of surfaces M_t with M_0 as above are illustrated in Figure 6.

REMARK 6.2 (Note on the Figures 6, 8, 9, 10, 11 and 12) In these diagrams the radius of the 5-point contact circle varies continuously along each parameterized segment of the V-curve. Thus in a neighbourhood of a cusp of Gauss g, the local branches denoted by V^1 and V^2 in (12) and Proposition 4.7 have cubic tangency at g, and the left-rightness changes on each side of g. Of course, left-rightness stays constant along a parameterized segment of the V-curve away from cusps of Gauss.

6.2. *D_4 transition on the parabolic curve ('flat umbilic')*

The label 'D_4' refers to contact between M_0 and its tangent plane to M_0 at the origin. In this case the quadratic terms of the Monge form of the surface M_0 are absent. Such a point is also called a *flat umbilic* in contrast to a generic umbilic which has quadratic terms of the form $\kappa(x^2+y^2), \kappa \neq 0$.

By rotating the coordinates and scaling equally in all directions we may assume that the Monge form of M_0 is

$$z = x^2y + b_2xy^2 + b_3y^3 + c_0x^4 + c_1x^3y + c_2x^2y^2 + c_3xy^3 + c_4y^4 + \dots, \tag{17}$$

with one root of the cubic terms along the x-axis $y=0$. The two cases are distinguished by

$$D_4^+ \text{(one real root)} : b_2^2 < 4b_3; \qquad D_4^- \text{(three real roots)} : b_2^2 > 4b_3.$$

We assume from now on that $b_2^2 \neq 4b_3$, so that the contact between M_0 and the tangent plane T_pM_0 at p is 'no worse' than D_4.

The parabolic curve of M_0 has the form $x^2 + b_2xy + (b_2^2 - 3b_3)y^2 + \text{h.o.t.} = 0$ with discriminant of the quadratic terms equal to $-3(b_2^2 - 4b_3)$. Thus (see [7, Fig.4, p.298], noting that the labels $D_4^\pm$ on Figure 4 are the wrong way round):
D_4^+: one branch of $M_0 \cap T_pM$ and the parabolic curve has a crossing of smooth branches
D_4^-: three transverse branches of $M_0 \cap T_pM_0$ and the parabolic curve has an isolated point.

Looking for circles in the (x,y)-plane which have 5-point contact with M_0 at the origin we find that for each real branch of $z=0$ there is a circle centred on the line perpendicular to that branch; for the branch tangent to $y=0$ the centre of this circle is at $\left(0, -\frac{1}{2c_0}\right)$ (and the radius is of course $\frac{1}{2|c_0|}$).

We shall make the generic assumption $c_0 \neq 0$ in what follows.

But the second circle having 5-point contact and centre $(0,r)$ shrinks to a 'circle of radius $r=0$'. We shall see in Section 6.2.2 that indeed there is at least one branch of the V-curve through the origin on which the radius of the 5-point contact circle tends to 0 at the origin.

To analyse this situation for the transitional surface M_0 we shall adopt the 'alternative' approach to the contact function as described in Section 2.2, using the mapping $H = (H_1, H_2)$ described there. The V-curve consists of those points $(x_0, y_0, f(x_0, y_0)) \in M_0$ for which u, v exist such that H is $\mathcal{K}$-equivalent to an $A_{\geq 4}$ singularity at $p = q = 0$.

Osculating circle of radius $1/|2c_0|$

We consider here the branch of the intersection of M_0 with its tangent plane $z=0$ at the origin which is tangent to $y=0$. For D_4^+ this is the only real branch of the intersection while for D_4^- the same argument applies to each of the three real branches of the intersection.

Consider the circle having 5-point contact with M_0 at the origin and centre $(u,v) = \left(0, -\frac{1}{2c_0}\right)$. We shall expand H about $(x_0, y_0, u, v, p, q) = \left(0, 0, 0, -\frac{1}{2c_0}, 0, 0\right)$, substituting $v = V - \frac{1}{2c_0}$ so that V is small. The coefficient of q in H_2 then works out as $\frac{1}{c_0} \neq 0$ so that we can solve $H_2 = Q$ say for a function $q = Q(p, x_0, y_0, u, V)$. Substituting in H_1 we can then put $Q = 0$ in H_1 since we are classifying H up to contact equivalence and terms containing Q can therefore be removed from H_1. The result is a mapping $(\overline{H}_1, Q)$ say where $\overline{H}_1$ is a function of p, x_0, y_0, u, V. It is a straightforward matter to check that $\overline{H}_1$ is divisible by p^2, corresponding to the fact that the circle always has at least 2-point contact with M_0 at $(x_0, y_0, f(x_0, y_0))$. Then the second, third and fourth derivatives of $\overline{H}_1$ evaluated at $p = 0$ give three equations in x_0, y_0, u, V the solution to which is the preimage of one branch of the V-curve; the V-curve itself is the projection of this set to the (x_0, y_0) plane. In fact in this case an argument similar to that in Section 3.1 shows that, provided $c_0 \neq 0$ as above, the solution set in (x_0, y_0, u, v)-space is smooth, parameterized locally by x_0, and its projection to the (x_0, y_0) plane is therefore smooth. In fact the V-curve is tangent to the x-axis, with local parameterization in the (x_0, y_0) plane of the form $y_0 = -2c_0x_0^2 + \text{h.o.t.}$

Assume, in the notation of (17), that $b_2^2 \neq 4b_3$ and $c_0 \neq 0$. Then we have the following.

PROPOSITION 6.3 *In a generic 1-parameter family of smooth surfaces M_t having a $D_4^{\pm}$ transition at $t = 0$, each smooth branch of $M_0 \cap T_pM_0$* (the local intersection of M_0 with its tangent plane at p) *has a tangent smooth branch of the V-curve.*

The corresponding 5-point contact circle or circles at p have nonzero radius; for the branch of $M_0 \cap T_pM_0$ tangent to the x-axis this radius is $1/|2c_0|$ in the notation of (17).

Degenerate osculating circle of (limiting) radius 0

The 5-point contact degenerate circle of radius zero is obtained by expanding H as power series in x_0, y_0, u, v, p, q, all of which are small. We find the following:

$$\begin{aligned} H_1 &= -y_0p^2 - 2(x_0 + b_2y_0)pq - (b_2x_0 + 3b_3y_0)q^2 - p^2q - b_2pq^2 - b_3q^3 + \text{ degree} \geq 4, \\ H_2 &= -2(u - x_0)p - 2(v - y_0)q + p^2 + q^2 + \text{degree} \geq 6. \end{aligned} \tag{18}$$

The V-curve consists of those points $(x_0, y_0, f(x_0, y_0)) \in M_0$ for which u, v exist such that H is $\mathcal{K}$-equivalent to an $A_{\geq 4}$ singularity at $p = q = 0$. It is convenient to substitute $U = u - x_0$, $V = v - y_0$ so that the quadratic terms of H_2 take the form $p^2 + q^2 - 2pU - 2qV$.

Evaluating H at $x_0 = y_0 = u = v = 0$ we obtain $(-q(p^2 + b_2q + b_3q^2) + \ldots, p^2 + q^2 + \ldots)$, which in the *complex* $\mathcal{K}$ classification of [**11**] is equivalent to the $\mathcal{K}$-simple germ $B_{3,3} : (x, y) \mapsto (xy, x^3 + y^3)$. According to the list of specializations in [**11**, p.278] this singularity has A_5 singularities in its neighbourhood; however the unfolding by parameters x_0, y_0, u, v will not be versal; in our situation of a generic 1-parameter family of surfaces we do not expect to find more degenerate singularities than A_4.

We shall approach this case by neglecting terms of degree ≥ 6 in H_2 and solving $H_2 = Q$ say exactly for q as a function of x_0, y_0, U, V, p, Q in order to reduce H_2 to Q. When this is done we can replace Q by 0 and H takes the form $(\overline{H}_1, Q)$, say where $\overline{H}_1$ is a function of x_0, y_0, U, V, p. Both this function and its derivative with respect to p vanish at $p = 0$, since the circle always has at least 2-point contact with M_0 at $(x_0, y_0, f(x_0, y_0))$. The conditions we want to impose are, as usual, that the second, third and fourth derivatives vanish at $p = 0$.

When this is done we find (after a rather tedious calculation) that x_0, y_0 can be expressed in terms of U, V but that the relation between U, V has lowest terms a homogeneous quintic:

$$\begin{aligned} & b_2(2b_2^2-7b_3)U^5+(4b_2^2b_3-5b_2^2-12b_3^2+14b_3)U^4V \\ -& b_2(2b_2^2-b_3-3)U^3V^2+(3b_2^2+6b_3-2)U^2V^3-5b_2UV^4+2V^5 \quad = \quad 0. \end{aligned} \tag{19}$$

A solution $(U,V)=(k,1)$ of this quintic gives a branch of the V-curve with slope

$$\frac{k(2-b_2k)}{3b_3k^2-2b_2k+1},$$

in the (x_0,y_0)-plane. This slope cannot be zero provided $b_2^2 \neq 4b_3$ as above, so that the V-curve branch cannot be tangent to the x_0 axis and by symmetry cannot be tangent to any of the branches of $M_0 \cap T_pM_0$.

It can be shown that if $b_2^2 > 4b_3$ the quintic equation has negative discriminant (that is for D_4^-), which indicates three real branches in the (U,V)-plane and therefore three real branches of the V-curve in the (x_0,y_0)-plane, with slopes given as above by the three real roots of the quintic. But if $b_2^2 < 4b_3$ (D_4^+) there can be one, three or five real branches of the V-curve.

We sum up this situation as follows.

PROPOSITION 6.4 *In addition to the smooth branches listed in Proposition 6.3 the V-curve has other smooth branches, whose corresponding 5-point contact circle has radius* 0*:*

D_4^-: *one for each of the three real branches of $M_0 \cap T_pM_0$, in each case not tangent to the latter branch;*

D_4^+: *either 1, 3 or 5 such branches depending on the cubic terms of M_0 at the origin. For more information see below.*

This situation can be better illustrated by a change of normal form; in fact we shall adopt a procedure analogous to that used to separate the 'lemon/star/monstar' cases of a generic umbilic point, as explained in, for example, [20]. It is an elementary calculation to check that every real cubic form in x, y can be transformed, by rotation and scaling in the (x,y)-plane, and then writing $z = x + \mathrm{i}y$ $(\mathrm{i} = \sqrt{-1})$, into the special form

$$z^3 + 3\overline{\beta}z^2\overline{z} + 3\beta z\overline{z}^2 + \overline{z}^3 \tag{20}$$

where β is a complex number. The only exception is a cubic form $ax^3+bx^2y+axy^2+by^3$, which equals $(ax+by)(x^2+y^2)$ and so has only one real root $ax+by=0$. So far as the case D_4^+ is concerned we can ignore this exception, since the cubic form of M_0 has three real roots. The conditions for (19) to have 1, 3 or 5 real roots can then be expressed in terms of $\beta = \beta_1 + \mathrm{i}\beta_2$ and the resulting diagram Figure 7 in the β plane has a pleasing symmetry and compactness.

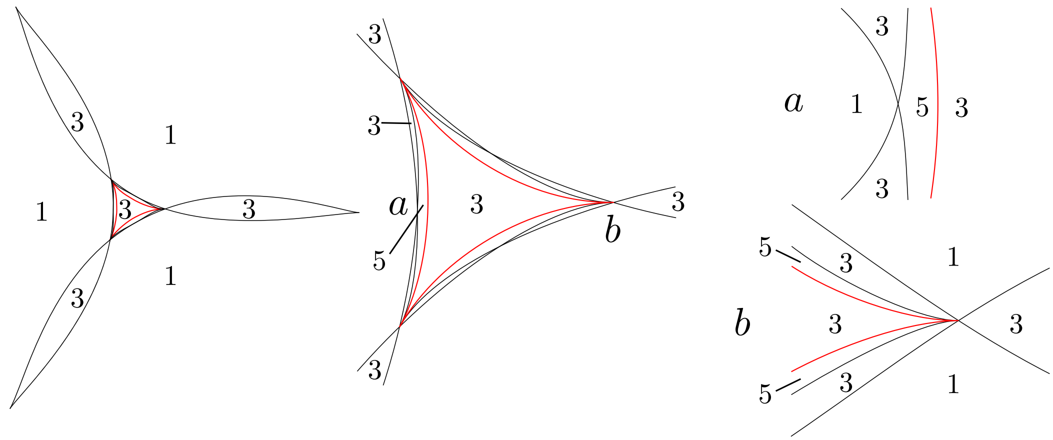


Figure 7. The regions of the β-plane corresponding to different numbers 1,3,5 of real roots of the quintic form (21). The 3-cusped inner curve is the discriminant of the cubic form (20) which forms part of the discriminant of (21). The middle diagram is an enlargement of the central area of the left-hand diagram and the details of two small areas a and b of the middle diagram are on the right. The region inside the inner 3-cusped curve corresponds to D_4^- and all the rest of the diagram to D_4^+.

For the record, the result of expressing the quintic form (19) in terms of β_1 and β_2 is

$$\begin{aligned} 3(\beta_1^3+\beta_1\beta_2^2-3\beta_1^2+5\beta_2^2+3\beta_1-1)U^5+3\beta_2(\beta_1^2+\beta_2^2-1)U^4V \\ +6(\beta_1-1)(\beta_1^2+\beta_2^2-1)U^3V^2+2\beta_2(3\beta_1^2+3\beta_2^2+40\beta_1+17)U^2V^3 \\ +(3\beta_1^3+3\beta_1\beta_2^2-29\beta_1^2+11\beta_2^2+17\beta_1+9)UV^4+\beta_2(3\beta_1^2+3\beta_2^2+16\beta_1+5)V^5. \end{aligned} \tag{21}$$

The discriminant of this quintic form is a product of two factors, one of which is the cube of the discriminant of the cubic form (20) of M_0 and the other has degree 10 in β_1 and β_2. Both are invariant under the rotation $\beta \mapsto \beta\exp(2\pi \mathrm{i}/3)$, as is clear from Figure 7. Figures 8, 9 and 10 illustrate the V-curve itself. The central diagram in each case follows from the calculations above and the outer diagrams, representing the evolution of the V-curve in a generic family of surfaces, are produced from an example.

Notation. *In the following figures the thin black curve is the parabolic curve, and the arrows show the direction of increasing absolute radius of the 5-point contact circle along the V-curve and 'Min' refers to a minimum of this radius. In some figures, there are also 6-point contact points (bi-vertices) labelled B. The various thick lines are the branches of the V-curve; see Remark 6.2 for further details.*

Lower diagram: an enlarged and a more enlarged picture of the curves in the broken circle in the right-hand upper diagram.

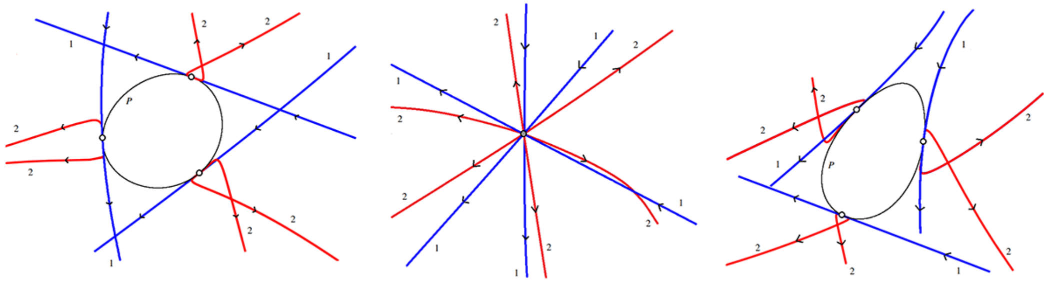


Figure 8. D_4^- case. The surface M_0 in this example is $z = x^3 - xy^2 - \frac{4}{5}x^2y + \frac{1}{7}y^4$ and the family M_t is obtained by adding a small term tx^2. For M_0 the curve labelled 1 is a branch of the V-curve along which the radius of the 5-point contact circle is nonzero. On the curves labelled 2 near each cusp of Gauss there is a local minimum of radius and also a bi-vertex which are not shown. The nearby flecnodal curves are also not included.

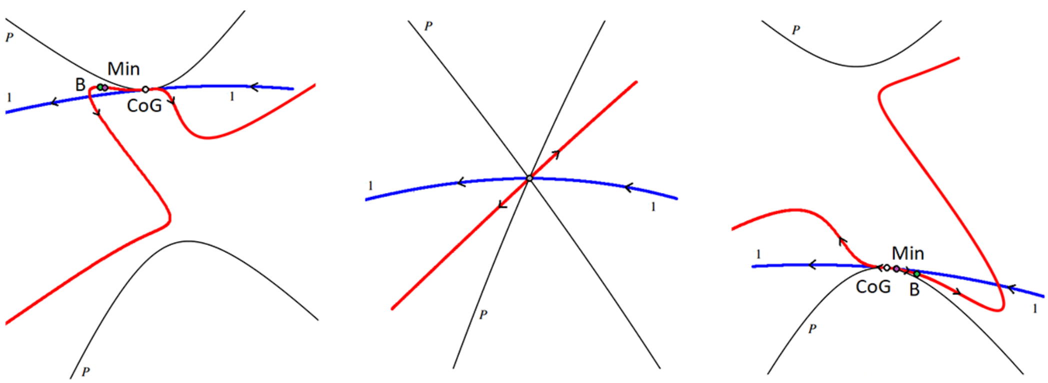


Figure 9. D_4^+ case with one real root of the quintic form (21), in a family of surfaces M_t where $t = 0$ is the middle diagram. For M_0 the curve labelled 1 is a branch of the V-curve along which the radius of the 5-point contact circle is nonzero; the other branch has this radius with limit 0 at the crossing. In this figure, the example surface M_0 chosen is $z = x^3 + xy^2 - \frac{4}{5}x^2y + \frac{1}{7}y^4$, and the family M_t is obtained by adding a small term tx^2.

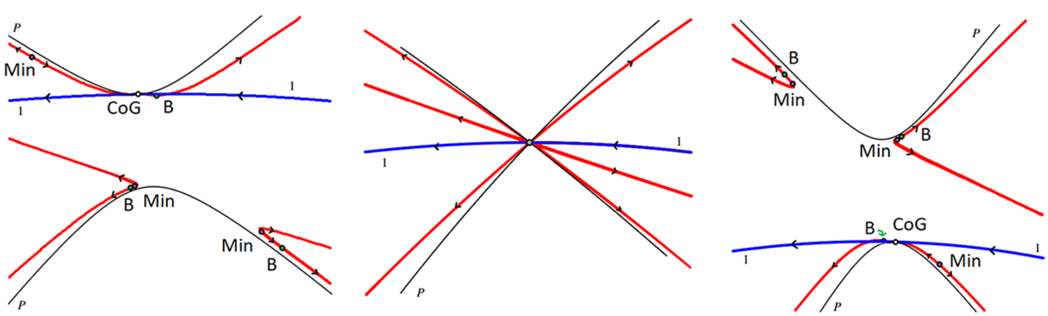
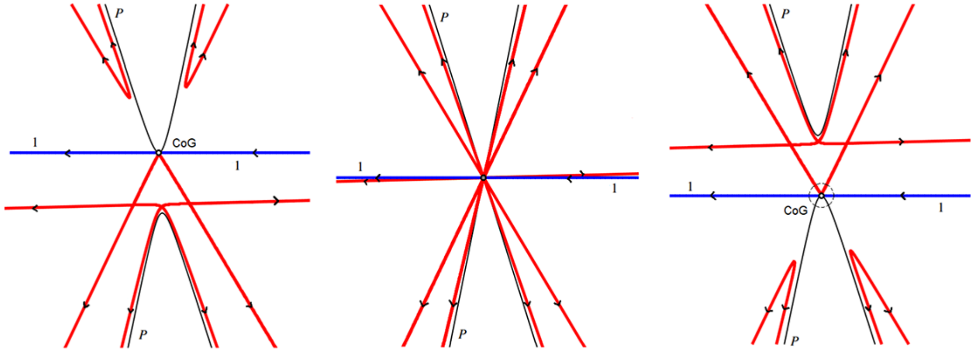


Figure 10. D_4^+ case with three real roots of the quintic form (21), in a family of surfaces M_t where $t = 0$ is the middle diagram. For M_0 the curve labelled 1 is the branch of the V-curve along which the radius of the 5-point contact circle is nonzero; the three other branches all have this radius with limit 0 at the crossing. The surface M_t in this example is $z = x^2y + xy^2 + 4y^3 + \frac{1}{7}x^4 + tx^2$.



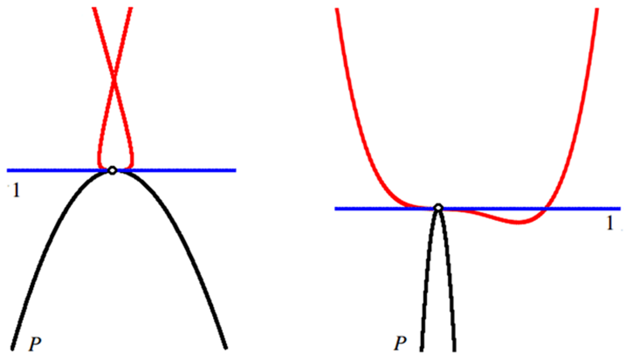


Figure 11. Upper diagram: D_4^+ case with five real roots of the quintic form (21), in a family of surfaces M_t where $t = 0$ is the middle diagram. The curve labelled 1 for M_0 is again the branch of the V-curve along which the radius of the 5-point contact circle is nonzero. The family of surfaces used in this example is $z = -x^2y + \frac{9}{2}xy^2 - \frac{51}{10}y^3 + \frac{1}{7}x^4 + ty^2$.

6.3. *Double cusp of Gauss (bigodron)*

This case refers to higher (A_4) contact between M_0 and its tangent plane at the origin and the Monge form of M_0 takes the form

$$z = y^2 - \sigma x^2 y + b_2 xy^2 + b_3 y^3 + \tfrac{1}{4}\sigma^2 x^4 + \ldots, \tag{22}$$

where $\sigma^2 b_2 + 2\sigma c_1 + 4d_0 \neq 0$. In this case circles in the plane $z = 0$ having 5-point contact with M_0 at the origin must have their centres on the y-axis but there is only *one* solution to the position of the centre, namely $(0, \frac{1}{\sigma}, 0)$. (Recall that $\sigma > 0$.) We shall see that this coincidence of solutions gives a smooth branch of the V-curve and also a singular branch.

The parabolic curve has the form $y = \frac{1}{2}\sigma x^2 + \ldots$; locally, the hyperbolic region is parameterized by $\{(x, y) : y > 0\}$.

Writing down the equations for the V-curve as usual we find in this case that x_0 and y_0 can be expressed in terms of u and $V = v - \frac{1}{\sigma}$ and that in the (u, V)-plane there is a locus whose lowest terms take the form

$$0 = \sigma^3 uV + \frac{\sigma(6\sigma^3 b_2^2 + 17\sigma^2 b_2 c_1 + 40\sigma b_2 d_0 + 6\sigma c_1^2 + 20 c_1 d_0)}{5(\sigma^2 b_2 + 2\sigma c_1 + 4d_0)} u^2.$$

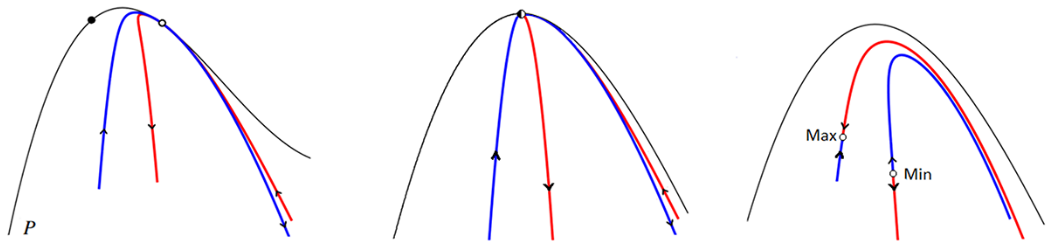


Figure 12. A_4 transition on the V-curve. The surface M_0 is $z = y^2 + x^2y + \frac{1}{4}x^4 - \frac{1}{5}x^5 - x^2y^2 + 2x^3y + 3xy^3$, and the family is given by adding a small multiple of xy. The thin black line is the parabolic curve and the thick lines are the two branches of the V-curve. On M_0 one branch has a 'rhamphoid cusp' (the two close together curves on the right-hand side of the middle diagram) and the other branch is smooth. As before the arrows indicate the direction of increasing absolute radius of the 5-point contact circle. On the left figure the filled-in circle is an elliptic cusp of Gauss and the white circle is a hyperbolic cusp of Gauss. The flecnodal curve is not shown.

The denominator of the fraction is nonzero by the assumption of exactly A_4 contact of M_0 with the plane $z = 0$. Thus there are two branches of the locus in the (u, V)-plane. One of them has the form $u = \text{constant} \times V + \ldots$ and the other $u = \text{constant} \times V^3 + \ldots$, with in fact no term in V^2. The first of these leads to a 'parabola' local component of the V-curve, of the form $y_0 = \text{constant} \times x_0^2 + \ldots$ but the other gives a local component whose initial terms, parameterized by V, are

$$\begin{aligned} x_0 &= -\frac{\sigma^4}{5(\sigma^2 b_2 + 2\sigma c_1 + 4d_0)}V^2, \\ y_0 &= \frac{\sigma^9}{50(\sigma^2 b_2 + 2\sigma c_1 + 4d_0)^2}V^4 - \frac{2\sigma^{10}(\sigma^2 b_2 + 6\sigma c_1 + 20d_0)}{125(\sigma^2 b_2 + 2\sigma c_1 + 4d_0)^3}V^5. \end{aligned}$$

Of course, both components lie in the hyperbolic region of M_0.

Using $\mathcal{A}$-equivalence, the above singularity is not in fact equivalent to the standard 'rhamphoid cusp' (t^2, t^5) but to (t^2, t^7). Of course this equivalence is not Euclidean invariant.

6.4. *Singular V-curve*

This is the special case where both components of the vector (7) are zero. In that situation generically the V-curve will have a non-degenerate quadratic form for its 2-jet, corresponding to the component of the intersection $M_0 \cap T_pM_0$ tangent to the x-axis in the hyperbolic case. Thus the V-curve will have an unstable crossing or isolated point.

In the case of a crossing this can be interpreted as saying the following. Corresponding to *one* of the branches of the intersection $M_0 \cap T_pM_0$, having a 5-point contact circle tangent to this branch, there are *two* distinct directions in which $p \in M_0$ can move away from the origin and still have a 5-point contact circle tangent to the intersection of M_0 with its tangent plane at p.

In the case of an isolated point, which in the family M_t will open out into a closed loop, this is a way in which the V-curve can acquire a single left or right loop, in contrast to the situation depicted in Figure 6 where two loops appear on the V-curve.

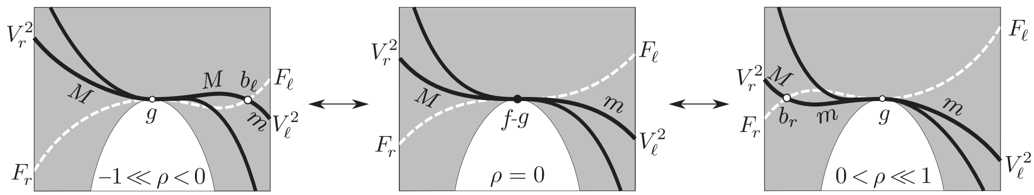


Figure 13. A flecgodron transition.

This is not the same situation as a V-curve crossing. There, *each* branch of $M \cap T_pM$ contributes a smooth branch (one left and one right) of the V-curve, whose crossing is stable under small perturbations of M.

6.5. *Flecgodrons: the coincidence of a cusp of Gauss and a biflecnode*

At a simple cusp of Gauss g with $\rho = 0$ ($c_0 = 0$), exactly one value of r (in Lemma 4.3) is zero. This means that the corresponding 'circle' is a straight line having 5-point contact with M at g (exactly 5-point contact requires $d_0 \neq 0$). Thus g is a cusp of Gauss and is also a biflecnode.

Flecgodron. A simple cusp of Gauss at which the asymptotic tangent line and the surface M have 5-point contact is called *a flecgodron*.

A surface in general position has no flecgodron: under any small generic deformation of a surface M having a flecgodron the condition $\rho = 0$ is destroyed. Perturbing M inside a generic 1-parameter family of surfaces (Figure 13) $\{M_t\}$, there is an isolated parameter value t_0 (near 0) whose corresponding surface M_{t_0} has a simple flecgodron.

In Figure 13, a left biflecnode b_ℓ approaches g as $\rho \to 0^-$, it coincides with g when $\rho = 0$ and, as ρ is growing, this point leaves g as a right biflecnode b_r. By Remark 4.6 and Proposition 4.7, a biflecnode near g is the intersection point of the flecnodal curve F with the local component V^2 of the V-curve (V^2 has the same local orientation right-to-left as F near g).

7. Further investigations

In Section 6 we have formally investigated the 'transitional moment' M_0 of the families studied, and explained the transitions by means of examples, but we reserve for further work a formal classification of the families themselves, including the behaviour of the V-curve relative to the flecnodal curve during these transitions. Global results about the V-curve on a compact surface are also for further investigation. We do not know whether there are interesting affinely invariant generalizations, say to contact of surfaces with conics in their tangent planes. There are also questions concerning the symmetry sets or medial axes of the families of curves obtained as plane sections of a smooth surface parallel to the tangent plane (as mentioned in Introduction); these will be investigated elsewhere.

Funding

Peter Giblin and Ricardo Uribe-Vargas acknowledge support from the Research Centre in Mathematics and Modelling at the University of Liverpool (spring 2008) and from the Engineering and

Physical Sciences Research Council (autumn 2008, grant number EP/G000786/1) when this problem was originally studied. Graham Reeve acknowledges support from Liverpool Hope University to deliver a talk on this subject at the 6th International Workshop on Singularities in Generic Geometry and its Applications, Valencia, October 2019. Ricardo Uribe-Vargas acknowledges support from Laboratory Solomon Lefschetz UMI2001 CNRS, Universidad Nacional Autonoma de México and the hospitality of the University of Liverpool during his two visits. The Institut de Mathématiques de Bourgogne receives support from the EIPHI Graduate School (contract ANR-17-EURE-0002).

References

1. V. I. Arnold, S. M. Gusein-Zade and A. N. Varchenko, *Singularities of Differentiable Maps*, Vol. I, Monographs in Mathematics, Vol. 82, Birkhäuser Boston Inc., Boston, MA, 1985.

2. T. Banchoff, T. Gaffney and C. McCrory, *Cusps of Gauss Mappings*, Pitman Advanced Publishing Program, London, 1982.

3. J. W. Bruce, Generic geometry and duality, *Singularities (Lille, 1991)*, Vol. 201, London Math. Soc. Lecture Note Ser., Cambridge Univ. Press, Cambridge, 1994, 29–59.

4. J. W. Bruce, Lines, circles, focal and symmetry sets, *Math. Proc. Cambridge Philos. Soc.* **118** no. 3 (1995), 411–436.

5. J. W. Bruce and P. J. Giblin, Generic geometry, *Amer. Math. Monthly* **90** no. 8 (1983), 529–545.

6. J. W. Bruce and P. J. Giblin, Growth, motion and one-parameter families of symmetry sets, *Proc. Royal Soc. Edinburgh* **104A** no. 3-4 (1986), 179–204.

7. J. W. Bruce, P. J. Giblin and F. Tari, Parabolic curves of evolving surfaces, *Int. J. Computer Vision* **17** no. 3 (1996), 291–306.

8. J. W. Bruce, P. J. Giblin and F. Tari, Families of surfaces: height functions, Gauss maps and duals, *Real and Complex Singularities* (Eds. W. L. Marar), Vol. 333, Pitman Research Notes in Mathematics, London, 1995, 148–178.

9. J. W. Bruce, P. J. Giblin and F. Tari, Families of surfaces: focal sets, ridges and umbilics, *Math. Proc. Camb. Phil. Soc.* **125** no. 2 (1999), 243–268.

10. A. Diatta and P. J. Giblin, Vertices and inflexions of plane sections of surfaces in $\mathbb{R}^3$, *Trends in Mathematics, Real and Complex Singularities*, Birkhäuser, Basel, 2006, 71–97.

11. A. Dimca and C. G. Gibson, On contact germs from the plane to the plane, *Proc. Symp. in Pure Math.* **40** no. 1 (1983), 277–282.

12. T. Fukui, M. Hasegawa and K. Nakagawa, Contact of a regular surface in Euclidean 3-space with cylinders and cubic binary differential equations, *J. Math. Soc. Japan* **69** no. 2 (2017), 819–847.

13. P. L. Hallinan, G. G. Gordon, A. L. Yuille, P. Giblin and D. Mumford, *Two-and Three-Dimensional Patterns of the Face*, (Eds. A. K. Peters), CRC Press, Boca Raton, 1999.

14. S. Izumiya, M. C. Romero-Fuster, M. A. S. Ruas and F. Tari, *Differential Geometry From a Singularity Theory Viewpoint*, Singapore: World Scientific Pub. Co., 2015.

15. M. Kazarian and R. Uribe-Vargas, Characteristic points, fundamental cubic form and Euler characteristic of projective surfaces, *Moscow Math. J.* **20** no. 3 (2020), 511–530.

16. J. J. Koenderink, *Solid Shape*, Cambridge, MA, USA: M.I.T. Press, 1990.

17. J. A. Montaldi, On contact between submanifolds, *Michigan Math. J.* **33** no. 2 (1986), 195–199.

18. J. A. Montaldi, Surfaces in 3-space and their contact with circles, *J. Diff. Geom.* **23** no. 2 (1986), 109–126.

19. J. Montaldi, On generic composites of maps, *Bull. London Math. Soc.* **23** no. 1 (1991), 81–85.

20. I. R. Porteous, *Geometric Differentiation*, 2nd edn., Cambridge University Press, Cambridge, 2001.

21. R. Uribe-Vargas, A projective invariant for swallowtails and godrons, and global theorems on the flecnodal curve, *Moscow Math. J.* **6** no. 4 (2006), 731–768.

22. R. Uribe-Vargas, *Surface evolution, implicit differential equations and pairs of Legendrian fibrations*, Preprint, 2002, A revised version has been submitted for publication (2020).

23. R. Uribe-Vargas, On projective umbilics: a geometric invariant and an index, *J. Singularities* **17** (2018), 81–90.