Optimization or Bayesian strategy? Performance of the Bhattacharyya distance in different algorithms of stochastic model updating

Sifeng Bi[1,*] , Michael Beer[2,3,4], Jingrui Zhang[1], Lechang Yang[5], Kui He[1]

1. School of Aerospace Engineering, Beijing Institute of Technology, 100081 Beijing, China
2. Institute for Risk and Reliability, Leibniz University Hannover, 30167 Hannover, Germany
3. Institute for Risk and Uncertainty, University of Liverpool, Liverpool L69 7ZF, the UK
4. International Joint Research Center for Engineering Reliability and Stochastic Mechanics, Tongji University, Shanghai 200092, China
5. School of Mechanical Engineering, University of Science and Technology Beijing, 100083Beijing, China

* Corresponding Author: sifeng.bi@bit.edu.cn

Abstract

The Bhattacharyya distance has been developed as a comprehensive uncertainty quantification metric by capturing multiple uncertainty sources from both numerical predictions and experimental measurements. This work pursues a further investigation of the performance of the Bhattacharyya distance in different methodologies for stochastic model updating, and thus to prove the universality of the Bhattacharyya distance in various currently popular updating procedures. The first procedure is the Bayesian model updating where the Bhattacharyya distance is utilized to define an approximate likelihood function and the transitional Markov chain Monte Carlo algorithm is employed to obtain the posterior distribution of the parameters. In the second updating procedure, the Bhattacharyya distance is utilized to construct the objective function of an optimization problem. The objective function is defined as the Bhattacharyya distance between the samples of numerical prediction and the samples of the target data. The comparison study is performed on a four degree-of-freedoms mass-spring system. A challenging task is raised in this example by assigning different distributions to the parameters with imprecise distribution coefficients. This requires the stochastic updating procedure to calibrate not the parameters themselves, but their distribution properties. The second example employs the GARTEUR SM-AG19 benchmark structure to demonstrate the feasibility of the Bhattacharyya distance in the presence of practical experiment uncertainty raising from measuring techniques, equipment, and subjective randomness. The results demonstrate the Bhattacharyya distance as a comprehensive and universal uncertainty quantification metric in stochastic model updating.

Keywords: Bayesian model updating, optimization model updating, Bhattacharyya distance, uncertainty quantification, Markov chain Monte Carlo

1 Introduction

The deterministic model updating has been developed as a typical technique since the 50s of the last century [1,2], while in recent decades the tendency of uncertainty analysis promotes the research on this topic from the deterministic sense to the stochastic sense [3]. The stochastic model updating [4,5] refers to the procedure to calibrate not only the model parameters themselves but also their uncertainty characteristics, such that the prediction of the numerical model is not committed to the maximum fidelity to a single set of experimental data, but to represent the uncertainty characteristics of multiple sets of measurements.

The uncertainties during stochastic model updating, i.e. the reasons for the discrepancy between the numerical predictions and experimental measurements, can be summarized as follows.

- Parameter uncertainty. The input parameters of the numerical model, e.g. material properties and geometric sizes, are not fully determined because of manufacturing tolerance, material inhomogeneity, novel composites, etc.
- Modeling uncertainty. The numerical models contain inevitable approximations and simplifications of the physical system.
- Experiment uncertainty. The experimental measurements are used as the target of model updating. However, the measurements themselves are uncertain because of the hard-to-control randomness during experiment processes, such as environmental noise, subject judgment, etc.

To appropriately handle the above uncertainties, the uncertainty quantification (UQ) metric (also known as calibration metric) is especially important for stochastic model updating to comprehensively and quantitatively measure the difference between the numerical prediction and experimental measurement. Ref. [6] proposes a series of distance-based UQ metrics using the Euclidean distance, Mahalanobis distance, and Bhattacharyya distance, whose performances in model updating and validation are investigated in detail.

The Euclidean distance is a typically deterministic distance focusing on the geometry difference between two discrete points, and thus no uncertainty information is taken into account by the direct Euclidean distance in a single-experiment-single-simulation strategy. Going a step further, limited uncertainty information of the covariance is considered by the Mahalanobis distance, where the covariance matrix between two random variables is taken as the weighting factor based on the Euclidean distance. However, the Mahalanobis distance is proven to be inappropriate for model updating [6] since the covariance information merely measures the dependency between two datasets but not the similarity between these two datasets. This is demonstrated in Ref. [6] where both the point-to-population Mahalanobis distance and the population-to-population (also termed as “pooled”) Mahalanobis distance are assessed in both updating and validation processes.

Other more comprehensive statistical distances are the Bhattacharyya distance and the Kullback-Leibler divergence, which are capable of measuring the difference between two probability distributions. Considering a distribution of simulation data $\mathbf{Y}_{sim}$ and a distribution of experimental data $\mathbf{Y}_{exp}$, the Kullback-Leibler divergence [7] $D_{KL}(\mathbf{Y}_{sim}, \mathbf{Y}_{exp})$ is the expectation of the logarithmic difference between the $\mathbf{Y}_{sim}$ and $\mathbf{Y}_{exp}$, where the expectation is taken using the probability of $\mathbf{Y}_{sim}$. However, the Kullback-Leibler divergence has a defect that it is a non-symmetric measure, i.e. $D_{KL}(\mathbf{Y}_{sim}, \mathbf{Y}_{exp}) \neq D_{KL}(\mathbf{Y}_{exp}, \mathbf{Y}_{sim})$. This non-symmetric property leads to instability of the model updating process where the bidirectional evaluation between the simulation data and the experimental data is performed. As a comparison, the Bhattacharyya distance is a symmetric measure between two probabilistic distributions and thus is specifically suitable for the random sampling process in stochastic model updating. Furthermore, in practice application only the limited data samples are available instead of the continuous probabilistic distribution. The evaluation of the Bhattacharyya distance can make full use of the discrete data samples by flexibly selecting the bin width in the binning algorithm [8]. Consequently, the Bhattacharyya distance is employed as the UQ metric in the following updating techniques.

Although the Bhattacharyya distance has the potential to be utilized as a feasible and convenient metric in model updating, the direct Monte Carlo procedure utilized, for example, in Refs. [6] demands large calculation cost, which obstructs the generalization of the Bhattacharya distance in practical applications. As an alternative of the direct Monte Carlo procedure, the Bayesian approach is popular for its adaptability in cases with limited data and large calculation demand [9,10]. The Bayesian approach provides the possibility to integrate model verification, validation, and calibration activities for overall UQ in different types of engineering systems [11]. Ref. [8] proposes a specific Bayesian framework with newly developed approximate likelihood functions employing the Markov Chain Monte Carlo (MCMC) algorithm [12], where the stochastic model updating is performed in a similar framework as the deterministic model updating by simply replacing the Euclidean distance with the Bhattacharyya distance in the likelihood function. This treatment leads to a largely reduced calculation cost, even when solving the complex NASA UQ challenge problem [13].

Besides the Bayesian updating approach, one of the popular updating approaches is the sensitivity-based technique [14,15], which constructs the model updating problem into an optimization procedure to minimize the discrepancy between the numerical prediction and experimental measurement. A representative work of the sensitivity-based optimization approach is the covariance model updating [16,17] where both the parameter mean vector and the covariance matrix are employed to construct the objective function. The Euclidean distance is used to measure the difference between the mean vectors, and the Frobenius norm is used to measure the difference between the covariance matrices of the experimental and simulated data. This approach is an improvement of the Mahalanobis distance where the covariance information merely measures the dependency between two data sets. This why the covariance updating approach performs well when calibrating the variance/covariance information among parameters, while the Mahalanobis distance is proven to be inappropriate in Ref. [6].

As a summary, the sensitivity-based optimization updating employs a simpler principle, compared with the Bayesian updating, to directly define an objective function using the distance between the predictions and measurements while the parameters (or their uncertainty characteristics) are calibrated as optimal variables in the problem. In this context, the Bhattacharyya distance can be conveniently embedded into the optimization problem by acting as the objective function.

In this work, a comparison study between the Bayesian updating and optimization updating approaches are performed with both procedures using the Bhattacharyya distance as the calibration metric. This comparison study is performed in both simulated and experimental examples. The simulated example employs a four Degree-of-Freedoms (DoFs) spring-mass system whose input parameters are assigned to obey different distributions. Moreover, the distribution coefficients are uncertain. That is to say, the calibration objects are not the parameters themselves but their distribution coefficients, such as the mean, variance, interval bounds, etc. The second example is performed on the GARTEUR SM-AG19 Benchmark structure. Multiple measurements tested by different organizations using different techniques and equipment are employed as the updating reference. The application of the multiple measurement fully presents various sources of experimental uncertainty, and thus presents a pertinent task for stochastic model updating. The performance of the Bhattacharyya distance in the Bayesian and optimization procedures are compared in different aspects, such as the precision of the inputs, the prediction of the output distributions, etc. The final results confirm the feasibility and generalization of the Bhattacharyya distance as a UQ metric in different updating methodologies.

2 Theories and methods

2.1 Bhattacharyya distance between uncertain data samples

A numerical simulation process with *n* input parameters and *m* output features can be described as a deterministic function:

$$\mathbf{y} = f(\mathbf{x}), \tag{1}$$

where $\mathbf{x} = [x_1, x_2, \dots, x_n]^T$ and $\mathbf{y} = [y_1, y_2, \dots, y_m]^T$ are the input and output vectors, respectively. The deterministic function between the inputs and the outputs can be either a sophisticated finite element (FE) model or a meta-model with simple mathematical relation, whereas any numerical model has inevitable modeling uncertainty as described in the Introduction section.

During the deterministic model updating, a single-test-single-simulation scenario is employed, implying only one set of experimental data is used. In contrast, the stochastic updating employs the multiple-test-multiple-simulation scenario to represent the uncertainties in both simulation and experiment. Suppose the number of the numerical simulation is executed N_{sim} times, the numerical prediction data appears as a matrix $\mathbf{Y}_{sim} \in \mathbb{R}^{N_{sim} \times m}$:

$$\mathbf{Y}_{sim} = \begin{bmatrix} y_{11} & \cdots & y_{1m} \\ \vdots & \ddots & \vdots \\ y_{N_{sim}1} & \cdots & y_{N_{sim}m} \end{bmatrix}. \tag{2}$$

Similarly, the experimental data set $\mathbf{Y}_{exp} \in \mathbb{R}^{N_{exp} \times m}$ has the structure as

$$\mathbf{Y}_{exp} = \begin{bmatrix} y_{11} & \cdots & y_{1m} \\ \vdots & \ddots & \vdots \\ y_{N_{exp}1} & \cdots & y_{N_{exp}m} \end{bmatrix}. \tag{3}$$

After the simulated and experimental data is available, the calibration metric is defined based on the two data sets with as much as possible uncertainty information captured. The most typical metric is the Euclidean distance considering only the mean of the data with the definition as:

$$d_E\left(\mathbf{Y}_{exp}, \mathbf{Y}_{sim}\right) = \sqrt{(\overline{\mathbf{Y}_{exp}} - \overline{\mathbf{Y}_{sim}})(\overline{\mathbf{Y}_{exp}} - \overline{\mathbf{Y}_{sim}})^T}, \tag{4}$$

where $\overline{\mathbf{Y}_{\blacksquare}} \in \mathbb{R}^{1 \times m}$ is a row vector of the means of the matrix. The direct evaluation of the Euclidean distance using

Eq. (4) is insufficient to measure the similarity between two probability distributions. A more comprehensive distance containing more information of uncertainties, i.e. the Bhattacharyya distance, is proposed herein to give a quantitative measure of the discrepancy between the two data sets. The original definition of the Bhattacharyya distance is given as

$$d_B\left(\mathbf{Y}_{exp}, \mathbf{Y}_{sim}\right) = -\log\left\{\int_{\mathbb{Y}} \sqrt{P(y_{exp})P(y_{sim})}\mathrm{d}y\right\}, \tag{5}$$

where $P(y_{\blacksquare})$ is the Probability Density Function (PDF) of the sample; $\int_{\mathbb{Y}} \blacksquare \mathrm{d}y$ denotes the integration over the whole output space; $\mathbb{Y}$ is the m-dimensional joint-probability space of all outputs. Eq. (5) indicates the Bhattacharyya distance is essentially a measure of the overlap between two probability distributions. Consequently, not only the mean value but the whole distribution properties among multivariable are taken into account. However, the direct evaluation of Eq. (5) is impractical because the precise estimation of the PDF is generally unavailable, especially when the amount of experimental samples is limited. To overcome this obstruction, an approximate Bhattacharyya distance based on probability mass functions (PMFs) of discrete distributions is proposed as

$$d_B\left(\mathbf{Y}_{exp}, \mathbf{Y}_{sim}\right) = -\log\left(\sum_{k=1}^{N_{bin}} \sqrt{PM_{exp}^{(k)} PM_{sim}^{(k)}}\right), \tag{6}$$

where $PM_{\blacksquare}^{(k)}$ is the probability mass of the k^{th} bin. The bins are generated as a grid within the lower and upper bounds of all values of the experimental and simulated samples $\mathbf{Y}_{exp}$ and $\mathbf{Y}_{sim}$. Note that, in the case of multivariable, the grid is created in a m-dimensional space to replace the m-dimensional joint-PDF. More detailed information of the binning algorithm can be referred to Ref. [8]. The binning algorithm makes the Bhattacharyya distance especially appropriate for stochastic model updating even when the distribution of the outputs cannot be precisely estimated.

2.2 Bayesian updating employing the MCMC algorithm

The Bayesian model updating approach is based on the Bayes' theorem [9,18]

$$P\left(\mathbf{X}\middle|\mathbf{Y}_{exp}\right) = \frac{P_L\left(\mathbf{Y}_{exp}\middle|\mathbf{X}\right)P(\mathbf{X})}{P\left(\mathbf{Y}_{exp}\right)}, \tag{7}$$

where

- $P(\mathbf{X})$ is the prior distribution of the inputs determined based on the prior knowledge of the system and expert experience;
- $P\left(\mathbf{X}\middle|\mathbf{Y}_{exp}\right)$ is the posterior distribution of the inputs conditional to the experimental data;
- $P\left(\mathbf{Y}_{exp}\right)$ is the normalization factor ensuring the integral of the posterior PDF of the input $P\left(\mathbf{X}\middle|\mathbf{Y}_{exp}\right)$ equal to one;
- $P_L\left(\mathbf{Y}_{exp}\middle|\mathbf{X}\right)$ is the likelihood function, theoretically defined as the probability of the experimental data conditional to each instance of the inputs.

The likelihood function is theoretically expressed as

$$P_L\left(\mathbf{Y}_{exp}\middle|\mathbf{X}\right) = \prod_{k=1}^{N_{exp}} P(\mathrm{y}_k|\mathbf{x}), \tag{8}$$

where N_{exp} is the number of the experimental samples; $P(\mathrm{y}_k|\mathbf{x})$ is the PDF of the k-th experimental sample conditional to the instance of inputs. Eq. (8) requires the estimation of the PDF for each of the N_{exp} experimental observations, which introduce considerable calculation cost. Furthermore, the precise estimation of the PDFs requires a large number of numerical model evaluations to generate a large number of simulated output samples. Hence, the calculation cost for precise evaluation of Eq. (8) is generally unrealistic for practical applications.

Consequently, it is important to define an alternative likelihood function, as long as the new likelihood still contains the information of the existing experimental outputs and the numerical inputs. Considering the Bhattacharyya distance defined in Section 2.1, a customized likelihood based on the Gaussian function is defined as

$$P_L\left(\mathbf{Y}_{exp}\middle|\mathbf{X}\right) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left\{-\frac{d_B\left(\mathbf{Y}_{exp},\mathbf{Y}_{sim}\right)^2}{2\sigma^2}\right\} \tag{9}$$

where $d_B(\mathbf{Y}_{exp},\mathbf{Y}_{sim})$ is the Bhattacharyya distance defined in Section 2.1, σ is a pre-defined coefficient controlling the centralization of the resulting posterior distribution of the Bayesian procedure. A large σ would lead to a flat PDF of the posterior distribution (e.g. the histogram of σ_2 in Figure 3), and inversely, a small σ leads to peaked poles of the distributions (e.g. the histogram of μ_3 in Figure 3). A peaked posterior distribution implies the parameter is converging to an explicit value, and thus the outcome of Bayesian updating is satisfying. However, a too small σ would lead to convergence difficulty and requires more calculation effort. The determination of σ is hence based on specific applications, and a suggested interval of σ is $[10^{-3},10^{-1}]$. The customized likelihood serves as a nice connection between the Bhattacharyya distance and the Bayesian updating framework. Furthermore, the new likelihood function is capable of capturing comprehensive uncertainty information from both the experimental measurements and the numerical simulation, with largely reduced calculation burden compared with the theoretical likelihood function shown in Eq. (8).

Another important element of the Bayes' theorem in Eq. (6) is the normalization factor $P\left(\mathbf{Y}_{exp}\right)$. A direct evaluation of the factor requires integral processes on the PDFs of the experimental data and the posterior distribution of the inputs, which lead to intractable difficulties in the overall Bayesian procedure. Hence, the transitional MCMC algorithm [19] is proposed in this work by introducing a series of intermediate PDFs of the inputs, which gradually converge to the posterior distribution, and thus avoiding the direct evaluation of the integral. More applications of the transitional MCMC algorithm can be found, for example, in model selection [20], structural health monitoring [21], and other industrial applications [22].

2.3 Optimization-based updating employing the genetic algorithm

A more direct principle to utilize the Bhattacharyya distance in model updating is to solve an optimization problem where the discrepancy between the experimental measurements and numerical simulations is minimized. The optimization problem is formulated as:

Find $\widehat{\mathbf{X}}$, minimizing the objective function

$$f_{obj}\left(\mathbf{Y}_{exp}, f(\mathbf{X})\right) = d_B\left(\mathbf{Y}_{exp}, f(\mathbf{X})\right), \tag{10}$$

with the constraints

$$\mathbf{X} \in [\underline{\mathbf{X}}, \overline{\mathbf{X}}], \tag{11}$$

where $[\underline{\mathbf{X}}, \overline{\mathbf{X}}]$ is the interval of the inputs determined based on the prior knowledge of the system.

However, one challenge when using the Bhattacharyya distance within an optimization framework is that the distance itself is random. The evaluation of the Bhattacharyya distance requires a set of predicted outputs with the size as N_{sim}, as shown in Eq. (2). These output samples are obtained from N_{sim} times model evaluations using N_{sim} input samples generated by the Monte Carlo simulation, which is the source of the randomness of the Bhattacharyya distance.

Because of the randomness of the objective function, the typical gradient-based methods and simplex methods tend to be incapable of solving the optimization problem herein. Consequently, the genetic algorithm is employed in this work. The genetic algorithm is a population-based searching algorithm inspired by natural evolution functions such as selection, crossover, and mutation. The genetic algorithm is popular in academic and industrial fields because of its simple principle, extensive application range, and feasibility of solving large-scale, strong-nonlinear, and complex problems. It is capable of finding a global solution in the overall search space [23]. The performance of the Bhattacharyya distance within the genetic algorithm optimization framework is assessed in the following case study section.

3 Case study I: The 4-DoF simulated system

3.1 Problem description

The case study is performed on a 4-DoF mass-spring system with four masses and eight springs, as shown in Figure 1. The stiffness coefficients k_1-k_4 and masses m_1-m_2 are supposed to be uncertain with the uncertainty characteristics shown in Table 1. Different from the deterministic updating task, the uncertain parameters in this problem are assumed to be random variables obeying probability distributions. Moreover, the distributions are presented as imprecise probability, whose distribution coefficients are not fully determined but fall within pre-defined intervals as listed in the 3rd column of Table 1. This means the updating objects are not the uncertain parameters themselves, but their distribution coefficients. The total number of the coefficients to be updated in this problem is 12. Besides the uncertain parameters in Table 1, the remaining parameters are set to be constants with determined values: $k_{5\text{-}8}$=5.0 N/m, m_3=0.3 kg, and m_4=0.5kg.

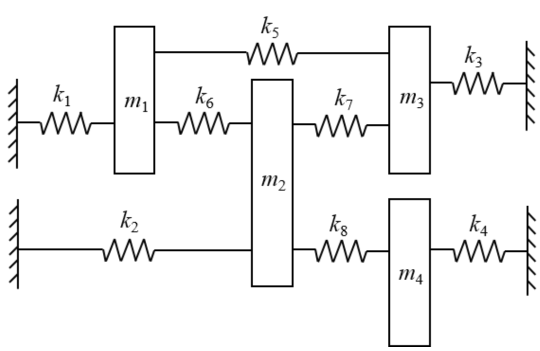


Figure 1: The proposed 4-DoF mass-spring system

The outputs of the system are the four eigen-frequencies: f_1-f_4. The experimental measurements, i.e. the target outputs, are generated from multiple sets of input parameters through multiple model evaluations. Here the model refers to the mathematic equations to calculate the frequencies using the masses and springs as illustrated in Figure 1. The multiple sets of parameters are generated by assigning a set of "target values" of the distribution coefficients of the input parameters as shown in the 4th column of Table 1. Based on these target distribution coefficients, 500 random samples of the input parameters are generated, and subsequently 500 outputs samples are obtained through the model evaluations. In the last column of Table 1, a set of initial values of the distribution coefficients is arbitrarily assigned within the pre-defined intervals but different from the true values. Based on these initial coefficients, 1000 random samples of the input parameters are generated, and similarly 1000 samples of the initial outputs are obtained. Figure 2 illustrates the relative positions of the target outputs and the initial outputs. The diagonal subfigures compare the histograms of the target and initial frequencies. As shown in Table 1, the initial values and the target values of the distribution coefficients are different. Hence the scatters of the initial outputs are clearly apart from the targets as illustrated in Figure 2.

Table 1: The uncertainty properties of the input parameters

Parameter	**Distribution format**	**Distribution coefficient**	**Target value**	**Initial value**
k_1	Gaussian	$\mu_1 \in [3.0, 8.0]$	$\mu_1 = 7.5$	$\mu_1 = 7.2043$
		$\sigma_1 \in [0.0, 0.5]$	$\sigma_1 = 0.1$	$\sigma_1 = 0.0924$
k_2	Gaussian	$\mu_2 \in [5.0, 10]$	$\mu_2 = 6.0$	$\mu_2 = 7.8050$
		$\sigma_2 \in [0.0, 0.5]$	$\sigma_2 = 0.45$	$\sigma_2 = 0.3442$
k_3	Gaussian	$\mu_3 \in [7.0, 12]$	$\mu_3 = 8.0$	$\mu_3 = 11.869$
		$\sigma_3 \in [0.0, 0.5]$	$\sigma_3 = 0.25$	$\sigma_3 = 0.2860$
k_4	Gaussian	$\mu_4 \in [4.0, 9.0]$	$\mu_4 = 5.0$	$\mu_4 = 8.2932$
		$\sigma_4 \in [0.0, 0.5]$	$\sigma_4 = 0.4$	$\sigma_4 = 0.4944$
m_1	Uniform	$a_1 \in [0.2, 0.5]$	$a_1 = 0.25$	$a_1 = 0.2571$
		$b_1 \in [0.6, 0.9]$	$b_1 = 0.8$	$b_1 = 0.6767$
m_2	Uniform	$a_2 \in [0.4, 0.7]$	$a_2 = 0.6$	$a_2 = 0.4132$
		$b_2 \in [1.0, 1.5]$	$b_2 = 1.3$	$b_2 = 1.3797$

The case study presents an extremely challenging task to calibrate the distribution coefficients of input parameters, with the objective to tune the predicted joint-distributions of outputs to the target distributions. The target distributions are not the standard distribution functions, e.g. the normal or Gaussian distributions, but the complicated distributions with implicit functions, from which the sample scatters become irregular, as illustrated in Figure 2. This task requires a comprehensive calibration metric to capture as much as possible information of the distributions in a stochastic model updating framework. Figure 2 illustrates the relative position between the scatters of the initial simulation and the target samples. On the upper-triangular part, the Euclidean distance and Bhattacharyya distance between the initial simulations and the target samples are provided, which can be compared with the updated ones in Figures 4 and 6 in the following sections.

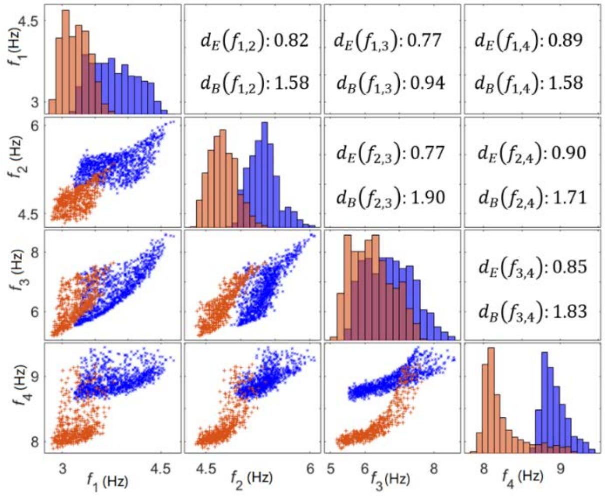


Figure 2: The scatters and the Euclidean and Bhattacharyya distances between the target samples (in orange) and the initial samples (in blue) with the unit in Hz (frequency)

3.2 Updating result of the Bayesian procedure

The Bayesian updating is performed using the Bhattacharyya distance to construct the likelihood function as shown in Eq. (9). In this example, the Bayesian updating employing the transitional MCMC algorithm converges after 13 iterations with the histograms of the updated samples of the 12 coefficients as illustrated in Figure 3.

The prior distributions of all the 12 coefficients are set to be uniform distributions within the pre-defined intervals as shown in Table 1. After the Bayesian updating procedure, if a posterior distribution has a strong centralization effect (e.g. μ_3 in Figure 3), it means that the Bayesian procedure is well capable of calibrating this coefficient. In contrast, if a posterior distribution is still close to uniform such as the one of σ_2 in Figure 3, it means the Bayesian procedure is ineffective in calibrating this coefficient.

The calibrations result of the coefficients in Figures 3 shows that the Bayesian updating procedure employing the Bhattacharyya distance is effective for most of the 12 coefficients, except σ_2. The possible explanation of the ineffectiveness for σ_2 is that the sensitivity of σ_2 is relatively low to the outputs investigated in this example, and thus it is intractable to update a coefficient which is insensitive to the target.

The insensitivity of σ_2 is also demonstrated in Figure 4, where the comparison between the target and updated scatters and histograms of the outputs are illustrated. Although σ_2 is not appropriately calibrated in the Bayesian updating, the updated distributions of the outputs can still fit with the target distributions, implying σ_2 is insensitive to the output distributions. Comparing Figure 4 and Figure 2, it can be obviously concluded that the distribution properties of the outputs have been precisely updated to capture the uncertainty information of the target samples. In Figure 4, it is illustrated that the updated histograms fit well with the target histograms, where the brown histogram denotes the overlap part of these two histograms.

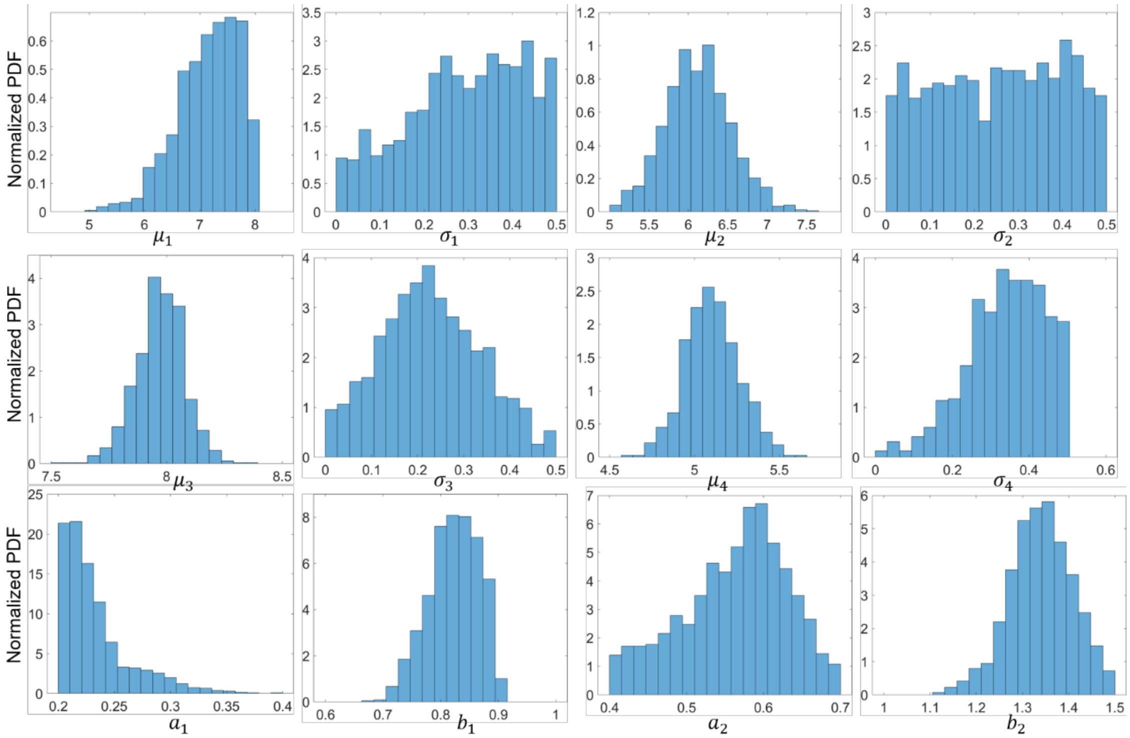


Figure 3: Posterior distributions of the 12 updating coefficients presented in histograms

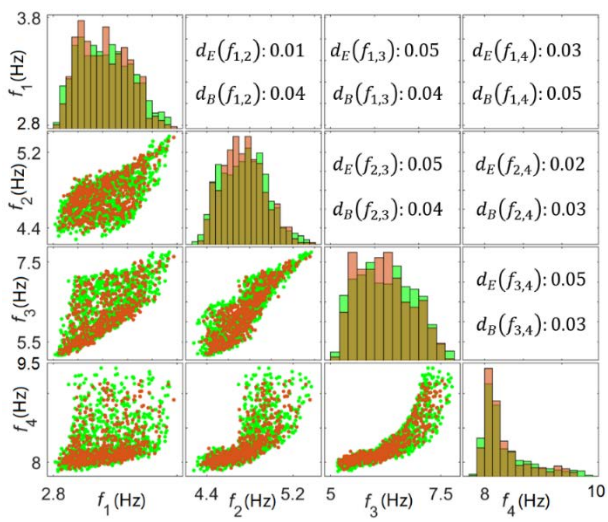


Figure 4: The scatters and the Euclidean and Bhattacharyya distances between the target samples (in orange) and the updated samples (in green)

The updated values of the coefficients are obtained by evaluating the means of the posterior distributions in Figures 3 and 4, and listed in Table 2. The percentage errors compared with the target values are provided in the parentheses after the updated values. The Absolute Mean Error (AME) of all the 12 coefficients are given in the last row of the table as 23.98%, which is relatively large for a general model updating procedure. Note that, however, most of the means of Gaussian distributions and bounds of the uniform distributions have quite high updating precisions (with absolute errors less than 7%). The high AME value is caused by the errors of the standard deviations, especially for σ_1 and σ_2 (highlighted in red and bold in Table 2). This fulfills the general experience that the variance information of the distribution is more difficult to be precisely updated than the mean. Nevertheless, the final updated distributions of the outputs are coincident with the target distributions as shown in Figure 4, demonstrating the feasibility of the Bhattacharyya distance within the Bayesian updating procedure to capture comprehensive uncertainty information in both predicted and target data.

Table 2: Updating results of the input distribution coefficients using the Bayesian and optimization approaches

Updating coefficient	Target value	Updated value [a]	
		Bayesian	Optimization
μ_1	7.5	7.151 (-4.65)	7.845 (4.59)
σ_1	0.1	**0.292 (192)**	**0.220 (120)**
μ_2	6.0	6.121 (2.02)	6.101 (1.68)
σ_2	0.45	**0.256 (-43.2)**	**0.148 (-67.2)**
μ_3	8.0	7.972 (-0.35)	8.013 (0.16)
σ_3	0.25	0.229 (-8.60)	0.201 (-19.5)
μ_4	5.0	5.103 (2.06)	4.886 (-2.28)
σ_4	0.4	0.341 (-14.8)	0.414 (3.42)
a_1	0.25	0.233 (-6.72)	0.207 (-17.2)
b_1	0.8	0.823 (2.86)	0.811 (1.42)
a_2	0.6	0.563 (-6.17)	0.556 (-7.38)
b_2	1.3	1.340 (3.07)	1.352 (4.00)
Absolute mean error		23.89%	20.74%

[a] Percentage errors compared with the target values in parentheses

3.3 Updating result of the optimization procedure

The optimization updating procedure is performed employing the genetic algorithm with the Bhattacharyya distance as the objective function as shown in Eqs. (10) and (11). The convergence procedure of the genetic algorithm is illustrated in Figure 5, where the genetic algorithm converges after 49 populations and the total number of model evaluations is 10000. The calculation cost with 10000 model evaluations herein is acceptable because each running of the model is extremely quick. For practical applications where large FE models are employed, application of parallel computing and highly efficient surrogate models can help to release the calculation cost. As a comparison, the number of model evaluations in the Bayesian procedure (Sec. 3.2) is 13,000. The calculation effort of the Bayesian approach is slightly higher, but total calculation times of these two approaches are comparable.

The optimized values of the updating coefficients are listed in the last column of Table 2. The results are similar to the ones of Bayesian updating. The coefficients about the mean and bounds of the distributions are optimized with high precision compared with the target values, while the optimized standard deviations are relatively less precise (highlighted in red and bold in Table 2). The AME value of the optimization results has a comparable level (20.74%) compared with the AME of the Bayesian results (23.89%).

The distributions of the outputs are illustrated in Figure 6, where both target data and optimized data are presented. The dark-red histogram in Figure 6 represents the overlap of the optimized histogram (in orange) and the target histogram (in black). It is shown that not only the histograms but also the scatters are coincident well with the target objects. Although the scatters appear as complex shapes with implicit distribution functions, the Bhattacharyya distance is feasible to capture comprehensive distribution information, and thus the updated output scatters share the same profile as the target ones.

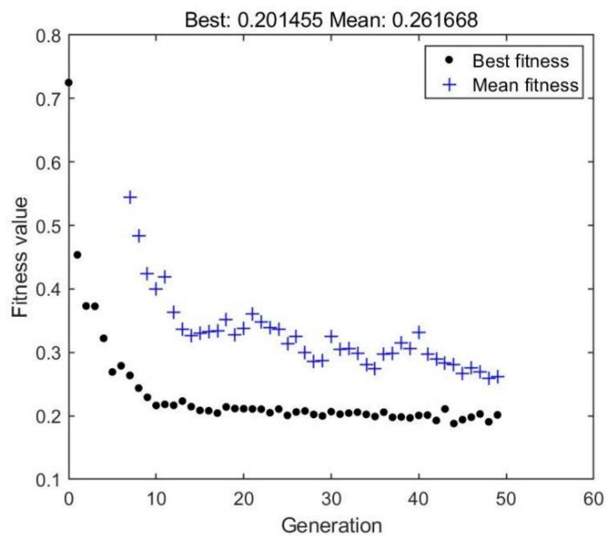


Figure 5: The converging process of the genetic algorithm

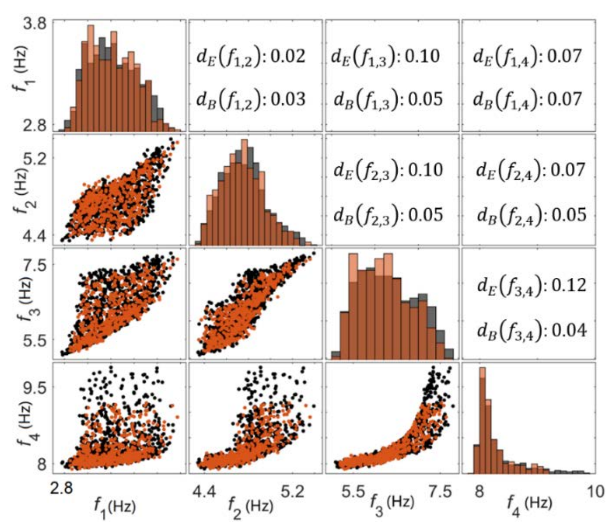


Figure 6: The scatters and the Euclidean and Bhattacharyya distances between the target samples (in orange) and the updated samples (in black)

4 Case study II: The GARTEUR SM-AG19 benchmark structure

4.1 Problem description

Within this section, a practical experiment setup is employed to demonstrate the feasibility of the Bhattacharyya distance in both Bayesian and optimization updating procedures. As illustrated in Figure 7, the GARTEUR SM-AG19 testbed [24–26], also see its replica the DLR AIRcraft MODel (AIRMOD) [14,17,27], is a benchmark structure widely used for various model updating studies. The natural frequencies of the GARTEUR structure are measured by various organizations using different configurations of sensors and different positions of excitations. These multiple measurements on the same structure by different people at different time using different equipment and techniques fully present various sources of the experimental uncertainty. The four sets of measurements (by University of Manchester, DLR, Imperial College, and University of Swansea) in Ref. [24] and two sets of measurements in Ref. [25] are listed in Table 3. All the six sets are employed as the target data in this example to raise the stochastic updating problem, i.e. how to calibrate the uncertainty properties of the input parameters such that the model output can represent the uncertainty property of the six sets of measurements.

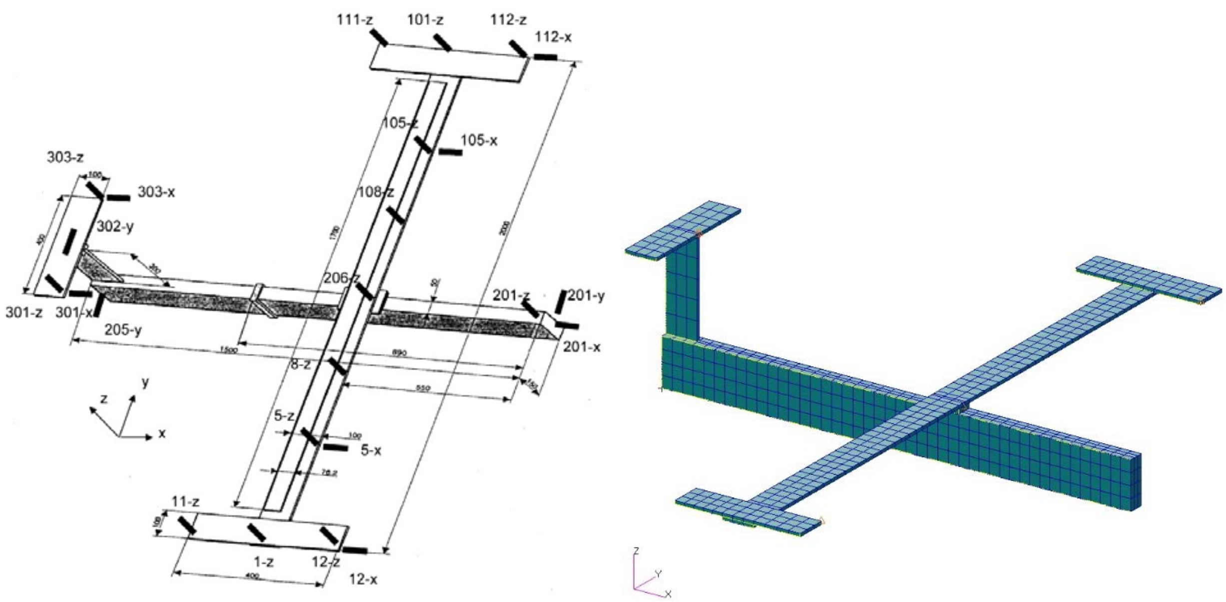


Figure 7: The geometry details [24] and the FE model of the GARTEUR SM-AG19 structure

Table 3: Multiple sets of measurements of the GARTEUR structure from the literature

Frequency order	Measurements #1	Measurements #2	Measurements #3	Measurements #4	Measurements #5	Measurements #6
1	6.38	6.51	6.54	6.55	6.4	6.7
2	16.10	16.37	16.55	16.61	16.1	16.2
3	33.13	33.44	34.86	34.88	33.1	33.4

4	33.53	33.97	35.30	35.36	33.5	33.8
5	35.65	36.17	36.53	36.71	35.6	35.5
6	48.38	49.41	49.81	50.09	48.4	48.3
7	49.43	50.20	50.63	50.72	49.4	49.4
8	55.08	55.61	56.39	56.44	55.1	54.8

The FE model of the GARTEUR structure is illustrated in the right part of Figure 7 with 1704 nodes, 846 hexahedron elements, and 6 lumped mass points. Based on the geometry and material details of the structure [24], the six parameters of the FE model is selected as the uncertain parameters waiting to be calibrated, as described in Table 4. It is important to give a prior assumption that all six parameters follow the Gaussian distribution, with the initial mean and variance as listed in Table 4. Both the mean and standard deviation are set to be uncertain and their intervals are given in Table 4 to represent the imprecise probability. The initial values of the parameter mean and standard deviation are listed as the last column of Table 4. 1000 samples of the input parameters are generated from the initial distribution coefficients and then 1000 model evaluations are executed to generate the 1000 samples of natural frequencies. The relative position between the six experiments and the 1000 simulations in the plane of the 1st and 2nd frequencies is illustrated in Figure 8. The task in this example is to update the distribution property of the input parameter, so that to tune the initial scatter of the frequencies to the six experimental data points. The reference range of the simulation scatter in Figure 8 denotes the 95% confidence interval of the sample distribution. Note that, although the input parameters are set to follow Gaussian distribution, the output frequencies are not necessarily following the Gaussian distribution. Hence the reference range is not a standard ellipse.

Table 4: Description and uncertainty properties of the input parameters of the FE model

Parameters	**Description**	**Distribution coefficient**	**Initial value**
E_w (10^{11}Kg)	Elastic modulus of the wing	$\mu_1 \in [0.5, 0.9]$ $\sigma_1 \in [0.01, 0.05]$	$\mu_1 = 0.7$ $\sigma_1 = 0.03$
μ	Poisson's ratio of the wing	$\mu_2 \in [0.2, 0.4]$ $\sigma_2 \in [0.01, 0.05]$	$\mu_1 = 0.3$ $\sigma_1 = 0.03$
ρ (10^3Kg/m^3)	Mass density of the wing	$\mu_3 \in [2.5, 3.1]$ $\sigma_3 \in [0.01, 0.1]$	$\mu_1 = 2.8$ $\sigma_1 = 0.03$
E_v (10^{11}Kg)	Elastic modulus of the vertical tail	$\mu_4 \in [0.5, 0.9]$ $\sigma_4 \in [0.01, 0.05]$	$\mu_1 = 0.7$ $\sigma_1 = 0.03$
M_t (Kg)	Lumped mass of the connection between the fuselage and the vertical tail	$\mu_5 \in [0.1, 0.5]$ $\sigma_5 \in [0.01, 0.05]$	$\mu_1 = 0.3$ $\sigma_1 = 0.03$

M_d (Kg)	Lumped masses located at the winglets	$\mu_6 \in [0.1, 0.5]$ $\sigma_6 \in [0.01, 0.05]$	$\mu_1 = 0.3$ $\sigma_1 = 0.03$

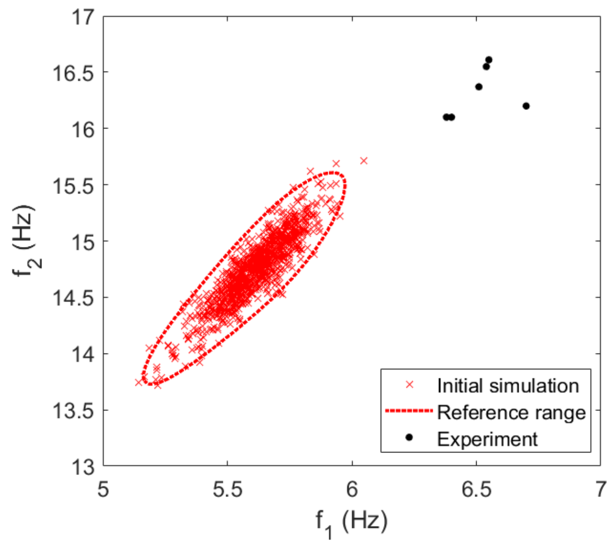


Figure 8: The relative position between the experiments and the initial simulation before updating

4.2 Updating result of the Bayesian procedure

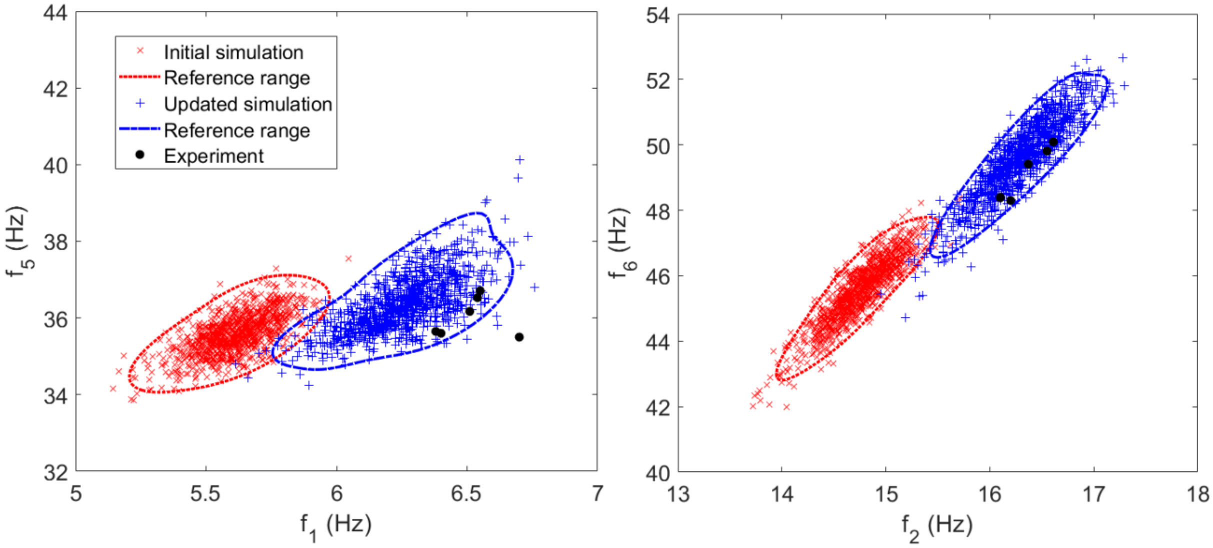
In the Bayesian updating procedure, the Bhattacharyya distance between the six experimental points and the 1000 simulation points are evaluated in each iteration. The procedure converged after 11 iterations, and thus the total number of model evaluations is 11,000. The change tendency of the Bhattacharyya distance in each iteration is shown in Table 5. The Bhattacharyya distance between the experimental points and the initial 1000 simulation points is infinite, implying the initial simulations are too far from the experimental data to have any overlap in the initial space, as illustrated in Figure 8.

Table 5: The change tendency of the Bhattacharyya distance in each iteration

Iteration	1	2	3	4	5	6	7	8	9	10	11
Bhattacharyya distance	Inf	5.0980	3.8008	3.0599	3.0270	1.8634	1.6302	1.6907	1.4115	1.3387	1.2556

Figure 9 presents the relative position of the updated simulation scatters and the experiments. Because of the limitation of the space, only the planes of $f_1\ vs.\ f_5$ and $f_2\ vs.\ f_6$ are presented. The figures of other frequency planes are available upon request by the readers. Figure 9(b) demonstrates that the simulation scatter has been tuned from the initial position to the new position to cover all the six experimental points. However, the result in Figure 9(a) is not as good as Figure 9(b). Although both the position and shape of the scatter are adjusted to be closed to the experiment, there is one experimental point out of the reference range. This can be explained from the complexity of the practical example, where the frequencies are measured by different people using different equipment in different environments.

This means the uncertainties come from not only the structure itself, but also the environment randomness and human subjective judgement. The updating process herein can only use the parameter to compensate the multiple sources of uncertainty. This compensation effect leads it very difficult to achieve the perfect fitness between the simulation and the experiment.



(a) In the plane of f_1 $vs.$ f_5 (b) In the plane of f_2 $vs.$ f_6

Figure 9: The relative position of the experiment scatter and the updated simulation in Bayesian procedure

4.3 Updating result of the optimization procedure

Similar to Sec. 3.3, the optimization updating procedure employs the genetic algorithm where the Bhattacharyya distance constructs the objective function. The convergence process is illustrated in Figure 10 where the genetic algorithm converges after 67 populations and the total number of model evaluations is 13,600. The number of model evaluations in the Bayesian approach (Sec. 4.2) is 11,000, implying the calculation effort of the optimization is larger, but the difference is not that much in this example. The Bhattacharyya distance between the updated simulation samples and the six experimental points is 1.1727, which is equivalent to the result of the Bayesian procedure.

The relative positions of the updated simulation scatters are illustrated in Figure 11. From the cross-comparison between Figures 9 and 11, it is shown that the optimization approach has the similar effort as the Bayesian approach to change not only the position but also the shape of the simulation scatters to fit with the experimental points. As explained in Sec. 4.2, the complexity of the experimental uncertainty leads the updated reference range cannot completely cover the experimental points. However, the results demonstrate the effect of the Bhattacharyya distance in both updating procedures to comprehensively measure the distribution property of the datasets, and hence is capable of updating not only of the position but also the shape of the simulation scatters.

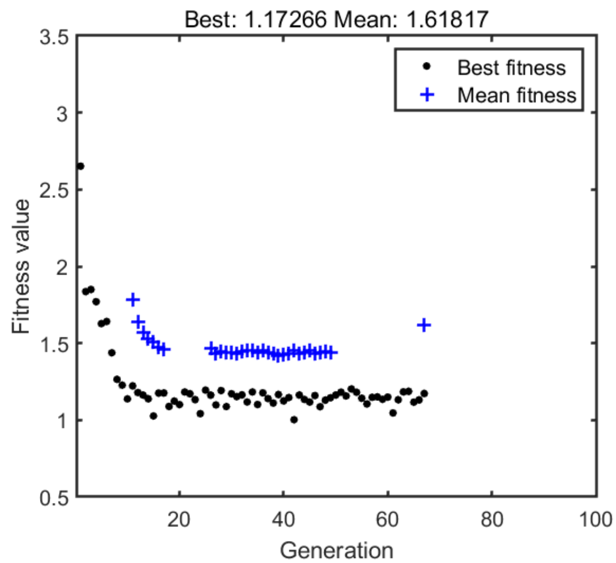


Figure 10: The converging process of the genetic algorithm

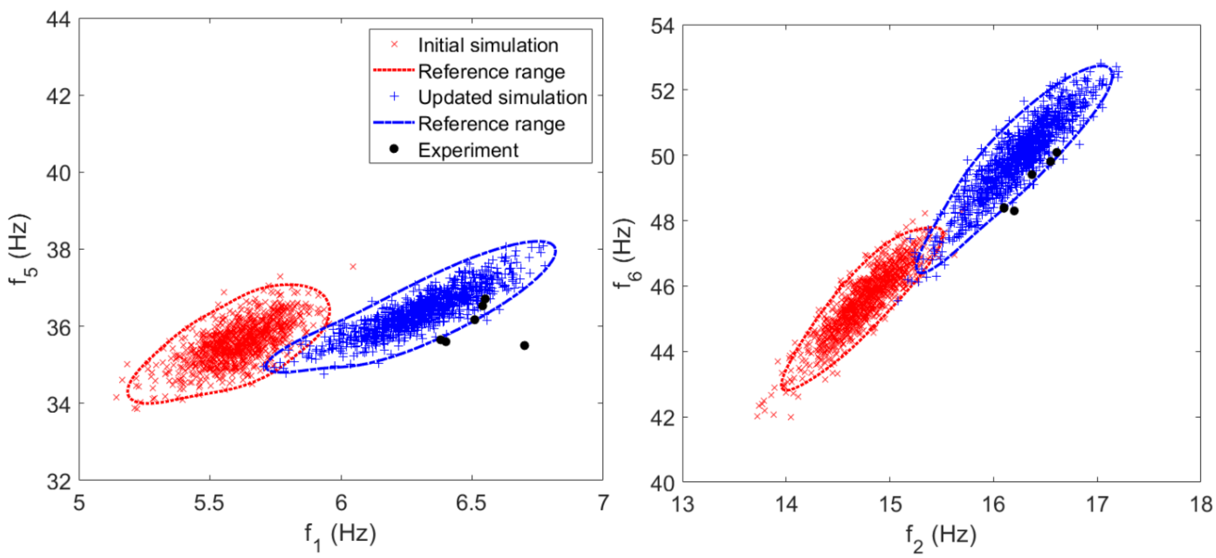


Figure 11: The relative position of the experiment scatter and the updated simulation in optimization procedure

5 Conclusions

The Bhattacharyya distance is proposed as a comprehensive uncertainty quantification (UQ) metric and utilized in stochastic model updating through the Bayesian and optimization procedures, respectively. Instead of calibrating the input parameters themselves, this work focuses on updating the uncertainty characteristics of the parameters, such that the calibrated numerical model can represent the probability distribution of the outputs as same as the multiple experimental measurements.

The Bayesian updating approach employs the transitional MCMC algorithm, using the Bhattacharyya distance to construct an approximate likelihood function. Because of the newly defined likelihood function, the Bhattacharyya distance can be conveniently embedded into the Bayesian updating framework, and the calculation cost is largely reduced compare with directly employing the original definition of the likelihood function.

The optimization updating approach proposes a more straightforward principle, which directly uses the Bhattacharyya distance as the objective function. Because of the randomness of the Bhattacharyya distance, the typical gradient-based optimizers are infeasible for this work. However, the genetic algorithm is found to be well appropriate to solve the optimization problem. Both the simulated and experimental examples demonstrate that the Bhattacharyya distance is capable of capturing the comprehensive uncertainty information from the limited experimental datasets. The Bayesian and optimization updating procedures have similar calculation cost and have equivalent updating effect. This demonstrates the feasibility and generality of the Bhattacharyya distance as a universal UQ metric in the currently general methodologies for stochastic model updating. And hence the application of the Bhattacharyya distance would further promote the acceptance of the uncertainty analysis and stochastic updating in practical engineering.

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Table caption list

Table 1: The uncertainty properties of the input parameters

Table 2: Updating results of the input distribution coefficients using the Bayesian and optimization approaches

Table 3: Multiple sets of measurements of the GARTEUR structure from the literature

Table 4: Description and uncertainty properties of the input parameters of the FE model

Table 5: The change tendency of the Bhattacharyya distance in each iteration

Figure caption list

Figure 1: The proposed 4-DoF mass-spring system

Figure 2: The scatters and the Euclidean and Bhattacharyya distances between the target samples (in orange) and the initial samples (in blue) with the unit in Hz (frequency)

Figure 3: Posterior distributions of the 12 updating coefficients presented in histograms

Figure 4: The scatters and the Euclidean and Bhattacharyya distances between the target samples (in orange) and the updated samples (in green)

Figure 5: The converging process of the genetic algorithm

Figure 6: The scatters and the Euclidean and Bhattacharyya distances between the target samples (in orange) and the updated samples (in black)

Figure 7: The geometry details and the FE model of the GARTEUR SM-AG19 structure

Figure 8: The relative position between the experiments and the initial simulation before updating

Figure 9: The relative position of the experiment scatter and the updated simulation in Bayesian procedure

Figure 10: The converging process of the genetic algorithm

Figure 11: The relative position of the experiment scatter and the updated simulation in optimization procedure