Application of a Reduced Order Model for Fuzzy Analysis of Linear Static Systems

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ABSTRACT

This contribution proposes a strategy for performing fuzzy analysis of linear static systems applying α-level optimization. In order to decrease numerical costs, full system analyses are replaced by a reduced order model that projects the equilibrium equations to a small-dimensional space. The basis associated with the reduced order model is constructed by means of a single analysis of the system plus a sensitivity analysis. This reduced basis is enriched as the α-level optimization strategy progresses in order to protect the quality of the approximations provided by the reduced order model. A numerical example shows that with the proposed strategy, it is possible to produce an accurate estimate of the membership function of the response of the system with a limited number of full system analyses.

1 Introduction

The performance of complex engineering systems can be predicted by means of appropriate numerical models [1]. A crucial step for the correct setup of such models is the characterization of input parameters such as material properties, loads, etc. Nonetheless, precise identification of such parameters may not be straightforward in view of uncertainties, which may arise due to randomness (aleatory uncertainty) or due to issues such as lack of knowledge, vagueness and imprecision (epistemic uncertainty). For the latter case, uncertainties can be best described in terms of the so-called non traditional approaches for uncertainty quantification [2]. Among these, interval and fuzzy analysis have shown their usefulness in many applications, see e.g. [2–8].

Characterization of uncertainty by means of intervals consists of associating lower and upper bounds to the input parameters of a model, without assigning any relative likelihood to the values in between. As the input parameters are intervals, the response of the numerical model normally becomes an interval as well. Nonetheless, the interval associated with the response can be seldom determined explicitly, as the response of the model is calculated point-wise for crisp values of the input parameters. Therefore, determination of the interval of the response is usually carried out applying either interval arithmetic or optimization [9]. Interval arithmetic propagates the uncertainty from the input parameters to the structural responses by applying interval arithmetic operations, as discussed in e.g. [6, 10–12]. The major challenge when implementing approaches based on interval arithmetic is keeping track of dependencies between parameters (that is, the *dependency problem*, see e.g. [10, 13]). Optimization computes the interval of the response by identifying the minimum and maximum of the structural response for crisp values of the input parameters within their respective intervals by means of an appropriate numerical search algorithm, see e.g. [14–17]. While the implementation of optimization approaches for interval analysis is straightforward, numerical costs may grow rapidly due to the necessity of carrying out repeated system analyses for locating the extrema of the response.

Fuzzy analysis can be interpreted as a collection of intervals that are indexed by a membership function [2]. In this way, it is possible to assess the sensitivity of the response with respect to the magnitude of imprecision of the input parameters [18]. Although fuzzy analysis offers valuable information, its practical implementation is extremely demanding from a numerical viewpoint, as it adds an additional analysis loop (associated with the membership function) when compared to interval analysis. Hence, practical fuzzy analysis must be carried out in combination with specialized strategies that involve, for example, approximation concepts [12, 19, 20], substructuring [21], surrogate models [22–24], multi-fidelity approaches [25], nonlinear programming formulations [26], etc.

This contribution proposes a strategy for performing fuzzy analysis of a type of problem, namely linear systems subject to static load. Uncertainties are propagated from the input parameters to the response by α-level optimization [2]. In order to reduce the numerical costs associated with the optimization step, the response of the system is approximated by means of a reduced order model (see, e.g. [27]) which projects the equilibrium equations of the full model onto a reduced order basis in a Galerkin sense. The reduced basis associated with this reduced order model is constructed by taking advantage of the structure of the problem, by performing a single system analysis followed by a sensitivity analysis. The quality of approximation of the reduced order model is constantly monitored throughout the execution of α-level optimization and, whenever required, the reduced basis is enriched with additional system analyses. In this way, it is possible to produce accurate estimates of the membership function associated with the response at reduced numerical costs. The novelty of the paper comes into the integration of a reduced order model [28] with an adaptive strategy for basis enrichment [29] within the context of fuzzy analysis.

The rest of this paper is organized as follows. The specific problem considered in this contribution as well as its solution by means of α-level optimization are discussed in Section 2. The proposed strategy for conducting α-level optimization is discussed in Section 3, while its application is illustrated by means of an example in Section 4. Section 5 closes this work with conclusions and an outlook for future research challenges.

2 Formulation of the Problem

2.1 Uncertain Linear System Under Static Load

This contribution focuses on the analysis of steady-state linear systems subject to static load. This type of systems can represent a number of practical problems in structural mechanics, confined seepage, steady-state heat transfer, etc. [1]. The associated numerical model is formulated within the framework of the finite element method, where some of the input parameters are uncertain. These parameters are collected in a vector $\boldsymbol{\theta} = [\theta_1, \ldots, \theta_{n_\theta}]^T$ of dimension $n_\theta \times 1$, where $(\cdot)^T$ denotes transpose of the argument. Under such assumptions, the behavior of the system is described in terms of the following set of equations:

$$\boldsymbol{K}(\boldsymbol{\theta})\,\boldsymbol{u}(\boldsymbol{\theta}) = \boldsymbol{f}(\boldsymbol{\theta}) \tag{1}$$

where $\boldsymbol{K}(\boldsymbol{\theta})$ is a $n_d \times n_d$ matrix associated with the system's properties; $\boldsymbol{f}(\boldsymbol{\theta})$ is a $n_d \times 1$ vector representing external load; and $\boldsymbol{u}(\boldsymbol{\theta})$ is a $n_d \times 1$ vector that describes the system's response. As noted from eq. (1), the uncertainty affecting the system's matrix $\boldsymbol{K}(\boldsymbol{\theta})$ and the external load $\boldsymbol{f}(\boldsymbol{\theta})$ propagates to the response $\boldsymbol{u}(\boldsymbol{\theta})$ of the system. It is considered that both $\boldsymbol{K}(\boldsymbol{\theta})$ and $\boldsymbol{f}(\boldsymbol{\theta})$ are continuous for all values that $\boldsymbol{\theta}$ may assume; moreover, it is also considered that $\boldsymbol{K}(\boldsymbol{\theta})$ is positive-definite and that $\boldsymbol{f}(\boldsymbol{\theta}) \neq \mathbf{0}$.

As an additional assumption, it is considered that matrix $\boldsymbol{K}(\boldsymbol{\theta})$ admits the following parametric representation [30, 31]:

$$\boldsymbol{K}(\boldsymbol{\theta}) = \boldsymbol{K}_0 + \sum_{k=1}^{n_K} \boldsymbol{K}_k p_k(\boldsymbol{\theta}) \tag{2}$$

where $\boldsymbol{K}_k$, $k = 0, \ldots, n_K$ are matrices of dimension $n_d \times n_d$ that are not affected by the uncertain input parameters $\boldsymbol{\theta}$; and $p_k(\boldsymbol{\theta})$, $k = 1, \ldots, n_K$ are scalar functions that depend on the uncertain input parameter vector $\boldsymbol{\theta}$.

Due to design or decision-making purposes, it is of interest monitoring a certain response $r(\boldsymbol{\theta})$ of the system. It is assumed that such response of interest can be calculated in terms of the response vector of the system, that is:

$$r(\boldsymbol{\theta}) = \boldsymbol{\gamma}^T \boldsymbol{u}(\boldsymbol{\theta}) \tag{3}$$

where $\boldsymbol{\gamma}$ is a vector of constant coefficients of dimension $n_d \times 1$. In addition, it should be noted that for practical applications, it is expected that the number of degrees-of-freedom n_d of the numerical model is large and thus, repeated solution of eq. (1) can become demanding from a numerical viewpoint due to the necessity of factorizing $\boldsymbol{K}(\boldsymbol{\theta})$ (that is, calculating the inverse of the matrix) .

2.2 Characterization of Uncertainty by means of Fuzzy Sets

It is assumed that the sources of uncertainty affecting each of the parameters θ_i, $i = 1, \ldots, n_\theta$ stem out of issues such as lack of knowledge, imprecision, vagueness, etc. A possible way to characterize this epistemic uncertainty is by means of fuzzy sets. Thus, the fuzzy set $\tilde{\theta}_i$ associated with the i-th input parameter is:

$$\tilde{\theta}_i = \left\{ \left(\theta_i, \mu_{\tilde{\theta}_i}(\theta_i)\right) : (\theta_i \in \Theta_i) \wedge \left(\mu_{\tilde{\theta}_i}(\theta_i) \in [0,1]\right) \right\}, \quad i = 1, \ldots, n_\theta \tag{4}$$

where Θ_i denotes the fundamental set that contains all physical values that the input parameter θ_i may assume; and $\mu_{\tilde{\theta}_i}(\theta_i)$ is the membership function. Note that the role of the $\mu_{\tilde{\theta}_i}(\theta_i)$ is allowing a gradual assessment of the membership of θ_i in the set $\tilde{\theta}_i$. Thus, $\mu_{\tilde{\theta}_i}(\theta_i) = 0$ indicates that θ_i is not included in $\tilde{\theta}_i$ while $\mu_{\tilde{\theta}_i}(\theta_i) = 1$ indicates that θ_i is fully included in $\tilde{\theta}_i$. Furthermore, $0 < \mu_{\tilde{\theta}_i}(\theta_i) < 1$ indicates that θ_i is partially included in $\tilde{\theta}_i$. In addition, it is assumed that there is only one element for which $\mu_{\tilde{\theta}_i}(\theta_i) = 1$ and that the membership function $\mu_{\tilde{\theta}_i}(\theta_i)$ is quasiconcave [9, 18], that is:

$$\mu_{\tilde{\theta}_i}\left(\theta_i^C\right) \geq \min\left(\mu_{\tilde{\theta}_i}\left(\theta_i^L\right), \mu_{\tilde{\theta}_i}\left(\theta_i^R\right)\right), \ \forall\, \theta_i^L, \theta_i^C, \theta_i^R \in \Theta_i \tag{5}$$

such that $\theta_i^L \leq \theta_i^C \leq \theta_i^R,\ i = 1,\ldots,n_\theta$. Figure 1 provides a schematic representation of a membership function under the assumptions described above.

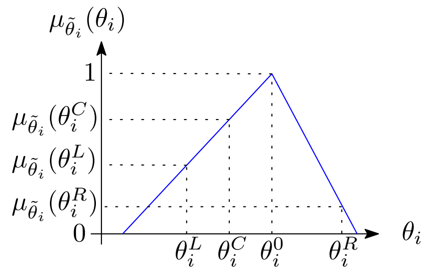

Fig. 1. Convex fuzzy set – schematic representation

As the input parameters of the model are characterized as fuzzy, it is clear from eqs. (1) and (3) that the response of interest becomes fuzzy as well, that is $\tilde{r}$. Nonetheless, for situations of practical interest, the membership function associated with $\tilde{r}$ cannot be expressed in closed form, as the solution of eq. (1) is known point-wise only for crisp values $\boldsymbol{\theta}$. Hence, the membership function must be calculated by means of specialized numerical procedures [9, 32], such as α-level optimization (see, e.g. [18]), as described in the following.

2.3 α-Level Optimization

A feasible means for determining the membership function of the response of interest in a discrete manner is applying α-level optimization [9, 18]. α-level optimization consists of constructing crisp sets of the input parameters by selecting subsets of the support of the associated fuzzy set that possess a membership value equal or larger than a certain threshold $\alpha \in (0,1]$, where α denotes the membership level under consideration. The crisp set associated with the i-th input parameter is:

$$\underline{\theta}_{i,\alpha_j} = \left\{ \theta_i \in \Theta_i : \mu_{\tilde{\theta}_i}(\theta_i) \geq \alpha_j \right\},\ i = 1,\ldots,n_\theta,\ \alpha_j \in (0,1] \tag{6}$$

where $\alpha_j,\ j = 1,\ldots,n_c$ denotes the α-cut value under consideration and n_c is the number of discrete cuts considered in the analysis; $\underline{\theta}_{i,\alpha_j}$ denotes the set of possible values that θ_i may assume for the membership value α_j. It should be noted that $\underline{\theta}_{i,\alpha_j}$ is actually the interval associated with θ_i for $\alpha = \alpha_j$. Hence, it becomes evident that the representation of uncertainty by means of fuzzy sets can be interpreted as a collection of intervals indexed by the membership function. A schematic representation of the interval $\underline{\theta}_{i,\alpha_j}$ is shown on the left hand side of Figure 2.

The above discussion highlights that for a given membership value α_j, the uncertainty associated with the input parameters is actually represented in terms of an interval (or crisp set). Hence, for that membership value, it is also possible to identify an interval associated with the response of interest, which is denoted as $\underline{r}_{\alpha_j}$ and is mathematically defined as follows.

$$\begin{aligned} \underline{r}_{\alpha_j} = \Big\{ r : \Big(\theta_i \in \underline{\theta}_{i,\alpha_j},\ i = 1,\ldots,n_\theta \Big) \wedge \\ r = r(\boldsymbol{\theta}) = \boldsymbol{\gamma}^T \boldsymbol{K}(\boldsymbol{\theta})^{-1} \boldsymbol{f}(\boldsymbol{\theta}) \Big\} \end{aligned} \tag{7}$$

The interval $\underline{r}_{\alpha_j}$ is represented schematically on the right hand side of Figure 2. As the sets $\underline{\theta}_{i,\alpha_j},\ i = 1,\ldots,n_\theta$ are compact and convex (due to the assumption on quasiconcavity of the membership function), these sets are fully characterized by their respective lower and upper bounds, which are denoted with superscripts $(\cdot)^L$ and $(\cdot)^R$, respectively, as shown in Figure 2. Furthermore, as there is a continuous mapping between the input parameters and the response of interest (that is, $r(\boldsymbol{\theta}) = \boldsymbol{\gamma}^T \boldsymbol{K}(\boldsymbol{\theta})^{-1} \boldsymbol{f}(\boldsymbol{\theta})$), the interval $\underline{r}_{\alpha_j}$ is also fully described by its lower and upper bounds, as shown schematically in Figure 2 with superscripts $(\cdot)^L$ and $(\cdot)^R$, respectively. The bounds of the response can be determined by solving the following two optimization problems [9].

$$r_{\alpha_j}^L = \min_{\boldsymbol{\theta}} \left(r(\boldsymbol{\theta}) \right),\ \theta_i \in \underline{\theta}_{i,\alpha_j},\ i = 1,\ldots,n_\theta \tag{8}$$

$$r_{\alpha_j}^R = \max_{\boldsymbol{\theta}} \left(r(\boldsymbol{\theta}) \right),\ \theta_i \in \underline{\theta}_{i,\alpha_j},\ i = 1,\ldots,n_\theta \tag{9}$$

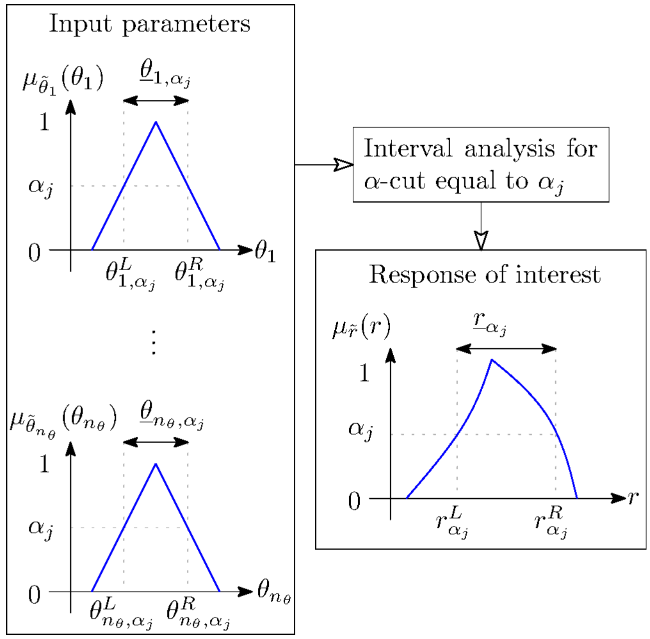

Fig. 2. Schematic representation of α-level optimization strategy

The above discussion described an interval analysis for a given membership value α_j. By repeating this interval analysis for a total of n_c α-cut levels, it is then possible to establish the intervals of the response for different membership levels and construct a discrete approximation of the membership function of the response of interest $\mu_{\tilde{r}}(r)$.

The α-level optimization scheme described above provides a means for approximating the membership function of the response of interest. Nonetheless, its practical application can be quite demanding from a numerical viewpoint, as it is necessary to solve $2n_c$ optimization problems involving the response of interest, which is calculated by means of eqs. (1) and (3). Therefore, the remaining part of this work formulates a strategy for reducing the numerical efforts associated with α-level optimization by means of an approximate representation of the response of interest.

3 Proposed Strategy

3.1 General Remarks

The strategy for solving the α-level optimization problem consists of replacing the response of interest $r(\boldsymbol{\theta})$ with an approximate response $r^{\mathrm{A}}(\boldsymbol{\theta})$ whose calculation is straightforward from a numerical viewpoint. The approximate response $r^{\mathrm{A}}(\boldsymbol{\theta})$ is formulated resorting to a reduced order model, as described in detail in Section 3.2. Naturally, the use of an approximate response induces errors, whose quantification is discussed in Section 3.3. Moreover, as the α-level optimization process progresses through the different α-cut levels $\alpha_j,\ j = 1,\ldots,n_c$, it is possible that the estimation errors grow beyond an acceptable threshold. In that case, it is necessary to improve the quality of the reduced order model by means of a basis enrichment, as examined in Section 3.4. The integration of the reduced order model, error estimation and basis enrichment for performing α-level optimization is discussed in Section 3.5.

3.2 Reduced Order Model

A reduced order model (ROM) allows approximating the system's response $\boldsymbol{u}(\boldsymbol{\theta})$ with decreased numerical efforts [27,33]. Thus, the response vector is approximated as:

$$\boldsymbol{u}(\boldsymbol{\theta}) \approx \boldsymbol{u}^{\mathrm{A}}(\boldsymbol{\theta}) = \boldsymbol{\Phi}\boldsymbol{\beta}(\boldsymbol{\theta}) \tag{10}$$

where $\boldsymbol{\Phi}$ is a $n_d \times n_r$ matrix whose columns contain the vectors $\boldsymbol{\phi}_i, i = 1,\ldots,n_r$ that conform the reduced basis; and $\boldsymbol{\beta}(\boldsymbol{\theta})$ is a $n_r \times 1$ vector whose components depend on the uncertain input parameters $\boldsymbol{\theta}$. Taking into account the approximate representation of the response vector, the equilibrium equation (that is, eq. (1)) is projected onto the reduced basis following a Galerkin approach [27,33], leading to the following reduced order model:

$$\boldsymbol{K}_R(\boldsymbol{\theta})\boldsymbol{\beta}(\boldsymbol{\theta}) = \boldsymbol{f}_R(\boldsymbol{\theta}) \tag{11}$$

where $\boldsymbol{K}_R(\boldsymbol{\theta}) = \boldsymbol{\Phi}^T\boldsymbol{K}_0\boldsymbol{\Phi} + \sum_{k=1}^{n_K}\boldsymbol{\Phi}^T\boldsymbol{K}_j\boldsymbol{\Phi}p_k(\boldsymbol{\theta})$ is the system's reduced matrix; and $\boldsymbol{f}_R(\boldsymbol{\theta}) = \boldsymbol{\Phi}^T\boldsymbol{f}(\boldsymbol{\theta})$ is the reduced load vector. Thus, the response of interest is approximated as:

$$r(\boldsymbol{\theta}) \approx r^{\mathrm{A}}(\boldsymbol{\theta}) = \boldsymbol{\gamma}^T\boldsymbol{\Phi}\boldsymbol{\beta}(\boldsymbol{\theta}) \tag{12}$$

In case that $n_r \ll n_d$, the numerical solution of the reduced order model as shown in eq. (11) is considerably less demanding than the solution of the original equilibrium equation (see eq. (1)). The selection of the reduced basis $\boldsymbol{\Phi}$ is of paramount importance for ensuring that $r^{\mathrm{A}}(\boldsymbol{\theta})$ approximates $r(\boldsymbol{\theta})$ with sufficient accuracy. Hence, different approaches have been proposed for its construction, see e.g. [33, 34]. In this contribution, the reduced basis $\boldsymbol{\Phi}$ is selected following the concepts proposed in [28], where the basis is constructed by taking the response vector plus its first- (and possibly second-) order derivatives evaluated at a nominal point $\boldsymbol{\theta}^0$. The advantage of such an approach is that all those quantities are derived from a single matrix factorization (that is, system analysis) at the aforementioned nominal point $\boldsymbol{\theta}^0$, as discussed in detail Appendix A. Thus, the basis is constructed as:

$$\boldsymbol{\Phi} = \mathrm{orth}\left(\left[\boldsymbol{u}\left(\boldsymbol{\theta}^0\right), \frac{\partial\boldsymbol{u}\left(\boldsymbol{\theta}^0\right)}{\partial\theta_1}, \ldots, \frac{\partial^2\boldsymbol{u}\left(\boldsymbol{\theta}^0\right)}{\partial\theta_{n_\theta}^2}\right]\right) \tag{13}$$

where $\mathrm{orth}(\boldsymbol{X})$ denotes that orthogonalization over the column space of $\boldsymbol{X}$ and where $\partial\boldsymbol{u}\left(\boldsymbol{\theta}^0\right)/\partial\theta_{i_1},\ i_1 = 1,\ldots,n_\theta$ and $\partial^2\boldsymbol{u}\left(\boldsymbol{\theta}^0\right)/\partial\theta_{n_{i_1}}\partial\theta_{n_{i_2}},\ i_1 = i_2 = 1,\ldots,n_\theta$ denote first and second order partial derivatives of the system's response. The orthogonalization is carried out by means of a singular value decomposition. Eventually, the number of vectors associated with the reduced basis can be decreased by discarding those with associated singular value below a certain threshold [34]. Numerical results as reported in [28] indicate that the procedure for calculating the basis as presented in eq. (13) leads to accurate approximations of the system's response.

The expansion point $\boldsymbol{\theta}^0$ can be selected according to different criteria, e.g. midpoint of a representative interval associated with a certain α-cut, etc. In this contribution, the expansion point is selected such that $\mu_{\tilde{\theta}_i}(\theta_i^0) = 1,\ i = 1,\ldots,n_\theta$.

3.3 Error Estimation

The approximate response $r^{\mathrm{A}}(\boldsymbol{\theta})$ calculated by means of the reduced order model described previously may strongly decrease the numerical effort associated with α-level optimization. However, this decrease in numerical effort comes at a price, as the system's response is calculated only approximately. Therefore, it is necessary to monitor the error associated with the proposed approximation.

The error introduced by the approximation for a given value $\boldsymbol{\theta}$ of the input parameters is equal to the difference between the *exact* and approximate response, that is $e(\boldsymbol{\theta}) = r(\boldsymbol{\theta}) - r^{\mathrm{A}}(\boldsymbol{\theta})$. Naturally, it is not practical to calculate such error within α-level optimization, as it demands performing a full system analysis (see eq. (1)). Hence, an alternative error measure must be applied.

Following the ideas proposed in [29], the error measure ε is selected as the Euclidean norm of the residual associated with eq. (1), considering the approximate response of the system, normalized by the Euclidean norm of the external load. Mathematically, the error measure ε is defined as:

$$\begin{aligned}\varepsilon(\boldsymbol{\theta}) &= \frac{\left\|\boldsymbol{K}(\boldsymbol{\theta})\boldsymbol{u}^{\mathrm{A}}(\boldsymbol{\theta}) - \boldsymbol{f}(\boldsymbol{\theta})\right\|}{\|\boldsymbol{f}(\boldsymbol{\theta})\|} \\ &= \frac{\|\boldsymbol{K}(\boldsymbol{\theta})\boldsymbol{\Phi}\boldsymbol{\beta}(\boldsymbol{\theta}) - \boldsymbol{f}(\boldsymbol{\theta})\|}{\|\boldsymbol{f}(\boldsymbol{\theta})\|}\end{aligned} \tag{14}$$

where $\|\cdot\|$ denotes Euclidean norm. This error measure does not involve a matrix factorization and hence, it can be computed with reduced numerical effort.

Note that the error measure in eq. (14) does not directly control the error associated with the approximation $r^{\mathrm{A}}(\boldsymbol{\theta})$. However, numerical experience as reported in [29] indicates that the normalized error norm associated with the residual exhibits a good correlation with the error associated with the response. Hence, it serves the purposes of the current work.

3.4 Basis Enrichment

The practical implementation of α-level optimization consists of replacing the exact response $r(\boldsymbol{\theta})$ with its approximation $r^{\mathrm{A}}(\boldsymbol{\theta})$. For this purpose, the approximate response is calculated considering a reduced basis constructed from a single system analysis plus a sensitivity analysis, as already described in Section 3.2. Figure 3 provides an schematic representation of the α-level optimization process, where $n_c = 2$ for illustration purposes. In this Figure, *Initial Step* denotes the stage where a single system analysis is carried out for the value $\boldsymbol{\theta}^0$ for constructing the reduced order model. After this step is completed, α-level optimization is carried out for membership values α_1 and α_2, in *Step 1* and *Step 2*, respectively. As *Step 1* possesses a membership value close to 1, it is expected that the approximate response is quite close to the exact one. This is due to the fact that all values contained within the intervals $\underline{\theta}_{i,\alpha_1}$, $i = 1,\dots,n_\theta$ are expected to lie relatively close to the expansion point θ_i^0, $i = 1,\dots,n_\theta$. However, for *Step 2*, it is expected that the approximation quality decreases, as the support of the intervals $\underline{\theta}_{i,\alpha_2}$, $i = 1,\dots,n_\theta$ may contain values which lie far away from the expansion point. While the situation illustrated in Figure 3 has been examined in qualitative terms, it evidences the fact that it is expected that the quality of the approximate response deteriorates as the values of the α-cuts under analysis decrease.

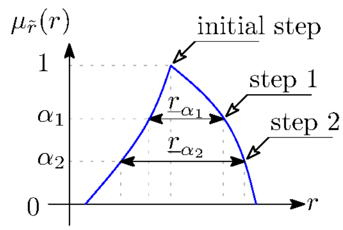

Fig. 3. Steps of α-level optimization – schematic representation

In order to protect the quality of the approximate response $r^{\mathrm{A}}(\boldsymbol{\theta})$, the following strategy is adopted, which has been adapted from [29]. As discussed in Section 3.3, the error measure $\varepsilon(\boldsymbol{\theta})$ is monitored for every single value $\boldsymbol{\theta}$ explored at the optimization stage of any α-cut value under analysis. Whenever this error exceeds a prescribed threshold value ε_t for a given value $\boldsymbol{\theta}^*$ of the input parameters, the following two actions are taken. First, an *exact* system analysis is carried out. That is, eq. (1) is solved in order to calculate $\boldsymbol{u}(\boldsymbol{\theta}^*)$. This allows in turn to calculate the response of interest $r(\boldsymbol{\theta}^*)$. In the second place, the reduced basis $\boldsymbol{\Phi}$ is enriched with $\boldsymbol{u}(\boldsymbol{\theta}^*)$. In other words, this exact system's response is included in the basis as an additional vector by means of the Gram-Schmidt process. In this context, recall that the Gram-Schmidt process allows orthonormalizing a set of vectors in an inner Euclidean product space (see, e.g. [35]), favoring numerical stability of the reduced order model.

The strategy described above ensures that the error measure is always kept below a threshold. Numerical experience as reported in Section 4 indicates that such strategy is indeed effective for protecting the quality of the approximations. Based on the recommendations in [29], the error threshold is chosen as $\varepsilon_t \in [10^{-4}, 10^{-3}]$.

3.5 Summary of the Proposed Strategy

The strategy for performing fuzzy analysis via α-level optimization as described above can be summarized in the following steps.

1. Set up the numerical model in terms of its equilibrium equation (eq. (1)) and the response of interest (eq. (3)).
2. Identify the uncertain input parameters of the model and describe their uncertainty by means of fuzzy sets (eq. (4)) whose membership function is quasiconcave and that possess a single element for which the membership value is equal to one (Section 2.2). Select a number of n_c α-cuts and define the membership values $\alpha_1 > \alpha_2 > \ldots > \alpha_{n_c}$ to be analyzed within the α-level optimization process. Select an error threshold ε_t.
3. Identify the expansion point $\boldsymbol{\theta}^0$. Solve the equilibrium equation for this expansion point in order to calculate $\boldsymbol{u}(\boldsymbol{\theta}^0)$ and perform a sensitivity analysis (eq. (1) and Appendix A). Construct the reduced basis $\boldsymbol{\Phi}$ with those results (eq. (13)). Set $j = 1$.
4. Solve the optimization problems in eqs. (8) and (9) considering $\alpha = \alpha_j$ by means of any suitable algorithm. For evaluating the response for a given $\boldsymbol{\theta}$, follow these steps.

(a) Calculate the approximate response $r^A(\boldsymbol{\theta})$ by means of eqs. (11) and (12).

(b) Compute the error measure $\varepsilon(\boldsymbol{\theta})$ by means of eq. (14). In case $\varepsilon(\boldsymbol{\theta}) \leq \varepsilon_t$, return to the optimizer the response value $r^A(\boldsymbol{\theta})$. Otherwise, solve eqs. (1) and (3) in order to calculate $\boldsymbol{u}(\boldsymbol{\theta})$ and $r(\boldsymbol{\theta})$. Enrich the reduced basis $\boldsymbol{\Phi}$ with $\boldsymbol{u}(\boldsymbol{\theta})$ via Gram-Schmidt process and return to the optimizer the response value $r(\boldsymbol{\theta})$.

5. In case $j = n_c$, stop the process. Otherwise, return to step 4 with $j = j + 1$.

Regarding step 4 described above, any suitable optimization algorithm can be applied for solving the optimization problems in eqs. (8) and (9). As it is usually not known whether or not the response of interest is a convex function, algorithms with global search capabilities should be preferred. For the particular case of this contribution, global search is conducted via Particle Swarm, which is an evolutionary algorithm that is well documented in the literature, see e.g. [36].

4 Example

4.1 Description

This example is partially based on an example presented in [20, 37]. It comprises a reinforced concrete slab simply supported on its edges, resting over an elastic soil modeled using a Winkler foundation. The slab supports a uniformly distributed load. Figure (4) depicts a schematic representation of the slab.

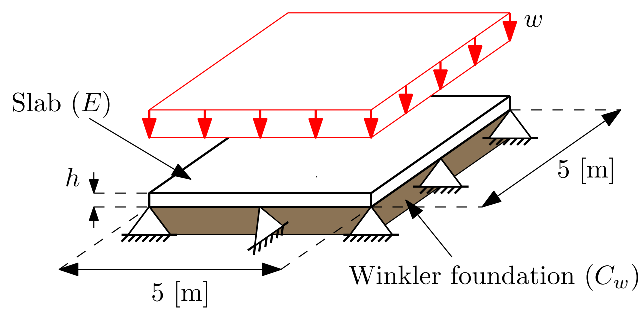

Fig. 4. Reinforced concrete slab resting on a Winkler foundation

The fuzzy variables of the model are the Young's modulus of the concrete (E), the thickness of the slab (h), the modulus of the Winkler foundation model (C_w) and the uniformly distributed load (w). The membership functions associated with these input parameters are shown in fig. (5). The objective is to determine the membership function associated with the vertical displacement of the slab at its center point. This displacement is determined by means of a finite element model comprising 900 quadrilateral Melosh-Zienkiewicz-Cheung plate elements [38] and 2763 degrees-of-freedom. The Winkler foundation model is included by means of equivalent springs located in the nodes of the elements, which are deduced based on the underlying weak formulation of the equations of equilibrium [38].

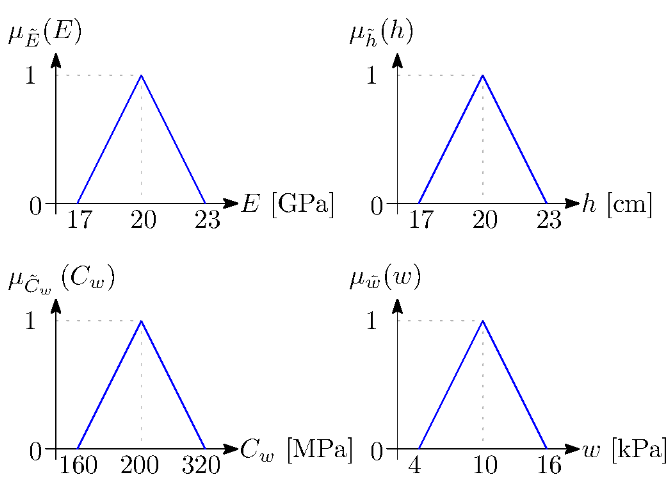

Fig. 5. Membership function associated with fuzzy input parameters

4.2 Solution Considering $\varepsilon_t \to \infty$

The membership function associated with the displacement at the center of the slab (which is denoted as r) is calculated by means of the strategy described in Section 3 by selecting $n_c = 10$ α-cut values for analysis and a threshold level for the error such that $\varepsilon_t \to \infty$. The latter selection is made in order to examine the application of the strategy when no enrichment of the reduced basis is considered. Two different cases are studied: in the first one, the reduced basis is constructed considering only the nominal response plus its first order derivatives. The results associated with such basis are denoted in the following as RB1. In addition, a second case (denoted as RB2) is considered, where the reduced basis is constructed considering the nominal solution plus its first- and second-order derivatives. The results produced by means of these two cases are compared to the *exact* results, where the system's response is calculated by means of the equilibrium equation (eq. (1)).
Figure 6 presents the results of the estimation of the membership function associated with the response. It can be readily seen that the results produced with the reduced basis RB1 and RB2 provide a good match with the reference result (denoted as exact). Nonetheless, it can be seen that for the case of the membership function estimated using the reduced basis RB1, some differences can be noted for low values of the membership. Such issue can be attributed to the fact that for those membership values, the support of the associated intervals of the input parameters covers a wide range of values which are far away from the expansion point.

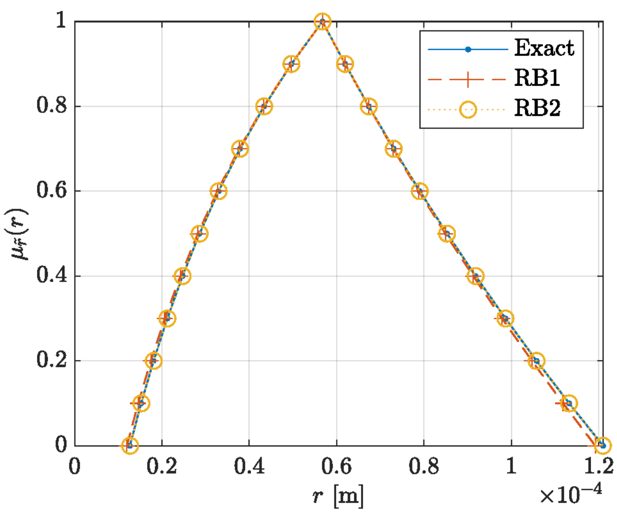

Fig. 6. Membership function associated with the response for $\varepsilon_t \to \infty$

Figure 7 shows the maximum value of the error measure ε registered when assessing the bounds of the interval of the response for a given α-cut value for the two reduced bases RB1 and RB2 considered in the analysis. Two observations can be readily seen from the figure. First, the error measure associated with RB1 is always larger than the one associated with RB2. This was expected, as RB2 should provide a better approximation given that its reduced basis involves more basis terms. Second, as the α-cut under analysis progresses, the error increases. This was also expected: recall that the α-cut values under analysis are sorted such that $\alpha_1 > \alpha_2 > \ldots > \alpha_{n_c}$. Hence, for larger values of j in Figure 7, the support of the intervals is larger than for smaller values of j.

4.3 Solution Considering $\varepsilon_t = 10^{-4}$

This Section repeats the analysis performed in Section 4.2, except that the threshold for the error measure is set as $\varepsilon = 10^{-4}$. This implies that the error measure associated with the approximation of the response is constantly monitored along α-level optimization and the reduced basis is enriched in case it is necessary.
The analysis is conducted considering again the exact response (eqs. (1) and (3)) and reduced bases RB1 and RB2. The results obtained for the estimate of the membership function are shown in Figure 8. As noted from the Figure, there is a perfect match between the reference results and those produced by RB1 and RB2. This is clearly a consequence of the basis enrichment conducted when applying α-level optimization.

The evolution of the error measure across the different steps of α-level optimization is shown in Figure 9. It is seen that the curves associated with RB1 and RB2 present a sawtooth pattern. That is, the error grows as the steps progress, but it presents a sudden decrease whenever it approaches the threshold level ε_t. Such sudden decrease is explained as an additional term has been included in the reduced basis, in order to protect the quality of the approximations.

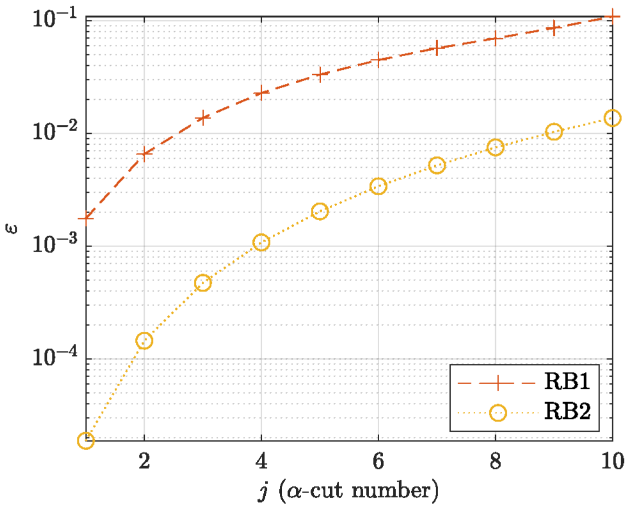

Fig. 7. Error measure ε associated with response calculated at different α-cut levels for $\varepsilon_t \to \infty$

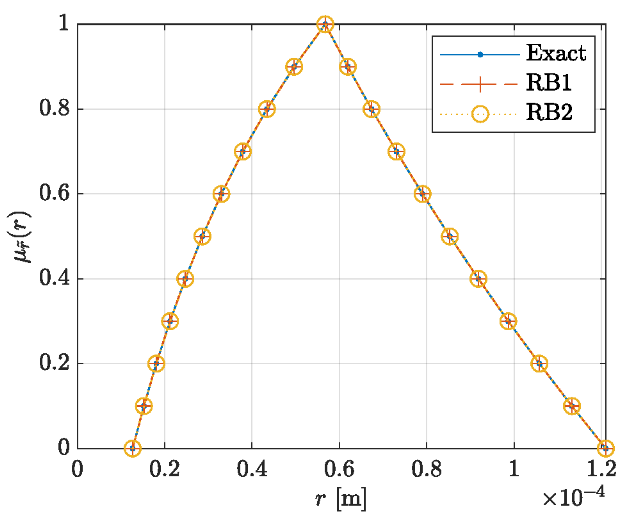

Fig. 8. Membership function associated with the response for $\varepsilon_t = 10^{-4}$

Figure 10 presents the number of accumulated additional system analyses required for basis enrichment as a function of the α-cut level number j. It can be observed that whenever there is an increase in the number of additional analyses in Figure 10, there is a sharp decrease of the error measure in Figure 9. Such behavior was expected due to basis enrichment. Furthermore, it is noted from Figure 10 that the analysis conducted with RB1 required a total of three additional system analyses, while RB2 required only two. Such difference is explained as at the initial stage, RB2 possesses more basis terms than RB1. Hence, it is expected that RB2 requires less additional terms for keeping the error measure below a certain threshold.

Finally, it is important to note that the proposed strategy for performing α-level optimization brings important benefits from the point of view of computation time. Such benefit is measured in terms of the speedup factor, which is equal to the execution time associated with the exact (reference) solution divided by the execution time associated with the proposed strategy (considering either RB1 or RB2). It is found that the speedup factor associated with RB1 is equal to 12.2 while the speedup factor associated with RB2 is 9.6. Note that these speedup factors include the time for constructing the reduced basis. The difference between the two different cases can be attributed to the initial size of the reduced basis, which is larger in the case of RB2 than that of RB1.

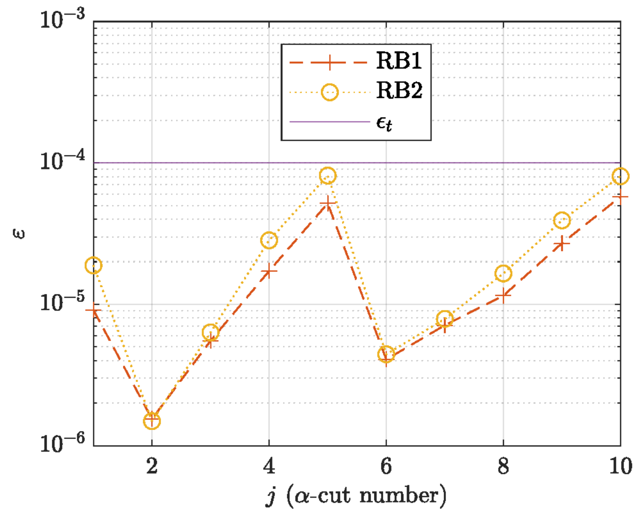

Fig. 9. Error measure ε associated with response calculated at different α-cut levels for $\varepsilon_t = 10^{-4}$

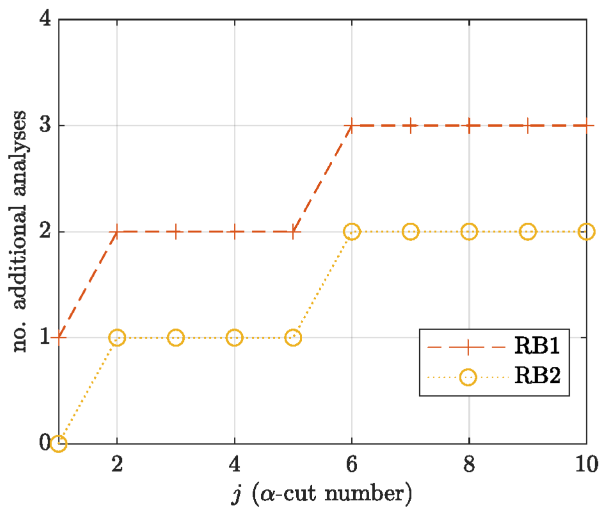

Fig. 10. Number of additional system analyses conducted at different α-cut levels for $\varepsilon_t = 10^{-4}$

5 Conclusions and Outlook

This contribution has presented a strategy for performing fuzzy analysis of linear system under static load applying α-level optimization. The strategy is based on a reduced order model, which provides a means for approximating the response of interest with reduced numerical efforts. As the quality of the approximate model may decrease during α-level optimization, the basis associated with the reduced order model is enriched adaptively, based on an error measure which is monitored as different α-cut levels are explored.

The results presented in the numerical example indicate that accurate estimates of the membership function associated with the response of interest can be obtained at reduced numerical efforts. This is quite remarkable, as the overall numerical efforts are brought down by one order of magnitude without sacrificing the quality of the final results.

Future research efforts will aim at expanding the range of application of the strategy reported here. Possible specific paths of development include, for example, considering other types of responses (such as forces or stresses), large-scale applications and extensions towards analysis of dynamic loading. These issues are currently being investigated by the authors.

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A First- and Second-Order Derivatives of the System's Response

The system's response and its first and second order derivatives are given by the following expressions (see, e.g. [39]).

$$\boldsymbol{u}(\boldsymbol{\theta}) = \boldsymbol{K}(\boldsymbol{\theta})^{-1}\boldsymbol{f}(\boldsymbol{\theta}) \tag{15}$$

$$\frac{\partial \boldsymbol{u}(\boldsymbol{\theta})}{\partial \theta_{i_1}} = \boldsymbol{K}(\boldsymbol{\theta})^{-1}\left(\frac{\partial \boldsymbol{f}(\boldsymbol{\theta})}{\partial \theta_{i_1}} - \sum_{k=1}^{n_K}\boldsymbol{K}_k \frac{\partial p_k(\boldsymbol{\theta})}{\partial \theta_{i_1}}\boldsymbol{u}(\boldsymbol{\theta})\right), \quad i_1 = 1,\ldots,n_\theta \tag{16}$$

$$\frac{\partial^2 \boldsymbol{u}(\boldsymbol{\theta})}{\partial \theta_{i_1}\partial \theta_{i_2}} = \boldsymbol{K}(\boldsymbol{\theta})^{-1}\left(\frac{\partial^2 \boldsymbol{f}(\boldsymbol{\theta})}{\partial \theta_{i_1}\partial \theta_{i_2}} - \sum_{k=1}^{n_K}\boldsymbol{K}_k \frac{\partial p_k(\boldsymbol{\theta})}{\partial \theta_{i_1}}\frac{\partial \boldsymbol{u}(\boldsymbol{\theta})}{\partial \theta_{i_2}} - \sum_{k=1}^{n_K}\boldsymbol{K}_l \frac{\partial p_k(\boldsymbol{\theta})}{\partial \theta_{i_2}}\frac{\partial \boldsymbol{u}(\boldsymbol{\theta})}{\partial \theta_{i_1}} - \sum_{k=1}^{n_K}\boldsymbol{K}_k \frac{\partial^2 p_k(\boldsymbol{\theta})}{\partial \theta_{i_1}\partial \theta_{i_2}}\boldsymbol{u}(\boldsymbol{\theta})\right), \quad i_1, i_2 = 1,\ldots,n_\theta \tag{17}$$

The calculation of the above expressions demands performing a single system analysis (matrix factorization) [39]. These partial derivatives are calculated analytically following a direct method [39].