# Energy flux in thin plates 

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#### Abstract

A formula for energy flux due to vibrations in a thin plate modelled by Kirchhoff theory is derived by considering a Mindlin plate and taking the low frequency (or low thickness) limit. It is shown that a term which is usually neglected in Kirchhoff theory persists close to free edges. This term does not affect the transverse displacement, but it does affect the energy flux. The new flux formula conserves energy and evaluates to zero along fixed, free and simply supported edges. An example problem, in which edge waves are excited by a point source located in a semi-infinite plate, is considered. Numerical calculations show that the energy radiated into the far field matches the energy introduced by the source.


## 1 Introduction

Energy flux has numerous applications in linear wave problems. It can be used to derive conservation of energy conditions [1, 2], to determine the proportion of incident wave energy converted into different phenomena in the scattered field [3] and to derive a radiation condition for Bloch waves $[4,5]$. In a time-harmonic problem, the general form for energy flux across a contour $\mathcal{C}$ (surface in three dimensions) in one period is

$$
\begin{equation*}
\langle\mathcal{E}\rangle=-\beta \operatorname{Im} \int_{\mathcal{C}} u \frac{\partial \bar{u}}{\partial n} \mathrm{~d} s \tag{1.1}
\end{equation*}
$$

Here, $\beta$ is a positive constant that depends on the physical context, the overbar represents a complex conjugate (as it does throughout), and the differentiation is in a direction orthogonal to $\mathcal{C}$, i.e.

$$
\begin{equation*}
\frac{\partial u}{\partial n}=\hat{\mathbf{n}} \cdot \nabla u \tag{1.2}
\end{equation*}
$$

where $\hat{\mathbf{n}}$ is a unit normal vector. The complex wavefunction $u$ depends only on position $\mathbf{r}$ and satisfies the Helmholtz equation

$$
\begin{equation*}
\left(\nabla^{2}+k^{2}\right) u=0 \tag{1.3}
\end{equation*}
$$

It is related to the physical wave field via

$$
\begin{equation*}
U(\mathbf{r} ; t)=\operatorname{Re}\left[u(\mathbf{r}) \mathrm{e}^{-\mathrm{i} \omega t}\right], \tag{1.4}
\end{equation*}
$$

where $\omega$ represents frequency. The relationship between the positive wavenumber $k$ and the frequency again depends on the physical context; often this is simply $k=\omega / c$, where $c$ is the phase speed. The convention that a function represented by an upper case letter depends on both position and time, and is related to a function represented by the corresponding lower case letter via (1.4) will be used throughout this paper. Homogeneous linear equations often apply to $u$ and $U$ interchangeably; for example the fact that $u$ satisfies (1.3) automatically means that $U$ is a solution. This does not apply to (1.1), since the integrand is nonlinear in $u$.

If $\mathcal{C}$ is a closed contour containing no sources, then $\langle\mathcal{E}\rangle=0$, and (1.1) can be obtained by substituting $u$ and $\bar{u}$ into Green's second identity. In this way, we find that

$$
\begin{equation*}
\int_{A}\left(u \nabla^{2} \bar{u}-\bar{u} \nabla^{2} u\right) \mathrm{d} A=\int_{\mathcal{C}}\left(u \frac{\partial \bar{u}}{\partial n}-\bar{u} \frac{\partial u}{\partial n}\right) \mathrm{d} s \tag{1.5}
\end{equation*}
$$

where $A$ is the region enclosed by $\mathcal{C}$. The Laplacian operators can be eliminated using (1.3), immediately showing that the left-hand side is zero. More generally, (1.1) can be derived directly from the physics of the problem (see [6, appendix B] for several examples), and a positive value for $\langle\mathcal{E}\rangle$ means that the net flow of energy across $\mathcal{C}$ is in the direction of $\hat{\mathbf{n}}$. Note that there is no contribution to the energy flux from any section of $\mathcal{C}$ that coincides with a boundary on which a homogeneous Neumann or Dirichlet condition applies, because there the integrand in (1.1) vanishes. Similarly, a Robin condition of the form

$$
\begin{equation*}
\frac{\partial u}{\partial n}+p u=0, \quad p \in \mathbb{R} \tag{1.6}
\end{equation*}
$$

allows us to write $u \partial \bar{u} / \partial n=-p|u|^{2}$. Consequently, the integrand in (1.1) is real on sections of $\mathcal{C}$ where (1.6) applies, again meaning there is no contribution to energy flux.

In contrast to the cases discussed above, no satisfactory formula for energy flux in a thin plate modelled by Kirchhoff theory seems to have appeared in the literature to date. A number of relevant results are given in a paper by Norris \& Vemula [7] but, shortly after this was published, Bobrovnitskii pointed out that certain equations have terms missing [8]. We will see later (sections 4 and 10) that the absence of these terms causes Norris \& Vemula's formula to violate conservation of energy in certain cases. However, the inclusion of Bobrovnitskii's terms leads to a formula that predicts nonzero flux across free edges. Therefore both formulas can produce results that are not physically plausible. In their reply to Bobrovnitskii [9], Norris \& Vemula refer to this situation as a 'recurring dilemma.' It was also noted in both [8] and [9] that Bobrovnitskii's correction terms disappear if the flux across a closed contour is calculated, and that Norris \& Vemula's final results are correct, even though certain intermediate formulae are not. However, both Norris \& Vemula and Bobrovnitskii make the implicit assumption that the contour is smooth. If the contour is closed but not smooth (an important case because rectangles are often used for conservation of energy calculations) then Bobrovnitskii's correction terms do not disappear in general.

The mains goal of this paper are to derive a formula which predicts zero net flux across closed contours containing no sources and also across fixed, free and simply supported edges, and to investigate its relationship to the formulae proposed by Norris \& Vemula and Bobrovnitskii. We obtain the new flux formula by taking the limit as frequency or plate thickness tend to zero in the equations for Mindlin plate theory [10], retaining only leading-order terms. Since the dilemma noted by Norris \& Vemula does not occur in Mindlin theory, we can expect this procedure to produce physically correct results. Indeed, we find that Kirchhoff theory is retrieved, with an extra term which corrects the energy flux but has no effect on the transverse displacement. We then consider a simple boundary value problem in which a point source is placed in a semi-infinite plate with a free edge. The source excites edge waves, which propagate without loss along the free edge, but decay exponentially in the orthogonal direction. Numerical results confirm that Norris \& Vemula's formula violates conservation of energy, whereas including Bobrovnitskii's correction terms leads to energy appearing to 'leak' across the free edge. The extra term from Mindlin theory corrects both of these flaws.

## 2 Note concerning line integrals

Integral representations for energy flux generally involve a line integral of a scalar field, which does not depend on contour orientation. Specifically, if $\mathcal{C}$ is parametrised by $\mathbf{r}(v)$, with $a \leq v \leq b$


Figure 1: The region $A$ and contour $\mathcal{C}$. The outgoing normal $\hat{\mathbf{n}}$ and tangent $\hat{\mathbf{s}}$ are such that $(n, s, z)$ forms a right-handed coordinate system, with $\hat{\mathbf{z}}$ directed out of the page. The orientation of $\mathcal{C}$ is determined by the direction of $\hat{\mathbf{s}}$.
then

$$
\begin{equation*}
\int_{\mathcal{C}} f(\mathbf{r}) \mathrm{d} s=\int_{a}^{b} f(\mathbf{r}(v))\left|\frac{\mathrm{d} \mathbf{r}}{\mathrm{~d} v}\right| \mathrm{d} v \tag{2.1}
\end{equation*}
$$

Note that this requires the existence of $\mathbf{r}^{\prime}(v)$, and in this case we will say that $\mathcal{C}$ is smooth. A piecewise smooth contour can be separated into smooth parts and (2.1) applied to each.

If a tangential derivative appears in the integrand (as is the case in many subsequent equations), this generates a line integral that does depend on contour orientation. Let $A$ be a simply connected region of the plane $z=0$, and let $\mathcal{C}$ be part of the boundary of $A$, traversed anticlockwise. Suppose further that $\mathcal{C}$ begins and ends at the points with position vectors $\mathbf{c}_{0}$ and $\mathbf{c}_{1}$, respectively. Let $\hat{\mathbf{n}}$ represent a unit outgoing normal that begins on $\mathcal{C}$ and let $\hat{\mathbf{s}}$ represent the unit tangent, starting at the same point. Finally, let $\hat{\mathbf{z}}$ be chosen so that $(n, s, z)$ forms a local right-handed coordinate system, as in figure 1. A line integral of the form

$$
\begin{equation*}
\mathrm{J}=\int_{\mathcal{C}} \frac{\partial}{\partial s} f(\mathbf{r}) \mathrm{d} s \tag{2.2}
\end{equation*}
$$

can be evaluated using the gradient theorem [11, section 6.2, theorem 3]. Thus,

$$
\begin{equation*}
\mathrm{J}=\int_{\mathcal{C}} \hat{\mathbf{s}} \cdot \nabla f(\mathbf{r}) \mathrm{d} s=\int_{\mathcal{C}} \nabla f(\mathbf{r}) \cdot \mathrm{d} \mathbf{s}=\Delta_{\mathcal{C}}[f(\mathbf{r})] \tag{2.3}
\end{equation*}
$$

where we have introduced the notation

$$
\begin{equation*}
\Delta_{\mathcal{C}}[f(\mathbf{r})]=f\left(\mathbf{c}_{1}\right)-f\left(\mathbf{c}_{0}\right) \tag{2.4}
\end{equation*}
$$

which we will use throughout. Note that we have implicitly adopted the convention that there is no conjugation of the first operand in a scalar product in (2.3). Integration by parts can be applied to line integrals in the usual way, but only if the contour is smooth. This can be proved by writing $f(\mathbf{r})=g_{1}(\mathbf{r}) g_{2}(\mathbf{r})$ in (2.2), and then deriving equivalent expressions by using the product rule and via (2.3).

## 3 Classical plate theory and Green's identity

For a plate modelled by Kirchhoff theory [12, chapter 4] [13, chapter 4], also known as classical theory, the transverse displacement $W$ satisfies fourth order equation

$$
\begin{equation*}
\nabla^{4} W+\frac{\rho h}{D} \frac{\partial^{2} W}{\partial t^{2}}=0 \tag{3.1}
\end{equation*}
$$

Here, $\rho, h$ and $D$ are the the plate density, thickness and bending stiffness, respectively. The latter is related to the Young modulus $E$ and Poisson ratio $\nu$ via

$$
\begin{equation*}
D=\frac{E h^{3}}{12\left(1-\nu^{2}\right)} . \tag{3.2}
\end{equation*}
$$

Assuming time-harmonic motion, with $w$ and $W$ related as in (1.4), the displacement may be separated into two components; thus

$$
\begin{equation*}
w=w_{1}+w_{2}, \tag{3.3}
\end{equation*}
$$

where the two fields $w_{1}$ and $w_{2}$ satisfy the Helmholtz equations

$$
\begin{equation*}
\left(\nabla^{2}+k^{2}\right) w_{1}=0 \quad \text { and } \quad\left(\nabla^{2}-k^{2}\right) w_{2}=0 . \tag{3.4}
\end{equation*}
$$

The wavenumber $k$ is given by

$$
\begin{equation*}
k=\left(\frac{\rho h \omega^{2}}{D}\right)^{1 / 4}=\left(\frac{12\left(1-\nu^{2}\right) \rho \omega^{2}}{E h^{2}}\right)^{1 / 4} \tag{3.5}
\end{equation*}
$$

Strictly, the equation for $w_{2}$ is a modified Helmholtz equation, but we can also consider this to be a Helmholtz equation with an imaginary wavenumber. There are three common types of boundary condition for Kirchhoff plates. These are the fixed edge, on which

$$
\begin{equation*}
W=\frac{\partial W}{\partial n}=0 \tag{3.6}
\end{equation*}
$$

the simply supported edge, where

$$
\begin{equation*}
W=M_{n}=0 \tag{3.7}
\end{equation*}
$$

and the free edge, on which

$$
\begin{equation*}
M_{n}=V_{n}=0 . \tag{3.8}
\end{equation*}
$$

In the above conditions, $M_{n}$ is the bending moment and $V_{n}$ is the Kirchhoff shear force. These may be expressed in terms of the displacement via

$$
\begin{equation*}
M_{n}=-D\left(\frac{\partial^{2} W}{\partial n^{2}}+\nu \frac{\partial W}{\partial s^{2}}\right) \tag{3.9}
\end{equation*}
$$

and

$$
\begin{equation*}
V_{n}=-D \frac{\partial}{\partial n}\left(\frac{\partial^{2} W}{\partial n^{2}}+(2-\nu) \frac{\partial W}{\partial s^{2}}\right) \tag{3.10}
\end{equation*}
$$

Various boundary integral formulae for waves in thin plates can be derived using Green's second identity. For example, substituting $w_{j}$ and $\bar{w}_{j}$ as in (1.5) yields

$$
\begin{equation*}
\operatorname{Im} \int_{\mathcal{C}} w_{j} \frac{\partial \bar{w}_{j}}{\partial n} \mathrm{~d} s=0 \tag{3.11}
\end{equation*}
$$

for a closed contour $\mathcal{C}$ containing no sources. However, the boundary conditions apply to $w$ rather than $w_{1}$ and $w_{2}$ individually, so there is no obvious way to simplify this expression if $\mathcal{C}$ runs along a free or simply supported edge. For a fixed edge, taking the difference between the $j=1$ and $j=2$ cases in (3.11) produces an integrand that vanishes on the boundary, since there $w_{1}=-w_{2}$ and $\partial w_{1} / \partial n=-\partial w_{2} / \partial n$. To generate an integral that simplifies on any boundary, we substitute $w$ and $\nabla^{2} \bar{w}$ into Green's identity, and in this way we find that

$$
\begin{equation*}
\int_{A}\left(w \nabla^{4} \bar{w}-\nabla^{2} w \nabla^{2} \bar{w}\right) \mathrm{d} A=\int_{\mathcal{C}}\left(w \frac{\partial}{\partial n} \nabla^{2} \bar{w}-\frac{\partial w}{\partial n} \nabla^{2} \bar{w}\right) \mathrm{d} s \tag{3.12}
\end{equation*}
$$

where $A$ is the region bounded by $\mathcal{C}$. After using (3.3) and (3.4) to eliminate the Laplacian operators (noting that $\nabla^{4} w=k^{4} w$ ), the left-hand side simplifies to

$$
\begin{equation*}
k^{4} \int_{A}\left(w \bar{w}-\left(w_{1}-w_{2}\right)\left(\bar{w}_{1}-\bar{w}_{2}\right)\right) \mathrm{d} A=4 k^{4} \operatorname{Re} \int_{A} w_{1} \bar{w}_{2} \mathrm{~d} A . \tag{3.13}
\end{equation*}
$$

Consequently, taking the imaginary part of (3.12) yields $\mathcal{K}=0$ for a closed contour containing no sources, where

$$
\begin{align*}
\mathcal{K} & =\operatorname{Im} \int_{\mathcal{C}}\left(w \frac{\partial}{\partial n} \nabla^{2} \bar{w}-\frac{\partial w}{\partial n} \nabla^{2} \bar{w}\right) \mathrm{d} s  \tag{3.14}\\
& =2 k^{2} \operatorname{Im} \int_{\mathcal{C}}\left(\bar{w}_{1} \frac{\partial w_{1}}{\partial n}-\bar{w}_{2} \frac{\partial w_{2}}{\partial n}\right) \mathrm{d} s \tag{3.15}
\end{align*}
$$

having used (3.3) and (3.4) in the last line. Let us now consider the situation in which $\mathcal{C}$ is a smooth (but not necessarily closed) contour that coincides with an edge. If this edge is fixed, then we have $w=\partial w / \partial n=0$ on $\mathcal{C}$, so it follows immediately that $\mathcal{K}=0$. For other types of edge, we observe that the condition $M_{n}=0$ can be rewritten as

$$
\begin{equation*}
\nabla^{2} w=(1-\nu) \frac{\partial^{2} w}{\partial s^{2}} \tag{3.16}
\end{equation*}
$$

Since $w=0$ on a simply supported edge its tangential derivatives must also vanish. Therefore both terms in (3.14) vanish, and $\mathcal{K}=0$ for a simply supported edge as well. The case of a free edge is more complicated. First, we rewrite the condition $V_{n}=0$ as

$$
\begin{equation*}
\frac{\partial}{\partial n} \nabla^{2} w=(\nu-1) \frac{\partial^{3} w}{\partial n \partial s^{2}} \tag{3.17}
\end{equation*}
$$

Then,

$$
\begin{equation*}
\mathcal{K}=(\nu-1) \operatorname{Im} \int_{\mathcal{C}}\left(w \frac{\partial^{3} \bar{w}}{\partial n \partial s^{2}}+\frac{\partial w}{\partial n} \frac{\partial \bar{w}}{\partial s^{2}}\right) \mathrm{d} s \tag{3.18}
\end{equation*}
$$

and integrating by parts yields

$$
\begin{equation*}
\mathcal{K}=(\nu-1) \operatorname{Im}\left\{\Delta_{\mathcal{C}}\left[w \frac{\partial^{2} \bar{w}}{\partial n \partial s}+\frac{\partial w}{\partial n} \frac{\partial \bar{w}}{\partial s}\right]-\int_{\mathcal{C}}\left(\frac{\partial w}{\partial s} \frac{\partial^{2} \bar{w}}{\partial n \partial s}+\frac{\partial^{2} w}{\partial n \partial s} \frac{\partial \bar{w}}{\partial s}\right) \mathrm{d} s\right\} . \tag{3.19}
\end{equation*}
$$

Evidently the imaginary part of the remaining integral is zero. Therefore if $\mathcal{C}$ is a free edge, then

$$
\begin{equation*}
\mathcal{K}=(\nu-1) \operatorname{Im} \Delta_{\mathcal{C}}\left[\frac{\partial}{\partial n}\left(w \frac{\partial \bar{w}}{\partial s}\right)\right] . \tag{3.20}
\end{equation*}
$$

In summary, the integral (3.14) (or equivalently (3.15)) evaluates to zero if $\mathcal{C}$ is a closed contour containing no sources, or if $\mathcal{C}$ coincides with a fixed or simply supported edge. However, if $\mathcal{C}$ coincides with a free edge then the value of the integral is given by (3.20), which is not zero in general. Consequently, (3.14) cannot be a conservation of energy condition because there can be no flux across a free edge.

## 4 Review of existing flux formulae for Kirchhoff plates

In this section, we examine the properties of the energy flux formulae for Kirchhoff plates, due to Norris \& Vemula [7] and Bobrovnitskii [8]. For the component of flux in the $n$ direction, Norris \& Vemula give the (incomplete) formula

$$
\begin{equation*}
F_{n}=-V_{n} \frac{\partial W}{\partial t}+M_{n} \frac{\partial^{2} W}{\partial n \partial t} \tag{4.1}
\end{equation*}
$$

Evidently, $F_{n}$ vanishes on a fixed, free or simply supported edge. Writing the bending moment and Kirchhoff shear force in terms of the displacement using (3.9) and (3.10) yields

$$
\begin{equation*}
\frac{F_{n}}{D}=\frac{\partial W}{\partial t} \frac{\partial}{\partial n} \nabla^{2} W-\frac{\partial^{2} W}{\partial n \partial t} \nabla^{2} W+(1-\nu)\left[\frac{\partial^{2} W}{\partial n \partial t} \frac{\partial^{2} W}{\partial s^{2}}+\frac{\partial W}{\partial t} \frac{\partial^{3} W}{\partial n \partial s^{2}}\right] \tag{4.2}
\end{equation*}
$$

Next, Norris \& Vemula integrate over a smooth, closed contour $\mathcal{C}$, and then integrate by parts twice in the last term. We follow the same process, but drop the assumption that $\mathcal{C}$ is closed. Thus, defining

$$
\begin{equation*}
\mathcal{E}=\int_{\mathcal{C}} F_{n} \mathrm{~d} s \tag{4.3}
\end{equation*}
$$

we find that

$$
\begin{align*}
\mathcal{E}=D \int_{\mathcal{C}}\left(\frac{\partial W}{\partial t} \frac{\partial}{\partial n} \nabla^{2} W-\frac{\partial^{2} W}{\partial n \partial t} \nabla^{2} W+(1-\nu)\right. & \left.\frac{\partial}{\partial t}\left[\frac{\partial W}{\partial n} \frac{\partial^{2} W}{\partial s^{2}}\right]\right) \mathrm{d} s \\
& +D(1-\nu) \Delta_{\mathcal{C}}\left[\frac{\partial W}{\partial t} \frac{\partial^{2} W}{\partial n \partial s}-\frac{\partial^{2} W}{\partial s \partial t} \frac{\partial W}{\partial n}\right] . \tag{4.4}
\end{align*}
$$

Finally, we take a time-average using the definition

$$
\begin{equation*}
\langle\mathcal{E}\rangle=\frac{\omega}{2 \pi} \int_{0}^{2 \pi / \omega} \mathcal{E} \mathrm{d} t \tag{4.5}
\end{equation*}
$$

The last term in the integral in (4.4) is eliminated by this operation, because $W$ has period $2 \pi / \omega$. For the remaining terms we use the standard result for products of time-harmonic fields, that is

$$
\begin{equation*}
\frac{\omega}{2 \pi} \int_{0}^{2 \pi / \omega} \operatorname{Re}\left[f(\mathbf{r}) \mathrm{e}^{-\mathrm{i} \omega t}\right] \operatorname{Re}\left[g(\mathbf{r}) \mathrm{e}^{-\mathrm{i} \omega t}\right] \mathrm{d} t=\frac{1}{2} \operatorname{Re}[f(\mathbf{r}) \bar{g}(\mathbf{r})] \tag{4.6}
\end{equation*}
$$

to obtain

$$
\begin{equation*}
\langle\mathcal{E}\rangle=\frac{\omega D}{2} \operatorname{Im}\left\{\int_{\mathcal{C}}\left(w \frac{\partial}{\partial n} \nabla^{2} \bar{w}-\frac{\partial w}{\partial n} \nabla^{2} \bar{w}\right) \mathrm{d} s+(1-\nu) \Delta_{\mathcal{C}}\left[\frac{\partial}{\partial n}\left(w \frac{\partial \bar{w}}{\partial s}\right)\right]\right\} . \tag{4.7}
\end{equation*}
$$

If $\mathcal{C}$ coincides with a fixed or simply supported edge, then the integral in (4.7) evaluates to zero as in section 3, and the end-point term disappears because $w=\partial w / \partial s=0$. If $\mathcal{C}$ coincides with a free edge, then the integral is given by (3.20), and this cancels the end-point term. On the other hand, if $\mathcal{C}$ is a closed contour containing no sources then the integral evaluates to zero according to Green's theorem (see (3.13) and (3.14)) but the end-point term does not disappear unless $\mathcal{C}$ is also smooth. For a piecewise smooth closed contour, it is necessary to apply (4.7) separately on each section. Green's theorem still applies, but there is no reason to suppose that the end-point contribution from the terminus of one section will cancel the contribution from the beginning of the next, because the directions of the derivatives will be different. Consequently, (4.7) may violate conservation of energy, and is not a valid formula for energy flux.

Bobrovnitskii [8] introduced a correction to (4.1) by including the additional term

$$
\begin{equation*}
-\frac{\partial}{\partial s}\left(M_{n s}^{\mathrm{K}} \frac{\partial W}{\partial t}\right)=-D(1-\nu) \frac{\partial}{\partial s}\left(\frac{\partial^{2} W}{\partial n \partial s} \frac{\partial W}{\partial t}\right) \tag{4.8}
\end{equation*}
$$

where $M_{n s}^{K}$ is the twisting moment for Kirchhoff theory. We have included the superscript ' K ' because the twisting moment for Mindlin theory (which we will need later) is defined in a slightly different way. The leading-order expression for the twisting moment in Mindlin theory at low frequency (or low thickness) is $-M_{n s}^{\mathrm{K}}$. See equations (4.2.20) and (8.3.9) with (8.3.3) in [13]. No such discrepancy occurs in the definition of the bending moment. It should be also noted that Bobrovnitskii's notation for bending and twisting moments is somewhat different to Norris \& Vemula's notation. We have used the latter for consistency. Evidently (4.8) can be integrated using the gradient theorem, so its inclusion affects only the end-point term in (4.7). Integrating over $\mathcal{C}$ and taking a time-average using (4.5) and (4.6), we find that

$$
\begin{equation*}
-D(1-\nu)\left\langle\int_{\mathcal{C}} \frac{\partial}{\partial s}\left(\frac{\partial^{2} W}{\partial n \partial s} \frac{\partial W}{\partial t}\right) \mathrm{d} s\right\rangle=\frac{\omega D}{2}(1-\nu) \operatorname{Im} \Delta_{\mathcal{C}}\left[\frac{\partial^{2} w}{\partial n \partial s} \bar{w}\right] \tag{4.9}
\end{equation*}
$$

and with this included, (4.7) becomes

$$
\begin{equation*}
\langle\mathcal{E}\rangle=\frac{\omega D}{2} \operatorname{Im}\left\{\int_{\mathcal{C}}\left(w \frac{\partial}{\partial n} \nabla^{2} \bar{w}-\frac{\partial w}{\partial n} \nabla^{2} \bar{w}\right) \mathrm{d} s+(1-\nu) \Delta_{\mathcal{C}}\left[\frac{\partial w}{\partial n} \frac{\partial \bar{w}}{\partial s}\right]\right\} . \tag{4.10}
\end{equation*}
$$

In general, this formula does not predict zero flux across a free edge, because the integral given by (3.20) no longer cancels the end-point terms. However, it does evaluate to zero for a piecewise smooth closed contour containing no sources. In this case, the integral vanishes due to Green's theorem, exactly as before. To prove the result for the second term, set $\hat{\mathbf{n}}=[\cos \Theta, \sin \Theta]$ for an arbitrary angle $\Theta$, so that $\hat{\mathbf{s}}=[-\sin \Theta, \cos \Theta]$. Then

$$
\begin{align*}
\frac{\partial w}{\partial n} \frac{\partial \bar{w}}{\partial s} & =\hat{\mathbf{n}} \cdot \nabla w \hat{\mathbf{s}} \cdot \nabla \bar{w}  \tag{4.11}\\
& =\left(\cos \Theta \frac{\partial w}{\partial x}+\sin \Theta \frac{\partial w}{\partial y}\right)\left(-\sin \Theta \frac{\partial \bar{w}}{\partial x}+\cos \Theta \frac{\partial \bar{w}}{\partial y}\right) . \tag{4.12}
\end{align*}
$$

The imaginary part of this expression is

$$
\begin{equation*}
\operatorname{Im}\left[\frac{\partial w}{\partial n} \frac{\partial \bar{w}}{\partial s}\right]=\frac{\partial w}{\partial x} \frac{\partial \bar{w}}{\partial y} \tag{4.13}
\end{equation*}
$$

which does not depend on $\Theta$. Consequently, the end-point contribution from the terminus of one section of a piecewise contour will always cancel the contribution from the beginning of the next section; the change in the orientation of the derivatives has no effect. Note that this calculation does not depend on the absence of sources inside $\mathcal{C}$. This is only needed so that Green's theorem yields zero for the integral.

## 5 Mindlin plate theory

Consider a plate plate modelled by Mindlin theory [10], [13, §8.3], with the undisturbed midplane occupying $z=0$. The deformation is given by [14]

$$
\begin{equation*}
\mathbf{U}=z \mathbf{\Psi}(\mathbf{r}, t)+W(\mathbf{r}, t) \hat{\mathbf{z}}, \tag{5.1}
\end{equation*}
$$

where $\mathbf{r}$ is a position vector in the plane $z=0, \boldsymbol{\Psi}$ is the in-plane vector of rotations and $W$ is the transverse displacement (note that the corresponding equation in [15] contains a typographical
error). As before, we will assume time-harmonic motion and use the convention that a function represented by an upper case letter depends on position and time, whereas the corresponding lower case letter represents a function defined from the former via (1.4), and dependent on position alone. Then the deformation can be further decomposed by writing

$$
\begin{equation*}
w=w_{1}+w_{2} \quad \text { and } \quad \boldsymbol{\psi}=A_{1} \nabla w_{1}+A_{2} \nabla w_{2}-\hat{\mathbf{z}} \times \nabla \phi, \tag{5.2}
\end{equation*}
$$

where $w_{j}$ and $\phi$ are solutions to the Helmholtz equations

$$
\begin{equation*}
\left(\nabla^{2}+k_{j}^{2}\right) w_{j}=0, \quad j=1,2 \quad \text { and } \quad\left(\nabla^{2}+k_{3}^{2}\right) \phi=0 . \tag{5.3}
\end{equation*}
$$

The three wavenumbers and the constants $A_{j}$ can be expressed in terms of the physical parameters of the plate by first writing

$$
\begin{equation*}
c_{s}=\kappa\left(\frac{\mu}{\rho}\right)^{1 / 2}, \quad c_{p}=\left[\frac{E}{\rho\left(1-\nu^{2}\right)}\right]^{1 / 2}, \quad k_{s}=\frac{\omega}{c_{s}} \quad \text { and } \quad k_{p}=\frac{\omega}{c_{p}} . \tag{5.4}
\end{equation*}
$$

Here, $\mu$ is a Lamé constant defined via

$$
\begin{equation*}
\mu=\frac{E}{2(1+\nu)}, \tag{5.5}
\end{equation*}
$$

and $E, \rho$ and $\nu$ are as defined in section 3. The constant $\kappa$, required for Mindlin theory, may be chosen to optimise the approximation of shear forces [13, pp. 484, 492-3]. We then have

$$
\begin{align*}
& k_{j}^{2}=\frac{k_{p}^{2}+k_{s}^{2}}{2}+(-1)^{j+1} \sqrt{k^{4}+\frac{\left(k_{p}^{2}-k_{s}^{2}\right)^{2}}{4}}, \quad j=1,2,  \tag{5.6}\\
& k_{3}^{2}=\frac{\kappa^{2} k_{1}^{2} k_{2}^{2}}{k_{p}^{2}}=\frac{2 k_{1}^{2} k_{2}^{2}}{(1-\nu) k_{s}^{2}} \tag{5.7}
\end{align*}
$$

and

$$
\begin{equation*}
A_{j}=-1+\frac{k_{s}^{2}}{k_{j}^{2}}, \tag{5.8}
\end{equation*}
$$

where $k$ is the wavenumber for Kirchhoff plate theory, as in (3.5). Clearly, $k_{1}^{2}>0$, but for the parameter regime in which Mindlin theory is valid, $k_{2}^{2}<0$ [15], which also means that $k_{3}^{2}<0$. We take $k_{2}$ and $k_{3}$ to be positive imaginary.

The boundary conditions at the edge of a Mindlin plate require that one term in each of the three pairs

$$
\begin{equation*}
\left(\frac{\partial \Psi_{n}}{\partial t}, M_{n}\right), \quad\left(\frac{\partial \Psi_{s}}{\partial t}, M_{n s}\right) \quad \text { and } \quad\left(\frac{\partial W}{\partial t}, Q_{n}\right) \tag{5.9}
\end{equation*}
$$

must vanish [10]. For time-harmonic waves, the condition $\partial W / \partial t=0$ is equivalent to $W=0$, and likewise for the conditions on $\Psi_{s}$ and $\Psi_{n}$. As before $M_{n}$ and $M_{n s}$ represent the bending and twisting moments, respectively, and $Q_{n}$ is the shear force. For Mindlin theory, these may be expressed in terms of the displacements as

$$
\begin{align*}
M_{n} & =D\left(\frac{\partial \Psi_{n}}{\partial n}+\nu \frac{\partial \Psi_{s}}{\partial s}\right)  \tag{5.10}\\
M_{n s} & =\frac{D(1-\nu)}{2}\left(\frac{\partial \Psi_{s}}{\partial n}+\frac{\partial \Psi_{n}}{\partial s}\right) \tag{5.11}
\end{align*}
$$

and

$$
\begin{equation*}
Q_{n}=\kappa^{2} \mu h\left(\frac{\partial W}{\partial n}+\Psi_{n}\right) \tag{5.12}
\end{equation*}
$$

|  | $k$ | $k_{s}$ | $k_{p}$ | $k_{j}(j=1,2)$ | $k_{3}$ | $A_{j}$ | $1+A_{j}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\omega \rightarrow 0$ | $O\left(\omega^{1 / 2}\right)$ | $O(\omega)$ | $O(\omega)$ | $O\left(\omega^{1 / 2}\right)$ | $O(1)$ | $O(1)$ | $O(\omega)$ |
| $h \rightarrow 0$ | $O\left(h^{-1 / 2}\right)$ | $O(1)$ | $O(1)$ | $O\left(h^{-1 / 2}\right)$ | $O\left(h^{-1}\right)$ | $O(1)$ | $O(h)$ |

Table 1: Asymptotic orders for various constants in the limits $\omega \rightarrow 0$ and $h \rightarrow 0$.

To obtain expressions in terms of $W_{j}$ and $\Phi$, we must first evaluate the vector product in (5.2). Using the basic properties of the vector product, and the right-handed nature of the local coordinate system $(n, s, z)$, we find that

$$
\begin{equation*}
\hat{\mathbf{z}} \times \nabla \Phi=\hat{\mathbf{z}} \times\left(\hat{\mathbf{n}} \frac{\partial \Phi}{\partial n}+\hat{\mathbf{s}} \frac{\partial \Phi}{\partial s}\right)=\hat{\mathbf{s}} \frac{\partial \Phi}{\partial n}-\hat{\mathbf{n}} \frac{\partial \Phi}{\partial s} \tag{5.13}
\end{equation*}
$$

It then follows that

$$
\begin{equation*}
\Psi_{n}=\frac{\partial}{\partial n}\left(A_{1} w_{1}+A_{2} w_{2}\right)+\frac{\partial \Phi}{\partial s} \quad \text { and } \quad \Psi_{s}=\frac{\partial}{\partial s}\left(A_{1} w_{1}+A_{2} w_{2}\right)-\frac{\partial \Phi}{\partial n} \tag{5.14}
\end{equation*}
$$

meaning that

$$
\begin{align*}
& M_{n}=D\left(\frac{\partial^{2}}{\partial n^{2}}+\nu \frac{\partial^{2}}{\partial s^{2}}\right)\left(A_{1} W_{1}+A_{2} W_{2}\right)+D(1-\nu) \frac{\partial^{2} \Phi}{\partial n \partial s},  \tag{5.15}\\
& M_{n s}=D(1-\nu)\left(\frac{\partial^{2}}{\partial n \partial s}\left(A_{1} W_{1}+A_{2} W_{2}\right)+\frac{\partial^{2} \Phi}{\partial s^{2}}+\frac{k_{3}^{2}}{2} \Phi\right) \tag{5.16}
\end{align*}
$$

and

$$
\begin{equation*}
Q_{n}=\kappa^{2} \mu h\left(\frac{k_{s}^{2}}{k_{1}^{2}} \frac{\partial W_{1}}{\partial n}+\frac{k_{s}^{2}}{k_{2}^{2}} \frac{\partial W_{2}}{\partial n}+\frac{\partial \Phi}{\partial s}\right) \tag{5.17}
\end{equation*}
$$

having used the Helmholtz equation for $\Phi$ in (5.16). The shear force $Q_{n}$ can be expressed in a more convenient manner by observing that

$$
\begin{equation*}
k_{s}^{2} \kappa^{2} \mu h=\rho h \omega^{2}=D k^{4} \tag{5.18}
\end{equation*}
$$

which follows from (3.5) and (5.4). Using this identity, we obtain

$$
\begin{equation*}
Q_{n}=D k^{4}\left(\frac{1}{k_{1}^{2}} \frac{\partial W_{1}}{\partial n}+\frac{1}{k_{2}^{2}} \frac{\partial W_{2}}{\partial n}+\frac{1}{k_{s}^{2}} \frac{\partial \Phi}{\partial s}\right) . \tag{5.19}
\end{equation*}
$$

Of the eight possible combinations in (5.9), three have particular physical importance. These are a fixed edge, on which $W=\Psi_{n}=\Psi_{s}=0$, a simply supported edge, where [16] $W=M_{n}=\Psi_{s}=0$, and a free edge, on which $M_{n}=M_{n s}=Q_{n}=0$.

## 6 Kirchhoff theory as a limit of Mindlin theory

In this section, we retrieve the equations for classical plate theory by letting $\omega$ or $h$ tend to zero, and retaining only leading-order terms. In what follows, the symbol ' $\sim$ ' indicates an expression that is correct at leading order in either limit, and the notation $O(\cdot)$ may refer to a power of $\omega$ or of $h$, as appropriate. We begin by establishing the asymptotic order of the various parameters that appear in Mindlin theory. These results can be obtained from (3.5), (5.4) and (5.6)-(5.8), and are shown in table 1. Now, from (5.4) and (5.6), it is clear that $k_{1} \sim k$ and $k_{2} \sim \mathrm{i} k$, so that the equations for $w_{1}$ and $w_{2}$ reduce to (3.4) at leading order. Using the limiting forms of $k_{1}$ and $k_{2}$ in (5.7), we find that the leading-order behaviour of $k_{3}$ is given by

$$
\begin{equation*}
k_{3}^{2} \sim-12 \kappa^{2} / h^{2} . \tag{6.1}
\end{equation*}
$$

However, this does not appear to yield any significant simplifications, so we leave the symbol $k_{3}$ 'as is'. We must also consider the boundary conditions. We make the assumption that the field incoming toward any boundary consists of modes with wavenumbers $k_{1}$ and $k_{2}$ only. Modes with wavenumbers $k_{2}$ and $k_{3}$ are exponentially damped, but table 1 shows that $\left|k_{2}\right| \ll\left|k_{3}\right|$, so that the rate of decay in the latter is much stronger. In this way, we allow for the possibility that $\Phi$ may persist in a thin strip close to a boundary but is negligible elsewhere. If the geometry is such that modes with wavenumber $k_{3}$ excited at one boundary can have a significant effect at another boundary then $\Phi$ cannot be neglected, and Kirchhoff theory is invalid.

Our next objective is to find means of satisfying all three Mindlin boundary conditions at leading order, by making appropriate choices for $\Phi$ for fixed, free and simply supported edges. If $W=0$ on the boundary then the tangential derivative $\partial W / \partial s$ must also vanish, so that

$$
\begin{equation*}
\Psi \sim-\hat{\mathbf{n}} \frac{\partial W}{\partial n}-\hat{\mathbf{z}} \times \nabla \Phi \tag{6.2}
\end{equation*}
$$

We can then eliminate $\Psi_{s}$ on the boundary by setting $\Phi=0$, thus satisfying a second boundary condition from (5.9) at leading order. For a fixed edge, we then require that $\partial W / \partial n=0$, which is the second condition for Kirchhoff theory. The choice $\Phi=0$ also works for the simply supported edge, because we then have

$$
\begin{equation*}
M_{n} \sim-D\left(\frac{\partial^{2} W}{\partial n^{2}}+\nu \frac{\partial^{2} W}{\partial s^{2}}\right) \tag{6.3}
\end{equation*}
$$

and if this vanishes on the boundary then the conditions for Kirchhoff theory are satisfied. In contrast, it is not possible to satisfy all three free edge boundary conditions at leading order with the choice $\Phi=0$. To resolve this issue, we begin by observing that the fields must vary on different length scales, in the low frequency limit because

$$
\begin{equation*}
\left(\nabla^{2} W_{j}\right) / W_{j}=O(\omega) \quad \text { and } \quad\left(\nabla^{2} \Phi\right) / \Phi=O(1) \quad \text { as } \quad \omega \rightarrow 0 \tag{6.4}
\end{equation*}
$$

However, the length scales for $W_{j}$ and $\Phi$ in the direction tangential to the edge must be the same, or else the boundary conditions cannot be satisfied. Similarly,

$$
\begin{equation*}
\left(\nabla^{2} W_{j}\right) / W_{j}=O\left(h^{-1}\right) \quad \text { and } \quad\left(\nabla^{2} \Phi\right) / \Phi=O\left(h^{-2}\right) \quad \text { as } \quad h \rightarrow 0, \tag{6.5}
\end{equation*}
$$

but again, the scales parallel to the edge must be the same. In view of this, we introduce dimensionless spatial variables $n_{*}$ and $s_{*}$ via

$$
\begin{equation*}
W_{j}(n, s ; t)=W_{j}\left(k_{1}^{-1} n_{*}, k_{1}^{-1} s_{*} ; t\right) \quad \text { and } \quad \Phi(n, s ; t)=\Phi\left(\left|k_{3}\right|^{-1} n_{*}, k_{1}^{-1} s_{*} ; t\right) \tag{6.6}
\end{equation*}
$$

Retaining only the leading-order contributions to $A_{j}$, we then find that

$$
\begin{equation*}
M_{n} \sim-D k_{1}^{2}\left(\frac{\partial^{2} W}{\partial n_{*}^{2}}+\nu \frac{\partial^{2} W}{\partial s_{*}^{2}}\right)+D(1-\nu) k_{1}\left|k_{3}\right| \frac{\partial^{2} \Phi}{\partial n_{*} \partial s_{*}} \tag{6.7}
\end{equation*}
$$

and

$$
\begin{equation*}
M_{n s} \sim-D(1-\nu)\left(k_{1}^{2} \frac{\partial^{2} W}{\partial n_{*} \partial s_{*}}-k_{1}^{2} \frac{\partial^{2} \Phi}{\partial s_{*}^{2}}-\frac{k_{3}^{2}}{2} \Phi\right) \tag{6.8}
\end{equation*}
$$

No terms are discarded from the shear force at this stage, so

$$
\begin{equation*}
Q_{n}=D k^{4}\left(\frac{1}{k_{1}} \frac{\partial W_{1}}{\partial n_{*}}+\frac{k_{1}}{k_{2}^{2}} \frac{\partial W_{2}}{\partial n_{*}}+\frac{k_{1}}{k_{s}^{2}} \frac{\partial \Phi}{\partial s_{*}}\right) . \tag{6.9}
\end{equation*}
$$

Assume without loss of generality that $W_{j}=O(1)$. Then, if $\Phi=O\left(\omega^{1 / 2}\right)$ as $\omega \rightarrow 0$, the terms in (6.7) are of equal magnitude, but $\Phi$ dominates in (6.8) and (6.9). Therefore we reject this
possibility. On the other hand, if $\Phi=O(\omega)$ then it disappears from (6.7) at leading order, but contributes to both (6.8) and (6.9). Similar reasoning leads to the conclusion that $\Phi=O(h)$ near the free edge in the limit $h \rightarrow 0$. Returning to dimensional variables, we again have (6.3), but now also

$$
\begin{equation*}
M_{n s} \sim D(1-\nu)\left(\frac{k_{3}^{2}}{2} \Phi-\frac{\partial^{2} W}{\partial n \partial s}\right) . \tag{6.10}
\end{equation*}
$$

For $Q_{n}$, we eliminate $k_{s}$ from (5.19) using (5.7) and then take the low frequency (or low thickness) limit to obtain

$$
\begin{align*}
Q_{n} & \sim-D\left(-k^{2} \frac{\partial W_{1}}{\partial n}+k^{2} \frac{\partial W_{2}}{\partial n}+\frac{1-\nu}{2} k_{3}^{2} \frac{\partial \Phi}{\partial s}\right)  \tag{6.11}\\
& \sim-D\left(\frac{\partial}{\partial n} \nabla^{2} W+\frac{1-\nu}{2} k_{3}^{2} \frac{\partial \Phi}{\partial s}\right), \tag{6.12}
\end{align*}
$$

having used the Helmholtz equations for $W_{1}$ and $W_{2}$ in the last line. Eliminating $\Phi$ from (6.10) and (6.12), we find that

$$
\begin{equation*}
Q_{n}+\frac{\partial M_{n s}}{\partial s} \sim-D \frac{\partial}{\partial n}\left(\nabla^{2} W+(1-\nu) \frac{\partial^{2} W}{\partial s^{2}}\right) \tag{6.13}
\end{equation*}
$$

The right-hand side is now the Kirchhoff shear force (3.10). Thus, if we require that this must vanish on a free edge along with the leading-order contribution to $M_{n}$ (6.3), then we have two boundary conditions for $W$ without reference to $\Phi$. However, we did not set $\Phi=0$. In fact, $\Phi$ should be chosen to satisfy (6.10) or (6.12) on the free edge. Since (6.13) is obtained by combining these, the two possibilities are equivalent, but (6.10) is more convenient because this determines $\Phi$ itself, rather than a derivative. Therefore, to satisfy all three boundary conditions at leading order, we set

$$
\begin{equation*}
k_{3}^{2} \Phi=2 \frac{\partial^{2} W}{\partial n \partial s} \tag{6.14}
\end{equation*}
$$

on the free edge. The consequences of this are that $\Phi$ is partially decoupled, and $W$ may be calculated according to the usual governing equations and boundary conditions for Kirchhoff theory. However, if the energy flux is to be calculated then $\Phi$ must be included in the vicinity of free edges. It can be determined as an outgoing solution to the Helmholtz equation with wavenumber $k_{3}$, satisfying (6.14) as a boundary condition. An example is given in section 8 .

## 7 Energy flux derived from Mindlin theory

For a plate modelled by Mindlin theory, the component of the energy flux vector in the $n$ direction is given by [15]

$$
\begin{equation*}
F_{n}=-\left(Q_{n} \frac{\partial W}{\partial t}+M_{n} \frac{\partial \Psi_{n}}{\partial t}+M_{n s} \frac{\partial \Psi_{s}}{\partial t}\right) \tag{7.1}
\end{equation*}
$$

Now we already have leading-order expressions for $Q_{n}, M_{n}$ and $M_{n s}$ in (6.12), (6.3) and (6.10), and the leading-order contributions to $\Psi_{s}$ and $\Psi_{n}$ are clearly $-\partial W / \partial s$ and $-\partial W / \partial n$, respectively. Using these, we reach the crucial result

$$
\begin{equation*}
\frac{F_{n}}{D} \sim \frac{\partial W}{\partial t} \frac{\partial}{\partial n} \nabla^{2} W-\nabla^{2} W \frac{\partial^{2} W}{\partial n \partial t}+(1-\nu)\left[\left(\frac{\partial^{2} W}{\partial s^{2}} \frac{\partial^{2} W}{\partial n \partial t}-\frac{\partial^{2} W}{\partial s \partial n} \frac{\partial^{2} W}{\partial s \partial t}\right)+\frac{k_{3}^{2}}{2} \frac{\partial}{\partial s}\left(\Phi \frac{\partial W}{\partial t}\right)\right] . \tag{7.2}
\end{equation*}
$$

This expression includes Norris \& Vemula's original formula (4.2) with Bobrovnitskii's correction (4.8), but the last term on the right-hand side is new. This arises because we have not set $\Phi=0$


Figure 2: The source and image point, along with the three contours used for the conservation of energy calculation. The plate occupies the region $y>0$, and has a free edge along $y=0$.
throughout the plate. The result of integrating along a smooth contour $\mathcal{C}$ and taking a time average can be deduced by noting that we need only add the contribution due to the new term to (4.10). After applying the gradient theorem and using (4.6), we find that

$$
\begin{equation*}
\langle\mathcal{E}\rangle \sim \frac{\omega D}{2} \operatorname{Im}\left\{\int_{\mathcal{C}}\left(w \frac{\partial}{\partial n} \nabla^{2} \bar{w}-\frac{\partial w}{\partial n} \nabla^{2} \bar{w}\right) \mathrm{d} s+(1-\nu) \Delta_{\mathcal{C}}\left[w \frac{k_{3}^{2}}{2} \bar{\phi}-\frac{\partial w}{\partial s} \frac{\partial \bar{w}}{\partial n}\right]\right\} . \tag{7.3}
\end{equation*}
$$

Finally, we eliminate the Laplacian operators from (7.3) using (3.4) to obtain

$$
\begin{equation*}
\langle\mathcal{E}\rangle \sim \frac{\omega D}{2} \operatorname{Im}\left\{2 k^{2} \int_{\mathcal{C}}\left(\bar{w}_{1} \frac{\partial w_{1}}{\partial n}-\bar{w}_{2} \frac{\partial w_{2}}{\partial n}\right) \mathrm{d} s+(1-\nu) \Delta_{\mathcal{C}}\left[w \frac{k_{3}^{2}}{2} \bar{\phi}-\frac{\partial w}{\partial s} \frac{\partial \bar{w}}{\partial n}\right]\right\} . \tag{7.4}
\end{equation*}
$$

Both terms on the right-hand side of (7.4), or equivalently (7.3), vanish if $\mathcal{C}$ coincides with a fixed or simply supported edge (see section 3). For a free edge, the value of the integral is given by (3.20), and $\phi$ is related to $w$ by (6.14). The two terms cancel each other. For a piecewise smooth closed contour containing no sources the integral vanishes, as does the last term, according to the proof in section 4 . Since the term involving $\phi$ has no derivatives this vanishes in a similar way - the contribution from the final point of each section cancels the contribution from the first point of the next section. Consequently, if the flux across a piecewise smooth closed contour is calculated with $\phi$ omitted, then the contributions to the flux from individual sections may be incorrect but the total will be correct. Since (7.3) and (7.4) give the correct leading-order (low frequency or low thickness) contribution to the time-averaged energy flux over the contour $\mathcal{C}$, they may be treated as exact results for Kirchhoff theory.

## 8 A point source in a semi-infinite Kirchhoff plate

As an example of the theory developed in the preceding sections, we now consider a boundary value problem in which a thin Kirchhoff plate occupies the region $x \in \mathbb{R}, y>0$ with a free edge on the line $y=0$. A point source is placed at the point $(x, y)=(0, a)$; see figure 2 for an illustration. The field incident on the free edge is given by the Green's function for the plate [17], that is

$$
\begin{equation*}
w^{\mathrm{i}}=\frac{\mathrm{i} b}{8}\left[\mathrm{H}_{0}^{(1)}\left(k \sqrt{x^{2}+(y-a)^{2}}\right)-\mathrm{H}_{0}^{(1)}\left(\mathrm{i} k \sqrt{x^{2}+(y-a)^{2}}\right)\right], \tag{8.1}
\end{equation*}
$$

where $\mathrm{H}_{0}^{(1)}(\cdot)$ represents a Hankel function of the first kind. The value of the constant $b$ is often given as $k^{-2} D^{-1}$, which does not have the correct dimensions, since $w$ is a displacement. This


Figure 3: The complex plane, with the integration contour, poles at $\alpha= \pm \alpha_{\mathrm{e}}$ and branch points at $\alpha= \pm k$ and $\alpha= \pm \mathrm{i} k$. The additional poles at $\alpha= \pm \mathrm{i} \alpha_{\mathrm{e}}$ are not shown, since these play no role in the analysis.
apparent discrepancy occurs when the force per unit area is equated to a Dirac delta function whose argument has dimensions of length in the derivation of the Green's function (see e.g. [18]). Here we leave $b$ as an arbitrary constant, but note that it has dimensions of length, and this gives the correct dimensions in all subsequent equations. The Green's function also has the integral representation

$$
\begin{equation*}
w^{\mathrm{i}}=\frac{b}{8 \pi} \int_{\Gamma}\left(\frac{\mathrm{e}^{-\gamma(\alpha)|y-a|}}{\gamma(\alpha)}-\frac{\mathrm{e}^{-\lambda(\alpha)|y-a|}}{\lambda(\alpha)}\right) \mathrm{e}^{-\mathrm{i} \alpha x} \mathrm{~d} \alpha, \tag{8.2}
\end{equation*}
$$

where

$$
\begin{equation*}
\gamma(\alpha)=\left(\alpha^{2}-k^{2}\right)^{1 / 2} \quad \text { and } \quad \lambda(\alpha)=\left(\alpha^{2}+k^{2}\right)^{1 / 2} \tag{8.3}
\end{equation*}
$$

and the contour $\Gamma$ is the real line traversed from left to right, with an indentation above $\alpha=-k$ and below $\alpha=k$; see figure 3. The branches of the functions $\lambda$ and $\gamma$ are such that when $\alpha$ is real,

$$
\lambda(\alpha)>0 \quad \text { and } \quad \gamma(\alpha)= \begin{cases}\sqrt{\alpha^{2}-k^{2}} & \text { if }|\alpha| \geq k,  \tag{8.4}\\ -\mathrm{i} \sqrt{k^{2}-\alpha^{2}} & \text { if }|\alpha|<k .\end{cases}
$$

In both representations for the Green's function, the first term is a solution to the Helmholtz equation with wavenumber $k$ (except at the source point), whereas the second has wavenumber $\mathrm{i} k$.

The scattered field may be expressed in the form

$$
\begin{equation*}
w^{\mathrm{s}}=\frac{b}{8 \pi} \int_{\Gamma}\left(B(\alpha) \mathrm{e}^{-\gamma(\alpha)(y+a)}+C(\alpha) \mathrm{e}^{-\lambda(\alpha)(y+a)}\right) \mathrm{e}^{-\mathrm{i} \alpha x} \mathrm{~d} \alpha \tag{8.5}
\end{equation*}
$$

which introduces an image point at $(0,-a)$. The functions $B$ and $C$ are to be determined using the fact that the total field (i.e. $w^{\mathrm{i}}+w^{\mathrm{s}}$ ) must satisfy $M_{n}=V_{n}=0$ on $y=0$, where $M_{n}$ and $V_{n}$ are the bending moment and Kirchhoff shear force, given by (3.9) and (3.10), respectively. The unit outgoing normal on the boundary is $\hat{\mathbf{n}}=-\hat{\mathbf{y}}$ and the tangent is $\hat{\mathbf{s}}=\hat{\mathbf{x}}$. Resolving the modulus sign in (8.2) by noting that $-|y-a|=y-a$ in the vicinity of the $y=0$, we find that the boundary conditions yield

$$
\begin{equation*}
(1+\gamma B)\left(G-2 k^{2}\right) \lambda \mathrm{e}^{-\gamma a}=(1-\lambda C) G \gamma \mathrm{e}^{-\lambda a} \tag{8.6}
\end{equation*}
$$

and

$$
\begin{equation*}
(1-\gamma B) G \mathrm{e}^{-\gamma a}=(1+\lambda C)\left(G-2 k^{2}\right) \mathrm{e}^{-\lambda a} . \tag{8.7}
\end{equation*}
$$

Here,

$$
\begin{equation*}
G(\alpha)=\alpha^{2}(1-\nu)+k^{2}, \tag{8.8}
\end{equation*}
$$

and we have omitted the argument $\alpha$ from the functions $B, C, G, \gamma$ and $\lambda$ for brevity. Solving for $B$ and $C$ now yields

$$
\begin{equation*}
B=\frac{\left(G-2 k^{2}\right)\left[2 \gamma G \mathrm{e}^{(\gamma-\lambda) a}-\left(G-2 k^{2}\right) \lambda\right]-\gamma G^{2}}{\left(G-2 k^{2}\right)^{2} \lambda \gamma-\gamma^{2} G^{2}} \tag{8.9}
\end{equation*}
$$

and

$$
\begin{equation*}
C=\frac{\left(G-2 k^{2}\right) \lambda\left[2 G \mathrm{e}^{(\lambda-\gamma) a}-\left(G-2 k^{2}\right)\right]-\gamma G^{2}}{\left(G-2 k^{2}\right)^{2} \lambda^{2}-\gamma \lambda G^{2}} . \tag{8.10}
\end{equation*}
$$

Note that both $B$ and $C$ have a factor $\left(G-2 k^{2}\right)^{2} \lambda-\gamma G^{2}$ in the denominator. Equating the two terms and squaring leads to the polynomial

$$
\begin{equation*}
(\nu+3)(\nu-1)^{3} \alpha^{8}+(6 \nu-2)(\nu-1) \alpha^{4} k^{4}+k^{8}=0 . \tag{8.11}
\end{equation*}
$$

This is the dispersion relation for edge waves [19, 20]. Since $-1<\nu<0.5$, the leading and constant coefficients have opposite signs, so there is one solution for $\alpha^{4}<0$ and one for $\alpha^{4}>0$. The former is a spurious solution generated by squaring (it is not possible to have $\arg \gamma=\arg \lambda$ if $\alpha^{2}$ is imaginary). The positive root is given by

$$
\begin{equation*}
\alpha_{\mathrm{e}}^{4}=\frac{1-3 \nu+2 \sqrt{\nu^{2}+(1-\nu)^{2}}}{(\nu+3)(1-\nu)^{2}} k^{4}, \tag{8.12}
\end{equation*}
$$

and we take $\alpha_{\mathrm{e}}>0$. Since the left-hand side of (8.11) evaluates to $k^{8}$ if $\alpha=0$ and $k^{8} \nu^{4}$ if $\alpha=k$, it follows that $\alpha_{\mathrm{e}}>k$. The contour of integration in (8.5) must be indented to avoid the poles on the real line. To ensure outgoing waves in the far field, it must pass above the pole at $\alpha=-\alpha_{\mathrm{e}}$, and below the pole at $\alpha=\alpha_{\mathrm{e}}$, as in figure 3. Setting $y=0$ in (8.5) and applying Jordan's lemma [21, theorem 5.6] to close the contour shows that an edge wave propagating to the right or to the left is included for $x>0$ or $x<0$, respectively. See section 9 for further details.

The additional mode $\phi$, predicted to exist near the free edge in section 6 , can easily be calculated. It may be represented by the Fourier integral

$$
\begin{equation*}
\phi=\frac{b}{8 \pi} \int_{\Gamma} P(\alpha) \mathrm{e}^{-\left(\alpha^{2}-k_{3}^{2}\right)^{1 / 2} y} \mathrm{e}^{-\mathrm{i} \alpha x} \mathrm{~d} \alpha, \tag{8.13}
\end{equation*}
$$

where $\left(\alpha^{2}-k_{3}^{2}\right)^{1 / 2}>0$ for real $\alpha$. Clearly this satisfies the Helmholtz equation with wavenumber $k_{3}$, and decays exponentially as $y$ is increased. The function $P(\alpha)$ may be determined by using (6.14) on the free edge. Thus, from (8.2) and (8.5), We find that

$$
\begin{equation*}
P(\alpha)=\frac{2 \mathrm{i} \alpha}{k_{3}^{2}}\left([1-\gamma(\alpha) B(\alpha)] \mathrm{e}^{-\gamma(\alpha) a}-[1+\lambda(\alpha) C(\alpha)] \mathrm{e}^{-\lambda(\alpha) a}\right) . \tag{8.14}
\end{equation*}
$$

Note that the substitution $\alpha=k \eta$ applied to (8.14), (8.2) and (8.5) shows that if $w$ is $O(1)$ in the low frequency or low thickness limit, then $\phi$ is $O(\omega)$ or $O(h)$, respectively. This is consistent with the general analysis in section 6 .

## 9 Far field due to a source in a semi-infinite Kirchhoff plate

To approximate (8.5) in the far field, we employ the method of steepest descents [22, chapter 3]. We begin by introducing polar coordinates $x=r \cos \theta$ and $y=r \sin \theta$. Then, for the term with $\gamma$ in the exponent, which is the scattered component of $w_{1}$, we write

$$
\begin{equation*}
\chi_{1}(\alpha)=\gamma(\alpha) \sin \theta+\mathrm{i} \alpha \cos \theta \tag{9.1}
\end{equation*}
$$

so that

$$
\begin{equation*}
w_{1}^{\mathrm{s}}=\frac{b}{8 \pi} \int_{\Gamma} B(\alpha) \mathrm{e}^{-a \gamma(\alpha)} \mathrm{e}^{-r \chi_{1}(\alpha)} \mathrm{d} \alpha \tag{9.2}
\end{equation*}
$$

Here, the factor $\mathrm{e}^{-a \gamma(\alpha)}$ has been separated from the rest of the exponent to avoid the necessity of working with shifted coordinates. A simple saddle point is located at

$$
\begin{equation*}
\alpha_{s}=-k \cos \theta, \tag{9.3}
\end{equation*}
$$

and here we have $\chi_{1}\left(\alpha_{s}\right)=-\mathrm{i} k$. The contribution from the saddle point therefore represents an outgoing circular wave, with no exponential decay. We denote the contribution by $w_{1}^{\mathrm{c}}$. To find the leading-order term in $w_{1}^{\mathrm{c}}$ for large $r$, we may use the standard saddle point formula, or write

$$
\begin{equation*}
w_{1}^{\mathrm{s}}=\frac{b}{8 \pi} \int_{\Gamma}\left(f(\alpha)-f\left(\alpha_{s}\right)\right) \frac{\mathrm{e}^{-r \chi_{1}(\alpha)}}{\gamma(\alpha)} \mathrm{d} \alpha+\frac{b f\left(\alpha_{s}\right)}{8 \pi} \int_{\Gamma} \frac{\mathrm{e}^{-r \chi_{1}(\alpha)}}{\gamma(\alpha)} \mathrm{d} \alpha, \tag{9.4}
\end{equation*}
$$

where $f(\alpha)=B(\alpha) \gamma(\alpha) \mathrm{e}^{-a \gamma(\alpha)}$. The first integral does not contribute to the far field at leading order, because the integrand vanishes at the saddle point. The second is a standard representation for a Hankel function; it may be evaluated by equating the first term in (8.1) to the first term in (8.2). Then, using [23, eqn. 10.17.5], we obtain

$$
\begin{equation*}
w_{1}^{\mathrm{c}}=d(\theta) \frac{b \mathrm{e}^{\mathrm{i} \pi / 4} \mathrm{e}^{\mathrm{i} k r}}{4 \sqrt{2 \pi k r}}+O\left(r^{-3 / 2}\right), \quad \text { where } \quad d(\theta)=B\left(\alpha_{s}\right) \gamma\left(\alpha_{s}\right) \mathrm{e}^{\mathrm{i} k a \sin \theta} \tag{9.5}
\end{equation*}
$$

Note that the factor $\gamma\left(\alpha_{s}\right)$ cancels with a corresponding factor in the denominator of $B(\alpha)$ (see (8.9)). We must also consider the possibility that singularities of $B(\alpha)$ may contribute to the far field. To this end, we begin by observing that the steepest descents path may be parametrised by writing $\chi_{1}(\alpha)=v^{2}-\mathrm{i} k$ for $v \in \mathbb{R}$. After some rearrangement, this leads to the formula

$$
\begin{equation*}
\alpha=-\left(k+\mathrm{i} v^{2}\right) \cos \theta+v\left(v^{2}-2 \mathrm{i} k\right)^{1 / 2} \sin \theta . \tag{9.6}
\end{equation*}
$$

Here, we take the principal value for the square root, so that $v<0$ and $v>0$ correspond to the branches of the path that lie predominantly in the left and right half planes, respectively. The steepest descents path crosses the imaginary axis at the point $\alpha=-\mathrm{i} k \cot \theta$, and so requires a diversion around a branch cut on the imaginary axis if $\theta<\pi / 4$ or $\theta>3 \pi / 4$. On the faces of the cut, $\gamma$ is pure imaginary, so

$$
\begin{equation*}
\operatorname{Re}\left[\chi_{1}(\alpha)\right]=|\alpha \cos \theta|>k \cot \theta \cos \theta>k / \sqrt{2}, \tag{9.7}
\end{equation*}
$$

so the contribution from this diversion is exponentially small. Finally, we account for the poles at $\alpha= \pm \alpha_{\mathrm{e}}$ by observing that the steepest descents path crosses the real axis twice: at the saddle and at the point $\alpha=-k \sec \theta$. If $\cos \theta>k \alpha_{\mathrm{e}}^{-1}$ then the pole at $\alpha=-\alpha_{\mathrm{e}}$ contributes to the far field. Similarly, if $\cos \theta<-k \alpha_{\mathrm{e}}^{-1}$ then the pole at $\alpha=\alpha_{\mathrm{e}}$ must be taken into account. The residues are calculated at the end of this section.

Next consider the term in (8.5) with $\lambda$ in its exponent. Denote this by $w_{2}^{\mathrm{s}}$. Writing

$$
\begin{equation*}
\chi_{2}(\alpha)=\lambda(\alpha) \sin \theta+\mathrm{i} \alpha \cos \theta, \tag{9.8}
\end{equation*}
$$

we find that

$$
\begin{equation*}
w_{2}^{\mathrm{s}}=\frac{b}{8 \pi} \int_{\Gamma} C(\alpha) \mathrm{e}^{-a \lambda(\alpha)} \mathrm{e}^{-r \chi_{2}(\alpha)} \mathrm{d} \alpha . \tag{9.9}
\end{equation*}
$$

In this case the saddle point is located at $\alpha=\mathrm{i} \alpha_{s}$, and here we have $\chi_{2}\left(\mathrm{i} \alpha_{s}\right)=k$. The contribution from this point is therefore exponentially small, and need not be calculated further. However, we must still consider the possibility of singularity contributions. Writing $\chi_{2}(\alpha)=k+v^{2}$ for $v \in \mathbb{R}$ and rearranging, we find that the steepest descents path is parametrised by the formula

$$
\begin{equation*}
\alpha=-\mathrm{i} \cos \theta\left(k+v^{2}\right)+v \sin \theta \sqrt{v^{2}+2 k} . \tag{9.10}
\end{equation*}
$$

This path lies entirely in the upper half plane if $\theta>\pi / 2$ and in the lower half plane if $\theta<\pi / 2$. Consequently, the residues from the poles at $\pm \alpha_{\mathrm{e}}$ must be taken into account, along with diversions around the branch cuts emanating from $\alpha= \pm k$. Since $\alpha= \pm k$ are not branch points of the exponent, they are (from the perspective of the steepest descents method) end-points, which have a lower order contribution than the saddle. To see this, consider the situation in which $\theta=\pi$, so that $\operatorname{Re}\left[\chi_{2}(k)\right]=0$, meaning there is no exponential decay in the branch point contribution. Closing the contour in the upper half plane and making the substitution $\alpha=k+\mathrm{i} k \eta$, we find that the branch point contribution is

$$
\begin{equation*}
w_{2}^{\mathrm{b}}=\frac{\mathrm{i} b k \mathrm{e}^{\mathrm{i} k r}}{8 \pi} \int_{0}^{\infty}\left[C_{r}(k+\mathrm{i} k \eta)-C_{\ell}(k+\mathrm{i} k \eta)\right] \mathrm{e}^{-a \lambda(k+\mathrm{i} k \eta)} \mathrm{e}^{-r k \eta} \mathrm{~d} \eta, \tag{9.11}
\end{equation*}
$$

where the function $C$ is defined in (8.10) and the subscripts ' $r$ ' and ' $\ell$ ' refer to evaluation on the left and right faces of the cut, respectively. Now $C(k+\mathrm{i} k \eta)$ remains bounded as $\eta \rightarrow 0$, because this limit corresponds to $\alpha \rightarrow k$ in (8.10). The difference $C_{r}(k+\mathrm{i} k \eta)-C_{\ell}(k+\mathrm{i} k \eta)$ switches sign if $\eta$ winds once around the branch point at the origin. Therefore the function

$$
\begin{equation*}
g(\eta)=\frac{k}{\sqrt{\eta}}\left[C_{r}(k+\mathrm{i} k \eta)-C_{\ell}(k+\mathrm{i} k \eta)\right] \mathrm{e}^{-a \lambda(k+\mathrm{i} k \eta)} \tag{9.12}
\end{equation*}
$$

has a convergent Taylor series about the point $\eta=0$. Hence,

$$
\begin{equation*}
w_{2}^{\mathrm{b}}=\frac{\mathrm{i} b g(0) \mathrm{e}^{\mathrm{i} k r}}{16 \sqrt{\pi}(k r)^{3 / 2}}+O\left(r^{-5 / 2}\right), \tag{9.13}
\end{equation*}
$$

having used the fact that

$$
\begin{equation*}
\int_{0}^{\infty} \sqrt{\eta} \mathrm{e}^{-k r \eta} \mathrm{~d} \eta=\frac{\sqrt{\pi}}{2}(k r)^{-3 / 2} \tag{9.14}
\end{equation*}
$$

Consequently, only the poles at $\pm \alpha_{\mathrm{e}}$ can make significant contributions to the far field in this case. Recalling that $k_{3}$ is imaginary, the integral representation for $\phi$ (8.13) can be analysed in much the same way, and again, only the poles need to be considered.

Finally, we calculate the residue terms which give rise to edge waves in the scattered field. Consider the pole at $\alpha=-\alpha_{\mathrm{e}}$, and denote its contribution to (8.5) by $w_{+}^{\mathrm{e}}$. This must be included near the edge for $x>0$. Since $\gamma, \lambda$ and $G$ are even functions, We find that

$$
\begin{equation*}
w_{+}^{\mathrm{e}}=-\frac{\mathrm{i} b \mathcal{L}}{4}\left[\hat{B}\left(\alpha_{\mathrm{e}}\right) \mathrm{e}^{-\gamma\left(\alpha_{\mathrm{e}}\right)(y+a)}+\hat{C}\left(\alpha_{\mathrm{e}}\right) \mathrm{e}^{-\lambda\left(\alpha_{\mathrm{e}}\right)(y+a)}\right] \mathrm{e}^{\mathrm{i} \alpha_{\mathrm{e}} x}, \tag{9.15}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathcal{L}=\operatorname{Res}_{\alpha=-\alpha_{e}} \frac{1}{\left(G-2 k^{2}\right)^{2} \lambda-G^{2} \gamma} \tag{9.16}
\end{equation*}
$$

and the functions $\hat{B}$ and $\hat{C}$ are given by

$$
\begin{equation*}
\hat{B}(\alpha)=\left(G-2 k^{2}\right)\left[2 G \mathrm{e}^{(\gamma-\lambda) a}-(\lambda / \gamma)\left(G-2 k^{2}\right)\right]-G^{2} \tag{9.17}
\end{equation*}
$$

and

$$
\begin{equation*}
\hat{C}(\alpha)=\left(G-2 k^{2}\right)\left[2 G \mathrm{e}^{(\lambda-\gamma) a}-\left(G-2 k^{2}\right)\right]-(\gamma / \lambda) G^{2} . \tag{9.18}
\end{equation*}
$$

Notably, $\lambda\left(\alpha_{\mathrm{e}}\right), \gamma\left(\alpha_{\mathrm{e}}\right)$ and $G\left(\alpha_{\mathrm{e}}\right)$ are all real, so $\hat{B}\left(\alpha_{\mathrm{e}}\right), \hat{C}\left(\alpha_{\mathrm{e}}\right)$ and $\mathcal{L}$ are also real. The value of $\mathcal{L}$ can be determined using L'Hôpital's rule. Thus,

$$
\begin{align*}
\mathcal{L} & =\lim _{\alpha \rightarrow-\alpha_{\mathrm{e}}} \frac{\alpha-\alpha_{\mathrm{e}}}{\left(G-2 k^{2}\right)^{2} \lambda-\gamma G^{2}}  \tag{9.19}\\
& =\left.\frac{1}{\left(G-2 k^{2}\right)^{2} \lambda^{\prime}+2\left(G-2 k^{2}\right) \lambda G^{\prime}-2 \gamma G G^{\prime}-\gamma^{\prime} G^{2}}\right|_{\alpha=-\alpha_{\mathrm{e}}}, \tag{9.20}
\end{align*}
$$

where $\gamma^{\prime}=\alpha / \gamma$ and $\lambda^{\prime}=\alpha / \lambda$ from (8.3) and $G^{\prime}=2 \alpha(1-\nu)$ from (8.8). The contribution to (8.5) from the pole at $\alpha=\alpha_{\mathrm{e}}$, which is required for $x<0$, can be deduced by symmetry. It is the same as (9.15) except that $\mathrm{e}^{\mathrm{i} \alpha_{\mathrm{e}} x}$ is replaced by $\mathrm{e}^{-\mathrm{i} \alpha_{\mathrm{e}} x}$. In a similar way, the edge wave contribution to (8.13) for $x>0$ is found to be

$$
\begin{equation*}
\phi_{\mathrm{e}}^{+}=\frac{\alpha_{\mathrm{e}} b \mathcal{L}}{2 k_{3}^{2}}\left[\gamma\left(\alpha_{\mathrm{e}}\right) \hat{B}\left(\alpha_{\mathrm{e}}\right) \mathrm{e}^{-\gamma\left(\alpha_{\mathrm{e}}\right) a}+\lambda\left(\alpha_{\mathrm{e}}\right) \hat{C}\left(\alpha_{\mathrm{e}}\right) \mathrm{e}^{-\lambda\left(\alpha_{\mathrm{e}}\right) a}\right] \mathrm{e}^{-\left(\alpha_{\mathrm{e}}^{2}-k_{3}^{2}\right) y} \mathrm{e}^{\mathrm{i} \alpha_{\mathrm{e}} x} . \tag{9.21}
\end{equation*}
$$

It is easy to verify that (9.15) and (9.21) satisfy (6.14). The contribution for $x<0$ is the same as (9.21), but with $\mathrm{e}^{\mathrm{i} \alpha_{\mathrm{e}} x}$ replaced by $\mathrm{e}^{-\mathrm{i} \alpha_{\mathrm{e}} x}$. Since $k_{3}$ is imaginary, and $\left|k_{3}\right| \gg k$ in the low frequency limit, this mode is very strongly localised to the vicinity of the free edge.

## 10 Conservation of energy for a source in a semi-infinite Kirchhoff plate

To perform a conservation of energy calculation for the problem discussed in sections 8-9, we must first determine the amount of energy radiating from the source. Since the resulting quantity is the total energy present in the system, we denote it by $\left\langle\mathcal{E}_{\mathrm{t}}\right\rangle$. To calculate this, we apply equation (7.4) on the circular contour $\mathcal{C}_{1}$ shown in figure 2 . The end-point terms disappear because $\mathcal{C}_{1}$ is smooth and closed. Therefore, after separating the incident and scattered waves, we have

$$
\begin{equation*}
\left\langle\mathcal{E}_{\mathrm{t}}\right\rangle=k^{2} \omega D \operatorname{Im} \int_{\mathcal{C}_{1}}\left(\left(\bar{w}_{1}^{\mathrm{i}}+\bar{w}_{1}^{\mathrm{s}}\right) \frac{\partial}{\partial r_{0}}\left(w_{1}^{\mathrm{i}}+w_{1}^{\mathrm{s}}\right)-\left(\bar{w}_{2}^{\mathrm{i}}+\bar{w}_{2}^{\mathrm{s}}\right) \frac{\partial}{\partial r_{0}}\left(w_{2}^{\mathrm{i}}+w_{2}^{\mathrm{s}}\right)\right) \mathrm{d} s, \tag{10.1}
\end{equation*}
$$

where

$$
\begin{equation*}
r_{0}=\sqrt{x^{2}+(y-a)^{2}} . \tag{10.2}
\end{equation*}
$$

Writing the incident components explicitly in terms of Hankel functions, and using [23, eqn. 10.6.3], we find that

$$
\begin{gather*}
w_{1}^{\mathrm{i}}=\frac{\mathrm{i} b}{8} \mathrm{H}_{0}^{(1)}\left(k r_{0}\right), \quad w_{2}^{\mathrm{i}}=-\frac{\mathrm{i} b}{8} \mathrm{H}_{0}^{(1)}\left(\mathrm{i} k r_{0}\right), \\
\frac{\partial w_{1}^{\mathrm{i}}}{\partial r_{0}}=-\frac{\mathrm{i} k b}{8} \mathrm{H}_{1}^{(1)}\left(k r_{0}\right) \quad \text { and } \quad \frac{\partial w_{2}^{\mathrm{i}}}{\partial r_{0}}=-\frac{k b}{8} \mathrm{H}_{1}^{(1)}\left(\mathrm{i} k r_{0}\right) . \tag{10.3}
\end{gather*}
$$

Two simplifications to (10.1) are immediate. First, [23, eqn. 10.27.8] shows that $w_{2}^{\mathrm{i}}$ and its derivatives are real so that $\operatorname{Im}\left[\bar{w}_{2}^{\mathrm{i}} \partial w_{2}^{\mathrm{i}} / \partial r_{0}\right]=0$. Second, the Wronskian relation [23, eqn. 10.5.5] with $\nu=0$ shows that

$$
\begin{equation*}
\operatorname{Im}\left[\bar{w}_{1}^{\mathrm{i}} \frac{\partial w_{1}}{\partial r_{0}}\right]=\frac{|b|^{2}}{32 \pi r_{0}}, \tag{10.4}
\end{equation*}
$$

which is constant on $\mathcal{C}_{1}$. To simplify the remaining terms, let the contour radius be $\varepsilon$ and consider the effect of taking the limit $\varepsilon \rightarrow 0$. Since the integration path length is $2 \pi \varepsilon$, a factor $\varepsilon^{-1}$ is required to obtain a nonzero result. Such a term is present in the Hankel functions of order one, but not those of order zero. No such term can be present in the components of the scattered field or their derivatives since all of these are bounded throughout the plate. Consequently,

$$
\begin{equation*}
\left\langle\mathcal{E}_{\mathrm{t}}\right\rangle=\frac{k^{2} \omega D}{8}\left[\frac{|b|^{2}}{2}+k \operatorname{Im} \lim _{\varepsilon \rightarrow 0} \int_{\mathcal{C}_{1}} b\left(\mathrm{H}_{1}^{(1)}(\mathrm{i} k \varepsilon) \bar{w}_{2}^{\mathrm{s}}-\mathrm{i} \mathrm{H}_{1}^{(1)}(k \varepsilon) \bar{w}_{1}^{\mathrm{s}}\right) \mathrm{d} s\right] . \tag{10.5}
\end{equation*}
$$

Finally, we use [23, eqns. 10.7.7] to obtain

$$
\begin{equation*}
\left\langle\mathcal{E}_{\mathrm{t}}\right\rangle=\frac{k^{2} \omega D}{2}\left[\frac{|b|^{2}}{8}+\operatorname{Im}\left(\bar{b} w^{\mathrm{s}}(0, a)\right)\right] . \tag{10.6}
\end{equation*}
$$

Next consider the far field. Here we take advantage of the fact that (7.4) always yields zero flux across the free edge, so we need not integrate along $y=0$. Elsewhere, there are two important contributions to include: the edge wave and the circular wave. These phenomena can be considered separately, despite the nonlinearity in (7.4). A proof that 'cross terms', in which a part of the circular wave is multiplied by a part of the edge wave, make no contribution to the integral can be found in [3, section 6]. Note that the end-point terms in (7.4) disappear in the limit $r \rightarrow \infty$ unless they contain only edge waves, since the amplitude of the circular wave is proportional to $r^{-1 / 2}$.

For the circular wave, we integrate along the semi-circular $\operatorname{arc} \mathcal{C}_{2}$, which has radius $R$ and centre $(x, y)=(0,0)$ (see figure 2). The incident field (also a circular wave) must be included in this calculation, so we express this in terms of polar coordinates centred at the origin. To achieve this, we use [23, eqn. 10.17.5] in (8.1), to obtain

$$
\begin{equation*}
w^{\mathrm{i}}=\frac{b \mathrm{e}^{\mathrm{i} \pi / 4} \mathrm{e}^{\mathrm{i} k r_{0}}}{4 \sqrt{2 \pi k r_{0}}}+O\left(r_{0}^{-3 / 2}\right) \tag{10.7}
\end{equation*}
$$

Next, we observe that

$$
\begin{equation*}
r_{0}=r \sqrt{1-2(a / r) \sin \theta+(a / r)^{2}}=r-a \sin \theta+O\left(r^{-1}\right) \tag{10.8}
\end{equation*}
$$

and hence $r_{0}^{-1 / 2}=r^{-1 / 2}+O\left(r^{-3 / 2}\right)$, meaning that

$$
\begin{equation*}
w^{\mathrm{i}}=\frac{b \mathrm{e}^{\mathrm{i} \pi / 4} \mathrm{e}^{\mathrm{i} k(r-a \sin \theta)}}{4 \sqrt{2 \pi k r}}+O\left(r^{-3 / 2}\right) \tag{10.9}
\end{equation*}
$$

Finally, we apply (7.4) to the sum of the incident field and the reflected circular wave, integrating along $\mathcal{C}_{2}$. Since the path length is $\pi R$, all $O\left(R^{-3 / 2}\right)$ and smaller terms can be removed by taking the limit $R \rightarrow \infty$. Hence, the contribution from the circular wave is given by

$$
\begin{align*}
\left\langle\mathcal{E}_{\mathrm{c}}\right\rangle & =\frac{\omega D|b k|^{2}}{32 \pi} \lim _{R \rightarrow \infty} \frac{1}{R} \int_{\mathcal{C}_{2}}\left|d(\theta)+\mathrm{e}^{-\mathrm{i} k a \sin \theta}\right|^{2} \mathrm{~d} s  \tag{10.10}\\
& =\frac{\omega D|b k|^{2}}{32 \pi} \int_{0}^{\pi}\left|d(\theta)+\mathrm{e}^{-\mathrm{i} k a \sin \theta}\right|^{2} \mathrm{~d} \theta, \tag{10.11}
\end{align*}
$$

where $d(\theta)$ is defined in (9.5).
Now consider the energy radiated by the edge waves, which we denote by $\left\langle\mathcal{E}_{\mathrm{e}}\right\rangle$. For convenience, we separate this into two components, writing

$$
\begin{equation*}
\left\langle\mathcal{E}_{\mathrm{e}}\right\rangle=\left\langle\mathcal{E}_{\mathrm{e}}^{1}\right\rangle+\left\langle\mathcal{E}_{\mathrm{e}}^{2}\right\rangle, \tag{10.12}
\end{equation*}
$$

where the first term on the right-hand side comes from the integral in (7.4) and the second is due to the end-point terms. We will apply (7.4) on the contour $\mathcal{C}_{3}$, which consists of three edges, joining the points $\left(R^{1+\delta}, 0\right),\left(R^{1+\delta}, R\right),\left(-R^{1+\delta}, R\right)$ and $\left(-R^{1+\delta}, 0\right)$ for a constant $\delta>0$, as shown in figure 2. The exponential decay in the edge waves means there will be no contributions to $\left\langle\mathcal{E}_{\mathrm{e}}\right\rangle$ from the upper edge of $\mathcal{C}_{3}$ in the limit $R \rightarrow \infty$. The contributions from the left and right edges are the same by symmetry. Now on the right edge of $\mathcal{C}_{3}$, we have $\theta \leq \arctan \left(R^{-\delta}\right)$ so that both the $w_{1}$ and $w_{2}$ components of the edge wave are present for sufficiently large $R$. Hence, from (9.15),

$$
\begin{align*}
\left\langle\mathcal{E}_{\mathrm{e}}^{1}\right\rangle & =\frac{\omega D|b k|^{2}}{8} \alpha_{\mathrm{e}} \mathcal{L}^{2} \lim _{R \rightarrow \infty} \int_{0}^{R}\left(\left[\hat{B}\left(\alpha_{\mathrm{e}}\right)\right]^{2} \mathrm{e}^{-2 \gamma\left(\alpha_{\mathrm{e}}\right)(y+a)}-\left[\hat{C}\left(\alpha_{\mathrm{e}}\right)\right]^{2} \mathrm{e}^{-2 \lambda\left(\alpha_{\mathrm{e}}\right)(y+a)}\right) \mathrm{d} y  \tag{10.13}\\
& =\frac{\omega D|b k|^{2}}{16} \alpha_{\mathrm{e}} \mathcal{L}^{2}\left(\left[\hat{B}\left(\alpha_{\mathrm{e}}\right)\right]^{2} \frac{\mathrm{e}^{-2 a \gamma\left(\alpha_{\mathrm{e}}\right)}}{\gamma\left(\alpha_{\mathrm{e}}\right)}-\left[\hat{C}\left(\alpha_{\mathrm{e}}\right)\right]^{2} \frac{\mathrm{e}^{-2 a \lambda\left(\alpha_{\mathrm{e}}\right)}}{\lambda\left(\alpha_{\mathrm{e}}\right)}\right) \tag{10.14}
\end{align*}
$$

Crucially, we cannot neglect the end-point terms in this case. The edge wave propagates without loss in the $x$ direction, so there are nonzero contributions from the lower ends of the vertical sides of $\mathcal{C}_{3}$. The extra mode $\phi$ must be included in these terms since this also has an edge wave component, given by (9.21). However, it is not necessary to actually use (9.21) because we need only find $\phi$ at the single point $(x, y)=\left(R^{1+\delta}, 0\right)$. Some care is then needed with the directions of the derivatives. The relationship (6.14) applies on the free edge, where the outgoing normal is $-\hat{\mathbf{y}}$ and the unit tangent is $\hat{\mathbf{x}}$; hence

$$
\begin{equation*}
\frac{k_{3}^{2}}{2} \phi(x, 0)=-\left.\frac{\partial^{2} w}{\partial x \partial y}\right|_{y=0} \tag{10.15}
\end{equation*}
$$

On the other hand, on the right edge of $\mathcal{C}_{3}$ the tangent is $\hat{\mathbf{y}}$ and the normal is $\hat{\mathbf{x}}$. The total end-point term in (7.4) (accounting for contributions from both the left and right edges of $\mathcal{C}_{3}$ and the fact that $\left(R^{1+\delta}, 0\right)$ is the first point on the right edge) is therefore given by

$$
\begin{equation*}
\left\langle\mathcal{E}_{\mathrm{e}}^{2}\right\rangle=\omega D(1-\nu) \operatorname{Im} \lim _{R \rightarrow \infty}\left[\frac{\partial}{\partial y}\left(w \frac{\partial \bar{w}}{\partial x}\right)\right]_{(x, y)=\left(R^{1+\delta}, 0\right)} \tag{10.16}
\end{equation*}
$$

Since only the edge wave persists in this limit, we may replace $w$ by $w_{\mathrm{e}}^{+}$, which is given by (9.15). The derivative in $x$ reduces to multiplication by $-\mathrm{i} \alpha_{\mathrm{e}}$, and the factors $\mathrm{e}^{ \pm \mathrm{i} \alpha_{\mathrm{e}} x}$ cancel each other, so that

$$
\begin{equation*}
\left\langle\mathcal{E}_{\mathrm{e}}^{2}\right\rangle=-\omega D(1-\nu) \alpha_{\mathrm{e}}\left[\frac{\partial}{\partial y}\left|w_{\mathrm{e}}^{+}\right|^{2}\right]_{y=0} \tag{10.17}
\end{equation*}
$$

A straightforward calculation using (9.15) now yields

$$
\begin{align*}
\left\langle\mathcal{E}_{\mathrm{e}}^{2}\right\rangle=\frac{1-\nu}{8} \omega D|b|^{2} \alpha_{\mathrm{e}} \mathcal{L}^{2}\left(\hat{B}\left(\alpha_{\mathrm{e}}\right) \gamma\left(\alpha_{\mathrm{e}}\right) \mathrm{e}^{-a \gamma\left(\alpha_{\mathrm{e}}\right)}+\right. & \left.\hat{C}\left(\alpha_{\mathrm{e}}\right) \lambda\left(\alpha_{\mathrm{e}}\right) \mathrm{e}^{-a \lambda\left(\alpha_{\mathrm{e}}\right)}\right) \\
& \times\left(\hat{B}\left(\alpha_{\mathrm{e}}\right) \mathrm{e}^{-a \gamma\left(\alpha_{\mathrm{e}}\right)}+\hat{C}\left(\alpha_{\mathrm{e}}\right) \mathrm{e}^{-a \lambda\left(\alpha_{\mathrm{e}}\right)}\right) . \tag{10.18}
\end{align*}
$$

Some numerical results are shown in table 2. The values in the first column are computed using (10.6). This necessitates the evaluation of the scattered field at the source point $(0, a)$. To achieve this, we move the contour of integration in (8.5) away from the singularities on the real line and then apply the composite trapezium rule, which in this case converges exponentially toward the correct value as the subinterval width decreases [24]. For $w_{1}$, it is possible to use the steepest descents path for quadrature, as in [25]. However, for $w_{2}$ with $x=0$ the steepest descents path is the real line. For simplicity, we use the path on which $\alpha=\mathrm{e}^{-\mathrm{i} \pi / 4} v$, for $v \in \mathbb{R}$. This is the tangent to the steepest descents path for $w_{1}$ at $\alpha=\alpha_{s}$, and also gives adequate

| $k a$ | $\left\langle\mathcal{E}_{\mathrm{t}}\right\rangle /\left(\|b k\|^{2} \omega D\right)$ | $\left\langle\mathcal{E}_{\mathrm{c}}\right\rangle /\left\langle\mathcal{E}_{\mathrm{t}}\right\rangle$ | $\left\langle\mathcal{E}_{\mathrm{e}}^{1}\right\rangle /\left\langle\mathcal{E}_{\mathrm{t}}\right\rangle$ | $\left\langle\mathcal{E}_{\mathrm{e}}^{2}\right\rangle /\left\langle\mathcal{E}_{\mathrm{t}}\right\rangle$ | $\left\langle\mathcal{E}_{\mathrm{c}}+\mathcal{E}_{\mathrm{e}}^{1}\right\rangle /\left\langle\mathcal{E}_{\mathrm{t}}\right\rangle$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0.5 | 0.1168895762 | 0.8964038598 | 0.1014680042 | 0.0021281360 | 0.9978718640 |
| 1.0 | 0.0582786783 | 0.8167996782 | 0.1794369075 | 0.0037634143 | 0.9962365857 |
| 1.5 | 0.0445877192 | 0.7803643590 | 0.2151237498 | 0.0045118912 | 0.9954881088 |
| 2.0 | 0.0569557876 | 0.8389613594 | 0.1577304853 | 0.0033081554 | 0.9966918446 |
| 2.5 | 0.0718820694 | 0.8791968419 | 0.1183215448 | 0.0024816132 | 0.9975183868 |
| 3.0 | 0.0737271824 | 0.8878782666 | 0.1098184594 | 0.0023032740 | 0.9976967260 |
| 3.5 | 0.0626687753 | 0.8740798424 | 0.1233334279 | 0.0025867298 | 0.9974132702 |
| 4.0 | 0.0507303923 | 0.8512972498 | 0.1456480064 | 0.0030547439 | 0.9969452561 |
| 4.5 | 0.0494347095 | 0.8540162035 | 0.1429849071 | 0.0029988895 | 0.9970011105 |
| 5.0 | 0.0596491678 | 0.8842190011 | 0.1134025540 | 0.0023784449 | 0.9976215551 |
| 5.5 | 0.0717410902 | 0.9078580969 | 0.0902490671 | 0.0018928360 | 0.9981071640 |
| 6.0 | 0.0749177794 | 0.9155376342 | 0.0827272876 | 0.0017350782 | 0.9982649218 |

Table 2: Numerical results showing the distribution of energy for $\nu=0.3$. The far field components are divided by $\left\langle\mathcal{E}_{\mathrm{t}}\right\rangle$ so that they sum to unity. The right-most column shows the effect of omitting $\left\langle\mathcal{E}_{\mathrm{e}}^{2}\right\rangle$.
results for $w_{2}$. Taking into account the fact that the integrand is an even function, we apply quadrature to the integral

$$
\begin{equation*}
w^{\mathrm{s}}(0, a)=\frac{b \mathrm{e}^{-\mathrm{i} \pi / 4}}{4 \pi} \int_{0}^{\infty}\left[B\left(v \mathrm{e}^{-\mathrm{i} \pi / 4}\right) \mathrm{e}^{-2 a \gamma\left(v \mathrm{e}^{-\mathrm{i} \pi / 4}\right)}+C\left(v \mathrm{e}^{-\mathrm{i} \pi / 4}\right) \mathrm{e}^{-2 a \lambda\left(v \mathrm{e}^{-\mathrm{i} \pi / 4}\right)}\right] \mathrm{d} v . \tag{10.19}
\end{equation*}
$$

The second column in table 2 shows the energy in the circular wave, as a proportion of the total. This is calculated by applying quadrature to (10.11). The third and fourth columns give the proportion of energy carried by the edge wave, calculated using (10.14) and (10.18). In each case, the values in the second, third and fourth columns sum to unity, showing that energy is conserved. Had we used (4.7) (i.e. Norris \& Vemula's formula, adjusted to allow for the fact that the contour $\mathcal{C}_{3}$ is not smooth), we would obtain the incorrect result $\left\langle\mathcal{E}_{\mathrm{e}}^{2}\right\rangle=0$. To see this, note that on the vertical edges of $\mathcal{C}_{3}$ we have $\hat{\mathbf{n}}= \pm \hat{\mathbf{x}}$, and $\hat{\mathbf{s}}= \pm \hat{\mathbf{y}}$, but the product $w_{\mathrm{e}}^{+} \partial \bar{w}_{\mathrm{e}}^{+} / \partial y$ does not depend on $x$. Consequently, the total for the far field would be $\left\langle\mathcal{E}_{\mathrm{c}}+\mathcal{E}_{\mathrm{e}}^{1}\right\rangle$. The final column of table 2 shows that conservation of energy is then violated because a small proportion of the incident wave energy $(<0.5 \%)$ appears to be lost. On the other hand, had we used (4.10) (i.e. including Bobrovnitskii's correction terms but omitting $\phi$ ), then the total for the far field would be correct, but the flux across the edge would not be zero. To determine the magnitude of this spurious effect, we return to (10.16) and observe that applying the product rule produces two terms, one from each of the end-point terms in (7.4). After simplification, these terms turn out to be identical in (10.17). Thus, if $\phi$ is omitted, the end-point contribution $\left\langle\mathcal{E}_{2}\right\rangle$ is split into two equal parts: half appears to be transmitted across the vertical edges of $\mathcal{C}_{3}$ by the edge wave, but the other half appears to 'leak' across the free edge.

## 11 Concluding remarks

By carefully applying the low frequency (or low thickness) limit to Mindlin theory, we have retrieved Kirchhoff plate theory, but with an extra term, which we denote by $\phi$. This term contributes to the in-plane vector of rotation in Mindlin theory, but is asymptotically smaller than the transverse displacement $w$ for small $\omega$ (or small $h$ ). The main features of Kirchhoff plate theory are not affected by $\phi$ in any way - the governing equations and boundary conditions remain the same, and $\phi$ can be calculated from $w$ after the latter has been determined. However, $\phi$ has a crucial effect on the energy flux in narrow regions adjacent to free edges; only by including it can we achieve both conservation of energy and zero flux across free edges. Where
$\phi$ appears in a line integral representing flux across a contour, its contributions are always in a form that facilitates evaluation by the gradient theorem. Consequently, terms involving $\phi$ are determined explicitly by evaluation at one or more discrete points, which in turn means that limits (such as the expansion of an integration contour toward infinity) can be applied very easily. If the flux across a piecewise smooth closed contour is calculated, the overall contribution from $\phi$ is always zero, but its contribution on individual sections is not.

As an example application for the new flux formula, we have considered a simple boundary value problem in which a point source is located in a semi-infinite thin plate, with a free edge on $y=0$. The scattered field includes edge waves, which propagate without loss along the edge, but decay exponentially in the orthogonal direction. Calculating the energy radiated by the source and comparing this with the energy radiated into the far field shows that Norris \& Vemula's flux formula [7] does not conserve energy. On the other hand, if we include Bobrovnitskii's correction terms [8] then the far field energy is equal to the energy radiated by the source, but the flux across the free edge is not zero. This physically implausible effect is cancelled when $\phi$ is also included, and we find that the energy is in fact carried by the edge wave.

## References

[1] V. Twersky. On the scattering of waves by an infinite grating. IRE Trans. on Antennas and Propagation, 4:330-345, 1956.
[2] J. D. Achenbach, Y.-C. Lu, and M. Kitahara. 3-D reflection and transmission of sound by an array of rods. J. Sound Vib., 125(3):463-476, 1988.
[3] I. Thompson and C. M. Linton. On the excitation of a closely spaced array by a line source. IMA J. Appl. Math., 72(4):476-497, 2007.
[4] N. Tymis and I. Thompson. Low-frequency scattering by a semi-infinite lattice of cylinders. Q. J. Mech. Appl. Math., 64(2):171-195, 2011.
[5] N. Tymis and I. Thompson. Scattering by a semi-infinite lattice of cylinders and the excitation of Bloch waves. Q. J. Mech. Appl. Math., 67(3):469-503, 2014.
[6] R. I. Brougham and Thompson I. A direct method for Bloch wave excitation by scattering at the edge of a lattice. Part II: Finite size effects. Q. J. Mech. Appl. Math., 72(3):387-414, 2019.
[7] A. N. Norris and C. Vemula. Scattering of flexural waves on thin plates. J. Sound Vib., 181(1):115125, 1995.
[8] Y. I. Bobrovnitskii. Calculation of the power flow in flexural waves on thin plates. J. Sound Vib., 194(1), 1996.
[9] A. N. Norris and C. Vemula. Calculation of the power flow in flexural waves on thin plates - reply. J. Sound Vib., 194(1):106, 1996.
[10] R. D. Mindlin. Influence of rotatory inertia and shear on flexural motion of isotropic, elastic plates. J. Appl. Mech., 18:31-38, 1951. Does not have a DOI (last checked November 2020).
[11] J. E. Marsden and A. J. Tromba. Vector Calculus. W. H. Freeman, San Francisco, 1981.
[12] S. P. Timoshenko and S. Woinowsky-Krieger. Theory of Plates and Shells. McGraw-Hill, 2nd edition, 1959.
[13] K. F. Graff. Wave Motion in Elastic Solids. Dover, New York, 1991.
[14] A. N. Norris, V. V. Krylov, and I. D. Abrahams. Flexural edge waves and comments on "A new bending wave solution for the classical plate equation" [J. Acoust. Soc. Am. 104, 2220-2222 (1998)]. J. Acoust. Soc. Am., 107(3):1781-1784, 2000.
[15] C. Vemula and A. N. Norris. Flexural wave propagation and scattering on thin plates using Mindlin theory. Wave Motion, 26:1-12, 1997.
[16] H. Zhong and C. Gu. Buckling of simply supported rectangular Reissner-Mindlin plates subjected to linearly varying in-plane loading. J. Engrg. Mech., 132(5):578-581, 2006.
[17] D. V. Evans and R. Porter. Flexural waves on a pinned semi-infinite thin elastic plate. Wave Motion, 45:745-757, 2008.
[18] R. Gunda, S. M. Vijayakar, R. Singh, and Farstad J. E. Harmonic green's functions of a semi-infinite plate with clamped or free edges. J. Acoust. Soc. Am., 103(2):888-899, 1998.
[19] Y. K. Konenkov. A Rayleigh-type flexural wave. Sov. Phys. Acoust., 6:122-123, 1960.
[20] R. N. Thurston and J. McKenna. Flexural acoustic waves along the edge of a plate. IEEE Trans. Son. Ultrason., 21(4):296-297, 1974.
[21] A. D. Osborne. Complex Variables and their Applications. Addison-Wesley, Harlow, UK, 1999.
[22] J. D. Murray. Asymptotic Analysis. Springer-Verlag, New York, 1984.
[23] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark. NIST Handbook of Mathematical Functions. Cambridge University Press, Cambridge, UK, 2010.
[24] L. N. Trefethen and J. A. C. Weideman. The exponentially convergent trapezoidal rule. SIAM Review, 56(3):385-458, 2014.
[25] C. M. Linton and I. Thompson. Oblique Rayleigh wave scattering by a cylindrical cavity. Q. J. Mech. Appl. Math., 68(3):235-261, 2015.

