Duality in Optimal Impulse Control

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Abstract

We consider the optimal impulse control of a dynamical system defined by a fixed uncontrolled flow. This problem is associated with two pairs of linear programs for which we prove the solvability and the absence of the duality gaps. Finally, we show how to retrieve the optimal control strategy from the solutions of those linear programs. The theoretical issues are illustrated by the meaningful example on the controlled epidemic model.

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1 Introduction

Optimal impulse control of dynamical systems (sometimes called singular control [23] or control with concentrations [5, 10]) attracts attention of many researchers. Note also that optimal stopping [9] is also an example of impulse control. The underlying system can be described in terms of ordinary differential equations, see [4, 5, 10, 13, 14, 16, 17, 23], or by a fixed flow in an Euclidean space or in an abstract Borel space, see [9, 18, 19, 20]. An impulse or an intervention means an instantaneous change of the state of the system. The target is to optimize an objective functional, typically having the shape of the sum of the impulse costs and the integral with respect to time of the running cost rate. The popular methods of attack to such problems include dynamic programming, see [9, 17, 18], and Pontryagin maximum principle, see [14, 16]. When the total number of impulses is fixed over a finite horizon, the impulse control problem can be treated as a parameter optimization problem, see [13]. The linear programming approach appeared in [4, 5, 10, 19, 20].

Similarly to [9, 13, 14, 17, 18, 19, 20], we consider the uncontrolled flow, that is, the purely impulsive control. Such a problem can be reformulated as a specific discrete-time control problem, where one step corresponds to the choice of the interval until the next impulse, along with the impulse itself. After that, one can use the dynamic programming method, see [18, 19], leading to the Bellman equation similar to the optimality equation for a Markov decision process [2, Prop.9.8], [11, §4.2], and [12, §9.5]. We call such equation 'integral' (see (7)). Due to its special form, this equation, under mild conditions, is equivalent to the 'differential' Bellman equation [9, 18] (see (21)).

In the framework of Markov decision processes, it is known that the (integral) Bellman equation gives rise to the so called dual linear program in the space of functions [11, (6.3.27)], [12, (12.3.22)]. The corresponding primal linear program is in the space of what is called occupation measures [8, 11, 12]. The similar dual pair of linear programs can be introduced for the differential Bellman equation. The target of the current article is to define and study the two above mentioned pairs of linear programs. Note that the dynamic programming approach (and the associated dual linear programs) is widely used in the standard optimal control problems [9, 11, 12, 17, 18], while the primal linear programs are effective in the case of constrained control problems [8, 19, 20].

The present article, although self-contained, is the last part of the project started in [9, 18, 19, 20]. In the articles [9, 18], the dynamic programming approach to the optimal impulse control was developed without investigating any linear programs. The so called primal linear programs in the spaces of occupation measures and of aggregated occupation measures (see (10) and (27)) were formulated and investigated in [19] and in [20] correspondingly for the constrained version of the optimal impulse control. In the special cases, the primal linear program (27) can be transformed to the linear programs introduced in [4, 5, 10]: this relationship was discussed in depth in [20]. The dual linear programs were not introduced in [4, 5, 9, 10, 18, 19, 20].

The novelty of the present article is in the following.

(a) Based on the abstract theory of linear programs [21], we formulate the primal linear programs as in [19, 20] and the corresponding dual linear programs (for the standard, unconstrained optimal impulse control) in terms of the Lagrangian L. Duality in such a general aspect was not studied in the above mentioned works.

(b) We prove the solvability of the dual programs and show that the Bellman function as in [9, 18] provides those solutions.

(c) We show that there is no duality gap in the both pairs of linear programs.

(d) We explain how to build the optimal control strategy based on the solutions to the investigated linear programs. As for the second primal linear program, here the reasoning is different and simpler than that presented in [20, §7]. Note also that in [20] the more challenging constrained version of the optimal impulse control was considered.

The rest of this article is organized as follows. The problem statement is described in Section 2. The main conditions and preliminary observations are presented in Section 3. Section 4 provides the background on the abstract dual pairs. The first and second pairs of linear programs are investigated in Sections 5 and 6 correspondingly. In Section 7, we explain how to retrieve the optimal strategy from the solutions to the investigated linear programs. Brief comments about possible numerical methods are given in Section 8. Example in Section 9 illustrates the presented results. In Section 10, we formulate the conclusion and mention several related open problems. The proofs of lemmas are postponed to the Appendix.

Throughout this paper, we use the following notations. $:=$ means the equality by definition. $\bar{\mathbb{R}}^0_+ := [0,\infty]$, $\mathbb{R}^0_+ := [0,\infty)$, $\mathbb{R}_+ := (0,\infty)$. The term 'measure' will always refer to a countably additive $\bar{\mathbb{R}}^0_+$-valued set function, equal to zero on the empty set. $\mathcal{P}(E)$ is the space of all probability measures on a measurable space $(E,\mathcal{B}(E))$. On the time axis $\mathbb{R}^0_+$ the expression 'for almost all u' is understood in the sense of the Lebesgue measure. By default, the σ-algebra on $\mathbb{R}^0_+$ is just the Borel one. If $(E,\mathcal{B}(E))$ is a measurable space then, for $Y \in \mathcal{B}(E)$, $\mathcal{B}(Y) := \{X \cap Y,\ X \in \mathcal{B}(E)\}$ is the restriction (trace) of the σ-algebra $\mathcal{B}(E)$. If it is obvious which σ-algebra is fixed on the space Y, we say that μ is a measure on Y (rather than on $\mathcal{B}(Y)$), for brevity. If $b=\infty$ then the Lebesgue integrals $\int_{[a,b]} f(u)du$ are taken over the open interval (a,∞). Expressions like 'positive, negative, increasing, decreasing' are understood in the non-strict sense, like 'nonnegative' etc. $\mathbb{I}\{\cdot\}$ is the indicator function; $\delta_y(dx)$ is the Dirac measure at the point y. For $b,c \in [-\infty,+\infty]$, $b^+ := \max\{b,0\}$, $b^- := -\min\{b,0\}$, $b \wedge c := \min\{b,c\}$, $b \vee c := \max\{b,c\}$. $\inf \emptyset := +\infty$. The integrals like $\int_E f(e)\mu(de)$ are calculated separately for the positive and negative parts f^+ and f^-, with the convention $+\infty - \infty := +\infty$; $0 \times \infty := 0$.

2 Impulse Control Problem

We will deal with a control model defined through the following elements.

- $\mathbf{X}$ is the state space, which is a topological Borel space.

- $\phi(\cdot,\cdot): \mathbf{X}\times\mathbb{R}^0_+ \to \mathbf{X}$ is the measurable flow possessing the semigroup property $\phi(x,t+s) = \phi(\phi(x,s),t)$ for all $x\in\mathbf{X}$ and $(t,s)\in(\mathbb{R}^0_+)^2$; $\phi(x,0)=x$ for all $x\in\mathbf{X}$. Between the impulses, the state changes according to the flow.

- $\mathbf{A}$ is the action space, again a topological Borel space.

- $l(\cdot,\cdot): \mathbf{X}\times\mathbf{A}\to\mathbf{X}$ is the mapping describing the new state after the corresponding action/impulse is applied.

- $C^g(\cdot): \mathbf{X}\to\mathbb{R}$ is the (gradual) cost rate.

- $C^I(\cdot,\cdot): \mathbf{X}\times\mathbf{A}\to\mathbb{R}$ is the cost function associated with the actions/impulses applied in the corresponding states.

All the mappings ϕ, l, C^g and C^I are assumed to be measurable. The initial state $x_0\in\mathbf{X}$ is fixed.

We are going to formulate the optimal impulse control problem as a Markov decision process.

Let $\mathbf{X}_\Delta := \mathbf{X}\cup\{\Delta\}$, where Δ is an isolated artificial point describing the case that the controlled process is over and no future costs appear: $C^g(\Delta)=C^I(\Delta,a):=0$. The dynamics (trajectory) of the system can be represented as one of the following sequences

$$\begin{array}{l} x_0 \to (\theta_1,a_1)\to x_1\to(\theta_2,a_2)\to\ldots; \qquad \theta_i<+\infty \text{ for all } i\in\{1,2,\ldots\}, \\ \text{or} \\ x_0\to(\theta_1,a_1)\to\ldots\to x_n\to(+\infty,a_{n+1})\to\Delta\to(\theta_{n+2},a_{n+2})\to\Delta\to\ldots, \end{array} \tag{1}$$

where $x_0\in\mathbf{X}$ is the initial state of the controlled process and $\theta_i<+\infty$ for all $i=1,2,\ldots,n$. For the state $x_{i-1}\in\mathbf{X}$, $i\in\{1,2,\ldots\}$, the pair $(\theta_i,a_i)\in\bar{\mathbb{R}}^0_+\times\mathbf{A}$ is the control at the step i: after θ_i time units, the impulsive action $a_i\in\mathbf{A}$ will be applied leading to the new state

$$x_i = \begin{cases} l(\phi(x_{i-1},\theta_i),a_i), & \text{if } \theta_i<+\infty; \\ \Delta, & \text{if } \theta_i=+\infty. \end{cases}$$

After $\theta_{n+1}=+\infty$ appears for the first time, the values of $a_{n+1},\theta_{n+2},a_{n+2},\ldots$ play no role. The state Δ will appear forever, after it appeared for the first time, i.e., it is absorbing, and $\phi(\Delta,t)\equiv\Delta$. Thus, the transition probability is defined as

$$Q(dy|x,(\theta,a)) := \begin{cases} \delta_{l(\phi(x,\theta),a)}(dy), & \text{if } x\neq\Delta,\ \theta\neq+\infty; \\ \delta_\Delta(dy) & \text{otherwise,} \end{cases} \qquad x,y\in\mathbf{X}_\Delta;\ (\theta,a)\in\bar{\mathbb{R}}^0_+\times\mathbf{A}. \tag{2}$$

After each impulsive action a_{i-1}, if $\theta_1,\theta_2,\ldots,\theta_{i-1}<+\infty$, the decision maker has in hand the complete information about the history, that is, the sequence

$$x_0,(\theta_1,a_1),x_1,\ldots,(\theta_{i-1},a_{i-1}),x_{i-1}.$$

The selection of the next control (θ_i,a_i) is based on this information, and we also allow the selection of the pair (θ_i,a_i) to be randomized.

For $x_{i-1}\neq\Delta$, the cost accumulated on the coming interval of the length $\theta_i\in[0,\infty]$ equals

$$C(x_{i-1},(\theta_i,a_i)) := \int_{[0,\theta_i)} C^g(\phi(x_{i-1},u))du + \mathbb{I}\{\theta_i<+\infty\}C^I(\phi(x_{i-1},\theta_i),a_i). \tag{3}$$

The next state x_i has the distribution $Q(dy|x_{i-1},(\theta_i,a_i))$ given by (2).

In the space of all the trajectories

$$\begin{array}{rcl}\Omega & := & \cup_{n=1}^{\infty}[\mathbf{X}\times((\mathbb{R}^0_+\times\mathbf{A})\times\mathbf{X})^n\times(\{+\infty\}\times\mathbf{A})\times\{\Delta\}\times((\bar{\mathbb{R}}^0_+\times\mathbf{A})\times\{\Delta\})^{\infty}]\\ & & \cup[\mathbf{X}\times((\mathbb{R}^0_+\times\mathbf{A})\times\mathbf{X})^{\infty}],\end{array}$$

we fix the natural σ-algebra $\mathcal{F}$. Finite sequences $h_i=(x_0,(\theta_1,a_1),x_1,(\theta_2,a_2),\dots,x_i)$ will be called (finite) histories; $i=0,1,2,\dots$, and the space of all such histories will be denoted as $\mathbf{H}_i$; $\mathcal{F}_i:=\mathcal{B}(\mathbf{H}_i)$ is the restriction of $\mathcal{F}$ to $\mathbf{H}_i$. Capital letters X_i,T_i,Θ_i,A_i and H_i denote the corresponding functions of $\omega\in\Omega$, i.e., random elements.

Definition 2.1 *A control strategy $\pi=\{\pi_i\}_{i=1}^{\infty}$ is a sequence of stochastic kernels π_i on $\bar{\mathbb{R}}^0_+\times\mathbf{A}$ given $\mathbf{H}_{i-1}$. The set of all strategies is denoted as Π. A Markov strategy is defined by stochastic kernels $\{\pi_i(d\theta\times da|x_{i-1})\}_{i=1}^{\infty}$. A strategy is called stationary if it is Markov and i-independent, defined by the single stochastic kernel $\pi(d\theta\times da|x)$. Every measurable mapping $f:\ \mathbf{X}_\Delta\to\bar{\mathbb{R}}^0_+\times\mathbf{A}$ defines a deterministic stationary strategy, which is given by $\pi_i(d\theta\times da|h_{i-1}):=\delta_{f(x_{i-1})}(d\theta\times da)$, and identified with f. The values of a strategy on the histories of the form $(x_0,(\theta_1,a_1),\dots,\Delta)$ are of no importance.*

For a given initial state $x_0\in\mathbf{X}$ and a strategy π, there is a unique probability measure $P^\pi_{x_0}(\cdot)$ on Ω constructed using the Ionescu-Tulcea Theorem [2, Prop.7.28] and satisfying the following relations for all $i\in\{1,2,\dots\}$, $\Gamma\in\mathcal{B}(\bar{\mathbb{R}}^0_+\times\mathbf{A})$, $\Gamma_X\in\mathcal{B}(\mathbf{X}_\Delta)$:

$$\begin{array}{rcl}P^\pi_{x_0}(X_0\in\Gamma_X) & = & \delta_{x_0}(\Gamma_X);\qquad P^\pi_{x_0}((\Theta_i,A_i)\in\Gamma|H_{i-1})=\pi_i(\Gamma|H_{i-1});\\ P^\pi_{x_0}(X_i\in\Gamma_X|H_{i-1},(\Theta_i,A_i)) & = & \begin{cases}\delta_{l(\phi(X_{i-1},\Theta_i),A_i)}(\Gamma_X), & \text{if } X_{i-1}\in\mathbf{X},\ \Theta_i<+\infty;\\ \delta_\Delta(\Gamma_X) & \text{otherwise.}\end{cases}\end{array}$$

This is a standard definition of a strategic measure in Markov decision processes; $E^\pi_{x_0}$ is the corresponding mathematical expectation.

The optimal control problem under study is the following one:

$$\text{Minimize with respect to } \pi \qquad \mathcal{V}(x_0,\pi) \tag{4}$$

$$:=\ E^\pi_{x_0}\left[\sum_{i=1}^{\infty}\mathbb{I}\{X_{i-1}\neq\Delta\}\left\{\int_{[0,\Theta_i]}C^g(\phi(X_{i-1},u))du\ +\mathbb{I}\{\Theta_i<+\infty\}\ C^I(\phi(X_{i-1},\Theta_i),A_i)\right\}\right].$$

Under Condition 3.1 below, this expression is well defined. In what follows, $\mathcal{V}^*(x_0):=\inf_{\pi\in\Pi}\mathcal{V}(x_0,\pi)$.

3 Conditions and Preliminaries

We have formulated the problem as the Markov decision process

$$(\mathbf{X}_\Delta:=\mathbf{X}\cup\{\Delta\},\ \bar{\mathbb{R}}^0_+\times\mathbf{A},\ Q,\ C). \tag{5}$$

Below, we briefly present the dynamic programming approach.

Condition 3.1 *The functions $C^g(\cdot)$ and $C^I(\cdot)$ are $\mathbb{R}^0_+$-valued.*

One can introduce the following set

$$V^c:=\{x\in\mathbf{X}:\ \int_{[0,\infty)}C^g(\phi(x,u))\ du=0\}\cup\{\Delta\},$$

which is obviously measurable. Clearly, under Condition 3.1, as soon as $X_{i-1}\in V^c$, it is reasonable not to apply any impulses in the future, i.e., to apply the control $(\infty,\hat{a})$ with the immaterial value of $\hat{a}\in\mathbf{A}$ being arbitrarily fixed. All other actions cannot improve the objective. As the result, all the further reasoning, conditions and equations can be for the state space having been reduced to $V:=\mathbf{X}_\Delta\setminus V^c$.

Condition 3.2 *(a) The space* $\mathbf{A}$ *is compact.*

(b) The mapping $(x,a) \in \boldsymbol{X} \times \boldsymbol{A} \to l(x,a)$ *is continuous.*

(c) The mapping $(x,\theta) \in \boldsymbol{X} \times \mathbb{R}^0_+ \to \phi(x,\theta)$ *is continuous.*

(d) The function $(x,a) \in \boldsymbol{X} \times \boldsymbol{A} \to C^I(x,a)$ *is lower semicontinuous.*

(e) The function $x \in \boldsymbol{X} \to C^g(x)$ *is lower semicontinuous.*

Remark 3.1 *Since the function* $C^g(\cdot)$ *appears only in the integrals of the type* $\int_{[0,t]} C^g(\phi(x,u))du$*, all the statements in this article remain valid if* $C^g(\cdot)$ *is such that, for all* $x \in \mathbf{X}$*, for almost all* $u \in \mathbb{R}^0_+$*,* $C^g(\phi(x,u)) = \hat{C}^g(\phi(x,u))$*, where the function* $\hat{C}^g(\cdot)$ *satisfies the formulated conditions.*

Condition 3.3 *(a)* $\sup_{x\in\mathbf{X}} \int_{[0,\infty)} |C^g(\phi(x,u))|du < \infty$.

(b) $C^I(\cdot) \geq \delta > 0$.

For example, Condition 3.3(a) is satisfied in the discounted model, when $\mathbf{X} = \tilde{\mathbf{X}} \times \mathbb{R}^0_+$,

$$\phi((\tilde{x},u),t) = (\tilde{\phi}(\tilde{x},t), u+t); \quad l((\tilde{x},u),a) = (\tilde{l}(\tilde{x},a),u); \quad C^g((\tilde{x},u)) = e^{-\alpha u}\tilde{C}^g(\tilde{x}), \tag{6}$$

and $\sup_{x\in\mathbf{X}} |\tilde{C}^g(\tilde{x})| < \infty$. Here $\tilde{\phi}, \tilde{l}$ and $\tilde{C}^g$ are similar to ϕ, l and C^g, but defined on the Borel space $\tilde{\mathbf{X}}$; $\alpha > 0$ is the discount factor.

If Conditions 3.1 and 3.3(a) are satisfied, then the positive function $\mathcal{V}^*(\cdot)$ is bounded because $\mathcal{V}(x_0,\varphi) \leq \sup_{x\in\mathbf{X}} \int_{(0,\infty)} C^g(\phi(x,u))du < \infty$ for the deterministic stationary strategy $\varphi(x) \equiv (\infty, \hat{a})$ with an arbitrarily fixed $\hat{a} \in \mathbf{A}$.

Condition 3.3(b) means, the impulses are expensive.

If Condition 3.3(a) is satisfied, then, for each cycle such that $\phi(x,\tau) = x$ for some $x \in \mathbf{X}$ and $\tau \in \mathbb{R}_+$, necessarily $C^g(\phi(x,u)) = 0$ for almost all $u \in \mathbb{R}^0_+$.

Proposition 3.1 *If Conditions 3.1, 3.2 and 3.3 are satisfied, then the Bellman function* $\mathcal{V}^*(\cdot)$ *is the unique bounded measurable solution to the following (integral) Bellman equation*

$$\begin{aligned} W(\Delta) &= 0, \\ W(x) &= \inf_{(\theta,a)\in\bar{\mathbb{R}}^0_+\times\boldsymbol{A}} \left\{ C(x,(\theta,a)) + \int_{\mathbf{X}_\Delta} W(y)Q(dy|x,(\theta,a)) \right\} \quad \forall\, x \in \boldsymbol{X}, \end{aligned} \tag{7}$$

and there is a measurable mapping $f^* : \mathbf{X} \to \bar{\mathbb{R}}^0_+ \times \mathbf{A}$ *providing here the infimum, which defines the optimal strategy in problem (4):* $\mathcal{V}^*(x_0) = \mathcal{V}(x_0, f^*)$ *for all* $x_0 \in \mathbf{X}$ *. This solution to equation (7) is necessarily nonnegative and lower semicontinuous.*

The proof follows from [18, Thm.1] and from the proof of Proposition 1 of [18].

Along with the above dynamic programming approach, another method based on the linear programming, is useful. For each $\pi \in \Pi$, let

$$\mu^\pi(dx \times d\theta \times da) := E^\pi_{x_0}\left[\sum_{i=0}^{\infty} \mathbb{I}\{X_i \in dx, \Theta_{i+1} \in d\theta, A_{i+1} \in da\}\right]$$

be the occupation measure on $\mathbf{X} \times \bar{\mathbb{R}}^0_+ \times \mathbf{A}$: see [12, §9.4]. Then, under Condition 3.1,

$$\mathcal{V}(x_0,\pi) = \int_{\mathbf{X}\times\bar{\mathbb{R}}^0_+\times\mathbf{A}} C(x,(\theta,a))\mu^\pi(dx \times d\theta \times da). \tag{8}$$

If Conditions 3.1 and 3.3 are satisfied, then we can ignore the strategies with $\mu^\pi(\mathbf{X} \times \bar{\mathbb{R}}^0_+ \times \mathbf{A}) = \infty$ because, for such strategies, with positive $P^\pi_{x_0}$ probability, there are infinite number of real-valued $\Theta_i < \infty$ which leads to $\mathcal{V}(x_0, \pi) = \infty$, while $\inf_{\pi \in \Pi} \mathcal{V}(x_0, \pi) = \mathcal{V}^*(x_0) < \infty$. As a result of this observation, the key properties of the occupation measures of our interest can be written down as follows:

$$\begin{cases} \mu(dx \times \bar{\mathbb{R}}^0_+ \times \mathbf{A}) = \delta_{x_0}(dx) + \displaystyle\int_{\mathbf{X} \times \mathbb{R}^0_+ \times \mathbf{A}} \mathbb{I}\{l(\phi(y,\theta), a) \in dx\} \mu(dy \times d\theta \times da); \\ \mu(\mathbf{X} \times \bar{\mathbb{R}}^0_+ \times \mathbf{A}) < \infty. \end{cases} \tag{9}$$

See [12, Lemma 9.4.3].

Assuming that Conditions 3.1, 3.2 and 3.3 are satisfied, the first linear program of our interest, traditionally called 'primal', has the form

$$\int_{\mathbf{X} \times \bar{\mathbb{R}}^0_+ \times \mathbf{A}} C(x, (\theta, a)) \mu(dx \times d\theta \times da) \to \inf_\mu \tag{10}$$

subject to (9).

The similar linear program, for the case of the constrained impulse control, was investigated in [19].

Any finite measure μ on $\mathbf{X} \times \bar{\mathbb{R}}^0_+ \times \mathbf{A}$ can be written in the form

$$\mu(dx \times d\theta \times da) = p_T(d\theta | x, a) p_A(da | x) \mu(dx \times \bar{\mathbb{R}}^0_+ \times \mathbf{A}), \tag{11}$$

where $p_T(\cdot)$ and $p_A(\cdot)$ are stochastic kernels on $\bar{\mathbb{R}}^0_+$ and $\mathbf{A}$ correspondingly: see [2, Prop.7.27]. The dependence of p_T and p_A on μ is not explicitly indicated here. Hence, using the Tonelli Theorem (see [1, Thm.11.28]), straightforward calculations imply that

$$\begin{aligned} & \int_{\mathbf{X} \times \bar{\mathbb{R}}^0_+ \times \mathbf{A}} \left\{ \int_{[0,\theta]} C^g(\phi(x,u)) du \right\} \mu(dx \times d\theta \times da) \\ = \; & \int_{\mathbf{X}} \int_{\mathbf{A}} \int_{\bar{\mathbb{R}}^0_+} \int_{[0,\theta]} C^g(\phi(x,u)) du \; p_T(d\theta | x, a) p_A(da | x) \mu(dx \times \bar{\mathbb{R}}^0_+ \times \mathbf{A}) \\ = \; & \int_{\mathbf{X}} \int_{\mathbf{A}} \int_{\mathbb{R}^0_+} C^g(\phi(x,u)) p_T([u, \infty] | x, a) \; du \; p_A(da | x) \mu(dx \times \bar{\mathbb{R}}^0_+ \times \mathbf{A}). \end{aligned}$$

After we introduce the following measure on $\mathbf{X}$

$$\begin{aligned} \eta(dy \times \square) \; &:= \; \int_{\mathbf{X}} \int_{\mathbb{R}^0_+} \delta_{\phi(x,u)}(dy) \left(\int_{\mathbf{A}} p_T([u,\infty] | x, a) p_A(da | x) \right) \; du \; \mu(dx \times \bar{\mathbb{R}}^0_+ \times \mathbf{A}) \\ &= \; \int_{\mathbb{R}^0_+} \int_{\mathbf{X}} \delta_{\phi(x,u)}(dy) \mu(dx \times [u, \infty] \times \mathbf{A}) du, \end{aligned} \tag{12}$$

we may write

$$\int_{\mathbf{X} \times \bar{\mathbb{R}}^0_+ \times \mathbf{A}} \left\{ \int_{[0,\theta]} C^g(\phi(x,u)) du \right\} \mu(dx \times d\theta \times da) = \int_{\mathbf{X}} C^g(y) \eta(dy \times \square).$$

Here $\square$ is the artificial isolated point to be added to the space $\mathbf{A}$ for notational convenience. Roughly speaking, if the measure $\eta(dy \times \square)$ comes from the occupation measure μ^π, then $\eta(\Gamma \times \square)$ equals the total time the process spends in the set $\Gamma \in \mathcal{B}(\mathbf{X})$ under the control strategy π.

Similarly to the above, we have that

$$\int_{\mathbf{X}\times\bar{\mathbb{R}}_+^0\times\mathbf{A}} \mathbb{I}\{\theta<\infty\}C^I(\phi(x,\theta),a)\mu(dx\times d\theta\times da) \quad = \quad \int_{\mathbf{X}\times\mathbf{A}} C^I(y,a)\eta(dy\times da),$$

where

$$\eta(dy\times da) \quad := \quad \int_{\mathbf{X}}\int_{\mathbb{R}_+^0} \delta_{\phi(x,\theta)}(dy)\mu(dx\times d\theta\times da) \tag{13}$$

is a finite measure on $\mathbf{X}\times\mathbf{A}$, since the measure μ is finite.

Now

$$\int_{\mathbf{X}\times\bar{\mathbb{R}}_+^0\times\mathbf{A}} C(x,(\theta,a))\mu(dx\times d\theta\times da) = \int_{\mathbf{X}\times\mathbf{A}_\Box} \mathcal{C}(x,a)\eta(dx\times da). \tag{14}$$

Here and below

$$\mathbf{A}_\Box := \mathbf{A}\cup\{\Box\}; \qquad \mathcal{C}(x,a) := \begin{cases} C^g(x), & \text{if } a=\Box;\\ C^I(x,a), & \text{if } a\in\mathbf{A}, \end{cases} \tag{15}$$

and the measure η on $\mathbf{X}\times\mathbf{A}_\Box$ is as in the following definition.

Definition 3.1 *For a measure μ on $\mathbf{X}\times\bar{\mathbb{R}}_+^0\times\mathbf{A}$ satisfying (9), the measure η on $\mathbf{X}\times\mathbf{A}_\Box$ defined by*

$$\eta(\Gamma_X\times\Gamma_A) := \eta(\Gamma_X\times(\Gamma_A\cap\mathbf{A})) + \eta(\Gamma_X\times\Box)\mathbb{I}\{\Box\in\Gamma_A\}, \quad \Gamma_X\in\mathcal{B}(\mathbf{X}),\ \Gamma_A\in(\mathbf{A}_\Box), \tag{16}$$

where the measures $\eta(dy\times\Box)$ on $\mathbf{X}$ and $\eta(dy\times da)$ on $\mathbf{X}\times\mathbf{A}$ were introduced in (12) and (13), is called the aggregated occupation measure (induced by μ).

The primal linear program (10) now can be rewritten as

$$\int_{\mathbf{X}\times\mathbf{A}_\Box} \mathcal{C}(x,a)\eta(dx\times da) \to \inf_\mu \qquad \text{subject to (9), (12), (13).}$$

In what follows, we will characterize the aggregated measures η without references to the measures μ: see (26).

Remark 3.2 *As was explained, for a strategy π with the finite objective (4), we have $\mathcal{V}(x_0,\pi) = \int_{\mathbf{X}\times\mathbf{A}_\Box}\mathcal{C}(x,a)\eta^\pi(dx\times da)$, where η^π is the aggregated occupation measure induced by μ^π: see (8) and (14).*

4 Abstract Linear Programs

There are different ways for defining abstract primal/dual linear programs. In the current article, we follow [21], see problems (1.5) and (1.6) therein. Namely, suppose $\mathcal{M}$ and $\mathcal{W}$ are two linear spaces, $\mathcal{M}^+$ is a convex cone in $\mathcal{M}$, and $L(\mu,W)$ is a well defined bilinear form on $\mathcal{M}^+\times\mathcal{W}$, taking values in $(-\infty,+\infty]$ (or in $[-\infty,+\infty)$). This means that

$$\forall W\in\mathcal{W}\ \forall\mu_1,\mu_2\in\mathcal{M}^+\ \forall a,b\in\mathbb{R}_+^0\ \ L(a\mu_1+b\mu_2,W) = aL(\mu_1,W)+bL(\mu_2,W)$$

and

$$\forall\mu\in\mathcal{M}^+\ \forall W_1,W_2\in\mathcal{W}\ \forall a,b\in\mathbb{R}\ \ L(\mu,aW_1+bW_2) = aL(\mu,W_1)+bL(\mu,W_2).$$

Now the primal and dual linear programs look as follows:

$$\sup_{W\in\mathcal{W}} L(\mu,W) \to \inf_{\mu\in\mathcal{M}^+}; \qquad \inf_{\mu\in\mathcal{M}^+} L(\mu,W) \to \sup_{W\in\mathcal{W}}.$$

Note that the value of the primal program cannot be smaller than the value of the dual program:

$$\inf_{\mu\in\mathcal{M}^+}\sup_{W\in\mathcal{W}} L(\mu,W) \ge \sup_{w\in\mathcal{W}}\inf_{\mu\in\mathcal{M}^+} L(\mu,W),$$

see [21, p.3]. If the last inequality is strict, then we say that there is a duality gap in the pair of linear programs. Sometimes, the functional L is called 'lagrangian' [15, Ch.8].

Lemma 4.1 *A pair (μ^*, W^*) is a saddle point of the function $L(\cdot)$, i.e.,*

$$L(\mu^*,W) \le L(\mu^*,W^*) \le L(\mu,W^*) \quad \forall \mu\in\mathcal{M}^+,\ W\in\mathcal{W}, \tag{17}$$

if and only if μ^* is a solution to the primal linear program, W^* is a solution to the dual linear program, there is no duality gap, and the common (optimal) value of the primal and dual programs equals

$$L(\mu^*,W^*) = \sup_{W\in\mathcal{W}} L(\mu^*,W) = \inf_{\mu\in\mathcal{M}^+} L(\mu,W^*).$$

For the proof, see [21, Thm.2].

Remark 4.1 *Sometimes, it is problematic to embed the cone $\mathcal{M}^+$ in a linear space, but $\mathcal{M}^+$ is a projection of a convex cone $\tilde{\mathcal{M}}^+$ in a linear space $\tilde{\mathcal{M}}$, like in Subsection 6.2. Note also that the presented above assertions are valid for an arbitrary $[-\infty,+\infty]$-valued function $L(\cdot)$ on arbitrary sets, $\mathcal{M}^+$, $\mathcal{W}$.*

5 First Pair of Linear Programs

In this section, we assume that Conditions 3.1, 3.2 and 3.3 are satisfied.

Let $\mathcal{M}_1^+$ be the space of finite measures μ on $\mathbf{X}\times\bar{\mathbb{R}}_+^0\times\mathbf{A}$ (the positive cone in the space $\mathcal{M}_1$ of finite signed measures μ on $\mathbf{X}\times\bar{\mathbb{R}}_+^0\times\mathbf{A}$), $\mathcal{W}_1$ be the space of bounded measurable functions $W(\cdot)$ on $\mathbf{X}$, and

$$\begin{aligned} L_1(\mu,W) \quad := \quad & \int_{\mathbf{X}\times\bar{\mathbb{R}}_+^0\times\mathbf{A}} C(x,(\theta,a))\mu(dx\times d\theta\times da) \\ & +W(x_0) + \int_{\mathbf{X}\times\bar{\mathbb{R}}_+^0\times\mathbf{A}} W(y)Q(dy|x,(\theta,a))\mu(dx\times d\theta\times da) - \int_{\mathbf{X}} W(x)\mu(dx\times\bar{\mathbb{R}}_+^0\times\mathbf{A}) \end{aligned}$$

be the $(-\infty,+\infty]$-valued bilinear function on $\mathcal{M}_1^+\times\mathcal{W}_1$, where the $\mathbb{R}_+^0$-valued function $C(\cdot)$ was defined in (3) and the substochastic kernel Q on $\mathbf{X}$ was introduced in (2). The function $L_1(\cdot)$ can take value $+\infty$ if the function $C^I(\cdot)$ is unbounded. Now the primal program (10) can be rewritten as

$$\sup_{W\in\mathcal{W}_1} L_1(\mu,W) \longrightarrow \inf_{\mu\in\mathcal{M}_1^+}.$$

Indeed, for each measure $\mu\in\mathcal{M}_1^+$, if it does not satisfy equality (9), then $\sup_{W\in\mathcal{W}_1} L_1(\mu,W)=+\infty$, and for finite measures μ satisfying (9), $L_1(\mu,W)=\int_{\mathbf{X}\times\bar{\mathbb{R}}_+^0\times\mathbf{A}} C(x,(\theta,a))\mu(dx\times d\theta\times da)$.

The dual program

$$\inf_{\mu\in\mathcal{M}_1^+} L_1(\mu,W) \longrightarrow \sup_{W\in\mathcal{W}_1}$$

can be rewritten as

$$\begin{aligned} &\text{Maximize} \quad && W(x_0) \ \text{ over } W\in\mathcal{W}_1 \\ &\text{subject to} \quad && C(x,(\theta,a)) + \int_{\mathbf{X}} W(y)Q(dy|x,(\theta,a)) - W(x) \ge 0 \qquad \forall (x,\theta,a)\in\mathbf{X}\times\bar{\mathbb{R}}_+^0\times\mathbf{A}. \end{aligned} \tag{18}$$

Indeed, for the functions $W\in\mathcal{W}_1$ satisfying (not satisfying) the presented inequality in program (18), $\inf_{\mu\in\mathcal{M}_1^+} L_1(\mu,W) = L_1(0,W) = W(x_0)$ ($\inf_{\mu\in\mathcal{M}_1^+} L_1(\mu,W)=-\infty$). Here 0 is the zero measure.

Theorem 5.1 *Suppose Conditions 3.1, 3.2 and 3.3 are satisfied. Then the following statements hold.*

(a) The solution to the dual program (18) is provided by the Bellman function $\mathcal{V}^*(\cdot) \in \mathcal{W}_1$, which is the unique bounded solution to the integral Bellman equation (7).

(b) There exists a deterministic stationary strategy f^* (optimal in problem (4)), and μ^*, the occupation measure corresponding to f^*, solves the primal program (10).

(c) The optimal values of the primal program (10) and the dual program (18) coincide (i.e., there is no duality gap) and equal $L_1(\mu^*, \mathcal{V}^*) = \mathcal{V}^*(x_0)$, and $(\mu^*, \mathcal{V}^*)$ is a saddle-point of $L_1(\cdot)$:

$$L_1(\mu^*, W) \leq L_1(\mu^*, \mathcal{V}^*) \leq L_1(\mu, \mathcal{V}^*) \quad \text{for all } \mu \in \mathcal{M}_1^+, \ W \in \mathcal{W}_1.$$

Proof. According to Propositon 3.1, $\mathcal{V}^*(\cdot)$ is the unique bounded nonnegative lower semicontinuous solution to equation (7), and there is a measurable mapping $f^* : \mathbf{X} \to \bar{\mathbb{R}}_+^0 \times \mathbf{A}$ providing the infimum, which defines the optimal strategy in problem (4). The corresponding occupation measure μ^*, satisfies conditions (9).

Let us denote the optimal values of linear programs (10) and (18) as $Val(10)$ and $Val(18)$ correspondingly.

For the measure μ^* we have

$$Val(10) \leq \int_{\mathbf{X}\times\bar{\mathbb{R}}_+^0\times\mathbf{A}} C(x, (\theta, a))\mu^*(dx \times d\theta \times da),$$

and for the function $\mathcal{V}^*(\cdot)$ we have

$$Val(18) \geq \mathcal{V}^*(x_0).$$

But

$$\mathcal{V}^*(x_0) = \mathcal{V}(x_0, f^*) = \int_{\mathbf{X}\times\bar{\mathbb{R}}_+^0\times\mathbf{A}} C(x, (\theta, a))\mu^*(dx \times d\theta \times da).$$

Since, in any case, $Val(10) \geq Val(18)$, we conclude that $Val(10) = Val(18) = \mathcal{V}^*(x_0)$, the measure μ^* solves the linear program (10), and the function $\mathcal{V}^*(\cdot)$ solves the linear program (18).

Items (a) and (b) are proved.

To prove Item (c), note that we already showed that

$$Val(10) = \sup_{W\in\mathcal{W}_1} L_1(\mu^*, W) = Val(18) = \inf_{\mu\in\mathcal{M}_1^+} L_1(\mu, \mathcal{V}^*) = \mathcal{V}^*(x_0).$$

Since

$$\sup_{W\in\mathcal{W}_1} L_1(\mu^*, W) \geq L_1(\mu^*, \mathcal{V}^*) \geq \inf_{\mu\in\mathcal{M}_1^+} L_1(\mu, \mathcal{V}^*),$$

we conclude that

$$Val(10) = Val(18) = L_1(\mu^*, \mathcal{V}^*) = \sup_{W\in\mathcal{W}_1} L_1(\mu^*, W) = \inf_{\mu\in\mathcal{M}_1^+} L_1(\mu, \mathcal{V}^*).$$

Finally, for all $\mu \in \mathcal{M}_1^+$ and $W \in \mathcal{W}_1$,

$$L_1(\mu^*, W) \leq \sup_{W\in\mathcal{W}_1} L_1(\mu^*, W) = L_1(\mu^*, \mathcal{V}^*) = \inf_{\mu\in\mathcal{M}_1^+} L_1(\mu, \mathcal{V}^*) \leq L_1(\mu, \mathcal{V}^*).$$

The proof is completed. □

Remark 5.1 *The linear programs (10) and (18) can have many solutions, for example in the case when, under an optimal strategy, a subset* $\hat{\mathbf{X}} \subset \mathbf{X}$ *is not reachable from the initial point* x_0*. In such situation, the function* $W(\cdot)$ *may be arbitrary enough on* $\hat{\mathbf{X}}$*: the only requirement is that* $W \in \mathcal{W}_1$ *and the constraint-inequality in (18) is satisfied. Similarly, the measure* μ *may be arbitrary enough on the set* $\{(x,\theta,a) \in \hat{\mathbf{X}} \times \bar{\mathbb{R}}_+^0 \times \mathbf{A} :\ C(x,(\theta,a)) = 0\}$*: only condition (9) must be satisfied. Therefore, even if the control problem (4) has a unique optimal control strategy* π*, there can exist many different solutions to the linear programs (10) and (18).*

6 Second Pair of Linear Programs

6.1 Additional Conditions and Preliminaries

In this subsection, we show that, under appropriate conditions, the integral Bellman equation (7) is equivalent to another equation (see (21)) which can be called 'differential Bellman equation'.

Recall that a function $w : \mathbf{X} \to \mathbb{R}$ is said to be absolutely continuous along the flow ϕ if, for all $x \in \mathbf{X}$, the function $t \mapsto w(\phi(x,t))$, $t \in \mathbb{R}_+^0$ is absolutely continuous. It is called increasing (decreasing), or measurable along the flow if so is the function $t \to w(\phi(x,t))$, $t \in \mathbb{R}_+^0$ for all $x \in \mathbf{X}$.

If, for each $x \in \mathbf{X}$, there is a constant $G(x)$ such that

$$|w(\phi(x,\tau_1)) - w(\phi(x,\tau_2))| \leq G(x)(\tau_2 - \tau_1), \qquad \forall 0 \leq \tau_1 < \tau_2 < \infty, \tag{19}$$

then the function $w(\cdot)$ is absolutely continuous along the flow ϕ.

Lemma 6.1 and its proof are similar to Lemma 2.2 in [6], where the authors assumed that $\mathbf{X}$ was a subset of an Euclidean space. For the general case, see [20, Lemma A.1].

Lemma 6.1 *Let* $\mathbf{X}$ *be an arbitrary set and* $\phi : \mathbf{X} \times \mathbb{R}_+^0 \to \mathbf{X}$ *be a flow in* $\mathbf{X}$ *possessing the semi-group property. Suppose function* $w(\cdot)$ *is absolutely continuous along the flow* ϕ*. Then the following assertions are valid.*

(a) There exists a function $\chi w : \mathbf{X} \to \mathbb{R}$ *such that, for any* $x \in \mathbf{X}$*, the function* $\chi w(\phi(x,s))$ *is Lebesgue integrable with respect to* s *on any finite interval* $[0,t] \subset \mathbb{R}_+^0$ *and*

$$w(\phi(x,t)) - w(x) = \int_{[0,t]} \chi w(\phi(x,s))\, ds \tag{20}$$

for all $x \in \mathbf{X}$ *and* $t \geq 0$*.*

(b) If, additionally, $\mathbf{X}$ *is a measurable space (that is, equipped with a* σ*-algebra of subsets), the function* $w(\cdot)$ *is measurable, and the functions* $\phi(\cdot,t) : \mathbf{X} \to \mathbf{X}$ *are measurable for all* $t \geq 0$*, then the function* χw *satisfying (a) can be chosen measurable.*

Note that the function $\chi w(\cdot)$ in (a) is not unique, but if $\chi w^1(\cdot)$ and $\chi w^2(\cdot)$ are two functions satisfying assertion (a) of Lemma 6.1, then, for each $x \in \mathbf{X}$, the functions $\chi w^1(\phi(x,s))$ and $\chi w^2(\phi(x,s))$ coincide for almost all $s \in \mathbb{R}_+^0$. We call $\chi w^1(\cdot)$ and $\chi w^2(\cdot)$ 'versions' of the function $\chi w(\cdot)$.

Condition 6.1 *(a) The Bellman function* $\mathcal{V}^*(\cdot)$ *is continuous on* $\mathbf{X}$ *and absolutely continuous along the flow* ϕ*.*

(b) The function $\chi\mathcal{V}^*(\phi(x,\cdot))$ *is integrable over* $[0,\infty)$ *for each* $x \in \mathbf{X}$*, and* $\sup_{x\in\mathbf{X}} |\int_{[0,\infty)} \chi\mathcal{V}^*(\phi(x,u))du| < \infty$*.*

Below, we provide sufficient conditions for this.

Condition 6.2 *(a) The functions* $C^g(\cdot)$ *and* $C^I(\cdot)$ *are continuous.*

(b) *The function $C^g(\cdot)$ is bounded, and, for each $\varepsilon > 0$, there is $T \in \mathbb{R}^0_+$ such that*

$$\sup_{x\in\mathbf{X}} \int_{(T,\infty)} |C^g(\phi(x,u))|du < \varepsilon.$$

(c) *The mapping $l : \mathbf{X} \times \mathbf{A} \to \mathbf{X}$ does not depend on the component $x \in \mathbf{X}$.*

(d) *The function $C^I(\cdot, a)$ is absolutely continuous along the flow ϕ for each fixed value of $a \in \mathbf{A}$. Moreover, for each $a \in \mathbf{A}$ and $x \in \mathbf{X}$,*

$$C^I(\phi(x,t),a) - C^I(x,a) = \int_{[0,t]} g(x,\phi(x,u),a)du, \quad \forall t > 0,$$

where $g(\cdot)$ is a fixed measurable function on $\{(x,y,a) : \ x \in \mathbf{X}, y = \phi(x,u)$ for some $u \geq 0, a \in \mathbf{A}\} \subset \mathbf{X}^2 \times \mathbf{A}$, bounded for each fixed $x \in \mathbf{X}$.

(e) *The function $G(x) := \sup_{y=\phi(x,u):u\geq 0} C^g(y) \vee \sup_{y=\phi(x,u):u\geq 0} \sup_{a\in\mathbf{A}} |g(x,y,a)|$ on $\mathbf{X}$ is measurable and such that $\sup_{x\in\mathbf{X}} \int_{[0,\infty)} G(\phi(x,u))du < \infty$.*

Note that Condition 6.2(b) is satisfied, e.g., in the discounted model (6) with the bounded function $\tilde{C}^g(\cdot)$. Conditions 6.2(d,e) are fulfilled if, for example, the function $C^g(\cdot)$ satisfies Condition 3.3(a) and decreases along the flow ϕ, and the function $C^I(\cdot)$ does not depend on x.

Lemma 6.2 *Suppose Conditions 3.1, 3.2, 3.3 and 6.2(a,b) are satisfied. Then the Bellman function $\mathcal{V}^*(\cdot)$ is bounded and continuous.*

The proofs of this and subsequent lemmas are presented in the Appendix.

Lemma 6.3 *Suppose Conditions 3.1, 3.2, 3.3 and 6.2(a-d) are satisfied. Then the Bellman function $\mathcal{V}^*(\cdot)$ is absolutely continuous along the flow ϕ. If additionally Condition 6.2(e) is satisfied, then Condition 6.1(b) is satisfied for any version of the function $\chi\mathcal{V}^*(\cdot)$.*

Theorem 6.1 *Suppose Conditions 3.1, 3.2, 3.3 and 6.1(a) are satisfied and the function $C^I(\cdot)$ is continuous. Then, for a bounded measurable function $W(\cdot)$ on $\mathbf{X}$, the following statements are equivalent.*

(a) *The function $W(\cdot)$ satisfies equation (7).*

(b) *The function $W(\cdot)$ is continuous on $\mathbf{X}$, absolutely continuous along the flow ϕ, satisfies equation*

$$\min \left\{ \chi W(\phi(x,t)) + C^g(\phi(x,t)); \right.$$
$$\left. \inf_{a\in\mathbf{A}} \{C^I(\phi(x,t),a) + W(l(\phi(x,t),a)) - W(\phi(x,t))\} \right\} = 0 \tag{21}$$
$$\text{for all } x \in \mathbf{X}, \text{ for almost all } t \geq 0,$$

and $\lim_{t\to\infty} W(\phi(x,t)) = 0$ for all $x \in \mathbf{X}$.

(c) *$W(x) = \mathcal{V}^*(x)$ for all $x \in \mathbf{X}$.*

Expression (21) is equivalent to the so-called 'quasi-variational inequalities' [7, 22]; we call it 'differential' Bellman equation.

Before proceeding to the proof, we formulate the following proposition justified in Subsection 3.1 of [9].

Proposition 6.1 *Suppose the integral $\int_{[0,\infty)} C^g(\phi(x,u))du$ is finite for all $x \in \mathbf{X}$ and $W(\cdot)$ is a function, absolutely continuous along the flow ϕ, such that $\underline{\lim}_{t\to\infty} W(\phi(x,t)) \geq 0$ for all $x \in \mathbf{X}$. Then the following statements are equivalent.*

1. The function $W(\cdot)$ satisfies equation (7) and the infimum with respect to θ in equation

$$W(x) = \inf_{\theta \in \bar{R}_+^0} \Big\{ \int_{[0,\theta]} C^g(\phi(x,u))\, du + I\{\theta < +\infty\} \inf_{a\in\mathbf{A}} \{C^I(\phi(x,\theta),a) + W(l(\phi(x,\theta),a))\}\Big\}, \quad x \in \mathbf{X} \tag{22}$$

is attained on a nonempty set $\Theta(x) \subset \bar{\mathbb{R}}_+^0$, and $\Theta(x)$ contains its infimum for each $x \in \mathbf{X}$.

2. The function $W(\cdot)$ is such that assertions (a), (b) and (c) below are valid.

(a) For each $y \in \mathbf{X}$,

$$\begin{array}{lll} \textit{either (i)} & \left\{\begin{array}{l} \mathcal{F}_+^W(y) = 0 \quad \textit{and} \\ \inf_{a\in\mathbf{A}} \left\{C^I(y,a) + W(l(y,a)) - W(y)\right\} > 0 \end{array}\right. & \\ \textit{or,} & \textit{if the Assertion (i) fails to hold (e.g., } \mathcal{F}_+^W(y) \textit{ does not exist), then} & (23) \\ \quad \textit{(ii)} & \left\{\begin{array}{l} \underline{\mathcal{F}}_-^W(y) \subset [0,\infty] \quad \textit{and} \\ \inf_{a\in\mathbf{A}} \left\{C^I(y,a) + W(l(y,a)) - W(y)\right\} = 0. \end{array}\right. & \end{array}$$

Here

$$\mathcal{F}_+^W(y) := \lim_{t\to 0+} \Big[\frac{W(\phi(y,t)) - W(y)}{t} + \frac{1}{t}\int_{[0,t]} C^g(\phi(y,u))\, du\Big],$$

provided that this limit exists;

$$\begin{aligned} \underline{\mathcal{F}}_-^W(y) \;:=\; & \Big\{\underline{\lim}_{t\to 0+} \Big[\frac{W(y) - W(\phi(\tilde{y}, s-t))}{t} \\ & + \frac{1}{t}\int_{[-t,0]} C^g(\phi(\tilde{y}, s+u))\, du\Big] : \quad (\tilde{y}, s) \in \mathbf{X}\times\mathbb{R}_+,\ \phi(\tilde{y},s) = y\Big\} \subset \mathbb{R}\cup\{\pm\infty\}. \end{aligned}$$

If $\{(\tilde{y},s) \in \mathbf{X}\times\mathbb{R}_+ :\ \phi(\tilde{y},s) = y\} = \emptyset$, that is, if the point $y \in \mathbf{X}$ is 'singular', we put $\underline{\mathcal{F}}_-^W(y) = \emptyset$.

(b) For the set $\mathcal{L}$, defined as

$$\mathcal{L} := \{x \in \mathbf{X} : \inf_{a\in\mathbf{A}} \big[C^I(x,a) + W(l(x,a)) - W(x)\big] = 0\}, \tag{24}$$

the set $\{t \in \mathbb{R}_+^0 :\ \phi(x,t) \in \mathcal{L}\}$, for each $x \in \mathbf{X}$, if not empty, contains its infimum.

(c) For each $x \in \mathbf{X}$, $\lim_{t\to\infty} W(\phi(x,t)) = 0$.

This proposition also appeared in [18], where a slightly stronger condition on the function $W(\cdot)$ was imposed.

Proof of Theorem 6.1. Equivalence of the statements (a) and (c) under Conditions 3.1, 3.2 and 3.3 was justified in Section 3.

Suppose a bounded measurable function $W(\cdot)$ satisfies equation (7). Then $W(\cdot) = \mathcal{V}^*(\cdot)$ and hence the function $W(\cdot)$ is continuous on $\mathbf{X}$ and absolutely continuous along the flow ϕ by Condition 6.1(a). Moreover, the infimum with respect to θ in equation (22) is attained on a nonempty set $\Theta(x) \subset \bar{\mathbb{R}}_+^0$, and $\Theta(x)$ contains its infimum by Corollary 2 of [18]. Finally, $W(\cdot) = \mathcal{V}^*(\cdot) \geq 0$, so that we have Statement 1 of Proposition 6.1 leading to Statement 2: $\lim_{t\to\infty} W(\phi(x,t)) = 0$ and, for each $y \in \mathbf{X}$, (23) is valid.

Since the function $W(\cdot)$ is absolutely continuous along the flow ϕ, for each $x \in \mathbf{X}$, for almost all $t \geq 0$,

$$\frac{dW(\phi(x,t))}{dt} = \chi W(\phi(x,t)) \qquad \text{and} \qquad \frac{d}{dt}\int_{[0,t]} C^g(\phi(x,u))du = C^g(\phi(x,t)).$$

Now (23) implies the following statement:

for each $x \in \mathbf{X}$, for almost all $t \geq 0$, (for $y = \phi(x,t)$),

$$\text{either (i)} \quad \begin{cases} \chi W(\phi(x,t)) + C^g(\phi(x,t)) = 0 \text{ and} \\ \inf_{a\in\mathbf{A}} \left\{ C^i(\phi(x,t),a) + W(l(\phi(x,t),a)) - W(\phi(x,t)) \right\} > 0 \end{cases}$$

or, if the Assertion (i) fails to hold, then

$$\text{(ii)} \quad \begin{cases} \chi W(\phi(x,t)) + C^g(\phi(x,t)) \geq 0 \text{ and} \\ \inf_{a\in\mathbf{A}} \left\{ C^i(\phi(x,t),a) + W(l(\phi(x,t),a)) - W(\phi(x,t)) \right\} = 0. \end{cases}$$

Therefore, statement (b) of Theorem 6.1 is valid.

Suppose statement (b) is valid and prove statement (a).

Under the imposed conditions, for the continuous on $\mathbf{X}$ function $W(\cdot)$, the function $\inf_{a\in\mathbf{A}} \left[C^I(x,a) + W(l(x,a)) - W(x) \right]$ is continuous by [2, Prop.7.32]. Hence the set (24) is closed. Therefore, for each $x \in \mathbf{X}$, the set $\{t \in \mathbb{R}^0_+ : \phi(x,t) \in \mathcal{L}\}$, if not empty, is closed and hence contains its infimum, because the flow ϕ is continuous.

Let us show that, for each $y \in \mathbf{X}$, expression (23) holds. According to (21), there is a sequence $t_i \downarrow 0$ such that

$$F_1(t_i) := \inf_{a\in\mathbf{A}} \{C^I(\phi(y,t_i),a) + W(l(\phi(y,t_i),a) - W(\phi(y,t_i))\} \geq 0.$$

The function $F_1(\cdot)$ is continuous by [2, Prop.7.32]. Thus,

$$F_1(0) = \inf_{a\in\mathbf{A}} \{C^I(y,a) + W(l(y,a)) - W(y)\} \geq 0.$$

(i) Suppose

$$F_2(y) := \inf_{a\in\mathbf{A}} \{C^I(y,a) + W(l(y,a)) - W(y)\} > 0.$$

Since again the function $F_2(\cdot)$ is continuous, there exists $\varepsilon > 0$ such that $F_2(\phi(y,t)) > 0$ for all $t \in [0,\varepsilon]$, and hence

$$\chi W(\phi(y,u)) + C^g(\phi(y,u)) = 0$$

for almost all $u \in [0,\varepsilon]$. Therefore, for all $t \in [0,\varepsilon]$,

$$W(\phi(y,t)) = W(y) - \int_{[0,t]} C^g(\phi(y,u))du,$$

and $\mathcal{F}^W_+(y) = 0$.

(ii) Suppose $F_2(y) = 0$ and the point y is not singular: $y = \phi(\tilde{y},s)$ for some $\tilde{y} \in \mathbf{X}$, $s > 0$. For an arbitrarily fixed $t \in [0,s]$, we integrate the inequality

$$\chi W(\phi(\tilde{y},r)) + C^g(\phi(\tilde{y},r)) \geq 0,$$

valid for almost all $r \in [s-t,s]$:

$$W(\phi(\tilde{y},s)) - W(\phi(\tilde{y},s-t)) + \int_{[s-t,s]} C^g(\phi(\tilde{y},r))dr = W(y) - W(\phi(\tilde{y},s-t)) + \int_{[-t,0]} C^g(\phi(\tilde{y},s+u))du \geq 0.$$

Hence, $\underline{\mathcal{F}}^W_-(y) \geq 0$, and equation (23) is proved.

According to Proposition 6.1, since Statement 2 is valid, the function $W(\cdot)$ satisfies equation (7). $\square$

Corollary 6.1 *If Conditions 3.1, 3.2, 3.3 and 6.1(a) are satisfied and the function $C^I(\cdot)$ is continuous, then $\lim_{t\to\infty} \mathcal{V}^*(\phi(x,t)) = 0$ for all $x \in \mathbf{X}$.*

For the proof, note that the Bellman function $\mathcal{V}^*(\cdot)$ is the unique bounded measurable solution to equation (7) by Proposition 3.1.

6.2 Linear Programs

Definition 6.1 *$\mathcal{W}_2$ is the linear space of measurable bounded functions $W(\cdot)$ on $\mathbf{X}$, absolutely continuous along the flow ϕ, such that $\lim_{t\to\infty} W(\phi(x,t)) = 0$ for all $x \in \mathbf{X}$, the function $\chi W(\phi(x,\cdot))$ is integrable over $[0,\infty)$ for each $x \in \mathbf{X}$ and $\sup_{x\in\mathbf{X}} |\int_{[0,\infty)} \chi W(\phi(x,u))du| < \infty$.*

Under appropriate conditions, the Bellman function $\mathcal{V}^*(\cdot)$ belongs to $\mathcal{W}_2$ by Lemmas 6.2, 6.3 and Theorem 6.1.

Definition 6.2 *A measure $\hat{\eta}$ on $\mathbf{X}$ is called normal if*

$$\hat{\eta}(\Gamma_X) = \int_{\mathbf{X}} \int_{\mathbb{R}^0_+} \delta_{\phi(x,u)}(\Gamma_X)\psi(x,u)du\; m(dx), \qquad \Gamma_X \in \mathcal{B}(\mathbf{X}),$$

where $\psi(\cdot) \geq 0$ is a measurable function on $\mathbf{X} \times \mathbb{R}^0_+$, m is a probability measure on $\mathbf{X}$, and, for the measure $\bar{m}(dx \times du) := m(dx) \times du$ on $\mathbf{X} \times \mathbb{R}^0_+$, the following assertion holds true:

$$\exists S \in \mathbb{R}_+ : \;\; \bar{m}(\{(x,u) : \; |\psi(x,u)| > S\}) = 0. \tag{25}$$

A measure η on $\mathbf{X} \times \mathbf{A}_\Box$, where $\mathbf{A}_\Box = \mathbf{A} \cup \{\Box\}$ is as introduced in (15), is called normal if $\eta(\mathbf{X} \times \mathbf{A}) < \infty$ and the measure $\eta(dx \times \Box)$ is normal.

The set of all normal measures on $\mathbf{X} \times \mathbf{A}_\Box$ is denoted as $\mathcal{M}_2^+$.

Remark 6.1 *Let $\phi_x(u) := \phi(x,u) : \;\mathbb{R}^0_+ \to \mathbf{X}$ be a measurable mapping (for each fixed $x \in \mathbf{X}$). Then $\breve{\eta}(\Gamma_x|x) := \int_{\mathbb{R}^0_+} \delta_{\phi_x(u)}(\Gamma_X)\psi(x,u)du$ is the measurable kernel on $\mathbf{X}$ given $\mathbf{X}$, which, for each $x \in \mathbf{X}$, coincides with the image of the measure $\psi(x,u)du$ on $\mathbb{R}^0_+$ with respect to the mapping $\phi_x(\cdot)$, and $\hat{\eta}(\Gamma_X) = \int_{\mathbf{X}} \breve{\eta}(\Gamma_X|x)m(dx)$.*

For example, every aggregated occupation measure η, induced by a finite measure μ as in Definition 3.1, is normal. A normal measure can equal $+\infty$ at a stationary point x, where $\phi(x,t) \equiv x$.

Note that, for every function $W(\cdot) \in \mathcal{W}_2$ and every normal measure $\hat{\eta}$ on $\mathbf{X}$, the integral $\int_{\mathbf{X}} \chi W(x)\hat{\eta}(dx)$ is well defined, independent of the version of the function $\chi W(\cdot)$, and finite.

The normal measures on $\mathbf{X}$ are not finite, and it is problematic to embed the space $\mathcal{M}_2^+$ in a linear space. But we will show that $\mathcal{M}_2^+$ is a convex cone, equal to a projection of a (positive) cone $\tilde{\mathcal{M}}^+$ in some linear space $\tilde{\mathcal{M}}$. See Corollary 6.2 and Remark 4.1.

Lemma 6.4 *Suppose m_1 and m_2 are two probability measures on $\mathbf{X}$ and $\psi_1(\cdot)$, $\psi_2(\cdot)$ are two real-valued measurable functions on $\mathbf{X} \times \mathbb{R}^0_+$ such that, for $j = 1, 2$, for the measure $\bar{m}_j(dx \times du) := m_j(dx) \times du$ on $\mathbf{X} \times \mathbb{R}^0_+$ and the function $\psi_j(\cdot)$, the requirement (25) is satisfied.*

Then, for arbitrarily fixed $a_1, a_2 \in \mathbb{R}$, there exist a probability measure m on $\mathbf{X}$ and a measurable function $\psi(\cdot)$ on $\mathbf{X} \times \mathbb{R}^0_+$ such that the following assertions are valid.

(a) For the measure $\bar{m}(dx \times du) := m(dx) \times du$ on $\mathbf{X} \times \mathbb{R}^0_+$ and the function $\psi(\cdot)$, the requirement (25) is satisfied.

(b) For each $i \in \{1, 2, \ldots\}$, the finite signed measures on $\mathbf{X} \times [i-1, i)$

$$M^i_j(dx \times du) := \mathbb{I}\{u \in [i-1,i)\}\psi_j(x,u)\bar{m}_j(dx \times du), \quad j = 1, 2,$$

and

$$M^i(dx \times du) := \mathbb{I}\{u \in [i-1,i)\}\psi(x,u)\bar{m}(dx \times du)$$

are such that

$$M^i = a_1 M^i_1 + a_2 M^i_2$$

set-wise.

(c) The function $\psi(\cdot)$ is positive if so are the functions $\psi_1(\cdot)$, $\psi_2(\cdot)$ and the numbers a_1, a_2.

The sequences $\{M^i\}_{i=1}^\infty$ of finite signed measures as in Lemma 6.4, associated with the probability m and the function $\psi(\cdot)$, form the linear space $\tilde{\mathcal{M}}_1$, and one can introduce the linear space $\tilde{\mathcal{M}} := \tilde{\mathcal{M}}_1 \times \tilde{\mathcal{M}}_2$, where $\tilde{\mathcal{M}}_2$ is the linear space of finite signed measures on $\mathbf{X} \times \mathbf{A}$. The natural positive cone in $\tilde{\mathcal{M}}$ is denoted as $\tilde{\mathcal{M}}^+$; it contains the corresponding positive measures. Now one can see that $\mathcal{M}_2^+$ is just a (linear) projection of the convex cone $\tilde{\mathcal{M}}^+$.

Corollary 6.2 from Lemma 6.4. *If η_1 and η_2 are two normal measures on $\mathbf{X}\times\mathbf{A}_\Box$, then $a_1\eta_1+a_2\eta_2$ is again a normal measure on $\mathbf{X} \times \mathbf{A}_\Box$ for all $a_1, a_2 \in \mathbb{R}_+^0$, that is, the space $\mathcal{M}_2^+$ is a convex cone.*

The proof is obvious.

Theorem 6.2 *Suppose Conditions 3.1, 3.2 and 3.3 are satisfied and a (finite) measure μ on $\mathbf{X} \times \bar{\mathbb{R}}_+^0 \times \mathbf{A}$ satisfies requirements (9). Then the aggregated occupation measure η as in Definition 3.1, induced by μ, satisfies equation*

$$0 = W(x_0) + \int_{\mathbf{X}} \chi W(x)\eta(dx \times \Box) - \int_{\mathbf{X}} W(x)\eta(dx \times \mathbf{A}) + \int_{\mathbf{X}\times\mathbf{A}} W(l(x,a))\eta(dx \times da) \qquad (26)$$

for all functions $W(\cdot) \in \mathcal{W}_2$. This equation is valid for any version of the function $\chi W(\cdot)$.

<u>Proof.</u> According to Lemma 6.1, for each fixed $x \in \mathbf{X}$, $\theta \in \mathbb{R}_+^0$,

$$W(\phi(x,\theta)) = W(x) + \int_{[0,\theta]} \chi W(\phi(x,s))ds.$$

After we integrate this equation over $\mathbf{X} \times \mathbb{R}_+^0$ with respect to the measure μ, represented as in (11), we obtain the following equality, according to the definition (13):

$$\begin{aligned}\int_{\mathbf{X}} W(y)\eta(dy \times \mathbf{A}) \;&=\; \int_{\mathbf{X}} W(x)\hat{p}(\bar{\mathbb{R}}_+^0|x)\mu(dx \times \bar{\mathbb{R}}_+^0 \times \mathbf{A}) - \int_{\mathbf{X}} W(x)\hat{p}(\{\infty\}|x)\mu(dx \times \bar{\mathbb{R}}_+^0 \times \mathbf{A}) \\ &\quad + \int_{\mathbf{X}}\int_{\mathbb{R}_+^0}\int_{[0,\theta]} \chi W(\phi(x,s))ds\; \hat{p}(d\theta|x)\mu(dx \times \bar{\mathbb{R}}_+^0 \times \mathbf{A}),\end{aligned}$$

where $\hat{p}(d\theta|x) := \int_{\mathbf{A}} p_T(d\theta|x,a)p_A(da|x)$. All the integrals here and below are finite.

After we apply the Fubini Theorem to the last integral, we obtain:

$$\begin{aligned}\int_{\mathbf{X}} W(y)\eta(dy \times \mathbf{A}) \;&=\; \int_{\mathbf{X}} W(x)\mu(dx \times \bar{\mathbb{R}}_+^0 \times \mathbf{A}) - \int_{\mathbf{X}} W(x)\hat{p}(\{\infty\}|x)\mu(dx \times \bar{\mathbb{R}}_+^0 \times \mathbf{A}) \\ &\quad + \int_{\mathbf{X}}\int_{\mathbb{R}_+^0}\int_{[s,\infty)} \chi W(\phi(x,s))\hat{p}(d\theta|x)\; ds\; \mu(dx \times \bar{\mathbb{R}}_+^0 \times \mathbf{A}) \\ &= \int_{\mathbf{X}} W(x)\mu(dx \times \bar{\mathbb{R}}_+^0 \times \mathbf{A}) - \int_{\mathbf{X}} W(x)\hat{p}(\{\infty\}|x)\mu(dx \times \bar{\mathbb{R}}_+^0 \times \mathbf{A}) \\ &\quad + \int_{\mathbf{X}}\int_{\mathbb{R}_+^0} \chi W(\phi(x,s))\hat{p}([s,\infty)|x)\; ds\; \mu(dx \times \bar{\mathbb{R}}_+^0 \times \mathbf{A}) \\ &= \int_{\mathbf{X}} W(x)\mu(dx \times \bar{\mathbb{R}}_+^0 \times \mathbf{A}) - \int_{\mathbf{X}} W(x)\hat{p}(\{\infty\}|x)\mu(dx \times \bar{\mathbb{R}}_+^0 \times \mathbf{A}) \\ &\quad + \int_{\mathbf{X}} \chi W(y)\eta(dy \times \Box) - \int_{\mathbf{X}}\int_{\mathbb{R}_+^0} \chi W(\phi(x,s))\hat{p}(\{\infty\}|x)\; ds\; \mu(dx \times \bar{\mathbb{R}}_+^0 \times \mathbf{A}).\end{aligned}$$

The last equality is by the definition (12) of the measure $\eta(dy\times\Box)$.

Since

$$\lim_{t\to\infty} W(\phi(x,t)) = W(x) + \int_{\mathbb{R}^0_+} \chi W(\phi(x,s))ds \equiv 0,$$

we conclude that

$$\int_{\mathbf{X}} W(x)\hat{p}(\{\infty\}|x)\mu(dx\times\bar{\mathbb{R}}^0_+\times\mathbf{A}) + \int_{\mathbf{X}}\int_{\mathbb{R}^0_+} \chi W(\phi(x,s))\hat{p}(\{\infty\}|x)\ ds\ \mu(dx\times\bar{\mathbb{R}}^0_+\times\mathbf{A}) = 0.$$

Finally,

$$\begin{aligned}\int_{\mathbf{X}} W(y)\eta(dy\times\mathbf{A}) &= \int_{\mathbf{X}} W(x)\mu(dx\times\bar{\mathbb{R}}^0_+\times\mathbf{A}) + \int_{\mathbf{X}}\chi W(y)\eta(dy\times\Box)\\ &= W(x_0) + \int_{\mathbf{X}\times\mathbb{R}^0_+\times\mathbf{A}} W(l(\phi(y,\theta),a))\mu(dy\times d\theta\times da) + \int_{\mathbf{X}}\chi W(y)\eta(dy\times\Box)\end{aligned}$$

by (9), and the required formula (26) follows from the definition (13). $\Box$

If $\phi(x,\tau)=x$ for some $x\in\mathbf{X}$ and $\tau\in\mathbb{R}_+$, then, on the cycle $\{\phi(x,t):\ 0\le t\le\tau\}$, any function $W(\cdot)\in\mathcal{W}$ is identical zero. Hence equation (26) does not provide any information about the measure η on that cycle.

Below, we assume that Conditions 3.1, 3.2, 3.3 and 6.1 are satisfied and the function $C^I(\cdot)$ is continuous. Let

$$\begin{aligned}L_2(\eta,W) &:= \int_{\mathbf{X}\times\mathbf{A}} C^I(x,a)\eta(dx\times da) + \int_{\mathbf{X}} C^g(x)\eta(dx\times\Box) + W(x_0)\\ &\quad + \int_{\mathbf{X}}\chi W(x)\eta(dx\times\Box) - \int_{\mathbf{X}} W(x)\eta(dx\times\mathbf{A}) + \int_{\mathbf{X}\times\mathbf{A}} W(l(x,a))\eta(dx\times da)\end{aligned}$$

be the $(-\infty,+\infty]$-valued bilinear function on $\mathcal{M}^+_2\times\mathcal{W}_2$. This function can take value $+\infty$ if the function $C^I(\cdot)$ is unbounded; all the other terms are finite.

The primal linear program $\sup_{W\in\mathcal{W}_2} L_2(\eta,W)\to\inf_{\eta\in\mathcal{M}^+_2}$ can be rewritten as follows:

$$\left\{\begin{array}{l}\displaystyle\int_{\mathbf{X}\times\mathbf{A}} C^I(x,a)\eta(dx\times da) + \int_{\mathbf{X}} C^g(x)\eta(dx\times\Box) = \int_{\mathbf{X}\times\mathbf{A}_\Box}\mathcal{C}(x,a)\eta(dx\times da) \to \inf_{\eta\in\mathcal{M}^+_2}\\ \text{subject to (26).}\end{array}\right. \tag{27}$$

(The function $\mathcal{C}(\cdot)$ was introduced in (15).) Indeed, if the equality (26) is not satisfied, then, for some function $W\in\mathcal{W}_2$, the righthand side of (26) is positive (one can change the sign of $W(\cdot)$ if needed) and, hence can be made arbitrarily large if we multiply $W(\cdot)$ by a large constant. Therefore, $\sup_{W\in\mathcal{W}_2} L_2(\eta,W)=+\infty$. In case the equality (26) is valid, $L_2(\eta,W)=\int_{\mathbf{X}\times\mathbf{A}} C^I(x,a)\eta(dx\times da)+\int_{\mathbf{X}} C^g(x)\eta(dx\times\Box)$. The similar to (27) linear program, for the case of the constrained optimal impulse control, was investigated in [20].

The dual linear program $\inf_{\eta\in\mathcal{M}^+_2} L_2(\eta,W)\to\sup_{W\in\mathcal{W}_2}$ can be rewritten as follows:

$$\begin{array}{rl} & W(x_0)\to\displaystyle\sup_{W\in\mathcal{W}_2} \\ \text{subject to} & \chi W(\phi(x,t)) + C^g(\phi(x,t))\ge 0 \text{ for all } x\in\mathbf{X} \text{ and for almost all } t\ge 0;\\ & C^I(x,a) + W(l(x,a)) - W(x)\ge 0\ \ \forall (x,a)\in\mathbf{X}\times\mathbf{A}.\end{array} \tag{28}$$

Indeed, suppose there is $y\in\mathbf{X}$ such that the measurable set

$$I := \{t\ge 0:\ \chi W(\phi(y,t)) + C^g(\phi(y,t)) < 0\}$$

has the positive Lebesgue measure. Then, after we take the normal measure $\eta(dx \times \Box)$ associated with $m(dz) := \delta_y(dz)$ and $\psi(z,u) := K\mathbb{I}\{u \in I\}$ with large $K > 0$, we see that the expression

$$\int_{\mathbf{X}} C^g(x)\eta(dx \times \Box) + \int_{\mathbf{X}} \chi W(x)\eta(dx \times \Box)$$

can be made arbitrarily big negative. Similarly, if $C^I(y,b) + W(l(y,b)) - W(y) < 0$ for some $(y,b) \in \mathbf{X} \times \mathbf{A}$, then one can take $\eta(dx \times da) := K\delta_{(y,b)}(dx \times da)$ leading to the arbitrarily big negative value of the expression

$$\int_{\mathbf{X}\times\mathbf{A}} C^I(x,a)\eta(dx \times da) + \int_{\mathbf{X}\times\mathbf{A}} W(l(x,a))\eta(dx \times da) - \int_{\mathbf{X}} W(x)\eta(dx \times \mathbf{A})$$

when the constant $K > 0$ increases. Therefore, if the constraints in (28) are not satisfied, then $\inf_{\eta \in \mathcal{M}_2^+} L_2(\eta, W) = -\infty$. If they are satisfied, then $\inf_{\eta \in \mathcal{M}_2^+} L_2(\eta, W) = W(x_0)$, and this infimum is attained at $\eta = 0$.

Theorem 6.3 *Suppose Conditions 3.1, 3.2, 3.3 and 6.1 are satisfied and the function $C^I(\cdot)$ is continuous. Then the following statements hold.*

- *(a) The solution to the dual program (28) is provided by the Bellman function $\mathcal{V}^*(\cdot) \in \mathcal{W}_2$.*
- *(b) There exists a deterministic stationary strategy f^* (optimal in problem (4)), and the aggregated occupation measure η^*, induced by the occupation measure μ^* corresponding to f^*, solves the primal program (27).*
- *(c) The optimal values of the primal program (27) and the dual program (28) coincide (i.e., there is no duality gap) and equal $L_2(\eta^*, \mathcal{V}^*) = \mathcal{V}^*(x_0)$, and $(\eta^*, \mathcal{V}^*)$ is a saddle-point of $L_2(\cdot)$:*

$$L_2(\eta^*, W) \le L_2(\eta^*, \mathcal{V}^*) \le L_2(\eta, \mathcal{V}^*) \quad \textit{for all } \eta \in \mathcal{M}_2^+,\ W \in \mathcal{W}_2.$$

Proof. We denote the optimal values of linear programs (27) and (28) as $Val(27)$ and $Val(28)$ correspondingly. The Bellman function $\mathcal{V}^*(\cdot)$ is bounded by Proposition 3.1; $\lim_{t\to\infty} \mathcal{V}^*(\phi(x,t)) = 0$ for all $x \in \mathbf{X}$ by Theorem 6.1. Hence, $\mathcal{V}^*(\cdot) \in \mathcal{W}_2$ according to Condition 6.1. All the inequalities in (28) are valid for the function $\mathcal{V}^*(\cdot)$ by Theorem 6.1. For the last inequality, note that equation (21) implies inequality

$$C^I(\phi(x,t_i),a) + \mathcal{V}^*(l(\phi(x,t_i),a) - \mathcal{V}^*(\phi(x,t_i)) \ge 0, \quad i = 1,2,\dots,$$

valid for all $(x,a) \in \mathbf{X} \times \mathbf{A}$ and for some sequence $t_i \downarrow 0$. It remains to pass here to the limit using the continuity of the functions $C^I(\cdot)$, $\mathcal{V}^*(\cdot)$ and the mappings $l(\cdot)$, $\phi(\cdot)$. Therefore,

$$Val(28) \ge \mathcal{V}^*(x_0).$$

The existence of the strategy f^* follows from Proposition 3.1. The aggregated occupation measure η^*, induced by the measure $\mu^* = \mu^{f^*}$ (which satisfies requirements (9)), is normal and satisfies equation (26) by Theorem 6.2. Therefore,

$$Val(27) \le \int_{\mathbf{X}\times\mathbf{A}_\Box} \mathcal{C}(x,a)\eta^*(dx \times da).$$

But, according to (8) and (14)

$$\mathcal{V}^*(x_0) = \mathcal{V}(x_0, f^*) = \int_{\mathbf{X}\times\mathbf{A}_\Box} \mathcal{C}(x,a)\eta^*(dx \times da).$$

Since, in any case, $Val(27) \geq Val(28)$, we conclude that $Val(27) = Val(28) = \mathcal{V}^*(x_0)$, the measure η^* solves the linear program (27), and the function $\mathcal{V}^*(\cdot)$ solves the linear program (28).

Items (a) and (b) are proved.

The proof of Item (c) coincides with the end of the proof of Theorem 5.1. □

Similarly to Remark 5.1, the linear programs (27) and (28) can have many solutions.

7 Construction of the Optimal Strategy

The dual linear programs (18) and (28) can help to obtain the Bellman function $\mathcal{V}^*(\cdot)$: see Theorems 5.1 and 6.3. After that, the optimal strategy comes from Proposition 3.1. At the same time, the straightforward dynamic programming approach is more widely used, and it immediately leads to the optimal strategy in problem (4): see Proposition 3.1.

Below, we concentrate at the primal programs (10) and (27). They are most popular in the constrained problems of the form

$$\mathcal{V}_0(x_0, \pi) \to \inf_{\pi \in \Pi} \text{ subject to } \mathcal{V}_j(x_0, \pi) \leq d_j, \; j = 1, 2, \ldots, , J, \tag{29}$$

where the objectives $\mathcal{V}_j(\cdot)$ are associated with different cost functions $C^g_j(\cdot)$ and $C^I_j(\cdot)$, and d_j are fixed numbers. See, e.g., [8, 19].

As for the first primal linear program (10), the optimal measure μ^* gives rise to the stationary strategy π^*:

$$\mu^*(dx \times d\theta \times da) = \mu^*(dx \times \bar{\mathbb{R}}^0_+ \times \mathbf{A})\pi^*(d\theta \times da|x),$$

which is optimal in problem (4). This is a standard reasoning for Markov decision processes with nonnegative costs [8], see also [19, Prop.3.2].

The case of the second primal linear program (27) is more tricky. Below, we look at it more attentively and impose some reasonable additional conditions and definitions. At the same time, some conditions, required in Subsection 6.2, are not needed. The flow ϕ is assumed to be continuous.

Definition 7.1 *For a fixed $z \in \mathbf{X}$, the set*

$${}_z\mathcal{X} := \{\phi(z, t) : \; t \in \mathbb{R}^0_+\}$$

can be called the (partial) orbit of the point z.

Condition 7.1 *For arbitrarily fixed $z_1, z_2 \in \mathbf{X}$,*

- *either ${}_{z_1}\mathcal{X} \cap {}_{z_2}\mathcal{X} = \emptyset$,*
- *or ${}_{z_1}\mathcal{X} \subset {}_{z_2}\mathcal{X}$,*
- *or ${}_{z_2}\mathcal{X} \subset {}_{z_1}\mathcal{X}$.*

Definition 7.2 *If Condition 7.1 is satisfied, then the set*

$${}_z\bar{\mathcal{X}} := {}_z\mathcal{X} \cup \{ {}_y\mathcal{X} : \; y \in \mathbf{X}, \; {}_z\mathcal{X} \subset {}_y\mathcal{X}\}$$

is called the full orbit of the point $z \in \mathbf{X}$.

Clearly, ${}_z\bar{\mathcal{X}} = {}_y\bar{\mathcal{X}}$ for each point $y \in {}_z\bar{\mathcal{X}}$, and the set of all full orbits is just a stratification of the space $\mathbf{X}$.

If the flow ϕ is continuous, then one can easily show that each full orbit is measurable. We assume below that Condition 7.1 is satisfied, which is the usual case if the flow comes from an ordinary differential equation. But it seems that all the proofs can be adjusted for a more general case.

At the end of a full orbit ${}_z\bar{\mathcal{X}}$, there may be a cycle, i.e., 'cyclic' points x such that $\phi(x,t) = x$ for some $t > 0$. In this case ${}_z\bar{\mathcal{X}}^c$ is the part of ${}_z\bar{\mathcal{X}}$ excluding that cycle. Each set ${}_z\bar{\mathcal{X}}^c$ is measurable (and certainly may be empty).

Definition 7.3 *Suppose Condition 7.1 is satisfied and consider a particular full orbit* ${}_z\bar{\mathcal{X}}$.

(a) A point $x \in {}_z\bar{\mathcal{X}}^c$ *is called a predecessor of a point* $y \in {}_z\bar{\mathcal{X}}$*, if there is* $t > 0$ *such that* $y = \phi(x,t)$*. In this case, a (closed) arc* $[x,y]$ *is defined as* $\{\phi(x,u) : 0 \le u \le t\}$*. Here* $t = \inf\{\theta > 0 : y = \phi(x,\theta\}$ *in case the point* y *is cyclic. A particular point* $x \in {}_z\bar{\mathcal{X}}^c$ *is also an arc denoted as* $[x,x]$*. Open arcs, for the points* x *and* y *as above, are denoted as* $(x,y) := \{\phi(x,u) : 0 < u < t\}$*, including the case* $(x, "\infty") := \{\phi(x,u) : 0 < u < \infty\}$*;* $[x, "\infty"] := \{\phi(x,u) : 0 \le u < \infty\}$*. An arc* $[x,y]$ *is called finite if* $y \neq "\infty"$*.*

(b) If x *is a predecessor of* $y = \phi(x,t)$ *on the full orbit* ${}_z\bar{\mathcal{X}}^c$*, then* $D(x,y) := t$ *will be called the 'time-distance' between* x *and* y *(and also between* y *and* x*). The time-distance between* $x \in {}_z\bar{\mathcal{X}}^c$ *and* x *equals zero.*

(c) The 'basic' measure $\eta^b(dx)$ *on* ${}_z\bar{\mathcal{X}}^c$ *is defined by its values on the finite arcs* $[x,y] \subset {}_z\bar{\mathcal{X}}^c$*:*

$$\eta^b([x,y]) := D(x,y) = t, \text{ where } t \text{ is such that } y = \phi(x,t).$$

In other words, the basic measure on ${}_z\bar{\mathcal{X}}^c$ is the image of the Lebesgue measure subject to the transformation $t \to \phi(z,t)$. Here t can take negative values: $\phi(z,-t) = \tilde{z}$, if $\phi(\tilde{z},t) = z$ for $t > 0$ and for some $\tilde{z} \in \mathbf{X}$. Now for any arc $(x,y) \subset {}_z\bar{\mathcal{X}}$ (open or not), for a basic measure $\eta^b(dx)$, for any positive measurable function f

$$\int_{(x,y)} f(s)\eta^b(ds) = \int_{[0,t)} f(\phi(x,u))du, \tag{30}$$

where t is such that $y = \phi(x,t)$. In case the flow ϕ is continuous, for a sequence $x_i \in {}_z\bar{\mathcal{X}}^c$, $x_i \to x \in {}_z\bar{\mathcal{X}}^c$ if and only if t_i, the time-distance between x_i and x, approaches zero. The measure η^b is well defined on each set ${}_z\bar{\mathcal{X}}^c$ and is non-atomic.

Assume that Conditions 3.1, 3.2 and 3.3 are satisfied. According to Proposition 3.1, there exists an optimal deterministic stationary strategy f^* in problem (4). The occupation measure μ^{f^*} (under the fixed initial state $x_0 \in \mathbf{X}$) gives rise to the aggregated occupation measure η^{f^*} which is normal and exhibits the following obvious property: the measure $\eta^{f^*}(dx \times da)$ on $\mathbf{X} \times \mathbf{A}$ is a finite sum of Dirac measures concentrated at the points $\{(y_i, a_i)\}_{i=1}^I$, $I \ge 0$.

Note that $I \le \sup_{x\in\mathbf{X}} \int_{[0,\infty)} C^g(\phi(x,u))du/\delta$, where δ comes from Condition 3.3(b), and y_i cannot belong to a cycle because, as explained below Condition 3.3, staying in that cycle provides no future cost, while any impulse results in the positive cost.

Following this observation, without loss of generality, we supplement the primal linear program (27) with the following requirement:

$$\begin{aligned}&\text{the measure } \eta(dx \times da) \text{ on } \mathbf{X} \times \mathbf{A} \text{ is a finite sum of Dirac measures,}\\ &\text{concentrated on the finite set } \mathcal{Y} = \{(y_i, a_i)\}_{i=1}^I,\ I \ge 0.\end{aligned} \tag{31}$$

Now

$$\int_{\mathbf{X}\times\mathbf{A}_\square}\mathcal{C}(x,a)\eta(dx\times da)=\int_{\mathbf{X}}C^g(x)\eta(dx\times\square)+\sum_{i=1}^{I}C^I(y_i,a_i)=\int_{\mathbf{X}}C^g(x)\eta(dx\times\square)+\sum_{(y,a)\in\mathcal{Y}}C^I(y,a).$$

Roughly speaking, the optimal measure $\eta^*(dx\times da)$ in the program (27), (31), being decomposed as $\eta^*(dy\times\mathbf{A})\pi^*(da|y)$, will define the optimal feedback which will be deterministic: $\pi^*(da|y)=\delta_{\hat{f}_a(y)}(da)$. As soon as an 'active' point y is reached, the impulse $\hat{f}_a(y)$ should be applied. (a is not an argument, just the notation of the function $\hat{f}_a(\cdot)$.) The set of active points $\hat{Y}^*$ is finite and also comes from the measure $\eta^*(dx\times da)$. The rigorous statement is as follows.

Theorem 7.1 *Suppose Conditions 3.1, 3.2, 3.3 and 7.1 are satisfied. Then the following statements are valid.*

(a) The modified primal linear program (27), (31) is solvable.

(b) For any solution η^* with the measure $\eta^*(dx\times da)$ being concentrated on the set $\mathcal{Y}^*:=\{(y_i,a_i)\}_{i=1}^I$, denote $\hat{Y}^*:=\{y_i\}_{i=1}^I$. Then all the points y_i are non-cyclic, and the deterministic stationary strategy g^*, defined by

$$\begin{aligned}&g^*_\theta(x):=\inf\{\theta:\ \phi(x,\theta)\in\hat{Y}^*\}\in\bar{\mathbb{R}}^0_+\ \ \textit{(the infimum, if smaller than } \infty\textit{, is attained!)};\\&g^*_a(x)\ \ \textit{is such that } (\phi(x,g^*_\theta(x)),g^*_a(x))\in\mathcal{Y}^*,\\&\qquad\textit{(or, in case } g^*_\theta(x)=\infty,\ g^*_a(x)=\hat{a}\ \textit{with the immaterial value of } \hat{a}\in\mathbf{A}),\end{aligned}$$

is optimal in problem (4).

(c) The optimal values of the modified primal linear program (27), (31) and of the initial problem (4) coincide.

Before proceeding to the proof, we need some definitions and preliminary observations.

Let us explain why the mappings g^*_θ and g^*_a are measurable, if the flow ϕ is continuous and the sets $\mathcal{Y}^*$, $\hat{Y}^*$ are finite. It is sufficient to consider a particular full orbit ${}_z\bar{\mathcal{X}}^c$. If $Y:=\hat{Y}^*\cap{}_z\bar{\mathcal{X}}^c$, then $g^*_\theta(\cdot)$ equals the minimum over all $y\in Y$ of the following functions (if $Y=\emptyset$, then $g^*_\theta(x)\equiv+\infty$):

$$g_y(x):=\begin{cases}D(x,y) & \text{if } x \text{ is a predecessor of } y \text{ (see Definition 7.3)};\\ 0, & \text{if } x=y;\\ \infty & \text{otherwise,}\end{cases}$$

which exhibit the following properties:

(i) if $x_i\to x$ and x is a predecessor of y, then $g_y(x_i)\to g_y(x)$;

(ii) if $x_i\to y$, then $g_y(y)=0\le\overline{\lim}_{i\to\infty}g_y(x_i)$;

(iii) if $x_i\to x$ and y is a predecessor of x, then $g_y(x)=\lim_{i\to\infty}g_y(x_i)=\infty$.

Thus, each function $g_y(\cdot)$ and hence $g^*_\theta(\cdot)$ is lower semicontinuous. For each a_i, the set $\{x\in{}_z\bar{\mathcal{X}}^c:\ g^*_a(x)=a_i\}$ is the finite union of measurable parts of the full orbit ${}_z\bar{\mathcal{X}}^c$ of the form

$$\{y'\}\cup\{x:\ x \text{ is a predecessor of } y'\}$$

and of the form

$$\{y'\}\cup\{x:\ x \text{ if a predecessor of } y' \text{ and } y\text{" is a predecessor of } x\}$$

with $y', y" \in \hat{Y}^*$.

Suppose a normal measure η on $\mathbf{X} \times \mathbf{A}_\square$ is admissible in the modified linear program (27), (31). Then $\hat{Y} := \{y_1, y_2, \dots, y_I\}$ is the set of active points; the corresponding actions (impulses) a_i are also known. We calculate $x_i := l(y_i, a_i)$, $i = 1, 2, \dots, I$ and denote $\hat{X} := \{x_0\} \cup \{x_i\}_{i=1}^I$. Below, $\hat{X}_c$ ($\hat{Y}_c$) is the collection of all the points from $\hat{X}$ ($\hat{Y}$), which do not belong to any cycles; similarly, $\mathcal{Y}_c := \{(y_i, a_i) : \ y_i \in \hat{Y}_c\}$. Note, there may be several identical points in $\hat{X}$ and in $\hat{X}_c$, but all points in $\hat{Y}$ are different.

Lemma 7.1 *Suppose the flow ϕ is continuous, Condition 7.1 is satisfied, a normal measure η satisfies conditions (26) and (31), and the sets $\hat{X}_c, \hat{Y}_c$ are as described above. Then, on any one full orbit ${}_z\bar{\mathcal{X}}^c$, the measure $\eta(dx \times \square)$ exhibits the following properties.*

(a) If ${}_z\bar{\mathcal{X}}^c \cap \hat{X}_c = \emptyset$, then $\eta({}_z\bar{\mathcal{X}}^c \times \square) = 0$ and ${}_z\bar{\mathcal{X}}^c \cap \hat{Y}_c = \emptyset$.

(b) For any finite arc $[x, y]$ in the sense of Definition 7.3(a) with $x \neq y$, the set $[x, y] \cap {}_z\bar{\mathcal{X}}^c$ will be called the 'elementary' arc, if it does not contain points from $\hat{X}_c \cup \hat{Y}_c$. Now, the value of the measure $\eta(dx \times \square)$ of each elementary arc coincides with the value of the measure $\bar{k}\eta^b(dx)$, where the basic measure η^b is as in Definition 7.3(c) and $\bar{k} := k_X - k_Y$; k_X (k_Y) is the number of points from $\hat{X}_c$ (from $\hat{Y}_c$) which are the predecessors of x. The case $\bar{k} < 0$ contradicts equation (26).

Of course, the value of $\bar{k}$ depends on the elementary arc, but we do not indicate that dependence for brevity. Remember, $\bar{k}$ is bounded by the total number of points in $\hat{X}_c$. Roughly speaking, each predecessor of a point x, belonging to $\hat{X}_c$ (to $\hat{Y}_c$) increases (decreases) the measure $\eta(dx \times \square)$.

Definition 7.4 *Suppose Condition 7.1 is satisfied and $\mathcal{Y} = \{(y_i, a_i)\}_{i=1}^I \subset \mathbf{X} \times \mathbf{A}$ is such that all points y_i are different. Denote $\hat{Y} := \{y_i\}_{i=1}^I$, $\hat{X} := \{x_0\} \cup \{l(y_i, a_i)\}_{i=1}^I$ and let $\hat{X}_c$ ($\hat{Y}_c$) be the set of all the points from $\hat{X}$ ($\hat{Y}$) which are not cyclic. If $\bar{k}$ is positive for each elementary arc in any full orbit ${}_z\bar{\mathcal{X}}^c$ with $z \in \hat{X}_c$, then we say that the system $\mathcal{Y}$ is consistent and a non-atomic measure $\tilde{\eta}(dx)$ on $\mathbf{X}_c^{\mathcal{Y}} := \bigcup_{z \in \hat{X}_c} {}_z\bar{\mathcal{X}}^c$ is induced by $\mathcal{Y}$ if it coincides with $\bar{k}\eta^b(dx)$ on all elementary arcs. The induced measure is always complemented by the zero measure on each full orbit ${}_z\bar{\mathcal{X}}^c$ which does not contain points from $\hat{X}_c$.*

Note that any non-atomic measure on ${}_z\bar{\mathcal{X}}^c$ is uniquely defined by its values on the elementary arcs.

According to Lemma 7.1, for any normal measure η on $\mathbf{X} \times \mathbf{A}_\square$, satisfying conditions (26) and (31), the system $\mathcal{Y}$ is consistent, and the measure $\eta(dx \times \square)$ on $\mathbf{X}$ is induced by $\mathcal{Y}$.

If $\mathcal{Y}$ is a consistent system, then the induced measure $\tilde{\eta}$ is undefined on the set of cyclic points (which may be non-measurable). But the integrals $\int_{\mathbf{X}} C^g(x)\tilde{\eta}(dx)$ and $\int_{\mathbf{X}} \chi W(x)\tilde{\eta}(dx)$ for $W \in \mathcal{W}_2$ are well defined because there is a version of $\chi W(\cdot)$ such that $\chi W(x) = 0$ and $C^g(x) = 0$ if the point x is cyclic. That is why, with some abuse of notation, we say that each consistent system $\mathcal{Y}$ defines the induced measure $\tilde{\eta}$ on $\mathbf{X}$. In particular, for any normal measure η on $\mathbf{X} \times \mathbf{A}_\square$, satisfying conditions (26) and (31), the measure $\eta(dx \times \square)$ is induced by the corresponding system $\mathcal{Y}$. Similarly, the expression 'the induced by $\mathcal{Y}_1$ measure $\tilde{\eta}_1$ is set-wise smaller/equal/bigger than the induced by $\mathcal{Y}_2$ measure $\tilde{\eta}_2$' only corresponds to the measurable subsets of the set $\mathbf{X} \setminus \{\text{cyclic points}\}$.

Remark 7.1 *One can define in the similar way the consistent system $(\hat{X}_c, \hat{Y}_c)$, where $\hat{X}_c$ and $\hat{Y}_c$ are the sets of non-cyclic points from $\mathbf{X}$, and introduce the induced non-atomic measure on $\bigcup_{z \in \hat{X}_c} {}_z\bar{\mathcal{X}}^c$, equal to $\bar{k}\eta^b(dx)$ on all the elementary arcs. In this connection, if $(\hat{X}_c^1, \hat{Y}_c^1)$ and $(\hat{X}_c^2, \hat{Y}_c^2)$ are two consistent systems, then the system $(\hat{X}_c^1 \cup \hat{X}_c^2, \hat{Y}_c^1 \cup \hat{Y}_c^2)$ is consistent, and the measure, induced by it, is set-wise bigger than the measure, induced by $(\hat{X}_c^1, \hat{Y}_c^1)$, and than the measure, induced by $(\hat{X}_c^2, \hat{Y}_c^2)$.*

Proof of Theorem 7.1. We assume that the point x_0 is not cyclic. Otherwise, the situation is trivial: the only optimal solution to the modified linear program (27), (31) corresponds to the case when $I = 0$ in (31), and the optimal values of the program (27), (31) and of the problem (4) equal zero.

According to Proposition 3.1, there exists an optimal deterministic stationary strategy f^* in problem (4). It gives rise to the aggregated occupation measure η^{f^*}, which is normal and satisfies conditions (26) and (31). Suppose a normal measure η satisfies conditions (26) and (31) and

$$\int_{\mathbf{X}\times\mathbf{A}_\square} \mathcal{C}(x,a)\eta(dx\times da) \le \int_{\mathbf{X}\times\mathbf{A}_\square} \mathcal{C}(x,a)\eta^{f^*}(dx\times da). \tag{32}$$

We will show that this inequality cannot be strict.

Let $\hat{X}, \hat{Y}, \hat{X}_c, \hat{Y}_c$ and $\mathcal{Y}_c$ be as described above Lemma 7.1. According to Lemma 7.1, the system $\mathcal{Y}$, associated with η, is consistent, and the measure $\eta(dx\times\square)$ is induced by $\mathcal{Y}$. We construct consecutively the new systems $\mathcal{Y}_c^j \subset \mathcal{Y}_c$, $j = 0, 1, \ldots$, according to the following algorithm.

(0) $\mathcal{Y}_c^0 := \emptyset$, $\hat{Y}_c^0 := \emptyset$, $j := 0$.

(1) $\theta_{j+1} := \inf\{\theta \in \mathbb{R}_+^0 :\ \phi(x_j,\theta) \in \hat{Y}_c \setminus \hat{Y}_c^j\}$.

(2) If $\theta_{j+1} = \infty$, then stop.

(3) Otherwise, calculate $y_{j+1} := \phi(x_j, \theta_{j+1})$.

(4) If the point y_{j+1} is cyclic, then stop.

(5) Otherwise, let a_{j+1} be the point associated with y_{j+1} in the system $\mathcal{Y}_c$; put $\hat{Y}_c^{j+1} := \hat{Y}_c^j \cup \{y_{j+1}\}$, $\mathcal{Y}_c^{j+1} := \mathcal{Y}_c^j \cup \{(y_{j+1}, a_{j+1})\}$, calculate $x_{j+1} := l(y_{j+1}, a_{j+1})$ and increase j by one.

(6) If the point x_j is cyclic, then stop.

(7) Otherwise, go to step (1).

The final value of j, denoted as J, equals the number of points in the last sets $\hat{Y}_c^J$ and $\mathcal{Y}_c^J$. The number of points in the set $\hat{X}_c^J$, coming from $\mathcal{Y}_c^J$ and including x_0, can equal J, if the algorithm terminated at step (6), or $J + 1$, if the algorithm terminated at step (2) or (4). On each step $j = 0, 1, \ldots, J$, the system $\mathcal{Y}_c^j$ is consistent and all the points $\{y_j\}_{j=1}^J$ are non-cyclic.

The possible results of the presented algorithm look like on Fig.1, and it will be shown that the algorithm cannot terminate on step (4).

Let us explain, why the induced by $\mathcal{Y}_c^J$ (equivalently, induced by $(\hat{Y}_c^J, \hat{Y}_c^J)$) measure $\tilde{\eta}(dx)$ is setwise smaller than $\eta(dx\times\square)$. As mentioned above, the measure $\eta(dx\times\square)$ is induced by the system $\mathcal{Y}$ which is consistent (equivalently, induced by the system $(\hat{X}_c, \hat{Y}_c)$ which is consistent as well: see Remark 7.1). Consider the system $\tilde{\mathcal{Y}} := \mathcal{Y} \setminus \mathcal{Y}_c^J$, the corresponding sets $\tilde{X} := \{l(y,a) :\ (y,a) \in \tilde{\mathcal{Y}}\}$, $\tilde{Y} := \{y :\ (y,a) \in \tilde{\mathcal{Y}}\}$ and $\tilde{X}_c, \tilde{Y}_c$ (the sets of points in $\tilde{X}, \tilde{Y}$, excluding all cyclic points). Clearly, $\tilde{X}_c = \hat{X}_c \setminus \hat{X}_c^J$, $\tilde{Y}_c = \hat{Y}_c \setminus \hat{Y}_c^J$.

Firstly, we show that the system $(\tilde{X}_c, \tilde{Y}_c)$ is consistent. Assume for contradiction that there is an elementary arc $[x,y] \cap {}_z\bar{\mathcal{X}}^c$ with $\bar{k} < 0$ for the system $(\tilde{X}_c, \tilde{Y}_c)$. If $\hat{X}_c^J \cap {}_z\bar{\mathcal{X}}^c = \emptyset$, then $\hat{Y}_c^J \cap {}_z\bar{\mathcal{X}}^c = \emptyset$ and the original system $(\hat{X}_c, \hat{Y}_c)$ is not consistent. Suppose $\hat{X}_c^J \cap {}_z\bar{\mathcal{X}}^c \ne \emptyset$ and let $x^1, y^1, x^2, y^2, \ldots, x^L, y^L$ be the full ordered list of the points on ${}_z\bar{\mathcal{X}}^c$ generated by the algorithm. If the algorithm terminated on step (2) or (4), then $y^L :=$ "∞" in case $x^L = x_J$. Now we have the collection of arcs on ${}_z\bar{\mathcal{X}}^c$

$$[x^1, y^1],\ [x^2, y^2], \ldots, [x^L, y^L]. \tag{33}$$

(We are not sure at the moment that these arcs are disjoint.) All the points from $\tilde{Y}_c \cap {}_z\bar{\mathcal{X}}^c$ can be ordered: $\tilde{y}_{i+1} = \phi(\tilde{y}_i, \tau_i)$ with $\tau_i > 0$. Suppose $\tilde{y}$ is the last one among the predecessors of x: $\tilde{y}$ exists

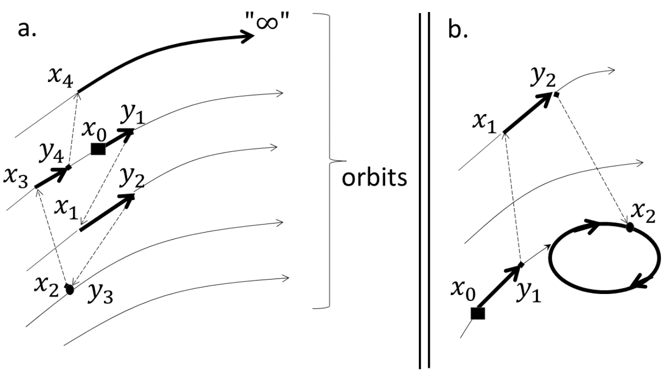

Figure 1: In case a, $J = 4$ and the algorithm terminated on step (2); here $\theta_3 = 0$. In case b, $J = 2$ and the algorithm terminated on step (6).

because $\bar{k} < 0$. The point $\tilde{y}$ cannot belong to any of the introduced arcs $[x^l, y^l]$, $l = 1, 2, \ldots, L$. Thus, it belongs to one of the open arcs (y^l, x^{l+1}), $1 \le l < L$, or to $(y^L, \text{"}\infty\text{"})$ if $y^L \ne \text{"}\infty\text{"}$. One can build a (small enough) elementary arc $[\tilde{x} := \phi(\tilde{y}, \varepsilon_1), \phi(\tilde{y}, \varepsilon_1 + \varepsilon_2)] \subset (y^l, x^{l+1})$ with $\varepsilon_1, \varepsilon_2 > 0$, which does not contain any points from $\tilde{Y}_c$. (Recall that all points in $\tilde{Y}_c$ are different.) Let k_X (k_Y) be the number of predecessors of $\tilde{x}$ from the set $\hat{X}_c^J$ ($\hat{Y}_c^J$) and $\tilde{k}_X$ ($\tilde{k}_Y$) be the number of predecessors of $\tilde{x}$ from the set $\tilde{X}_c$ ($\tilde{Y}_c$). Since $k_X = k_Y$ and $\tilde{k}_X < \tilde{k}_Y$ (because initially we assumed that $\bar{k} < 0$ for $[x, y] \cap {}_z\bar{\mathcal{X}}^c$), the total value of $\bar{k}$ for $\tilde{x}$, calculated using the full original system $(\hat{X}_c, \hat{Y}_c)$, equals $k_X + \tilde{k}_X - k_Y - \tilde{k}_Y$ and is negative, which contradicts the fact that $(\hat{X}_c, \hat{Y}_c)$ is consistent.

According to Remark 7.1, the induced by $\mathcal{Y}_c^J$ (equivalently, by $(\hat{X}_c^J, \hat{Y}_c^J)$) measure $\tilde{\eta}(dx)$ is set-wise smaller than $\eta(dx \times \Box)$, the measure induced by $\mathcal{Y}$ (equivalently, by $(\hat{X}_c, \hat{Y}_c)$). Therefore, if $\mathcal{Y}_c^J$ is the proper subset of $\mathcal{Y}$, then

$$\int_{\mathbf{X}} C^g(x)\tilde{\eta}(dx) + \sum_{(y,a)\in\mathcal{Y}_c^J} C^I(y,a) < \int_{\mathbf{X}} C^g(x)\eta(dx \times \Box) + \sum_{(y,a)\in\mathcal{Y}} C^I(y,a) = \int_{\mathbf{X}\times\mathbf{A}_\Box} \mathcal{C}(x,a)\eta(dx \times da): \tag{34}$$

recall that $C^I(y,a) \ge \delta > 0$. Clearly, we have equality here, if $\mathcal{Y}_c^J = \mathcal{Y}$.

Suppose the arcs (33) on a full orbit ${}_z\bar{\mathcal{X}}^c$ intersect: $[x^{l_1}, y^{l_1}] \cap [x^{l_2}, y^{l_2}] \ne \emptyset$, where $l_1 < l_2$ and $y^{l_2} \ne \text{"}\infty\text{"}$. According to the algorithm (0)–(7), y^{l_1} is a predecessor of y^{l_2} and

$$[x^{l_1}, y^{l_1}] \cap [x^{l_2}, y^{l_2}] = \begin{cases} \text{either} & [x^{l_1}, y^{l_1}], \\ \text{or} & [x^{l_2}, y^{l_1}]. \end{cases}$$

We have a loop in the algorithm: the point y^{l_1} appeared again at a later step as the point on the arc $[x^{l_2}, y^{l_2}]$. Let $j+1$ be the number of the point y^{l_1}, as it was enumerated on step (3) of the algorithm. We eliminate the loop by deleting the points $(y^{l_1} = y_{j+1}, a_{j+1}), (y_{j+2}, a_{j+2}), \ldots, (y_{j+K}, a_{j+K})$, up to and excluding the end of the loop y^{l_2}, from the system $\mathcal{Y}$. Simultaneously, the points $x_{j+1}, x_{j+2}, \ldots, x_{j+K} = x^{l_2}$ also disappear. The algorithm will provide the shorter sequence of points (J decreases), and in (33) we will have

$$[x^1, y^1],\ [x^2, y^2], \ldots, [x^{l_1}, y^{l_2}], \ldots$$

instead ot

$$[x^1, y^1],\ [x^2, y^2], \ldots, [x^{l_1}, y^{l_1}], [x^{l_2}, y^{l_2}] \ldots.$$

In the case $y^{l_2} = \text{"}\infty\text{"}$, we do the same and obtain the sequence

$$[x^1, y^1],\ [x^2, y^2], \ldots, [x^{l_1}, \text{"}\infty\text{"}].$$

After we eliminate all the loops in this way, we finish with the new system $\mathfrak{Y} \subset \mathcal{Y}_c^J$, which is also consistent. The measure Υ, induced by $\mathfrak{Y}$, is set-wise smaller than $\tilde{\eta}$ – the measure, induced by $\mathcal{Y}_c^J$. Therefore, if at least two arcs as in (33) intersect, then

$$\int_{\mathbf{X}} C^g(x)\Upsilon(dx \times \Box) + \sum_{(y,a)\in\mathfrak{Y}} C^I(y,a) < \int_{\mathbf{X}} C^g(x)\tilde{\eta}(dx) + \sum_{(y,a)\in\mathcal{Y}_c^J} C^I(y,a). \tag{35}$$

We have equality here if all arcs in (33) are disjoint (there are no loops) and $\mathfrak{Y} = \mathcal{Y}_c^J$.

Consider the deterministic stationary strategy g as in the statement (b) of Theorem 7.1, coming from the system $\mathfrak{Y}$. The corresponding aggregated occupation measure η^g is such that

$$\eta^g(dx \times \Box) = \Upsilon(dx), \quad \eta^g(dx \times da) = \sum_{(y_i,a_i)\in\mathfrak{Y}} \delta_{(y_i,a_i)}(dx \times da).$$

Since the strategy f^* is optimal in problem (4), we have the following relations, based on the inequalities (35),(34) and (32):

$$\begin{aligned} \mathcal{V}(x_0,f^*) &\le \mathcal{V}(x_0,g) = \int_{\mathbf{X}\times\mathbf{A}_\Box} \mathcal{C}(x,a)\eta^g(dx \times da) \le \int_{\mathbf{X}\times\mathbf{A}_\Box} \mathcal{C}(x,a)\eta(dx \times da) \\ &\le \int_{\mathbf{X}\times\mathbf{A}_\Box} \mathcal{C}(x,a)\eta^{f^*}(dx \times da) = \mathcal{V}(x_0,f^*). \end{aligned} \tag{36}$$

The equalities are valid according to Remark 3.2.

We have shown that strict inequalities in (34) and (35) are excluded, i.e., $\mathfrak{Y} = \mathcal{Y}_c^J = \mathcal{Y}$. Moreover, the strict inequality in (32) is also excluded, i.e., η^{f^*} is a solution to the modified primal linear program (27), (31).

If η^* is any solution to the program (27), (31), then it transforms (32) to equality and, for the described in part (b) deterministic stationary strategy g^*, we have

$$\mathcal{V}(x_0,g^*) = \int_{\mathbf{X}\times\mathbf{A}_\Box} \mathcal{C}(x,a)\eta^*(dx \times da) = \int_{\mathbf{X}\times\mathbf{A}_\Box} \mathcal{C}(x,a)\eta^{f^*}(dx \times da) = \mathcal{V}(x_0,f^*).$$

All the statements (a),(b) and (c) are proved. $\Box$

8 Computational Methods

Although numerical algorithms are beyond the scope of the present paper, we give some relevant comments.

First of all, under Conditions 3.1, and 3.2 the integral Bellman equation (7) can be solved by iterations

$$W_{n+1}(x) = \inf_{(\theta,a)\in\mathbb{R}_+^0\times\mathbf{A}} \left\{ C(x,(\theta,a)) + \int_{\mathbf{X}_\Delta} W_n(y)Q(dy|x,(\theta,a)) \right\}, \quad \forall x \in \mathbf{X}$$

starting from $W_0(x) \equiv 0$: $W_n(\cdot) \uparrow \mathcal{V}^*(\cdot)$. For details, see [18, Thm.1]. The optimal strategy is as in Proposition 3.1. Note, this method gives the optimal strategy for all possible initial states $x_0 \in \mathbf{X}$.

In case the initial state $x_0 \in \mathbf{X}$ is fixed, working with primal linear programs is computationally advantageous. Assume that Conditions 3.1, 3.2, 3.3 and 7.1 are satisfied and the initial state $x_0 \in \mathbf{X}$ is fixed. Then the total number of finite intervals θ_i (i.e. the total number of impulses $a_i \in \mathbf{A}$) up to the absorption at the state Δ is not greater than

$$\mathbf{I} := \left\lfloor \frac{\int_{[0,\infty)} C^g(\phi(x,u))du}{\delta} \right\rfloor \quad \text{(integer part of the ratio)} .$$

Below we explain how the linear program (27),(31) can be solved numerically.

The reasonable measures $\eta(dx \times da)$ as in (31) can have $I \leq \mathbf{I}$ unit atoms at points from

$$\mathcal{Y} = \{(y_1 = \phi(x_0, \theta_1), a_1),\ (y_2 = \phi(l(y_1, a_1), \theta_2), a_2), \ldots, (y_I = \phi(l(y_{I-1}, a_{I-1}), \theta_I), a_I)\}.$$

Due to Theorem 7.1, one can require that all the points from $\hat{Y} = \{y_i\}_{i=1}^I$ are not cyclic. According to the notations introduced above Lemma 7.1, $x_i = l(y_i, a_i)$, $i = 1, 2, \ldots, I$, and all the points $x_0, x_1, \ldots, x_{I-1}$ are not cyclic, too.

After these observations, one can try $I = 0, 1, \ldots, \mathbf{I}$ and minimize the objective (27) which has the form

$$O(\{\theta_i, a_i\}_{i=1}^I) := \sum_{i=1}^{I} C^I(y_i, a_i) + \int_{\mathbf{X}} C^g(x)\eta(dx \times \Box) \longrightarrow \inf_{\{\theta_i, a_i\}_{i=1}^I}. \tag{37}$$

The sequence $\{\theta_i, a_i\}_{i=1}^I$ defines the sets $\hat{Y} = \{y_i\}_{i=1}^I$ and $\mathcal{Y} = \{(y_i, a_i)\}_{i=1}^I$ as described above, and

$$\int_{\mathbf{X}} C^g(x)\eta(dx \times \Box) = \int_{(x_0, y_1)} C^g(\eta(dx \times \Box) + \int_{(x_1, y_2)} C^g(\eta(dx \times \Box) + \ldots + \int_{(x_I, \text{"}\infty\text{"})} C^g(\eta(dx \times \Box)$$

due to Lemma 7.1(a). Here (x_{i-1}, y_i) are arcs in the full orbits ${}_{x_{i-1}}\bar{\mathcal{X}}$: see Definitions 7.1 and 7.2. One can take open arcs because the measure $\eta(dx \times \Box)$ is normal, hence $\eta(\{z\} \times \Box) = 0$ for all singletons $\{z\}$ with non-stationary z. Note that only x_I can be stationary, but in this case $C^g(x_I) = 0$ and the last term in the formula above vanishes. According to Lemma 7.1(b), if, e.g., an arc (x_{i-1}, y_i) does not overlap with all other arcs, then

$$\int_{(x_{i-1}, y_i)} C^g(\eta(dx \times \Box) = \int_{[0, \theta_i)} C^g(\phi(x_{i-1}, t))dt$$

because the measure $\eta(dx \times \Box) = \eta^b(dx)$ is basic in the sense of Definition 7.3(c): see (30). Here $\bar{k} = 1$. In general, the arcs can overlap leading to $\bar{k} > 1$ for some elementary arcs, but, in any case,

$$\int_{\mathbf{X}} C^g(x)\eta(dx \times \Box) = \sum_{i=1}^{I+1} \int_{[0, \theta_i)} C^g(\phi(x_{i-1}, t))dt,$$

where $\theta_{I+1} = \infty$. Note that the measure

$$\mu(dx \times d\theta \times da) := \sum_{i=1}^{I} \delta_{x_{i-1}}(dx)\delta_{\theta_i}(d\theta)\delta_{a_i}(da) + \delta_{X_I}(dx)\delta_\infty(d\theta)\delta_{a_{I+1}}(da)$$

satisfies equation (9). (The value of a_{I+1} is of no importance.) The objective (37) coincides with the objective (10) meaning that, after solving the linear program (27), we also obtain the solution to the linear program (10). Quite formally, the program (27) is less dimensional than (10) because the component θ is absent.

Now, under the introduced conditions, the objective (37) is a continuous function of the controls $\{(\theta_i, a_i)\}_{i=1}^I$ and, e.g., in case $\mathbf{A} \subset \mathbb{R}^k$ with the Euclidean topology, the problem (37) is a smooth finite-dimensional minimization problem for which standard numerical methods can be applied. After it is solved for all $I = 0, 1, \ldots, \mathbf{I}$, one has in hand the optimal I^* and $\{\theta_i^*, a_i^*\}_{i=1}^{I^*}$ providing the absolute infimum

$$\inf_{I \in \{0, 1, \ldots, \mathbf{I}\}, \{\theta_i, a_i\}_{i=1}^I} O(\{\theta_i, a_i\}_{i=1}^I).$$

The sequence $\{\theta_i^*, a_i^*\}_{i=1}^{I^*}$ defines the set $\mathcal{Y}^* = \{(y_i, a_i)\}_{i=1}^{I^*}$, and the optimal control strategy can be built as is shown in Theorem 7.1(b). The sequence

$$(\theta_1^*, a_1^*), (\theta_2^*, a_2^*), \ldots, (\theta_{I^*}^*, a_{I^*}^*), (+\infty, a_{I^*+1}), \ldots$$

defines the optimal dynamics as in (1). The values of $a_{I^*+1}, a_{I^*+2}, \ldots$ and $\theta_{I^*+2}, \theta_{I^*+3}, \ldots$ are of no importance.

9 Example

The main goal of this section is to illustrate the introduced concepts including the form of the measures μ^* and η^* and the Bellman function $\mathcal{V}^*(\cdot)$.

Consider the susceptible-infected-removed (SIR) model of epidemics described by the following differential equations

$$\begin{cases} \dot{x}(t) = -\beta \dfrac{x(t)y(t)}{x(t)+y(t)}; \\ \dot{y}(t) = \beta \dfrac{x(t)y(t)}{x(t)+y(t)} - \gamma y(t). \end{cases} \tag{38}$$

Here $x(t) > 0$ and $y(t) > 0$ denote the numbers of susceptible and infective individuals in the closed population of the size N at time t; the initial values $x(0) = x_0$, $y(0) = y_0$ are fixed; $\beta, \gamma > 0$. The number of removed (dead or recovered with immunity) individuals at time t equals $N - x(t) - y(t)$. Similar models were considered in [3, 17, 18]. The flow ϕ on the space $\{(x,y):\ x>0,\ y>0,\ x+y \le N\}$ comes from (38). At any moment, the decision maker can isolate all infectives, so that $\mathbf{A} = \{1\}$ is a singleton. As the result, the epidemic terminates, that is, the system moves to the isolated absorbing cemetery $\blacksquare$ with no future cost. The state space is

$$\mathbf{X} = \{\vec{x} = (x,y):\ x > 0,\ y > 0,\ x+y \le N\} \cup \{\blacksquare\};$$

the topology in $\mathbf{X} \cap \mathbb{R}^2$ is just the trace of the standard Euclidean topology. The generic elements of $\mathbf{X}$ are denoted as $\vec{x}$; $\phi(\blacksquare, t) \equiv \blacksquare$ and $l(\vec{x}, a) \equiv \blacksquare$ for all $\vec{x} \in \mathbf{X}$. The costless cemeteries Δ and $\blacksquare$ have different meaning: $\blacksquare$ means that the process was stopped at a finite time moment, and Δ is the fictitious state meaning that no impulses will be applied, and the process will never be stopped.

The gradual cost rate is the infection rate

$$C^g(x,y) = \beta \frac{xy}{x+y} \ \text{ for } (x,y) \in \mathbb{R}^2; \qquad C^g(\blacksquare) = 0,$$

which, after integration along the flow, results in the total number of new infectives. Here and below we omit one pair of brackets in the expressions like $C^g(\vec{x}) = C^g((x,y))$ for $\vec{x} \in \mathbb{R}^2$. The cost function associated with impulses, also called 'interventions', equals

$$C^I(x,y,a) = \delta + cy \ \text{ for } \vec{x} \in \mathbb{R}^2,$$

where $\delta > 0$ is the cost of the initialization of the isolation process, and $c > 0$ is the cost of the isolation of one unit of infectives. For consistency, we put $C^I(\blacksquare, a) = \delta$.

The complete solution to this optimal impulse control problem can be found in [17], where it was allowed to isolate any number of existing infectives. But the optimal solution prescribes at any moment either do nothing, or isolate ALL infectives immediately.

Below, we consider the most interesting case $\beta < \gamma$, $c < \frac{\beta}{\gamma - \beta}$. If there are no impulses, then

$$x(t) = x_0 \frac{\left(1 + \frac{y_0}{x_0} e^{-(\gamma-\beta)t}\right)^{\frac{\beta}{\gamma-\beta}}}{\left(1+\frac{y_0}{x_0}\right)^{\frac{\beta}{\gamma-\beta}}}; \qquad y(t) = y_0 \frac{\left(1 + \frac{y_0}{x_0} e^{-(\gamma-\beta)t}\right)^{\frac{\beta}{\gamma-\beta}}}{\left(1+\frac{y_0}{x_0}\right)^{\frac{\beta}{\gamma-\beta}} e^{(\gamma-\beta)t}}; \tag{39}$$

$$\lim_{t\to\infty} y(t) = 0; \quad \lim_{t\to\infty} x(t) = x_0 \left(\frac{x_0}{x_0+y_0}\right)^{\beta/(\gamma-\beta)}; \quad w(t) := \frac{y(t)}{x(t)} = \frac{y_0}{x_0} e^{-(\gamma-\beta)t}.$$

Conditions 3.1, 3.2 and 3.3 are satisfied:

$$\int_{[0,\infty)} C^g(\phi(\vec{x},u))du = \begin{cases} x\left(1-\left(\frac{x}{x+y}\right)^{\beta/(\gamma-\beta)}\right), & \text{if } \vec{x}=(x,y)\in\mathbb{R}^2; \\ 0, & \text{if } \vec{x}=\blacksquare \end{cases} \leq N.$$

The optimal control strategy and the shape of the function $\mathcal{V}^*(\cdot)$ are illustrated on Fig.2.

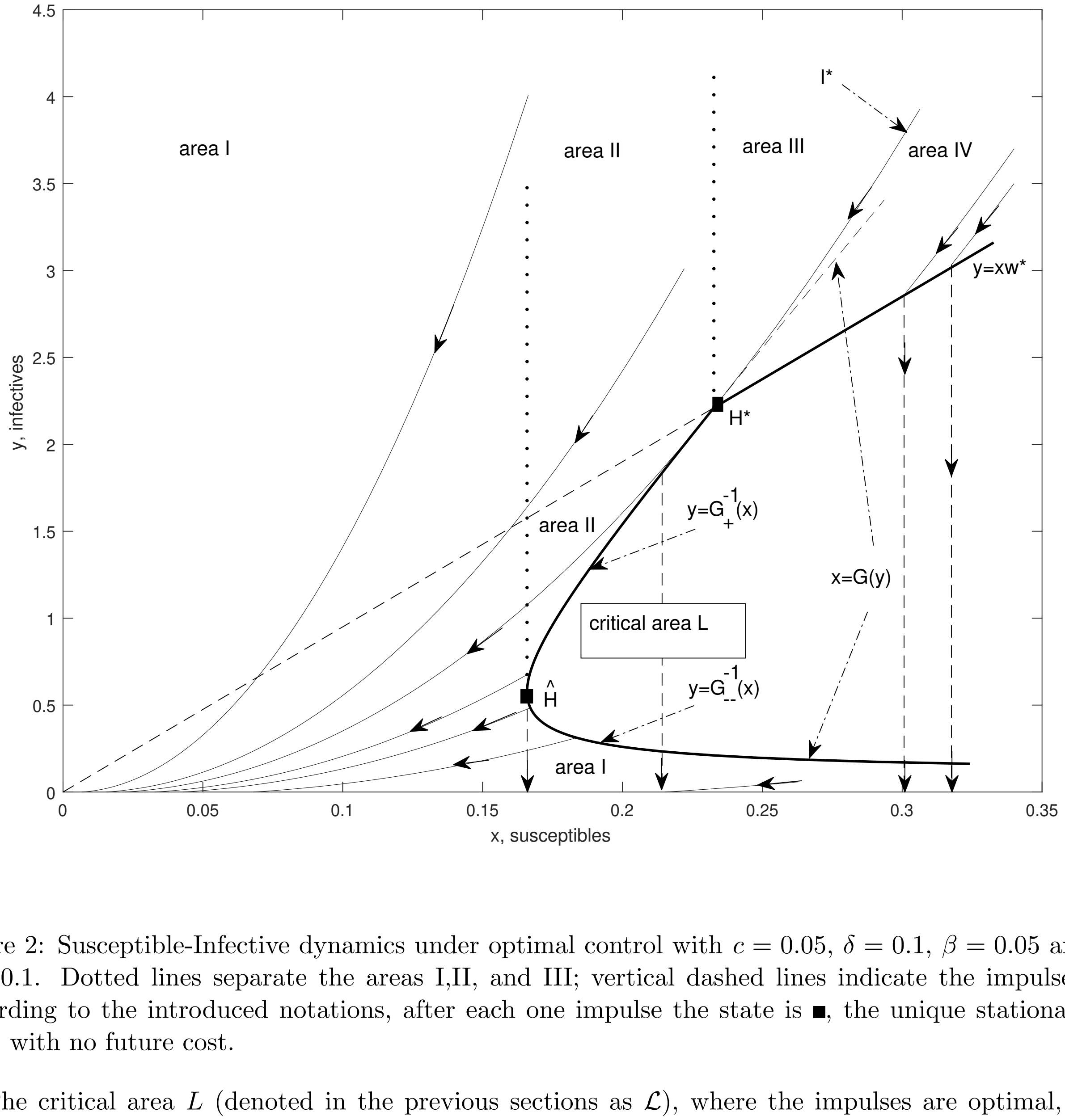

Figure 2: Susceptible-Infective dynamics under optimal control with $c = 0.05$, $\delta = 0.1$, $\beta = 0.05$ and $\gamma = 0.1$. Dotted lines separate the areas I,II, and III; vertical dashed lines indicate the impulses. According to the introduced notations, after each one impulse the state is $\blacksquare$, the unique stationary point with no future cost.

The critical area L (denoted in the previous sections as $\mathcal{L}$), where the impulses are optimal, is defined as follows:

$$L := \left\{(x,y):\ x>0, y>0,\ x+y<N,\ x\geq \max\left\{G(y),\ \frac{y}{w^*}\right\}\right\},$$

where $w^* := \frac{\beta+c\beta-c\gamma}{c\gamma}$ and $G(y)$ is the solution to the equation

$$\delta + cy = x\left[1 - \left(1 + \frac{y}{x}\right)^{-\frac{\beta}{\gamma-\beta}}\right]$$

with respect to x. The points $\hat{H}(\hat{x},\hat{y})$ and $H^*(x^*,y^*)$ have coordinates

$$\hat{x} := \frac{\delta}{1 - \left(\frac{c(\gamma-\beta)}{\beta}\right)^{\frac{\beta}{\gamma}} - c\left[\left(\frac{\beta}{c(\gamma-\beta)}\right)^{\frac{\gamma-\beta}{\gamma}} - 1\right]}; \quad \hat{y} := \hat{x}\left(\left(\frac{\beta}{c(\gamma-\beta)}\right)^{\frac{\gamma-\beta}{\gamma}} - 1\right);$$

$$x^* = \frac{\delta}{1 - cw^* - (1+w^*)^{-\frac{\beta}{\gamma-\beta}}}; \quad y^* = w^* x^*.$$

The trajectory I^* is defined in the parametric form by equalities (39), where $x_0 = x^*$, $y_0 = y^*$ and $t \in (-\infty,+\infty)$, $x(t) + y(t) \le N$. Cf. Definition 7.2: I^* is the full orbit of the point H^*. By the way, in this example, Condition 7.1 is satisfied. All the above presented expressions were justified in [17].

Now in the areas I, II and III, which form one open set, no impulses are needed, so that

$$\mathcal{V}^*(x,y) = \int_{[0,\infty)} C^g(\phi(\vec{x},u))du = x\left(1 - \left(\frac{x}{x+y}\right)^{\beta/(\gamma-\beta)}\right).$$

Similarly, if $\vec{x} = \blacksquare$, the process is over (stopped), no more impulses are needed, and $\mathcal{V}^*(\blacksquare) = 0$.

In the critical area L

$$\mathcal{V}^*(x,y) = \delta + cy.$$

In the area IV, which can be represented as

$$x > x^*; \quad w^* x < y \le x\left[(1+w^*)\left(\frac{x}{x^*}\right)^{\frac{\gamma-\beta}{\beta}} - 1\right]; \quad x + y \le N,$$

one has to wait until the trajectory touches the critical area L, that is, the time interval up to the impulse/intervention is

$$\theta_1 := \frac{1}{\gamma-\beta}\ln\left(\frac{y/x}{w^*}\right). \tag{40}$$

Further,

$$\mathcal{V}^*(x,y) = \int_{[0,\theta_1)} C^g(x(t),y(t))dt + \delta + cy(\theta_1) = x + (cw^*-1)x\left[\frac{1+w^*}{1+\frac{y}{x}}\right]^{\frac{\beta}{\gamma-\beta}} + \delta, \tag{41}$$

where $x(t)$ and $y(t)$ are given by (39) with $x_0 = x$, $y_0 = y$. Here $x - x\left[\frac{1+w^*}{1+\frac{y}{x}}\right]^{\frac{\beta}{\gamma-\beta}}$ is the total number of the new infectives over the time interval θ_1; $y(\theta_1) = w^* x\left[\frac{1+w^*}{1+\frac{y}{x}}\right]^{\frac{\beta}{\gamma-\beta}}$ is the number of infectives to be isolated at the moment of intervention.

The optimal strategy is deterministic stationary:

$$f^*(\vec{x}) = \begin{cases} (\infty,1), & \text{if } \vec{x} \text{ is in the areas I, II, or III or } \vec{x} = \blacksquare; \\ (0,1), & \text{if } \vec{x} \text{ is in the critical area L;} \\ (\theta_1,1), & \text{if } \vec{x} \text{ is in the area IV.} \end{cases}$$

Conditions 6.2(a,c,d) are satisfied, but Conditions 6.2(b,e) are not. Nevertheless, Condition 6.1 is satisfied. Explanations are in [17]; note also that $\mathcal{V}^*(x,y) \le x \le N$ for $(x,y) \in \mathbb{R}^2$ because no more than x susceptibles can become infected, and the positive Bellman function $\mathcal{V}^*(\cdot)$ is smaller.

All the theorems and propositions from the previous sections hold.

The above presented Bellman function $\mathcal{V}^*(\cdot)$ belongs to $\mathcal{W}_1$ and also to $\mathcal{W}_2$ and provides a solution to the dual linear programs (18) and (28).

As for the primal programs (10) and (27), the optimal measures μ^* and η^*, coming from the optimal strategy f^*, depend on the initial condition $\vec{x}_0$.

(a) If $\vec{x}_0 = \blacksquare$, then the process is over (stopped), no more impulses are needed, and $\mu^*(d\vec{x} \times d\theta \times da) = \delta_\blacksquare(d\vec{x})\delta_\infty(d\theta)\delta_1(da)$; $\eta^*(\blacksquare \times \Box) = \infty$; $\eta^*(\blacksquare \times \mathbf{A}) = 0$; $\eta^*((\mathbf{X} \cap \mathbb{R}^2) \times \mathbf{A}_\Box) = 0$.

(b) Suppose $\vec{x}_0 = (x_0, y_0)$ belongs to the open set represented on Fig.2 as the union of the areas I, II and III. Then the optimal strategy prescribes not to isolate any infectives at all, because the amount of susceptibles x_0 is not large, and preventing their infection is not profitable compared with the cost of the isolation. The key expressions are as follows:

$$\mu^*(d\vec{x} \times d\theta \times da) = \delta_{\vec{x}_0}(d\vec{x})\delta_\infty(d\theta)\delta_1(da),$$

and

$$\int_{\mathbf{X}\times\mathbb{R}^0_+\times\mathbf{A}} C(\vec{x},(\theta,a))\mu^*(d\vec{x} \times d\theta \times da) = x_0\left(1 - \left(\frac{x_0}{x_0+y_0}\right)^{\beta/(\gamma-\beta)}\right)$$

because

$$C(\vec{x},(\infty,1)) = \int_{[0,\infty)} C^g(\phi(\vec{x},u))du = x\left(1 - \left(\frac{x}{x+y}\right)^{\beta/(\gamma-\beta)}\right) \quad \text{for } \vec{x} = (x,y) \in \mathbb{R}^2.$$

For the measure η^*, induced by μ^*, we have $\eta^*(\mathbf{X} \times \mathbf{A}) = 0$; the component $\eta^*(d\vec{x} \times \Box)$ is concentrated on the trajectory of the system (38) starting from $\vec{x}_0$

$${}_{\vec{x}_0}\mathcal{X} := \{(x(u),y(u)) :\ u \ge 0,\ x(0) = x_0,\ y(0) = y_0\} = \{\phi(\vec{x}_0,u) :\ u \in \mathbb{R}^0_+\} :$$

see Definition 7.2. It is defined by its values on the arcs: if $\vec{x}_1 = \phi(\vec{x}_0,u)$; $\vec{x}_2 = \phi(\vec{x}_1,t) = \phi(\vec{x}_0,u+t)$, then

$$\eta^*(\{\phi(\vec{x}_0,s) :\ u \le s \le u+t\} \times \Box) =: D(\vec{x}_1,\vec{x}_2) = t$$

is the 'time-distance' between $\vec{x}_1$ and $\vec{x}_2$. See Remark 6.1, where $m(d\vec{x}) = \delta_{\vec{x}_0}(d\vec{x})$ and $\psi(\vec{x},u) = 1$: the measure $\eta^*(d\vec{x} \times \Box)$ on ${}_{\vec{x}_0}\mathcal{X}$ is the image of the Lebesgue measure with respect to the mapping $\phi(\vec{x}_0,u) :\ \mathbb{R}^0_+ \to {}_{\vec{x}_0}\mathcal{X}$. See also Definition 7.3: the measure $\eta^*(d\vec{x} \times \Box)$ is basic.

$$\begin{aligned}\int_{\mathbf{X}\times\mathbf{A}_\Box} \mathcal{C}(\vec{x},a)\eta^*(d\vec{x} \times da) &= \int_{{}_{\vec{x}_0}\mathcal{X}} \beta\frac{xy}{x+y}\eta^*(d\vec{x} \times \Box) = \int_{[0,\infty)} \left[-\frac{dx}{dt}\right] dt \\ &= x_0 - \lim_{t\to\infty} x(t) = x_0\left(1 - \left(\frac{x_0}{x_0+y_0}\right)^{\beta/(\gamma-\beta)}\right).\end{aligned}$$

(c) Suppose $\vec{x}_0 = (x_0, y_0)$ belongs to the critical area L. Then the optimal strategy prescribes to isolate immediately all the infectives. The key expressions are as follows:

$$\mu^*(d\vec{x} \times d\theta \times da) = \delta_{\vec{x}_0}(d\vec{x})\delta_0(d\theta)\delta_1(da) \ \text{ for } \vec{x} \in \mathbb{R}^2; \qquad \mu^*(\blacksquare \times d\theta \times da) = \delta_\infty(d\theta) \times \delta_1(da),$$

and

$$\int_{\mathbf{X}\times\mathbb{R}^0_+\times\mathbf{A}} C(\vec{x},(\theta,a))\mu^*(d\vec{x} \times d\theta \times da) = \delta + cy_0$$

because

$$C(\vec{x},(0,1)) = C^I(\vec{x},1) = \delta + cy \quad \text{for } \vec{x}=(x,y)\in\mathbb{R}^2.$$

For the measure η^*, induced by μ^*, we have

$$\eta^*((\mathbf{X}\cap\mathbb{R}^0_+)\times\Box)=0;\quad \eta^*(d\vec{x}\times da)=\delta_{\vec{x}_0}(d\vec{x})\delta_1(da)\ \text{ for } \vec{x}\in\mathbb{R}^2;\quad \eta^*(\blacksquare\times\Box)=\infty;\quad \eta^*(\blacksquare\times\mathbf{A})=0:$$

the process spends infinite time in the cemetery $\blacksquare$, and no impulses are applied;

$$\int_{\mathbf{X}\times\mathbf{A}_\Box}\mathcal{C}(\vec{x},a)\eta^*(d\vec{x}\times da)=\delta+cy_0.$$

(d) Now suppose $\vec{x}_0=(x_0,y_0)$ belongs to the area IV. Then the optimal strategy prescribes to wait and to apply the impulse θ_1 time units later. The key expressions are as follows:

$$\mu^*(d\vec{x}\times d\theta\times da)=\delta_{\vec{x}_0}(d\vec{x})\delta_{\theta_1}(d\theta)\delta_1(da)\ \text{ for } \vec{x}\in\mathbb{R}^2;\qquad \mu^*(\blacksquare\times d\theta\times da)=\delta_\infty(d\theta)\delta_1(da),$$

and

$$\int_{\mathbf{X}\times\bar{\mathbb{R}}^0_+\times\mathbf{A}} C(\vec{x},(\theta,a))\mu^*(d\vec{x}\times d\theta\times da)=x_0+(cw^*-1)x_0\left[\frac{1+w^*}{1+\frac{y_0}{x_0}}\right]^{\frac{\beta}{\gamma-\beta}}+\delta$$

because

$$C(\vec{x}_0,(\theta_1,1))=\int_{[0,\theta_1]}\left[-\frac{dx(t)}{dt}\right]dt+C^I(x(\theta_1),y(\theta_1),1),$$

where $x(t)$ and $y(t)$ are given by the equalities (39), so that

$$C(\vec{x}_0,(\theta_1,1))=x_0-x(\theta_1)+\delta+cy(\theta_1),$$

where

$$x(\theta_1)=x_0\left[\frac{1+w^*}{1+\frac{y_0}{x_0}}\right]^{\frac{\beta}{\gamma-\beta}};\quad y(\theta_1)=w^*x_0\left[\frac{1+w^*}{1+\frac{y_0}{x_0}}\right]^{\frac{\beta}{\gamma-\beta}},$$

see also (41). Below, $\vec{x}(\theta_1):=(x(\theta_1),y(\theta_1))$.

For the measure η^*, induced by μ^*, we have

$$\eta^*(d\vec{x}\times da)=\delta_{\vec{x}_{\theta_1}}(d\vec{x})\delta_1(da)\ \text{ for } \vec{x}\in\mathbb{R}^2;\quad \eta^*(\blacksquare\times\Box)=\infty;\quad \eta^*(\blacksquare\times da)=0:$$

the process spends infinite time in the cemetery $\blacksquare$, and no impulses are applied. The component $\eta^*(d\vec{x}\times\Box)$, considered on $\mathbb{R}^2$, is basic, concentrated on the part of the trajectory ${}_{\vec{x}_0}\mathcal{X}$ from $\vec{x}_0$ up to $\vec{x}(\theta_1)$, cf. (b). It coincides with the image of the Lebesgue measure on $[0,\theta_1]$ with respect to the mapping $\phi(\vec{x}_0,u):\ [0,\theta_1]\to{}_{\vec{x}_0}\mathcal{X}$. Therefore,

$$\begin{aligned}\int_{\mathbf{X}\times\mathbf{A}_\Box}\mathcal{C}(\vec{x},a)\eta^*(d\vec{x}\times da) &= \int_{[0,\theta_1]}\left[-\frac{dx(t)}{dt}\right]dt+\delta+cy(\theta_1)\\ &= x_0-x(\theta_1)+\delta+cy(\theta_1).\end{aligned}$$

In each case the occupation measure $\mu^{f^*}=\mu^*$ satisfies the characteristic equation (9) and solves the first primal program (10). The aggregated occupation measure η^*, induced by μ^*, satisfies the characteristic equation (26) and solves the second primal program (27). As for the Bellman function $\mathcal{V}^*(\cdot)$, it can be modified arbitrarily enough outside the trajectory ${}_{\vec{x}_0}\mathcal{X}$ (e.g., one can put it there equal to zero); the resulting function will still solve the dual linear programs (18) and (28). See Remark 5.1.

10 Conclusion and Discussion

Linear programming proved its effectiveness in many optimal control problems. In this article, we investigated two pairs of linear programs coming from the optimal impulse control with total cost. The dual programs (18) and (28) actually represent the dynamic programming approach in its integral and differential form: see equations (7) and (21) correspondingly.

The primal linear programs (10) and (27) are useful in the case of constrained optimization (29) which was not discussed in this article. In this case, the dynamic programming approach is problematic, but the primal linear programs (10) and (27) remain the same: one simply has to supplement them with inequalities

$$\int_{\mathbf{X}\times\bar{\mathbb{R}}^0_+\times\mathbf{A}} C_j(x,(\theta,a))\mu(dx\times d\theta\times da) \le d_j;$$
$$\int_{\mathbf{X}\times\mathbf{A}_\square} \mathcal{C}_j(x,a)\eta(dx\times da) \le d_j$$

correspondingly ($j=1,2,\ldots,J$).

The conditions imposed on the primitives of the model are not very restrictive. Several of them probably can be relaxed without much efforts. For example, instead of Condition 3.3(a), one can require that, after applying a bounded number of impulses, one can reach, from any initial state x_0, such a point $\hat{x}\in\mathbf{X}$ that $\int_{[0,\infty)}|C^g(\phi(\hat{x},u))du|<\infty$. As mentioned in Section 7, it seems that one can omit Condition 7.1.

As for Conditions 3.1 and 3.3(b), the situation, when the cost functions can be positive and negative, is more challenging. Investigation of such models is an interesting open problem.

11 Appendix

Proof of Lemma 6.2. According to Proposition 3.1, the Bellman function $\mathcal{V}^*(\cdot)$ is bounded and lower semicontinuous. To show that it is upper semicontinuous, we consider $+\infty\in\bar{\mathbb{R}}^0_+$ as the isolated point. The action space $\bar{\mathbb{R}}^0_+\times\mathbf{A}$ in the Markov decision process (5) is not compact, but the original transition probability (2) is continuous.

Let us show that the cost function $C(x,(\theta,a))$ given by (3) is continuous on $\mathbf{X}\times\bar{\mathbb{R}}^0_+\times\mathbf{A}$. The second term $\mathbb{I}\{\theta<\infty\}C^I(\phi(x,\theta),a)$ is continuous because so are the function $C^I(\cdot)$ and the flow ϕ. (Remember, $\theta=+\infty$ is the isolated point in $\bar{\mathbb{R}}^0_+$.)

Consider the first term $\int_{[0,\theta]}C^g(\phi(x,u))du$ and suppose $x_j\to x$ and $\theta_j\to\theta<\infty$. Then

$$\begin{aligned}&\left|\int_{[0,\theta]}C^g(\phi(x,u))du-\int_{[0,\theta_j]}C^g(\phi(x_j,u))du\right|\\ \le\ &\left|\int_{[0,\theta]}C^g(\phi(x,u))du-\int_{[0,\theta]}C^g(\phi(x_j,u))du\right|+\sup_{Y\in\mathbf{X}}|C^g(y)|\times|\theta-\theta_j|\to 0.\end{aligned}$$

Here the first term approaches zero by the dominated convergence theorem.

Suppose $\theta=\infty$ and $x_j\to x$. Then, for an arbitrary $\varepsilon>0$, for T such that

$$\sup_{y\in\mathbf{X}}\int_{(T,\infty)}|C^g(\phi(y,u))|du<\frac{\varepsilon}{4}$$

we have

$$\begin{aligned}
&\left|\int_{[0,\infty)} C^g(\phi(x,u))du - \int_{[0,\infty)} C^g(\phi(x_j,u))du\right| \\
\leq\ &\left|\int_{[0,T]} C^g(\phi(x,u))du - \int_{[0,T]} C^g(\phi(x_j,u))du\right| + \frac{\varepsilon}{2} \leq \varepsilon
\end{aligned}$$

for all x_j close enough to x: for fixed T,

$$\lim_{x_j\to x}\left|\int_{[0,T]} C^g(\phi(x,u))du - \int_{[0,T]} C^g(\phi(x_j,u))du\right| = 0$$

by the dominated convergence theorem, as above. Therefore, the cost function $C(\cdot)$ is continuous.

Now we approximate the solution to equation (7) as follows:

$$\begin{aligned}
W_n(\Delta) &= 0; \\
W_0(x) &:= \int_{[0,\infty)} C^g(\phi(x,u))du, \quad x \in \mathbf{X}; \\
W_{n+1}(x) &= \inf_{(\theta,a)\in\bar{\mathbb{R}}^0_+\times\mathbf{A}} \left\{C(x,(\theta,a)) + \int_{\mathbf{X}_\Delta} W_n(y)Q(dy|x,(\theta,a))\right\}, \quad x \in \mathbf{X}.
\end{aligned}$$

As usual, the stochastic kernel Q is given by (2), and, as mentioned above, it is continuous. The sequence $W_n(\cdot) \geq 0$ decreases point-wise and, for each $n = 0, 1, \ldots$, the function $W_n(\cdot)$ is bounded and upper semicontinuous by the inductive argument. (See [2, Prop.7.31;7.32].) Therefore, the function $W_\infty(x) = \lim_{n\to\infty} W_n(x) = \inf_{n\in\{0,1,\ldots\}} W_n(x)$ on $\mathbf{X}_\Delta$ is bounded and upper semicontinuous according to [2, Prop.7.32].

It remains to show that the function $W_\infty(\cdot)$ satisfies equation (7) for all $x \in \mathbf{X}$. Since, for all $n = 0, 1, \ldots$, $x \in \mathbf{X}$,

$$\begin{aligned}
W_{n+1}(x) &= \inf_{(\theta,a)\in\bar{\mathbb{R}}^0_+\times\mathbf{A}} \left\{C(x,(\theta,a)) + \int_{\mathbf{X}_\Delta} W_n(y)Q(dy|x.(\theta,a))\right\} \\
&\geq \inf_{(\theta,a)\in\bar{\mathbb{R}}^0_+\times\mathbf{A}} \left\{C(x,(\theta,a)) + \int_{\mathbf{X}_\Delta} W_\infty(y)Q(dy|x.(\theta,a))\right\},
\end{aligned}$$

we conclude that

$$W_\infty(x) = \lim_{n\to\infty} W_{n+1}(x) \geq \inf_{(\theta,a)\in\bar{\mathbb{R}}^0_+\times\mathbf{A}} \left\{C(x,(\theta,a)) + \int_{\mathbf{X}_\Delta} W_\infty(y)Q(dy|x,(\theta,a))\right\}.$$

On the other hand, for all $n = 0, 1, \ldots$, $x \in \mathbf{X}$,

$$W_\infty(x) \leq W_{n+1}(x) \leq C(x,(\theta,a)) + \int_{\mathbf{X}_\Delta} W_n(y)Q(dy|x,(\theta,a))$$

for all $(\theta, a) \in \bar{\mathbb{R}}^0_+ \times \mathbf{A}$. After we pass to the limit as $n \to \infty$, by the dominated (or monotone) convergence theorem, we obtain:

$$W_\infty(x) \leq C(x,(\theta,a)) + \int_{\mathbf{X}_\Delta} W_\infty(y)Q(dy|x,(\theta,a))$$

for all $x \in \mathbf{X}$, $(\theta, a) \in \bar{\mathbb{R}}^0_+ \times \mathbf{A}$. Therefore,

$$W_\infty(x) \leq \inf_{(\theta,a)\in\bar{\mathbb{R}}^0_+\times\mathbf{A}} \left\{ C(x,(\theta,a)) + \int_{\mathbf{X}_\Delta} W_\infty(y) Q(dy|x,(\theta,a)) \right\},$$

and the bounded function $W_\infty(\cdot)$ satisfies equation (7). Since the bounded solution to equation (7) is unique according to Proposition 3.1, we conclude that the Bellman function $\mathcal{V}^*(\cdot) = W_\infty(\cdot)$ is upper semicontinuous.

The proof is completed. □

Proof of Lemma 6.3. According to Proposition 3.1, there exists a measurable mapping $f^* : \mathbf{X} \to \bar{\mathbb{R}}^0_+ \times \mathbf{A}$ providing the infimum in (7) and defining the optimal strategy in problem (4), which is presented below as $(f^*_\theta(y), f^*_a(y))$ for $y \in \mathbf{X}$.

Let $x \in \mathbf{X}$ be arbitrarily fixed. We will show that inequality (19) holds for the Bellman function $\mathcal{V}^*(\cdot)$, for the constant

$$G(x) := \sup_{y=\phi(x,u):u\geq 0} C^g(y) \vee \sup_{(y,a)\in\{\phi(x,u):u\geq 0\}\times\mathbf{A}} |g(x,y,a)|.$$

Suppose $y_1 = \phi(x,\tau_1)$ and $y_2 = \phi(x,\tau_2)$ are two arbitrarily fixed points with $0 \leq \tau_1 < \tau_2 < \infty$.

(a) If $(\tau_2 - \tau_1) < f^*_\theta(y_1)$, then

$$\mathcal{V}^*(y_1) = \int_{[0,\tau_2-\tau_1]} C^g(\phi(y_1,u))du + \mathcal{V}^*(y_2)$$

by Corollary 1 of [18]. The same equality holds also in case $\tau_2 - \tau_1 = f^*_\theta(y_1)$, because the function $\mathcal{V}^*(\cdot)$ is continuous by Lemma 6.2: one should pass to the limit as $t \uparrow (\tau_2 - \tau_1)$ in the equality

$$\mathcal{V}^*(y_1) = \int_{[0,t]} C^g(\phi(y_1,u))du + \mathcal{V}^*(\phi(y_1,t)).$$

Therefore,

$$|\mathcal{V}^*(y_1) - \mathcal{V}^*(y_2)| \leq G(x)(\tau_2 - \tau_1).$$

(b) Suppose now that $(\tau_2 - \tau_1) > f^*_\theta(y_1)$. Here $0 \leq f^*_\theta(y_1) < \infty$.

According to (7),

$$\begin{aligned}
\mathcal{V}^*(y_1) &\leq \int_{[0,\tau_2-\tau_1]} C^g(\phi(y_1,u))du + \inf_{(\theta,a)\in\bar{\mathbb{R}}^0_+\times\mathbf{A}} \left\{ \int_{[0,\theta]} C^g(\phi(y_2,u))du + \mathbb{I}\{\theta < +\infty\} C^I(\phi(y_2,\theta),a) \right\} \\
&= \int_{[0,\tau_2-\tau_1]} C^g(\phi(y_1,u))du + \mathcal{V}^*(y_2) \leq \mathcal{V}^*(y_2) + G_x(\tau_2 - \tau_1). \qquad (42)
\end{aligned}$$

Furthermore,

$$\mathcal{V}^*(y_2) \leq C^I(y_2, f^*_a(y_1)) + \mathcal{V}^*(l(y_2, f^*_a(y_1))).$$

This inequality holds because we substituted specific values $(\theta = 0, f^*_a(y_1))$ in the formula (7) for y_2. Since

$$\mathcal{V}^*(y_1) = \int_{[0,f^*_\theta(y_1)]} C^g(\phi(y_1,u))du + C^I(\phi(y_1, f^*_\theta(y_1)), f^*_a(y_1)) + \mathcal{V}^*(l(\phi(y_1, f^*_\theta(y_1)), f^*_a(y_1)))$$

and $l(y_2, f^*_a(y_1)) = l(\phi(y_1, f^*_\theta(y_1)), f^*_a(y_1))$, by Condition 6.2(c,d), we see that

$$\begin{aligned}
\mathcal{V}^*(y_2) &\leq C^I(y_2, f^*_a(y_1)) + \mathcal{V}^*(y_1) - \int_{[0,f^*_\theta(y_1)]} C^g(\phi(y_1,u))du - C^I(\phi(y_1, f^*_\theta(y_1)), f^*_a(y_1)) \\
&\leq \mathcal{V}^*(y_1) + |C^I(\phi(y_1, \tau_2 - \tau_1), f^*_a(y_1)) - C^I(\phi(y_1, f^*_\theta(y_1)), f^*_a(y_1))| \\
&\leq \mathcal{V}^*(y_1) + \int_{[f^*_\theta(y_1),\tau_2-\tau_1]} |g(x, \phi(y_1,u), f^*_a(y_1))|du.
\end{aligned}$$

Here, the semigroup property of the flow ϕ was used several times.

Therefore, $\mathcal{V}^*(y_2) \le \mathcal{V}^*(y_1) + G(x)(\tau_2 - \tau_1)$ and, taking into account (42), we see that again

$$|\mathcal{V}^*(y_1) - \mathcal{V}^*(y_2)| \le G(x)(\tau_2 - \tau_1).$$

Thus, the Bellman function $\mathcal{V}^*(\cdot)$ is absolutely continuous along the flow ϕ.

For the last statement, note that, for all $x \in \mathbf{X}$, $|\chi\mathcal{V}^*(\phi(x,u))| \le G(\phi(x,u))$ for any version of the function $\chi\mathcal{V}^*(\cdot)$, for almost all $u \ge 0$. $\square$

Proof of Lemma 6.4. The case $a_1 = a_2 = 0$ is trivial: it is sufficient to fix an arbitrary probability m on $\mathbf{X}$ and put $\psi(x,u) \equiv 0$. In case $a_1 \neq 0$, $a_2 = 0$, one should take $m = m_1$ and put $\psi(x,u) := a_1\psi_1(x,u)$. The case $a_1 = 0$, $a_2 \neq 0$ is symmetric.

Suppose $a_1 \neq 0$ and $a_2 \neq 0$.

We take

$$m(dx) := \frac{|a_1|m_1(dx) + |a_2|m_2(dx)}{|a_1| + |a_2|} \text{ and } \psi(x,u) := a_1\psi_1(x,u)d_1(x) + a_2\psi_2(x,u)d_2(x),$$

where $d_j(\cdot) := \frac{dm_j}{dm}$ is the Radon-Nikodym derivative ($j = 1,2$).

(a) For an arbitrarily fixed $S \in \mathbb{R}_+$, denote

$$D^S := \{(x,u) : \ |\psi(x,u)| > S\} \subset \mathbf{X} \times \mathbb{R}^0_+.$$

Since $|\psi(x,u)| \le |a_1\psi_1(x,u)|d_1(x) + |a_2\psi_2(x,u)|d_2(x)$ for each $(x,u) \in D^S$ we have

$$\text{either } |a_1\psi_1(x,u)|d_1(x) > \frac{S}{2}, \text{ or } |a_2\psi_2(x,u)|d_2(x) > \frac{S}{2};$$

hence $D^S \subset D_1^{S/2} \cup D_2^{S/2}$, where

$$D_j^{S/2} := \{(x,u) : \ |a_j\psi_j(x,u)|d_j(x) > \frac{S}{2}\}, \quad j = 1,2.$$

Note that, for $j = 1,2$, $m(\{x \in \mathbf{X} : \ |a_j|d_j(x) > |a_1| + |a_2|\}) = 0$ because, otherwise, for the set $E_j := \{x \in \mathbf{X} : \ |a_j|d_j(x) > |a_1| + a_2|\}$ we would have had $m_j(E_j) = \int_{E_j} d_j(x)m(dx) > (|a_1|+|a_2|)m(E_j)/|a_j|$ which is impossible. Furthermore, for each $x \in D_j^{S/2}$, $d_j(x) > 0$ because $S > 0$. Thus,

$$\begin{aligned}
\bar{m}(D_j^{S/2}) &= \int_{\mathbf{X}}\int_{\mathbb{R}^0_+} \mathbb{I}\{0 < d_j(x) \le (|a_1| + |a_2|)/|a_j|\}\frac{\mathbb{I}\{|a_j\psi_j(x,u)|d_j(x) > S/2\}}{d_j(x)} du\ dm_j(x) \\
&\le \int_{\mathbf{X}\times\mathbb{R}^0_+} \mathbb{I}\{0 < d_j(x) \le (|a_1| + |a_2|)/|a_j|\}\frac{\mathbb{I}\{(|a_j + |a_2|)|\psi_j(x,u)| > S/2\}}{d_j(x)} \bar{m}_j(dx \times du).
\end{aligned}$$

The last expression equals zero if

$$\int_{\mathbf{X}\times\mathbb{R}^0_+} \mathbb{I}\left\{|\psi_j(x,u)| > \frac{S/2}{|a_1| + |a_2|}\right\} \bar{m}_j(dx \times du) = 0.$$

Let $S_j > 0$ be a constant in (25) corresponding to the measure $\bar{m}_j$ and the function $\psi_j(\cdot)$ ($j = 1,2$). Then $\bar{m}(D_j^{S/2}) = 0$ as soon as $S \ge 2S_j(|a_1| + |a_2|)$ and, for the constant $\hat{S} := \frac{2(S_1 \vee S_2)}{|a_1|+|a_2|}$, we have

$$\bar{m}(D^{\hat{S}}) \le \bar{m}(D_1^{\hat{S}/2}) + \bar{m}(D_2^{\hat{S}/2}) = 0,$$

that is, the requirement (25) is satisfied for $\hat{S} \in \mathbb{R}_+$.

(b) The proof of this part is straightforward: for each $i = 1, 2, \ldots$, $\Gamma \in \mathcal{B}(\mathbf{X} \times [i-1, i))$,

$$\begin{aligned} M^i(\Gamma) &= \int_{[i-1,i)} \int_{\mathbf{X}} \mathbb{I}\{(x,u) \in \Gamma\}[a_1\psi_1(x,u)d_1(x) + a_2\psi_2(x,u)d_2(x)]m(dx)\ du \\ &= a_1 \int_{[i-1,i)} \int_{\mathbf{X}} \mathbb{I}\{(x,u) \in \Gamma\}\psi_1(x,u)m_1(dx)\ du \\ &\quad + a_2 \int_{[i-1,i)} \int_{\mathbf{X}} \mathbb{I}\{(x,u) \in \Gamma\}\psi_2(x,u)m_2(dx)\ du = a_1 M_1^i(\Gamma) + a_2 M_2^i(\Gamma). \end{aligned}$$

All three functions, integrated here, are bounded $\bar{m}$-a.s., $\bar{m}_1$-a.s. and $\bar{m}_2$-a.s. correspondingly.

(c) This assertion is obvious. □

Proof of Lemma 7.1. (a) For an arbitrarily fixed arc $[x, y] \subset {}_z\bar{\mathcal{X}}^c$ with $y \neq$ "∞", take the following function $W(\cdot) \in \mathcal{W}_2$:

$$W(z) = \begin{cases} -D(x,y), & \text{if } z \text{ is a predecessor of } x \text{ on } {}_z\bar{\mathcal{X}}^c; \\ -D(z,y), & \text{if } z \in [x,y]; \\ 0 & \text{otherwise,} \end{cases}$$

where D is the time-distance. (See Definition 7.3(b).) Now

$$\chi W(z) = \begin{cases} 1, & \text{if } z \in [x,y]; \\ 0 & \text{otherwise,} \end{cases}$$

and equality (26) implies that $\eta([x,y] \times \Box) = 0$, because $W(\cdot) \leq 0$, $W(x_0) = 0$, and $W(l(y,a)) = 0$ for all points $(y,a) \in \mathcal{Y}$. Hence $\eta({}_z\bar{\mathcal{X}}^c \times \Box) = 0$, because ${}_z\bar{\mathcal{X}}^c$ can be represented (up to a finite number of points, where $\eta = 0$) as a union of arcs $[x,y] \subset {}_z\bar{\mathcal{X}}^c$ with x, y from the countable set, everywhere dense in $\mathbf{X}$.

Assume for contradiction that there is $x \in {}_z\bar{\mathcal{X}}^c \cap \hat{Y}_c$. Since x is non-cyclic, one can repeat the presented above construction for a (small enough) arc $[x, \phi(x,\varepsilon)] \subset {}_z\bar{\mathcal{X}}^c$ with $\varepsilon > 0$. In equality (26) we will obtain the strictly positive value on the right-hand side.

(b) The proof follows from the same analysis of the similar functions $W(\cdot) \in \mathcal{W}_2$. The only difference is that the elementary arc can be half-closed: $[x,y) = \{\phi(x,u) : 0 \leq u < t\}$, where $t = D(x,y)$, and the point y is cyclic. Now, like previously, equality (26) implies that

$$\left.\begin{array}{l} \eta([x,y] \times \Box) \\ \eta([x,y) \times \Box) \end{array}\right\} = \int_{\mathbf{X}} \chi W(x)\eta(dx \times \Box) = \int_{\mathbf{X}} W(y)\eta(dy \times \mathbf{A}) - \left[W(x_0) + \int_{\mathbf{X}\times\mathbf{A}} W(l(y,a))\eta(dy \times da)\right] = D(x,y)[k_X - k_Y] = \bar{k} \left\{ \begin{array}{l} \eta^b([x,y]) \\ \eta^b([x,y)). \end{array}\right.$$

□

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