Receptance-based antiresonant frequency assignment of an uncertain dynamic system using interval multiobjective optimization method

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**Abstract:** This study addresses the antiresonant frequency assignment problem of vibrating systems with uncertainty, which is solved using an interval multi-objective optimization method. The study accounts for several uncertain factors, including in the structural parameters and test data, and defines them as interval variables. The bounds of interval variation of the receptance (frequency response function or FRF) are determined from a small number of numerical or experimental results; thus, it is unnecessary to collect a large amount of data to construct the sample space. The antiresonant frequency assignment problem of uncertain systems is formulated as an interval multi-objective optimization. Optimal solutions (the structural modifications that assign the desired antiresonant frequency frequencies) are determined through the MI-NSGA-II algorithm proposed in this work. It employs the basic framework of the interval multi-objective genetic algorithm INSGA-II and redefines the interval confidence level to evaluate the merits and demerits between individual feasible solutions and improve the stability of INSGA-II. The obtained modifications not only accurately achieve the goal of antiresonant frequency assignment but also stably maximize the robustness of the antiresonant frequency frequencies after modifications. From the perspective of enhancing the applicability of the receptance method, the proposed method can overcome the problems of unsuccessful antiresonant frequency assignment due to measurement inaccuracies and the uncertainty of physical parameters to a certain extent. Numerical and experimental examples have been provided to demonstrate the effectiveness of the proposed method.

**Keywords:** Uncertain dynamic system; antiresonant frequency assignment; receptance method; structural modification.

## 1. Introduction

Achieving an optimal design in structural dynamics is an effective approach to address vibration control that arises in engineering [1]. The receptance method currently adopted — an inverse structural modification method — has attracted significant attention of many researchers[2,3]. In this method, the desired resonant frequency (natural frequency), the antiresonant frequency, or the modal shape is realized through suitable structural modification based on the knowledge of some frequency response functions (FRFs).

The receptance method was first formally presented by Mottershead and Ram [4] in a literature review. In this method, it is not required to establish the numerical model of the system or calculate the entire mass, stiffness, and damping matrices; only a few FRF measurements are required to complete the modification calculations. However, it might be physically impossible to implement modifications obtained using the early version of the receptance method; this restricts the applicability of the receptance method in practice. To enhance the realizability of receptance-based structural modifications, Ouyang and his colleagues [5,6] introduced the concept of convex optimization to determine the optimal modification values and transformed the inverse structural modification into an optimization problem. Their method allowed the number of modifications (design variables) to be greater than those of the desired frequencies (objectives); thus, a set of physically realizable solutions could be obtained in a reasonable solution domain. In a recent study, Liu et al. [7] summarized the higher-rank modification theory of the receptance method and extended it to subsystem modification. Richiedei et al. [8] applied the unit-rank output feedback control technique to solve the assignment problem of both point- and cross- receptances. Their work managed to lower the difficulty in realizing higher-rank structural modifications and makes the receptance method more feasible in practice.

In recent years, the receptance method has gained popularity in practice due to advancements in testing equipment and measurement technologies. Mottershead et al. [9] were able to assign the first two natural frequencies of an aircraft tail by mass modification. Zarraga et al. [10] successfully shifted the natural frequency of a brake clutch with measured FRFs and suppressed friction-induced noise in the system. Similar work has been carried out by Tsai and Ouyang [11], who applied the receptance method to a bearing–rotor–gear system. Their method has also been extended and applied in the field of active control. Tehrani et al. [12] applied the method to a lightweight glass–fibre beam with smart material sensors and actuators and achieved pole assignment in this structure by controlling the output force of the actuators. Singh et al. [13] and Mokrani et al. [14] used the receptance method on aeroelastic wings with multiple control surfaces.

The appealing advantages of the receptance method have made it a popular inverse modification method for optimizing structural dynamic performance. However, structural modification is mainly realized on deterministic systems and rarely involves uncertain systems. In practice, uncertainty is common in engineering, which can reduce the accuracy and reliability of the structural modification results [15]. Therefore, there is an urgent need to conduct research on structural modification of uncertain dynamic systems. Uncertainty in practice generally can be divided into two types: uncertainty of structural parameters and uncertainty of test data [16]. The former is caused by the uncertainty of physical parameters, such as errors in manufacturing, machining, and assembly, which lead to uncertainty in mass and stiffness. The latter lies in errors of measurement data that can be caused by environmental noise, testing position error, inherent inadequacy of an instrument and an imperfect testing method, which can cause a large scatter of test data and poor repeatability of tests.

In order to minimize the effects of uncertain factors on assignment results, a robust design optimization (RDO) strategy, which renders the modification results insensitive to uncertain factors, can be introduced to address the difficulties in finding the right structural modification of uncertain dynamic systems. The first researcher to introduce RDO in the receptance method was Tehrani [17], who placed emphasis on the local sensitivity of desired poles to noise and obtained poles with the best robustness by minimizing the infinite norm of each row of eigenvalue sensitivities. Tehrani’s work was further extended by Liang et al. [18], who introduced uncertainty of physical parameters in the computation of eigenvalue sensitivity and adopted a genetic algorithm (GA) to minimize the Frobenius norm of closed-loop poles. However, the aforementioned RDO strategy was implemented based on local sensitivity, which only provided a local optimal solution, not a global one. To obtain a global optimal solution for robust optimization, Adamson et al. [19] employed variance to describe the effect of uncertain factors and adopted the differential evolution algorithm to minimize the variance of the real and imaginary parts of each pole, thereby improving the robustness of the placed poles. It is worth noting that in Adamson’s method, variance computation was based on the polynomial chaos (PC) method, a typical non-statistical probabilistic method that requires a fair amount of test data for constructing data samples and obtaining more accurate results [20]. Subsequently, Bai et al. [21,22], Franklin et al. [23], and Xie [24] focused on extending the receptance method to uncertain systems and introduced different theories and methods.

The receptance method offers the advantage of requiring smaller amounts of test data to implement structural modifications [25]; thus, introducing the PC method can undoubtedly weaken its superiority. In particular, it is difficult in practice to construct suitable data samples due to data deficiencies [26]. Uncertain parameters or factors are usually uncertain but bounded, and measurement values (e.g., FRFs) of the uncertain system are usually interval-based [27]. Therefore, it is easier to obtain the variation interval of experimental data than its probability density functions [28], thereby offering distinct advantages when defining the FRFs of a system with bounded uncertain factors in an interval form.

In this study, the interval math is integrated into the receptance method to define bounded but uncertain FRFs to further strengthen the significant advantages offered by the receptance method in the structural modification of uncertain systems. The interval confidence level is employed to redefine the non-dominated sorting of the interval multi-objective optimization algorithm, INSGA-II [29], and a new interval multi-objective optimization algorithm called MI-NSGA-II is developed. The optimal solutions obtained through MI-NSGA-II can help in realizing the structural modification of uncertain systems while also increasing the robustness of the objectives with respect to parameter uncertainty. Compared with the previous structural modification methods for uncertain systems, the approach proposed in this study is more general. On the one hand, the approach can realize structural modification with optimal robustness through not only active control but also passive modification; on the other hand, the proposed method can extend the structural modification of deterministic systems and further enhance the reliability of the receptance method in engineering by minimizing the effect of measurement errors on the modification results.

In order to fully demonstrate the superiority of the proposed approach, this method is employed to realize the antiresonant frequency assignment of uncertain dynamic systems since an antiresonant frequency is more sensitive to uncertainty and more challenging to assign accurately than a natural frequency [30,31]. Importantly, the targets of structural modifications in this study are multiple antiresonant frequency frequencies. The proposed approach is validated through two numerical examples and an experiment. These involve both discrete and continuous systems, and the uncertain factors include the system parameters and the test data.

The remainder of this paper is structured as follows: Section 2 introduces the theoretical development of the higher-rank receptance method and formulizes the antiresonant frequency assignment equation of uncertain systems as the interval objective function in accordance with the interval math. Section 3 presents the improved interval multi-objective genetic algorithm MI-NSGA-II with details. Section 4 provides two numerical verification examples to highlight the stability and effectiveness of this method. Section 5 describes how the experimental validation of this method is carried out—assigning two antiresonant frequency frequencies of a complicated continuous beam with uncertain distributed loads, showing the good performance of the proposed method in practice. Finally, the concluding remarks are provided in Section 6.

## 2. Theory

### 2.1 Receptance method and problem formulation

A viscous damped linear system with *n*-degrees of freedom (DOFs) is considered. The Laplace transform of its motion equation is expressed as

|  |  |
| --- | --- |
|  | (1) |

where **M**, **C**, and **K** are *n*×*n* mass, damping, and stiffness matrices, respectively; **f**(*s*) is the vector of generalized force applied to the system, and **u**(*s*) is the generalized coordinate vector.

After changing some structural parameters of the original system (denoted by **v**), the modification matrices can be expressed as ,, and. The motion equation of the modiﬁed system can then be written as

|  |  |
| --- | --- |
|  | (2) |
|  | (3) |

is called the dynamic stiffness modification matrix.

By premultiplying both sides of Eq. (3) by the receptance matrix and inverting the resultant matrix on the left-hand side, Eq. (3) can be expressed as [4]

|  |  |
| --- | --- |
|  | (4) |

The receptance matrix  of the modified system can be expressed as

|  |  |
| --- | --- |
|  | (5) |

where **∆B** can be written as

|  |  |
| --- | --- |
|  | (6) |

where  is *x*th unit vector of *n* dimensions, and  is a non-zero entry in ∆**B**. The set can be defined as , and the number of distinct elements in the set is *r*. After re-arranging, Eq. (5) can be expanded as [7]

|  |  |
| --- | --- |
|  | (7) |

After simplifying, Eq. (7) can be written as

|  |  |
| --- | --- |
|  | (8) |

Eq. (8) could be simpliﬁed as

|  |  |
| --- | --- |
|  | (9) |

Therefore, for an arbitrary element  in , it can be expressed as

|  |  |
| --- | --- |
|  | (10) |

The desired antiresonant frequencies are assigned by setting Eq. (11) to zero.

|  |  |
| --- | --- |
|  | (11) |

The equation can be solved to find the required dynamic stiffness matrix . It should be noted that the number of desired antiresonant frequencies *m* is usually smaller than *r*, ensuring that all the targets are physically and technically feasible.

The above formulation has demonstrated how antiresonant frequencies are assigned using only receptances obtained experimentally. This method does not directly involve the mass, stiffness, and damping matrices of the system, thereby no model-form error is introduced.

### 2.2 Uncertain dynamic system and the interval objective function

Here, the uncertainty of the dynamic system is considered. By assuming that the system contains several uncertain parameters, the receptance matrix of the system becomes an uncertain matrix. Eq. (10) can then be rewritten as

|  |  |
| --- | --- |
|  | (12) |

where **θ** is the vector of the uncertain variables. It should be noted that the uncertainty of the receptance matrix may arise from the uncertainty of structural parameters. Considering automobile powertrain mounting systems as an example. The Young's modulus of the material and the support stiffness of the mount components may contain uncertain factors affected by manufacturing, assembly, and complex working environments [32]. Moreover, uncertainty may reside in measured FRF data, which may be caused by environmental noise or inconsistency between the positions of the measuring points and the theoretical settings.

The uncertain factors are usually constrained by feasible domains [33]. Hence, the receptance matrix  is uncertain but bounded (interval). The arbitrary FRF in the matrix varies within an interval range and can be expressed as

|  |  |
| --- | --- |
|  | (13) |

where  is an FRF of an uncertain system, and  and  are the lower and upper bounds of the FRF, respectively, which can be either be obtained experimentally or theoretically. Therefore, the FRF of an uncertain system can be redefined as an interval number or a function in accordance with the interval analysis [34]

|  |  |
| --- | --- |
|  | (14) |

where  is an interval FRF, which depends only on the desired frequency and the locations of excitation and response and is not affected directly by uncertain factors. The interval FRF can thus tackle the problem of uncertainty without relying on the data sample.

To facilitate subsequent calculations, some key information about the interval FRF needs to be introduced

|  |  |
| --- | --- |
|  | (15) |

where  is the midpoint of the interval FRF,  is the radius of the interval FRF, and *un* is the uncertainty level of the interval FRF. After obtaining the center of all the related interval FRFs  , the center matrix of the interval receptance matrix  in Eq. (12) can be constructed. The center value (mean) of the interval FRFs of the modified system can thus be derived and expressed as

|  |  |
| --- | --- |
|  | (16) |

Apparently, Eq. (16) is difficult to simplify, and the interval natural expansion formula [34] is thus employed to tackle this problem. The upper and lower bounds of the interval FRF of the modified system can be expressed as

|  |  |
| --- | --- |
|  | (17) |

Problem (12) can be rewritten in a more compact interval form

|  |  |
| --- | --- |
|  | (18) |

Eq. (18) is an interval function whose midpoint and radius represent the average value and the limit value of its ﬂuctuation range, respectively. From the perspective of structural modification, a smaller midpoint modulus indicates that the assigned antiresonant frequency has higher accuracy, and a smaller interval radius shows that the assigned antiresonant frequency has higher robustness. This fact has motivated researchers to convert the interval optimization objective into two determinate optimization sub-objectives: the midpoint of the target interval and its width (e.g., [35,36]). Such deterministic optimization problems theoretically admit a set of optimal solutions, achieving both accuracy and robustness by minimizing these two sub-objectives simultaneously. However, the two sub-objectives with different magnitudes may cause numerical instability, and hence lead to insufficient accuracy of the calculation results [37].

In order to make the optimization process efficient and stable, a possibility-based interval optimization algorithm, which can directly address the numerical and robustness problems of Eq. (18), in the framework of interval arithmetic, is introduced in this study. The details of the proposed algorithm are presented in Section 3. The objective function form should be properly reconstructed, beforehand, as the interval form to make it suitable for interval optimization since the definition of the objective function directly affects the modification result [38]. The interval objective function of the antiresonant frequency assignment can be defined as

|  |  |
| --- | --- |
|  | (19) |

It should be stressed that the minimization in Eq. (19) specifically indicates the minimization of the distance between the upper and lower bounds of Eq. (19), and the reference point (origin), which is equivalent to simultaneously minimizing the midpoint and radius of the objective interval. Such an interval-form objective function can boost the accuracy and robustness of the assigned antiresonant frequency. Furthermore, when multiple antiresonant frequencies need to be assigned (via higher-rank modifications), the structural modification problem will become an interval multi-objective minimization problem. The corresponding objective function can then be written as

|  |  |
| --- | --- |
|  | (20) |

where  is an interval objective function, and  is the set of interval objective functions. The to-be-determined **v** can be obtained by synthetically minimizing the interval objective vector **Ψ**. To realize the minimization of the interval vector, a specific interval multi-objective optimization algorithm has been introduced later in this study.

## 3 Interval multiobjective optimization method

### 3.1 Algorithm overview

In this study, the antiresonant frequency assignment of an uncertain system represents a typical interval multi-objective optimization problem (MOP), which is a kind of expensive and complex optimization problem. The traditional deterministic evolution algorithm (e.g., NSGA-II [39]) is not suitable to overcome this issue, especially in practical applications. It thus makes sense to develop an efﬁcient and effective algorithm for interval MOPs.

In order to solve the interval MOPs, Gong et al. [40] employed the NSGA-II framework and presented an effective interval optimization algorithm called INSGA-II. Within INSGA-II, both the dominance relation (reﬂecting the quality of the optimal solution) and the crowding distance (reﬂecting the diversity of solutions) are redeﬁned as forms suitable for interval MOPs. The dominated sorting of INSGA-II is implemented based on the confidence level of interval evolutionary individuals since the size relationship between two interval numbers cannot be directly compared [41]. The crowding distance is defined based on the overlapping degree of evolutionary individuals. In the selection stage of evolutionary individuals, the individuals with higher ranks of non-dominated sorting should be preferentially selected. When the ranks are the same, the individual with a larger crowding distance should be preferentially selected. Such treatment ensures that the next-generation population exhibits high quality and good distribution and facilitates the search of a group of Pareto optimal solutions in a single running process [42].

The aforementioned appealing advantages have helped INSGA-II to become a potential algorithm to solve the interval MOPs and provide a useful aid for designers to determine an implementable optimal solution. However, the performance of INSGA-II is sometimes unstable due to premature convergence when solving the minimization optimization problem in practical applications [43]. The reason for premature convergence is that the definition of confidence level cannot reflect the difference in the convergence levels and the dominant partial order relation of evolutionary individuals. Therefore, this study redefined the interval confidence level of INSGA-II to more comprehensively mine the evolutionary interval information of interval individuals and improve the possibility of subsequent high-quality evolution. The modified INSGA-II is named MI-NSGA-II in this paper and its algorithm flowchart is illustrated in Fig. 1, where the improvements to INSGA-II have been denoted in the blue block. Moreover, the reﬂecting boundary method is employed to handle the bounds of design variables during the optimization to prevent the algorithm from falling into a local optimum [44].

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Fig. 1. Algorithm flowchart of MI-NSGA-II.

In order to better understand the proposed algorithm, the dominance relation based on redefined confidence level and the interval crowding distance, the core ideas of MI-NSGA-II, are introduced in detail in subsequent Sections 3.2 and 3.3.

### 3.2 Redefinition of dominance relation based on interval confidence level

Dominated sorting essentially involves a magnitude comparison between individual fitness functions. However, the size relationship between two interval numbers cannot be directly compared, especially when one interval encloses the other. Therefore, an interval confidence level is usually introduced to rank interval numbers in INSGA-II, among which Lin's method is the most popular [37]. For two intervals  and , the interval confidence level of the interval *a* being larger than or equal to *b*, denoted as , is defined as

|  |  |
| --- | --- |
|  | (21) |

where and  are the widths of the two intervals. For minimization problem, individual *a* being superior to *b* is equivalent to  or , where  [40].

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Fig. 2. A position relationship between interval individuals.

During an interval optimization calculation of the antiresonant frequency of an uncertain system, the following scenario in Fig. 2 is a frequent occurrence, where the midpoint of individual *a* is slightly larger than that of *b*, while the width of *a* is far smaller than that of *b*. It is apparent from Fig. 2 that individual *a* has higher robustness than *b* with almost the same precision as individual *b*. According to the robust requirement of the antiresonant frequency, theoretically, *a* should be superior to *b,* and the interval confidence level  should be larger than 0.5. However,  calculated by Eq.(21) is equal to  and smaller than 0.5, which contradicts the theoretical conclusion from the robust requirement. Therefore, the definition of confidence level obtained by Lin's method (Eq.21) is improper and may lead to the wrong dominance relation of individuals and then low-quality evolution when meeting situations like Fig. 2. This is also the reason why the performance of INSGA-II is unstable.

In order to boost the population evolution toward the optimal direction, this study introduced ∆*P* to improve Lin’s method for finding robust individuals. The redefined confidence level of the interval *a* being larger than or equal to *b* can be expressed as

|  |  |
| --- | --- |
|  | (22) |

Similarly, interval *b* being larger than or equal to *a* can be denoted as

|  |  |
| --- | --- |
|  | (23) |

where ∆*P* is a correction term introduced in this paper to Lin’s formula and it is defined as

|  |  |
| --- | --- |
|  | (24) |

where *ac* and *bc* are the midpoints of individuals *a* and *b*, respectively.

It is clear that the redefined confidence level can correctly evaluate the interval dominance relation in Fig. 2 to preserve the potential good genes and steer the population to move toward the optimal direction. Therefore, introducing a correction term can increase the stability of the whole algorithm in the framework of interval optimization algorithm INSGA-II.

According to Fig. 1, non-dominant sorting should be performed after obtaining the dominant relationship of individuals. Non-dominated sorting is defined as follows.

**Definition 1:** For the two solutions  and  of Eq. (18), the *i*th objective functions corresponding to them are  and , respectively. By defining the interval confidence level of  as larger than and equal to  as , the following expression is obtained.

|  |  |
| --- | --- |
|  | (25) |

Similarly,

|  |  |
| --- | --- |
|  | (26) |

**Definition 2:**  dominates , denoted as , provided that the following condition is satisﬁed.

|  |  |
| --- | --- |
|  | (27) |

If neither  nor  dominates each other,  and  are mutually non-dominated, denoted as .

**Definition 3: x\***,, can be considered as a Pareto optimal solution if and only if there is no solution dominating **x\*** in the decision space **S**.

**Definition 4:** The Pareto optimal solution set is the set of all Pareto optimal solutions, denoted by **X\***. The Pareto front is the set of the objective function intervals (or the hyper-cuboid), corresponding to the Pareto optimal solutions in **X\***.

It is worth noting that the traditional Pareto front is a hyper-surface (or several hyper-surfaces), comprising some points that are the objective function values of these Pareto optimal solutions. However, the Pareto front defined in this study is a larger hyper-cuboid (or several larger hyper-cuboids) comprising some smaller hyper-cuboids that are the objective function intervals of these Pareto optimal solutions.

### 3.3 Interval crowding distance

After sorting all the fitness functions by using interval confidence level, individuals with different ranks are obtained. For individuals having the same rank, the crowding distance is employed to sort those individuals further and obtain evenly distributed Pareto front and Pareto optimal solutions [42]. However, the previous crowding distance is only suitable for determinate objectives, not the interval objectives. Consequently, appropriate redefinition is required on the crowding distance of the interval objectives.

Compared to the point objectives of a deterministic problem, the overlap of interval objectives should be considered when calculating the crowding distance of the interval objectives. Thus, Gong et al. [40] proposed a novel crowding distance procedure suitable for interval objectives, calculated according to the overlap degree, the generalized interval volume, and the distance between the midpoints of the intervals.

For the two evolutionary individuals  and  with the same rank, the *i*th objective functions corresponding to them are  and , respectively, where *i*=1, 2, ⋯, *m*. Their overlap interval is defined as  and the width is defined . The overlap degree of the individuals  and  can be expressed as

|  |  |
| --- | --- |
|  | (28) |

If the hyper-volumes of  and  are defined as  and , respectively, the distance between the intervals x1 and x2 can be written as

|  |  |
| --- | --- |
|  | (29) |

Assuming that the two evolutionary individuals with the same rank nearest to  (obtained through Eq. (29)) are  and , respectively, the crowding distance of  can be expressed as

|  |  |
| --- | --- |
|  | (30) |

This calculation of crowding distance not only includes the hyper-volume of the interval objectives but also considers the overlap degree between the different interval objectives. The crowding distance of  is determined by the objectives of the two individuals closest to .

## 4. Numerical examples

### 4.1 Uncertainty of FRF data

In order to numerically evaluate the robustness of the proposed procedure that can deal with the uncertainty in measurement data, the method is applied to the antiresonant frequency assignment of the ﬁve-DOF system depicted in Fig. 3 (adopted from Ref [5]). The values of the original system parameters are listed in Table 1, where the same stiffness value  is assumed for all the ground springs and N/A indicates that the parameter cannot be changed.

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Fig. 3. Five-DOF undamped system in Ref [5].

Table 1. System parameter nominal values.

|  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| parameters | (kg) | (kg) | (kg) | (kg) | (kg) | (kN/m) | (kN/m) | (kN/m) | (kN/m) | (kN/m) |
| Nominal values | 1.73 | 5.21 | 8.21 | 2.61 | 1.34 | 98.9 | 73.6 | 68.2 | 73.5 | 82.1 |
| Lower bound | 0 | 0 | 0 | 0 | 0 | 0 | N/A | N/A | N/A | N/A |
| Upper bound | 2 | 2 | 2 | 2 | 2 | 300 | N/A | N/A | N/A | N/A |

To highlight the superiority of the proposed method in dealing with the uncertainty of data, all related FRFs are assumed to contain noise. These FRFs with uncertainty in the vibration system can be written as

|  |  |
| --- | --- |
|  | (31) |

where  is the FRF with uncertainty (noise),  is the noise-free FRF, which can be directly computed by inverting the matrix  at the frequency range of interest, and *un* is the uncertainty level (assumed as 5% in this study). The lower and upper bounds of the uncertain FRF can then be expressed as

|  |  |
| --- | --- |
|  | (32) |

As far as the previous research on antiresonant frequency assignment is concerned, the structural modifications of a real system are directly obtained by using a deterministic optimization algorithm, where the influence of measurement errors is neglected in the optimization calculation [11,31]. Therefore, a comparison is made between the method proposed and the deterministic optimization technology to highlight the effect of measure errors on structural modification determination and evaluate the effectiveness of the proposed approach. Among the deterministic optimization methods reported in the literature, NSGA-II proposed by Parmar et al. is one of the most popular MOP methods used for finding the optimal solutions [29]. In the comparison proposed, the convergence condition was set as the fitness function of all the sub-objects less than 2e-6. (The fitness function of MI-NSGA-II is the average of the lower and upper bounds of the objective interval.)

In this study, the antiresonant frequencies of point FRF  of the original system are listed in Table 2, and are required to change. The goal is to assign the two antiresonant frequencies mentioned in the second column of Table 3. It is assumed that all the masses and grounded springs are modiﬁable, leading to ten design variables. The feasible domain of each design variable is listed in rows 3–4 of Table 1. The same bounds are also adopted when applying the NSGA-II method.

Table 2. Antiresonant frequency for point-FRF  of the original system.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| Antiresonant frequency | 1 | 2 | 3 | 4 |
| Original frequency (Hz) | 25.03 | 42.90 | 50.36 | 64.67 |

It is important to note that all the antiresonant frequencies presented in this study are directly identified from the eigenvalues of ; thus, the measurement noise does not affect the identified results.  and  are the matrices obtained by removing the second row and the second column from matrices **M** and **K**, respectively.

Table 3. Desired frequency and modiﬁed antiresonant frequency of .

|  |  |  |  |
| --- | --- | --- | --- |
| Antiresonant frequency | Desired frequency (Hz) | Assigned frequency (Hz) | |
| MI-NSGA-II | NSGA-II |
| 1 | 30 | 29.98 | 28.83 |
| 2 | 60 | 60.01 | 59.67 |

With the aforementioned configuration parameters, the two methods yield the optimal modifications or solutions (those values behind +) listed in Table 4.

Table 4. System parameter after modification.

|  |  |  |
| --- | --- | --- |
|  | MI-NSGA-II | NSGA-II |
| (kg) | 1.73+0.53 | 1.73+0.46 |
| (kg) | 5.21+0.81 | 5.21+1.08 |
| (kg) | 8.21+0.86 | 8.21+0.16 |
| (kg) | 2.61+0.57 | 2.61+1.44 |
| (kg) | 1.34+1.35 | 1.34+1.51 |
| (kN/m) | 98.9+147.74 | 98.9+134.47 |
| (kN/m) | 98.9+209.48 | 98.9+118.88 |
| (kN/m) | 98.9+102.01 | 98.9+71.14 |
| (kN/m) | 98.9+149.22 | 98.9+58.76 |
| (kN/m) | 98.9+166.02 | 98.9+112.08 |

The antiresonant frequency frequencies of the five-DOF system, updated by applying the system parameters in Table 5, are shown in Table 4. What stands out in Table 4 is that the improved algorithm (MI-NSGA-II) provides more accurate results compared with those provided by NSGA-II. As a piece of evidence to verify the results obtained, Fig. 4 shows the amplitudes of the noise-free FRF  of the original system and the systems modiﬁed by employing MI-NSGA-II and NSGA-II.

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Fig. 4. FRF comparison between the two algorithms.

On average, the algorithms’ performance is reﬂected through the antiresonant frequency mismatches. To find out how uncertainty levels of FRFs affect optimal solutions, FRFs with four different uncertainty levels are used in antiresonant frequency assignment. For each uncertainty level, 500 system modifications for the same tasks are carried out. The antiresonant frequency variability for all uncertainty levels is displayed in Fig. 5. The distribution of antiresonant frequencies obtained through the MI-NSGA-II is more concentrated than those obtained through the NSGA-II. Notably, NSGA-II cannot ensure the accuracy of the obtained antiresonant frequency, and the uncertainty of FRFs usually leads to poor structural modification solutions; in contrast, the assigned antiresonant frequency obtained by employing MI-NSGA-II shows higher accuracy and better robustness, regardless of the uncertainty levels. This means that the MS-NSGA-II algorithm introduced in this paper can tolerate poor FRF measurements and shows higher reliability in actual noisy environments, making receptance-based inverse structural modifications easier to realize.

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Fig. 5. Variability of antiresonant frequencies.

### 4.2 Uncertainty of system parameters

A U-shaped frame has fixed ends and a hollow section with internal loads (fluid). It is connected to the ground using five elastic supports, P1 to P5. The overall dimensions of the frame are shown in Fig. 6. Its material is 304L stainless steel with Young’s modulus of 185 GPa and density of 7930 kg/m3, which can be obtained experimentally. Other properties of the frame are presented in the first and second columns of Table 5. The elastic support consists of a point mass element weighing 0.25 kg and a spring element with a stiffness of 20 N/mm (as shown in the partial-view sketch of Fig. 6).

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Fig. 6. Complex beam model with elastic supports.

Table 5. Parameters of the complex beam system.

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Hollow annular section | |  | Elastic support | |  | Internal fluid | |
| Parameter | Outer diameter | Thickness |  | Mass | Stiffness |  | Lower bound | Upper bound |
| Nominal value | 16 mm | 1.5 mm |  | 0.25 kg | 20 N/mm |  | 1.6 kg | 2.1 kg |

The internal fluid presents a distributed load and additional mass. It should be noted that the magnitude of the mass varies in the range of 1.6 kg–2.1 kg (as shown in columns five and six of Table 5), which is intentionally introduced as a type of uncertainty associated with the system parameters and can be described as an interval variable.

Using the aforementioned parameters, the finite element model of the frame is established using the ANSYS Multiphysics simulation software [45]. Subsequently, the cross- and point-FRFs of the observation points, P1–P5, in the Z-direction (also the direction of the gravitation) are calculated by employing harmonic analysis. The stiffness of all the five elastic supports could be changed in the range of [0, 320], which means that the upper and lower bounds of stiffness modification are -20 and 300, respectively. The point-FRF  is assumed as the target of structural modification to assign antiresonant frequencies. Fig. 7 displays the magnitude variability of  for different internal loads. The desired antiresonant frequency frequencies of the uncertain system are 40 Hz and 100 Hz and are marked in this figure.

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Fig. 7. Amplitude variability of point-FRF  in Z-direction for different internal loads.

The receptance method is employed to establish the interval objective functions, and the interval optimization algorithm is used to determine the optimal solutions (modifications). In order to assess the superiority of MI-NSGA-II in effectiveness and stability, INSGA-II is employed to obtain optimal modifications too, as a contrasting validation. The population of both algorithms is set to 100 and the number of running steps is set to 2000. More especially, HV-indicator (hypervolume-indicator) and U-indicator (uncertainty-indicator) are introduced in this paper to compare the performance of the two algorithms. HV-indicator represents the overall accuracy of the population at each step of running the algorithm and can be understood as the number of individuals in the evolutionary population whose dominance is superior to that of the reference point ((1e-4, 1e-4) in this work); U-indicator represents the overall uncertainty of the population and can be expressed as the sum of the widths of all individual fitness function intervals (Eq. (33)) [41].

|  |  |
| --- | --- |
|  | (33) |

In order to highlight the difference between the two interval optimization algorithm on stability, 20 optimization calculations for the same tasks are carried out, and the best and worst cases are selected based on the HV-indicator and U-indicator and shown in Fig. 8. From the HV-indicator at generation 2000, the accuracy of almost all individuals of the two algorithms exceeds 1e-4, whether the case is good or bad. This means that both algorithms have equal excellent performance in terms of accuracy. However, the U-indicator at generation 2000 of the best case and the worst case of MI-NSGA-II is almost equal, while that of the worst case of INSGA-II is twice that of the best case. This clearly shows that MI-NSGA-II has better stability, compared with INSGA-II.

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Fig. 8. Performance comparison of the two algorithms: (a) HV-indicator; (b) U-indicator.

To better explain the effect of the instability on the results of antiresonant frequency assignment, the modifications of the worst and best cases of the two algorithms are obtained and shown in Table 6, and the corresponding variability of  after modification is shown in Fig. 9.

Table 6. Stiffness parameters of the updated elastic supports.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| Stiffness of elastic support | | (N/mm) | (N/mm) | (N/mm) | (N/mm) | (N/mm) |
| INSGA-II | Worst case | 20.00+107.34 | 20.00+0.00 | 20.00+245.64 | 20.00+11.07 | 20.00+190.27 |
| Best case | 20.00+27.85 | 20.00+0.00 | 20.00+44.29 | 20.00+205.07 | 20.00+48.39 |
| MI-NSGA-II | Worst case | 20.00+3.67 | 20.00+0.00 | 20.00+40.92 | 20.00+199.05 | 20.00+53.83 |
| Best case | 20.00+12.31 | 20.00+0.00 | 20.00+28.38 | 20.00+291.36 | 20.00-20.00 |

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Fig. 9. Variability of point-FRF  after modification: (a) worst case of INSGA-II; (b) best case of INSGA-II; (c) worst case of MI-NSGA-II; (d) best case of MI-NSGA-II.

Table 7 displays some key information about the antiresonant frequency assignment of the two different algorithms. The errors in the assigned antiresonant frequencies by MI-NSGA-II are all, clearly, not larger than those obtained by INSGA-II, and the antiresonant frequencies assigned by MI-NSGA-II have a smaller change interval (interval width) compared with those assigned by INSGA-II. This means that the modifications computed through MI-NSGA-II can render the assigned antiresonant frequencies insensitive to uncertainty caused by the additional mass. In other words, these results indicate that the assigned antiresonant frequencies obtained by MI-NSGA-II have better robustness compared with those obtained by INSGA-II. Moreover, it is clear from Fig. 9 and Table 7 that the assigned antiresonant frequencies of the best and worst cases obtained by using the MI-NSGA-II algorithm have a smaller difference, compared with the INSGA-II algorithm. These results demonstrate that the proposed MI-NSGA-II solves the instability problem of the traditional INSGA-II and improves the reliability of the interval optimization algorithm when dealing with antiresonant frequency assignment.

Table 7. Results of antiresonant frequency assignment.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| After modification | | | Desired frequency | Frequency interval | Errors | Interval width |
| INSGA-II | Worst case | First target | 40.00 Hz | [39.75, 41.50] Hz | [-0.6%, +3.8%] | 1.75 Hz |
| Second target | 100.00 Hz | [98.17, 102.00] Hz | [-1.8%, +2.0%] | 3.83 Hz |
| Best case | First target | 40.00 Hz | [39.25, 40.77] Hz | [-1.9%, +1.9%] | 1.52 Hz |
| Second target | 100.00 Hz | [99.32, 101.75] Hz | [-0.7%, +1.8%] | 3.43 Hz |
| MI-NSGA-II | Worst case | First target | 40.00 Hz | [39.50, 41.02] Hz | [-1.2%, +2.5%] | 1.52 Hz |
| Second target | 100.00 Hz | [99.57, 102.00] Hz | [-0.4%, +2.0%] | 3.43 Hz |
| Best case | First target | 40.00 Hz | [39.25, 40.70] Hz | [-1.9%, +1.8%] | 1.45 Hz |
| Second target | 100.00 Hz | [99.50, 102.96] Hz | [-0.5%, +3.0%] | 3.46 Hz |

## 5. Experimental validation of the proposed method

### 5.1 Experimental setup

The frame structure shown in Fig. 6 is employed for the experimental validation of the method in this study. It is connected to a very rigid test bench by five elastic supports (as shown in Fig. 10). The wire spring of each elastic support provides effectively the same vertical (Z-direction) stiffness as the spring element in Fig. 6, and a third of the mass of the wire spring and the mass of the cover provide an equivalent point mass of the whole spring element. Furthermore, the frame body is installed in parallel to the bench so that the elastic supports always remain perpendicular to the bench, so that the horizontal stiffness of the supports does not affect the dynamic performance of the system during the frame’s vertical vibration. All other relevant dimensions and parameters shown in Fig. 10 are already given in Fig .6.

The internal mass comes from filled water, which affects only the mass of the frame, but not its stiffness, since the fluid-structure interaction (FSI) is negligible in this condition [46]. Another advantage of using the filled water is that it is easy to change this mass parameter. Therefore, the setup, very accurately, represents the uncertain continuous system that has already been depicted in Fig. 6.

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Fig. 10. Frame and constraints.

In the antiresonant frequency assignment problem, the wire springs of all the elastic supports are modiﬁable. All point- and cross-FRFs of modification-related locations (measurement points P1–P5) could be experimentally determined by carrying out impact tests [47]. The identiﬁcation is performed by exciting the measurement points with an impact hammer. The impact hammer used is an LC02, which can deliver an impact force of up to 5 kN to the system. The excitation force is measured by a load cell (Model 3A102) embedded inside the hammer. Each measurement point is equipped with a Donghua piezoelectric accelerometer (Model 1A116E). All signals are recorded through a DH5902 signal acquisition instrument connected to a PC. A dynamic signal acquisition and analysis software system is adopted to process the recorded signals and identify antiresonant frequency by applying the least squares method outlined in Ref. [11].

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Fig. 11. Point- and cross- FRFs of all measurement points (P1–P5) in the Z-direction.

In order to acquire the FRFs of the observation points mentioned in Section 4.2, the directions of excitation and response should remain perpendicular to the ground as much as possible. All the required point- and cross-FRFs of the original system were then measured (as shown in Fig. 11). It should be noted that the measurements only involve the system FRFs under the extreme values of the mass parameter (+1.6 kg and +2.1 kg) since the uncertainty due to this mass and the obtained FRFs are in interval form. Apart from the uncertainty of the physical parameters, the experimental FRFs contain uncertainty, given the unavoidable presence of measurement noise, unmodelled structural damping and possible nonlinearities.

### 5.2 Parameter modification and results in discussion

The goal of structural modiﬁcation is to assign the antiresonant frequency (40 Hz and 100 Hz) reported in Table 7 to FRF  of the frame with uncertain parameters and measurement errors. In order to ensure the physical realizability of the modiﬁed system, the lower and upper bounds of the solutions (modifications) must be defined (see the third and fourth columns of Table 8). The proposed MI-NSGA-II and the classic INSGA-II are then adopted to solve this interval MOPs, respectively. For each interval optimization algorithm, 500 optimization calculations for the same tasks are carried out, and the corresponding best case is selected as the performable optimal modifications by employing the HV-indicator and U-indicator. The obtained best modifications by the two algorithms are listed in the fifth and seventh columns of Table 8, respectively. It should be noted that the experimental modification results differ somewhat from the numerical results outlined in Section 4.2, likely due to the fabrication and installation errors of the test model.

Table 8. Modiﬁcation bounds and values.

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | | Lower bound | Upper bound | INSGA-II | | MI-NSGA-II | |
|  | |  |  | Computed | Applied | Computed | Applied |
| Modifications | (kN/m) | -20 | 300 | +69.54 | +64 | +2.19 | 0 |
|  | (kN/m) | -20 | 300 | -5.51 | 0 | -4.3 | 0 |
|  | (kN/m) | -20 | 300 | +106.29 | +104 | +36.66 | +37 |
|  | (kN/m) | -20 | 300 | +32.71 | +37 | +221.37 | +224 |
|  | (kN/m) | -20 | 300 | +174.34 | +171 | -2.68 | 0 |

The physical implementation of the optimal modifications leads to some approximations of the computed results, as a result of the restricted practical availability of stiffness. The applied modifications are shown in the sixth and eighth columns of Table 8. The structural modiﬁcations are realized by replacing the original wire springs with new ones with the required stiffness (), which enables the closest approximation for the theoretically determined stiffness modiﬁcation. In this study, the stiffness calculation and structural design of wire springs are not investigated, and the updated wire springs are directly selected from the spring design standard in Ref [48].

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Fig. 12. Experimental FRF  of the modified complex beam system with uncertain loads: (a) INSGA-II; (b) MI-NSGA-II.

The procedure described in Section 5.1 is implemented to measure the FRFs  of the modiﬁed system at various water mass values (as displayed in Fig. 12). The antiresonant frequency variability values are then calculated and recorded in Table 9. The modifications calculated using INSGA-II lead to the assigned antiresonant frequencies with significant errors. In contrast, the effect of uncertainty on the calculation results of MI-NSGA-II is small and can be neglected, and the obtained modifications can be used to realize the objectives of antiresonant frequency assignment. More importantly, the antiresonant frequency frequencies implemented using the modifications determined using MI-NSGA-II show a better robust compared with those obtained by INSGA-II. Such impressive comparison results demonstrate that the proposed MI-NSGA-II has a better performance in accuracy and stability than INSGA-II when dealing with the robust assignment of antiresonant frequency of a real uncertain system.

Table 9. Results of antiresonant frequency assignment.

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | |  | INSGA-II | |  | MI-NSGA-II | |
|  | First target | Second target |  | First target | Second target |
| Desired frequency (Hz) | | | 40.00 | 100.00 |  | 40.00 | 60.00 |
| After modification | Frequency interval | Value (Hz) | [37.6, 39.53] | [96.88, 100.64] |  | [39.06, 40.47] | [97.57, 101.09] |
| Error | [-6.0%, -1.2%] | [-3.1%, +0.6%] |  | [-2.4%, +1.2%] | [-2.4%, +1.1%] |
| Interval width | Value (Hz) | 1.93 | 3.76 |  | 1.41 | 3.52 |
| Difference | N/A | N/A |  | -26.9% | -6.4% |

## 6. Conclusions

In this study, a receptance-based interval multi-objective optimization method for assigning antiresonant frequency to uncertain dynamic systems is developed. This method describes the uncertainty of the system in an interval form; thus, it is unnecessary to collect a large amount of data to construct a data sample. Uncertainty in the physical parameters and measurement data of a system can thus be handled. A series of evidence, including numerical and experimental verifications, of the proposed method's effectiveness, is provided. The proposed method can overcome the problems of unsuccessful structural modification due to such uncertainty to a certain extent. This study enhances the chance of success in finding the right structure modification in practice.

In this work, the optimal solutions (modifications) are obtained by using the proposed MI-NSGA-II algorithm, and by physically implementing those optimal solutions, the desired antiresonant frequencies can be assigned accurately, and the robustness of the assigned antiresonant frequencies to uncertain factors can also be optimized simultaneously. MI-NSGA-II employs the basic framework of the interval multi-objective genetic algorithm, INSGA-II, and redefines interval confidence level by introducing a correction term to overcome the premature convergence of INSGA-II. When dealing with the antiresonant frequency assignment of a system with uncertain parameters, the proposed MI-NSGA-II is more stable and the assigned antiresonant frequencies are also more robust compared with the classic INSGA-II.

Finally, some limitations of this work should be noted: on the one hand, the uncertainty of FRF, which is assumed to be known in this paper, is difficult to quantify in practice, and so interval quantification of FRF uncertainty should be studied to ensure the effectiveness of structural modification in practice; in addition, it is difficult for the proposed MI-NSGA-II algorithm to achieve convergence for high-dimensional problems, which will be dealt with in future research.

## Author Contributions

**Lin Zhang:** Methodology, Formal analysis, Investigation, Visualization, Software, Writing - original draft. **Tao Zhang:** Investigation, Supervision, Software, Funding acquisition, Resources, Writing - review & editing. **Huajiang Ouyang:** Conceptualization, Methodology, Formal analysis, Validation, Writing - review & editing. **Tianyun Li:** Conceptualization, Validation, Funding acquisition, Data curation.

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## Conflict of interest

We declare that we do not have any commercial or associative interest that represents a conflict of interest connected with the work submitted.

## Appendix A

The established receptance matrices with extreme load parameters atanddescribed in Section 5 are presented in the following, , , and .









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