

Active exterior cloaking of an inclusion with evanescent multipole devices for flexural waves in thin plates

by F. J. P. Allison, Ö. Selsil and S. G. Haslinger

(Department of Mathematical Sciences, University of Liverpool,
Liverpool L69 7ZL)

Abstract

We present an active exterior cloak for flexural waves propagating in a Kirchhoff plate of infinite extent. The evanescent multipole devices are characterised by Macdonald functions K_n of required order, which, assuming time-harmonic vibrations, are solutions of the fourth-order biharmonic equation. It is shown that in the region of interfering waves, which emanate from the devices, a field is recreated which cancels the incident wave to yield a region of “stillness”. An inclusion is then positioned in this region for further investigation, with additional attention given to the boundary condition.

1 Introduction

Thousands of articles have been published since the groundwork of the science of optical illusion was laid. Transformational optics began with Dolin [1] who demonstrated the form-invariance of the Maxwell equations under a space-deforming transformation and proposed a physical application whereby a plane wave passes through a spherical material inhomogeneity without distortion (also see [2]). Much more recently, transformational cloaking was re-established with Greenleaf *et al.* [3] who studied the invariant form of the conductivity equations under coordinate transformations, and thereafter the invariance of the Maxwell equations were investigated independently by Leonhardt [4] and Pendry *et al.* [5], both articles displaying how to guide electromagnetic waves around a region to be cloaked. Other contemporaneous studies included the work by Alù and Engheta [6], which presented a design of lossless metamaterial coatings to drastically reduce the scattering cross section of spherical and cylindrical objects, making them nearly invisible. However, it is well known that the idea of neutral inclusions goes back to Mansfield [7] who examined how to make certain reinforced holes without altering the stress distribution in a uniformly stressed plate. Similar ideas were soon broadened from electromagnetic waves to sound waves, water waves and elastic waves to count a few. More recently, passive cloaks that surround the object to be hidden from the incoming wave were replaced by active ones which adjust to different frequencies, shape of the inclusion to be cloaked, as well as the boundary conditions posed on the inclusion itself. Milton and his co-workers’ seminal work [8, 9, 10] on an active exterior cloak, where the cloaking region lies outside the cloaking devices, in contrast to an interior cloak, where a cloaking device surrounds the object to be cloaked, was introduced for waves governed by the two-dimensional Laplace and Helmholtz equations. The generalisation of their novel idea to three-dimensions followed shortly after in [11], and their explorations for the dynamic case were consecutively complemented by Norris *et al.* in [12] where analytical expressions for the device amplitude coefficients for general incidence were given.

It was recently shown by O’Neill *et al.* in [13] that a small number of active monopole sources were sufficient to cloak a clamped inclusion in a Kirchhoff plate. A different approach, based on [8, 9, 10], using multipole devices was devised by Futhazar *et al.* [14] to create a finite “still” region and to ensure that only the incident field was present in the far field. O’Neill *et al.* later extended their investigation to the cloaking of coated inclusions in thin plates for frequency ranges in which scattering resonances occur [15] and to the cloaking of finite clusters of pins in thin plates [16]. Here, we present the idea of using active evanescent multipole devices to cloak and ‘shield’ an inclusion in a thin plate. In particular, we make use of the fact that the solution to the time-harmonic biharmonic equation consists of not only Hankel functions of the first kind, $H_n^{(1)}$, but also Macdonald functions K_n (see [17]). By shielding we convey a finite region of “stillness” in the vicinity of the inclusion. The reason we use the quotation marks is to emphasise, here and

in what follows, the approximate nature of the stillness (or silence for acoustics) of these regions. It should be noted that, although the Green’s function for the biharmonic equation is in fact bounded, we utilise solely the Macdonald functions to construct our elastic cloak and therefore consent to infinite fields at the device centres since $K_0(z) \sim \log(z)$ and $K_n(z) \sim 1/z^n$ as $z \rightarrow 0$. Unlike the Hankel functions however, (used as a basis to formulate the acoustic exterior cloak in [8, 9, 10]), the fields emanating from the evanescent devices decay at infinity and thus require less attention than their acoustic analogue. It is known that clamped inclusions in thin plates possess the most scattering signature, as illustrated in Fig. 1(a). We preview our results with the use of three active evanescent devices in Fig. 1(b).

The structure of the article is as follows. In Section 2, we formulate the problem for an inclusion in an infinite Kirchhoff plate and present the general solution. Section 3 introduces the evanescent multipole devices of which, three are initially used to create a region of “stillness” in our first simulation. Next, we employ a second simulation to compare our previous results with the outcome of a configuration of two devices. We conclude that the former is preferable, although the latter may deserve more attention in the future. The study of cloaking and shielding a circular inclusion is presented in Section 4. Finally, we conclude and briefly discuss future plans in Section 5.

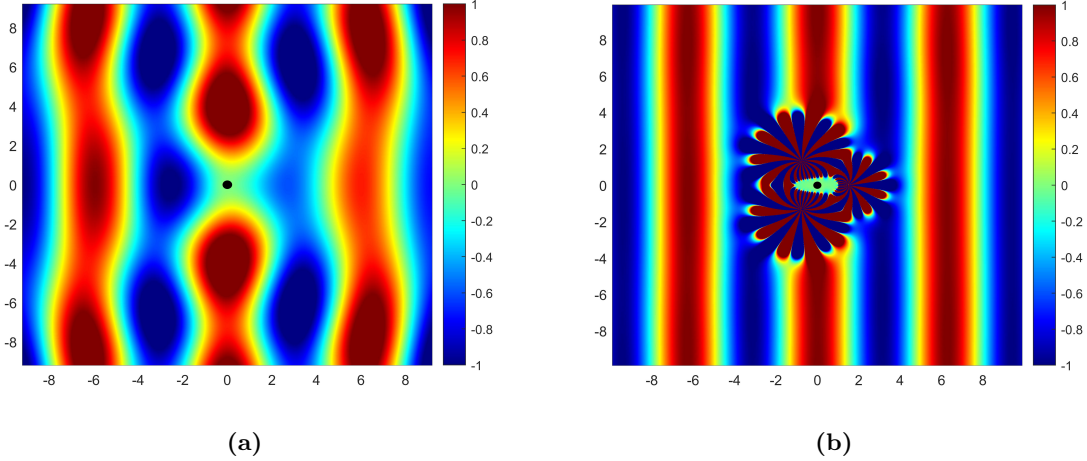


Figure 1: (a) The total field u_{tot} obtained for a plane wave incident on a clamped inclusion. (b) The total field after the introduction of three active evanescent devices creating an effective cloak and a finite region of “stillness” surrounding the inclusion, represented by the green zone.

2 Problem formulation

The out-of-plane elastic displacement $W(\mathbf{x}; t)$ in an infinite Kirchhoff plate satisfies the equation of motion

$$D\nabla^4 W + \rho h \frac{\partial^2 W}{\partial t^2} = 0, \quad \mathbf{x} \in \mathbb{R}^2, t \in (0, \infty),$$

where $\mathbf{x} = (x_1, x_2)$, $\nabla^4 = \Delta^2$ is the biharmonic operator, ρ, h and $D = Eh^3/[12(1 - \nu^2)]$ are the mass density, thickness of the plate and plate’s flexural rigidity, respectively, with E and ν representing the Young’s modulus and the Poisson’s ratio of the elastic material. Assuming time-harmonic vibrations, $W(\mathbf{x}; t) = w(\mathbf{x}) \exp(i\omega t)$, the governing equation above may be reduced to

$$(\nabla^4 - \beta^4) w = (\nabla^2 + \beta^2) (\nabla^2 - \beta^2) w = 0, \quad \mathbf{x} \in \mathbb{R}^2, \quad (2.1)$$

where $\beta^2 = \omega \sqrt{\rho h / D}$ is the spectral parameter. The general solution of (2.1) is a linear combination of solutions to the Helmholtz and modified Helmholtz equations, and may be written in polar

coordinates r and θ as

$$w = \sum_{n=-\infty}^{\infty} [A_n J_n(\beta r) + E_n H_n^{(1)}(\beta r) + B_n I_n(\beta r) + F_n K_n(\beta r)] e^{in\theta}, \quad (2.2)$$

where J_n are the Bessel functions, $H_n^{(1)}$ are the Hankel functions and I_n , K_n are the modified Bessel functions of first kind and the Macdonald functions of order n , respectively. In the presence of an inclusion \mathcal{D} , we propose to consider clamped boundary conditions, that is

$$w = \frac{\partial w}{\partial \mathbf{n}} = 0, \quad \mathbf{x} \in \partial \mathcal{D}. \quad (2.3)$$

Here \mathbf{n} is the unit outward normal. In this case, equation (2.1) is satisfied for $\mathbf{x} \in \mathbb{R}^2 \setminus \overline{\mathcal{D}}$, where $\overline{\mathcal{D}}$ denotes the inclusion together with its boundary.

3 Evanescent multipole devices

The objective is to cultivate a region of “stillness” in the vicinity of the origin of the Kirchhoff plate where the incident wave does not propagate; in theory, within this region an inclusion may be situated without inducing a scattered field. We follow the footsteps of the ground-breaking work by Milton and his co-workers (see [8, 9, 10]), however, as discussed in the problem formulation, here we are dealing with the fourth-order biharmonic equation instead of the second-order Helmholtz equation. The presence of the modified Bessel functions in the solution of the biharmonic equation provides us with additional flexibility. By choosing only the Macdonald functions, K_n , as the basis for the multi-polar sources, and neglecting the Hankel functions, we construct the so-called evanescent multipole devices. As a result, the analogous treatment of the active exterior cloak for a thin plate somewhat simplifies. In fact, we only require that the devices cancel the incident wave u_{inc} inside a small closed region Ω surrounding the origin, since the constraint of decay at infinity is automatically satisfied by the nature of the evanescent devices.

We denote the sum of the fields generated by Q evanescent devices located at a distance $|\mathbf{x}_s^{(q)}| = \delta$ from the origin (outside the region Ω) by $u_{\text{dev}}(r, \theta)$ and require that $u_{\text{dev}} = -u_{\text{inc}}$ for $r \in \Omega$. The desired field may be actualised by taking a linear combination of outgoing waves emanating from the devices’ centres $\mathbf{x}_s^1, \dots, \mathbf{x}_s^Q$ as

$$u_{\text{dev}} = \sum_{q=1}^Q \sum_{n=-\infty}^{\infty} B_{q,n} K_n(\beta r^{(q)}) e^{in\theta^{(q)}}, \quad (3.1)$$

where $B_{q,n}$ are the amplitudes associated with the q^{th} device of n^{th} multipole order, and $r^{(q)} = |\mathbf{x} - \mathbf{x}_s^{(q)}|$ and $\theta^{(q)} = \arg(r^{(q)})$. We truncate the infinite summation from $-N$ to N and denote the total number of poles in use as $M = 2N + 1$. In order to determine the coefficients $B_{q,n}$ numerically, as in [8, 9], we first discretise the boundary of Ω into P points $\mathbf{p}_1, \dots, \mathbf{p}_P$. The resulting linear equations are $\mathbf{A}\mathbf{b} \approx -\mathbf{u}_{\text{inc}}$, where $\mathbf{b} \in \mathbb{C}^{QM}$ is a vector with the coefficients $B_{q,n}$, the matrix $\mathbf{A} \in \mathbb{C}^{P \times QM}$ is constructed so that $(\mathbf{A}\mathbf{b})_k = u_{\text{dev}}(\mathbf{p}_k)$ and $-\mathbf{u}_{\text{inc}} \in \mathbb{C}^P$ is built by using discrete values of a given incident field, and as such $(\mathbf{u}_{\text{inc}})_k = u_{\text{inc}}(\mathbf{p}_k)$. In fact, u_{inc} is chosen to be an incident plane wave defined explicitly and in terms of its Jacobi-Anger expansion as

$$u_{\text{inc}} = e^{i\beta r \cos \theta} = \sum_{n=-\infty}^{\infty} i^n J_n(\beta r) e^{in\theta}. \quad (3.2)$$

As we work with the fourth order biharmonic equation, it is necessary to note that the normal derivative of the total field on the boundary of Ω must be small, and this has been verified numerically. Also note that, unlike in [8, 9], a condition imposed sufficiently far away from the devices, to uphold the decay of the radiating fields is not necessary.

3.1 Simulations

An overdetermined system of equations with $P > QM$ may be solved, in the least squares sense, using the singular value decomposition (SVD) method (see, for example, [18] ch. 12). Utilising SVD we may re-write the coefficient matrix as $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\dagger$, where the superscript \dagger denotes the Hermitian transpose. Here \mathbf{U} and \mathbf{V} are unitary matrices and $\mathbf{\Sigma}$ is a rectangular diagonal matrix, the elements of which are the singular values of \mathbf{A} . Multiplying through by the Moore-Penrose inverse $\mathbf{A}^+ = \mathbf{V}\mathbf{\Sigma}^{-1}\mathbf{U}^\dagger$, we solve for an optimal choice of coefficients for given parameters, δ , β , M and the geometrical dimensions of Ω . In Fig. 2(a), we illustrate the combined field $u_{\text{com}} = u_{\text{dev}} + u_{\text{inc}}$ (see (3.1),(3.2)) for $\beta = 1$, with the device centres distributed evenly on a circle of radius $\delta = 1.5$, and its close-up is shown in Fig. 2(b). For all the figures in this section the cloaked region Ω is chosen to be elliptical with minor and major axes 0.3 and 1, respectively, and a boundary discretised into $P = 200$ points. For these given parameters we observe that assigning $M = 31$

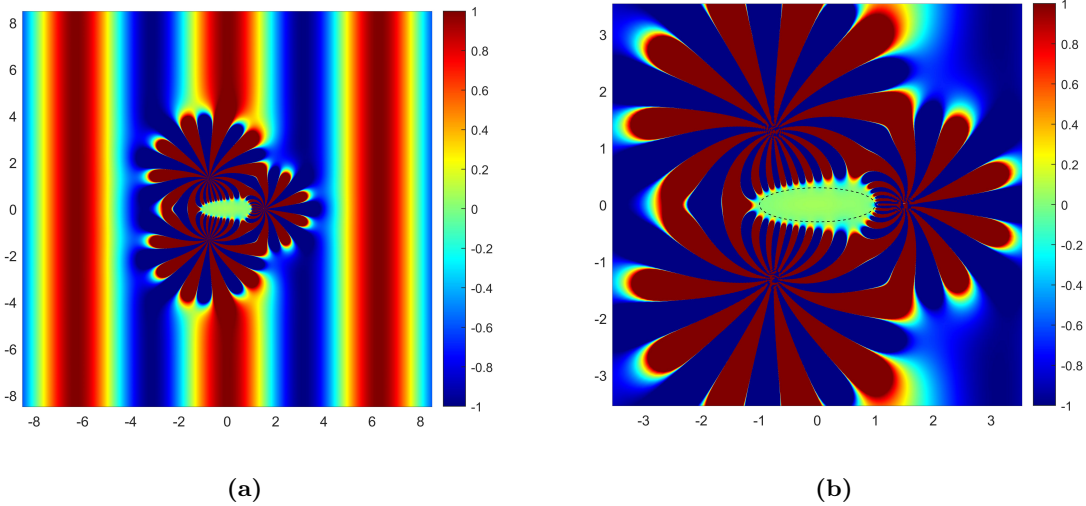


Figure 2: (a) The real part of the field $u_{\text{com}} = u_{\text{dev}} + u_{\text{inc}}$ with three devices. (b) A close-up of (a) with the superimposed dashed black line indicating the boundary of the elliptical region we discretise and impose the condition $u_{\text{dev}}(\mathbf{p}_k) \approx -u_{\text{inc}}(\mathbf{p}_k)$. The parameters are $\beta = 1$, $Q = 3$, $N = 15$ and $\delta = 1.5$.

unique amplitudes to each device yields the optimum silent region; introducing additional poles requires re-evaluation of the scaling of the problem. Note that It is possible to observe from Fig. 2(a) that the evanescent cloak only perturbs the local fields whilst leaving the propagating far-field intact. The green zone at the omphalos of the figure is a “still” region where the plate is immune from the effects of the incident wave, and therefore a haven to cloak an object from detection. In [8, 9, 10] it has been proven that the configuration must consist of at least three devices to act as an exterior cloak, the silent region of which ‘is the complement of the union of the three disks’ [10]. However, we experiment with the idea of using fewer than three devices and in Fig. 3(a) we present a cloaking attempt with the same parameters as in Fig. 2(a) but for two, rather than three, devices.

In Fig. 3(a), two devices with up-down symmetry, visibly create a region of “stillness” that looks comparable to the three-device setup. Quantitatively, we may depict the displacement inside the cloaked region by measuring the field $u_{\text{com}} = u_{\text{dev}} + u_{\text{inc}}$ on two concentric circles, purposefully zoomed in and superimposed on Fig. 3(a) (see the inset). The dashed lines in Fig. 3(b) pertain to the two-device configuration in Fig. 3(a) and indicate that the field becomes larger toward the centre of the region of “stillness”. We attribute this growth to the evanescent nature of the cloak, in the sense that u_{dev} captures $-u_{\text{inc}}$ less well the further it propagates into the epicentre, interestingly exposing the advantage, decay at infinity, and the disadvantage, reconstructing the field $-u_{\text{inc}}$ across a larger region, of the evanescent cloak. In Fig. 3(b), we also plot the solid lines

which represent the field measured on identical concentric circles inside the “still” region of Fig. 2. It is clear that, although the displacement generated by the two devices is small, the three-device configuration provides a better protection from the incident wave. Whether the scattering from an inclusion positioned at the origin due to the remnant field inside the “still” region is negligible or not is discussed in the next section.

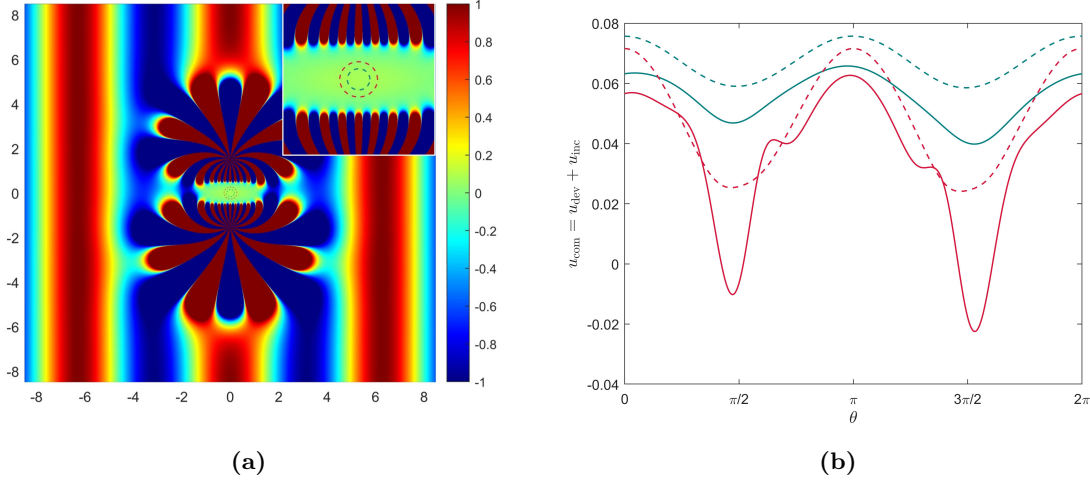


Figure 3: (a) The real part of the field $u_{\text{com}} = u_{\text{dev}} + u_{\text{inc}}$ generated by two devices positioned with up-down symmetry. The blue and red dashed concentric circles of radii 0.15 and 0.25, respectively, have been magnified and superimposed for clarity. (b) A plot of the displacements measured on the perimeter of the circles. The colour-coded dashed lines correspond to the dashed circles in (a) and the solid lines represent the field measured on identical concentric circles inside the “still” region of Fig. 2. The parameter values are exactly the same as in Fig. 2.

4 Cloaking of a clamped inclusion with evanescent multipole devices

In this section, we introduce an inclusion to the region of “stillness” and demonstrate that the active evanescent devices are effective tools to cloak even a clamped inclusion in a Kirchhoff plate. In contrast to this, as shown in [19], such a task proves impossible for a passive cloak for flexural waves. We also note that in [14] an active cloak for flexural waves in a thin plate was constructed with four devices with large multipole expansions and a reduction in these expansions had to be compensated by an increase in the number of devices themselves. Moreover, the efficacy of the devices to cloak an inclusion in the “still” region was left uninvestigated, despite this being an important aspect. A successful cloaking technique based on a small number of active monopoles in a thin plate was presented in [13]. For the sake of convenience, let us assume that the inclusion we would like to cloak is a circular disk of radius r_d . In agreement with the notations of Section 2, we let w denote the sum of the combined field $u_{\text{com}} = u_{\text{dev}} + u_{\text{inc}}$ in the “still” region and the resulting scattering u_{sca} due to u_{com} . As the disk is clamped, we simply refer to the boundary conditions (2.3), and note that the normal derivative is now in the radial direction. We are thus left with

$$-u_{\text{com}}(r_d) = u_{\text{sca}}(r_d), \quad -u'_{\text{com}}(r_d) = u'_{\text{sca}}(r_d), \quad (4.1)$$

where prime $'$ is used for derivative in the radial direction. The left-hand sides of these two equations are determined numerically and approximated by respective complex Fourier series, the coefficients of which are then used to solve for outgoing wave coefficients E_n, F_n (see (2.2)) as the

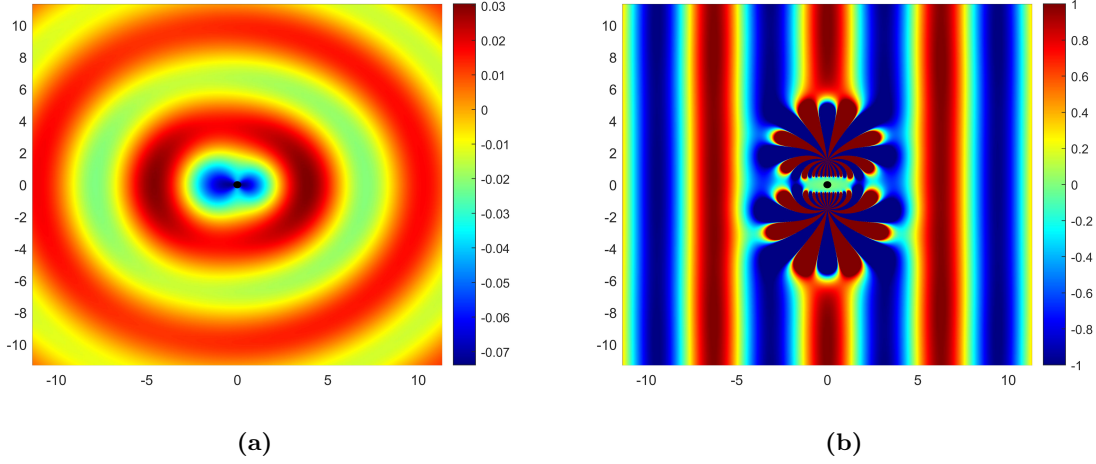


Figure 4: (a) The real part of the scattered field u_{sca} due to the combined field $u_{\text{com}} = u_{\text{dev}} + u_{\text{inc}}$ in the “still” region. The circular inclusion of radius $r_d = 0.2$ is coloured black and positioned with its centre at the origin. (b) The real part of the total field $u_{\text{tot}} = u_{\text{com}} + u_{\text{sca}}$ generated by two devices positioned with up-down symmetry. The elliptical region and other parameter values are exactly the same as in Fig. 3(a).

scattered field u_{sca} is explicitly given by

$$u_{\text{sca}} = \sum_{n=-\infty}^{\infty} [E_n H_n^{(1)}(\beta r) + F_n K_n(\beta r)] e^{in\theta}. \quad (4.2)$$

Comparing the coefficients and retaining the above series from $n = -\mathcal{N}$ to \mathcal{N} , a matrix equation $\mathbf{K}\mathbf{s} = \mathbf{c}$ is constructed where (below prime ' denotes derivative with respect to the argument)

$$\mathbf{K} = \begin{bmatrix} H_{-\mathcal{N}}^{(1)}(\beta r_d) & 0 & \dots & 0 & K_{-\mathcal{N}}(\beta r_d) & 0 & \dots & 0 \\ 0 & \ddots & & \vdots & 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 & \vdots & & \ddots & 0 \\ 0 & \dots & 0 & H_{\mathcal{N}}^{(1)}(\beta r_d) & 0 & \dots & 0 & K_{\mathcal{N}}(\beta r_d) \\ \beta H_{-\mathcal{N}}^{(1)'}(\beta r_d) & 0 & \dots & 0 & \beta K_{-\mathcal{N}}'(\beta r_d) & 0 & \dots & 0 \\ 0 & \ddots & & \vdots & 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 & \vdots & & \ddots & 0 \\ 0 & \dots & 0 & \beta H_{\mathcal{N}}^{(1)'}(\beta r_d) & 0 & \dots & 0 & \beta K_{\mathcal{N}}'(\beta r_d) \end{bmatrix},$$

\mathbf{s} is the column vector with scattering coefficients E_n, F_n to be found and \mathbf{c} is the column vector with the complex Fourier coefficients already determined.

In Fig. 4(a), we plot the scattered field u_{sca} given in equation (4.2) for an inclusion of radius $r_d = 0.2$ with its centre positioned at the origin and in Fig. 4(b) the total field u_{tot} which encompasses the incident and scattered fields as well as the fields generated by the multipole devices. We note that, as expected, the inclusion lies entirely inside the elliptical region Ω . It is visually clear from a comparison of Fig. 1(a) and Fig. 4(b) that even two active evanescent devices are perfectly capable of cloaking the inclusion when the non-zero scattering is taken into account (also see the configuration with three devices presented in Fig. 1(b)).

5 Conclusion

Our emphasis is on the adoption of evanescent multipole devices as the foundation for an active cloak in a thin plate. This novel idea, exclusive to the biharmonic equation, exploits the rapidly

decaying nature of the Macdonald functions K_n to simplify the treatment of what is otherwise a difficult cloaking problem. In this introductory paper, we establish the underlying concept and demonstrate that as few as two multipole devices successfully cloak and shield a clamped inclusion. We achieve an incremental improvement with three multipole devices and present a comparison between the two configurations. In fact, our results follow as a development to initial simulations in Section 3 that display a region of “stillness” in the absence of an inclusion.

Immediate future work will consist of a rigorous approach to understand the operation of the evanescent devices and their applications in different cloaking problems.

Acknowledgements

We would like to express our heartfelt gratitude to A. B. Movchan for the wonderful idea of active evanescent devices and for encouraging us to develop it into this article. F. J. P. Allison gratefully acknowledges EPSRC for funding his studentship.

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