Reconstruction theorems for genus 2 Gromov-Witten invariants

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by

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Abstract

We use Pixton's relations to prove a reconstruction theorem for genus 2 Gromov-Witten invariants in the style of Kontsevich-Manin (genus 0) and Getzler (genus 1). We also calculate genus 2 (descendant) Gromov-Witten invariants of $\mathbb{P}^2$ blown up in a finite number of points in general position.

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Contents

1 Introduction

The field of enumerative geometry is about counting geometric object subject to some specified constraints. For example, we can ask how many lines pass through 2 points in the plane. Euclid's first axiom states that there is 1 such line (assuming the two points are generic, i.e. *distinct*).

A less trivial question would be to ask how many lines meet 4 given lines in general position in 3-dimensional projective space. The modern approach to these kind of questions is to construct a (moduli) space that parametrizes the geometrical objects and consider the loci corresponding to the constraints. For the question we were considering there is a 4-dimensional moduli space of lines in 3-dimensional space. The constraint of meeting a line corresponds to a codimension 1 constraint in the moduli space. The fact that the 4 lines are in general position means that intersecting the 4 loci lowers the dimension by 4, and we obtain a finite number of points as the answer, which turns out to be 2. In general an enumerative question is well posed when the sum of the codimensions of the constraints equals the dimension of the moduli space.

This thesis is about Gromov-Witten theory. Gromov-Witten theory is a curve counting theory originating from theoretical physics and symplectic-differential geometry. In this theory the answers to certain enumerative questions on a complex smooth projective variety X become *Gromov-Witten invariants* of X. We will now give an overview of some of the important results in Gromov-Witten theory.

1.1 Gromov-Witten theory

The axioms of Gromov-Witten theory in algebraic geometry were set out in the nineties in the foundational paper paper [39] by Kontsevich-Manin. In loc.cit. the author explain how to construct a *compact* moduli space of stable maps to some target variety X and how to define Gromov-Witten invariants as intersection numbers on this moduli space. The problem with this approach is that the moduli space of stable maps is generally not smooth nor equidimensional or irreducible. So to do intersection theory one needs to construct a virtual fundamental class to replace the usual fundamental class. Such a construction was given in algebraic geometry by Behrend and Fantechi [8] and in [7] Behrend proved that Gromov-Witten invariants defined using this virtual fundamental class satisfy the axioms of Kontsevich and Manin.

Just before Kontsevich and Manin published their foundational paper, Ruan and Tian [57] established a mathematical foundation for the quantum cohomology ring of a semi-positive symplectic manifold. The quantum cohomology ring is constructed out of genus 0 Gromov-Witten invariants and it is the local picture of a Frobenius manifold enriching the cohomology of the target space. A suitable virtual class for the symmetric approach was constructed by Li and Tian [44] for symplectic manifolds, at the same time as Behrend-Fantechi. In [43] they proved that for a smooth projective variety the symplectic and algebraic invariants are equivalent.

Besides the quantum cohomology, the genus 0 case has a special place in the theory because for many target varieties the moduli space of stable maps of genus 0 is *unobstructed*: this in particular means that the virtual fundamental class is just the regular fundamental class. Smooth projective varieties whose genus 0 Gromov-Witten theory is unobstructed are called *convex*. For example any homogeneous variety is convex, e.g. projective space, Grassmannians, and flag varieties.

Besides intersecting loci of maps whose image meets a certain subvariety of X, it is natural to also consider intersections with ψ-classes on the moduli space of stable maps. Gromov-Witten invariants without ψ-classes are called primitive invariants while invariants with ψ-classes are called descendant invariants. In [28] an enumerative interpretation is given for genus 1 and 2 descendant invariants of the projective plane. This is done by relating the ψ-classes to tangency conditions. However, for most descendant invariants we do not have an enumerative interpretation.

Because of their abstract construction, even a primitive Gromov-Witten invariant of X is not necessarily an answer to the corresponding enumerative question about counting curves in X. This is caused by many issues, including

1. Gromov-Witten invariants count maps to X that have a curve as image: we are not just counting embedded curves in X but also covers of curves in X.

2. Somewhat related, the theory arises by calculating integrals on a moduli space where points may have nontrivial automorphisms, which could produce rational numbers as outputs.

3. Two constraints in the moduli space might fail to intersect properly and/or transversally, in which case excess intersections may arise, and the numbers can be negative.

Despite these and other issues, there are still many varieties for which the Gromov-Witten invariants are *enumerative*, i.e. they are equal to some corresponding enumerative count although sometimes in a nontrivial manner: Just as for descendant invariants, in some specific cases where the primitive invariants are not enumerative, one can prove that they are the answer to a more subtle enumerative question.

A more refined version of Gromov-Witten theory counts *reduced* Gromov-Witten invariants (which in first approximation we can think of as being "more geometric"). These invariants are defined using a modular desingularization of the main component of moduli space of stable maps. This is highly nontrivial for genus higher than 0. Reduced Gromov-Witten invariants have been defined for general target varieties for genus 1 by Vakil and Zinger in [61] and [66]. Their approach was extended to genus 2 by Battistella and Carocci in [4]. K3 surfaces are an interesting example for the theory: The Gromov-Witten theory of a K3 surface is trivial, yet its reduced Gromov-Witten theory is very interesting. The Yau-Zaslow conjecture [65] describes the genus 0 reduced Gromov-Witten potential of a K3 surface in terms of quasimodular forms. A proof of the conjecture was given by Beauville in [6].

In the special case where the target variety X is a point, all maps are trivial and the Gromov-Witten invariants are just intersections of ψ-classes on the moduli space of stable curves. Witten's conjecture from [64] provides an algorithm to compute all such intersection numbers. The conjecture was later proven by Kontsevich [40] and many others, including Okounkov-Pandharipande [51], Kazarian-Lando [38], and Mirzakhani [50].

Due to the origins of the theory in physics, it is customary to combine all the Gromov-Witten invariants in a generating function called *the Gromov-Witten potential*. In this language, Witten's conjecture computes the Gromov-Witten potential of a point. A very important generalization of Witten's conjecture is the *Virasoro conjecture*. Eguchi, Hori, Xiong ([18]) and Katz formulated a series of differential operators that they conjectured should annihilate the Gromov-Witten potential of any target space. The name "Virasoro conjecture" comes from the fact that this collection of operators forms a Lie subalgebra of the Virasoro algebra. A proof of the annihilation of the genus 0 part of the potential was given by Liu and Tian [47].

As we mentioned before, the quantum cohomology of a projective manifold forms a Frobenius algebra. The Gromov-Witten theory of the manifold is much better understood under the additional hypothesis that this algebra is semisimple. There are quite a few spaces that have semisimple quantum cohomology, for example projective spaces and Grassmannians. Bayer [5] proved that if the (p, p)-quantum cohomology of a projective variety is semisimple, then the same is true of its blowup at any number of points. Hertling, Manin, and Teleman [33] proved that the cohomology of a projective variety with semisimple quantum cohomology is of Hodge-Tate type (i.e. all of (p, p) type). This simplifies the Virasoro conjecture because the definition of the differential operators involves the Hodge decomposition. Proofs of the Virasoro conjecture for targets with semisimple quantum cohomology were given for genus 1 by Dubrovin and Zhang [17] and for genus 2 around the same time by Lee [42] and by Liu [46].

The (genus 0) quantum cohomology of a projective manifold X extends to a cohomological field theory, a structure that encapsulates the Gromov-Witten theory of all genera. While Gromov-Witten theory is the motivating example, there are many other cohomological field theories that do not come from Gromov-Witten theory. Givental developed a group action that turns one cohomological field theory into another. Using this group action Teleman [59] gave a full classification of cohomological field theories with unit with the property that the genus 0 part is semisimple. This proved two important conjectural results by Givental: One is a proof of the Virasoro conjecture (for all genera) for symplectic manifolds with semisimple quantum cohomology (see [24]). The other is an algorithm to obtain all Gromov-Witten invariants of a projective manifold with semisimple quantum cohomology from those of genus 0 (see [25]).

For most spaces with semisimple quantum cohomology we still do not know the Gromov-Witten theory very explicitly, because the calculations involved in applying Givental's method or the Virasoro conjecture are highly nontrivial.

Another approach to reconstruct Gromov-Witten invariants from genus 0 invariants is given by Costello's work. In [15] Costello proves that one can reconstructs all Gromov-Witten invariants of any projective variety X from the genus 0 Gromov-Witten invariants of the symmetric powers of X.

The Virasoro conjecture for target curves was proven by Okounkov and Pandharipande [53]. Together with their work in [52] this completely determines the Gromov-Witten theory of target curves. Because we are counting maps from a curve to a curve, this can be related to Hurwitz theory. To make this connection to Hurwitz theory we want to count curves with a specified ramification profile. This can be done using relative Gromov-Witten theory, which is a variation of the theory that counts curves with tangency conditions along a specified subvariety. In fact Okounkov and Pandharipande determined the full relative Gromov-Witten theory of target curves.

An important motivation behind the mathematical formulation of Gromov-Witten theory was the use of mirror symmetry by the physicists in [11] to predict the genus 0 Gromov-Witten invariants of the Calabi-Yau quintic threefold (the degree 5 hypersurface in $\mathbb{P}^4$). This prediction was extended to genus 1 and 2 in [11] and all the way up to genus 51 in [34]. Mathematical proofs have been given for genus 0 in [45], for genus 1 in [67], and for genus 2 in [30]. For higher genus there is currently no mathematical proof.

Another very important result is the virtual localization formula by Graber and Pandharipande [29] for the virtual fundamental class of the moduli space of stable maps to a variety with a $\mathbb{C}^*$-action. When the moduli space of stable maps is unobstructed this reduces to the classical Atiyah-Bott localization formula.

In the recent ground-breaking paper [3] by Argüz, Bousseau, Pandharipande, and Zvonkine an algorithm is provided to compute the Gromov-Witten invariants of any smooth complete intersection in projective space.

The approach to Gromov-Witten theory that we follow in this thesis is the use of *tautological relations*: relations in the tautological ring of the moduli space of stable curves. The tautological ring is a subalgebra of the Chow ring of the moduli space of stable curves that contains most classes arising from geometric constructions. (One can also consider the tautological ring in cohomology, as the image of the Chow tautological ring under the cycle map.) The reason it is convenient to work with this subalgebra is that the full Chow (or cohomology) ring is too complicated to be handled explicitly. Given a tautological relation, we obtain a relation in the Chow ring of the moduli space of stable maps by pulling it back along the forgetful map that remembers and stabilizes the source curve only. The splitting lemma in Gromov-Witten theory then expresses these pulled back relations in terms of Gromov-Witten invariants.

1.2 Reconstructing Gromov-Witten invariants

One of the early and most famous results in Gromov-Witten theory is Kontsevich's recursive formula for the number of rational curves of degree d in $\mathbb{P}^2$ through $3d-1$ points in general position. Before this result the number was only known for $d \leq 6$. The way Kontsevich obtained this formula was by pulling back a tautological relation in genus 0 to the moduli space of stable maps.

In their foundational paper [39] Kontsevich and Manin generalize this approach to prove their first reconstruction theorem. It states that for any target varieties X whose cohomology is generated by $H^2(X)$, the genus 0 Gromov-Witten invariants can be computed using recursive formulas from the 3-pointed invariants as initial values. (The number of points of a Gromov-Witten invariant is the number of constraints, i.e. the number of subvarieties in the target X that the image of the map has to meet.) In [22] Getzler discovered a new tautological relation in genus 1 and used it to proof a similar reconstruction theorem. For target varieties X whose primitive cohomology is in $H^{\leq 2}(X)$, all genus 1 invariants can be computed from the 1-pointed genus 1 invariants and all genus 0 invariants. Later Belorousski and Pandharipande [9] found a new tautological relation in genus 2. They tried to use this relation to prove a general reconstruction theorem for genus 2 invariants but were not able to do so. They were able to use it for the specific target space $\mathbb{P}^2$, for which they calculated the genus 2 Gromov-Witten invariants.

In [46] Liu expresses all genus 2 Gromov-Witten invariants of projective varieties with semisimple quantum cohomology in terms of genus 0 and 1 invariants. However there are many varieties that are not semisimple yet satisfy Getzler's hypothesis.

The main result of this thesis is a reconstruction theorem for genus 2 invariants:

Theorem 11.1. *If $P^i(X) = 0$ for $i > 2$, then all (including descendant) genus two Gromov-Witten invariants can be reconstructed recursively from genus two invariants with at most two points and invariants of lower genus.*

The reason why we are able to prove the theorem now while Belorousski and Pandharipande could not, is that enormous progress has been made in the study of tautological relations in recent years.

In [56] Pixton gave a conjectural description of a large family of tautological relations. The relations were proven to hold in cohomology by Pandharipande, Pixton, and Zvonkine [54] and shortly after were proven to also hold in Chow by Janda [36]. The conjecture that the Pixton's relations are in fact all tautological relations remains open.

Pixton's relations are described by explicit formulas that are quite complicated. Pixton wrote a computer program to compute these relations, which was then expanded upon by Schmitt and others in [16]. We use this program to obtain our relations, and have expanded upon it.

Our approach is similar to the approach of Getzler. In [22] Getzler defines the *symbol* of a tautological relation by first pulling back along the forgetful map to obtain a relation between Gromov-Witten invariants, and then setting most terms to zero. There is a total order on Gromov-Witten invariants such that the symbol consists exactly of those terms that are of maximal order. So when one solves a system of equations obtained from taking symbols, one can then express the solved invariants in terms of lower order invariants. This is what Getzler uses to prove his reconstruction theorem for genus 1, where he manages to reduce to the case of 1-pointed invariants.

In genus 0 and 1, there is a generating basis for the tautological ring that does not include ψ-classes. This means that any descendant Gromov-Witten invariant can easily be expressed in terms of primitive invariants. So the proofs of the reconstruction theorems by Kontsevich-Manin and Getzler only needed to consider primitive invariants. In genus 2 one can not avoid ψ-classes and thus when one obtains a relation between Gromov-Witten invariants from a tautological relation, some of those invariants will be descendant invariants. To deal with this we extend Getzler's notion of symbol to the case where there are ψ-classes.

The program [16] by Pixton, Schmitt, and others can calculate the tautological relations for a given genus, number of points, and codimension. Our extension of the program can convert these tautological relations into symbols and do linear algebra calculations with these symbols. We used this to find the three relations that we use to prove our reconstruction theorem.

In [56] Pixton gives a formal definition of a *new* relation as one that can not be obtained from other relations by natural operations such as multiplying with a cocycle class or pulling back along the forgetful map that forgets a point. The relation found by Belorousski and Pandharipande in [9] is a new relation in the tautological ring of the moduli space of 3-pointed genus 2 curves. There is also a new genus 2 relation in the moduli space of 6-pointed curves that they did not have access to. Our proof uses this 6-pointed new relation plus two old relations with 4 and 5 points. It is an interesting phenomenon that old tautological relations can give rise to "new" information about Gromov-Witten invariants. This is due to the fact that multiplication with a cocycle class does not respect our ordering on Gromov-Witten invariants.

Since Pixton's relations are defined for any genus, one might ask if it is possible to obtain a reconstruction theorem for general genus. However, this is for now out of reach for us. Giving a reconstruction theorem for genus 3 using our methods might be possible, but it is a lot harder than in genus 2. Besides the problem becoming computationally more difficult, in genus 2 the computer program is not able to do calculations of all the tautological rings we need, because it runs out of memory. We have improved the part of the program which calculates relations that are symmetric in the numbering of the points. This allows us to compute the new 8-pointed relation in genus 3. However even with access to this new relation we have only been able to obtain some partial results for genus 3, analogous to partial results in our proof of the genus 2 construction.

We have tried to apply our theorem to a simple yet nontrivial example but the requirement to know all 2-pointed genus 2 Gromov-Witten invariants turns out to be quite demanding. We have tried to look at spaces for which all but a finite number of 2-pointed genus 2 invariants are zero by a dimension argument. For some of these spaces the Gromov-Witten theory is either quite well known such as for $\mathbb{P}^1 \times \mathbb{P}^1$ and for $\mathbb{P}^2$ blown up in a point. For these two spaces we have written a program that calculates the genus 2 Gromov-Witten invariants using the algorithm prescribed by the theorem. One of the next spaces that are natural to consider is $\mathbb{P}^3$ blown up in a point, but although it has only finitely many nonzero 2-pointed Gromov-Witten invariants, we do not know how to compute them.

When we blow up $\mathbb{P}^2$ in more than 1 point, there are infinitely many nonzero 2-pointed invariants and a direct application of the theorem seems impractical. If we use instead the tautological relations that the theorem prescribes as ingredients, we find a new algorithm specific to X_r, the blowup of $\mathbb{P}^2$ at r points in general position.

Theorem 13.1. *We can reconstruct all genus 0,1, and 2 Gromov-Witten invariants of X_r from the finitely many initial cases in Lemma 13.4 and* $< pt^2 >_{0,H}^{X_r}= 1$.

The genus 0 case has already been done by Göttsche and Pandharipande in [26]. We have added calculations for genus 1 and 2. Our approach is an extension of theirs and of the computation of the genus 2 invariants of $\mathbb{P}^2$ by Belorousski and Pandharipande in [9].

The projective plane blown up in finitely many points has semisimple quantum cohomology so in theory one could apply the Virasoro conjecture or Givental's group action. But the computational difficulty of applying these methods is very high. However it would be quite reasonable to apply Liu's result from [46] expressing genus 2 Gromov-Witten invariants in terms of lower genus invariants for projective varieties with semisimple quantum cohomology.

When $r \leq 3$, X_r is a del Pezzo surface. The *primitive* Gromov-Witten invariants of del Pezzo surfaces are enumerative and algorithms to compute them in any genus are found in [60], [58], and [10]. In the unpublished paper [55] Parker describes a method to calculate primitive Gromov-Witten invariants of X_r in any genus. Our algorithm seems to be the only existing one that computes the full theory, including descendant invariants, in genus 2. We have written a computer program that implements the algorithm and the results agree with those stated in the literature for del Pezzo surfaces.

It would be interesting to see if our reconstruction theorem can be applied to more complicated examples. In particular examples for which the quantum cohomology is not semisimple because for the semisimple case Liu's [46] Theorem 0.1 is more powerful. Even when we cannot apply the theorem directly, we can still use it as a suggestion of what relations to use (like in the case of our computation of the Gromov-Witten invariants of X_r). An interesting target would be the Enriques surfaces, though first the genus 1 theory would have to be completed. Maulik and Pandharipande [49] gave a conjectural formula of the genus 1 invariants of an Enriques surface which has not yet been proven. The newer paper [13] by Ciliberto-Dedieu-Galati-Knutsen might provide us with the tools we need.

2 Conventions

We will work over the field of complex numbers $\mathbb{C}$. When we say genus we always mean the arithmetic genus (unless otherwise specified). Unless otherwise specified all our Chow and cohomology rings will be with rational coefficients. Throughout X will be a smooth complex projective variety. When we talk about cohomology classes on X we implicitly take them to be homogeneous.

We let $T_0, \dots T_r$ be a homogeneous basis of $H^*(X)$. By our convention $T_0 = 1$ is the fundamental class. The intersection numbers $g_{ef} := \int_X T_e \cup T_f$ form a matrix (g_{ef}). We write (g^{ef}) for the inverse matrix.

We will make use of intersection theory as described in the book of Fulton [20]. Much of this theory is extended to the Chow rings of Deligne-Mumford stacks in [62]. In particular for a proper Deligne-Mumford stack $\mathcal{X}$ we have Chow groups with rational coefficients $A_*(\mathcal{X})$ and $A^*(\mathcal{X})$. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of Deligne-Mumford stacks $f : \mathcal{X} \to \mathcal{Y}$. When f is proper we have a pushforward map $f^* : A_*(\mathcal{X}) \to A_*(\mathcal{Y})$. When f is flat we have a pullback map $f^* : A_*(\mathcal{Y}) \to A_*(\mathcal{X})$. When f is a regular embedding we have a Gysin pullback map which we will write in the same way $f^* : A_*(\mathcal{Y}) \to A_*(\mathcal{X})$. For these maps see Definition 3.6 and 3.11 in [62]. See Section 5 in [62] for the equivalent maps on A^* rather than A_*. Given a cartesian diagram

$$\begin{array}{ccc}
\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} & \xrightarrow{f'} & \mathcal{Y} \\
\downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\
\mathcal{X} & \xrightarrow{f} & \mathcal{Z},
\end{array}$$

we have $f'_* \circ g'^* = g^* \circ f_*$. We have the usual cap product

$$\cap : A^*(\mathcal{X}) \times A_*(\mathcal{X}) \to A_*(\mathcal{X}),$$

and cup product

$$\cup : A^*(\mathcal{X}) \times A^*(\mathcal{X}) \to A^*(\mathcal{X}).$$

These obey the projection formula: Let $f : \mathcal{X} \to \mathcal{Y}$ be a proper flat morphism of Deligne-Mumford stacks, let $\alpha \in A_*(\mathcal{X})$, $\beta \in A^*(\mathcal{Y})$, and $\gamma \in A^*(\mathcal{X})$, then

$$f_*(f^*\beta \cap \alpha) = \beta \cap f_*\alpha, \qquad f_*(f^*\beta \cup \gamma) = \beta \cup f_*\gamma.$$

In Section 2.2 of [1] the (co)homology of a proper Deligne-Mumford stack is defined as the (co)homology of its coarse moduli space. There are definitions for pullbacks and pushforwards and we have a cap product and a cup product. The cap product satisfies the product rule. We also have a cycle map $A_*(\mathcal{X}) \to H_*(\mathcal{X})$.

We will work on the level of Chow whenever possible and implicitly use the cycle map when we combine Chow classes with (co)homology classes.

3 The moduli space of stable curves

We give a brief introduction to the moduli space of stable curves. A full introduciton can be found in [31] and [2].

Definition 3.1. A *curve* is a proper 1-dimensional scheme over $\mathbb{C}$. A curve C is called *nodal* if every closed point $p \in C$ is either nonsingular or has complete local ring $\hat{\mathcal{O}}_{C,p} \cong \mathbb{C}[x, y]/(xy)$.

Definition 3.2. A *family of n-pointed nodal curves of genus g over a base scheme B* is a flat proper morphism $f : \mathbf{C} \to B$ together with disjoint smooth sections $\sigma_i : B \to \mathbf{C}$ for $1 \leq i \leq n$, such that for every geometric point $b \in B$, the fiber $f^{-1}(b)$ is a nodal curve of arithmetic genus g. The sections σ_i and their images in the fibers are called the *marked points*.

Definition 3.3. An irreducible component of arithmetic genus g' of a nodal curve is called *stable* if it contains at least $3 - 2g'$ points that are either a marked point or in the intersection with another component. A family of n-pointed nodal curves of genus g is called stable if every irreducible component in every fiber is stable.

A nodal curve can be *stabilized* by contracting its unstable components.
Consider the contravariant functor from schemes to sets

$$B \to \left\{ \begin{matrix} \text{families } (\mathbf{C}/B, \sigma_1, \ldots, \sigma_n) \text{ of } n\text{-pointed} \\ \text{stable curves of genus } g \text{ over } B \end{matrix} \right\} / \sim, \tag{1}$$

where two families of curves

$$(\mathbf{C}/B, \sigma_1, \ldots, \sigma_n), \quad (\mathbf{C}'/B, \sigma_1', \ldots, \sigma_n'),$$

are isomorpic if there is an isomorphism of B-schemes $\tau : \mathbf{C} \xrightarrow{\sim} \mathbf{C}'$ such that $\tau \circ \sigma_i = \sigma_i'$ for $1 \leq i \leq n$.

Theorem 3.4. *There is a smooth proper Deligne-Mumford stack $\overline{\mathcal{M}}_{g,n}$ representing the functor (1).*

Proof. For an overview of the construction see Chapters XII and for the properness see Chapter XIV in [2]. □

It has dimension

$$\dim(\overline{\mathcal{M}}_{g,n}) = 3g - 3 + n.$$

The moduli space of curves comes with the following natural maps: There are gluing maps

$$q : \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2},$$
$$\delta : \overline{\mathcal{M}}_{g,n+2} \to \overline{\mathcal{M}}_{g+1,n},$$

where q glues the last marked point of each of the two curves together and δ glues the last two markings of the same curve together. There is also a forgetful map

$$\pi_i : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n},$$

that forgets the i-th marked point followed by stabilization. See Section XII.10 of [2] for a discussion of these maps and the universal family: The stack $\overline{\mathcal{M}}_{g,n}$ has a universal family and it is isomorphic to the forgetful map

$$\begin{tikzcd} \overline{\mathcal{M}}_{g,n+1} \arrow[d, "\pi_{n+1}"] \\ \overline{\mathcal{M}}_{g,n} . \arrow[u, bend left, "\sigma_1 \ldots \sigma_n"] \end{tikzcd}$$

Let $\omega_{\pi_{n+1}}$ be the relative dualizing sheaf of the universal family. We define the *ψ-classes* ψ_i for $1 \leq i \leq n$ as

$$\psi_i := c_1(\sigma_i^* \omega_{\pi_{n+1}}) \in A^1(\overline{\mathcal{M}}_{g,n}),$$

where c_1 is the first Chern class.

By forgetting the point corresponding to the ψ-class we obtain a *κ-class*

$$\kappa_i := \pi_{j\,*}(\psi_j^{i+1}) \in A^i(\overline{\mathcal{M}}_{g,n}),$$

for $i \geq 0$. (Note that this definition does not depend on j.)

There is an action of the symmetric group $\mathfrak{S}_n$ on $\overline{\mathcal{M}}_{g,n}$ that permutes the markings $\sigma_1, \ldots, \sigma_n$.

4 The moduli space of stable maps

Similar to the moduli space of stable curves we now introduce the moduli space of stable maps. A Gromov-Witten invariant will then be an intersection number on the moduli space of stable maps. For a more detailed construction of the moduli space of stable maps see for example Chapter V in [48].

Definition 4.1. Let X be a non-singular projective variety. A *family of n-pointed stable maps of genus g over a base scheme B with target variety X* consists of a commutative diagram

$$\begin{tikzcd} \mathbf{C} \arrow[rr, "f"] \arrow[dr, "g"'] & & X \arrow[dl] \\ & B, & \end{tikzcd}$$

together with sections $\sigma_i : B \to \mathbf{C}$ for $1 \leq i \leq n$ such that the map g together with the sections σ_i form a family of n-pointed nodal curves. Furthermore for every geometric point $b \in B$ we require that if f is constant on an irreducible component of $g^{-1}(b)$, then that component is stable.

Let $\beta \in H_2(X, \mathbb{Z})$ and consider the contravariant functor from schemes to sets

$$B \to \left\{ \begin{matrix} \text{families } ((\mathbf{C} \xrightarrow{f} X)/B, \sigma_1, \ldots, \sigma_n) \text{ of } n\text{-pointed stable maps of genus } g \\ \text{over } B, \text{ such that } f_*[g^{-1}(b)] = \beta \text{ for every geometric point } b \in B \end{matrix} \right\} / \sim, \quad (2)$$

where two families of stable maps

$$((\mathbf{C} \xrightarrow{f} X)/B, \sigma_1, \ldots, \sigma_n), \quad ((\mathbf{C}' \xrightarrow{f'} X)/B, \sigma'_1, \ldots, \sigma'_n),$$

are isomorpic if there is an isomorphism of B-schemes $\tau : \mathbf{C} \xrightarrow{\sim} \mathbf{C}'$ such that $\tau \circ \sigma_i = \sigma'_i$ for $1 \leq i \leq n$ and $f' \circ \tau = f$.

Theorem 4.2. *There is a proper Deligne-Mumford stack $\overline{\mathcal{M}}_{g,n}(X, \beta)$ representing the functor (2).*

Proof. This is proven in 5.2, 5.7, and 5.8 of Chapter V of [48]. □

Remark 4.3. We have $\overline{\mathcal{M}}_{g,n}(X, 0) \cong \overline{\mathcal{M}}_{g,n} \times X$. In particular taking X to be a point recovers the moduli space of stable curves.

Just as with the moduli space of stable curves we have forgetful maps

$$\pi_i : \overline{\mathcal{M}}_{g,n+1}(X, \beta) \to \overline{\mathcal{M}}_{g,n}(X, \beta),$$

forgetting the i-th point followed by stabilization (see V.4.6 in [48]).

For every $1 \leq i \leq n$ we also have an evaluation morphism

$$\mathrm{ev}_i : \overline{\mathcal{M}}_{g,n}(X, \beta) \to X,$$

sending a map $f : (C; p_1, \ldots, p_n) \to X$ to $f(p_i)$ (see V.4.2.2 in [48]). These morphisms combine into a single evaluation morphism

$$\mathrm{ev} : \overline{\mathcal{M}}_{g,n}(X, \beta) \to X^n.$$

Now over $\overline{\mathcal{M}}_{g,n}(X, \beta)$ there is a universal family (V.4.5 in [48])

$$\begin{tikzcd} \overline{\mathcal{M}}_{g,n+1}(X, \beta) \arrow[r, "\mathrm{ev}_{n+1}"] \arrow[d, "\pi_{n+1}"] & X \\ \overline{\mathcal{M}}_{g,n}(X, \beta) \arrow[u, bend left, "\sigma_1 \ldots \sigma_n"] & \end{tikzcd}$$

where the fiber over an n-pointed map is the map itself. For each $0 \leq i \leq n$ we have a section σ_i that sends an n-map curve to its i-th marked point.

As for stable curves, we let $\omega_{\pi_{n+1}}$ be the relative dualizing sheaf of the universal family. We define the $\tilde{\psi}$*-class*

$$\tilde{\psi}_i := c_1(\sigma_i^* \omega_{\pi_{n+1}}) \in A^1(\overline{\mathcal{M}}_{g,n}(X, \beta)),$$

where c_1 is the first Chern class.

There is also a forgetful map

$$F : \overline{\mathcal{M}}_{g,n}(X, \beta) \to \overline{\mathcal{M}}_{g,n},$$

that only remembers and stabilizes the source curve (V.4.9 in [48]).

The moduli space of stable maps is not as well behaved as the moduli space of stable curves. It is compact but in general it is not smooth or equidimensional.

Example 4.4. Consider $\overline{\mathcal{M}}_{1,0}(\mathbb{P}^2, 3)$. The space of smooth maps $M_{1,0}(\mathbb{P}^2, 3)$ is the space of smooth degree 3 plane cubics and is of the expected dimension 9. There is a component with maps having a genus 1 component mapping to a point and a rational tail mapping with degree 3. This has dimension

$$\dim \overline{\mathcal{M}}_{0,1}(\mathbb{P}^2, 3) + \dim \overline{\mathcal{M}}_{1,1} = 9 + 1 = 10.$$

To work around this problem we replace the fundamental class of the moduli space of stable maps by an equidimensional *virtual fundamental class*

$$[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\mathrm{vir}} \in A_k(\overline{\mathcal{M}}_{g,n}(X, \beta)),$$

where

$$k = \int_\beta c_1(T_X) + (\dim(X) - 3)(1 - g) + n$$

is the *expected dimension* or *virtual dimension* of $\overline{\mathcal{M}}_{g,n}(X, \beta)$. For an overview of the construction of the virtual fundamental class and its properties, see [7].

Definition 4.5. Let $\beta \in H_2(X)$ and $\gamma_1, \ldots, \gamma_n \in H^*(X)$. We define a *Gromov-Witten invariant* of X to be

$$< \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n) >_{g,\beta}^X := \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} \tilde{\psi}_1^{a_1} \cup \cdots \cup \tilde{\psi}_n^{a_n} \cup \mathrm{ev}^*(\gamma_1 \otimes \cdots \otimes \gamma_n).$$

If a_i is zero then we may simply write γ_i in place of $\tau_0(\gamma_i)$. Usually we also leave out the X in the notation. Invariants without ψ-classes are called *primary invariants* while invariants with ψ-classes are called *descendant invariants.*

Note that while we define Gromov-Witten invariants for all $\beta \in H_2(X)$, for a Gromov-Witten invariant to be nonzero, β needs to be effective.

Just as on $\overline{\mathcal{M}}_{g,n}$, we have an action of the symmetric group $\mathfrak{S}_n$ on $\overline{\mathcal{M}}_{g,n}(X,\beta)$ that permutes the marked points. This extends to an action on $H^*(\overline{\mathcal{M}}_{g,n}(X,\beta))$. Permuting the points in a Gromov-Witten invariant means we have to permute the cohomology classes on X and so Gromov-Witten invariants inherit the supercommutativity of the cohomology of X, i.e.

$$\left\langle \cdots \tau_{k_i}(\gamma_i)\tau_{k_{i+1}}(\gamma_{i+1}) \cdots \right\rangle_{g,\beta}^X = (-1)^{|\gamma_i||\gamma_{i+1}|} \left\langle \cdots \tau_{k_{i+1}}(\gamma_{i+1})\tau_{k_i}(\gamma_i) \cdots \right\rangle_{g,\beta}^X.$$

A naive interpretation of the primitive Gromov-Witten invariant $\langle \gamma_1 \cdots \gamma_n \rangle_{g,\beta}^X$ is the enumerative count of curves of class β that pass through subvarieties of classes $\gamma_1, \ldots, \gamma_n$ in general position in X. In some cases this naive interpretation is actually the thruth and the enumerative count is equal to the Gromov-Witten invariant. In these cases we say that the primitive Gromov-Witten invariants are *enumerative.* For example the primitive Gromov-Witten invariants of $\mathbb{P}^2$ are enumerative and $< pt^5 >_{0,2H}^{\mathbb{P}^2} = 1$ is the well known fact that there is one rational curve of degree 2 through 5 points in general position in $\mathbb{P}^2$.

5 The tautological ring

The Chow and cohomology rings of the moduli space of stable curves are in general very complicated. Therefore it is convenient to restrict to the tautological ring, which is a subring that contains most classes arising from geometric constructions. The tautological ring is generated by decorated strata classes which are cohomology classes consisting of all curves of a specified topological type together with a polynomial in ψ- and κ-classes.

An equivalent definition is the following.

Definition 5.1. The system of *tautological rings* of the moduli space of stable curves is the minimal system of $\mathbb{Q}$-subalgebras $R^*(\overline{\mathcal{M}}_{g,n}) \subset \{A^*(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})\}$ that is closed under pushforward along all the natural gluing and forgetful maps.

Remark 5.2. The tautological rings can also be defined as a family of subalgebras of $H^{2*}(\overline{\mathcal{M}}_{g,n}, \mathbb{Q})$. Taking the image of the Chow tautological ring along the cycle map then gives the cohomological tautological ring. Everything we say about the tautological ring holds in both Chow and cohomology.

Definition 5.3. An n-pointed stable graph Γ consists of data

$$\Gamma := (V, H, E, L, g : V \to \mathbb{Z}_{>0}, \eta : H \to V, i : H \to H),$$

with properties:

(i) V is a set of vertices and g is the geometric genus function.

(ii) H is a set half edges with a vertex assignment η.

(iii) i is an involution.

(iv) The set of edges E consists of the size 2 orbits of i.

(v) The set of legs L consists of the fixed points of i. There is a bijective labeling $L \leftrightarrow \{1 \ldots n\}$.

(vi) The graph consisting of the data $V, E, \eta|_E$ is a finite connected graph.

(vii) For every vertex $v \in V$ the stability condition holds:

$$2g(v) - 2 + n(v) > 0,$$

where $n(v) := |\eta^{-1}(v)|$ is the valence at v.

From a stable curve we obtain its *dual stable graph* or *dual graph* by taking the irreducible components as vertices and nodes as edges. The two half edges that make up an edge correspond to the inverse images of a node under the normalization map of the curve. Legs correspond to marked points. The genus of a vertex is the geometric genus of the corresponding irreducible component.

To an n-pointed stable graph Γ we associate the product of moduli spaces

$$\overline{\mathcal{M}}_\Gamma := \prod_{v \in V} \overline{\mathcal{M}}_{g(v),n(v)} \,. \tag{3}$$

There is a canonical gluing map

$$\xi_\Gamma : \overline{\mathcal{M}}_\Gamma \to \overline{\mathcal{M}}_{g,n},$$

where $g = h^1(\Gamma) + \sum_{v \in V} g(v)$ is called the genus of Γ. (The first Betti number h^1 of a graph is the number of independent loops.) The map ξ_Γ corresponds to gluing the curves together in the manner prescribed by the involution i. This can be interpreted as a repeated application of the gluing maps q and δ from Section 3.

Because of this we obtain from the definition that $\xi_{\Gamma *}[\overline{\mathcal{M}}_\Gamma]$ is an element of the tautological ring. The image of ξ_Γ is the closure of the set of curves with dual graph Γ. It factors through the action of $\mathrm{Aut}(\Gamma)$, where an automorphism of Γ is a graph automorphism that respects the legs and genus assignments.

For every vertex $v \in V$ we have monomials in ψ- and κ-classes, i.e. expressions of the form

$$\theta_v = \prod_{i=1}^{n} \psi_i^{a_i} \prod_i \kappa_i^{b_i} \in A^*(\overline{\mathcal{M}}_{g(v),n(v)}),$$

where the a_i and b_i are any natural numbers. these combine into a monomial in ψ- and κ-classes in $A^*(\overline{\mathcal{M}}_\Gamma)$

$$\boxtimes_{v\in V} \theta_v.$$

Definition 5.4. A *decorated stable graph* Γ_θ is a stable graph Γ together with a monomial θ in κ- and ψ-classes in $A^*(\overline{\mathcal{M}}_\Gamma)$. Now

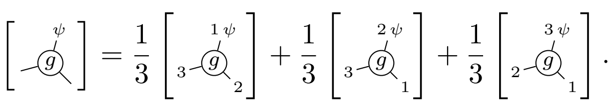
$$[\Gamma_\theta] := \frac{1}{|\mathrm{Aut}(\Gamma)|}\xi_{\Gamma *}(\theta) \in A^*(\overline{\mathcal{M}}_{g,n})$$

is the *decorated stratum class* corresponding to the decorated graph. We write $[\Gamma] := [\Gamma_1]$ for a decorated stratum class with trivial decoration. We define the *degree* of a decorated stratum class to be

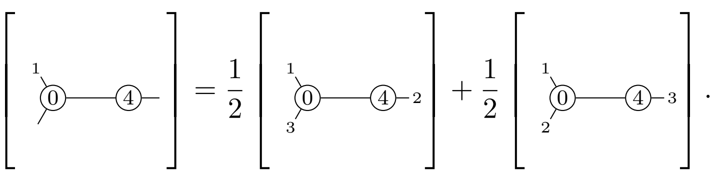
$$\deg([\Gamma_\theta]) := \deg_{\mathbb{C}}(\theta) + \#E_\Gamma.$$

When we draw a decorated graph we write the ψ-classes at the corresponding half-edges and the κ-classes at corresponding vertices.

If we draw a decorated graph without specifying a numbering of the legs, it corresponds to the $\mathfrak{S}_n$-invariant sum over all possible ways to number the legs, divided by the size of the $\mathfrak{S}_n$-orbit. For example when $n = 3$ and the graph is trivial, we have



We can also leave out $n' < n$ points in the notation, which corresponds to the $\mathfrak{S}_{n'}$-invariant sum divided by the size of the $\mathfrak{S}_{n'}$ orbit. For example



Theorem 5.5 (Proposition 11 in [27]). *The tautological ring $R^*(\overline{\mathcal{M}}_{g,n})$ is generated additively by the decorated strata class $[\Gamma_\theta]$ for all n-pointed genus g stable graphs Γ and all monomials θ in κ- and ψ-classes in $A^*(\overline{\mathcal{M}}_\Gamma)$.*

Note that this includes all ψ- and κ-classes since if Γ is the trivial graph with one vertex and no edges, then ξ_Γ is the identity map and $[\Gamma_\theta] = \theta$.

By imposing restrictions on the dual graph of a curve we obtain the following open subsstacks of the moduli space of stable curves

$$\mathcal{M}_{g,n} \subset \mathcal{M}_{g,n}^{\mathrm{rt}} \subset \mathcal{M}_{g,n}^{\mathrm{ct}} \subset \overline{\mathcal{M}}_{g,n}\,.$$

- The moduli space of (smooth) curves $\mathcal{M}_{g,n}$ consists of curves whose dual graph consists of one vertex and no edges.
- The moduli space of curves with rational tail $\mathcal{M}_{g,n}^{\mathrm{rt}}$ consists of curves that have a vertex whose genus is equal to the genus of the whole graph.
- The moduli space of curves of compact type $\mathcal{M}_{g,n}^{\mathrm{ct}}$ consists of curves whose dual graph has no loops.

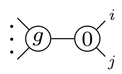
For each of these subspaces we obtain a corresponding tautological ring by restriction.

We have the following description for the pullback of a decorated stratum class along the forgetful map.

Lemma 5.6 (Lemma 17.4.28 in [2]). *i)* $\pi_j^*(\kappa_i) = \kappa_i - \psi_j^i$,

ii)

$$\pi_j^*(\psi_i) = \psi_i - \Big[\quad\Big],$$



iii) $\pi_j^*([\Gamma]) = \sum_{v\in\Gamma}[\Gamma_v]$, *where $[\Gamma_v]$ is the graph obtained by adding the j-th leg to the vertex v.*

A conjecture by Faber that was later proven by Ionel gives us a bound on the number of κ-classes that are needed to generate the tautological ring.

Theorem 5.7 (Theorem 1.5 in [35]). *The tautological ring $R^*(\mathcal{M}_g)$ is generated by the classes $\kappa_1, \ldots, \kappa_{\lfloor g/3\rfloor}$.*

Using Lemma 5.6 and the the long exact sequence of cohomology on an open subspace we obtain the following corollary.

Corollary 5.8. *Any class $\kappa_i \in R^*(\overline{\mathcal{M}}_{g,n})$ with $i > \lfloor g/3\rfloor$ can be expressed in terms of decorated strata classes whose decoration only contains ψ-classes and k_j for all $j \leq \lfloor g/3\rfloor$.*

We also have a bound on the ψ-classes required to generate the tautological ring.

Theorem 5.9 (Proposition 2.5 in [21])**.** *Any monomial of ψ-classes of degree at least* $\max(g, 1)$ *in* $R^*(\overline{\mathcal{M}}_{g,n})$ *can be expressed in terms of the boundary strata classes that involve no κ-classes, that is, in terms of the dual graphs with at least one edge, decorated only by ψ-classes.*

6 Tautological relations

We already obtain some relations on the tautological ring from Corollary 5.8 and Theorem 5.9.

There is a large family of relations called Pixton's relations. The description of these relations is quite involved and it is impractical to calculate the relations manually. In later chapters we will use a computer program by Pixton to calculate the relations we need (see Appendix B).

Definition 6.1. Define the *strata algebra* $S_{g,n}$ to be the free $\mathbb{Q}$-algebra generated by the n-pointed genus g decorated strata classes $[\Gamma_\theta]$, for which

$$\deg([\Gamma_\theta]) \leq \dim_{\mathbb{C}}(\overline{\mathcal{M}}_{g,n}) = 3g - 3 + n.$$

The multiplication on $S_{g,n}$ is the one inherited from the intersection theory of $\overline{\mathcal{M}}_{g,n}$, as described by Equation (11) in [27].

We define $R_{g,n}$, the $\mathbb{Q}$-algebra of *tautological relations*, by the short exact sequence

$$0 \to R_{g,n} \to S_{g,n} \to R^*(\overline{\mathcal{M}}_{g,n}) \to 0.$$

As expected we denote the restriction to a degree r by

$$0 \to R^r_{g,n} \to S^r_{g,n} \to R^r(\overline{\mathcal{M}}_{g,n}) \to 0.$$

We similarly define tautological relations on $\mathcal{M}^{\mathrm{ct}}_{g,n}, \mathcal{M}^{\mathrm{rt}}_{g,n}$, and $\mathcal{M}_{g,n}$.

Remark 6.2. Corollolary 5.8 and Theorem 5.9 state the existence of certain tautological relations.

In [56] Pixton gave a conjectural description for the tautological relations of $R^*(\overline{\mathcal{M}}_{g,n})$. This was an extension of the conjectural relations of $R^*(\mathcal{M}_g)$ by Faber and Zagier. In [54] it was proven that Pixton's conjectural relations are actually relations in cohomology. Shortly afterwards Janda [36] gave a proof in Chow. We will write $P_{g,n} \subseteq R_{g,n}$ for the subalgebra of Pixton's relations. The conjecture that $P_{g,n} = R_{g,n}$ is still open.

The definition of Pixton's relations can also be applied to $\mathcal{M}^{\mathrm{ct}}_{g,n}, \mathcal{M}^{\mathrm{rt}}_{g,n}$, and $\mathcal{M}_{g,n}$. The relations obtained this way are the restriction of Pixton's relations on $\overline{\mathcal{M}}_{g,n}$. Therefore we have

Lemma 6.3. *Let R be a Pixton relation on $\mathcal{M}^{\mathrm{ct}}_{g,n}, \mathcal{M}^{\mathrm{rt}}_{g,n}$, or $\mathcal{M}_{g,n}$, then R extends to a tautological relation on $\overline{\mathcal{M}}_{g,n}$.*

The analogue of this is not known for elements of $R_{g,n}$ in general. Even though every tautological relation on $\mathcal{M}^{\mathrm{ct}}_{g,n}, \mathcal{M}^{\mathrm{rt}}_{g,n}$, or $\mathcal{M}_{g,n}$ has an extension to $A^*(\overline{\mathcal{M}}_{g,n})$, we do not know if this extension lives in $R^*(\overline{\mathcal{M}}_{g,n})$.

Definition 6.4. Let R^{old} consist of $S^{>0}_{g,n} \cdot R_{g',n'}$, $q_*(R_{g,n} \otimes R_{g',n'})$, and $\pi^*(R_{g,n})$ for all natural gluing and forgetful maps q and π and any $g, n, g', n' \in \mathbb{Z}_{\geq 0}$. We define $R^{\mathrm{new}}_{g,n} := R_{g,n}/(R_{g,n} \cap R^{\mathrm{old}})$ to be the *new* relations.

From Proposition 2 in [56] we obtain the following.

Lemma 6.5. *Let $g > 0$ and let $[R] \in P_{g,n} \cap R^{\mathrm{new}}_{g,n}$, then $[R]$ has a representative $R' \in P_{g,n}$ that is symmetric, i.e. R' is fixed by the action of $\mathfrak{S}_n$ permuting the points.*

7 Pulling back tautological relations

We obtain a relation on the moduli space of stable maps by pulling back a tautological relation along the forgetful map that remembers and stabilizes the source curve. There is a splitting lemma that allows us to express the restriction of a Gromov-Witten invariant to the pullback of a decorated stratum class $[\Gamma_\theta]$ in terms of Gromov-Witten invariants with smaller genus and number of points, corresponding to the vertices of Γ. This way the restriction of a Gromov-Witten invariant to the pullback of a tautological relation gives us a relation between Gromov-Witten invariants.

Definition 7.1. Let $S^{\mathrm{nk}}_{g,n}$ be the linear subspace of $S_{g,n}$ generated by decorated strata classes that do not have any κ-classes in their decoration. For every $\beta \in H_2(X)$ we define a map

$$\mathcal{T}_\beta : S^{\mathrm{nk}}_{g,n} \otimes (H^*(X))^{\otimes n} \to \mathbb{Q},$$

by taking

$$\mathcal{T}_\beta(S, \gamma_1, \dots, \gamma_n) := \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} F^*(p(S)) \cup \mathrm{ev}^*(\gamma_1 \otimes \cdots \otimes \gamma_n), \qquad (4)$$

where p is the map $S_{g,n} \twoheadrightarrow R^*(\overline{\mathcal{M}}_{g,n})$ and $F : \overline{\mathcal{M}}_{g,n}(X, \beta) \to \overline{\mathcal{M}}_{g,n}$ is the forgetful map that only remembers and stabilizes the source curve.

We will now show how to describe the right-hand side of (4) as a polynomial in Gromov-Witten invariants.

Let χ_Γ be the pullback of ξ_Γ along the forgetful map F that only remembers and stabilizes the source curve. We have the fiber square

$$\begin{tikzcd} F^*(\overline{\mathcal{M}}_\Gamma) \arrow[r, "\chi_\Gamma"] \arrow[d, "F_\Gamma"] & \overline{\mathcal{M}}_{g,n}(X,\beta) \arrow[d, "F"] \\ \overline{\mathcal{M}}_\Gamma \arrow[r, "\xi_\Gamma"] & \overline{\mathcal{M}}_{g,n}, \end{tikzcd}$$

where

$$F_\Gamma^*(\overline{\mathcal{M}}_\Gamma) = \bigoplus_{\sum_v \beta(v)=\beta} \prod_{v\in V} \overline{\mathcal{M}}_{g(v),n(v)}(X,\beta(v))$$

is a direct sum over all the ways to divide the degree β among the vertices. Since flat pullback commutes with proper pushforward on fiber squares we have

$$F^*[\Gamma_\theta] = \frac{1}{|\mathrm{Aut}(\Gamma)|}\chi_{\Gamma *} F_\Gamma^*(\theta).$$

The projection formula now allows us to rewrite the terms of $\mathcal{T}_\beta(S,\gamma_1,\ldots,\gamma_n)$:

$$\begin{aligned} \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} \chi_{\Gamma *} F_\Gamma^*(\theta) \cup \mathrm{ev}^*(\gamma_1\otimes\cdots\otimes\gamma_n) = \\ \int F_\Gamma^*(\theta) \cup \chi_\Gamma^*\left(\mathrm{ev}^*(\gamma_1\otimes\cdots\otimes\gamma_n)\cap[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}\right). \end{aligned} \tag{5}$$

We can then calculate $F_\Gamma^*(\theta)$ using

Lemma 7.2 (See VI.3.6 in [48])**.** *Let $D_i \in A^1(\overline{\mathcal{M}}_{g,n}(X,\beta))$ be defined by taking the closure of the locus of maps whose source curve has two irreducible components such that one of these components is rational, has only the i-th marked point, and the restriction of the map to this component is constant. Then*

$$F^*(\psi_i) = \tilde{\psi}_i - D_i.$$

The *splitting lemma* 7.3 gives us a way to express the remaining part of (5),

$$\chi_\Gamma^*\left(\mathrm{ev}^*(\gamma_1\otimes\cdots\otimes\gamma_n)\cap[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}\right),$$

as a sum of products of Gromov-Witten invariants in $F^*(\overline{\mathcal{M}}_\Gamma)$. The morphism χ_Γ is the morphism that glues moduli spaces of stable maps. Similar to the situation for stable curves, these gluing morphisms can be constructed from the base cases

$$\begin{aligned} \tilde{q} : \overline{\mathcal{M}}_{g_1,n_1+1}(X,\beta_1)\times_X \overline{\mathcal{M}}_{g_2,n_2+1}(X,\beta_2) \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}(X,\beta_1+\beta_2), \\ \tilde{\delta} : \overline{\mathcal{M}}^{\delta}_{g,n+2}(X,\beta) \to \overline{\mathcal{M}}_{g+1,n}(X,\beta). \end{aligned} \tag{6}$$

We can only glue maps when the points being glued map to the same point in X, i.e. recalling that ev_i is the evaluation at the i-th point, we have fiber squares

$$\begin{tikzcd}
\overline{\mathcal{M}}_{g_1,n_1+1}(X,\beta_1)\times_X\overline{\mathcal{M}}_{g_2,n_2+1}(X,\beta_2) \arrow[r, hook] \arrow[d] & \overline{\mathcal{M}}_{g_1,n_1+1}(X,\beta_1)\times\overline{\mathcal{M}}_{g_2,n_2+1}(X,\beta_2) \arrow[d, "\mathrm{ev}_{n_1+1}\times\mathrm{ev}_{n_1+2}"] \\
\Delta \arrow[r, hook] & X\times X,
\end{tikzcd}$$

and

$$\begin{tikzcd}
\overline{\mathcal{M}}^{\delta}_{g,n+2}(X,\beta) \arrow[r, hook] \arrow[d] & \overline{\mathcal{M}}_{g,n+2}(X,\beta) \arrow[d, "\mathrm{ev}_{n+1}\times\mathrm{ev}_{n+2}"] \\
\Delta \arrow[r, hook] & X\times X.
\end{tikzcd}$$

We can now pull back the formula for the decomposition of the diagonal to obtain an expression for the pullback along $\tilde{q}$ and $\tilde{\delta}$. Coupled with Theorem 13 in [23] this gives the Splitting Lemma:

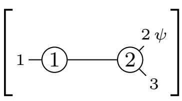
Lemma 7.3 (Splitting lemma). *Let $\tilde{q}$ be the gluing map (6) for some given $\beta_1+\beta_2=\beta\in H_2(X)$, we have*

$$\begin{aligned}
\tilde{q}^*\left(\mathrm{ev}^*(\gamma_1\otimes\cdots\otimes\gamma_{n_1+n_2})\cap[\overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}(X,\beta)]^{vir}\right) = \\
\sum_{e,f} g^{ef}\left(\mathrm{ev}^*(\gamma_1\otimes\cdots\otimes\gamma_{n_1}\otimes T_e)\cap[\overline{\mathcal{M}}_{g_1,n_1+1}(X,\beta_1)]^{vir}\right)\otimes \\
\left(\mathrm{ev}^*(T_f\otimes\gamma_{n_1+1}\otimes\cdots\gamma_{n_1+n_2})\cap[\overline{\mathcal{M}}_{g_2,n_2+1}(X,\beta_2)]^{vir}\right),
\end{aligned}$$

and for the self-gluing $\tilde{\delta}$, we have

$$\begin{aligned}
\tilde{\delta}^*\left(\mathrm{ev}^*(\gamma_1\otimes\cdots\otimes\gamma_n)\cap[\overline{\mathcal{M}}_{g+1,n}(X,\beta)]^{vir}\right) = \\
\sum_{e,f} g^{ef}\,\mathrm{ev}^*(\gamma_1\otimes\cdots\otimes\gamma_n\otimes T_e\otimes T_f)\cap[\overline{\mathcal{M}}_{g,n+2}(X,\beta)]^{vir}.
\end{aligned}$$

Example 7.4. When we apply the splitting lemma to

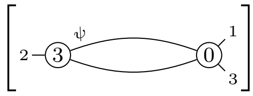
$\mathcal{T}_\beta($  $,\gamma_1,\gamma_2,\gamma_3)$

we obtain

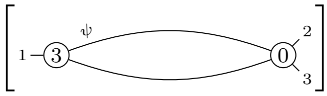
$$\sum_{\beta_1+\beta_2=\beta}\sum_{e,f} <\gamma_1 T_e>_{1,\beta_1} g^{ef} <T_f\tau_1(\gamma_2)\gamma_3>_{2,\beta_2}.$$

The following example illustrates how to deal with more complicated graphs.

Example 7.5. We want to apply the splitting lemma to

$\mathcal{T}_\beta($  $, \gamma_1, \gamma_2, \gamma_3).$

But the ordering of the points is different from the one prescribed by the splitting lemma. To fix this we reorder the points which introduces a sign

$(-1)^{|\gamma_2||\gamma_3|}\mathcal{T}_\beta($  $, \gamma_2, \gamma_1, \gamma_3).$

Now we contract the edges one at a time to obtain

$$(-1)^{|\gamma_2||\gamma_3|} \sum_{\beta_1+\beta_2=\beta} \sum_{e,f} \sum_{e',f'} < \gamma_1\gamma_3 T_{e'} T_e >_{0,\beta_1} g^{ef} g^{e'f'} < T_f \tau_1(T_{f'})\gamma_2 >_{3,\beta_2} .$$

Note that the result does not depend on the choice of reordering or the choice of which edge to contract first. This can be seen from the following two facts: First the total degree of a nonzero Gromov-Witten is equal to the virtual dimension of the corresponding moduli space, hence even. Second for g^{ef} to be nonzero $|T_e| + |T_f|$ has to be even.

Definition 7.6. Let $L \in S_{g,n}$ be a linear combination of decorated strata classes that can have single κ_i classes in the decoration. We construct $L' \in S^{\mathrm{nk}}_{g,n}$ from L by replacing every κ_i by a new marked point at the corresponding vertex with a ψ^{i+1} decoration. We define

$$\mathcal{T}_\beta(L, \psi_1, \dots, \psi_n) := \mathcal{T}_\beta(L', \psi_1, \dots, \psi_n, 1).$$

The idea behind this definition is

$$\begin{aligned}\int F^*(\psi_{n+1}^{i+1}) \cap \left[\overline{\mathcal{M}}_{g,n+1}(X,\beta)\right]^{\mathrm{vir}} &= \int \pi_{n+1\,*}\left(F^*(\psi_{n+1}^{i+1}) \cap \left[\overline{\mathcal{M}}_{g,n+1}(X,\beta)\right]^{\mathrm{vir}}\right) \\ &= \int F^* \pi_{n+1\,*}(\psi_{n+1}^{i+1}) \cap \left[\overline{\mathcal{M}}_{g,n}(X,\beta)\right]^{\mathrm{vir}} \\ &= \int F^*(\kappa_i) \cap \left[\overline{\mathcal{M}}_{g,n}(X,\beta)\right]^{\mathrm{vir}} .\end{aligned}$$

Remark 7.7. When there is a polynomial in κ-classes rather than a single κ_i, replacing it requires a few more steps. The interested reader is referred to Section 2.1 in [19]. We will only consider genus at most 3. When $g \le 2$ then by Corollary 5.8 we can express all tautological relations in terms of decorated strata classes that have no κ-classes. And it turns out that when the genus is 3, we only need a single κ_1-class.

By construction we have the following Proposition.

Proposition 7.8. *Let $L \in R_{g,n}$ and $\gamma_1, \ldots, \gamma_n \in H^*(X)$, then $\mathcal{T}_\beta(L, \gamma_1, \ldots, \gamma_n)$ gives a relation among Gromov-Witten invariants, i.e.*

$$\mathcal{T}_\beta(L, \gamma_1, \ldots, \gamma_n) = 0.$$

Giving the full formal proof in the case where there are κ-classes is similar to proving the dilaton equation, which is the special case that follows from the fact that $\kappa_0 = 2g - 2 + n$.

Lemma 7.9 (Dilaton Equation, VI.5.3 in [48]). *For Gromov-Witten invariants that have a single ψ-class we have*

$$\langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n)\tau_1(T_0)\rangle_{g,\beta} = (2g-2+n)\,\langle \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n)\rangle_{g,\beta}\,,$$

except when $\beta = 0$, $g = 1$ and $n = 2$, in which case we have

$$\langle \tau_1(\gamma)\rangle^X_{1,0} = -\frac{1}{24}\int_X c_{\dim(X)}(X) \cup \gamma.$$

8 The symbol of a relation

In this section we develop the notion of a symbol of a tautological relation $L \in R_{g,n}$. This is done by taking $\mathcal{T}_\beta(L, \gamma_1, \ldots, \gamma_n)$ and forgetting those terms that can be considered of a lower order in some specific sense. This will later allow us to prove reconstruction theorems by using induction on this order. In order to make this ordering precise we will now describe our framework. It relies on the simple description of Gromov-Witten invariants on $\overline{\mathcal{M}}_{0,3}(X, 0)$, which is included as a special case in the following lemma.

Lemma 8.1 (Proposition 12 in [23]). *Let $a \in H^*(X)$ with $|a| \leq 2$ then*

$$\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n)\, a\rangle_{g,\beta} = \int_\beta a \;\cdot \langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n)\rangle_{g,\beta}$$

$$+ \sum_{i=1}^n \langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_i - 1}(\gamma_i \cup a) \cdots \tau_{k_n}(\gamma_n)\rangle_{g,\beta}\,.$$

(Here it is understood that the terms of the sum where $k_i - 1$ would become negative are zero.) except when $\beta = 0$ and $(g, n) \in \{(0,2), (1,0)\}$, in which case the only (possibly) nonzero invariants are

$$\langle \gamma_1\gamma_2\gamma_3\rangle^X_{0,0} = \int_X \gamma_1 \cup \gamma_2 \cup \gamma_3,$$
$$\langle \gamma\rangle^X_{1,0} = -\frac{1}{24}\int_X c_{\dim(X)-1}(X) \cup \gamma.$$

If $\alpha \in H^*(X)$ is a cohomology class we can write it in terms of the generating basis $T_0 \dots T_n$, i.e. $\alpha = \sum \alpha_i T_i$. We can rewrite further as

$$\alpha = \sum_{i,f} \alpha_i \cdot \delta_{if} T_f = \sum_{i,e,f} \alpha_i \cdot g_{ie} \cdot g^{ef} T_f = \sum_{e,f} \left(\int_X \alpha \cup T_e \right) g^{ef} T_f .$$

Using this and the special case of Lemma 8.1 we obtain

$$\sum_{e,f} \langle \gamma_1 \gamma_2 T_e \rangle_{0,0} \, g^{ef} \, \langle \tau_{a_0}(T_f) \tau_{a_3}(\gamma_3) \cdots \tau_{a_n}(\gamma_n) \rangle_{g,\beta} = \langle \tau_{a_0}(\gamma_1 \cup \gamma_2) \tau_{a_3}(\gamma_3) \cdots \tau_{a_n}(\gamma_n) \rangle_{g,\beta} . \quad (7)$$

Definition 8.2. An n-pointed *formal Gromov-Witten invariant* of X,

$$\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n) \rangle_{g,\beta}^X ,$$

consists of the data $k_1, \dots, k_n, g \in \mathbb{Z}_{\geq 0}$, $\beta \in H_2(X)$, and $\gamma_1, \dots, \gamma_n \in H^*(X)$. Here the data

$$\gamma_1, \dots, \lambda\gamma_i, \dots, \gamma_j, \dots, \gamma_n$$

is equivalent to

$$\gamma_1, \dots, \gamma_i, \dots, \lambda\gamma_j, \dots, \gamma_n$$

for any $\lambda \in \mathbb{Q} \setminus \{0\}$ and any $1 \leq i, j \leq n$. Furthermore the data $\gamma_1, \dots, \gamma_n$ and any permutation of it are equivalent up to a sign. This sign is determined by supercommutativity, i.e. swapping γ_i and γ_{i+1} introduces a factor $(-1)^{|\gamma_i||\gamma_{i+1}|}$.

Definition 8.3. We define $\mathbb{Q}[GW(X)]$ to be the $\mathbb{Q}$-algebra of polynomials generated by the n-pointed formal Gromov-Witten invariants of X where n ranges over all nonnegative numbers. Let $G_{g,\beta}^X \subset \mathbb{Q}[GW(X)]$ be the $\mathbb{Q}$-vector space of linear combinations of formal degree β genus g Gromov-Witten invariants of X.

We now define a new grading on $\mathrm{G}_{g,\beta}^X$ where the degree of an invariant $\langle \tau_{k_1}(\gamma_1) \cdots \tau_{k_n}(\gamma_n) \rangle_{g,\beta}^X$ is given by $n - \sum_i k_i$.

We have a realization map

$$\eta : \mathbb{Q}[GW(X)] \to \mathbb{Q}$$

that send a formal Gromov-Witten invariant to the value of its corresponding Gromov-Witten invariant.

Note that this new grading is different from the usual cohomological grading $\sum_i |\gamma_i| + k_i$.

Definition 8.4. For $S \in S^r_{g,n}$, we define S_{prim}, the *primitive part* of S, by restricting it to those terms that are strata classes with rational tail for which we have

$$n(v) - |\theta| = n - r,$$

where v is a vertex of genus g and θ is the decoration at v.

Remark 8.5. The *primitive part* of S consists exactly of those terms that are decorated strata classes with rational tail where every vertex besides a single genus g vertex has exactly 3 special points and no decoration.

The map $\mathcal{T}_\beta$ from Section 7 factors through a map $\mathcal{T}^{\text{form}}_\beta$

$$\begin{tikzcd} S^{\text{r}}_{g,n} \otimes (H^*(X))^{\otimes n} \arrow[r, "\mathcal{T}^{\text{form}}_\beta"] \arrow[rr, bend left, "\mathcal{T}_\beta"] & \mathbb{Q}[GW(X)] \arrow[r, "\eta"] & \mathbb{Q}. \end{tikzcd}$$

Definition 8.6. We fix the degree $\beta \in H_2(X)$ and we define a map

$$\Sigma : S^r_{g,n} \otimes (H^*(X))^{\otimes n} \to G^X_{g,\beta}$$

by taking $\Sigma(S)(\gamma_1, \ldots, \gamma_n)$, the *symbol* of $S \in S^r_{g,n}$, to be the restriction of $\mathcal{T}^{\text{form}}_\beta(S_{\text{prim}}, \gamma_1, \ldots, \gamma_n)$ to those terms for which the degree β is concentrated in one genus g Gromov-Witten invariant.

Since we have restricted to S_{prim}, we can apply (7) to express every term of $\mathcal{T}^{\text{form}}_\beta(S_{\text{prim}}, \gamma_1, \ldots, \gamma_n)$ as a single Gromov-Witten invariant. So the map to $G^X_{g,\beta}$ is well defined as the symbol is indeed a linear combination of genus g invariants. The image of $S^r_{g,n}$ is homogeneous of degree $n - r$.

Remark 8.7. Our definition of a symbol is a generalization of the definition of symbol for genus 1 in [22].

Notation 8.8. To save space we will not write the cup products for Gromov-Witten invariants in $G^X_{g,\beta}$ (e.g. $< \gamma_1\gamma_2 >=< \gamma_1 \cup \gamma_2 >$). To prevent the resulting ambiguity we also change the notation for invariants in $G^X_{g,\beta}$ from

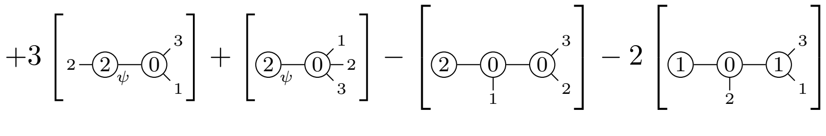
$$< \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n) >_{g,\beta}$$

to

$$< \psi^{a_1}\gamma_1, \ldots, \psi^{a_n}\gamma_n >.$$

Since the genus g and degree β are fixed we can leave them out.

Example 8.9. Consider the linear combination of decorated strata classes in $S^2_{2,3}$



The primitive part consists of the first and third term. The symbol is

$$(-1)^{|\gamma_2||\gamma_3|} \cdot 3 \langle \psi\gamma_1\gamma_3, \gamma_2\rangle - \langle \gamma_1\gamma_2\gamma_3\rangle \,.$$

Definition 8.10. We introduce an equivalence relation on $G^X_{g,\beta}$ as follows. Let $x \in G^X_{g,\beta}$ be homogeneous of degree d. We say $x{\sim}0$ if there is a $P \in \mathbb{Q}[GW(X)]$ such that $\eta(x-P) = 0$ and P is a polynomial in formal Gromov-Witten invariants

$$< \tau_{a_1}(\gamma_1)\cdots\tau_{a_m}(\gamma_m) >_{g',\beta'}$$

for which $g' \leq g$, $\beta' \leq \beta$, or $m - \sum_i a_i \leq d$, and at least one of these three inequalities is not an equality.

(For $\beta, \beta' \in H_2(X)$ we say that $\beta' < \beta$ if there exists a $\beta'' \in H_2(X)$ such that β'' is the image under the cycle map of a nonzero effective 1-cycle and $\beta' + \beta'' = \beta$.)

By construction, if $R \in R^r_{g,n}$, then $\Sigma(R)(\gamma_1, \ldots, \gamma_n) \sim 0$.

Remark 8.11. Lemma 8.1 is known as the string equation when $|a| = 0$ and the divisor equation when $|a| = 2$. They are usually considered separately as it feels a bit artificial to write the integral when $|a| = 0, 1$. However this concern disappears when we work with the $\sim$ equivalence relation:

Corollary 8.12. *Let $a \in H^*(X)$ with $|a| \leq 2$ then*

$$\langle \tau_{k_1}(\gamma_1)\cdots\tau_{k_n}(\gamma_n)\; a\rangle_{g,\beta} \sim \sum_{i=1}^{n} \langle \tau_{k_1}(\gamma_1)\cdots\tau_{k_i-1}(\gamma_i \cup a)\cdots\tau_{k_n}(\gamma_n)\rangle_{g,\beta}\,,$$

except when $\beta = 0$ and $(g,n) \in \{(0,2),(1,0)\}$.

Definition 8.13. Let $S \in S^r_{g,n}$, by abuse of notation we define the *pullback of* $\Sigma(S)(\gamma_1, \ldots, \gamma_n)$ *with cohomology class* γ_{n+1} to be

$$\Sigma(\pi^*(S))(\gamma_1, \ldots, \gamma_n, \gamma_{n+1}).$$

By the string equation (here we need the analogue for the moduli space of stable curves, rather than the string equation for Gromov-Witten invariants, see 2.59 in [31]), the pullback of $\langle \psi^{a_1}\gamma_1, \ldots, \psi^{a_n}\gamma_n\rangle$ with class ψ_{n+1} is

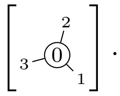
$$\langle \psi^{a_1}\gamma_1, \ldots, \psi^{a_n}\gamma_n, \gamma_{n+1}\rangle - \sum_{i=1}^{n} \left\langle \psi^{a_1}\gamma_1, \ldots, \psi^{a_i-1}\gamma_i\gamma_{n+1}, \ldots, \psi^{a_n}\gamma_n\right\rangle.$$

Remark 8.14. One could ask the question if there is a concept of a *new symbol* that corresponds to the concept of a new relation. New relations are defined using pullback along forgetful maps, pushforward along gluing maps, and multiplication by a decorated stratum class.

We have a corresponding notion of pullback in Definition 8.13.

Pushing a relation forward along gluing maps usually results in a relation that has a vanishing symbol. The only cases where the symbol does not vanish corresponds to substitution of the variables γ_i by multiple of themselves. The simplest example of this substitution occurs when we glue a relation $R \in R^r_{g,n}$ to a class

$$[\Gamma] =$$

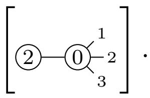


We have

$$\Sigma(q_*(R, [\Gamma])(\gamma_1, \ldots, \gamma_n, \gamma_{n+1}) = \Sigma(R)(\gamma_1, \ldots, \gamma_n\gamma_{n+1}).$$

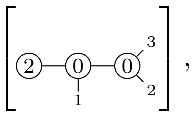
However, multiplication by a general decorated stratum class does not correspond to anything on the symbol side. To illustrate why this is the case we consider the class

$$[\Gamma] =$$



When we multiply by ψ_1 we obtain

$$[\Gamma] \cdot \psi_1 =$$



$$[\Gamma] \cdot \psi_1^2 = 0,$$

and so

$$\begin{aligned} \Sigma([\Gamma])(\gamma_1, \gamma_2, \gamma_3) &= 0, \\ \Sigma([\Gamma] \cdot \psi_1)(\gamma_1, \gamma_2, \gamma_3) &= \langle \gamma_1\gamma_2\gamma_3 \rangle, \\ \Sigma([\Gamma] \cdot \psi_1^2)(\gamma_1, \gamma_2, \gamma_3) &= 0. \end{aligned}$$

So we can only define a *new symbol* as a symbol that is not obtained from doing a pullback or substitution of a variable by a multiple of itself. With such a definition there will be old relations that give new symbols. This is consistent with our experience since we will use the symbols of 3 different relations in genus 2, but only one of them is a new relation (see Remark 11.7).

Remark 8.15. Pushing forward along the forgetful map also has a simple corresponding operation on the symbol side, namely

$$\Sigma(\pi_{n\,*}(R))(\gamma_1, \ldots, \gamma_{n-1}) = \Sigma(R)(\gamma_1, \ldots, \gamma_{n-1}, 1).$$

9 Reconstruction for genus 0

We will write down the proof of the first reconstruction theorem in [39] using the language of symbols. This serves as an example for the more complicated cases of genus 1 and 2. Unlike the original statement we will also allow for odd cohomology.

We say a Gromov-Witten invariant can be *reconstructed* from a set of Gromov-Witten invariants S if it can be calculated recursively by taking the set S as initial values and applying the following relations:

- Relations obtained from $\mathcal{T}_\beta(R, \gamma_1, \ldots, \gamma_n)$ for any $R \in R_{g,n}$ and any $\gamma_1, \ldots, \gamma_n \in H^*(X)$,
- Lemma 7.9 (the dilaton equation and its special case),
- Lemma 8.1 (string and divisor equations and the special case).

If $g \sim 0$ for every Gromov-Witten invariant g that is not in S, then we have such a reconstruction.

Theorem 9.1. *If $H^*(X)$ is generated as a ring by $H^{\leq 2}(X)$ then all genus zero Gromov-Witten invariants can be reconstructed recursively from primitive invariants with at most 2 points.*

Proof. Because we are in genus zero, by Theorem 5.9, we only need to consider primitive invariants. Let us have a 3 pointed Gromov-Witten invariant $< a, b, c >$, $a, b, c \in H^*(X)$. We claim that

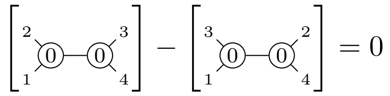
$$0 \sim < a, b, c > . \tag{8}$$

We construct symbols from relations and by using Lemmas 7.9 and 8.12. The equations in these Lemmas still hold when pulled back, so we can pull back (8) and insert cohomology classes $\gamma_4, \ldots, \gamma_n$ to obtain

$$0 \sim < a, b, c, \gamma_4, \ldots, \gamma_n > .$$

So if the claim holds then any primitive genus zero Gromov-Witten invariant with more than 2 points can be reconstructed from those with lower degree or lower number of points, which proves the theorem.

To prove the claim we use induction on $|c|$. If $|c| \leq 2$ then we apply Corollary 8.12. If $|c| > 2$, then using the hypothesis on X, we can write $c = c'd$ with $0 < |d| \leq 2$. We now take the tautological relation S in $R^1(\overline{\mathcal{M}}_{0,4})$ known as the WDVV relation:



We take its symbol $\Sigma(S)(a,b,c',d)$ to obtain

$$0 \sim < a,b,c'd > + < ab,c',d > -(-1)^{|b||c'|}(< a,c',bd > + < ac',b,d > .$$

Using Corollary 8.12 this becomes

$$0 \sim < a,b,c'd > -(-1)^{|b||c'|} < a,c',bd > .$$

Since $|c'| < |c|$ this ends our induction step. $\square$

10 Reconstruction for genus 1

In [22] Getzler found a new tautological relation in $R^2(\overline{\mathcal{M}}_{1,4})$ and used it to prove the following reconstruction theorem for genus 1. There is a step missing in the proof, namely in Corollary 3.3 of [22] the initial conditions for the difference equation are missing. So in this section I will give my own proof of the theorem. This proof uses the same type of methods as are used by Getzler.

As in genus 0, in genus 1 it is sufficient to consider only primitive invariants by Theorem 5.9.

We fix an ample divisor $\omega \in H^2(X)$ and define the primitive cohomology of X to be

$$P^i(X) := \operatorname{coker}(H^{i-2}(X,\mathbb{C}) \xrightarrow{\cdot\cup\omega} H^i(X,\mathbb{C})).$$

Theorem 10.1. *If $P^i(X) = 0$ for $i > 2$, then all genus one Gromov-Witten invariants can be reconstructed recursively from primitive genus 1 invariants with at most 1 point and all genus zero invariants.*

Remark 10.2. Note that the hypothesis on X is more demanding than in Theorem 9.1. Rather than requiring that any cohomology class $a \in H^*(X)$ can be written as a linear combination of products of cohomology classes of degree at most 2, we now require that $a = a'\omega^k$ for some $k \in \mathbb{Z}_{\geq 0}$ and $a' \in H^{\leq 2}(X)$.

We will use the following very basic result about difference equations.

Definition 10.3. Let V is some vector space over a field K. Let I be the set $\{0,1,\ldots,N\}$ where N is either a positive integer or infinity. A linear *difference equation* consists of data $a_0,\ldots,a_k \in K$, $h \in V$. If $h = 0$ we say the difference equation is *homogeneous*. A *solution* to the difference equation is a map $f : I \to V$ satisfying the equation

$$a_k f(i+k) + \ldots + a_1 f(i+1) + a_0 f(i) + h = 0, \tag{9}$$

for all $0 \leq i \leq N-k$. When f is a solution of the diffence equation, we will also refer to the equation (9) as the difference equation.

Lemma 10.4. *Let f, h be as in Definition 10.3 and let*

$$f(i+2) - 2f(i+1) + f(i) - h = 0$$

be a difference equation. Let $l \in I$, then f satisfies the formula

$$lf(i) = if(l) + (l-i)f(0) - \frac{li(l-i)}{2}h.$$

Proof. By subtracting a shifted version of the difference equation we obtain the homogeneous difference equation

$$f(i+3) - 3f(i+2) + 3f(i+1) - f(i) = 0.$$

This has a solution of the form

$$f(i) = c_0 + c_1 i + c_2 i^2.$$

We substitute $i = 0, 1, 2$ and compare the results with the original difference equation for $i = 0$. From this we obtain

$$f(i) = if(1) - (i-1)f(0) + \binom{i}{2}h.$$

In particular

$$lf(1) = f(l) + (l-1)f(0) - \binom{l}{2}h,$$

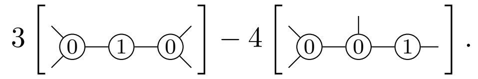
so

$$\begin{aligned} lf(i) =& i(f(l) + (l-1)f(0) - \binom{l}{2}h) - l(i-1)f(0) + l\binom{i}{2}h \\ =& if(l) + (l-i)f(0) - \frac{li(l-i)}{2}h. \end{aligned}$$

$\square$

Now we can proof the reconstruction theorem:

Proof of Theorem 10.1. The primitive part of Getzler's relation $L_{1,4}$ in $R^2(\overline{\mathcal{M}}_{1,4})$ is



Let $a, b \in H^{\leq 2}(X)$. Fix an integer $l \geq 2$, we define

$$f(i) := \left\langle a\omega^i, b\omega^{l-i} \right\rangle,$$

and

$$h := \left\langle ab\omega^{l-2}, \omega^2 \right\rangle.$$

Applying Corollary 8.12 to the symbol of $L_{1,4}$ gives

$$\Sigma(L_{1,4})(a\omega^k, b\omega^{l-2-k}, \omega, \omega) \sim f(k+2) - 2f(k+1) + f(k) - h,$$

for every $0 \leq k \leq l-2$. Again by Corollary 8.12 $f(0) \sim f(l) \sim 0$ so Lemma 10.4 becomes

$$< a\omega^i, b\omega^j > \sim \frac{ij}{2} < ab\omega^{i+j-2}, \omega^2 > . \tag{10}$$

We substitute $b \to \omega, j \to 1$ or $b \to 1, j \to 2$ to obtain

$$< a\omega^i, \omega^2 > \sim \frac{i}{2} < a\omega^i, \omega^2 > \sim i < a\omega^i, \omega^2 >,$$

so

$$< a\omega^i, \omega^2 > \sim 0. \tag{11}$$

Using our hypothesis on X we can write $ab\omega^{i+j-2} = c\omega^k$ for some $k \in \mathbb{Z}_{\geq 0}$ and $c \in H^{\leq 2}(X)$. So we can apply (11) to the right-hand side of (10) to obtain

$$< a\omega^i, b\omega^j > \sim 0.$$

□

11 Reconstruction for genus 2

In this section we will prove a reconstruction theorem for genus 2 Gromov-Witten invariants. In the 90s Belorousski and Pandharipande had attempted to prove such a reconstruction theorem for genus 2 using a new relation in $R^2_{2,3}$ that they found in [9]. In that paper they computed the invariants of $\mathbb{P}^2$ using their new relation, but they were not able to prove a general reconstruction theorem. This is probably due to the fact that they did not have the new relation in $R^3(\overline{\mathcal{M}}_{2,6})$ to work with. (See remark 11.7 about new relations in genus 2.)

By Corollary 5.8 and Theorem 5.9, in genus 2 the tautological ring is generated by decorated strata classes that have no decorations and decorated strata classes that have a single ψ-class at one half-edge on a genus 2 vertex.

In this section we will often apply Corollary 8.12 without explicitly mentioning it every time.

Theorem 11.1. *If $P^i(X) = 0$ for $i > 2$, then all (including descendant) genus two Gromov-Witten invariants can be reconstructed recursively from genus two invariants with at most two points and invariants of lower genus.*

Proof. Let $a, b, c \in H^*(X)$ such that $|a|, |b|, |c| \leq 2$ and let $i, j, k \geq 0$, then

$$\left\langle a\omega^i, b\omega^j, c\omega^k \right\rangle \sim 0,$$

and

$$\left\langle \psi a\omega^i, b\omega^j, c\omega^k \right\rangle \sim \text{a linear combination of 2 pointed primitive invariants.}$$

This is proven in Lemmas 11.6 and 11.5.

By pulling back the two relations above we can always express an invariant with 3 or more points as a linear combination of invariants with lower degree, genus or number of points. □

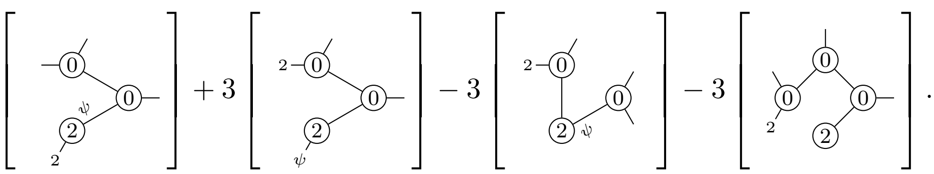
We will now prove the Lemmas required for Theorem 11.1. This requires some buildup where for simplicity we work with Gromov-Witten invariants that have only one or two points. We then do a pullback of these invariants to help us obtain the results we want for 3 pointed invariants. The tautological relations we use are all obtained using the computer program in Appendix B.

Our first aim is to express descendant invariants in terms of primitive invariants.

Lemma 11.2. *Let $a, b \in H^{\leq 2}(X)$, the following holds for all $i, j \geq 0$*

$$\begin{aligned}(i+j)\left\langle \psi a\omega^i, b\omega^j \right\rangle \sim & i\left\langle ab\omega^{i+j}\right\rangle + j\left\langle \psi a, b\omega^{i+j}\right\rangle \\ & - \frac{ij(i+j)}{2}\Big(\left\langle \psi\omega^2, ab\omega^{i+j-2}\right\rangle \\ & - 2\left\langle \psi\omega, ab\omega^{i+j-1}\right\rangle + 3\left\langle ab\omega^{i+j}\right\rangle \Big).\end{aligned}$$

Proof. Let us fix the number $l := i + j$. For $i = 0$ or $j = 0$ the statement follows from Corollary 8.12 so we can assume $l \geq 2$. Using our computer program, we find a relation $L_{2,4}$ in $R^3_{2,4}$ with primitive part



We define

$$f(i) := \left\langle \psi a\omega^i, b\omega^{l-i} \right\rangle,$$

and

$$h := \left\langle \psi\omega^2, ab\omega^{l-2} \right\rangle - 2\left\langle \psi\omega, ab\omega^{l-1} \right\rangle + 3\left\langle ab\omega^l \right\rangle,$$

in order to obtain

$$\Sigma(L_{2,4})(a\omega^k, b\omega^{l-2-k}, \omega, \omega) \sim f(k+2) - 2f(k+1) + f(k) - h,$$

for every $0 \leq k \leq l-2$. Applying Lemma 10.4 gives the desired formula. □

In order to get rid of the term with $\psi\omega^2$ in Lemma 11.2, we do the following:

Corollary 11.3. *Let $b \in H^{\leq 2}(X)$, the following holds for all $j \geq 0$*

$$(j+1)(j+2)\left\langle \psi\omega^2, b\omega^j \right\rangle \sim 2j(j+2)\left\langle \psi\omega, b\omega^{j+1} \right\rangle - (3j^2+3j-2)\left\langle b\omega^{i+2} \right\rangle .$$

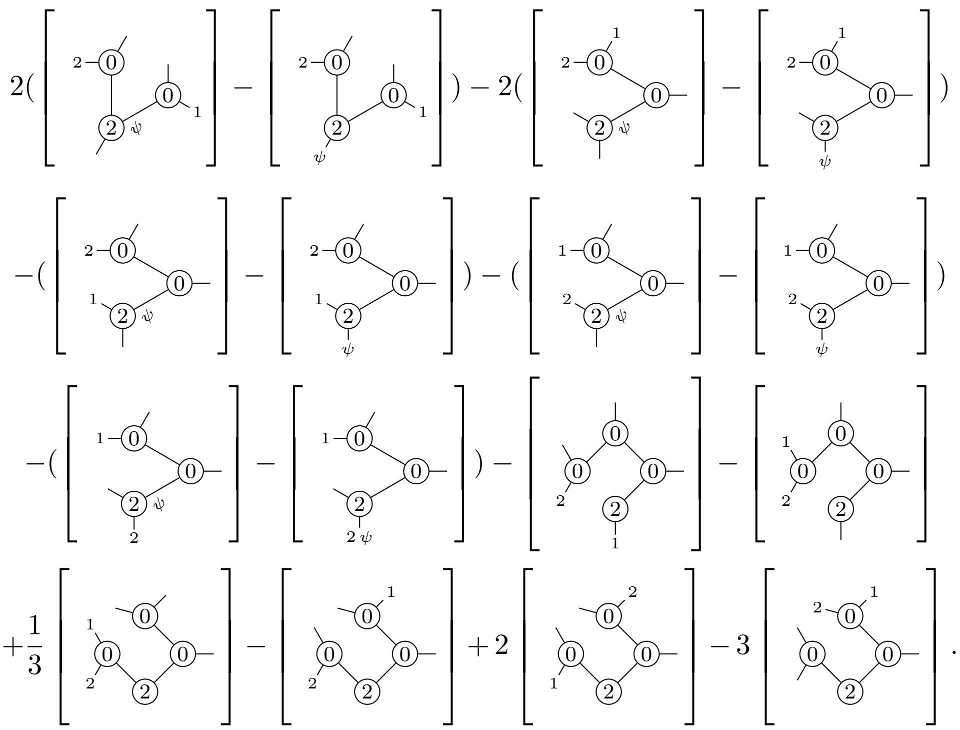
Proof. Substitute $a \to \omega$ and $i \to 1$ in Lemma 11.2. $\square$

Now we want to get rid of the term in Lemma 11.2 with $\psi\omega$ in it. However, in order to get rid of terms with a $\psi\omega$ we need to consider 3-pointed invariants.

Lemma 11.4. *Let $a, b \in H^{\leq 2}(X)$, the following holds for all $i, j \geq 0$*

$$\begin{aligned}(i+j)\left\langle \psi\omega, a\omega^i, b\omega^j \right\rangle \sim & 2(i+j)\left\langle a\omega^{i+1}, b\omega^j \right\rangle + 2(i+j)\left\langle a\omega^i, b\omega^{j+1} \right\rangle \\ & - i\left\langle a\omega^{i+j}, b\omega \right\rangle - j\left\langle a\omega, b\omega^{i+j} \right\rangle \\ & - \frac{ij(i+j)}{6}\Big(3\left\langle ab\omega^{i+j-1}, \omega^2 \right\rangle - \left\langle ab\omega^{i+j-2}, \omega^3 \right\rangle\Big).\end{aligned}$$

Proof. Let us write $l := i+j$. For $i = 0$ or $j = 0$ it is trivial so we can assume $l \geq 2$. Using our computer program, we find a relation $L_{2,5}$ in $R^3_{2,5}$ with primitive part



We define

$$f(i) := \left\langle \psi\omega, a\omega^i, b\omega^{l-i} \right\rangle - 2\left\langle a\omega^{i+1}, b\omega^{l-i} \right\rangle - 2\left\langle a\omega^i, b\omega^{l-i+1} \right\rangle ,$$

$$h := \left\langle ab\omega^{l-1}, \omega^2 \right\rangle - \frac{1}{3}\left\langle ab\omega^{l-2}, \omega^3 \right\rangle,$$

in order to obtain

$$\Sigma(L_{2,5})(a\omega^k, b\omega^{l-2-k}, \omega, \omega, \omega) \sim f(k+2) - 2f(k+1) + f(k) - h,$$

for $0 \leq k \leq l$. Applying Lemma 10.4 gives the desired formula $\square$

We now have all the ingredients we need to express descendant invariants in terms of primitive invariants.

Lemma 11.5. *Let $a, b, c \in H^{\leq 2}(X)$, the following holds for all $i, j, k \geq 0$*

$$\left\langle \psi a\omega^i, b\omega^j, c\omega^k \right\rangle \sim \text{a linear combination of 2 pointed primitive invariants.}$$

Proof. Let us define the equivalence relation $\sim\sim$ by saying that a linear combination of Gromov-Witten invariants is equivalent to zero if it is equivalent to zero for the $\sim$ equivalence relation or if it is a linear combination of 2-pointed primitive invariants.

By Lemma 11.4 we have

$$\left\langle \psi\omega, a\omega^i, b\omega^j \right\rangle \sim\sim 0.$$

Now pulling back Corollary 11.3 and inserting γ gives

$$\left\langle \psi\omega^2, b\omega^j, \gamma \right\rangle \sim\sim 0.$$

By the hypothesis on X we can rewrite $ab\omega^{i+j-2}$ as $a'\omega^{k'}$ for some $a' \in H^{\leq 2}(X)$, $k' \in \mathbb{Z}_{\geq 0}$. So we can apply the above formulas to the pullback of Lemma 11.2 (where we insert γ) to obtain

$$(i+j)\left\langle \psi a\omega^i, b\omega^j, \gamma \right\rangle \sim\sim j\left\langle \psi a, b\omega^{i+j}, \gamma \right\rangle.$$

Repeatedly applying this formula gives

$$\left\langle \psi a\omega^i, b\omega^j, c\omega^k \right\rangle \sim\sim jk\left\langle \psi a\omega^{i+j+k-2}, b\omega, c\omega \right\rangle. \tag{12}$$

So it is sufficient to proof that

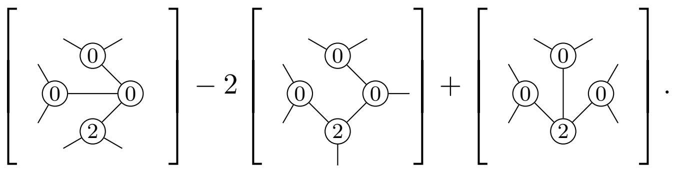
$$\left\langle \psi a\omega^i, b\omega, c\omega \right\rangle \sim\sim 0$$

We apply (12) to $L_{2,5}$, the relation from the proof of Lemma 11.4, to obtain

$$\begin{aligned}(-1)^{|a|(|b|+|c|)}\Sigma(L_{2,5})(b\omega^i, c, a, \omega, \omega) &\sim\sim \left\langle \psi a, b\omega^i, c\omega^2 \right\rangle - 2\left\langle \psi a, b\omega^{i+1}, c\omega \right\rangle \\ &\sim\sim (2i - 2(i+1))\left\langle \psi a\omega^i, b\omega, c\omega \right\rangle.\end{aligned}$$

$\square$

What is left is to find an expression for primitive invariants. Using our computer program we find a symmetric relation $L_{2,6}$ in $R^3_{2,6}$ that has the following primitive part:



We write $\Phi_k(\gamma_1, \gamma_2, \gamma_3)$ for the system of equations

$$\left\{\Sigma(L_{2,6})(\gamma_1\omega^{k_1}, \gamma_2\omega^{k_2}, \gamma_3\omega^{k_3}, \omega^{k_4+1}, \omega^{k_5+1}, \omega^{k_6+1}) \sim 0\right\}_{k_1+k_2+k_3+k_4+k_5+k_6=k}. \tag{13}$$

Since the degree k will be split over 3 points rather than 2, we can no longer use a simple difference equation to find a general expression. We have an infinite series of matrices $\Phi_k(a, b, c)$ for $a, b, c \in H^{\leq 2}(X)$. However, we can reduce to the case where all of the degree is concentrated in the first point, i.e. invariants of the form

$$\left\langle a\omega^i, b\omega, c\omega\right\rangle$$

for $i \geq 0$. This means we will only need to consider $\Phi_k(a\omega^j, b, c)$ where $j \geq 0$ is an unspecified variable and k is small.

We take Equation (11.3) and divide it by $(j+1)(j+2)$. We also pull it back and insert the cohomology class $b\omega^k$. Finally we apply Lemma 11.4 to obtain a sum of primitive invariants on the right-hand side.

$$\begin{aligned}
\left\langle \psi\omega^2, a\omega^j, b\omega^k\right\rangle &\sim \left\langle a\omega^j, b\omega^{k+2}\right\rangle - \frac{3j^2+3j-2}{(j+1)(j+2)}\left\langle a\omega^{j+2}, b\omega^k\right\rangle \\
&\quad + \frac{2j}{j+1}\left\langle \psi\omega, a\omega^{j+1}, b\omega^k\right\rangle - \frac{2j}{j+1}\left\langle a\omega^{j+1}, b\omega^{k+1}\right\rangle \\
&\sim \left\langle a\omega^j, b\omega^{k+2}\right\rangle + \frac{j^2+5j+2}{(j+1)(j+2)}\left\langle a\omega^{j+2}, b\omega^k\right\rangle \\
&\quad + \frac{2j}{j+1}\left\langle a\omega^{j+1}, b\omega^{k+1}\right\rangle - \frac{2j}{j+k+1}\left\langle a\omega^{j+k+1}, b\omega\right\rangle \\
&\quad - \frac{2jk}{(j+1)(j+k+1)}\left\langle a\omega, b\omega^{j+k+1}\right\rangle \\
&\quad - \frac{jk}{3}\left(3\left\langle ab\omega^{j+k}, \omega^2\right\rangle - \left\langle ab\omega^{j+k-1}, \omega^3\right\rangle\right).
\end{aligned}$$

We swap the 2nd and 3rd point (i.e. we swap $a\omega^i$ and $b\omega^j$) and subtract the resulting formula.

$$0 \sim -\frac{2k}{(k+1)(k+2)}\left\langle a\omega^j, b\omega^{k+2}\right\rangle + \frac{2j}{(j+1)(j+2)}\left\langle a\omega^{j+2}, b\omega^k\right\rangle + \frac{2(j-k)}{(j+1)(k+1)}\left\langle a\omega^{j+1}, b\omega^{k+1}\right\rangle - \frac{2j}{(k+1)(j+k+1)}\left\langle a\omega^{j+k+1}, b\omega\right\rangle + \frac{2k}{(j+1)(j+k+1)}\left\langle a\omega, b\omega^{j+k+1}\right\rangle. \tag{14}$$

Let $\Psi_{j,k}(a,b)$ denote the right-hand side of (14). We define

$$\Theta_{j,k}(a,b) := (j+2)(k+1)\Big((j+1)\Psi_{j,k}(a,b) - \frac{k(j+2)}{(k-1)}\Psi_{j+1,k-1}(a,b)\Big).$$

We have

$$\begin{aligned}\Theta_{j,k}(a,b) = &-\frac{2k(j+1)(j+2)}{k+2}\left\langle a\omega^j, b\omega^{k+2}\right\rangle \\ &+\frac{2(jk-k^2+k+2)(j+2)}{k-1}\left\langle a\omega^{j+1}, b\omega^{k+1}\right\rangle \\ &-\frac{2(j^2-2jk+3j-2k)(k+1)}{k-1}\left\langle a\omega^{j+2}, b\omega^{k}\right\rangle \\ &-\frac{2k(j+1)(j+2)(k+1)}{(k-1)(j+3)}\left\langle a\omega^{j+3}, b\omega^{k-1}\right\rangle \\ &+\frac{4(j+1)(j+2)}{k-1}\left\langle a\omega^{j+k+2}, b\omega\right\rangle.\end{aligned}$$

Lemma 11.6. *Let $a,b,c \in H^{\leq 2}(X)$, the following holds for all $i,j,k \geq 0$*

$$\left\langle a\omega^i, b\omega^j, c\omega^k\right\rangle \sim 0$$

Proof. Using Θ we can always reduce to the case $j,k \leq 2$.

We write $\Theta_{j,k}(a,b)\langle\gamma\rangle$ to denote the pullback of $\Theta_{j,k}(a,b)$ where we insert the cohomology class γ.

The rest of the proof consists of proving that certain systems of equations are full rank. We only describe the equations here. The full matrices and their determinants are listed in Appendix A

The relations

$$\Phi_1(a\omega^i,\omega,\omega), \quad \Theta_{1,2}(1,\omega)\left\langle a\omega^i\right\rangle, \quad \Theta_{i,2}(a,1)\left\langle \omega^2\right\rangle.$$

make up a full rank system of 4 unique equations in 4 variables, which proves that

$$\left\langle a\omega^{i+2},\omega^2,\omega^2\right\rangle \sim \left\langle a\omega^{i+1},\omega^3,\omega^2\right\rangle \sim \left\langle a\omega^{i},\omega^3,\omega^3\right\rangle \sim 0,$$

for $i \geq 0$. Coupled with the one relation in $\Phi_0(a,\omega,\omega)$,

$$\Sigma(L_{2,6})(a,\omega,\omega,\omega,\omega,\omega) \sim 60\left\langle a\omega,\omega^2,\omega^2\right\rangle \sim 0,$$

this proves that $\left\langle \gamma,\omega^i,\omega^j\right\rangle \sim 0$ for all $\gamma \in H^*(X)$, $i,j \geq 0$. For the remainder of this proof we will set all invariants of this form to zero.

Consider the equations

$$\Phi_0(a,b,\omega), \quad \Phi_1(a,b,\omega), \quad \Phi_2(a\omega^i,b,\omega),$$
$$(-1)^{|a||b|}\Theta_{1,2}(\omega,b)\left\langle a\omega^i\right\rangle, \quad \Theta_{i+1,2}(a,1)\left\langle b\omega\right\rangle, \quad \Theta_{i,2}(a,1)\left\langle b\omega^2\right\rangle.$$

Together these equations imply $\left\langle a\omega^i,b\omega^j,\omega^k\right\rangle \sim 0$, for $i \geq 0$, $0 \leq j \leq 2$, and $0 \leq k \leq 3$. this proves that $\left\langle \gamma_1,\gamma_2,\omega^i\right\rangle \sim 0$ for all $\gamma_1,\gamma_2 \in H^*(X)$, $i \geq 0$. For the remainder of this proof we will set all invariants of this form to zero.

Consider the equations

$$\Phi_0(a,b,c), \quad \Phi_1(a,b,c), \quad \Phi_2(a\omega^i,b,c),$$
$$\Theta_{i,2}(a,b)\left\langle c\omega\right\rangle, \quad (-1)^{|b||c|}\Theta_{i,2}(a,c)\left\langle b\omega\right\rangle.$$

Together these equations imply $\left\langle a\omega^i,b\omega^j,c\omega^k\right\rangle \sim 0$, for $i \geq 0$, $0 \leq j \leq 2$, and $0 \leq k \leq 2$. $\square$

Remark 11.7. There are new relations in genus two that express κ-classes and degree two monomials ψ-classes in terms of decorated strata classes with at most one ψ-class. Besides these relations the only known new relations are in $R^2_{2,3}$ and $R^3_{2,6}$. The new relation in $R^2_{2,3}$ was first discovered by Belorousski and Pandharipande in [9]. It is uniquely represented by a single relation that we will denote by L_{BP}.

We use the 3 relations $L_{2,i}$ for $4 \leq i \leq 6$ in our proof of 11.1. We have

$$L_{\mathrm{BP}} = \pi_{2*}L_{2,4},$$

and the relation $L_{2,6}$ represents the new relation in $R^3_{2,6}$. The relations $L_{2,4}$ and $L_{2,5}$ are old and we can recover them from L_{BP}.

For any $0 \leq i \leq 4$, the relation $\pi_1^*(L_{\mathrm{BP}}) \cdot \psi_i$ has terms with genus 2 components that have a degree 2 monomial in ψ-classes. By 5.9 we can express these terms using decorated strata classes that have at most one ψ-class. A formula for how to do this is given by Getzler in [23]. Modulo these simplifications we have for the primitive part, and also modulo pullbacks along gluing morphisms of the new relation in $R^1(\overline{\mathcal{M}}_{0,4})$ we have

$$2L_{2,4} = \pi_1^*(L_{\mathrm{BP}})(-\psi_1 + 3\psi_2 + \psi_3 + \psi_4).$$

Similarly, for $L_{2,5}$ we have

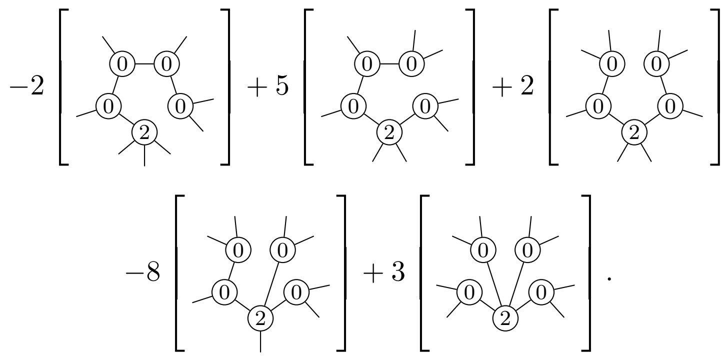
$$6L_{2,5} = \pi_{1,2}^*(L_{\mathrm{BP}})(3\psi_1 - 4\psi_2 - 9\psi_4) + \pi_{1,4}^*(L_{\mathrm{BP}}) \cdot (-7\psi_1 + 28\psi_2 + 4\psi_4 - 16\psi_5)$$
$$+\pi_{2,4}^*(L_{\mathrm{BP}}) \cdot (-3\psi_4 + 12\psi_5) + 3\pi_{4,5}^*(L_{\mathrm{BP}}) \cdot \psi_4.$$

Note that these two formulas give specific representatives in equivalence classes of relations that have the same symbol. Taking a different representative does not change the validity of the proof of Theorem 11.1.

12 Reconstruction in higher genus

We have tried to copy the methods of Section 11 to find a similar reconstruction theorem in genus 3. Sadly this has not been succesful. In this section we will describe a few partial results we obtained.

Part of the problem is that the program we use to calculate relations runs out of memory when the number of points and the codimension grow too large. In genus 2 we are still able to calculate all of the relations we need without running out of memory. But in genus 3 the relations we are able to directly calculate are insufficient to obtain a reconstruction theorem using the methods that we used in genus 2. To combat this issue we have rewritten part of the program so that the calculation of symmetrical relations is more efficient (see Appendix B.1). This enabled us to calculate 12 linearly independent representatives of a new relation in $R^4_{3,8}$. But we have not been able to apply this new relation to obtain a reconstruction result. Just like for the new relation in $R^3_{2,6}$, there is a representation whose primite part has no ψ-classes. It is given by



By Corollary 5.8 and Theorem 5.9, in genus 3 the tautological ring restricted to rational tail is generated by decorated strata classes that have no decorations outside of the genus 3 vertex. The decoration on the genus 3 vertex can be reduced to a monomial in ψ-classes of degree at most two, times a power of κ_1.

As in Section 11, the following relations are all obtained using the computer program in Appendix B.

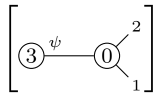
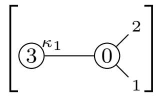
In $R^2(\mathcal{M}^{rt}_{3,1})$ we have the relation

$$\kappa_1^2 = 10\psi_1^2.$$

In $R^2(\mathcal{M}^{rt}_{3,2})$ there are relations

$$\kappa_1\psi_1 = 5\psi_1^2,$$

and

$6\psi_1\psi_2 = 5\psi_1^2 + 5\psi_2^2 - 5$  $+ 25$  .

So we can further restrict to the case where the decoration at the genus 3 vertex is either κ_1 or ψ_i^2 for any point i.

By Definition 7.6, the class κ_1 will result in a Gromov-Witten invariant of the form $< \psi^2, \dots >$. We can express invariants of this form in terms of invariants that have at most a single ψ class:

Lemma 12.1. *For genus 3, let $a, b \in H^{\leq 2}(X)$, the following holds for all $i, j \geq 0$. If $|a| + |b| > 2$, then*

$$\begin{aligned}
2(i+j)\left\langle \psi^2, a\omega^i, b\omega^j \right\rangle \sim & 5(i+j)\left\langle \psi a\omega^i, b\omega^j \right\rangle + 5(i+j)\left\langle \psi b\omega^j, a\omega^i \right\rangle \\
& - 3i\left\langle \psi b, a\omega^{i+j} \right\rangle - 3j\left\langle \psi a, b\omega^{i+j} \right\rangle \\
& - \frac{3ij(i+j+1)}{2}\left\langle \psi\omega, ab\omega^{i+j-1} \right\rangle \\
& + \frac{3ij}{i+j+1}\left\langle \psi ab, \omega^{i+j} \right\rangle + \frac{10ij(i+j)}{i+j+1}\left\langle ab\omega^{i+j} \right\rangle .
\end{aligned}$$

If $|a| + |b| \leq 2$, then

$$\begin{aligned}
2(i+j)\left\langle \psi^2, a\omega^i, b\omega^j \right\rangle \sim & 5(i+j)\left\langle \psi a\omega^i, b\omega^j \right\rangle + 5(i+j)\left\langle \psi b\omega^j, a\omega^i \right\rangle \\
& - 3i\left\langle \psi b, a\omega^{i+j} \right\rangle - 3j\left\langle \psi a, b\omega^{i+j} \right\rangle \\
& - \frac{3ij(i+j)^2}{2(i+j-1)}\left\langle \psi\omega, ab\omega^{i+j-1} \right\rangle \\
& + \frac{3ij}{i+j-1}\left\langle \psi ab, \omega^{i+j} \right\rangle + 10ij\left\langle ab\omega^{i+j} \right\rangle .
\end{aligned}$$

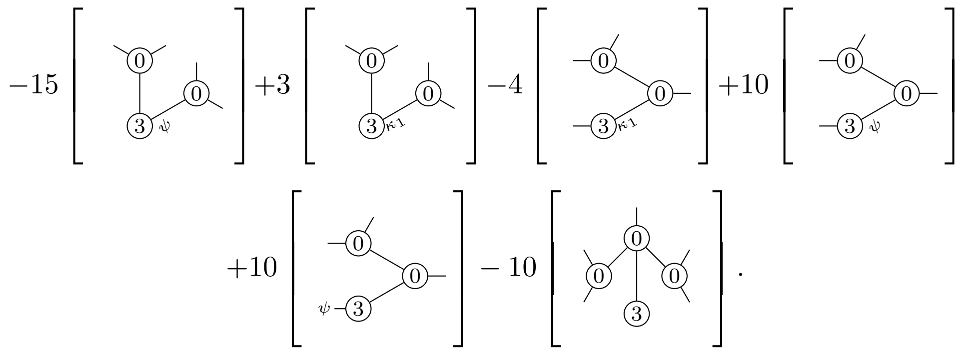
Proof. Let us fix the number $l := i + j$. When $i = 0$ or $j = 0$ the formulas hold by Corollary 8.12 so we can assume $l \geq 2$. We define

$$f(i) := 2\left\langle \psi^2, a\omega^i, b\omega^{l-i} \right\rangle - 5\left\langle \psi a\omega^i, b\omega^{l-i} \right\rangle - 5\left\langle \psi b\omega^{l-i}, a\omega^i \right\rangle ,$$

and

$$\begin{aligned}
h := & 2\left\langle \psi^2, ab\omega^{l-2}, \omega^2 \right\rangle - 5\left\langle \psi ab\omega^{l-2}, \omega^2 \right\rangle \\
& - 5\left\langle \psi\omega^2, ab\omega^{l-2} \right\rangle + 6\left\langle \psi\omega, ab\omega^{l-1} \right\rangle - 10\left\langle ab\omega^l \right\rangle .
\end{aligned}$$

Using the computer program we find a relation R in $R^3_{3,4}$ with primitive part



Its symbol is given by

$$\Sigma(R)(a\omega^k, b\omega^{l-2-k}, \omega, \omega) \sim f(k+2) - 2f(k+1) + f(k) - h,$$

for every $0 \le k \le l-2$. We apply Lemma 10.4 to obtain

$$\begin{aligned} 2(i+j)\left\langle \psi^2, a\omega^i, b\omega^j \right\rangle \sim & 5(i+j)\left\langle \psi a\omega^i, b\omega^j \right\rangle + 5(i+j)\left\langle \psi b\omega^j, a\omega^i \right\rangle \\ & - 3i\left\langle \psi b, a\omega^{i+j} \right\rangle - 3j\left\langle \psi a, b\omega^{i+j} \right\rangle \\ & - \frac{ij(i+j)}{2}\Big(2\left\langle \psi^2, ab\omega^{i+j-2}, \omega^2 \right\rangle \\ & - 5\left\langle \psi ab\omega^{i+j-2}, \omega^2 \right\rangle - 5\left\langle \psi\omega^2, ab\omega^{i+j-2} \right\rangle \\ & + 6\left\langle \psi\omega, ab\omega^{i+j-1} \right\rangle - 10\left\langle ab\omega^{i+j} \right\rangle.\Big). \end{aligned} \tag{15}$$

We substitute $b \to \omega$ and $j \to 1$ to obtain

$$\begin{aligned} 2(i+1)\left\langle \psi^2, a\omega^i, \omega^2 \right\rangle \sim & 5(i+1)\left\langle \psi a\omega^i, \omega^2 \right\rangle + 5(i+1)\left\langle \psi\omega^2, a\omega^i \right\rangle \\ & - 3i\left\langle \psi\omega, a\omega^{i+1} \right\rangle - 3\left\langle \psi a, \omega^{i+2} \right\rangle \\ & - \frac{i(i+1)}{2}\Big(2\left\langle \psi^2, a\omega^i, \omega^2 \right\rangle - 5\left\langle \psi a\omega^i, \omega^2 \right\rangle \\ & - 5\left\langle \psi\omega^2, a\omega^i \right\rangle + 6\left\langle \psi\omega, a\omega^{i+1} \right\rangle - 10\left\langle a\omega^{i+2} \right\rangle.\Big), \end{aligned}$$

which multiplied by two becomes

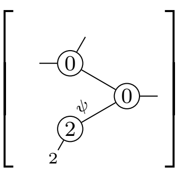
$$\begin{aligned} (i+1)(i+2)\Big(2\left\langle \psi^2, a\omega^i, \omega^2 \right\rangle - 5\left\langle \psi a\omega^i, \omega^2 \right\rangle - 5\left\langle \psi\omega^2, a\omega^i \right\rangle\Big) \sim \\ - 3i(i+2)\left\langle \psi\omega, a\omega^{i+1} \right\rangle - 6\left\langle \psi a, \omega^{i+2} \right\rangle + 10i(i+1)\left\langle a\omega^{i+2} \right\rangle. \end{aligned} \tag{16}$$

If $|a|+|b| > 2$ then $ab = c\omega$ for some $c \in H^{\le 2}(X)$. We substitute $i \to i+j-1$ and $a \to c$ in (16) and then apply the result to (15).

For the case where $|a|+|b| <= 2$ we have $ab = c$ for some $c \in H^{\le 2}(X)$. Similarly we now substitute $i \to i+j-2$ and $a \to c$ in (16) and then apply the result to (15). This way we obtain the desired formulas.

□

The relation $L_{2,4}$ extends to genus 3 in the following sense: The primitive part of $L_{2,4}$ is



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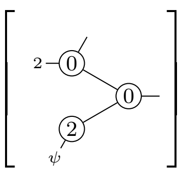

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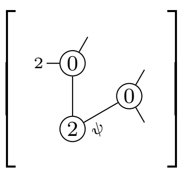

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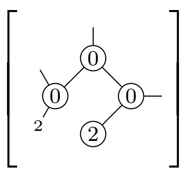
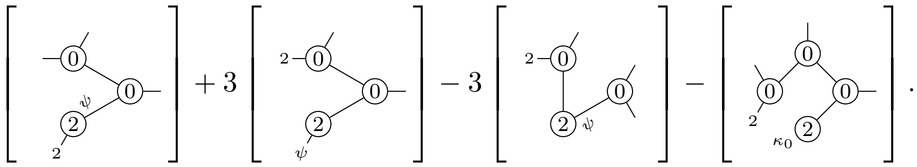
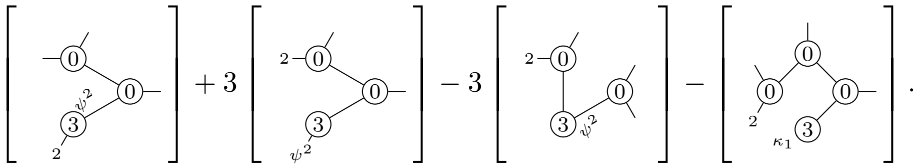


,

which is equivalent to



The computer tells us there is a relation in $R^4_{3,4}$ that has primitive part



We checked using the computer that $L_{2,4}$ extends in the same manner to genus up to 8. Therefore we can also extend Lemma 11.2.

Lemma 12.2. *Let $2 \leq g \leq 8$ where g is the genus and let $a, b \in H^{\leq 2}(X)$, the following holds for all $i, j \geq 0$*

$$\begin{aligned}(i+j)\left\langle \psi^{g-1} a\omega^i, b\omega^j \right\rangle \sim\, & i \left\langle \psi^{g-2} ab\omega^{i+j} \right\rangle + j \left\langle \psi^{g-1} a, b\omega^{i+j} \right\rangle \\ & - \frac{ij(i+j)}{2} \Big(\left\langle \psi^{g-1}\omega^2, ab\omega^{i+j-2} \right\rangle \\ & - 2 \left\langle \psi^{g-1}\omega, ab\omega^{i+j-1} \right\rangle + \left\langle \psi^{g-1}, ab\omega^{i+j} \right\rangle \Big).\end{aligned}$$

Of course the corresponding Corollary 11.3 also extends to genus 3.

Corollary 12.3. *Let $2 \leq g \leq 8$ where g is the genus and let $b \in H^{\leq 2}(X)$, the following holds for all $j \geq 0$*

$$\begin{aligned}(j+1)(j+2)\left\langle \psi^{g-1}\omega^2, b\omega^j \right\rangle \sim\, & 2j(j+2)\left\langle \psi^{g-1}\omega, b\omega^{j+1} \right\rangle + 2\left\langle \psi^{g-2} b\omega^{i+2} \right\rangle \\ & - j(j+1)\left\langle \psi^{g-1}, b\omega^{i+2} \right\rangle.\end{aligned}$$

Remark 12.4. This leads one to wonder if the relation $L_{2,4}$ extends this way for all genera. We can obtain expressions for general genus using formulas such as the formula for the DR cycle (see [37]) or the half-spin formula in [41]. But these expressions will contain degree d monomials of ψ-classes. We know that we can express these monomials in terms of lower invariants using 5.9, but there is no formula that lets us do this for every genus. So obtaining a reconstruction theorem for general genus from methods is far out of reach.

Following our method for genus 2, next we would like to express invariants of the form $\left\langle \psi^2\omega, \ldots \right\rangle$ in terms of invariants with at most a single ψ-class.

Lemma 12.5. *For genus 3, let $a, b \in H^{\leq 2}(X)$, the following holds for all $i, j \geq 0$. Invariants of the form $\langle \psi^2\omega, a\omega^i, b\omega^j \rangle$ are equivalent ($\sim$) to a linear combination of invariants with a single ψ-class and primitive invariants.*

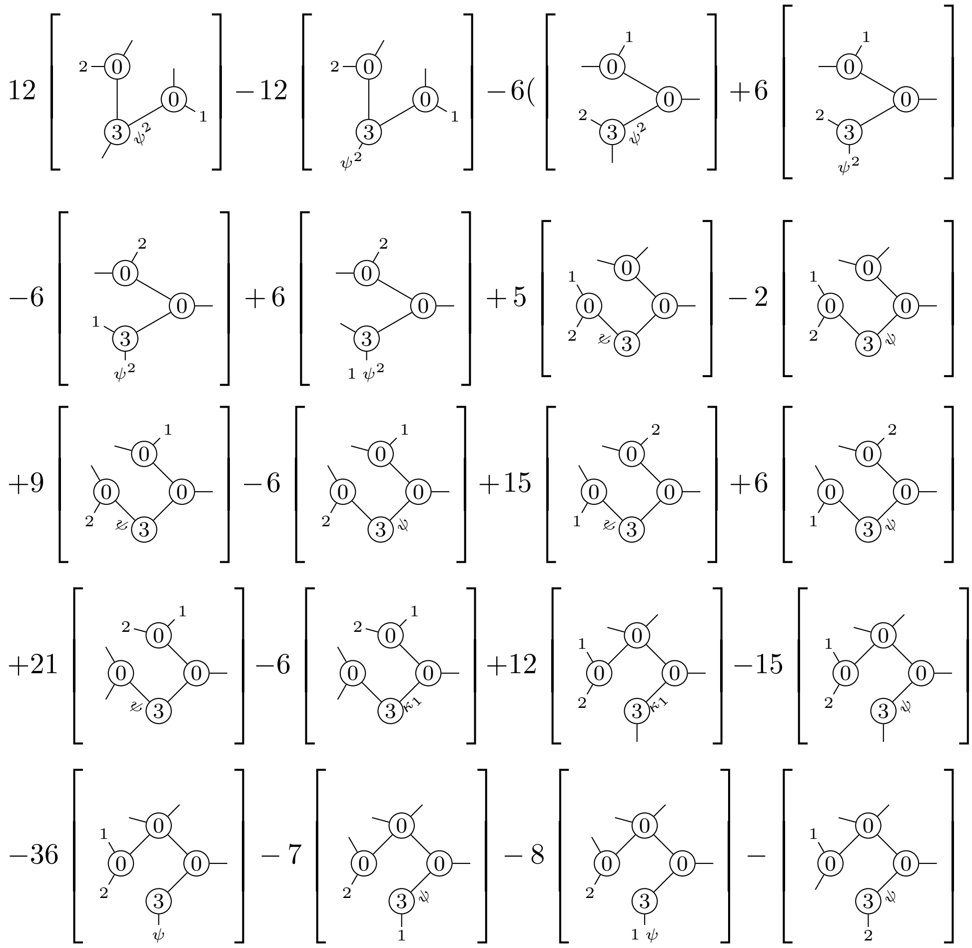
Proof. Let us fix the number $l := i + j$. When $i = 0$ or $j = 0$ the formulas hold by Corollary 8.12 so we can assume $l \geq 2$. We define

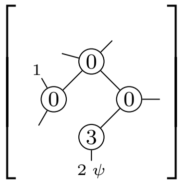
$$\begin{aligned} f(i) := {} & 6\left\langle \psi^2\omega, a\omega^i, b\omega^{l-i} \right\rangle - 8\left\langle \psi a\omega^i, b\omega^{l-i+1} \right\rangle - 7\left\langle \psi a\omega^{i+1}, b\omega^{l-i} \right\rangle \\ & - 8\left\langle \psi b\omega^{l-i}, a\omega^{i+1} \right\rangle - 7\left\langle \psi b\omega^{l-i+1}, a\omega^i \right\rangle, \end{aligned}$$

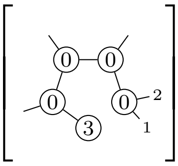
and

$$\begin{aligned} h := {} & 6\left\langle \psi^2\omega, ab\omega^{l-2}, \omega^2 \right\rangle - 5\left\langle \psi ab\omega^{l-2}, \omega^3 \right\rangle + 2\left\langle \psi\omega^3, ab\omega^{l-2} \right\rangle \\ & - 21\left\langle \psi\omega^2, ab\omega^{l-1} \right\rangle + 24\left\langle \psi\omega, ab\omega^l \right\rangle - 50\left\langle \psi\omega, ab\omega^l \right\rangle. \end{aligned}$$

The computer lets us obtain a relation R in $R^4_{3,5}$ with primitive part



-8 

$+65$ 

.

Its symbol is given by

$$\Sigma(R)(a\omega^k, b\omega^{l-2-k}, \omega, \omega, \omega) \sim f(k+2) - 2f(k+1) + f(k) - h,$$

for every $0 \leq k \leq l-2$. Now we obtain the desired result by doing the same kind of calculation as in the proof of Lemma 12.1. $\square$

13 Genus 2 Gromov-Witten invariants of blowups of the projective plane

We calculate descendant Gromov-Witten invariants of X_r, $\mathbb{P}^2$ blown up at points $p_1, \ldots, p_r$ in general position.

Theorem 13.1. *We can reconstruct all genus* 0*,1, and 2 Gromov-Witten invariants of* X_r *from the finitely many initial cases in Lemma 13.4 and* $< pt^2 >_{0,H}^{X_r} = 1$.

We have an explicit algorithm described in Sections 13.4, 13.5, and 13.6 that calculates these invariants. We also have written a computer program that applies the algorithm (see Appendix B.2).

We can not simply apply Theorem 11.1 to the spaces X_r unless $r = 0$ or $r = 1$. This is because the condition of knowing all invariants with at most 2 points is quite demanding. However even for target spaces where we cannot use the theorem directly, we can still use the relations prescribed by the theorem towards obtaining a reconstruction. We just need some extra ingredient, likely one obtained from the geometry of the target space in question.

13.1 Applying Theorem 11.1 to some example variety

We would like to apply Theorem 11.1 to some simple but nontrivial example. By the hard Lefschetz theorem (see II.6 in [63]) all Kähler surfaces have no primitive cohomology after degree two. Most of these spaces have infinitely many invariants with at most 2 points. So the required base cases cannot be calculated on a case by case basis and some additional strategy is needed. But finding a strategy that calculates invariants with at most 2 points might not be much easier than finding a strategy that calculates all points.

For example let us consider the case of a ruled surface X with invariant e. (Ruled surfaces are discussed in Chapter V.2 of [32].) Effective classes on ruled surfaces can be expressed in the form $\beta = aS + bF$ with either $\beta = F$,

$\beta = S$, or $a > 0$ and $b \geq e$ if $e \geq 0$ and $b \geq \frac{1}{2}ae$ if $e < 0$. We have $S^2 = -e$, $F^2 = 0$, and $SF = 1$ and the canonical divisor can be expressed as

$$K_X = -2S + (2g_X - 2 - e)F,$$

where g_X is the genus of the ruled surface. So the virtual dimension of $\overline{\mathcal{M}}_{2,n}(X, aS + bF)$ is

$$\int_{aS+bF} -K_X + (\dim(X) - 3)(1 - 2) + n = 2b + (2 - 2g_X - e)a + 1 + n.$$

To get a nonzero Gromov-Witten invariant with possibly one ψ-class we need

$$2n \leq 2b + (2 - 2g_X - e)a + 1 + n \leq 2n + 1,$$

so to be in the range of 0, 1, or 2 points we have

$$-1 \leq 2b + (2 - 2g_X - e)a \leq 2. \tag{17}$$

If $e < 0$ then $-g \leq e \leq 2g - 2$. So the only cases where there are a finite number of effective divisors satisfying (17) are $g = e = 0$ or $g = 0, e = 1$. Which respectively correspond to $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ blown up at a point. We have written a computer program that uses Theorem 11.1 to calculate the invariants of these two spaces. See Appendix B.3 for the program.

Both of these spaces are del Pezzo surfaces and their primitive Gromov-Witten invariants are already known. However I was not able to find a calculation of their descendant invariants in the literature.

13.2 Known results about Gromov-Witten invariants of X_r

In the range $r \leq 8$, the space X_r is Fano so it is a del Pezzo surface. In [26], Göttsche and Pandharipande find formulas for all r for genus 0, we will extend their approach to genus 1 and 2. In the range $r \leq 8$, the space X_r is Fano and we call it a del Pezzo surface. It turns out that the Gromov-Witten invariants of Fano surfaces are enumerative (see 4.3 in [60]). Various authors have computed the enumerative invariants of del Pezzo surfaces: In [12], Caporaso and Harris consider counts of curves that have a prescribed intersection with a given line in $\mathbb{P}^2$. From this they derive a recursive formula that counts curves in $\mathbb{P}^2$ for any genus. This approach has been extended by Vakil in [60] to X_r with $r \leq 6$. Using a tropical counting method, in [58] Shoval and Shustin further extended this to $r \leq 7$. And finally it was extended to $r \leq 8$ by Brugallé in [10] using floor diagrams. There is also an unpublished paper [55] by Parker in which a method is described to calculate the primitive Gromov-Witten invariants for all $r \geq 0$.

We have checked our numbers against the ones listed in the papers mentioned above and they agree.

13.3 Setting up the reconstruction for X_r.

A cohomology basis for X_r is given by $1, pt, H, E_1, \ldots, E_r$, where H is the hyperplane class and E_i is the class of the exceptional divisor at the i-th blown up point. We have

$$H \cdot H = 1, \qquad H \cdot E_i = 0, \qquad E_i \cdot E_j = \delta_{i,j},$$

for any $1 \leq i, j \leq r$.

Lemma 13.2. *Let C be an effective curve on X_r of class β. Then either β is a multiple of E_i for some $1 \leq i \leq r$, or $\beta = dH - \sum_{i=1}^{r} \alpha_i E_i$ with $0 \leq \alpha_i \leq d$. In the second case the coefficient a_i is equal to the multiplicity of $\pi(C)$ at p_i, where $\pi : X_r \to \mathbb{P}^2$ is the blow-up map.*

Proof. This is a special case of Lemma 2.3 in [14]. □

The anticanonical divisor of X_r is given by $-K_{X_r} = 3d - \sum_{i=1}^{r} E_i$ so the virtual dimension of $\overline{\mathcal{M}}_{g,n}(X_r, \beta)$ is

$$\int_{dH-\sum_{i=1}^{r}\alpha_i E_i} -K_{X_r} + (\dim(X) - 3)(1-g) + n = 3d - \sum_{i=1}^{r} \alpha_i + g - 1 + n.$$

Remark 13.3. Genus 2 invariants with at most two points will need to satisfy

$$-1 \leq 3d - \sum_{i=1}^{r} \alpha_i \leq 2,$$

which has infinitely many solutions corresponding to effective clases when $r > 1$. So we see that we can not simply apply Theorem 11.1. Because X_r is only of dimension 2 and has relatively simple cohomology, we only need one of the three tautological relations used to prove Theorem 11.1: We will use the relations $L_{2,4}$ and $L_{\mathrm{BP}} = \pi_{2*} L_{2,4}$ to obtain recursive formulas to calculate all the Gromov-Witten invariants up to genus 2. It might even be possible to calculate the genus 2 invariants using only the relation L_{BP}. But this would require some (possibly straightforward but tedious) additional trick.

Looking at the virtual dimension, we see that all nonzero invariants where $d \neq 0$ are of the form

$$\begin{aligned}
N^{(g)}_{d,\alpha} &:= \left\langle pt^{3d-\sum\alpha_i+g-1} \right\rangle_{g,dH-\sum\alpha_i E_i}, \\
H^{(2)}_{d,\alpha} &:= \left\langle \tau_1(H) \cdot pt^{3d-\sum\alpha_i} \right\rangle_{2,dH-\sum\alpha_i E_i}, \\
P^{(2)}_{d,\alpha} &:= \left\langle \tau_1(pt) \cdot pt^{3d-\sum\alpha_i-1} \right\rangle_{2,dH-\sum\alpha_i E_i}, \\
K^{(2)}_{d,\alpha} &:= \left\langle \tau_1(E_1) \cdot pt^{3d-\sum\alpha_i} \right\rangle_{2,dH-\sum\alpha_i E_i}.
\end{aligned}$$

The cases with a $\tau_1(E_i)$ for $i > 1$ are covered by the fact that

$$K^{(2)}_{d,\alpha} = \left\langle \tau_1(E_{\sigma(1)}) \cdot pt^{3d-\sum \alpha_i} \right\rangle_{2,dH-\sum \alpha_{\sigma(i)} E_{\sigma(i)}},$$

σ By the local nature of blowing-up we have

$$N^{(g)}_{d,(\alpha_1,\ldots,\alpha_{r-1},0)} = N^{(g)}_{d,(\alpha_1,\ldots,\alpha_{r-1})}.$$

The same thing holds for $H^{(2)}_{d,\alpha}$ and $P^{(2)}_{d,\alpha}$. It holds for $K^{(2)}_{d,\alpha}$ only when $r > 1$.

Lemma 13.4. *For genus up to 2, the only nontrivial invariants with $d = 0$ are*

$$N^{(0)}_{0,E_i} = 1, \quad \langle H \rangle_{1,0} = -\frac{1}{8}, \quad \langle E_i \rangle_{1,0} = -\frac{1}{24},$$
$$H^{(2)}_{0,0} = -\frac{1}{960}, \quad K^{(2)}_{0,0} = -\frac{1}{2880}.$$

Proof. By looking at the virtual dimension it follows that (up to linear combinations) these are the only possible nonzero invariants. The exceptional divisor E_i is itself a rigid curve of genus 0. The genus 1 cases follow from Lemma 8.1.

For genus 2 we have $\overline{\mathcal{M}}_{2,1}(X_r, 0) = \overline{\mathcal{M}}_{2,1} \times X_r$ and by VI.6.3 in [48]

$$\left[\overline{\mathcal{M}}_{2,1} \times X_r\right]^{\mathrm{vir}} = c_4(\mathbb{E}^\vee \boxtimes TX_r), \tag{18}$$

where $\mathbb{E}$ is the Hodge bundle and TX_r is the tangent bundle of X_r.

The space $\overline{\mathcal{M}}_{2,1}$ is four-dimensional and we have the one-dimensional class ψ_1 times the projection of $c_4(\mathbb{E}^\vee \boxtimes TX_r)$ to $\overline{\mathcal{M}}_{2,1}$. So to obtain something nonzero the contribution of $c_4(\mathbb{E}^\vee \boxtimes TX_r)$ to $\overline{\mathcal{M}}_{2,1}$ must be of degree 3. In particular the contributions of any $c_i(\mathbb{E}^\vee \boxtimes TX_r)$ is zero for $i < 3$. Since

$$\mathrm{ch}_4(\mathcal{E}) = \frac{c_1(\mathcal{E})^4 - 4c_1(\mathcal{E})^2 c_2(\mathcal{E}) + 4c_1(\mathcal{E})c_3(\mathcal{E}) - 4c_4(\mathcal{E})}{24},$$

we can replace the right-hand side of Equation (18) by

$$-6\mathrm{ch}_4(\mathbb{E}^\vee \boxtimes TX_r).$$

To calculate the Chern character of a product we apply the product rule

$$\mathrm{ch}(\mathbb{E}^\vee \boxtimes TX_r) = \mathrm{ch}(\mathbb{E}^\vee)\mathrm{ch}(TX_r).$$

Since the contribution to $\overline{\mathcal{M}}_{2,1}$ must be of degree 3, the only term that contributes to $\mathrm{ch}_4(\mathbb{E}^\vee \boxtimes TX_r)$ is $\mathrm{ch}_3(\mathbb{E}^\vee)\mathrm{ch}_1(TX_r)$. Using definition $\lambda_i := c_i(\mathbb{E})$ we obtain

$$\int_{\left[\overline{\mathcal{M}}_{2,1} \times X_r\right]^{\mathrm{vir}}} \psi \boxtimes D, = \int_{\overline{\mathcal{M}}_{2,1}} \psi(\lambda_1^3 - 3\lambda_1\lambda_2) \int_{X_r} -K_{X_r} D$$

which equals $-\frac{1}{960}$ when $D = H$ and $-\frac{1}{2880}$ when $D = E_i$. $\square$

So our recursive strategy will rely on computing invariants modulo invariants with lower d.

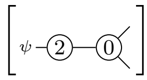
Let $L \in R_{2,n}$ be a relation, we define for given degree β

$$L(\gamma_1, \dots, \gamma_n) := \mathcal{T}_\beta(\pi^*_{\{n+1,\dots,n+m\}}(L), \gamma_1, \dots, \gamma_n, pt, pt, \dots, pt),$$

where for m, which is the number of points to add, we take the unique choice such that the resulting invariants are not all zero for dimensional reasons.

We write $\alpha + [j]$ for the tuple α' where $\alpha'_i = \alpha_i$ for $i \neq j$ and $\alpha'_j = \alpha_j + 1$.

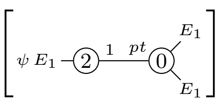
Example 13.5. The decorated stratum class



occurs in L_{BP}. We will describe its contribution to $L_{\mathrm{BP}}(E_1, E_1, E_1)$. Fixing d and α, to obtain a relation between nonzero Gromov-Witten invariants we need to add $3d - \sum \alpha_i - 1$ points. By applying Lemmas 8.1 and 13.4, we can write the result as

$$-N^{(2)}_{d,\alpha+[1]} - K^{(2)}_{d,\alpha} + \alpha_1 K^{(2)}_{d,\alpha+[1]} + \sum_{\substack{d'+d''=d \\ d',d''>0 \\ \alpha'+\alpha''=\alpha}} \alpha''^2_1 N^{(0)}_{d'',\alpha''} \Big(\alpha'_1 \binom{3d-\alpha-1}{3d'-\alpha'+1} N^{(2)}_{d',\alpha'}$$
$$+ \alpha''_1 \binom{3d-\alpha-1}{3d'-\alpha'} N^{(2)}_{d',\alpha'} + (d'd'' - \sum_i \alpha'_i \alpha''_i) \binom{3d-\alpha-1}{3d'-\alpha'} K^{(2)}_{d',\alpha'} \Big),$$

for $3d - \sum \alpha_i + 1 \geq 2$. We decribe how to obtain the first term inside the sum: Recall that in the splitting lemma we sum over all pairs of choices of cohomology classes from the chosen basis. These then get assigned to the pair of half edges that makes up the edge. Consider the following case:



We apply Lemma 8.1 to the 1, which makes the ψ disappear, then we apply it to each E_1 which gives a coefficient of $\alpha'_1 \alpha''^2_1$. Finally the only case where we obtain the nonzero product of invariants $N^{(0)}_{d'',\alpha''} N^{(2)}_{d',\alpha'}$ is when out of the $3d - \alpha - 1$ added points, $3d' - \alpha' + 1$ are on the genus 2 component.

Definition 13.6. We define an equivalence relation $\approx$, where a linear combination of Gromov-Witten invariants is equivalent to zero if it can be expressed in terms of invariants with lower d or lower genus.

Remark 13.7. Note that the equivalence relation $\approx$ is not the same as our equivalence relation $\sim$ from the section about general reconstructions since $N^{(2)}_{d,\alpha}$ and $N^{(2)}_{d,\alpha+[1]}$ have the same genus and d, but a different β and number of points.

13.4 Reconstructing genus 0 Gromov-Witten invariants

For the genus 0 invariants we use the method described in [26]. They use the relation $L_{0,4}$ in $R^1(\overline{\mathcal{M}}_{0,4})$ to get two recursive formulas: For $d - \sum \alpha_i - 1 \geq 3$, $L_{0,4}(H, H, pt, pt)$ gives

$$N^{(0)}_{d,\alpha} \approx 0,$$

and when $d - \sum \alpha_i - 1 \geq 1$, from $L_{0,4}(H, H, E_1, E_1)$ we obtain

$$\alpha_1 d^2 N^{(0)}_{d,\alpha+[1]} + (\alpha_1^2 - d^2) N^{(0)}_{d,\alpha} \approx 0,$$

When $d - \sum \alpha_i - 1 = 1$ the invariant $N^{(0)}_{d,\alpha+[1]}$ has zero points. So from these two formulas and the base case $N^{(0)}_{1,0} = 1$ we can recursively calculate all genus 0 invariants.

13.5 Reconstructing genus 1 Gromov-Witten invariants

For genus 1 we use Getzler's relation $L_{1,4}$ from [22]. For $3d - \sum \alpha_i \geq 2$, $L_{1,4}(H, H, H, H)$ gives

$$N^{(1)}_{d,\alpha} \approx 0,$$

and when $3d - \sum \alpha_i \geq 0$, from $L_{1,4}(E_1, E_1, E_1, E_1)$ we obtain

$$\frac{(\alpha_1 + 2)(\alpha_1 + 3)}{3} N^{(1)}_{d,\alpha+2[1]} + \frac{2\alpha_1}{3} N^{(1)}_{d,\alpha+[1]} - N^{(1)}_{d,\alpha} \approx 0.$$

Together with the genus 0 invariants this is sufficient to recursively calculate all genus 1 invariants.

13.6 Reconstructing genus 2 Gromov-Witten invariants

In [9] the genus 2 Gromov-Witten invariants of $\mathbb{P}^2$ are calculated using the relation L_{BP}. So we can assume that $r > 0$ and we always have an exeptional divisor E_1.

We calculate the invariants of type H using $L_{2,4}(H, E_1, E_1, E_1)$, which gives

$$-(\alpha_1 + 2) H^{(2)}_{d,\alpha+[2]} + H^{(2)}_{d,\alpha+[1]} \approx 0.$$

for $3d - \sum \alpha_i + 1 \geq 3$.

For type P we use $L_{\mathrm{BP}}(H, H, H)$ to obtain

$$d P^{(2)}_{d,\alpha} - H^{(2)}_{d,\alpha} \approx 0.$$

for $3d - \sum \alpha_i + 1 \geq 2$.

For type N we use the linear combination

$$\frac{3d}{2} L_{\mathrm{BP}}(E_1, H, H) - L_{\mathrm{BP}}(E_1, E_1, H) + \frac{d}{2} L_{\mathrm{BP}}(H, H, H),$$

which gives the formula

$$-(\alpha_1+2)dN^{(2)}_{d,\alpha+2[1]}+dN^{(2)}_{d,\alpha+[1]}-dP^{(2)}_{d,\alpha}+H^{(2)}_{d,\alpha}-(\alpha_1+1)H^{(2)}_{d,\alpha+[1]}\approx 0,$$

for $3d-\sum\alpha_i+1\geq 2$.

Finally for type K we take $L_{\mathrm{BP}}(E_1,E_1,E_1)$, which gives

$$-2(\alpha_1+2)N^{(2)}_{d,\alpha+2[1]}+\frac{2-3(\alpha_1+1)}{2}N^{(2)}_{d,\alpha+[1]}+\frac{3\alpha_1}{2}P^{(2)}_{d,\alpha}-\frac{3}{2}K^{(2)}_{d,\alpha}-K^{(2)}_{d,\alpha+[1]}\approx 0.$$

for $3d-\sum\alpha_i+1\geq 2$. And when we take $L_{\mathrm{BP}}(E_1,E_1,E_1)$ with $\alpha_1=-1$, we obtain

$$K^{(2)}_{d,(0,\alpha_2,\ldots,\alpha_r)}=-N^{(2)}_{d,(0,\alpha_2,\ldots,\alpha_r)}.$$

13.7 Optimization and some results

To speed up the calculations we use the following result from [26].

Lemma 13.8. *When $3d-\sum_{i=1}^{r-1}\alpha_i+1\geq 0$, we have*

$$N^{(0)}_{d,(\alpha_1,\ldots,\alpha_{r-1},1)}=N^{(0)}_{d,(\alpha_1,\ldots,\alpha_{r-1})}.$$

Proof. We use induction on d. By taking $L_{0,4}(H,H,pt,pt)$ for $d,(\alpha_1,\ldots,\alpha_{r-1},1)$ we obtain an expression of $N^{(0)}_{d,(\alpha_1,\ldots,\alpha_{r-1},1)}$ in terms of invariants with lower d. In the coefficients of this expression, α_r only appears as part of the multinomials, or in the form $\sum\alpha'_i\alpha''_i$. Because $\alpha_r=1$ there is zero contribution to the latter case, and for multinomials we have the basic identity

$$\binom{n}{k_1,\ldots,k_m}=\binom{n-1}{k_1-1,\ldots,k_m}+\ldots+\binom{n-1}{k_1,\ldots,k_m-1}.$$

So using our induction hypothesis, we see that this expression is the same as the one we obtain when we take $L_{0,4}(H,H,pt,pt)$ for $d,(\alpha_1,\ldots,\alpha_{r-1})$. $\square$

Remark 13.9. The proof of Lemma 13.8 does not work in higher genus. This is because of the appearance of graphs with self-edges, graphs with genus 0 vertices without legs, and because of the existence of nonzero Gromov-Witten invariants $\langle E_i\rangle_{1,0}$ and $\langle\tau_1(E_i)\rangle_{2,0}$.

Lemma 13.10. *When $3d-\sum_{i=1}^{r-1}\alpha_i+1\geq 0$ and $r\leq 8$, we have*

$$N^{(g)}_{d,(\alpha_1,\ldots,\alpha_{r-1},1)}=N^{(g)}_{d,(\alpha_1,\ldots,\alpha_{r-1})}.$$

Proof. When $r\leq 8$, X_r is a del Pezzo surface and the primitive invariants are enumerative. Let $\beta=dH-\sum_{i=1}^{r-1}\alpha_iE_i$. The number $N^{(g)}_{d,(\alpha_1,\ldots,\alpha_{r-1},1)}$ is the enumerative count of curves in X_r of class $\beta-E_r$ through $3d-\sum_{i=1}^{r-1}\alpha_i+g-2$ points in general position. By Lemma 13.2, after blowing down the r-th point, these are exactly the curves in X_{r-1} of class β through $3d - \sum_{i=1}^{r-1} \alpha_i + g - 2$ points in general position that also pass through the point p_r. So it is the count of curves through $3d - \sum_{i=1}^{r-1} \alpha_i + g - 1$ in general position, which is $N^{(g)}_{d,(\alpha_1,\ldots,\alpha_{r-1})}$. $\square$

For the numbers we calculated,

$$N^{(g)}_{d,(\alpha_1,\ldots,\alpha_{r-1},1)} = N^{(g)}_{d,(\alpha_1,\ldots,\alpha_{r-1})},$$

holds even when $r > 8$. The same also holds for $H^{(2)}_{d,\alpha}$ and $P^{(2)}_{d,\alpha}$. It holds for $K^{(2)}_{d,\alpha}$ only when $r > 1$.

As described in Section 5.1 of [26], we can use the Cremona transformation that sends the class $dH - \sum_i \alpha^r_{i=1} E_i$ to

$$(2d-\alpha_1-\alpha_2-\alpha_3)H-(d-\alpha_2-\alpha_3)E_1-(d-\alpha_1-\alpha_3)E_2-(d-\alpha_1-\alpha_2)E_3-\sum_{i=4}^{r} \alpha_i E_i,$$

which we write as d', α'. Using this transformation we have

$$\begin{aligned} N^{(g)}_{d,\alpha} &= N^{(g)}_{d',\alpha'}, \\ P^{(2)}_{d,\alpha} &= P^{(2)}_{d',\alpha'}, \\ H^{(2)}_{d,\alpha} &= 2H^{(2)}_{d',\alpha'} - K^{(2)}_{d',\alpha'} - K^{(2)}_{d',\sigma_{12}(\alpha')} - K^{(2)}_{d',\sigma_{13}(\alpha')}, \\ K^{(2)}_{d,\alpha} &= K^{(2)}_{d',\alpha'} - K^{(2)}_{d',\sigma_{12}(\alpha')} - K^{(2)}_{d',\sigma_{13}(\alpha')}. \end{aligned}$$

We use this to speed up our computation when $\alpha_1 + \alpha_2 + \alpha_3 > d$.

We will list some results. To keep the notation short we use the exponential notation for partitions, i.e. $N^{(2)}_{6,2^2} = N^{(2)}_{6,2,2}$.

$$\begin{aligned} P^{(2)}_{4,2} &= -\frac{2}{3}, & K^{(2)}_{3,1} &= -\frac{1}{12}, & H^{(2)}_{4,2} &= -\frac{5}{3}, \\ H^{(2)}_{4,2^2} &= -\frac{1}{3}, & H^{(2)}_{5,3} &= 72, & H^{(2)}_{6,2^4} &= 157689, \\ N^{(2)}_{6,2^9} &= 0, & N^{(2)}_{7,2^8} &= 190172, & N^{(2)}_{7,2^9} &= 25992, \\ N^{(2)}_{7,2^{10}} &= 3113, & N^{(2)}_{7,2^{11}} &= 313, & N^{(2)}_{8,2^9} &= 685599264, \\ N^{(2)}_{8,2^{10}} &= 135998195, & N^{(2)}_{8,2^{11}} &= 25721212, & N^{(2)}_{8,2^{12}} &= 4604976. \end{aligned}$$

A Matrices

In the proof of Theorem 11.1 we use the fact that certain systems of equations are of full rank. In this appendix we write down the corresponding matrices and show that their determinants are nonzero. The computer program in Appendix B does the calculations involved. For example the matrix below is constructed using the function `construct_matrix_a()`. See the description of the program in the readme file for the other matrices.

We have

$$\begin{aligned}\Theta_{j,2}(a,b) = &-(j+1)(j+2)\left\langle a\omega^{j}, b\omega^{4}\right\rangle + 4j(j+2)\left\langle a\omega^{j+1}, b\omega^{3}\right\rangle \\ &- 6(j^2-j-4)\left\langle a\omega^{j+2}, b\omega^{2}\right\rangle + \frac{4j(j+1)(j+2)}{(j+3)}\left\langle a\omega^{j+3}, b\omega\right\rangle .\end{aligned}$$

We present the matrices from the proof of 11.6 and their determinants. For the first matrix the rows correspond to the relations

$$\Phi_1(a\omega^i,\omega,\omega), \quad \Theta_{1,2}(1,\omega)\left\langle a\omega^i\right\rangle, \quad \Theta_{i,2}(a,1)\left\langle \omega^2\right\rangle .$$

Remember from (13) that Φ_k is not a single relation but a system of relations parametrized by distributions of the integer k. The rows for $\Phi_1(a\omega^i,\omega,\omega)$ in order are given by the distrubtions $[1,0,0,0,0,0]$ and $[0,1,0,0,0,0]$. The columns correspond to

$$\left\langle a\omega^{i+2},\omega^2,\omega^2\right\rangle, \left\langle a\omega^{i+1},\omega^3,\omega^2\right\rangle, \left\langle a\omega^{i},\omega^4,\omega^2\right\rangle, \left\langle a\omega^{i},\omega^3,\omega^3\right\rangle .$$

$$\begin{pmatrix} -30 & 10 & 0 & 0 \\ 0 & -20 & 4 & 4 \\ 0 & 0 & 0 & 24 \\ 6(i^2-i-4) & 4i(i+2) & -(i+1)(i+2) & 0 \end{pmatrix}$$

This matrix has determinant $-2880(i^2-9i-18)$, which has no integer roots.

For the next matrix the rows correspond to the relations

$$\Phi_2(a\omega^i,b,\omega), \quad (-1)^{|a||b|}\Theta_{1,2}(\omega,b)\left\langle a\omega^i\right\rangle, \quad \Theta_{i+1,2}(a,1)\left\langle b\omega\right\rangle, \quad \Theta_{i,2}(a,1)\left\langle b\omega^2\right\rangle .$$

Here the ordering of the rows for Φ_2 is given by the folliwing distributions of 2.

$$[2,0,0,0,0,0],[1,1,0,0,0,0],[1,0,1,0,0,0],[0,2,0,0,0,0],$$
$$[0,1,1,0,0,0],[0,0,2,0,0,0],[0,0,1,1,0,0].$$

The columns correspond to

$$\left\langle a\omega^{i+3}, b\omega, \omega^2\right\rangle, \left\langle a\omega^{i+2}, b\omega^2, \omega^2\right\rangle, \left\langle a\omega^{i+2}, b\omega, \omega^3\right\rangle, \left\langle a\omega^{i+1}, b\omega^3, \omega^2\right\rangle, \left\langle a\omega^{i+1}, b\omega^2, \omega^3\right\rangle$$
$$\left\langle a\omega^{i+1}, b\omega, \omega^4\right\rangle, \left\langle a\omega^{i}, b\omega^4, \omega^2\right\rangle, \left\langle a\omega^{i}, b\omega^3, \omega^3\right\rangle, \left\langle a\omega^{i}, b\omega^2, \omega^4\right\rangle, \left\langle a\omega^{i}, b\omega, \omega^5\right\rangle.$$

$$\begin{pmatrix}
-24 & 6 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & -24 & 4 & 6 & 4 & -2 & 0 & 0 & 0 & 0 \\
-3 & -3 & -12 & 1 & 4 & 3 & 0 & 0 & 0 & 0 \\
0 & 6 & 0 & -24 & 4 & 0 & 6 & 4 & -2 & 0 \\
1 & -3 & 4 & -3 & -12 & 3 & 1 & 4 & 3 & -2 \\
-6 & 0 & 3 & -6 & 3 & -12 & 3 & -1 & 3 & 3 \\
4 & 0 & -8 & 4 & -8 & -4 & -3 & 4 & 3 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & -6 & 12 & 12 & 6 \\
6(i^2+i-4) & 0 & 4(i+1)(i+3) & 0 & 0 & -(i+2)(i+3) & 0 & 0 & 0 & 0 \\
0 & 6(i^2-i-4) & 0 & 0 & 4i(i+2) & 0 & 0 & 0 & -(i+1)(i+2) & 0
\end{pmatrix}$$

This matrix has determinant $60825600(i^4+22i^3+56i^2-33i-121)$, which does not have any integer roots.

We have

$$\Phi_0(a,b,\omega) \sim -30\left\langle a\omega, b\omega, \omega^2\right\rangle.$$

The system of relations $\Phi_1(a,b,\omega)$ is given by the following matrix where the rows correspond to

$$[1,0,0,0,0,0], [0,1,0,0,0,0], [0,0,1,0,0,0]$$

and the columns correspond to

$$\left\langle a\omega^2, b\omega, \omega^2\right\rangle, \left\langle a\omega, b\omega^2, \omega^2\right\rangle, \left\langle a\omega, b\omega, \omega^3\right\rangle$$

$$\begin{pmatrix}
-24 & 6 & 4 \\
6 & -24 & 4 \\
-3 & -3 & -12
\end{pmatrix},$$

which has nonzero determinant.

Consider the equations

$$\Phi_0(a,b,c), \quad \Phi_1(a,b,c), \quad \Phi_2(a\omega^i,b,c),$$

$$\Theta_{i,2}(a,b)\left\langle c\omega\right\rangle, \quad (-1)^{|b||c|}\Theta_{i,2}(a,c)\left\langle b\omega\right\rangle.$$

Together these equations imply $\left\langle a\omega^i, b\omega^j, c\omega^k\right\rangle \sim 0$, for $i \geq 0$, $0 \leq j \leq 2$, and $0 \leq k \leq 2$. The system of 11 relations $\Psi_2(a,b,c)$ has rank 8. So we only take 8 relations corresponding to the distributions

$$[2,0,0,0,0,0],[1,1,0,0,0,0],[1,0,1,0,0,0],[1,0,0,1,0,0],$$

$$[0,2,0,0,0,0],[0,1,1,0,0,0],[0,1,0,1,0,0],[0,0,2,0,0,0].$$

These form our rows together with

$$\Theta_{i,2}(a,b)\left\langle c\omega\right\rangle, \quad (-1)^{|b||c|}\Theta_{i,2}(a,c)\left\langle b\omega\right\rangle.$$

The columns correspond to

$$\left\langle a\omega^{i+3}, b\omega, c\omega\right\rangle, \left\langle a\omega^{i+2}, b\omega^2, c\omega\right\rangle, \left\langle a\omega^{i+2}, b\omega, c\omega^2\right\rangle, \left\langle a\omega^{i+1}, b\omega^3, c\omega\right\rangle, \left\langle a\omega^{i+1}, b\omega^2, c\omega^2\right\rangle$$
$$\left\langle a\omega^{i+1}, b\omega, c\omega^3\right\rangle, \left\langle a\omega^{i}, b\omega^4, c\omega\right\rangle, \left\langle a\omega^{i}, b\omega^3, c\omega^2\right\rangle, \left\langle a\omega^{i}, b\omega^2, c\omega^3\right\rangle, \left\langle a\omega^{i}, b\omega, c\omega^4\right\rangle.$$

$$\begin{pmatrix}
-12 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & -12 & 3 & 3 & 3 & -2 & 0 & 0 & 0 & 0 \\
3 & 3 & -12 & -2 & 3 & 3 & 0 & 0 & 0 & 0 \\
-4 & -4 & -4 & 2 & 2 & 2 & 0 & 0 & 0 & 0 \\
0 & 3 & 0 & -12 & 3 & 0 & 3 & 3 & -2 & 0 \\
-2 & 3 & 3 & 3 & -12 & 3 & -2 & 3 & 3 & -2 \\
2 & -4 & 2 & -4 & -4 & 2 & 2 & 2 & 2 & -2 \\
0 & 0 & 3 & 0 & 3 & -12 & 0 & -2 & 3 & 3 \\
\frac{4i(i+1)(i+2)}{i+3} & 6(i^2-i-4) & 0 & 4i(i+2) & 0 & 0 & -(i+1)(i+2) & 0 & 0 & 0 \\
\frac{4i(i+1)(i+2)}{i+3} & 0 & 6(i^2-i-4) & 0 & 0 & 4i(i+2) & 0 & 0 & 0 & -(i+1)(i+2)
\end{pmatrix}$$

This matrix has determinant $19008000\frac{(i^2-3)(i^3+3i^2-2i-7)}{(i+3)}$, which does not have any nonnegative integer zeros or poles.

We have

$$\Phi_0(a,b,c) \sim -12\left\langle a\omega, b\omega, c\omega\right\rangle.$$

The system of relations $\Phi_1(a,b,c)$ is given by the following matrix where the rows correspond to

$$[1,0,0,0,0,0],[0,1,0,0,0,0],[0,0,1,0,0,0]$$

and the columns correspond to

$$\left\langle a\omega^2, b\omega, c\omega\right\rangle \ \left\langle a\omega, b\omega^2, c\omega\right\rangle \ \left\langle a\omega, b\omega, c\omega^2\right\rangle$$

$$\begin{pmatrix} -12 & 3 & 3 \\ 3 & -12 & 3 \\ 3 & 3 & -12 \\ -4 & -4 & -4 \end{pmatrix},$$

which has rank 3.

B Computer program

Our work relies heavily on computer calculations. These calculations are done by a program we have written in SageMath. It can be downloaded from `https://github.com/Wennink/reconstructinggromovwitteninvariants` together with installation instructions.

We build upon the SageMath program *admcycles* by Delacroix, Pixton, Schmitt, Zachhuber, and van Zelm. It can be found at `https://gitlab.com/jo314schmitt/admcycles` and is documented in [16]. We display tautological strata classes in the same way as it is done in *admcycles*, which is explained in Section 3.1 of [16].

Our program contains functions that calculate specific ingredients we use in our proofs such as for example `getL25()` from Lemma 11.4 or the following from Appendix A:

```
sage: calculate_discriminant_a()
(-2880) * (i^2 - 9*i - 18)
```

For a full list of these functions see the readme file of the program.

B.1 Computing tautological relations

Memory usage is the main bottleneck when it comes to computing tautological relations. But when working with symmetric relations, the program only needs to store a single element representing an orbit of the $\mathcal{S}_n$-action. Because of this symmetric relations can be computed in a larger range of g, n, r than computations of all (unsymmetric) relations. However we were still not able to compute the symmetric relations in $P^4_{3,8}$ on our computer.

We have written a new version of the code that computes symmetric relations using improvements where we use $\mathcal{S}'_n$-actions for $n' < n$. For example one of the steps the old version takes in the process of calculating the symmetric relations of $P^r_{g,n}$ is to calculate a generating basis of (nonsymmetric) relations in $P^{r-1}_{g,n}$ and then multiply by ψ_i for $1 \leq i \leq n$. It would be more efficient to calculate relations that are fixed by the $\mathcal{S}_{n-1}$-action on the first $n-1$ points, and then multiply by ψ_n.

We can compare the old and new versions of computing symmetric relations.

The old version:

```
sage: m = get_memory_usage()
sage: %time a=derived_rels(3,4,6,1)
CPU times: user 2min 3s, sys: 628 ms, total: 2min 4s
Wall time: 2min 4s
sage: get_memory_usage()-m
1368.984375
```

The new version:

```
sage: m = get_memory_usage()
sage: %time a=derived_rels_SS(3,4,6,1)
CPU times: user 1min 40s, sys: 416 ms, total: 1min 41s
Wall time: 1min 41s
sage: get_memory_usage()-m
940.66796875
```

The new verion is faster but most importantly it uses less memory. The higher the number of points n, the bigger these differences become:

The old version:

```
sage: m = get_memory_usage()
sage: %time a=derived_rels(3,4,7,1)
CPU times: user 14min 15s, sys: 4.06 s, total: 14min 19s
Wall time: 14min 20s
sage: get_memory_usage()-m
9712.96484375
```

The new version:

```
sage: m = get_memory_usage()
sage: %time a=derived_rels_SS(3,4,7,1)
CPU times: user 7min 30s, sys: 2.19 s, total: 7min 32s
Wall time: 7min 32s
sage: get_memory_usage()-m
4823.75390625
```

This difference in memory usage is what allows us to calculate the new relation in $P^4_{3,8}$.

We have now got a method to calculate partially symmetric relations. In particular we can use this to calculate nonsymmetric relations. Below we list some comparison results between our method `pre_processed_fzm` and the method `DR.FZ_matrix` (denoted by `pre` and `dr` respectively). All results are for stable curves.

	$P^2_{3,7}$	$P^2_{2,7}$	$P^3_{2,5}$	$P^4_{2,4}$
`dr` time	2m12s	2m19s	7m23s	out of memory
`pre` time	17m50s	10m19s	1m57s	2m50s
`dr` memory usage	1242	1078	4906	more than 14500
`pre` memory usage	3109	2734	1379	1674

It seems that `DR.FZ_matrix` performs better at low codimension and high genus or high number of points, while `pre_processed_fzm` is more efficient for higher codimension.

B.2 Computing Gromov-Witten invariants of $\mathbb{P}^2$ blown up in finitely many points

The part of the computer program that calculates these Gromov-Witten invariants is built around two functions we have written. The first is `compute_gw_formula`, which computes the expression $\mathcal{T}_\beta(L, \gamma_1, \dots, \gamma_n)$ for any $L \in S_{g,n}$ for $g \leq 2$ and any choice of classes from the generating basis $\{pt, H, E_1, \dots, E_r\}$. The second function is `apply_form` which evaluates this obtained formula.

To make the program calculate a Gromov-Witten invariant of degree $\beta = dH - \sum_i \alpha_i$, we write `gw_inv(g,psiprofile,[d,alpha_1,alpha_2,...])`, where the psiprofile is

- `[]` to calculate $N^{(g)}_{d,\alpha}$,
- `[(1,-1)]` to calculate $P^{(g)}_{d,\alpha}$,
- `[(1,0)]` to calculate $H^{(g)}_{d,\alpha}$, and
- `[(1,1)]` to calculate $K^{(g)}_{d,\alpha}$.

For example we can calcuate $N^{(0)}_4$ and $H^{(2)}_{4,2}$ as follows:

```
sage: gw_inv(0,[],[4])
620
sage: gw_inv(2,[(1,0)],[4,2])
-5/3
```

B.3 Computing Gromov-Witten invariants of X_1 and $\mathbb{P}^1 \times \mathbb{P}^1$

Because the core of the code for X_r is independent of a choice of relation, one can easily use different tautological relations to obtain a different reconstruction. In this way we have constructed a function `gw_X1` that calculates the invariants of X_1 using the algorithm prescribed by Theorem 11.1.

```
sage: gw_X1(2,[(1,0)],[4,2])
-5/3
```

We cannot use the same program to calculate invariants of $\mathbb{P}^1 \times \mathbb{P}^1$ but we can use a slightly different version of it specifically written for $\mathbb{P}^1 \times \mathbb{P}^1$.

```
sage: gw_P1xP1(2,[(1,0)],[4,2])
68
```

B.4 Symbols of tautological relations

We describe some of most important functions in the program related to symbols.

For $g \leq 3$, the function `get_primitive_part_fzm(g,n,r,nonsym)` gives a matrix of (symmetric) relations in $P^r_{g,n}$ modulo restriction to the primitive part. This matrix comes together with a legend that represents the columns as decorated strata classes.

```
sage: prim = get_primitive_part_fzm(2,3,2,false)
sage: prim[0]
[   1   -1 -2/3]
sage: list_tg(prim[1])
[0] : Graph :      [0, 2] [[2, 3, 5], [1, 6]] [(5, 6)]
Polynomial : 1*psi_1^1
[1] : Graph :      [0, 2] [[2, 3, 5], [1, 6]] [(5, 6)]
Polynomial : 1*psi_6^1
[2] : Graph :      [0, 0, 2] [[2, 3, 6], [1, 7, 8], [9]] [(6, 7), (8, 9)]
Polynomial : 1*
```

We can obtain a tautological class ("tautclass" in the program) from a row in this matrix.

```
sage: tcbp = partial_to_tautclass(vec=prim[0][0],taut_gens=prim[1])
sage: tcbp
Graph :      [0, 2] [[2, 3, 5], [1, 6]] [(5, 6)]
Polynomial : 1*psi_1^1

Graph :      [0, 2] [[2, 3, 5], [1, 6]] [(5, 6)]
Polynomial : (-1)*psi_6^1

Graph :      [0, 0, 2] [[2, 3, 6], [1, 7, 8], [9]] [(6, 7), (8, 9)]
Polynomial : (-2/3)*
```

Since we set `nonsym` to `false` we are working with symmetric relations and the program only stores one element in each $\mathcal{S}_n$-orbit. To obtain the full expression for the tautological relation we "unsymmetrize" it.

```
sage: unsymtc = smarter_unsym_tc(tc=tcbp,n=3)
```

We now obtain the symbol which the program displays as follows.

```
sage: sbp = symbol_from_tautclass(g=2,tc=unsymtc)
sage: sbp
-1/3 [PP*aa*bb, cc]
-1/3 [PP*aa*cc, bb]
1/3  [PP*aa, bb*cc]
-1/3 [PP*bb*cc, aa]
1/3  [PP*bb, aa*cc]
1/3  [PP*cc, aa*bb]
-2/3 [aa*bb*cc]
```

We can do substitutions of classes in symbols and do vector space operations on symbols such as multiplying by a scalar. There are also methods to apply the string, dilaton, or divisor equations from Lemmas 7.9 and 8.12.

```
sage: 3*sbp.subs([aa,W,W])
-2 [PP*aa*W, W]
1  [PP*aa, W^2]
-1 [PP*W^2, aa]
2  [PP*W, aa*W]
-2 [aa*W^2]
sage: 3*sbp.subs([aa,W,W]).applydivisor()
1  [PP*aa, W^2]
2  [PP*W, aa*W]
-5 [aa*W^2]
```

The above is the program's way of displaying the symbol

$$< \psi a, \omega^2 > +2 < \psi\omega, a\omega > -5 < a\omega^2 > .$$

Glossary of notations

$< \psi^{a_1}\gamma_1, \ldots, \psi^{a_n}\gamma_n >$	Gromov-Witten invariant using symbol notation. 28
$< \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n) >^X_{g,\beta}$	Gromov-Witten invariant. 17
$G^X_{g,\beta}$	linear combinations of formal Gromov-Witten invariants. 27
$N^{(g)}_{d,\alpha}$,$H^{(g)}_{d,\alpha}$,$P^{(g)}_{d,\alpha}$,$K^{(g)}_{d,\alpha}$	Gromov-Witten invariants of X_r. 48
$P_{g,n}$	algebra of Pixton's relations. 21
$R^*(\overline{\mathcal{M}}_{g,n})$	tautological ring of the moduli space of curves. 17, 20
$R_{g,n}$	algebra of relations. 21
$R^{\text{new}}_{g,n}$	new relations. 22
S_{prim}	the primitive part of a relation S. 27
$S_{g,n}$	the strata algebra. 21
$T_0, \ldots T_r$	homogeneous basis of $H^*(X)$. 12

X	the target variety. 12
X_r	the blowup of $\mathbb{P}^2$ in r points. 45
$[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathbf{vir}}$	virtual fundamental class. 16
Γ	stable graph. 18
Γ_θ	decorated stable graph. 19
$\overline{\mathcal{M}}_\Gamma$	product of Moduli spaces associated to Γ. 18
$\overline{\mathcal{M}}_{g,n}$	moduli space of stable curves. 14
$\overline{\mathcal{M}}_{g,n}(X,\beta)$	moduli space of stable maps. 15
$\Phi_k(\gamma_1,\gamma_2,\gamma_3)$	system of equations in $G^X_{g,\beta}$. 37
$\Sigma(S)(\gamma_1,\ldots,\gamma_n)$	the symbol map. 27
$\Theta_{j,k}(a,b)$	an element of $G^X_{g,\beta}$. 38
δ	gluing map that glues a curve to itself. 14
ev_i	evaluation map at the i-th point of a stable map. 16
κ	a tautological class. 14
$\mathbb{Q}[GW(X)]$	polynomial ring in formal Gromov-Witten invariants. 27
$\mathcal{M}^{\mathrm{ct}}_{g,n}$	moduli space of curves of compact type. 20
$\mathcal{M}^{\mathrm{rt}}_{g,n}$	moduli space of curves with rational tail. 20
$\mathcal{M}_{g,n}$	moduli space of smooth curves. 20
$\mathcal{T}_\beta$	map to polynomials in Gromov-Witten invariants. 22, 25
π_i	forgetful map forgetting the i-th point. 14, 15
ψ	a tautological class. 14
$\sim$	equivalence relation on $G^X_{g,\beta}$. 28
$\tilde{\psi}$	cocycle on the moduli space of stable maps. 16
ξ_Γ	glueing map associated to Γ. 18
g^{ef}	inverse of intersection product. 12
q	gluing map that glues two curves together. 14

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