ON HERMITE'S PROBLEM, JACOBI-PERRON TYPE ALGORITHMS, AND DIRICHLET GROUPS

OLEG KARPENKOV

ABSTRACT. A well-known result of Lagrange (1770) characterises quadratic irrationalities as those real numbers that can be written as periodic continued fractions. Hermite asked in 1848 if there exists some way to write cubic irrationalities periodically. To approach this problem, Jacobi and Perron generalised the classical continued fraction algorithm to the three-dimensional case; this algorithm is called now the Jacobi-Perron algorithm. It is known only to provide periodicity for some cubic irrationalities.

In this paper we introduce two new algorithms in the spirit of the Jacobi-Perron algorithm: the heuristic algebraic periodicity detecting algorithm and the $\sin^2$-algorithm. The heuristic algebraic periodicity detecting algorithm is a very fast and efficient algorithm, its output is periodic for numerous examples of cubic irrationalities, however its periodicity for cubic irrationalities is not proven. The $\sin^2$-algorithm is limited to the totally-real cubic case (all the roots of cubic polynomials are real numbers). Recently we proved the periodicity of the $\sin^2$-algorithm for all cubic totally-real irrationalities. To the best of our knowledge this is the first Jacobi-Perron type algorithm for which the cubic periodicity is proven. The $\sin^2$-algorithm provides an answer to Hermite's problem for the totally real case (let us mention that the case of cubic algebraic numbers with complex conjugate roots remains open).

We conclude this paper with one important application of Jacobi-Perron type algorithms to the computation of independent elements in the maximal groups of commuting matrices of algebraic irrationalities.

CONTENTS

1. Euclid's algorithm for quadratic irrationalities 4
2. On periodicity of cubic irrationalities 6
2.1. Jacobi-Perron algorithm 6
2.2. A few words about Gauss-Kuzmin statistics 8
2.3. Heuristic algebraic periodicity detecting algorithm 9
2.4. Technical remarks on the heuristic APD-algorithm 13
2.5. A remark on the periodicity of the $\sin^2$-algorithm in the cubic totally real case 13

Date: 5 January 2022.
2020 *Mathematics Subject Classification.* Primary 11R16; Secondary 11A05.
Key words and phrases. Jacobi-Perron algorithm, Klein continued fractions, Dirichlet group.

3. The situation in degree greater than 3 15
4. Dirichlet groups 17
4.1. Magic of integer commuting matrices 17
4.2. Dirichlet groups 17
4.3. Dirichlet's unit theorem 18
4.4. Several questions that we can answer 19
5. Jacobi-Perron type algorithms and Dirichlet groups 19
5.1. Jacobi-Perron algorithm in matrix form 19
5.2. Matrices with prescribed cubic eigenvectors 19
5.3. Answers to Questions 1–3 21
References 23

This paper is dedicated to periodic representations of algebraic numbers. Recall that a number α is algebraic if it is a root of some polynomial with integer coefficients. The smallest degree of any integer polynomial with root α is called the *degree* of α. It is well known that decimal representations for all rational numbers are eventually periodic or finite, so the case of algebraic numbers of degree 1 is straightforward. Let us consider a similar question for algebraic numbers of higher degrees.

It turns out that the study of this question has a rich history. Our journey starts in ancient Greece with the invention of Euclid's algorithm about 300 BC. Euclid's algorithm was originally developed for computing the greatest common divisor of two integers. It was two millennia after its invention when Euclid's algorithm was used in the study of quadratic irrationals (i.e. algebraic numbers of degree 2). An important stage here was the introduction of the concept of regular continued fractions by J. Wallis in 1695, that finally linked Euclid's algorithm to irrational numbers in general and to quadratic irrationalities in particular. In 1770 J.-L. Lagrange proved the periodicity of continued fractions for quadratic irrationalities, closing the question for the quadratic case (see Section 1).

For the first time the problem of the generalisation of the Lagrange theorem on periodicity of continued fractions for quadratic irrationalities to the case of algebraic numbers of degree three was posed by Ch. Hermite in 1848 in a very general setting. Hermite was wondering if there is a periodic description to cubic irrationalities. There are many different interpretations of this question that led to remarkable theories in geometry and dynamics of numbers (see a small survey on various multidimensional generalisations of ordinary continued fractions in Chapter 23 in [18]).

For this paper we restrict ourselves entirely to the algorithmic approach to the problem that was initiated by C. G. J. Jacobi in 1868 and further developed by O. Perron in 1907. They developed the multidimensional continued fraction algorithm, now known as the Jacobi-Perron algorithm. The Jacobi-Perron algorithm generalises the Euclidean algorithm and provides a sequence of pairs of integers similar to the regular continued fractions provided by the Euclidean algorithm. The output of the algorithm is periodic for certain cubic numbers, however it is believed to be non-periodic for some others. For that reason the Jacobi-Perron algorithm does not provide a complete solution to Hermite's problem, however it suggests that an algorithmic approach might be beneficial to the question. A similar situation occurs with many other Jacobi-Perron type algorithms, that are neither proved nor disproved to produce a periodic output.

Recall that Jacobi-Perron type algorithms in the three-dimensional case work with single vectors $(1, \alpha, \beta)$ as an input. Informally speaking one vector, even if it is cubic, does not represent any algebraic cone (while in fact the periodic structure is derived from the cone). As a result such algorithms should not always produce periodic sequences. In order to tackle the periodicity problem we suggest to start with triples of vectors rather than with single ones. Once these vectors are conjugate for some cubic extension we should get periodicity.

In this paper we introduce two new modifications of the Jacobi-Perron algorithm. We call the first one the heuristic algebraic periodicity detecting algorithm (or heuristic APD-algorithm for short) and the second — the $\sin^2$-algorithm. The heuristic APD-algorithm demonstrates periodicity in numerous experiments and is conjectured to be periodic for all cubic numbers. The $\sin^2$-algorithm works only in the totally real case (all three roots of the polynomial are real numbers). For the $\sin^2$-algorithm we were able to prove periodicity for triples of cubic conjugate vectors in [19]. (To the best of our knowledge, this is the first complete proof of periodicity for Jacobi-Perron type algorithms.) So the $\sin^2$-algorithm provides an answer to Hermite's problem in the form of Jacobi-Perron type algorithm for the totally real cubic case. The non-totally-real case remains open, however we believe that the techniques of the proof for the $\sin^2$-algorithm can be adapted for that case as well. Both the heuristic APD-algorithm and the $\sin^2$-algorithm are discussed in Section 2.

In Section 3 we say a few words regarding higher dimensional cases, which currently remains open.

Further we address an important application of Jacobi-Perron type algorithms. It turns out that such algorithms provide a simple way to write independent (with respect to the matrix multiplication operation) commuting pairs of matrices for the corresponding Dirichlet group. Recall that groups of commuting matrices (so called Dirichlet -groups) are described by the mysterious Dirichlet's unit theorem (we formulate and discuss it later in Subsection 4.3), whose complete understanding will probably cast light on the periodicity of generalised Euclidean algorithms. Classical proofs of Dirichlet's unit theorem provide huge estimates on the coefficients of the generators of the Dirichlet groups. The brute force algorithms provided by this theorem are very slow and seem to have no practical value. We discuss a simple and fast approach to the problem in the last two sections of the paper.

1. Euclid's algorithm for quadratic irrationalities

As we have already mentioned, the periodicity of quadratic irrationalities is closely related to Euclid's algorithm. Recall that the classical Euclid's algorithm computes the greatest common divisor of two integer numbers. Let us first write down a slightly extended form such that it can be applied to arbitrary numbers (not necessarily integers).

Extended Euclid's algorithm

Input: We start with a pair of real numbers $(p, q) = (p_0, q_0)$ such that $q_0 > 0$.

Step of the algorithm: Assume that we have found two real numbers (p_i, q_i) with $q_i \geq 0$. Then the next step is:

$$(p_i, q_i) \mapsto (p_{i+1}, q_{i+1}) = (q_i, p_i - \lfloor p_i/q_i \rfloor q_i)$$

Here we call the value $a_i = \lfloor p_i/q_i \rfloor$ the *i-th element* of the algorithm.

Termination of the algorithm: In case that we have arrived at the pair (p_i, q_i) with $q_i = 0$ we do not proceed further. Here the algorithm terminates.

Remark 1.1. Note that in the case of a pair of integers (p, q) with $q > 0$ we have the classical Euclid's algorithm. Here the algorithm terminates in a finite number of steps and at the last step we get $(\gcd(p,q),0)$ where $\gcd(p,q)$ is the greatest common divisor of p and q.

Remark 1.2. The algorithm terminates if p/q is a rational number, and it does not terminate otherwise.

Example 1.3. Let us apply the algorithm to the pair $(21,15)$. We have:

$$(21,15) \mapsto (15,6) \mapsto (6,3) \mapsto (3,0).$$

The output of the algorithm is as follows:

$$a_1 = 1, \quad a_2 = 2, \quad \text{and} \quad a_3 = 2.$$

Note that

$$\gcd(21,15) = 3 \quad \text{and} \quad \frac{21}{15} = 1 + \frac{1}{2+1/2}.$$

Now let us focus on the case of pairs $(\alpha, 1)$ where α is any real number. In this case the extended Euclid's algorithm generates a remarkable sequence of numbers a_i. If α is a rational number, then the algorithm terminates (on the n-th step for some integer n) and the output sequence (a_i) satisfies the following identity:

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}$$

The expression on the right hand side is called a *regular continued fraction* for α and denoted by $[a_0; a_1 : \cdots : a_n]$. (The term continued fraction was introduced by J. Wallis in 1695.)

The above identity for rational α is extended to the case of irrational α by the following limit

$$\lim_{k\to\infty} [a_0; a_1 : \cdots : a_k],$$

which we call the *regular continued fraction* for α and denote by $[a_0; a_1 : \cdots]$. Let us just notice that this limit always exists and distinct sequences converge to distinct irrational numbers. (For the details of the classical theory of continued fractions we refer e.g. to [21].)

We are finally arriving at a very non-trivial theorem on periodicity of continued fractions for quadratic irrationalities. This theory was introduced by J.-L. Lagrange in 1770, almost a century after the studies by J. Wallis (see in Chapter 34 of [28]).

Theorem 1.4. (J.-L. Lagrange.) *A regular continued fraction of* α *is periodic if and only if* α *is a quadratic irrationality (i.e.* $\dfrac{a+b\sqrt{c}}{d}$ *for some integers* a*,* b*,* c*, and* d*, where* $b \neq 0$*,* $c > 1$*,* $d > 0$*, and* c *is square-free).*

Probably we should note that Lagrange proved the *if* statement, while the *only if* statement was proved by Euler. This theorem gives a complete answer to the question of periodic representations for quadratic irrationalities.

Example 1.5. Let us apply the extended Euclidean algorithm to $(2\sqrt{5}, 1)$. We have

$$(2\sqrt{5}, 1) \mapsto c_1(1+\sqrt{5}/2, 1) \mapsto c_2(4+2\sqrt{5}, 1) \mapsto c_3(1+\sqrt{5}/2, 1) \mapsto \ldots$$

where

$$c_1 = 2\sqrt{5} - 4, \quad c_2 = 9 - 4\sqrt{5}, \quad c_3 = 34\sqrt{5} - 76, \quad \ldots$$

Note that the vectors obtained on the first and on the third step are proportional. Hence the output of the Euclidean algorithm is periodic with one element in the pre-period and two elements in the period. Here we have

$$a_1 = 4, \quad a_{2k} = 2, \quad \text{and} \quad a_{2k+1} = 8$$

for all integer $k \geq 1$. We obtain

$$2\sqrt{5} = [4; 2:8:2:8:2:8:\ldots].$$

2. On periodicity of cubic irrationalities

The problem of exhibiting periodicity for cubic irrationalities was posed by Ch. Hermite in 1848 (see e.g. [33], [14]), where he was asking whether *there exists some way to write cubic irrationalities periodically?* In this section we discuss the Jacobi-Perron algorithmic approach and recent advances in it.

2.1. **Jacobi-Perron algorithm.** The Jacobi-Perron algorithm is one of the possible ways to generalise the extended Euclid's algorithm to higher dimensions. It was proposed by C. G. J. Jacobi in 1868 in [15] and further developed by O. Perron in 1907, see [32]. The algorithm is as follows.

Jacobi-Perron algorithm

Input: We start with triples of real numbers (x, y, z).
Step of the algorithm: In the previous step we have constructed (x_i, y_i, z_i). Then we proceed with the following iteration:

$$(x_i, y_i, z_i) \mapsto (x_{i+1}, y_{i+1}, z_{i+1}) = \Big(y_i, z_i - \Big\lfloor \frac{z_i}{y_i} \Big\rfloor y_i, x_i - \Big\lfloor \frac{x_i}{y_i} \Big\rfloor y\Big).$$

Here the *i-th element* of the corresponding multidimensional continued fraction is set to be the pair of integers

$$\Big(\Big\lfloor\frac{z_i}{y_i}\Big\rfloor, \Big\lfloor\frac{x_i}{y_i}\Big\rfloor\Big).$$

Termination of the algorithm: If we have arrived to a triple (x_i, y_i, z_i) where $y_i = 0$, then the algorithm terminates.

Remark 2.1. (How to generate cubic vectors.) As we have seen the input data for the Jacobi-Perron algorithm is a triple of numbers. Let us discuss how to write a cubic vector starting from a cubic number α. For the first two coordinates of this vector we take 1 and α. Now it remains to find out how to pick the last coordinate of this vector. There is a natural answer to this question. Consider an arbitrary polynomial q of degree 2 with integer coefficients and let us take the vector

$$(1, \alpha, q(\alpha)).$$

The simplest choice here would be $(1, \alpha, \alpha^2)$, taking the polynomial $q(x) = x^2$.

In general, one can pick three numbers in $\mathbb{Q}(\alpha)$ that form a basis of the linear space $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$. Different choices of the basis of $\mathbb{Q}(\alpha)$ will result in different outputs of the Jacobi-Perron algorithm. The problem of *describing all possible periods for continued fraction algorithms for different vectors in* $\mathbb{Q}(\alpha)$ remains open *for every single* α. In particular the sets of available periods for the classical case of regular continued fractions of quadratic irrationalities in $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{3})$, $\mathbb{Q}(\sqrt{5})$ are unknown.

Let us continue with the following example.

Example 2.2. Let ξ be a real root of the polynomial $x^3 + 2x^2 + x + 4$, namely

$$\xi = -\frac{(53+6\sqrt{78})^{1/3}}{3} - \frac{1}{3(53+6\sqrt{78})^{1/3}} - \frac{2}{3}.$$

Now consider the vector

$$(1, \xi, \xi^2 + \xi).$$

Then the Jacobi-Perron algorithm will generate the following periodic output.

	1	2	3	4	5	6	$2k+1$	$2k+2$
$\lfloor x/y \rfloor$	-1	1	1	1	2	6	3	7
$\lfloor z/y \rfloor$	-2	0	0	0	2	4	1	1

(Here $k \geq 3$.) After the first 6 steps of the algorithm the sequence starts to be periodic with period 2.

The question of periodicity for the Jacobi-Perron algorithm is known in mathematical folklore as Jacobi's Last Theorem.

Problem 2.3. (Jacobi's Last Theorem.) *Let K be a totally real cubic number field. Consider arbitrary elements y and z of K satisfying $0 < y, z < 1$ such that 1, y, and z are independent elements over $\mathbb{Q}$. Is it true that the Jacobi-Perron algorithm generates an eventually periodic continued fraction with starting data $v = (1, y, z)$?*

The answer to the question of Jacobi's Last Theorem seems to be negative. Let us consider another example to see this.

Example 2.4. Let us consider the vector

$$v = (1, \sqrt[3]{4}, \sqrt[3]{16}).$$

Numerical computations suggest that the output of the Jacobi-Perron algorithm for this vector is not eventually periodic. Here we show the output elements for the first several steps of the algorithm (for further numerical computations and discussions we refer to [12]).

	1	2	3	4	5	6	7	8	9	10	11	12	...	94	...
$\lfloor x/y \rfloor$	0	1	13	1	6	1	1	3	2	3	4	1	...	476	...
$\lfloor z/y \rfloor$	1	1	9	1	2	0	0	2	0	1	1	1	...	388	...

2.2. **A few words about Gauss-Kuzmin statistics.** It follows a brief informal discussion of the last example. The sequence of the last example seems to be non-periodic. One could notice that the sizes of the elements are relatively small; we have a few bumps only. In fact, this is a rather predictible behaviour for non-periodic sequences. Let us consider the first 30 digits for the regular continued fraction for π. We have

$$\pi = [3 : 7; 15; 1; 292; 1; 1; 1; 2; 1; 3; 1; 14; 2; 1; 1; 2; 2; 2; 2; 1; 84; 2; 1; 1; 15; \ldots].$$

As we see, the most frequent element is 1; the next frequent element is 2; etc. This phenomenon is described by the Gauss-Kuzmin theorem stating that the frequency of an element k is

$$\frac{1}{\ln(2)} \ln\left(1 + \frac{1}{k(k+1)}\right).$$

For the first time this was proved by R. O. Kuzmin in 1928 in [25] (see also in [26]). It is interesting to notice that the Gauss-Kusmin statistics have a projective nature; it can be written in terms of cross-ratios:

$$\frac{1}{\ln(2)} \ln\left(1+\frac{1}{k(k+1)}\right) = \frac{\ln[-1,0,k,k+1]}{\ln[-1,0,1,\infty]}.$$

It remains to say that *the analogues of the Gauss-Kuzmin theorem for Jacobi-Perron type algorithms in higher dimensions are not known*, however we might expect a similar behaviour for the elements in higher dimensions as well.

For further discussions and the first successful generalisation of Gauss-Kuzmin theorem to higher dimensional case we refer to [24] and [16]; see also Chapter 19 in [18].

2.3. **Heuristic algebraic periodicity detecting algorithm.** Computations by L. Elsner and H. Hasse [12] suggest that the output of the Jacobi-Perron algorithm for the cubic vector $(1, \sqrt[3]{4}, \sqrt[3]{16})$ of Example 2.4 is non-periodic. However *the proof of this fact is missing*, there is only a strong belief that the sequence is indeed not periodic.

Let us informally say a few words on the reason for the Jacobi-Perron algorithm potentially to be non-periodic for cubic vectors. In fact, any cubic vector has a pair of algebraically conjugate vectors that are completely defined by the original vector. The pairs of these three vectors generate an arrangement of three planes with an action of the corresponding Dirichlet groups that we will discuss in Section 4. In some sense the choice of the elements in the classical Jacobi-Perron algorithm is blind to the action of the corresponding Dirichlet group; it follows more the Euclidean distances to nearest integers. The latter seems to be not appropriate for cubic vectors.

Let us introduce an important ternary form related to triples of vectors. We will use it for triples of cubic conjugate vectors.

Definition 2.5. Consider three vectors

$$u = (u_1, u_2, u_3), \quad v = (v_1, v_2, v_3), \quad \text{and} \quad w = (w_1, w_2, w_3)$$

in $\mathbb{C}^3$. The following ternary form

$$\det\begin{pmatrix} x & y & z \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} \cdot \det\begin{pmatrix} u_1 & u_2 & u_3 \\ x & y & z \\ w_1 & w_2 & w_3 \end{pmatrix} \cdot \det\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ x & y & z \end{pmatrix}$$

in three variables x, y, and z is called the *Markov-Davenport characteristic* of (x, y, z) with respect to the vectors u, v, w. Denote it by $\chi_{u,v,w}$.

Remark 2.6. Markov-Davenport characteristics were first studied in the context of their minima in a series of works [7, 8, 9, 10] by H. Davenport in the middle of the 20th century. These minima generalise two-dimensional Markov minima introduced by A. Markov in 1879 in [29] (for more details see the very nice book [1] by M. Aigner and also the paper [20]).

Remark 2.7. Note that the set of zeroes for a Markov-Davenport characteristic is the union of all invariant planes in $\mathbb{C}^3$. The Markov-Davenport characteristic $\chi_{u,v,w}$ measures *how close a point is to the union of planes spanned by pairs of vectors* (u, v)*,* (v, w)*, and* (w, u).

Now we would like to introduce a modification of the Jacobi-Perron algorithm that will be aiming to minimise the Markov-Davenport characteristic (rather than the Euclidean distance to the nearest integer vector). We would like to call this algorithm the *heuristic algebraic periodicity detecting algorithm* or the *heuristic APD-algorithm*, for short.

Heuristic APD-algorithm

Input: One starts with triples of real vectors (ξ, ν, μ) where the last coordinate of $\xi = (x_0, y_0, z_0)$ is positive (i.e. $z_0 > 0$),

Step 0: First of all let us make all the coordinates of ξ positive by applying the following integer lattice preserving transformation:

$$T_0 : (x, y, z) \mapsto \Big(x - \Big\lfloor \frac{x}{z} \Big\rfloor z, y - \Big\lfloor \frac{y}{z} \Big\rfloor z, z\Big),$$

namely we consider

$$(\xi_1, \nu_1, \mu_1) = \big(T_0(\xi), T_0(\nu), T_0(\mu)\big).$$

Step i for $i \geq 1$: In the previous step we have constructed (ξ_i, ν_i, μ_i) with positive coordinates of $\xi_i = (x_i, y_i, z_i)$.

- **Stage 1: Determination of the element of the continued fraction:** The new element of the heuristic continued fraction (a_i, b_i) is defined by the following four conditions:

  — $0 \leq a_i \leq \lfloor x_i/z_i \rfloor$;

  — $0 \leq b_i \leq \lfloor y_i/z_i \rfloor$;

  — $(a_i, b_i) \neq (0, 0)$ (with only one exception: we choose $(0, 0)$ if $\lfloor x_i/z_i \rfloor < 1$ and $\lfloor y_i/z_i \rfloor < 1$);

  — the triple $(a_i, b_i, 1)$ provides the minimal possible value for the absolute value of the Markov-Davenport characteristic $\chi_{\xi,\nu,\mu}$ among the vectors $(a, b, 1)$ whose first two coordinates satisfy the first three conditions.
- **Stage 2: Iteration step:**

  $$T_i : (x, y, z) \mapsto (y - b_i z, z, x - a_i z).$$

  Here we construct

  $$(\xi_{i+1}, \nu_{i+1}, \mu_{i+1}) = \big(T_i(\xi_i), T_i(\nu_i), T_i(\mu_i)\big)$$

Termination of the algorithm. In the case that the last coordinate of ξ_i is zero (i.e. $z_i = 0$) we do not proceed further and the algorithm terminates.

Remark 2.8. As one can see, the iteration step of the heuristic APD-algorithm resembles the Jacobi-Perron algorithm. Here the main difference between the algorithms is as follows. The Jacobi-Perron algorithm takes maximal possible values for a_i and b_i, which would be the best to approximate coordinatewise. However, as Example 2.4 shows, the coordinatewise approximation is not a good approximation with respect to the Markov-Davenport characteristic. In the heuristic APD-algorithm we are aiming to minimise the Markov-Davenport characteristic, which owing to Remark 2.7 (informally speaking) provides a simultaneous approximation.

Remark 2.9. If the last coordinate of the vector (x_i, y_i, z_i) is greater than the other two coordinates, then the continued fraction element generated at this step is $(0, 0)$, and the corresponding transformation T_i is simply a cyclic coordinate permutation. Hence the value of the continued fraction on the next step will be distinct to $(0, 0)$. In fact the algorithm does not produce two consecutive $(0, 0)$ pairs (disregarding Step 0).

Remark 2.10. The idea of the heuristic APD-algorithm appeared during a study of Klein polyhedra by the author. Klein polyhedra were introduced in 1895 by F. Klein in [22, 23] (roughly at the same time when the Jacobi-Perron algorithm appeared for the first time). Theory of Klein polyhedra represents the combinatorial periodicity of algebraic cones. They are known to be doubly periodic for the case of totally real cubic numbers. Not much is known about the link between periodicity of Klein polyhedra and Jacobi-Perron type algorithms, however they are both related to Dirichlet groups discussed briefly below. (For further details of Klein polyhedra we refer to [3, 2, 18].)

The heuristic APD-algorithm is designed to work with triples of cubic conjugate vectors. For simplicity we define conjugate vectors using the following property.

Definition 2.11. Let M be a matrix with integer elements, and let the characteristic polynomial of M be irreducible over $\mathbb{Q}$. Then the triples of linearly independent eigenvectors of M are said to be *cubic conjugate vectors.*

Remark 2.12. (How to generate triples of cubic conjugate vectors from a single cubic number.) Let α be a cubic number and let p be any polynomial with integer coefficients of degree 3 having α as a root. (We assume that p is irreducible over $\mathbb{Q}$.) Similarly to the case of single cubic vectors (see Remark 2.1) we can construct triples of conjugate cubic vectors. In order to do this we additionally pick an arbitrary degree 2 polynomial q with integer coefficients. Now a triple of conjugate vectors can be naturally derived from (α, p, q). Namely, let α, β, and γ be distinct roots of p. Then we set

$$\begin{aligned} \xi &= \big(1, \alpha, q(\alpha)\big); \\ \nu &= \big(1, \beta, q(\beta)\big); \\ \mu &= \big(1, \gamma, q(\gamma)\big). \end{aligned}$$

It turns out that these vectors are eigenvectors of some integer matrix, and hence they are cubic conjugate vectors.

Example 2.13. Let us now consider the cubic vector of Example 2.4 for which we did not see any periodicity of the Jacobi-Perron algorithm output:

$$\xi = (1, \sqrt[3]{4}, \sqrt[3]{16}).$$

Note that $\sqrt[3]{4}$ is a root of $x^3 - 4$. Note also that

$$\sqrt[3]{16} = (\sqrt[3]{4})^2$$

Let β and γ be two other complex roots of $x^3 - 4$. Consider two vectors:

$$\nu = (1, \beta, \beta^2) \quad \text{and} \quad \mu = (1, \gamma, \gamma^2).$$

Then the output of the heuristic APD-algorithm for the triple of vectors (ξ, ν, μ) is as follows (here $k \geq 1$):

	0	1	2	3	4	5	6	$4k+3$	$4k+4$	$4k+5$	$4k+6$
a_1	0	0	1	0	0	0	1	1	1	0	0
b_1	0	0	2	1	1	1	5	0	1	1	6

Note that Step 0 does not change the triple. At Step 1 we have a situation when the last element is the greatest, so we have the exception $(a_1, b_1) = (0, 0)$. Hence we simply do the cyclic permutation of the coordinates. After 6 steps of the pre-period we have a periodic sequence with period 4.

Let us continue with the following conjecture.

Conjecture 2.14. The heuristic APD-algorithm is periodic for all triples of cubic conjugate vectors as in Definition 2.11.

Remark 2.15. The conjecture can be considered separately for both totally-real and complex cases. This conjecture has a straightforward generalisation to the case of d-tuples of conjugate algebraic vectors of degree d (see some further discussions in Section 3).

2.4. **Technical remarks on the heuristic APD-algorithm.** When trying to calculate the continued fractions for particular examples of quadratic numbers, one can use the following two approaches. The first one is more symbolic: we represent quadratic numbers as surds and do all our computations with them (e.g. see Example 1.5). The second approach is to work with sufficiently precise rational decimal approximations of quadratic numbers and find the period using their continued fractions.

Similar to the quadratic case, cubic and quartic cases have a symbolic approach due to Cardano's formula and Ferrari's method. It is easier to detect periodicity using sufficiently good approximation of cubic vectors. For instance, in order to find a periodic representation for the vector of Example 2.4, it is enough to know only the first 5 decimal digits of the coordinates of this vector. In general, the smaller the periods and the pre-period are, and the smaller the elements of the pre-period and the period are, the smaller the rate of approximation that is sufficient for constructing the period and the pre-period.

Remark 2.16. (On Gauss-Kuzmin statistics.) Practically the algorithm is very fast if the obtained elements of continued fractions are small, and it is starting to be slower with the growth of the elements. Here we should mention that in practice large elements occur very seldom, so on average the algorithm works fast. As in the case of the Jacobi-Perron algorithm (see Subsection 2.2) the *distribution of frequencies is unknown for the case of the heuristic APD-algorithm.*

2.5. **A remark on the periodicity of the $\sin^2$-algorithm in the cubic totally real case.** It seems that the heuristic APD-algorithm establishes periodicity of cubic vectors. In addition it works very fast, so heuristically it solves effectively the problem of finding the pre-periods and the periods for cubic vectors. Thus practically it can be effectively used for the computations of the independent elements in Dirichlet groups (or units in orders of algebraic fields). We discuss this later in Section 4. Currently, the main concern regarding this algorithm is that a proof for cubic periodicity is missing.

Recently we have developed an algorithm similar to the heuristic APD-algorithm that is designed for the totally real case (i.e. when the corresponding cubic extension of rational numbers is embeddable in the real line) and proved its periodicity (see [19]). As far as we are aware, this is the first complete proof of periodicity for Jacobi-Perron type algorithms.

Let us outline the idea of the algorithm. Given three real vectors (ξ, ν, μ), in each step of the algorithm we aim to maximise the $\sin^2$ of the angle between the planes spanned by pairs of vectors (ξ, ν) and (ξ, μ). One can say that the value of $\sin^2$ here replaces the Markov-Davenport characteristic in the heuristic APD-algorithm.

$\sin^2$-algorithm

Input: We are given three vectors ξ, ν, μ such that
— ξ has positive coordinates (x, y, z) satisfying $x > y > z > 0$;
— all three coordinates for both ν and μ are neither simultaneously positive nor simultaneously negative.

Step of the algorithm: Let us apply the following linear transformation

$$(\xi_i, \nu_i, \mu_i) \to (\Phi_i(\xi_i), \Phi_i(\nu_i), \Phi_i(\mu_i))$$

with

$$\Phi_i = T_i M_i.$$

Here M_i is taken to be the maximiser of the value of $\sin^2$ for the angle between the plane spanned by $M_i(\xi_i)$ and $M_i(\nu_i)$ and the plane spanned by $M_i(\xi_i)$ and $M_i(\mu_i)$. The maximisation is done among all the transformations

$$N_{\alpha,\beta,\gamma} : (x, y, z) \mapsto \big(x - \alpha z - \gamma(y - \beta z), y - \beta z, z\big)$$

satisfying

$$0 \le \alpha \le \left\lfloor \frac{x_i}{z_i} \right\rfloor, \quad 0 \le \beta \le \left\lfloor \frac{y_i}{z_i} \right\rfloor, \quad \text{and} \quad 0 \le \gamma \le \left\lfloor \frac{x_i/z_i - \alpha}{y_i/z_i - \beta} \right\rfloor,$$

and the transformation

$$N_0 = (x, y, z) \mapsto \big(x - y, y, z - (x - y)\big),$$

which is considered only in case

$$z_i > x_i - y_i > 0.$$

After M_i is constructed we set T_i as a permutation of the basis vectors that puts the coordinates of $M_i(\xi)$ in decreasing order.

At each step the algorithm returns Φ_i as an output.

Termination of the algorithm: In the case that the last coordinate of ξ_i is zero (i.e. $z_i = 0$) we do not proceed further and the algorithm terminates.

Remark 2.17. Note that the triples in the input of the algorithm have some initial conditions. There is a rather simple way to change the coordinates of an arbitrary triple of vectors (by an integer lattice preserving linear transformation) such that in the new basis this triple fulfills the input conditions. We omit the technical details here. For further details and the proof of periodicity for cubic vectors we refer the interested reader to [19].

Remark 2.18. Finally we would like to refer to several research papers on cubic periodicity in some other settings. Cubic periodicity was also studied for the cases of the following generalised continued fractions: for Klein polyhedra [27, 13], Minkovski-Voronoi polyhedra [35, 30, 5], triangle sequences [6], and ternary continued fractions (or bifurcating continued fractions) [31].

3. The situation in degree greater than 3

This section is rather short as *almost nothing is known in the cases of degree greater than 3*. Currently the main source of ideas that are applied in the higher degree case come from the study of cubic vectors.

We should notice that *there is a straightforward generalisation of the heuristic APD-algorithm, which is likely to demonstrate periodicity for irrationalities of degree* $d > 3$. Let us briefly formulate it.

Consider an irreducible polynomial of degree d and let $\xi_1, \ldots \xi_d$ be the set of its roots. Let $q_1, \ldots, q_d$ be a basis of the space of polynomials of degree less than d with rational coefficients.

Consider vectors

$$(q_1(\xi_i), q_2(\xi_i), \ldots, q_d(\xi_i)) \quad \text{for } i = 1, \ldots, d.$$

The Markov-Davenport characteristic for these vectors is now written as a product of d matrices of size $d \times d$:

$$\xi(x_1, \ldots, x_d) = \prod_{k=1}^{d} M_k$$

where $M_k(\xi)$ is obtained from the matrix

$$\begin{pmatrix} q_1(\xi_1) & q_1(\xi_2) & \ldots & q_1(\xi_d) \\ q_2(\xi_1) & q_2(\xi_2) & \ldots & q_2(\xi_d) \\ \vdots & \vdots & \ddots & \vdots \\ q_d(\xi_1) & q_d(\xi_2) & \ldots & q_d(\xi_d) \end{pmatrix}$$

by replacing its k-th column by the column of variables $(x_1, \ldots, x_d)$. It is interesting to note that after multiplication by some constant the coefficients of the Markov-Davenport characteristic are all integers; see e.g. Chapter 21.4 in [18].

The multidimensional heuristic APD-algorithm will be as follows.

Multidimensional heuristic APD-algorithm

Input: One starts with d-tuples of conjugate real vectors $(\xi_{1,0}, \xi_{2,0}, \ldots, \xi_{d,0})$ generated as above, where the last coordinate of $\xi_{1,0} = (x_{1,1,0}, x_{1,2,0}, \ldots, x_{1,d,0})$ is positive (i.e. $x_{1,d,0} > 0$),

Step 0: First of all let us make all of the coordinates of $\xi_{1,0}$ positive by applying the following integer lattice preserving transformation:

$$T_0 : (x_1, x_2, \ldots, x_d) \mapsto \Big(x_1 - \Big\lfloor \frac{x_1}{x_d} \Big\rfloor x_d, x_2 - \Big\lfloor \frac{x_2}{x_d} \Big\rfloor x_d, \ldots, x_2 - \Big\lfloor \frac{x_{d-1}}{x_d} \Big\rfloor x_d, x_d\Big),$$

namely we consider

$$(\xi_{1,1}, \xi_{2,1}, \ldots, \xi_{d,1}) = \big(T_0(\xi_{1,0}), T_0(\xi_{2,0}), \ldots, T_0(\xi_{d,0})\big).$$

Step i for $i \geq 1$: In the previous step we have constructed the d-tuple

$$(\xi_{1,i}, \xi_{2,i}, \ldots, \xi_{d,i})$$

with positive coordinates of $\xi_{1,i} = (x_{1,1,i}, x_{1,2,i}, \ldots, x_{1,d,i})$. In addition from Step 2 on we have $x_{1,d-1,i} > x_{1,k,i}$ for all $k < d-1$ and $k = d$.

- **Stage 1: Determination of the element of the continued fraction:** The new element of the heuristic continued fraction $(a_{1,i}, a_{2,i}, \ldots, a_{d-1,i})$ is defined by the following conditions:
  — $0 \leq a_{k,i} \leq \lfloor x_{1,k,i}/x_{1,d,i} \rfloor$ for $k = 1, \ldots, d-1$;
  — $(a_{1,i}, a_{2,i}, \ldots, a_{d-1,i}) \neq (0, 0, \ldots, 0)$ (with only one exception: we choose $(0, 0, \ldots, 0)$ if $\lfloor x_{1,k,i}/x_{1,d,i} \rfloor < 1$ for $k = 1, \ldots, d-1$);
  — the d-tuple $(a_{1,i}, a_{2,i}, \ldots, a_{d-1,i}, 1)$ provides the minimal possible absolute value of the Markov-Davenport characteristic among the d-tuples $(a_1, \ldots, a_{d-1}, 1)$ whose first two coordinates satisfy the first two conditions.
- **Stage 2: Iteration step:**

$$T_i : (x_1, x_2, \ldots, x_d) \mapsto (x_2 - a_{2,i}x_d, \ldots, x_{d-1} - a_{d-1,i}x_d, x_d, x_1 - a_{1,i}x_d).$$

Here we construct

$$(\xi_{1,i+1}, \xi_{2,i+1}, \ldots, \xi_{d,i+1}) = \big(T_i(\xi_{1,i}), T_i(\xi_{2,i}), \ldots, T_i(\xi_{d,i})\big).$$

Termination of the algorithm. In the case that the last coordinate of $\xi_{1,i}$ is zero we do not proceed further and the algorithm terminates.

Remark 3.1. The algorithm was tested for quartic irrationalities, in all the examples the algorithm produced periodic output.

4. Dirichlet groups

In order to understand better the reason for periodicity, let us study maximal commutative subgroups of $\mathrm{SL}(d, \mathbb{Z})$. Such subgroups are called *Dirichlet groups.*

4.1. Magic of integer commuting matrices.

Let us start with a simple exercise.

Example 4.1. Let

$$A = \begin{pmatrix} 2 & 5 & -1 \\ 3 & 6 & 1 \\ 4 & 7 & 1 \end{pmatrix}.$$

Find an integer matrix with unit determinant that commutes with A?

The obvious solution to this exercise is the identity matrix, but let us disregard it. Let us first peek the answer to this question. The first matrix that we are able to find is

$$B = \begin{pmatrix} 88778750433916 & 1881948516620816 & -1642359549748757 \\ -77918418013751 & -849278651461089 & 759124773173459 \\ 534000559063825 & -721564227716990 & 360094549931638 \end{pmatrix}$$

The sizes of the elements of this matrix are rather impressive, are they not? It is most likely that a brute force algorithm would find a solution of this problem only in the next millennium.

Even if one notices that the matrix B is, in fact, a polynomial in A with integer coefficients, namely

$$B = -147205796095883A^2 + 1347947957556991A - 399030223241821,$$

the brute force search for the coefficients of such a polynomial is still very long. This is very much in contrast to the complexity of the input matrix A, each element of which requires 4 bits only.

Let us discuss how to find the answer efficiently.

4.2. Dirichlet groups.

First we give some necessary definitions.

Definition 4.2. Let A be an $n \times n$ matrix with real coefficients. Denote by $\Gamma(A)$ the set of all integer matrices commuting with A.

(i) The *Dirichlet group* $\Xi(A)$ is the subset of invertible matrices in $\Gamma(A)$.

(ii) The *positive Dirichlet group* $\Xi_+(A)$ is the subset of $\Xi(A)$ that consists of all matrices with positive real eigenvalues.

4.3. **Dirichlet's unit theorem.** The first questions that one might ask before approaching Example 4.1 is *whether such a matrix does exist.* Could it be that all unit determinant integer matrices commuting with A are in fact powers of A? In terms of Dirichlet groups, we ask wether the group $\Xi(A)$ is isomorphic to $\mathbb{Z}$ or not.

The answer to this question is provided by the Dirichlet's unit theorem. A precise formulation of the theorem is as follows.

Dirichlet's unit theorem. *Let K be a field of algebraic numbers of degree $n = s + 2t$, where s is the number of real roots and $2t$ is the number of complex roots for the minimal integer polynomial of any irrational element of K. Consider an arbitrary order D in K. Then D contains units $\varepsilon_1, \ldots, \varepsilon_r$ for $r = s + t - 1$ such that every unit ε in D has a unique decomposition of the form*

$$\varepsilon = \xi \varepsilon_1^{a_1} \cdots \varepsilon_r^{a_r} \ ,$$

where $a_1, \ldots, a_r$ are integers and ξ is a root of 1 contained in D.

So what is hidden behind Dirichlet's unit theorem? Rather than to go forward with all the formal definitions involved in the formulation of this theorem we prefer to reformulate this theorem simply in terms of matrices. (For necessary definitions and the proof of the theorem we refer an interested reader e.g. to the book [4]; for the justification of the reformulation we refer e.g. to Chapter 17 of [18]. Also there is a lot of related material specifically on algebraic cubic and quartic fields in the book [11].)

Dirichlet's unit theorem in the matrix form. *Let A be an integer matrix whose characteristic polynomial is irreducible over the field of rational numbers $\mathbb{Q}$. Let it have s real and $2t$ complex eigenvalues. Then there exists a finite Abelian group G such that*

$$\Xi(A) \cong G \oplus \mathbb{Z}^{s+t-1}.$$

For the positive Dirichlet group we have:

$$\Xi_+(A) \cong \mathbb{Z}^{s+t-1}.$$

Example 4.3. In the three-dimensional case we have two possible situations.

- **Complex case:** If the characteristic polynomial has two complex roots, then both Dirichlet and positive Dirichlet groups are isomorphic to $\mathbb{Z}$.

- **Totally real case:** If all the roots of the characteristic polynomial are real numbers, we have

$$\Xi(A) \cong G \oplus \mathbb{Z}^2 \quad \text{and} \quad \Xi_+(A) \cong \mathbb{Z}^2.$$

4.4. **Several questions that we can answer.** The technique discussed in this paper gives a constructive approach to the following questions.

Question 1. Find an $\mathrm{SL}(3,\mathbb{Z})$-matrix commuting with a given integer matrix with irreducible characteristic polynomial over $\mathbb{Q}$.

Question 2. Find an $\mathrm{SL}(3,\mathbb{Z})$-matrix with a given cubic vector as an eigenvector.

Question 3. Let M be any $\mathrm{SL}(3,\mathbb{Z})$-matrix whose characteristic polynomial is irreducible over $\mathbb{Q}$. Find an $\mathrm{SL}(3,\mathbb{Z})$-matrix commuting with M that is not a power of M. (Note that this question is interesting only in the totally real case as otherwise $\Xi(A) = \mathbb{Z}$.)

Remark 4.4. Currently the $\sin^2$-algorithm establishes the answers for the totally real case only. Once Conjecture 2.14 if proven we have the complete answers to all these three questions.

In the next section we rewrite Jacobi-Perron type algorithms in matrix form and give answers to these three questions.

5. Jacobi-Perron type algorithms and Dirichlet groups

5.1. **Jacobi-Perron algorithm in matrix form.** Notice that Jacobi-Perron type algorithms can be formulated in terms of matrix multiplication. This concerns both the Jacobi-Perron algorithm, the heuristic APD-algorithm, the $\sin^2$-algorithm, and many other similar algorithms (various types of such algorithms are collected in the book of F. Schweiger [34], see also Chapter 23.4 in [18]). If an algorithm produces an eventually periodic output, the answers to Questions 1—3 above can be obtained from the matrix form as we explain in the next subsection.

5.2. **Matrices with prescribed cubic eigenvectors.** Assume that the Jacobi-Perron algorithm is eventually periodic and its pre-period and period for a given vector are respectively as follows:

$$\left(\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \ldots, \begin{pmatrix} a_n \\ b_n \end{pmatrix} \right) \quad \text{and} \quad \left(\begin{pmatrix} c_1 \\ d_1 \end{pmatrix}, \ldots, \begin{pmatrix} c_m \\ d_m \end{pmatrix} \right).$$

Denote

$$M_1=\prod_{i=1}^{n}\begin{pmatrix} a_i & 0 & 1\\ 1 & 0 & 0\\ b_i & 1 & 0\end{pmatrix} \quad \text{and} \quad M_2=\prod_{j=1}^{m}\begin{pmatrix} c_j & 0 & 1\\ 1 & 0 & 0\\ d_j & 1 & 0\end{pmatrix}.$$

and set

$$M=M_1M_2(M_1)^{-1}.$$

Then M *is an* $\mathrm{SL}(3,\mathbb{Z})$*-matrix with the original cubic vector as the eigenvector whose absolute value of the eigenvalue is the greatest among all the absolute values of the eigenvalues of* M.

Remark 5.1. There are similar representations for both the heuristic APD-algorithm and the $\sin^2$-algorithm. We omit them here, as they literally repeat the representation for the Jacobi-Perron algorithm with obvious changes to the matrices.

Example 5.2. Let us discuss the vector

$$(1,\xi,\xi^2+\xi)$$

considered in Example 2.2 above. Recall that ξ is a real root of the polynomial

$$x^3+2x^2+x+4.$$

The Jacobi-Perron algorithm generates a periodic sequence with 6 steps of the pre-period and 2 steps of the period:

	1	2	3	4	5	6	$2k+1$	$2k+2$
$\lfloor x/y\rfloor$	-1	1	1	1	2	6	3	7
$\lfloor z/y\rfloor$	-2	0	0	0	2	4	1	1

First of all, we write matrices for the pre-period and the period:

$$M_1=\begin{pmatrix} \mathbf{-1} & 0 & 1\\ 1 & 0 & 0\\ \mathbf{-2} & 1 & 0\end{pmatrix}\cdot\begin{pmatrix} \mathbf{1} & 0 & 1\\ 1 & 0 & 0\\ \mathbf{0} & 1 & 0\end{pmatrix}^3\cdot\begin{pmatrix} \mathbf{2} & 0 & 1\\ 1 & 0 & 0\\ \mathbf{2} & 1 & 0\end{pmatrix}\cdot\begin{pmatrix} \mathbf{6} & 0 & 1\\ 1 & 0 & 0\\ \mathbf{4} & 1 & 0\end{pmatrix}$$

$$=\begin{pmatrix} -22 & -1 & -3\\ 51 & 2 & 7\\ -67 & -3 & -9\end{pmatrix};$$

$$M_2=\begin{pmatrix} \mathbf{3} & 0 & 1\\ 1 & 0 & 0\\ \mathbf{1} & 1 & 0\end{pmatrix}\cdot\begin{pmatrix} \mathbf{7} & 0 & 1\\ 1 & 0 & 0\\ \mathbf{1} & 1 & 0\end{pmatrix}=\begin{pmatrix} 22 & 1 & 3\\ 7 & 0 & 1\\ 8 & 0 & 1\end{pmatrix}.$$

Finally we get

$$M=M_1M_2(M_1)^{-1}=\begin{pmatrix} 5 & -4 & 3\\ -12 & 9 & -7\\ 16 & -12 & 9\end{pmatrix}.$$

This concludes the computation of M.

Example 5.3. Let us consider once more the vector

$$v = (1, \sqrt[3]{4}, \sqrt[3]{16}).$$

As we have seen in Example 2.4, we are unable to get a periodic sequence generated by the Jacobi-Perron algorithm, and hence we cannot find a requested matrix using the Jacobi-Perron algorithm.

Let us use the heuristic APD-algorithm instead (similarly multiplying the corresponding matrices for linear maps used in the algorithm). Following the results of the continued fraction computations in Example 2.13 we find that v is an eigenvector of the matrix

$$M = \begin{pmatrix} 5 & 8 & 12 \\ 3 & 5 & 8 \\ 2 & 3 & 5 \end{pmatrix}.$$

5.3. **Answers to Questions 1–3.** Finally we discus the answers to Questions 1–3.

Answer to Question 1. Let us show how to construct an $\mathrm{SL}(3,\mathbb{Z})$-matrix commuting with a given integer matrix A (assuming that the characteristic polynomial of A is irreducible over $\mathbb{Q}$). What we should do is take a basis of eigenvectors of A and run the heuristic APD-algorithm for it (or the $\sin^2$-algorithm in the totally real case). The algorithm will generate a heuristically eventually periodic sequence. From its period and pre-period sequences one computes the required matrix M (as discussed in Subsection 5.2). An example is shown in Example 4.1 above.

Answer to Question 2. The second question is very similar to the first one. Here we are requested to find an $\mathrm{SL}(3,\mathbb{Z})$-matrix having a given cubic vector as an eigenvector. Assume we are given a cubic vector

$$(1, \xi, q(\xi)),$$

where ξ is a root of some cubic polynomial p. Then we construct the other two conjugate vectors (following discussions of Remark 2.12), apply the heuristic APD-algorithm (or $\sin^2$-algorithm in the totally real case) and write the matrix from the period and pre-period of the algorithm.

Remark 5.4. Usually a cubic vector is defined by the corresponding polynomials p and q. However, this might not be the case, and we might have an expression for a root ξ of a cubic polynomial in the style of Kordano's formula instead. In this case the polynomials p and q can be guessed from rational approximations of ξ^3 and $q(\xi)$ and their approximate formulae in terms of approximations of ξ^2 and ξ and 1.

Answer to Question 3. Finally we write a matrix commuting with a given matrix $A \in SL(3,\mathbb{Z})$ that is not a power of A. As we have mentioned, this question only makes sense for the totally-real case (in the complex case the Dirichlet group of A is isomorphic to $\mathbb{Z}$). Let ξ, ν, and μ be eigenvectors of A. First of all we construct $\mathrm{SL}(3,\mathbb{Z})$ matrices M_ξ, M_ν and M_μ following the computations for the triple vectors

$$(\xi,\nu,\mu), \quad (\nu,\mu,\xi), \quad \text{and} \quad (\mu,\xi,\nu)$$

respectively. The main feature that we use further is that *both the heuristic APD-algorithm, and* $\sin^2$*-algorithm construct a matrix whose maximal absolute value of the eigenvalue corresponds to the first cubic vector in the corresponding triple.*

Notice that maximal absolute values of the eigenvalues of A^n correspond simultaneously to the same eigenline for $n>0$; and to the same eigenline for $n<0$. Hence one of the vectors ξ, ν, and μ is not on these two eigenlines. Therefore, the required matrix M can be chosen from M_ξ, M_ν, and M_μ by comparing the sizes of absolute values of eigenvalues.

Let us illustrate the answers to Questions 2 and 3 with the following example.

Example 5.5. Consider an irreducible cubic polynomial

$$p(x) = 2x^3 - 4x^2 - 7x - 2$$

with three real roots denoted by α, β, and γ. Our goal is to compute two independent (with respect to matrix multiplication) $\mathrm{SL}(3,\mathbb{Z})$-matrices with eigenvectors

$$\xi = (1,\alpha,\alpha^2), \quad \nu = (1,\beta,\beta^2), \quad \text{and} \quad \mu = (1,\gamma,\gamma^2).$$

Direct computations using the heuristic APD-algorithm applied to triples

$$(\xi,\nu,\mu), \quad (\nu,\mu,\xi), \quad \text{and} \quad (\mu,\xi,\nu)$$

result in the following three matrices:

$$A = \begin{pmatrix} 55 & 210 & 176 \\ 176 & 671 & 562 \\ 562 & 2143 & 1795 \end{pmatrix}; \quad B = \begin{pmatrix} -497 & -1122 & 400 \\ 400 & 903 & -322 \\ -322 & -727 & 259 \end{pmatrix};$$

$$C = \begin{pmatrix} 185 & 172 & -72 \\ -72 & -67 & 28 \\ 28 & 26 & -11 \end{pmatrix}.$$

Brute force search of the powers of matrices show that

$$A^3B^5C^7 = \mathrm{Id},$$

where Id is the identity matrix. Since at least two of these matrices are independent, we have that any two of these matrices are linearly independent.

Remark 5.6. Clearly the triples of matrices M_ξ, M_ν, and M_μ generate a finite index sublattice in the positive Dirichlet groups. There exists a technique to find the basis using constructions of multidimensional Klein continued fractions and observing the combinatorics of their periods. We do not discuss it in this paper and refer an interested reader to Chapter 20 of [18] (see also [17]).

Remark 5.7. The method described in this section works well for $\mathrm{SL}(3,\mathbb{Z})$-matrices but it has some limitations for $\mathrm{SL}(d,\mathbb{Z})$-matrices with $d>3$. For instance, in the totally real case of the quartic case ($d=4$) the corresponding positive Dirichlet group is three-dimensional. A direct application of the current method potentially can output 4 matrices spanning $\mathbb{Z}^2$, and not $\mathbb{Z}^3$ as expected. The author is not aware of any examples where this does occur, if it does, such examples must be rare.

Acknowledgements. The author is grateful to P. Giblin, A. Pratoussevitch, H. Řada, C. Series, H. Servatius, M. van-Son, A. Ustinov, A. Veselov, and the unknown reviewer for useful comments and remarks.

References

[1] M. Aigner. *Markov's theorem and 100 years of the uniqueness conjecture*. Springer, Cham, 2013. A mathematical journey from irrational numbers to perfect matchings.

[2] V. I. Arnold. Higher-dimensional continued fractions. *Regul. Chaotic Dyn.*, 3(3):10–17, 1998. J. Moser at 70 (Russian).

[3] V. I. Arnold. *Continued fractions (In Russian)*. Moscow: Moscow Center of Continuous Mathematical Education, 2002.

[4] A. I. Borevich and I. R. Shafarevich. *Number theory*. Translated from the Russian by Newcomb Greenleaf. Pure and Applied Mathematics, Vol. 20. Academic Press, New York, 1966.

[5] G. Bullig. Zur Kettenbruchtheorie im Dreidimensionalen (Z 1). *Abh. math. Sem. Hansische Univ.*, 13:321–343, 1940.

[6] K. Dasaratha, L. Flapan, T. Garrity, Ch. Lee, C. Mihaila, N. Neumann-Chun, S. Peluse, and M. Stoffregen. Cubic irrationals and periodicity via a family of multi-dimensional continued fraction algorithms. *Monatsh. Math.*, 174(4):549–566, 2014.

[7] H. Davenport. On the product of three homogeneous linear forms. I. *Proc. London Math. Soc.*, 13:139–145, 1938.

[8] H. Davenport. On the product of three homogeneous linear forms. II. *Proc. London Math. Soc.(2)*, 44:412–431, 1938.

[9] H. Davenport. On the product of three homogeneous linear forms. III. *Proc. London Math. Soc.(2)*, 45:98–125, 1939.

[10] H. Davenport. On the product of three homogeneous linear forms. IV. *Proc. Cambridge Philos. Soc.*, 39:1–21, 1943.

[11] B. N. Delone and D. K. Faddeev. *The theory of irrationalities of the third degree*. Translations of Mathematical Monographs, Vol. 10. American Mathematical Society, Providence, R.I., 1964.

[12] L. Elsner and H. Hasse. Numerische Ergebnisse zum Jacobischen Kettenbruchalgorithmus in rein-kubischen Zahlkörpern. *Math. Nachr.*, 34:95–97, 1967.

[13] O. N. German and E. L. Lakshtanov. On a multidimensional generalization of Lagrange's theorem for continued fractions. *Izv. Math.*, 72(1):47–61, 2008. Russian Version: Izv. Ross. Akad. Nauk Ser. Mat., 72(1), 2008, 51–66.

[14] C. Hermite. Extraits de lettres de M. Ch. Hermite à M. Jacobi sur différents objects de la théorie des nombres. (Continuation). *J. Reine Angew. Math.*, 40:279–315, 1850.

[15] C. G. J. Jacobi. Allgemeine Theorie der Kettenbruchähnlichen Algorithmen, in welchen jede Zahl aus drei vorhergehenden gebildet wird (Aus den hinterlassenen Papieren von C. G. J. Jacobi mitgetheilt durch Herrn E. Heine). *Journal für die Reine und Angewandte Mathematik*, 69(1):29–64, 1868.

[16] O. Karpenkov. On an invariant Möbius measure and the Gauss-Kuz'min face distribution. *Proc. Steklov Inst.*, 258:74–86, 2007. Russian Version: Tr. Mat. Inst. Steklova 258 (2007), 79–92.

[17] O. Karpenkov. Constructing multidimensional periodic continued fractions in the sense of Klein. *Math. Comp.*, 78(267):1687–1711, 2009.

[18] O. Karpenkov. *Geometry of Continued Fractions*. Algorithms and Computation in Mathematics, 26. Springer-Verlag, Berlin, 2013.

[19] O. Karpenkov. On a periodic Jacobi-Perron type algorithm (preprint), arXiv:2101.12627 [math.nt]. 2021.

[20] O. Karpenkov and M. van Son. Generalised Markov numbers. *J. Number Theory*, 213:16–66, 2020.

[21] A. Ya. Khinchin. *Continued fractions*. Moscow, FISMATGIS, 1961.

[22] F. Klein. Ueber eine geometrische Auffassung der gewöhnliche Kettenbruchentwicklung. *Nachr. Ges. Wiss. Göttingen Math-Phys. Kl.*, 3:352–357, 1895.

[23] F. Klein. Sur une représentation géométrique de développement en fraction continue ordinaire. *Nouv. Ann. Math.*, 15(3):327–331, 1896.

[24] M. L. Kontsevich and Yu. M. Suhov. Statistics of Klein polyhedra and multidimensional continued fractions. In *Pseudoperiodic topology*, volume 197 of *Amer. Math. Soc. Transl. Ser. 2*, pages 9–27. Amer. Math. Soc., Providence, RI, 1999.

[25] R. O. Kuzmin. On one problem of Gauss. *Dokl. Akad. Nauk SSSR Ser. A*, pages 375–380, 1928.

[26] R. O. Kuzmin. On a problem of Gauss. *Atti del Congresso Internazionale dei Matematici, Bologna*, 6:83–89, 1932.

[27] G. Lachaud. Polyèdre d'Arnol'd et voile d'un cône simplicial: analogues du théorème de Lagrange. *C. R. Acad. Sci. Paris Sér. I Math.*, 317(8):711–716, 1993.

[28] J.-L. Lagrange. Additions au m'emoire sur la r'esolution des 'equations num'eriques. In *M'em. Acad. royale sc. et belles-lettres*, volume 24. Berlin, 1770.

[29] A. Markoff. Sur les formes quadratiques binaires indéfinies. *Math. Ann.*, 15(3-4):381–406, 1879.

[30] H. Minkowski. *Gesammelte Abhandlungen (pp. 293–315).* AMS-Chelsea, 1967.

[31] N. Murru. On the periodic writing of cubic irrationals and a generalization of Rédei functions. *Int. J. Number Theory*, 11(3):779–799, 2015.

[32] O. Perron. Grundlagen für eine Theorie des Jacobischen Kettenbruchalgorithmus. *Math. Ann.*, 64(1):1–76, 1907.

[33] É. Picard. L'œuvre scientifique de Charles Hermite. *Ann. Sci. École Norm. Sup. (3)*, 18:9–34, 1901.

[34] F. Schweiger. *Multidimensional continued fractions.* Oxford Science Publications. Oxford University Press, Oxford, 2000.

[35] G. F. Voronoĭ. *On a Generalization of the Algorithm of Continued Fraction. Collected works in three volumes (In Russian)*. USSR Ac. Sci., Kiev., 1952.

(Oleg Karpenkov) UNIVERSITY OF LIVERPOOL, MATHEMATICAL SCIENCES BUILDING, LIVERPOOL L69 7ZL, UNITED KINGDOM

Email address: `karpenk@liverpool.ac.uk`