Twisted Hodge Diamonds give rise to Non-Fourier-Mukai Functors

Thesis submitted in accordance with the requirements of the University of Liverpool for the degree of Doctor in Philosophy by

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Liverpool
June 19, 2022

Abstract

We apply computations of twisted Hodge diamonds to construct an infinite number of non-Fourier-Mukai functors with well behaved target and source spaces.

To accomplish this we first study the characteristic morphism introduced in [BF08] in order to control it for tilting bundles. Then we continue by applying twisted Hodge diamonds of hypersurfaces embedded in projective space to compute the Hochschild dimension of these spaces. This allows us to compute the kernel of the embedding into the projective space in Hochschild cohomology. Finally we use the above computations to apply the construction in [RVdBN19] of non-Fourier-Mukai functors and verify that the constructed functors indeed cannot be Fourier-Mukai for odd dimensional quadrics.

Using this approach we prove that there are a large number of Hochschild cohomology classes that can be used for the construction of [RVdBN19]. Furthermore, our results allow the application of computer-based calculations to construct candidate functors for arbitrary degree hypersurfaces in arbitrary high dimensions. Verifying that these are not Fourier-Mukai still requires the existence of a tilting bundle.

In particular we prove that there is at least one non-Fourier-Mukai functor for every odd dimensional smooth quadric.

Acknowledgements

First of all I would like to thank my supervisor Alice Rizzardo from all my heart for all the incredible support and help throughout my PhD. Thank you so much! I still hear "don't spend too much time on it!" in my head whenever I work on organisatorial tasks. Thank you for all the knowledge on mathematics and life you shared with me. I also would like to thank my second supervisor Jon Woolf for helping with organisational work and the occasional mathematical clarification.

Secondly I would like to thank my friends and family for the invaluable support, especially during times of plague and isolation. Be it my parents, Gabriela K. and Josef K. and godmother Anneliese L. who are always there when I need them and never stopped supporting me. My brother Erik K., my deer friends Linda L. and Sabine L. who always found the right words when I found myself in a hole again. My office colleagues, in fact and in spirit, who helped me finally understand the cheesy Liverpool sentence "you never walk alone!", of particular note in this regard are, Manuela F., Lydia R., Lukas d. T. and Rhys W. with their pure and kind honesty one sadly misses in so many people.

Finally I would like to thank the former 5th floor of the red house who showed me that no matter how long you have been gone, there are always people that feel like home.

To all of you, thank you so much, I couldn't have done this without you!

Felix

Contents

Chapter 1

Introduction

1.1 Background and Results

The concept of Fourier-Mukai functors generalizes the idea of a correspondence to the categorical level.

Definition 1.1.1. A functor $f : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ between bounded derived categories of schemes is called Fourier-Mukai if there exists an object $M \in \mathcal{D}^b(Y \times X)$ such that $f \cong \Phi_M := \mathrm{L}\pi_{Y,*}\left(M \overset{L}{\otimes} \mathrm{R}\pi_X^*(\_)\right)$. In this case M is called the Fourier-Mukai kernel.

In particular these functors can be understood geometrically as Φ_M admits a complete charactersiation by $M \in \mathcal{D}^b(Y \times X)$.

It also turns out that most functorial constructions done in algebraic geometry are Fourier-Mukai. This means that understanding the property of being Fourier-Mukai, respectively of not being Fourier-Mukai, is essential for understanding which functors between derived categories of sheaves may arise from geometric constructions and which do not. Another indicator of the geometric nature of Fourier-Mukai functors are the following results by V. Orlov and B. Toën:

Theorem. *[Orl97] Let X and Y be smooth projective schemes. Then every fully faithful exact functor $\Psi_M : \mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ is a Fourier-Mukai functor for some Fourier-Mukai kernel $M \in \mathcal{D}^b(X \times Y)$.*

Theorem. *[Toe07] Let X and Y be smooth projective schemes. Then a functor $\mathcal{D}^b(X) \to \mathcal{D}^b(Y)$ is precisely Fourier-Mukai if it is induced by a dg-functors between the canonical dg-enhancements.*

The above results show that a lot of functors between derived categories of smooth projective schemes are Fourier-Mukai. So Bondal, Larsen and Lunts [BLL04] conjectured nearly 20 years ago that every exact functor between such derived categories admits a description as a Fourier-Mukai functor.

This conjecture was disproven fifteen years later when A. Rizzardo, M. Van den Bergh and A. Neeman [RVdBN19] constructed the first non-Fourier-Mukai functor

$$\Psi_\eta : \mathcal{D}^b(Q_3) \hookrightarrow \mathcal{D}^b\left(\mathbb{P}^4\right),$$

where Q_3 denotes the smooth three dimensional quadric in $\mathbb{P}^4$. Shortly thereafter V. Vologodsky constructed in a note [Vol16] another class of non-Fourier-Mukai functors over a field of characteristic p. However, Vologodsky's functor turns out to be liftable to a $\mathbb{Z}_p$-linear dg-level, whereas the example from [RVdBN19] can be proven to not even have a lift to the spectral level if one works over the rational numbers.

In this work we generalize the result from [RVdBN19] to higher dimensions. In particular we will work over a closed field of characteristic zero in order to show that even in the nicest possible case there is an abundance of non-Fourier-Mukai functors.

We then verify that in the case of a smooth odd dimensional quadric we can apply our result to get a non-Fourier-Mukai functors in arbitrary high dimensions.

Theorem. *Let $Q \hookrightarrow \mathbb{P}^{2k}$ be the embedding of a smooth odd dimensional quadric for $k > 2$. Then we have an exact functor*

$$\Psi_\eta : \mathcal{D}^b(Q) \to \mathcal{D}^b(\mathbb{P}^n)$$

that cannot be Fourier-Mukai.

1.2 Proof strategy

Generally we follow the ideas from [RVdBN19]. In order to conclude that we can construct more non-Fourier-Mukai functors we include an auxiliary results on the kernel of the push forward in Hochschild cohomology. Furthermore we will use more general objects, degrees and indices. We need to do this as the proof in [RVdBN19] is very specialized to the three dimensional quadric and one needs to take care when generalizing their strategy to a more general setting.

Recall that the construction in [RVdBN19] proceeds in two steps:

1. First the authors construct a prototypical non-Fourier-Mukai functor between not necessarily geometric dg-categories.

2. Using behaviour of Hochschild cohomology under embeddings this functor is turned into a geometric functor.

More precisely, in step (1) [RVdBN19] construct a functor

$$L : \mathcal{D}^b(X) \to \mathcal{D}_\infty(\mathcal{X}_\eta)$$

for a smooth scheme X and $\eta \in \mathrm{HH}^{\geq \dim X+3}$, where $\mathcal{D}_\infty(\mathcal{X}_\eta)$ is the derived category of an $\mathcal{A}_\infty$-category arising as infinitesimal deformation in the η-direction.

In step (2) the construction of L is turned into a geometric one. In [RVdBN19] this is achieved by showing that the canonical $\eta \in \mathrm{HH}^{2\dim Q_3}\left(Q_3, \omega_{Q_3}^{\otimes 2}\right)$ is annihilated by the embedding $Q_3 \hookrightarrow \mathbb{P}^4$, which allows the passing from the algebraic world to the geometric world. The authors then define Ψ_η to be L composed with the pushforward into the geometric category $\mathcal{D}^b(\mathbb{P}^4)$.

Although the construction in [RVdBN19] is very general, it has two major drawbacks:

The first is that although L is constructed to be prototypical non-dg it is not obvious that the composition with the pushforward is again non-Fourier-Mukai. One usually handles these complications by applying an inductive obstruction theory that gets unwieldy quickly as one needs to keep track of inductively chosen lifts. Indeed [RVdBN19] only gives a single example of a non-Fourier-Mukai functor although the construction given in step (1) and (2) is very general in nature.

We are able to solve this issue by restricting to Hochschild cohomology classes in degree $\dim(X) + 3$, this leads to the first obstruction vanishing and so we do not need to control the previous lifts in order to conclude that the pushed forward obstruction does not vanish.

The second drawback is that the results in [RVdBN19] rely heavily on the existence of a tilting bundle in order to conclude that the prototypical functor L cannot be dg. Furthermore in [RRVdB19] T. Raedschelders, A. Rizzardo and M. Van den Bergh construct an infinite amount of non-Fourier-Mukai functors using the prototypical L mentioned above. However, to do this they apply a geometrification result by Orlov and hence lose control over the target space. In particular the above mentioned geometrification result relies even more on the existence of a tilting bundle. Although our concrete examples still require the existence of a tilting bundle we study the naturality of the characteristic morphism, which might in future allow results using more general generators. In particular we phrase our main result such that a non-vanishing characteristic morphism suffices, which is guaranteed for tilting objects.

Altogether this PhD improves on the construction from [RVdBN19] to prove the existence of non-Fourier-Mukai functors

$$\Psi_\eta : \mathcal{D}^b(Q) \to \mathcal{D}^b\left(\mathbb{P}^{n+1}\right),$$

for Q a smooth quadric in arbitrary high dimension.

Furthermore one can use our results to calculate the dimensions of choices for constructing candidate non-Fourier-Mukai functors as entries in twisted Hodge diamonds. For instance, if one wants to deform a smooth degree 6 hypersurface $f : X \hookrightarrow \mathbb{P}^{n+1}$ along the Hochschild cohomology of $\mathcal{O}_X(-8)$ in a way that might gives rise to a non-Fourier-Mukai functor, we may pick an η in a 20993-dimensional space:

$$\begin{array}{ccccccccccc}
 & & & & & 0 & & & & & \\
 & & & & 0 & & 0 & & & & \\
 & & & 0 & & 0 & & 0 & & & \\
 & & 0 & & 0 & & 0 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & & 0 & \\
2996 & & 20993 & & 15267 & & 917 & & 0 & & 0 \\
 & 1575 & & 0 & & 0 & & 0 & & 0 & \\
 & & 5775 & & 0 & & 0 & & 0 & & \\
 & & & 10395 & & 0 & & 0 & & & \\
 & & & & 9002 & & 0 & & & & \\
 & & & & & 2996. & & & & &
\end{array}$$

1.3 Structure

1.3.1 Preliminaries: $\mathcal{A}_\infty$-Structures and their Deformations

We start by collecting a few basic notions about $\mathcal{A}_\infty$-structures and their deformations which we will later use to deform a scheme in a non-geometric way. In particular we focus on the definitions of $\mathcal{A}_\infty$- respectively $\mathcal{A}_n$-categories and modules over them. We use these to control lifts of Γ-objects to the level of equivariant sheaves. Furthermore we discuss how one can use Hochschild cohomology to deform small $\Bbbk$-linear categories into an $\mathcal{A}_\infty$-category by introducing a higher composition morphism.

1.3.2 Equivariant Sheaves and the Characteristic Morphism

In this section we define the structure of equivariant objects in a category and discuss that a Fourier-Mukai functor is compatible with equivariant sheaves and base change. We will use this later to produce a contradiction to our functors being Fourier-Mukai.

We then introduce the (geometric) equivariant characteristic morphism and study a categorical incarnation of it. Using this approach we are able to prove that it admits naturality with respect to push forwards, this may eventually be used to weaken the assumption on the existence of tilting bundles.

1.3.3 Preliminaries: Schemes and $\mathbb{k}$-linear Categories

We continue by recalling the construction of W. Lowen and M. Van den Bergh of a $\mathbb{k}$-linear category associated to a compact scheme in order to embed the derived category of sheaves in the derived category of modules over a $\mathbb{k}$-linear category which we can then deform. Furthermore we recall that the essential image of the direct image under the diagonal embedding corresponds under this construction to bimodules. We will later use these constructions to pass from the geometric world to the algebraic one, while still being able to compute the Hochschild cohomology of certain bimodules. This allows us to later deform the constructed $\mathbb{k}$-linear category along a Hochschild cohomology class.

1.3.4 Twisted Hodge Diamonds give Kernels in Hochschild Cohomology

We apply computations of twisted Hodge diamonds in order to compute the Hochschild cohomology of ample sheaves on a degree d hypersurface. We then continue by computing the Hochschild cohomology of the direct image under the embedding of an ample sheaf. We then use these computations, together with a long exact sequence, to compute the kernel in Hochschild cohomology of the embedding. This we will use in order to pass back to the geometric world.

1.3.5 Non-Fourier-Mukai Functors

We follow the construction in [RVdBN19] of non-Fourier-Mukai functors and then apply the results from the previous sections in order to conclude that in the case of a smooth degree d hypersurface we indeed we have plenty of choices for constructing candidate functors.

We then continue by proving that under assumptions regarding the characteristic morphism and mild conditions on some Ext-groups these functors indeed cannot be Fourier-Mukai. In particular we verify, using tilting bundles, that there exist non-Fourier-Mukai functors for smooth quadrics in arbitrary high dimensions.

Notation

Throughout this work we consider $\Bbbk$ to be a closed field of characteristic zero and all schemes, algebras, $\mathcal{A}_\infty$-categories and dg-categories are considered to be over $\Bbbk$. We will assume all $\mathcal{A}_\infty$-structures to be strictly unital and graded cohomologically.

Furthermore the bounded derived category of coherent sheaves over a scheme X will be denoted by $\mathcal{D}^b(X)$ or $\mathcal{D}^b(\mathrm{coh}(X))$ depending on the context, wherever we need to pass to the category $\mathcal{D}^b_{\mathrm{coh}X}(\mathrm{Qch}(X))$ using [Huy06, Proposition 3.5] we will indicate this.

Chapter 2

Preliminaries: $\mathcal{A}_\infty$-Structures and their Deformations

In this chapter we will collect a few basic facts we will need later. In particular we will focus on $\mathcal{A}_\infty$-categories, $\mathcal{A}_\infty$-modules and $\mathcal{A}_\infty$-deformations of $\Bbbk$-linear categories.

2.1 Modules over $\Bbbk$-linear Categories

We start by recalling the definition of modules over a $\Bbbk$-linear category and the relationship between those and the classical notion of modules.

The idea of generalizing the notion of modules over rings to categories first was introduced by B. Mitchell [Mit72]. All in all the idea is that one can interpret a $\Bbbk$-algebra as a $\Bbbk$-linear category with one object and under that interpretation a module corresponds to a functor from the $\Bbbk$-linear category to the category of $\Bbbk$-vector-spaces.

Remark 2.1.1. Recall that a $\Bbbk$-linear category $\mathcal{C}$ is a category such that every morphism space $\mathcal{C}(M, N)$ is a $\Bbbk$-vectorspace and composition defines a $\Bbbk$-linear map $\_ \circ \_ : \mathcal{C}(M', M) \otimes \mathcal{C}(M'', M') \to \mathcal{C}(M'', M)$.

Definition 2.1.2 ([Mit72])**.** Let $\mathcal{X}$ be a small $\Bbbk$-linear category. A $\mathcal{X}$-module is a $\Bbbk$-linear functor

$$\mathcal{M} : \mathcal{X} \to \mathrm{Vect}(\Bbbk).$$

A morphism of $\mathcal{X}$-modules is a natural transformation between two $\mathcal{X}$-modules $\mathcal{N}$ and $\mathcal{M}$:

$$f : \mathcal{N} \to \mathcal{M}.$$

We refer to the category of $\mathcal{X}$-modules $\mathcal{X}-\mathrm{mod}$.

Lemma 2.1.3. *Let $\mathcal{X}$ be a $\Bbbk$-linear category. Then we have that the category $\mathcal{X}-\mathrm{mod}$ is a $\Bbbk$-linear abelian category.*

Proof. By Definition 2.1.2 we have $\mathcal{X}-\mathrm{mod} = \mathrm{Fun}_{\mathbb{k}}(\mathcal{X}, \mathrm{Vect}(\mathbb{k}))$. In particular we have immediately a canonical $\mathbb{k}$-action on the morphism spaces. As kernels and cokernels can be computed objectwise in the target category [Wei94, A.4.3.] we have that $\mathrm{Fun}_{\mathbb{k}}(\mathcal{X}, \mathrm{Vect}(\mathbb{k}))$ is also abelian. In particular we ger that $\mathcal{X}-\mathrm{mod}$ is abelian $\mathbb{k}$-linear. □

Remark 2.1.4. Let $\mathit{\Gamma}$ be a $\mathbb{k}$-algebra. Then we have that a classically defined $\mathit{\Gamma}$-module M consists of a $\mathbb{k}$-vector space V together with a $\mathbb{k}$-algebra morphism $\gamma : \mathit{\Gamma} \to \mathrm{End}(V)$.

On the other hand, if we consider $\mathit{\Gamma}$ to be a $\mathbb{k}$-linear category with one object $*$, then $\mathcal{M}$ consists by Definition 2.1.2 also of a vectorspace $V = \mathcal{M}(*)$ together with a morphism of $\mathbb{k}$-algebras (a map of morphism spaces)

$$\mathit{\Gamma} \to \mathrm{End}(V) = \mathrm{Vect}(\mathbb{k})(\mathcal{M}(*), \mathcal{M}(*)).$$

In particular in this case the two notions of modules over $\mathit{\Gamma}$ coincide.

Similarly the notion of natural transformation captures in this case precisely the commuting with the $\mathit{\Gamma}$ action.

Definition 2.1.5 ([Mit72])**.** Let $\mathcal{X}$ be a $\mathbb{k}$-linear category. We define the derived category of $\mathcal{X}$-modules (respectively bounded, bounded below or bounded above) derived category, to be the derived category (respectively bounded, bounded below or bounded above derived category) of the abelian category $\mathcal{X}-\mathrm{mod}$.

$$\mathcal{D}^{\natural}(\mathcal{X}) := \mathcal{D}^{\natural}(\mathcal{X}-\mathrm{mod}),$$

for $\natural \in \{\_, b, -, +\}$.

As we will later define a $\mathbb{k}$-linear category corresponding to a scheme and then model morphisms of schemes also as functors between $\mathbb{k}$-linear categories we will denote the restriction of scalar functors in the following way:

Definition 2.1.6. Let $f : \mathcal{X} \to \mathcal{Y}$ be a $\mathbb{k}$-linear functor and let $\mathcal{M}$ be a $\mathcal{Y}$-module. Then we define the module $f_*\mathcal{M}$ to be the $\mathcal{X}$-module defined by

$$f_*\mathcal{M} := \mathcal{M} \circ f.$$

Remark 2.1.7. We choose the notation f_* over f^* as we will later model the category of sheaves on a projective scheme by modules over a $\mathbb{k}$-linear category, and under this construction the functor f_* corresponds to the direct image and so the notation turns out to be more consistent and less confusing throughout this work.

Lemma 2.1.8. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a $\Bbbk$-linear functor. Then the assignment $\mathcal{M} \mapsto f_*\mathcal{M}$ defines a left exact functor $f_* : \mathcal{Y}-\mathrm{mod} \mapsto \mathcal{X}-\mathrm{mod}$.*

Proof. As kernels and images are computed on the target category we do not need to worry about left-exactness. It also defines a functor as it is just precomposition with a functor and so it has to be functorial. □

2.2 $\mathcal{A}_\infty$-Structures

Throughout this section we follow [Kel01] and [Sei08], in particular we will use the sign conventions from [Kel01]. Although B. Keller only talks about $\mathcal{A}_\infty$-algebras, the sign conventions can also be applied to $\mathcal{A}_\infty$-categories and are equivalent to the sign conventions in the book by P. Seidel which is considering $\mathcal{A}_\infty$-categories throughout. Furthermore K. Lefèvre-Hasegawa [LH03] covers the case of $\mathcal{A}_\infty$-categories using the same signs as Keller, however we primarily refer to [Sei08] for the category case, as [LH03] is in French.

2.2.1 $\mathcal{A}_\infty$-Categories and their Functors

Since we will repeatedly use dg-categories as examples for $\mathcal{A}_\infty$-categories we recall the definition of a dg-category

Definition 2.2.1. A dg category $\mathcal{C}$ is a category such that we have for all $M, N \in \mathcal{C}$ a chain complex $\mathcal{C}^*(M, N)$, such that the Leibnitz rule holds

$$d(x \circ y) = dx \circ y + x \circ dy.$$

Definition 2.2.2 ([Kel01, 3.1.])**.** Let $n \in \mathbb{N} \cup \{\infty\}$. An $\mathcal{A}_n$-category $\mathcal{X}$ over a field $\Bbbk$ consists of a class of objects $\mathrm{obj}(\mathcal{X})$ and $\mathbb{Z}$-graded $\Bbbk$-vector-spaces as morphism spaces

$$\mathcal{X}(a, b),$$

for $a, b \in \mathrm{obj}(\mathcal{X})$, together with compositions

$$\mathrm{m}_i : \underbrace{\mathcal{X}(a_i, a_{i-1}) \otimes_\Bbbk \mathcal{X}(a_{i-1}, a_{i-2}) \otimes \ldots \otimes \mathcal{X}(a_1, a_0)}_{i} \to \mathcal{X}(a_i, a_0)$$

of degree $2 - i$ for $1 \leq i \leq n$ and $a_0, \ldots, a_i \in \mathrm{obj}(\mathcal{X})$ such that

$$\sum_{r+s+t=k} (-1)^{r+st} \mathrm{m}_u \circ \left(\mathrm{Id}^{\otimes r} \otimes \mathrm{m}_s \otimes \mathrm{Id}^{\otimes t}\right) = 0 \qquad (*_k)$$

holds for all $k \leq n$, where $u = r + 1 + t$.

We will sometimes denote $a \in \mathrm{obj}(\mathcal{X})$ by $a \in \mathcal{X}$ to avoid clumsy notation.

Definition 2.2.3 ([Sei08, (2a)])**.** An $\mathcal{A}_n$-category $\mathcal{X}$ is called unital if every object $a \in \mathrm{obj}(\mathcal{X})$ admits a unit $\mathrm{Id} \in \mathcal{X}(a,a)^0$ such that

$$\begin{aligned}
&\mathrm{m}_1(\mathrm{Id}) = 0 \\
&\mathrm{m}_2(x,\mathrm{Id}) = x = \mathrm{m}_2(\mathrm{Id},x) \\
&\mathrm{m}_i(x_i,...,\mathrm{Id},....,x_1) = 0 && i \neq 2.
\end{aligned}$$

Remark 2.2.4. Observe that the first few incarnations of ($*_k$) give:

$k=1$: In this case ($*_1$) gives

$$\mathrm{m}_1 \circ \mathrm{m}_1 = 0.$$

This means that m_1 defines a differential on $\mathcal{X}(a,b)$.

$k=2$: Here ($*_2$) boils down to

$$\mathrm{m}_1 \circ \mathrm{m}_2 = \mathrm{m}_2(\mathrm{m}_1 \circ \mathrm{Id} + \mathrm{Id} \circ \mathrm{m}_1),$$

which is the Leibnitz rule $d(x \circ y) = dx \circ y + x \circ dy$.

$k=3$: And ($*_3$) gives

$$\begin{aligned}
&\mathrm{m}_2 \circ (\mathrm{Id} \otimes \mathrm{m}_2 - \mathrm{m}_2 \otimes \mathrm{Id}) = \\
&= \mathrm{m}_1 \circ \mathrm{m}_3 + \mathrm{m}_3 \otimes (\mathrm{m}_1 \otimes \mathrm{Id} \otimes \mathrm{Id} + \mathrm{Id} \otimes \mathrm{m}_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes \mathrm{Id} \otimes \mathrm{m}_1),
\end{aligned}$$

which means that m_2 is associative up to a homotopy given by m_3. More generally one can think of an $\mathcal{A}_n$-category as a category that is homotopy-associative up to degree n.

By Definition 2.2.2 every $\mathcal{A}_n$ category defines an $\mathcal{A}_m$-category for all $m \leq n$ just by forgetting the higher actions.

Definition 2.2.5 ([Sei08, (1a)])**.** Let $\mathcal{X}$ be an $\mathcal{A}_n$-category for $n \geq 3$. Then the category $\mathrm{H}^*(\mathcal{X})$ is the graded $\mathbb{k}$-linear category consisting of the same objects as $\mathcal{X}$ and morphism spaces

$$\mathrm{H}^*(\mathcal{X})(a,b) := \mathrm{H}^*(\mathcal{X}(a,b)).$$

Where we use Remark 2.2.4 to consider $\mathcal{X}(a,b)$ as a chain complex with differential m_1.

The $\Bbbk$-linear category $\mathrm{H}^0(\mathcal{X})$ is the category with the same objects as $\mathcal{X}$ and morphism spaces

$$\mathrm{H}^0(\mathcal{X})(a,b) := \mathrm{H}^0(\mathcal{X}(a,b)).$$

We have by Remark 2.2.4 that $\mathrm{H}^*(\mathcal{X})$ defines a graded $\Bbbk$-linear category and $\mathrm{H}^0(\mathcal{X})$ defines an ordinary $\Bbbk$-linear category.

Definition 2.2.6 ([Sei08, (2a)])**.** An $\mathcal{A}_n$-category is called homologically unital if $\mathrm{H}^0(\mathcal{X})$ admits a unit morphism $\mathrm{Id} \in \mathrm{H}^0(\mathcal{X})(a,a)$ for all $a \in \mathrm{obj}(\mathcal{X})$.

Definition 2.2.7. An $\mathcal{A}_n$-category is called small if its objects form a set. It is called essentially small if the isomorphism classes of objects form a set.

Definition 2.2.8 ([Kel01, 3.1.])**.** An $\mathcal{A}_n$-category is an $\mathcal{A}_n$-algebra if $\mathrm{obj}(\mathcal{X})$ consists of only one object for $n \in \mathbb{N} \cup \{\infty\}$.

Example 2.2.9. There are a few obvious examples of $\mathcal{A}_\infty$-categories:

- Let $\mathcal{X}$ be a $\Bbbk$-linear category, then it is an $\mathcal{A}_\infty$ category via

$$\mathrm{m}_i = \begin{cases} (\_) \circ (\_) & i = 2 \\ 0 & i \neq 2. \end{cases}$$

- More generally, let $\mathcal{X}$ be a dg-category, then $\mathcal{X}$ is an $\mathcal{A}_\infty$-category with

$$\mathrm{m}_i = \begin{cases} d & i = 1 \\ (\_) \circ (\_) & i = 2 \\ 0 & i \notin \{1,2\}. \end{cases}$$

Definition 2.2.10 ([Kel01, 3.4.])**.** An $\mathcal{A}_n$-functor between two $\mathcal{A}_n$-categories $f : \mathcal{X} \to \mathcal{Y}$ is given by a map on objects

$$f : \mathrm{obj}(\mathcal{X}) \to \mathrm{obj}(\mathcal{Y})$$

and a set of morphisms

$$\{f_i : \mathcal{X}(a_i, a_{i-1}) \otimes \mathcal{X}(a_{i-1}, a_{i-2}) \otimes \ldots \otimes \mathcal{X}(a_1, a_2) \to \mathcal{Y}(f(a_i), f(a_0))\}$$

of degree $1-i$ for every $i \leq n$ and $a_i, \ldots, a_0 \in \mathrm{obj}(\mathcal{X})$ such that

$$\sum_{r+s+t=k} (-1)^{r+st} f_u \left(\mathrm{Id}^{\otimes r} \otimes \mathrm{m}_s \otimes \mathrm{Id}^{\otimes t}\right) = \sum_{\substack{1 \leq l \leq n \\ k = i_1 + \ldots + i_l}} (-1)^m \mathrm{m}_r (f_{i_1} \otimes f_{i_2} \otimes \ldots \otimes f_{i_l}) \qquad (**_k)$$

holds, where $u = r + 1 + t$ and

$$m = (l-1)(i_1 - 1) + (l-2)(i_2 - 1) + \dots + 2(i_{l-2} - 1) + (i_{l-1} - 1).$$

Remark 2.2.11. Again we compute the first few incarnations of ($**_k$):

$k = 1$: In this case we have

$$f_1 \circ \mathrm{m}_1 = m_1 \circ f_1,$$

in particular f_1 defines a morphism of chain complexes.

$k = 2$: Here we get

$$f_1 \circ \mathrm{m}_2 = \mathrm{m}_2 \circ (f_1 \otimes f_1) + \mathrm{m}_1 \circ f_2 + f_2 (\mathrm{m}_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes \mathrm{m}_1),$$

so f_1 commutes with m_2 up to a homotopy given by f_2.

More generally one can think of an $\mathcal{A}_n$-morphism f as commuting with the $\mathcal{A}_n$-structure up to higher homotopies, whose information f includes in form of the higher f_i.

Definition 2.2.12 ([Kel01, 3.1.])**.** An $\mathcal{A}_n$-functor between two unital $\mathcal{A}_n$-algebras is called an $\mathcal{A}_n$-morphism for $n \in \mathbb{N} \cup \{\infty\}$.

Definition 2.2.13 ([Kel01, 3.1.])**.** An $\mathcal{A}_\infty$-functor $f : \mathcal{A} \to \mathcal{B}$ is a quasi-equivalence if

$$f : \mathrm{obj}(\mathcal{A}) /_{\cong} \to \mathrm{obj}(\mathcal{B}) /_{\cong}$$

is surjective and all f_1 induce isomorphisms on cohomology

$$\mathrm{H}^*(f_1) : \mathrm{H}^*(\mathcal{A}(a, a')) \xrightarrow{\sim} \mathrm{H}^*(\mathcal{B}(fa, fa')).$$

Proposition 2.2.14 ([LH03, Proposition 3.2.1])**.** *Every homologically unital $\mathcal{A}_\infty$-category is quasi-equivalent to an unital one.*

Definition 2.2.15 ([Kel01, 3.4.])**.** A quasi-equivalence between two $\mathcal{A}_\infty$-algebras is called a quasi-isomorphism.

Theorem 2.2.16 ([Kad80])**.** *Let $\mathcal{X}$ be an $\mathcal{A}_\infty$-category. Then the cohomology $\mathrm{H}^*(\mathcal{X})$ has an $\mathcal{A}_\infty$-category structure such that*

- $\mathrm{m}_1 = 0$
- *there is a quasi-equivalence $\mathrm{H}^*\mathcal{X} \xrightarrow{\sim} \mathcal{X}$ lifting the identity on $\mathrm{H}^*\mathcal{X}$.*

Moreover, this structure is unique up to (non-unique) isomorphism of $\mathcal{A}_\infty$-categories.

Remark 2.2.17. From now on we will assume that the cohomology $\mathrm{H}^*(\mathcal{X})$ of an $\mathcal{A}_\infty$-category is equipped with the $\mathcal{A}_\infty$-structure arising by Theorem 2.2.16 instead of just regarding it as a graded category interpreted as an $\mathcal{A}_\infty$-category. The $\mathcal{A}_\infty$-category constructed in Theorem 2.2.16 is also referred to as the minimal $\mathcal{A}_\infty$-model of $\mathcal{X}$.

2.2.2 $\mathcal{A}_\infty$-Modules and their Functors

Definition 2.2.18 ([Kel01, 4.2.])**.** Let $\mathcal{X}$ be a small $\mathcal{A}_n$-category for $n \in \mathbb{N} \cup \{\infty\}$. An $\mathcal{A}_n$-module over $\mathcal{X}$ consists of a $\mathbb{Z}$-graded space

$$\mathcal{M}(a, b)$$

for every pair of objects $a, b \in \mathrm{obj}\mathcal{X}$ and higher composition morphisms

$$\mathrm{m}_i : \underbrace{\mathcal{M}(a_i, a_{i-1}) \otimes \mathcal{X}(a_{i-1}, a_{i-2}) \otimes \ldots \otimes \mathcal{X}(a_1, a_0)}_{i} \to \mathcal{M}(a_i, a_0)$$

of degree $2 - i$ such that the following equation holds

$$\sum_{r+s+t=k} (-1)^{r+st}\, \mathrm{m}_u \circ \left(\mathrm{Id}^{\otimes r} \otimes \mathrm{m}_s \otimes \mathrm{Id}^{\otimes t}\right) = 0, \qquad (**_k)$$

where depending on the input m_i needs to be considered as the ith higher composition morphism of $\mathcal{X}$ or $\mathcal{M}$.

Remark 2.2.19. We again compute a few incarnations of $(**_k)$ to give some intuition on the modelled structure.

$k = 1$: In this case we get

$$\mathrm{m}_1^{\mathcal{M}} \circ \mathrm{m}_1^{\mathcal{M}} = 0.$$

So m_1 defines a differential.

$k = 2$: Here we get

$$\mathrm{m}_1^{\mathcal{M}} \circ \mathrm{m}_2^{\mathcal{M}} = \mathrm{m}_2^{\mathcal{M}} \circ \left(\mathrm{m}_1^{\mathcal{M}} \otimes \mathrm{Id}_{\mathcal{M}} + \mathrm{Id}_{\mathcal{M}} \otimes \mathrm{m}_1^{\mathcal{A}}\right),$$

which means that m_2 suffices the Leibnitz rule.

$k = 3$: For this we get similar to the $\mathcal{A}_\infty$-algebra case that the action of $\mathcal{M}$ induced by m_2 is associative up to a homotopy, which is given by m_3.

So one can think about an $\mathcal{A}_\infty$-module as a homotopy coherent module over $\mathcal{X}$.

Example 2.2.20. We collect once more the standard examples.

- Let $\mathcal{M}$ be a graded module over a $\Bbbk$-linear category $\mathcal{X}$, then it is an $\mathcal{A}_\infty$-module over $\mathcal{X}$ via

$$\mathrm{m}_i = \begin{cases} (\_) \circ (\_) & i = 2 \\ 0 & i \neq 2. \end{cases}$$

- Let $\mathcal{M}$ be a dg-module over a dg-algebra $\mathcal{X}$. Then it defines an $\mathcal{A}_\infty$-module over $\mathcal{X}$ via

$$\mathrm{m}_i = \begin{cases} d_{\mathcal{M}} & i = 1 \\ (\_) \circ (\_) & i = 2 \\ 0 & i \notin \{1, 2\}. \end{cases}$$

Definition 2.2.21 ([Kel01, 4.2.]). Let $\mathcal{M}, \mathcal{N}$ be $\mathcal{A}_n$-modules over an $\mathcal{A}_n$-category $\mathcal{X}$ for $n \in \mathbb{N} \cup \{\infty\}$. A morphism of $\mathcal{A}_n$-modules consists of a set of morphisms:

$$f_i : \underbrace{\mathcal{M}(a_i, a_{i-1}) \otimes \mathcal{X}(a_{i-1}, a_{i-2}) \otimes \ldots \otimes \mathcal{X}(a_1, a_0)}_{i} \to \mathcal{N}(a_i, a_0)$$

of degree $1 - i$ for $i \leq n$, such that we have for every $k < n$

$$\sum_{r+s+t} (-1)^{r+st} f_u \circ \left(\mathrm{Id}^{\otimes r} \otimes \mathrm{m}_s \otimes \mathrm{Id}^{\otimes t}\right) = \sum_{n=r+s} (-1)^{(r-1)s} \mathrm{m}_{u'} \left(f_r \otimes \mathrm{Id}^s\right) \qquad (**_k)$$

where $u = r + s + t$ and $u' = 1 + s$.

Example 2.2.22. We compute again $(**_k)$ for small k:

$k = 1$: Similar to the cases above $(**_1)$ boils down to

$$f_1 \circ \mathrm{m}_1 = \mathrm{m}_1 \circ f_1,$$

which means that f_1 defines a morphism of chain complexes.

$k = 2$: Here we get

$$f_1 \circ \mathrm{m}_2 - f_2 \circ (\mathrm{m}_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes \mathrm{m}_1) = \mathrm{m}_2 \circ (f_1 \otimes \mathrm{Id}_{\mathcal{X}}) + \mathrm{m}_1 \circ f_2.$$

This means that similarly to the case of an $\mathcal{A}_n$-functor between $\mathcal{A}_n$-categories the equation $(**_2)$ encodes that f_1 is compatible with the action induced by m_2 up to a homotopy given by f_2.

These examples are another reason one can think about $\mathcal{A}_\infty$-structure as a notion for inductive homotopy coherent algebraic structures.

Definition 2.2.23 ([Kel01, 4.2.]). An $\mathcal{A}_n$-morphism $f : \mathcal{M} \to \mathcal{N}$ is a quasi-isomorphism if it induces an isomorphism on cohomology

$$\mathrm{H}^*(f) : \mathrm{H}^*\mathcal{M} \xrightarrow{\sim} \mathrm{H}^*\mathcal{N}.$$

Definition 2.2.24 ([Kel01, 4.2.]). Let $f : \mathcal{M} \to \mathcal{M}'$ and $g : \mathcal{M}' \to \mathcal{M}''$ be morphisms of $\mathcal{A}_\infty$-modules over a homologically unital $\mathcal{A}_\infty$-algebra $\mathcal{X}$. Then the composition $f \circ g : \mathcal{M} \to \mathcal{M}''$ is given by

$$(f \circ g)_n = \sum_{n=r+s} (-1)^{(r-1)s} f_u \left(g_r \otimes \mathrm{Id}^{\otimes s}\right),$$

where we put $u = 1 - s$.

Definition 2.2.25 ([Kel01, 4.2.]). Let $\mathcal{X}$ be a homologically unital $\mathcal{A}_\infty$-algebra, then we define the category of $\mathcal{A}_\infty$-modules $\mathcal{C}_\infty(\mathcal{X})$ to be the category consisting of $\mathcal{A}_\infty$-modules and morphisms given by $\mathcal{A}_\infty$-morphisms.

Remark 2.2.26. The identity of an object in $\mathcal{C}_\infty(\mathcal{X})$ is given by

$$\mathrm{Id} = (\mathrm{Id}, 0, \ldots).$$

Definition 2.2.27 ([Sei08, 1k]). Let $f : \mathcal{X} \to \mathcal{Y}$ be an $\mathcal{A}_i$-functor. Then the functor

$$f_* : \mathcal{C}_\infty(\mathcal{Y}) \to \mathcal{C}_\infty(\mathcal{X})$$

is given on modules by

$$f_*\mathcal{M}(a) := \mathcal{M}(f(a))$$

for objects $a \in \mathrm{obj}(\mathcal{X})$. Higher compositions are given by

$$\mathrm{m}_k(m, x_{k-1}, \ldots, x_1) = \sum_{l<k} \sum_{s_1,\ldots,s_l} \mathrm{m}_l(m, f_{s_l}(x_{k-1}, \ldots, x_{k-s_l}), \ldots, f_{s_1}(a_{s_1}, \ldots, a_1)).$$

On morphisms f^* is given by

$$f_*\varphi_k(m, x_{k-1}, \ldots, x_1) = \sum_{l<k} \sum_{s_1,\ldots,s_l} \varphi_l(m, f_{s_l}(x_{k-1}, \ldots, x_{k-s_l}), \ldots, f_{s_1}(a_{s_1}, \ldots, a_1)).$$

Remark 2.2.28. We again choose the notation f_* over f^* as we will later model the category of sheaves on a projective scheme by modules over a $\Bbbk$-linear category, and under this construction the functor f_* corresponds to the direct image and so the notation turns out to be more consistent and less confusing throughout this work.

Definition 2.2.29 ([Kel01, 4.2.])**.** Let $\mathcal{X}$ be a homologically unital small $\mathcal{A}_\infty$-category. Then we define the category

$$\mathcal{D}_\infty(\mathcal{X}) := \mathcal{C}_\infty(\mathcal{X})\left[\{\mathcal{A}_\infty - \text{quasi-isomorphism}\}^{-1}\right].$$

Remark 2.2.30 ([Kel01, 4.2.])**.** More generally one could consider $\mathcal{A}_\infty$-categories over commutative rings instead of a field $\Bbbk$. In this case we would have to distinguish between the derived category of $\mathcal{A}_\infty$-modules, as we defined it, and the category of $\mathcal{A}_\infty$-modules up to homotopy. However, over a field one can prove that actually every quasi-isomorphism of $\mathcal{A}_\infty$-modules is a homotopy equivalence and vice versa. In particular in this case the naively derived category arising by formally inverting quasi-isomorphisms and the category of $\mathcal{A}_\infty$-modules up to homotopy coincide.

The interpretation of $\mathcal{D}_\infty(\mathcal{X})$ as arising via $\mathcal{A}_\infty$-modules up to homotopy immediately gives that $\mathcal{D}_\infty(\mathcal{X})$ is well-defined and there are no set-theoretic issues arising.

2.3 $\mathcal{A}_\infty$-Categories and their Deformations

2.3.1 Deformations

In this section we will recall some facts about $\mathcal{A}_\infty$-categories and their deformations along a Hochschild cohomology class.

Definition 2.3.1 ([DVM96])**.** Let $\mathcal{X}$ be a $\Bbbk$-linear category. Then we call a $\mathcal{X}$-bimodule $\Bbbk$-central if the $\Bbbk$-action induced by the left $\mathcal{X}$-action is the same as the $\Bbbk$-action induced by the right $\mathcal{X}$-action.

Definition 2.3.2 ([?])**.** Let $\mathcal{X}$ be a $\Bbbk$-linear category and $\mathcal{M}$ a $\Bbbk$-central $\mathcal{X}$-bimodule. Then we denote by

$$\mathrm{C}^n(\mathcal{X},\mathcal{M}) := \mathrm{Hom}_{\mathcal{X}\otimes_\Bbbk\mathcal{X}^{\mathrm{op}}}\Big(\underbrace{\mathcal{X}\otimes_\Bbbk \dots \otimes_\Bbbk \mathcal{X}}_{n}, \mathcal{M}\Big)$$

$$\begin{aligned} d_{\mathrm{HH}} : \mathrm{C}^n(\mathcal{X}, M) &\to \mathrm{C}^{n+1}(\mathcal{X},\mathcal{M}) \\ f : \left(\mathcal{X}^{\otimes n}\right) \to M &\mapsto x_1 f(x_2\otimes \dots \otimes x_{n+1}) \\ &\quad + \sum_{i=1}^{n} (-1)^i f(x_1\otimes\dots\otimes x_i x_{i+1}\otimes\dots\otimes x_n) \\ &\quad + (-1)^{n+1} f(x_1\otimes\dots\otimes x_n)\, x_{n+1} \end{aligned}$$

the Hochschild complex.

A Hochschild cochain in degree ≥ 2 is called normalized if we have

$$f\left(x_1 \otimes ... \otimes \mathrm{Id} \otimes ... \otimes x_n\right) = 0.$$

We denote the subcomplex of normalized Hochschild cochains by $\overline{\mathrm{C}}^*\left(\mathcal{X}, \mathcal{M}\right)$.

The Hochschild cohomology $\mathrm{HH}^*\left(\mathcal{X}, \mathcal{M}\right)$ is the cohomology of the Hochschild complex.

Proposition 2.3.3 ([Lod98, §1.5.7])**.** *The inclusion* $\overline{\mathrm{C}}^*\left(\mathcal{X}, \mathcal{M}\right) \hookrightarrow \mathrm{C}^*\left(\mathcal{X}, \mathcal{M}\right)$ *is a quasi-isomorphism.*

Now we may define the deformation of a $\Bbbk$-linear category by a normalized Hochschild cocycle:

Definition 2.3.4 ([RVdB14, 6.1])**.** Let $\mathcal{X}$ be a $\Bbbk$-linear category, $\mathcal{M}$ a $\Bbbk$-central bimodule and $\eta \in \mathrm{C}^n\left(\mathcal{X}, \mathcal{M}\right)$ with $n > 2$. Then we define $\mathcal{X}_\eta$ as the $\mathcal{A}_\infty$-category consisting of the same objects as $\mathcal{X}$, morphism spaces given by

$$\mathcal{X}_\eta\left(a, b\right) := \mathcal{X}\left(a, b\right) \oplus \mathcal{M}[n-2]\left(a, b\right)$$

and higher compositions given by:

$$\begin{aligned} \mathrm{m}_2\left(\left(x, m\right), \left(y, n\right)\right) &:= \left(\mathrm{m}_2\left(x, y\right), \mathrm{m}_2\left(x, n\right) + \mathrm{m}_2\left(m, y\right)\right) \\ \mathrm{m}_n\left(\left(x_1, m_1\right), ..., \left(x_n, m_n\right)\right) &:= \left(0, \eta\left(x_1, ..., x_n\right)\right) \\ \mathrm{m}_i &:= 0 \quad i \notin \{2, n\} \end{aligned}$$

This construction comes together with a canonical $\mathcal{A}_\infty$-functor

$$\pi : \mathcal{X}_\eta \to \mathcal{X} \tag{2.3.1}$$

acting via the identity on objects and given on morphism spaces by

$$\pi : \mathcal{X}_\eta\left(a, b\right) = \mathcal{X}\left(a, b\right) \oplus \mathcal{M}[n-2] \xrightarrow{(\mathrm{Id}, 0)} \mathcal{X}\left(a, b\right).$$

The following Theorem 2.3.5 shows that this construction indeed defines an $\mathcal{A}_\infty$-structure:

Theorem 2.3.5. *Let* $\mathcal{X}$ *be a* $\Bbbk$*-linear category,* $\mathcal{M}$ *a* $\Bbbk$*-central* $\mathcal{X}$*-bimodule and* *let $n \geq 3$. Then there is a bijective identification of the form*

$$\begin{aligned}\varphi : \mathrm{HH}^n(\mathcal{X}, \mathcal{M}) &\to \left\{\begin{array}{c}\text{Isomorphism classes of } \mathcal{A}_\infty\text{-structures on} \\ \mathcal{X} \oplus \mathcal{M}[n-2] \text{ such that } \mathrm{m}_i = 0 \text{ for } i \notin \{2, n\} \\ \text{and } \mathrm{m}_2\left((x,m),(y,n)\right) = (xy, xn + my)\end{array}\right\} \\ \overline{\eta} &\mapsto \overline{\mathcal{X}_\eta},\end{aligned}$$

where we denote by $\overline{\mathcal{X}_\eta}$ the isomorphism class of $\mathcal{X}_\eta$ and by $\overline{\eta}$ the cohomology class of a representative η.

Proof. First we prove that the map $\widehat{\varphi} : \ker(d_{\mathrm{HH}^n}) \ni \eta \mapsto \mathcal{X}_\eta$ is well-defined and surjective, and then we will show that the kernel is $\mathrm{im}(d_{\mathrm{HH}})$.

So consider $\eta \in \ker(d_{\mathrm{HH}^n})$. For this to induce an $\mathcal{A}_\infty$-stucture via $\mathrm{m}_n := \eta$ and

$$\mathrm{m}_2(x,m) \otimes (x', m') = (xx', xm' + mx')$$

we need to show that the equation ($*_k$)

$$\sum (-1)^{ji+l} \mathrm{m}_c \circ (\mathrm{Id}^{\otimes j} \otimes \mathrm{m}_i \otimes \mathrm{Id}^{\otimes l}) = 0$$

holds for all $j + i + l = k$ with $c = j + 1 + l$. Since only m_2 and m_n are non trivial we can focus on $c, i = n$ and $c, i = 2$ while ($*_{2n}$) immediately holds as either the source or target space has to vanish.

So consider first $j + i + l = 4$, in this case the equation boils down to

$$\mathrm{m}_2(\mathrm{Id} \otimes \mathrm{m}_2) - \mathrm{m}_2(\mathrm{m}_2 \otimes \mathrm{Id}) = 0.$$

This is the associator, since evaluation at $(x,m) \otimes (x',m') \otimes (x'',m'')$ yields

$$\begin{aligned}&(x,m)((x',m')(x'',m'')) - ((x,m)(x',m'))(x'',m'') = \\ &= (x,m)(x'x'', x'm'' + m'x'') - (xx', xm' + mx')(x'',m'') \\ &= (xx'x'', xx'm'' + xm'x'' + mx'x'') - (xx'x'', xx'm'' + xm'x'' + mx'x'') \\ &= 0.\end{aligned}$$

In the case of $j + i + l = n + 2$ the equation ($*_k$) is given by:

$$\mathrm{m}_2(\mathrm{Id} \otimes m_n) + \sum_{i=0}^{n-1} (-1)^i \mathrm{m}_n(\mathrm{Id}^{\otimes i} \otimes \mathrm{m}_2 \otimes \mathrm{Id}^{\otimes n-i-1}) + (-1)^n \mathrm{m}_2(\mathrm{m}_n \otimes \mathrm{Id}) = 0.$$

Observing that m_2 corresponds to multiplication and substituting m_n with η

yields:

$$x_0\eta(x_1,...,x_n) + \sum_{i=0}^{n-1}(-1)^i\eta(x_0,...,x_ix_{i+1},...,x_n) + (-1)^n\eta(x_0,...,x_{n-1})x_n = 0,$$

which is literally $d_{\mathrm{HH}}(\eta) = 0$.

To prove surjectivity let $(\mathcal{X} \oplus \mathcal{M}[n-2], \mathrm{m}_i)$ be an $\mathcal{A}_\infty$-algebra such that m_i is only nontrivial for $i = 2, n$ and m_2 is as above. Then we get by the defining equations of $\mathcal{A}_\infty$-algebras

$$\mathrm{m}_2(\mathrm{Id} \otimes m_n) + \sum_{i=0}^{n-1}(-1)^i\mathrm{m}_n(\mathrm{Id}^{\otimes k} \otimes \mathrm{m}_2 \otimes \mathrm{Id}^{\otimes n-k-1}) + (-1)^n\mathrm{m}_2(\mathrm{m}_n \otimes \mathrm{Id}) = 0.$$

In particular we may consider $\mathrm{m}_n \in \mathrm{C}^k(\mathcal{X}^{\otimes n}, \mathcal{M})$ and the above equation implies $d_{\mathrm{HH}}(\mathrm{m}_n) = 0$ and hence $\widehat{\varphi}$ is surjective.

Now consider two isomorphic $\mathcal{A}_\infty$-categories of the desired form

$$(\mathcal{X} \oplus \mathcal{M}[n-2], \mathrm{m}_i), (\mathcal{X} \oplus \mathcal{M}[n'-2], \mathrm{m}'_i).$$

In particular we have an $\mathcal{A}_\infty$-equivalence

$$f : (\mathcal{X} \oplus \mathcal{M}[n-2], \mathrm{m}_i) \to (\mathcal{X} \oplus \mathcal{M}[n'-2], \mathrm{m}'_i).$$

In order for f to be an isomorphism f_1 has to be an isomorphism of chain complexes over $\Bbbk$ since there exists some f^{-1} such that

$$(f \circ f^{-1})_1 = f_1 \circ f_1^{-1} = \mathrm{Id} = f_1^{-1} \circ f_1 = (f^{-1} \circ f)_1.$$

In particular we get $n = n'$.

Let $g : (\mathcal{X} \oplus \mathcal{M}[n-2], \mathrm{m}_i) \to (\mathcal{X} \oplus \mathcal{M}[n-2], \mathrm{m}''_i)$ denote the A_∞-isomorphism such that $g_1 = f_1^{-1}$ and $g_i = 0$ for $i > 1$ and $\mathrm{m}''_n := f_1 \circ \mathrm{m}_n$ and $\mathrm{m}''_i = \mathrm{m}_i$ for $i \neq n$. We may interpret $g \circ f$ as a morphism with $(g \circ f)_1 = \mathrm{Id}$ and $\mathrm{m}'_2 = \mathrm{m}_2$. Where the latter equality holds since by the defining property of $\mathcal{A}_\infty$-functors we have:

$$\begin{aligned} \mathrm{m}'_2(f_1, f_1) &= f_1(m_2) + \mathrm{m}'_1(f_2) + f_2(\mathrm{m}_1 \otimes \mathrm{Id} + \mathrm{Id} \otimes \mathrm{m}_1) \\ &= f_1 m_2 \qquad\qquad \mathrm{m}_1 = 0 = \mathrm{m}'_1. \end{aligned}$$

Now $g \circ f$ is an $\mathcal{A}_\infty$-equivalence. Therefore the following equation has to hold:

$$\begin{aligned}
&(g \circ f)_1 (\mathrm{m}_n) + \sum_{l=0}^{n-2} (-1)^l ((g \circ f)_{n-1} (\mathrm{Id}^{n-2-l} \otimes \mathrm{m}_2 \otimes \mathrm{Id}^{\otimes l}) = \\
&= \mathrm{m}'_n(((g \circ f)_1 \otimes ... \otimes ((g \circ f)_1) \\
&\quad + \mathrm{m}'_2((g \circ f)_{n-1} \otimes ((g \circ f)_1) \\
&\quad + (-1)^n \mathrm{m}'_2(((g \circ f)_1 \otimes ((g \circ f)_{n-1}).
\end{aligned}$$

This gives after rearranging and plugging in $(g \circ f)_1 = \mathrm{Id}$,

$$\begin{aligned}
&\mathrm{m}_n - \mathrm{m}'_n \\
&= (-1)^n \left(\mathrm{m}_2(\mathrm{Id} \otimes f_{n-1}) + (-1)^n \mathrm{m}_2(f_{n-1} \otimes \mathrm{Id}) + \sum_{j=1}^{n-2} (-1)^j (\mathrm{Id}^{j-1} \otimes \mathrm{m}_2 \otimes \mathrm{Id}^{n-j=2} \right) \\
&= (-1)^n d_{\mathrm{HH}^{n-1}}(f_{n-1}).
\end{aligned}$$

In particular the kernel of $\widehat{\varphi}$ is precisely the module of Hochschild coboundaries. □

The following Corollaries allow us to work with $\mathcal{X}_{\bar{\eta}}$ for $\bar{\eta}$ a Hochschild cohomology class and to assume that $\mathcal{X}_\eta$ is strictly unital.

Corollary 2.3.6. *The equivalence class of $\mathcal{X}_\eta$ depends only on $\bar{\eta} \in \mathrm{HH}^n(\mathcal{X}, \mathcal{M})$.*

Proof. This is just the fact that the bijection in Theorem 2.3.5 is well-defined. □

Corollary 2.3.7. *Up to quasi-isomorphism we may assume $\mathcal{X}_\eta$ to be unital.*

Proof. By the construction of $\mathcal{X}_\eta$ it suffices for η to be a normalized Hochschild cochain. By Corollary 2.3.6 we may choose a different representative in the same cohomology class to get an equivalent $\mathcal{A}_\infty$-category and by Proposition 2.3.3 we can choose a representative η' that is normalized. □

2.3.2 Tensoring with a $\Bbbk$-Algebra

Definition 2.3.8 ([RVdBN19, 6.2])**.** Let $\mathcal{X}$ be a $\Bbbk$-linear category, $\mathcal{M}$ a $\Bbbk$-central bimodule and Γ a $\Bbbk$-algebra. Then we define the morphism

$$\begin{aligned}
\mathrm{C}^*(\mathcal{X}, M) &\to \mathrm{C}^*(\mathcal{X} \otimes \Gamma, \mathcal{M} \otimes \Gamma) \\
\eta &\mapsto \eta \cup \mathrm{Id}
\end{aligned}$$

to send a Hochschild cochain η in degree n to the degree n morphism

$$\begin{gathered}(\mathcal{X} \otimes \Gamma)^{\otimes n} \to (\mathcal{M} \otimes \Gamma) \\ (x_1 \otimes \gamma_1) \otimes ... \otimes (x_n \otimes \gamma_n) \mapsto \eta(x_1 \otimes ... \otimes x_n) \otimes (\gamma_1 ... \gamma_n) .\end{gathered}$$

Proposition 2.3.9 ([RVdBN19, 6.2])**.** *The morphism $\eta \mapsto \eta \cup \mathrm{Id}$ induces a morphism*

$$\mathrm{HH}^*(\mathcal{X}, \mathcal{M}) \to \mathrm{HH}^*(\mathcal{X} \otimes \Gamma, \mathcal{M} \otimes \Gamma) .$$

Proof. We only need to check that $\eta \cup \mathrm{Id}$ is compatible with the Hochschild differential. For this observe the following:

$$\begin{aligned}
&d_{\mathrm{HH}}(\eta \cup \mathrm{Id})((x_1 \otimes \gamma_1) \otimes ... \otimes (x_{n+1} \otimes \gamma_{n+1})) \\
&\quad = (x_1 \otimes \gamma_1)(\eta \cup \mathrm{Id})((x_2 \otimes \gamma_2) \otimes ... (x_{n+1} \otimes \gamma_{n+1})) \\
&\quad + \sum_{i=1}^{n} (-1)^i (\eta \cup \mathrm{Id})((x_1 \otimes \gamma_1) \otimes ... (x_i \otimes \gamma_i)(x_{i+1} \otimes \gamma_{i+1}) ... \otimes (x_{n+1} \otimes \gamma_{n+1})) \\
&\quad + (-1)^{n+1} (\eta \cup \mathrm{Id})((x_1 \otimes \gamma_1) \otimes ... \otimes (x_n \otimes \gamma_n))(x_{n+1} \otimes \gamma_{n+1}) \\
&= (x_1 \otimes \gamma_1)(\eta \cup \mathrm{Id})((x_2 \otimes \gamma_2) \otimes ... \otimes (x_{n+1} \otimes \gamma_{n+1})) \\
&\quad + \sum_{i=1}^{n} (-1)^i (\eta \cup \mathrm{Id})((x_1 \otimes \gamma_1) \otimes ... (x_i x_{i+1} \otimes \gamma_i \gamma_{i+1}) ... \otimes (x_{n+1} \otimes \gamma_{n+1})) \\
&\quad + (-1)^{n+1} (\eta \cup \mathrm{Id})((x_1 \otimes \gamma_1) \otimes ... \otimes (x_n \otimes \gamma_n))(x_{n+1} \otimes \gamma_{n+1}) \\
&= (x_1 \otimes \gamma_1)(\eta(x_2 \otimes ... \otimes x_{n+1}) \otimes \gamma_2 ... \gamma_{n+1}) \\
&\quad + \sum_{i=1}^{n} (-1)^i \eta(x_1 \otimes ... \otimes x_i x_{i+1} \otimes ... \otimes x_{n+1}) \otimes \gamma_1 ... \gamma_{n+1} \\
&\quad + (-1)^{n+1} (\eta(x_1 \otimes ... \otimes x_n) \otimes \gamma_1 ... \gamma_n)(x_{n+1} \otimes \gamma_{n+1}) \\
&= (x_1 \eta(x_2 \otimes ... \otimes x_{n+1}) \otimes \gamma_1 \gamma_2 ... \gamma_{n+1}) \\
&\quad + \sum_{i=1}^{n} (-1)^i \eta(x_1 \otimes ... \otimes x_i x_{i+1} \otimes ... \otimes x_{n+1}) \otimes \gamma_1 ... \gamma_{n+1} \\
&\quad + (-1)^{n+1} (\eta(x_1 \otimes ... \otimes x_n) x_{n+1} \otimes \gamma_1 ... \gamma_n \gamma_{n+1}) \\
&= d_{\mathrm{HH}}(\eta)(x_1 \otimes ... \otimes x_{n+1}) \otimes \gamma_1 ... \gamma_{n+1} .
\end{aligned}$$

So the morphism is compatible with the differential and induces a morphism on cohomology, which is the Hochschild cohomology. □

Definition 2.3.10 ([RVdBN19, 6.2])**.** Let $\mathcal{X}$ be an $\mathcal{A}_\infty$-category and let Γ be a $\Bbbk$-algebra. Then we define the $\mathcal{A}_\infty$-category $\mathcal{X} \otimes_{\Bbbk} \Gamma$ to consist of objects

$$\mathrm{obj}(\mathcal{X} \otimes \Gamma) := \mathrm{obj}(\mathcal{X}) ,$$

and morphisms

$$\mathcal{X} \otimes_{\Bbbk} \mathit{\Gamma}(a,b) := \mathcal{X}(a,b) \otimes_{\Bbbk} \mathit{\Gamma},$$

with compositions

$$\mathrm{m}_{i,\mathcal{X}\otimes_{\Bbbk}\mathit{\Gamma}}\left((x_1 \otimes_{\Bbbk} \gamma_1) \otimes ... \otimes (x_i \otimes_{\Bbbk} \gamma_i)\right) := \mathrm{m}_{i,\mathcal{X}}\left(x_1 \otimes ... \otimes x_i\right) \otimes_{\Bbbk} \gamma_1 ... \gamma_i,$$

for composable $x_1, ..., x_i$.

We will drop the subscript $\Bbbk$ if it is clear from context in order to prevent clumsy notation.

Remark 2.3.11. Observe that since we are considering a $\Bbbk$-algebra $\mathit{\Gamma}$ every $\gamma \in \mathit{\Gamma}$ is in degree zero and so there are no signs arising.

Proposition 2.3.12. *Let $\mathcal{X}$ be a $\Bbbk$-linear category, $\mathcal{M}$ a $\Bbbk$-central $\mathcal{X}$-bimodule, $\eta \in \mathrm{HH}^n(\mathcal{X},\mathcal{M})$ with $n \geq 3$ and $\mathit{\Gamma}$ a $\Bbbk$-algebra. Then we have*

$$\mathcal{X}_\eta \otimes \mathit{\Gamma} \cong (\mathcal{X} \otimes \mathit{\Gamma})_{\eta \cup \mathrm{Id}}.$$

Proof. Consider the $\mathcal{A}_\infty$-functor

$$F : \mathcal{X}_\eta \otimes \mathit{\Gamma} \to (\mathcal{X} \otimes \mathit{\Gamma})_{\eta \cup \mathrm{Id}}$$

defined on objects by the identity

$$\mathrm{obj}(\mathcal{X}) = \mathrm{obj}(\mathcal{X}_\eta) = \mathrm{obj}(\mathcal{X}_\eta \otimes \mathit{\Gamma}) \ni a \overset{F}{\mapsto} a \in \mathrm{obj}(\mathcal{X} \otimes \mathit{\Gamma})_{\eta \cup \mathrm{Id}} = \mathrm{obj}(\mathcal{X} \otimes \mathit{\Gamma}) = (\mathrm{obj}\mathcal{X}).$$

And on morphisms as the distributor

$$\begin{gathered}\mathcal{X}(a,b) \oplus \mathcal{M}[n-2](a,b) \otimes \mathit{\Gamma} \xrightarrow{F} (\mathcal{X} \otimes \mathit{\Gamma})(a,b) \oplus (\mathcal{M} \otimes \mathit{\Gamma})[n-2](a,b) \\ (x,y) \otimes \gamma \mapsto (x \otimes \gamma, y \otimes \gamma).\end{gathered}$$

Then we immediately get that this defines a bijection on objects and morphism spaces. However, we still need to check that F defines an $\mathcal{A}_\infty$-functor. For this observe that on both categories we have by definition as the only nontrivial compositions m_2 and m_n. In particular it suffices to check that F preserves these two:

m_2: We have by the definition of $\mathcal{X}_\eta$ and $\otimes_{\Bbbk}$ the following

$$\begin{aligned}
&\mathrm{m}_2\left(F\left((x,y)\otimes\gamma\right)\otimes F\left((x',y')\otimes\gamma'\right)\right)\\
&=\mathrm{m}_2\left((x\otimes\gamma,y\otimes\gamma)\otimes(x'\otimes y',\beta\otimes\gamma')\right)\\
&=\left(\mathrm{m}_2\left((x\otimes y)\otimes(x'\otimes y')\right),\mathrm{m}_2\left((x\otimes\gamma)\otimes(y'\otimes\gamma')\right)+\mathrm{m}_2\left((y\otimes\gamma)\otimes(x'\otimes\gamma')\right)\right)\\
&=\left(\mathrm{m}_2\left((x\otimes x')\otimes\gamma\gamma'\right),\mathrm{m}_2\left(x\otimes y'\right)\otimes\gamma\gamma'+\mathrm{m}_2\left(y\otimes x'\right)\otimes\gamma\gamma'\right)\\
&=F\circ\mathrm{m}_2\left((x\otimes\gamma,y\otimes\gamma)\otimes(x'\otimes\gamma',y'\otimes\gamma')\right).
\end{aligned}$$

m_n: Using the definitions of $\mathcal{X}_\eta$ and $\otimes_{\Bbbk}$ we can compute

$$\begin{aligned}
&\mathrm{m}_n\left(F\left((x_1,y_1)\otimes\gamma_1\right)\otimes\ldots\otimes F\left((x_n,y_n)\otimes\gamma_n\right)\right)\\
&=\mathrm{m}_n\left((x_1\otimes\gamma_1,y_1\otimes\gamma_1)\otimes\ldots\otimes(x_n\otimes\gamma_n,y_n\otimes\gamma_n)\right)\\
&=\left(0,(\eta\cup\mathrm{Id})\left((x_1\otimes\gamma_1)\otimes\ldots\otimes(x_n\otimes\gamma_n)\right)\right)\\
&=\left(0,\eta\left(x_1\otimes\ldots\otimes x_n\right)\otimes\gamma_1\ldots\gamma_n\right)\\
&=F\left(\left(0,\eta\left(x_1\otimes\ldots\otimes x_n\right)\right)\otimes\gamma_1\ldots\gamma_n\right)\\
&=F\left(\mathrm{m}_n\left((x_1,y_1)\otimes\ldots\otimes(x_n,y_n)\right)\otimes\gamma_1\ldots\gamma_n\right)\\
&=F\circ\mathrm{m}_n\left(((x_1,y_1)\otimes\gamma_1)\otimes\ldots((x_n,y_n)\otimes\gamma_n)\right).
\end{aligned}$$

So F respects also m_n which means that it respects every structure morphism and hence is an $\mathcal{A}_\infty$-functor. Since it is bijective it is an equivalence.

Observe again that in the above calculations there is no sign arising as we are working with a $\Bbbk$-linear algebra $\mathit{\Gamma}$ and so every $\gamma\in\mathit{\Gamma}$ is in degree zero. □

2.3.3 The characteristic Morphism of a $\Bbbk$-linear Category

Definition 2.3.13 ([Low08])**.** Let $\mathcal{X}$ be a $\Bbbk$-linear category, $\mathcal{M}$ a $\Bbbk$-central $\mathcal{X}$-bimodule and let $\mathcal{N}\in\mathcal{D}(\mathcal{X})$. Then we define the (algebraic) characteristic morphism to be given by

$$\begin{aligned}
c_{\mathcal{N}}(\mathcal{M}):\mathrm{HH}^*(\mathcal{X},\mathcal{M})=\mathrm{Ext}^*_{\mathcal{X}\otimes\mathcal{X}^{\mathrm{op}}}(\mathcal{X},\mathcal{M})&\to\mathrm{Ext}^*_{\mathcal{X}}(\mathcal{N},\mathcal{M}\otimes_{\mathcal{X}}\mathcal{N})\\
\eta&\mapsto\eta\otimes_{\mathcal{X}}\mathrm{Id}_{\mathcal{N}}.
\end{aligned}$$

and the dual (algebraic) characteristic morphism to be

$$\begin{aligned}
c^*_{\mathcal{N}}(\mathcal{M}):\mathrm{HH}^*(\mathcal{X},\mathcal{M})=\mathrm{Ext}^*_{\mathcal{X}\otimes\mathcal{X}^{\mathrm{op}}}(\mathcal{X},\mathcal{M})&\to\mathrm{Ext}^*_{\mathcal{X}}(\mathrm{RHom}_{\mathcal{X}}(\mathcal{M},\mathcal{N}),\mathcal{N})\\
\eta&\mapsto\mathrm{RHom}(\eta,\mathrm{Id}_{\mathcal{N}}).
\end{aligned}$$

Lemma 2.3.14 ([RVdBN19, 6.3.1])**.** *Let $\mathcal{X}$ be a $\Bbbk$-linear category, $\mathcal{N}\in\mathcal{D}(\mathcal{X})$,*

$\mathcal{M}$ an invertible $\Bbbk$-central $\mathcal{X}$-bimodule and denote its dual by $\mathcal{M}^D$. Then we have the commutative diagram

$$\begin{tikzcd} & \operatorname{Ext}^*_{\mathcal{X}}(\mathcal{N}, \mathcal{M} \otimes_{\mathcal{X}} \mathbb{N}) \arrow[dd, "\wr\ \mathcal{M}^D \otimes_{\mathcal{X}} -"] \\ \operatorname{HH}^*(\mathcal{X}, \mathcal{M}) \arrow[ur, "c_{\mathcal{N}}"] \arrow[dr, "c^*_{\mathcal{N}}"'] & \\ & \operatorname{Ext}^*_{\mathcal{X}}(\operatorname{RHom}_{\mathcal{X}}(\mathcal{M}, \mathcal{N}), \mathcal{N}). \end{tikzcd}$$

In particular this allows us to pass freely from $c_{\mathcal{N}}$ to $c^*_{\mathcal{N}}$ for an invertible bimodule $\mathcal{M}$.

2.3.4 Deformations of Objects

Given these notions of deformation we will now recall some results on the existence of compatible lifts and colifts of modules.

Definition 2.3.15 ([RVdBN19, 6.4])**.** Let $\mathcal{X}$ be a $\Bbbk$-linear category, $\mathcal{M}$ a $\Bbbk$-central $\mathcal{X}$-bimodule, $\eta \in \operatorname{HH}^n(\mathcal{X}, \mathcal{M})$ with $n \geq 3$ and $U \in \mathcal{D}(\mathcal{X})$.

- Assume that $\mathcal{M}$ is right flat, then we call a pair (V, ψ) consisting of an object $V \in \mathcal{D}_\infty(\mathcal{X}_\eta)$ and an isomorphism of graded $\operatorname{H}^*(\mathcal{X}_\eta)$-modules $\psi : \operatorname{H}^*(V) \xrightarrow{\sim} \operatorname{H}^*(\mathcal{X}_\eta) \otimes_{\mathcal{X}} U$ a lift of U to $\mathcal{X}_\eta$.
- Assume that $\mathcal{M}$ is left projective, then we call a pair (V, ψ) consisting of an object $V \in \mathcal{D}_\infty(\mathcal{X}_\eta)$ and an isomorphism of graded $\operatorname{H}^*(\mathcal{X}_\eta)$-modules $\psi : \operatorname{H}^*(V) \xrightarrow{\sim} \mathcal{X}(\operatorname{H}^*(\mathcal{X}_\eta), U)$ a colift of U to $\mathcal{X}_\eta$.

Lemma 2.3.16 ([RVdBN19, 6.4.1])**.** *Let $\mathcal{X}$ be a $\Bbbk$-linear category, $\mathcal{M}$ a $\Bbbk$-central $\mathcal{X}$-bimodule, $\eta \in \operatorname{HH}^n(\mathcal{X}, \mathcal{M})$ with $n \geq 3$ and $U \in \mathcal{D}(\mathcal{X})$. The object U has a lift to $\mathcal{X}_\eta$ if and only if $c_U(\eta) = 0$ and a colift to $\mathcal{X}_\eta$ if and only if $c^*_U(\eta) = 0$.*

The proof of Lemma 2.3.16 consists of trying to lift the $\mathcal{A}_2$-morphism given by the action on $\operatorname{Ext}^*_{\mathcal{X}}(M, M)$ to the $\mathcal{A}_\infty$-level. While studying this one observes that ($**_k$) for $k = n$ is the only obstruction arising and one can compute that this is measured by $c^*_U(\eta)$.

2.3.5 $\mathcal{A}_\infty$-Obstructions

In this subsection we collect obstructions against lifting a morphisms or modules from the $\mathcal{A}_n$-level to the $\mathcal{A}_{n+1}$-level. In particular we will discuss that, if we can find these lifts for every $n > 1$, then we get that the morphism is in particular liftable to the $\mathcal{A}_\infty$-level. We follow for this the discussion in [RVdBN19, §7].

Lemma 2.3.17 ([RVdBN19, Lemma 7.2.1])**.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be an $\mathcal{A}_i$-functor between two $\mathcal{A}_{i+1}$-categories. Then there exists a well-defined obstruction*

$$o_{i+1}(f) \in \mathrm{HH}^{i+1}(\mathrm{H}^*\mathcal{X}, \mathrm{H}^*\mathcal{Y})_{1-i}$$

such that there exists an $\mathcal{A}_{i+1}$-functor $\widehat{f} : \mathcal{X} \to \mathcal{Y}$ with $\widehat{f}_j = f_j$ for $j < i$ and $\widehat{f}_i = f_i + \delta$ with $[m_1, \delta] = 0$ if and only if $o_{i+1}(f)$ vanishes.

Furthermore $o_{i+1}(f)$ is natural in the sense that if we have $\mathcal{A}_n$ functors $g : \mathcal{X}' \to \mathcal{X}$ and $h : \mathcal{Y} \to \mathcal{Y}'$, then

$$o_{i+1}(h \circ f \circ g) = \mathrm{H}^*(h) \circ o_{i+1}(f) \circ \mathrm{H}^*(g).$$

The prove of the above Lemma 2.3.17 consists of a tedious verification that the equation $(**_k)$ for $k = i + 1$ defines an element in

$$o_{i+1}(f) \in \mathrm{HH}^{i+1}(\mathrm{H}^*\mathcal{X}, \mathrm{H}^*\mathcal{Y})_{1-i}.$$

So one can find a lift as the preimage of $o_{i+1}(f)$ under the differential of $\mathrm{Ext}^*(\mathrm{H}^*\mathcal{X}, \mathrm{H}^*\mathcal{Y})$ if and only if

$$0 = o_{i+1}(f) \in \mathrm{HH}^{i+1}(\mathrm{H}^*\mathcal{X}, \mathrm{H}^*\mathcal{Y})_{1-i}.$$

All in all this is a very similar verification to Lemma 2.3.16, as that Lemma also tries to lift an $\mathcal{A}_2$-morphism to the $\mathcal{A}_\infty$-level. But as the spaces are different it is hard to compare the two obstructions directly.

Corollary 2.3.18 ([RVdBN19, Corollary 7.2.4])**.** *Let $f : \mathcal{X} \to \mathcal{Y}$ be an $\mathcal{A}_i$-functor between $\mathcal{A}_\infty$-categories. Then there exists an $\mathcal{A}_\infty$-functor $\widetilde{f} : \mathcal{X} \to \mathcal{Y}$ with $\widetilde{f}_j = f_j$ for $j < i$ if and only if there exists a sequence of obstructions and subsequent lifts $\widehat{f}_i$ such that $o_n\left(\widehat{f}_{n-1}\right) = 0$ for all $n > i$. In particular such a sequence also gives a lift of the desired form $\widetilde{f}$.*

Corollary 2.3.19 ([RVdBN19, Corollary 7.2.4])**.** *Let $\mathcal{X}$ be a $\Bbbk$-linear category, let $f : \mathcal{X} \to \mathrm{H}^0(\mathcal{Y})$ be a $\Bbbk$-linear functor and let $-n < 0$ be maximal such that $\mathrm{H}^{-n}(\mathcal{Y}) \neq 0$. Then*

$$o_3(f) = o_4(f) = \ldots = o_{n+1}(f) = 0$$

and $o_{n+2}(f)$ does not depend on any choices.

Proof. Consider the obstructions introduced in Lemma 2.3.17. Then we have that the space $\mathrm{Hom}\left(\mathrm{H}^*\mathcal{X}, \mathrm{H}^*\mathcal{Y}\right) \cong \mathrm{Hom}\left(\mathcal{X}, \mathrm{H}^*\mathcal{Y}\right)_{1-i}$ vanishes for $i < n+1$. In particular all obstructions and possible lifts we could choose for $i \leq n+1$ have to be 0. And so we only have one choice for a lift to the $\mathcal{A}_{n+1}$-level which is independent of choices as it is unique up to coboundaries in $\underline{\mathrm{Hom}}\left(\mathcal{X}, \mathcal{Y}\right)^{\cdot}$. □

Lemma 2.3.20 ([RVdBN19, Lemma 7.3.1]). *1. Let $\mathcal{X}$ be a dg-category , let Γ be a $\Bbbk$-linear category and let $T \in \mathcal{D}\left(\mathcal{X}\right)_{\Gamma}$. Then there is a sequence of obstructions*

$$o_{i+2} \in \mathrm{HH}^{i+2}\left(\Gamma, \mathrm{Ext}^{-i}_{\mathcal{X}}\left(T, T\right)\right)$$

for $i > 0$ such that T lifts to an object in $\mathcal{D}\left(\mathcal{X} \otimes \Gamma\right)$ if and only if all obstructions vanish. More precisely o_{i+1} is only defined if $o_3\left(T\right), ..., o_i\left(T\right)$ vanish and it depends on choices.

2. If $f : \mathcal{X} \to \mathcal{Y}$ is a dg-functor and $f_* : \mathcal{D}\left(\mathcal{Y}\right) \to \mathcal{D}\left(\mathcal{X}\right)$ the corresponding change of rings functor, then after having made choices for T we may make corresponding choices for $f_*\left(T\right)$ in such a way that

$$f_*\left(o_{i+2}\left(T\right)\right) = o_{i+2}\left(f_*\left(T\right)\right).$$

Remark 2.3.21. The proof of Lemma 2.3.20 similarly to the case of lift and colift consists of trying to lift the action morphism $\Gamma \to \mathrm{Ext}^*_{\mathcal{X}}\left(T, T\right)$. However as one is now in the setting of a general dg-category one needs to actually try to inductively lift it via Lemma 2.3.17.

Chapter 3

Equivariant Sheaves and the Characteristic Morphism

In this chapter we define Γ-equivariant sheaves on a scheme X, for a $\Bbbk$-algebra Γ. We will use this in order to study the (geometric) Γ-equivariant characteristic morphism.

3.1 Equivariant Sheaves and Fourier-Mukai Functors

In this section we introduce equivariant sheaves and prove that the equivariant structure is compatible with Fourier-Mukai functors. In particular we can use this later to get a contradiction to being Fourier-Mukai.

Definition 3.1.1. [LVdB06, §4] Let Γ be a $\Bbbk$-algebra and $\mathcal{C}$ a $\Bbbk$-linear category. Then we define the category $\mathcal{C}_\Gamma$ to consist of objects

$$\operatorname{obj}(\mathcal{C}_\Gamma) := (M, \psi : \Gamma \to \operatorname{End}_{\mathcal{C}}(M)),$$

where $\mathcal{M} \in \mathcal{C}$ and ψ is a morphism of $\Bbbk$-algebras, and morphisms

$$\mathcal{C}_\Gamma((M,\psi),(N,\varphi)) := \{\alpha \in \mathcal{C}(F,G) \,|\, \alpha \circ \psi(\gamma) = \varphi(\gamma) \circ \alpha \in \mathcal{C}(M,N) \quad \forall \gamma \in \Gamma\}.$$

We will mostly denote (T, φ) by T if the action is clear from context to avoid clumsy notation.

Example 3.1.2. We give a few examples to illustrate the above Definition 3.1.1:

- Since $\mathcal{C}$ is required to be $\Bbbk$-linear we have that $\operatorname{End}_{\mathcal{C}}(\_)$ comes with a canonical $\Bbbk$-action and so we have

$$\mathcal{C}_\Bbbk \cong \mathcal{C}.$$

- Let $\mathcal{C}$ be a $\Bbbk$-linear category and $M \in \mathcal{C}$, then we have canonically

$$M = (M, \mathrm{Id}) \in \mathcal{C}_{\mathrm{End}_{\mathcal{C}}(M)}.$$

- Let $F : \mathcal{C} \to \mathcal{C}'$ be a $\Bbbk$-linear functor between $\Bbbk$-linear categories and let Γ be a (possibly non-commutative) $\Bbbk$-algebra. Then we can extend F canonically to a functor

$$\begin{aligned} F : \mathcal{C}_\Gamma &\to \mathcal{C}'_\Gamma \\ M &\mapsto FM := (FM, F \circ \gamma) \in \mathcal{C}'_\Gamma. \end{aligned}$$

- Consider the point $* = \mathrm{Spec}\,(\Bbbk)$ and a $\Bbbk$-algebra Γ. Then $(M, \varphi) \in \mathrm{coh}\,(*)_\Gamma$ consists of $M \in \mathrm{coh}\,(*) \cong \mathrm{Vect}_\Bbbk$ and a $\Bbbk$-algebra morphism $\varphi : \Gamma \to \mathrm{End}_\Bbbk\,(M)$. Which means that

$$\mathrm{coh}\,(*)_\Gamma \cong \Gamma\mathrm{-mod}.$$

- Let $T \in \mathrm{coh}\,(X)$ be tilting for X smooth projective and set $\Gamma := \mathrm{End}_X\,(T)$. Then we have

$$\mathcal{D}^b\,(X) \cong \mathcal{D}^b\,(\Gamma) \cong \mathcal{D}^b\,(\mathrm{coh}\,(*)_\Gamma)\,.$$

  We will prove in Lemma 3.2.12 that this equivalence is compatible with products of schemes under mild conditions.

  We will use the next specific version of the second example throughout this work:

- Let $\mathcal{F} \in \mathrm{coh}\,(X)$ be a coherent sheaf on a scheme. Then we have canonically

$$F = (\mathcal{F}, \mathrm{Id}) \in \mathrm{coh}\,(X)_{\mathrm{End}_X(\mathcal{F})}\,. \tag{3.1.1}$$

Remark 3.1.3. The categories $\mathcal{D}\,(\mathcal{C}_\Gamma)$ and $\mathcal{D}\,(\mathcal{C})_\Gamma$ may seem very similar in notion, however, they do not coincide. An object in $M \in \mathcal{D}\,(\mathcal{C}_\Gamma)$ can be interpreted as a complex of equivariant objects, i.e. it admits an action in every degree and a differential that is compatible with these actions. On the other hand an object in $\mathcal{D}\,(\mathcal{C})_\Gamma$ can be interpreted as a complex of sheaves together with an action on the whole complex that suffices the relations given by the Γ-action up to homotopy. The difference between these two notions essentially boils down to the difference between commutative diagrams up to homotopy not coinciding with homotopy commutative diagrams, which also led to the development of derivators [Gro11]. For some more information on this interplay we refer to [RVdB14].

By the above discussions there is a canonical forgetful functor

$$\begin{aligned} \pi : D\left(\mathcal{C}_{\Gamma}\right) &\to \mathcal{D}\left(\mathcal{C}\right)_{\Gamma} \\ \overline{\left(M^{\cdot}, \varphi^{\cdot}\right)} &\mapsto \left(\overline{M}^{\cdot}, \overline{\varphi}^{\cdot}\right), \end{aligned}$$

where we denote by $\overline{(\_)}$ an equivalence class of $(\_)$. One can think of the above functor as forgetting that Γ acts on every degree separately. However, this functor is neither essentially injective nor surjective in general, which we will use later.

Remark 3.1.4 ([Huy06, Remark 2.51])**.** Let $f : \mathcal{A} \to \mathcal{B}$ be a left or right exact functor betweed derived categories. Recall that an object $M \in \mathcal{A}$ is called f-adapted if $\mathrm{R}^i f\left(M\right) \cong 0$ respectively $\mathrm{L}^i f\left(M\right) \cong 0$ for $i > 0$.

Lemma 3.1.5. *Let $f : \mathcal{C} \to \mathcal{C}'$ be a right or left exact functor between abelian $\Bbbk$-linear categories such that $\mathcal{C}$ has enough f-adapted objects and let Γ be a $\Bbbk$-algebra. Then the canonical functor*

$$\begin{aligned} f : \mathcal{D}^{\natural}\left(\mathcal{C}_{\Gamma}\right) &\to \mathcal{D}^{\natural}\left(\mathcal{C}'\right)_{\Gamma} \\ \overline{\left(M, \psi\right)} &\mapsto \left(\overline{M}, \overline{f \circ \psi}\right) \end{aligned}$$

admits a lift

$$f_{\Gamma} : \mathcal{D}^{\natural}\left(\mathcal{C}_{\Gamma}\right) \to \mathcal{D}^{\natural}\left(\mathcal{C}'_{\Gamma}\right),$$

with $\natural \in \{b, +, -, \}$. In the case $\natural = b$, respectively $\natural = -$ for left exact and $\natural = +$ for right exact functors, we assume that every $M \in \mathcal{C}$ admits a bounded f-adapted resolution.

Proof. We have by Example 3.1.2 a canonical functor

$$\begin{aligned} f_{\Gamma} : \mathcal{C}_{\Gamma} &\to \mathcal{C}'_{\Gamma} \\ \left(M, \psi\right) &\mapsto \left(fM, f \circ \psi\right). \end{aligned}$$

Now as $\mathcal{C}_{\Gamma}$ and $\mathcal{C}'_{\Gamma}$ are abelian with kernels and cokernels computed on objects we get that f_{Γ} has the same exactness as f, and as cohomology also is computed on M only we get that every f-adapted object is also f_{Γ} adapted.

Since we have enough f-adapted objects we may consider for $M \in \mathcal{D}^{\natural}\left(\mathcal{C}\right)$ an f-adapted replacement, which by assumption is also finite for $\natural = b$ respectively if f is left exact and $\natural = -$ or f being right exact and $\natural = +$. In particular we may invoke [Wei94, Theorem 10.5.9] in order to find a well-defined derived functor:

$$\begin{aligned} f_{\Gamma} : \mathcal{D}^{\natural}\left(\mathcal{C}_{\Gamma}\right) &\to \mathcal{D}^{\natural}\left(\mathcal{C}'_{\Gamma}\right) \\ \overline{\left(M, \psi\right)^{\cdot}} &\mapsto \overline{\left(fM, f \circ \psi\right)^{\cdot}}. \end{aligned}$$

Furthermore [Wei94, Theorem 10.5.9] allows us to freely use f-adapted resolutions to compute f_Γ on the derived category, i.e. we will assume from now on that every $M^\cdot$ is f-adapted.

Recall the functor from Remark 3.1.3

$$\begin{aligned} \pi : \mathcal{D}^\natural\left(\mathcal{C}'_\Gamma\right) &\to \mathcal{D}^\natural\left(\mathcal{C}'\right)_\Gamma \\ \overline{(F,\psi)^\cdot} &\mapsto \left(\overline{F^\cdot}, \overline{\psi^\cdot}\right). \end{aligned}$$

We now just need to verify that the diagram

$$\begin{tikzcd} \mathcal{D}^\natural\left(\mathcal{C}_\Gamma\right) \arrow[r, "f_\Gamma"] \arrow[dr, "f"'] & \mathcal{D}^\natural\left(\mathcal{C}'_\Gamma\right) \arrow[d, "\pi"'] \\ & \mathcal{D}^\natural\left(\mathcal{C}'\right)_\Gamma \end{tikzcd}$$

commutes.

We indeed get

$$\begin{aligned} \pi \circ f_\Gamma \overline{(M, f\circ\psi)^\cdot} &= \pi \overline{(fM, f\circ\psi)^\cdot} \\ &= \left(\overline{fM^\cdot}, \overline{f\circ\psi^\cdot}\right) \\ &= f\left(\overline{M^\cdot}, \overline{\psi^\cdot}\right) \end{aligned}$$

as claimed. □

We will drop the Γ in f_Γ if it is clear from context, respectively from the target or source categories.

Lemma 3.1.6. *Let $f : X \to Y$ be a morphism of finite-dimensional noetherian $\Bbbk$-schemes, Γ a $\Bbbk$-algebra and $M \in \mathcal{D}^b(X)$. Then we have the following:*

- *If f is proper, then the functor $f_* : \mathcal{D}^b\left(\mathrm{coh}\left(X\right)_\Gamma\right) \to \mathcal{D}^b\left(\mathrm{coh}\left(Y\right)\right)_\Gamma$ admits a canonical lift*

$$f_{*,\Gamma} : \mathcal{D}^b\left(\mathrm{coh}\left(X\right)_\Gamma\right) \to \mathcal{D}^b\left(\mathrm{coh}\left(Y\right)_\Gamma\right).$$

- *If f is flat, then the functor $f^* : \mathcal{D}^b\left(\mathrm{coh}\left(Y\right)_\Gamma\right) \to \mathcal{D}^b\left(\mathrm{coh}\left(X\right)\right)_\Gamma$ admits a canonical lift*

$$f^*_\Gamma : \mathcal{D}^b\left(\mathrm{coh}\left(Y\right)_\Gamma\right) \to \mathcal{D}^b\left(\mathrm{coh}\left(X\right)_\Gamma\right).$$

- *If X is regular, then the functor $M \otimes \_ : \mathcal{D}^b\left(\mathrm{coh}\left(X\right)_\Gamma\right) \to \mathcal{D}^b\left(\mathrm{coh}\left(X\right)\right)_\Gamma$*

admits a canonical lift

$$M \otimes_{\Gamma} \_ : \mathcal{D}^b\left(\mathrm{coh}\left(X\right)_{\Gamma}\right) \to \mathcal{D}^b\left(\mathrm{coh}\left(X\right)_{\Gamma}\right).$$

Proof. We check the cases separately:

f_*: By Lemma 3.1.5 it suffices to show that every coherent sheaf M admits an f_*-adapted finite resolution in $\mathrm{coh}\left(X\right)$. By [Huy06, Theorem 3.22] the object M admits an f_*-adapted resolution of finite length of quasi-coherent sheaves. By [Huy06, Theorem 3.23] these quasi-coherent sheaves can be picked to be coherent for f proper.

So we can find by Lemma 3.1.5 a lift

$$f_{*,\Gamma} : \mathcal{D}^b\left(\mathrm{coh}\left(X\right)_{\Gamma}\right) \to \mathcal{D}^b\left(\mathrm{coh}\left(Y\right)_{\Gamma}\right).$$

f^*: As f is flat f^* is exact and does not need to be derived. In particular we get by Lemma 3.1.5 immediately a lift

$$f^*_{\Gamma} : \mathcal{D}^b\left(\mathrm{coh}\left(Y\right)_{\Gamma}\right) \to \mathcal{D}^b\left(\mathrm{coh}\left(X\right)_{\Gamma}\right).$$

$M \otimes (\_)$: By [Huy06, Proposition 3.26] we have that every $\mathcal{F} \in \mathrm{coh}\left(X\right)$ admits a bounded locally free resolution, which is in particular $M \otimes (\_)$ adapted. So we get by Lemma 3.1.5 that $M \otimes (\_)$ admits a lift

$$M \otimes_{\Gamma} (\_) : \mathcal{D}^b\left(\mathrm{coh}\left(X\right)_{\Gamma}\right) \to \mathcal{D}^b\left(\mathrm{coh}\left(X\right)_{\Gamma}\right)$$

as claimed. □

Corollary 3.1.7. *Let $f : \mathcal{D}^b\left(X\right) \to \mathcal{D}^b\left(Y\right)$ be a Fourier-Mukai functor between finite-dimensional smooth projective $\Bbbk$-schemes and let Γ be a $\Bbbk$-algebra. Then we have that the induced functor*

$$f : \mathcal{D}^b\left(\mathrm{coh}\left(X\right)_{\Gamma}\right) \to \mathcal{D}^b\left(\mathrm{coh}\left(Y\right)\right)_{\Gamma}$$

admits a lift:

$$f_{\Gamma} : \mathcal{D}^b\left(\mathrm{coh}\left(X\right)_{\Gamma}\right) \to \mathcal{D}^b\left(\mathrm{coh}\left(Y\right)_{\Gamma}\right).$$

Proof. Observe first that X and Y being smooth projective immediately gives that

$$\begin{aligned} \pi_1 : X \times Y \to X &\text{ is proper,} \\ \pi_2 : X \times Y \to Y &\text{ is flat} \\ \text{and } X \times Y &\text{ is regular.} \end{aligned}$$

As f is a Fourier-Mukai functor it has the form $\pi_{1,*}\left(M \otimes \pi_2^*\left(\_\right)\right)$ for some $M \in \mathcal{D}^b\left(Y \times X\right)$. So we get by Lemma 3.1.6 that $\pi_{1,*}$, π_2^* and $M \otimes \_$ admit canonical lifts. In particular f admits a canonical lift

$$f_\Gamma := \pi_{1,*,\Gamma}\left(M \otimes_\Gamma \pi_{2,\Gamma}^*\left(\_\right)\right)$$

as claimed. □

3.2 Hochschild Cohomology and the characteristic Morphism

As we want to study the characteristic morphism we start by recalling the definition of the (geometric) Hochschild cohomology:

Definition 3.2.1. [Swa96] Let X be a separated scheme and M a sheaf on X. Then the Hochschild cohomology of X with coefficients in M is given by

$$\mathrm{HH}^*\left(X, M\right) := \mathrm{Ext}^*_{X \times X}\left(\mathcal{O}_\Delta, \Delta_* M\right),$$

where $\Delta : X \hookrightarrow X \times X$ is the diagonal embedding.

For the definition of the (geometric) characteristic morphism below we follow [Low08] and [BF08, §3.3].

Definition 3.2.2. Let X, Y be regular schemes, Γ a $\Bbbk$-algebra and $M, T \in \mathrm{coh}\left(X\right)$. Then the (geometric) characteristic morphism is defined to be

$$\begin{aligned} c_T\left(M\right) : \mathrm{HH}^*\left(X, M\right) = \mathrm{Ext}^*_{X \times X}\left(\mathcal{O}_\Delta, \Delta_* M\right) &\to \mathrm{Ext}^*_X\left(T, M \otimes T\right) \\ \left(\alpha : \mathcal{O}_\Delta \to \Sigma^n \Delta_* M\right) &\mapsto \left(T \xrightarrow{\pi_{1*}\left(\alpha \otimes \pi_2^* \mathrm{Id}\right)} \Sigma^n M \otimes T\right), \end{aligned}$$

where we use $T \cong \pi_{1*}\left(\mathcal{O}_\Delta \otimes \pi_2^* T\right)$ and $M \otimes T \cong \pi_{1*}\left(\Delta_* \Sigma^n M \otimes \pi_2^* T\right)$. If we have a Γ-action on T, i.e. $\left(T, \varphi\right) \in \mathrm{coh}\left(X\right)_\Gamma$, there also exists a Γ-equivariant characteristic morphism

$$\begin{aligned} c_{T,\Gamma}(M) : \mathrm{HH}^*(X,M) = \mathrm{Ext}^*_{X\times X}(\mathcal{O}_\Delta, \Delta_* M) &\to \mathrm{Ext}^*_{\mathrm{coh}(X)_\Gamma}(T, M\otimes T) \\ \alpha : (\mathcal{O}_\Delta \to \Sigma^n \Delta_* M) &\mapsto \left(T \xrightarrow{\pi_{1*}(\alpha\otimes\pi_2^*\mathrm{Id})} \Sigma^n M\otimes T\right), \end{aligned}$$

where we consider $M\otimes T$ as on object in $\mathrm{coh}(X)_\Gamma$ via the functor $M\otimes(\_)$, i.e.

$$\begin{aligned} \psi : \Gamma &\to \mathrm{End}(M\otimes T) \\ \gamma &\mapsto \mathrm{Id}\otimes\varphi(\gamma). \end{aligned}$$

To study the characteristic morphism for special T we will define the following functor realizing the characteristic morphism on a categorical level.

Definition 3.2.3. Let X, Y be projective schemes and let $T = (T,\varphi) \in \mathrm{coh}(Y)_\Gamma$. Then we define the functor:

$$\begin{aligned} C_T^X : D^b(X\times Y) &\to \mathcal{D}^b(\mathrm{coh}(X)_\Gamma) \\ M &\mapsto (\pi_{1*}(M\otimes\pi_2^*T), \gamma\mapsto \pi_{1*}(\mathrm{Id}\otimes\pi_2^*\varphi(\gamma))) \\ (\alpha : M\to N) &\mapsto C_T^X(\alpha) = \pi_{1*}(\alpha\otimes\pi_2^*T). \end{aligned}$$

Remark 3.2.4. One can think of the functor C_T^X to send an object $M \in \mathcal{D}^b(X\times Y)$ to the image of T under the Fourier-Mukai functor with kernel M, equipped with the action induced by $\Phi_{M,\Gamma}$, i.e.

$$\begin{aligned} C_T^X : D^b(X\times Y) &\to \mathcal{D}^b(\mathrm{coh}(X)_\Gamma) \\ M &\mapsto \Phi_M(T). \end{aligned}$$

The functor C_T^X allows us to compute $c_{T,\Gamma}$ on a categorical level.

Proposition 3.2.5. *Let X be a scheme and $T\in\mathrm{coh}(X)_\Gamma$ and consider*

$$C_T^X : \mathcal{D}^b(X\times X) \to \mathcal{D}^b(\mathrm{coh}(X)_\Gamma).$$

Then we have that the equivariant characteristic morphism $c_{T,\Gamma}(M)$ is given by evaluating the functor C_T^X on the morphism space $\mathrm{Ext}_{X\times X}(\mathcal{O}_\Delta, \Delta_* M)$,

$$c_{T,\Gamma}(M) = C_T^X : \mathrm{Ext}^*_{X\times X}(\mathcal{O}_\Delta, \Delta_* M) \to \mathrm{Ext}^*_{\mathrm{coh}(X)_\Gamma}(T, M\otimes T).$$

Proof. By Definition 3.2.3 we have

$$C_T^X : \operatorname{Ext}^*_{X\times X}\left(\mathcal{O}_\Delta, \Delta_* M\right) \to \operatorname{Ext}_{\operatorname{coh}(X)_\Gamma}\left(C_T^X\left(\mathcal{O}_\Delta\right), C_T^X\left(\Delta_* M\right)\right)$$
$$\alpha \mapsto \pi_{1*}\left(\alpha \otimes \pi_2^* T\right).$$

We now have $C_T^X\left(\mathcal{O}_\Delta\right) \cong T$ and $C_T^X\left(\Delta_* M\right) \cong M \times T$. So the above turns by Definition 3.2.2 into

$$c_{T,\Gamma}\left(M\right) : \operatorname{Ext}^*_{X\times X}\left(\mathcal{O}_\Delta, \Delta_* M\right) \to \operatorname{Ext}^*_{\operatorname{coh}(X)_\Gamma}\left(T, M \otimes T\right)$$

as claimed. □

Definition 3.2.6 ([BvdB03, §2.1])**.** Let $\mathcal{C}$ be a pointed category, i.e. a category admitting a zero object. An object $G \in \mathcal{C}$ is a generator if $\mathcal{C}\left(G, M\right) = 0$ implies $M = 0$.

Remark 3.2.7. In a pointed category $\mathcal{C}$ the equation $\mathcal{C}\left(M, N\right) = 0$ for two objects $M, N \in \mathcal{C}$ means that the only morphism between M and N is the unique morphism factoring over 0.

Furthermore we have for an object $M \in \mathcal{C}$ that if $\mathcal{C}\left(M, M\right) = 0$ means that $M \cong 0$ as in that case $0 = \operatorname{Id}$ and so the unique morphisms $0 \to M$ and $M \to 0$ define isomorphisms.

Proposition 3.2.8. *Let $f_* : \mathcal{C} \to \mathcal{D}$ be a faithful functor between pointed categories with a left adjoint $f^* : \mathcal{D} \to \mathcal{C}$ and let $T \in \mathcal{D}$ be a generator. Then f^*T is a generator.*

Proof. Let M be such that $\mathcal{C}\left(f^*T, M\right) = 0$. Then we have that $\mathcal{C}\left(f^*T, M\right) = \mathcal{D}\left(T, f_*M\right) = 0$, in particular $f_*M \cong 0$. Now $\mathcal{D}\left(f_*M, f_*M\right) = 0$ and so $\mathcal{C}\left(M, M\right) = 0$. This can only hold if $M \cong 0$ and so f^*T is a generator. □

Proposition 3.2.9. *Let $\mathcal{C}$ be a $\Bbbk$-linear category that admits a generator G and let Γ be a $\Bbbk$-algebra. Then*

$$\left(G \otimes \Gamma, \psi : \gamma' \mapsto \left(G \otimes \Gamma \xrightarrow{g\otimes\gamma \mapsto g\otimes\gamma'\gamma} G \otimes \Gamma\right)\right)$$

defines a generator of $\mathcal{C}_\Gamma$. Where we denote by $G \otimes \Gamma$ the sheaf arising by tensoring locally with the $\Bbbk$-algebra Γ as $\Bbbk$-vectorspaces and acting exclusively on Γ.

Proof. Let $\left(X, \varphi\right) \in C_\Gamma$ and let $f : G \to X$ be a morphism. Then we have the following morphism in $\mathcal{C}_\Gamma$,

$$\begin{aligned}\widehat{f} : G \otimes \Gamma &\to X \\ g \otimes \gamma &\mapsto \varphi(\gamma) \circ f(g).\end{aligned}$$

This indeed defines a morphism in $\mathcal{C}_\Gamma$ as

$$\begin{aligned}\varphi(\gamma') \circ \widehat{f}(g \otimes \gamma) &= \varphi(\gamma') \circ \varphi(\gamma) \circ f(g) \\ &= \varphi(\gamma'\gamma) \circ f(g) \\ &= \widehat{f}(g \otimes \gamma'\gamma) \\ &= \widehat{f} \circ \psi(\gamma')(g \otimes \gamma).\end{aligned}$$

We can compute that if $\widehat{f}$ vanishes then f has to vanish as well since

$$\begin{aligned}0 &= \widehat{f}(g \otimes \mathrm{Id}) \\ &= \varphi(\mathrm{Id}) \circ f(g) \\ &= \mathrm{Id} \circ f(g) \\ &= f(g).\end{aligned}$$

This means that if the morphism space $\mathcal{C}_\Gamma(G \otimes \Gamma, X)$ vanishes, that also $\mathcal{C}(G, X)$ vanishes.

Now assume that $\mathcal{C}_\Gamma(G \otimes \Gamma, (X, \varphi)) = 0$. Then we have by the above discussion that $\mathcal{C}(G, X) = 0$. As G is a generator we get that X has to be a zero object. And so (X, φ) has to be a zero object as well. In particular we get that $G \otimes \Gamma$ is indeed a generator of $\mathcal{C}_\Gamma$. □

Remark 3.2.10. Proposition 3.2.9 is a consequence of $M \otimes \Gamma$ being the free object in $\mathcal{C}_\Gamma$ over M.

Remark 3.2.11. Recall that an object $\mathcal{T}$ in an abelian category $\mathcal{A}$ is called tilting if $\mathcal{T}$ is a generator and $\mathrm{Ext}^i(T, T) \cong 0$ for all $i > 0$.

Lemma 3.2.12. *Let X, Y be smooth projective schemes, such that X admits a generator $G \in \mathrm{coh}(X)$ with $\mathrm{RHom}^i(G, G)$ finite-dimensional for all i, let Y be such that it admits a tilting object $T \in \mathrm{coh}(Y)$ and set $\Gamma := \mathrm{End}(T)$. Then*

$$C_T^X : \mathcal{D}^b(X \times Y) \to \mathcal{D}^b(\mathrm{coh}(X)_\Gamma)$$

is an equivalence of derived categories.

Proof. Throughout this proof we denote by $T^D := \mathrm{R}\mathscr{H}\mathrm{om}_Y(T, \mathcal{O}_Y)$, the dual of T and by $\pi_S : S \to \mathrm{Spec}(\Bbbk)$ the unique projection from a scheme S to the point

Spec $(\Bbbk)$. Observe that by [Huy06, Proposition 3.26] we have $T^D \in \mathcal{D}^b(Y)$ as smooth schemes are in particular regular. Furthermore, we will use the following diagram for flat base change twice

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X \arrow[dr, "\pi_X"'] & & Y \arrow[dl, "\pi_Y"] \\ & \mathrm{Spec}(\Bbbk). & \end{tikzcd}$$

Since T is tilting, it is a generator of $\mathcal{D}^b(Y)$ and T^D is generating $\mathcal{D}^b(Y)$ by [RVdBN19, Lemma 8.9.1]. So we get that $G \boxtimes T^D$ generates $\mathcal{D}^b(X \times Y)$ by [BvdB03, Lemma 3.4.1]. Furthermore, we have by [BH13, 1.10] that $\Gamma = \mathrm{End}_Y(T)$ is finite-dimensional.

We first show that $C_T^X\left(G \boxtimes T^D\right)$ is isomorphic to $G \otimes \Gamma$:

$$\begin{aligned} C_T^X\left(G \boxtimes T^D\right) &= \pi_{1*}\left(\left(G \boxtimes T^D\right) \otimes \pi_2^* T\right) && \text{definition of } C_T^X \\ &\cong \pi_{1*}\left(\pi_1^* G \otimes \pi_2^* T^D \otimes \pi_2^* T\right) && \text{definition of } \boxtimes \\ &\cong \pi_{1*}\left(\pi_1^* G \otimes \pi_2^*\left(T \otimes T^D\right)\right) && \text{[Huy06, (3.12)]} \\ &\cong \pi_{1*}\left(\pi_1^* G \otimes \pi_2^* \mathrm{R}\mathscr{H}om_Y(T,T)\right) && \text{definition of } T^D \\ &\cong G \otimes \pi_{1*}\pi_2^*\left(\mathrm{R}\mathscr{H}om_Y(T,T)\right) && \text{[Huy06, (3.11)]} \\ &\cong G \otimes \pi_X^* \pi_{Y,*} \mathrm{R}\mathscr{H}om_Y(T,T) && \text{flat base change} \\ &\cong G \otimes \Gamma. && T \text{ has no higher Ext-groups} \end{aligned}$$

The above computation is compatible with the Γ-action as all isomorphisms involved are natural isomorphism. In particular replacing $\pi_2^* T$ by $\pi_2^* \gamma$ yields multiplication with γ in Γ.

As by Proposition 3.2.9 $G \otimes \Gamma$ is a generator for $\mathrm{coh}(X)_\Gamma$ the functor C_T^X sends a generator to a generator. So it suffices to prove that

$$\mathrm{RHom}_{X \times Y}\left(G \boxtimes T^D, G \boxtimes T^D\right) \xrightarrow{C_X^T} \mathrm{RHom}_{\mathrm{coh}(X)_\Gamma}\left(C_T^X\left(G \boxtimes T^D\right), C_T^X\left(G \boxtimes T^D\right)\right)$$

is an isomorphism.

To do that we first compute the source and target spaces:

$$
\begin{aligned}
&\mathrm{RHom}_{X\times Y}\left(G\boxtimes T^D, G\boxtimes T^D\right)\cong && \\
&\cong \mathrm{RHom}_{X\times Y}\left(\pi_1^*G\otimes\pi_2^*T^D,\pi_1^*G\otimes\pi_2^*T^D\right) && \text{definition of }\boxtimes\\
&\cong \mathrm{RHom}_{X\times Y}\left(\pi_1^*G,\mathrm{R}\mathscr{H}\mathrm{om}_{X\times Y}\left(\pi_2^*T^D,\pi_1^*G\otimes\pi_2^*T^D\right)\right) && \text{[Huy06, (3.14)]}\\
&\cong \mathrm{RHom}_{X\times Y}\left(\pi_1^*G,\pi_1^*G\otimes\mathrm{R}\mathscr{H}\mathrm{om}_{X\times Y}\left(\pi_2^*T^D,\pi_2^*T^D\right)\right) && \text{([Huy06, 3.13)]}\\
&\cong \Gamma_{X\times Y}\mathrm{R}\mathscr{H}\mathrm{om}_{X\times Y}\left(\pi_1^*G,\pi_1^*G\otimes\mathrm{R}\mathscr{H}\mathrm{om}_{X\times Y}\left(\pi_2^*T^D,\pi_2^*T^D\right)\right) && \text{[Huy06, p.85]}\\
&\cong \pi_{\Bbbk,*}\mathrm{R}\mathscr{H}\mathrm{om}_{X\times Y}\left(\pi_1^*G,\pi_1^*G\otimes\mathrm{R}\mathscr{H}\mathrm{om}_{X\times Y}\left(\pi_2^*T^D,\pi_2^*T^D\right)\right) && \Gamma_{X\times Y}\cong\pi_{X\times Y,*}\\
&\cong \pi_{X\times Y,*}\left(\mathrm{R}\mathscr{H}\mathrm{om}_{X\times Y}\left(\pi_1^*G,\pi_1^*G\right)\otimes\mathrm{R}\mathscr{H}\mathrm{om}_{X\times Y}\left(\pi_2^*T^D,\pi_2^*T^D\right)\right) && \text{[Huy06, (3.13)]}\\
&\cong \pi_{X\times Y,*}\left(\pi_1^*\mathrm{R}\mathscr{H}\mathrm{om}_{X}\left(G,G\right)\otimes\pi_2^*\mathrm{R}\mathscr{H}\mathrm{om}_{Y}\left(T^D,T^D\right)\right) && \text{[Huy06, (3.13)]}\\
&\cong \pi_{X,*}\circ\pi_{1,*}\left(\pi_1^*\mathrm{R}\mathscr{H}\mathrm{om}_{X}\left(G,G\right)\otimes\pi_2^*\mathrm{R}\mathscr{H}\mathrm{om}_{Y}\left(T^D,T^D\right)\right) && \pi_{X\times Y}=\pi_X\circ\pi_1\\
&\cong \pi_{X,*}\circ\left(\pi_{1,*}\left(\pi_1^*\mathrm{R}\mathscr{H}\mathrm{om}_{X}\left(G,G\right)\otimes\pi_2^*\mathrm{R}\mathscr{H}\mathrm{om}_{Y}\left(T^D,T^D\right)\right)\right) && \circ\text{ is associative}\\
&\cong \pi_{X,*}\left(\mathrm{R}\mathscr{H}\mathrm{om}_{X}\left(G,G\right)\otimes\pi_{1,*}\pi_2^*\mathrm{R}\mathscr{H}\mathrm{om}_{Y}\left(T^D,T^D\right)\right) && \text{[Huy06, (3.11)]}\\
&\cong \pi_{X,*}\left(\mathrm{R}\mathscr{H}\mathrm{om}_{X}\left(G,G\right)\otimes\pi_X^*\pi_{Y,*}\mathrm{R}\mathscr{H}\mathrm{om}_{Y}\left(T^D,T^D\right)\right) && \text{flat base change}\\
&\cong \pi_{X,*}\left(\mathrm{R}\mathscr{H}\mathrm{om}_{X}\left(G,G\right)\otimes\pi_X^*\Gamma^{\mathrm{op}}\right) && \text{[Huy06, p.85]}\\
&\cong \pi_{X,*}\mathrm{R}\mathscr{H}\mathrm{om}_{X}\left(G,G\right)\otimes\Gamma^{\mathrm{op}} && \text{[Huy06, (3.11)]}\\
&\cong \mathrm{RHom}_{X}\left(G,G\right)\otimes\Gamma^{\mathrm{op}}. && \text{[Huy06, p.85]}
\end{aligned}
$$

Now for $\mathrm{RHom}_{\mathrm{coh}(X)_\Gamma}\left(G\otimes\Gamma,G\otimes\Gamma\right)$ we have

$$
\begin{aligned}
\mathrm{RHom}_{\mathrm{coh}(X)_\Gamma}\left(G\otimes\Gamma,G\otimes\Gamma\right) &= \mathrm{RHom}_{\mathrm{coh}(X)}\left(G,G\right)\otimes\mathrm{RHom}_{\Gamma-\mathrm{mod}}\left(\Gamma,\Gamma\right)\\
&\cong \mathrm{RHom}_X\left(G,G\right)\otimes\Gamma^{\mathrm{op}}.
\end{aligned}
$$

As the two spaces are isomorphic and in particular degree-wise isomorphic, it suffices to prove bijectivity on $\mathrm{RHom}^i_{X\times Y}\left(G\boxtimes T^D,G\boxtimes T^D\right)$. Since

$$\mathrm{RHom}^i_{X\times Y}\left(G\boxtimes T^D,G\boxtimes T^D\right)\cong\mathrm{RHom}^i_X\left(G,G\right)\otimes\Gamma$$

we know that $\mathrm{RHom}^i_{X\times Y}\left(G\boxtimes T^D,G\boxtimes T^D\right)$ is finite-dimensional as tensor product of finite-dimensional vector spaces. So it suffices to check that C^X_T is surjective. For this let

$$\alpha\otimes\beta\in\mathrm{RHom}^i_{\mathrm{coh}(X)_\Gamma}\left(G\otimes\Gamma,G\otimes\Gamma\right)\cong\mathrm{RHom}^i\left(G,G\right)\otimes\Gamma^{\mathrm{op}}.$$

Then we can pick $\alpha \boxtimes \beta \in \mathrm{RHom}_{X\times Y}\left(G \boxtimes T^D, G \boxtimes T^D\right)$ and get

$$\begin{aligned} C_T^X(\alpha \boxtimes \beta) &\cong \pi_{1,*}\left(\alpha \boxtimes \beta \otimes \pi_2^* \mathrm{Id}_T\right) \\ &\cong \pi_{1,*}(\alpha \boxtimes \beta) \\ &\cong \alpha \otimes \beta \in \mathrm{RHom}_{\mathrm{coh}(X)_\Gamma}(G \otimes \Gamma, G \otimes \Gamma). \end{aligned}$$

This means that C_T^X is surjective on the generating set of morphisms of the form $\alpha \otimes \beta$. In particular C_T^X is surjective and an isomorphism as it is surjective between vector spaces of the same dimension which finishes the proof. □

Lemma 3.2.13. *Let $f : X \to Y$ be a proper morphism of schemes, Γ a $\Bbbk$-algebra and $T \in \mathrm{coh}(Y)_\Gamma$. We have $f^*T \in \mathrm{coh}(X)_\Gamma$. Consider the two functors:*

$$\begin{aligned} C_{f^*T}^X : \mathcal{D}^b(X \times X) &\to \mathcal{D}^b\left(\mathrm{coh}(X)_\Gamma\right) \\ M &\mapsto \left(\pi_{1*}\left(M \otimes \pi_2^* f^* T\right)\right) \\ (\alpha : M \to N) &\mapsto \pi_{1*}\left(\alpha \otimes \pi_2^* f^* T\right) \end{aligned}$$

and

$$\begin{aligned} C_T^X \circ (\mathrm{Id} \times f)_* : \mathcal{D}^b(X \times X) \to \mathcal{D}^b(X \times Y) &\to \mathcal{D}^b\left(\mathrm{coh}(X)_\Gamma\right) \\ M \mapsto (\mathrm{Id} \times f)_* M &\mapsto \left(\pi_{1*}(\mathrm{Id} \times f)_* M \otimes \pi_2^* T\right) \\ (\alpha : M \to N) \mapsto (\mathrm{Id} \times f)_* \alpha &\mapsto \pi_{1*}\left((\mathrm{Id} \times f)_* \alpha \otimes \pi_2^* T\right). \end{aligned}$$

Then we have a natural isomorphism $C_{f^*T}^X \cong C_T^X \circ (\mathrm{Id} \times f)_*$.

Proof. Observe that $\mathrm{Id} \times f$ is proper as product of proper morphisms and so by [Huy06, Theorem 3.23]

$$(\mathrm{Id} \times f)_* : \mathcal{D}^b(X \times X) \to \mathcal{D}^b(X \times Y)$$

is well-defined. We will use the following two commutative diagrams in order to construct the isomorphism

$$\begin{tikzcd} X \times X \arrow[r, "\mathrm{Id} \times f"] \arrow[d, "\pi_1"'] & X \times Y \arrow[d, "\pi_1'"] \\ X \arrow[r, "\mathrm{Id}"'] & X \end{tikzcd} \qquad \begin{tikzcd} X \times X \arrow[r, "\mathrm{Id} \times f"] \arrow[d, "\pi_2"'] & X \times Y \arrow[d, "\pi_2'"] \\ X \arrow[r, "f"'] & Y, \end{tikzcd}$$

where we distinguish between the projections from $X \times X$ and $X \times Y$ in order to avoid confusion. This means that in this notation $C_T^X = \pi_{1*}'\left((\_) \otimes \pi_2'^* T\right)$ and

$C^X_{f^*T} = \pi_{1*}\left(\left(\_\right) \otimes \pi_2^* f^* T\right)$.

On objects and morphisms we have the following sequence of natural isomophisms

$$\begin{aligned}
C^X_{f^*T}\left(_\right) &= \pi_{1*}\left(\left(_\right) \otimes \pi_2^* f^* T\right) && \text{Definition 3.2.3} \\
&\cong \pi'_{1*}\left(\mathrm{Id} \times f\right)_*\left(\left(_\right) \otimes \left(\mathrm{Id} \times f\right)^* \pi_2'^* T\right) && \begin{matrix} \pi_1 = \pi'_1 \circ \left(\mathrm{Id} \times f\right) \\ f \circ \pi_2 = \pi'_2 \circ \left(\mathrm{Id} \times f\right) \end{matrix} \\
&\cong \pi'_{1*}\left(\left(\mathrm{Id} \times f\right)_*\left(_\right) \otimes {\pi'_2}^* T\right) && \text{projection formula} \\
&= C^X_T \circ \left(\mathrm{Id} \times f\right)_*\left(_\right). && \text{Definition 3.2.3}
\end{aligned}$$

Both functors also induce the same Γ-action as we get analogously

$$\begin{aligned}
\pi_{1*}\left(\mathrm{Id} \otimes \pi_2^* f^* \gamma\right) &\cong \pi'_{1*}\left(\mathrm{Id} \times f\right)_*\left(\mathrm{Id} \otimes \left(\mathrm{Id} \times f\right)^* \pi_2'^* \gamma\right) && \begin{matrix} \pi_1 = \pi'_1 \circ \left(\mathrm{Id} \times f\right) \\ f \circ \pi_2 = \pi'_2 \circ \left(\mathrm{Id} \times f\right) \end{matrix} \\
&\cong \pi'_{1*}\left(\left(\mathrm{Id} \times f\right)_* \mathrm{Id} \otimes {\pi'_2}^* \gamma\right). && \text{projection formula}
\end{aligned}$$

This means that the actions match up along the same natural isomorphisms and so

$$C^X_{f^*T} \cong C^X_T \circ \left(\mathrm{Id} \times f\right)_*$$

as claimed. □

Remark 3.2.14. The above Lemma 3.2.13 can be interpreted very naturally using Remark 3.2.4. As C^X_T sends an M to the image of T under the Fourier-Mukai functor Φ_M and we have by [Huy06, Exercise 5.12] $\Phi_{(\mathrm{Id} \times f)_* M} \cong \Phi_M \circ f^*$. In particular the two functors $C^X_{f^*T}$ and $C^X_T \circ \left(f \times \mathrm{Id}\right)_*$ should be isomorphic.

Proposition 3.2.15. *Let $f : X \to Y$ be a proper morphism of schemes and $T \in \mathrm{coh}\left(Y\right)_\Gamma$. Then we have*

$$c_{f^*T,\Gamma}\left(M\right) = C^X_T \circ \left(\mathrm{Id} \times f\right)_* : \mathrm{Ext}^*_{X \times X}\left(\mathcal{O}_\Delta, \Delta_* M\right) \to \mathrm{Ext}^*_{\mathrm{coh}(X)_\Gamma}\left(f^*T, M \otimes f^*T\right).$$

Proof. By Proposition 3.2.5 we have

$$c_{f^*T,\Gamma}\left(M\right) = C^X_{f^*T} : \mathrm{Ext}^*_{X \times X}\left(\mathcal{O}_\Delta, \Delta_* M\right) \to \mathrm{Ext}^*_{\mathrm{coh}(X)_\Gamma}\left(T, M \otimes T\right)$$

and by Lemma 3.2.13 we get

$$c_{f^*T,\Gamma}\left(M\right) = C^X_{f^*T} = C^X_T \circ \left(\mathrm{Id} \times f\right)_* : \mathrm{Ext}^*_{X \times X}\left(\mathcal{O}_\Delta, \Delta_* M\right) \to \mathrm{Ext}^*_{\mathrm{coh}(X)_\Gamma}\left(f^*T, M \otimes f^*T\right)$$

as claimed. $\square$

Remark 3.2.16. The above result could be used to compute injectivity of the characteristic morphism if one can find an $(f, \mathrm{Id}) : X \times X \to X \times Y$ that is injective on $\mathrm{Ext}^*_{X \times X}(\mathcal{O}_\Delta, \Delta_* M)$ and such that Y admits a tilting bundle. However, the existence of such a morphism is not straight forward. In particular a closed immersion of a divisor $f : X \hookrightarrow \mathbb{P}^n$ is in general not injective on $\mathrm{Ext}^i_X(\_, \_)$ as by the Grothendieck-Serre spectral sequence there might be correction terms arising in degrees $i > 1$.

Chapter 4

Preliminaries: Schemes and $\Bbbk$-linear Categories

In this chapter we collect results from [RVdBN19, §8] in order to pass from the geometric category $\operatorname{coh}(X)$ for a quasi-compact scheme X to modules over a $\Bbbk$-linear category $\mathcal{X}$. We will later use this to deform the scheme $\mathcal{X}$ in a non-geometric fashion.

We do this here as we will be using equivariant sheaves, which were introduced in the previous section.

Remark 4.0.1. Recall that one denotes by $\mathcal{D}_{\mathcal{C}}(\mathcal{A})$ the full sub-category of $\mathcal{D}(\mathcal{A})$ consisting of objects with cohomology objects in $\mathcal{C}$ for $\mathcal{C}$ a full subcategory of an abelian category $\mathcal{A}$.

4.1 Ordinary Presheaves and Sheaves

We start by discussing ordinary sheaves and presheaves, and how we can turn them into modules over a small $\Bbbk$-linear category for quasi-compact schemes.

4.1.1 Presheaves

We follow mostly [Low08, GS88] while using left modules instead of right modules, so some conventions might be slightly different. The notation is set up to be compatible with [RVdBN19].

Remark 4.1.1. Recall that a presheaf of algebras $\mathcal{O}$ on a poset $(I, \leq)$ consists of an algebra $\mathcal{O}_i$ for all $i \in I$ and a restriction morphism $\rho_{j,i} : \mathcal{O}_j \to \mathcal{O}_i$ for all $i \leq j$.

Definition 4.1.2 ([Low08, §2.2])**.** Let $(I, \leq)$ be a poset and $\mathcal{O}$ a presheaf of $\Bbbk$-algebras on $(I, \leq)$. Then define the small $\Bbbk$-linear category $\widetilde{\mathcal{O}}$ to consist of objects

$$\operatorname{obj}\left(\widetilde{\mathcal{O}}\right) = I$$

and morphisms

$$\widetilde{\mathcal{O}}(i,j) = \begin{cases} \mathcal{O}(j) & \text{if } j \leq i \\ 0 & \text{else,} \end{cases}$$

with composition is given by multiplication and restriction

$$\widetilde{\mathcal{O}}(j,k) \otimes_{\Bbbk} \widetilde{\mathcal{O}}(i,j) \xrightarrow{\mu \circ \rho_{j,k}} \widetilde{\mathcal{O}}(i,k),$$

where we denoted by $\mu : \mathcal{O}(k) \otimes \mathcal{O}(k) \to \mathcal{O}(k)$ the multiplication and by $\rho_{j,k} : \mathcal{O}(j) \to \mathcal{O}(k)$ the restriction for $k \leq j$.

Lemma 4.1.3. *Let $(I, \leq)$ be a poset and let $\mathcal{O}$ be a presheaf of $\Bbbk$-algebras on $(I, \leq)$. Then we have an equivalence of categories*

$$\pi^* : \mathcal{O}\text{−mod} \longleftrightarrow \widetilde{\mathcal{O}}\text{−mod} : \pi_*,$$

,here π^* sends an $\mathcal{O}$-module M to the functor $\pi^* M$ defined on objects by

$$i \mapsto M(i)$$

and on morphisms by

$$\widetilde{\mathcal{O}}(i,j) = \begin{cases} \mathcal{O}(j) & \xrightarrow{M} M(j) \quad j \leq i \\ 0 & \xrightarrow{0} 0 \qquad \textit{else,} \end{cases}$$

while π_* sends a $\widetilde{\mathcal{O}}$-module M to the $\mathcal{O}$-module defined by

$$\pi^* M(i) = M(i)$$

with restriction maps given via $\widetilde{\mathcal{O}}(i,j) = \mathcal{O}(j)$. More precisely the restriction map is

$$\rho_{i,j}^{\pi_* M} := M\left(\mathrm{Id}_{M(j)}\right) : \pi_* M(i) \to \pi_* M(j).$$

Proof. First observe that π_* and π^* send a morphism $M(i) \to N(i)$ to a morphism $M(i) \to N(i)$. In particular we get that both π_* and π^* define functors that act by the identity on morphisms of modules. Now to see that they are indeed mutually inverse equivalences we compute their compositions

$\pi_* \circ \pi^*$: Consider an $\mathcal{O}$-module M. Then we immediately have on objects:

$$\pi_* \circ \pi^* M(i) = M(i).$$

Furthermore we have the restriction maps for $i \geq j$

$$\begin{aligned} \rho_{i,j}^{\pi_* \circ \pi^* M} &= \pi^* M (\mathrm{Id}) && \text{Definition of } \pi_* \\ &= \mathrm{Id} \circ \rho_{i,j}^M && \text{Definition of } \pi^* \\ &= \rho_{i,j}^M. \end{aligned}$$

In particular we get $\pi_* \circ \pi^* = \mathrm{Id}$.

$\pi^* \circ \pi_*$: Consider an $\widetilde{\mathcal{O}}$-module M. We again have immediately on objects

$$\pi^* \circ \pi_* M (i) = M (i) .$$

So we only need to check that

$$M \overset{!}{=} \left(\pi^* \circ \pi_* M : \widetilde{\mathcal{O}} (i, j) \to \mathrm{Vect}_{\Bbbk} (\pi_* \circ \pi^* M (i) , \pi_* \circ \pi^* M (j))\right) .$$

For this we compute

$$\begin{aligned} \pi^* \circ \pi_* M (\gamma) &= \pi_* M (\gamma) \circ \rho_{i,j}^{\pi_*} M) && \text{Definition of } \pi^* \\ &= M (\gamma) \circ M (\mathrm{Id}) && \text{Definition of } \pi_* \\ &= M (\gamma) . \end{aligned}$$

So we get $\pi^* \circ \pi_* = \mathrm{Id}$.

So

$$\pi^* : \mathcal{O}{-}\mathrm{mod} \longleftrightarrow \widetilde{\mathcal{O}}{-}\mathrm{mod} : \pi_*$$

defines an equivalence of categories. $\square$

Definition 4.1.4. Let $\mathcal{O}$ and $\mathcal{O}'$ be two presheaves of rings on a poset $(I, \leq)$. Then we define the category of bimodules over $\mathcal{O}$ and $\mathcal{O}'$ to be

$$\mathrm{Bimod}_{\Bbbk} (\mathcal{O}, \mathcal{O}') := \mathcal{O} \otimes_{\Bbbk} \mathcal{O}'^{\mathrm{op}}{-}\mathrm{mod}.$$

Lemma 4.1.5 ([LVdB11, Lemma 5.2])**.** *Let $\mathcal{O}, \mathcal{O}'$ be presheaves of rings on a poset $(I, \leq)$. Then there exsits a fully faithful functor*

$$\mathit{\Pi} : \mathcal{D} (\mathrm{Bimod}_{\Bbbk} (\mathcal{O}, \mathcal{O}')) \to \left(\mathrm{Bimod}_{\Bbbk} \left(\widetilde{\mathcal{O}}, \widetilde{\mathcal{O}'}\right)\right)$$

such that

$$\mathit{\Pi}^* \mathcal{M} (i, j) = \begin{cases} \mathcal{M} (j) & \text{if } j \leq i \\ 0 & \text{else.} \end{cases}$$

Lemma 4.1.6 ([RVdBN19, Lemma 8.2.1])**.** *Let $\mathcal{O}, \mathcal{O}'$ be preasheaves on a poset $(I, \leq)$, let $U \in \mathcal{D}(\mathcal{O}')$ and let $\mathcal{M} \in \mathcal{D}(\mathrm{Bimod}_{\Bbbk}(\mathcal{O}, \mathcal{O}'))$. Then there is a natural isomorphism in $\mathcal{D}\left(\widetilde{\mathcal{O}}\right)$*

$$\pi^*(\mathcal{M} \otimes_{\mathcal{O}'} U) \cong \mathit{\Pi}^*(\mathcal{M}) \otimes_{\widetilde{\mathcal{O}'}} \pi^* U.$$

4.1.2 Sheaves

We continue following [RVdBN19] in order to construct a $\Bbbk$-linear category corresponding to a quasi-compact $\Bbbk$-seperated scheme.

Remark 4.1.7. Recall that by [BN93] we have for a quasi-separated scheme X

$$\mathcal{D}(\mathrm{Qch}(X)) \cong \mathcal{D}_{\mathrm{Qch}}(\mathcal{O}_X\mathrm{-mod}).$$

We will now improve on this and pass for a quasi-compact scheme X from $\mathcal{D}(\mathrm{Qch}(X))$ to the derived category of modules over a small $\Bbbk$-linear category.

Definition 4.1.8 ([RVdBN19, § 8.3])**.** Let X be a quasi-compact seperated $\Bbbk$-scheme with an affine covering

$$X = \bigcup_{i \in I} U_i.$$

Define the following:

- For $I' \subset I$ the set
$$U_{I'} := \bigcap_{i \in I'} U_i.$$
- The poset $(\mathcal{I}, \leq)$ of subsets of I ordered such that
$$I' \leq I'' \iff I' \supset I''.$$
- The presheaf of rings $\widehat{\mathcal{O}_X}^I$ on $(\mathcal{I}, \leq)$ associated to $\mathcal{O}_X$, that is
$$\widehat{\mathcal{O}_X}^I(I') := \mathcal{O}_{U_{I'}}$$
with the canonical restriction morphisms:
$$\rho_{I',I''} = \rho_{U_{I'},U_{I''}} : \widehat{\mathcal{O}_X}^I(I') = \mathcal{O}_{I'} \to \mathcal{O}_{I''} = \widehat{\mathcal{O}_X}^I(I'')$$
as $I'' \leq I'$ implies $U_{I''} \subset U_{I'}$ via $I'' \supset I'$.

- Let
$$\epsilon^* : \mathcal{D}(X) \to \mathcal{D}\left(\widehat{\mathcal{O}_X}^I\text{−mod}\right)$$
be the functor induced by sending a quasi-coherent sheaf M to its corresponding presheaf $\epsilon^* M$.

We will drop the I for legibility whenever it does not impact the discussion.

Lemma 4.1.9 ([LVdB05, Theorem 7.6.6])**.** *Let X be a quasi-compact separated $\Bbbk$-scheme. Then there is an adjunction*

$$\epsilon^* : \mathcal{D}(X) \longleftrightarrow \mathcal{D}\left(\widehat{\mathcal{O}_X}^I\right) : \epsilon_*$$

such that $\epsilon_* \circ \epsilon^* \cong \mathrm{Id}$. In particular ϵ^* is fully faithful and its essential image is $\mathcal{D}_{\epsilon^*\mathrm{coh}(X)}\left(\widehat{\mathcal{O}_X}^I\right)$, i.e. objects $\mathcal{M}$ such that $\mathcal{H}^i(\mathcal{M}) \in \epsilon^*\mathrm{coh}(X)$.

Lemma 4.1.10 ([RVdBN19, Lemma 8.4.2])**.** *Let X be a quasi-separated quasi-compact scheme and let $M, N \in \mathrm{Qch}(X)$. Then*

$$\epsilon^*(M \otimes_{\mathcal{O}_X} N) \cong \epsilon^* M \otimes_{\widehat{\mathcal{O}_X}^I} \epsilon^* N.$$

Definition 4.1.11 ([RVdBN19, § 8.3])**.** Let X be a quasi-compact separated scheme. Then we denote by the corresponding curly letter the associated $\Bbbk$-linear category, that is
$$\mathcal{X} := \widetilde{\widehat{\mathcal{O}_X}}^I.$$

This comes together with a fully faithful embedding by Lemma 4.1.3 and Lemma 4.1.9
$$w : \mathcal{D}(X) \xhookrightarrow{\epsilon^*} \mathcal{D}\left(\widehat{\mathcal{O}_X}^I\text{−mod}\right) \xrightarrow[\sim]{\pi^*} \mathcal{D}(\mathcal{X}). \tag{4.1.1}$$

Lemma 4.1.12 ([RVdBN19, Lemma 8.7.1])**.** *Let $f : X \hookrightarrow Y$ be a closed immersion of quasi-compact separated schemes and $Y = \bigcup_{i\in I} U_i$ an affine cover. Then there exists an affine covering $X = \bigcup_{i\in I} f^{-1}U_i$ f such that f induces a $\Bbbk$-linear functor*

$$\mathfrak{f} : \mathcal{Y} \to \mathcal{X}$$

making the diagram

$$\begin{tikzcd}
\mathcal{D}^b(X) \arrow[r, "f_*"] \arrow[d, "w"', "\wr"] & \mathcal{D}^b(Y) \arrow[d, "\wr"', "w"] \\
\mathcal{D}(\mathcal{X}) \arrow[r, "\mathfrak{f}_*"'] & \mathcal{D}(\mathcal{Y})
\end{tikzcd}$$

commute.

Proof. Since f is a closed immersion $\bigcup_{i=1}^{n} f^{-1}U_i = X$ is an affine covering as well. In particular we can define the functor $\mathfrak{f} : \mathcal{Y} \to \mathcal{X}$ on objects to be the identity and on morphism spaces we can define it by $\mathfrak{f} := f^{\#} : \mathcal{O}_Y(U_I) \to \mathcal{O}_X(U_I)$. Now we just need to verify that this indeed defines a functor, but as $f^{\#}$ is a morphism of ringed spaces it is in particular a morphism of rings. Since composition in $\mathcal{X}$ and $\mathcal{Y}$ are defined via restriction composed with multiplication the compatibility of $f^{\#}$ with these structures gives that $\mathfrak{f}$ is a $\Bbbk$-linear functor.

Now for the diagram

$$\begin{tikzcd} \mathcal{D}^b(X) \arrow[r, "f_*"] \arrow[d, "w"'] & \mathcal{D}^b(Y) \arrow[d, "w"] \\ \mathcal{D}(\mathcal{X}) \arrow[r, "\mathfrak{f}_*"'] & \mathcal{D}(\mathcal{Y}). \end{tikzcd}$$

As all functors are induced from functors on abelian categories, it suffices to check the commutativity on the level of sheaves:

Let $\mathcal{F}$ be a sheaf on X, then we have that $w\mathcal{F}$ is the module associated to $I \mapsto \mathcal{F}(f^{-1}U_I)$ and in particular $\mathfrak{f}_* w\mathcal{F}$ is the $\mathcal{Y}$-module defined by $I \mapsto \mathcal{F}(f^{-1}U_I)$ with action of $\mathcal{Y}$ via $\mathfrak{f} := f^{\#}$. On the other hand $f_*\mathcal{F}$ is the sheaf over Y defined by $\mathcal{F}(U) = \mathcal{F}(f^{-1}U)$, which after applying w gets sent to the module $I \mapsto \mathcal{F}(f^{-1}U_I)$ with action $\mathcal{Y}$ via $f^{\#} : \mathcal{O}_Y U_I \to \mathcal{O}_X(f^{-1}U_i)$. In particular the two compositions coincide and the diagram indeed commutes. $\square$

Definition 4.1.13 ([RVdBN19, § 8.3])**.** Let X be a scheme. Then define $\Delta_*\mathcal{D}(X)$ to be the full subcategory of $\mathcal{D}(X \times X)$ consisting of objects isomorphic to $\Delta_* M$ for $M \in \mathcal{D}(X)$, where we denote by $\Delta : X \to X \times X$ the diagonal.

We will use the above defined notation even though it is not closed under cones and Δ_* is not full, however it makes the following diagrams and concepts clearer.

Proposition 4.1.14 ([RVdBN19, § 8.3])**.** *Let $X = \bigcup_{i \in I} U_i$ be an open covering of a quasi-compact seperated scheme X and set*

$$Z := \bigcup_{i \in I} U_i \times U_i \subset X \times X.$$

Then

$$\iota_Z^* : \Delta_*\mathcal{D}^b(X) \hookrightarrow \mathcal{D}^b(Z)$$

is fully faithful.

Proof. Since X is seperated we have that

$$\varDelta : X \hookrightarrow Z \subset X \times X$$

is a closed embedding and so all objects in $\varDelta_* \mathcal{D}^b(\mathrm{coh}(X))$ are supported on Z and we can restrict to Z and get that the functor

$$\iota_* : \varDelta_* \mathcal{D}^b(X) \hookrightarrow \mathcal{D}^b(Z)$$

is fully faithful. □

Lemma 4.1.15 ([RVdBN19, Lemma 8.4.5])**.** *Let X be quasi-compact separated and let $M, N \in \mathcal{D}(X)$. Then we have*

$$\epsilon^* \varDelta_* M \otimes_{\widehat{\mathcal{O}_X}^I} \epsilon^* N \cong \epsilon^* (M \otimes_{\mathcal{O}_X} N). \tag{4.1.2}$$

Lemma 4.1.16 ([RVdBN19, § 8.3])**.** *Let X be a quasi-compact separated scheme. Then there is a fully faithful embedding*

$$W : \varDelta_* \mathcal{D}(X) \xrightarrow{\iota_Z^*} \mathcal{D}(Z) \xrightarrow{\epsilon^*} \mathcal{D}\left(\widehat{\mathcal{O}_X}^I \otimes \widehat{\mathcal{O}_X}^I\right) \xrightarrow{\varPi^*} \mathcal{D}(\mathcal{X} \times \mathcal{X}). \tag{4.1.3}$$

Proof. By Proposition 4.1.14 and Lemma 4.1.5 we have that $\varDelta_*$ and $\varPi^*$ are fully faithful embeddings, in particular we only need to worry about ϵ^*, but for this it suffices to observe

$$\widehat{\mathcal{O}_Z}^I \cong \widehat{\mathcal{O}_X}^I \otimes \widehat{\mathcal{O}_X}^I.$$

So W is a fully faithful embedding as a composition of fully faithful embeddings. □

Corollary 4.1.17. *Let X be a separated scheme and let M be a sheaf on X. Then we have*

$$\mathrm{HH}^*(X, M) \cong \mathrm{HH}^*(\mathcal{X}, W(M)). \tag{4.1.4}$$

Proof. By Lemma 4.1.16 we have that W is fully faithful. In particular

$$W : \mathrm{HH}^*(X, M) = \mathrm{Ext}^*_{X \times X}(\mathcal{O}_\varDelta, \varDelta_* M) \xrightarrow{\sim} \mathrm{Ext}^*_{\mathcal{X} \times \mathcal{X}}(\mathcal{X}, WM)$$

is an isomorphism. □

4.2 Equivariant Sheaves

We apply the above discussion on passing from sheaves to modules to the case of $\varGamma$-equivariant sheaves, for a $\Bbbk$-algebra $\varGamma$.

Lemma 4.2.1. *Let X be a quasi-compact scheme and let Γ be a $\Bbbk$-algebra. Then we have an embedding*

$$w : \mathcal{D}^b\left(\operatorname{coh}\left(X\right)_\Gamma\right) \hookrightarrow \mathcal{D}\left(\mathcal{X} \otimes \Gamma\right).$$

Proof. By Lemma 4.1.3 we have an embedding $\operatorname{Qch}\left(X\right) \to \widehat{\mathcal{O}_X}^I-\mathrm{mod}$. In particular if we restrict the embedding to coherent sheaves we get an embedding $\operatorname{coh}\left(X\right) \hookrightarrow \widehat{\mathcal{O}_X}^I-\mathrm{mod}$.

So we get for quasi-compact schemes an embedding $\operatorname{coh}\left(X\right)_\Gamma \hookrightarrow \widehat{\mathcal{O}_X}^I-\mathrm{mod}_\Gamma$ which turns after deriving into

$$\mathcal{D}\left(\operatorname{coh}\left(X\right)_\Gamma\right) \hookrightarrow \mathcal{D}\left(\widehat{\mathcal{O}_X}^I-\mathrm{mod}_\Gamma\right) \cong \mathcal{D}\left(\widehat{\mathcal{O}_X}^I \otimes \Gamma\right).$$

Now we can finally apply Lemma 4.1.9 to get an embedding

$$\mathcal{D}\left(\operatorname{coh}\left(X\right)_\Gamma\right) \hookrightarrow \mathcal{D}\left(\widetilde{\widehat{\mathcal{O}}^I \otimes \Gamma}\right).$$

Since the construction of $\widetilde{(\_)}$ actually just translates the local data on a poset to the data of a $\Bbbk$-linear category this is compatible with tensoring with a $\Bbbk$-linear category. In particular we get a functor

$$w : \mathcal{D}\left(\operatorname{coh}\left(X\right)_\Gamma\right) \hookrightarrow \mathcal{D}\left(\mathcal{X} \otimes \Gamma\right),$$

where we used Definition 4.1.11 $\widetilde{\widehat{\mathcal{O}_X}}^I = \mathcal{X}$. $\square$

4.2.1 Actions of Bimodules

We will now recall that under the equivalences w and W the functor

$$\begin{aligned} \Phi_{\_}\left(\_\right) : \Delta_*\mathcal{D}^b\left(\operatorname{Qch}\left(X\right)\right) \times \mathcal{D}^b\left(\operatorname{Qch}\left(X\right)\right) &\to \mathcal{D}^b\left(\operatorname{Qch}\left(X\right)\right) \\ \left(M, N\right) &\mapsto \Phi_M\left(N\right) \end{aligned}$$

corresponds to

$$\_ \otimes_\mathcal{X} \_ : \mathcal{D}\left(\mathcal{X} \times \mathcal{X}^{\mathrm{op}}\right) \times \mathcal{D}\left(\mathcal{X}\right) \to \mathcal{D}\left(\mathcal{X}\right),$$

where $\Phi_M\left(\_\right)$ denotes the Fourier-Mukai functor with kernel M.

Lemma 4.2.2 ([RVdBN19, Lemma 8.4.1])**.** *Let X be a smooth quasi-compact and* *separated scheme. Then we have the following commutative diagram*

$$\begin{tikzcd}
\Delta_* \mathcal{D}^b(\mathrm{Qch}(X)) \arrow[d, hook, "W"] & \times & \mathcal{D}(\mathrm{Qch}(X)) \arrow[d, hook, "w"'] \arrow[r, "\Phi_{_}(_)"] & \mathcal{D}(\mathrm{Qch}(X)) \arrow[d, hook, "w"] \\
\mathcal{D}(\mathcal{X} \otimes_{\Bbbk} \mathcal{X}^{\mathrm{op}}) & \times & \mathcal{D}(X) \arrow[r, "_ \otimes_{\mathcal{X}} _"'] & \mathcal{D}(X).
\end{tikzcd}$$

Corollary 4.2.3. *Let X be a smooth quasi-compact and separated scheme. Then we have the following commutative diagram*

$$\begin{tikzcd}
\Delta_* \mathcal{D}(\mathrm{coh}(X)) \arrow[d, hook, "W"] & \times & \mathcal{D}(\mathrm{coh}(X)) \arrow[d, hook, "w"'] \arrow[r, "\Phi_{_}(_)"] & \mathcal{D}(\mathrm{coh}(X)) \arrow[d, hook, "w"] \\
\mathcal{D}(\mathcal{X} \otimes_{\Bbbk} \mathcal{X}^{\mathrm{op}}) & \times & \mathcal{D}(X) \arrow[r, "_ \otimes_{\mathcal{X}} _"'] & \mathcal{D}(X).
\end{tikzcd}$$

Proof. We have by [Huy06, Proposition 3.5] an embedding $\mathcal{D}(X) \hookrightarrow \mathcal{D}(\mathrm{Qch}X)$. In particular we can restrict the diagram from Lemma 4.2.2 to $\mathcal{D}(X)$. □

Lemma 4.2.4 ([RVdBN19, (8.3)])**.** *Let X be a smooth quasi-compact and separated scheme and Γ a $\Bbbk$-algebra. Then we have the following commutative diagram*

$$\begin{tikzcd}
\Delta_* \mathcal{D}(\mathrm{Qch}(X)) \arrow[d, hook, "W"] & \times & \mathcal{D}(\mathrm{Qch}(X)_{\Gamma}) \arrow[d, hook, "w"'] \arrow[r, "\Phi_{_}(_)"] & \mathcal{D}(\mathrm{Qch}(X)_{\Gamma}) \arrow[d, hook, "w"] \\
\mathcal{D}(\mathcal{X} \otimes_{\Bbbk} \mathcal{X}^{\mathrm{op}}) & \times & \mathcal{D}(\mathcal{X} \otimes_{\Bbbk} \Gamma) \arrow[r, "_ \otimes_{\mathcal{X}} _"'] & \mathcal{D}(\mathcal{X} \otimes_{\Bbbk} \Gamma).
\end{tikzcd}$$

Corollary 4.2.5. *Let X be a smooth quasi-compact and separated scheme and Γ a $\Bbbk$-algebra. Then we have the following commutative diagram*

$$\begin{tikzcd}
\Delta_* \mathcal{D}(\mathrm{coh}(X)) \arrow[d, hook, "W"] & \times & \mathcal{D}(\mathrm{coh}(X)_{\Gamma}) \arrow[d, hook, "w"'] \arrow[r, "\Phi_{_}(_)"] & \mathcal{D}(\mathrm{coh}(X)_{\Gamma}) \arrow[d, hook, "w"] \\
\mathcal{D}(\mathcal{X} \otimes_{\Bbbk} \mathcal{X}^{\mathrm{op}}) & \times & \mathcal{D}(\mathcal{X} \otimes_{\Bbbk} \Gamma) \arrow[r, "_ \otimes_{\mathcal{X}} _"'] & \mathcal{D}(\mathcal{X} \otimes_{\Bbbk} \Gamma).
\end{tikzcd}$$

Proof. Similar to Corollary 4.2.3 we can embed $\mathcal{D}(\mathrm{coh}(X)_{\Gamma}) \hookrightarrow \mathcal{D}(\mathrm{Qch}(X)_{\Gamma})$. So we can again restrict the diagram from Lemma 4.2.4. □

Chapter 5

Twisted Hodge Diamonds give Kernels in Hochschild Cohomology

We will show how twisted Hodge diamonds, and in particular their interior, can be used to understand the pushforward of Hochschild cohomology under the closed embedding of a smooth projective hypersurface of degree d.

Throughout this chapter we will follow Brückmann‘s paper ”Zur Kohomologie von projektiven Hyperflächen” [Brü74] for computations.

Definition 5.0.1. Let X be a projective scheme of dimension n and let $\mathcal{O}_X(1)$ be a very ample line bundle. Then we define the twisted Hodge numbers of X to be

$$\mathrm{h}_p^{i,j}(X) := \dim \mathrm{H}^j\left(X, \Omega_X^i(p)\right).$$

Similarly to ordinary Hodge numbers the twisted Hodge numbers can be arranged in a twisted Hodge diamond:

$$\begin{tikzcd}
 & \mathrm{h}_p^{n,n}(X) \arrow[dl, dash] \arrow[dr, dash] & \\
\mathrm{h}_p^{n,0}(X) \arrow[dr, dash] & & \mathrm{h}_p^{0,n}(X) \arrow[dl, dash] \\
 & \mathrm{h}_p^{0,0}(X). &
\end{tikzcd}$$

We will drop the X if the space is clear from context.

Lemma 5.0.2. *Let X be a smooth projective scheme of dimension n with canonical sheaf of form $\mathcal{O}_X(t)$. Then we have*

$$\mathrm{HH}^m(X, \mathcal{O}_X(p)) \cong \bigoplus_{i=0}^{n} \mathrm{H}^{i-m+n}\left(X, \Omega_X^i\left(t-p\right)\right).$$

In particular this gives

$$\dim \mathrm{HH}^m(X, \mathcal{O}_X(p)) = \sum_{i=0}^{n} \mathrm{h}_{t-p}^{i,i-m+n}(X).$$

Proof. We compute, using $\omega_X \cong \mathcal{O}_X(t)$ and the Hochschild-Kostant-Rosenberg (HKR) isomorphism [Swa96]

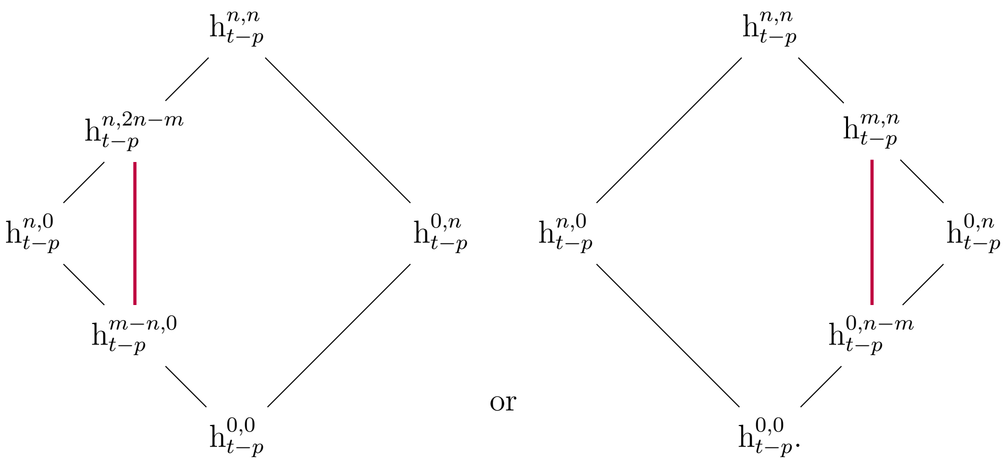
$$\begin{aligned}
\mathrm{HH}^m(X, \mathcal{O}_X(p)) &\cong \bigoplus_{i=0}^{n} \mathrm{Ext}_X^{m-i}\left(\Omega_X^i, \mathcal{O}_X(p)\right) && \text{HKR} \\
&\cong \bigoplus_{i=0}^{n} \mathrm{Ext}_X^{n-m+i}\left(\mathcal{O}_X(p), \Omega_X^i(t)\right)^* && \text{Serre Duality} \\
&\cong \bigoplus_{i=0}^{n} \mathrm{Ext}_X^{n-m+i}\left(\mathcal{O}_X, \Omega_X^i(t-p)\right)^* && \text{twisting on both sides} \\
&\cong \bigoplus_{i=0}^{n} \mathrm{H}^{n-m+i}\left(X, \Omega_X^i(t-p)\right)^*. && \mathrm{Ext}_X^j(\mathcal{O}_X, _) \cong \mathrm{H}^j(X, _)
\end{aligned}$$

Applying dimension on both sides gives

$$\dim \mathrm{HH}^m(X, \mathcal{O}_X(p)) = \sum_{i=0}^{n} \mathrm{h}_{t-p}^{i,i-m+n}(X)$$

as desired. □

Remark 5.0.3. By Lemma 5.0.2 one can compute $\dim \mathrm{HH}^m(X, \mathcal{O}_X(p))$ as the sum over the m-th column in the $t-p$ twisted hodge diamond,



5.1 The Hochschild Cohomology of a smooth Hypersurface

We will use the computations in [Brü74] and Lemma 5.0.3 to compute the Hochschild cohomology of X.

Lemma 5.1.1. *Let $X \hookrightarrow \mathbb{P}^{n+1}$ be a smooth degree d hypersurface. Then*

$$\mathrm{h}_p^{i,j}(X) = 0$$

if (i,j) is not of the form $(i,0),(i,n),(i,n-i),(i,i)$, with $0 \leq i \leq n$. And we have for (i,i):

$$\mathrm{h}_p^{i,i}(X) = \delta_{p,0} \qquad \text{if } i \notin \left\{0, \frac{n}{2}, n\right\}.$$

Moreover we get

$$\mathrm{h}_p^{i,n-i}(X) = \sum_{\mu=0}^{n+2} (-1)^\mu \binom{n+2}{\mu}\binom{-p+id-(\mu-1)(d-1)}{n+1} + \delta_{p,0}\delta_{i,n-i}. \quad (5.1.1)$$

Proof. First of all we can assume $0 \leq i,j \leq n$ as outside of that range we have $\Omega_X^i(p) = 0$, respectively $\mathrm{H}^j(X, \Omega_X^i(p)) = 0$ for dimension reasons.

By [Brü74, Satz 2,(42),(40),(38) and (39)] we have for $0 < i < n$

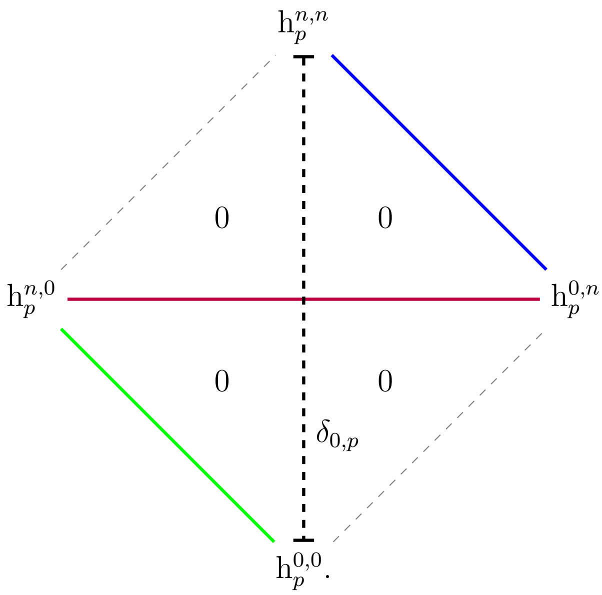
$$\mathrm{h}_p^{i,j}(X) = \begin{cases} \binom{-p-1}{n-i}\binom{-p+1+i}{1+i} + \sum_{\mu=1}^{n-i+1}(-1)^\mu\binom{n+2}{\mu}\binom{-p-\mu(d-1)+i}{n+1} & \text{if } j = n \\ \sum_{\mu=0}^{n+2}(-1)^\mu\binom{n+2}{\mu}\binom{-p+id-(\mu-1)(d-1)}{n+1} + \delta_{p,0}\delta_{i,j} & \text{if } i+j=n \\ \binom{p-1}{i}\binom{p+n+1-i}{n+1-i} + \sum_{\mu=1}^{i+1}(-1)^\mu\binom{n+2}{\mu}\binom{p+n-\mu(d-1)-i}{n+1} & \text{if } j = 0 \\ \delta_{p,0} & \text{if } i = j \notin \{0,n\} \\ 0 & \text{else.} \end{cases}$$

So only the cases for $i \in \{0,n\}$ remain. Now [Brü74, Lemma 5] gives for $j \notin \{0,n\}$

$$\mathrm{h}_p^{0,j}(X) = 0 = \mathrm{h}_p^{n,j}(X).$$

Which finishes the claim. □

Remark 5.1.2. By Lemma 5.1.1 the p-twisted Hodge diamond of a smooth degree d hypersurface has the shape:



In particular the only non-trivial entries appear along the indicated lines. More precisely we have along the blue lines the values for $\mathrm{h}^{i,n}(X)$, along the red lines the values $\mathrm{h}^{i,n-i}(X)$ and along the green line $\mathrm{h}^{i,0}(X)$. Furthermore the dashed line disappears if $p \neq 0$ as these are the Kronecker deltas $\delta_{p,0}$.

Proposition 5.1.3. *Let $X \hookrightarrow \mathbb{P}^{n+1}$ be the embedding of a smooth degree d hypersurface. Then the following formulas hold.*

$$\begin{aligned}
\mathrm{h}_p^{i,0}(X) &= \mathrm{h}_{-p}^{n-i,n}(X) && \\
\mathrm{h}_p^{i,n-i}(X) &= \mathrm{h}_{p-d}^{i-1,n+1-i}(X) && i \notin \{0,1,n\},\ p \neq 0 \\
\mathrm{h}_{p-d}^{i,n+1}\left(\mathbb{P}^{n+1}\right) - \mathrm{h}_p^{i,n+1}\left(\mathbb{P}^{n+1}\right) &= \mathrm{h}_p^{i,n}(X) + \mathrm{h}_{p-d}^{i-1,n}(X) && i \notin \{0,1,n\} \\
\mathrm{h}_p^{i,0}\left(\mathbb{P}^{n+1}\right) - \mathrm{h}_{p-d}^{i,0}\left(\mathbb{P}^{n+1}\right) &= \mathrm{h}_p^{i,0}(X) + \mathrm{h}_{p-d}^{i-1,0}(X) && i \notin \{0,1,n\}.
\end{aligned}$$

Proof. We compute for the first equation:

$$\begin{aligned}
\mathrm{h}_p^{i,0}(X) &= \dim \mathrm{H}^0\left(X, \Omega_X^i(p)\right) && \text{definition} \\
&= \dim \mathrm{Ext}^0\left(\mathcal{O}_X, \Omega_X^i(p)\right) && \mathrm{Ext}^*(\mathcal{O}_X, _) \cong \mathrm{H}^*(X, _) \\
&= \dim \mathrm{Ext}^n\left(\Omega_X^i(p), \Omega_X^n\right) && \text{Serre Duality} \\
&= \dim \mathrm{Ext}^n\left(\mathcal{O}_X, \Omega_X^{n-i}(-p)\right) && \\
&= \dim \mathrm{H}^n\left(X, \Omega_X^{n-i}(-p)\right) && \mathrm{Ext}^*(\mathcal{O}_X, _) \cong \mathrm{H}^*(X, _) \\
&= \mathrm{h}_p^{n-i,0}(X). && \text{definition}
\end{aligned}$$

For the second equation we we have by (5.1.1) the following identity

$$\begin{aligned}
\mathrm{h}_p^{i,n-i}(X) &= \sum_{\mu=0}^{n+2}(-1)\binom{n+2}{\mu}\binom{-p+id-(\mu-1)(d-1)}{n+1} \\
&= \sum_{\mu=0}^{n+2}(-1)\binom{n+2}{\mu}\binom{-p+d-d+id-(\mu-1)(d-1)}{n+1} \\
&= \sum_{\mu=0}^{n+2}(-1)\binom{n+2}{\mu}\binom{-p+d+(i-1)d-(\mu-1)(d-1)}{n+1} \\
&= \mathrm{h}_{p-d}^{i-1,n+1-i}(X).
\end{aligned}$$

And for the last two Brückmann gives the formula [Brü74, (31)], which together with [Brü74, Satz 2] gives both

$$\begin{aligned}
\mathrm{h}_{p-d}^{i,n}(X) &= \mathrm{h}_{p-d}^{i+1,n}\left(\mathbb{P}^{n+1}\right)-\mathrm{h}_p^{i+1,n}\left(\mathbb{P}^{n+1}\right)-\mathrm{h}_p^{i+1,n}(X) \\
\mathrm{h}_p^{i,0}(X) &= \mathrm{h}_p^{i,0}\left(\mathbb{P}^{n+1}\right)-\mathrm{h}_p^{i,0}\left(\mathbb{P}^{n+1}\right)-\mathrm{h}_{p-d}^{i-1,0}(X).
\end{aligned}$$

After rearranging, these are

$$\begin{aligned}
\mathrm{h}_{p-d}^{i,n}(X)+\mathrm{h}_p^{i+1,n}(X) &= \mathrm{h}_{p-d}^{i+1,n}\left(\mathbb{P}^{n+1}\right)-\mathrm{h}_p^{i+1,n}\left(\mathbb{P}^{n+1}\right) \\
\mathrm{h}_p^{i,0}(X)+\mathrm{h}_{p-d}^{i-1,0}(X) &= \mathrm{h}_p^{i,0}\left(\mathbb{P}^{n+1}\right)-\mathrm{h}_p^{i,0}\left(\mathbb{P}^{n+1}\right).
\end{aligned}$$

Index shifting in the first equation gives

$$\begin{aligned}
\mathrm{h}_{p-d}^{i,n}\left(\mathbb{P}^{n+1}\right)-\mathrm{h}_p^{i,n}\left(\mathbb{P}^{n+1}\right) &= \mathrm{h}_p^{i,n}(X)+\mathrm{h}_{p-d}^{i-1,n}(X) \\
\mathrm{h}_p^{i,0}\left(\mathbb{P}^{n+1}\right)-\mathrm{h}_{p-d}^{i,0}\left(\mathbb{P}^{n+1}\right) &= \mathrm{h}_p^{i,0}(X)+\mathrm{h}_{p-d}^{i-1,0}(X),
\end{aligned}$$

as claimed. □

We can use Lemma 5.1.1 together with Lemma 5.0.2 to compute the dimensions of $\mathrm{HH}^m(X,\mathcal{O}_X(p))$:

Corollary 5.1.4. *Let X be a smooth n-dimensional hypersurface of degree d and let $t=d-n-2$. Then we have that* $\dim \mathrm{HH}^m(X,\mathcal{O}_X(p))$ *is given by*

$$\begin{cases}
\mathrm{h}_{t-p}^{0,n}(X) & \textit{for } m=0 \\
\mathrm{h}_{t-p}^{m-n,0}(X)+\mathrm{h}_{t-p}^{\frac{m}{2},n-\frac{m}{2}}(X)+\mathrm{h}_{t-p}^{m,n}(X)+(n-2)\,\delta_{t,p}\delta_{m,n} & \textit{for } 0<m<2n \textit{ even} \\
\mathrm{h}_{t-p}^{m-n,0}(X)+\mathrm{h}_{t-p}^{m,n}(X)+(n-1)\,\delta_{t,p}\delta_{m,n} & \textit{for } 0<m<2n \textit{ odd} \\
\mathrm{h}_{t-p}^{n,0}(X) & \textit{for } m=2n \\
0 & \textit{else.}
\end{cases}$$

Proof. Observe that we have $\mathcal{O}_X(t) \cong \omega_X$ and that we can assume $0 \leq m \leq 2n$ for dimension reasons. By Lemma 5.0.2 we have

$$\dim \mathrm{HH}^m(X, \mathcal{O}_X(p)) = \sum_{i=0}^{n} \mathrm{h}^{i,i-m+n}_{t-p}(X).$$

So we can use Lemma 5.1.1 to compute every summand. In particular we get for $n = 0$ and $n = 2n$

$$\dim \mathrm{HH}^0(X, \mathcal{O}_X(p)) = \mathrm{h}^{0,n}_{t-p}(X)$$
$$\dim \mathrm{HH}^{2n}(X, \mathcal{O}_X(p)) = \mathrm{h}^{n,0}_{t-p}(X)$$

as there only one summand appears.

Now for $0 < m < 2n$ we can use Lemma 5.1.1 to get for $m \neq n$

$$\begin{aligned}\dim \mathrm{HH}^m(X, \mathcal{O}_X(p)) &= \sum_{i}^{m-n} \mathrm{h}^{i,i-n+m}_{t-p}(X)\\ &= \sum_{i-j=m-n} \mathrm{h}^{i,j}_{t-p}(X)\\ &= \begin{cases} \mathrm{h}^{m-n,0}_{t-p}(X) + \mathrm{h}^{\frac{2n-m}{2},\frac{m}{2}}_{t-p}(X) + \mathrm{h}^{m,n}_{t-p}(X) & \text{for } m \text{ even}\\ \mathrm{h}^{m-n,0}_{t-p}(X) + \mathrm{h}^{m,n}_{t-p}(X) & \text{for } m \text{ odd.}\end{cases}\end{aligned}$$

For $m = n$ all the above calculations still hold, however, we get for all (i,i) with $0 < i < n$ and $(i,i) \neq \left(\frac{n}{2}, \frac{n}{2}\right)$ an additional $\delta_{0,t-p} = \delta_{t,p}$, which means that $\dim \mathrm{HH}^n(X, \mathcal{O}_X(p))$ is given by

$$\begin{cases} \mathrm{h}^{0,0}_{t-p}(X) + \mathrm{h}^{\frac{n}{2},\frac{n}{2}}_{t-p}(X) + \mathrm{h}^{n,n}_{t-p}(X) + (n-2)\,\delta_{t,p}\delta_{m,n} & \text{for } m \text{ even}\\ \mathrm{h}^{0,0}_{t-p}(X) + \mathrm{h}^{n,n}_{t-p}(X) + (n-1)\,\delta_{t,p}\delta_{m,n} & \text{for } m \text{ odd}\end{cases}$$

as claimed. $\square$

5.2 The Hochschild Cohomology of the direct Image

Since we want to control the pushforward in Hochschild cohomology we will use computations by [Brü74] to understand the Hochschild dimensions of the direct image of a line bundle under a smooth embedding.

Lemma 5.2.1. *Let $f : X \hookrightarrow Y$ be an embedding of a smooth n-dimensional degree*

d hypersurface and set $t = d - n - 2$. Then we have

$$\mathrm{HH}^m \left(Y, f_* \mathcal{O}_X (p)\right) \cong \bigoplus_{i=0}^{\dim Y} \mathrm{H}^{n-m+i} \left(X, f^* \Omega_Y^i (t-p)\right).$$

Proof. We can compute, using $\omega_X \cong \mathcal{O}_X (t)$ and the Hochschild-Kostant-Rosenberg Isomorphism (HKR) [Swa96]

$$\begin{aligned}
\mathrm{HH}^m \left(Y, f_* \mathcal{O}_X (p)\right) &\cong \bigoplus_{i=0}^{\dim Y} \mathrm{Ext}_Y^{m-i} \left(\Omega_Y^i, f_* \mathcal{O}_X (p)\right) && \text{HKR} \\
&\cong \bigoplus_{i=0}^{\dim Y} \mathrm{Ext}_X^{m-i} \left(f^* \Omega_Y^i, \mathcal{O}_X (p)\right) && f^* \dashv f_* \\
&\cong \bigoplus_{i=0}^{\dim Y} \mathrm{Ext}_X^{n-m+i} \left(\mathcal{O}_X (p), f^* \Omega_Y^i (t)\right)^* && \text{Serre Duality} \\
&\cong \bigoplus_{i=0}^{\dim Y} \mathrm{Ext}_X^{n-m+i} \left(\mathcal{O}_X, f^* \Omega_Y^i (t-p)\right)^* && \text{twisting on both sides} \\
&\cong \bigoplus_{i=0}^{\dim Y} \mathrm{H}^{n-m+i} \left(X, f^* \Omega_Y^i (t-p)\right)^* && \mathrm{Ext}_X^j \left(\mathcal{O}_X, _\right) \cong \mathrm{H}^j \left(X, _\right)
\end{aligned}$$

as desired. ☐

Lemma 5.2.2. *Let $f : X \hookrightarrow \mathbb{P}^{n+1}$ be a closed embedding of a smooth degree d hypersurface. Then we have for $(i, j) \notin \{(0,0), (0,n), (0,n+1), (n,1) (n,n), (n,n+1)\}$*

$$\dim \mathrm{H}^j \left(X, f^* \Omega_{\mathbb{P}^{n+1}}^i (p)\right) = \begin{cases} \mathrm{h}_p^{i,0} (X) + \mathrm{h}_{p-d}^{i-1,0} (X) & \textit{if } j{=}0 \\ \mathrm{h}_p^{i,n} (X) + \mathrm{h}_{p-d}^{i-1,n} (X) & \textit{if } j = n \\ \delta_{p,0} & \textit{if } i = j \notin \{0, n\} \\ \delta_{p,d} & \textit{if } i - 1 = j \notin \{0, n\} \\ 0 & \textit{else.} \end{cases}$$

Moreover, we get

$$\begin{aligned}
\dim \mathrm{H}^0 \left(X, f^* \Omega_{\mathbb{P}^{n+1}}^n (p)\right) &= \mathrm{h}_p^{n,0} (X) + \mathrm{h}_{p-d}^{n-1,0} (X) - \mathrm{h}_{p-d}^{n-1,1} (X) \\
\dim \mathrm{H}^n \left(X, f^* \Omega_{\mathbb{P}^{n+1}}^1 (p)\right) &= \mathrm{h}_p^{1,n} (X) + \mathrm{h}_{p-d}^{0,n} (X) - \mathrm{h}_p^{1,n-1} (X) \\
\dim \mathrm{H}^0 \left(X, f^* \Omega_{\mathbb{P}^{n+1}}^0 (p)\right) &= \mathrm{h}_p^{0,0} (X) \\
\dim \mathrm{H}^n \left(X, f^* \Omega_{\mathbb{P}^{n+1}}^0 (p)\right) &= \mathrm{h}_p^{0,n} (X) \\
\dim \mathrm{H}^0 \left(X, f^* \Omega_{\mathbb{P}^{n+1}}^{n+1} (p)\right) &= \mathrm{h}_{p-d}^{n,0} (X) \\
\dim \mathrm{H}^n \left(X, f^* \Omega_{\mathbb{P}^{n+1}}^{n+1} (p)\right) &= \mathrm{h}_{p-d}^{n,n} (X).
\end{aligned}$$

Proof. First observe that $\Omega_{\mathbb{P}^{n+1}}^i = 0$ for $i > n+1$ and $i < 0$. In particular we can assume $0 \leq i \leq n+1$ and for dimension reasons we can additionally assume $0 \leq j \leq n$. Furthermore, [Brü74, Lemma 5 and Lemma 6] give for $0 < j < n$

$$\begin{aligned} \dim \mathrm{H}^j \left(X, f^* \Omega^0 (p)\right) &= 0 \\ \dim \mathrm{H}^j \left(X, f^* \Omega^{n+1} (p)\right) &= 0. \end{aligned}$$

By [Brü74, Satz 1, Lemma 6, (21) and (25)] we have for $0 < i < n+1$

$$\dim \mathrm{H}^j \left(X, f^* \Omega^i_{\mathbb{P}^{n+1}} (p)\right) = \begin{cases} \mathrm{h}^{i,0}_p \left(\mathbb{P}^{n+1}\right) - \mathrm{h}^{i,0}_{p-d} \left(\mathbb{P}^{n+1}\right) & \text{if j=0} \\ \mathrm{h}^{i,n+1}_p \left(\mathbb{P}^{n+1}\right) - \mathrm{h}^{i,n+1}_{d-p} \left(\mathbb{P}^{n+1}\right) & \text{if } j = n \\ \delta_{p,0}\delta_{i,j} + \delta_{p,d}\delta_{i-1,j} & \text{if } j \notin \{0, n\}. \end{cases}$$

Which gives after applying Proposition 5.1.3 for $i \notin \{1, n\}$

$$\dim \mathrm{H}^j \left(X, f^* \Omega^i_{\mathbb{P}^{n+1}} (p)\right) = \begin{cases} \mathrm{h}^{i,0}_p (X) + \mathrm{h}^{i-1,0}_{p-d} (X) & \text{if j=0} \\ \mathrm{h}^{i,n}_p (X) + \mathrm{h}^{i-1,n}_{d-p} (X) & \text{if } j = n \\ \delta_{p,0}\delta_{i,j} + \delta_{p,d}\delta_{i-1,j} & \text{if } j \notin \{0, n\}. \end{cases}$$

Now for the special cases:

We start with the case of $i \in \{1, n\}$. By the discussion above we have

$$\begin{aligned} \dim \mathrm{H}^0 \left(X, f^* \Omega^n_{\mathbb{P}^{n+1}} (p)\right) &= \mathrm{h}^{n,0}_p \left(\mathbb{P}^{n+1}\right) - \mathrm{h}^{n,0}_{p-d} \left(\mathbb{P}^{n+1}\right) \\ \dim \mathrm{H}^n \left(X, f^* \Omega^1_{\mathbb{P}^{n+1}} (p)\right) &= \mathrm{h}^{1,n+1}_p \left(\mathbb{P}^{n+1}\right) - \mathrm{h}^{1,n+1}_{p-d} \left(\mathbb{P}^{n+1}\right). \end{aligned}$$

This turns, using [Brü74, (31), (33), Satz 2 and Lemma 5] into

$$\begin{aligned} \dim \mathrm{H}^0 \left(X, f^* \Omega^n_{\mathbb{P}^{n+1}} (p)\right) &= \mathrm{h}^{n,0}_p (X) + \mathrm{h}^{n-1,0}_{p-d} (X) - \mathrm{h}^{n-1,1}_{p-d} (X) \\ \dim \mathrm{H}^n \left(X, f^* \Omega^1_{\mathbb{P}^{n+1}} (p)\right) &= \mathrm{h}^{1,n}_p (X) + \mathrm{h}^{0,n}_{p-d} (X) - \mathrm{h}^{1,n-1}_p (X). \end{aligned}$$

So only the cases for $i = 0$ and $i = n+1$ remain:

For $i = 0$ we have $f^* \Omega^0_{\mathbb{P}^{n+1}} (p) \cong \mathcal{O}_X (p)$, so we can apply Lemma 5.0.1 to get:

$$\begin{aligned} \dim \mathrm{H}^0 \left(X, \mathcal{O}_X (p)\right) &= \mathrm{h}^{0,0}_p (X) \\ \dim \mathrm{H}^n \left(X, \mathcal{O}_X (p)\right) &= \mathrm{h}^{0,n}_p (X). \end{aligned}$$

Now for $i = n+1$ we have

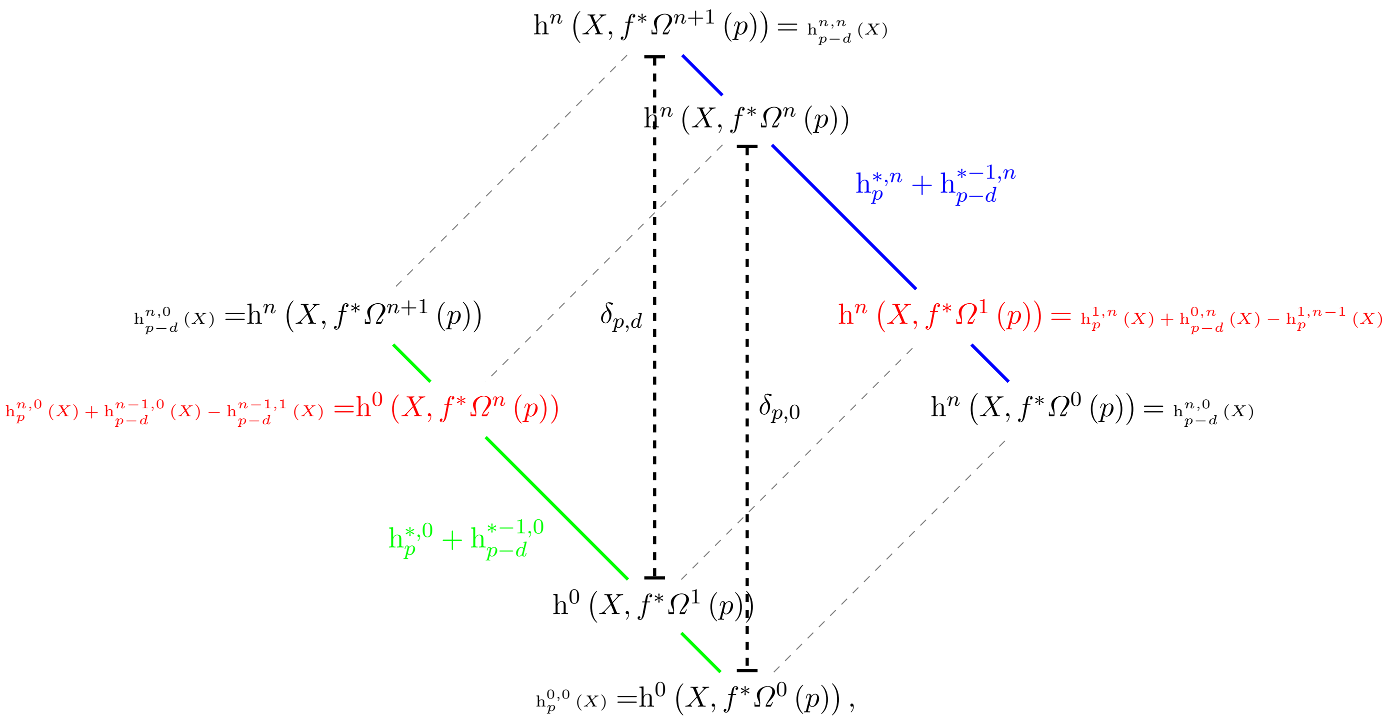
$$\begin{aligned} f^*\Omega^{n+1}_{\mathbb{P}^{n+1}}(p) &\cong f^*\mathcal{O}_X(p-n-2) \\ &\cong \mathcal{O}_X(d-n-2+p-d) \\ &\cong \Omega^n_X(p-d). \end{aligned}$$

And so we get by Definition 5.0.1

$$\begin{aligned} \dim \mathrm{H}^0\left(X, \Omega^n_X(p-d)\right) &= \mathrm{h}^{n,0}_{p-d}(X) \\ \dim \mathrm{H}^n\left(X, \Omega^n_X(p-d)\right) &= \mathrm{h}^{n,n}_{p-d}(X) \end{aligned}$$

as claimed. $\square$

Remark 5.2.3. If we arrange the computation of the cohomology dimensions from Lemma 5.2.2 analogously to a twisted hodge diamond, we get that it is of the shape



where apart from the two special cases

$$\begin{aligned} \mathrm{h}^0\left(X, f^*\Omega^n_{\mathbb{P}^{n+1}}(p)\right) &= \mathrm{h}^{n,0}_p(X) + \mathrm{h}^{n-1,0}_{p,d}(X) - \mathrm{h}^{n-1,1}_d(X) \\ \mathrm{h}^n\left(X, f^*\Omega^1_{\mathbb{P}^{n+1}}(p)\right) &= \mathrm{h}^{1,n}_p(X) + \mathrm{h}^{0,n}_{p-d}(X) - \mathrm{h}^{1,n-1}_{p-d}(X) \end{aligned}$$

the only non-trivial entries are along the indicated lines. There we have

$$\mathrm{h}^0\left(X, f^*\Omega^i_{\mathbb{P}^{n+1}}\right) = \mathrm{h}^{*,0}_p(X) + \mathrm{h}^{*-1,0}_{p-d}(X)$$
$$\mathrm{h}^n\left(X, f^*\Omega^i_{\mathbb{P}^{n+1}}\right) = \mathrm{h}^{*,n}_p(X) + \mathrm{h}^{*-1,n}_{p-d}(X)$$

and along the two vertical diagonals we have

$$\mathrm{h}^i\left(X, f^*\Omega^i_{\mathbb{P}^{n+1}}\right) = \delta_{p,0}$$
$$\mathrm{h}^i\left(X, f^*\Omega^{i+1}_{\mathbb{P}^{n+1}}\right) = \delta_{p,d}.$$

Observe that this has the shape of the p and $p-d$ twisted hodge diamond for X laid on top of each other with the interior middle line removed.

Since we will focus on the case $p > d$ we will be able to ignore the dashed lines.

Proposition 5.2.4. *Let $f : X \hookrightarrow \mathbb{P}^{n+1}$ be a smooth n-dimensional hypersurface of degree d and set $t = d - n - 2$. Then we have for $m \notin \{1, 2n\}$:*

$$\begin{aligned}&\dim \mathrm{HH}^m\left(\mathbb{P}^{n+1}, f_*\mathcal{O}_X(p)\right) = \\ &= \mathrm{h}^{0,m-n}_{t-p}(X) + \mathrm{h}^{0,m-n-1}_{t-p-d}(X) + \mathrm{h}^{n,m}_{t-p}(X) + \mathrm{h}^{n,m-1}_{t-p-d}(X) + (n-1)\left(\delta_{d,p}\delta_{m,n+1} + \delta_{0,p}\delta_{m,n}\right)\end{aligned}$$

and for $m = 1$, $m = 2n$:

$$\dim \mathrm{HH}^1\left(\mathbb{P}^{n+1}, f_*\mathcal{O}_X(p)\right) = \mathrm{h}^{1,n}_{t-p}(X) + \mathrm{h}^{0,n}_{t-p-d}(X) - \mathrm{h}^{1,n-1}_{t-p}(X)$$
$$\dim \mathrm{HH}^{2n}\left(\mathbb{P}^{n+1}, f_*\mathcal{O}_X(p)\right) = \mathrm{h}^{n,0}_{t-p}(X) + \mathrm{h}^{n-1,0}_{t-p-d}(X) - \mathrm{h}^{n-1,1}_{t-p-d}(X).$$

Proof. We will compute the cases separately using $\mathcal{O}_X(t) \cong \omega_X$:

We can use Lemma 5.2.1 to get for $m \notin \{1, 2n\}$

$$\begin{aligned}&\dim \mathrm{HH}^m\left(\mathbb{P}^{n+1}, f_*\mathcal{O}_X(p)\right) = \\ &= \sum_{i=0}^{n} \dim \mathrm{H}^{n-m+i}\left(X, f^*\Omega^i_{\mathbb{P}^{n+1}}(t-p)\right) \\ &= \sum_{i-j=m-n} \dim \mathrm{H}^j\left(X, f^*\Omega^i_{\mathbb{P}^{n+1}}(t-p)\right) \\ &= \mathrm{h}^{0,m-n}_{t-p}(X) + \mathrm{h}^{0,m-n-1}_{t-p-d}(X) + \mathrm{h}^{n,m}_{t-p}(X) + \mathrm{h}^{n,m-1}_{t-p-d}(X) + (n-1)\left(\delta_{d,p}\delta_{m,n+1} + \delta_{0,p}\delta_{m,n}\right).\end{aligned}$$

For $m = 1$ we similarly get

$$\begin{aligned}\dim \mathrm{HH}^1\left(\mathbb{P}^{n+1}, f_*\mathcal{O}_X(p)\right) &= \sum_{i=0}^{n+1} \dim \mathrm{H}^{n-1+i}\left(X, f^*\Omega^i_{\mathbb{P}^{n+1}}(t-p)\right)\\ &= \dim \mathrm{H}^n\left(X, f^*\Omega^1_{\mathbb{P}^{n+1}}(t-p)\right)\\ &= \mathrm{h}^{1,n}_{t-p}(X) + \mathrm{h}^{0,n}_{t-p-d}(X) - \mathrm{h}^{1,n-1}_{t-p}(X).\end{aligned}$$

And for $m = 2n$

$$\begin{aligned}\dim \mathrm{HH}^{2n}\left(\mathbb{P}^{n+1}, f_*\mathcal{O}_X(p)\right) &= \sum_{i=0}^{n+1} \dim \mathrm{H}^{-n-1+i}\left(X, f^*\Omega^i_{\mathbb{P}^{n+1}}(t-p)\right)\\ &= \dim \mathrm{H}^0\left(X, f^*\Omega^{n+1}_{\mathbb{P}^{n+1}}(t-p)\right)\\ &= \mathrm{h}^{n,0}_{t-p}(X) + \mathrm{h}^{n-1,0}_{t-p-d}(X) - \mathrm{h}^{n-1,1}_{t-p-d}(X)\end{aligned}$$

which finishes the claim. □

Remark 5.2.5. Since we will be able to assume $p \notin \{0, d\}$ in the next section we will exclude these cases. However, all of the following proofs and arguments still hold in these cases, one just needs to keep track of the Kronecker deltas in $\dim \mathrm{HH}^n(X, \mathcal{O}_X(p))$.

Proposition 5.2.6. *Let $f : X \hookrightarrow \mathbb{P}^{n+1}$ be a smooth n-dimensional hypersurface of degree d, let $t = d - n - 2$. Then we have for all $p \in \mathbb{Z}$ such that $t - p \notin \{0, d\}$ that $\dim \mathrm{HH}^m(\mathbb{P}^{n+1}, f_*\mathcal{O}_X(p))$ is given by:*

$$\begin{cases} \dim \mathrm{HH}^0(X, \mathcal{O}_X(p)) & m = 0\\ \dim \mathrm{HH}^1(X, \mathcal{O}_X(p)) + \dim \mathrm{HH}^0(X, \mathcal{O}_X(p+d)) - \mathrm{h}^{1,n-1}_{t-p}(X) & m = 1\\ \dim \mathrm{HH}^m(X, \mathcal{O}_X(p)) + \dim \mathrm{HH}^{m-1}(X, \mathcal{O}_X(p+d)) - \mathrm{h}^{\frac{m}{2}, n-\frac{m}{2}}_{t-p}(X) & \begin{array}{c}1 < m < 2n\\ \textit{even}\end{array}\\ \dim \mathrm{HH}^m(X, \mathcal{O}_X(p)) + \dim \mathrm{HH}^{m-1}(X, \mathcal{O}_X(p+d)) - \mathrm{h}^{\frac{m+1}{2}, n-\frac{m+1}{2}}_{t-p}(X) & \begin{array}{c}1 < m < 2n\\ \textit{odd}\end{array}\\ \dim \mathrm{HH}^{2n}(X, \mathcal{O}_X(p)) + \dim \mathrm{HH}^{2n-1}(X, \mathcal{O}_X(p+d)) - \mathrm{h}^{n-1,1}_{t-p-d}(X) & m = 2n\\ \dim \mathrm{HH}^{2n}(X, \mathcal{O}_X(p+d)) & m = 2n+1\\ 0 & \textit{else.}\end{cases}$$

Proof. For dimension reasons we immediately get $\dim \mathrm{HH}^m(\mathbb{P}^{n+1}, f_*\mathcal{O}_X(p)) = 0$ for $m < 0$ respectively $2n + 1 < m$. Now for the computations:

$m = 0$: We compute

$$\begin{aligned} \dim \mathrm{HH}^0\left(\mathbb{P}^{n+1}, f_*\mathcal{O}_X(p)\right) &= \mathrm{h}^{0,n}_{t-p}(X) && \text{Proposition 5.2.4} \\ &= \dim \mathrm{HH}^0(X, \mathcal{O}_X(p)). && \text{Corollary 5.1.4} \end{aligned}$$

$m = 1$: We get by Proposition 5.2.4 and Corollary 5.1.4

$$\begin{aligned} &\dim \mathrm{HH}^1\left(\mathbb{P}^{n+1}, f_*\mathcal{O}_X(p)\right) = \\ &= \mathrm{h}^{1,n}_{t-p}(X) + \mathrm{h}^{0,n}_{t-p-d}(X) - \mathrm{h}^{1,n-1}_{t-p}(X) \\ &= \dim \mathrm{HH}^1(X, \mathcal{O}_X(p)) + \dim \mathrm{HH}^0(X, \mathcal{O}_X(p+d)) - \mathrm{h}^{1,n-1}_{t-p}(X). \end{aligned}$$

$1 < m < 2n$: In this case we have by Corollary 5.1.4 and Proposition 5.2.4

$$\begin{aligned} &\dim \mathrm{HH}^m\left(\mathbb{P}^{n+1}, f_*\mathcal{O}_X(p)\right) = \\ &= \mathrm{h}^{0,m-n}_{t-p}(X) + \mathrm{h}^{0,m-n-1}_{t-p-d}(X) + \mathrm{h}^{n,m}_{t-p}(X) + \mathrm{h}^{n,m-1}_{t-p-d}(X) \\ &= \mathrm{h}^{0,m-n}_{t-p}(X) + \mathrm{h}^{n,m}_{t-p}(X) + \mathrm{h}^{0,m-n-1}_{t-p-d}(X) + \mathrm{h}^{n,m-1}_{t-p-d}(X) \\ &= \begin{cases} \dim \mathrm{HH}^m(X, \mathcal{O}_X(p)) + \dim \mathrm{HH}^{m-1}(X, \mathcal{O}_X(p+d)) - \mathrm{h}^{\frac{m}{2}, n-\frac{m}{2}}_{t-p}(X) & m \text{ even} \\ \dim \mathrm{HH}^m(X, \mathcal{O}_X(p)) + \dim \mathrm{HH}^{m-1}(X, \mathcal{O}_X(p+d)) - \mathrm{h}^{\frac{m}{2}, n-\frac{m+1}{2}}_{t-p}(X) & m \text{ odd.} \end{cases} \end{aligned}$$

$m = 2n$: Here we get by Proposition 5.2.4 and Corollary 5.1.4

$$\begin{aligned} &\dim \mathrm{HH}^{2n}\left(\mathbb{P}^{n+1}, f_*\mathcal{O}_X(p)\right) = \\ &= \mathrm{h}^{n,0}_{t-p}(X) + \mathrm{h}^{n-1,0}_{t-p-d}(X) - \mathrm{h}^{n-1,1}_{t-p}(X) \\ &= \dim \mathrm{HH}^{2n}(X, \mathcal{O}_X(p)) + \dim \mathrm{HH}^{2n-1}(X, \mathcal{O}_X(p+d)) - \mathrm{h}^{n-1,1}_{t-p-d}(X). \end{aligned}$$

$m = 2n+1$: We compute

$$\begin{aligned} \dim \mathrm{HH}^{2n+1}\left(\mathbb{P}^{n+1}, f_*\mathcal{O}_X(p)\right) &= \mathrm{h}^{n,0}_{t-p-d}(X) && \text{Proposition 5.2.4} \\ &= \dim \mathrm{HH}^n(X, \mathcal{O}_X(p+d)) && \text{Corollary 5.1.4.} \end{aligned}$$

So we covered all cases and the statement holds. □

Proposition 5.2.7 ([RVdBN19, Proposition 9.5.1])**.** *Consider the embedding of a smooth n-dimensional degree d hypersurface $X \overset{f}{\hookrightarrow} \mathbb{P}^{n+1}$. Then we have a long exact sequence of the form:*

$$\cdots \to \mathrm{HH}^{i-2}(X, \mathcal{O}_X(p+d)) \to \mathrm{HH}^i(X, \mathcal{O}_X(p)) \xrightarrow{f_*} \mathrm{HH}^i\left(\mathbb{P}^{n+1}, f_*\mathcal{O}_X(p)\right) \to \cdots.$$

Theorem 5.2.8. *Let $f : X \hookrightarrow \mathbb{P}^{n+1}$ be the embedding of a smooth degree d hypersurface and set $t = d - n - 2$. Then we have for all $p \in \mathbb{Z}$ such that $t - p \notin \{0, d\}$*

$$\dim \ker \left(f_* : \mathrm{HH}^m\left(X, \mathcal{O}_X\left(p\right)\right) \to \mathrm{HH}^m\left(\mathbb{P}^{n+1}, f_*\mathcal{O}_X\left(p\right)\right)\right) = \begin{cases} \mathrm{h}_{t-p}^{\frac{m}{2}, n-\frac{m}{2}}\left(X\right) & 0 < m < 2n \text{ } \textit{even} \\ \mathrm{h}_{t-p-d}^{n-1,1}\left(X\right) & m = 2n \\ 0 & \textit{else.} \end{cases}$$

Proof. For dimension reasons we may assume $0 \leq m \leq 2n$. In the diagrams for this prove we will denote $\mathcal{O}_X$ by $\mathcal{O}$ and $\mathbb{P}^{n+1}$ by $\mathbb{P}$ in order to avoid clumsy notation.

We will proceed by induction over l with $2l = m$ using the long exact sequence from Proposition 5.2.7:

$$\cdots \to \mathrm{HH}^{m-2}\left(X, \mathcal{O}_X\left(p + d\right)\right) \to \mathrm{HH}^m\left(X, \mathcal{O}_X\left(p\right)\right) \xrightarrow{f_*} \mathrm{HH}^m\left(\mathbb{P}^{n+1}, f_*\mathcal{O}_X\left(p\right)\right) \to \cdots.$$

This way we can cover the odd case $2l - 1$ and even case $2l$ in the induction step simultaneously:

We will start with $l = 1$ as induction start and include the case of $m = 0$ to cover the cases for $m = 0, 1, 2$:

We compute all the dimensions in the long exact sequence in Proposition 5.2.7 using Proposition 5.2.6 and proceed by diagram chase. Consider the following diagram, where we denote the spaces on the left and their dimensions on the right. We will use the arrows on the right-hand side to indicate that the dimensions to the right of their tail are the dimensions to the left of their tip.

$$
\begin{tikzcd}
0 \arrow[d] \\
\mathrm{HH}^0\left(X,\mathcal{O}\left(p\right)\right) \arrow[d, "f_*"] \\
\mathrm{HH}^0\left(\mathbb{P},f_*\mathcal{O}\left(p\right)\right) \arrow[d] \\
\mathrm{HH}^{-1}\left(X,\mathcal{O}\left(p+d\right)\right) \arrow[d] \\
\mathrm{HH}^1\left(X,\mathcal{O}\left(p\right)\right) \arrow[d, "f_*"] \\
\mathrm{HH}^1\left(\mathbb{P},f_*\mathcal{O}\left(p\right)\right) \arrow[d] \\
\mathrm{HH}^0\left(X,\mathcal{O}\left(p+d\right)\right) \arrow[d] \\
\mathrm{HH}^2\left(X,\mathcal{O}\left(p\right)\right)
\end{tikzcd}
\qquad
\begin{array}{rcl}
 & & 0 \\
0 & + & \dim\mathrm{HH}^0\left(X,\mathcal{O}\left(p\right)\right) \\
\dim\mathrm{HH}^0\left(X,\mathcal{O}\left(p\right)\right) & + & 0 \\
0 & + & 0 \\
0 & + & \dim\mathrm{HH}^1\left(X,\mathcal{O}\left(p\right)\right) \\
\dim\mathrm{HH}^1\left(X,\mathcal{O}\left(p\right)\right) & + & \dim\mathrm{HH}^0\left(X,\mathcal{O}\left(p+d\right)\right) - \mathrm{h}^{1,n-1}_{t-p} \\
\dim\mathrm{HH}^0\left(X,\mathcal{O}\left(p+d\right)\right) - \mathrm{h}^{1,n-1}_{t-p} & + & \mathrm{h}^{1,n-1}_{t-p} \\
\mathrm{h}^{1,n-1}_{t-p} & - & \mathrm{h}^{1,n-1}_{t-p} + \dim\mathrm{HH}^2\left(X,\mathcal{O}\left(p\right)\right).
\end{array}
$$

By the above diagram chase we get that the image of the last arrow on the left has dimension $\mathrm{h}^{1,n-1}_{t-p}\left(X\right)$. So by the exactness of the sequence from Proposition 5.2.7 we get that this is also the dimension of the kernel of

$$f_* : \mathrm{HH}^2\left(X,\mathcal{O}_X\left(p\right)\right) \to \mathrm{HH}^2\left(\mathbb{P}^n, f_*\mathcal{O}_X\left(p\right)\right).$$

And so we get:

$$
\begin{aligned}
\dim\ker\Big(f_* : \mathrm{HH}^0\left(X,\mathcal{O}_X\left(p\right)\right) \to \mathrm{HH}^0\left(\mathbb{P}^n, f_*\mathcal{O}_X\left(p\right)\right)\Big) &= 0 \\
\dim\ker\Big(f_* : \mathrm{HH}^1\left(X,\mathcal{O}_X\left(p\right)\right) \to \mathrm{HH}^1\left(\mathbb{P}^n, f_*\mathcal{O}_X\left(p\right)\right)\Big) &= 0 \\
\dim\ker\Big(f_* : \mathrm{HH}^2\left(X,\mathcal{O}_X\left(p\right)\right) \to \mathrm{HH}^2\left(\mathbb{P}^n, f_*\mathcal{O}_X\left(p\right)\right)\Big) &= \mathrm{h}^{1,n-1}_{t-p}\left(X\right)
\end{aligned}
$$

as expected.

For the induction step we will cover the cases $m = 2l-1$ and $m = 2l$ simultaneously. Assume that

$$\dim\ker\Big(f_* : \mathrm{HH}^{2l-2}\left(X,\mathcal{O}_X\left(p\right)\right) \to \mathrm{HH}^{2l-2}\left(\mathbb{P}^n, f_*\mathcal{O}_X\left(p\right)\right)\Big) = \mathrm{h}^{n-l+1,l-1}_{t-p}\left(X\right).$$

We compute again the dimensions in the long exact sequence from Proposition 5.2.7 using our computations in Proposition 5.2.6. We write the long exact sequence on the left and the dimensions on the right. We draw the arrows on the right hand side from left to right to indicate that the dimensions to the right of their tail are the dimensions to the left of their tip.

$$
\begin{tikzcd}[column sep=small]
\ker\left(f_*\right)^{2l-2} \arrow[d, hook] & & & \mathrm{h}_{t-p}^{l-1,n-l+1} \arrow[dll, dashed] \\
\mathrm{HH}^{2l-2}\left(X,\mathcal{O}\left(p\right)\right) \arrow[d, "f_*"] & \mathrm{h}_{t-p}^{l-1,n-l+1} & - & \mathrm{h}_{t-p}^{l-1,n-l+1}+\dim\mathrm{HH}^{2l-2}\left(X,\mathcal{O}\left(p\right)\right) \arrow[dll, dashed] \\
\mathrm{HH}^{2l-2}\left(\mathbb{P},f_*\mathcal{O}\left(p\right)\right) \arrow[d] & -\mathrm{h}_{t-p}^{l-1,n-l+1}+\dim\mathrm{HH}^{2l-2}\left(X,\mathcal{O}\left(p\right)\right) & + & \dim\mathrm{HH}^{2l-2}\left(X,\mathcal{O}\left(p+d\right)\right) \arrow[dll, dashed] \\
\mathrm{HH}^{2l-3}\left(X,\mathcal{O}\left(p+d\right)\right) \arrow[d] & \dim\mathrm{HH}^{2l-2}\left(X,\mathcal{O}\left(p+d\right)\right) & + & 0 \arrow[dll, dashed] \\
\mathrm{HH}^{2l-1}\left(X,\mathcal{O}\left(p\right)\right) \arrow[d, "f_*"] & 0 & + & \dim\mathrm{HH}^{2l-1}\left(X,\mathcal{O}\left(p\right)\right) \arrow[dll, dashed] \\
\mathrm{HH}^{2l-1}\left(\mathbb{P},f_*\mathcal{O}\left(p\right)\right) \arrow[d] & \dim\mathrm{HH}^{2l-1}\left(X,\mathcal{O}\left(p\right)\right) & + & \dim\mathrm{HH}^{2l-2}\left(X,\mathcal{O}\left(p+d\right)\right)-\mathrm{h}_{t-p}^{l-1,n-l+1} \arrow[dll, dashed] \\
\mathrm{HH}^{2l-2}\left(X,\mathcal{O}\left(p-d\right)\right) \arrow[d] & \dim\mathrm{HH}^{2l-2}\left(X,\mathcal{O}\left(p+d\right)\right)-\mathrm{h}_{t-p}^{l,n-l} & + & \mathrm{h}_{t-p}^{l,n-l}\left(X\right) \arrow[dll, dashed] \\
\mathrm{HH}^{2l}\left(X,\mathcal{O}\left(p\right)\right) & \mathrm{h}_{t-p}^{l,n-l} & - & \mathrm{h}_{t-p}^{l,n-l}+\dim\mathrm{HH}^{2l}\left(X,\mathcal{O}\left(p\right)\right).
\end{tikzcd}
$$

By the exactness of the sequence this means that

$$
\begin{aligned}
\dim\ker\left(f_*:\mathrm{HH}^{2l-1}\left(X,\mathcal{O}_X\left(p\right)\right)\to\mathrm{HH}^{2l-1}\left(\mathbb{P}^n,f_*\mathcal{O}_X\left(p\right)\right)\right)&=0\\
\dim\ker\left(f_*:\mathrm{HH}^{2l}\left(X,\mathcal{O}_X\left(p\right)\right)\to\mathrm{HH}^{2l}\left(\mathbb{P}^n,f_*\mathcal{O}_X\left(p\right)\right)\right)&=\mathrm{h}_{t-p}^{n-l,l}\left(X\right).
\end{aligned}
$$

Now finally for the case of $l=n$:

By the above induction we have

$$
\dim\ker\left(f_*:\mathrm{HH}^{2n-2}\left(X,\mathcal{O}_X\left(p\right)\right)\to\mathrm{HH}^{2n-2}\left(\mathbb{P}^n,f_*\mathcal{O}_X\left(p\right)\right)\right)=\mathrm{h}_{t-p}^{1,n-1}\left(X\right).
$$

We apply again diagram chase along long exact sequence from Proposition 5.2.7 using the computations in Proposition 5.2.6. We continue to write the long exact sequence on the left and the dimensions on the right. The diagonal arrows on the right again symbolize that the dimensions to the right of their tail are the dimensions of the kernel to the left of their tip:

$$
\begin{tikzcd}[column sep=small]
\ker\left(f_*\right)^{2n-2} \arrow[d, hook] & & \mathrm{h}^{n-1,1}_{t-p} \arrow[dl, dashed] \\
\mathrm{HH}^{2n-2}\left(X,\mathcal{O}\left(p\right)\right) \arrow[d, "f_*"] & \mathrm{h}^{n-1,1}_{t-p} \; - & \mathrm{h}^{n-1,1}_{t-p} + \dim \mathrm{HH}^{2n-2}\left(X,\mathcal{O}\left(p\right)\right) \arrow[dl, dashed] \\
\mathrm{HH}^{2n-2}\left(\mathbb{P},f_*\mathcal{O}\left(p\right)\right) \arrow[d] & -\mathrm{h}^{n-1,1}_{t-p} + \dim \mathrm{HH}^{2n-2}\left(X,\mathcal{O}\left(p\right)\right) \; + & \dim \mathrm{HH}^{2n-3}\left(X,\mathcal{O}\left(p+d\right)\right) \arrow[dl, dashed] \\
\mathrm{HH}^{2n-3}\left(X,\mathcal{O}\left(p+d\right)\right) \arrow[d] & \dim \mathrm{HH}^{2n-2}\left(X,\mathcal{O}\left(p+d\right)\right) \; + & 0 \arrow[dl, dashed] \\
\mathrm{HH}^{2n-1}\left(X,\mathcal{O}\left(p\right)\right) \arrow[d, "f_*"] & 0 \; + & \dim \mathrm{HH}^{2n-1}\left(X,\mathcal{O}\left(p\right)\right) \arrow[dl, dashed] \\
\mathrm{HH}^{2n-1}\left(\mathbb{P},f_*\mathcal{O}\left(p\right)\right) \arrow[d] & \dim \mathrm{HH}^{2n-1}\left(X,\mathcal{O}\left(p\right)\right) \; + & \dim \mathrm{HH}^{2n-2}\left(X,\mathcal{O}\left(p+d\right)\right) - \mathrm{h}^{n-1,1}_{t-p} \arrow[dl, dashed] \\
\mathrm{HH}^{2n-2}\left(X,\mathcal{O}\left(p-d\right)\right) \arrow[d] & \dim \mathrm{HH}^{2n-2}\left(X,\mathcal{O}\left(p+d\right)\right) - \mathrm{h}^{n-1,1}_{t-p-d} \; + & \mathrm{h}^{n-1,1}_{t-p-d} \arrow[dl, dashed] \\
\mathrm{HH}^{2n}\left(X,\mathcal{O}\left(p\right)\right) & \mathrm{h}^{n-1,1}_{t-p-d} \; + & \dim \mathrm{HH}^{2n}\left(X,\mathcal{O}\left(p\right)\right) - \mathrm{h}^{n-1,1}_{t-p-d}.
\end{tikzcd}
$$

This diagram gives us

$$
\begin{aligned}
\dim\ker\left(f_*:\mathrm{HH}^{2n-1}\left(X,\mathcal{O}_X\left(p\right)\right)\to\mathrm{HH}^{2n-1}\left(\mathbb{P}^n,f_*\mathcal{O}_X\left(p\right)\right)\right) &= 0\\
\dim\ker\left(f_*:\mathrm{HH}^{2n}\left(X,\mathcal{O}_X\left(p\right)\right)\to\mathrm{HH}^{2n}\left(\mathbb{P}^n,f_*\mathcal{O}_X\left(p\right)\right)\right) &= \mathrm{h}^{n-1,1}_{t-p-d}\left(X\right).
\end{aligned}
$$

So we covered the case for $m = 2n$ and are done as for $m < 0$ and $m > 0$ the source space is trivial. □

We now finally state the following in order to guarantee the existence of non-trivial kernels of pushforwards of Hochschild cohomology.

Proposition 5.2.9. *Let $f : X \hookrightarrow \mathbb{P}^{2k}$ be an embedding of a smooth odd dimensional degree $d > 1$ hypersurface of dimension $n = 2k-1$ for $k > 2$ and let $p = -kd-d$. Then we have*

$$
\ker\left(\mathrm{HH}^{n+3}\left(X,\mathcal{O}_X\left(p\right)\right)\to\mathrm{HH}^{n+3}\left(\mathbb{P}^{n+1},f_*\mathcal{O}_X\left(p\right)\right)\right)\cong\Bbbk.
$$

Proof. By Theorem 5.2.8 we have

$$
\dim\ker\left(\mathrm{HH}^{n+3}\left(X,\mathcal{O}_X\left(p\right)\right)\to\mathrm{HH}^{n+3}\left(\mathbb{P}^{n+1},f_*\mathcal{O}_X\left(p\right)\right)\right)=\mathrm{h}^{k+1,k-2}_{t-p}\left(X\right)
$$

with $t = d - n - 2 = d - 2k - 1$.

So it suffices to compute that $\mathrm{h}^{k+1,k-2}_{t-p}(X) = 1$, with $t - p = kd + 2d - 2k - 1$. By (5.1.1) this is

$$
\begin{aligned}
&\mathrm{h}^{k+1,k-2}_{t-p}(X) = \\
&= \sum_{\mu=0}^{2k-1+2} (-1)^{\mu} \binom{2k-1+2}{\mu} \binom{-kd - 2d + 2k + 1 + (k+1)\,d - (\mu - 1)(d-1)}{2k - 1 + 1} \\
&= \sum_{\mu=0}^{2k+1} (-1)^{\mu} \binom{2k+1}{\mu} \binom{-kd - 2d + 2k + 1 + kd + d - \mu d + d + \mu - 1}{2k} \\
&= \sum_{\mu=0}^{2k+1} (-1)^{\mu} \binom{2k+1}{\mu} \binom{2k - \mu d + \mu}{2k} \\
&= \binom{2k+1}{0} \binom{2k}{2k} \\
&= 1.
\end{aligned}
$$

Here we used that for $\mu > 1$ we have $2k - \mu d + \mu < 2k$ as $d > 1$, which means that the terms $\binom{2k+1}{\mu}\binom{2k+\mu d+\mu}{2k}$ vanish for $\mu \geq 1$.

So we get

$$
\dim \ker \left(\mathrm{HH}^{n+3}(X, \mathcal{O}_X(p)) \to \mathrm{HH}^{n+3}\left(\mathbb{P}^{n+1}, f_* \mathcal{O}_X(p) \right) \right) = 1
$$

as claimed. □

5.3 Examples

We collect a few examples of twisted Hodge diamonds that were computed using the Sage package by Pieter Belmans and Piet Glas [BG].

The first two examples illustrate the general shape as given in Lemma 5.1.1 and the third will be an explicit example of Proposition 5.2.9.

Example 5.3.1. Let $f : X \hookrightarrow \mathbb{P}^6$ be a smooth degree 7 hypersurface then the 8-twisted hodge diamond is

$$\begin{array}{c}
0 \\
0 \quad 0 \\
0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \\
2996 \quad 20993 \quad 15267 \quad 917 \quad 0 \quad 0 \\
1575 \quad 0 \quad 0 \quad 0 \quad 0 \\
5775 \quad 0 \quad 0 \quad 0 \\
10395 \quad 0 \quad 0 \\
9002 \quad 0 \\
2996.
\end{array}$$

And so we have by Theorem 5.2.8, since $t-8=-8$,

$$\begin{aligned}
&\dim\ker\left(f_*:\mathrm{HH}^4\left(X,\mathcal{O}_X\left(-8\right)\right)\to\mathrm{HH}^4\left(X,f_*\mathcal{O}_X\left(-8\right)\right)\right)=917\\
&\dim\ker\left(f_*:\mathrm{HH}^6\left(X,\mathcal{O}_X\left(-8\right)\right)\to\mathrm{HH}^6\left(X,f_*\mathcal{O}_X\left(-8\right)\right)\right)=15267\\
&\dim\ker\left(f_*:\mathrm{HH}^8\left(X,\mathcal{O}_X\left(-8\right)\right)\to\mathrm{HH}^8\left(X,f_*\mathcal{O}_X\left(-8\right)\right)\right)=20993.
\end{aligned}$$

Example 5.3.2. Let $f:X\hookrightarrow\mathbb{P}^8$ be smooth degree 5 hypersurface then the -7-twisted hodge diamond is

$$\begin{array}{c}
6390 \\
0 \quad 20511 \\
0 \quad 0 \quad 25704 \\
0 \quad 0 \quad 0 \quad 16840 \\
0 \quad 0 \quad 0 \quad 0 \quad 4950 \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 720 \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 36 \\
0 \quad 0 \quad 0 \quad 486 \quad 13051 \quad 30276 \quad 8451 \quad 165 \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \quad 0 \\
0 \quad 0 \quad 0 \\
0 \quad 0 \\
0.
\end{array}$$

And so we have by Theorem 5.2.8, since $t + 7 = 3$,

$$\dim\ker\Big(f_* : \mathrm{HH}^2\left(X, \mathcal{O}_X(3)\right) \to \mathrm{HH}^2\left(X, f_*\mathcal{O}_X(3)\right)\Big) = 8451$$
$$\dim\ker\Big(f_* : \mathrm{HH}^4\left(X, \mathcal{O}_X(3)\right) \to \mathrm{HH}^4\left(X, f_*\mathcal{O}_X(3)\right)\Big) = 15267$$
$$\dim\ker\Big(f_* : \mathrm{HH}^6\left(X, \mathcal{O}_X(3)\right) \to \mathrm{HH}^6\left(X, f_*\mathcal{O}_X(3)\right)\Big) = 13051$$
$$\dim\ker\Big(f_* : \mathrm{HH}^8\left(X, \mathcal{O}_X(3)\right) \to \mathrm{HH}^8\left(X, f_*\mathcal{O}_X(3)\right)\Big) = 486.$$

The next example illustrates a case of Proposition 5.2.9:

Example 5.3.3. Let $f : X \hookrightarrow \mathbb{P}^{10}$ be a smooth degree 5 hypersurface and consider $\mathcal{O}_X(-30)$. Then we can compute, using Theorem 5.2.8

$$\dim\ker\left(f_* : \mathrm{HH}^m\left(X, \mathcal{O}_X(-30)\right) \to \mathrm{HH}^m\left(X, f_*\mathcal{O}_X(-30)\right)\right).$$

To do this we need to compute the $t - p = 24$ twisted Hodge-diamond

$$\begin{array}{c}
0\\
0 \quad 0\\
0 \quad 0 \quad 0\\
0 \quad 0 \quad 0 \quad 0\\
0 \quad 0 \quad 0 \quad 0 \quad 0\\
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\\
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\\
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\\
0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\\
11979044 \quad 100298 \quad 2882 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\\
107439618 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\\
523445109 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\\
1580020794 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\\
3149538513 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\\
4236318471 \quad 0 \quad 0 \quad 0 \quad 0\\
3815626243 \quad 0 \quad 0 \quad 0\\
2209626573 \quad 0 \quad 0\\
744650346 \quad 0\\
111098130.
\end{array}$$

And as expected by Proposition 5.2.9 we get

$$\dim\ker\Big(f_* : \mathrm{HH}^{12}\left(X, \mathcal{O}_X(-30)\right) \to \mathrm{HH}^{12}\left(X, f_*\mathcal{O}_X(-30)\right)\Big) = 1.$$

Chapter 6

Non-Fourier-Mukai Functors

In this chapter we follow the ideas from [RVdBN19] to construct candidate non-Fourier-Mukai functors for hypersurfaces of arbitrary degree. We then verify that under assumptions on the characteristic morphisms and some concentrated Ext-groups these indeed cannot be Fourier-Mukai. We finish the chapter by computing that these assumptions are satisfied when the source category is the derived category of an odd dimensional quadric, which gives concrete non-Fourier-Mukai functors between well behaved spaces in arbitrary high dimensions.

Since we follow the approach from [RVdBN19] we will consider functors of a similar form:

$$\Psi_\eta : \mathcal{D}^b(X) \xrightarrow{L} \mathcal{D}^b_{\mathrm{wcoh}(X)}\left(\mathcal{X}_\eta^{dg}\right) \xrightarrow{\psi_{\mathcal{X},\eta,*}} \mathcal{D}^b_{\mathrm{wcoh}(X)}(\mathcal{X}_\eta) \xrightarrow{\widetilde{f}_*} \mathcal{D}^b\left(\mathbb{P}^{n+1}\right), \tag{6.0.1}$$

where $\mathcal{X}_\eta^{dg}$ denotes the dg-hull of $\mathcal{X}_\eta$ and $\psi_{\mathcal{X},\eta,*}$ is the induced comparison functor.

6.1 Constructing candidate non-Fourier-Mukai Functors

We start by collecting a few results from [RVdBN19], which are central for our construction. We refer the interested reader to [RVdBN19] for an in depth discussion.

In order to apply the construction from Definition 4.1.11 we fix for every quasi-compact scheme a finite affine covering $X = \bigcup_{i\in I} U_i$.

We use the following construction from [RVdBN19] as the core of our candidate functors:

Proposition 6.1.1. *Let X be smooth projective of dimension n and let $\eta \in \mathrm{HH}^{\geq n+3}(X, M)$ for M a coherent sheaf. Then there exists an exact functor*

$$\mathcal{D}^b(X) \xrightarrow{L} \mathcal{D}^b_{\mathrm{wcoh}(X)}\left(\mathcal{X}_\eta^{dg}\right)$$

such that $\underline{\mathrm{RHom}}_{\mathcal{X}_\eta}(\mathcal{X}, L(\_)) \cong w$*, where* $\underline{\mathrm{RHom}}$ *denotes the derived internal Hom functor.*

Proof. First observe that by Lemma 4.1.16 we have an isomorphism

$$\mathrm{HH}^*(X, M) \cong \mathrm{HH}^*(\mathcal{X}, WM)$$

and so we may consider $\eta \in \mathrm{HH}^{\geq n+3}(\mathcal{X}, WM)$.

By [RVdBN19, Lemma 10.1] and since $\mathrm{Qch}(X)$ has global dimension n and $\mathrm{H}^i(\mathcal{X}_\eta)$ vanishes in the right degrees we can apply [RVdBN19, Proposition 5.3.1] with $\mathcal{A} = w\mathrm{Qch}(X)$ and $\mathfrak{c} = \mathcal{X}_\eta$ to get a functor

$$L' : \mathcal{D}^b(\mathrm{Qch}(X)) \cong \mathcal{D}^b(w\mathrm{Qch}X) \to \mathcal{D}^b_{w\mathrm{Qch}(X)}\left(\mathcal{X}_\eta^{dg}\right).$$

Now we can use [Huy06, Proposition 3.5] to turn this into a functor

$$L : \mathcal{D}^b(X) \xrightarrow{\sim} \mathcal{D}^b_{\mathrm{coh}(X)}(\mathrm{Qch}(X)) \xrightarrow{L'} \mathcal{D}^b\left(\mathcal{X}_\eta^{dg}\right)$$

with the desired property.

Finally, by [RVdBN19, Corollary 10.4] we know that the essential image of this functor is contained in $\mathcal{D}^b_{w\mathrm{coh}X}(\mathcal{X}_\eta)$. □

Remark 6.1.2. Similarly to the notation $\mathcal{X}$ we will denote the $\Bbbk$-linear category corresponding to $\mathbb{P}^{n+1}$ by $\mathcal{P}^{n+1}$.

We will also use the following notation from [RVdBN19] for $\widetilde{f}$.

Proposition 6.1.3. *[RVdBN19, Proposition 7.2.6] Let* $f : \mathcal{P}^{n+1} \to \mathcal{X}$ *be a functor of* $\Bbbk$*-linear categories and* $\eta \in \mathrm{HH}^k(\mathcal{X}, \mathcal{M})$ *such that* $f_*\eta = 0$*. Then there exists an* $\mathcal{A}_\infty$*-functor* $\widetilde{f}$ *making the diagram*

$$\begin{tikzcd} & \mathcal{X}_\eta \arrow[dl, "\pi"'] & \\ \mathcal{X} & & \mathcal{P}^{n+1} \arrow[ll, "f"] \arrow[ul, dashed, "\widetilde{f}"'] \end{tikzcd}$$

commute. In particular we have

$$\pi \circ \widetilde{f} = f. \tag{6.1.1}$$

Now we construct a candidate functor Ψ_η for $\eta \in \mathrm{HH}^{\geq n+3}(X, \mathcal{O}_X(p))$.

Construction 6.1.4. Let $X \hookrightarrow \mathbb{P}^{n+1}$ be the embedding of a smooth n-dimensional scheme with $n \geq 3$ and let

$$0 \neq \eta \in \ker\left(f_* : \mathrm{HH}^m\left(X, \mathcal{O}_X(p)\right) \to \mathrm{HH}^m\left(\mathbb{P}^{n+1}, f_*\mathcal{O}_X(p)\right)\right)$$

for $m > n+2$.

Then a functor of the form (6.0.1) is constructed to be,

$$\Psi_\eta : \mathcal{D}^b\left(\mathrm{coh}\left(X\right)\right) \xrightarrow{L} \mathcal{D}^b_{\mathrm{wcoh}(X)}\left(\mathcal{X}^{dg}_\eta\right) \xrightarrow{\psi_{\mathcal{X}_\eta,*}} \mathcal{D}^b_{\mathrm{wcoh}(X)}\left(\mathcal{X}_\eta\right) \xrightarrow{\widetilde{f}_*} \mathcal{D}^b\left(\mathrm{coh}\left(\mathbb{P}^{n+1}\right)\right),$$

where we have the functor L by Proposition 6.1.1, $\psi_{\mathcal{X}_\eta,*}$ is the comparison functor between the $\mathcal{A}_\infty$-category $\mathcal{X}_\eta$ and its dg-hull $\mathcal{X}^{dg}_\eta$ constructed in [RVdBN19, § D.1] and $\widetilde{f}_*$ exists by Proposition 6.1.3.

Now we may use the results from Chapter 5 to construct candidate functors. Observe that these are just candidate functors, for verifying that they are not Fourier-Mukai we will need to assume that a equivariant characteristic morphism of η does not vanish and that $m = n+3$.

Corollary 6.1.5. *Let $f : X \to \mathbb{P}^{n+1}$ be the embedding of a degree d hypersurface and let $m > n+2$ then we have a $\mathrm{h}_p^{\frac{m}{2}, n-\frac{m}{2}}(X)$-dimensional space of choices to construct a candidate functor*

$$\Psi_\eta : \mathcal{D}^b\left(X\right) \to \mathcal{D}^b\left(\mathbb{P}^{n+1}\right).$$

Proof. In order for Construction 6.1.4 to work we need

$$0 \neq \eta \in \ker\left(f_* : \mathrm{HH}^m\left(X, \mathcal{O}_X(p)\right) \to \mathrm{HH}^m\left(\mathbb{P}^{n+1}, f_*\mathcal{O}_X(p)\right)\right).$$

By Theorem 5.2.8 $\ker\left(f_* : \mathrm{HH}^m\left(X, \mathcal{O}_X(p)\right) \to \mathrm{HH}^m\left(\mathbb{P}^{n+1}, f_*\mathcal{O}_X(p)\right)\right)$ has dimension $\mathrm{h}_p^{\frac{m}{2}, n-\frac{m}{2}}(X)$ which finishes the claim. □

Now we can state our main Theorem, which we will prove throughout § 6.2.

Theorem 6.1.6. *Let $f : X \hookrightarrow \mathbb{P}^{n+1}$ be an embedding of a smooth degree d hypersurface of dimension $n \geq 3$ and let*

$$0 \neq \eta \in \ker\left(f_* : \mathrm{HH}^{n+3}\left(X, \mathcal{O}_X(p)\right) \to \mathrm{HH}^{n+3}\left(\mathbb{P}^{n+1}, f_*\mathcal{O}_X(p)\right)\right)$$

such that there exists a $\Bbbk$*-algebra* Γ *and* $G \in \mathcal{D}^b(\mathrm{coh}(X)_\Gamma)$ *with*

$$\begin{aligned} &c_{G,\Gamma}(\eta) \neq 0 \\ &\mathrm{Ext}^i_X(G(-p), T) = 0 && \textit{for } i \neq n \\ &\mathrm{Ext}^{n-1}_X(G, G(p+d)) \cong \mathrm{Ext}^{n-2}_X(G, G(p+d)) \cong 0. \end{aligned}$$

Then we have that the functor

$$\Psi_\eta : \mathcal{D}^b(\mathrm{coh}(X)) \to \mathcal{D}^b\left(\mathrm{coh}\left(\mathbb{P}^{n+1}\right)\right)$$

is well-defined and not a Fourier-Mukai functor.

6.2 Proving Theorem 6.1.6

We fix for the rest of this section an embedding of a smooth degree d hypersurface $f : X \hookrightarrow \mathbb{P}^{n+1}$, a non-vanishing Hochschild cohomology class $\eta \in \mathrm{HH}^{n+3}(X, \mathcal{O}(p))$, such that $f_*\eta = 0$, Γ a $\Bbbk$-algebra and $G \in \mathcal{D}(\mathrm{coh}(X)_\Gamma)$ such that

$$\begin{aligned} &c_{G,\Gamma}(\eta) \neq 0 && && (6.2.1) \\ &\mathrm{Ext}^i_X(G(-p), T) = 0 && \text{for } i \neq n && (6.2.2) \\ &\mathrm{Ext}^{n-1}_X(G, G(p+d)) \cong \mathrm{Ext}^{n-2}(G, G(p+d)) \cong 0. && && (6.2.3) \end{aligned}$$

Observe first that by Construction 6.1.4 Ψ_η is well-defined and even unique up to a choice of $\widetilde{f}$. So we may focus for the rest of this section on verifying that Ψ_η cannot be Fourier-Mukai.

We follow mostly the ideas from [RVdBN19].

We start by using the assumptions on G to prove that the negative part of $\mathrm{Ext}^*_{\mathcal{X}_\eta}(LG, LG)$ is concentrated in degree -1 which allows us to control which $\mathcal{A}_\infty$-obstruction does not vanish. This obstruction we will then push forward to prove that Ψ_η cannot be Fourier-Mukai. In order to avoid clumsy notation we start by setting

$$\mathcal{G} := wG \in \mathcal{X}\text{−mod and } \widetilde{\mathcal{G}} := L(G). \tag{6.2.4}$$

Remark 6.2.1. We have by [RVdBN19, § D.1] an equivalence $\psi_{\mathcal{X}_\eta} : \mathcal{X}^{dg}_\eta \xrightarrow{\sim} \mathcal{X}_\eta$ and by Definition 2.3.4 a canonical functor $\pi : \mathcal{X}_\eta \to \mathcal{X}$. So we will denote the functor

$$\psi^{-1}_{\mathcal{X}_\eta,*} \circ \pi_* : \mathcal{D}(\mathcal{X}) \to \mathcal{D}_\infty(\mathcal{X}_\eta) \to \mathcal{D}\left(\mathcal{X}^{dg}_\eta\right)$$

simply by π_* and

$$\psi_{\mathcal{X}_\eta,*} \circ \pi^* : \mathcal{D}\left(\mathcal{X}_\eta^{dg}\right) \to \mathcal{D}_\infty\left(\mathcal{X}_\eta\right) \to \mathcal{D}\left(\mathcal{X}\right)$$

by π^* to avoid clumsy and confusing notation.

Definition 6.2.2. Consider the distinguished triangle in $\mathcal{D}\left(\mathcal{X}_\eta^{dg}\right)$ [RVdBN19, Lemma 10.3]:

$$\mathcal{G} \xrightarrow{\alpha} \widetilde{\mathcal{G}} \xrightarrow{\beta} \Sigma^{-n-1}\mathcal{G} \otimes w\mathcal{O}_X\left(-p\right) \xrightarrow{\gamma} \Sigma\mathcal{G}, \tag{6.2.5}$$

where $\mathcal{G}$ is considered as an $\mathcal{X}_\eta^{dg}$-module via $\pi_* : \mathcal{D}^b\left(\mathcal{X}\right) \to \mathcal{D}\left(\mathcal{X}_\eta^{dg}\right)$. Then define the morphism φ by:

$$\begin{aligned}\varphi : \operatorname{Ext}_X^{n+1+i}\left(G\left(-p\right), G\right) &\to \operatorname{Ext}^i_{\mathcal{X}_\eta^{dg}}\left(\widetilde{\mathcal{G}}, \widetilde{\mathcal{G}}\right)\\ \left(g : \Sigma^{-n-1-i}G\left(-p\right) \to G\right) &\mapsto \alpha \circ \pi_*\left(w\left(g\right)\right) \circ \Sigma^{-i}\beta : \left(\Sigma^{-i}\widetilde{\mathcal{G}} \to \widetilde{\mathcal{G}}\right).\end{aligned}$$

Lemma 6.2.3. *For $i < 0$ the morphism*

$$\varphi : \operatorname{Ext}_X^{n+1+i}\left(G\left(-p\right), G\right) \cong \operatorname{Ext}^i_{\mathcal{X}_\eta^{dg}}\left(\widetilde{\mathcal{G}}, \widetilde{\mathcal{G}}\right)$$

is an isomorphism.

Proof. We will check that for $i < 0$ the morphisms involved in the definition of

$$\begin{aligned}\varphi : \operatorname{Ext}_X^{n+1+i}\left(G\left(-p\right), G\right) &\to \operatorname{Ext}^i_{\mathcal{X}_\eta^{dg}}\left(\widetilde{\mathcal{G}}, \widetilde{\mathcal{G}}\right)\\ \left(g : \Sigma^{-n-1-i}G\left(-p\right) \to G\right) &\mapsto \left(\alpha \circ \pi_*\right)\left(w\left(g\right)\right) \circ \Sigma^{-i}\beta : \left(\Sigma^{-i}\widetilde{\mathcal{G}} \to \widetilde{\mathcal{G}}\right)\end{aligned}$$

are isomorphisms.

w: By (4.1.1) $w : \mathcal{D}^b\left(X\right) \to \mathcal{D}^b\left(\mathcal{X}\right)$ is a fully faithful embedding, in particular, using $\mathcal{G} = wG$ (6.2.4) we have

$$w : \operatorname{Ext}_X^{n+1+i}\left(G, G\right) \xrightarrow{\sim} \operatorname{Ext}_{\mathcal{X}}^{n+1+i}\left(\mathcal{G}, \mathcal{G}\right).$$

$\alpha \circ \pi_*\left(\_\right)$: We have by [RVdBN19, Corollary 5.3.2] an adjunction:

$$\operatorname{RHom}_{\mathcal{X}_\eta^{dg}}\left(\mathcal{G} \otimes w\mathcal{O}_X\left(-p\right), \widetilde{\mathcal{G}}\right) \cong \operatorname{RHom}_{\mathcal{X}}\left(\mathcal{G} \otimes w\mathcal{O}_X\left(-p\right), \mathcal{G}\right)$$

This isomorphism can be computed explicitly to be:

$$\alpha \circ \pi_* : \operatorname{RHom}_{\mathcal{X}_\eta^{dg}}\left(\mathcal{G} \otimes w\mathcal{O}_X\left(-p\right), \widetilde{\mathcal{G}}\right) \cong \operatorname{RHom}_{\mathcal{X}}\left(\mathcal{G} \otimes w\mathcal{O}_X\left(-p\right), \mathcal{G}\right),$$

see [RVdBN19, (11.6)].

$\_\circ\beta$: Consider the distinguished triangle (6.2.5) in $\mathcal{D}\left(\mathcal{X}_\eta^{dg}\right)$:

$$\mathcal{G} \xrightarrow{\alpha} \widetilde{\mathcal{G}} \xrightarrow{\beta} \Sigma^{-n-1}\mathcal{G} \otimes w\mathcal{O}_X(-p).$$

Apply $\mathrm{RHom}_{\mathcal{X}_\eta^{dg}}\left(\_,\widetilde{\mathcal{G}}\right)$ to get the distinguished triangle:

$$\mathrm{RHom}_{\mathcal{X}_\eta^{dg}}\left(\Sigma^{-n-1}\mathcal{G}\otimes w\mathcal{O}_X(-p),\widetilde{\mathcal{G}}\right) \xrightarrow{\_\circ\beta} \mathrm{RHom}_{\mathcal{X}_\eta^{dg}}\left(\widetilde{\mathcal{G}},\widetilde{\mathcal{G}}\right) \to \mathrm{RHom}_{\mathcal{X}_\eta^{dg}}\left(\mathcal{G},\widetilde{\mathcal{G}}\right).$$

Now we may use [RVdBN19, Corollary 5.3.2] and Proposition 4.1.11,

$$\mathrm{RHom}_{\mathcal{X}_\eta^{dg}}\left(\mathcal{G},\widetilde{\mathcal{G}}\right) \cong \mathrm{RHom}_{\mathcal{X}_\eta}\left(\mathcal{G},\widetilde{\mathcal{G}}\right) \cong \mathrm{RHom}_{\mathcal{X}}(\mathcal{G},\mathcal{G}) \cong \mathrm{RHom}_X(G,G),$$

to get

$$\mathrm{RHom}_{\mathcal{X}_\eta^{dg}}\left(\Sigma^{-n-1}\mathcal{G}\otimes w\mathcal{O}_X(-p),\widetilde{\mathcal{G}}\right) \xrightarrow{\_\circ\beta} \mathrm{RHom}_{\mathcal{X}_\eta^{dg}}\left(\widetilde{\mathcal{G}},\widetilde{\mathcal{G}}\right) \to \mathrm{RHom}_X(G,G).$$

Applying H^i turns this into the long exact sequence:

$$\cdots \to \mathrm{Ext}_X^{i-1}(G,G) \to \mathrm{Ext}_{\mathcal{X}_\eta^{dg}}^{n+1+i}\left(\mathcal{G}\otimes w\mathcal{O}_X(-p),\widetilde{\mathcal{G}}\right) \xrightarrow{\_\circ\beta} \mathrm{Ext}_{\mathcal{X}_\eta^{dg}}^{i}\left(\widetilde{\mathcal{G}},\widetilde{\mathcal{G}}\right) \to \cdots.$$

And as G is a sheaf on X, specializing to $i<0$ yields the long exact sequence:

$$\cdots \to 0 \to \mathrm{Ext}_{\mathcal{X}_\eta^{dg}}^{n+1+i}\left(\mathcal{G}\otimes w\mathcal{O}_X(-p),\widetilde{\mathcal{G}}\right) \xrightarrow{\_\circ\beta} \mathrm{Ext}_{\mathcal{X}_\eta^{dg}}^{i}\left(\widetilde{\mathcal{G}},\widetilde{\mathcal{G}}\right) \to 0 \to \cdots.$$

In particular

$$\_\circ\beta : \mathrm{Ext}_{\mathcal{X}_\eta^{dg}}^{n+1+i}\left(\mathcal{G}\otimes w\mathcal{O}_X(-p),\widetilde{\mathcal{G}}\right) \xrightarrow{\sim} \mathrm{Ext}_{\mathcal{X}_\eta^{dg}}^{i}\left(\widetilde{\mathcal{G}},\widetilde{\mathcal{G}}\right)$$

is an isomorphism for $i<0$.

So altogether we get that

$$\begin{aligned}\varphi : \mathrm{Ext}_X^{n+1+i}(G(-p),G) &\xrightarrow{\sim} \mathrm{Ext}_{\mathcal{X}_\eta^{dg}}^{i}\left(\widetilde{\mathcal{G}},\widetilde{\mathcal{G}}\right)\\ \left(g:\Sigma^{-n-1-i}G(-p)\to G\right) &\mapsto \left(\alpha\circ\pi_*(w(g))\circ\Sigma^{-i}\beta:\Sigma^{-i}\widetilde{\mathcal{G}}\to\widetilde{\mathcal{G}}\right)\end{aligned}$$

is indeed an isomorphism for $i<0$ as it is a composition of isomorphisms. $\square$

Corollary 6.2.4. *Let $p<-n-1$ and $i>1$. Then* $\mathrm{Ext}_{\mathcal{X}_\eta}^{-i}\left(\widetilde{\mathcal{G}},\widetilde{\mathcal{G}}\right)=0$.

Proof. By (6.2.2) we have that $\mathrm{Ext}_X^*(G(-p),G)$ is concentrated in degree n and so we have by Lemma 6.2.3

$$\mathrm{Ext}_{\mathcal{X}_\eta^{dg}}^{-i}\left(\widetilde{\mathcal{G}},\widetilde{\mathcal{G}}\right) \cong \mathrm{Ext}_X^{n+1-i}(G(-p),G) \cong 0$$

for $i > 1$.

And since we have a quasi-equivalence $\mathcal{X}_\eta \cong \mathcal{X}_\eta^{dg}$ we get

$$\mathrm{Ext}^{-i}_{\mathcal{X}_\eta}\left(\widetilde{\mathcal{G}}, \widetilde{\mathcal{G}}\right) \cong \mathrm{Ext}^{n+1-i}_X \left(G\left(-p\right), G\right)$$

as claimed. □

Definition 6.2.5 ([RVdBN19, Lemma 11.4])**.** Let $\mathcal{X}$ be a $\Bbbk$-linear category, Γ a $\Bbbk$-algebra and let $\mathcal{M}$ be a $\Bbbk$-central $\mathcal{X}$-bimodule. Then we have for a Γ-equivariant $\mathcal{X}$-module $\mathcal{G}$,i.e. $\mathcal{G} \in \mathcal{X}-\mathrm{mod}_\Gamma$, the (algebraic) Γ-equivariant characteristic morphism

$$\begin{aligned} c_{\mathcal{G},\Gamma} : \mathrm{HH}^*\left(\mathcal{X}, \mathcal{M}\right) = \mathrm{Ext}^*_{\mathcal{X}\otimes\mathcal{X}^{\mathrm{op}}}\left(\mathcal{X}, \mathcal{M}\right) &\to \mathrm{Ext}^*_{\mathcal{X}\otimes\Gamma}\left(\mathcal{G}, \mathcal{G}\otimes\mathcal{M}\right) \\ \eta &\mapsto \mathcal{G}\otimes_{\mathcal{X}}\eta \end{aligned}$$

Observe that this morphism factors naturally as

$$c_{\mathcal{G},\Gamma} : \mathrm{HH}^*\left(\mathcal{X}, \mathcal{M}\right) \xrightarrow{\eta\mapsto\eta\cup 1} \mathrm{HH}^*\left(\mathcal{X}\otimes\Gamma, \mathcal{M}\otimes\Gamma\right) \xrightarrow{c_{\mathcal{G}}} \mathrm{Ext}^*_{\mathcal{X}\otimes\Gamma}\left(\mathcal{G}, \mathcal{G}\otimes\mathcal{M}\right),$$

where $c_{\mathcal{G}} : \mathrm{HH}^*\left(\mathcal{X}\otimes\Gamma, \mathcal{M}\otimes\Gamma\right) \to \mathrm{Ext}^*_{\mathcal{X}\otimes\Gamma}\left(\mathcal{G}, \mathcal{G}\otimes\mathcal{M}\right)$ is the (algebraic) characteristic morphism for $\mathcal{G} \in \mathcal{D}\left(\mathcal{X}\otimes\Gamma\right)$, see Definition 2.3.13.

Lemma 6.2.6. *There is a commutative diagram:*

$$\begin{tikzcd} \mathrm{HH}^{n+3}\left(X, \mathcal{O}_X\left(p\right)\right) \arrow[r, "c_{G,\Gamma}"] \arrow[d, "\wr"'] & \mathrm{Ext}^{n+3}_{\mathrm{coh}(X)_\Gamma}\left(G, G\left(p\right)\right) \arrow[d, "\wr"] \\ \mathrm{HH}^{n+3}\left(\mathcal{X}, w\mathcal{O}_X\left(p\right)\right) \arrow[r, "c_{\mathcal{G},\Gamma}"] & \mathrm{Ext}^{n+3}_{\mathcal{X}\otimes\Gamma}\left(\mathcal{G}, \mathcal{G}\otimes w\mathcal{O}_X\left(p\right)\right), \end{tikzcd}$$

where $c_{G,\Gamma}$ is the (geometric) equivariant characteristic morphism discussed in § 3 and $c_{\mathcal{G},\Gamma}$ is the (algebraic) characteristic morphism from Definition 6.2.5.

Proof. By [RVdBN19, (8.13)] we have the commutative diagram

$$\begin{tikzcd} \Delta_*\mathcal{D}\left(X\right) \arrow[r, "\pi_{1,*}\left(\_\otimes\pi_2^*G\right)"] \arrow[d, "\wr"', "W"] & \mathcal{D}\left(\mathrm{coh}\left(X\right)_\Gamma\right) \arrow[d, "w"', "\wr"] \\ \mathcal{D}\left(\mathcal{X}\otimes\mathcal{X}^{\mathrm{op}}\right) \arrow[r, "\mathcal{G}\otimes_{\mathcal{X}}\_"'] & \mathcal{D}\left(\mathcal{X}\otimes\Gamma\right), \end{tikzcd}$$

where we denote by $\Delta_* \mathcal{D}(X) \subset \mathcal{D}(X \times X)$ the essential image of the direct image along the diagonal embedding $\Delta : X \to X \times X$.

Considering the induced diagram on morphism spaces for

$$\mathrm{HH}^{n+3}(X, \mathcal{O}(p)) = \mathrm{Ext}^{n+3}_{X \times X}(\mathcal{O}_\Delta, \mathcal{O}_\Delta(p)) = \mathrm{Ext}^{n+3}_{\Delta_* \mathcal{D}(X)}(\mathcal{O}_\Delta, \mathcal{O}_\Delta(p))$$

gives that the diagram

$$\begin{tikzcd} \mathrm{HH}^{n+3}(X, \mathcal{O}_X(p)) \arrow[r, "c_{G,\Gamma}"] \arrow[d, "\wr"'] & \mathrm{Ext}^{n+3}_{\mathrm{coh}(X)_\Gamma}(G, G(p)) \arrow[d, "\wr"] \\ \mathrm{HH}^{n+3}(\mathcal{X}, w\mathcal{O}_X(p)) \arrow[r, "c_{\mathcal{G},\Gamma}"] & \mathrm{Ext}^{n+3}_{\mathcal{X} \otimes \Gamma}(\mathcal{G}, \mathcal{G} \otimes w\mathcal{O}_X(p)) \end{tikzcd}$$

commutes. □

Since $G \in \mathcal{D}^b(\mathrm{coh}(X)_\Gamma)$ we get a Γ-action on $\mathcal{G}$ and $\widetilde{\mathcal{G}}$ via the functors w and L, i.e. $\widetilde{\mathcal{G}} \in \mathcal{D}\left(\mathcal{X}_\eta^{dg}\right)_\Gamma$. So Lemma 2.3.20 gives well-defined obstructions against $\widetilde{\mathcal{G}} \in \mathcal{D}_\infty(\mathcal{X}_\eta)_\Gamma \cong \mathcal{D}\left(\mathcal{X}_\eta^{dg}\right)_\Gamma$ admitting a lift to an $\mathcal{A}_\infty$-module in $\mathcal{D}_\infty(\mathcal{X}_\eta \otimes \Gamma)$:

$$o_i\left(\widetilde{\mathcal{G}}\right) \in \mathrm{HH}^i\left(\Gamma, \mathrm{Ext}^{2-i}_{\mathcal{X}_\eta}\left(\widetilde{\mathcal{G}}, \widetilde{\mathcal{G}}\right)\right) \quad \text{for } i > 2.$$

Remark 6.2.7. The next Lemma will use the obstruction obtained from the equivariant characteristic morphism (6.2.1) in order to conclude that the first $\mathcal{A}_\infty$-obstruction against an equivariant lift of $\widetilde{\mathcal{G}}$ cannot vanish. We do this by observing that a colift of $\mathcal{G}$ to $\mathcal{X}_\eta$ would also give an equivariant lift of $\widetilde{\mathcal{G}}$. The control of $o_3\left(\widetilde{\mathcal{G}}\right)$ is necessary as we want to push forward the obstruction from $\mathcal{X}_\eta$ to $\mathcal{P}^{n+1}$ which cannot be done with the obstruction arising by the characteristic morphism.

Lemma 6.2.8. *We have:*

$$0 \neq o_3\left(\widetilde{\mathcal{G}}\right) \in \mathrm{HH}^3\left(\Gamma, \mathrm{Ext}^{-1}_{\mathcal{X}_\eta}\left(\widetilde{\mathcal{G}}, \widetilde{\mathcal{G}}\right)\right).$$

Proof. Assume $o_3\left(\widetilde{\mathcal{G}}\right)$ vanishes. Then by Corollary 6.2.4 $\mathrm{Ext}^{-i}_{\mathcal{X}_\eta}\left(\widetilde{\mathcal{G}}, \widetilde{\mathcal{G}}\right) = 0$ for $i > 1$ and so:

$$o_i\left(\widetilde{\mathcal{G}}\right) \in \mathrm{HH}^i\left(\Gamma, \mathrm{Ext}^{2-i}_{\mathcal{X}_\eta}\left(\widetilde{\mathcal{G}}, \widetilde{\mathcal{G}}\right)\right) = 0$$

for all $i > 2$.

So $\widetilde{\mathcal{G}}$ would admit a lift, i.e. an object

$$\widehat{\mathcal{G}} \in \mathcal{D}\left(\mathcal{X}_\eta^{\mathrm{dg}} \otimes \Gamma\right) \cong \mathcal{D}_\infty(\mathcal{X}_\eta \otimes \Gamma)$$

with $\widehat{\mathcal{G}} \cong \widetilde{\mathcal{G}}$ in $\mathcal{D}(\mathcal{X}_\eta)_\Gamma$.

Consider the triangle (6.2.5) in $\mathcal{D}_\infty(\mathcal{X}_\eta) \cong \mathcal{D}\left(\mathcal{X}_\eta^{dg}\right)$

$$\mathcal{G} \to \widetilde{\mathcal{G}} \cong \widehat{\mathcal{G}} \to \Sigma^{n+1}\mathcal{G} \otimes w\mathcal{O}_X(-p) \to \Sigma\mathcal{G},$$

where we use the shorthand $\mathcal{G}$ for $\pi_*\mathcal{G}$. This gives:

$$\mathrm{H}^*\left(\widehat{G}\right) \cong \mathcal{G} \oplus \Sigma^{n+1}\mathcal{G} \otimes w\mathcal{O}_X(-p).$$

By the construction of the triangle (6.2.5) in [RVdBN19, § 10], the above isomorphism is compatible with the $\mathcal{X}_\eta$-action. So by Definition 2.3.15 $\widehat{\mathcal{G}}$ is a colift of $\mathcal{G} \in \mathcal{D}(\mathcal{X} \otimes_{\Bbbk} \Gamma)$ to $\mathcal{D}_\infty\left((\mathcal{X}_\eta \otimes_{\Bbbk} \Gamma)_{\eta\cup 1}\right)$.

By Lemma 2.3.16, the obstruction against such a colift is the image of $\eta \cup 1$ under the characteristic morphism

$$\mathrm{HH}^{n+3}(\mathcal{X} \otimes \Gamma, w\mathcal{O}_X(p) \otimes \Gamma) \to \mathrm{Ext}^{n+3}_{\mathcal{X}\otimes\Gamma}(\mathcal{G}, \mathcal{G} \otimes w\mathcal{O}_X(p)).$$

However, this obstruction cannot vanish. As if we consider the equivariant characteristic morphism $c_{\mathcal{G},\Gamma}$:

$$\mathrm{HH}^{n+3}(\mathcal{X}, w\mathcal{O}_X(p)) \xrightarrow{\mu\mapsto\mu\cup 1} \mathrm{HH}^{n+3}(\mathcal{X} \otimes \Gamma, w\mathcal{O}_X(p) \otimes \Gamma) \xrightarrow{c_{\mathcal{G}}} \mathrm{Ext}^{n+3}_{\mathcal{X}\otimes\Gamma}(\mathcal{G}, \mathcal{G} \otimes w\mathcal{O}_X(p)),$$

we have the commutative diagram from Lemma 6.2.6:

$$\begin{tikzcd}
\mathrm{HH}^{n+3}(X, \mathcal{O}_X(p)) \arrow[r, "c_{G,\Gamma}"] \arrow[d, "\wr"'] & \mathrm{Ext}^{n+3}_{\mathrm{coh}(X)_\Gamma}(G, G(p)) \arrow[d, "\wr"] \\
\mathrm{HH}^{n+3}(\mathcal{X}, w\mathcal{O}_X(p)) \arrow[r, "c_{\mathcal{G},\Gamma}"] & \mathrm{Ext}^{n+3}_{\mathcal{X}\otimes\Gamma}(\mathcal{G}, \mathcal{G} \otimes w\mathcal{O}_X(p)).
\end{tikzcd}$$

By assumption (6.2.1) we have that $c_{G,\Gamma}(\eta) \neq 0$. So $c_{\mathcal{G}}(\eta \cup 1) \neq 0$, which means that such a colift of $\widetilde{\mathcal{G}}$ to $(\mathcal{X} \otimes \Gamma)_{\eta\cup 1}$ cannot exist. Now by the discussion above this means that a lift of $\widetilde{\mathcal{G}}$ to $\mathcal{D}_\infty(\mathcal{X}_\eta \otimes \Gamma)$ cannot exist and so $o_3\left(\widetilde{\mathcal{G}}\right)$ cannot be zero. □

Lemma 6.2.9. *There is a commutative diagram*

$$\begin{tikzcd}
\mathrm{Ext}^n_X(G(-p), G) \arrow[r, "\varphi"] \arrow[d, "f_*"'] & \mathrm{Ext}^{-1}_{\mathcal{X}_\eta}\left(\widetilde{\mathcal{G}}, \widetilde{\mathcal{G}}\right) \arrow[d, "\widetilde{f}_* \circ \psi_{\mathcal{X}_\eta,*}"] \\
\mathrm{Ext}^n_{\mathbb{P}^{n+1}}(f_*(G(-p)), f_*(G)) \arrow[r, "\widetilde{f}_*\varphi"'] & \mathrm{Ext}^{-1}_{\mathcal{P}^{n+1}}\left(\widetilde{f}_*\left(\widetilde{\mathcal{G}}\right), \widetilde{f}_*\left(\widetilde{\mathcal{G}}\right)\right),
\end{tikzcd} \tag{6.2.6}$$

where the lower morphism is given by

$$\begin{aligned}\widetilde{f}_*\varphi : \operatorname{Ext}^n_{\mathbb{P}^{n+1}}\left(f_*G\left(-p\right), f_*G\right)\right) &\to \operatorname{Ext}^{-1}_{\mathcal{P}^{n+1}}\left(\widetilde{f}_*\widetilde{\mathcal{G}}, \widetilde{f}_*\widetilde{\mathcal{G}}\right)\\ g &\mapsto \widetilde{f}_*\alpha \circ w\left(g\right) \circ \widetilde{f}_*\beta.\end{aligned}$$

Proof. Recall that by Definition 6.2.2 the morphism φ is given by

$$\begin{aligned}\varphi : \operatorname{Ext}^{n+1+i}_X\left(G\left(-p\right), G\right) &\to \operatorname{Ext}^{-1}_{\mathcal{X}^{dg}_\eta}\left(\widetilde{\mathcal{G}}, \widetilde{\mathcal{G}}\right)\\ \left(g : \Sigma^{-n-1-i}G\left(-p\right) \to G\right) &\mapsto \left(\alpha \circ \pi_*\left(wg\right) \circ \beta : \Sigma^i\widetilde{\mathcal{G}} \to \widetilde{\mathcal{G}}\right),\end{aligned}$$

where α and β are the first and second morphisms in the distinguished triangle (6.2.5) in $\mathcal{D}\left(\mathcal{X}^{dg}_\eta\right)$

$$\mathcal{G} \xrightarrow{\alpha} \widetilde{\mathcal{G}} \xrightarrow{\beta} \Sigma^{-n-1}\mathcal{G} \otimes w\mathcal{O}_X\left(-p\right) \xrightarrow{\gamma} \Sigma\mathcal{G}.$$

Applying the exact functor $\widetilde{f}_*\circ\psi_{\mathcal{X}_\eta,*}$ gives the distinguished triangle in $\mathcal{D}\left(\mathcal{P}^{n+1}\right)$

$$\widetilde{f}_*\mathcal{G} \xrightarrow{\widetilde{f}_*\alpha} \widetilde{f}_*\widetilde{\mathcal{G}} \xrightarrow{\widetilde{f}_*\beta} \Sigma^{-n-1}\widetilde{f}_*\mathcal{G} \otimes w\mathcal{O}_X\left(-p\right) \xrightarrow{\widetilde{f}_*\gamma} \Sigma\widetilde{f}_*\mathcal{G},$$

which is a shorthand for

$$\widetilde{f}_*\pi_*\mathcal{G} \xrightarrow{\widetilde{f}_*\alpha} \widetilde{f}_*\widetilde{\mathcal{G}} \xrightarrow{\widetilde{f}_*\beta} \Sigma^{-n-1}\widetilde{f}_*\pi_*\mathcal{G} \otimes w\mathcal{O}_X\left(-p\right) \xrightarrow{\widetilde{f}_*\gamma} \Sigma\widetilde{f}_*\pi_*\mathcal{G}.$$

So we may use $\pi \circ \widetilde{f} = f$ to get

$$f_*\mathcal{G} \xrightarrow{\widetilde{f}_*\alpha} \widetilde{f}_*\widetilde{\mathcal{G}} \xrightarrow{\widetilde{f}_*\beta} \Sigma^{-n-1}f_*\mathcal{G} \otimes w\mathcal{O}_X\left(-p\right) \xrightarrow{\widetilde{f}_*\gamma} \Sigma f_*\mathcal{G}. \tag{6.2.7}$$

In particular

$$\begin{aligned}\widetilde{f}_*\varphi : \operatorname{Ext}^n_{\mathbb{P}^{n+1}}\left(f_*\left(w\mathcal{O}_X\left(p\right) \otimes \mathcal{G}\right), f_*\left(\mathcal{G}\right)\right) &\to \operatorname{Ext}^{-1}_{\mathcal{P}^{n+1}}\left(\widetilde{f}_*\left(\widetilde{\mathcal{G}}\right), \widetilde{f}_*\left(\widetilde{\mathcal{G}}\right)\right)\\ g &\mapsto \widetilde{f}_*\alpha \circ wg \circ \widetilde{f}_*\beta\end{aligned}$$

is well-defined.

Now we compute

$$\begin{aligned}
\widetilde{f}_* \circ \varphi(g) &= \widetilde{f_*}(\alpha \circ \pi_*(wg) \circ \beta) && \text{Definition of } \varphi \\
&= \widetilde{f}_*\alpha \circ \left(\widetilde{f}_* \circ \pi_*\right)(wg) \circ \widetilde{f}_*\beta && \widetilde{f}_* \text{ is a functor} \\
&= \widetilde{f}_*\alpha \circ (f_* \circ w)(g) \circ \widetilde{f}_*\beta && \widetilde{f}_* \circ \pi_*(g) = f_* g \\
&= \widetilde{f}_*\alpha \circ (w \circ f_*)(g) \circ \widetilde{f}_*\beta && \text{[RVdBN19, Lemma 8.7.1]} \\
&= \widetilde{f}_*\varphi(f_* g) && \text{Definition of } \widetilde{f}_*\varphi
\end{aligned}$$

and the diagram indeed commutes. □

Corollary 6.2.10. *The right map in the diagram* (6.2.6)

$$\widetilde{f}_* \circ \psi_{\eta,*} : \mathrm{Ext}^{-1}_{\mathcal{X}^{dg}_\eta}\left(\widetilde{\mathcal{G}}, \widetilde{\mathcal{G}}\right) \xrightarrow{\sim} \mathrm{Ext}^{-1}_{\mathcal{P}^{n+1}}\left(\widetilde{f}_*\left(\widetilde{\mathcal{G}}\right), \widetilde{f}_*\left(\widetilde{\mathcal{G}}\right)\right)$$

is an isomorphism.

Proof. Since $G, G(-p)$ are coherent sheaves on X and f_* is exact we have

$$\begin{aligned}
&\mathrm{Ext}^1_{\mathcal{P}^{n+1}}\left(f_*\left(\Sigma^{-n-1}\mathcal{G} \otimes w\mathcal{O}_X(-p)\right), f_*(\mathcal{G})\right) \cong \\
&\cong \mathrm{Ext}^1_{\mathbb{P}^{n+1}}\left(f_*\left(\Sigma^{-n-1}G \otimes \mathcal{O}_X(-p)\right), f_*(G)\right) && (4.1.1) \\
&\cong \mathrm{Ext}^{n+2}_{\mathbb{P}^{n+1}}\left(f_*(G(-p)), f_*(G)\right) && \mathrm{Ext}^i\left(\Sigma^{-j}_, _\right) \cong \mathrm{Ext}^{i+j}(_, _) \\
&= 0. && \dim \mathbb{P}^{n+1} = n+1
\end{aligned}$$

So in the distinguished triangle (6.2.7)

$$f_*\mathcal{G} \xrightarrow{\widetilde{f}_*\alpha} \widetilde{f}_*\widetilde{\mathcal{G}} \xrightarrow{\widetilde{f}_*\beta} \Sigma^{-n-1}f_*\mathcal{G} \otimes w\mathcal{O}_X(-p) \xrightarrow{\widetilde{f}_*\gamma} \Sigma f_*\mathcal{G}$$

$\widetilde{f}_*\gamma$ vanishes, and we have:

$$\widetilde{f}_*\left(\widetilde{\mathcal{G}}\right) \cong f_*(\mathcal{G}) \oplus f_*\left(\Sigma^{-n-1}w\mathcal{O}_X(-p) \otimes_{\mathcal{X}} \mathcal{G}\right)$$

via the splitting morphisms $\widetilde{f}_*\alpha$ and $\widetilde{f}_*\beta$.

This means that both, the top morphism, by Lemma 6.2.3, and the lower morphism, by splitting, in (6.2.6) are isomorphsims. So by Lemma 6.2.9 it suffices to prove that

$$f_* : \mathrm{Ext}^n_X(\mathcal{O}_X(-p) \otimes G, G) \to \mathrm{Ext}^n_{\mathbb{P}^{n+1}}\left(f_*(\mathcal{O}_X(-p) \otimes G), f_*(G)\right)$$

is an isomorphism.

As tensoring with $\mathcal{O}_X(p)$ is an autoequivalence this is equivalent to

$$f_* : \mathrm{Ext}^n_X(G, G(p)) \to \mathrm{Ext}^n_{\mathbb{P}^{n+1}}(f_*G, f_*G(p))$$

being an isomorphism. Consider the long exact sequence associated to a divisor [RVdBN19, (9.13)]:

$$\cdots \to \mathrm{Ext}^{n-2}_X(G, G(p+d)) \to \mathrm{Ext}^n_X(G, G(p)) \xrightarrow{f_*} \mathrm{Ext}^n_{\mathbb{P}^{n+1}}(f_*(G), f_*G(p)) \to \cdots .$$

By assumption (6.2.3) we have

$$\mathrm{Ext}^{n-2}_X(G, G(p+d)) \cong 0 \quad \text{and} \quad \mathrm{Ext}^{n-1}_X(G, G(p+d)) \cong 0,$$

so the long exact sequence has the shape

$$\cdots \to 0 \to \mathrm{Ext}^n_X(G, G(p)) \xrightarrow{f_*} \mathrm{Ext}^n_{\mathbb{P}^{n+1}}(f_*(G), f_*G(p)) \to 0 \to \cdots .$$

By exactness that immediately gives that

$$f_* : \mathrm{Ext}^n_X(G(-p), G) \xrightarrow{\sim} \mathrm{Ext}^n_{\mathbb{P}^{n+1}}(f_*G(-p), f_*G)$$

is an isomorphism, which finishes the proof. □

Lemma 6.2.11. *The obstruction* $o_3(\Psi_\eta(G)) \in \mathrm{HH}^3\left(\Gamma, \mathrm{Ext}^{-1}_{\mathbb{P}^{n+1}}(\Psi(G), \Psi(G))\right)$ *against lifting to* $\mathcal{D}(\mathcal{P}^{n+1} \otimes \Gamma)$ *from Lemma 2.3.20 does not vanish.*

Proof. By part (2) of Lemma 2.3.20 we have

$$o_3(\Psi(G)) = \left(\widetilde{f}_* \circ \psi_{\mathcal{X}_\eta,*}\right) o_3\left(\widetilde{\mathcal{G}}\right) \in \mathrm{HH}^3\left(\Gamma, \mathrm{Ext}^{-1}_{\mathbb{P}^{n+1}}(\Psi(G), \Psi(G))\right).$$

Furthermore, as o_3 is the first obstruction we do not need to keep track of any choices. So we can use Corollary 6.2.10 to get that $\widetilde{f}_* \circ \psi_{\mathcal{X}_\eta,*}$ induces an isomorphism in degree -1 and by Lemma 6.2.8 we have $0 \neq o_3\left(\widetilde{\mathcal{G}}\right)$. So altogether

$$0 \neq \left(\widetilde{f}_* \circ \psi_{\mathcal{X}_\eta,*}\right) o_3\left(\widetilde{\mathcal{G}}\right) = o_3(\Psi(G)) \in \mathrm{HH}^3\left(\Gamma, \mathrm{Ext}^{-1}_{\mathbb{P}^{n+1}}(\Psi(G), \Psi(G))\right). \quad \square$$

Now we can finally finish the proof of Theorem 6.1.6.

Proof. Assume Ψ_η is Fourier-Mukai. Then by Corollary 3.1.7 Ψ_η admits a lift

$$\Psi_{\eta,\Gamma} : \mathcal{D}^b(\mathrm{coh}(X)_\Gamma) \to \mathcal{D}^b\left(\mathrm{coh}\left(\mathbb{P}^{n+1}\right)_\Gamma\right).$$

This means that $\Psi_\eta(G) \in \mathcal{D}^b(\mathbb{P}^{n+1})_\Gamma$ has a lift to $\mathcal{D}^b(\mathrm{coh}(\mathbb{P}^{n+1})_\Gamma) \hookrightarrow \mathcal{D}_\infty(\mathcal{P}^{n+1} \otimes_{\Bbbk} \Gamma)$.

Since we have by Lemma 6.2.11

$$o_3\left(\Psi_\eta\left(G\right)\right) \neq 0$$

such a lift cannot exist.

So Ψ_η cannot be Fourier Mukai. □

6.3 Application: odd dimensional Quadrics

We will show that the tilting bundle G for an odd dimensional quadric hypersurface and its endomorphism algebra Γ satisfy the assumptions of Theorem 6.1.6. For this we start by recalling that quadrics admit an exceptional sequence, which gives rise to a tilting bundle.

Theorem 6.3.1 ([Bö5, Corollary 3.2.8])**.** *Let $Q \hookrightarrow \mathbb{P}^{2k}$ be the embedding of a smooth quadric. Then Q admits an exceptional sequence:*

$$\left(S\left(-2k+1\right), \mathcal{O}_Q\left(-2k+2\right), ..., \mathcal{O}_Q\left(-1\right), \mathcal{O}_Q\right),$$

where S denotes the spinor bundle.

In particular we may consider for the embedding of a smooth quadric $f : Q \hookrightarrow \mathbb{P}^{2k}$ the tilting bundle:

$$G := S\left(-2k+1\right) \oplus \bigoplus_{l=0}^{-2k+2} \mathcal{O}_Q\left(-l\right) \text{ and } \Gamma := \operatorname{End}\left(G\right).$$

Now we need to verify the assumptions on the concentration of $\operatorname{Ext}^*_Q\left(G(-p), G\right)$ and $\operatorname{Ext}^*_Q\left(G, G\left(p+d\right)\right)$. We will use $p = -2k-2$ and

$$0 \neq \eta \in \ker\left(f_* : \operatorname{HH}^{n+3}\left(X, \mathcal{O}_Q\left(-2k-2\right)\right) \to \operatorname{HH}^{n+3}\left(\mathbb{P}^{2k}, f_*\mathcal{O}_Q\left(-2k-2\right)\right)\right)$$

as we know by Proposition 5.2.9 that

$$f_* : \operatorname{HH}^{n+3}\left(Q, \mathcal{O}_Q\left(-2k-2\right)\right) \to \operatorname{HH}^{n+3}\left(\mathbb{P}^{n+1}, f_*\mathcal{O}_Q\left(-2k-2\right)\right)$$

has one-dimensional kernel.

For the Ext-calculations we will need the following statement which also holds for even quadrics. However, as in the even case we would need to track the different spinor bundles depending on the equivalence class of the dimension modulo four, we will restrict to the odd case for legibility.

Lemma 6.3.2. *Let $Q \hookrightarrow \mathbb{P}^{2k}$ be a smooth odd dimensional quadric and let S be the spinor bundle. Then the following hold:*

1. *We have for $i \notin \{0, 1, n\}$*

$$\operatorname{Ext}^i_Q(S, S(m)) \cong \operatorname{Ext}^{i-1}_Q(S, S(m+1)).$$

2. *If $m \leq -1$ we have additionally*

$$\operatorname{Ext}^i_Q(S, S(m)) \cong \operatorname{Ext}^{i-1}_Q(S, S(m+1))$$
$$\operatorname{Ext}^i_Q(S, S(m)) \cong \operatorname{Ext}^{i-1}_Q(S, S(m+1)).$$

Proof. Consider the short exact sequence [Ott88, Theorem 2.8]

$$0 \mapsto S \mapsto \mathcal{O}_Q^{2^{k+1}} \to S(1) \to 0$$

which gives after applying $\operatorname{Ext}^i_Q(\_, S(m+1))$ the long exact sequence

$$\cdots \longrightarrow \operatorname{Ext}^{i-1}_Q\left(\mathcal{O}_Q^{2^{k+1}}, S(m+1)\right) \to \operatorname{Ext}^{i-1}_Q(S, S(m+1)) \to \operatorname{Ext}^i_Q(S(1), S(m+1)) \to \operatorname{Ext}^i_Q\left(\mathcal{O}_Q^{2^{k+1}}, S(m+1)\right) \longrightarrow \cdots.$$

In particular we have

$$\operatorname{Ext}^i_Q(S, S(m)) \cong \operatorname{Ext}^i_Q(S(1), S(m+1)) \cong \operatorname{Ext}^{i+1}_Q(S, S(m-1))$$

if we have for $j \in \{i, i-1\}$

$$\operatorname{Ext}^j\left(\mathcal{O}_Q^{2^{k+1}}, S(m+1)\right) \cong \bigoplus_{l=0}^{2^{k+1}} \operatorname{Ext}^j(\mathcal{O}_Q, S(m+1)) \cong \bigoplus_{l=0}^{2^{k+1}} \mathrm{H}^j(X, S(m+1)) = 0.$$

By [Ott88, Theorem 2.3] we have $\mathrm{H}^j(X, S(m+1)) = 0$ for $j \notin \{0, n\}$ which implies 1.

If $m \leq -1$ we have $m+1 \leq 0$ and so we get by [Ott88, Theorem 2.3] $H^0(X, S(m+1)) = 0$, which gives 2. $\square$

Proposition 6.3.3. *Let $i \neq 2k-1$. Then we have*

$$\operatorname{Ext}^*_Q\left(G\left(2k+2\right), G\right) = 0$$

Proof. Since G is a sheaf we may assume $0 \leq i \leq 2k-2$ for dimension reasons. By definition of G and additivity of Ext we have

$$\begin{aligned}
&\operatorname{Ext}^i_Q\left(G(2k+2), G\right) = \\
&= \operatorname{Ext}^i_Q\left(\left(S\left(-2k+1\right) \oplus \bigoplus_{l=0}^{2k-2} \mathcal{O}_Q\left(-l\right)\left(2k+2\right)\right), S\left(-2k+1\right) \oplus \bigoplus_{l=0}^{2k-2} \mathcal{O}_Q\left(-l\right)\right) \\
&\cong \bigoplus_{h,l=0}^{2k-2} \operatorname{Ext}^i_Q\left(\mathcal{O}_Q\left(2k+2-l\right), \mathcal{O}_Q\left(-h\right)\right) \\
&\oplus \bigoplus_{l=0}^{2k-2} \operatorname{Ext}^i_Q\left(\mathcal{O}_Q\left(2k+2-l\right), S\left(-2k+1\right)\right) \\
&\oplus \bigoplus_{l=0}^{2k-2} \operatorname{Ext}^i_Q\left(S\left(2k+2-2k+1\right), \mathcal{O}_Q\left(-l\right)\right) \\
&\oplus \operatorname{Ext}^i_Q\left(S\left(-2k+1+2k+2\right), S\left(-2k+1\right)\right)
\end{aligned}$$

In particular we can compute these Ext-groups one by one.

We start with $\operatorname{Ext}^i_Q\left(\mathcal{O}_Q\left(2k+2-l\right), \mathcal{O}_Q\left(-h\right)\right)$ for which we get

$$\begin{aligned}
&\operatorname{Ext}^i_Q\left(\mathcal{O}_Q\left(2k+2-l\right), \mathcal{O}_Q\left(-h\right)\right) \cong && \\
&\cong \operatorname{Ext}^i_Q\left(\mathcal{O}_Q, \mathcal{O}_Q\left(l-h-2k-2\right)\right) && \text{twisting on both sides} \\
&\cong \mathrm{H}^i\left(Q, \mathcal{O}_Q\left(l-h-2k-2\right)\right) && \operatorname{Ext}^i_Q\left(\mathcal{O}_Q, _\right) \cong \mathrm{H}^i\left(Q, _\right) \\
&\cong 0. && l-h-2k-2 < 0
\end{aligned}$$

Since we have $l-4k-1 \leq 0$ we get

$$\begin{aligned}
&\operatorname{Ext}^i_Q\left(\mathcal{O}_Q\left(2k+2-l\right), S\left(-2k+1\right)\right) \cong && \\
&\cong \operatorname{Ext}^i_Q\left(\mathcal{O}_Q, S\left(l-4k-1\right)\right) && \text{twisting on both sides} \\
&\cong \mathrm{H}^i\left(Q, S\left(l-4k-1\right)\right) && \operatorname{Ext}^i_Q\left(\mathcal{O}_Q, _\right) \cong \mathrm{H}^i\left(Q, _\right) \\
&\cong 0. && \text{[Ott88, Theorem 2.3]}
\end{aligned}$$

By [Ott88, Theorem 2.8] we have $S^\vee \cong S(1)$ which we may use to compute

$$\begin{aligned}
\mathrm{Ext}^i_Q\left(S(3), \mathcal{O}_Q(-l)\right) &\cong \mathrm{Ext}^i_Q\left(S, \mathcal{O}_Q(-3-l)\right) && \text{twisting on both sides}\\
&\cong \mathrm{Ext}^i_Q\left(\mathcal{O}_Q, S^\vee(-3-l)\right) && \text{dualizing}\\
&\cong \mathrm{Ext}^i_Q\left(\mathcal{O}_Q, S(-2-l)\right) && S^\vee \cong S(1)\\
&\cong \mathrm{H}^i\left(Q, S(-2-l)\right) && \mathrm{Ext}^i_Q\left(\mathcal{O}_Q, _\right) \cong \mathrm{H}^i\left(Q, _\right)\\
&\cong 0. && \text{[Ott88, Theorem 2.3]}
\end{aligned}$$

We may use $i \leq 2k-1$ to get

$$\begin{aligned}
\mathrm{Ext}^i_Q\left(S(3), S(-2k+1)\right) &\cong \mathrm{Ext}^i_Q\left(S, S(-2k-2)\right) && \text{twisting on both sides}\\
&\cong \mathrm{Ext}^1_Q\left(S, S(-2k-3+i)\right) && \text{Lemma 6.3.2}\\
&\cong \mathrm{Ext}^{-1}_Q\left(S, S(-2k-1+i)\right) && \text{Lemma 6.3.2}\\
&\cong 0. && S \text{ is a sheaf}
\end{aligned}$$

So every direct summand vanishes, and in particular

$$\mathrm{Ext}^i_Q\left(G(2k+2), G\right) = 0 \text{ for } i \neq n$$

as desired. $\square$

Proposition 6.3.4. *Let $i \notin \{0, 2k-1\}$. Then we have*

$$\mathrm{Ext}^i_Q\left(G, G(-2k)\right) \cong 0.$$

Proof. Since Q has dimension $2k-1$ and G is a sheaf we may assume $0 < i < 2k-1$.

By definition of G and additivity of $\mathrm{Ext}^i_Q\left(\_,\_\right)$ we have:

$$\begin{aligned}
&\mathrm{Ext}^i_Q\left(G, G(-2k)\right)\\
&= \mathrm{Ext}^i_Q\left(S(-2k+1) \oplus \bigoplus_{l=0}^{2k-2} \mathcal{O}_Q(-l), S(-4k+1) \oplus \bigoplus_{h=0}^{2k-2} \mathcal{O}_Q(-2k-h)\right)\\
&\cong \bigoplus_{l,h=0}^{2k-2} \mathrm{Ext}^i_Q\left(\mathcal{O}_Q(-l), \mathcal{O}_Q\left(\mathcal{O}_Q(-2k-h)\right)\right)\\
&\oplus \bigoplus_{l=0}^{2k-2} \mathrm{Ext}^i_Q\left(\mathcal{O}_Q(-l), S(-4k+1)\right)\\
&\oplus \bigoplus_{l=0}^{2k-2} \mathrm{Ext}^i_Q\left(S(-2k+1), \mathcal{O}_Q(-2k-l)\right)\\
&\oplus \mathrm{Ext}^i_Q\left(S(-2k+1), S(-4k+1)\right).
\end{aligned}$$

As above we can compute the cases separately.

We start with $\mathrm{Ext}^i_Q\left(\mathcal{O}_Q\left(-l\right),\mathcal{O}_Q\left(-2k-h\right)\right)$. For this we get:

$$\begin{aligned}
&\mathrm{Ext}^i_Q\left(\mathcal{O}_Q\left(-l\right),\mathcal{O}_Q\left(-2k-h\right)\right)\cong && \\
&\cong \mathrm{Ext}^i_Q\left(\mathcal{O}_Q,\mathcal{O}_Q\left(l-2k-h\right)\right) && \text{twisting on both sides}\\
&\cong \mathrm{H}^i\left(Q,\mathcal{O}_Q\left(1-2k-h\right)\right) && \mathrm{Ext}^i_Q\left(\mathcal{O}_Q,_\right)\cong \mathrm{H}^i\left(Q,_\right)\\
&\cong 0. && i\notin\{0,2k-1\}
\end{aligned}$$

For $\mathrm{Ext}^i_Q\left(\mathcal{O}_Q\left(-l\right),S\left(-4k+1\right)\right)$ we get

$$\begin{aligned}
&\mathrm{Ext}^i_Q\left(\mathcal{O}_Q\left(-l\right),S\left(-4k+1\right)\right)\cong && \\
&\cong \mathrm{Ext}^i\left(\mathcal{O}_Q,S\left(l-4k+1\right)\right) && \text{twisting on both sides}\\
&\cong \mathrm{H}^i\left(Q,S\left(l-4k+1\right)\right) && \mathrm{Ext}^i_Q\left(\mathcal{O}_Q,_\right)\cong \mathrm{H}^i\left(Q,_\right)\\
&\cong 0. && i\notin\{0,2k-1\}\ \text{[Ott88, Theorem 2.3]}
\end{aligned}$$

While for $\mathrm{Ext}^i_Q\left(S\left(-2k+1\right),\mathcal{O}_Q\left(-2k-l\right)\right)$ one can compute:

$$\begin{aligned}
&\mathrm{Ext}^i_Q\left(S\left(-2k+1\right),\mathcal{O}_Q\left(-2k-l\right)\right)\cong && \\
&\cong \mathrm{Ext}^i_Q\left(S,\mathcal{O}_Q\left(-1-l\right)\right) && \text{twisting on both sides}\\
&\cong \mathrm{Ext}^i_Q\left(\mathcal{O}_Q,S^\vee\left(-1-l\right)\right) && \text{dualizing}\\
&\cong \mathrm{Ext}^i_Q\left(\mathcal{O}_Q,S\left(-l\right)\right) && \text{[Ott88, Theorem 2.8]}\\
&\cong \mathrm{H}^i\left(Q,S\left(-l\right)\right) && \mathrm{Ext}^i_Q\left(\mathcal{O}_Q,_\right)\cong \mathrm{H}^i\left(Q,_\right)\\
&= 0. && i\notin\{0,2k-1\}\ \text{[Ott88, Theorem 2.3]}
\end{aligned}$$

Finally for $\mathrm{Ext}^i_Q\left(S\left(-2k+1\right),S\left(-4k+1\right)\right)$ we get

$$\begin{aligned}
&\mathrm{Ext}^i_Q\left(S\left(-2k+1\right),S\left(-4k+1\right)\right)\cong && \\
&\cong \mathrm{Ext}^i_Q\left(S,S\left(-2k\right)\right) && \text{twisting on both sides}\\
&\cong \mathrm{Ext}^1_Q\left(S,S\left(-2k+i-1\right)\right) && \text{Lemma 6.3.2(1)}\\
&\cong \mathrm{Ext}^0_Q\left(S,S\left(-2k-i\right)\right) && \text{Lemma 6.3.2(2)}\\
&\cong \mathrm{Ext}^{-1}_Q\left(S,S\left(-2k+1-i\right)\right) && \text{Lemma 6.3.2(2)}\\
&= 0, && S \text{ is a sheaf}
\end{aligned}$$

where we used $i<2k-1$ and so $-2k+i\leq -1$, respectively $-2k+1+i\leq -1$ for the last two lines.

So all the direct summands of $\mathrm{Ext}^i_Q\left(G,G\left(-2k\right)\right)$ vanish for $i\notin\{0,2k-1\}$ as claimed. □

So altogether we can now phrase the following Theorem 6.3.5 which also recovers the result from [RVdBN19] when specialized to the case $k = 2$.

Theorem 6.3.5. *Let $Q \hookrightarrow \mathbb{P}^{2k}$ be the embedding of a smooth odd dimensional quadric for $k \geq 2$. Then we have an exact functor:*

$$\Psi_\eta : \mathcal{D}^b(Q) \to \mathcal{D}^b(\mathbb{P}^n)$$

that cannot be Fourier-Mukai.

Proof. We want to apply Theorem 6.1.6.

First of all we have by Proposition 5.2.9 for $k > 2$ an

$$0 \neq \eta \in \mathrm{HH}^{2k+2}(Q, \mathcal{O}_Q(-2k-2))$$

that is in the kernel of $f_* : \mathrm{HH}^{n+3}(Q, \mathcal{O}(-2k-2)) \to \mathrm{HH}^{n+3\left(\mathbb{P}^{2k}, f_*\mathcal{O}(-2k-2)\right)}$.

For $k = 2$ we get that the top Hochschild cohomology is $\mathrm{HH}^{n+3}(Q, \mathcal{O}(-2k-2))$, and so by Theorem 5.2.8 we have

$$\dim\ker\left(f_* : \mathrm{HH}^{n+3}(Q, \mathcal{O}(-6)) \to \mathrm{HH}^{n+3}\left(\mathbb{P}^4, f_*\mathcal{O}(-6)\right)\right) = \mathrm{h}_1^{2,1}(Q).$$

Using the formula (5.1.1) we compute

$$\begin{aligned}
\mathrm{h}_1^{2,1}(Q) &= \sum_{\mu=0}^{5} (-1)^\mu \binom{6}{\mu} \binom{-1+4-(\mu-1)(2-1)}{4} \\
&= \sum_{\mu=0}^{5} (-1)^\mu \binom{6}{\mu} \binom{-1+4-(\mu-1)}{4} \\
&= \sum_{\mu=0}^{5} (-1)^\mu \binom{6}{\mu} \binom{4-\mu}{4} \\
&= (-1)^0 \binom{6}{0} \binom{4}{4} \\
&= 1,
\end{aligned}$$

where we used that $\binom{4-\mu}{4}$ only can be nonzero if $\mu = 0$. In particular we get a one-dimensional kernel from which we may pick an $\eta \neq 0$.

We now collect the other assumptions which we verified above.

By Theorem 6.3.1 Q admits a tilting bundle G and by Lemma 3.2.12 we know that for $\Gamma := \mathrm{End}(G)$ the functor $C^Q_{G,\Gamma}$ is an equivalence. In particular we get by Proposition 3.2.5 $c_{G,\Gamma}(\eta) \neq 0$, which is assumption (6.2.1). Now finally we need to verify that the corresponding Ext-groups are suitably concentrated, which is verified in Proposition 6.3.3 for assumption (6.2.2) and Proposition 6.3.4 for assumption (6.2.3).

So we may apply Theorem 6.1.6 to get a non-Fourier-Mukai functor Ψ_η. $\square$

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