Graph parameters, implicit representations and factorial properties

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Abstract. A representation of an n-vertex graph G is implicit if it assigns to each vertex of G a binary code of length $O(\log n)$ so that the adjacency of two vertices is a function of their codes. A necessary condition for a hereditary class $\mathcal{X}$ of graphs to admit an implicit representation is that $\mathcal{X}$ has at most factorial speed of growth. This condition, however, is not sufficient, as was recently shown in [10]. Several sufficient conditions for the existence of implicit representations deal with boundedness of some parameters, such as degeneracy or clique-width. In the present paper, we analyze more graph parameters and prove a number of new results related to implicit representation and factorial properties.

Keywords: Graph parameter · Hereditary class · Implicit representation · Factorial property

1 Introduction

A representation of an n-vertex graph G is implicit if it assigns to each vertex of G a binary code of length $O(\log n)$ so that the adjacency of two vertices is a function of their codes. The idea of implicit representation was introduced in [11]. Its importance is due to various reasons. First, it is order-optimal. Second, it allows one to store information about graphs locally, which is crucial in distributed computing. Finally, it is applicable to graphs in various classes of practical or theoretical importance, such as graphs of bounded vertex degree, of bounded clique-width, planar graphs, interval graphs, permutation graphs, line graphs, etc.

To better describe the area of applicability of implicit representations, let us observe that if graphs in a class $\mathcal{X}$ admit an implicit representation, then the number of n-vertex labelled graphs in $\mathcal{X}$, also known as the *speed* of $\mathcal{X}$, must be $2^{O(n \log n)}$, since the number of graphs cannot be larger than the number of binary words representing them. In the terminology of [5], hereditary classes containing $2^{\Theta(n \log n)}$ n-vertex labelled graphs have *factorial* speed of growth. The family of factorial classes, i.e. hereditary classes with a factorial speed of growth, is rich and diverse. In particular, it contains all classes mentioned earlier and a variety of other classes, such as unit disk graphs, classes of graphs of bounded arboricity, of bounded functionality [1], etc. The authors of [11], who introduced the notion of implicit representation, ask whether *every* hereditary class of speed $2^{O(n \log n)}$ admits such a representation.

Recently, this question was answered in the negative in [10] by proving the existence of a factorial class of bipartite graphs that does not admit an implicit representation. This negative result raises the following question: if the speed is not responsible for implicit representation, then what is responsible for it?

Looking for an answer to this question, we observe that most positive results on implicit representations deal with classes where certain graph parameters are bounded. In an attempt to produce more positive results, in Section 3 we analyze more graph parameters and in Section 4 we reveal new classes of graphs that admit an implicit representation.

In spite of the negative result in [10], factorial speed remains a necessary condition for an implicit representation in a hereditary class $\mathcal{X}$, and determining the speed of $\mathcal{X}$ is the first natural step towards deciding whether such a representation exists. A new result on this topic is presented in Section 5. All relevant preliminary information can be found in Section 2.

2 Preliminaries

All graphs in this paper are simple, i.e. undirected, without loops or multiple edges. The vertex set and the edge set of a graph G are denoted $V(G)$ and $E(G)$, respectively. The neighbourhood of a vertex $x \in V(G)$, denoted $N(x)$, is the set of vertices adjacent to x, and the degree of x, denoted $\deg(x)$, is the size of its neighbourhood. The codegree of x is the number of vertices non-adjacent to x.

As usual, K_n, P_n and C_n denote a complete graph, a chordless path and a chordless cycle on n vertices, respectively. By nG we denote the disjoint union of n copies of G.

The subgraph of G induced by a set $U \subseteq V(G)$ is denoted $G[U]$. If G does not contain an induced subgraph isomorphic to a graph H, we say that G is H-free and that H is a forbidden induced subgraph for G. A *homogeneous set* is a subset U of $V(G)$ such that $G[U]$ is either complete or edgeless.

A graph $G = (V, E)$ is *bipartite* if its vertex set can be partitioned into two independent sets. A bipartite graph given together with a bipartition of its vertex set into two independent sets A and B will be denoted $G = (A, B, E)$. The *bipartite complement* of a bipartite graph $G = (A, B, E)$ is the bipartite graph $\widetilde{G} := (A, B, (A \times B) - E)$. By $K_{n,m}$ we denote a complete bipartite graph with parts of size n and m. The graph $K_{1,n}$ is called a *star*. The *bi-codegree* of a vertex x in a bipartite graph $G = (A, B, E)$ is the degree of x in $\widetilde{G}$.

Given two bipartite graphs $G_1 = (A_1, B_1, E_1)$ and $G_2 = (A_2, B_2, E_2)$, we say that G_1 does not contain a one-sided copy of G_2 if all induced occurrences of G_2 in G_1 have the vertices of A_2 in the same part of G_1.

2.1 Graph classes

A class of graphs is *hereditary* if it is closed under taking induced subgraphs. It is well known that a class $\mathcal{X}$ is hereditary if and only if $\mathcal{X}$ can be described by a set of minimal forbidden induced subgraphs. In this section, we introduce a few hereditary classes that play an important role in this paper.

Motivated by the negative result in [10], which proves the existence of a factorial class of *bipartite* graphs that does not admit an implicit representation, we focus on hereditary subclasses of bipartite graphs. In particular, we study *monogenic* classes of bipartite graphs, i.e. classes defined by a single forbidden induced bipartite subgraph. The results in [2] and [13] provide a complete dichotomy for monogenic classes of bipartite graphs with respect to their speed. This dichotomy is presented in Theorem 1 below, where $S_{1,2,3}$ and $F_{t,p}$ are the graphs represented in Figure 1.

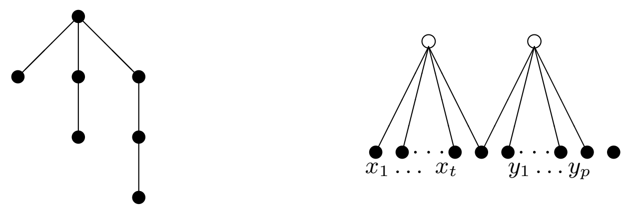

Fig. 1. The graphs $S_{1,2,3}$ (left) and $F_{t,p}$ (right)

Theorem 1. *For a bipartite graph H, the class of H-free bipartite graphs has at most factorial speed of growth if and only if H is an induced subgraph of one of the following graphs: P_7, $S_{1,2,3}$ and $F_{t,p}$.*

2.2 Tools

Several useful tools to produce an implicit representation have been introduced in [3]. In this section, we mention two such tools, and generalise one of them.

The first result deals with the notion of locally bounded coverings, which can be defined as follows. Let G be a graph. A set of graphs $H_1, \ldots, H_k$ is called a *covering* of G if the union of $H_1, \ldots, H_k$ coincides with G, i.e. if $V(G) = \bigcup_{i=1}^{k} V(H_i)$ and $E(G) = \bigcup_{i=1}^{k} E(H_i)$.

Theorem 2. [3] *Let $\mathcal{X}$ be a class of graphs and c a constant. If every graph $G \in \mathcal{X}$ can be covered by graphs from a class $\mathcal{Y}$ admitting an implicit representation in such a way that every vertex of G is covered by at most c graphs, then $\mathcal{X}$ also admits an implicit representation.*

The second result deals with the notion of partial coverings and can be stated as follows.

Theorem 3. [3] *Let $\mathcal{X}$ be a hereditary class. Suppose there is a constant d and a hereditary class $\mathcal{Y}$ which admits an implicit representation such that every graph $G \in \mathcal{X}$ contains a non-empty subset $A \subseteq V(G)$ with the properties that $G[A] \in \mathcal{Y}$ and each vertex of A has at most d neighbours or at most d non-neighbours in $V(G) - A$. Then $\mathcal{X}$ admits an implicit representation.*

Next we provide a generalisation of Theorem 3 that will be useful later.

Theorem 4. *Let $\mathcal{X}$ be a hereditary class. Suppose there is a constant d and a hereditary class $\mathcal{Y}$ which admits an implicit representation so that every graph $G \in \mathcal{X}$ contains a non-empty subset $A \subseteq V(G)$ with the following properties:*

(1) $G[A] \in \mathcal{Y}$,
(2) $V(G) - A$ can be split into two subsets B_1 and B_2 with no edges between them, and
(3) every vertex of A has at most d neighbours or at most d non-neighbours in B_1 and at most d neighbours or at most d non-neighbours in B_2.

Then $\mathcal{X}$ admits an implicit representation.

Proof. Let G be an n-vertex graph in $\mathcal{X}$. We assign to the vertices of G pairwise distinct *indices* recursively as follows. Let $\{1, 2, \ldots, n\}$ be the *index range* of G, and let A, B_1, and B_2 be the partition of $V(G)$ satisfying the conditions (1)-(3) of the theorem. We assign to the vertices in A indices from the interval $\{|B_1|+1, |B_1|+2, \ldots, n-|B_2|\}$ bijectively in an arbitrary way. We define indices of the vertices in B_1 recursively by decomposing $G[B_1]$ and using the interval $\{1, 2, \ldots, |B_1|\}$ as its index range. Similarly, we define indices of the vertices in B_2 by decomposing $G[B_2]$ and using the interval $\{n-|B_2|+1, n-|B_2|+2, \ldots, n\}$ as its index range.

Now, for every vertex $v \in A$ its label consists of six components:

1. the label of v in the implicit representation of $G[A] \in \mathcal{Y}$;
2. the index of v;
3. the index range of B_1, which we call the *left index range* of v;
4. the index range of B_2, which we call the *right index range* of v;
5. a boolean flag indicating whether v has at most d neighbours or d non-neighbours in B_1 and the indices of those at most d vertices;
6. a boolean flag indicating whether v has at most d neighbours or d non-neighbours in B_2 and the indices of those at most d vertices.

For the third and the fourth component we store only the first and the last elements of the ranges, and therefore the total label size is $O(\log n)$. The labels of the vertices in B_1 and B_2 are defined recursively.

Note that two vertices can only be adjacent if either they have the same left and right index ranges or the index of one of the vertices is contained in the left or right index range of the other vertex. In the former case, the adjacency of the vertices is determined by the labels in the first components of their labels. In the latter case, the adjacency is determined using the information stored in the components 5 and 6 of the labels. □

In the context of bipartite graphs, Theorem 4 can be restated as follows.

Theorem 5. *Let $\mathcal{X}$ be a hereditary class of bipartite graphs. Suppose there is a constant d and a hereditary class $\mathcal{Y}$ which admits an implicit representation so that every graph $G \in \mathcal{X}$ contains a non-empty subset $A \subseteq V(G)$ with the following properties:*

(1) $G[A] \in \mathcal{Y}$,
(2) $V(G) - A$ *can be split into two subsets* B_1 *and* B_2 *with no edges between them, and*
(3) every vertex of A has at most d neighbours or at most d non-neighbours in the opposite *part of B_1 and at most d neighbours or at most d non-neighbours in the* opposite *part of B_2.*

Then $\mathcal{X}$ admits an implicit representation.

3 Graph parameters

It is easy to see that classes of bounded vertex degree admit an implicit representation. More generally, bounded degeneracy in a class provides us with an implicit representation, where the *degeneracy* of a graph G is the minimum k such that every induced subgraph of G contains a vertex of degree at most k.

Spinrad showed in [14] that bounded clique-width also yields an implicit representation. The recently introduced parameter *twin-width* generalizes clique-width in the sense that bounded clique-width implies bounded twin-width, but not vice versa. It was shown in [6] that bounded twin-width also implies the existence of an implicit representation.

The notion of graph functionality, introduced in [1], generalizes both degeneracy and twin-width in the sense that bounded degeneracy or bounded twin-width implies bounded functionality, but not vice versa. As we mentioned earlier, graphs of bounded functionality have at most factorial speed of growth. However, whether they admit an implicit representation is wide-open. To approach this question, in Section 3.1 we analyse a parameter intermediate between twin-width and functionality. Then in Section 3.2, we introduce one more parameter.

3.1 Symmetric difference

Let G be a graph. Given two vertices x, y, we define the *symmetric difference* of x and y as the number of vertices in $G - \{x, y\}$ adjacent to exactly one of x and y, and we denote it by $\mathrm{sd}(x, y)$. We define the symmetric difference $\mathrm{sd}(G)$ of G as the smallest number such that any induced subgraph of G has a pair of vertices with symmetric difference at most $\mathrm{sd}(G)$.

In [1], it is shown that bounded clique-width implies bounded symmetric difference and a number of classes of bounded symmetric difference are identified. Below we show that symmetric difference is bounded for $F_{t,p}$-free bipartite graphs (see Figure 1 for an illustration of $F_{t,p}$). These classes have unbounded clique-width for all $t, p \geq 2$. To show that they have bounded symmetric difference, we assume without loss of generality that $t = p$.

Theorem 6. *For each $t \geq 2$, every $F_{t,t}$-free bipartite graph $G = (B, W, E)$ has symmetric difference at most $2t$.*

Proof. It is sufficient to show that G has a pair of vertices with symmetric difference at most $2t$. For two vertices x, y, we denote by $\mathrm{dd}(x, y)$ the degree difference $|\deg(x) - \deg(y)|$ and for a subset $U \subseteq V(G)$, we write $\mathrm{dd}(U) := \max\{\mathrm{dd}(x, y) : x, y \in U\}$. Assume without loss of generality that $\mathrm{dd}(W) \leq \mathrm{dd}(B)$ and let x, y be two vertices in B with $\mathrm{dd}(x, y) = \mathrm{dd}(B)$, $\deg(x) \geq \deg(y)$.

Write $X := N(x) - N(y)$. Clearly, $\mathrm{dd}(B) \leq |X|$. If $|X| \leq 2$, then $\mathrm{sd}(x, y) \leq 4 \leq 2t$ and we are done.

Now assume $|X| \geq 3$. Since $\mathrm{dd}(X) \leq \mathrm{dd}(W) \leq \mathrm{dd}(B) \leq |X|$, the set X contains two vertices p and q with $\mathrm{dd}(p, q) \leq 1$. Then $\mathrm{sd}(p, q) \leq 2t$, since otherwise both $P := N(p) - N(q)$ and $Q := N(q) - N(p)$ have size at least t, in which case x, y, p, q together with t vertices from P and t vertices from Q induce the forbidden graph $F_{t,t}$. □

Symmetric difference is also bounded in the class of $S_{1,2,3}$-free bipartite graphs, since these graphs have bounded clique-width [12]. For the remaining class from Theorem 1, i.e. the class of P_7-free bipartite graphs, boundedness of symmetric difference is an open question.

Conjecture 1. Symmetric difference is bounded in the class of P_7-free bipartite graphs.

We also conjecture that graphs of bounded symmetric difference admit an implicit representation and verify this conjecture for the classes of $F_{t,p}$-free bipartite graphs in Section 4.

Conjecture 2. Any class of graphs of bounded symmetric difference admits an implicit representation.

3.2 One more parameter

Let us say that a class $\mathcal{X}$ of bipartite graphs is *double-star-free* if there is a constant p such that no graph G in $\mathcal{X}$ contains an unbalanced copy of $2K_{1,p}$, i.e. an induced copy of $2K_{1,p}$ in which the centres of both stars belong to the same part of the bipartition of G. In particular, every class of *double-star-free* graphs is $F_{t,p}$-free for some t, p.

We will say that a class $\mathcal{X}$ of graphs is of bounded *double-star partition number* if there are constants k and p such that the vertices of every graph in $\mathcal{X}$ can be partitioned into at most k homogeneous subsets so that the edges between any pair of subsets form a bipartite graph that does not contain an unbalanced copy of $2K_{1,p}$.

Classes of bounded double-star partition number have been defined in the previous paragraph through two constants, k and p. By taking the maximum of the two, we can talk about a single constant, which can be viewed as a graph parameter defining the family of classes of bounded double-star partition number. This parameter has never been formally defined in the literature. Our motivation is based on the results in [4], where the author identifies ten minimal hereditary classes of graphs, which, in our terminology, have unbounded double-star partition number. One of them is the class $\mathcal{S}$ of star forests in which the centers of all stars belong to the same part of the bipartition. One more class is the class of bipartite complements of graphs in $\mathcal{S}$. Moreover, $\mathcal{S}$ and the class of bipartite complements of graphs in $\mathcal{S}$ are the only two minimal hereditary classes of *bipartite* graphs of unbounded double-star partition number.

Theorem 7. [4] *A hereditary class $\mathcal{X}$ of bipartite graphs is of bounded double-star partition number if and only if $\mathcal{X}$ excludes a graph from $\mathcal{S}$ and the bipartite complement of a graph from $\mathcal{S}$.*

Our interest to this parameter is due to the fact that any class of bounded double-star partition number admits an implicit representation, as we show in Section 4.

4 Implicit representations

In this section, we identify a number of new hereditary classes of graphs that admit an implicit representation.

4.1 $F_{t,p}$-free bipartite graphs

In this section we show that $F_{t,p}$-free bipartite graphs admit an implicit representation for any t and p. Together with Theorem 6 this verifies Conjecture 2 for these classes.

Without loss of generality we assume that $t = p$ and split the analysis into several intermediate steps. The first step deals with the case of double-star-free bipartite graphs.

Theorem 8. *Let $G = (A, B, E)$ be a $2K_{1,t}$-free bipartite graph. Then G has a vertex of degree at most $t-1$ or bi-codegree at most $(t-1)(t^2-4t+5)$.*

Proof. Let $x \in A$ be a vertex of maximum degree. Write Y for the set of neighbours of x, and Z for its set of non-neighbours in B (so $B = Y \cup Z$). We may assume $|Y| \geq t$ and $|Z| \geq (t-1)(t^2-4t+5)+1$, since otherwise we are done.

Note that any $w \in A$ is adjacent to fewer than t vertices in Z. Indeed, if $w \in A$ has t neighbours in Z, then it must be adjacent to all but at most $t-1$ vertices in Y (since otherwise a $2K_{1,t}$ appears), so its degree is greater than that of x, a contradiction.

We now show that Z has a vertex of degree at most $t-1$. Pick members $z_1 \dots z_{t-1} \in Z$ in a non-increasing order of their degrees, and write W_i for the neighbourhood of z_i. Since G is $2K_{1,t}$-free and $\deg(z_{i+1}) \leq \deg(z_i)$, for all

$1 \leq i \leq t-2$, $|W_{i+1} - W_i| \leq t-1$. It is not difficult to see that in fact, $|W_{i+1} - \bigcap_{s=1}^{i} W_s| \leq (t-1)i$, and in particular, $|W_{t-1} - \bigcap_{i=1}^{t-2} W_i| \leq (t-1)(t-2)$.

With this, we can compute an upper bound on the number of vertices in Z which have neighbours in W_{t-1}: by the degree condition given above, each vertex in $W_{t-1} \cap \bigcap_{i=1}^{t-2} W_i$ is adjacent to no vertices in Z other than $z_1, \ldots, z_{t-1}$. Each of the at most $(t-1)(t-2)$ vertices in $W_{t-1} - \bigcap_{i=1}^{t-2} W_i$ has at most $t-2$ neighbours in Z other than z_{t-1}. This accounts for a total of at most $(t-1)+(t-1)(t-2)^2 = (t-1)(t^2-4t+5)$ vertices which have neighbours in W_{t-1}, including z_{t-1} itself. By assumption on the size of Z, there must be a vertex $z \in Z$ which has no common neighbours with z_{t-1}. Since $2K_{1,t}$ is forbidden, one of z and z_{t-1} has degree at most $t-1$, as claimed. □

An immediate implication of this result, combined with Theorem 5, is that double-star-free bipartite graphs admit an implicit representation.

Corollary 1. *The class of $2K_{1,t}$-free bipartite graphs admits an implicit representation for any fixed t.*

Together with Theorem 2 this corollary implies one more interesting conclusion.

Corollary 2. *Classes of graphs of bounded double-star partition number admit an implicit representation.*

In the context of bipartite graphs, this corollary together with Theorem 7 implies the following generalization of Corollary 1.

Corollary 3. *Any class of bipartite graphs excluding a star forest and the bipartite complement of a star forest admits an implicit representation.*

Our next step towards implicit representations of $F_{t,t}$-free bipartite graphs deals with the case of $F^1_{t,t}$-free bipartite graphs, where $F^1_{t,t}$ is the graph obtained from $F_{t,t}$ by deleting the isolated vertex.

Theorem 9. *The class of $F^1_{t,t}$-free bipartite graphs admits an implicit representation.*

Proof. It suffices to prove the result for connected graphs (this follows for instance from Theorem 2). Let G be a connected $F^1_{t,t}$-free bipartite graph and let v be a vertex of maximum degree in G. We denote by V_i the set of vertices at distance i from v.

First, we show that the subgraph $G[V_1 \cup V_2]$ admits an implicit representation. To this end, we denote by u a vertex of maximum degree in V_1, by U the neighbourhood of u in V_2, $W := V_2 - U$, and $V'_1 := V_1 - \{u\}$.

Let x be a vertex in V'_1 and assume it has t neighbours in W. Then x has at least t non-neighbours in U (due to maximality of u), in which case the t neighbours of x in W, the t non-neighbours of x in U together with x, u and

v induce an $F^1_{t,t}$. This contradiction shows that every vertex of V_1' has at most $t-1$ neighbours in W, and hence the graph $G[V_1' \cup W]$ admits an implicit representation by Theorem 5.

To prove that $G[V_1' \cup U]$ admits an implicit representation, we observe that this graph is $2K_{1,t}$-free. Indeed, if the centers of the two stars belong to V_1', then they induce an $F^1_{t,t}$ together with vertex v, and if the centers of the two stars belong to U, then they induce an $F^1_{t,t}$ together with vertex u. Therefore, the graph $G[V_1 \cup V_2]$ can be covered by at most three graphs (one of them being the star centered at u), each of which admits an implicit representation, and hence by Theorem 2 this graph admits an implicit representation.

To complete the proof, we observe that every vertex of V_2 has at most $t-1$ neighbours in V_3. Indeed, if a vertex $x \in V_2$ has t neighbours in V_3, then x has at least t non-neighbours in V_1 (due to maximality of v), in which case the t neighbours of x in V_3, the t non-neighbours of x in V_1 together with x, v, and any neighbour of x in V_1 (which must exist by definition) induce an $F^1_{t,t}$.

Now we apply Theorem 3 with $A = \{v\} \cup V_1 \cup V_2$ to conclude that G admits an implicit representation, because every vertex of A has at most $t-1$ neighbours outside of A. □

The last step towards implicit representations of $F_{t,t}$-free bipartite graphs is similar to Theorem 9 with some modifications

Theorem 10. *The class of $F_{t,t}$-free bipartite graphs admits an implicit representation.*

Proof. By analogy with Theorem 9 we consider a *connected* $F_{t,t}$-free bipartite graph G, denote by v a vertex of maximum degree in G and by V_i the set of vertices at distance i from v. Also, we denote by u a vertex of maximum degree in V_1, by U the neighbourhood of u in V_2, $W := V_2 - U$, and $V_1' := V_1 - \{u\}$.

Let x be a vertex in V_1' and assume it has t neighbours and one non-neighbour y in W. Then x has at least t non-neighbours in U (due to maximality of u), in which case the t neighbours of x in W, the t non-neighbours of x in U together with x, y, u and v induce an $F_{t,t}$. This contradiction shows that every vertex of V_1' has either at most $t-1$ neighbours or at most 0 non-neighbours in W, and hence the graph $G[V_1' \cup W]$ admits an implicit representation by Theorem 5.

To prove that $G[V_1' \cup U]$ admits an implicit representation, we show that this graphs is $\widetilde{F}^1_{t,t}$-free. Indeed, if the centers of the two stars of $\widetilde{F}^1_{t,t}$ belong to V_1', then $\widetilde{F}^1_{t,t}$ together with vertex v induce an $F_{t,t}$, and if the centers of the two stars of $\widetilde{F}^1_{t,t}$ belong to U, then $\widetilde{F}^1_{t,t}$ together with vertex u induce an $F_{t,t}$. Therefore, the graph $G[V_1 \cup V_2]$ can be covered by at most three graphs, each of which admits an implicit representation, and hence by Theorem 2 this graph admits an implicit representation.

To complete the proof, we observe that every vertex of V_2 has either at most $t-1$ neighbours or 0 non-neighbours in V_3. Indeed, if a vertex $x \in V_2$ has t neighbours and one non-neighbour y in V_3, then x has at least t non-neighbours in V_1 (due to maximality of v), in which case the t neighbours of x in V_3, the t

non-neighbours of x in V_1 together with x, y, v, and any neighbour of x in V_1 induce an $F_{t,t}$.

Finally, we observe that if a vertex $x \in V_2$ has t neighbours in V_3, then V_5 (and hence V_i for any $i \geq 5$) is empty, because otherwise an induced $F_{t,t}$ arises similarly as in the previous paragraph, where vertex y can be taken from V_5. Now we apply Theorem 5 with $A = \{v\} \cup V_1 \cup V_2$ to conclude that G admits an implicit representation. Indeed, if each vertex of V_2 has at most $t-1$ neighbours in V_3, then each vertex of A has at most $t-1$ neighbours outside of A, and if a vertex of V_2 has at least t neighbours in V_3, then $V_i = \emptyset$ for $i \geq 5$ and hence every vertex of A has at most $t-1$ neighbours or at most 0 non-neighbours in the *opposite* part outside of A. □

4.2 One-sided forbidden induced bipartite subgraphs

In the context of bipartite graphs, some hereditary classes are defined by forbidding one-sided copies of bipartite graphs. For instance, the class of star forests in which the centers of all stars have the same colour, say black, is defined by forbidding a P_3 with a white center. Very little is known about implicit representations for classes defined by one-sided forbidden induced bipartite subgraphs. It is known, for instance, that bipartite graphs without a one-sided P_5 admit an implicit representation. This is not difficult to show and also follows from the fact P_6-free bipartite graphs have bounded clique-width and hence admit an implicit representation (note that P_6 is symmetric with respect to swapping the bipartition). Below we strengthen the result for one-sided forbidden P_5 to one-sided forbidden $F_{t,1}$, where again $F^1_{t,1}$ is the graph obtained from $F_{t,1}$ by deleting the isolated vertex.

Lemma 1. *The class of bipartite graphs containing no one-sided copy of $F^1_{t,1}$ admits an implicit representation.*

Proof. Let $G = (U, V, E)$ be a bipartite graph containing no copy of $F^1_{t,1}$ with the vertex of largest degree in U. Let u be a vertex of maximum degree in U. We split the vertices of V into the set V_1 of neighbours and the set V_0 of non-neighbours of u. Consider a vertex $x \in U$ such that x has a neighbour in V_1 and a neighbour in V_0 and denote by V_{10} the set of non-neighbours of x in V_1 and by V_{01} the set of neighbours of x in V_0. We note that $|V_{01}| \leq |V_{10}|$, since $deg(x) \leq deg(u)$. Besides, $|V_{10}| < k$, since otherwise k vertices in V_{10}, a vertex in V_{01} and a common neighbour of u and x (these vertices exist by assumption) together with u and x induce a forbidden copy of $F^1_{t,1}$. Therefore, x has at most $k-1$ non-neighbours in V_1 and at most $k-1$ neighbours in V_0.

Now we define three subsets A, B_1, B_2 as follows:

A is the set of vertices of U that have non-neighbours both of x in V_1 and in V_0,
B_1 consists of V_1 and the vertices of U that have neighbours only in V_1,
B_2 consists of V_0 and the vertices of U that have neighbours only in V_0.

With this notation, the result follows from Theorem 5. □

Theorem 11. *The class of bipartite graphs containing no one-sided copy of $F_{t,1}$ admits an implicit representation.*

Proof. Let $G = (U, V, E)$ be a connected bipartite graph containing no one-sided copy of $F_{t,1}$ with the vertex of largest degree in U. Let v be a vertex in V and let V_i the set of vertices at distance i from v. Then the graph $G_1 := G[V_1 \cup V_2]$ does not contain a one-sided copy of $\widetilde{F}^1_{t,1}$ with the vertex of largest degree in V_1, since otherwise together with v this copy would induce a one-sided copy of $F_{t,1}$ with the vertex of largest degree in U. Therefore, by Lemma 1 the graph G_1 admits an implicit representation.

For any $i > 1$, the $G_i := G[V_i \cup V_{i+1}]$ does not contain a one-sided copy of $F^1_{t,1}$ with the vertex of largest degree in V_i (for odd i) or with the vertex of largest degree in V_{i+1} (for even i), since otherwise together with v this copy would induce a one-sided copy of $F_{t,1}$ with the vertex of largest degree in U. Therefore, by Lemma 1 the graph G_i admits an implicit representation for all $i > 1$. Together with Theorem 2 this implies an implicit representation for G. □

For larger indices of one-sided forbidden $F_{t,p}$ the question remains open. Moreover, it remains open even for one-sided forbidden $2P_3$. It is interesting to note that if we forbid $2P_3$ with black centers and all black vertices have incomparable neighbourhoods, then the graph has bounded clique-width [7] and hence admits an implicit representation. However, in general the clique-width of $2P_3$-free bipartite graphs is unbounded and the question of implicit representation for one-sided forbidden $2P_3$ remains open.

5 Factorial properties

We repeat that bounded functionality implies at most factorial speed of growth. Whether the reverse implication is also valid was left as an open question in [1]. It turns out that the answer to this question is negative. This is witnessed by the class $\mathcal{Q}$ of induced subgraphs of hypercubes. Indeed, in [1] it was shown that $\mathcal{Q}$ has unbounded functionality. On the other hand, it was proved in [8] that there exists an implicit representation for $\mathcal{Q}$ and, in particular, the class is factorial; in fact, a result from recent work [9] implies that, more generally, the hereditary closure of Cartesian products of any finite set of graphs admits an implicit representation. These results however are non-constructive and they provide neither explicit labeling schemes, nor specific factorial bounds on the number of graphs. Below we give a concrete bound on the speed of $\mathcal{Q}$.

Theorem 12. *There are at most n^{2n} n-vertex graphs in $\mathcal{Q}$.*

Proof. Let Q_n denote the n-dimensional hypercube, i.e. the graph with vertex set $\{0, 1\}^n$, in which two vertices are adjacent if and only if they differ in exactly one coordinate. To obtain the desired bound, we will produce, for each labelled

n-vertex graph in $\mathcal{Q}$, a sequence of $2n$ numbers between 1 and n which allows us to retrieve the graph uniquely.

As a preliminary, let $G \in \mathcal{Q}$ be a connected graph on n vertices. By definition of $\mathcal{Q}$, G embeds into Q_m for some m. We claim that, in fact, G embeds into Q_{n-1}. If $m < n$, this is clear. Otherwise, using an embedding into Q_m, each vertex of G corresponds to an m-digit binary sequence. For two adjacent vertices, the sequences differ in exactly one position. From this, it follows inductively that the n vertices of G all agree in at least $m-(n-1)$ positions. The coordinates on which they agree can simply be removed; this produces an embedding of G into Q_{n-1}. Additionally, by symmetry, if G has a distinguished vertex r, we remark that we may find an embedding sending r to $(0, 0, \ldots, 0)$.

We are now ready to describe our encoding. Let $G \in \mathcal{Q}$ be any labelled graph with vertex set $\{x_1, \ldots, x_n\}$. We start by choosing, for each connected component C of G:

– a spanning tree T_C of C;
– a root r_C of T_C;
– an embedding φ_C of T_C into Q_{n-1} sending r_C to $(0, 0, \ldots, 0)$.

Write C^i for the component of x_i. We define two functions $p, d : V(G) \to \{1, \ldots, n\}$ as follows:

$$p(x_i) = \begin{cases} i, & \text{if } x_i = r_{C^i}; \\ j, & \text{if } x_i \neq r_{C^i}, \text{ and } x_j \text{ is the parent of } x_i \text{ in } T_{C^i}. \end{cases}$$

$$d(x_i) = \begin{cases} 1, & \text{if } x_i = r_{C^i}; \\ j, & \text{if } x_i \neq r_{C^i}, \text{ and } \varphi(x_i) \text{ and } \varphi(p(x_i)) \text{ differ in coordinate } j. \end{cases}$$

One easily checks that the above maps are well-defined; in particular, when x_i is not a root, the embeddings of x_i and of its parent do, indeed, differ in exactly one coordinate. The reader should also know that the value of d on the roots is, in practice, irrelevant – setting it to 1 is an arbitrary choice.

We now claim that G can be restored from the sequence $p(1), d(1), \ldots, p(n), d(n)$. To do so, we first note that this sequence allows us to easily determine the partition of G into connected components. Moreover, for each connected component, we may then determine its embedding φ_C into Q_{n-1}: $\varphi_C(r_C)$ is by assumption $(0, 0, \ldots, 0)$; we may then identify its children using p, then compute their embeddings using d; we may then proceed inductively. This information allows us to determine the adjacency in G as claimed, and the encoding uses $2n$ integers between 1 and n as required. □

Problem 1. Find specific implicit representation for the class $\mathcal{Q}$ of induced subgraphs of hypercubes.

References

1. B. Alecu, A. Atminas, V. Lozin, Graph functionality, *J. Combinatorial Theory B*, 147 (2021) 139–158.

2. P. Allen, Forbidden induced bipartite graphs. *J. Graph Theory* 60 (2009) 219–241.
3. A. Atminas, A. Collins, V. Lozin, and V. Zamaraev, Implicit representations and factorial properties of graphs, *Discrete Mathematics*, 338 (2015) 164–179.
4. A. Atminas, Classes of graphs without star forests and related graphs, *arXiv preprint arXiv:1711.01483*
5. J. Balogh, B. Bollobás, D. Weinreich, The speed of hereditary properties of graphs, *J. Combin. Theory Ser. B* 79 (2000) 131–156.
6. É. Bonnet, C. Geniet, E. J. Kim, S. Thomassé, R. Watrigant, Twin-width II: small classes. *SODA 2021*: 1977-1996
7. E. Boros, V. Gurvich, M. Milanic, Characterizing and decomposing classes of threshold, split, and bipartite graphs via 1-Sperner hypergraphs. *J. Graph Theory* 94 (2020) 364–397.
8. N. Harms, Universal Communication, Universal Graphs, and Graph Labeling, *ITCS 2020*
9. N. Harms, S. Wild, V. Zamaraev, Randomized Communication and the Implicit Graph Conjecture *arXiv preprint arXiv:2111.03639*
10. H. Hatami, P. Hatami, The implicit graph conjecture is false, *arXiv preprint arXiv:2111.13198*
11. S. Kannan, M. Naor, S. Rudich, Implicit representation of graphs, *SIAM J. Discrete Mathematics*, 5 (1992) 596–603.
12. V. Lozin, Bipartite graphs without a skew star. *Discrete Math.* 257 (2002) 83–100.
13. V. Lozin and V. Zamaraev, The structure and the number of P_7-free bipartite graphs, *European J. Combinatorics*, 65 (2017) 143–153.
14. J.P. Spinrad, Efficient graph representations. Fields Institute Monographs, 19. American Mathematical Society, Providence, RI, 2003. xiii+342 pp.