

Limit Theorems for the Maximum Path Weight in a Directed Graph on the Line with Random Weights of Edges

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Received November 19, 2020; revised February 1, 2021; accepted February 8, 2021

Abstract—We consider an infinite directed graph with vertices numbered by integers $\dots, -2, -1, 0, 1, 2, \dots$, where any pair of vertices $j < k$ is connected by an edge (j, k) that is directed from j to k and has a random weight $v_{j,k} \in [-\infty, \infty)$. Here $\{v_{j,k}, j < k\}$ is a family of independent and identically distributed random variables that take either finite values (of any sign) or the value $-\infty$. A path in the graph is a sequence of connected edges $(j_0, j_1), (j_1, j_2), \dots, (j_{m-1}, j_m)$ (where $j_0 < j_1 < \dots < j_m$), and its weight is the sum $\sum_{s=1}^m v_{j_{s-1}, j_s} \geq -\infty$ of the weights of the edges. Let $w_{0,n}$ be the maximal weight of all paths from 0 to n .

Assuming that $\mathbf{P}(v_{0,1} > 0) > 0$, that the conditional distribution $\mathbf{P}(v_{0,1} \in \cdot \mid v_{0,1} > 0)$ is nondegenerate, and that $\mathbf{E} \exp(Cv_{0,1}) < \infty$ for some $C = \text{const} > 0$, we study the asymptotic behaviour of the random sequence $w_{0,n}$ as $n \rightarrow \infty$. In the domain of the normal and moderately large deviations we obtain a local limit theorem when the distribution of random variables $v_{i,j}$ is arithmetic and an integro-local limit theorem if this distribution is non-lattice.

Key words: directed graph, maximal path weight, skeleton and renewal points, normal and moderate large deviations, integro-local limit theorem.

DOI: 10.1134/S00329460210200??

1. INTRODUCTION, MAIN NOTATION AND THE MAIN RESULT

We consider an infinite directed graph $G(\mathbb{Z}, E)$, with vertices indexed by all integers $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$, whose edges $E = \{e = (j, k), j < k, j, k \in \mathbb{Z}\}$ are all edges directed from smaller to bigger vertices. We assume that there no directed edges from bigger to smaller vertices and that there is no loops of the form (j, j) .

Every edge $e \in E$ gets a weight v_e , that may be either a number (positive or negative) or $-\infty$. We assume that the random variables $\{v_{j,k}, j < k\}$ are mutually independent and distributed as a random variable v taking values in $[-\infty, \infty)$. Let $p = \mathbf{P}(v > -\infty)$ and $p^+ = \mathbf{P}(v > 0)$. Let v^+ be a random variable with distribution

$$\mathbf{P}(v^+ < t) = \mathbf{P}(v < t \mid v > 0), \quad t > 0. \quad (1.1)$$

Throughout the paper, we assume that the following conditions hold:

$$p^+ > 0, \quad \mathbf{P}(v^+ = c) < 1 \quad \text{for any } c > 0, \quad \mathbf{E} e^{Cv^+} < \infty \quad \text{for some } C > 0, \quad (1.2)$$

i.e. the random variable v takes positive values with positive probability, its distribution on the positive halfline is non-degenerate, and the right tail of its distribution is relatively light.

In the literature, one can also find another interpretation of the model: if $v_e = -\infty$, then one can say that there is no edge e , and if $v_e > -\infty$, then the edge exists and its weight equals v_e . In the latter setting, we get two independent “randomnesses”: an edge may either exist or not, and if it exists, then its weight is an independent of everything random variable with distribution $\mathbf{P}(v \in \cdot | v > -\infty)$.

A path π of length $L(\pi) = m$ is a sequence of m connected edges $e_1 = (j_0, j_1), e_2 = (j_1, j_2), \dots, e_m = (j_{m-1}, j_m)$ where the end vertex of each edge coincides with the initial vertex of the next edge and $j_0 < j_1 < \dots < j_m$, and we say that this is a path from j_0 to j_m and write $e_i \in \pi, i = 1, \dots, L(\pi)$. The weight $w(\pi)$ of the path is defined as the sum of the weights of its edges, i.e.

$$w(\pi) = \sum_{s=1}^{L(\pi)} v_{j_{s-1}, j_s} = \sum_{e \in \pi} v_e.$$

Clearly, the path weight is finite if and only if all weights of its edges are finite.

For $j < k$, let $\Pi_{j,k}$ be the family of all paths from j to k having finite weights (i.e. $w(\pi) > -\infty$ for all $\pi \in \Pi_{j,k}$) and let $w_{j,k}$ be the maximal weight of all paths from j to k . Then

$$w_{j,k} = \max_{\pi \in \Pi_{j,k}} w(\pi)$$

with probability 1, since we follow the standard convention that the maximum over empty set is $-\infty$. We also let $w_{j,j} = 0$, for all j .

Such graphs with random weights naturally appear in various applications. For example, if the edge weight takes only two values, 1 (the edge exists) or $-\infty$ (no edge), i.e.

$$p = \mathbf{P}(v = 1) = 1 - \mathbf{P}(v = -\infty), \quad (1.3)$$

then such a graph may describe ordering of jobs in a computer network (see, e.g. [1, 2]), where vertices represent jobs and edges their time constraints (if $v_{j,k} = 1$, then service of job k cannot start before service of job j ends); or functioning of biological models (see, e.g., [3, 4]) where vertices represent types of animals and paths describe “food chains”: if $v_{j,k} = 1$, then type k may be considered as a food for type j).

Introduce two mutually exclusive conditions:

[R] *The distribution of the random variable v is non-lattice*, this means that, for any a and $h > 0$, the probability that v takes values in the lattice of span h shifted by a is strictly smaller than 1, $\sum_{s=-\infty}^{\infty} \mathbf{P}(v = a + sh) < 1$.

[Z] *The distribution of random variable v is arithmetic*, i.e. $\sum_{s=-\infty}^{\infty} \mathbf{P}(v = sh) = 1$, for some $h > 0$.

Without loss of generality, we may assume further that the lattice span is $h = 1$, this means that v is an integer-valued random variable and the greatest common divisor of the set $\{k \geq 1 : \mathbf{P}(v = k) > 0\}$ equals one.¹

¹ Conditions [R] and [Z] may be also formulated in terms of the characteristic function $f(z) = \mathbf{E}e^{izv}$ of random variable v . Namely,

[R] $|f(2\pi z)| < 1$ for all $z \neq 0$;

[Z] $f(2\pi z) = 1$ for all $z \in \mathbb{Z}$ and $|f(2\pi z)| < 1$ for all $z \in \mathbb{R} \setminus \mathbb{Z}$.

Notice that we have excluded the case $v^+ = \text{const}$ (see (1.2)) and the case of lattice, but non-arithmetic distributions.

We are interested in the asymptotic behaviour of the random sequence $w_{0,n}$ as $n \rightarrow \infty$. We will consider the domain of the normal and moderately large deviations and prove a local limit theorem under condition **[Z]** and an integro-local limit theorem under condition **[R]**.

The proof of these statements is split into two steps. In the first step (Section 2) we first introduce an embedded regenerative sequence with corresponding “weights” (using the methods developed in the papers [5, 6]) and then show that both the lengths of regenerative cycles τ_k and the corresponding cycle weights ζ_k have finite exponential moments (precise definitions of these variables are given in Section 3). Note that the sequence $\{(\tau_k, \zeta_k)\}_{k=1}^\infty$ contains independent, for $k \geq 1$, and identically distributed, for $k \geq 2$, two-dimensional random vectors that have, for $k \geq 2$, a common distribution with a random vector (τ, ζ) , whose coordinates do typically depend on each other.

In the second step of the proof (Section 3), we note that vectors (τ_k, ζ_k) form a stationary compound renewal process (CRP), and one can apply to its study methods and results from the paper [7]. Then we prove that, in the limit theorems, the asymptotics of the sequence $w_{0,n}$ coincides with that of the introduced CRP, and this completes the proof of our results.

In order to formulate our results, it is left to introduce the rate function for the CRP driven by random vector $(\tau, \zeta) \stackrel{d}{=} (\tau_2, \zeta_2)$. For $(\lambda, \mu) \in \mathbb{R}^2$, let

$$A(\lambda, \mu) := \ln \mathbf{E} e^{\lambda\tau + \mu\zeta}. \tag{1.4}$$

Introduce the convex set

$$\mathcal{A}^{\leq 0} := \{(\lambda, \mu) : A(\lambda, \mu) \leq 0\}$$

and let

$$D(\alpha) := \sup_{(\lambda, \mu) \in \mathcal{A}^{\leq 0}} \{\lambda + \mu\alpha\}.$$

The rate function $D(\alpha)$ plays a certain role in the description of the logarithmic asymptotics of the large deviations probabilities for the CRP determined by vector (τ, ζ) , and it has been studied quite thoroughly (see, e.g., [8]). Notice that this is a convex non-negative function that takes value 0 at only one point $\alpha = a$ where

$$a = \frac{\mathbf{E} \zeta}{\mathbf{E} \tau} > 0. \tag{1.5}$$

Under our assumptions, function $D(\alpha)$ is analytic in a neighbourhood of point $\alpha = a$ and, further,

$$D(a) = 0, \quad D'(a) = 0, \quad D''(a) = \frac{1}{\sigma^2},$$

where

$$\sigma^2 := \frac{\mathbf{E}(\zeta - a\tau)^2}{\mathbf{E} \tau}. \tag{1.6}$$

We present now the main result of the paper.

Theorem 1. *Assume that conditions (1.2) hold.*

- I. *If the random variable v satisfies conditions **[Z]**, then, for any sequence $x = x_n \in \mathbb{Z}$ such that $\alpha := x/n \rightarrow a$ as $n \rightarrow \infty$, the following asymptotic relation holds:*

$$\mathbf{P}(w_{0,n} = x) \sim \frac{1}{\sigma\sqrt{2\pi n}} e^{-nD(\alpha)}. \tag{1.7}$$

If, in addition, $y_n := x - an = o(n^{2/3})$, then we get

$$\mathbf{P}(w_{0,n} = x) \sim \frac{1}{\sigma\sqrt{2\pi n}} e^{-\frac{y_n^2}{2n\sigma^2}}.$$

II. If the random variable v satisfies conditions $[\mathbf{R}]$, then, for a certain sequence of positive numbers $\Delta_n^{(0)} = o(1)$ and for any sequence $x = x_n \in \mathbb{R}$ such that $\alpha := x/n \rightarrow a$ $n \rightarrow \infty$, the following asymptotic relation holds:

$$\mathbf{P}(w_{0,n} \in [x, x + \Delta_n]) \sim \frac{\Delta_n}{\sigma\sqrt{2\pi n}} e^{-nD(\alpha)}, \tag{1.8}$$

where sequence $\Delta_n = o(1)$ satisfies relations $\Delta_n \geq \Delta_n^{(0)}$ (i.e. converges to 0 sufficiently slowly). If, in addition, $y_n := x - an = o(n^{2/3})$, then we get

$$\mathbf{P}(w_{0,n} \in [x, x + \Delta_n]) \sim \frac{\Delta_n}{\sigma\sqrt{2\pi n}} e^{-\frac{y_n^2}{2n\sigma^2}}.$$

Remark 1. One can strengthen the results of Theorem 1 by considering, along with the normal and moderately large deviations of the type $\alpha := x/n \rightarrow a$, the large deviations of the type $|\alpha - a| \leq \delta$ for some (generally speaking, small) $\delta > 0$. In this case the constant $\frac{1}{\sigma\sqrt{2\pi}}$ that appears in the right-hand sides of relations (1.7), (1.8) should be replaced by a more complicated function that depends on parameter $\alpha = x/n$. However, for determining this function, one needs to produce additional complicated constructions. This is why we have decided to restrict our consideration in Theorem 1 to the normal and moderately large deviations.

Remark 2. The statements of Theorem 1 are presented in terms of rate function $D(\alpha)$ that is determined by the distribution of random vector (τ, ζ) that is introduced implicitly and depends on parameters c_1 and c_2 that are chosen arbitrarily from a certain interval (see Lemma 1). In Theorem 3 below, we will show that the results of Theorem 1 do not depend on a particular choice of these parameters.

The asymptotic properties of sequence $w_{0,n}$, as $n \rightarrow \infty$, have been studied earlier in the papers [5,6,9,10]. In the paper [9], the authors consider the case $\mathbf{P}(v > 0) = 1$ and prove the strong law of large numbers and the central limit theorem assuming that the third moment of random variable v is finite, and limit theorems of another type if the latter condition is violated. The central limit theorem in the case of signed random variable v is proved in [10]. In the earlier papers [5,6], the case of weights (1.3) was studied. We should mention also the paper [11] where the asymptotics for the *minimal* path length from 0 to n were considered, as $n \rightarrow \infty$, in the case where the weights are constant, but the probabilities of existence of edges depend on distances between the vertices.

The rest of the paper includes three Sections. In Sections 2 and 3 we provide the proofs of our results following the scheme presented above, and Section 4 contains an auxiliary result.

2. CONSTRUCTION OF REGENERATIVE SEQUENCE AND ITS PROPERTIES

In this Section, we introduce a construction that allows us to determine an a.s. infinite random set of vertices $\{\Gamma_i\}_{i \in \mathbb{Z}}$ (in what follows, we call them *renewal vertices* – see Definition 3 below), where $\dots < \Gamma_{-2} < \Gamma_{-1} < 0 \leq \Gamma_0 < \Gamma_1 < \dots$, such that:

1. Sequence of two-dimensional vectors

$$(\Gamma_n - \Gamma_{n-1}, w_{\Gamma_{n-1}, \Gamma_n}), \quad n \neq 0, \tag{2.1}$$

consists of independent and identically distributed vectors that do not depend on $(\Gamma_{-1}, \Gamma_0, w_{\Gamma_{-1}, 0}, w_{0, \Gamma_0})$. Using the terminology of the theory of point processes, one can say that sequence $\{(\Gamma_n, w_{\Gamma_{n-1}, \Gamma_n})\}$ forms a stationary marked point process with marks $\{w_{\Gamma_{n-1}, \Gamma_n}\}$, that determines the CRP.

2. For some $C > 0$, all four exponential moments

$$\mathbf{E} \exp(C\Gamma_0), \quad \mathbf{E} \exp(C(\Gamma_1 - \Gamma_0)), \quad \mathbf{E} \exp(Cw_{0,\Gamma_0}), \quad \mathbf{E} \exp(Cw_{\Gamma_0,\Gamma_1}) \tag{2.2}$$

are finite. Then, for $C_1 = C/2$, the following moments are finite with necessity, too:

$$\mathbf{E} \exp(C_1(\Gamma_0 + w_{0,\Gamma_0})) \quad \text{and} \quad \mathbf{E} \exp(C_1(\Gamma_1 - \Gamma_0 + w_{\Gamma_0,\Gamma_1})). \tag{2.3}$$

3. For any $0 \leq m \leq n$, if $\Gamma_m \leq n$, then

$$w_{0,n} = w_{0,\Gamma_0} + w_{\Gamma_0,\Gamma_1} + \dots + w_{\Gamma_{m-1},\Gamma_m} + w_{\Gamma_m,n} \tag{2.4}$$

(recall that we assume $w_{j,j} = 0$, for $j \in \mathbb{Z}$).

We need a number of auxiliary statements. We will partially follow the scheme of the proof of one of the main results in [9] where the weights were assumed to take either positive values or value $-\infty$ and where conditions for existence of the first and second moments of random variables Γ_0 and w_{0,Γ_0} were studied.

2.1. Construction of skeleton and renewal points

We will introduce consequently four random subsets of the set \mathbb{Z} of vertices: the set \mathcal{S} of skeleton, the set \mathcal{S}^+ of skeleton-plus, the set \mathcal{R} of renewal, and the set \mathcal{R}^+ of renewal-plus vertices.

Definition 1. Vertex $x \in \mathbb{Z}$ is called a *skeleton* vertex if it is connected to any other vertex by a path of finite weight, i.e. for any $j < x$ and $k > x$ inequalities $w_{j,x} > -\infty$ and $w_{x,k} > -\infty$ hold. Denote by \mathcal{S} the random set of skeleton vertices.

If $p = \mathbf{P}(v > -\infty) = 1$, then every vertex $x \in \mathbb{Z}$ is a skeleton vertex. If $p \in (0, 1)$, then Lemmas 5–7 from [6] imply the following five statements.

1. The probability for vertex x to be a skeleton vertex is strictly positive and is the same for all $x \in \mathbb{Z}$.
2. The sequence of events $\{x \in \mathcal{S}\}$, $x = \dots, -2, -1, 0, 1, 2, \dots$ is stationary ergodic and, therefore, with probability one there are infinitely many skeleton points $\{t_i\}$.
3. The sequence $\{t_i\}$ (where $\dots t_{-2} < t_{-1} < 0 \leq t_0 < t_1 < \dots$) forms a stationary renewal process (in discrete time) and, in particular, the lengths of the intervals $\{t_i - t_{i-1}, i \in \mathbb{Z}, i \neq 0\}$ are independent and identically distributed random variables that do not depend on the pair of random variables (t_{-1}, t_0) , and the latter random variables depend on each other and t_0 has the same distribution as $|t_{-1}| - 1$. Further, $\mathbf{P}(t_{-1} = -i) = \mathbf{P}(t_1 - t_0 \geq i) / \mathbf{E}(t_1 - t_0)$, $i = 1, 2, \dots$

4. For some $C > 0$,

$$\mathbf{E} e^{Ct_0} < \infty \quad \text{and hence} \quad \mathbf{E} e^{C(t_1-t_0)} < \infty. \tag{2.5}$$

5. For any $j < k$ denote by

$$L_{j,k} = \max_{\pi \in \Pi_{j,k}} L(\pi) \quad (\text{where the maximum over an empty set is } -\infty)$$

the maximal path length among the paths from $\Pi_{j,k}$. Then, for each $n > 0$, if $L_{0,n} > 0$, then any path of length $L_{0,n}$ from 0 to n has to include all intermediate skeleton points (if there are any). Namely, assume that $0 \leq t_0 < t_1 < \dots < t_m \leq n < t_{m+1}$, for some $m \geq 0$. Then any path of maximal length from 0 to n , that belongs to the set $\Pi_{0,n}$, has to include every vertex t_0, \dots, t_m . Further, with necessity, all values $L_{t_0,t_1}, \dots, L_{t_{m-1},t_m}$ are strictly positive and

$$L_{0,n} = L_{0,t_0} + L_{t_0,t_1} + \dots + L_{t_{m-1},t_m} + L_{t_m,n}.$$

Along with the set of paths $\Pi_{j,k}$ (introduced in Section 1) that include only edges of finite weights, we introduce also the set of paths $\Pi_{j,k}^+$ from j to k that include only edges with positive weights (i.e. $v_e > 0$ for every $e \in \pi$, given $\pi \in \Pi_{j,k}^+$) and let

$$w_{j,k}^+ := \max_{\pi \in \Pi_{j,k}^+} w(\pi).$$

Definition 2. Vertex $x \in \mathbb{Z}$ is a *skeleton-plus* vertex if it is connected to any other vertex by a path with edges of positive weights only, i.e., for and $j < x$ and $k > x$, there is a path π from j to x and a path $\tilde{\pi}$ from x to k such that $v_e > 0$ for all $e \in \pi$ and all $e \in \tilde{\pi}$ and, in particular, inequalities $w_{j,x}^+ > 0$ and $w_{x,k}^+ > 0$ hold. Denote by $\mathcal{S}^+ = \{t_i^+\}$ the set of skeleton-plus vertices.

Notice that since we assume that $p^+ > 0$, the results from the paper [6] are also applicable to the set \mathcal{S}^+ and, in particular, exponential moments in (2.5) stay finite if one replaces $\{t_i\}$ by $\{t_i^+\}$.

Now we introduce the sets \mathcal{R} of *renewal* and \mathcal{R}^+ of *renewal-plus* vertices. Let $c_1 \geq c_2 > 0$ be two fixed numbers. For $x \in \mathbb{Z}$, introduce the following events:

$$\begin{aligned} A_x^r(c_1) &= \bigcap_{i=1}^{\infty} \{w_{x,x+i} \geq c_1 i\}, \\ A_x^0(c_2) &= \bigcap_{j,i=1}^{\infty} \{v_{x-j,x+i} < c_2(j+i)\}, \\ A_x^l(c_1) &= \bigcap_{j=1}^{\infty} \{w_{x-j,x} \geq c_1 j\}. \end{aligned}$$

Here event $A_x^r(c_1)$ means that vertex x is connected to any vertex on its right by a path of a finite weight and, moreover, all corresponding maximal path weights are strictly positive and increase at least linearly (with speed not smaller than c_1) with growth of distance from x . Similarly, event $A_x^l(c_1)$ means that vertex x is connected to any vertex on its left by a path of a finite weight and, moreover, all corresponding maximal path weights are strictly positive and grow at least linearly with growth of distance from x . In particular, if both events $A_x^r(c_1)$ and $A_x^l(c_1)$ occur, then vertex x is necessarily a skeleton vertex and, in addition, $w_{x-j,x+i} \geq c_1(j+i)$, for all $j, i \geq 0$. If, in addition, event $A_x^0(c_2)$ takes place, then (by $c_2 \leq c_1$), for any $j, i \geq 0$, any path of maximal weight from vertex $x-j$ to vertex $x+i$ must necessarily include vertex x .

Definition 3. Vertex x is a *renewal* vertex if all three events $A_x^l(c_2)$, $A_x^0(c_1)$ and $A_x^r(c_2)$ occur. Let $\mathcal{R} = \{\Gamma_i\}$ be the random set of renewal vertices.

Since the events $\{x \in \mathcal{R}\}$ form a stationary ergodic sequence, the “0-1” law holds: with probability one, the set \mathcal{R} is either infinite or empty. Assume it is infinite, then its elements may be ordered as

$$\dots < \Gamma_{-1} < 0 \leq \Gamma_0 < \Gamma_1 < \dots,$$

and representation (2.4) holds. It is this representation that helps us to study the asymptotics of sequence $w_{0,n}$, as n grows to infinity.

In analogy to the notation above, we introduce events

$$\begin{aligned} A_x^{r+}(c_1) &= \bigcap_{i=1}^{\infty} \{w_{x,x+i}^+ \geq c_1 i\}, \\ A_x^{0+}(c_2) \equiv A_x^0(c_2) &= \bigcap_{j,i=1}^{\infty} \{v_{x-j,x+i} < c_2(j+i)\}, \end{aligned}$$

$$A_x^{l+}(c_1) = \bigcap_{j=1}^{\infty} \{w_{x-j,x}^+ \geq c_1 j\}.$$

Here the event $A_x^{r+}(c_1)$ means that vertex x is connected to any vertex on its right by a path with edges of positive weights only and, moreover, all related maximal path weights grow at least linearly with the distance from vertex x . Similarly, the event $A_x^{l+}(c_1)$ means that vertex x is connected to any vertex on its left by a path with edges of positive weights only and, moreover, all related maximal path weights grow at least linearly with the distance from vertex x . Necessarily, $A_x^{l+}(c_1) \subseteq A_x^l(c_1)$ and $A_x^{r+}(c_1) \subseteq A_x^r(c_1)$, for any $c_1 > 0$.

Definition 4. Call x a *renewal-plus* vertex if $A_x^{l+}(c_1) \cap A_x^{0+}(c_2) \cap A_x^{r+}(c_1)$ occurs, and denote by \mathcal{R}^+ the set of all renewal-plus vertices. Note that this set is also either infinite a.s. or empty a.s.

The following relations hold:

$$\mathcal{R}^+ \subseteq \mathcal{R} \subseteq \mathcal{S} \quad \text{and} \quad \mathcal{R}^+ \subseteq \mathcal{S}^+ \subseteq \mathcal{S}. \tag{2.6}$$

In addition, the sets \mathcal{R} and \mathcal{R}^+ increase as c_2 increases and c_1 decreases.

2.2. Regenerative structure and existence of exponential moments

For formulation of the next statement, we need the distribution of random variable v^+ that was defined in (1.1). Let $\text{ess inf } v^+ = \inf\{t > 0 : \mathbf{P}(v^+ < t) > 0\}$ and $V = \mathbf{E} \min_{t_0^+ \leq i < j \leq t_1^+} v_{i,j}^+$. Clearly, $\text{ess inf } v^+ < V$ if the distribution of random variable v^+ does not degenerate. Let $\gamma^+ = \frac{1}{\mathbf{E}(t_1^+ - t_0^+)}$.

From Lemmas 5–7 in [6], we get the following result.

Lemma 1. Assume that condition (1.2) holds. If

$$\gamma^+ \text{ess inf } v^+ < c_2 \leq c_1 < \gamma^+ V, \tag{2.7}$$

then, for any $x \in \mathbb{Z}$, $\mathbf{P}(A_x^{l+}(c_1) \cap A_x^{0+}(c_2) \cap A_x^{r+}(c_1)) > 0$ and the set \mathcal{R}^+ is infinite with probability 1. Therefore, the set \mathcal{R} is infinite with probability 1, too.

Remark 3. In paper [6], renewal points have been introduced in the case $c_1 = c_2$ and corresponding results have been proved in this case only. However, one can easily check that the proofs of these results remain unchanged (apart from minor changes in notation) under the more general conditions (2.7).

Introduce the following cycles

$$\mathcal{C}_k^+ := \left(\Gamma_k^+ - \Gamma_{k-1}^+; \left\{ v_{\Gamma_{k-1}^+ + j, \Gamma_{k-1}^+ + i}^+, 0 \leq j < i \leq \Gamma_k^+ - \Gamma_{k-1}^+ \right\} \right), \quad k \in \mathbb{Z},$$

and

$$\mathcal{C}_k := \left(\Gamma_k - \Gamma_{k-1}; \left\{ v_{\Gamma_{k-1} + j, \Gamma_{k-1} + i}, 0 \leq j < i \leq \Gamma_k - \Gamma_{k-1} \right\} \right), \quad k \in \mathbb{Z}.$$

The following results take place.

Lemma 2. Under conditions (1.2) and (2.7), the following two statements hold:

- (I) Random elements $\{\mathcal{C}_k^+, k \in \mathbb{Z}\}$ are mutually independent and random elements $\{\mathcal{C}_k^+, k \in \mathbb{Z} \setminus \{0\}\}$ are identically distributed. The process $\left(\Gamma_k^+, w_{\Gamma_{k-1}^+, \Gamma_k^+}^+ \right)_{k \in \mathbb{Z}}$ is a stationary marked point process in discrete time and it generates a stationary CRP, i.e. its first coordinates Γ_k^+ form a stationary point process with corresponding marks $w_{\Gamma_{k-1}^+, \Gamma_k^+}^+$.

(II) *The previous statement remains valid for cycles \mathcal{C}_k , with natural replacements of Γ_k^+ by Γ_k and $w_{\Gamma_{k-1}^+, \Gamma_k^+}$ by $w_{\Gamma_{k-1}, \Gamma_k}$.*

Note that the first statement of Lemma 2 is a direct consequence of Lemma 3.8 from [9] and that the schemes of the proofs of both statements of Lemma 2 are identical.

We will formulate and prove now the first of the main statements of this Section.

Lemma 3. *Assume that conditions (1.2) and (2.7) hold. Then there exists constant $C > 0$ such that*

$$\mathbf{E} e^{C\Gamma_0^+} < \infty \quad \text{and hence} \quad \mathbf{E} e^{C(\Gamma_1^+ - \Gamma_0^+)} < \infty. \tag{2.8}$$

Since the sequence $\{\Gamma_n^+\}$ is a subsequence of $\{\Gamma_n\}$, the statement (2.8) remains valid after the replacement of Γ_0^+ and Γ_1^+ by, correspondingly, Γ_0 and Γ_1 , i.e. both first expectations in (2.2) are finite, for some $C > 0$.

Proof. We borrow from [9] a number of auxiliary constructions. Introduce a set $\mathcal{U} = \{x \in \mathbb{Z} : \mathbf{I}(A_x^{l+}(c_1)) = 1\}$. It is not difficult to see that $\mathcal{R}^+ \subseteq \mathcal{U}$. Enumerate the elements of the set \mathcal{U} in the increasing order $\dots, \rho_{-1}, \rho_0, \rho_1, \dots$ where ρ_0 is its smallest non-negative element. Define a new sequence of cycles: for $k \in \mathbb{Z}$,

$$\mathcal{D}_k = (\rho_k - \rho_{k-1}; \{v_{\rho_{k-1}+j, \rho_{k-1}+i}^+, 0 \leq j < i \leq \rho_k - \rho_{k-1}\}).$$

The next result follows from Lemma 3.10 of [9].

Lemma 4. *Assume that conditions (1.2) and (2.7) hold. Then cycles $(\mathcal{D}_k, k \in \mathbb{Z})$ are mutually independent random elements where $(\mathcal{D}_k, k \in \mathbb{Z} \setminus \{0\})$ are identically distributed, and sequence $\rho_n, n \in \mathbb{Z}$ forms a stationary point process that generated the corresponding CRP.*

The statement of Lemma 4 may be easily explained in simple terms. For that, introduce for $d > 0$ events

$$A_{x,d}^{l+}(c_1) := \bigcap_{j=1}^d \{w_{x-j,x}^+ \geq c_1 j\}.$$

Note that, for any integers $k \geq 0$ and $0 \leq s_0 < s_1 < \dots < s_k$, the event $\{\rho_0 = s_0, \dots, \rho_k = s_k\}$ is uniquely determined by the collection of random variables $\mathcal{B}_{s_k} := \{v_{i,j}, i < j \leq s_k\}$ and, on this event, the equality $\rho_{k+1} - \rho_k = m$ holds if and only if $m = \min\{d > 0 : \mathbf{I}(A_{s_k, s_k+d}^{l+}(c_1)) = 1\}$, and the latter is determined by the random variables $\mathcal{B}_{s_k, s_k+m} := \{v_{i,j}^+, s_k \leq i < j \leq m\}$. It is not difficult to see that the families of random variables \mathcal{B}_{s_k} and \mathcal{B}_{s_k, s_k+m} are mutually independent and that the distribution of random variables from \mathcal{B}_{s_k, s_k+m} does not depend on k and s_k . Essentially these facts imply the statement of Lemma 4.

For $d > 0$, introduce auxiliary events

$$A_{x,d}^{r+}(c_1) := \bigcap_{i=1}^d \{w_{x,x+i}^+ \geq c_1 i\},$$

$$A_{x,d}^{0+}(c_2) := \bigcap_{1 \leq i \leq d, j \geq 1} \{v_{x-j, x+i} < c_2(j+i)\}.$$

Let $\mu = \inf\{d > 0 : \mathbf{I}(A_{0,d}^{0+}(c_2) \cap A_{0,d}^{r+}(c_1)) = 0\}$. Note that $\mathbf{P}(\mu = \infty) = \mathbf{P}(A_0^{0+}(c_2) \cap A_0^{r+}(c_1)) > 0$. Define recursively random variables $\sigma_0, \mu_0, \dots, \sigma_K, \mu_K$, where $K = \min\{k \geq 0 : \mu_k = \infty\}$. Let

$$\sigma_0 = \rho_0, \quad \mu_0 = \inf\{d > 0 : \mathbf{I}(A_{\sigma_0, \sigma_0+d}^{0+}(c_2) \cap A_{\sigma_0, \sigma_0+d}^{r+}(c_1)) = 0\},$$

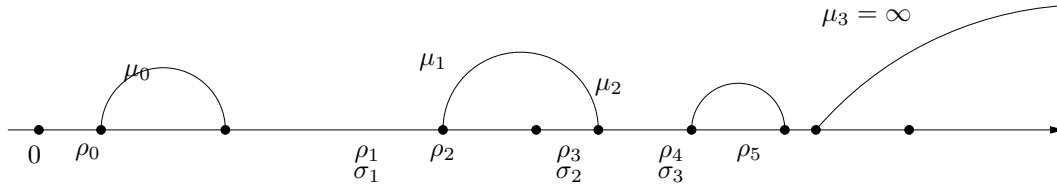


Fig. 1. The process of construction of sequence σ_k . Here $K = 3$ and $\sigma_3 = \rho_4$ is the moment of regeneration that gives an upper bound for Γ_0

and, for $k = 0, 1, \dots$, if $\mu_k < \infty$, then

$$\begin{aligned} \sigma_{k+1} &= \inf\{x \in \mathcal{U} : x \geq \sigma_k + \mu_k\}, \\ \mu_{k+1} &= \inf\{d > 0 : \mathbf{I}(A_{\sigma_k, \sigma_k+d}^{0+}(c_2) \cap A_{\sigma_k, \sigma_k+d}^{r+}(c_1)) = 0\}. \end{aligned}$$

The process of construction of this sequence is presented more transparently in Figure 1.

As it follows from the construction, $\sigma_k \in \mathcal{U}$ for $k \leq K$ and $\sigma_K \in \mathcal{R}^+$ and, therefore, $\sigma_K \geq \Gamma_0^+$ a.s. Moreover, the random variables $\{\mu_k\}$ form a sequence of independent and identically distributed random variables having a common distribution with μ . Therefore, the random variable K is geometrically distributed with parameter q and, in particular, its exponential moments $\mathbf{E}e^{CK}$ are finite for $C < -1/\ln q$. In addition, for any $k \geq 1$, given $\{K = k\}$, the random variables μ_0, \dots, μ_{k-1} are conditionally independent and distributed as $\mathbf{P}(\mu_0 \in \cdot | \mu_0 < \infty)$.

Since $\rho_k - \rho_{k-1} \geq 1$,

$$\sigma_K \leq \rho_M = \rho_0 + \sum_{k=1}^M (\rho_k - \rho_{k-1}), \tag{2.9}$$

where $M := \sum_{j=0}^{K-1} \mu_j$.

Therefore, if we show that the term on the right-hand side of (2.9) has a finite exponential moment, then σ_K also has a finite exponential moment, and this in turn implies the statement of Lemma 3. Given Lemma 9 from Section 4 and the elementary inequality $e^{x+y} < e^{2x} + e^{2y}$ for $x, y \geq 0$, it is sufficient to show that, firstly, random variable $\rho_1 - \rho_0$ has also a finite exponential moment (then the same holds for ρ_0) and, secondly, the probabilities $\mathbf{P}(\mu_0 = m | \mu_0 < \infty)$ decay exponentially fast in m .

Due to the independence of ρ_0 and $\rho_1 - \rho_0$ and since, for any natural $n > 0$, given occurrence of event $A_0^{l+}(c_1)$, events $A_n^{l+}(c_1)$ and $A_{n,n}^{l+}(c_1)$ either occur or not simultaneously, we get that the distributions of random variables $\rho_1 - \rho_0$ and $\nu := \min\{n : \mathbf{I}(A_{n,n}^{l+}(c_1)) = 1\}$ coincide:

$$\mathbf{P}(\rho_1 - \rho_0 = m) = \mathbf{P}(\rho_1 - \rho_0 = m | \rho_0 = 0) = \mathbf{P}(\nu = m | \rho_0 = 0) = \mathbf{P}(\nu = m), \quad m = 1, 2, \dots,$$

and existence of an exponential moment of random variable ν follows from Proposition 3.12 in [9].

Further,

$$\begin{aligned} \mathbf{P}(\mu = d) &= \mathbf{P}\left((A_{0,d}^{0+}(c_2) \cap A_{0,d}^{r+}(c_1))^c \cap (A_{0,d-1}^{0+}(c_2) \cap A_{0,d-1}^{r+}(c_1))\right) \\ &\leq \mathbf{P}\left((A_{0,d}^{0+}(c_2))^c \cap A_{0,d-1}^{0+}(c_2)\right) + \mathbf{P}\left((A_{0,d}^{r+}(c_1))^c \cap A_{0,d-1}^{r+}(c_1)\right) \\ &\leq \mathbf{P}\left(\sup_{j \geq 1} (v_{-j,d}^+ - c_2 j) > c_2 d\right) + \mathbf{P}(w_{0,d}^+ < c_1 d) \end{aligned}$$

$$\leq \sum_{j=1}^{\infty} \mathbf{P}(v > c_2(d + j)) + \mathbf{P}(w_{0,d}^+ < c_1d).$$

In the last line, the sum of probabilities $\sum_{j=1}^{\infty} \mathbf{P}(v > c_2(d + j))$ decays in d exponentially fast, thanks to (1.2). In order to show that the very last probability decreases exponentially fast in d too, we choose $\varepsilon > 0$ such that $\tilde{c} := c_1(1 + 3\varepsilon)$ also satisfies (2.7). Let $\eta(d) = \max\{k : t_k^+ \leq d\}$ (where, by convention, the maximum over empty set is $-\infty$). Then, for $r = \gamma^+(1 + \varepsilon)^{-1}$,

$$\begin{aligned} \mathbf{P}(w_{0,d}^+ < c_1d) &\leq \mathbf{P}(t_0^+ > d) + \mathbf{P}\left(w_{t_0^+, t_{\eta(d)}^+}^+ < c_1d, t_0^+ \leq d\right) \\ &\leq \mathbf{P}(t_0^+ > d) + \mathbf{P}(\eta(d) < [rd]) + \mathbf{P}\left(\sum_{i=1}^{[rd]} w_{t_{i-1}, t_i}^+ < c_1d\right) \\ &\leq \mathbf{P}(t_0^+ > d) + \mathbf{P}\left(\sum_1^{[rd]} (t_i^+ - t_{i-1}^+) > d\right) + \mathbf{P}\left(\sum_{i=1}^{[rd]} w_{t_{i-1}, t_i}^+ < c_1d\right), \end{aligned}$$

where $[rd]$ is the integer part of rd . In the last line of these inequalities all three summands decrease exponentially fast in d : the first summand since t_0^+ has a finite exponential moment, the second summand since the increments $t_i^+ - t_{i-1}^+$ have a finite exponential moment, $\mathbf{E}(t_1^+ - t_0^+)r < 1$, and, by the exponential Chebyshev inequality with $h > 0$,

$$\mathbf{P}\left(\sum_{i=1}^{[rd]} (t_i^+ - t_{i-1}^+) > d\right) \leq \left(\mathbf{E} \exp(h(t_1^+ - t_0^+))\right)^r e^{-hd},$$

where the right-hand side decays exponentially fast in d , if one takes $h > 0$ sufficiently small. Finally, the third summand decays exponentially fast because of a well-known fact: for any sequence X, X_1, X_2, \dots of independent and identically distributed positive random variables with finite mean $\mathbf{E}X$ and for any $\delta \in (0, 1)$ the probabilities $\mathbf{P}\left(\sum_{i=1}^n X_i < (1 - \delta)n \mathbf{E}X\right)$ decrease exponentially fast as n grows. In our case, $n = [rd] \geq rd - 1$, $X_i = w_{t_{i-1}, t_i}^+$, $\mathbf{E}w_{t_0^+, t_1^+}^+ \geq V$ and $c_1d \leq c_1(1 + n)(1 + \varepsilon)/\gamma^+ \leq c_1(1 + 2\varepsilon)nV/\gamma^+ < (1 - \delta)nV$ for all sufficiently large n , where $\delta = \varepsilon/(1 + 3\varepsilon)$.

Thus, the probabilities $\mathbf{P}(\mu = d)$ and, therefore, the probabilities $\mathbf{P}(\mu = d | \mu < \infty)$ decrease exponentially fast as d grows. This completes the proof of Lemma 3. \triangle

We will proceed now with the proof of finiteness of the two last mathematical expectations in (2.2).

Lemma 5. *Assume that conditions (1.2) and (2.7) hold. Then $\mathbf{E} \exp(Cw_{0,\Gamma_0}) < \infty$ and, therefore, $\mathbf{E} \exp(Cw_{\Gamma_0,\Gamma_1}) < \infty$, for some $C > 0$.*

Proof. Choose any path π from vertex 0 to vertex Γ_0 and assume that it includes $d + 1$ vertices, $0 = x_0 < x_1 < \dots < x_d = \Gamma_0$. Since $\sum_{k=1}^d (x_k - x_{k-1}) = \Gamma_0$, we get

$$\begin{aligned} w_{0,\Gamma_0} &= \sum_{k=1}^d v_{x_{k-1}, x_k} \leq \Gamma_0 + \sum_{k=1}^d (v_{x_{k-1}, x_k} - (x_k - x_{k-1}))^+ \\ &\leq \Gamma_0 + \sum_{0 \leq x < y \leq \Gamma_0} (v_{x,y} - (y - x))^+ \\ &\leq \Gamma_0 + \sum_{x=0}^{\Gamma_0-1} Z_x, \end{aligned} \tag{2.10}$$

where $\{Z_x := \max_{y>x}(v_{x,y} - (y - x))^+\}_{x \in \mathbb{Z}}$ is a sequence of independent and identically distributed random variables. By condition (1.2), the tail distribution

$$\mathbf{P}(Z_0 > m) \leq \sum_{k=1}^{\infty} \mathbf{P}(v > m + k)$$

decays exponentially fast in m . To complete the proof, it is enough to use the inequality $e^{x+y} < e^{2x} + e^{2y}$ and Lemma 9. \triangle

We complete Section 2 with a short proof of a simple fact.

Lemma 6. *Let $p \in (0, 1]$ and assume that conditions (1.2) and (2.7) hold. Then $\mathbf{P}(\Gamma_1 - \Gamma_0 = 1, w_{\Gamma_0, \Gamma_1} \geq y) > 0$ for any $y \in (c_2, \text{ess sup } v^+)$.*

Proof. The following two events coincide:

$$\{\Gamma_0 = 0, \Gamma_1 - \Gamma_0 = 1, w_{\Gamma_0, \Gamma_1} \geq y\} = A_0^l(c_1) \cap A_{0,1}^{0+}(c_2) \cap \{v_{0,1} \geq y\} \cap B_{1,1}(c_2) \cap A_1^r(c_1), \quad (2.11)$$

where

$$B_{1,1}(c_2) = \bigcap_{j=1}^{\infty} \{v_{0,1+j} < c_2(1 + j)\}.$$

Since all five events in the right-hand side of (2.11) are mutually independent and each of them has positive probability, the result follows. \triangle

3. ANALYSIS OF THE COMPOUND RENEWAL PROCESS AND PROOF OF THE MAIN THEOREM

We analyse here the CRP determined by the stationary marked process $(\Gamma_k, w_{\Gamma_{k-1}, \Gamma_k})$, $k \in \mathbb{Z}$. Using results from the previous Section and from [7], and also the classical Stone’s theorem [12], we will show that, for any admissible constants c_1, c_2 , the corresponding CRP has the same exact asymptotics with sequence $w_{0,n}$ in the domain of the normal and moderately large deviations (this is the result of Theorem 1). Next, we apply a corresponding change of measure in order to remove the “defect” of the CRP. Then we conclude with the statement that parameters α , σ^2 and $D(\alpha)$ that appear in Theorem 1, in fact, do not depend on constants c_1 and c_2 .

In what follows, it will be convenient to us to introduce some notation that in correspondence with notation from [7]:

$$(\tau_k, \mathbf{u}_k) := (\tau_k, (u_{k,1}, \dots, u_{k,\tau_k})), \quad k = 1, 2, \dots, \quad (3.1)$$

where

$$\begin{aligned} (\tau_1, (u_{1,1}, \dots, u_{1,\tau_1})) &:= (\Gamma_0, (w_{0,1}, \dots, w_{0,\Gamma_0})), \\ (\tau_k, (u_{k,1}, \dots, u_{k,\tau_k})) &:= (\Gamma_{k-1} - \Gamma_{k-2}, (w_{\Gamma_{k-2}, \Gamma_{k-2}+1}, \dots, w_{\Gamma_{k-2}, \Gamma_{k-1}})), \quad \text{for } k \geq 2. \end{aligned}$$

It was shown in Section 2 that vectors (τ_k, \mathbf{u}_k) , $k \geq 2$, are indentially distributed, and we will use notation

$$(\tau, \mathbf{u}) := (\tau, (u_1, \dots, u_\tau)) \quad (3.2)$$

for any vector having this distribution. We let also $\zeta = u_\tau$ and $\zeta_k = u_{\tau_k}$, for $k \geq 1$. Then, in particular, $\{(\tau_k, \zeta_k)\}$ is a sequence of independent random vectors that have the same distribution for $k \geq 2$ with vector (τ, ζ) .

We will list now statements from Section 2 (based on Lemmas 2, 3, 5 and 6) that we need here. Let $p \in (0, 1]$ and let c_1, c_2 satisfy condition (2.7).

S_I . The sequence (3.1) consists of independent random vectors, and, for $k \geq 2$, the random vectors (τ_k, \mathbf{u}_k) have the same distribution.

S_{II} . The random variables u_{1,τ_1} and u_1, \dots, u_τ are positive, and

$$\max\{u_1, \dots, u_{\tau-1}\} \leq u_\tau.$$

S_{III} . For some $C > 0$,

$$\mathbf{E} e^{C\tau_1} < \infty, \quad \mathbf{E} e^{C\tau} < \infty, \quad \mathbf{E} e^{Cu_{1,\tau_1}} < \infty, \quad \mathbf{E} e^{Cu_\tau} < \infty.$$

S_{IV} . The probability $\mathbf{P}(\tau = 1, u_\tau \geq y)$ is strictly positive for any $y \in (c_2, \text{ess sup } v^+)$.

Denote

$$(\tau, \zeta) := (\tau, u_\tau), \quad (\tau_k, \zeta_k) := (\tau_k, u_{k,\tau_k}), \quad k = 1, 2, \dots \tag{3.3}$$

Then the sequence $\{(\tau_k, \zeta_k)\}$ consists of independent random vectors that have, for $k \geq 2$, a common distribution with vector (τ, ζ) . The results above imply

Corollary 1. *Let conditions (1.2) and (2.7) hold. Then the following statements take place.*

(I) *If the random variable v satisfies condition [Z], then*

[ZZ] *The distribution of the random vector (τ, ζ) is arithmetic and is concentrated on the lattice of span 1 for each of its coordinates.*

(II) *If the random variable v satisfies condition [R], then (τ, ζ)*

[ZR] *The marginal distribution of the first coordinate of vector (τ, ζ) is arithmetic with span 1 and the marginal distribution on the second coordinate is non-lattice.²*

We turn now to the proof of the main result.

Proof of Theorem 1. Consider sequence $\{(\tau_k, \zeta_k)\}_{k=1}^\infty$ of independent random vectors, having for $k \geq 2$ the same distribution as (τ, ζ) (see formula (3.2) and notation after it). Introduce sequences of partial sums

$$T_n := \sum_{k=0}^n \tau_k, \quad Z_n := \sum_{k=0}^n \zeta_k, \quad n \geq 0,$$

where $(\tau_0, \zeta_0) := (0, 0)$. Let

$$\begin{aligned} \eta_+(n) &:= \min\{k \geq 1 : T_k > n\}, \\ \nu_+(n) &:= \max\{k \geq 0 : T_k \leq n\} = \eta_+(n) - 1, \\ \gamma_+(n) &:= n - \nu_+(n). \end{aligned}$$

Introduce our CRP (we will call it the “first CRP”) by

$$Z_+(n) := \sum_{k=0}^{\nu_+(n)} \zeta_k.$$

Let us clarify that we consider here notation $Z_+(n)$ with low case + just to reconcile our notation with that from the paper [7], where, in addition to notation $Z_+(n)$, $\nu_+(n)$, $\gamma_+(n)$, further notation

² In terms of the characteristic function $f(z, l) := \mathbf{E} e^{iz\tau + il\zeta}$ of random vector (τ, ζ) , these two conditions may be presented as:

[ZZ] $f(2\pi z, 2\pi t) = 1$ for any $(z, t) \in \mathbb{Z}^2$ and $|f(z, t)| < 1$ for any $(z, t) \notin \mathbb{Z}^2$.

[ZR] $f(2\pi z, 0) = 1$ for any $z \in \mathbb{Z}$, $|f(2\pi z, 0)| < 1$ for any $z \notin \mathbb{Z}$ and $|f(0, t)| < 1$ for any $t \neq 0$.

$Z(n), \nu(n), \gamma(n)$ have been used. We like to point out that thus “plus” does not relate to that in notation $w_{j,m}^+$.

Using the introduced notation, we obtain the representation

$$w_{0,n} = Z_+(n) + w_{n-\gamma_+(n),n} = Z_{\nu_+(n)} + w_{\nu_+(n),n}.$$

Consider a random vector (of random length), given by formula (3.2):

$$(\tau, (u_1, u_2, \dots, u_\tau)),$$

whose coordinates clearly satisfy the following constraints:

$$\tau \geq 1, \quad \min_{1 \leq i \leq \tau} u_i \geq c_1 > 0, \quad \max_{1 \leq i \leq \tau} u_i \leq u_\tau.$$

Next, introduce a random vector (τ^*, ζ^*) , taking values $(i, y) \in \mathbb{Z} \times \mathbb{R}, i \geq 0, y \geq 0$, by

$$\mathbf{P}(\tau^* = i, \zeta^* \in dy) := \frac{1}{Q} \mathbf{P}(\tau \geq i + 1, u_i \in dy),$$

(we assume $u_0 = 0$ a.s.), where

$$Q := \sum_{i=0}^{\infty} \int_0^{\infty} \mathbf{P}(\tau \geq i + 1, u_i \in dy) = \sum_{i=0}^{\infty} \mathbf{P}(\tau \geq i + 1) = \mathbf{E}\tau.$$

It follows from condition S_{III} that there exists a constant $C > 0$ such that

$$\mathbf{E}e^{C\tau^*} < \infty, \quad \mathbf{E}e^{C\zeta^*} < \infty.$$

Along with the sequence $\{(\tau_k, \zeta_k)\}$ that determines CRP $Z_+(n)$ and the functionals $\nu_+(n), \gamma_+(n)$, we introduce another sequence $\{(\tau_k^*, \zeta_k^*)\}$ by

$$(\tau_1^*, \zeta_1^*) := (\tau_1, \zeta_1) + (\tau^*, \zeta^*), \quad (\tau_k^*, \zeta_k^*) := (\tau_k, \zeta_k) \quad \text{for } k \geq 2,$$

where the random vector (τ^*, ζ^*) does not depend on the sequence $\{(\tau_k, \zeta_k)\}$. The new sequence $\{(\tau_k^*, \zeta_k^*)\}$ determines a new CRP $Z_+^*(n)$ and new functionals $\nu_+^*(n)$ and $\gamma_+^*(n)$.

Lemma 7. *Assume that conditions S_I-S_{IV} hold. Then, for any $n \geq 2$ and any real $x \geq c_1$ and $\Delta > 0$, the following equality takes place:*

$$\mathbf{P}(Z_+(n) + w_{n-\gamma_+(n),n} \in [x, x + \Delta), \tau_1 \leq n) = Q \mathbf{P}(Z_+^*(n) \in [x, x + \Delta), \gamma_+^*(n) = 0). \quad (3.4)$$

Proof. We have

$$\begin{aligned} P_n &:= \mathbf{P}(Z_+(n) + w_{n-\gamma_+(n),n} \in [x, x + \Delta), \tau_1 \leq n) \\ &= \sum_{k=1}^{\infty} \mathbf{P}(T_k = n, Z_k \in [x, x + \Delta)) \\ &\quad + \sum_{k=1}^{\infty} \sum_{i=1}^n \int_0^{\infty} \mathbf{P}(T_k = n - i, Z_k + y \in [x, x + \Delta), \tau_{k+1} \geq i + 1, u_{k+1,i} \in dy). \end{aligned}$$

Since $\mathbf{P}(\tau \geq 1, u_0 = 0) = 1$ and since $(\tau_{k+1}, u_{k+1,i})$ and (T_k, Z_k) are independent for each k ,

$$P_n = \sum_{k=1}^{\infty} \mathbf{P}(T_k = n, Z_k \in [x, x + \Delta)) \mathbf{P}(\tau \geq 1, u_0 = 0)$$

$$\begin{aligned}
 & + \sum_{k=1}^{\infty} \sum_{i=1}^n \int_0^{\infty} \mathbf{P}(T_k = n - i, Z_k + y \in [x, x + \Delta]) \mathbf{P}(\tau \geq i + 1, u_i \in dy) \\
 & = Q \sum_{k=1}^{\infty} \sum_{i=0}^n \int_0^{\infty} \mathbf{P}(T_k = n - i, Z_k + y \in [x, x + \Delta]) \frac{1}{Q} \mathbf{P}(\tau \geq i + 1, u_i \in dy) \\
 & = Q \sum_{k=1}^{\infty} \sum_{i=0}^n \int_0^{\infty} \mathbf{P}(T_k = n - i, Z_k + y \in [x, x + \Delta]) \mathbf{P}(\tau^* = i, \zeta^* \in dy) \\
 & = Q \sum_{k=1}^{\infty} \mathbf{P}(T_k^* = n, Z_k^* \in [x, x + \Delta]) \\
 & = Q \mathbf{P}(Z_+^*(n) \in [x, x + \Delta], \gamma_+^*(n) = 0, \nu_+^*(n) \geq 1) \\
 & = Q \mathbf{P}(Z_+^*(n) \in [x, x + \Delta], \gamma_+^*(n) = 0) - Q \mathbf{P}(Z_+^*(n) \in [x, x + \Delta], \gamma_+^*(n) = 0, \nu_+^*(n) = 0) \\
 & = Q \mathbf{P}(Z_+^*(n) \in [x, x + \Delta], \gamma_+^*(n) = 0),
 \end{aligned}$$

where the last equality follows from the fact that, for $x > 0$,

$$\mathbf{P}(Z_+^*(n) \in [x, x + \Delta], \nu_+^*(n) = 0) = 0. \quad \Delta$$

In the particular case where v has an arithmetic distribution, we $\Delta = 1$ in Lemma 7 and take $x \in \mathbb{Z}$ to obtain the following corollary of Lemma 7.

Lemma 8. *Assume that conditions $S_{\text{I}}-S_{\text{IV}}$ hold and let v satisfy condition $[\mathbf{Z}]$. Then, for any integers $n \geq 2$ and $x \geq 1$, we get*

$$\mathbf{P}(Z_+(n) + w_{n-\gamma_+(n),n} = x, \tau_1 \leq n) = Q \mathbf{P}(Z_+^*(n) = x, \gamma_+^*(n) = 0). \tag{3.5}$$

We continue now with the proof of Theorem 1.

I. Consider the arithmetic case first. In order to apply results from the paper [7], we need to introduce, in addition to CRP $Z_+(n)$, another CRP $Z(n)$, since the main results in [7] are obtained for CRP $Z(n)$.

For $n \geq 1$, let

$$\nu(n) := \max\{k \geq 1 : T_k < n\}, \quad \gamma(n) := n - \nu(n).$$

Then

$$Z(n) := Z_{\nu(n)}.$$

It is easy to see (we assume that the processes $Z(n)$ and $Z_+(n)$ are constructed on a common probability space based on the same sequence $\{(\tau_k, \zeta_k)\}$) that, for any $n \geq 1$,

$$\nu_+(n) = \nu(n + 1), \quad Z_+(n) = Z(n + 1), \quad \gamma_+(n) = \gamma(n) - 1. \tag{3.6}$$

In particular, the defect $\gamma(n)$ takes values $\{1, 2, \dots\}$ and the defect $\gamma_+(n)$ takes values $\{0, 1, 2, \dots\}$. We will use similar notation for CRP $\{(\tau_k^*, \zeta_k^*)\}$ and the corresponding functionals, with adding an extra upper-case “*”, for example:

$$\nu^*(n), \quad \nu_+^*(n), \quad Z^*(n), \quad Z_+^*(n), \quad \gamma^*(n), \quad \gamma_+^*(n), \quad \text{etc.}$$

The following relations in the domain of the normal and moderately large deviations follow from formulae (3.6) of Theorem 2.1, Corollary 2.1 and Theorem 2.1* of the paper [7] in the case where $x \in \mathbb{N}$, $x - na = o(n)$: as $n \rightarrow \infty$,

$$\mathbf{P}(Z(n) = x) \sim \mathbf{P}(Z_+(n) = x) \sim \mathbf{P}(Z^*(n) = x) \sim \mathbf{P}(Z_+^*(n) = x) \sim \frac{1}{\sigma\sqrt{2\pi n}} e^{-nD(\frac{x}{n})}, \tag{3.7}$$

$$\mathbf{P}(Z_+^*(n) = x, \gamma_+^*(n) = 0) \sim \mathbf{P}(Z_+(n) = x, \gamma_+(n) = 0) \sim \frac{1}{\mathbf{E}\tau} \mathbf{P}(Z_+^*(n) = x). \tag{3.8}$$

Here equivalences (3.7) show that, under our assumptions, differences between processes $Z(n)$ and $Z_+(n)$, as well as differences related to inhomogeneity disappear for local theorems in the domain of normal and moderately large deviations.

Applying Lemma 8 and noticing that $Q = \mathbf{E}\tau$ and that, for some $h > 0$,

$$\begin{aligned} & \mathbf{P}(Z_+(n) + w_{n-\gamma_+(n),n} = x) \\ &= \mathbf{P}(Z_+(n) + w_{n-\gamma_+(n),n} = x, \tau_1 > n) + \mathbf{P}(Z_+(n) + w_{n-\gamma_+(n),n} = x, \tau_1 \leq n) \\ &= \mathbf{P}(Z_+(n) + w_{n-\gamma_+(n),n} = x, \tau_1 \leq n) + O(e^{-nh}), \end{aligned}$$

we obtain from (3.7) and (3.8) the statement of part I of Theorem 1:

$$\mathbf{P}(Z_+(n) + w_{n-\gamma_+(n),n} = x) \sim \frac{1}{\sigma\sqrt{2\pi n}} e^{-nD(\frac{x}{n})}.$$

II. Our proof of part II of Theorem 1 is based on the integro-local theorem in the domains of the normal, moderately large and large deviations obtained by Stone [12]. We formulate this theorem in notation that is convenient to us. We define the rate function for random vector (τ, ζ) as

$$\Lambda(\theta, \alpha) := \sup_{\lambda, \mu} \{ \lambda\theta + \mu\alpha - A(\lambda, \mu) \}.$$

Next, we denote by $|\Lambda''(\theta, \alpha)|$ the determinant of matrix $\Lambda''(\theta, \alpha)$ of the second derivatives of the rate function $\Lambda(\theta, \alpha)$.

Theorem 2 [12]. *Let the distribution of vector (τ_1, ζ_1) coincide with the distribution of (τ, ζ) and let conditions S_{III} and [ZR] hold. Then, for some $\delta > 0$ and for a certain sequence $\Delta^{(0)} := \Delta_n^{(0)} > 0$ that tends to zero as $n \rightarrow \infty$, and for any $(x, y) \in \mathbb{Z} \times \mathbb{R}$ such that $|\theta - a_\tau| + |\alpha - a_\zeta| \leq \delta$, where $(\theta, \alpha) := (\frac{x}{n}, \frac{y}{n})$, the following equality holds:*

$$\mathbf{P}(T_n = x, Z_n \in [y, y + \Delta]) = \frac{\Delta\sqrt{|\Lambda''(\theta, \alpha)|}}{2\pi n} \exp\{-n\Lambda(\theta, \alpha)\}(1 + o(1)),$$

where $\Delta := \Delta_n \geq \Delta_n^{(0)}$ and $\Delta_n \rightarrow 0$ as $n \rightarrow \infty$, and the remaining term $o(1) = \varepsilon_n(x, y)$ satisfies the relation

$$\lim_{n \rightarrow \infty} \sup_{\substack{(x,y) \in \mathbb{Z} \times \mathbb{R} \\ |\theta - a_\tau| + |\alpha - a_\zeta| \leq \delta}} |\varepsilon_n(x, y)| = 0.$$

Note that, by applying statements $S_{\text{I}}-S_{\text{IV}}$ and repeating all phases of the proofs of Theorems 2.1 and 2.1* and of Corollary 2.1 from [7], we obtain natural analogues of these statements where symbols $= x$ are replaced by $\in [x, x + \Delta)$ and an additional coefficient Δ appears in the right-hand side. The rest of the proof of part II completely repeats the corresponding piece of the proof of part I. \triangle

We formulate and prove now the last results.

Theorem 3. *The characteristics a, σ and $D(\alpha)$ that are used in Theorem 1 do not depend on a choice of admissible constants c_1 and c_2 .*

Proof. We have shown that the local theorems in the domains of the normal and moderately large deviations for processes $w_{0,n}$ and $Z_+(n)$ look identical, and the formulation of Theorem 1

includes characteristics a, σ and $D(\alpha)$ that are uniquely determined by vector (τ, ζ) (see (1.4)–(1.6)) that “drives” CRP $Z_+(n)$. Since random vector (τ, ζ) is defined in terms of arbitrarily chosen constants c_1 and c_2 that satisfy condition (2.7), one can guess that the characteristics a, σ and $D(\alpha)$ may depend on these constants too. However, we will show now that this is not the case. Namely, we will show that, for any other pair of constants $\tilde{c}_2 \leq \tilde{c}_1$ satisfying (2.7), the corresponding characteristics $\tilde{a}, \tilde{\sigma}$ and $\tilde{D}(\alpha)$ determined by vector $(\tilde{\tau}, \tilde{\zeta})$ coincide, in fact, with a, σ and $D(\alpha)$.

Based on what have we proved already, one can see that, for the processes $w_{0,n}$ and $\tilde{Z}_+(n)$, a similar local theorem holds, with corresponding characteristics $\tilde{a}, \tilde{\sigma}$ and $\tilde{D}(\alpha)$. This means that, for the process $w_{0,n}$, two local theorems take place, and their statements differ in characteristics $a, \sigma, D(\alpha)$ and $\tilde{a}, \tilde{\sigma}, \tilde{D}(\alpha)$ only. The local theorems in the domain of the normal deviations clearly lead to the corresponding laws of large numbers: for any $\varepsilon > 0$, one has

$$\lim_{n \rightarrow \infty} \mathbf{P}(|w_{0,n} - an| \leq n\varepsilon) = 1, \quad \lim_{n \rightarrow \infty} \mathbf{P}(|w_{0,n} - \tilde{a}n| \leq n\varepsilon) = 1.$$

Then $\tilde{a} = a$, with necessity.

Next, we may rewrite the statements of these local theorems using equality $\tilde{a} = a$ and arrive at the equivalence

$$\frac{1}{\sigma\sqrt{n}} e^{-nD(\frac{x}{n})} \sim \frac{1}{\tilde{\sigma}\sqrt{n}} e^{-n\tilde{D}(\frac{x}{n})},$$

that is valid for any $x = x_n \in \mathbb{Z}$ in the area $|\frac{x}{n} - a| = o(1)$. Again, necessarily, the latter equivalence implies equality $\sigma = \tilde{\sigma}$ and identity of analytic (in a neighbourhood of point $\alpha = a$) functions $\tilde{D}(\alpha) = D(\alpha)$. \triangle

4. AUXILIARY RESULT

In our proofs, we use an auxiliary result that is not original. We include its proof because it is very short.

Lemma 9. *Let $S_n = \sum_{i=1}^n X_i, n = 1, 2, \dots$ be a sequence of independent and identically distributed non-negative random variables $\{X_i\}$ and let N be a counting random variable. Assume that*

$$\mathbf{E} e^{CX_1} < \infty \text{ and } \mathbf{E} e^{CN} < \infty. \text{ for some } C > 0.$$

Then one can choose a constant $b > 0$ such that

$$\mathbf{E} e^{bS_N} < \infty.$$

Proof. Take any $a > \mathbf{E} X_1$. Then

$$S_N = \sum_{i=1}^N (X_i - a) + aN \leq \sup_{n \geq 0} (S_n - na) + aN \equiv R + aN, \tag{4.1}$$

here we let $S_0 = 0$. From (4.1) and from the elementary inequality $e^{x+y} \leq e^{2x} + e^{2y}$, for any $b > 0$ we have

$$e^{bS_N} \leq e^{2bR} + e^{2baN} \leq 1 + \sum_{n=1}^{\infty} e^{2b(S_n - na)} + e^{2baN}$$

and, therefore,

$$\mathbf{E} e^{bS_N} \leq 1 + \sum_{n=1}^{\infty} (\mathbf{E} e^{2b(X_1 - a)})^n + \mathbf{E} e^{2baN}. \tag{4.2}$$

Since $a > \mathbf{E} X_1$, then one can choose $b > 0$ so small that $2b \max(1, a) < C$ and $\mathbf{E} e^{2b(X_1 - a)} < 1$. For such b , the right-hand side of (4.2) is finite too. \triangle

FUNDING

The paper is supported by the Mathematical Center in Akademgorodok, grant 075-15-2019-1675 by the Ministry of Science and Higher Education (A.V. Logachov, A.A. Mogul'skii and S.G.Foss); and also by a joint Russian-French grant RFBR 19-51-15001 – CNRS 193-382 (S.G. Foss and T. Konstantopoulos).

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