EQUILIBRIUM STRESSABILITY OF MULTIDIMENSIONAL FRAMEWORKS

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ABSTRACT. We prove an equilibrium stressability criterium for trivalent multidimensional tensegrities. The criterium appears in different languages: (1) in terms of stress monodromies, (2) in terms of surgeries, (3) in terms of exact discrete 1-forms, and (4) in Cayley algebra terms.

CONTENTS

1. Introduction

In the previous century, Fuller [9] coined the term *tensegrity*, a combination of 'tension' and 'integrity', to describe networks of rods and cables, such as those created by artist Kenneth Snelson, in which the tension of the cables and the compression in the rods combine to yield structural integrity to the whole. More generally, the word tensegrity is used to describe a variety of practical and abstract structures, e.g. bicycle tires and tents, whose rigidity follows from the balance of *tension* and *compression*, in the mathematical literature the *stress*, on the members.

Practically, structures exhibiting tensegrity may be generated and analyzed using conventional techniques of structural engineering [17]. Theoretically, tensegrity is often considered as part of the study of geometric constraint systems, [2, 15]. The classical tensegrity model consists of a set of vertices V, and two graphs, (V, S) and (V, C), the graph of *struts* and *cables*, and a placement function $\mathbf{p} : V \to \mathbb{R}^d$. One looks for a motion of the placed vertices such that the distances between pairs of vertices connected by struts do not fall below their initial values, and such that the distances between pairs of vertices connected by cables do not expand beyond their initial value. If no such motion exists, apart from the rigid motions of the space itself, the system is said to be *rigid*. A *stress* is a function $s : S \cup C \to \mathbb{R}$, with $s(t) \geq 0$ and $s(c) \leq 0$ for all cables, $c \in C$, and struts, $t \in S$. A stress is an *equilibrium stress* if for each vertex v

$$\sum_{(v,w)\in C\cup S} s((v,w))(\mathbf{p}(v) - \mathbf{p}(w)) = \mathbf{0}.$$

There are two avenues in which the existence of a proper, i.e. nowhere zero, equilibrium stress may allow one to establish structural integrity. A result of Roth and Whitely [12] states that if a tensegrity has a proper equilibrium stress, and if the placement for $(V, C \cup S)$ is statically rigid as a bar and joint framework, then that tensegrity must be first order rigid and hence rigid. More delicately, if the proper equilibrium stress passes the *second-order stress test* of Connelley and Whiteley [3], then the tensegrity structure is second order rigid, hence rigid.

Another important aspect of proper equilibrium stresses is the connection between the existence of an equilibrium stress and the lifting of embedded graphs into higher dimensions, as provided by the theory

of Maxwell-Cremona. It was understood by Lee, Ryshkov, Rybnikov, and some others that the connection between equilibrium stresses and lifts extends in certain cases to arbitrary CW-complexes M realized in any dimension d, also not necessarily embedded. Although the subject can be traced earlier (see [8, 19]), the first systematic study of multidimensional stresses, liftings and reciprocal diagrams was undertaken by Rybnikov in [13, 14].

In the present paper we introduce d-frameworks and their equilibrium stresses, i.e. self-stresses, in a slightly broader way than it was done by Rybnikov (Section 2). We introduce face paths and stress transition along such face paths in d-frameworks in Section 3. We also give (Section 4) necessary and sufficient conditions for the equilibrium stressability of trivalent frameworks. This result appears as a generalization of equilibrium stressability criteria for classical tensegrities [7]. The conditions are equivalently expressed in terms of exact discrete multiplicative 1-forms, Cayley algebra, or, in terms of some surgeries introduced in Section 4.

In Section 5, we explain how Rybnikov's frameworks (R-frameworks, for short) arise in the proposed context. We give an equilibrium stressability criterion and derive some examples that demonstrate similarities and differences between planar and multidimensional tensegrities.

2. Main definitions and constructions

Our model for tensegrity in this paper will be based on the following structure. Let $D > d \geq 1$ be two integers. In the sequel, the term *plane* means an affine subspace in $\mathbb{R}^D$.

Definition 2.1. A *d-framework* $\mathcal{F} = (E, F, I, \mathbf{n})$ consists of E, a collection of $(d-1)$-dimensional planes in $\mathbb{R}^D$; F, a collection of d-dimensional planes in $\mathbb{R}^D$; a subset $I \subset \{(p, q) \in (E \times F) \mid p \subset q\}$; a function $\mathbf{n}$ assigning to each pair (e, f) with $e \in E$ and $(e, f) \in I$, a unit vector $\mathbf{n}(e, f)$ which is contained in f and which is normal to e. We call planes from F *faces* and planes from E *edges*. The set I is called the *set of incidences*.

A d-framework is called *generic* if, for every $e \in E$, all the planes f with $(e, f) \in I$ are distinct.

Let $\mathcal{F} = (E, F, I, \mathbf{n})$ be a d-framework. A *stress* s on $\mathcal{F}$ is any function $s : F \to \mathbb{R}$. A framework $\mathcal{F}$ together with a stress s is said to be in *equilibrium* if for every $e \in E$ we have

$$(1) \qquad \sum_{(e,f)\in I} s(f)\mathbf{n}(e, f) = 0.$$

Such a stress is called an *equilibrium stress*, a *self-stress* or sometimes a *prestress* for $\mathcal{F}$.

Definition 2.2. A d-framework is said to be *self-stressable* or a *tensegrity* if there exists a non-zero self-stress on it.

Example 2.3. The simplest non-trivial example here is the classical case of graphs in the plane ($D = 2, d = 1$).

We say that a d-framework is *trivalent* if each element of E is incident (i.e., contained in a pair in I) to precisely 3 elements of F.

Surface based tensegrities of this type are models for minimal surfaces (or, more generally, harmonic surfaces) which meet at edges, such as, soap bubbles or tents. In this model, in the $D = 3$ case, the surfaces are flat, and we can think of them as rigid plates, each having an expansion or contraction coefficient, say caused by heat or cooling, for which the equilibrium condition indicates that the forces cancel on the edges, so that the framework does not deform.

Example 2.4. Let $d = 2$ and $D = 3$. Consider a 2-framework whose edges, E, and faces, F, correspond to the edges and triangles of the graph K_5, embedded in $\mathbb{R}^3$. If four of the vertices, $\{1, 2, 3, 4\}$, of K_5 are placed as vertices of a regular tetrahedron and the fifth one as their centroid, then the resulting 2-framework is generic in our sense. For each edge-face pair choose the normal to be a unit vector pointing into the interior of the face-triangle. Note that the chosen normals sum to the zero vector around the lines corresponding to edges $\{i, 5\}$, so choosing equal stress on these interior triangles leaves those edges equilibrated. Then, it is easy to see that choosing stresses on the exterior and interior triangles in the ratio $-\sqrt{6}/4$ yields an equilibrium stress.

In this example one may imagine the interior expanding triangles exerting an outward force balanced by the contracting "skin" of the exterior triangles.

Example 2.5. Again, let $d = 2$ and $D = 3$. We may create a different 2-framework based on the graph K_5 in $\mathbb{R}^3$ by keeping E as before, and associating the faces F to the K_4 subgraphs of K_5, with the usual incidence relation. Since any two K_4's intersect in 3 edges, their face planes must be identical, and the 5 vertices of the embedded K_5 must be coplanar.

Since the normal vectors all lie in the plane of the K_5, it is no loss of generality to assume that the vertices lie on a regular pentagon, and it is quickly checked that only the zero stress satisfies Equation (1).

Example 2.6. Let $d = 2$ and $D = 3$. Consider the vertices of a regular cube and set E to be the set of all lines joining a pair of non-antipodal vertices (see Figure 1). The face planes F consist of all six planes containing the faces of the cube, together with the six planes containing antipodal pairs of cube edges, as well as the eight planes of the dual tetrahedra. Let incidences be induced by containment. Since
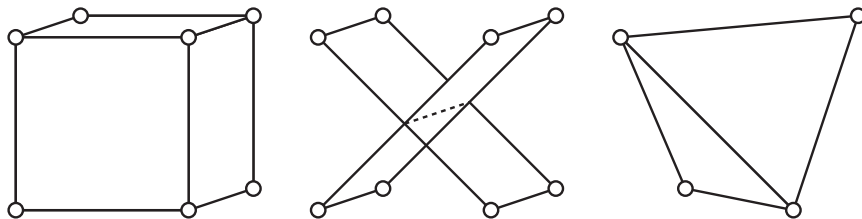

FIGURE 1. A 3-framework based on the cube with three types of faces.

each plane contains a polygon of edges supported by incident lines of the structure, we may take the normals to be inwardly pointing unit vectors. It is easy to check that the self-stresses on the three types of faces are in the ratio $1 : -\sqrt{2} : \sqrt{3}/4$.

Example 2.7. This example has two versions, both with $d = 2$ and $D = 3$. Consider the vertices of an octahedron, regularly embedded in $\mathbb{R}^3$. The set of 12 edge lines E lie along the edges of the octahedron,
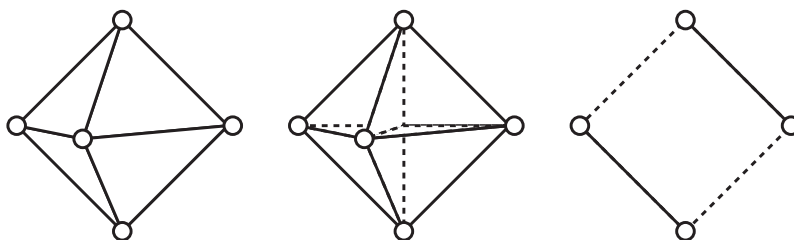
FIGURE 2. Three face types: triangles, squares, or faces with just two edges.

and the set F of 11 face planes will consist of those eight supporting triangles of the octahedron, together with the three planes which pass through four coplanar vertices. Let the incidences be all those induced by containment and, as before, let all normals be chosen inwardly pointing with respect to the triangle or square to which they belong. Then this is easily computed to be stressable, and hence a tensegrity. In fact, the self-stress is unique, since each each line is incident to three distinct planes.

FIGURE 3. A face path contained in the framework described in Example 2.4 with $f_{0,1} = \{1,2,4\}$, $f_{1,2} = \{2,3,4\}$, $f_{2,3} = \{3,4,5\}$, $f_{3,4} = \{1,3,5\}$.

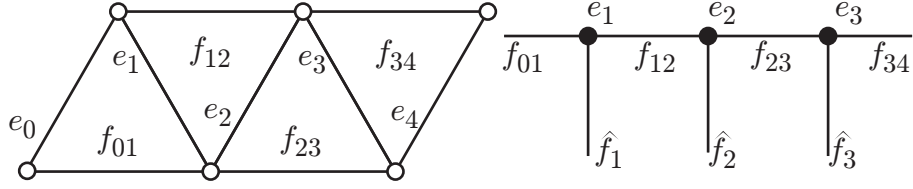As an alternative, we can take each of the three planes containing four vertices to have multiplicity 2, with each one incident to a different pair of opposite lines, and with the same choice of normals. This structure consisting of 12 lines and 14 planes is also a tensegrity.

3. Self-stressability of frameworks.

In this section we study self-stressability of trivalent d-frameworks in $\mathbb{R}^{d+1}$, so starting from now on, we assume that $D = d + 1$.

3.1. Self-stressability of face-paths and face-cycles. Let us start with the following general definition.

3.1.1. *Face-path and face-cycle.* Let $E = (e_1, \ldots, e_k)$ be a sequence of distinct $(d-1)$-dimensional planes in $\mathbb{R}^D$; $F = (f_{0,1}, f_{1,2}, \ldots, f_{k,k+1})$ be a sequence of d-dimensional planes in $\mathbb{R}^D$; $\hat{F} = (\hat{f}_1, \ldots, \hat{f}_k)$ be a sequence of d-dimensional planes in $\mathbb{R}^D$. Then the collection $(E, F, \hat{F})$ is said to be a *face-path* if for $i = 1, \ldots, k$ we have

$$ e_i \subset f_{i,i+1}, \quad e_i \subset f_{i-1,i}, \quad \text{and} \quad e_i \subset \hat{f}_i. $$

For example, removal of two antipodal triangles of an octahedron leaves a face-cycle with six faces.

Denote the set of pairs defined by these inclusions by I. For a particular choice of normals $\mathbf{n}$ we obtain a d-framework $\mathcal{F} = (E, F \cup \hat{F}, I, \mathbf{n})$ which is called a *face-path d-framework*, see Figure 3. It is called a *face-cycle d-framework* if $f_{0,1} = f_{k,k+1}$. In this case it is denoted by $C(E, F, \hat{F}, \mathbf{n})$, see Figure 4.

3.1.2. *Self-stressability of face-path d-frameworks.*

Proposition 3.1. Any generic face-path d-framework which contains no face-cycle has a one-dimensional space of self-stresses. All the stresses for all the planes of F and $\hat{F}$ are either simultaneously zero, or simultaneously non-zero.
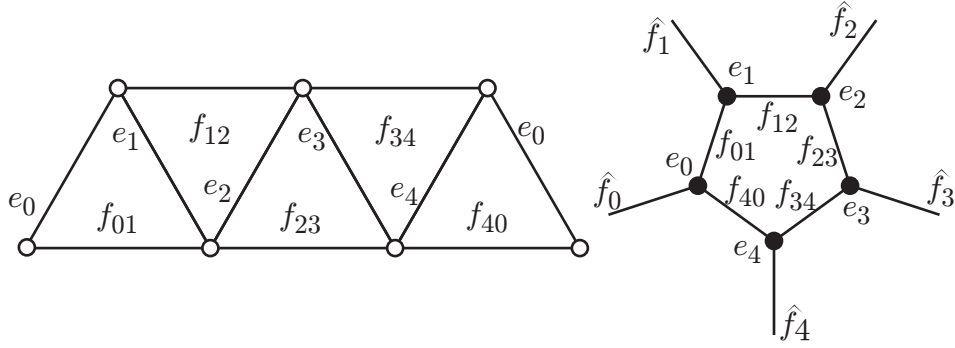
FIGURE 4. A face path contained in the framework described in Example 2.4, with labeling as before and $f_{4,0} = \{1, 2, 5\}$.

Proof. Setting $s(f_{0,1}) = 1$ we inductively define all stresses for all other planes using Equation (1). Therefore a non-zero self-stress exists. By construction the obtained stress is non-zero at all planes of F and $\hat{F}$.

Once we know any of the stresses at one of the planes of F and $\hat{F}$, we reconstruct the remaining stresses uniquely using Equation (1). Hence the space of stresses is at most one-dimensional. Therefore, all self-stresses are proportional to a self-stress that is non-zero at all planes of F and $\hat{F}$. $\qquad\square$

3.1.3. *Edge-orientation transition.* Suppose we have a face path whose edges are $e_1, e_2, \ldots$, and we are given an orientation of e_i by declaring a frame spanning e_i as positive. We pass over this positive orientation on e_i to a positive orientation on e_{i+1} by requiring that the given frame of e_i together with $\mathbf{n}(e_i, f_{i,i+1})$ and a new (chosen to be positive) frame of e_{i+1} together with $\mathbf{n}(e_{i+1}, f_{i,i+1})$ differ by an orientation reversing automorphism on $f_{i,i+1}$. We call this the *edge-orientation transition.*

A face-cycle d-framework $C(E, F, \hat{F}, \mathbf{n})$ is said to be *edge-orientable* if the edge-orientation transition around the cycle returns to the starting edge in its initial orientation.

Non-orientable face-cycles are a usual phenomenon in frameworks. Indeed, we see them even in small examples like Example 2.4. Figure 5 depicts such a face-cycle. Here the first and the last edges coincide, but are oppositely oriented.

Note that the edge-orientability of a cycle depends neither on the choice of the first element $e_1 \in E$, nor the choice of direction in the cycle. We observe the following proposition.

Proposition 3.2. A face-cycle d-framework $C = C(E, F, \hat{F}, \mathbf{n})$ has the following properties.
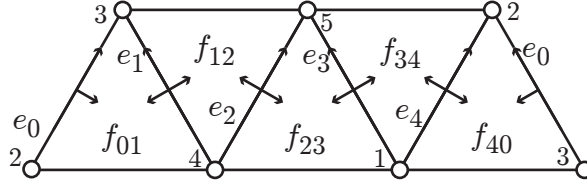
FIGURE 5. A face-cycle contained in K_5 which is not edge-orientable. For this particular example faces correspond to triangles, and we choose all the normals to point inward.

(i) Reversing simultaneously all the normals at a single $e_i \in E$ (namely $\mathbf{n}(e_i, f_{i-1,i})$, $\mathbf{n}(e_i, f_{i,i+1})$, and $\mathbf{n}(e_i, \hat{f}_i)$) does not change the self-stressability or orientability of C.

(ii) Reversing simultaneously the normals at $f_{i,i+1} \in F$ (namely $\mathbf{n}(e_i, f_{i,i+1})$ and $\mathbf{n}(e_{i+1}, f_{i,i+1})$) does not change the self-stressability or orientability of C.

(iii) Reversing the normal $\mathbf{n}(e_i, \hat{f}_i)$ does not change the self-stressability or orientability of C.

Proof. In all the items we change altogether an even number of normals for all the faces in F. Therefore, orientability is preserved.

The change in (i) does not change the equations of self-stressability, so it preserves self-stressability. For (ii) and (iii) the change of the signs of stresses $s(f_{i,i+1})$ and $s(\hat{f}_i)$ respectively delivers the equivalence of the conditions of self-stressability. $\square$

3.1.4. *Self-stressability of face-cycles of length 3.* Let us consider a trivalent cycle $C(E, F, \hat{F}, \mathbf{n})$ of length 3 with

$$F = \{f_1, f_2, f_3\}, \hat{F} = \{\hat{f}_1, \hat{f}_2, \hat{f}_3\}, E = \{e_1, e_2, e_3\}$$

(for a schematic sketch see Figure 8). Create a new plane $\hat{f}_3'$ by the following Cayley algebra algorithm (see Figure 6):

(i) $g_1 = (e_1 \vee e_2) \wedge \hat{f}_3$,
(ii) $g_2 = f_{2,3} \wedge \hat{f}_1$,
(iii) $g_3 = (g_1 \vee g_2) \wedge f_{3,1}$,
(iv) $g_4 = (g_2 \vee e_1) \wedge (g_3 \vee e_2)$,
(v) $\hat{f}_3' = e_3 \vee g_4$.

In this notation, the stressability conditions are given by the following.

Proposition 3.3. A face-cycle d-framework

$$C\big((e_1, e_2, e_3), (f_{1,2}, f_{2,3}, f_{3,1}), (\hat{f}_1, \hat{f}_2, \hat{f}_3), \mathbf{n}\big)$$

is self-stressable if and only if

$$\dim(\hat{f}_1 \cap \hat{f}_2 \cap \hat{f}_3) = d - 1$$

if C is edge-orientable, and

$$\dim(\hat{f}_1 \cap \hat{f}_2 \cap \hat{f}_3') = d - 1$$

if C is non-edge-orientable, where $\hat{f}_3'$ is constructed as above in step (v) (see also Figure 6).
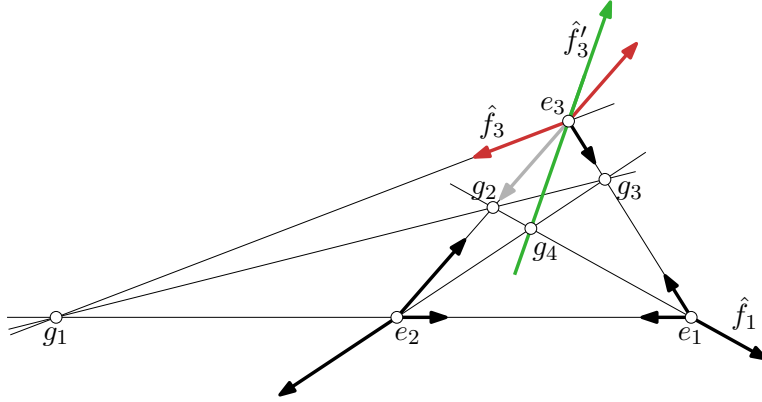
FIGURE 6. The plane $\hat{f}_3'$ can be constructed using Cayley algebra since the two lines $f_{2,3}, f_{3,1}$ separate the two lines $\hat{f}_3, \hat{f}_3'$ harmonically.

Proof. Let us first examine the case where $d = 1$ and $D = 2$, with stresses at all edges of the triangles equal to 1. The triangle in Figure 7 (left) corresponds to an edge-oriented d-framework and therefore $\hat{f}_1 \cap \hat{f}_2 \cap \hat{f}_3$ is not empty. Note that this intersection is empty for the

FIGURE 7. Orientable (*left*) and non-orientable (*right*) self-stressed d-frameworks.

triangle in Figure 7 (right), which corresponds to a non-edge-oriented choice of normals. The condition for non-edge-oriented tensegrities is more complicated as we will see below.

Let us consider a self-stressed trivalent cycle $C(E, F, \hat{F}, \mathbf{n})$ of length 3 (for a schematic sketch see Figure 8). Then let us change the direction of the normal $\mathbf{n}(e_3, f_{2,3})$. Consequently, its orientability changes and the old stress for this new cycle is not a self-stress. However, by changing the d-plane $\hat{f}_3$ to a new d-plane $\hat{f}'_3$ we can resolve the stresses around e_3 again to reobtain a self-stressed framework.
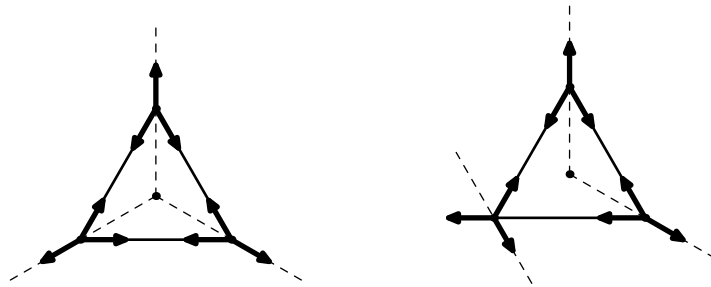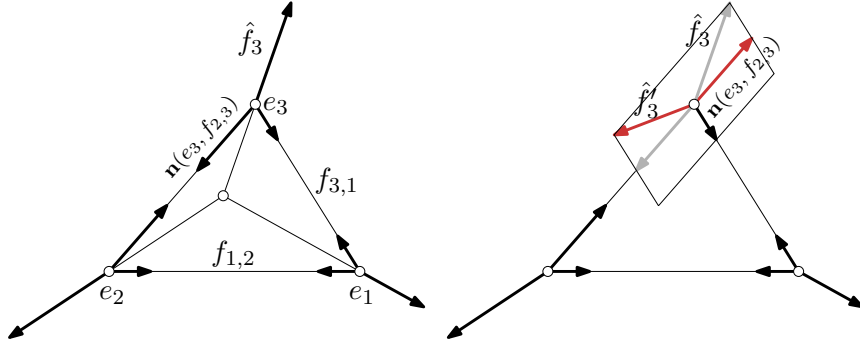

FIGURE 8. Self-stressed orientable and non-orientable face-cycles of length 3. *Left*: An orientable face-cycle. *Right*: Reversing the normal $\mathbf{n}(e_3, f_{2,3})$ yields a non-orientable face-cycle which is still self-stressable after replacing $\hat{f}_3$ by $\hat{f}'_3$.

Let us consider the following two cases in the planar situation as depicted by Figure 8. The classical tensegrity (Figure 8), which corresponds to the orientable case, has the property that the lines $\hat{f}_1, \hat{f}_2, \hat{f}_3$ meet in a point (see, e.g., [6]). Now changing the orientation of $\mathbf{n}(e_3, f_{2,3})$ yields $\hat{f}'_3$ as the new d-plane (see Figure 8 right). From the parallelogram in Figure 8 (right) we derive the condition for the *non-orientable* case. Standard projective geometry implies [11] that the two lines $f_{2,3}, f_{3,1}$ separate the two lines $\hat{f}_3, \hat{f}'_3$ harmonically. This property is characterized by incidence relations of points and lines and therefore expressible in terms of Cayley algebra.

Next we describe how to reduce any dimension d to the above one-dimensional case. Denote by Π the intersection $\Pi = f_{1,2} \cap f_{2,3} \cap f_{3,1}$. Observe that $\dim \Pi = d - 2$. It it well known that self-stressability is a projective invariant, see [18], so we can consider the plane Π to be at infinity.

Fix a two-dimensional plane π orthogonal to e_1, e_2, and e_3 (this is possible since Π is at infinity).

Now the face-cycle d-framework C is a Cartesian product of $\mathbb{R}^{d-3}$ with the two-dimensional tensegrity $\mathcal{F} = C \cap \pi$ in the plane π (see Figure 9). Stresses of $\mathcal{F}$ are in a bijection with the stresses of the

initial tensegrity. So the problem is reduced to the planar situation, i.e., to a 1-framework in the two-dimensional plane.

Let us now consider the edge-orientable case. According to Proposition 3.2 the problem has been reduced to the case of normals in Figure 9. The necessary and sufficient condition in the plane is that the three lines

$$\hat{f}_1 \cap \pi, \qquad \hat{f}_2 \cap \pi, \quad \hat{f}_3 \cap \pi,$$

intersect in one point, say a (see, e.g., in [6]). Therefore C is self-stressable if and only if the three planes $\hat{f}_1$, $\hat{f}_2$, and $\hat{f}_3$ intersect in a common $(d-2)$-plane (i.e., the plane that spans a and Π).
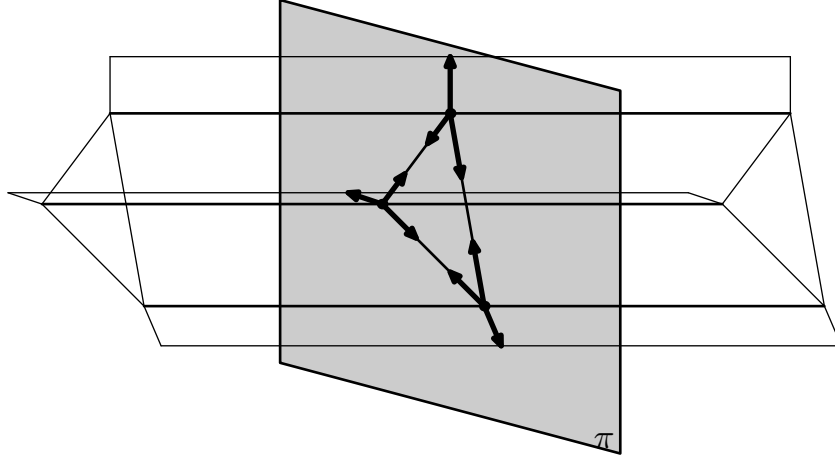
FIGURE 9. A face-cycle d-framework C and the corresponding tensegrity $C \cap \pi$.

The non-edge-orientable case is reduced to the edge-orientable in the following way. Let us make our three-cycle orientable by changing the last normal $n(e_3, f_3)$ (denote the resulting set of normals by $\mathbf{n}'$). In this case, in order to preserve the property of self-stressability condition for at the edge e_3, we should also change the sign of one of the coordinates for the plane $\hat{f}_3$. The resulting plane is the plane $\hat{f}_3'$, whose Cayley algebra expression is described above (see step (v)). Now the stressability of the original non-edge-orientable cycle is equivalent to the stressability of an edge-oriented cycle

$$C\big((e_1, e_2, e_3), (f_{1,2}, f_{2,3}, f_{3,1}), (\hat{f}_1, \hat{f}_2, \hat{f}_3'), \mathbf{n}'\big).$$

This concludes the proof. □

3.2. **HΦ-surgeries.** In this section we discuss HΦ-surgeries and elementary surgery-flips on face-paths d-frameworks and face-cycle d-frameworks which preserve self-stressability in $\mathbb{R}^{d+1}$.

Definition 3.4. Let $1 \leq i \leq n$ be positive integers ($n \geq 4$), and let

$$C = \big((e_1, \ldots, e_n), (f_{0,1}, f_{1,2}, \ldots, f_{n,1}), (\hat{f}_1, \ldots, \hat{f}_n), \mathbf{n})\big)$$

be a face-path (or a face-cycle if $f_{n,1} = f_{0,1}$) d-framework. Denote

$$\hat{f}_i' = \langle f_{i-1,i} \cap f_{i,i+1}, \hat{f}_i \cap \hat{f}_{i+1} \rangle.$$

We say that the $\mathrm{H}\Phi_i$-*surgery* of C is the following face-path (face-cycle) d-framework (see Figure 10)

$$\mathrm{H}\Phi_i(C) = \big((e_1, \ldots, e_{i-1}, \quad f_{i-1,i} \cap f_{i+1,i+2}, \quad e_{i+2}, \ldots, e_n)$$
$$(f_{1,2}, \ldots, f_{i-1,i}, \qquad\qquad\qquad f_{i+1,i+2}, \ldots, f_{n,1});$$
$$(\hat{f}_1, \ldots, \hat{f}_{i-1}, \qquad \hat{f}_i', \qquad \hat{f}_{i+2}, \ldots, \hat{f}_n), \mathbf{n}'\big).$$

The normals $\mathbf{n}'$ coincide with the normals of $\mathbf{n}$ for the same adjacent
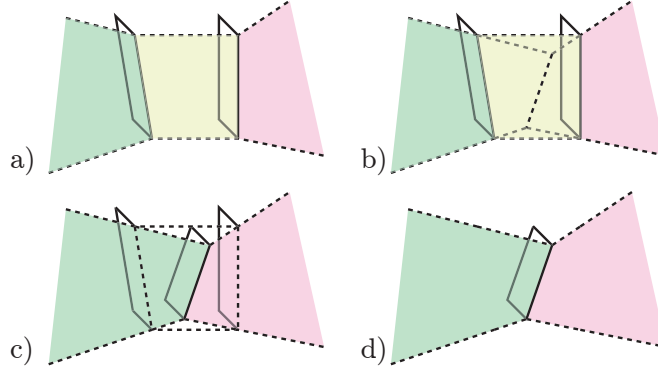
FIGURE 10. An $\mathrm{H}\Phi_i$-surgery.

pairs. We have three extra normals in $\mathbf{n}'$ to the new element in E:

$$\mathbf{n}'(f_{i-1,i} \cap f_{i,i+1}, f_{i-1,i}), \quad \mathbf{n}'(f_{i-1,i} \cap f_{i,i+1}, f_{i,i+1}), \quad \text{and}$$
$$\mathbf{n}'(f_{i-1,i} \cap f_{i,i+1}, \hat{f}_i).$$

The first two are defined by the fact that the cycle:

$$\big((e_i, f_{i-1,i} \cap f_{i,i+1}, e_{i+1}), (f_{i-1,i}, f_{i+1,i+2}, f_{i,i+1}), (\hat{f}_i, \hat{f}_i', \hat{f}_{i+1}), \mathbf{n}'\big)$$

of length 3 is edge-orientable. The orientation of $\hat{f}_i$ does not play any role here (and hence can be chosen arbitrarily).

Alternatively, in terms of Cayley algebra $\hat{f}_i'$ reads

$$\hat{f}_i' = (f_{i-1,i} \wedge f_{i,i+1}) \vee (\hat{f}_i \wedge \hat{f}_{i+1}).$$

In the planar case we have precisely $\mathrm{H}\Phi$-surgeries on framed cycles (i.e., face-cycle 1-frameworks) that were used for the conditions of planar tensegrities (for further details see [6]).

In order to have a well-defined HΦ-surgery, one should consider several simple conditions on the elements of F and $\hat{F}$.

Definition 3.5. We say that an $H\Phi_i(C)$-surgery is *admissible* if
  (i) the planes $f_{i-1,i}$ and $f_{i,i+1}$ do not coincide;
  (ii) the planes $\hat{f}_i$ and $\hat{f}_{i+1}$ do not coincide;
  (iii) the planes $f_{i-1,i} \wedge f_{i,i+1}$ and $\hat{f}_i \wedge \hat{f}_{i+1}$ do not coincide.

For an admissible $H\Phi_i(C)$-surgery we have
$$\dim f_{i-1,i} \cap f_{i+1,i+2} = d - 1;$$
$$\dim \hat{f}'_i = d.$$
$$\dim \hat{f}_i \cap \hat{f}_{i+1} = d - 1.$$
where the last follows from Items (i) and (ii) above. Then by Item (iii) we have $\dim \hat{f}'_i \geq d$. Since the d-planes $f_{i-1,i}$, $f_{i+1,i+2}$, $\hat{f}_i$, and $\hat{f}_{i+1}$ by construction share a $(d-2)$-plane, we have $\dim \hat{f}'_i \leq d$.

Let us distinguish the following elementary surgery-flips.

Definition 3.6. An *elementary surgery-flip* is one of the following surgeries.
  • An admissible $H\Phi_i$-surgery or its inverse.
  • Removing or adding consecutive duplicates at position i. Here we say that we have *a duplicate at position i* if
$$e_i = e_{i+1}, \quad f_{i,i+1} = f_{i+1,i+2}, \quad \text{and} \quad \hat{f}_i = \hat{f}_{i+1}.$$
  • Removing or adding a loop of length 2. Here we say that we have *a simple loop of length 2 at position i* if
$$e_i = e_{i+2}, \quad f_{i,i+1} = f_{i+2,i+3}, \quad \text{and} \quad \hat{f}_i = \hat{f}_{i+2}.$$

Proposition 3.7. Assuming that a surgery is admissible, a face-cycle d-framework C is self-stressable if and only if the face-cycle d-framework $H\Phi_i(C)$ is self-stressable.

Proof. Assume that C has a non-zero self-stress s. Let us show that $H\Phi_i(C)$ has a self-stress.

Consider the face-cycle d-framework
$$C_i = \big((e_i, f_{i-1,i} \cap f_{i,i+1}), (f_{i-1,i}, f_{i+1,i+2}, f_{i,i+1}), (\hat{f}_i, \hat{f}'_i, \hat{f}_{i+1}), \mathbf{n}\big),$$
where
$$\hat{f}'_i = \big\langle f_{i-1,i} \cap f_{i,i+1}, \hat{f}_i \cap \hat{f}_{i+1} \big\rangle.$$
and $\mathbf{n}$ is constructed according Definition 3.4. This face-cycle d-framework admits a self-stress by Proposition 3.3 since three d-planes of (f_i, f'_i, f_{i+1}) intersect in a plane of dimension $d - 2$. Now let us add C_i

to C taking the self-stress s_i which negates the stress at $e_{i,i+1}$. Then the stresses at $\hat{f}_i$ for C and C_i negate each other; and the stresses at $f_{i-1,i}$ for C and C_i coincide. For the same reason the stresses at $f_{i+1,i+2}$ for C and C_i coincide. Therefore, the constructed self-stress is in fact a non-zero self-stress on $H\Phi_i(C)$.

The same reasoning works for the converse statement. In fact adding the C_i to C provides an isomorphism between the space of self-stresses on C and the space of self-stresses on $H\Phi_i(C)$. $\square$

Remark 3.8. It is possible to describe one $H\Phi$-surgery in terms of Cayley algebra. Consider a face-cycle d-framework C with admissible $H\Phi_i(C)$-surgery. We have only one new plane $\hat{f}'_i$ in this case, and its Cayley expression is

$$\hat{f}'_i = (f_{i-1,i} \wedge f_{i,i+1}) \vee (e_i \wedge e_{i+1}).$$

3.3. **Stress transition and stress monodromy.** We will now adapt to our setting the notion of "quality transfer" due to Rybnikov [13].

Definition 3.9. Let Γ be a generic face-path d-framework with starting plane $f_a \in F$ and ending plane $f_z \in F$. Assign some stress s to the first plane. Due to genericity, it uniquely defines the stress on the second face. The stress on the second face uniquely defines the stress on the third face, and so on. So the stress on f_a uniquely defines the stress on f_z. This is called the *stress transition* along the face path.

If $f_a = f_z$, that is, we have a face-cycle, we arrive eventually at some stress s' assigned to f_a again. The ratio $s(f_a)/s(f_z)$ is called the *stress-monodromy* along C. A stress monodromy of 1 is *trivial*.

It is clear that:

Lemma 3.10.   (1) A generic face-cycle is self-stressable if and only if the stress monodromy is trivial.
  (2) The monodromy does not depend on the choice of the first face.
  (3) Reversal of the direction of the cycle takes monodromy m to $1/m$.
  (4) Monodromy behaves multiplicatively with respect to homological addition: the monodromy of the homological sum is the product of monodromies. $\square$

3.4. **Face-path equivalence.** Let us now introduce the notion of equivalent face-path d-frameworks.

Definition 3.11.   Two face-path (face-cycle) d-frameworks Γ_1 and Γ_2 starting from the plane f_a and ending at the plane f_z are *equivalent* if there exists a sequence of elementary surgery-flips taking Γ_1 to Γ_2.

It turns out that equivalent face-path d-frameworks have equivalent stress-transitions.

Proposition 3.12. The stress-transition of two equivalent face-path d-frameworks coincide.

Proof. It is enough to prove this statement for any elementary surgery-flip. In case of HΦ-surgeries we must show that the face-path d-frameworks

$$C = \big((e_1, e_2), (f_{0,1}, f_{1,2}, f_{2,3}), (\hat{f}_1, \hat{f}_2), N\big),$$

and

$$\mathrm{H}\Phi_1(C) = \big((e_3), (f_{0,1}, f_{2,3}), (\hat{f}_3), N'\big)$$

have the same stress-transition (see Figure 11). This is equivalent to
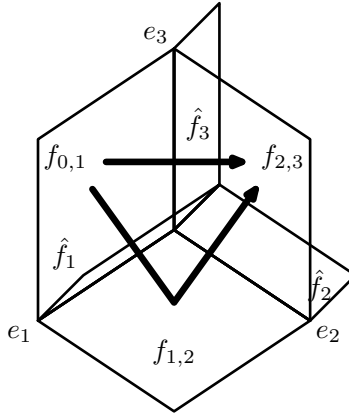
FIGURE 11. An elementary flip.

the fact that the face-cycle d-framework

$$\big((e_1, e_2, e_3), (f_{0,1}, f_{1,2}, f_{2,3}), (\hat{f}_1, \hat{f}_2, \hat{f}_3), \mathbf{n}''\big)$$

(where $\mathbf{n}''$ is as in the cycle of Definition 3.4) has a unit stress-transition (i.e. trivial monodromy or, equivalently, is self-stressable).

By the construction of Definition 3.4 we get that this cycle is edge-orientable, and that the intersection

$$\hat{f}_1 \cap \hat{f}_2 \cap \hat{f}_3 \neq \emptyset.$$

Therefore, by Proposition 3.3 it is self-stressable.

The cases of removing duplicates or loops of length 2 are straight-forward. $\qquad\square$

3.5. **Face-path d-frameworks in d-frameworks.** In this subsection we briefly discuss face-path d-frameworks and face-cycle d-frameworks that are parts of a larger d-framework.

Definition 3.13. Let $\mathcal{F}$ be a trivalent d-framework. Then for every (cyclic) sequence of adjacent d-planes γ we naturally associate a face-path d-framework (face-cycle d-framework) $\Gamma(T, \gamma)$ with

- F is the sequence of the planes spanned by the corresponding d-planes of γ;
- E is the sequence of the intersections of the d-planes of the above F;
- $\hat{F}$ is the sequence of planes of $\mathcal{F}$ that are adjacent to the planes of E and distinct to the faces already considered in F;
- $\mathbf{n}$ is the corresponding sequence of normals defined by the normals of $\mathcal{F}$.

We say that a face-path d-framework (face-cycle d-framework) $\Gamma(T, \gamma)$ is *induced* by γ on $\mathcal{F}$.

Induced face-path and face-cycle d-frameworks have a natural homotopy relation, which is defined as follows.

Definition 3.14.
- Two induced face-path d-frameworks Γ_1 and Γ_2 for $\mathcal{F}$ starting from the plane f_a and ending at the plane f_z are *face-homotopic* if there exists a sequence of elementary surgery-flips taking Γ_1 to Γ_2 and such that after each surgery-flip we have an induced face-path d-framework for $\mathcal{F}$.
- Two face-cycle d-frameworks Γ_1 and Γ_2 are *face-homotopic* if there exists a sequence of elementary surgery-flips taking Γ_1 to Γ_2 and such that after each surgery-flip we have an induced face-path d-framework for $G(M)$.

Finally we formulate the following important property of face-homotopic face-path and face-cycle d-frameworks.

Proposition 3.15. Face-homotopic face-path (face-cycle) d-frameworks have the same stress-transition (stress-monodromy).

Proof. The proof directly follows from Proposition 3.12. $\qquad\square$

4. Geometric Characterizations of self-stressability for Trivalent d-Frameworks

In this section we discuss the practical question of writing geometric conditions for cycles. We characterize self-stressable trivalent d-frameworks in terms of exact discrete multiplicative 1-forms and in

terms of resolvable cycles. Before that we show that a trivalent d-framework is self-stressable if and only if every path and every loop is self-stressable.

Definition 4.1. A face-cycle d-framework is called a *face-loop d-framework* if it contains no repeating planes.

Let us formulate the following general theorem.

Theorem 4.2. *Consider a generic face-connected trivalent d-framework. Then the following three statements are equivalent.*

(i) *$\mathcal{F}$ has a non-zero self-stress (which is in fact non-zero at any d-plane).*
(ii) *For every two d-planes f_a, f_z in $\mathcal{F}$ the stress-transition does not depend on the choice of an induced face-path d-framework on $\mathcal{F}$.*
(iii) *Every induced face-loop d-framework on $\mathcal{F}$ is self-stressable.*

Proof. (ii) $\Leftrightarrow$ (i): Item (i) tautologically implies Item (ii). Let us show that Item (ii) implies Item (i). Fix a starting face f_a and put a stress $s(f_a) = 1$ on it. Expand the stress to all the other faces. By assumption this can be done uniquely. Therefore, this stress is a self-stress. (Indeed, if we do not have the equilibrium condition at some plane e, then at the planes incident to e we have more than one possible stress-transition.)

(ii) $\Rightarrow$ (iii): Indeed any simple face-cycle d-framework on $\mathcal{F}$ can be considered as one long face-path d-framework with $f_a = f_z$. By condition of Items (ii) the stress-transition equals 1, and therefore this face-cycle d-framework is self-stressable. The last is equivalent to Item (iii).

(iii) $\Rightarrow$ (ii): Let us use reductio ad absurdum. Suppose Item (iii) is true while Item (ii) is false. If Item (ii) is false then there exist at least two face-path d-frameworks with the same f_a and f_z where the stress-transitions fail to be the same. Now the union C of the first face-path d-framework and the inverse second is a induced face-cycle d-framework on $G(M)$ with non-unit stress-transition. Let us split C into consecutive loops $C_1, \ldots, C_k$. At least one of them should have a non-unit translation. Therefore, Item (iii) is false as well, a contradiction.

For completeness of the last proof we should add the following two observations regarding cycles of small length. Firstly, the stress-transition remains constant at planes that repeat successively two or more times. This happens due to genericity of $\mathcal{F}$: there are zero contributions from $\hat{f}_i$ in case if $f_{i-1,i} = f_{i,i+1}$. And secondly, if it happens that

$f_{i-1,i} = f_{i+1,i+2}$ then we immediately have $e_i = e_i + 1$ and therefore again the stress-transitions at $f_{i-1,i}$ and at $f_{i+1,i+2}$ coincide.   □

4.1. **Ratio condition for self-stressable multidimensional trivalent frameworks.** In this section we characterize generic trivalent d-frameworks $\mathcal{F}$ with respect to their self-stressability in terms of specific products of ratios. More precisely, we equip each d-framework with a so called discrete multiplicative 1-form which turns out to be exact if and only if the d-framework is self-stressable. Let us start with the definition of discrete multiplicative 1-forms (see, e.g., [1]).

Definition 4.3. A real valued function $q : \vec{E}(G) \to \mathbb{R} \setminus \{0\}$ (where $\vec{E}(G)$ denotes the set of oriented edges of the graph G) is called a *discrete multiplicative* 1-*form*, if $q(-a) = 1/q(a)$ for every $a \in \vec{E}(G)$. It is called *exact* if for every cycle $a_1, \dots, a_k$ of directed edges the values of the 1-form multiply to 1, i.e.,

$$q(a_1) \cdot \ldots \cdot q(a_k) = 1.$$

Now, as a next step we will equip any general trivalent d-framework with a discrete multiplicative 1-form q. However, we will not define q directly on the d-framework but on what we call its dual graph.

Definition 4.4. The vertices of the *dual graph of a d-framework* are the d-dimensional planes and the edges "connect d-dimensional planes" that are sharing a $(d-1)$-dimensional plane.

Consequently, the edges of the dual graph of $\mathcal{F}$ can be identified with triples of successive $(d-1)$-planes $a_i := (e_{i-1}, e_i, e_{i+1})$ (where $e_i \in E$). So let us now equip the dual graph of $\mathcal{F}$ with a discrete multiplicative 1-form. For an illustration see Figure 12.
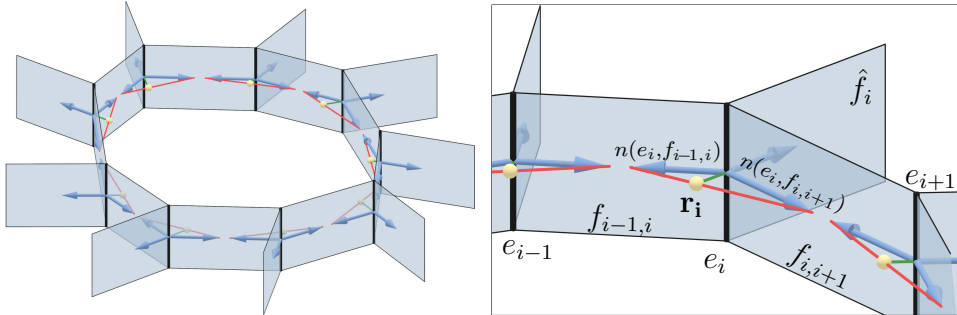

FIGURE 12. Illustration of some edges of the dual graph in a face loop of a d-framework. The values of the discrete multiplicative 1-form $q(a_i)$ (cf. Eqn. (3)) is the affine ratio $q(a_i) = \big(\mathbf{n}(e_i, f_{i-1,i}) - \mathbf{r_i}\big) : \big(\mathbf{r_i} - \mathbf{n}(e_i, f_{i,i+1})\big)$.

As our d-framework is trivalent the $(d-1)$-plane e_i is contained in three d-planes $f_{i-1,i}, f_{i,i+1}, \hat{f}_i$ and therefore the corresponding normals $\mathbf{n}(e_i, f_{i-1,i})$, $\mathbf{n}(e_i, f_{i,i+1})$, $\mathbf{n}(e_i, \hat{f}_i)$ are linearly dependent, i.e., lie in a 2-plane. This together with the fact that the d-framework is generic implies that there are $\lambda_{i-1,i}, \lambda_{i,i+1}, \hat{\lambda} \in \mathbb{R} \setminus \{0\}$ such that

$$(2) \qquad \lambda_{i-1,i}\mathbf{n}(e_i, f_{i-1,i}) + \lambda_{i,i+1}\mathbf{n}(e_i, f_{i,i+1}) + \hat{\lambda}\mathbf{n}(e_i, \hat{f}) = 0.$$

Now we are in position to define our discrete multiplicative 1-form on the oriented dual graph by

$$(3) \qquad q(a_i) = q(e_{i-1}, e_i, e_{i+1}) := \frac{\lambda_{i,i+1}}{\lambda_{i-1,i}},$$

since clearly $q(-a_i) = 1/q(a_i)$ is fulfilled. The geometric meaning of $q(a_i)$ is the following (cf. Figure 12 right). Denote by $\mathbf{r_i}$ the intersection point of the straight line with direction $\mathbf{n}(e_i, \hat{f})$ and intersect it with the line through $\mathbf{n}(e_i, f_{i-1,i})$ and $\mathbf{n}(e_i, f_{i,i+1})$. A simple computation shows

$$\mathbf{r_i} = \frac{\lambda_{i-1,i}}{\lambda_{i-1,i} + \lambda_{i,i+1}}\mathbf{n}(e_i, f_{i-1,i}) + \frac{\lambda_{i,i+1}}{\lambda_{i-1,i} + \lambda_{i,i+1}}\mathbf{n}(e_i, f_{i,i+1}).$$

Thus $q(a_i)$ is the affine ratio of the three points $\mathbf{n}(e_i, f_{i-1,i}), \mathbf{r_i}, \mathbf{n}(e_i, f_{i,i+1})$, i.e., $q(a_i) = \big(\mathbf{n}(e_i, f_{i-1,i}) - \mathbf{r_i}\big) : \big(\mathbf{r_i} - \mathbf{n}(e_i, f_{i,i+1})\big)$.

With that definition of a discrete multiplicative 1-form we can now characterize self-stressable d-frameworks.

Theorem 4.5. *A generic trivalent d-framework is self-stressable if and only if the discrete multiplicative 1-form defined by (3) is exact.*

Proof. Suppose the d-framework has a self-stress s. Therefore Equation (1) implies

$$s(f_{i-1,i})\mathbf{n}(e_i, f_{i-1,i}) + s(f_{i,i+1})\mathbf{n}(e_i, f_{i,i+1}) + s(\hat{f})\mathbf{n}(e_i, \hat{f}) = 0.$$

Comparison with Equation (2) implies that the coefficients in both equations are just a multiple of each other, i.e.,

$$(s(f_{i-1,i}), s(f_{i,i+1}), s(\hat{f})) = \mu(\lambda_{i-1,i}, \lambda_{i,i+1}, \hat{\lambda}).$$

Consequently, the value of the discrete multiplicative 1-form is the ratio of neighboring stresses:

$$q(a_i) = \frac{\lambda_{i,i+1}}{\lambda_{i-1,i}} = \frac{s(f_{i,i+1})}{s(f_{i-1,i})}.$$

Therefore it is easy to see that the product of values $q(a_i)$ along any closed loop in the dual graph multiplies to 1:

$$q(a_1) \cdot \ldots \cdot q(a_k) = \frac{s(f_{1,2})}{s(f_{k,1})} \frac{s(f_{2,3})}{s(f_{1,2})} \cdot \ldots \cdot \frac{s(f_{k-1,k})}{s(f_{k-2,k-1})} \frac{s(f_{k,1})}{s(f_{k-1,k})} = 1.$$

Now conversely, let us assume that the discrete multiplicative 1-form q is exact. By Theorem 4.2 it is sufficient to show that each loop of the form $e_1, \ldots, e_k$ of $(d-1)$-planes is self-stressable. Choose an arbitrary stress $s(f_{1,2}) \in \mathbb{R} \setminus \{0\}$ for the first d-plane. Equation (1) and the d-framework being generic then uniquely determines the stresses of the two other d-planes incident to e_1, that is, $s(f_{k,1})$ and $s(\hat{f}_1)$. Continuing, determining stresses this way defines all stresses along the loop including the last stress that we now denote by $\tilde{s}(f_{k,1})$ because it was defined before. However, the exactness of q gives

$$
\begin{aligned}
1 &= q(a_1) \cdot \ldots \cdot q(a_k) = \frac{\lambda_{1,2}}{\lambda_{k,1}} \frac{\lambda_{2,3}}{\lambda_{1,2}} \cdot \ldots \cdot \frac{\lambda_{k-1,k}}{\lambda_{k-2,k-1}} \frac{\lambda_{k,1}}{\lambda_{k-1,k}} \\
&= \frac{s(f_{1,2})}{s(f_{k,1})} \frac{s(f_{2,3})}{s(f_{1,2})} \cdot \ldots \cdot \frac{s(f_{k-1,k})}{s(f_{k-2,k-1})} \frac{\tilde{s}(f_{k,1})}{s(f_{k-1,k})} \\
&= \frac{\tilde{s}(f_{k,1})}{s(f_{k,1})},
\end{aligned}
$$

so $\tilde{s}(f_{k,1}) = s(f_{k,1})$. Consequently, we can consistently define a non-zero stress. $\qquad\square$

4.2. **Cayley algebra conditions.** Let us start with the following important definition.

Definition 4.6.
- A face-cycle d-framework of length 3 is *in general position* if all 6 planes in the sequences F and $\hat{F}$ are pairwise distinct.
- A face-cycle d-framework is *resolvable* if there exists a sequence of HΦ-surgeries transforming it to a face-cycle d-framework of length 3 in general position.
- A d-framework is *resolvable* if all its simple induced face-cycle d-frameworks are resolvable.

We continue with the following definition.

Definition 4.7 (Cayley algebra condition for a single face-cycle resolvable d-framework). Consider a resolvable face-cycle d-framework

$$C = (E, F, \hat{F}, \mathbf{n})$$

and any sequence of HΦ-surgeries transforming it to a face-cycle d-framework

$$C' = (E', F', \hat{F}', \mathbf{n}')$$

of length 3 in general position.

- Let us write all elements of F' and $\hat{F}'$ in C' as Cayley algebra expressions of the elements of F and $\hat{F}$ in C. The resulting expressions are compositions of expressions of Remark 3.8.
- Finally, we use the dimension condition of Proposition 3.3 for C' to determine if C' is stressable or not.

The composition of the above two items gives an existence condition for nonzero self-stresses on C. We call this condition a *Cayley algebra geometric condition* for C to admit a non-zero self-stress.

Note that one can write distinct Cayley algebra geometric condition for C using different sequences of HΦ-surgeries transforming C to a face-cycle d-framework of length 3 in general position.

As we have already mentioned in the above definition the Cayley algebra geometric conditions detect self-stressability on cycles. So we can write self-stressability conditions for a general trivalent resolvable d-framework. Namely we have the following theorem.

Theorem 4.8. *Let $\mathcal{F}$ be a trivalent resolvable d-framework and let further $C_1, \ldots, C_n$ be all pairwise non-face-homotopic face-loop d-frameworks on $\mathcal{F}$. Then $\mathcal{F}$ has a self-stress if and only if it fulfills Cayley algebra geometric conditions for cycles $C_1, \ldots, C_n$ as in Definition 4.7.*

Proof. First of all the self-stressability $C_1, \ldots, C_n$ is equivalent to self-stressability of all face-loops on $\mathcal{F}$. It follows directly from definition of face-homotopic paths. Hence by Theorem 4.2 the self-stressability of all $C_1, \ldots, C_n$ is equivalent to self-stressability of $\mathcal{F}$ itself.

Finally the geometric conditions for face-loop d-frameworks C_i ($i = 1, \ldots, n$) are described in Definition 4.7. $\qquad\square$

Corollary 4.9. Each realization of K_5 in $\mathbb{R}^3$ is self-stressable. Here we mean a realization of a 2-framework associated with K_5, in the spirit of Example 2.4. Namely, the edges are all the edges of K_5, and faces are all the associated triangles.

Indeed, each face-loop in K_5 is face-homotopic to a face-loop of length three. For them, the stressability condition is automatic. Another argument of stressability of K_5 will appear later in Remark 5.11 as a consequence of Theorem 5.10.

5. R-frameworks and their self-stressability. Examples.

In this section we work in the settings of Rybnikov's papers [13, 14].

5.1. **R-frameworks.** Informally, R-frameworks are PL (piecewise linear) realizations of CW-complexes in $\mathbb{R}^{d+1}$. To make this precise, let us start with a reminder about CW complexes.

A finite CW-complex is constructed inductively by defining its skeleta (for details see, e.g., [5]). The zero skeleton sk_0 is a finite set of points called *vertices*. Once the $(k-1)$-skeleton sk_{k-1} is constructed, a finite collection of closed k-balls B_i (called *cells*) is attached by some continuous mappings $\phi_i : \partial B_i \to sk_{k-1}$. The images of B_i in the complex are called *closed cells*.

Definition 5.1. A *regular* CW-complex is a CW-complex such that
  (i) For each k-cell B_i, the mapping ϕ_i is a homeomorphism between ∂B_i and a subcomplex of the skeleton sk_{k-1}.
  (ii) The intersection of two closed cells is either empty or some single closed cell of this CW-complex.

Let M be a regular finite CW-complex with no cells of dimension greater than d.

Its faces of dimension d will be called *d-faces*. The $(d-1)$-faces are called the *d-edges*. The $(d-2)$-faces are called the *d-vertices*.

Example 5.2. Let $\overline{M}$ be a regular finite CW-complex whose support[1] $|\overline{M}|$ is a connected $(d+1)$-manifold, either closed or with boundary. Let M be its d-skeleton.

In this setting we also have cells of $\overline{M}$ of dimension $d+1$. These will be called *chambers*.

Definition 5.3. Assume that a mapping $p : \text{Vert}(M) \to \mathbb{R}^{d+1}$ is such that the image of the vertex set of each k-cell spans some affine k-plane.

We say that p *realizes* M in $\mathbb{R}^{d+1}$; we also say that the pair (M, p) is a *realization* of M, or a *Rybnikov-framework*, or *R-framework*, for short.

Notation: given a cell $f \in M$, we abbreviate the image of the vertex set $p(\text{Vert}(F))$ as $p(f)$ and denote by $\langle p(f) \rangle$ its affine span.

Definition 5.4. An R-framework is *generic* if, whenever two d-faces f_1 and f_2 share a d-edge, then $\langle p(f_1) \rangle \neq \langle p(f_2) \rangle$.

---

[1]The support of a CW-complex is the topological space represented by the complex. That is, one forgets the combinatorics and leaves the topology only.

From now on we assume that all R-frameworks we deal with are generic.

As before, we say that an R-framework is *trivalent* if each d-edge is incident to exactly three d-faces.

An R-framework is $(3, 4)$-*valent* if each d-vertex is incident to exactly four d-edges (and therefore, to six d-faces).

Informal remark: by construction, vertices are mapped to points. One may also imagine that the 1-cells of the complex are mapped to line segments. Therefore 2-cells are mapped to some closed planar broken lines (polygons). Here self-intersections may occur. As the dimension of faces grows, the more complicated the associated geometrical object is.

However, there exist nice examples with convex polyhedra as images of the faces. In particular, if a face is a (combinatorial) simplex, one may think that its image is a simplex lying in $\mathbb{R}^d$.

Example 5.5.
  (i) *The Schlegel diagram* [20] of a convex $(d + 2)$-polytope K is a realisation of the boundary complex of K.
  (ii) More generally, the projection of a $(d+2)$-dimensional polyhedral body K (that is, of a body with piecewise linear boundary) to $\mathbb{R}^{d+1}$ yields a realization (M, p) where $|M|$ is homeomorphic to ∂K.

5.2. **Self-stresses and liftings.** Now we turn to a particular notion of stresses, which is borrowed from Rybnikov's paper [13] and represents a special case of Definition 2.1. The principal difference is that in Rybnikov's setting the choice of normal vectors **n** is dictated by the R-framework.

As Examples 2.4 and 5.5 show, in certain cases, a realization (M, p) represents all the faces as convex polytopes. Let us call such a realization *non-crossing*. Otherwise, we say that the realization is *self-crossing*.

Let us start by introducing stresses for the **non-crossing version**:

Definition 5.6 (cf. [14])**.** Assume that a non-crossing realization (M, p) is fixed. Let us assign to each pair (f, e) where e is a d-edge contained in a d-face f a unit normal $\mathbf{n}(e, f)$ to $p(e)$ pointing inside the convex polytope $p(f)$.

A real-valued function s defined on the set of d-faces is called a *self-stress* if at each d-edge e of the complex,

$$\sum_{f \supset e} s(f)\mathbf{n}(e, f) = 0. \quad (*)$$

To relax the non-crossing condition, let us make some preparation following [14]. The informal idea is to triangulate the faces of the complex, since the representation of a simplex is never self-crossing.

Pick a (combinatorial) orientation of each of the cells of M, and a (combinatorial) triangulation of M without adding new vertices. So each d-face now is replaced by a collection of (combinatorial) simplices. The realization (M, p) yields a realization $(\overline{M}, p)$ of the new CW-complex.

Definition 5.7 (cf. [14]). Assume that a generic realization (M, p) is fixed. Choose a triangulation $(\overline{M}, p)$ as is described above.

For a d-edge $e \in \overline{M}$ and a d-face f containing g, choose $\mathbf{n}(e, f)$ to be the unit normal to the oriented cell f at its simplicial face g whose orientation is induced by the orientation of f.

A real-valued function s on the set of d-cells of M is called a *self-stress* if for every d-edge e of $\overline{M}$, the condition $(*)$ is fulfilled.

This definition is proven to be independent on the choice of the combinatorial triangulation and also on the choice of the orientations of the faces.

The notion of stressed realizations has the following physical meaning. One imagines that the d-faces are realized by planar soap film. The faces are made of different types of soap, that is, with different physical property. Each of the faces creates a tension, which should be equilibrium at the d-edges. The tension is always orthogonal to the boundary of a face and lies in the affine hull of the face. A self-intersecting face produces both compression and tension as is depicted in Figure 13.
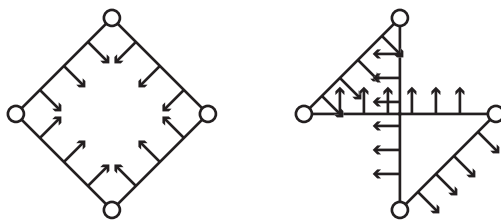

FIGURE 13. Tensions for a convex quadrilateral (left) and a self intersecting quadrilateral (right).

We say that a R-framework (M, p) is *self-stressable* whenever there exists a non-zero stress.

Proposition 5.8. The following two statements hold.

(i) Each R-framework yields a d-framework $(E, I, F, \mathbf{n})$ which agrees with the Definition 2.1. The incidences are dictated by combinatorics of M. Its self-stressability agrees with the Definition 2.2 $\qquad\square$

(ii) For $d = 1$, an R-framework is a planar realization of some graph. Its self-stressability agrees with the classical notion of self-stresses of graphs in the plane. $\qquad\square$

Assume now that M is the d-skeleton of some $(d+1)$-dimensional manifold $\overline{M}$, that is, the chambers are well-defined.

Definition 5.9. A *lift* of $(\overline{M}, p)$ is an assignment of a linear function $h_C : \mathbb{R}^d \to \mathbb{R}$ to each chamber C. By definition, a lift satisfies the following: whenever two chambers C and C' share a d-face f, the restrictions of h_C and $h_{C'}$ on the affine span $\langle p(f) \rangle$ coincide.

A lift is non-trivial if (at least some of) the functions h_C are different for different chambers.

Let us fix some chamber C. Lifts that are identically zero on C form a linear space $\mathrm{Lift}(\overline{M}, p)$.

Theorem 5.10 (cf. [14]). *Let M be the d-skeleton of some $(d+1)$-dimensional manifold $\overline{M}$.*

(i) *If the first homology group of $\overline{M}$ vanishes, that is,*

$$H_1(M, \mathbb{Z}_2) = 0,$$

then the linear spaces $\mathrm{Lift}(\overline{M}, p)$ and the space of self-stresses $\mathrm{Stress}(M, p)$ are canonically isomorphic.

(ii) *Liftability of $(\overline{M}, p)$ implies self-stressability of (M, p).* $\qquad\square$

Remark 5.11. The theorem gives another proof of Corollary 4.9 Indeed, each realization of K_5 can be viewed as a projection of a four-dimensional simplex. In other words, it is liftable, and hence stressable.

Each d-vertex v of a R-framework yields in a natural way a spherical framework via the following algorithm:

(1) We may assume that each face is a simplex, otherwise triangulate the faces.

(2) Take an affine h which is orthogonal to the affine span $\langle p(v) \rangle$. Clearly, we have $dim h = 3$.

(3) Take a small sphere S^2 lying in the plane e and centered at the intersection point $O = h \cap \langle p(v) \rangle$.

(4) For each d-face f incident to v, take the projection $Pr_h(f)$ to the plane h and the intersection $Pr_h(f) \cup S^2$. Since f is a simplex, the intersection is a geodesic arch.

This yields a framework $\mathcal{S}_v$ placed in the sphere S^2. Self-stressability of spherical graphs is well understood since it reduces to self-stressability of planar graphs (see [4, 10, 16]). A face loop is called *local with respect a d-vertex v* if all the d-faces and d-edges participating in the path are incident to v.

Lemma 5.12. The two statements are equivalent:

(i) The stress monodromy of each local (writh respect to some d-vertex v) face loop is trivial.
(ii) The spherical framework $\mathcal{S}_v$ is self-stressable.

Proof. Triviality of any local stress monodromy implies that stresses can be assigned to the faces incident to v in such a way that locally the equilibrium condition holds.

The same stress assignment gives a self-stress of $\mathcal{S}_v$, and vice versa. $\square$

Example 5.13. If a d-vertex v has exactly four incident d-edges, then $\mathcal{S}_v$ is stressable. Indeed, in this case $\mathcal{S}_v$ is a K_4 placed on the sphere, which is always stressable.

Theorem 5.14. *Assume that $\mathcal{R}$ is a trivalent R-framework.*

(i) *$\mathcal{R}$ is self-stressable iff for each face loop the stress monodromy is trivial.*
(ii) *$\mathcal{R}$ is self-stressable iff the two conditions hold:*
  (a) *For each vertex v, the induced spherical framework $\mathcal{S}_v$ is self-stressable.*
  (b) *For some generators $g_1, \ldots, g_k$ of the first homolog group $H_1(\mathcal{R})$, and some collection of representatives $\gamma_1, \ldots, \gamma_k$ that are face-cycles, all the stress monodromies are trivial. (One representative for one generator).*
(iii) *In particular, if $\mathcal{R}$ is one-connected, its self-stressability is equivalent to self-stressability of $\mathcal{S}_v$ for all the d-vertices.*

Proof. (i) Take any face, assign to it any stress, and extend it to other faces. Triviality of the monodromy guarantees that no contradiction will arise.

(ii) Keeping in mind Lemma 3.10, observe that any face loop is a linear combination of $\gamma_1, \ldots, \gamma_k$ and some local face loops. By Lemma 5.12, the monodromies of local loops are trivial. It remains to apply (i). Now (iii) follows. $\square$

5.3. **Some examples.** Let us start by two elementary examples.

(1) Take the Schlegel diagram of a 3-dimensional cube (that is the projection of the edges of a cube). It is a trivalent graph in the plane. It is self-stressable since it is liftable by construction.

However, one easily can redraw it keeping the combinatorics in such a way that the realization is no longer self-stressable. For this, it is sufficient to generically perturbe the positions of the vertices.

(2) Now let us work out an analogous example in $\mathbb{R}^3$. Take the Schlegel diagram of a 4-dimensional cube (that is, the projections of all the 2-faces). It is a $(3,4)$-valent R-framework. It is self-stressable since it is liftable by construction. But, unlike (1), by Theorem 5.14, any other its realization is self-stressable.

The above examples suggest general questions:

Are all trivalent R-frameworks self-stressable? Are all $(3,4)$-valent R-frameworks self-stressable?

The answer is negative, which is demonstrated through the below example, which is interesting for its own sake.

Example 5.15. Take a prism P in $\mathbb{R}^3$, a pyramid over P, and the projection of the 2-skeleton of the pyramid back to $\mathbb{R}^3$. We obtain a R-framework $\mathcal{R}$ which is defined with some freedom: firstly, one may alter the position of the vertex of the pyramid, and secondly, one may apply a projective transform to P.

Lemma 5.16. The R-framework $\mathcal{R}$ is self-stressable, and the space of stresses is one-dimensional.

Proof. The R-framework $\mathcal{R}$ is a projection of the 2-skeleton of 4-dimensional tetrahedron. In other words, $\mathcal{R}$ is liftable, and therefore, self-stressable. Since it is trivalent, the space of stresses is at most one-dimensional. $\square$

The prism P has two disjoint triangular faces. They are also faces of $\mathcal{R}$; let us call them *green*. The edges of these faces are also called *green*. The faces of $\mathcal{R}$ that are not green are called *white*. The edges that are not green are called *white*. Fix also one of the white faces, let us call it the *test face* for $\mathcal{R}$.

It is easy to check that any two of white faces of $\mathcal{R}$ are connected by a face path which uses white edges only.

Main construction, first step:

Take two copies of $\mathcal{R}$, say, $\mathcal{R}_1$ and $\mathcal{R}_2$ such that one of the green faces of $\mathcal{R}_1$ coincides with a green face of for $\mathcal{R}_2$. Patch $\mathcal{R}_1$ and $\mathcal{R}_2$ along these faces, and eliminate these green faces. Define the result by $\mathcal{R}_{12}$.

Lemma 5.17. $\mathcal{R}_{12}$ is self-stressable, and the space of the self-stresses has dimension 1.

Proof. Self-stressability: Take a self-stress s_1 of $\mathcal{R}_1$ and a stress s_2 of $\mathcal{R}_2$ such that $s_1 - s_2$ on the patched green faces vanishes. Then $s_1 - s_2$ represents a non-trivial stress of $\mathcal{R}_{12}$.

Dimension one: Consider a stress s on $\mathcal{R}_{12}$. Take a stress s_1 of $\mathcal{R}_1$ which agrees with s on the test face of F_1 and a stress s_2 of $\mathcal{R}_2$ which agrees with s on the test face of $\mathcal{R}_2$. Take $s - s_1 - s_2$. It is a stress on $\mathcal{R}_{12}$ plus the green face which is zero everywhere on $\mathcal{R}_{12}$, except, may be, the green face, which possible only if $s - s_1 - s_2$ vanishes everywhere. Therefore, each stress of $\mathcal{R}_{12}$ is a linear combination of two stresses of $\mathcal{R}_1$ and $\mathcal{R}_2$ that cancel each other on the green face. $\square$

Main construction, second step: We proceed in the same manner: we take one more copy of $\mathcal{R}$, which is called $\mathcal{R}_3$ and patch it to $\mathcal{R}_{12}$ along green faces, and eliminate the green face which was used for the patch. We get $\mathcal{R}_{123}$. Analogously, we have:

Lemma 5.18. R_{123} is self-stressable, and the space of the stresses has dimension 1. $\square$

Main construction, next steps: Now we have a chain of three copies of $\mathcal{R}$ patched together. Only two green faces survive. By adjusting the shapes of the components, we may assume that these green faces coincide. Patch the last two green faces and remove them. After that, we get a R-framework $\widetilde{\mathcal{R}}$. Generically we have:

Lemma 5.19. $\widetilde{\mathcal{S}}$ is not self-stressable, but it is locally self-stressable, that is $\mathcal{S}_v$ is self-stressable for each vertex v.

Proof. Non-self-stressability: Before the last step, the space of stresses was one-dimensional. After the last step (=after patching two last green faces), the dimension can only drop. Let us patch back the two last green faces and get a framework $\widetilde{\mathcal{R}}'$.

Assume $\overline{\mathcal{R}}$ is self-stressable. This means that $\widetilde{\mathcal{R}}'$ has a stress which sums up to zero on the last two green faces. By the above lemmata, the value of the stress on the test face of $\mathcal{R}_1$ uniquely defines the stress on the first green face and the stress on the test face of $\mathcal{R}_2$ (and all the faces of $\mathcal{R}_1$ and $\mathcal{R}_2$), this uniquely defines the stress on the test face of $\mathcal{R}_3$, and so on. We conclude that the stress on the second green face is also uniquely defined, and generically, the green stresses do not cancel each other.

Local self-stressability follows from the fact that putting back any of the green faces creates a self-stressable R-framework. $\square$

Let us observe that $\widetilde{\mathcal{R}}$ is trivalent, but not $(3, 4)$-valent. Each of the green edges has four incident faces, and there are vertices of valency

higher that 4. However one can prove that a number of local surgeries turns it to a $(3, 4)$-valent R-framework.

Eventually we arrive at a $(3, 4)$ R-framework which is locally self-stressable, but not globally self-stressable.

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