# Estimate Based Dynamic Implementation*

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September 17, 2015

**Abstract**

This paper introduces an $\varepsilon$-efficient mechanism in a setting with sequentially arriving agents who hold informative signals about future types. To reveal the information the principal organizes betting on future type reports. An agent's betting reward depends on how accurately the prior updated upon his report predicts the type reports observed in the following period. The mechanism satisfies participation constraints and generates no deficit after any reported history.

JEL: D61, D71, D82, D84

Keywords: Dynamic mechanisms, scoring rule, Bayesian learning, conjugate priors.

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*This paper is based on Chapter 3 of my doctoral dissertation at Toulouse School of Economics. I am grateful to Andrea Attar, Alex Gershkov, Alia Gizatulina, Ioanna Grypari, Martin Hellwig, Johannes Hörner, Rida Laraki, David Martimort, Benny Moldovanu, Tristan Tomala and Philipp Weinschenk, as well as seminar participants in Edinburgh, Paris and Bonn, whose comments and questions helped improve this paper. The financial support by the Max Planck Society and Toulouse School of Economics are gratefully acknowledged.

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# 1 Introduction

This study is motivated by a number of practical applications of dynamic mechanism design, where the principal lacks information about the future type distributions. In industrial regulation, pollution control or auctions such information is necessary to evaluate the intertemporal trade-offs, but often unavailable to the principal. Due to Mezzetti (2004), we know that full revelation of future-related information is achieved by delaying incentive payments until the ultimate resolution of uncertainty. However in practice payments may not possibly be delayed until the last stage in mechanisms that run for a very long time. Gershkov and Moldovanu (2009) are the first to study the problem of principal's learning under this timing constraint.[1] They look at a setting where the present agents' payoff types are informative of the future, therefore the principal can make inferences from the current type reports. This paper takes a different approach and assumes that the information about the future is independent of agents' own preferences. In this case, information cannot be revealed by a direct mechanism, where only payoff types are reported. By constructing an indirect *estimate based* mechanism I show that dynamic efficiency can be $\varepsilon-$implemented, if transfers can be delayed only until the next cohort's arrival.

Formally I consider a discrete time, finite horizon setting and agents who arrive sequentially in cohorts and live for two periods.[2] A member of cohort $t$ derives utility from allocation at $t$; in period $t+1$ he receives monetary transfers, and his total utility is the sum of both components. As standard in this literature, the agent has private information composing his *type*. Here, type has two indepen-

[1]Dynamic populations have also been extensively studied in the literature often referred to as online mechanism design: see the seminal works by Lavi and Nisan (2004) and Parkes and Singh (2003). Their approach is non-Bayesian, and therefore the problem of principal's learning does not emerge.

[2]The population is thus dynamic with static private information, similarly to Lavi and Nisan (2004), Parkes and Singh (2003), and Said (2012). Another strand of literature studies persistent population with dynamic types: Athey and Segal (2013), Bergemann and Välimäki (2010). Cavallo, Parkes and Singh (2009) study the general setting with arbitrary dynamic population and dynamic types.

dent components: (i) a one-dimensional preference parameter, referred to as the *payoff type*, and (ii) a multidimensional signal drawn from the future payoff type distribution, referred to as the *hyperbelief type*.[3] The payoff type pins down the agents' own utility function, while the hyperbelief type pins down the player's information about the future payoff types.

The parametric class of payoff type distributions is fixed, however the exact distribution is unknown. Let $p_t$ be the probability distribution with an unknown parameter $\alpha_t$ and $\Phi_t$ be a class of (hyper-)distributions over the possible values of $\alpha_t$. A hyperbelief type is then drawn from a future $p_t$ which leads to an update $\phi_t \in \Phi_t$. I assume that the class $\Phi_t$ is *conjugate* to $p_t$, meaning that all updated hyperdistributions belong to the same class. Conjugate classes have been found for most standard distributions, see Raiffa and Schlaifer (1961).

The estimate based mechanism introduced here features a two-part transfer aiming to elicit both components of type. Firstly, to reveal the payoff type I use a transfer of Vickrey-Clarke-Groves type (1961; 1971; 1973). The VCG transfer equals the externality imposed by the agent's payoff-type report on the current and the future generations, where the future welfare is taken in expectation conditional on the entire history of reported hyperbeliefs. To this end, I follow the existing literature on dynamic implementation, such as Bergemann and Välimäki (2010); Pavan, Segal and Toikka (2010).[4]

The second part of the transfer, novel to dynamic implementation, is the scoring reward used for the verification of signals. It is a function of agent's report of hyperbelief type (signal) and the next cohort's type reports. More precisely, the agent receives a (negative) transfer equal to the log-likelihood of the profile of type reports in the following period, where the likelihood is evaluated according

[3]The term *hyperbelief* comes from *hyperdistribution*, a term borrowed from the theory of conjugate priors used in the present analysis (see Raiffa and Schlaifer (1961)). It should not be confused with higher-order belief as it describes the probabilistic view of the future environment, and not of other agents' beliefs.

[4]In a setting with persistent population and dynamic types, Athey and Segal (2013) design a dynamic extension of the expected externality mechanism d'Aspremont, Cremer and Gerard-Varet (2004) satisfying the exact budget balance, unlike the VCG.

to the distribution updated conditionally on *his* report. Reporting the hyperbelief to the mechanism is thus similar to placing a bet on the next period type reports. Given that signals and payoff-types are drawn independently, matching the empirical distribution of payoff types is equivalent to matching the empirical distribution of signals. The expected value of such scoring transfer is maximized when the hyperbelief types are reported truthfully. In static implementation mutual verification of reports has been used in the literature stemming from Crémer and McLean, 1985 – notably, Johnson, Pratt and Zeckhauser (1990), and Miller et al. (2007).[5] Several recent papers expand the use of scoring rules to various settings (Postlewaite and McLean (2015); Chambers and Lambert (2014)).

The estimate based mechanism features a linear combination of the VCG and scoring transfers. This guarantees $\varepsilon-$implementation of the efficient outcome in the following sense. For any given value $\varepsilon > 0$, the principal can scale up the scoring transfer such that truthful revelation of payoff *and* hyperbelief types is an $\varepsilon-$equilibrium: No agent can gain more than $\varepsilon$ in deviation from truth-telling. Moreover, since all possible deviations are in the neighborhood of truth-telling, the present concept of $\varepsilon-$implementation is stronger than implementation in $\varepsilon-$equilibrium. In this sense, $\varepsilon-$implementation is $\varepsilon-$close to exact implementation: best replies are in the neighborhood of truth-telling, therefore the allocation is arbitrary close to welfare maximization. The exact implementation cannot be achieved in a general setup with continuous types and allocations because the belief report marginally affects the present cohort's welfare; loosely speaking, there are second-order gains and first-order losses to misreporting. When the type space is discrete, the estimate based mechanism achieves exact implementation.

To complete the analysis, I show that the budget of bets can be exactly balanced without change to incentives. Assume that each cohort includes at least two

[5] Crémer and McLean (1985, 1988) were the first to design a mechanism with mutual report verification in a static setting. Their mechanism punishes for reports that appear contradictory given the known correlation between types, and thus reveals the types perfectly. McAfee and Reny (1992) provide an extension of the Crémer-McLean mechanism to continuous type spaces.

participants, and define an arbitrary derangement of the set of players in the cohort. The derangement defines cycles of payments between the agents in a cohort, such that the agents pay their betting rewards to each other, and the exact budget balance in bets is achieved in every period. The resulting transfer, the *balanced scoring* transfer, satisfies the individual rationality constraint. The intuition for this is that the ex ante distribution of signals is the same for all agents; therefore, the expected difference in betting rewards is zero. It follows that the entire *balanced* mechanism that comprises the VCG and the balanced scoring transfers requires no external funding at any period of time (generates no deficit after any history) and satisfies the participation constraint.

The rest of the paper is organized as follows. Section 2 considers the allocation of pollution permits in a simple setting with two periods. In this example the comparison of reports within one period (as in Crémer and McLean, 1985) fails to produce truthtelling as a unique and undominated equilibrium; therefore delayed verification is introduced. Section 3 presents a more general setting with an arbitrary, although finite number of periods. Section 4 presents the estimate based mechanism. Two subsections study two components of the transfer: VCG (4.1) and the scoring payment (4.2). In 4.1 I show that the VCG transfer induces the revelation of payoff-type, if the current and the future hyperbelief-types are truthfully reported (Lemma 1). In 4.2 I show that the scoring transfer alone ensures that hyperbelief types are truthfully reported, if the future payoff-type reports are truthful (Lemma 2). 4.3 uses the above results to show that truthful revelation of the entire type can be achieved in $\varepsilon-$equilibrium (Proposition 1). Finally, I construct a mechanism where the scoring transfer is exactly balanced. The balanced mechanism satisfies individual participation constraints, conditional on any available public history, and generates no deficit ex post (Proposition 2). A summary of notation is given in the Appendix.

## 2 Illustration: Pollution control.

Suppose an area accommodates $n$ firms that use a hazardous input Y for their production in period 1. A firm's cost of production decreases in its usage of Y:

$$C(y_i;\theta_i) = \frac{1}{\theta_i y_i}, \tag{1}$$

where $y_i > 0$ is the amount of Y used by firm $i = 1, 2, ...n$ and $\theta_i$ is a technology parameter privately known to the firm. The regulator's task is to allocate usage permits to firms $(y_1, y_2, ..y_n)$ such that the total cost of production and environmental damage are minimized.

The environmental damage in period 2 is linear in the total use of Y and equals:

$$\theta_d \sum_i y_i, \tag{2}$$

where $\theta_d$ is the damage per unit of Y used ($\theta_d$ is the *type* of environment at $t = 2$). I assume that $\theta_d$ is a Bernoulli random variable: it takes value $D > 0$ with probability $\alpha$, or value $0$ with probability $1 - \alpha$. $\theta_d$ is independent of the firm's types $\theta_i$, $i = 1, 2, ...n$. Denote $p(\theta_d;\alpha)$ the respective probability distribution function:

$$p(\theta_d;\alpha) = \begin{cases} \alpha, & \theta_d = D \\ 1 - \alpha, & \theta_d = 0 \end{cases}. \tag{3}$$

Contrary to the common assumption in mechanism design, I consider the probability $\alpha$ of high damage to be unknown. Each firm $i$ observes an informative signal,

$$x_i \in \{0, D\}, \tag{4}$$

drawn independently from distribution $p(\theta_d;\alpha)$. Note that any such draw is informative of $\alpha$. Let $X = (x_1, x_2, ..x_n)$ denote the vector of observed signals. $X$ represents all the information available at $t = 1$ about the damage at $t = 2$. The firms' signals are correlated with each other as they are drawn from the same distribution, however any firm's signal $x_i$ is uncorrelated with its payoff type $\theta_i$.[6]

The efficient allocation of permits in period 1 minimizes the expected total cost to all agents (I assume no discounting between periods), provided the information $X$:

$$\min_{(y_1,y_2,..y_n)} \int_{[0,1]} \left( \sum_i \frac{1}{\theta_i y_i} + \alpha D \sum_i y_i \right) \phi\left(\alpha\left|X\right.\right) d\alpha, \tag{5}$$

where the probability measure $\phi\left(\alpha\left|X\right.\right)$ is the Bayes-updated hyperprior $\phi^0(\alpha)$ given $X$:

$$\phi(\alpha|X) = \frac{\Pr[X|\alpha]\,\phi^0(\alpha)}{\int_{[0,1]} \Pr[X|\tilde{\alpha}]\,\phi^0(\tilde{\alpha})\,d\tilde{\alpha}}, \tag{6}$$

where $\Pr[X|\alpha] = \prod_i p(x_i|\alpha)$.

Suppose that the prior $\phi^0(\alpha)$ is uniform over the interval $[0,1]$ of the possible values of $\alpha$. Since the uniform distribution belongs to the class of Beta distributions, conjugate to the Bernoulli class, the update $\phi(\alpha|X)$ is also a Beta distribution.[7] If $h$ denotes the number of firms with a high signal $(i:\ x_i = D)$, and $l$ the number of firms with a low signal $(i:\ x_i = 0)$, then the updated probability (6) becomes:

$$\phi(\alpha|X) = \frac{\alpha^h (1-\alpha)^l \beta(\alpha;1,1)}{\int_{[0,1]} \tilde{\alpha}^h (1-\tilde{\alpha})^l \beta(\tilde{\alpha};1,1)\,d\tilde{\alpha}} = \beta(\alpha; 1+h, 1+l) \tag{7}$$

Note that $h$ and $l$ are sufficient statistics for data $X$. Given the updated prior, the efficient allocation of permits that solves the minimization problem (5) is the following:

[6]Therefore, the *beliefs determine preferences* (BDP) property is not satisfied. BDP is a necessary condition for full surplus extraction with Crémer-McLean mechanism (Neeman (2004)).

[7]The uniform distribution, or $\beta(\alpha;1,1)$, has the maximal entropy within the class of Beta distributions, and thus is the least informed prior within that class.

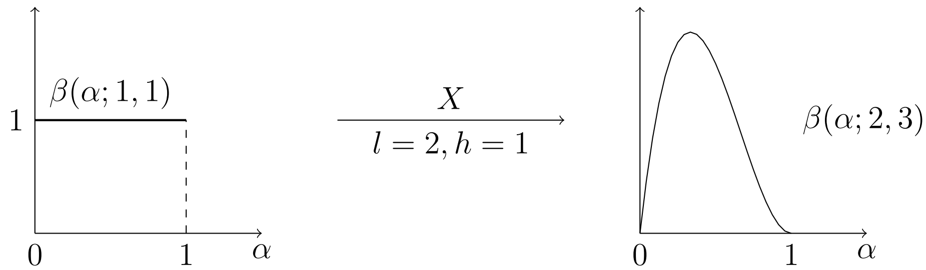


Figure 1: *The principal's initial hyperbelief* $\phi^0(\alpha) = \beta(\alpha;1,1)$*, the data* $X = \left(\theta_d^L, \theta_d^L, \theta_d^H\right)$*, and the updated hyperbelief* $\phi(\alpha|X) = \beta(\alpha;h+1,l+1) = \frac{\alpha^h(1-\alpha)^l}{B(2,3)} = \frac{\alpha(1-\alpha)^2}{B(2,3)}$.

$$f_i^*(\theta, X) = \sqrt{\frac{2+h+l}{\theta_i D(1+h)}}, \tag{8}$$

for all $i = 1, 2, ...n$.

The difficulty in implementing the efficient allocation $f^*$ is that there is no guarantee that firms will reveal the signals about the future truthfully (i.e. the numbers $h$ and $l$ may not represent the actual number of high and low signals received). This implies that the mechanism has to be incentive-compatible not only with regard to the elicitation of costs, but also of the signals. As a possible solution to this problem I first discuss an application of the idea of Crémer and McLean (1985) as the mutual verification of firms' reports in the first period (immediately). In the second step I show that welfare is improved by delaying verification to the second period when the damage realizes. In Sections 3 and 4 I extend the result to a repeated setting.

**Immediate Verification** First, note that the classic result of Crémer and McLean (1985) does not apply to the present setup, because the BDP property is not satisfied (see footnote 6). However we can use the idea of rewarding similarity and punishing divergence in the agents' reports in order to induce truthfulness. Consider a direct mechanism, where each firm reports its technology parameter $\theta_i$ and the signal about future damage $x_i$. The principal updates her prior $\phi^0$ on the firms' information $X = (x_i)_{i \in N}$ and assigns the efficient allocation $f^*$. With logarithmic scoring, we have the following two-part transfer to firm $i$:

$$\tau_i \left( \hat{\theta}_i, \hat{X} \right) = \underbrace{-\mathbb{E} \left[ \theta_d \middle| \hat{X} \right] \times f_i^* \left( \hat{\theta}_i, \hat{X} \right)}_{externality} + \lambda \times \underbrace{\ln \prod_{j \neq i} \Pr \left[ \hat{x}_j \left| \hat{x}_i \right. \right]}_{scoring}, \tag{9}$$

where $\hat{X} = (\hat{x}_i)_{i \in N}$ denotes the reported values, and $\Pr[\hat{x}_j |\hat{x}_i]$ is calculated as follows. If $\hat{x}_j = D$, $\Pr[\hat{x}_j |\hat{x}_i] = \int_{[0,1]} \alpha \phi(\alpha|\hat{x}_i) \, d\alpha$ and if $\hat{x}_j = 0$, $\Pr[\hat{x}_j |\hat{x}_i] = \int_{[0,1]} (1 - \alpha) \phi(\alpha|\hat{x}_i) \, d\alpha$.

The transfer consists of two parts that provide distinct incentives. The *externality* transfer induces the truthful revelation of the technology parameter, whereas the *scoring* transfer rewards similarity in reports. The mechanism thus constructed has the following undesirable property.

**Claim 1.** For any $\lambda > 0$ reporting no damage ($D = 0$) irrespective of the true signal is an equilibrium in the immediate verification mechanism. This equilibrium Pareto dominates the truthful revelation of signals.

(See proof in the Appendix) The claim could be re-stated as follows: regardless of whether truthtelling is equilibrium, it is impossible to find $\lambda > 0$ such that truthtelling is undominated. If immediate verification is used to assign rewards, the firms will be prone to report low damage $\hat{x}_i = 0$ even if their true signal suggests high damage $\hat{x}_i = D$. As a result of this manipulation, the regulator underestimates the probability of high damage and assigns larger permits $y_i$ in the first period, such that the firms save on production costs and create higher-than-efficient pollution. Note that truthful revelation could also be an equilibrium, however, since it is Pareto dominated the agents would be more likely to coordinate on reporting low damage. In the following paragraph I show that delaying verification rules out all equilibria but truthtelling.

**Delayed Verification**[8] The estimate based mechanism uses delayed verification, since it provides incentives for truthful revelation of signals. The change is made to the scoring transfer. Instead of rewarding similarity in firms' reports, the mechanism rewards similarity between a given report and the realized damage in period 2. The mechanism assigns the following transfers after the environmental uncertainty resolves. If the realized damage is high, $\theta_d = D$, then firm $i$ will receive:

$$\tau_i\left(\hat{\theta}_i, \hat{X}; D\right) = \underbrace{-D \times f_i^*\left(\hat{\theta}_i, \hat{X}\right)}_{externality} + \lambda \times \underbrace{\ln \int_{[0,1]} \alpha\phi\left(\alpha|\hat{x}_i\right) d\alpha}_{scoring}, \tag{10}$$

If the realized damage is low, $\theta_d = 0$, firm $i$'s transfer amounts to:

$$\tau_i\left(\hat{\theta}_i, \hat{X}; 0\right) = \underbrace{0 \times f_i^*\left(\hat{\theta}_i, \hat{X}\right)}_{externality} + \lambda \times \underbrace{\ln \int_{[0,1]} \left(1-\alpha\right)\phi\left(\alpha|\hat{x}_i\right) d\alpha}_{scoring}. \tag{11}$$

The scoring transfer is the "betting reward" that the firm gets for correctly predicting the realized damage. Observe that $\int_{[0,1]} \alpha\phi\left(\alpha|\hat{x}_i\right) d\alpha$ and $\int_{[0,1]} \left(1-\alpha\right)\phi\left(\alpha|\hat{x}_i\right) d\alpha$ are the conditional likelihoods of, respectively, $D$ and $0$ given $i$'s reported signal $x_i$. We make the following observation.[9]

**Claim 2.** Maximization of the scoring transfer yields truthful revelation of signal as a strict optimum.

When signals are reported truthfully, the externality (VCG) transfer represents the actual change in the social welfare due to the firm's use of hazardous input Y. Since the firm compensates the environmental damage its production has caused, its objective is aligned with the principal's program of total welfare maximization, and therefore it is optimal to report the true cost parameter $\theta_i$. Observe

[8] Parkes and Singh (2003) use a *delayed mechanism* to elicit true arrival times in problems that commonly arise in wireless networking and web-surfing (see also Friedman and Parkes, 2003). Their setup is different, in particular, there is no parameter uncertainty and learning.

[9] The following claim is a particular case of Lemma 2 stated in Section 4.2.

that its hyperbelief report $\hat{x}_i$ also enters the VCG transfer through the update of principal's prior $\phi(\alpha|X)$ used to estimate the future externality. In settings with continuous types, this leads to small deviations from truth-telling. In an attempt to marginally change the prior the agent slightly misreports. Significant deviations can be precluded by scaling up the scoring transfer. The proof of Proposition 1 demonstrates that for any given level of precision one can find $\lambda \in \mathbb{R}$ to scale up the scoring transfer and preclude deviations from truth-telling. This result implies $\varepsilon-$implementation of the efficient rule $f^*$.

**Three Periods** Now, suppose that there is a preliminary stage 0, when an expert makes a prediction of damage in period 2. If the expert is remunerated in period 2, his scoring transfer can be constructed as one of the firms' transfer, given by Equations (10) and (11). However, if the expert's remuneration cannot be delayed until then, it is impossible to test his estimate against the realization of damage. To solve this problem, the estimate based mechanism assigns the expert's reward *conditional on the firms' reports* in period 1. Denoting the expert's report $\hat{x}_0$ and his (scoring) transfer by $\tau_0$, we have the following:

$$\tau_0\left(\hat{x}_0; \hat{X}\right) = \ln \Pr\left(\hat{X}|\hat{x}_0\right) = \ln \int_{[0,1]} \Pr\left(\hat{X}|\alpha\right) \phi\left(\alpha|\hat{x}_0\right) d\alpha, \tag{12}$$

Observe that the scoring transfer rewards the expert for correctly predicting the firms' report of the future damage, and not its actual realization. Nevertheless, it gives the expert strong incentives to report his signal truthfully, provided that the firms will also be truthful in period 1. This truthfulness by induction, demonstrated in the example with three periods, is the main idea behind the estimate based mechanism.

# 3 The General Model

Consider a dynamic system with finite sequence of opening periods $\{1, 2, .., T\} \equiv \mathbb{T}$, where $T \geq 2$. The set of participants entering at period $t$ is called *cohort t* and denoted by $N_t$, each cohort includes at least two members. Let $N^t = \underset{s \leq t}{\cup} N_s$ denote the set of participants arriving at time $t$ or earlier.[10]

The principal chooses an allocation from a time-invariant space $\mathbb{Y}$ of alternatives; $\mathbb{Y}$ is a compact subset of a metric space. $y_t \in \mathbb{Y}$ is the decision taken by the principal at $t \in \mathbb{T}$. The history of allocations up to $t$ is denoted $y^t$.

**Utilities** Agent $i \in N_t$ derives utility $u_i$ from the allocation history $y^t$ at $t$ and receives monetary transfer at $t+1$. The agent's utility function is known up to parameter $\theta_i \in \Theta_t$, where $\theta_i$ (also denoted $x_{i0}$) is $i$'s private information referred to as $i$'s *payoff type.*[11]

$$u_i : \mathbb{Y}^t \times \Theta_t \to \mathbb{R}_+. \tag{13}$$

The utility functions $u_i$ are strictly concave and Lipschitz-continuous in $y_t$ for all $t$, $i \in N_t$. Additionally, I assume that non-participation yields zero utility to any agent in $N$.

Note that the utility functions are "observationally measurable"[12] in the sense that the utility does not depend on any variables unobserved by the agent. In particular, the utility is invariant in the future allocations and other agents' types.

**Types** The payoff-type $\theta_i \equiv x_{i0}$ of agent $i \in N_t$ is a random draw from a compact set $\Theta_t \subseteq \mathbb{R}$, $t \in \mathbb{T}$. The corresponding probability distribution function $p_t(\cdot; \alpha_t)$

[10]The generalization to setting where the agents live for more than two periods is straightforward, as long as the payoff type is persistent.

[11]Strictly speaking, there is no abuse of notation in writing $u_i$ and not $u_{i,t}$ since $i$ is an element of $N_t$ and not of, say, $\{1, 2, .., n_t\}$.

[12]As in Athey and Segal (2013)

over $\Theta_t$ belongs to a given parametric class $\mathcal{P}_t$; parameters $\alpha_t$ are independent across $t$. The class of priors conjugate to $\mathcal{P}_t$ is denoted $\Phi_t \subset \Delta(\mathcal{A}_t)$, where $\mathcal{A}_t$ is the set of possible values of parameter $\alpha_t$. Let $\phi_t^0 \in \Phi_t$ denote the initial prior at period 0. $\phi^0 = \prod_{t\in\mathbb{T}} \phi_t^0$ is the joint prior over the space of possible values $(\alpha_1, \alpha_2, ..\alpha_T)$ before the start of the game:

$$\phi^0 : \underset{t\in\mathbb{T}}{\times} \mathcal{A}_t \to \mathbb{R}_+. \tag{14}$$

To distinguish the elements of $\Phi_t$ from the elements of $\mathcal{P}_t$ I refer to $\phi_t \in \Phi_t$ as *hyperbeliefs* and to its parameters as *hyperparameters*. In the example of Section 2, $\mathcal{P}_t$ is the Bernoulli class, $\Phi_t$ is the conjugate Beta class and $(1,1)$ are the initial hyperparameters ($Beta(1,1)$ is the uniform distribution over the interval $[0,1]$).

**Information** Next to his own payoff type $\theta_i$ the agent holds private information about the future payoff-type distributions $\phi_t$. For every $t$ and $i \in N_t$ I define $x_{ik}$ as $i$'s signal about cohort $N_{t+k}$. $x_{ik}$ is a random draw from distribution $p_{t+k}(\cdot;\alpha_{t+k})$. Agent $i$'s entire private information, or *type*, is summarized in the vector $x_i$:

$$x_i = \begin{pmatrix} x_{i0} \equiv \theta_i \\ x_{i1} \\ \vdots \\ x_{iT-t} \end{pmatrix} \quad \begin{array}{l} \text{payoff-type} \\ \left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \text{hyperbelief-type} \end{array} \tag{15}$$

where for all $i \in N_t$:

$$x_i \in \underset{k=0}{\overset{T-t}{\times}} \Theta_{t+k}. \tag{16}$$

Let $X_t = (x_i)_{i\in N_t}$ denote the (matrix of) signals of cohort $N_t$ and $X^t = (X_1, X_2, ...X_t)$ the history of signals up to period $t$. In the following, I refer to $X^t$ as the *information available at t,* even though $X^t$ cannot be observed entirely by any single agent at $t$. $X^t$ comprises $X^{t-1}$, the information that is public at $t$, as well as $|N_t|$

pieces of private information. So far we only assume that the history of reports is public. Lemma 3 shows that there is a sense in which it is optimal for the principal to reveal the information about past reports: this increases the expected betting reward.

Note that an agent's hyperbelief and payoff types are independent. Therefore it is impossible to extract surplus with the Crémer-McLean mechanism as it requires the BDP property (see footnote 6).

**Sequential Updating** The hyperbeliefs are updated upon the arrival of new information. In a given period $t \in \mathbb{T}$, types $x_{i0}$ of the members of $N_t$ and their signals $x_{ik}$ about the future types are drawn. Given the hyperprior $\phi_s\left(\alpha_s|X^{t-1}\right)$, $s > t$, from the previous period[13] and the new data $X_t$, the updated hyperprior is derived by the Bayes rule as follows:[14]

$$\phi_s\left(\alpha_s|X^t\right) = \phi_s\left(\alpha_s|X_{t,s-t}, X^{t-1}\right) = \frac{\Pr\left[X_{t,s-t}|\alpha_s, X^{t-1}\right]\phi_s\left(\alpha_s|X^{t-1}\right)}{\int_{\mathcal{A}_s}\Pr\left[X_{t,s-t}|\tilde{\alpha}_s, X^{t-1}\right]\phi_s\left(\tilde{\alpha}_s|X^{t-1}\right)d\tilde{\alpha}_s}, \quad (17)$$

where

$$X_{t,s-t} = \left(x_{1,s-t}, x_{2,s-t}, ..x_{|N_t|,s-t}\right) \quad (18)$$

is the profile of the signals of received by cohort $t$ about the payoff-types at $s$, and

$$\Pr\left[X_{t,s-t}|\alpha_s, X^{t-1}\right] = \Pr\left[X_{t,s-t}|\alpha_s\right] = \prod_{i\in N_t} p_s\left(x_{i,s-t};\alpha_s\right) \quad (19)$$

is the probability of such signal profile conditional on parameter value $\alpha_s$.

[13]If $t = 0$ we have $\phi_s\left(\alpha_s|\text{Ø}\right) = \phi^0\left(\alpha_s\right)$ for all $s > 0$.

[14]The first equality in (17) holds due to the mutual independence of parameters.

It is easy to check that the update defined by Equation (17) is a probability measure. Moreover, by the conjugate prior property the updated hyperdistribution $\phi_s(\alpha_s|X^t)$ given by Equation (17) belongs to the same class as $\phi_s(\alpha_s|X^{t-1})$, for any $t \in \mathbb{T}$.

For $s \leq t$, the hyperprior is transferred from the previous period: $\phi_s(\alpha_s|X^t) = \phi_s(\alpha_s|X^s)$.[15] Finally, the joint hyperprior writes:

$$\phi\left(\boldsymbol{\alpha}|X^t\right) = \prod_{s\in\mathbb{T}} \phi_s\left(\alpha_s|X^t\right) \tag{20}$$

**Efficiency** The social welfare is defined as the sum[16] of all the agents' utilities:

$$W\left(X^T, y^T\right) = \sum_{t\in\mathbb{T}} \sum_{i\in N_t} u_i\left(y^t, x_{i0}\right) \tag{21}$$

A dynamic choice rule $\boldsymbol{f} = (f_1, f_2, ..f_T)$ is a finite sequence of functions $f_t$ mapping the information available up to $t$ into the set of allocations $\mathbb{Y}$. A dynamic choice rule $\boldsymbol{f}$ is *dynamically efficient* if for all $t \in \mathbb{T}$ and $X^t$:

$$f_t\left(X^t\right) \in \arg\max_{y_t\in\mathbb{Y}} \left\{\mathbb{E}\left[W\left(X^T, y^T\right)\middle|X^t\right]\right\} \tag{22}$$

Observe that efficiency requires that expectation be conditional on the entire information available at $t$, including the information which is private.

We can reformulate the problem to derive a notion of efficiency that is more operational for our purpose of designing transfers. The choice rule $\boldsymbol{f}$ is dynamically efficient if it solves the *stochastic optimal control* problem, where the allocation $y_t$ is the control variable and $X^t, y^{t-1}$ is the state with a stochastic component. The law of motion of the stochastic component $x_{ik}$ of state is given by:

$$x_{ik} \sim p_{t+k}\left(\cdot;\alpha_{t+k}\right)$$

[15]The implementation problem at $t$ does not require the hyperbelief over $\alpha_t$ to be updated.
[16]The discount factors $\delta^t$ can be subsumed in the utility functions $u_i$, $i \in N_t$.

$$(LM)$$

Provided the optimal control formalism, we can use the standard techniques to solve the dynamic problem (Bellman, 1966). Write the Bellman function as follows:

$$J_t\left(X^t, y^{t-1}\right) = \max_{y_t \in \mathbb{Y}} \left\{ \sum_{i \in N_t} u_i\left(y^t, x_{i0}\right) + \mathbb{E}\left[J_{t+1}\left(X^{t+1}, y^t\right) \middle| X^t\right] \right\}, \tag{23}$$

subject to $(LM)$ and the terminal condition $J_{T+1}\left(X^{T+1}, y^T\right) = 0$, further $(TC)$. The Bellman function is interpreted as the maximal future value of the mechanism from time $t$ on, given the past decisions and information. The value at $t$ includes the known utilities of cohort $N_t$, and the future cohorts' utilities in expectation over their types.

By the Bellman principle we maximize (23) with respect to the control variable $y_t$, conditional on the information available at $t$, $X^t$. The only relevant uncertainty at this stage is the uncertainty about $X_{t+1}$; all the subsequent payoff-type uncertainty is contained in the Bellman value at $t+1$.

The expected value of the Bellman function:[17]

$$\mathbb{E}\left[J_{t+1}\left(X^{t+1}, y^t\right) \middle| X^t\right] = \int_{\boldsymbol{\mathcal{A}}} \left( \int_{\mathbb{X}_{t+1}} J_{t+1}\left(X^{t+1}, y^t\right) \Pr\left[X_{t+1} | \tilde{\boldsymbol{\alpha}}, X^t\right] \boldsymbol{dX_{t+1}} \right) \phi\left(\tilde{\boldsymbol{\alpha}} | X^t\right) \boldsymbol{d\tilde{\alpha}} \tag{24}$$

The following definition of efficiency is then equivalent to the one introduced previously.

**Definition** The choice rule $\boldsymbol{f}$ is dynamically efficient, if for all $t \in \mathbb{T}$ it solves the maximization problem in Equation (23).

[17] Where $\Pr\left[X_{t+1} | \tilde{\boldsymbol{\alpha}}, X^t\right] = \prod_{s=t+1,..T} \Pr\left[X_{t+1,s-(t+1)} | \tilde{\alpha_s}\right] = \prod_{s=t+1,..T} \prod_{i \in N_{t+1}} p_s\left(x_{i,s-(t+1)}; \alpha_s\right)$ is defined in Equation (19).

**Mechanisms** A direct mechanism $(\boldsymbol{f}, \boldsymbol{\tau})$ is an observationally-measurable mapping from types into allocation and transfers. I focus on mechanisms where transfers can be delayed by one period. That is, cohort $N_t$ receives transfers at $t+1$ (Figure 2). A mechanism $\varepsilon$*–implements* social choice function $\boldsymbol{f}$ if for any $\varepsilon > 0$ truth-telling is an $\varepsilon-$equilibrium, and any profitable deviation that yields payoff less than $\varepsilon$ is in the neighborhood of truth-telling.[18]

Mechanism $(\boldsymbol{f}, \boldsymbol{\tau})$ satisfies the *participation constraint* conditional on public history if for all $t \in \mathbb{T}$, $i \in N_t$ and $X^{t-1}$ the following holds:

$$\mathbb{E}\left[u_i\left(f^t\left(X^t\right), x_{j0}\right) + \tau_i\left(X^{t+1}\right) \middle| X^{t-1}\right] \geq 0 \tag{25}$$

The condition postulates that under "the veil of ignorance", that is, before the agent observes his private information $x_i$, but after the observation of history $X^{t-1}$ – the participation in the mechanism yield a higher payoff in expectation than abstention.[19] I discuss the present participation constraint at the end of section 4.3.

Mechanism $(\boldsymbol{f}, \boldsymbol{\tau})$ satisfies *no deficit*, if for all $t \in \mathbb{T}$, and all report histories $X^{t+1}$: $\sum_{i \in N_t} \tau_i\left(X^{t+1}\right) \leq 0$.

Note that this condition is stronger than the requirement of no deficit after the final round. It postulates that the payments made at every given point in time generate no deficit in the principal's budget.

# 4 The Estimate Based Mechanism

The timing is described by the following iterations ($t = 1, 2, ..T-1$).

- In period $t$ agent $i \in N_t$ reports her information $x_i = (x_{i0}, x_{i1}, x_{i2}, ..x_{iT-t})$.

[18] A strategy profile is an $\varepsilon-$equilibrium, if there is no player and deviation that increases the player's payoff by at least $\varepsilon$.

[19] If the participation condition 25 holds for agent $i \in N_t$, one can achieve that his payoff at $t+1$ is positive as follows. At arrival in $t$, the agent buys a 0-interest bond from the principal and gets repayment at $t+1$.

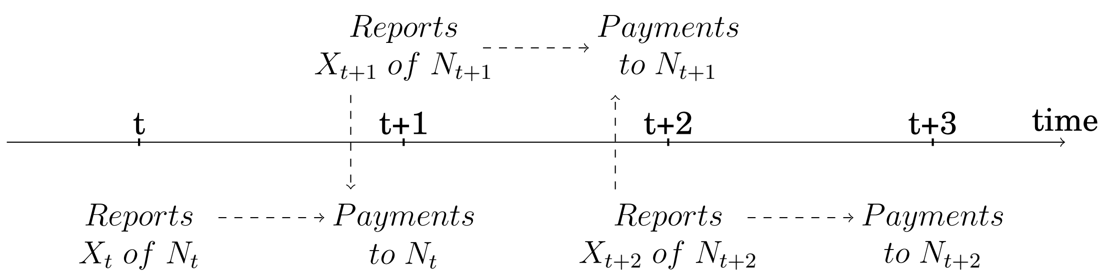


Figure 2: The timing of the mechanism.

- The principal merges new data $X_t = (x_i)_{i \in N_t}$ with history $X^{t-1}$ to obtain $X^t$, and updates his belief and implements allocation $f(X^t)$ at $t$.
- In period $t+1$ agent $i \in N_t$ receives the transfer and leaves the mechanism. *Etc.*

The *estimate based mechanism* $(\boldsymbol{f}, \boldsymbol{\tau})$ is a pair of efficient dynamic choice rule $\boldsymbol{f}$, defined by Equation (23), and transfer system $\boldsymbol{\tau}$, defined as follows:

$$\tau_i \left(X^{t+1}\right) = \tau_i^{VCG} \left(X^t\right) + \lambda \tau_i^{Sc.} \left(X^{t+1}\right), \tag{26}$$

for all $t \in \mathbb{T}$ and $i \in N_t$, where the components are given by Equations (28) and (31), respectively.

The two-part transfer $\tau_i$ serves to induce truthful revelation of the entire vector of private information $x_i$ and implement the efficient allocation. Essentially it formalizes the idea voting on preferences, but betting on beliefs (Hanson, 2013).

Next I discuss the construction of both parts of the transfer. Lemmas 1 and 2 state that the respective transfers yield truth-telling when applied separately to reveal types and hyperbeliefs; Lemma 3 is an auxiliary result on information disclosure by the principal. Proposition 1 states the $\varepsilon$-implementation result. Finally, I describe a way to balance the scoring budget and satisfy the ex-ante participation constraints, summarizing the result in Proposition 2. As before, all proofs are given in the Appendix.

## 4.1 VCG Transfer $\tau_i^{VCG}$

The Vickrey-Clarke-Groves payment equals the externality, expressed in money, that the agent's report imposes on the present and future agents. This transfer aligns the incentives of every agent with the principal's objective, since the maximization of his own utility in sum with the VCG transfer is equivalent to total welfare maximization. Therefore truthful report of payoff type $\theta_i = x_{i0}$ is optimal.

To construct the VCG transfer in our environment, I first introduce a family of choice rules $\boldsymbol{f}_{-i}$, $i \in N^T$, that are efficient with respect to a restricted player set $N^T/i$. That is, for a given $i$, $\boldsymbol{f}_{-i}$ maximizes the total welfare net of the utility of agent $i$:

$$W_{-i}\left(X^T, y^T\right) = \sum_{t\in\mathbb{T}} \sum_{j\in N_t/i} u_j\left(y^t, x_{j0}\right) \tag{27}$$

The VCG transfer writes:[20]

$$\begin{aligned}\tau_i^{VCG}\left(X^t\right) &= \sum_{j\neq i} u_j\left(f^t\left(X^t\right), x_{j0}\right) - \sum_{j\neq i} u_j\left(f_{-i}^t\left(X^t\right), x_{j0}\right) + \\ &\quad + \mathbb{E}\left[J_{t+1}\left(X^{t+1}, f^t\left(X^t\right)\right) \middle| X^t\right] - \mathbb{E}\left[J_{t+1}\left(X^{t+1}, f_{-i}^t\left(X^t\right)\right) \middle| X^t\right]\end{aligned} \tag{28}$$

$\forall t \in \mathbb{T}$, $\forall i \in N_t$.

Observe that because of independence the payoff-type component $x_{i0}$ does not affect the distribution of $X_{t+1}$, thus $\mathbb{E}\left[J_{t+1}\left(X^{t+1}, y^t\right)|X^t\right]$ is invariant in $x_{i0}$.

We have the following result:

**Lemma 1** For arbitrary period $t \in \mathbb{T}$, history $X^{t-1}$, agent $i \in N_t$, and payoff-type

[20] Bergemann and Välimäki (2010) construct a similar VCG transfer in a setting with persistent population and dynamic information. In their paper, similarly to the present one, the VCG transfer generates no deficit and satisfies the participation constraint.

$x_{i0}$, the expected sum of $i$'s utility and the VCG transfer (28) conditional on $i$'s information at $t$ is maximized if $i$ reveals $x_{i0}$ truthfully.

The lemma states that the optimal choice of type report by agent $i \in N_t$ is to tell the truth about his payoff type, regardless of the reported history. The proof demonstrates that given the VCG transfer the agent's problem becomes equivalent to the total expected welfare maximization. It follows that if the principal uses the accurate prior, then efficient implementation is achieved.

## 4.2 Scoring Transfer $\tau_i^{Sc.}$

The scoring transfer induces the truthful revelation of beliefs and is assigned upon state verification in the following period. To verify the report of agent $i$ of cohort $N_t$ we use the next cohort's report $X_{t+1}$. To construct the transfer, first calculate the probability of event $X_{t+1}$ implied by $i$'s information. His information is composed of the history of reports $X^{t-1}$ (the history of reports is public) plus the privately known type $x_i$. The conjugate update with respect to $i$'s information is:

$$\phi\left(\boldsymbol{\alpha}|x_i, X^{t-1}\right) \tag{29}$$

thus the implied probability of the event $X_{t+1}$ is

$$\Pr\left[X_{t+1}|x_i, X^{t-1}\right] = \int_{\boldsymbol{\mathcal{A}}} \Pr\left[X_{t+1}|\alpha, x_i, X^{t-1}\right] \phi\left(\alpha|x_i, X^{t-1}\right) \boldsymbol{d\alpha} \tag{30}$$

The principal assigns the scoring transfer equal to the natural[21] logarithm of this probability:

$$\tau_i^{Sc.}\left(X^{t+1}\right) = \ln \Pr\left[X_{t+1}|x_i, X^{t-1}\right] \tag{31}$$

[21] The choice of logarithm base is arbitrary.

Note that the probability of a given state - report by the next generation, $X_{t+1}$, accounts not only for the distribution of true types and beliefs of the next generation, as specified by the hyperbelief-types, but also for strategic communication of $N_{t+1}$. The following lemma states that if the next generation is truth-telling, then the player $i$ who faces the scoring transfer in expectation will report his belief truthfully.

**Lemma 2** Fix $t \in \mathbb{T}$, $i \in N_t$ and suppose that the next-cohort's report $X_{t+1}$ is truthful. Then the maximization of $\mathbb{E}\left[\tau_i^{sc.}\left(\hat{x}_i, X^{t+1}\right)|x_i, X^{t-1}\right]$, $i \in N_t$ induces a truthful belief-type report: $\hat{x}_{ik} = x_{ik}$, $\forall k = 1, ..T-t$.

The proof relies on the assumption of conditional independence of private signals between cohorts. If the future reports are truthful, an agent's incentives to "match" the reported and the true distributions coincide. The shape of the scoring function provides these incentives. The agent will make his best bet given his information, that is, he will report the true draw from the unknown distribution.

As an auxiliary result, observe that the transfer given by (31) is information-optimal in the following sense. Suppose the designer can choose how much information about past reports to reveal to the arriving agents. It turns out that disclosing all past information increases the ex ante expectation of the scoring transfer. Consider the reduction of past information $X^{t-1} \in \mathbb{X}^{t-1} = \underset{s=1}{\overset{t}{\times}} \underset{k=0}{\overset{T-s}{\times}} \Theta_{s+k}$ as an orthogonal projection on space $\underset{s \in S}{\times} \underset{k \in K(s)}{\times} \Theta_s$ of types for some $S \subseteq \{1, 2, ..t\}$ and $K(s) \subseteq \{1, 2, ..T-t\}$. The reduction subsumes cases such as no disclosure about the past ($S$ and $K(S)$ are empty) or the disclosure of only the previous generation's report. Then we have the following lemma.

**Lemma 3** For all $S$ and $K(S)$, the unconditional expectation of the reduced-information transfer is lower than the unconditional expectation of the full-information transfer:

$$\mathbb{E}\tau_i^{sc.}\left(x_i, proj_{S,K(S)}\left(X^{t-1}\right)\right) \leq \mathbb{E}\tau_i^{sc.}\left(x_i, X^{t-1}\right) \tag{32}$$

Furthermore, if the projection $proj_{S,K(S)}$ is different from identity then the inequality is strict.

To obtain the result note that both sides of Equation (32) coincide with the Shannon measure of entropy[22] of probability distributions $\Pr\left[X_{t+1}|x_i, proj_{S,K(S)}\left(X^{t-1}\right)\right]$ and $\Pr\left[X_{t+1}|x_i, X^{t-1}\right]$, respectively. The established properties of Shannon entropy with regard to conditional distributions produce the result. (See Appendix)

## 4.3 Main results

The main result of this paper is that by making transfers conditional on reports of two sequential generations, the principal can reconcile dynamic efficiency with the agents' incentive to misrepresent their information. This is achieved by the estimate based mechanism. The following proposition states the result.

**Proposition 1** For any $\varepsilon > 0$ there exists $\lambda \in \mathbb{R}_+$ such that truth-telling is an $\varepsilon-$equilibrium of the estimate based mechanism. Moreover, any profitable deviation that yields payoff less than $\varepsilon$ is in the neighborhood of truth-telling.

In this sense, the estimate based mechanism $\varepsilon$--implements the efficient choice rule. The appropriate transfer scaling yields truth-telling with arbitrary precision. The proof relies on Lemmas 1 and 2, and proceeds by induction starting at truth-telling at stage $T$. Note that $N_T$'s hyperbeliefs are void, and the Vickrey-Clarke-Groves transfer induce exact truthfulness of payoff types reports. The compactness of choice set $\mathbb{Y}$ and Lipschitz-continuity of the utility functions are the required assumptions.

It is possible to balance the budget of bets in the estimate based mechanism as follows. Assign to each agent $i$ a player $j$ of the same cohort and let $i$ pay $j$'s scoring transfer. Put differently, fix an arbitrary permutation $\rho$ on the set $N_t$,

[22] with the natural logarithm base.

such that for all $i \in N_t$, $\rho(i) \neq i$ ($\rho$ is a derangement). The balanced scoring transfer is defined as follows:

$$\tau_i^{Sc.B}\left(X^{t+1}\right) = \tau_i^{Sc.}\left(X^{t+1}\right) - \tau_{\rho(i)}^{Sc.}\left(X^{t+1}\right) \tag{33}$$

The *balanced estimate based mechanism* is a dynamic mechanism $(\boldsymbol{f}, \boldsymbol{\tau})$, where the allocation choice rule $\boldsymbol{f} = (f_1, f_2, ..f_T)$ is efficient:

$$f_t\left(X^t\right) \in \arg\max_{y_t \in \mathbb{Y}} \left\{ \sum_{i \in N_t} u_i\left(y^t, x_{i0}\right) + \mathbb{E}\left[J_{t+1}\left(X^{t+1}, y^t\right) \middle| X^t\right] \right\}, \tag{34}$$

and the transfer system $\boldsymbol{\tau}$ is given by:

$$\tau_i\left(X^{t+1}\right) = \tau_i^{VCG}\left(X^t\right) + \lambda \tau_i^{Sc.B}\left(X^{t+1}\right), \tag{35}$$

for all $t \in \mathbb{T}$ and $i \in N_t$.[23] In line with the previous literature, the budget of the VCG transfer does not generate a deficit. This property is inherited by the balanced mechanism.

**Proposition 2** The balanced estimate based mechanism satisfies the individual participation constraint at $t$, conditional on any public history at $t$, for any $t = 1, ..T - 1$, and generates no deficit *ex post*.

See proof in the Appendix. Note that contrary to the "classic" case where type distributions are known to the principal the stronger *interim* constraint is not satisfied in the present environment. This limitation is due to the necessity of revealing hyperbeliefs: as we can see from the proof, if one looked at payoff types in isolation the interim participation constraint would hold. However the expectation of the betting reward conditional on the hyperbelief type realization (signal) may not be positive. As a way to fix this flaw and achieve interim individual rationality one can use bond posting (see footnote 19).

[23] The scoring transfer to the last cohort $N_T$ is set to 0.

# 5 Discussion

This paper shows how $\varepsilon$-efficiency can be achieved in a setting with sequentially arriving agents that hold independent private values for the allocation as well as private information about the future distributions of type. Both the principal and the agents know the parametric class of the payoff-type distributions, however the distribution parameters are unknown. The difference in knowledge between the principal and the agents is that the latter observe informed signals about the underlying stochastic environment. Each agent receives a series of signals, drawn independently from the future type distributions. The signals reduce uncertainty about the parameter value. The principal's objective, achieved by the present mechanism, is to elicit the signals and update the hyperbelief. The classic result of Crémer and McLean (1985) does not apply to this setup because the necessary condition that beliefs determine preferences is not met: payoff and hyperbelief types are independent.

It is generally impossible to achieve exact implementation in a setup with continuous types and allocations by using continuously differentiable scoring rules. The impossibility is due to the fact that belief reports marginally affect externality payments through the mechanism's choice of allocation. Take truthful revelation as benchmark. Consider an agent who slightly misreports his signal about the future so as to shift the allocation in the direction of marginal increase in his utility (second-order gains versus first-order losses). By doing so, the agent loses an infinitesimal amount in the scoring transfer (continuous differentiability), but gains in the direct utility of allocation. Moreover, if members of the same cohort also benefit from the change in allocation, then there is further gain in the externality transfer. When type and allocation spaces are discrete, and further assumptions on the utility function are imposed, the estimate based mechanism proposed in this paper achieves exact efficiency, since infinitesimal deviations from truth-telling, such as the one described above, are not available to the agents.

The present information model allows for various degrees of initial uncertainty.

The shape of the hyperdistribution reflects the principal's guesses about the parameter, as well as the quality of his information. A high entropy hyperdistribution implies that the principal is uninformed, whereas a low entropy (in the classic case of mechanism design, degenerate) hyperdistribution corresponds to a substantial degree of certainty about the model. In this setting the agents hold additional private knowledge about the hyperdistribution, i.e., their information is strictly superior to the principal's.

I have assumed that the quality of information is homogeneous across the participants and the acquisition of information is costless. As a next step, one could study statistically efficient handling of information, which may be of different quality. In particular, one could allow for different betting budgets for different players, so as to provide incentives to those participants who observe more signals (and thus hold more accurate beliefs) to distinguish themselves from those with inferior information. Differentiating the betting budget across participants can be used to incentivize information acquisition, if it is costly.

# A Appendix

## A.1 Summary of notation

Agent $i$'s information is the following:

$$x_i = \begin{pmatrix} x_{i0} \equiv \theta_i \\ x_{i1} \\ \vdots \\ x_{iT-t} \end{pmatrix} \quad \begin{array}{l} \text{payoff-type} \\ \left.\begin{array}{l} \\ \\ \end{array}\right\} \text{hyperbelief-type} \end{array}$$

**Bold** face refers to multiple variables:

$$\int_{\boldsymbol{\mathcal{A}}} = \int_{\mathcal{A}_1} \int_{\mathcal{A}_2} \cdots \int_{\mathcal{A}_T} \quad ; \quad \boldsymbol{d\tilde{\alpha}} = d\tilde{\alpha}_1 d\tilde{\alpha}_2 \cdots d\tilde{\alpha}_T \quad ; \quad \mathbb{X}_t = \prod_{k=0}^{T-t} \Theta_{t+k} \quad ; \quad \boldsymbol{d}X_t = \prod_{i \in N_t} \prod_{k=0}^{T-t} dx_{ik}$$

$X_{t,s} = \begin{pmatrix} x_{1_t s-t} & x_{2_t s-t} & \cdots & x_{i,s-t} & \cdots & x_{|N_t|_t s-t} \end{pmatrix}$, the signals of cohort $N_t$ about period $s$, $t < s \leq T$, where $1_t$ denotes agent 1 in cohort $N_t$, $2_t$ agent 2 in cohort $N_t$ etc.

To single out agent $i$'s payoff-type report I use the following notation:

$$\left(\hat{x}_{i0}, X^t_{-i0}\right) \equiv \left( X_1, X_2, ..X_{t-1}, \begin{pmatrix} x_{1_t 0} & x_{2_t 0} & \cdots & \hat{x}_{i0} & \cdots & x_{|N_t|_t 0} \\ x_{1_t 1} & x_{2_t 1} & \cdots & x_{i1} & \cdots & x_{|N_t|_t 1} \\ x_{1_t 2} & x_{2_t 2} & \cdots & x_{i2} & \cdots & x_{|N_t|_t 2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{1_t (T-t)} & x_{2_t (T-t)} & \cdots & x_{i(T-t)} & \cdots & x_{|N_t|_t (T-t)} \end{pmatrix} \right)$$

## A.2 Proof of Claim 1

Recall from Equation (8) that the efficient level of permit :

$$y_i = \sqrt{\frac{2+h+l}{\theta_i D\,(1+h)}}, \tag{36}$$

for all $i$. Then agent $i$'s payoff net of the scoring payment is the following:

$$U_i + \tau_i^{CG} = -\frac{1}{\theta_i y_i} - y_i \frac{1+h}{2+h+l} D \tag{37}$$

$$= -\frac{1}{\theta_i}\sqrt{\frac{\theta_i D\,(1+h)}{2+h+l}} - \sqrt{\frac{2+h+l}{\theta_i D\,(1+h)}}\frac{1+h}{2+h+l} D \tag{38}$$

$$= -2\sqrt{\frac{D\,(1+h)}{\theta_i\,(2+h+l)}}, \tag{39}$$

for each $i$, where $h$ is the number of low, and $l$ the number of high signals. Equation (39) implies that all agents benefit from lower $h$ and higher $l$. In case of immediate verification we have the following:

$$\tau_i^{Sc.}(\hat{x}_i) = \ln\left(\Pr\left[D\,|\hat{x}_{i1}\right]\right)^{h_{-i}} + \ln\left(\Pr\left[0|\hat{x}_i\right]\right)^{l_{-i}} \tag{40}$$

$$= h_{-i} \ln \Pr\left[D\,|\hat{x}_i\right] + l_{-i} \ln \Pr\left[0|\hat{x}_i\right] \tag{41}$$

where $h_{-i}$ (or $l_{-i}$) is the number of agents, excluding $i$, who report high (respectively, low) signal. Since $\Pr\left[D\,|D\right] = \Pr\left[0\,|0\right] = \frac{2}{3}$, and $\Pr\left[0\,|D\right] = \Pr\left[D\,|0\right] = \frac{1}{3}$ we obtain the scoring transfers as follows. If $i$ reports a high signal ($D$),[24] then:

$$\tau_i^{Sc.}(D) = h_{-i} \ln \frac{2}{3} + l \ln \frac{1}{3}. \tag{42}$$

If $i$ reports a low signal ($0$).[25]

$$\tau_i^{Sc.}(0) = h \ln \frac{1}{3} + l_{-i} \ln \frac{2}{3}. \tag{43}$$

Thus the profile of reports $(\theta_i, 0)_{i\in N}$ (truthful payoff-type report, but "no damage" irrespective of the signal) is an equilibrium. In this equilibrium, player $i$'s payoff equals:

[24] Observe that if $i$ reports $D$, $l = l_{-i}$.

[25] If $i$ reports $0$, $h = h_{-i}$.

$$U_i + \tau_i^{CG} + \lambda\tau_i^{Sc.} = -2\sqrt{\frac{D\left(1+h\right)}{\theta_i\left(2+h+l\right)}} + \lambda\left(h_{-i}\ln\frac{2}{3} + l\ln\frac{1}{3}\right) \quad (44)$$

$$= -2\sqrt{\frac{D\left(1+n\right)}{\theta_i\left(2+n\right)}} + \lambda\left(\left(n-1\right)\ln\frac{2}{3}\right) \quad (45)$$

The equilibrium profile $(\theta_i, 0)_{i\in N}$ Pareto dominates the truthful profile $(\theta_i, x_i)_{i\in N}$. Suppose the expected transfer if the signal is high, $x_i = D$, and player $i$ reports truthfully:

$$\mathbb{E}\left[\tau_i^{Sc.}\left(\hat{X}_i\right)|x_i = D\right] = \sum_{h_{-i}} C_{n-1}^{h_{-i}}\left(\frac{2}{3}\right)^{h_{-i}}\left(\frac{1}{3}\right)^{n-1-h_{-i}}\left(h_{-i}\ln\frac{2}{3} + \left(n-1-h_{-i}\right)\ln\frac{1}{3}\right) \quad (46)$$

$$= \sum_{h_{-i}} C_{n-1}^{h_{-i}}\left(\frac{2}{3}\right)^{h_{-i}}\left(\frac{1}{3}\right)^{n-1-h_{-i}}\left(h_{-i}\ln 2 - \left(n-1\right)\ln 3\right) \quad (47)$$

$$= \left(n-1\right)\left(\frac{2}{3}\ln 2 - \ln 3\right) \quad (48)$$

Similarly, it can be shown that the expected transfer if the signal is low, $x_i = 0$ also equals $\left(n-1\right)\left(\frac{2}{3}\ln 2 - \ln 3\right)$. Since $\left(n-1\right)\ln\frac{2}{3} > \left(n-1\right)\left(\frac{2}{3}\ln 2 - \ln 3\right)$, player $i$'s utility when profile $(\theta_i, 0)_{i\in N}$ is played is greater than his utility in the truth-telling equilibrium; this holds for all $i$, thus the distortionary equilibrium Pareto dominates (for all $\lambda > 0$). ■

## A.3 Proof of Claim 2

Consider the maximization of the expected scoring transfer by agent $i$. The transfer writes:

$$\int_{[0,1]} \left( \Pr \left[ D|\tilde{\alpha} \right] \tau_i^{sc.} \left( \hat{x}_i, \theta_d^H \right) + \Pr \left[ 0|\tilde{\alpha} \right] \tau_i^{sc.} \left( \hat{x}_i, \theta_d^L \right) \right) \phi \left( \tilde{\alpha}|x_i \right) d\tilde{\alpha} \tag{49}$$

$$= \int_{[0,1]} \left( \tilde{\alpha} \ln \int_{[0,1]} \alpha \phi \left( \alpha|\hat{x}_i \right) d\alpha + \left( 1 - \tilde{\alpha} \right) \ln \int_{[0,1]} \left( 1 - \alpha \right) \phi \left( \alpha|\hat{x}_i \right) d\alpha \right) \phi \left( \tilde{\alpha}|x_i \right) d\tilde{\alpha} \tag{50}$$

$$= \int_{[0,1]} \tilde{\alpha} \phi \left( \tilde{\alpha}|x_i \right) \ln \int_{[0,1]} \alpha \phi \left( \alpha|\hat{x}_i \right) d\alpha + \int_{[0,1]} \left( 1 - \tilde{\alpha} \right) \phi \left( \tilde{\alpha}|x_i \right) d\tilde{\alpha} \ln \int_{[0,1]} \left( 1 - \alpha \right) \phi \left( \alpha|\hat{x}_i \right) d\alpha \tag{51}$$

Equation (51) boils down to:

$$p \ln \hat{p} + \left( 1 - p \right) \ln \left( 1 - \hat{p} \right), \tag{52}$$

where $p$ is the probability agent $i$ attaches to the high damage realization, $\hat{p}$ is the probability implied by his report $\hat{x}_i$, and $(1-p)$ and $(1-\hat{p})$ are the respective complementary probabilities. The first-order condition writes $\frac{p}{\hat{p}} = \frac{1-p}{1-\hat{p}}$ and has the solution $\hat{p} = p$. The second-order condition holds. This implies that the reported signal is true at the optimum, $\hat{x}_i = x_i$. ■

## A.4 Proof of Lemma 1

The problem of agent $i \in N_{t-1}$ writes:

$$\max_{\hat{x}_{i0} \in \Theta_t} \mathbb{E} \left[ \underbrace{u_i \left( f^t \left( \hat{x}_{i0}, X_{-i0}^t \right), x_{i0} \right) + \tau_i^{CG} \left( \hat{x}_{i0}, X_{-i0}^t \right)}_{=:U_i\left(\hat{x}_{i0}, X_{-i0}^t\right)} \Big| x_i, X^{t-1} \right], \tag{53}$$

where

$$\tau_i^{CG}\left(X^t\right) = \sum_{j\neq i} u_j\left(f^t\left(X^t\right), \hat{x}_{j0}\right) - \sum_{j\neq i} u_j\left(f^t\left(X_{-i}^t\right), x_{j0}\right) + \tag{54}$$

$$+\,\mathbb{E}\left[J_{t+1}\left(X^{t+1}, f^t\left(X^t\right)\right)\middle|X^t\right] - \mathbb{E}\left[J_{t+1}\left(X^{t+1}, f^t\left(X_{-i}^t\right)\right)\middle|X^t\right] \tag{55}$$

eliminate the components of transfer invariant in $\hat{\theta}_i$, the problem becomes equivalent to:

$$\max_{\hat{x}_{i0}\in\Theta_i}\left\{\mathbb{E}\left[\sum_j u_j\left(f^t\left(\hat{x}_{i0}, X_{-i0}^t\right), x_{j0}\right) + J_{t+1}\left(X^{t+1}, f^t\left(\hat{x}_{i0}, X_{-i0}^t\right)\right)\middle|X^t\right]\right\} \tag{56}$$

Recall that for all $X^t$:

$$f^t\left(X^t\right) \in \arg\max_{y_t\in\mathbb{Y}}\left\{\sum_j u_j\left(y_t; x_{j0}\right) + \mathbb{E}\left[J_{t+1}\left(X^{t+1}, y_t\right)\middle|X^t\right]\right\} \tag{57}$$

Thus

$$x_{i0} \in \arg\max_{\hat{x}_{i0}\in\Theta_i}\left\{\sum_j u_j\left(f^t\left(\hat{x}_{i0}, X_{-i0}^t\right), x_{j0}\right) + \mathbb{E}\left[J_{t+1}\left(X^{t+1}, f^t\left(\hat{x}_{i0}, X_{-i0}^t\right)\right)\middle|X^t\right]\right\} \tag{58}$$

$$= \arg\max_{\hat{x}_{i0}\in\Theta_i}\mathbb{E}\left[U_i\left(\hat{x}_{i0}, X_{-i0}^t\right)\middle|X^t\right] \tag{59}$$

for all $X^t$ and by the law of iterated expectations ($X^t$ contains strictly more information than $x_i, X^{t-1}$), hence

$$= \arg\max_{\hat{x}_{i0}\in\Theta_i}\mathbb{E}\left[U_i\left(\hat{x}_{i0}, X_{-i0}^t\right)\middle|x_i, X^{t-1}\right] \tag{60}$$

implying that truthful report $\hat{x}_{i0} = x_{i0}$ is the solution to the initial maximization problem. The strict convexity of the utility functions yields uniqueness of the solution. ■

## A.5 Proof of Lemma 2

We need to show that:

$$(x_{i1}, x_{i2}, ..x_{iT-t}) \in \underset{(\hat{x}_{i1}, \hat{x}_{i2}, ..\hat{x}_{iT-t})}{\arg\max} \mathbb{E}\left[\tau_i^{Sc.}\left(\hat{x}_i, X^{t+1}\right)|x_i, X^t\right] \tag{61}$$

where $\hat{x}_i = (x_{i0}, \hat{x}_{i1}, \hat{x}_{i2}, ..\hat{x}_{iT-t})$. Given that the report $X_{t+1}$ is truthful, the expectation of the scoring transfer conditional on the agent's information is given by:

$$\mathbb{E}\left[\tau_i^{Sc.}\left(\hat{x}_i, X^{t+1}\right)|x_i, X^t\right] = \int_{\mathbb{X}_{t+1}} \Pr\left[X_{t+1}|x_i, X^{t-1}\right] \ln \Pr\left[X_{t+1}|\hat{x}_i, X^{t-1}\right] \mathbf{d}X_{t+1} \tag{62}$$

The proof relies on the independence of value/signal draws within and between cohorts,

$$\Pr\left[X_{t+1}|x_i, X^{t-1}\right] = \prod_{j\in N_{t+1}} \prod_{k=0,1,..T-(t+1)} \Pr\left[x_{jk}|\hat{x}_i, X^{t-1}\right], \tag{63}$$

$$\Pr\left[x_{jk}|\hat{x}_i, X^{t-1}\right] = \Pr\left[x_{jk}|\hat{x}_{i,k+1}, X^{t-1}\right]. \tag{64}$$

Use Equations (63) and (64) to simplify the expected scoring transfer as follows:

$$\mathbb{E}\tau_i^{Sc.}\left(x_i, X^{t+1}\right) = \int_{\mathbb{X}_{t+1}} \Pr\left[X_{t+1}|x_i, X^{t-1}\right] \ln \Pr\left[X_{t+1}|\hat{x}_i, X^{t-1}\right] \mathbf{d}X_{t+1} \tag{65}$$

$$= \sum_k \int_{\mathbb{X}_{t+1}} \Pr\left[X_{t+1}|x_i, X^{t-1}\right] \ln \Pr\left[x_k|\hat{x}_{i,k+1}, X^{t-1}\right] \mathbf{d}X_{t+1} \tag{66}$$

where $x_k = (x_{jk})_{j\in N_{t+1}}$. The first-order condition with respect to $\Pr\left[x_k|\hat{x}_{i,k+1}, X^{t-1}\right]$ writes as follows:

$$\frac{\partial \mathbb{E}\tau_i^{Sc.}}{\partial \Pr\left[x_k|\hat{x}_{i,k+1}, X^{t-1}\right]} = \int_{\mathbb{X}_{t+1}} \frac{\Pr\left[X_{t+1}|x_i, X^{t-1}\right]}{\Pr\left[x_k|\hat{x}_{i,k+1}, X^{t-1}\right]} \mathbf{d}X_{t+1} \tag{67}$$

$$= \int \frac{\Pr\left[x_k | x_{i,k+1}, X^{t-1}\right]}{\Pr\left[x_k | \hat{x}_{i,k+1}, X^{t-1}\right]} \mathbf{d}x_k = 0 \tag{68}$$

The first order condition (68) implies that $\hat{x}_{i,k+1} \equiv x_{i,k+1}$, for $k = 0, 1, ..T - (t+1)$. The second-order condition holds globally:

$$\frac{\partial^2 \mathbb{E}\tau_i^{Sc.}}{\partial \left(\Pr\left[x_k | \hat{x}_{i,k+1}, X^{t-1}\right]\right)^2} = \int_{\mathbb{X}_{t+1}} -\frac{\Pr\left[x_k | x_{i,k+1}, X^{t-1}\right]}{\Pr\left[x_k | \hat{x}_{i,k+1}, X^{t-1}\right]^2} \mathbf{d}x_k < 0 \tag{69}$$

■

## A.6 Proof of Lemma 3

Denote the reduction in past information: $proj_{S,K(S)}\left(X^{t-1}\right) = \overline{X^{t-1}}$. That is, consider the case when only part of past information is disclosed to the agent. Compare the transfer with reduced information:

$$\mathbb{E}\tau_i^{Sc.}\left(x_i, \overline{X^{t-1}}\right) = \mathbb{E}\left[\mathbb{E}\left[\ln \Pr\left[X_{t+1} | x_i\right]\right] \middle| x_i, \overline{X^{t-1}}\right] \tag{70}$$

with the unconditional expectation of the scoring transfer with past and present information content:

$$\mathbb{E}\tau_i^{Sc.}\left(x_i, X^{t-1}\right) = \mathbb{E}\left[\mathbb{E}\left[\ln \Pr\left[X_{t+1} | x_i, X^{t-1}\right]\right] \middle| x_i, X^{t-1}\right] \tag{71}$$

The right-hand side of 71 is the negative of the Shannon entropy[26] of random variable $X_{t+1}$ conditional on random variables $x_i$ and $X^{t-1}$ that I denote $\mathcal{H}\left(X_{t+1} | x_i, X^{t-1}\right)$. Similarly, the right-hand side of Equation 70 is the negative of the Shannon entropy $X_{t+1}$ conditional on $x_i$ only, denoted $\mathcal{H}\left(X_{t+1} | x_i\right)$. This implies that:

[26] Here the entropy is defined with the natural number $e$ as the logarithm base. Choosing a different base does not change our analysis: the constant would cancel out in Equation 72.

$$\mathbb{E}\tau_i^{Sc.}\left(x_i, X^{t-1}\right) - \mathbb{E}\tau_i^{Sc.}\left(x_i\right) = \mathcal{H}\left(X_{t+1}|x_i, \overline{X^{t-1}}\right) - \mathcal{H}\left(X_{t+1}|x_i, X^{t-1}\right) > 0 \tag{72}$$

Observe that $X^{t-1}$ contains information about $X_{t+1}$ that is not contained in $\overline{X^{t-1}}$ (see Equation 17), therefore the entropy of $X_{t+1}$ conditional on $X^{t-1}$ and $x_i$ is lower than the entropy of $X_{t+1}$ conditional on $x_i$ and $\overline{X^{t-1}}$. This can also be observed from re-writing 72 in terms of unconditional entropies:

$$\begin{aligned} &\mathcal{H}\left(X_{t+1}|x_i, \overline{X^{t-1}}\right) - \mathcal{H}\left(X_{t+1}|x_i, X^{t-1}\right) = \\ &= \left(\mathcal{H}\left(X_{t+1}, x_i, \overline{X^{t-1}}\right) - \mathcal{H}\left(x_i, \overline{X^{t-1}}\right)\right) - \left(\mathcal{H}\left(X_{t+1}, x_i, X^{t-1}\right) - \mathcal{H}\left(x_i, X^{t-1}\right)\right) > 0 \end{aligned} \tag{73}$$

The marginal increase in entropy due to the addition of $X_{t+1}$ is greater when $X_{t+1}$ is added to $x_i$ and $\overline{X^{t-1}}$ than when $X_{t+1}$ is added to $x_i$ and $X^{t-1}$, since the latter pair has greater informational content. Thus we obtain that:

$$\mathbb{E}\tau_i^{Sc.}\left(x_i, X^{t-1}\right) > \mathbb{E}\tau_i^{Sc.}\left(x_i, \overline{X^{t-1}}\right) \tag{74}$$

The analysis is equivalent for arbitrary projections $proj_{S,K(S)}\left(X^{t-1}\right)$ different from the identity. Using the most information available increases the expected payoff.

■

## A.7 Proof of Proposition 1

The proof is by induction. The inductive hypothesis for $t = 1, 2, ..T - 1$ is the following: If $N_t$ report their types and beliefs truthfully, then $N_{t-1}$ report truthfully, too.

At the last stage $t = T$ the belief-type reports are void, and the payoff-type reports are truthful due to the ex-post VCG transfer. Thus at $t = T$ there is truthful revelation (trivially for the belief-type), and therefore it suffices to prove the inductive hypothesis.

Fix $t$. By Lemma 2 no player $i \in N_t$ can increase his *scoring* transfer by reporting anything different from the true type $x_i$. Thus the profit can be generated only in the remaining "welfare" part:

$$u_i\left(f^t\left(X^t\right), x_{i0}\right) + \tau_i^{CG}\left(X^t\right) =: w_i\left(X^t\right) \tag{75}$$

$w_i\left(X^t\right)$ is the total welfare change due to $i$'s report. Lemma 1 states that $w_i\left(X^t\right)$ is maximized at $\theta_i$, for any belief report $\hat{x}_{i1}, \hat{x}_{i2}, ..\hat{x}_{iT-t}$. Thus we can restrict attention to deviations in the belief report only.

If $X^t$ is the truthful report profile, let $D_i\left(X^t\right)$ denote a transformation of $X^t$ that replaces the hyperbelief of agent $i$ by some $\hat{x}_{i1}, \hat{x}_{i2}, ..\hat{x}_{iT-t}$ different from $x_{i1}, x_{i2}, ..x_{iT-t}$. Denote the class of such transformations $\mathcal{D}_i$ and let $\mathcal{D}_i^{\varepsilon}$ consist of all $D_i\left(X^t\right)$ that induce a welfare change greater or equal to $\varepsilon > 0$:

$$\mathcal{D}_i^{\varepsilon} = \left\{D_i \in \mathcal{D}_i : w_i\left(D_i\left(X^t\right)\right) - w_i\left(X^t\right) \geq \varepsilon\right\} \tag{76}$$

Consider a deviation of player $i$, where he distorts his hyperbelief. Under the equilibrium assumption, the profile of reports becomes $D_i\left(X^t\right) =: \hat{X}^t$. Since the choice set $\mathbb{Y}$ is compact, the change in the allocation $f^t(\cdot)$ is bounded, implying that $\exists c \in \mathbb{R}$:

$$\left\|f^t\left(X^t\right) - f^t\left(\hat{X}^t\right)\right\| < c \tag{77}$$

By the assumption that the utility functions $u_i$ are Lipschitz-continuous in $y^t$, for all $i$ there exists $K_i$ such that

$$\left|u_i\left(f^t\left(X^t\right); x_{i0}\right) - u_i\left(f^t\left(\hat{X}^t\right); x_{i0}\right)\right| < K_i \left\|f^t\left(X^t\right) - f^t\left(\hat{X}^t\right)\right\| \tag{78}$$

and thus $\left|u_i\left(f^t\left(X^t\right); x_{i0}\right) - u_i\left(f^t\left(\hat{X}^t\right); x_{i0}\right)\right| < cK_i$. Let $K := \max\left\{K_i\right\}_{i \in N^T/N^{t-1}}$. The welfare writes:

$$w_i\left(\hat{X}^t\right) = \sum_j u_j\left(f^t\left(\hat{X}^t\right), x_{j0}\right) - \sum_{j\neq i} u_j\left(f^t_{-i}\left(\hat{X}^t\right), x_{j0}\right) +$$

$$+\,\mathbb{E}\left[J_{t+1}\left(X^{t+1}, f^t\left(\hat{X}^t\right)\right)\middle|\hat{X}^t\right] - \mathbb{E}\left[J_{t+1}\left(X^{t+1}, f^t_{-i}\left(\hat{X}^t\right)\right)\middle|\hat{X}^t\right]. \tag{79}$$

Recall that the $J_t$ is a weighted sum of the future utilities. By a version of the Cauchy-Bunyakovsky inequality:

$$\left|\mathbb{E}\left[J_{t+1}\left(X^{t+1}, f^t\left(\hat{X}^t\right)\right)\middle|\hat{X}^t\right] - \mathbb{E}\left[J_{t+1}\left(X^{t+1}, f^t\left(X^t\right)\right)\middle|X^t\right]\right| \leq \tag{80}$$

$$\leq \int \left|\Pr\left[X^T\middle|\hat{X}^t\right] - \Pr\left[X^T\middle|X^t\right]\right| \times$$

$$\times \sum_{j\in N^T/N^t}\left|\bar{u}_j\left(f^T\left(X^T\right); x_{i0}\right) - \bar{u}_j\left(f^T\left(\hat{X}^T\right); x_{i0}\right)\right| \mathbf{d}X^T < \tag{81}$$

$$< 2\left(\left|N^T\right| - \left|N^t\right|\right) cK, \tag{82}$$

where $\bar{u}_j\left(y^T; x_{i0}\right) = u_j\left(y^s; x_{i0}\right)$ for $j \in N_s$. Similarly, for $\mathbb{E}\left[J_{t+1}\left(X^{t+1}, f^t_{-i}\left(\hat{X}^t\right)\right)\middle|\hat{X}^t\right]$ we obtain

$$\left|\mathbb{E}\left[J_{t+1}\left(X^{t+1}, f^t_{-i}\left(\hat{X}^t\right)\right)\middle|\hat{X}^t\right] - \mathbb{E}\left[J_{t+1}\left(X^{t+1}, f^t_{-i}\left(X^t\right)\right)\middle|X^t\right]\right| <$$

$$< 2\left(\left|N^T\right| - \left|N^t\right|\right) cK. \tag{83}$$

Summing up differences in the four components we derive that

$$\left|w_i\left(\hat{X}^t\right) - w_i\left(X^t\right)\right| < 4\left|N_t\right| cK + 4\left(\left|N^T\right| - \left|N^t\right|\right) cK$$

$$= 4cK\left(\left|N^T\right| - \left|N^{t-1}\right|\right). \tag{84}$$

We have shown that the gain that $i$ can achieve by misreporting the hyperbelief is bounded. Thus $\left|w_i\left(\hat{X}^t\right) - w_i\left(X^t\right)\right|$ is bounded for all $\hat{X}^t = D_i\left(X^t\right),\ D_i \in \mathcal{D}_i^\varepsilon$. Note that, on the other hand, for any $\hat{X}^t = D_i\left(X^t\right),\ D_i \in \mathcal{D}_i^\varepsilon$ the (negative) change in the scoring transfer, $\mathbb{E}\tau_i^{Sc.}\left(\hat{X}^t, X_{t+1}\right) - \mathbb{E}\tau_i^{Sc.}\left(X^{t+1}\right)$ is also bounded from above due to the strict concavity of the scoring rule. Thus we can choose $\lambda$ such that the (negative) change in the scoring transfer, $\mathbb{E}\tau_i^{Sc.}\left(\hat{X}^t, X_{t+1}\right) - \mathbb{E}\tau_i^{Sc.}\left(X^{t+1}\right)$ is less than $-4cK\left(\left|N^T\right| - \left|N^{t-1}\right|\right)/\lambda$ for any $\hat{X}^t = D_i\left(X^t\right),\ D_i \in \mathcal{D}_i^\varepsilon$, implying that $\hat{X}^t$ is not a profitable deviation. Going through all $x_i$, $i$ and $t$ choose the maximal $\lambda$. The maximum exists, since $|N^T|$ is finite. The inductive hypothesis is proven, hence the proposition. ■

## A.8 Proof of Proposition 2

### Participation constraint

The proof that the balanced mechanism satisfies the individual participation constraint, conditional on public history, proceeds in two steps. The first step is to show that the sum of the agent's utility and the VCG transfer is greater or equal to zero. The second step is to show that the expectation of the balanced scoring transfer equals zero, such that the entire participation constraint holds.

**Step 1.** Since choice rules $\boldsymbol{f}$ and $\boldsymbol{f}_{-i}$ are efficient by construction (see Equation (23)), the following holds for all $t$, $X^t$ and $i \in N_t$:

$$\begin{aligned} & \sum_j u_j\left(f^t\left(X^t\right), x_{j0}\right) \quad + \quad \mathbb{E}\left[J_{t+1}\left(X^{t+1}, f^t\left(X^t\right)\right) \middle| X^t\right] \\ \geq & \sum_{j \neq i} u_j\left(f_{-i}^t\left(X^t\right), x_{j0}\right) \quad + \quad \mathbb{E}\left[J_{t+1}\left(X^{t+1}, f_{-i}^t\left(X^t\right)\right) \middle| X^t\right] \end{aligned} \tag{85}$$

Recall the definition of Vickrey-Clarke-Groves transfer (Equation (28) in text):

$$\tau_i^{VCG}\left(X^t\right) = \sum_{j\neq i} u_j\left(f^t\left(X^t\right), x_{j0}\right) - \sum_{j\neq i} u_j\left(f_{-i}^t\left(X^t\right), x_{j0}\right) + \tag{86}$$
$$+\mathbb{E}\left[J_{t+1}\left(X^{t+1}, f^t\left(X^t\right)\right)\middle|X^t\right] - \mathbb{E}\left[J_{t+1}\left(X^{t+1}, f_{-i}^t\left(X^t\right)\right)\middle|X^t\right] \tag{87}$$

Thus, inequality (85) is equivalent to:

$$u_i\left(f^t\left(X^t\right), x_{i0}\right) + \tau_i^{VCG}\left(X^t\right) \geq 0. \tag{88}$$

By the law of iterated expectations:

$$\mathbb{E}\left[u_j\left(f^t\left(X^t\right), x_{j0}\right) + \tau_i^{CG}\left(X^t\right)\middle|X^{t-1}\right] \geq 0. \tag{89}$$

**Step 2.** First, observe that:

$$\begin{aligned}\Pr\left[X_{t+1}, x_j|X^{t-1}\right] &= \int_{\mathcal{A}} \Pr\left[X_{t+1}, x_j|\alpha, X^{t-1}\right]\phi\left(\alpha|X^{t-1}\right)\boldsymbol{d\alpha} \\ &= \int_{\mathcal{A}} \Pr\left[X_{t+1}|\alpha, X^{t-1}\right]\Pr\left[x_j|\alpha, X^{t-1}\right]\phi\left(\alpha|X^{t-1}\right)\boldsymbol{d\alpha}.\end{aligned} \tag{90}$$

$$\Pr\left[x_j|X^{t-1}\right] = \int_{\mathcal{A}} \Pr\left[x_j|\alpha, X^{t-1}\right]\phi\left(\alpha|X^{t-1}\right)\boldsymbol{d\alpha}. \tag{91}$$

Since signals and types are drawn independently (from the true $\alpha$ - distributions), $\Pr\left(x_j|\alpha, X^{t-1}\right)$ in both (90) and (91) is invariant in $j$. Therefore

$$\begin{aligned}&\ln\Pr\left[X_{t+1}, x_j|X^{t-1}\right] - \ln\Pr\left[x_j|X^{t-1}\right] \\ &= \ln\Pr\left[X_{t+1}|x_j, X^{t-1}\right] = \tau_j^{Sc.}\left(X^{t+1}\right)\end{aligned} \tag{92}$$

is also invariant in $j$. This implies that the expectations of one's own and another agent's scoring transfer are equal, conditional on the past history $X^{t-1}$:

$$\mathbb{E}\left[\tau_i^{Sc.}\left(X^{t+1}\right)\middle|X^{t-1}\right]=\mathbb{E}\left[\tau_{\rho(i)}^{Sc.}\left(X^{t+1}\right)\middle|X^{t-1}\right]. \tag{93}$$

Thus the expectation of the balanced scoring transfer is zero:

$$\mathbb{E}\left[\tau_i^{Sc.B}\left(X^{t+1}\right)\middle|X^{t-1}\right]=\mathbb{E}\left[\tau_i^{Sc.}\left(X^{t+1}\right)-\tau_{\rho(i)}^{Sc.}\left(X^{t+1}\right)\middle|X^{t-1}\right]=0. \tag{94}$$

We conclude that

$$\mathbb{E}\left[u_i\left(f^t\left(X^t\right),x_{j0}\right)+\tau_i^{CG}\left(X^t\right)+\tau_i^{Sc.B}\left(X^{t+1}\right)\middle|X^{t-1}\right]\geq 0, \tag{95}$$

thus the participation constraint holds. $\blacksquare(PC)$

### No deficit

From Equation (85) it immediately follows that the Vickrey-Clarke-Groves transfer to each agent is non-positive:

$\tau_i^{CG}\left(X^t\right)\leq 0$

The balanced scoring transfer satisfies $\sum_{i\in N_t}\tau_i^{Sc.B}\left(X^{t+1}\right)=0$ by construction (See Equation (33)). Therefore, the aggregate no-deficit constraint holds:

$\sum_{i\in N_t}\tau_i^{CG}\left(X^t\right)+\tau_i^{Sc.B}\left(X^{t+1}\right)\leq 0.$ $\blacksquare(ND)$

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