Level-strategyproof Belief Aggregation Mechanisms

Estelle Varloot, Rida Laraki

August 12, 2022

Abstract

In the problem of aggregating experts' probabilistic predictions or opinions over an ordered set of outcomes, we introduce the axiom of level-strategyproofness (level-SP) and argue that it is natural in several real-life applications and robust as a notion. It implies truthfulness in a rich domain of single-peaked preferences over the space of cumulative distributions. This contrasts with the existing literature, where we usually assume single-peaked preferences over the space of probability distributions instead. Our main results are (1) explicit characterizations of all level-SP methods with and without the addition of other axioms (certainty preservation, plausibility preservation, proportionality); (2) comparisons and axiomatic characterizations of two new and practical level-SP methods: the proportional-cumulative and the middlemost-cumulative; (3) an application of the proportional-cumulative to construct a new voting method that extends majority judgment and where voters can express their uncertainties/doubts about the merits/qualities of the candidates/alternatives to be ranked.

1 Introduction

This paper is interested in the aggregation of experts' probabilistic opinions in an incentive compatible way, without money transfers. It axiomatically characterizes new voting methods where reporting its most preferred output is the optimal strategy for every expert, for a large set of utility functions.

In many real life situations, even the most prominent experts are uncertain —their opinions or predictions are probabilistic— and may disagree in their judgments, even if they share a common interest with the regulator [Boy19]. Thus, a method is needed to pool their opinions.

For example, with no evidence-based information on the Covid-19 disease, the European Academy of Neurology (EAN) developed an ad-hoc three-round voting method[1] to reach a consensus [Oer+21]. When a potentially dangerous

---

[1] "In round 1, statements were provided by EAN scientific panels (SPs). In round 2, these statements were circulated to SP members not involved in writing them, asking for

volcano becomes restless, civil authorities turn to scientists for help in anticipating risks. To do so, volcanologists developed elaborate mathematical models to elicit and aggregate experts' probabilistic opinions [Asp06; Chr+18]. The Technical University of Delft has developed a software EXCALIBUR and a successor ANDURYL [LM18]. This software is extensively used in forecasting the weather, in calculating the risks to manned spaceflight due to collision with space debris, or in estimating the future of the polar bear population (see [ATA05; ONe+08] and chapter 15 in [Coo+91]).[2]

Aggregating experts' probabilistic opinions is a well-studied mathematical/social choice problem [Arm01; Saa08; SM17; Coo+91], sometimes referred to as belief aggregation or opinion pooling. The formal model is this: each expert $i \in N$ is asked to provide to the regulator his prior probability distribution $p_i \in \mathcal{P} = \Delta(\Lambda)$ over a set of outcomes Λ. The objective of the regulator is to design a PAF = Probability Aggregation Function $\psi : \mathcal{P}^N \to \mathcal{P}$ satisfying some desirable properties.

Most of the literature described so far assumes honest reporting by the experts of their desired outcome. In practice, judges may have strategic incentives.

For example, the FDA uses advisory committees and follows their recommendations 70% of the time, and there have been several controversies around conflict of interest in the advisory committees [PP15]. Hence, we wish for the PAF to be incentive compatible (IC), e.g. reporting its preferred output is an optimal strategy for every expert.

When monetary transfers are possible (e.g. the experts can be paid after some realizations of the random variable), the problem has been well studied and several incentive compatible "scoring rules" have been designed [Goo52; McC56; Win69; DF83]. Our paper deals with situations where monetary transfers are impossible because the realization of the uncertainty is far in the future and/or the consequences of a bad decision are potentially catastrophic (volcanic eruption, irreversible global warming, etc).

To the best of our knowledge, incentive compatible belief aggregation without monetary transfers has been studied only as a special case of single-peaked domain restrictions and only when the set of outcomes Λ is finite. First of all, for strategyproofness to be stated formally, one needs assumptions about the individual preferences over the set $\mathcal{P}$ of probability distributions over Λ. If the voters' preferences are unrestricted, Gibbard [Gib73] and Satterthwaite's [Sat75] theorems apply and only a dictatorial method is strategyproof. Unfortunately, the Gibbard-Satterthwaite negative conclusion still holds even if one restricts the domain provided that it is "rich" enough (e.g. the class of all convex preferences over $\mathcal{P}$ [Zho91] or the class of generalized single-peaked preferences, including all additively separable convex preferences over $\mathcal{P}$ [NP07]). Fortunately, under a more severe restriction, the possibility of an anonymous aggregation has

---

agreement/disagreement. Items with agreement $> 70\%$ were retained for round 3, in which SP co-chairs rated importance on a five-point Likert scale. Results were graded by importance and reported as consensus statements."

[2]We refer the reader to Roger Cooke web page: `http://rogermcooke.net` which contains useful references, real data, a wide range of applications and links to software.

been proved recently [Goe+16; Fre+19] in the domain of the L_1-metric single-peaked preferences on $\mathcal{P}$.[3] As such, Goel et al. [Goe+16] proved the existence of a Pareto optimal strategyproof PAF and Freeman et al. [Fre+19] identified a large family of strategyproof PAF in the spirit of Moulin's characterization [Mou80] where phantom functions replace phantom voters. Nevertheless, as in [Goe+16; Fre+19] the main motivating problem is not the probability but the budget aggregation problem, their methods are neutral with respect to Λ, that is, they are invariant to permutations of the elements of Λ. In many applications, neutrality is not a desirable property.

This last point is best explained by an example. Suppose that the outcome space Λ is a finitely ordered set $\{a_m \succ \cdots \succ a_2 \succ a_1\}$ (such as the Richter or the volcanic scale). To define the utility functions, we need to compute the cost for expert i if the aggregate probability p is different from its forecast p_i. The usual way is by measuring the distances using some L_q-distance on $\mathcal{P} \subset \mathcal{R}^m$. For example [Goe+16; Fre+19] uses the L_1-distance on $\mathcal{P}$. This is not a good measure in our context. For example, imagine that p_i (the peak of i) is the Dirac mass δ_{a_1} at the smallest alternative a_1 and that p (the output) is the Dirac mass δ_{a_m} at the highest alternative $a_m \in \Lambda$. If our measure was good, then we should obtain that any other probability is better from the perspective of expert i than δ_{a_m}, especially the Dirac mass at a_2 as it is much closer to a_1 than a_m. However, no L_q-distance on $\mathcal{P} = \Delta(\Lambda) \subset \mathcal{R}^m$ will distinguish $\|\delta_{a_1} - \delta_{a_m}\|_{L_q}$ from $\|\delta_{a_1} - \delta_{a_2}\|_{L_q}$ as both are equal (those metrics are neutral).

To capture that $\mathrm{dist}(\delta_{a_1}, \delta_{a_m})$ must be bigger than $\mathrm{dist}(\delta_{a_1}, \delta_{a_2})$, the most natural way is to measure the distances in the space of cumulative distribution functions (CDFs) $\mathcal{C} = \Sigma(\Lambda)$. Characterizing strategyproof PAF when the preferences are single-peaked in the CDF space has, to our knowledge, never been done before. This is what our article does by fully characterizing all strategyproof methods under various combinations of axioms. **Our main contributions are:**

- We define the new concept of level-strategyproofness (level-SP) and prove it to imply incentive compatibility (IC) for a rich class of single-peaked preferences in the space of CDFs.[4]
- We prove several characterizations of level-SP methods in combination or not with other axioms and explore the boundaries of our characterizations by establishing some impossibilities.
- We characterize and compare two new methods: the middlemost and proportional cumulatives.
- We use the proportional-cumulative to construct a method to rank alternatives where voters can express their uncertainties/doubts about the qualities/merits of each alternative.

---

[3]Single-peaked preferences on $\Delta(\Lambda)$ with Λ finite under the L_1-metric is a very small domain because every peak corresponds exactly to one preference ordering.

[4]Being IC for a rich class of preferences is a desirable robustness property in social choice theory. Unfortunately, it is very rarely satisfied. For example, the mechanisms in [Fre+19] are IC for the L_1 distance on $\mathcal{P}$ but they are not IC for the L_2-distance on $\mathcal{P}$, nor are level-SP as shown in section 8.

The intuition behind level-SP is simple. Suppose the regulator decision depends on the likelihood of crossing a certain threshold, for example, the probability of having a major natural hazard. Then, experts incentives will also depend on the probability of crossing that threshold. If the aggregation rule is level-SP, then whatever the threshold is and even if it is not known in advance, no expert, by misrepresenting his truly desired probability distribution, can obtain that the probability of exceeding the given threshold is closer to what he wanted.

In addition to level-SP, a natural axiom to satisfy is *certainty preservation*. It says that if all experts agree that some event can (or cannot) happen then the aggregation preserves that property.[5] This axiom is classical in the literature and is also called the *zero preservation property* in [SM17] or *consensus preservation* in [DL17]. In addition, the designer may want to satisfy the *plausibility preservation* axiom: whenever all experts agree that some interval has positive probability,[6] so does society. We prove that the order-cumulatives (e.g. the middlemost-cumulative) are the unique anonymous level-SP PAF that are certainty and plausibility preserving. Their main drawback is their lack of diversity: they are such that whenever the experts' inputs dominate one another, the output is one of the inputs (and not a combination of their opinions).

Our second main method (we call the proportional-cumulative) solves this drawback: it is more diversified as it is such that whenever the inputs dominate one another, it agrees with each expert in proportion to their weight, and it is characterized as the unique level-SP method satisfying a "proportionality axiom", namely, if all inputs are the Dirac distributions $\{\delta_{a_i}\}_{i=1,...,n}$, then the output is their weighted average $\sum_{i=1}^{n} w_i \delta_{a_i}$ (where w_i is the given weight of expert i).

The paper is organized as follows. Section 2 introduces the model, the new notion of level-SP, and its implications together with a quick summary of the fundamental results of Moulin [Mou80] in the one-dimensional framework. Section 3 characterizes all level-SP methods. Section 4 isolates certainty preserving level-SP methods. Section 5 focuses on plausibility preserving level-SP methods. Section 6 combines all the axioms and characterizes the middlemost-cumulative. Section 7 is dedicated to diversity, and it characterizes the proportional cumulative. Section 8 compares our new two cumulative methods. Section 9 applies the proportional cumulative to propose an extension of majority judgment [BL11] to electing and ranking problems with uncertain voters. Section 10 concludes.

2   Models and Concepts

We first recall the classical characterizations of strategyproof aggregation rules when voters have one-dimensional single-peaked preferences (Moulin [Mou80]).

---

[5]Formally, certainty preservation is defied as as follows: for every Borel measurable event A, if $p_i(A) = 0$ for every $i \in N$ then $p(A) = 0$ or equivalently if $p_i(A) = 1$ for every $i \in N$ then $p(A) = 1$.

[6]If one asks this property for Borel measurable events, we reach an impossibility.

Then we describe our probability aggregation model, introduce the notion of level-strategyproofness (level-SP) and prove it to imply classical strategyproofness for a rich family of single-peaked utility functions over the CDF space.

In the sequel, $N = \{1, \ldots, n\}$ denotes the set of voters or experts, and n their number. For any set Z, $\mathbf{z} = (z_1, \ldots, z_n)$ denotes an element of Z^N (interpreted as a voting profile where each voter i submits the input z_i). For a voting profile $\mathbf{z} \in Z^N$, we let $\mathbf{z}_{-i}(z_i')$ denote the profile which differs from $\mathbf{z}$ only in dimension i which is replaced by z_i'. This is a standard notation in social choice theory and it corresponds to the notation (z_i', z_{-i}) in game theory.

2.1 One-dimensional StrategyProofness (One-SP)

We recall Moulin's results [Mou80] and characterizations. They are central to understand our results.

Definition 1 (Single-peaked preference). *A preference order T (represented with $\preccurlyeq$) over the set of alternatives $[0, 1]$ is **single-peaked** if there exists $x \in [0, 1]$ such that for all $y, z \in [0, 1]$, $z \leq y \leq x \Rightarrow z \preccurlyeq y$ and $x \leq y \leq z \Rightarrow z \preccurlyeq y$. The value x is known as the **peak**.*

In other words, an ordinal preference on the line $[0, 1]$ is single-peaked iff the cardinal utility function representing it is weakly increasing until the peak, then it is weakly decreasing.

Definition 2 (One-SP). *A voting rule $g : [0, 1]^N \to [0, 1]$ is a one-dimensional strategyproof (**one-SP**) iff whenever all experts have single-peaked preferences and all submit their peaks to be aggregated by g, no expert can obtain a strictly better alternative by reporting a fake peak.*

It can be proved that one-SP is equivalent to the following property called **uncompromisingness** in [BJ83]. A voting rule g is one-SP iff for all experts $i \in N$ and for all peak profiles $\mathbf{r} \in [0, 1]^N$:

$$r_i < g(\mathbf{r}) \Rightarrow g(\mathbf{r}) \leq g(\mathbf{r}_{-i}(r_i'))$$

and

$$r_i > g(\mathbf{r}) \Rightarrow g(\mathbf{r}) \geq g(\mathbf{r}_{-i}(r_i')).$$

Our level-SP definition below is a natural extension of uncompromisingness when the input space is the set of probability distributions, and it will be proved to imply classical strategyproofness for a large class of single-peaked preferences in the cumulative space.

One may be interested in voting rules such that whenever all experts agree on a peak, society chooses that peak. This axiom is called unanimity. We formulate it now for a general input space X as it will be used later for our own characterizations.

Axiom 1 (Unanimity). *$h : X^N \to X$ is **unanimous** if for all $x \in X$ we have $h(x, x, \ldots, x) = x$.*

Now we give two well-known characterizations of Moulin [Mou80], that we need in the next section.

Lemma 1. *[Moulin's max-min formula [Mou80]] A **voting rule** $g : [0,1]^N \to [0,1]$ is one-SP iff for each coalition of players $S \in 2^N$, there exists a unique value $\beta_S \in [0,1]$ called a "phantom" s.t.*

$$S \subseteq S' \text{ implies } \beta_S \leq \beta_{S'} \text{ and}$$

$$\forall \boldsymbol{r} \in [0,1]^N, g(\boldsymbol{r}) = \max_{S \subseteq N} \min \left(\beta_S, \min_{i \in S} r_i\right).$$

Moreover, the method is unanimous iff $\beta_\emptyset = 0$ and $\beta_N = 1$.

Moulin's most popular "median" formula was established in the anonymous case when the experts are treated equally by the rule. We formulate this axiom for any input space X as we will use it later.

Axiom 2 (Anonymity). *$h : X^N \to X$ is **anonymous** if $\forall \boldsymbol{x} \in X^N$ and all permutation σ over N:*

$$h(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = h(\boldsymbol{x}).$$

Lemma 2. *[Moulin's median formula [Mou80]] A voting rule $g : [0,1]^N \to [0,1]$ is one-SP and anonymous iff there exist $n+1$ "phantom voters" $\alpha_0 \leq \cdots \leq \alpha_n$ in [0,1] such that:*

$$\forall \boldsymbol{r} \in [0,1]^N, g(\boldsymbol{r}) = \text{median}(r_1, \ldots, r_n, \alpha_0, \ldots, \alpha_n).$$

Moreover, the method is unanimous if and only if $\alpha_0 = 0$ and $\alpha_n = 1$.

remark 1. *When g is anonymous, the α_k in the median characterization is equal to β_S in the max-min characterization whenever the cardinal of S is k.*

2.2 Level StrategyProofness (Level-SP)

This subsection describes our probability model and our level-SP notion. The next subsection links it to "classical" strategyproofness.

Society, a decision-maker, or the regulator wants to estimate the probability of a random variable X that ranges over a linearly ordered set of outcomes Λ that we identify with a Borel subset of the real line $\mathbf{R}$.[7] In practical applications, Λ is a finite set or an interval, two examples to keep in mind.

To construct society's estimation, each expert $i \in N = \{1, \ldots, n\}$ is asked to submit its subjective probability distribution[8] estimation $p_i \in \mathcal{P}$, where $\mathcal{P}$ denotes the set of Borel probability distributions over Λ. Our objective is to

---

[7] Borel sets are obtained from countable union, countable intersection, and relative complement of the intervals.

[8] A probability distribution is a σ-additive positive measure over the Borel algebra with total mass equals to 1.

design a Probability Aggregation Function (PAF) $\psi : \mathcal{P}^N \to \mathcal{P}$ satisfying some desirable properties. As some experts may be strategically behaving, we wish ψ to be Incentive Compatible (IC). Our IC notion is called Level-SP (level-strategyproof). It implies honest reporting when society's final decision is based on a threshold, that is to say, events that are below (or above) a certain level. In practical applications, situations that are represented with tipping points (e.g. climate change, the degree of a volcanic eruption damaging impact) are level-events.

Definition 3 (Level Events). *We define for any outcome $a \in \Lambda$, the **level event** $\mathcal{E}(a)$ as "the threshold a has not been crossed", e.g.*

$$\mathcal{E}(a) = \{x \in \Lambda : x \leq a\}$$

The next definition says that a PAF is IC with respect to level events if, no matter the threshold $a \in \Lambda$ is, no expert can by misreporting obtain that society's probability for the level event $\mathcal{E}(a)$ is closer to the one he/she wishes.

Axiom 3 (Level-SP). *A PAF $\psi : \mathcal{P}^N \to \mathcal{P}$ is **level-strategyproof** if for every expert $i \in N$, every input profile $\boldsymbol{p} \in \mathcal{P}^N$, any potential deviation $p'_i \in \mathcal{P}$ of expert i and any threshold $a \in \Lambda$:*

$$p_i(\mathcal{E}(a)) < \psi(\boldsymbol{p})(\mathcal{E}(a)) \Rightarrow \psi(\boldsymbol{p})(\mathcal{E}(a)) \leq \psi(\boldsymbol{p}_{-i}(p'_i))(\mathcal{E}(a))$$

and

$$p_i(\mathcal{E}(a)) > \psi(\boldsymbol{p})(\mathcal{E}(a)) \Rightarrow \psi(\boldsymbol{p})(\mathcal{E}(a)) \geq \psi(\boldsymbol{p}_{-i}(p'_i))(\mathcal{E}(a)).$$

This axiom is exactly the uncompromisingness property seen above to be satisfied for all level events. Formulated differently, let $\mathcal{C}$ be the set of **cumulative distribution functions** (CDF) over Λ. Let $\pi : \mathcal{P} \to \mathcal{C}$ be the mapping that transforms a probability distribution p over Λ into its CDF P:

$$\forall p \in \mathcal{P}, \forall a \in \Lambda, P(a) = \pi(p)(a) = p(\mathcal{E}(a)) = \int_{x \leq a} \mathrm{d}p(x).$$

A PAF $\psi : \mathcal{P}^N \to \mathcal{P}$ is associated with a unique **cumulative aggregation function (CAF)** $\Psi : \mathcal{C}^N \to \mathcal{C}$ that takes the CDFs $\{P_i = \pi(p_i)\}_{i \in N}$ of experts as inputs and returns the CDF of $\psi(\mathbf{p})$ as its output, that is, $\Psi(\pi(p_1), \ldots, \pi(p_n)) = \pi(\psi(p_1, \ldots, p_n))$. With this notation, saying that the PAF $\psi : \mathcal{P}^N \to \mathcal{P}$ is level-SP is equivalent to saying that the associated CAF $\Psi : \mathcal{C}^N \to \mathcal{C}$ satisfies: for every $a \in \Lambda$, every $\mathbf{P} \in \mathcal{C}^N$, $\mathbf{P'} \in \mathcal{C}^N$ and all $i \in N$:

$$P_i(a) < \Psi(\mathbf{P})(a) \Rightarrow \Psi(\mathbf{P})(a) \leq \Psi(\mathbf{P}_{-i}(P'_i))(a)$$

and

$$P_i(a) > \Psi(\mathbf{P})(a) \Rightarrow \Psi(P)(a) \geq \Psi(\mathbf{P}_{-i}(P'_i))(a).$$

This implies that for every $a \in \Lambda$, $\mathbf{P} \in \mathcal{C}^N$, $\mathbf{P'} \in \mathcal{C}^N$ and $i \in N$:

$$|\Psi(\mathbf{P}_{-i}(P'_i))(a) - P_i(a)| \geq |\Psi(\mathbf{P})(a) - P_i(a)| \tag{1}$$

remark 2. *Level SP is better formulated in the CAF space, and one may wonder why we don't just use the CAF model. The reason is: all the remaining axioms (namely certainty preservation, plausibility preservation, and proportionality) are more naturally formulated in the PAF model.*

2.3 The strong implications of level-SP

As the regulator does not usually know exactly the preferences of the experts, it is desirable to have truthful reporting property for a large family of reasonable utility functions. This section shows that level-SP is a robust IC concept as it implies "classical" strategyproofness for a large and natural class of single-peaked utility functions.

For each probability measure ν on Λ and each positive real $r \in \mathbf{R}_+$, we can define an L_r-distance on $\mathcal{C}$ as follows. $\|P - Q\|_{L_r} = [\int_\Lambda |P(a) - Q(a)|^r d\nu(a)]^{1/r}$, then, if ψ verifies level-SP, it also satisfies, $\forall i \in N$, $\forall \mathbf{P} \in \mathcal{C}^N$ and any $P_i' \in \mathcal{C}$:

$$\|\Psi(\mathbf{P}_{-i}(P_i')) - P_i\|_{L_r} = \left[\int_\Lambda |\Psi(\mathbf{P}_{-i}(P_i'))(a) - P_i(a)|^r d\nu(a)\right]^{1/r}$$
$$\geq \left[\int_\Lambda |\Psi(\mathbf{P})(a) - P_i(a)|^r d\nu(a)\right]^{1/r}$$
$$= \|\Psi(\mathbf{P}) - P_i\|_{L_r}$$

Where the inequality follows from (1). Consequently, if the utility of expert i is single-peaked and is measured by the use of some distance L_r on $\mathcal{C}$ to its peak, e.g.:

$$u_i(p) = -\|\pi(\psi(\mathbf{p})) - \pi(p_i)\|_{L_r} = -\|\Psi(\mathbf{P}) - P_i\|_{L_r}.$$

Thus, if level-SP is satisfied, then it is an optimal strategy for an expert with the utility function u_i defined just above to vote honestly. In the last section, we will prove that Level-SP is incompatible with strategyproofness w.r.t. the L_1 distance in the probability space (used in [Fre+19]).

3 All Level-SP methods

In this section, we will characterize all level-SP methods. They consist of aggregating the cumulatives instead of the probabilities and then using Moulin's formulae by replacing the phantoms with weakly increasing functions satisfying regularity conditions.

Recall that the cumulative distribution function (CDF) $P = \pi(p) : \Lambda \to [0, 1]$ associated with a probability distribution p on $\mathbf{R}$ is given by the formulae $P(a) = \int_{x \leq a} dp(x)$. The following lemma is well-known. It helps understand the regularity conditions in the next theorem.

Lemma 3. *A function $P : \Lambda \to [0, 1]$ is a cumulative distribution iff*

- *P is weakly increasing and right continuous;*
- *if $\sup \Lambda \notin \Lambda$ then $\lim_{a \to \sup \Lambda} P(a) = 1$, otherwise $P(\sup \Lambda) = 1$.*
- *if $\inf \Lambda \notin \Lambda$ then $\lim_{a \to \inf \Lambda} P(a) = 0$;*

For example, when $\Lambda = \mathbf{R}$ then $\inf \Lambda = -\infty \notin \Lambda$ and $\sup \Lambda = +\infty \notin \Lambda$. This in hand and thanks to Moulin max-min formula, we can prove the following characterization.

Theorem 1 (Level-SP: Max-min formula). *A PAF $\psi : \mathcal{P}^N \to \mathcal{P}$ is level-SP if and only if there exists for every $S \subseteq N$ a weakly increasing right continuous function $f_S : \Lambda \to [0, 1]$ that verifies the following properties:*
- *(1) for all $S \subseteq S'$ and $a \in \Lambda$ we have $f_S(a) \leq f_{S'}(a)$;*
- *(2) if $\sup \Lambda \notin \Lambda$, then $\lim_{a \to \sup \Lambda} f_N(a) = 1$ otherwise $f_N(\sup \Lambda) = 1$;*
- *(3) if $\inf \Lambda \notin \Lambda$ then $\lim_{a \to \inf \Lambda} f_\emptyset(a) = 0$;*
- *(4) $\Psi : \mathcal{C}^N \to \mathcal{C}$ the CAF associated with ψ is given by the formula:*

$$\forall a \in \Lambda, \Psi(\boldsymbol{P})(a) = \pi \circ \psi(\boldsymbol{p})(a) = \max_{S \subseteq N} \min \left(f_S(a), \min_{i \in S} P_i(a) \right).$$

Moreover, ψ is unanimous if and only if for every $a \in \Lambda$ $f_\emptyset(a) = 0$ and $f_N(a) = 1$ (in which case, (2) and (3) are automatically satisfied).
We will call the $\{a \to f_S(a)\}_{S \in 2^N}$ above the phantom functions associated with ψ.

The technical conditions (2) and (3) are necessary for Ψ to output a cumulative distribution. Conditions (1) and (4) are derived from to the Moulin max-min formula.

Proof. (Sketch of) In order to prove the theorem, we first need to prove that level-SP implies that for any $a \in \Lambda$, there is a one-SP function $g_a : [0, 1]^N \to [0, 1]$ such that (*) $\forall \mathbf{P}, \Psi(\mathbf{P})(a) = g_a(P_1(a), \ldots, P_n(a))$. The rest is a direct consequence of Lemma 1 and the characterization in Lemma 3.

Let us prove the existence of such $\{g_a\}_{a \in \Lambda}$ by *reductio ad absurdum*. Suppose that Ψ verifies level-SP but that there exists a level $a \in \Lambda$ such that no voting rule g_a (Level-SP or not) is such that (*) is satisfied. Then there must exist two CDF profiles $\mathbf{P}$ and $\mathbf{Q}$ such that for all voters $i \in N$, $P_i(a) = Q_i(a)$ and $\Psi(\mathbf{P})(a) \neq \Psi(\mathbf{Q})(a)$. We will show that switching experts inputs, one by one, from P_i to Q_i, one at a time, does not change the output. This will result in a contradiction with the assumption $\Psi(\mathbf{P})(a) \neq \Psi(\mathbf{Q})(a)$. Wlog, suppose $\Psi(\mathbf{P})(a) < \Psi(\mathbf{Q})(a)$.

Here is the proof for switching expert 1's opinion when $P_1(a) < \Psi(\mathbf{P})(a)$ (the proof for $P_1(a) > \Psi(\mathbf{P})(a)$) is symmetrical):
- By Level-SP, we have $\Psi(\mathbf{P})(a) \leq \Psi(\mathbf{P}_{-1}(Q_1))(a)$.
- Since $Q_1(a) = P_1(a)$ we therefore have $Q_1(a) < \Psi(\mathbf{P}_{-1}(Q_1))(a)$.
- By Level-SP, we therefore have $\Psi(\mathbf{P})(a) \geq \Psi(\mathbf{P}_{-1}(Q_1))(a)$. It follows that $\Psi(\mathbf{P})(a) = \Psi(\mathbf{P}_{-1}(Q_1))(a)$.

If remains to consider the case $\Psi(\mathbf{P})(a) = P_1(a) = Q_1(a) \neq \Psi(\mathbf{P}_{-1}(Q_1)(a))$. This would contradict level-SP since by switching expert 1's input from Q_1 to P_1 the output becomes the value Q_1.

Consequently, when we switch expert 1's input from P_1 to Q_1 we do not change the output. The same proof is then repeated for all other experts. It follows that our assumption was wrong: the family of voting rules $\{g_a\}_{a\in\Lambda}$ where (*) is satisfied exists. Due to the definition of Level-SP, each g_a must be one-SP. As such, from Lemma 1, there exist phantoms $(\beta_a^S)_{S\subset N}$ associated with each g_a. We define the f_S in the theorem as follows $f_S : a \to \beta_a^S$. The properties that f_S must verify are simply those needed so that the outcome of Ψ is a cumulative distribution (see lemma 3).

A more detailed and direct proof (not by contradiction) can be found in appendix A.1. $\qquad\square$

We now provide a similar characterization for the anonymous case.

Theorem 2 (Level-SP: the Median formula)**.** *A PAF $\psi : \mathcal{P}^N \to \mathcal{P}$ is level-SP and anonymous if and only if there exists $n+1$ weakly increasing right continuous function $f_k : \Lambda \to [0,1]$ that verifies the following properties:*
  - *for all $0 \le k \le n-1$ we have $f_k \le f_{k+1}$;*
  - *if $\sup \Lambda \notin \Lambda$, then $\lim_{a\to\sup\Lambda} f_n(a) = 1$ otherwise $f_n(\sup\Lambda) = 1$*
  - *if $\inf \Lambda \notin \Lambda$ then $\lim_{a\to\inf\Lambda} f_0(a) = 0$;*
  - *$\Psi : \mathcal{C}^N \to \mathcal{C}$ the CAF associated with ψ is given by the formula:*

$$\forall a \in \Lambda, \Psi(\boldsymbol{P})(a) = \text{median}\left(P_1(a),\ldots,P_n(a),f_0(a),\ldots,f_n(a)\right)$$

Moreover ψ is unanimous if and only if for all $a \in \Lambda$ $f_0(a) = 0$ and $f_n(a) = 1$. **We will call the $\{f_k\}_{k\in\{1,\ldots,n+1\}}$ the phantom functions associated with** ψ.

Proof. Essentially the same as in Theorem 1 except we use Lemma 2 instead of Lemma 1 (see appendix A.1). $\qquad\square$

remark 3. *As in Moulin, it can be shown that, under anonymity, the phantom function f_S in the max-min formula is equal to f_k in the median formula where $k = \#S$, the cardinal of S.*

Figure 1 below provides an example with $\Lambda = \mathbf{R}_+$ and 3 voters. The next sections refine the characterizations by adding desirable (and classical) axioms. Some combination of the axioms will single out only one method, or lead to an impossibility.

4 Certainty preservation Axiom

It is desirable, and somehow incontestable, to wish that when all experts agree that an event is certain to happen (or not to happen) then the aggregation of their input probabilities reflects that fact.

Axiom 4 (Certainty preservation)**.** *A PAF ψ **preserves certainty** iff for any probability profile $\boldsymbol{p}$ and all events $A \subseteq \Lambda$ Borel measurable:*

$$(p_i(A) = 1 \quad \forall i \in N \Rightarrow \psi(\boldsymbol{p})(A) = 1),$$
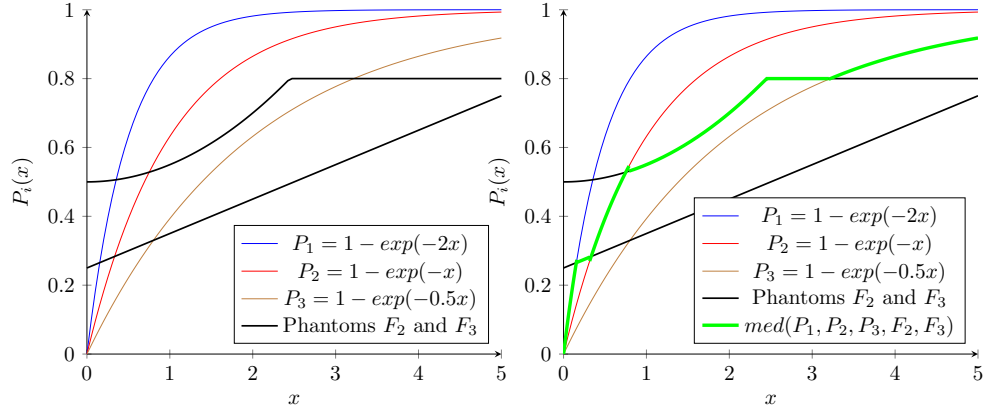$$\text{or equivalently, } (p_i(A) = 0 \quad \forall i \in N \Rightarrow \psi(\boldsymbol{p})(A) = 0).$$

Figure 1: A unanimous and anonymous Level-SP rule for $\Lambda = \mathbf{R}_+$. In left hand is drawn the cumulative functions P_1, P_2 and P_3 of 3 voters. In black is drawn two the phantom functions F_2 and F_3 ($F_1 = 0$ and $F4 = 1$ are not drawn). On the right hand, for each level x in the x-axe, the outcome in green is determined by the median formula: e.g. $median(P_1(x), P_2(x), P_3(x), F_2(x), F_3(x))$.

In other words, if an event is judged impossible (resp. certain) by all experts then it is judged impossible (resp. certain) by the aggregation function. This is a fairly standard axiom in the literature sometimes called the *zero preservation property* in [SM17] or *consensus preservation* in [DL17].

Proposition 1 (Certainty preservation characterization). *A level-SP PAF ψ is certainty preserving iff it is unanimous and its associated phantom functions f_S are constants.*

Proof. (Sketch of $\Rightarrow$:) For any interval $I = [a, b] \cap \Lambda$ and S. Select a profile $\mathbf{P}$ such that all experts in S agree that $\mathcal{E}(a)$ is certain almost surely and all other experts agree that $\mathcal{E}(b)$ is impossible almost surely. It follows that I is impossible almost surely so the outcome will be constant over I and that the outcome is equal to f_S over I. Therefore f_S is constant over I. It follows that all f_S are constant over Λ. For the complete proof, see appendix A.2. $\qquad\square$

In other words a level-SP ψ is certainty preserving if and only if there exists 2^n phantom constant values $\beta_S \in [0, 1]$ such that $\beta_\emptyset = 0$, $\beta_N = 1$ and $\Psi : \mathcal{C}^N \to \mathcal{C}$ the CAF associated with ψ verifies the following:

$$\forall a \in \Lambda, \Psi(\mathbf{P})(a) = \max_{S \subseteq N} \min \left(\beta_S, \min_{i \in S} P_i(a) \right).$$

In the anonymous case the characterization simplifies to:

$$\forall a \in \Lambda, \Psi(\mathbf{P})(a) = \mathrm{median}(P_1(a), \ldots, P_n(a), \alpha_1, \ldots, \alpha_{n-1}).$$

Where $\alpha_k = \beta_S$ whenever $\#S = k$. When ψ is Level-SP and certainty preserving, the one-SP function $g : \Lambda^n \to \Lambda$ such that:

$$\forall a \in \Lambda, \Psi(\mathbf{P})(a) = g(\mathbf{P}(a))$$

11

is called in the sequel **the one-SP rule associated with** Ψ. In the general case, $g(r_1, ..., r_n) = \max_{S \subseteq N} \min(\beta_S, \min_{i \in S} r_i)$ and in the anonymous case, $g(r_1, ..., r_n) = \text{median}(r_1, \ldots, r_n, \alpha_1, \ldots, \alpha_{n-1})$.

5   Plausibility Preservation Axiom

We now wish the PAF to satisfy that if all experts agree that an outcome may happen with positive probability then the PAF preserves that property.

Axiom 5 (Plausibility preservation). *A PAF $\psi : \mathcal{P}^N \to \mathcal{P}$ verifies plausibility preservation iff for any input profile $\boldsymbol{p} \in \mathcal{P}^N$ and any possible interval $I = [a, b] \cap \Lambda$:*

$$p_i(I) > 0 \quad \forall i \in N \implies \psi(\boldsymbol{p})(I) > 0.$$

Plausibility preservation implies that some monotonicity property is satisfied by the phantom functions in all intervals where their values are not 0 or 1.

Proposition 2 (Plausibility preservation characterization). *A Level-SP PAF $\psi : \mathcal{P}^N \to \mathcal{P}$ is plausibility preserving iff each associated phantom function f_S is strictly increasing on the interval where its value is not in $\{0, 1\}$ and $f_\emptyset(a) < 1$ for all $a < \sup \Lambda$ and $f_N(a) > 0$ for all a.*

Proof. (Sketch of $\Rightarrow$:) For any interval $I = [a, b] \cap \Lambda$, any $S \subseteq N$ and any $0 < \epsilon < 0.5$. Select a profile $\mathbf{P}$ such that all experts in $i \in S$ agree that $p_i(\mathcal{E}(a)) = 1 - \epsilon$, $p_i(\mathcal{E}(b)) = 1$ and all other experts j agree that $p_j(\mathcal{E}(a)) = 0$ and $p_j(\mathcal{E}(b)) = \epsilon$. It follows that all experts agree that $p_i(I) > 0$ Therefore $\Psi(\mathbf{P})(b) - \Psi(\mathbf{P})(a) > 0$. If $f_S(I) = x \in]0, 1[$, then by choosing ϵ such that $\epsilon < x < 1 - \epsilon$ we have $\Psi(\mathbf{P})(b) - \Psi(\mathbf{P})(a) = 0$. This is absurd since the outcome should have determined that I is possible, therefore we contradicted $f_S(I) = x \in]0, 1[$. The rest can be deduced from this.

A more detailed proof can be found in the appendix A.3.

$\square$

Figure 2 represents a PAF that is certainty preserving because the phantom functions $F_0 = 0, F_1 = 0.25, F_2 = 0.6, F_3 = 0.6$ are constant (as in Figure 1, we didn't draw F_0 and F_4, for simplicity and also because the median is the same with or without them). However, it is not plausibility preserving because $F_3 = 0.6$ is not in $\{0, 1\}$ in the interval $[0, 1]$ but is not strictly increasing (is constant). As such, the green function (the CDF of society) is constant on the interval [1,2] implying that the probability of [1,2] of society is 0 although all experts give a positive probability to $[1, 2]$.

Figure 3 represents a plausibility preserving PAF because two phantom functions (not drawn) $F_0 = 0$ and $F_4 = 1$ have values in $\{0, 1\}$ and the two others F_2 and F_3 are strictly increasing. Observe that F_3 is discontinuous at its left for $x = 2$ (which is allowed for phantom functions). Since F_2 is not a constant function, the PAF is not certainty preserving.
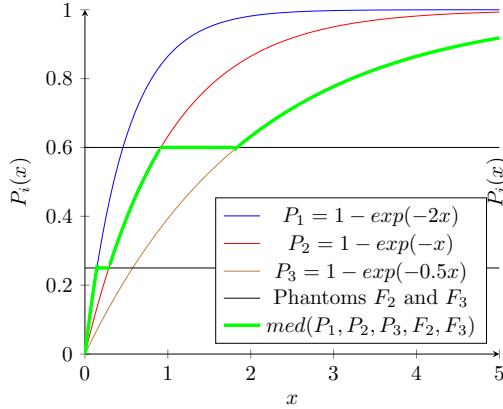
Figure 2: A certainty preserving but not plausibility preserving, unanimous and anonymous method.
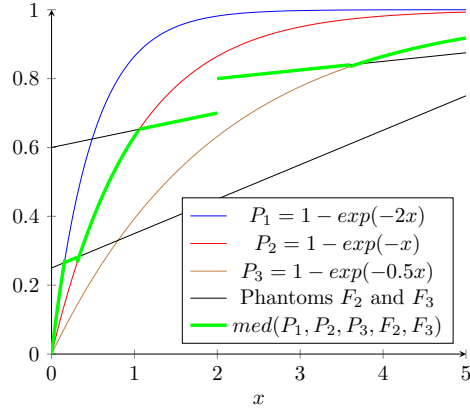
Figure 3: A plausibility preserving but not certainty preserving, unanimous and anonymous method.

It may seem asymmetrical that plausibility preservation is defined by putting a condition on the intervals while certainty preservation is defined by putting conditions on all Borel subsets. First, as the proof shows, if in the definition of certainty preservation, one replaces Borel sets with intervals, we still the exact same characterization. However, if we replace intervals with Borel subsets in the definition of plausibility, we reach an impossibility.

Axiom 6 (Strong plausibility preservation). *A probability aggregation function $\psi : \mathcal{P}^N \to \mathcal{P}$ verifies strong plausibility preservation iff for any input profiles $\boldsymbol{p} \in \mathcal{P}^N$ and any Borel measurable event $A \subset \Lambda$:*

$$p_i(A) > 0 \quad \forall i \in N \to \psi(\boldsymbol{p})(A) > 0;$$

that is, if all expert agrees that some Borel event is plausible, so does society.

Unfortunately, extending the plausibility preservation to Borel subsets is a too strong axiom.

Theorem 3 (Strong Plausibility Impossibility). *When Λ is a real interval, the unique probability aggregation functions that are level-SP, unanimous, and strong plausibility preserving are the dictatorials.*

Proof. (Sketch of $\Rightarrow$) This is done by reducing to the absurd. We build two disjoint intervals I_1 and I_2 in Λ and a profile $\mathbf{P}$, such that we can divide the experts into two groups S and T. Experts in S believe that I_1 is impossible almost surely and that I_2 is possible, experts in T believe that I_2 is impossible almost surely and that I_1 is possible. In the outcome, we select an expert of S opinion over I_1 and of T over I_2. By strong plausibility, we should have that the outcome claims $I_1 \cap I_2$ is possible. Therefore we reach a contradiction. See appendix C.2 for the complete proof. $\square$

6 Combining Plausibility preservation and Certainty preservation

6.1 Combination of the axioms in the general case

Having defined the previous two axioms and how they are characterized in the level-SP setting we may wish to understand the combination of the two with level-SP.

Proposition 3 (Combination 1st Characterization). *A level-SP Ψ is certainty preserving and plausibility preserving if and only if the output of its associated one-SP voting rule g always output one of its inputs (e.g. $g(r_1, ..., r_n) \subset \{r_1, ..., r_n\}$ for every $(r_1, ..., r_n) \in [0, 1]^N$).*

Proof. (Sketch of) By certainty preservation all phantoms f_S are constant (to some β_S). By plausibility preservation, all the β_S must take their values in $\{0, 1\}$. This easily implies the property we want to prove because this implies that the output of Ψ only depends on the ordering of the input. The exact details are found in appendix A.4. □

remark 4. *Hence, a dictatorship is level-SP, certainty preserving, and plausibility preserving.*

This characterization implies the following interesting action on dominated profiles, defined now.

Definition 4 (Dominated profiles). *A CDF P dominates Q iff $P(a) \geq Q(a)$ for all $a \in \Lambda$. Profile $\boldsymbol{P} = (P_1, .., P_n)$ is dominated if for any two experts i and j, P_i dominates P_j or P_j dominates P_i.*

Proposition 4 (Combination 2nd Characterization). *A level-SP ψ is certainty preserving and plausibility preserving iff for any dominated profile $\boldsymbol{P} = (P_1, .., P_n)$, $\psi(\boldsymbol{p}) \in \{p_1, ..., p_n\}$.*

Proof. (Sketch of) For a given one-SP voting rule g such that all the phantoms β_S are in $\{0, 1\}$, the outcome only depends on the ordering of the inputs. When the inputs of a CAF are dominated, the ordering is the same for all levels. As such for all $a \in \Lambda$, the same expert has his input selected (see appendix A.4 for the complete proof). □

Hence, the combination of certainty and plausibility preservation with level-SP implies that the output probability is one of the inputs whenever the input profile is dominated.

6.2 Combination of the axioms in the anonymous case

When anonymity is added to the combination, we obtain a nice class of PAF that can be generated from a well-known family of one-SP rules, sometimes called the order functions [BL11].

Definition 5 (Order-functions). *An **order-function** $g^k : [0,1]^n \to [0,1]$, (where k is in $\{1,\ldots,n\}$) is the one-SP voting rule that for a set of n values in $[0,1]$ return the kth smallest value.*

We therefore have $g^1(r_1,\ldots,r_n) = \min(r_1,\ldots,r_n)$, have that $g^n(r_1,\ldots,r_n) = \max(r_1,\ldots,r_n)$ and when n is odd

$$g^{(n+1)/2}(r_1,\ldots,r_n) = \mathrm{median}(r_1,\ldots,r_n)$$

.

Definition 6 (Order-cumulatives). *We denote as the kth **order-cumulative** $\Psi^k : \mathcal{C}^N \to \mathcal{C}$ the CAF which is defined by applying the kth order function at each level $a \in \Lambda$ in the CDF space:*

$$\forall a \in \Lambda, \Psi^k(\boldsymbol{P})(a) := g^k(P_1(a),\ldots,P_n(a)).$$

ψ^k *denotes the associated PAF that will also be called an order-cumulative.*

Consider a safe-to-dangerous scale Λ such as the Richter magnitude scale for earthquakes or the volcanic explosivity index. The min order-cumulative Ψ^1 is the most cautious response: since for each threshold, we consider the opinion of the most worried expert. On the other hand, the max order-cumulative Ψ^n is the less paranoid response.

Theorem 4 (Order Cumulatives Characterization). *The order-cumulatives are the unique level-SP PAF that are anonymous, certainty preserving, and plausibility preserving.*

Proof. (sketch) ψ is level-SP therefore their is an associated voting rule g. ψ is level-SP and plausibility preserving therefore the phantoms are equal to 0 or 1. As such there is a k such that:

$$\forall a, \Psi(\mathbf{P})(a) = med(P_1(a),\ldots,P_n(a),\overbrace{0,\ldots,0}^{n-k},\overbrace{1\ldots,1}^{k})$$

Therefore this is the $k-th$ order function (the detailed proof is in appendix A.4). $\square$

We are particularly interested in the **middlemost-cumulative**. That is to say, the order-cumulative defined by the median order-function when n is odd and when n is even, we have two middlemost-cumulatives: the lower $\Psi^{\frac{n}{2}}$ and the upper $\Psi^{\frac{n}{2}+1}$. It is easy and standard to show that they are **welfare maximizers** if experts' utilities are measured using the L_1 distance in $\mathcal{C}$ to the peak.

7 Diversity axioms

If there are three experts where two of them believe that $a_1 \in \Lambda$ will occur almost surely and the last one believes that $a_2 \in \Lambda$ will occur almost surely, then any order-cumulative (as well as any Level-SP method that satisfies certainty and plausibility preservation) will output a probability function that selects a_1 almost surely or a_2 almost surely. This lack of diversity in the output may sometimes be nonacceptable. For example, when all experts have equal weights and are equally competent, one may feel that the output where alternative a_1 has probability 2/3 to be chosen and alternative a_2 has probability 1/3 of being chosen is a better aggregation. More generally, one may wish that society's probability support contains all experts' probability supports as it means that all opinions are represented with some probability in the aggregation. We are going to formulate a weak diversity axiom, defined for single minded voters (next subsection), for which we can characterize a unique Level-SP rule. Then, we will show an impossibility result if one desires a stronger form of diversity.

7.1 Weighted Proportional-cumulative

Let us start by defining formally what it means to be single-minded.

Definition 7 (Single-minded). *A dirac mass δ_a is the probability law where alternative a is selected almost surely. An expert is single-minded if his input is a dirac mass.*

In the next (weak diversity) axiom, experts will have some given weights $(w_i)_{i \in N}$ (which is the case in most practical applications).

Axiom 7 (Weighted proportionality). *If experts are single-minded ($p_i = \delta_{a_i}$ for all $i \in N$), the aggregation must coincide with the weighted average:*

$$\forall (a_1, \ldots, a_n) \in \Lambda^N \quad \psi(\delta_{a_1}, \ldots, \delta_{a_n}) = \sum_{i \in N} w_i \delta_{a_i};$$

where $w_i \geq 0$ is the weight of expert $i \in N$ with $\sum_{j \in N} w_j = 1$.

In the anonymous case, all experts have the same weights (see next subsection).

The weighted average $\psi(p_1, \ldots, p_n) = \sum_i w_i p_i$ satisfies weighted proportionality but it is not level-SP. The next theorem shows that there is exactly one PAF satisfying level-SP and weighted proportionality. This function happens to be certainty preserving and can be uniquely described by a single one-SP voting rule $\mu_{\mathbf{w}} : [0,1]^n \to [0,1]$ defined as follows:

$$\forall \mathbf{r} = (r_1, \ldots, r_n), \mu_{\mathbf{w}}(\mathbf{r}) := \sup \left\{ y \;\middle|\; \sum_{i : r_i \geq y} w_i \geq y \right\}.$$

To our knowledge, this is a new voting rule. Before we state the main theorem, let us give an equivalent formulation of $\mu_{\mathbf{w}}$ when the weights are rational numbers.

Proposition 5 (Rational weights). *If all the weights are rationals $w_i = s_i/d$ then:*

$$\forall \boldsymbol{r} = (r_1, ..., r_n), \mu_{\boldsymbol{w}}(\boldsymbol{r}) := \text{median}(\overbrace{r_1, \ldots, r_1}^{s_1}, \ldots, \overbrace{r_n, \ldots, r_n}^{s_n}, 0, 1/d, \ldots, 1-1/d, 1)$$

Proof. The proof can be found in the appendix B.2. $\square$

Theorem 5 (Weighted Proportional Cumulative). *The unique PAF $\psi : \mathcal{P}^n \to \mathcal{P}$ that verifies level-SP and weighted proportionality is the unique certainty preserving one associated with the one-SP rule $\mu_{\boldsymbol{w}}$, that is:*

$$\forall a \in \Lambda, (P_1, \ldots, P_n) \in \mathcal{C}; \Psi(P_1, \ldots, P_n)(a) := \mu_{\boldsymbol{w}}(P_1(a), \ldots, P_n(a))$$

Initially, this theorem was our main objective when we started this paper. The theorem was first proved in the anonymous case (aka the equal weights case) and was stated and proved using more axioms than necessary (such as certainty preservation). We succeeded in eliminating all the unnecessary axioms and then simplified the proof to the one below, where we essentially verified that the CAF associated with μ_w must be the solution.

Proof. $\Rightarrow$: The weighted proportionality axiom can be rewritten in terms of CDF as follows:

$$\forall (a_1, \ldots, a_n) \in \Lambda^N, \Psi(\delta_{a \geq a_1}, \ldots, \delta_{a \geq a_n})(a) = \sum_{i : a_i \leq a} w_i.$$

Let ψ be level-SP and weighted proportional. Let Ψ be the CAF associated with ψ and f_S be its phantom functions. Let us first show that (w.l.o.g):

$$\forall S \subseteq N, \forall a \in \Lambda : f_S(a) = \sum_{i \in S} w_i.$$

Let us take any $S \subseteq N$ and alternatives $a < b$. Suppose that all experts $i \in S$ are single-minded in a (their input is the Dyrac mass at a) and that all the other experts are single-minded b. By weighted proportional we therefore have $f_S(a) = g_a(\mathbf{P(a)}) = \sum_{i \in S} w_i$. As such (wlog for $a = \sup \Lambda$) the phantoms associated with ψ are the constant phantom functions $f_S = \sum_{i \in S} w_i$. Consequently, g_a is independent on a (the phantoms as constant). Let g denote the common function (which is the one-SP rule associated with ψ).

It remains to show that $g = \mu_w$.
- Suppose that there is j such that $g(\mathbf{r}) = r_j$, then there is S and $S' = S - \{k | r_j = r_k\}$ such that for all $k \in S$, we have $r_j = \min(f_S, \min_{k \in S} r_k)$ and $r_j \leq f_{S'}$ (lemma 1).
  - For any $y > r_j$, we have $\sum_{i : r_i \geq y} w_i \leq \sum_{i : r_i > r_j} w_i = f_{S'} \leq r_j < y$. Therefore by definition of μ_w we have $\mu_w(\mathbf{r}) \leq r_j$.
  - For any $y \leq r_j$, we have $\sum_{i : r_i \geq y} w_i \geq \sum_{i : r_i \geq r_j} w_i = f_S \geq r_j \geq y$. Therefore by definition of μ_w, $\mu_w(\mathbf{r}) \geq r_j$.

By combining the above we obtain that if there is j such that $g(\mathbf{r}) = r_j$, then $\mu_w(\mathbf{r}) = r_j$.

- Suppose that there is no i such that $g(\mathbf{r}) = r_i$, then by lemma 1 there is an S such that $g(\mathbf{r}) = f_S = \min(f_S, \min_{i \in S} r_i)$. It follows that for any $k \in S$ we have $r_k > f_S$. Suppose there is an i such that if $k \notin S$ and $r_k > f_S$ then for $S' = S \cup r_k$ since $f'_S \geq f_S$ we have $\min(f_{S'}, \min_{i \in S'} r_i) \geq f_S$. It follows that $f_S = f_{S'}$. We can thus choose S such that for all $k \notin S$, $r_k < f_S$.

  - For any $y > f_S$ we have $\sum_{i:r_i \geq y} w_i \leq \sum_{i:r_i \geq f_S} w_i = f_S < y$, therefore $\mu_w(\mathbf{r}) \leq r_j$.
  - For $y \leq r_j$, we have $\sum_{i:r_i \geq y} w_i \geq \sum_{i:r_i \geq f_S} w_i = f_S \geq y$ therefore $\mu_w(\mathbf{r}) \geq f_S$. Therefore $\mu_w(\mathbf{r}) = f_S$.

  As such $g = \mu_w$. QED.

$\Leftarrow$: If Ψ is the level independent CAF associated with μ_w then for all $\mathbf{r} \in \{0, 1\}^N$, we have $\mu_w(\mathbf{r}) = \sum_{i:r_i = 1} w_i$. As such, Ψ verifies weighted proportionality.

$\square$

One of the interesting aspects of the weighted proportional-cumulative is how it beautifully aggregates dominated inputs. The next proposition shows that the experts will contribute in the aggregation proportionally to their weights on the segment that best describes their role in the group.

In the example of Figures 4 and 5, there are 3 experts, with weights 0.3, 0.5 and 0.2 respectively. The input is a dominated profile where P_1 (in blue) is dominating P_2 which is dominating P_3 (we can see it clearly on the left hand side, with the CDFs but not so easily with the probability density functions (PDFs)). In green is drawn the weighted cumulative that we can better understand on the right hand side. We first follow the PDF of the first voter 0.3 of the total mass, then the green follows the second voter PDF 0.5 of the total mass, then follow the last expert 0.2 of the total mass. This rule of computation is very general as the following proposition shows.

Proposition 6 (Proportional-cumulative for dominated profiles). *Suppose that Λ is an interval, and that all the P_i are continuous and verify for all $i \in N$, $P_i \geq P_{i+1}$. Then the weighted-proportional level-SP mechanism $\psi : \mathcal{P}^N \to \mathcal{P}$ of weight $\mathbf{w}$ can be computed for this input profile as follows:*

$$\psi(\boldsymbol{p})(a) = \begin{cases} p_i(a) & if \quad \sum_{k \leq i-1} w_k \leq P_i(a) < \sum_{k \leq i} w_k \\ 0 & else \end{cases}$$

Proof. According to Theorem 5 we have:

$$\forall a, \forall \mathbf{P}, \Psi(\mathbf{P})(a) = \sup \left\{ y \,\middle|\, \sum_{i:P_i(a) \geq y} w_i \geq y \right\}.$$

Suppose for all $i \in N$, $P_i \geq P_{i+1}$. Wlog we will also suppose that all the weights are strictly positive. Let us determine the value of $\Psi(\mathbf{P})(a)$ for any given a.
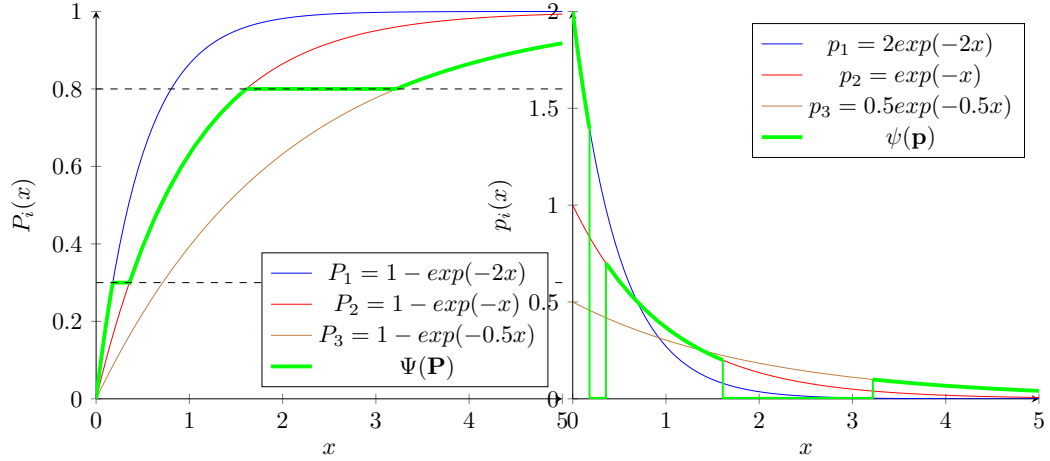
Figure 4: Weighted proportional cumulative (in green) with weights $w_1 = 0.3$, $w_2 = 0.5$ and $w_3 = 0.2$ (in cumulative space).

Figure 5: Weighted proportional cumulative (in green) with weights $w_1 = 0.3$, $w_2 = 0.5$ and $w_3 = 0.2$ (in probability space))

- Suppose $\sum_{k \leq i-1} w_k \leq P_i(a) < \sum_{k \leq i} w_k$.
  - For any $y > P_i(a)$ we have $\{\bar{k} : P_k(a) \geq y\} \subseteq \{k < i\}$. Therefore $\sum_{k:P_k(a) \geq y} w_k \leq \sum_{k \leq i-1} w_k \leq P_i(a) < y$. By definition of μ_w we therefore have $\Psi(\mathbf{P})(a) \leq P_i(a)$.
  - For any $y \leq P_i(a)$ we have $\{k \leq j\} \subseteq \{k : P_k(a) \geq y\}$. As such $\sum_{k:P_k(a) \geq y} w_k \geq \sum_{k \leq j} w_k \geq P_i(a) \geq y$. By definition of μ_w we therefore have $\Psi(\mathbf{P})(a) \geq P_i(a)$.

  We have therefore shown that $\Psi(\mathbf{P})(a) = P_i(a)$.

  By Bolzano's theorem there is a $a_1 \leq a$ such that $P_i(a_1) = \sum_{k \leq j-1} w_k$ and $a_2 > a$ such that $P_i(a_2) = \sum_{k \leq j} w_k$. Therefore on the interval $[a_1, a_2]$, $\Psi(\mathbf{P}) = P_i$. It follows that $\psi(\mathbf{p})(a) = p_i(a)$ on the interval $]a_1, a_2[$.

- Suppose that there is no i such that $\sum_{k \leq i-1} w_k \leq P_i(a) < \sum_{k \leq i} w_k$. Then there is an i such that, $P_{i+1}(a) < \sum_{k \leq i} w_k < P_i(a)$. We show that $\Psi(\mathbf{P})(a) = \sum_{k \leq i} w_k$ by comparing y to $\sum_{k \leq i} w_k$ just as we compared y to $P_i(a)$ in the previous section

  By Bolzano's theorem there is a value $a_1 < a$ such that $P_i(a_1) = \sum_{k \leq j} w_k$ and $a_2 > a_1$ such that $P_{i+1}(a_2) = \sum_{k \leq j} w_k$. We have $\Psi(\mathbf{P})(a_1) = \Psi(\mathbf{P})(a_2) = \sum_{k \leq i} w_k$.

  Therefore $\Psi(\mathbf{P})(a_1) = \Psi(\mathbf{P})(a_2)$. It follows that on the interval $]a_1, a_2[$ $\psi(\mathbf{p})(a) = 0$.

$\square$

7.2 Proportional Cumulative

When all experts have the same weight ($w_i = \frac{1}{n}$ for all $i \in N$), we obtain the proportional cumulative

Definition 8 (proportional-cumulative). *When all expert have the same weight, the proportional cumulative $\Psi : \mathcal{C}^N \to \mathcal{C}$ is the aggregation method defined as follows:*

$$\forall \boldsymbol{P} = (P_1, ..., P_n), \forall a \in \Lambda, \Psi(\boldsymbol{P})(a) = \text{median}\left(P_1(a), \ldots, P_n(a), \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}\right)$$

Hence the One-SP rule associated with the proportional cumulative is

$$g(r_1, ..., r_n) = \text{median}\left(r_1, \ldots, r_n, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}\right).$$

This function is called the uniform median [Bou+15].

Axiom 8 (proportionality). *If experts are single-minded (e.g. $p_i = \delta_{a_i}$ for all $i \in N$), the aggregation must coincide with the average:*

$$\forall (a_1, \ldots, a_n) \in \Lambda^N \quad \psi(\delta_{a_1}, \ldots, \delta_{a_n}) = \sum_{i \in N} \frac{1}{n} \delta_{a_i}$$

If Λ is finite, this corresponds to the proportionality axiom in [Fre+19] in the budgeting problem (that one can interpret as a probability aggregation problem). That's why we also call it proportionality.

Theorem 6 (Proportionality). *The proportional cumulative is the unique Level-SP PAF satisfying proportionality.*

This is an immediate consequence of Theorem 5 and Proposition 5 with $w_i = \frac{1}{n}$ for $i = 1, ..., n$.

7.3 A stronger diversity axiom

One may want to have diversity in the aggregation for all inputs and not only dirac inputs.

Axiom 9 (Diversity). *A PAF ψ is diverse if for every input probability profile $\boldsymbol{p} = (p_1, ..., p_n)$, the support of $\psi(\boldsymbol{p})$ contains the union of the supports of the probabilities p_i, $i = 1, ..., n$.*

The mean $\psi(p_1, ..., p_n) := \frac{1}{n} \sum_{i=1}^{n} p_i$ satisfies diversity but is not level-SP.
The axiom above implies that whenever a (Borel) event has a positive probability for at least one voter, so does society. In particular, if all voters agree that some event has positive probability, e.g. strong plausibility must be satisfied. Hence, from Theorem 3 we deduce the following result.

Theorem 7 (Diversity Impossibility). *If Λ is an interval, no level-SP unanimous PAF satisfies diversity.*

8 Comparing methods

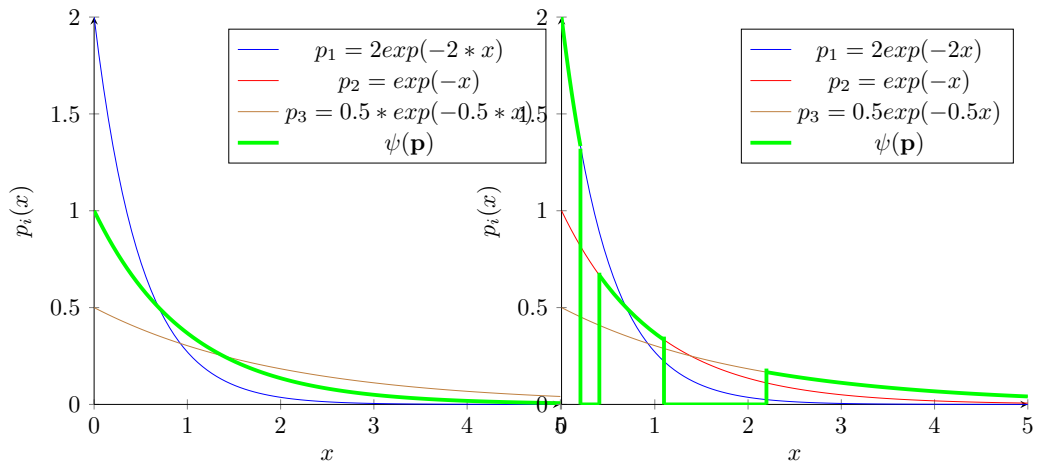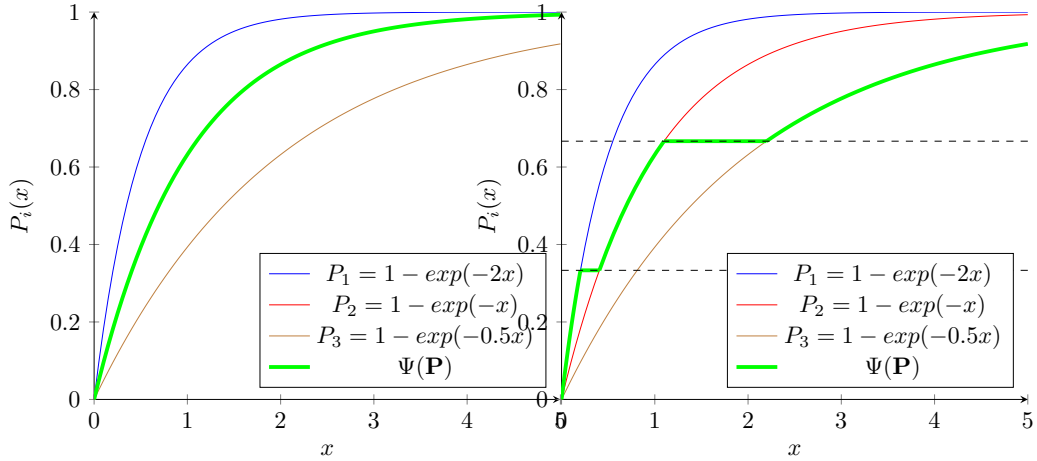## 8.1 Comparing the proportional and the middlemost cumulatives

In our view, the main drawback of the middlemost-cumulative is its lack of diversity. When the various supports S_i of the experts' subjective probabilities p_i are disjoint intervals I_i, or more generally, when the P_i's dominate one another (as in Fig 6), the outcome of middlemost-cumulative equals the view of the median-expert (see the green curve in Figure 6 (CDFs) and Figure 8 (PDFs)).

The proportional-cumulative takes into account all the experts' views in a natural way. As Figure 7 shows, the leftmost expert CDF is followed by the proportional-cumulative for one-third of the probability mass. This is a good property since the fact he is the leftmost expert means that the left third of his probability distribution is what best describes how he differs from the other experts. Similarly, the rightmost expert is followed by the proportional-cumulative for the rightmost third of his opinion, which also best represents how his opinion differs from the group. The middle expert is followed for his middle opinion which again is interesting for the same reason. As such, not only do experts contribute for exactly one n-th of the final outcome but proportional-cumulative ensures that the most sticking aspects of their opinion (compared to the group) are represented.

Given that the order-cumulatives are the unique anonymous level-SP PAF that satisfy certainty and plausibility preservations, the proportional-cumulative must violate one of the axioms. As it is anonymous and certainty preserving (because represented by a unique unanimous one-SP rule: the uniform median), it must violate plausibility preservation. That can be seen in Figure 9: the density of proportional-cumulative (in green) for the interval $[1, 2]$ is zero while the experts' densities for this interval are all strictly positive. We believe that the violation of this axiom is not problematic because it satisfies a "level" version of plausibility, namely if a level event has a positive probability for all experts, so does society. Hence, if we are interested in problems where the decision-maker's final decision is only based on a level-event, only the level plausibility preservation axiom is needed to be satisfied, and it is.

8.2 Comparing with the phantom moving mechanisms

In the context of Budget aggregation with finitely many alternatives in $\Lambda = \{a_1, ..., a_m\}$, [Fre+19] proposed a class of anonymous incentive compatible methods (the phantom moving mechanisms). Their notion of strategyproofness corresponds in our context to the optimality of truthful reporting when the experts have single-peaked preferences measured by the L_1 distance to the peak in the probability space $\mathcal{P}$, that we will denote by $\| \circ \|_1^{\mathcal{P}}$. More precisely, if $p = (p_k)_{k=1,...,m}$ and $q = (q_k)_{k=1,...,m}$ are probability distributions over $\Lambda = \{a_1, ..., a_m\}$, then their distance in the $\mathcal{P}$ space is computed as $d(p, q) = \|p - q\|_1^{\mathcal{P}} = \sum_{k=1}^{m} |p_k - q_k|$.

Figure 6: A dominated cumulative profile (P_1, P_2, P_3). Voter P_1 is more biased to the left, P_3 more biased to the right, $P2$ is the median voter. The middlemost-cumulative (green) coincide with P_2.

Figure 7: The proportional-cumulative (green) follows at its left the leftist voter P_1 for $\frac{1}{3}$ of the mass, at its right the rightist P_3 for $\frac{1}{3}$ of the mass, and at its middle the median voter P_2 for $\frac{1}{3}$ of the mass.

Figure 8: Probabilities densities of CDFs in Figure 6

Figure 9: Probability densities of CDFs in Figure 7

A voter is single-peaked with respect to $\| \circ \|_1^{\mathcal{P}}$ if, whenever p is the voter's peak and q is society's output then its utility is $-\|p - q\|_1^{\mathcal{P}}$. Let's call a method $L_1^{\mathcal{P}}$-SP if honesty is the optimal strategy when a voter's utility is single-peaked in for $\| \circ \|_1^{\mathcal{P}}$.

Theorem 8 (Certainty preserving L_1-SP Impossibility). *When $\#\Lambda \geq 4$, the only certainty preserving ψ that are both Level-SP and $L_1^{\mathcal{P}}$-SP are the dictatorials.*

Proof. The proof can be found in appendixes C.1 and D for additional results. ☐

Hence our methods and the one in [Fre+19; Goe+16] are IC in two different environments, but none is strategically robust in both environments, and no method can be IC in both.

9 Application: Electing and Ranking with Uncertain Voters

The objective of this section is to show that proportional-cumulative could be combined with a recent evaluation-based voting method (Majority Judgment [BL11; BL20], **MJ**) to construct a strategically robust voting method in situations where voters have uncertainties and doubts about the candidates. To elect one candidate/alternative or rank several [Arr51], existing voting methods (plurality, Borda, Condorcet, approval voting, etc) implicitly assume that individual voters are certain about their opinions or views. In practice, voters are mostly uncertain. In recruitment committees, members are often hesitating between a good safe candidate vs a risky one. On the day of the Brexit vote, no voter knew with certainty what the final deal between the UK and the EU would be, nor did they know the long-term consequences of such a deal. Similarly, when a reviewer is judging a paper for a conference, they are often uncertain about the quality of some of the papers. To capture those uncertainties, one can use an extension of majority judgment.

Let us first describe how majority judgment (MJ) works for a small number of voters. Suppose we have a numerical scale $\{0, \ldots, 9\}$ and a set of grades ordered from best to worse $A = (9, 7, 6, 5, 2)$ corresponding to the grades given by 5 voters to candidate A. Its majority value is obtained by iterating from the middlemost grade (the median), then down, up, down, etc, which gives us the 5 dimensional vector $v(A) = (6, 5, 7, 2, 9)$. Two ordered set of grades A and B are compared in lexicographic order by their majority values. For example if $B = (9, 8, 6, 4, 1)$ then $v(B) = (6, 4, 8, 1, 9)$ and so $v(A) > v(B)$ because $(6, 5, 7, 2, 9) \succcurlyeq (6, 4, 8, 1, 9)$. Majority judgment is an ordinal method as the ranking remains unchanged if the numerical grades are replaced by some qualitative set of grades such as {Great, Good, Average, Poor, Terrible}. The MJ ranking can be extended to a continuum of voters (e.g. a normalized distribution over the set of grades, see [BL11] chapter 14).

Imagine N voters who should rank M candidates or alternatives $\mathcal{A} = \{A_1, A_2, A_3, \ldots, A_M\}$. Majority Judgment under Uncertainty **MJU** works as follows:

- *Fixed by the designer:* a scale of grades Λ (such as {Great, Good, Average, Poor, Terrible} and a normalized positive vector $\mathbf{w} = (w_1, \ldots, w_N)$ where w_i is the weight of voter i.
- *Input from voters (the ballots):* each voter $i \in N$ is asked to submit for each candidate $A \in \mathcal{A}$ a probability distribution $p_i^A \in \mathcal{P} = \Delta(\Lambda)$. For example, the voter may think that A_k will be Good for sure, and that A_l will be Great with probability $\frac{2}{3}$ and Terrible with probability $\frac{1}{3}$.
- *Output for voters:* each candidate A is given an aggregate probability distribution p^A **computed using the weighted-proportional-cumulative** that is $P^A(a) = \mu_{\mathbf{w}}(P_1^A(a), \ldots, P_N^A(a))$.
- *Ranking of the alternatives:* classical **majority judgment is applied** to rank the distributions p^{A_1}, p^{A_2}, $\ldots$, p^{A_M} and consequently the candidates/alternatives in $\mathcal{A}$.

Main properties of MJU:

- *IIA*: adding or dropping a candidate does not change the ranking between the others. This is because MJ is transitive and the ranking between two candidates A and B by MJU depends only on their distributions p^A and p^B, which are only a function of the inputs regarding them.
- *Impartiality*: candidates are treated equally by the method. This is because MJ is impartial and MJU applies the same PAF (the weighted-proportional-cumulative) to compute each candidate's A distribution p^A. If we want the method to be *anonymous* (voters are treated equally) just set $w_1 = \cdots = w_N = \frac{1}{N}$.
- *Extends MJ*: if all voters are certain about their choices (i.e. their input are Dirac measures) and have equal weights (anonymity), MJU outputs the empirical distribution of inputs for each candidate, and so the ranking of MJU coincides with the ranking of MJ. Note that proportional-cumulative is the only level-SP method that induces an extension of MJ.
- *Resistance to strategic manipulations*: This is because the proportional-cumulative is level-SP and MJ was designed to counter strategic manipulations.

The philosophy behind MJ was to allow voters to better express themselves (compared to plurality for example) by submitting a grade on a scale Λ for each candidate. MJU goes further in this philosophy by allowing voters to express their uncertainties about the candidates on the same scale.

10 Conclusion

This paper studies the probability aggregation problem when the set of outcomes Λ is an ordered set. It defines level-strategyproofness and proves it to imply classical strategyproofness for a rich set of single-peaked preferences over the CDF space. Several characterizations are established when level-strategy-proofness is combined with other axioms and two methods are singled out:

middlemost-cumulative and proportional-cumulative. Both are easy to compute and can be extended to problems where experts are weighted. The paper gives several arguments supporting the claim that the weighted proportional-cumulative is perhaps the best of all level-SP methods. Its unique weakness is the non-satisfaction of plausibility preservation. Fortunately, it does satisfy a weaker version. Namely, if all experts agree that the probability of a level set is positive, so does society. Hence, if we are interested in problems where the final decision is level-based, then proportional-cumulative is –in our opinion– the best IC method. In practice, the weighted proportional-cumulative can be used to aggregate experts' beliefs in various applications going from voting, nuclear safety, investment banking, volcanology, public health, ecology, engineering to climate change and aeronautics/aerospace. Specific examples include calculating the risks to manned spaceflight due to collision with space debris and quantifying the uncertainty of a groundwater transport model used to predict future contamination with hazardous materials (see Cooke [Coo+91], Chapter 15, where those applications and others with real data are described and several aggregation methods are analyzed).

11 Acknowledgment

References

[Arr51]   Kenneth J Arrow. *Social Choice and Individual Values*. Yale University Press, 1951.

[Goo52]   I.J. Good. "Rational Decisions". In: *J. Roy. Stat. Soc.* 1.4 (1952), pp. 107–114.

[McC56]   J. McCarthy. "Measures of the value of information". In: *Proc. Nat. Acad. of Manag. Sc.* 42.9 (1956), pp. 645–655.

[Win69]   R.L. Winkler. "Scoring rules and the evaluation of probability assessors". In: *J. Amer. Stat. Assoc.* 64.327 (1969), pp. 1073–1078.

[Gib73]   Allan Gibbard. "Manipulation of voting schemes: a general result". In: *Econometrica: journal of the Econometric Society* (1973), pp. 587–601.

[Sat75]   Mark Allen Satterthwaite. "Strategy-proofness and Arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions". In: *Journal of economic theory* 10.2 (1975), pp. 187–217.

[Mou80]   Hervé Moulin. "On strategy-proofness and single peakedness". In: *Public Choice* 35.4 (1980), pp. 437–455.

[BJ83]   Kim C. Border and J. S. Jordan. "Straightforward Elections, Unanimity and Phantom Voters". In: *The Review of Economic Studies* 50.1 (1983), pp. 153–170.

[DF83]   M. De Groot and S. Fienberg. "Effective scoring rules for probabilities forecasts". In: *Manag. Sc.* 29.4 (1983), pp. 447–454.

[Coo+91] Roger Cooke et al. *Experts in uncertainty: opinion and subjective probability in science*. Oxford University Press on Demand, 1991.

[Zho91]   Lin Zhou. "Impossibility of strategy-proof mechanisms in economies with pure public goods". In: *The Review of Economic Studies* 58.1 (1991), pp. 107–119.

[Arm01]   Jon Scott Armstrong. *Principles of forecasting: a handbook for researchers and practitioners*. Vol. 30. Springer Science & Business Media, 2001.

[ATA05]   Nigel W Arnell, Emma L Tompkins, and W Neil Adger. "Eliciting information from experts on the likelihood of rapid climate change". In: *Risk Analysis* 25.6 (2005), pp. 1419–1431.

[Asp06]   WP Aspinall. "Structured elicitation of expert judgment for probabilistic hazard and risk assessment in volcanic eruptions". In: *Statistics in volcanology* 1 (2006), pp. 15–30.

[NP07]   Klaus Nehring and Clemens Puppe. "The structure of strategy-proof social choice—Part I: General characterization and possibility results on median spaces". In: *Journal of Economic Theory* 135.1 (2007), pp. 269–305.

[ONe+08]   Saffron J O'Neill et al. "Using expert knowledge to assess uncertainties in future polar bear populations under climate change". In: *Journal of Applied Ecology* 45.6 (2008), pp. 1649–1659.

[Saa08]   TL Saaty. "Revista de la real academia de ciencias exactas, fisicas y naturales". In: *Serie A: Matematicas.* 102.2 (2008), p. 251.

[BL11]   Michel Balinski and Rida Laraki. *Majority judgment: measuring, ranking, and electing.* MIT press, 2011.

[Bou+15]   Craig Boutilier et al. "Optimal social choice functions: A utilitarian view". In: *Artificial Intelligence* 227 (2015), pp. 190–213.

[PP15]   Urfalino Philippe and Costa Pascaline. "Secrecy and Publicity in Votes and Debates". In: Cambridge University Press, 2015. Chap. 8.

[Goe+16]   Ashish Goel et al. "Knapsack voting: Voting mechanisms for participatory budgeting". In: *Unpublished manuscript* (2016).

[DL17]   Franz Dietrich and Christian List. "Probabilistic opinion pooling generalized. Part one: general agendas". In: *Social Choice and Welfare* 48.4 (2017), pp. 747–786.

[SM17]   Jan Sprenger and Carlo Martini. "Opinion Aggregation and Individual Expertise". In: *Scientific Collaboration and Collective Knowledge* (2017).

[Chr+18]   Annemarie Christophersen et al. "Bayesian network modeling and expert elicitation for probabilistic eruption forecasting: Pilot study for Whakaari/White Island, New Zealand". In: *Frontiers in Earth Science* 6 (2018), p. 211.

[Boy19]   Thomas Boyer-Kassem. "Scientific expertise and risk aggregation". In: *Philosophy of Science* 86.1 (2019), pp. 124–144.

[Fre+19]   Rupert Freeman et al. "Truthful aggregation of budget proposals". In: *Proceedings of the 2019 ACM Conference on Economics and Computation.* 2019, pp. 751–752.

[BL20]   Michel Balinski and Rida Laraki. "Majority Judgment vs. Majority Rule". In: *Social Choice and Welfare* 54 (2020), pp. 429–461.

[Oer+21]   TJ von Oertzen et al. "EAN consensus statement for management of patients with neurological diseases during the COVID-19 pandemic". In: *European journal of neurology* 28.1 (2021), pp. 7–14.

[LM18]   G. Leontaris and O. Morales-Nápoles. "ANDURIL — A MATLAB toolbox for ANalysis and Decisions with UnceRtaInty: Learning from expert judgments". In: *SoftwareX* (2918), pp. 313–317.

Introduction to the appendix

Our main problem is to determine a PAF (probability aggregation function) $\psi : \mathcal{P}^N \to \mathcal{P}$ satisfying some desirable properties where Λ is a Borel subset of $\mathbf{R}$ and $\mathcal{P}$ is the set of probability distributions over Λ. Let $M = \sup \Lambda$. For example, if $\Lambda = \mathbf{R}$, then $M = +\infty$.

To each probability $p \in \mathcal{P}$ is associated a unique cumulative distribution function (or CDF) $P = \pi(p)$ and vice-versa where $P(a) = \int_{x \leq a} p(x) dx$. Let $\mathcal{C}$ denote the set of cumulative distributions on Λ (e.g. increasing and right continuous functions $Q : \Lambda \to [0,1]$ s.t. $\lim_{a \to M} Q(a) = 1$).

Our probability aggregation problem becomes equivalent to an aggregation of CDFs. More precisely, to each PAF ψ is associated a unique cumulative aggregation function (CAF) $\Psi := \pi \circ \psi : \mathcal{C}^N \to \mathcal{C}$ where $\Psi(P_1, ..., P_1) = \psi(\pi^{-1}(P_1), ..., \pi^{-1}(P_n))$ and vice-versa.

We could have defined our problem directly as a cumulative aggregation problem, but it is more convenient to define and use both formulations because: 1) the existing literature is only concerned with the probability aggregation one, 2) all our axioms (except one) are more naturally formulated in the probability space, 3) the level-SP axiom and all our characterizations are more naturally formulated in the cumulative space.

This supplementary material is organized as follows. Section 1 explores the main implications of Level-SP when combined or not with the certainty preservation and/or the plausibility axioms. Section 1.1 provides a step-by-step proof for the main characterizations of level-SP methods with and without anonymity. Section 1.2 (resp. 1.3) establishes the characterizations when one supposes, in addition to level-SP, certainty preservation (resp. plausibility preservation). Section 1.4 derives some additional characterizations when both certainty and plausibility preservations hold. Section 2 provides the rigorous proofs for the characterizations of level-SP methods that are proportional. Section 3 proves some impossibility theorems.

A Level-SP, certainty and plausibility characterizations

A.1 Level-SP characterizations

If a PAF ψ is level-SP, then its associated CAF Ψ verifies the following property.

Lemma 4 (level-SP, rewritten in terms of cumulative). *Suppose ψ is level-SP. Then, for any expert i, for any level $a \in \Lambda$ and any cumulative input votes $P_1, \ldots, P_n, P_i' \in \mathcal{C}$ we have:*

$$P_i(a) < \Psi(\boldsymbol{P})(a) \Rightarrow \Psi(\boldsymbol{P})(a) \leq \Psi(\boldsymbol{P}_{-i}(P_i'))(a)$$

and

$$P_i(a) > \Psi(\boldsymbol{P})(a) \Rightarrow \Psi(\boldsymbol{P})(a) \geq \Psi(\boldsymbol{P}_{-i}(P_i'))(a)$$

where $\Psi := \pi \circ \psi$.

We will now establish some lemmas pertinent to level-SP that are useful in proving the main characterizations of the paper.

Lemma 5 (Level-SP $\implies$ Monotonicity)**.** *If Ψ is level-SP then for all experts i, for all levels $a \in \Lambda$, and all cumulative votes $P_1, \ldots, P_n, P_i'$:*

$$P_i(a) \leq P_i'(a) \Rightarrow \Psi(\boldsymbol{P})(a) \leq \Psi(\boldsymbol{P}_{-i}(P_i'))(a)$$

Proof. We will use a *reductio ad absurdum* to reach our result. Suppose that Ψ verifies level-SP but is not monotonous in all $a \in \Lambda$. Then there is a P_i, P_i' and a such that $P_i(a) \leq P_i'(a)$ and $\Psi(\mathbf{P})(a) > \Psi(\mathbf{P}_{-i}(P_i'))(a)$.
If $\Psi(\mathbf{P}_{-i}(P_i'))(a) \geq P_i(a)$. Then:

$$\Psi(\mathbf{P})(a) > \Psi(\mathbf{P}_{-i}(P_i'))(a) \geq P_i(a)$$

This contradicts level-SP.
If $\Psi(\mathbf{P})(a) \leq P_i'(a)$. Then:

$$P_i'(a) \geq \Psi(\mathbf{P})(a) > \Psi(\mathbf{P}_{-i}(P_i'))(a)$$

Then replacing $P_i'(a)$ by $P_i(a)$ improves the output, this contradicts level-SP.
Else:
$$\Psi(\mathbf{P}_{-i}(P_i'))(a) < P_i(a) \leq P_i'(a) < \Psi(\mathbf{P})(a)$$

This also contradicts level-SP. Therefore level-SP implies monotonicity. $\square$

Lemma 6 (Level-SP $\implies$ Level-by-level Independence)**.** *If Ψ is level-SP then for all $a \in \Lambda$ the value of $\Psi(P_1, \ldots, P_n)(a)$ only depends on $P_1(a), \ldots, P_n(a)$.*

Proof. For any $a \in \Lambda$, suppose we have $P_1, \ldots, P_n$ and $P_1', \ldots, P_n'$ such that for all experts i, $P_i(a) = P_i'(a)$. Then according to the monotonicity lemma (Lemma 5) $\Psi(P_1, \ldots, P_n)(a) = \Psi(P_1', \ldots, P_n')(a)$. Therefore $\Psi(P_1, \ldots, P_n)(a)$ only depends of $P_1(a), \ldots, P_n(a)$. $\square$

Theorem 1 (Level-SP: Max-min formula)**.** *A PAF $\psi : \mathcal{P}^N \to \mathcal{P}$ is level-SP if and only if there exists for every $S \subseteq N$ a weakly increasing right continuous function $f_S : \Lambda \to [0, 1]$ that verifies the following properties:*
- *(1) for all $S \subseteq S'$ and $a \in \Lambda$ we have $f_S(a) \leq f_{S'}(a)$;*
- *(2) if $\sup \Lambda \notin \Lambda$, then $\lim_{a \to \sup \Lambda} f_N(a) = 1$ otherwise $f_N(\sup \Lambda) = 1$;*
- *(3) if $\inf \Lambda \notin \Lambda$ then $\lim_{a \to \inf \Lambda} f_\emptyset(a) = 0$;*
- *(4) $\Psi : \mathcal{C}^N \to \mathcal{C}$ the CAF associated to ψ is given by the formula:*

$$\forall a \in \Lambda, \Psi(\boldsymbol{P})(a) = \pi \circ \psi(\boldsymbol{p})(a) = \max_{S \subseteq N} \min \left(f_S(a), \min_{i \in S} P_i(a)\right).$$

Moreover, ψ is unanimous if and only if for every $a \in \Lambda$ $f_\emptyset(a) = 0$ and $f_N(a) = 1$ (in which case, (2) and (3) are automatically satisfied).
We will call the $\{a \to f_S(a)\}_{S \in 2^N}$ above the phantom functions associated to ψ.

Proof. ⇒:

- Let us show the existence of the phantom functions f_S.
  The level-by-level lemma (Lemma 6) gives us that if Ψ is level-SP then for all $a \in \Lambda$ we have that the value $\Psi(P_1, \ldots, P_n)(a)$ only depends on $P_1(a), \ldots, P_n(a)$ as such level-SP implies the existence of a one-SP function g_a such that:
  $$\forall \mathbf{P}, \Psi(\mathbf{P})(a) = g_a(P_1(a), \ldots, P_n(a)).$$

  Since g_a is one-SP, by Moulin [Mou80] classical result, we have that there are phantom values $b_S^a : S \subseteq N$ that are increasing with S such that:
  $$\forall \mathbf{r}, g_a(\mathbf{r}) = \max_{S \subseteq N} \min \left(b_S^a, \min_{i \in S}\{r_i\} \right)$$

  It follows that there are 2^n phantom functions $f_S : \Lambda \to [0, 1]$ that are increasing over the subsets of S such that:
  $$\forall a \in \Lambda, \forall \mathbf{P}, \Psi(\mathbf{P})(a) = \max_{S \subseteq N} \min \left(f_S(a), \min_{i \in S}\{P_i(a)\} \right)$$
  .

- Let us now show that phantom functions f_S are right continuous and increasing.
  Let us consider any phantom function f_S and any two alternatives $a < b$. Let us suppose that all experts $i \in S$ are single-minded in a (every vote p_i with $i \in S$ is equal to a Dirac mass at $a \in \Lambda$, e.g. $p_i = \delta_a, \forall i \in S$) and that all other experts are single-minded in b. Then the outcome of the interval $[a, b[$ is given by f_S. Since the outcome is a cumulative distribution it must be right continuous and increasing. As such f_S is right continuous and increasing on the interval $[a, b[$. Since this is true for all intervals $[a, b[$ we have that f_S is right continuous and increasing on Λ, and this is true for every S.

- Let us now show that $\lim_{a \to \sup \Lambda} f_N(a) = 1$.
  Since when inputs are cumulative distributions the outcome is a cumulative distribution we have:
  $$\lim_{a \to \sup \Lambda} \Psi(\mathbf{P})(a) = 1,$$

  therefore since for all i, $\lim_{a \to \sup \Lambda} P_i(a) = 1$, it follows that $\lim_{a \to \sup \Lambda} \Psi_a(\mathbf{1}) = \lim_{a \to \sup \Lambda} f_N(a) = 1$. Therefore:
  $$\lim_{a \to \sup \Lambda} f_N(a) = 1.$$

- The proof for if $\inf \Lambda \notin \Lambda$ then $\lim_{a \to \inf \Lambda} f_N(a) = 0$ is symetrical to the one for $\lim_{a \to \sup \Lambda} f_N(a) = 1$.

⇐: Let us fix a, let g_a be the one-SP voting rule defined as:
$$\forall \mathbf{r}, g_a(\mathbf{r}) = \max_{S \subseteq N} \min \left(f_S(a), \min_{i \in S}\{r_i\} \right).$$

Then by definition we have $\forall \mathbf{P}, g_a(P_1(a), \ldots, P_n(a)) = \Psi(\mathbf{P})(a)$. Since g_a is a one-SP rule, we have for all alternatives $a \in \Lambda$ and all profiles $\mathbf{P}$:

$$P_i(a) < \Psi(\mathbf{P})(a) \Rightarrow \Psi(\mathbf{P})(a) \leq \Psi(\mathbf{P}_{-i}(P_i'))(a)$$

and

$$P_i(a) > \Psi(\mathbf{P})(a) \Rightarrow \Psi(P)(a) \geq \Psi(\mathbf{P}_{-i}(P_i'))(a).$$

In other words, Ψ is level-SP.

The increasing right continuity of the f_S functions imply that the outcome is right continuous and increasing. Since $\lim_{a \to \sup \Lambda} f_N(a) = 1$, we know that $\lim_{a \to \sup \Lambda}, \Psi(\mathbf{P})(a) = 1$ by using the formula for the value S. As such, if the inputs are cumulative distributions the outcome is also a cumulative distribution.

$\circ$: Unanimity is immediate by the condition for a unanimous one-SP function in the Moulin's max-min characterization. $\qquad \square$

Theorem 2 (Level-SP: the Median formula). *A PAF $\psi : \mathcal{P}^N \to \mathcal{P}$ is level-SP and anonymous if and only if there exists $n+1$ weakly increasing right continuous function $f_k : \Lambda \to [0,1]$ that verifies the following properties:*
- *for all $0 \leq k \leq n-1$ we have $f_k \leq f_{k+1}$;*
- *if $\sup \Lambda \notin \Lambda$, then $\lim_{a \to \sup \Lambda} f_n(a) = 1$ otherwise $f_n(\sup \Lambda) = 1$*
- *if $\inf \Lambda \notin \Lambda$ then $\lim_{a \to \inf \Lambda} f_0(a) = 0$;*
- *$\Psi : \mathcal{C}^N \to \mathcal{C}$ the CAF associated to ψ is given by the formula:*

$$\forall a \in \Lambda, \Psi(\boldsymbol{P})(a) = \mathrm{median}\,(P_1(a), \ldots, P_n(a), f_0(a), \ldots, f_n(a))$$

Moreover ψ is unanimous if and only if for all $a \in \Lambda$ $f_0(a) = 0$ and $f_n(a) = 1$. **We will call the $\{f_k\}_{k \in \{1,\ldots,n+1\}}$ the phantom functions associated to** *ψ.*

Proof. The proof is essentially the same as the general case except one uses the anonymous characterization. The following only gives the details not given in the general proof.

$\Rightarrow$: Once we have shown the existence of g_a we must show that if ψ is anonymous then for all $a < \Lambda$ we have that g_a is anonymous. Let us do this by *reductio ad absurdum*. Suppose that g_a is not anonymous then there is a permutation σ and inputs $\mathbf{r}$ such that $g_a(\mathbf{r}) \neq g_a(r_{\sigma(1)}, \ldots, r_{\sigma(n)})$. It follows that for any $\mathbf{P}$ such that $\forall i, r_i = P_i(a)$, we have:

$$\Psi(\mathbf{P})(a) \neq \Psi(P_{\sigma(1)}, \ldots, P_{\sigma(n)})(a).$$

As such Ψ is not anonymous. It follows (by the bijection between the CDFs and the PDFs) that ψ is not anonymous. QED.

$\Leftarrow$: It remains to show that ψ is anonymous. It is immediate by the characterization that all g_a are anonymous. As such:

$$\forall a, \forall \mathbf{P}, \Psi(\mathbf{P})(a) = \Psi(P_{\sigma(1)}, \ldots, P_{\sigma(n)})(a).$$

Therefore:
$$\forall \mathbf{P}, \Psi(\mathbf{P}) = \Psi(P_{\sigma(1)}, \ldots, P_{\sigma(n)}).$$

Hence Ψ is anonymous, consequently, by the bijection between the CDFs and the PDFs, so is ψ.

$\square$

Lemma 7. *If ψ is anonymous then the two characterizations and definitions of phantom function (via max-min and median formula) are consistent and linked as follows:*
$$\forall S \subset N : f_S = f_{\#S}.$$

where $\#S$ denote the cardinal of S, the "f" on the left side denotes a phantom function in the max-min formula (general case) and the "f" in the right side it denotes a phantom function in the median formula (anonymous case).

Thus the terminology "phantom functions" is not ambiguous since, in both characterizations, they represent the "same functions."

Proof. Let ψ be a level-SP PAF and let $\{f_S : S \in N\}$ be the associated phantom functions. Let us suppose that $\forall k, \exists h_k \in \Lambda \to [0,1], \forall S \in N : k = \#S \Rightarrow f_S = h_k$. We wish to show that ψ is anonymous and that, in the anonymous characterizations, the phantom functions are $h_1, \ldots, h_n$.

Let us first show that ψ is anonymous. Let σ be a permutation of n elements.

$$\begin{aligned} \forall a, \Psi(P_{\sigma(1)}, \ldots, P_{\sigma(n)})(a) &= \max_{S \subseteq N} \min \left(f_S(a), \min_{i \in S} \{r_{\sigma(i)}\} \right) \\ &= \max_{k \leq n} \min \left(h_k(a), \min_{i \in S : \#S = k} \{r_{\sigma(i)}\} \right) \\ &= \max_{k \leq n} \min \left(h_k(a), \min_{i \in S : \#S = k} \{r_i\} \right) \\ &= \Psi(\mathbf{P})(a) \end{aligned}$$

Now let us show that when we consider the anonymous characterization for ψ we find that $h_k = f_k$.

Suppose that for any given $a < b \leq \sup \Lambda$, the set S of experts are single-minded in a and the rest are single-minded in b. It follows that according to the characterization in theorem 1, we have $\Psi(\mathbf{P})(a) = h_k(a)$ and according to the characterisation in lemma 2 we have $\Psi(\mathbf{P})(a) = f_k(a)$. As such $h_k = f_k$.

$\square$

A.2   Certainty preservation

Proposition 1 (Certainty preservation characterization)**.** *A level-SP PAF ψ is certainty preserving iff it is unanimous and its associated phantom functions f_S are constants.*

Proof. ⇒: Let us show that for a fixed $S \subset N$ we have that f_S is a constant. Consider any two alternatives $a < b < \sup \Lambda$, and suppose that all experts in S are single-minded in a and the rest are single-minded in b. By certainty preservation we have that:

$$f_S(a) = \Psi(\mathbf{P})(a) = \Psi(\mathbf{P})(b) = f_S(b).$$

Since a and b can be arbitrarily chosen, the phantom functions are constants.

⇐: Suppose the f_S are all constants then if for all i, and for any a, b $P_i(a) = P_i(b)$ we have $\Psi(\mathbf{P})(a) = \Psi(\mathbf{P})(b)$. Therefore ψ satisfies certainty preservation. $\square$

A.3 Plausibility Preservation

Proposition 2 (Plausibility preservation characterization). *A Level-SP PAF $\psi : \mathcal{P}^N \to \mathcal{P}$ is plausibility preserving iff each associated phantom function f_S is strictly increasing on the interval where its value is not in $\{0, 1\}$ and $f_\emptyset(a) < 1$ for all $a < \sup \Lambda$ and $f_N(a) > 0$ for all a.*

Proof. ⇒: Let us consider the phantom function f_S and two alternatives $a < b < \sup \Lambda$ such that f_S never takes the values 0 or 1 on the interval $[a, b[$. Suppose all experts that are part of S submit:

$$P_i(c) = f_S(c) + (1 - f_S(c))\frac{c - a}{b - a}.$$

All other experts submit:

$$P_i(c) = f_S(c)\frac{c - a}{b - a}.$$

Since phantom functions are non-decreasing, the inputs are strictly increasing. Since ψ is plausibility preserving we therefore have that $\Psi(\mathbf{P})$ must be strictly increasing on the interval $[a, b[$. Due to theorem 1 we know that on the interval $[a, b[$ we have $\Psi(\mathbf{P}) = f_S$. Therefore we have shown that f_S is strictly increasing on the interval $[a, b[$. Since this proof holds for any interval $[a, b[$ chosen where f_S is never worth 0 or 1 we have shown that f_S is strictly increasing when not worth 0 or 1.

Suppose that there is $a < \sup \Lambda$ such that $f_\emptyset(a) = 1$. Then if all experts agree that $P_i(a) < 1$ we have by plausibility preservation:

$$1 = \Psi(a) < \Psi(\sup \Lambda) = 1$$

This is a contradiction. Therefore we conclude that for all $a < \sup \Lambda$ $f_\emptyset(a) < 1$.

Similarly, if $f_N(a) = 0$ for any a then assuming all experts are single-minded in a. We have $p_i([a, a]) = 1$ and $\psi(\mathbf{p}) = 0$. This contradicts plausibility preservation. It follows that $f_N > 0$.

⇐: Suppose that all the phantom functions $(f_S)_{S \subset N}$ are strictly increasing when not in $\{0, 1\}$, and that $f_\emptyset < 1$ (except maybe in $\sup \Lambda$) and $f_N > 0$. For

any interval $[a, b]$, suppose that for all i, $p_i([a, b]) > 0$. As such for all experts i, we have $P_i(a) < P_i(b)$ and for all $S \subseteq N$, $f_S(a) \leq f_S(b)$ with equality iff $f_S(a) = 1$ or $f_S(b) = 0$.

We will now show by *reducto ad absurdm* we have plausibility preservation. Hence, suppose that:

$$\Psi(\mathbf{P})(a) = \max_{S \subseteq N} \min\left(f_S(a), \min_{i \in S}\{P_i(a)\}\right) = \max_{S \subseteq N} \min\left(f_S(b), \min_{i \in S}\{P_i(b)\}\right) = \Psi(\mathbf{P})(b).$$

Hence, there is a set T such that:

$$\max_{S \subseteq N} \min\left(f_S(a), \min_{i \in S}\{P_i(a)\}\right) = \min\left(f_T(a), \min_{i \in T}\{P_i(a)\}\right)$$

Let us now consider $\min\left(f_T(b), \min_{i \in T}\{P_i(b)\}\right)$. We have the following inequalities:

$$\Psi(\mathbf{P})(a) = \min\left(f_T(a), \min_{i \in T}\{P_i(a)\}\right) \leq \min\left(f_T(b), \min_{i \in T}\{P_i(b)\}\right) \leq \max_{S \subseteq N} \min\left(f_S(b), \min_{i \in S}\{P_i(b)\}\right) = \Psi(\mathbf{P})(b).$$

As such we have:

$$\min\left(f_T(a), \min_{i \in T}\{P_i(a)\}\right) = \min\left(f_T(b), \min_{i \in T}\{P_i(b)\}\right). \tag{2}$$

If there is an expert j such that $\min\left(f_T(b), \min_{i \in T}\{P_i(b)\}\right) = P_j(b)$. Then since $P_j(a) < P_j(b)$, we cannot have the equality in equation (2). As such we have $\Psi(\mathbf{P})(b) = f_T(b) < \min_{i \in T}\{P_i(b)\}$. It follows that $f_T(b) = \Psi(\mathbf{P})(a) \leq f_T(a)$. Therefore $f_T(a) = f_T(b)$. Therefore since f_T is strictly increasing when not equal to 0 or 1 we have $f_T(a) = 1$ or $f_T(b) = 0$.

If $f_T(a) = 1$ we contradict that $f_T(b) < \min_{i \in T}\{P_i(b)\}$. If $f_T(b) = 0$, then $f_T(b) < min(f_N(b), \min_{i \in N}\{P_i(b)\})$ which contradicts that T is the set such that $\Psi(\mathbf{P})(a) = min(f_T(b), \min_{i \in T}\{P_i(b)\})$.

As such we have reached a contradiction. It was absurd to assume that $\Psi(\mathbf{P})(a) = \Psi(\mathbf{P})(b)$. $\qquad\square$

A.4 Combining level-SP, certainty and plausibility preservations

Proposition 3 (Combination 1st Characterization)**.** *A Level-SP Ψ is certainty preserving and plausibility preserving if and only if the output of its associated one-SP voting rule g always output one of its inputs (e.g. $g(r_1, ..., r_n) \subset \{r_1, ..., r_n\}$ for every $(r_1, ..., r_n) \in [0, 1]^N$).*

Proof. $\Rightarrow$: Since ψ is level-SP and certainty preserving the phantom functions f_S are constants. Since ψ is level-SP and plausibility preserving the phantom functions f_S are strictly increasing when not worth 0 or 1. As such the phantoms functions f_S are constants in $\{0, 1\}$ with $f_\emptyset = 0$ and $f_N = 1$.

Let g denote the one-SP associated voting rule and consider any input $\mathbf{r}$. We will use *reducto ad absurdum*. Suppose that $g(\mathbf{r}) \notin \{r_i : i \in N\}$. Then according to theorem 1 we have that $g(\mathbf{r}) \in \{0, 1\}$.

- If $g(\mathbf{r}) = 1$, then according to theorem 1 there is an S such that $f_S = 1$ and if $S \neq \emptyset$ then for all $i \in S$, $r_i = 1$. Since by hypothesis, $1 = g(\mathbf{r}) \notin \{r_i : i \in N\}$, none of the r_i are equal to 1. As such $S = \emptyset$. However since $f_\emptyset = 0$ (certainty preservation) this is absurd, and so we cannot have $g(\mathbf{r}) = 1$.
- If $g(\mathbf{r}) = 0$, then according to theorem 1 $min(f_N, min_{i \in N}\{r_i\}) = 0$. Since $f_N = 1$ (certainty preservation) we therefore have that one of the r_i is equal to 0. However by hypothesis $0 = g(\mathbf{r}) \notin \{r_i : i \in N\}$. We have reached a contradiction.

As such our hypothesis was wrong and so:

$$\forall \mathbf{r}, g(\mathbf{r}) \in \{r_i : i \in N\}.$$

$\Leftarrow$: Suppose that ψ has an associated one-SP voting rule g such that for all $\mathbf{r}$, $g(r_1, ..., r_n) \subset \{r_1, ..., r_n\}$. Then ψ is certainty preserving and level-SP. We also have $\forall \mathbf{r} \in \{0,1\}^N, g(\mathbf{r}) \in \{0,1\}$. As such all phantoms functions are in $\{0,1\}$. Therefore ψ is plausibility preserving. $\square$

Proposition 4 (Combination 2nd Characterization). *A Level-SP ψ is certainty preserving and plausibility preserving iff for any dominated profile $\boldsymbol{P} = (P_1, .., P_n)$, $\psi(\boldsymbol{p}) \in \{p_1, ..., p_n\}$.*

Proof. Let r_i be selected. Then according to lemma 1 $f_{j:r_j \geq r_i} = 1$ and $f_{j:r_j > r_i} = 0$.

Suppose that there is k such that $r_i \neq r_k$ and $f_{j:r_j \geq r_k} = 1$ and $f_{j:r_j > r_k} = 0$. Then we reach a contradiction since f_S is increasing with S and that either $\{j : r_j \geq r_k\} \subseteq \{i : r_j > r_i\}$ or $\{j : r_j \geq r_i\} \subseteq \{i : r_j > r_k\}$.

It follows that $f_{j:r_j \geq r_i} = 1$ and $f_{j:r_j > r_i} = 0$, iff r_i is the outcome.

Since the profile is dominating we have a permutation σ such that $P_{\sigma(1)} \leq \cdots \leq P_{\sigma(n)}$.

Therefore since f_S is increasing with S and that $f_\emptyset = 0$ and $f_N = 1$ we have the existance of i such that $f_{j:P_j \geq P_i} = 1$ and $f_{j:P_j > P_i} = 0$.

A such no matter the alternative a we have $g(\mathbf{P(a)}) = P_i(a)$, therefore $\psi(\mathbf{p}) = p_i$.

$\square$

Theorem 3 (Order Cumulatives). *The order-cumulatives are the unique level-SP PAF that are anonymous, certainty preserving, and plausibility preserving.*

Proof. $\Rightarrow$: As shown above, all f_S are in $\{0,1\}$ and since $g(\mathbf{r}) = median(r_1, \ldots, r_n, f_0, \ldots, f_n)$ for some real values $f_0 \leq ... \leq f_n$, there is k such that $0 = f_1 = f_2 = ... = f_k < f_{k+1} = ... = 1$. Thus, we deduce that $g(\mathbf{r})$ is always the k-th greatest element of $\mathbf{r}$ and so g is an order function.

$\Leftarrow$: Since an order function can be written as $median(r_1, \ldots, r_n, f_0, \ldots, f_n)$ where all the phantoms are worth 0 or 1, the corresponding ψ is anonymous, certainty preserving, plausibility preserving and level-SP. $\square$

B Characterization of the weighted proportional cumulative

B.1 Main weighted theorem

Recall that the weighted proportionality axiom says the following: if all experts are single-minded (every i'vote p_i equals the dirac mass at $a_i \in \Lambda$, e.g. $p_i = \delta_{a_i}, \forall i \in N$), the aggregation must coincide with the weighted average:

$$\forall (a_1, ..., a_n) \in \Lambda^N \quad \psi(\delta_{a_1}, \ldots, \delta_{a_n}) = \sum_i w_i \delta_{a_i};$$

where $w_i \geq 0$ is given and is the normalized weight attributed to expert $i \in N$ (that is, $\sum_{j \in N} w_j = 1$).

That axiom can be rewritten in terms of CAF as follows:

$$\forall (a_1, \ldots, a_n) \in \Lambda^N, \Psi(\delta_{a \geq a_1}, \ldots, \delta_{a \geq a_n})(a) = \sum_{i: a_i \leq a} w_i.$$

This CAF formulation is useful in the proofs but the PAF formulation is more elegant and intuitive.

Theorem 4 (Weighted Proportional Cumulative). *The unique PAF $\psi : \mathcal{P}^n \to \mathcal{P}$ that verifies level-SP and weighted proportionality is the unique certainty preserving one associated to the one-SP rule $\mu_{\boldsymbol{w}}$, that is:*

$$\forall a \in \Lambda, (P_1, \ldots, P_n) \in \mathcal{C}; \Psi(P_1, \ldots, P_n)(a) := \mu_{\boldsymbol{w}}(P_1(a), \ldots, P_n(a))$$

Proof. $\Rightarrow$: Let ψ be level-SP and weighted proportional. Let Ψ be the cumulative aggregation function associated to ψ and f_S be its phantom functions. Let us first show that:
$$\forall S \subseteq N, \forall a \in \Lambda : f_S(a) = \sum_{i \in S} w_i.$$

Let us take any $S \subseteq N$ and $a < \sup \Lambda$. Suppose that all experts $i \in S$ are single-minded in a and all other experts are single-minded in any $b > a$. By weighted proportional we therefore have $g_a(P_1(a), \ldots, P_N(a)) = \sum_{i \in S} w_i$. By level-SP we have $g_a(P_1(a), \ldots, P_N(a)) = f_S(a)$. We conclude that for all alternatives $a < \sup \Lambda$ we have $f_S(a) = \sum_{i \in S} w_i$ and we can assume without loss of generality that this also holds for $a = \sup \Lambda$. Hence we conclude that the phantoms associated to ψ are the constant phantom functions $f_S = \sum_{i \in S} w_i$. Consequently, ψ is certainty preserving, and we denote by g the associated voting rule. We want to show that $g = \mu_w$.

- Suppose that there is j such that $g(\mathbf{r}) = r_j$, then there is S such that $r_j = min\{r_k : k \in S\}$ and $r_j \leq f_S$ (theorem 1). Let $S' = \{k : r_j < r_k\}$. By the same theorem we have that $f_{S'} \leq r_j$.
  - For any $y > r_j$, we have $\sum_{i:r_i \geq y} w_i \leq \sum_{i:r_i > r_j} w_i = f_{S'} \leq r_j < y$. Therefore by definition of μ_w we have $\mu_w(\mathbf{r}) \leq r_j$.

- For any $y \le r_j$, we have $\sum_{i:r_i \ge y} w_i \ge \sum_{i:r_i \ge r_j} \ge \sum_{i:r_i \in S} = f_S \ge r_j \ge y$. Therefore by definition of μ_w, $\mu_w(\mathbf{r}) \ge r_j$.

By combining the above we obtain that if there is j such that $g(\mathbf{r}) = r_j$, then $\mu_w(\mathbf{r}) = r_j$.

- Suppose that there is no i such that $g(\mathbf{r}) = r_i$, then by theorem 1 there is an S such that $g(\mathbf{r}) = f_S$ and $f_S \le \min_{i \in S} r_i$ therefore if $k \in S$ then $r_k > f_S$. Suppose there is an i such that if $k \notin S$ and $r_k > f_S$ then for $S' = S \cup r_k$ since $f'_S \ge f_S$ we have $\min(f_{S'}, \min_{i \in S'} r_i) \ge f_S$. Therefore according to the theorem, $f_S = f_{S'}$. It follows that we can choose S such that for all $k \notin S$, $r_k < f_S$.

  - For any $y > f_S$ we have $\sum_{i:r_i \ge y} w_i \le \sum_{i:r_i \ge f_S} w_i = f_S < y$, therefore $\mu_w(\mathbf{r}) \le r_j$.
  - For $y \le r_j$, we have $\sum_{i:r_i \ge y} w_i \ge \sum_{i:r_i \ge f_S} w_i = f_S \ge y$ therefore $\mu_w(\mathbf{r}) \ge f_S$. Therefore $\mu_w(\mathbf{r}) = f_S$.

  It follows that $g = \mu_w$.

$\Leftarrow$: If Ψ is the certainty preserving, level-SP CAF associated with μ_w then for $\mathbf{r} \in \{0,1\}^N$, we have $\mu_w(\mathbf{r}) = \sum_{i:r_i=1} w_i$. As such ψ verifies weighted proportionality.

$\square$

B.2   Rational weights

Proposition 5 (Rational weights)**.** *If all the weights are rationals $w_i = s_i/d$ then:*

$$\forall \boldsymbol{r} = (r_1, ..., r_n), \mu_{\boldsymbol{w}}(\boldsymbol{r}) := \mathrm{median}(\overbrace{r_1, \ldots, r_1}^{s_1}, \ldots, \overbrace{r_n, \ldots, r_n}^{s_n}, 0, 1/d, \ldots, 1{-}1/d, 1)$$

Proof. First let us consider the anonymous case (all weights are equal) and suppose that we have d experts. According to the weighted proportionality property $f_{\#S} = f_S = \frac{\#S}{d}$. As such:

$$\forall \mathbf{t} = (t_1, ..., t_d), \mu_d(\mathbf{t}) := \mu_{\mathbf{w}=(\frac{1}{d}, \ldots, \frac{1}{d})}(\mathbf{t}) := \mathrm{median}(t_1, \ldots, t_d, 0, 1/d, \ldots, 1{-}1/d, 1).$$

Now that we have established the characterization of μ_d, we wish to establish that if we have n experts such that each expert has weight s_i/d where $s_i \in \mathbf{N}$, then:

$$\forall \mathbf{r}, \mu_w(\mathbf{r}) = \mu_d(\overbrace{r_1, \ldots, r_1}^{s_1}, \ldots, \overbrace{r_n, \ldots, r_n}^{s_n}) = \mathrm{median}(\overbrace{r_1, \ldots, r_1}^{s_1}, \ldots, \overbrace{r_n, \ldots, r_n}^{s_n}, 1/d, \ldots, 1{-}1/d). \tag{3}$$

Intuitively, we simply consider that for each expert i, if expert i has weight $w_i = s_i/d$ then he is duplicated so as to appear s_i times in the anonymous case with d players.

Let us now provide a formal proof. We will show that

$$\forall \mathbf{r}, \lambda_{\mathbf{w}}(\mathbf{r}) = \mathrm{median}(\overbrace{r_1, \ldots, r_1}^{s_1}, \ldots, \overbrace{r_n, \ldots, r_n}^{s_n}, 1/d, \ldots, 1 - 1/d).$$

verifies the weighted property. The uniticity of a level-SP function that verifies the weighted property will then allow us to conclude that $\lambda_{\mathbf{w}} = \mu_{\mathbf{w}}$.

Let us consider $\mathbf{r}$ where all the values are 0 or 1. Let S be the set of experts i such that $r_i = 1$. Since we have s_i iterations of r_i, the number of iterations of 1 in the equation defining $\lambda_{\mathbf{w}}$ is $\sum_{i \in S} s_i$. Conversely the number of iterations of 0 is $\sum_{i \notin S} s_i = d - \sum_{i \in S} s_i$. It follows that the outcome of $\lambda_{\mathbf{w}}$ is the $k_t h$ phantom value. This element is $\frac{k}{n} = \sum_{i \in S} w_i$. This shows that $\lambda_{\mathbf{w}}$ verifies the weighted property. QED.

$\square$

Theorem 5 (Proportionality). *The proportional cumulative is the unique Level-SP PAF satisfying proportionality.*

Proof. Immediate using the previous proposition and theorem 5. $\square$

B.3   Dominated opinions

Proposition 6 (Proportional-cumulative for dominated profiles). *Suppose that Λ is an interval, and that all the P_i are continuous and verify for all $i \in N$, $P_i \geq P_{i+1}$. Then the weighted-proportional level-SP mechanism $\psi : \mathcal{P}^N \to \mathcal{P}$ of weight $\mathbf{w}$ can be computed for this input profile as follows:*

$$\psi(\boldsymbol{p})(a) = \begin{cases} p_i(a) & if \quad \sum_{k \leq i-1} w_k \leq P_i(a) < \sum_{k \leq i} w_k \\ 0 & else \end{cases}$$

Proof. According to proposition 5 we have:

$$\forall a, \forall \mathbf{P}, \Psi(\mathbf{P})(a) = \sup \left\{ y \,\middle|\, \sum_{i : P_i(a) \geq y} w_i \geq y \right\}.$$

Suppose for all $i \in N$, $P_i \geq P_{i+1}$. Wlog we will also suppose that all the weights are strictly positive. Let us determine the value of $\Psi(\mathbf{P})(a)$ for any given a.

- Suppose $\sum_{k \leq i-1} w_k \leq P_i(a) < \sum_{k \leq i} w_k$.
  - For any $y > P_i(a)$ we have $\{\bar{k} : P_k(a) \geq y\} \subseteq \{k < i\}$. Therefore $\sum_{k : P_k(a) \geq y} w_k \leq \sum_{k \leq i-1} w_k \leq P_i(a) < y$. By definition of μ_w we therefore have $\Psi(\mathbf{P})(a) \leq P_i(a)$.
  - For any $y \leq P_i(a)$ we have $\{k \leq j\} \subseteq \{k : P_k(a) \geq y\}$. As such $\sum_{k : P_k(a) \geq y} w_k \geq \sum_{k \leq j} w_k \geq P_i(a) \geq y$. By definition of μ_w we therefore have $\Psi(\mathbf{P})(a) \geq P_i(a)$.

  We have therefore shown that $\Psi(\mathbf{P})(a) = P_i(a)$.

  By Bolzano's theorem there is a $a_1 \leq a$ such that $P_i(a_1) = \sum_{k \leq j-1} w_k$ and $a_2 > a$ such that $P_i(a_2) = \sum_{k \leq j} w_k$. Therefore on the interval $[a_1, a_2]$, $\Psi(\mathbf{P}) = P_i$. It follows that $\psi(\boldsymbol{p})(a) = p_i(a)$ on the interval $]a_1, a_2[$.

- Suppose that there is no i such that $\sum_{k \leq i-1} w_k \leq P_i(a) < \sum_{k \leq i} w_k$. Then there is an i such that, $P_{i+1}(a) < \sum_{k \leq i} w_k < P_i(a)$. We show that $\Psi(\mathbf{P})(a) = \sum_{k \leq i} w_k$ by comparing y to $\sum_{k \leq i} w_k$ just as we compared y to $P_i(a)$ in the previous section

By Bolzano's theorem there is a value $a_1 < a$ such that $P_i(a_1) = \sum_{k \leq j} w_k$ and $a_2 > a_1$ such that $P_{i+1}(a_2) = \sum_{k \leq j} w_k$. We have $\Psi(\mathbf{P})(a_1) = \Psi(\mathbf{P})(a_2) = \sum_{k \leq i} w_k$.

Therefore $\Psi(\mathbf{P})(a_1) = \Psi(\mathbf{P})(a_2)$. It follows that on the interval $]a_1, a_2[$ $\psi(\mathbf{p})(a) = 0$.

$\square$

C   Impossibility results

C.1   Strategyproofness in the probability space

Theorem 6 (Certainty preserving L_1-SP Impossibility)**.** *When $\#\Lambda \geq 4$, the only certainty preserving ψ that are both Level-SP and $L_1^{\mathcal{P}}$-SP are the dictatorials.*

Proof. We will first prove our theorem for 2 experts.

Suppose that $f_{\{1\}} \leq f_{\{2\}} < 1$. We will reach a contradiction.

Let $A > 0$ and B be such that $f_{\{2\}} + A < B$ and $A + B < 1$. Let $p_1 := (0, f_{\{2\}} + A, 0, 1 - (f_{\{2\}} + A))$ and $p_2 := (f_{\{2\}} + A, 0, B - f_{\{2\}}, 1 - A - B)$.

$g(r_1, r_2) = \text{median}(r_1, r_2, f_{\{2\}})$ if $r_2 \geq r_1$

$g(r_1, r_2) = \text{median}(r_1, r_2, f_{\{1\}})$ if $r_2 \leq r_1$

$$p_1 = (0, f_{\{2\}} + A, B - f_{\{2\}} - A, 1 - B)$$
$$P_1 = (0, f_{\{2\}} + A, B, 1))$$
$$p_2 = (f_{\{2\}} + A, 0, B - f_{\{2\}}, 1 - A - B)$$
$$P_2 = (f_{\{2\}} + A, f_{\{2\}} + A, B + A, 1)$$
$$\Psi(\mathbf{P}) = (f_{\{2\}}, f_{\{2\}} + A, B, 1)$$
$$\psi(\mathbf{p}) = (f_{\{2\}}, A, B - f_{\{2\}} - A, 1 - B)$$
$$\|\psi(\mathbf{p}) - p_2\|_1^{\mathcal{P}} = |f_{\{2\}} - (f_{\{2\}} + A)| + |A - 0| + |(B - f_{\{2\}} - A) - (B - f_{\{2\}})|$$
$$+ |(1 - B) - (1 - B - A)|$$
$$\|\psi(\mathbf{p}) - p_2\|_1^{\mathcal{P}} = 4A.$$

Let us now consider what happens when the second expert lies and submits the probability function $p_2' = (f_1, 0, f_2 - f_1, 1 - f_2)$.

$$
\begin{aligned}
p_1 &= (0, f_{\{2\}} + A, B - f_{\{2\}} - A, 1 - B) \\
P_1 &= (0, f_{\{2\}} + A, B, 1)) \\
p_2' &= (f_{\{2\}}, 0, B - f_{\{2\}}, 1 - B) \\
P_2' &= (f_{\{2\}}, f_{\{2\}}, B, 1) \\
\Psi(\mathbf{P'}) &= (f_{\{2\}}, f_{\{2\}}, B, 1) \\
\psi(\mathbf{p'}) &= (f_{\{2\}}, 0, B - f_{\{2\}}, 1 - B) \\
\|\psi(\mathbf{p'}) - p_2\|_1^{\mathcal{P}} &= |f_{\{2\}} - (f_{\{2\}} + A)| + |0 - 0| \\
&\quad + |(B - f_{\{2\}}) - (B - f_{\{2\}})| + |(1 - B) - (1 - B - A)| \\
\|\psi(\mathbf{p'}) - p_2\|_1^{\mathcal{P}} &= 2A.
\end{aligned}
$$

Therefore our hypothesis is false. We therefore have that $f_{\{2\}} = 1$. A similar proof gives us that $f_{\{1\}} = 0$. As such:

$$
\forall \mathbf{r}, r_1 \leq r_2 \Rightarrow \mathrm{median}(r_1, r_2, 1) = r_2
$$

and

$$
\forall \mathbf{r}, r_1 \leq r_2 \Rightarrow \mathrm{median}(r_1, r_2, 0) = r_2.
$$

Therefore if $f_{\{1\}} \leq f_{\{2\}}$ we have that the second expert is a dictator. Else the first expert is a dictator.

It remains to show that we can add new experts.

Let us suppose that we have shown that if we have n experts then one is a dictator. Let us now consider $n + 1$ experts.

Let us consider a level-SP and certainty preserving ψ with $n + 1$ experts, let g be its associated voting rule. For any expert i there is a level-SP and certainty preserving φ_i (with associated voting rule g_i) such that:

$$
\forall \mathbf{P} \in \mathcal{C}^n, \varphi_i(\mathbf{P}) = \psi((\mathbf{P}), P_i).
$$

Since φ_i is certainty preserving and level-SP it is a dictatorship, let v_i be the dictator. As such so are all the g_i.

Suppose that there exists i such that $v_i \neq i$. Then for any $r_i \neq r_{v_i}$ we have $g(\mathbf{r}, r_i) = r_{v_i}$ therefore by level-SP for any value r_{n+1} we have $g(\mathbf{r}, r_{n+1}) = r_{v_i}$. Therefore ψ is a dictatorship with dictator v_i.

Else for all $i \leq n$ we have that $g(\mathbf{r}, r_i) = r_i$. As such for S such that $n + 1 \in S$ by considering $\mathbf{r}$ such that for all $i \in S$ we have $r_i = r_{n+1}$ and all experts $i \notin S$ verify $r_i = 0$ we wind that $f_S \geq r_{n+1}$. As such $f_S = 1$. Similarly if S does not have the expert $n+1$ then $f_S = 0$. This positioning of the phantoms corresponds to the dictatorship with dictator $n + 1$.

We have shown that no matter what ψ is a dictatorship.

$\square$

C.2 Strong plausibility

Certainty preservation was defined with respect to Borel sets, while plausibility was defined only with respect to intervals. One may wonder why we did such a choice?

Axiom 10 (Strong plausibility preservation). *A probability aggregation function $\psi : \mathcal{P}^N \to \mathcal{P}$ verifies strong plausibility preservation iff for any input profiles $\boldsymbol{p} \in \mathcal{P}^N$ and any Borel measurable event $A \subset \Lambda$:*

$$p_i(A) > 0 \quad \forall i \in N \to \psi(\boldsymbol{p})(A) > 0;$$

that is, if all expert agrees that some Borel event is plausible, so does society.

Unfortunately, extending the plausibility preservation to Borels is a too strong axiom.

Theorem 7 (Strong Plausibility Impossibility). *When Λ is a real interval, the unique probability aggregation functions that are level-SP, unanimous, and strong plausibility preserving are the dictatorials.*

Proof. Let ψ be a probability aggregation function that is level-SP, unanimous, and strong plausibility preserving. Suppose we can choose $\mathbf{p}$ and experts $i \neq j$ such that there is $a < b$ that verify $\Psi(\mathbf{P})(a) = P_i(a) = P_i(b) = \Psi(\mathbf{P})(b)$ and for all experts $k \neq i$ we have $P_k(a) \neq \Psi(\mathbf{P})(a)$ and $p_i([a,b]) > 0$. Suppose we can also have $[c, d[$ disjoint from $[a, b]$ such that $\Psi(\mathbf{P})(c) = P_j(c) = P_j(d) = \Psi(\mathbf{P})(d)$ and $P_i(c) \neq \Psi(\mathbf{P})(c)$ and $p_i([c, d[) > 0$.

For such an example, we contradict strong preservation for $A = [a, b] \cup [c, d[$. As such no such example exists.

Let us show that their exists an expert i such and an interval $[a, b]$ such that we can find $\mathbf{p}$ that verifies $\Psi(\mathbf{P})(a) = P_i(a) = P_i(b)\Psi(\mathbf{P})(b)$. By unanimity for all a we have $f_\emptyset(a) = 0$ and $f_N(a) = 1$. The previous proves that there is an expert i such that for all S without i there is a positive and finite number of alternatives a_S such that there exists S where $f_S(a_S) = 0$ and $f_{S \cup \{i\}}(a_S) > 0$ and for all $a > a_S$ $f_S(a) > 0$. As such since Λ is rich we can find an interval $[a, b]$ and an S without i such that for all $a' \in [a, b]$ we have $f_S(a') = 0$ and $f_{S \cup \{i\}}(a') > 0$. Let us choose $\mathbf{P}$ such that $0 < P_i(a) = P_i(b) < f_{S \cup \{i\}}(a)$ and $P_i(b) < P_k(a) < P_k(b)$ if $k \in S$ and $P_k(a) < P_k(b) < P_i(a)$ if $k \notin S$ and $k \neq i$. This profile verifies $\Psi(\mathbf{P})(a) = P_i(a) = P_i(b)\Psi(\mathbf{P})(b)$.

As such we can find an expert i and $a < b$ such that $\Psi(\mathbf{P})(a) = P_i(a) = P_i(b)\Psi(\mathbf{P})(b)$ and all experts $k \neq i$ verify $P_i(a) \neq \Psi(\mathbf{P})(a)$. Therefore either (1) we cannot find an expert $j \neq i$ and an interval $[c, d[$ disjoint from $[a, b]$ such that $\Psi(\mathbf{P})(c) = P_j(c) = P_j(d) = \Psi(\mathbf{P})(d)$ or (2) $P_i(c) = \Psi(\mathbf{P})(c)$.

Suppose that (1) is false. There is an interval $[c, d[$ disjoint of $[a, b]$ such that all experts $k \neq i$ are constant and equal on $[c, d]$. If $\Psi(c) = P_k(c) = P_k(d) = \Psi(\mathbf{P})(d)$ then for all sub-intervals $[c', d'[$ of $[c, d[$ this is true as such $P_i(c') = \Psi(\mathbf{P})(c')$. Therefore $\Psi = P_i$ for $[c, d[$.

Else $\Psi(\mathbf{P}(c) \neq P_k(c)$ or $\Psi(\mathbf{P})(d) \neq P_k(d)$.

Suppose that $\Psi(\mathbf{P}(c) \neq P_k(c)$. Then by Moulin's characterization $\Psi(\mathbf{P}(c) \in \{f_{N-i}(c), f_i(c), P_i(c)\}$ If $\Psi(\mathbf{P}(c) < P_i(c)$ then we select $f_{N-i}(c)$. By changing to $P'_k(c) = P'_k(d) > f_{N-i}(c)$ we can find a sub-interval $[c, d']$ of $[c, d]$ were $\Psi(\mathbf{P})(c) = P_j(c) = P_j(d') = \Psi(\mathbf{P})(d')$. This is absurd. A similar proof show that $\Psi(\mathbf{P}(c) = f_i(c)$ results in a contradiction. It follows that $P_i(c) = \Psi(\mathbf{p})(c)$.

Suppose that $\Psi(\mathbf{P}(c) = P_k(c)$. Then by considering considering subsets $[c, d'[$ or $[c, d[$ we find that for all d' we have $\Psi(\mathbf{P}(d') \neq P_k(d)$. Therefore $\Psi(\mathbf{P})(d') > P_k(d)$. As such by unanimity $P_i(d') > \Psi(\mathbf{P})(d')$.

By considering $[c', d[$ and using the above we find that for all c' we have $P_i(c') = \Psi(\mathbf{P})(c')$. By right continuity we therefore have $P_i(c) = \Psi(\mathbf{P})(c)$.

We have shown that no matter what $c \notin [a, b]$ $P_i(c) = \Psi(\mathbf{P})(c)$. By replacing $[a, b]$ by a subset we can show that for all $c \in \Lambda$ we have $P_i(c) = \Psi(\mathbf{P})(c)$.

Therefore i is a dictator. $\qquad\square$

C.3 Diversity

Theorem 8 (Diversity Impossibility). *If Λ is an interval, no Level-SP unanimous PAF satisfies diversity.*

Proof. Diversity implies strong plausibility. As such we can simply use the strong plausibility impossibility theorem (theorem 3. $\qquad\square$

C.4 Weak Diversity

Axiom 11 (Weak Diversity). *For every diracs inputs $(p_1, ..., p_n) = (\delta_{a_1}, ..., \delta_{a_n})$, there exists positive weights $w_1 > 0, ..., w_n > 0$ such that $\psi(\delta_{a_1}, ..., \delta_{a_n}) = \sum_{i=1}^{n} w_i \delta_{a_i}$.*

Why do we call it weak diversity? because the axiom requires that, all experts possible opinions (when they are degenerate) have a positive probability in the output.

Weighted proportional methods are weak diverse. But this axiom is much weaker than the weighted proportionality axiom because the expert's weights may depend on the input in the weak diversity axiom while in the case of proportionality, the expert's weights are the same for all inputs. This opens the door to a much larger class of methods, that can be characterized as follows.

Proposition 7. *The association of weak diversity with Level-SP is equivalent to certainty preservation $+$ that f_S is strictly increasing with respect to S.*

Proof. $\Rightarrow$: Suppose ψ verifies weak diversity and level-SP. Let $a < M$, and suppose that all experts are single minded, that is the input is $(\delta_{a_1}, ..., \delta_{a_n})$.

If no expert wanted elements in the interval $]a, b]$ ($a_i \notin]a, b]$ for all i) then by weak diversity $\Psi(a) = \Psi(b)$. As such for any S we have that if the experts i in S have $a_i \leq a$ and the rest have $b < a_i$ then $f_S(a) = \Psi(a) = \Psi(b) = f_S(b)$. Since we can consider any interval we can conclude that the f_S are constants. Therefore we are certainty preserving. Since whenever all experts are single

minded, no matter what alternative they choose the weight of each of those alternative must be felt ($w_i > 0$) then for all $S \neq \emptyset$ for all $i \in S$, $f_S > f_{S-i}$.

$\Leftarrow$: Suppose certainty preservation and that f_S is strictly increasing. Then, for any input of single-minded experts $(\delta_{a_1}, ..., \delta_{a_n})$ and all $a \in \Lambda$

$$\Psi(\mathbf{P})(a) = f_{\{i:a_i \leq a\}}(a)$$

Therefore, $\psi(\delta_{a_1}, ..., \delta_{a_n})(a) = f_{\{i:a_i \leq a\}} - f_{\{i:a_i < a\}}$ and so:

$$\psi(\delta_{a_1}, ..., \delta_{a_n}) = \sum_{i=1}^{n} (f_{\{i:a_j \leq a_i\}} - f_{\{i:a_j < a_i\}}) \delta_{a_i}$$

$\square$

By construction, the weighted proportional cumulative methods satisfy weak diversity. However, we have seen that they do not preserve plausibility. On the other hand, the order-cumulative PAFs are not weak diversified but they do preserve plausibility. Are there methods that are weak diversified and plausibility preserving? Unfortunately, no!

Theorem 9 (Weak diversity Impossibility). *For 3 or more experts, there are no level-SP probability aggregation functions that satisfy weak diversity and preserve plausibility.*

Proof. Suppose that $\psi : \mathcal{P}^N \to \mathcal{P}$ is a level-SP probability aggregation function that satisfies weak diversity and preserves plausibility. Then, since weak diversity implies certainty preservation and that ψ preserves plausibility, we must have $f_S \in \{0, 1\}^N$. On the other hand, weak diversity implies that f_S is strictly increasing with S. By pigeon-hole principle, we reach a contradiction. $\square$

Unfortunately, weighted proportional cumulative methods do satisfy a weaker version of plausibility, namely, if all voters agree that a level event has positive probability, so does society. This is good enough for the applications where the regulator's final decision only depends on the probability of some level events.

D   Combining the two notions of SP

Here we prove the existence of methods that are at the same time strategyproof in the cumulative (level-SP) and in the probability spaces ($L_1^{\mathcal{P}} - SP$ as in [Fre+19]).

Proposition 8. *For $\#\Lambda = 3$, any Level-SP PAF is also $L_1^{\mathcal{P}} - SP$.*

Proof. Suppose that Λ has three alternatives $\{a_1, a_2, a_3\}$.

$\|\psi(\mathbf{P}) - p_i\|_1 = |\Psi(\mathbf{P})(a_1) - P_i(a_1)| + |\Psi(\mathbf{P})(a_2) - \Psi(\mathbf{P})(a_1) - P_i(a_2) + P_i(a_1)| + |\Psi(\mathbf{P})(a_2) - P_i(a_2)|$

Let $\mathbf{Q}$ only differ from $\mathbf{P}$ in dimension i.

Suppose that $\|\psi(\mathbf{P}) - p_i\|_1 > \|\psi(\mathbf{Q}) - p_i\|_1$.

By level-SP it is impossible to decrease $|\Psi(\mathbf{P})(a_1) - P_i(a_1)|$ or $|\Psi(\mathbf{P})(a_2) - P_i(a_2)|$. Therefore we decreased $|\Psi(\mathbf{P})(a_2) - \Psi(\mathbf{P})(a_1) - P_i(a_2) + P_i(a_1)|$. If you did this by causing a change to the value of $\Psi(\mathbf{P})(a_2)$ then you increased $|\Psi(\mathbf{P})(a_2) - P_i(a_2)|$ by the same amount (resp for $\Psi(\mathbf{P})(a_1)$). Therefore we cannot decrease the value for the L_1 norm. We have reached our contradiction. $\square$

Proposition 9. *There is an infinite number of ψ functions that are Level-SP and L_1-SP for 4 alternatives.*

But such methods cannot be certainty preserving, unless dictatorial, as shown above.

Proof. Let us choose a Level-SP function Ψ such that, $\Psi_1 = min$ and $\Psi_3 = max$.
- Let us suppose that by changing P_1 player i can improve his output for $|\psi_j(a_2) - p_j(a_2)|$. Then he decreases Ψ_1 which is detrimental for $|\psi_j(a_1) - p_j(a_1)|$. Therefore changes to P_1 cannot improve the $\| \circ \|_1$ distance.
- Let us suppose that by changing P_3 player i can improve his output for:

$$|\psi_j(a_3) - p_j(a_3)|$$

  . Then he increases Ψ_3 which is detrimental for $|\psi_j(a_3) - p_j(a_3)|$. Therefore changes to P_3 cannot improve the $\| \circ \|_1$ distance.
- Suppose that $p_2 > \psi_2$ then since $p_1 \geq \psi_1$ we have $P_2 > \Psi_2$. Therefore by level SP it is impossible to improve $|p_2 - \psi_2|$.
- Similarly if $p_3 > \psi_3$ then by level SP it is impossible to improve $|p_2 - \psi_2|$.
- Suppose that $p_2 \leq \psi_2$ and $p_3 \leq \psi_3$. Therefore any changes to Ψ_2 is detrimental to one and beneficial to the other.

Therefore we are both level-SP and L_1-SP. $\square$