Stable Matching Games

Felipe Garrido-Lucero[1], *Rida Laraki*[1,2]

1. *LAMSADE (CNRS, UMR 7243), Université Paris Dauphine-PSL*
2. *University of Liverpool, Computer Science Department*

Abstract

Gale and Shapley introduced a matching problem between two sets of agents where each agent on one side has an exogenous preference ordering over the agents on the other side. They defined a matching as stable if no unmatched pair can both improve their utility by forming a new pair. They proved, algorithmically, the existence of a stable matching. Shapley and Shubik, Demange and Gale, and many others extended the model by allowing monetary transfers. We offer a further extension by assuming that matched couples obtain their payoff endogenously as the outcome of a strategic game they have to play in a usual non-cooperative sense (without commitment) or in a semi-cooperative way (with commitment, as the outcome of a bilateral binding contract in which each player is responsible for his/her part of the contract). Depending on whether the players can commit or not, we define in each case a solution concept that combines Gale-Shapley pairwise stability with a (generalized) Nash equilibrium stability. In each case, we give the necessary and sufficient conditions for the set of stable allocations to be non-empty, we study its lattice structure, and provide an algorithm that converges to its maximal element. Finally, we prove that our second model -with commitment- encompasses and refines most of the literature.

1 Introduction

The Gale and Shapley [5] two-sided market matching problem consists in finding a "stable" pairing between two different sets of agents M and W given that each agent on one side has an exogenous preference ordering over the agents on the other side.

The marriage problem focuses on a coupling μ that associates to each agent on one side, to at most one agent on the other side. The coupling μ is stable if no uncoupled pair of agents $(m, w) \in M \times W$, both prefer to be paired together rather than with their partners in μ. Gale and Shapley [5] used a "propose-dispose", also called "deferred-acceptance", algorithm to prove the existence of a stable matching for every instance. Knuth [10] proved a lattice structure over the set of stable matchings (mentioned in Gale and Shapley [5]). Gale and Sotomayor [7] showed that the algorithm in which men are proposing outputs the best stable matching for men.

Shapley and Shubik [13] extended the model by allowing monetary transfers. Demange and Gale [4] considered more general utility functions for money (non-quasi-linear), allowed monetary transfers on both sides (from buyer to seller and vice-versa) and proved that the set of stable allocations has a lattice structure (non-emptiness of this set has been proved in [3, 12]). Hatfield and Milgrom [8], extended the Demange-Gale model to a one-to-many setting by allowing couples to sign a "binding contract". Under monotonicity assumptions allowing the use of Tarski's fixed point theorem, they proved that the set of stable allocations is a non-empty lattice. Chiappori and Reny [2] studied a model where men and women must form couples and, simultaneously, determine a sharing rule for splitting their total income.

In real life bilateral markets, to be attractive, an agent can take actions that cannot be modeled by monetary transfers. When a firm hires a worker, it can combine the monetary transfer with employee perks: medical insurance, gym, extra time-off, flexible schedule, childcare assistance. The worker can promise to be flexible, work hard, learn new technologies, and be respectful of the company code of conduct. When a university hires a professor, it can reduce or increase its teaching duties, requires a minimum number of top publications, ask for some responsibilities in the department, etc. The professor can promise or not to publish in top journals, be an excellent teacher, apply to/win grants, accept some responsibilities, organize a seminar, and supervise Ph.D. students. All those actions are individual decisions that can be put explicitly or implicitly in a contract but each agent is responsible regarding her own part of the contract. Each agent will do what is needed to be accepted by the other party and will refuse to engage if he/she judges the partner's proposition is insufficient.

2 Matching Games Model

We extend the above matching models by supposing that individual members of a couple $(i, j) \in M \times W$, obtain their payoffs as the output of a strategic game $G_{i,j} = (X_i, Y_j, U_{i,j}, V_{i,j})$, that they have to play, where X_i is i's action/strategy set, Y_j is j's action/strategy set, and $U_{i,j}, V_{i,j} : X_i \times Y_j \to \mathbb{R}$ are the utility functions of i and j, respectively. Hence, if i and j are matched, i chooses to play x_i and j chooses to play y_j, i's and j's final utilities are $U_{i,j}(x_i, y_j)$ and $V_{i,j}(x_i, y_j)$, respectively. An outcome/allocation of the matching game, called a *matching profile*, is a triple (μ, x, y) with μ a matching between M and W, $x = (x_i)_{i\in M} \in \prod_{i\in M} X_i$ a strategy profile for all agents in M, and $y = (y_j)_{j\in W} \in \prod_{j\in W} Y_j$ a strategy profile for all agents in W. For example, a matching problem with linear transfers can be represented by a family of constant-sum games where the set of strategies are $X_i = Y_j = \mathbb{R}^+$, and the payoff functions are $U_{i,j}(x_i, y_j) = -x_i + y_j + a_{i,j}$ and $V_{i,j}(x_i, y_j) = x_i - y_j + b_{i,j}$, with $a_{i,j}$ and $b_{i,j}$ representing the utility of being with the partner when there is no transfer.[1]

Suppose that some centralized or decentralized process leads to a matching profile where agents (men/women or workers/firms) are matched in pairs and each matched player is intended to play some action. We want to formulate the necessary conditions for that matching profile to be sustainable (e.g. stable). We will consider two static stability notions which depend on the players' level of commitment before they play their game.

3 No Commitment - Results

The first studied case is when a matched couple (i, j) **cannot commit** on actions. In this case, for the players not to deviate from the intended actions, these last must constitute a Nash equilibrium of $G_{i,j}$. A matching profile (μ, x, y) will be called *Nash stable* if (a) all matched couples play a Nash equilibrium of their game and (b) no pair of agents (i', j'), that are not already a couple, can jointly deviate to some Nash strategy profile $(x'_{i'}, y'_{j'})$ in their game $G_{i',j'}$ that Pareto improves their payoffs. This last condition is a natural extension of the Gale-Shapley pairwise stability condition as condition (b) implies that matching profiles cannot have -Nash-blocking pairs. Using a propose-dispose algorithm, we prove that whenever all games $G_{i,j}$ admit a non-empty compact set of Nash equilibria, a Nash stable matching profile exists. In addition, we obtain a semi-lattice structure: the maximum between two Nash stable allocations with respect to men's preferences (resp. women's preferences) is Nash stable (although the minimum is not necessarily well defined). When all games $G_{i,j}$ have a unique Nash equilibrium payoff (as in strictly competitive games), the model becomes a classical Gale-Shapley problem and so, the lattice structure is recovered.

It is important to remark that games with linear transfers, like the ones described above, are constant-sum games where the unique Nash equilibrium is $(x^*_i, y^*_j) = (0, 0)$, as the null transfer is a strictly dominant strategy. For positive transfers to occur, players must be able to commit. This is implicitly assumed in the literature of matching with transfers.

[1] The game of transfers is constant sum because the sum of payoffs $U_{i,j}(x_i, y_j) + V_{i,j}(x_i, y_j) = a_{i,j} + b_{i,j}$ is independent on x_i or y_j. Strategically, this is equivalent to a zero-sum game and it is a particular instance of a strictly competitive game.

4 Commitment - Results

The second studied case corresponds to the one in which **players can commit (e.g. (1) by signing binding contracts or (2) because the game is infinitely repeated and so any deviation from the agreed stationary strategy profile at some stage is immediately punished at the next stage by breaking the relation, or (3) because they decide their strategies/types in a first round (e.g. investment in education), and then decide to form or not a couple in a second round, etc)**. A matching profile (μ, x, y) is called *externally stable* if no pair of agents $(i, j) \notin \mu$ can jointly deviate to some strategy profile (x'_i, y'_j) in their game $G_{i,j}$ that Pareto improves their payoffs (no blocking pairs). A similar propose-dispose algorithm to the one without commitment allows us to prove that, if all the strategic games $G_{i,j}$ have compact strategy spaces and continuous payoff functions, the matching game always admits an externally stable matching profile. As above, a semi-lattice structure holds as well: the maximum between two externally stable matching profiles with respect to men's preferences (resp. women's preferences) is externally stable. Even more, when all games $G_{i,j}$ are constant-sum games (or more generally, are strictly competitive games) a lattice structure holds. This extends Shapley-Shubik's and Demange-Gale's models as they are particular instances where the games $G_{i,j}$ are strictly competitive. In addition, as proved by Gale and Sotomayor [6] for the marriage problem, we prove that our algorithm outputs the highest element, with respect to the proposer side, of the lattice.

Players may choose their actions optimally. Thus, a **constrained Nash equilibrium** (CNE) condition must naturally hold. An externally stable matching profile (μ, x, y) is *internally stable* if any profitable deviation of a player in its game decreases the partner's payoff below his/her market outside option. Said differently, fixing y_j, x_i must maximize i's payoff under j's participation constraint, and vice-versa. Once again, in other words, players within a couple must best reply to the partner's strategy subject to guarantee his/her market outside option. If we adopt the two stages interpretation of Nöldeke-Samuelson's model [11] where players decide first on their strategies and then whether to form or not a couple, a CNE is just a Nash equilibrium of the two stages game in which partners of a couple agree to form in the second stage.

Putting all together, our solution concept combines a cooperative notion (Gale-Shapley pairwise stability) with a non-cooperative notion (a generalized Nash equilibrium). A similar solution concept is used in network formation games: fixing the network, each player's action must maximize its payoff, and for each link in the network, both players must agree to form it (see Jackson and Wolinsky [9], Bich and Morhaim [1]). Our model can be seen as a particular network game model where only bi-party graphs are possible and a link is formed if a man and a woman agree to match.

CNE are not always guaranteed to exist as they belong to the class of generalized Nash equilibria[2]. We define the class of **feasible games** as those who admits CNE's existence and prove it includes constant-sum, strictly competitive, potential, and infinitely repeated games, as well as: (a) when all games $G_{i,j}$ are feasible, a new algorithm, if it converges, reaches an externally and internally stable matching profile and (b) this new algorithm converges when all games are constant-sum, strictly competitive, potential or infinitely repeated, as well as combination of them. As strictly competitive games are feasible, Shapley-Shubik's and Demange-Gale's results are recovered and refined by our internal stability concept, which reduces drastically the set of stable outcomes in their model.

Proving that a game is feasible, as well as the convergence of the algorithm mentioned above, uses the particular properties of each of the games. In other words, it is a game dependent proof. More in detail, the convergence of the algorithm passes through the accurate choice of an oracle, which itself depends on the given class of games. It is still open the question of how large is the

[2]Indeed, there exist finite games in mixed strategies without CNE.

class of feasible games, as well as if the existence of externally and internally stable outcomes holds for any game in the class of feasible games.

References

[1] Bich, P., and Morhaim, L. On the existence of pairwise stable weighted networks. *Mathematics of Operations Research, to appear* (2017).

[2] Chiappori, P.-A., and Reny, P. J. Matching to share risk. *Theoretical Economics 11*, 1 (2016), 227–251.

[3] Crawford, V. P., and Knoer, E. M. Job matching with heterogeneous firms and workers. *Econometrica: Journal of the Econometric Society* (1981), 437–450.

[4] Demange, G., and Gale, D. The strategy structure of two-sided matching markets. *Econometrica: Journal of the Econometric Society* (1985), 873–888.

[5] Gale, D., and Shapley, L. S. College admissions and the stability of marriage. *The American Mathematical Monthly 69*, 1 (1962), 9–15.

[6] Gale, D., and Sotomayor, M. Ms. machiavelli and the stable matching problem. *The American Mathematical Monthly 92*, 4 (1985), 261–268.

[7] Gale, D., and Sotomayor, M. Some remarks on the stable matching problem. *Discrete Applied Mathematics 11*, 3 (1985), 223–232.

[8] Hatfield, J. W., and Milgrom, P. R. Matching with contracts. *American Economic Review 95*, 4 (2005), 913–935.

[9] Jackson, M., and Wolinsky, A. A strategic model of economic and social network. *J Econ Theory 71* (1996), 44–74.

[10] Knuth, D. E. Marriages stables. *Technical report* (1976).

[11] Nöldeke, G., and Samuelson, L. Investment and competitive matching. *Econometrica 83*, 3 (2015), 835–896.

[12] Quinzii, M. Core and competitive equilibria with indivisibilities. *International Journal of Game Theory 13*, 1 (1984), 41–60.

[13] Shapley, L. S., and Shubik, M. The assignment game I: The core. *International Journal of game theory 1*, 1 (1971), 111–130.