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Modelling Volatility with Markov-Switching GARCH Models

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Liverpool, August 2022

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Introduction

The relationship between risk and expected return, where risk is interpreted as financial uncertainty and volatility, plays a central role in financial economics. Indeed, volatility is a key input to many investment decisions, portfolio selection, pricing of derivative securities and risk management decisions; and has effects on the economy as a whole (Poon and Granger, 2003). Given the relevance of volatility forecasts in economic and finance, several approaches have been taken to produce these forecasts. Among the most popular time series volatility models is the autoregressive conditional heteroskedasticity (ARCH) introduced by Engle (1982) and its generalisation to GARCH by Bollerslev (1986). GARCH models specify the conditional volatility of returns at time t based on the information set at time $t-1$. Therefore, forecasts of the volatility can be obtained in a straightforward way using iterative procedures.

A good volatility model should be able to capture the empirical regularities of asset returns: volatility clustering, leptokurtosis, the asymmetric reaction of the volatility to past positive and negative returns ("leverage effect"), and the possibility of exogenous or predetermined variables influencing volatility (Engle and Patton, 2007). The volatility clustering property implies that large volatility changes tend to be followed by large changes, of either sign, and small changes tend to be followed by small changes (Mandelbrot, 1997). Consequently, volatility shocks today will influence the expectation of volatility many periods into the future. That is, there exists significant autocorrelation in volatility despite insignificant autocorrelation in returns. Furthermore, stock return distributions often exhibit excess kurtosis which could be explained by the presence of outliers. The standard GARCH model of Bollerslev (1986) successfully captures both volatility clustering and leptokurtosis. However, this model does not accommodate the "leverage effect" or asymmetry in volatility. The "leverage effect" first explored empirically by Black (1976) and Christie (1982) implies that negative returns increase future volatility by a larger amount than positive returns of the same magnitude. For this reason, multiple refinements of Bollerslev's (1986) model have been proposed to include volatility asymmetry in the GARCH context. The most popular asymmetric GARCH models include the exponential GARCH (EGARCH) process of Nelson (1991), the threshold GARCH (TGARCH) model of Zakoian (1994) and the GJRGARCH model of Glosten et al. (1993). In an important paper, Hentschel (1995) develops a family of models that nests virtually all symmetric and asymmetric GARCH models proposed in the literature.

A major drawback of single-regime GARCH models, however, is that they do not account for structural changes in the volatility process. An early study by Lamoureux and Lastrapes (1990) argues that neglected deterministic changes in the unconditional variance of the return process lead to strong persistence measured by GARCH models. Mikosch and Stărică (2004) and Hillebrand (2005) show that this high degree of persistence, known as "integrated GARCH effect", is caused by the nonstationary behaviour of very long return series. Indeed, financial markets often change their behaviour abruptly (Ang and Timmermann, 2012). For example, the S&P 500 index lost one-third of its value in March 2020 during the COVID-19 pandemic. As another example of sudden market changes, the implied volatility of the S&P 500 portfolio, which is referred to as the VIX index, rose in value by approximately 175% during September 2008 in response to the Lehman Brothers collapse (Schwert, 2011). Some changes are transitory ("jumps"). For instance, it did not take too long for the stock market to recover during the COVID-19 pandemic. At the beginning of June 2020, the S&P 500 was about 95% of its peak value in February 2020 (Capelle-Blancard and Desroziers, 2020). Oftentimes, the changed behaviour of the market continues for many periods; e.g., the mean and volatility patterns of October 2008 persisted during the global financial crisis of 2008-2009. Engle et al. (2013) divide their sample into sub-samples to deal with the structural breaks presented in their data. However, since most structural breaks are only known *a posteriori*, a model that incorporates the impact of structural breaks potentially improves forecasts of the volatility.

The Markov-switching model introduced by Hamilton (1989) is one of the most popular nonlinear time series models to capture the abrupt changes of behaviour in the market and the persistence of the changes for several time periods. This model allows different structures that describe the time series dynamics in different regimes. The process governing the switching between regimes follows a latent (i.e., unobservable) homogeneous first-order Markov chain. Thus, the probability of a change in regime depends on the past only through the value of the most recent regime. In this way, a volatility dynamic persists for a period of time and is replaced by another dynamic when a switching takes place. GARCH models have been implemented in Hamilton's (1989) Markov-switching framework. The first contributions to the regime-switching autoregressive heteroscedastic literature, however, argue that Markov-switching GARCH models are not tractable (Hamilton and Susmel, 1994; Cai, 1994). Indeed, the lag structure of GARCH models causes dependence of the conditional variance on the entire past history of the process. Since the regimes are unobservable, it is necessary to consider all possible regime paths when estimating the volatility which makes estimation extremely difficult as the number of possible paths increases exponentially with time. To solve this problem, Gray (1996) proposes an approximation method where conditional variances depend on the expectation of previous conditional variances. Past conditional variances are aggregated across regimes, hence resulting in a model that is not path dependent. Gray's (1996) collapsing procedure is later refined by Klaassen (2002) who considers the expectation of past conditional variances given all past information and the current regime.

Chapter 1: Markov-switching GARCH models: A unifying framework

In Chapter 1, we extend Hentschel's (1995) family of single-regime GARCH models to a regime-switching framework. We follow the collapsing procedure developed by Gray (1996)-Klaassen (2002). Thus, we propose a tractable Markov-switching GARCH model that includes many, if not all, symmetric and asymmetric Markov-switching GARCH models. We examine the properties of our model and provide necessary and sufficient conditions for it to be asymptotic stationary. Since the model constitutes a unifying framework in which several models are nested, our conditions apply to all the included models in the family. In this sense, we make a substantial contribution to the literature on the statistical properties of Markov-switching GARCH models. In fact, conditions for asymptotic stationarity of a Markov-switching GARCH model in Gray's (1996)-Klaassen's (2002) framework have only been studied for the symmetric standard GARCH model case (Klaassen, 2002; Abramson and

Cohen, 2007). In practice, most Markov-switching GARCH models are commonly restricted to impose stationarity conditions obtained for single-regime GARCH models separately within each regime (see Francq and Zakoian, 2019, Chapter 2, for a review on stationarity conditions for single-regime GARCH models). However, these restrictions may be too conservative as it is the interaction between the parameters in each volatility regime and the probabilities of transition between regimes which determines the stationarity of the Markov-switching GARCH models. Indeed, we show that our model is stationary as long as the single-regime conditions are satisfied on average with respect to the transition probabilities of the regimes even if the GARCH parameters in some regime are explosive (in the sense that do not satisfy stationarity conditions for single-regime models).

In an empirical application, we apply the model to daily returns of the S&P 500 Index between January 2000 and March 2019. Our unified framework is useful for model selection and tests of functional form. For this reason, we estimate the nested models for comparison. We also estimate the single-regime specifications of the models. We consider two regimes and use a fat-tailed distribution, the Student-t, to account for the excess kurtosis in the data. The results show that asymmetric Markov-switching GARCH models outperform single-regime models and symmetric models when estimating the volatility of returns. Moreover, we find that including asymmetry and regime-switching significantly improves out-of-sample performance in terms of forecasting Value-at-Risk (VaR).

Chapter 2: Forecasting Value-at-Risk and Expected Shortfall with Markov-switching GARCH models

In Chapter 2, the forecasting performance of the models introduced in Chapter 1 is further explored from a risk management perspective. In particular, we compare the accuracy of risk predictions in terms of Value-at-Risk (VaR) and Expected Shortfall (ES) of a total of twenty-six models including parametric, nonparametric and semi-parametric models. Apart from the single-regime and Markov-switching GARCH models studied in Chapter 1, we consider nonparametric models based on historical simulation. Historical simulations are the most popular techniques for forecasting risk among commercial banks, mainly due to their simplicity of calculation. We also include in our analysis the models recently proposed by Patton et al.

(2019). These models are semiparametric in that they impose a parametric structure on the dynamics of VaR and ES but make no assumptions about the conditional distribution of returns. To the best of our knowledge, this is the first paper that provides a comparison of regime-switching GARCH models with nonparametric and semiparametric models in terms of VaR and ES forecast accuracy.

Most studies of the implications of Markov-switching GARCH models for risk management focus on VaR. VaR measures the maximum potential loss in value of a risky asset or portfolio over a target horizon for a given probability (Jorion, 2007). VaR is easy to calculate and understand, however, it has important deficiencies. First, VaR is not a coherent risk measure as it does not meet the subadditivity axiom (Artzner et al., 1999). Thus, there may be cases in which, in contrast with the idea of diversification, the total VaR of a portfolio is larger than the sum of the VaRs of the portfolio components. In addition, VaR measures do not provide any information regarding the loss beyond the estimated VaR level. For these reasons, the Basel Committee on Banking Supervision (2019) recommends the use of ES instead of VaR. ES gives the expected return on an asset conditional on the return being below the VaR level. ES has the advantage that is a coherent measure. Recent literature on the performance of Markov-switching GARCH models in predicting VaR and ES includes a large empirical study using returns of stocks, equity indices and foreign exchange rates in Ardia et al. (2018). The authors find that regime-switching models significantly outperform single-regime GARCH models. Other papers such as Caporale and Zekokh (2019) and Maciel (2020) show the superiority of Markov-switching GARCH over single-regime GARCH models in the context of cryptocurrencies.

We use data for four international stock market indices: the S&P 500, Dow Jones Industrial Average, the NIKKEI 225 and the FTSE 100. Using traditional and recently developed backtesting procedures for VaR and ES, we show that asymmetric Markov-switching GARCH models significantly outperform nonparametric, semiparametric, single-regime GARCH and symmetric models. Furthermore, the Model Confidence Set (MCS) procedure of Hansen et al. (2011) identifies our most flexible Markov-switching GARCH specification as the best performing model for VaR and ES prediction at different tail probability values. Moreover, according to Diebold and Mariano's (1995) tests of equal predictive ability, the superiority of our model is significant and consistent across the four indices and probabilities level considered in our study.

Chapter 3: Uncertainty and volatility: A Markov-switching GARCH-MIDAS approach

In Chapter 3, we study the relationship between the volatility of the S&P 500 index and macroeconomic uncertainty. Despite the substantial progress on modelling time-varying volatility with GARCH models, the links between stock market volatility, observed on a daily basis, and macroeconomic variables observed at a lower frequency were not considered in the GARCH context until 2013 in the seminal paper by Engle et al. (2013). The authors introduce the MIxed Data Sampling (MIDAS) approach of Ghysels et al. (2004) into the GARCH class of models. In this way, the GARCH-MIDAS model permits the inclusion of macroeconomic variables (observed at a monthly, quarterly or even lower frequency) directly into the specification of the volatility dynamics and enables improved forecasts of the volatility.

We extend the GARCH-MIDAS model to account for regime changes. To the best of the author's knowledge, our proposed model encompasses all GARCH-MIDAS models already proposed in the literature. Besides regime changes and mixed-frequency data, the model has the advantage that it accommodates the asymmetric impact of innovations on volatility. To the best of the authors knowledge, no asymmetric Markov-switching GARCH-MIDAS model has been applied so far in the literature. The publications either account for regime-changes in symmetric GARCH-MIDAS (e.g., Pan et al., 2017) or asymmetries in single-regime GARCH-MIDAS (see e.g., Borup and Jakobsen, 2019; Conrad and Kleen, 2020; Wang et al., 2020). In this sense, our paper is the first to combine regime switches and asymmetries in GARCH-MIDAS models. We show that accounting for both aspects of the volatility leads to substantial improvements in empirical fit and predictive accuracy.

As explanatory variables, we use two measures of uncertainty observed at a monthly frequency: the Financial Uncertainty (FU) index of Ludvigson et al. (2020) and the Macroeconomic Uncertainty (MU) index of Jurado et al. (2015), and a measure of daily fluctuations in the market, the daily Chicago Board Options Exchange Volatility Index (VIX). Our results yield the following in-sample and out-of-sample conclusions. First, our model provides a significantly better in-sample fit than nested single-regime and regime-switching GARCH-MIDAS models proposed in the literature. Furthermore, the results suggest that the models that combine the monthly FU variable and the VIX index in the specification of the volatility give a superior characterisation of the data than models with just one explanatory variable or with the combination MU-VIX. In fact, the MU variable is not found to provide useful information when estimated along with the VIX.

In the out-of-sample exercise, we obtain volatility forecasts at horizons of 1 day, 2 weeks, 1 month, 2 months and 3 months. We show that our model provides more accurate forecasts than the remaining GARCH-MIDAS models at essentially all horizons. We also find that models with the FU variable outperform for longer forecast horizons of 2 weeks, 1 month, 2 months and 3 months. On the other hand, models that consider the VIX variable achieve more precise 1 day-ahead forecasts than those that incorporate only one of the monthly macroeconomic variables. Consequently, our results suggest that high-frequency uncertainty proxies such as the VIX variable outperform at one-day horizon and low-frequency uncertainty such as the FU index performs better at longer forecast horizons ranging from 2 weeks to 3 months. In this way, our findings provide useful indication for selecting the appropriate explanatory variable based on the required forecast horizon.

Chapter 1

Markov-switching GARCH models: A unifying framework

Abstract

This paper examines statistical properties of a family of Markov-switching GARCH (MSGARCH) models. The family is obtained by extension of Hentschel's (1995) nesting of single-regime GARCH to the regime-switching framework introduced by Gray (1996) and Klaassen (2002). The resulting specification nests a wide range of symmetric and asymmetric MSGARCH models already proposed in the literature. Consequently, our specification is useful for model selection and tests of functional form. We derive necessary and sufficient conditions for asymptotic stationarity of the general model. In an empirical application using daily returns of the S&P 500 index, we show that the nested asymmetric MSGARCH models provide superior empirical fit and Value-at-Risk (VaR) forecasts accuracy than single-regime and symmetric MSGARCH models.

1.1 Introduction

Volatility, as a measure of uncertainty, is central to much of modern finance theory. Since the introduction of GARCH (generalised autoregressive conditional heteroskedasticity) models by Engle (1982) and Bollerslev (1986), multiple extensions of the standard GARCH model have been proposed to capture stylised facts of financial time series such as leptokurtosis and asymmetric volatility (or "leverage effect") first noted by Black (1976). The most popular asymmetric GARCH models include the exponential GARCH (EGARCH) process of Nelson (1991), the GJRGARCH model of Glosten et al. (1993) and Zakoian's (1994) threshold GARCH (TGARCH) model[1]. Hentschel (1995) provides a parametric family of models that nests a wide range of symmetric and asymmetric GARCH models. The family is obtained by applying a Box-Cox (1964) transformation to the conditional volatility. This approach accommodates the inclusion of models in which the logarithm of the conditional variance is parameterized, such as the EGARCH model.

A major drawback of single-regime GARCH models is that they do not account for structural changes in the volatility process. In fact, applications using high-frequency data show that GARCH-type models often involve a high degree of persistence. This finding motivates the introduction of the integrated GARCH model (IGARCH) by Engle and Bollerslev (1986). In IGARCH processes, a shock to the conditional variance is persistent in the sense that it affects future forecasts of all horizons. However, Lamoureux and Lastrapes (1990) argue that it might be misleading to assume strong persistence in variance and show that it may be overestimated due to the existence of deterministic structural shifts in the variance process that are not taken into account. Failure to consider these structural breaks leads to poor forecasting performance (see e.g., Marcucci, 2005 and Ardia et al., 2018).

One possibility to allow for periods with different volatility dynamics is the use of Markov-switching GARCH (MSGARCH) models. Following Hamilton's (1989) regime-switching model, the parameters of the conditional variance equation can be viewed as the outcome of a latent discrete-state Markov process. However, as pointed out by Hamilton and Susmel (1994) and Cai (1994), combining the regime-switching model with GARCH leads to estimation problems as a result of path dependence. The lag structure of GARCH models causes dependence of the conditional variance on the entire history of states. Since the states are unobservable, it is necessary to consider all possible regime paths when estimating the volatility. This is infeasible in practice as the number of possible paths increases exponentially with time. Gray (1996) introduces a collapsing procedure that circumvents this problem. His idea is to integrate out the unobserved regime path by replacing lagged conditional variances with their expectations over the entire set of states; see also Dueker (1997) for a related approach. Klaassen (2002)

[1] Important surveys on GARCH models include Bollerslev et al. (1994), Bollerslev (2010), Andersen et al. (2006) and Francq and Zakoian (2019).

refines Gray's model by using the expectation conditional on all available information and the current regime. An alternative Markov-switching GARCH model avoiding path dependence is proposed by Haas et al. (2004). The authors assume that past conditional variances are in the same regime as the current conditional variance. More recently, Augustyniak et al. (2018) introduce a different concept of collapsing procedure based on the particle filter algorithm. This approach includes Gray's (1996), Dueker's (1997) and Klaassen's (2002) methods as special cases.

MSGARCH models are widely used in the field of econometrics both by practitioners and academics. Henry (2009) extends the collapsing procedure proposed by Gray (1996) to a regime-switching EGARCH model to investigate the relationship between equity returns and short-term interest rates. Reher and Wilfling (2016) compare the performance of asymmetric and symmetric models within Klaassen's (2002) regime-switching GARCH framework to estimate and forecast daily volatility of the German DAX index. Augustyniak and Boudreault (2012) analyse monthly returns of the S&P 500 price index using different models which include those proposed by Gray (1996), Klaassen (2002) and Haas et al. (2004). See Ané and Ureche-Rangau (2006) and Sajjad et al. (2008) for more applications of Gray's (1996) and Klaassen's (2002) collapsing procedures. Other papers have focused on simulation-based methods to estimate path-dependent MSGARCH models; e.g., see Bauwens et al. (2014), Augustyniak (2014), Dufays (2016) and Billio et al. (2016). In recent years, MSGARCH models have gained special attention in the context of cryptocurrencies (see e.g., Caporale and Zekokh, 2019; Maciel, 2020).

Statistical properties of path-dependent MSGARCH models are found in Francq et al. (2001), Francq and Zakoian (2005), Francq and Zakoïan (2006), Meitz and Saikkonen (2008) and Bauwens et al. (2010), among others. Regarding path-independent MSGARCH models, Klaassen (2002) develops necessary conditions for stationarity of his model in the case of two regimes and first-order GARCH processes. Haas et al. (2004) and Liu (2006) give necessary and sufficient conditions for stationarity of Haas et al.'s (2004) model in the special case of first-order GARCH. Abramson and Cohen (2007) derive stationarity conditions for Klaassen's (2002) and Haas et al.'s (2004) models in the general case of k-regimes and GARCH processes of any order. Liu (2007) extends the model of Haas et al.'s (2004) to allow for an asymmetric response of volatility to positive and negative shocks and obtains conditions for the existence of stationarity solution and high-order moments.

In this paper, we perform a theoretical and empirical analysis of Markov-switching GARCH (MSGARCH) models. The analysis is based on a family of MSGARCH models that incorporates Hentschel's (1995) nesting of single-regime symmetric and asymmetric GARCH models to the regime-switching framework introduced by Gray (1996) and later refined by Klaassen (2002). The resulting model nests virtually all symmetric and asymmetric MSGARCH models already proposed in the literature[2]. Thus, it is useful for model selection under a unifying framework. In the theoretical part of the paper, we derive necessary and sufficient conditions for asymptotic stationarity of the general model. Hence, since the model constitutes a unified framework in which several models are included, stationary conditions of the nested models are obtained as special cases. The development of these conditions makes a substantial contribution to the literature on statistical properties of MSGARCH models within Gray's (1996)-Klaassen's (2002) framework. Indeed, these conditions have only been examined by Klaassen (2002) and by Abramson and Cohen (2007) for the symmetric regime-switching standard GARCH model. We extend the proofs derived in Abramson and Cohen (2007) to obtain results for a wide range of symmetric and asymmetic regime-switching GARCH models. In the second part of the paper, we conduct an empirical application using daily returns of the S&P 500 index. We show that asymmetric MSGARCH models yield superior in-sample fit and out-of-sample forecasts of Value-at-Risk (VaR) than single-regime models and symmetric MSGARCH models. In this way, we confirm the existing evidence in the literature regarding asymmetry and regime changes in the dynamics of the volatility of stock returns.

This paper contains four sections. The next section introduces the family of MSGARCH models and provides necessary and sufficient conditions for stationarity. Section 1.3 describes the data and presents estimation and forecasting results. Section 1.4 concludes.

[2]Exceptions of GARCH models that do not fit in our model are the smooth transition GARCH (STGARCH) model of González-Rivera (1998) and the Volatility Switching GARCH (VSGARCH) model of Fornari and Mele (1997).

1.2 Nesting Markov-switching GARCH models

This section first presents a general single-regime model that nests many commonly used GARCH processes and then extends this model to a regime-switching framework following the collapsing procedure proposed by Gray (1996) and Klaassen (2002).

1.2.1 A family of GARCH models

GARCH(p, q) models consist of two elements: the mean equation and the variance equation. Since the focus of this paper is the volatility rather than the mean, we assume that returns can be described as

$$r_t = \mu + \varepsilon_t \tag{1.1}$$

where μ is constant representing the mean[3] and ε_t denotes the innovation process.

The variance equation specifies the variance at time t, conditional on the set of available information $\mathcal{I}_{t-1} = \{r_{t-i},\, i \geq 1\}$, as a function of p past innovations and q past conditional variances. Hentschel (1995) proposes the following GARCH$(1, 1)$ model for the innovation process[4]:

$$\begin{aligned}
\varepsilon_t &= z_t\,\sigma_t \mid \mathcal{I}_{t-1} \sim \mathcal{F}\left(0, \sigma_t^2, \nu\right) \\
\frac{\sigma_t^\lambda - 1}{\lambda} &= \omega + \alpha\, f^{\hat{\lambda}}(z_t)\sigma_{t-1}^\lambda + \beta\,\frac{\sigma_{t-1}^\lambda - 1}{\lambda}, \quad \lambda \geq 0, \;\; \hat{\lambda} > 0 \\
f^{\hat{\lambda}}(z_t) &= \left[|z_{t-1} - \psi| - \gamma\,(z_{t-1} - \psi)\right]^{\hat{\lambda}}
\end{aligned} \tag{1.2}$$

where $\{z_t\}$ is a sequence of independent and identically distributed (iid) zero-mean unit-variance random variables; $\mathcal{F}(0, \sigma_t^2, \nu)$ is a continuous distribution with mean zero, time-varying variance σ_t^2 and shape parameter ν. The parameters ω, α, β, γ and ψ are subject to appropriate constraints in order to guarantee that the conditional variance takes positive real values. For $\lambda = 0$ or $\lambda = \frac{1}{n}$ with n integer number, no constraints are required since the

[3] For modelling asset returns, the mean is often treated as a function of the conditional volatility. Assuming time-varying conditional mean does not make the formulas that follow essentially different.

[4] The model can be generalised to a higher-order model. For simplicity, we only present the first-order case. Furthermore, the first-order GARCH is the most commonly applied parameterization in practice.

conditional variance is found by exponentiation or by raising σ_t^λ to an even power and hence positivity is guaranteed. For the remaining cases, sufficient conditions for positivity of σ_t^λ, and therefore σ_t^2, are

$$\lambda\omega + 1 - \beta > 0, \qquad \alpha, \beta \geq 0, \qquad |\gamma| \leq 1 \tag{1.3}$$

By appropriately choosing the parameters λ, γ and ψ, model (1.1)-(1.2) nests a wide range of symmetric and asymmetric GARCH models. Hentschel (1995) illustrates the asymmetry in his model via the news impact curve of Pagan and Schwert (1990) and Engle and Ng (1993). The news impact curve is useful in understanding the impact of shocks on the conditional variance. It measures how news impacts on the conditional time-varying volatilities. The asymmetric models are those that allow at least one of the parameters γ or ψ to be nonzero. The parameter γ causes a rotation to the news impact curve while the parameter ψ causes a shift to the news impact curve. Most asymmetric GARCH models take into account the leverage effect by transforming the news impact curve of the standard GARCH[5]. As in Hentschel (1995), we introduce asymmetry by allowing a shift and a rotation of the news impact curve of the standard GARCH via the function

$$f^{\hat{\lambda}}(z_t) = \left[|z_{t-1} - \psi| - \gamma\,(z_{t-1} - \psi)\right]^{\hat{\lambda}}$$

As shown in Figure 1.1, for $\hat{\lambda} < 1$ the function $f^{\hat{\lambda}}(z_t)$ is concave while for $\hat{\lambda} > 1$ it is convex. A positive value of γ causes a rotation to the right (clockwise). If the news impact curve is clockwise rotated, negative shocks increase volatility by a larger amount than positive shocks of the same magnitude (asymmetry effect). Thus, we expect nonnegative values of γ. On the other hand, the parameter ψ causes a shift of the news impact curve. As shown in Figure 1.1, the plots are shifted by $|\psi|$ units to the right (left) when ψ is positive (negative).

[Figure 1.1 here]

The vast majority of the models considered in the literature do not allow for both a shift and a rotation simultaneously. To the best of the authors knowledge, a model that allows for

[5]The news impact curve of the standard GARCH model is symmetric centred on 0. Therefore, no distinction is made between the effects of positive and negative shocks on the conditional volatility.

both a shift and a rotation simultaneously are only applied in Hentschel (1995) and Reher and Wilfling (2016). In fact, most of the literature considering asymmetric models allows just for a rotation of the news impact curve. In particular, the EGARCH, TGARCH and GJRGARCH models are the most widely used asymmetric GARCH models. However, a combination of both a shift and a rotation can reinforce the asymmetric effect on the volatility. Furthermore, despite the fact that the terms asymmetry and leverage effect are used interchangeably in the GARCH literature, Caporin and Costola (2019) argue that both effects should not be treated as the same phenomenon. The authors contend that a GARCH model is consistent with the presence of leverage effect if negative (positive) shocks lead to an increase (decrease) in variance and show that the NAGARCH model causing a shift of the news impact curve is the only GARCH model allowing for local leverage effect. Therefore, by permitting both rotation and shift of the news impact curve we accommodate two types of asymmetry: (a) asymmetry in the sense that positive and negative news of the same magnitude have different effects on the volatility and (b) leverage effect as defined in Caporin and Costola (2019).

Below we briefly describe the nested models in (1.2). In order to formulate the nested models in a way that is consistent with the standard literature, we present an equivalent representation of (1.2). After reparametrization, (1.2) can be rewritten as:

$$\ln\sigma_t^2 = \dot{\omega} + \dot{\alpha}\,\left(|z_{t-1}| - \mathbb{E}\,|z_{t-1}| - \gamma\, z_{t-1}\right) + \beta\ln\sigma_{t-1}^2, \qquad \text{if } \lambda = 0,\ \hat{\lambda} = 1 \tag{1.4}$$

$$\sigma_t^{\lambda} = \ddot{\omega} + \left\{\ddot{\alpha}\,\left[|z_{t-1} - \psi| - \gamma\,(z_{t-1} - \psi)\right]^{\hat{\lambda}} + \beta\right\}\sigma_{t-1}^{\lambda}, \qquad \text{if } \lambda,\ \hat{\lambda} > 0 \tag{1.5}$$

where $\dot{\omega} = 2\,\omega + 2\,\alpha\,\mathbb{E}\,|z_{t-1}|$, $\dot{\alpha} = 2\,\alpha$, $\ddot{\alpha} = \lambda\,\alpha$ and $\ddot{\omega} = \lambda\,\omega + 1 - \beta$. Table 1.1 shows the parameter restrictions and the volatility specifications of the nested models[6,7].

[Table 1.1 here]

Exponential GARCH (EGARCH) model ($\lambda = 0$, $\hat{\lambda} = 1$, γ free, $\psi = 0$)

The value of $\frac{\sigma_t^{\lambda}-1}{\lambda}$ in (1.2) at $\lambda = 0$ is established using L'Hopital's rule whereby, if $f(\lambda)$ and

[6] Without risk of confusion, we drop the superscripts in the ω and α parameters in (1.4)-(1.5).

[7] Note that, for most of the models we have $\hat{\lambda} = \lambda$. Hence, our choice to label the exponent of the standardised shocks function $f(z_t) = |z_{t-1} - \psi| - \gamma\,(z_{t-1} - \psi)$ by $\hat{\lambda}$.

$g(\lambda)$ are two differentiable functions, then

$$\lim_{\lambda \to a} \frac{f(\lambda)}{g(\lambda)} = \lim_{\lambda \to a} \frac{f'(\lambda)}{g'(\lambda)}$$

Therefore, we have

$$\lim_{\lambda \to 0^+} \frac{\sigma_t^\lambda - 1}{\lambda} = \lim_{\lambda \to 0^+} \sigma^\lambda \ln \sigma_t = \ln \sigma_t$$

and the volatility equation in (1.2) becomes

$$\ln \sigma_t = \omega + \alpha \left[|z_{t-1} - \psi| - \gamma \left(z_{t-1} - \psi \right) \right]^{\hat{\lambda}} + \beta \ln \sigma_{t-1}$$

If $\hat{\lambda}$ and ψ are set to 1 and 0, respectively, and the constant unconditional mean of $|z_{t-1}|$ is subtracted from $|z_{t-1}| - \gamma\, z_{t-1}$ and added to the intercept, the variance equation[8] becomes

$$\begin{aligned} \ln \sigma_t^2 &= 2\,\omega + 2\,\alpha\,\mathbb{E}\,|z_{t-1}| + 2\,\alpha \left(|z_{t-1}| - \mathbb{E}\,|z_{t-1}| - \gamma\, z_{t-1} \right) + \beta \ln \sigma_{t-1}^2 \\ &= \dot{\omega} + \dot{\alpha} \left(|z_{t-1}| - \mathbb{E}\,|z_{t-1}| - \gamma\, z_{t-1} \right) + \beta \ln \sigma_{t-1}^2, \qquad \text{if } \lambda = 0,\; \hat{\lambda} = 1 \end{aligned}$$

where $\dot{\omega} = 2\,\omega + 2\,\alpha\,\mathbb{E}\,|z_{t-1}|$ and $\dot{\alpha} = 2\,\alpha$. And we get model (1.4) which is the same as Nelson's (1991) EGARCH model.

Absolute value GARCH (AVGARCH) model ($\lambda = 1,\ \hat{\lambda} = 1,\ \gamma = 0,\ \psi = 0$)

The absolute value GARCH (AVGARCH) model used in Schwert (1989) and Taylor (2008) parameterises the conditional volatility as a distributed lag of the absolute innovations and the lagged conditional volatilities. In this model both asymmetric parameters γ and ψ are set to 0. It is, therefore, a symmetric model in the sense that past negative shocks influence volatility by the same amount as positive shocks of the same magnitude:

$$\sigma_t = \omega + \left(\alpha\, |z_{t-1}| + \beta \right) \sigma_{t-1}$$

Threshold GARCH (TGARCH) model ($\lambda = 1,\ \hat{\lambda} = 1,\ |\gamma| \leq 1,\ \psi = 0$)

[8] We need to multiply both sides of the equality by 2 to obtain the variance in the equation instead of the volatility.

The threshold GARCH (TGARCH) model proposed by Zakoian (1994) extends the AV-GARCH model to allow the news impact curve to be rotated through the parameter γ:

$$\sigma_t = \omega + [\alpha (|z_{t-1}| - \gamma z_{t-1}) + \beta] \sigma_{t-1}$$

Standard GARCH (GARCH) model ($\lambda = 2$, $\hat{\lambda} = 2$, $\gamma = 0$, $\psi = 0$)

The standard GARCH model proposed by Bollerslev (1986) treats the conditional variance as a linear function of squared shocks and lagged conditional variances. It is obtained by setting both λ and $\hat{\lambda}$ to 2 and imposing γ and ψ to be equal 0:

$$\sigma_t^2 = \omega + \left(\alpha z_{t-1}^2 + \beta\right) \sigma_{t-1}^2$$

Glosten-Jagannathan-Runkle (GJRGARCH) model ($\lambda = 2$, $\hat{\lambda} = 2$, γ free, $\psi = 0$)

When $\lambda = \hat{\lambda} = 2$ and $\psi = 0$ but γ is freely estimated, we obtain the Glosten-Jagannathan-Runkle GARCH (GJRGARCH) model proposed by Glosten et al. (1993):

$$\sigma_t^2 = \omega + \left[\alpha (|z_{t-1}| - \gamma z_{t-1})^2 + \beta\right] \sigma_{t-1}^2$$

The GJRGARCH model allows the conditional variance to respond differently to the past negative and positive innovations by allowing a rotation of the news impact curve through the parameters γ.

Nonlinear asymmetric GARCH (NAGARCH) model ($\lambda = 2$, $\hat{\lambda} = 2$, $\gamma = 0$, ψ free)

When a shift of the news impact curve is allowed with $\gamma = 0$ but a free ψ, then the model reduces to the Engle and Ng's (1993) nonlinear asymmetric GARCH (NAGARCH) model for $\lambda = \hat{\lambda} = 2$:

$$\sigma_t^2 = \omega + \left[\alpha (z_{t-1} - \psi)^2 + \beta\right] \sigma_{t-1}^2$$

Nonlinear GARCH (NLGARCH) model (λ free, $\hat{\lambda} = \lambda$, $\gamma = 0$, $\psi = 0$)

The nonlinear GARCH (NLGARCH) model propodes by Higgins and Bera (1992) parameterizes the conditional standard deviation raised to the power λ as a function of the lagged conditional standard deviations and the lagged absolute innovations raised to the same power:

$$\sigma_t^\lambda = \omega + \left(\alpha\, |z_{t-1}|^\lambda + \beta\right) \sigma_{t-1}^\lambda$$

This formulation reduces to the standard GARCH model for $\lambda = \hat{\lambda} = 2$ and to the AV-GARCH model for $\lambda = \hat{\lambda} = 1$.

Asymmetric power (APGARCH) model (λ free, $\hat{\lambda} = \lambda$, $|\gamma| \leq 1$, $\psi = 0$)

The asymmetric power (APGARCH) model of Ding et al. (1993) is obtained by freely estimating $\lambda > 0$, setting $\hat{\lambda} = \lambda$ and imposing ψ to be equal 0:

$$\sigma_t^\lambda = \omega + \left[\alpha\, (|z_{t-1}| - \gamma\, z_{t-1})^\lambda + \beta\right] \sigma_{t-1}^\lambda$$

The APGARCH model, in turn, nests the AVGARCH model for $\lambda = \hat{\lambda} = 1$ and $\gamma = 0$ the TGARCH model for $\lambda = \hat{\lambda} = 1$, the standard GARCH model for $\lambda = \hat{\lambda} = 2$ and $\gamma = 0$, the GJRGARCH model for $\lambda = \hat{\lambda} = 2$ and the NLGARCH model for $\gamma = 0$.

In the next section, we extend model (1.1)-(1.2) to the Markov-switching framework of Gray (1996)-Klaassen (2002) and provide necessary and sufficient conditions for stationarity of the resulting model.

1.2.2 A family of Markov-switching GARCH models

Let $s_t \in \{1, \ldots, k\}$ be a first-order ergodic Markov chain representing the (unobserved) regime at time t. Consider the regime-switching version of Equations (1.1)-(1.2) as follows:

$$\begin{aligned} r_t &= \mu_{s_t} + \varepsilon_t \\ \varepsilon_t &= z_{t,s_t}\, \sigma_{t,s_t} \mid s_t, \mathcal{I}_{t-1} \sim \mathcal{F}\left(0, \sigma^2_{t,s_t}, \nu_{s_t}\right) \\ \frac{\sigma^\lambda_{t,s_t} - 1}{\lambda} &= \omega_{s_t} + \alpha_{s_t} \left[|z_{t-1,s_t} - \psi_{s_t}| - \gamma_{s_t}\,(z_{t-1,s_t} - \psi_{s_t})\right]^{\hat{\lambda}}\, \overline{\sigma^\lambda_{t-1,s_t}} + \beta_{s_t} \frac{\overline{\sigma^\lambda_{t-1,s_t}} - 1}{\lambda}, \end{aligned} \quad (1.6)$$

where $\lambda \geq 0$, $\hat{\lambda} > 0$, $z_{t,s_t} = \frac{r_t - \mu_{s_t}}{\sigma_{t,s_t}} \overset{iid}{\sim} \mathcal{F}(0, 1, \nu_{s_t})$ and

$$\overline{\sigma_{t-1,s_t}^{\lambda}} = \sum_{i=1}^{k} \mathbb{P}(s_{t-1} = i \mid s_t, \mathcal{I}_{t-1})\, \sigma_{t-1,i}^{\lambda} \tag{1.7}$$

That is, $\overline{\sigma_{t-1,s_t}^{\lambda}}$ denotes the expectation of $\sigma_{t-1,s_{t-1}}^{\lambda}$ over the set of states, conditional on past information $\mathcal{I}_{t-1}$ and the current regime s_t. The regime parameters ω_{s_t}, α_{s_t}, β_{s_t} and γ_{s_t} satisfy the conditions in (1.3) whenever $\lambda \neq 0, \frac{1}{n}$, $n \in \mathbb{N}$.

This model combines the nesting of single-regime GARCH models in (1.2) with Gray (1996)-Klaassen (2002) Markov-switching framework. A similar family of models is considered by Reher and Wilfling (2016) in an empirical analysis of the German stock market. The authors also extend Klaassen's (2002) Markov-switching GARCH framework by incorporating Hentschel's (1995) nesting of single-regime GARCH specifications. However, the authors assume a mixture of normal distributions for the distribution of returns and GARCH-in-mean. In contrast, we consider Student-t distributions to account for the excess kurtosis presented in our data in the empirical application in Section 1.3. Furthermore, since the focus of our paper is on the volatility rather than the mean, we do not consider GARCH-in-mean effects.

Asymptotic stationarity

We now consider stationarity of the MSGARCH family. We begin by introducing notation. The stationary and transition probabilities of the Markov chain $\{s_t\}$ will be denoted by $\pi_i = \mathbb{P}(s_t = i)$ and $p_{ij} = \mathbb{P}(s_t = j \mid s_{t-1} = i)$, $i, j \in \{1, \ldots, k\}$. Define the k-vectors

$$\boldsymbol{\Delta}_t = \left[\lambda\, \alpha_1 \left[|z_{t,1} - \psi_1| - \gamma_1 (z_{t,1} - \psi_1)\right]^{\hat{\lambda}}, \ldots, \lambda\, \alpha_k \left[|z_{t,k} - \psi_k| - \gamma_k (z_{t,k} - \psi_k)\right]^{\hat{\lambda}}\right]^{\top}, \tag{1.8}$$

$$\boldsymbol{\beta} = [\beta_1, \ldots, \beta_k]^{\top}, \tag{1.9}$$

where $\top$ denotes the transpose operator. Consider a $k \times k$ matrix $\mathbf{K}_n$ given by

$$\mathbf{K}_n = \mathbb{E}\left[\operatorname{diag}\left(\boldsymbol{\Delta}_t + \boldsymbol{\beta}\right)\right]^n, \quad n \in \mathbb{N} \tag{1.10}$$

and a $k \times k$ matrix $\mathbf{Q}$ with elements

$$\{\mathbf{Q}\}_{i,j} = \mathbb{P}(s_{t-1} = j \mid s_t = i) = \frac{\pi_j}{\pi_i} p_{ji}, \quad i,j = 1,\dots,k \tag{1.11}$$

Let $\rho(\cdot)$ denote the spectral radius of a matrix, i.e., its largest eigenvalue in modulus. We have the following theorem:

Theorem 1. *For the MSGARCH family, defined in* (1.6)-(1.7), $\rho(\boldsymbol{K}_1 \boldsymbol{Q}) < 1$ *is:*

1. *a sufficient condition for asymptotic stationarity when* $\lambda \geq 2$.

2. *a necessary condition for asymptotic stationarity when* $0 < \lambda \leq 2$.

Proof. See Appendix 1.5.2. □

Theorem 1 provides a sufficient and necessary condition for stationarity of the model when $\lambda = 2$. This condition can be found in Klaassen (2002) for the standard GARCH model ($\lambda = 2$ and $\gamma_{s_t}, \psi_{s_t} = 0$) with $k = 2$. Abramson and Cohen (2007) generalise Klaassen's (2002) condition to a Markov chain of higher order k.

The next theorem provides sufficient conditions for stationarity. Let $\lceil \cdot \rceil$ represent the ceiling function. We can state the following result:

Theorem 2. *The MSGARCH model in* (1.6)-(1.7) *is asymptotically stationary if* $\rho(\boldsymbol{K}_i \boldsymbol{Q}) < 1$ *for* $i = 1, \dots, \lceil \frac{2}{\lambda} \rceil$.

Proof. See Appendix 1.5.2. □

Note that for $\lambda \geq 2$ we have $\lceil \frac{2}{\lambda} \rceil = 1$ and the condition in Theorem 2 is that given in Theorem 1. As λ approaches zero, Theorem 2 yields a sharp stationarity condition as a corollary:

Corollary 1. *As* λ *approaches zero, a sufficient condition for asymptotic stationarity of the MSGARCH family is* $|\beta_{s_t}| \leq 1$ *for* $s_t = 1, \dots, k$ *with at least one* β_{s_t} *inside the unit circle.*

Proof. See Appendix 1.5.2. □

1.3 Data, Estimation and Volatility forecasting

1.3.1 Data

We apply our family of models to daily returns on the S&P 500 price index from January 2000 to March 2019. In total, we have 4840 observations. The plot of the time series of stock returns is presented in Figure 1.2.

[Figure 1.2 here]

In Table 1.2, we report summary statistics of returns. The daily average return is 0.0136%, with a minimum of −9.4695% and a maximum of 10.9572%, which shows the high variability of returns during this period with a standard deviation of 1.2034. The skewness is significant and negative indicating that negative returns are more likely to be far below the mean than positive returns. The kurtosis is significantly higher than 3, suggesting that fat-tailed distributions are appropriate to estimate the data. Indeed, the Jarque-Bera test for normality clearly rejects the null hypothesis that the data comes from a normal distribution. Furthermore, we apply the Lagrange Multiplier test for ARCH effects up to ten lags. The statistic from this test follows a $\chi^2(10)$ distribution. The null hypothesis of no ARCH effects is strongly rejected indicating the presence of ARCH effects.

[Table 1.2 here]

1.3.2 Estimation results

The MSGARCH family developed in Section 1.2 can be estimated by numerical maximum likelihood. Appendix 1.5.3 shows that the likelihood function follows a first-order recursive structure. We estimate the models in MATLAB by extending the MFE MATLAB toolbox (Sheppard, 2009) for estimation of single-regime GARCH models to our regime-switching frameworks. Starting values are chosen as follows: (1) estimate a Gaussian mixture model by using the expectation maximization algorithm; (2) calculate the most probable state sequence by applying Viterbi's (1967) algorithm and separate the data in k different vectors, one for each regime; (3) estimate the single-regime counterpart model for each of the vectors obtained in (2). We ensure that the parameter estimates respect the positivity of the variance and the stationarity of the process by imposing the constraints developed in Section 1.2.

Due to the high-kurtosis presented in the data, we consider a heavy-tail distribution, the Student-t distribution with state-dependent degrees of freedom. The use of the t-distribution is useful for regime-switching models because it improves the stability of the regimes. As pointed out by Klaassen's (2002), the use of the normal distribution could jeopardise regime stability since a large shock in the low-volatility regime will force a switch to the high-volatility regime, even if it is a single outlier in an otherwise moderate period.

We compare the performance of the nested models presented in Table 1.1 and the more general model where all the parameters are freely estimated. We consider two-regime and single-regime models. We refer to the general model as FGARCH when the number of regimes is one and MSFGARCH when we allow for two regimes. In total, we estimate 18 models. Table 1.3 compares goodness of fit statistics. The table is divided into four panels based on the assumed value of λ for each model in the first column (see Table 1.1). The second and third columns list, respectively, the number of parameters n and the maximized log-likelihood LL. The Akaike information criterion (AIC) and Bayesian information criterion (BIC) computed as AIC$= 2 \times (n - LL)$ and BIC $= -2\,LL + n\log(T)$, $T = 4840$ being the sample size, are displayed in the fourth and sixth columns. The AIC ranks the MSNAGARCH, MSFGARCH and MSEGARCH models as the three best specifications. According to the BIC, the best specifications are the MSNAGARCH, MSFGARCH and the more parsimonious single-regime FGARCH. Both criteria agree in categorising the symmetric models as providing the worst data fit. Overall, the results from both criteria suggest that the models that account for leverage effect with a shift to the news impact curve ($\psi \neq 0$) provide superior insample fit. This is remarkable since most of the empirical applications on asymmetric GARCH apply models such as the EGARCH, TGARCH or GJRGARCH which introduce asymmetry by rotating the news impact curve ($\gamma \neq 0$). We also apply Likelihood Ratio (LR) tests to formally test the performance of the asymmetric models against their symmetric counterparts. The statistics from these tests are reported in the antepenultimate and penultimate columns of Table 1.3. The null hypothesis of symmetry, corresponding to the model in the first column of the same row, is tested against the alternative that asymmetry is caused by a rotation or a shift. The LR test statistics for these tests are asymptotically distributed as a χ^2 random variable with degrees of freedom equal to the 1 for single-regime models and equal to 2 for regime-switching models[9]. The statistics far exceed the critical values at any confidence level, thus strongly rejecting the null of symmetry.

The last column in the table contains LR tests for the null of the models in the first row against the most general model with two regimes, the MSFGARCH model. All the Markov-switching models are rejecting at a 1% significance level against the MSFGARCH, except for the MSNAGARCH model. This suggests that allowing just for a shift of the news impact curve is enough to accommodate the asymmetry effects presented in the data. A difficulty arises when testing a Markov-switching model versus a single-regime model. The problem is known as the Davies's (1977) problem and occurs since the state parameter is unidentified under the null hypothesis. This problem is typically approached by treating the traditional likelihood ratio (LR) test statistic as a function of the unidentified parameter and obtaining an upper bound for the statistic across all possible values of the parameter (see Davies, 1977; Garcia et al., 1996; *inter alia*). Garcia (1998) follows Davies's (1977) approach and derives an upper bound for the significance level of the LR test under the nuisance parameters[10,11]. Given the LR statistic $LR = 2\,(LR_A - LR_0)$ where LR_0 is the log-likelihood under the null and LR_A the log-likelihood under the alternative, the upper bound proposed by Garcia (1998) is given as $\mathbb{P}\left(\chi^2_d > LR\right) + 2\,\left(\frac{LR}{2}\right)^{\frac{d}{2}}\,\exp\left(-\frac{LR}{2}\right)\,\left[\Gamma\left(\frac{d}{2}\right)\right]^{-1}$, where d is the difference in the number of parameters. For the values of the LR statistics corresponding to the null of single-regime models in the last columns of Table 1.3, the term $\exp\left(-\frac{LR}{2}\right)$ is essentially 0 so the upper bound for our tests is approximately equal to $\mathbb{P}\left(\chi^2_d > LR\right)$, the usual marginal level associated with the traditional LR test[12]. We obtain LR test statistics that far exceeds the marginal level associated with the traditional LR at any significance level, hence supporting two-regime specifications.

[Table 1.3 here]

[9] In the Markov-switching case, the parameters γ and ψ switch under the alternative.

[10] See Qu and Zhuo (2021) for an algorithm to simulate critical values of LR tests for Markov-switching models similar to that developed in Garcia (1998).

[11] The bounds holds assuming that the likelihood function has a single peak.

[12] The smallest LR test statistic for testing single-regime against the MSFGARCH model takes a value of 68.25 and corresponds to the null of the single-regime FGARCH model. Being $d = 9$ the difference in the number of parameters, we have $2\,\left(\frac{68.25}{2}\right)^{\frac{9}{2}}\,\exp\left(-\frac{68.25}{9}\right)\,\left[\Gamma\left(\frac{9}{2}\right)\right]^{-1} = 2.0602 \cdot 10^{-9} \approx 0$.

Table 1.4 exhibits parameter estimates for the nested models in the MSGARCH family with two regimes. Regime 1 corresponds to the high-mean low-volatility regime while regime 2 corresponds to a low-mean high-volatility state. The second and third rows list values of λ and $\hat{\lambda}$, respectively. These parameters are set to the established value according to each specification (see Table 1.1), except for the three last columns where is freely estimated. We calculate Bollerslev and Wooldridge's (1992) coefficient robust standard errors and perform Student-t tests to assess the statistical significance of the parameters. The estimates show that almost all the parameters are statistically significant, except for the mean parameter in the high-volatility state, μ_2. The models that allow asymmetric response to volatility shocks are estimated with highly significant asymmetry parameters γ_i or ψ_i, reinforcing the strong presence of asymmetry in the data. However, when we allow for both types of asymmetry in the MSFGARCH model, the shift parameter ψ is highly significant but the rotation parameter γ is not suggesting again that the shift alone appears enough to capture the asymmetry in returns. The state-dependent degrees of freedom are almost always greater than 4. Hence, in both regimes, we have fatter tails than the normal distribution. The last two rows concern the sufficient stationary conditions provided in Theorem 2. We have $\rho_i = \rho(\boldsymbol{K}_{i,\lambda}\mathbf{Q}) < 1$ for $i = 1, \ldots, \lceil \frac{2}{\lambda} \rceil$ indicating stationarity of the estimated models.

[Table 1.4 here]

The transition probabilities $p_{ii} = \mathbb{P}(s_t = i \mid s_{t-1} = i)$ can be used to estimate the persistence for regime i as $(1-p_{ii})^{-1}$, $i = 1, 2$. For the MSFGARCH model, for instance, persistence of regime 1 and 2 are estimated to be 568.66 and 496.51 days, respectively, which correspond to approximately 2 years. The unconditional probabilities, reported in the bottom panel of the table, are calculated as $\pi_1 = (1-p_{22})/(2-p_{11}-p_{22})$ and $\pi_2 = (1-p_{11})/(2-p_{11}-p_{22}) = 1-\pi_1$ (see Hamilton, 1994, Chapter 22). The estimated unconditional probabilities of being in the high-mean low-volatility regime are higher than 50% and thus the models estimate more than half of the sample in the low-volatility regime. The left panel of Figure 1.3 displays the smoothed probabilities of the high-mean low-volatility regime as estimated by the 9 Markov-switching models. Smoothed probabilities are obtained following Kim (1994)'s algorithm (see Appendix 1.5.3). Most models estimate the period from the dot-com crisis until 2006, the subprime crisis of 2008 and the second semester of 2011 in the high-volatility regime. The right panel of Figure 1.3 shows the conditional time-varying volatilities estimated by the models. The models display a similar conditional volatility pattern during the sample period. As expected, the periods where the smoothed probabilities of being in the high-mean low-volatility regime are near one coincide with those where the estimated conditional volatility is higher.

Table 1.5 contains the parameters estimates from the single-regime models. Again, we find the asymmetric parameters γ and ψ highly significant. Furthermore, for the FGARCH model with a rotated and shifted news impact curve, we find ψ highly significant but not γ. This reiterates the fact that the shift alone appears sufficient to capture the asymmetric effect on volatility.

[Table 1.5 here]

We perform Ljung and Box's (1978) tests on standardized squared residuals to test for the presence of ARCH effects. All the models pass the tests for different lag values (5, 10, 20).

1.3.3 Forecasting Value-at-Risk

The development of models that provide accurate volatility forecasts is essential to financial risk management. In particular, volatility forecasts are used as inputs to Value-at-Risk (VaR) calculations. VaR is one of the most popular risk measures for regulating and managing market risk. In this section, we evaluate the ability of the single-regime and Markov-switching regime models estimated in Section 1.3.2 to provide accurate one-step-ahead VaR forecasts.

VaR is defined as the 100α% quantile of the distribution of returns, such that, at time t there is a 100α% probability that the return will be lower than the VaR value. As in Section 1.3.2, we assume that returns following a Student-t distribution. The VaR at time t is given by

$$\mathrm{VaR}_t^{\alpha} = \sum_{i=1}^{2} \mathbb{P}\left(s_t = i \mid \mathcal{I}_{t-1}\right) \left[\mu_i + \sigma_{t,i} \sqrt{\frac{\nu_i - 2}{\nu_i}}\, t_{\nu_i}^{-1}(\alpha)\right]$$

where $\mathbb{P}\left(s_t = i \mid \mathcal{I}_{t-1}\right)$ is the so-called predicted probability which gives the probability of being in regime $i \in \{1, 2\}$ given the information set up to time $t-1$, $\mathcal{I}_{t-1}$; μ_i, $\sigma_{t,i}$ and ν_i are the estimated mean, conditional volatility and shape parameter from model (1.6) and $t_{\nu_i}^{-1}(\alpha)$ is the

α-quantile of the standard t-distribution (see e.g., McNeil et al., 2015, pp. 71). We consider the values $\alpha = 1\%$, $\alpha = 2.5\%$ and $\alpha = 5\%$ for the significance levels of VaR. The out-of-sample period starts on January 2007 and ends on March 2019. Each model is estimated on a rolling-window basis where the window size is 1759, the number of observations from January 2000 to December 2006. The model parameters are updated every 21 observations[13].

The most popular procedures for testing the accuracy of VaR forecasts are based on the so-called "hit sequence" of VaR violations,

$$\text{Hit}_t^\alpha = \mathbb{I}_{\{r_t \leq \text{VaR}_t^\alpha\}} \tag{1.12}$$

For a correctly specified model, we expect that the observed return values will only be worse than the VaR forecast $100\alpha\%$ of the time. Hence, ideally, the hit variable should have a mean value of α (see Kupiec, 1995). This property is known as Unconditional Coverage (UC). The hit sequence should also be completely unpredictable and, therefore, independently distributed as a Bernouilli random variable, $\text{Hit}_t^\alpha \overset{iid}{\sim} \text{Bernouilli}(\alpha)$. Furthermore, the occurrences of observations outside the interval provided by the VaR forecasts must be homogeneous over the sample and not come in clusters (Christoffersen, 1998). This property is called Conditional Coverage (CC). Christoffersen (1998) introduces a three-step procedure for testing the correct unconditional coverage, independence and conditional coverage. All three tests can be implemented in the likelihood ratio (LR) framework. The null hypothesis and the LR statistic of the tests are given by:

- Unconditional coverage test:

$$\begin{aligned} \text{H}_{\text{UC}}: \quad & \mathbb{E}\left(\text{Hit}_t^\alpha\right) = \alpha \\ \text{LR}_{\text{UC}} = & -2\log\left[\frac{\alpha^{n_1}(1-\alpha)^{n_0}}{\pi^{n_1}(1-\pi)^{n_0}}\right] \sim \chi_1^2 \end{aligned} \tag{1.13}$$

where n_1 and n_0 are, respectively, the number of 1's and 0's in the hit sequence and $\pi = \frac{n_1}{n_0+n_1}$ is the maximum likelihood estimate of α.

[13] The forecast performance of GARCH models does not change significantly when moving from a daily updating frequency to a monthly updating frequency (see Ardia and Hoogerheide, 2014).

- Test of independence:

$$
\begin{aligned}
\mathrm{H}_{\mathrm{Ind}}: &\quad \mathrm{Hit}_t^{\alpha} \overset{iid}{\sim} \text{Bernouilli}(\alpha) \\
\mathrm{LR}_{\mathrm{Ind}} &= -2\log\left[\frac{\pi^{n_1}(1-\pi)^{n_0}}{(1-\pi_{01})^{n_{00}}\pi_{01}^{n_{01}}(1-\pi_{11})^{n_{10}}\pi_{11}^{n_{11}}}\right] \sim \chi_1^2
\end{aligned} \tag{1.14}
$$

where n_{ij} is the number of observations with a j following an i, $\pi_{01} = \frac{n_{01}}{n_{00}+n_{01}}$ and $\pi_{11} = \frac{n_{11}}{n_{10}+n_{11}}$.

- Conditional coverage test:

$$
\begin{aligned}
\mathrm{H}_{\mathrm{CC}}: &\quad \mathbb{E}\left(\mathrm{Hit}_t^{\alpha} \mid \mathrm{Hit}_{t-1}^{\alpha}, \ldots, \mathrm{Hit}_1^{\alpha}\right) = \alpha \\
\mathrm{LR}_{\mathrm{CC}} &= -2\log\left[\frac{\alpha^{n_1}(1-\alpha)^{n_0}}{(1-\pi_{01})^{n_{00}}\pi_{01}^{n_{01}}(1-\pi_{11})^{n_{10}}\pi_{11}^{n_{11}}}\right] \sim \chi_2^2
\end{aligned} \tag{1.15}
$$

Note that the three LR tests are numerically related by the identity $\mathrm{LR}_{\mathrm{CC}} = \mathrm{LR}_{\mathrm{UC}} + \mathrm{LR}_{\mathrm{Ind}}$, which enables joint testing of randomness and correct coverage while retaining the individual hypotheses as subcomponents.

Table 1.6 presents the backtesting results for each model and significance level. The top panel contains the results for the single-regime models while the bottom panel contains the results for the Markov-switching models. Columns labelled PF provide the proportion of failures given by the number of ones in the hit sequence in (1.12) divided by the out-of-sample size, 3081, times 100. It is clear from the table that single-regime GARCH models result in a far higher proportion of failures than the nominal level α. On the other hand, the proportion of failures provided by the Markov-switching GARCH models are closer to the correct nominal value. The Unconditional Coverage test in (1.13) formally tests if the proportion of failures is significantly different from α. The test statistics from these tests are presented in the LR_{UC} columns. No single-regime model passes this test for any of the considered values of α. All the Markov-switching GARCH models pass the test for $\alpha = 1\%$ and $\alpha = 2.5\%$. When $\alpha = 5\%$, most of the asymmetric Markov-switching GARCH models pass the test. However, the symmetric regime-switching models and the MSGJRGARCH and MSNAGARCH models result in VaR levels that are rather conservative and provide too few VaR violations, thus failing the test. Both single-regime and Markov-switching GARCH models pass the independence test in (1.14) (column LR_{ind}). Finally, columns labelled LR_{CC} give the results from the Conditional Coverage tests. The tests reject all single-regime models for the three nominal levels. The symmetric MSGARCH and MSNLGARCH models and the asymmetric MSGJRGARCH and MSNAGARCH models also fail the test when $\alpha = 5\%$.

[Table 1.6 here]

On balance, we find that regime-switching GARCH models outperform single-regime GARCH models when forecasting VaR. This conclusion is in line with the results in Ardia et al. (2018). Furthermore, our results suggest that failure to account for the asymmetric effect on the volatility specification leads to poor out-of-sample VaR forecasts.

1.4 Conclusion

In this paper, we develop a parametric family of Markov-switching GARCH models. The family extends Hentschel's (1995) nesting of symmetric and asymmetric single-regime GARCH models to the Markov-switching framework introduced by Klaassen (2002), which extends that of Gray (1996). Necessary and sufficient conditions for asymptotic stationarity of the family are provided. The conditions are obtained by constraining the spectral radius of matrices whose elements are built from the model parameters.

In an empirical application, we apply the model to daily returns of the S&P 500 Index between January 2000 and March 2019. Our unified framework is useful for model selection and tests of functional form. For this reason, we estimate the nested models for comparison. We also estimate the single-regime specifications of the models. We consider two regimes and use a fat-tailed distribution, the Student-t, to account for the excess kurtosis in the data. The results show that asymmetric regime-switching GARCH models outperform single-regime models and symmetric models when estimating the volatility of returns. Moreover, we find that including asymmetry and regime-switching significantly improves out-of-sample forecasts of Value-at-Risk (VaR).

1.5 Appendix

1.5.1 Tables and Figures

Table 1.1: Nested GARCH models.

Model	λ	$\hat{\lambda}$	γ	ψ	Volatility Specification	Reference
EGARCH	0	1	free	0	$\ln\sigma_t^2 = \omega + \alpha\,(\lvert z_{t-1}\rvert - \mathbb{E}\lvert z_{t-1}\rvert - \gamma\, z_{t-1}) + \beta\,\ln\sigma_{t-1}^2$	Nelson (1991)
AVGARCH	1	1	0	0	$\sigma_t = \omega + (\alpha\,\lvert z_{t-1}\rvert + \beta)\,\sigma_{t-1}$	Schwert (1989); Taylor (2008)
TGARCH	1	1	$\lvert\gamma\rvert \leq 1$	0	$\sigma_t = \omega + [\alpha\,(\lvert z_{t-1}\rvert - \gamma\, z_{t-1}) + \beta]\,\sigma_{t-1}$	Zakoian (1994)
GARCH	2	2	0	0	$\sigma_t^2 = \omega + \left(\alpha\, z_{t-1}^2 + \beta\right)\sigma_{t-1}^2$	Bollerslev (1986)
GJRGARCH	2	2	free	0	$\sigma_t^2 = \omega + \left[\alpha\,(\lvert z_{t-1}\rvert - \gamma\, z_{t-1})^2 + \beta\right]\sigma_{t-1}^2$	Glosten et al. (1993)
NAGARCH	2	2	0	free	$\sigma_t^2 = \omega + \left[\alpha\,(z_{t-1} - \psi)^2 + \beta\right]\sigma_{t-1}^2$	Engle and Ng (1993)
NLGARCH	free	λ	0	0	$\sigma_t^\lambda = \omega + \left(\alpha\,\lvert z_{t-1}\rvert^\lambda + \beta\right)\sigma_{t-1}^\lambda$	Higgins and Bera (1992)*
APGARCH	free	λ	$\lvert\gamma\rvert \leq 1$	0	$\sigma_t^\lambda = \omega + \left[\alpha\,(\lvert z_{t-1}\rvert - \gamma\, z_{t-1})^\lambda + \beta\right]\sigma_{t-1}^\lambda$	Ding et al. (1993)*

*Nested if $\lambda > 0$.

Table 1.2: Descriptive statistics for S&P 500 daily returns

Mean	Std	Min	Max	Skewness	Kurtosis	Normality test	LM(10)
0.0136	1.2034	−9.4695	10.9572	−0.2165	11.5175	14668.2854***	1279.8932***

Note: Sample period: 2000:M1 to 2019:M3 (4840 observations).

Table 1.3: Model comparisons.

Model	n	LL	AIC		BIC		LRT		
			Value	Rank	Value	Rank	$H_A : \gamma$ free	$H_A : \psi$ free	H_A : MSFGARCH
Panel A : $\lambda = 0$									
EGARCH	6	-6409.23	12830.46	11	12869.36	9			122.58
MSEGARCH	14	-6373.50	12775.00	3	12865.79	7	231.86		51.12
Panel B : $\lambda = 1$									
AVGARCH	5	-6518.69	13047.38	18	13079.80	17			341.50
TGARCH	6	-6402.76	12817.52	10	12856.43	5	258.93		109.64
MSAVGARCH	12	-6504.08	13032.15	17	13109.97	18			312.27
MSTGARCH	14	-6374.61	12777.22	4	12868.01	8			53.34
Panel C : $\lambda = 2$									
GARCH	5	-6509.96	13029.92	15	13062.35	13	170.51	220.08	324.04
GJRGARCH	6	-6424.71	12861.42	12	12900.32	12			153.54
NAGARCH	6	-6399.92	12811.84	8	12850.75	4			103.96
MSGARCH	12	-6488.16	13000.31	14	13078.13	16	195.05	274.41	280.43
MSGJRGARCH	14	-6390.63	12809.26	7	12900.05	11			85.38
MSNAGARCH	14	-6350.95	12729.90	1	12820.69	1			6.02
Panel D : λ free									
NLGARCH	6	-6509.91	13031.81	16	13070.72	14	218.31		323.93
APGARCH	7	-6400.75	12815.50	9	12860.89	6			105.62
FGARCH	9	-6382.07	12782.13	6	12840.49	2			68.25
MSNLGARCH	13	-6481.33	12988.65	13	13072.95	15	213.47		266.77
MSAPGARCH	15	-6374.59	12779.18	5	12876.45	10			53.30
MSFGARCH	18	-6347.94	12731.88	2	12848.60	3			

Note: n denotes the number of parameters to be estimated and LL the maximized log-likelihood value of a given model. AIC and BIC are the Akaike information criterion and the Bayesian information criterion, respectively, computed as AIC $= -2\,(LL - n)$ and BIC $= -2\,LL + n\log(T)$ where $T = 4840$ is the sample size. The LRTs in the antepenultimate and penultimate columns are the likelihood test statistics for the following tests: H_0 : the symmetric specification corresponding to the model in the first column of the same row, versus H_A : the parameters γ or ψ are freely estimated. Under the null hypothesis, the LRT statistics are asymptotically χ^2-distributed with 1 degrees of freedom. The last column reports the LRT of the problem: H_0 : the specification corresponding to the model in the first column of the same row versus H_A : the most general model in (1.6), denoted here MSFGARCH model. *,** and *** denote statistical significance at 1%, 5% and 10% levels, respectively.

Table 1.4: Estimated MSGARCH models

	MSEGARCH	MSAVGARCH	MSTGARCH	MSGARCH	MSGJRGARCH	MSNAGARCH	MSNLGARCH	MSAPGARCH	MSFGARCH
λ	0	1	1	2	2	2	2.6684***	1.0207***	2.7802***
							(0.5094)	(0.1139)	(0.0015)
$\hat{\lambda}$	1	1	1	2	2	2			3.2607***
									(0.0030)
μ_1	0.0628***	0.0863***	0.0563***	0.0923***	0.0679***	0.0443***	0.0920***	0.0565***	0.0489***
	(0.0173)	(0.0122)	(0.0128)	(0.0137)	(0.0131)	(0.0118)	(0.0139)	(0.0123)	(0.0123)
μ_2	0.0106	0.0197	-0.0206	0.0267	-0.0177	0.0160	0.0270	-0.0206	-0.0198
	(0.0145)	(0.0208)	(0.0208)	(0.0195)	(0.0202)	(0.0002)	(0.0188)	(0.0202)	(0.0215)
ω_1	-0.0832***	0.0253***	0.0409***	0.0379**	0.0351***	0.0479***	0.0312**	0.0408***	0.0205***
	(0.0179)	(0.0076)	(0.0081)	(0.0168)	(0.0073)	(0.0085)	(0.0128)	(0.0080)	(0.0073)
ω_2	-0.0005	0.0107	0.0157***	0.0152***	0.0145***	0.0045***	0.0153***	0.0156***	0.0290***
	(0.0026)	(0.0045)	(0.0043)	(0.0049)	(0.0042)	(0.0142)	(0.0049)	(0.0042)	(0.0033)
α_1	0.0386	0.1489***	0.1293***	0.1701***	0.0778***	0.0814***	0.1441***	0.1287***	0.0063***
	(0.0420)	(0.0196)	(0.0171)	(0.0380)	(0.0215)	(0.0153)	(0.0430)	(0.0138)	(0.0008)
α_2	0.0642***	0.0887***	0.0641***	0.0870***	0.0367***	0.0438***	0.0716***	0.0639***	0.0038***
	(0.0144)	(0.0120)	(0.0092)	(0.0120)	(0.0091)	(0.0104)	(0.0187)	(0.0077)	(0.0004)
β_1	0.8764***	0.8647***	0.8535***	0.7819***	0.7994***	0.4071***	0.7629***	0.8524***	0.3160***
	(0.0176)	(0.0198)	(0.0192)	(0.0617)	(0.0273)	(0.0022)	(0.0747)	(0.0184)	(0.0123)
β_2	0.9930***	0.9226***	0.9371***	0.9062***	0.9187***	0.8534***	0.8997***	0.9366***	0.7422***
	(0.0022)	(0.0111)	(0.0090)	(0.0127)	(0.0106)	(0.0125)	(0.0138)	(0.0088)	(0.0015)
γ_1	0.3677***		0.9999***		0.9971***			0.9999***	0.0005
	(0.0427)		(0.1384)		(0.2234)			(0.0162)	(0.0003)
γ_2	0.1286***		0.9999***		0.9973***			0.9998***	0.0671
	(0.0121)		(0.1535)		(0.2101)			(0.2250)	(0.0140)
ψ_1						2.4281***			3.9511***
						(0.1417)			(0.0598)
ψ_2						1.5298***			2.8122***
						(0.0469)			(0.1358)
ν_1	7.1875***	4.5121***	5.0657***	4.0540***	4.5904***	6.3791***	3.9847***	5.0593***	6.1426***
	(1.2946)	(0.5095)	(0.6460)	(0.4895)	(0.5434)	(0.8684)	(0.4771)	(0.6235)	(0.7339)
ν_2	8.9030***	12.9438***	21.7877**	11.3146***	21.6618**	16.1935***	11.6159***	21.9176**	16.9923***
	(1.3357)	(3.6986)	(9.3938)	(2.5920)	(9.0882)	(5.4780)	(2.7015)	(9.5347)	(5.5298)
p_{11}	0.9983	0.9990	0.9984	0.9978	0.9980	0.9977	0.9978	0.9984	0.9982
p_{22}	0.9882	0.9989	0.9983	0.9962	0.9975	0.9974	0.9969	0.9983	0.9980
π_1	0.5009	0.5387	0.5169	0.6308	0.5560	0.5239	0.5844	0.5161	0.5339
π_2	0.4991	0.4613	0.4831	0.3692	0.4440	0.4761	0.4156	0.4839	0.4661
ρ_1	0.9930	0.9923	0.9867	0.9911	0.9900	0.9957	0.9935	0.9869	0.9999
ρ_2		0.9886	0.9807					0.9812	

Note: This table shows parameter estimates for the two-regime model in (1.6) and the nested models with two regimes. Standard errors are displayed in parenthesis; p-values are displayed as $[\cdot]$. *,** and *** denote statistical significance at 1%, 5% and 10% levels, respectively.

Table 1.5: Estimated GARCH models

	EGARCH	AVGARCH	TGARCH	GARCH	GJRGARCH	NAGARCH	NLGARCH	APGARCH	FGARCH
λ									2.9807***
							1.8146***	0.9857***	(0.8044)
$\hat{\lambda}$							(0.2339)	(0.0915)	2.2429***
									(0.3905)
μ	0.0386***	0.0670***	0.0331***	0.0660***	0.0379	0.0256**	0.0663***	0.0329***	0.0289***
	(0.0102)	(0.0102)	(0.0103)	(0.0105)	(0.0105)	(0.0107)	(0.0104)	(0.0102)	(0.0105)
ω	-0.0031	0.0134***	0.0203***	0.0102***	0.0147***	0.0168***	0.0110***	0.0204***	0.0125***
	(0.0027)	(0.0030)	(0.0029)	(0.0027)	(0.0025)	(0.0022)	(0.0028)	(0.0029)	(0.0038)
α	0.1321***	0.1141***	0.0887***	0.1060***	0.0480***	0.0791***	0.1115***	0.0890***	0.1840**
	(0.0137)	(0.0100)	(0.0080)	(0.0111)	(0.0109)	(0.0092)	(0.0120)	(0.0067)	(0.0802)
β	0.9814***	0.9022***	0.9114***	0.8921***	0.8918***	0.7623***	0.8926***	0.9118***	0.6420***
	(0.0030)	(0.0088)	(0.0074)	(0.0105)	(0.0092)	(0.0096)	(0.0105)	(0.0069)	(0.0772)
γ	0.1584***		0.9999***		0.9941***			0.9999***	-0.2807
	(0.0118)		(0.0928)		(0.2040)			(0.0000)	(0.1950)
ψ						1.3972***			1.5833***
						(0.1245)			(0.2636)
ν	7.0591***	6.1559***	7.3679***	6.3298***	7.2661***	7.5901***	6.3143***	7.3644***	7.5103***
	(0.6259)	(0.5603)	(0.7644)	(0.5897)	(0.7518)	(0.7381)	(0.5882)	(0.7642)	(0.7958)

Note: This table presents parameter estimates for the single-regime model in (3.12) and the nested models, presented in Table 1.2. Standard errors are displayed in parenthesis; p-values are displayed as [·]. *,** and *** denote statistical significance at 1%, 5% and 10% levels, respectively.

Table 1.6: Out-of-sample VaR forecasts evaluation

	VaR ($\alpha = 1\%$)				VaR ($\alpha = 2.5\%$)				VaR ($\alpha = 5\%$)			
	PF(%)	LR_{UC}	LR_{ind}	LR_{CC}	PF(%)	LR_{UC}	LR_{ind}	LR_{CC}	PF(%)	LR_{UC}	LR_{ind}	LR_{CC}
Single-regime models												
EGARCH	1.9156	20.5643***	0.6081	21.1724***	3.8961	21.1034***	0.1032	21.2066***	6.4610	12.7238***	1.5951	14.3189***
AVGARCH	1.9156	20.5643***	0.0550	20.6193***	3.7987	18.4334***	0.1494	18.5828***	6.0390	6.5826***	1.2847	7.8674***
TGARCH	1.7857	15.5726***	0.0364	15.6090***	3.7662	17.5788***	0.1707	17.7495***	6.4286	12.1855***	1.5078	13.6933***
GARCH	1.7532	14.4157***	3.0099	17.4256***	3.9286	22.0283***	0.2170	22.2452***	6.3961	11.6580***	0.7875	12.4455***
GJRGARCH	1.5260	7.4137***	0.1333	7.5470***	3.7987	18.4334***	0.1274	18.5608***	6.2662	9.6569***	1.1097	10.7665***
NAGARCH	1.6883	12.2156***	0.0508	12.2664***	3.6039	13.5773***	0.0734	13.6507***	6.0065	6.1888***	0.2546	6.4434***
NLGARCH	1.7208	13.2964***	3.1728	16.4692***	3.8961	21.1034***	0.1906	21.2940***	6.4286	12.1855***	1.5078	13.6933***
APGARCH	1.7208	13.2964***	0.0432	13.3396***	3.7662	17.5788***	0.1707	17.7495***	6.2987	10.1407***	1.1842	11.3249***
FGARCH	1.7857	15.5726***	0.0364	15.6090***	3.6364	14.3409***	0.2864	14.6273***	6.1688	8.2719***	0.4288	8.7007***
Markov-switching models												
MSEGARCH	1.1039	0.3228	0.7812	1.1040	2.6299	0.2070	0.8339	1.0409	4.8052	0.2533	0.2166	0.4699
MSAVGARCH	0.9091	0.2670	1.3045	1.5715	2.3377	0.3436	0.3856	0.7292	4.3182	3.1682**	0.3707	3.5388
MSTGARCH	1.2338	1.5777	0.4927	2.0704	2.5649	0.0515	0.0523	0.1038	4.7403	0.4503	0.7419	1.1922
MSGARCH	0.8117	1.1829	1.6640	2.8469	2.3052	0.4962	0.1258	0.6219	4.0260	6.5925***	2.5600	9.1526***
MSGJRGARCH	0.8442	0.8010	1.5368	2.3378	2.3377	0.3436	0.1064	0.4500	4.3182	3.1682**	1.7983	4.9665**
MSNAGARCH	1.2338	1.5777	0.4927	2.0704	2.4026	0.1234	0.0767	0.2001	4.1558	4.9079***	0.4721	5.3800**
MSNLGARCH	0.8442	0.8010	0.4597	1.2606	2.5000	0.0000	0.0537	0.0537	4.1234	5.3043***	1.2986	6.6029***
MSAPGARCH	1.0065	0.0012	1.0049	1.0061	2.4351	0.0551	0.5161	0.5712	4.6429	0.8539	0.1648	1.0188
MSFGARCH	1.1364	0.5514	0.6835	1.2349	2.7273	0.6296	0.2676	0.8972	4.3831	2.5831	0.2953	2.8784

Note: This table presents out-of-sample evaluation of one-step-ahead Value-at-Risk (VaR). PF(%) represents the proportion of failures in the hit sequence in (1.12); LR_{UC}, LR_{ind} and LR_{CC} are the statistics from the Likelihood Ratio (LR) tests of unconditional coverage in (1.13), independence in (1.14) and conditional coverage in (1.15), respectively.

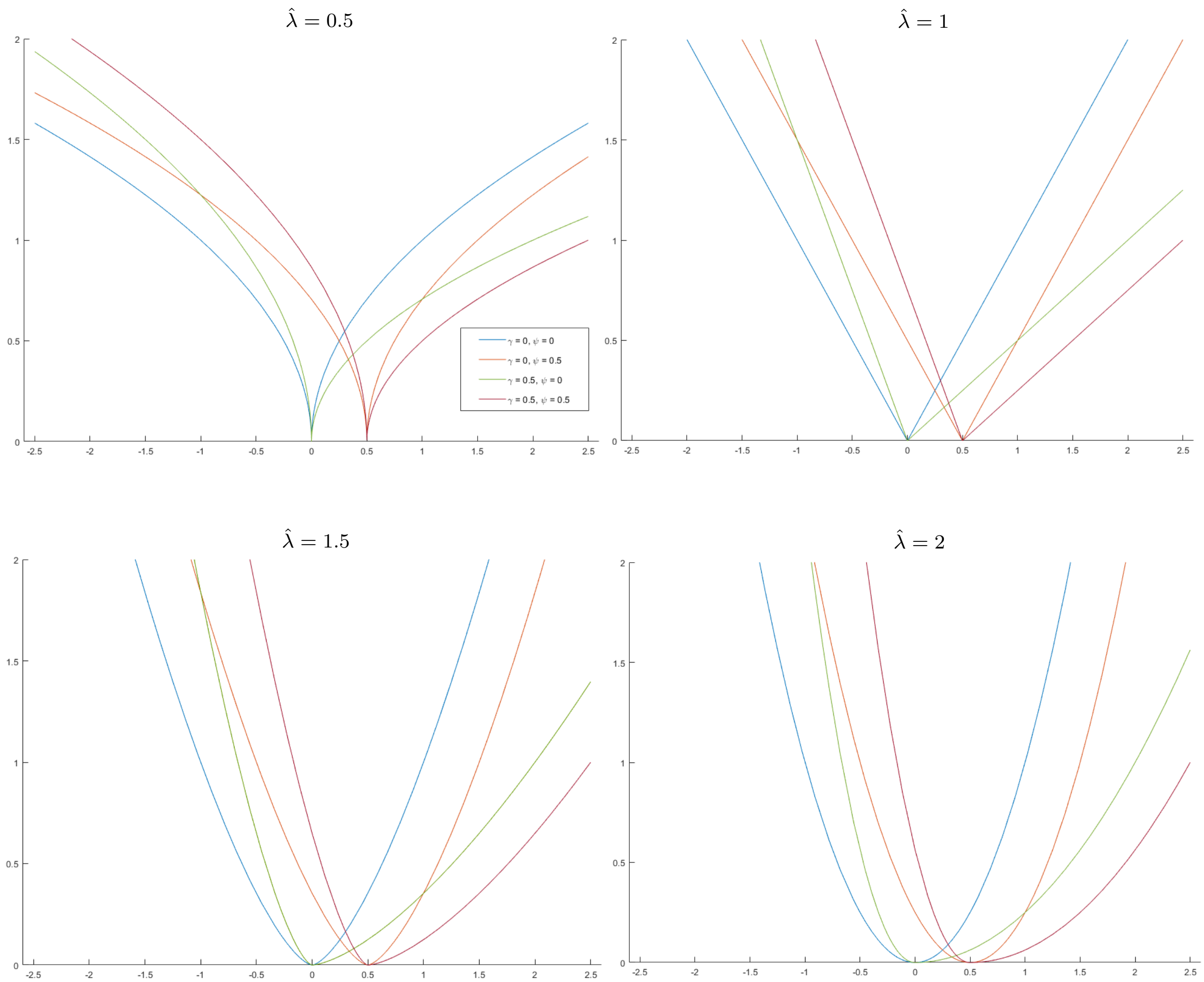

Figure 1.1: The Transformation $f^{\hat{\lambda}}(z_t)$

The panels show the behaviour of the transformation function

$$f^{\hat{\lambda}}(z_t) = [|z_t - \psi| - \gamma (z_t - \psi)]^{\hat{\lambda}}$$

for different values of $\hat{\lambda}$, γ and ψ.

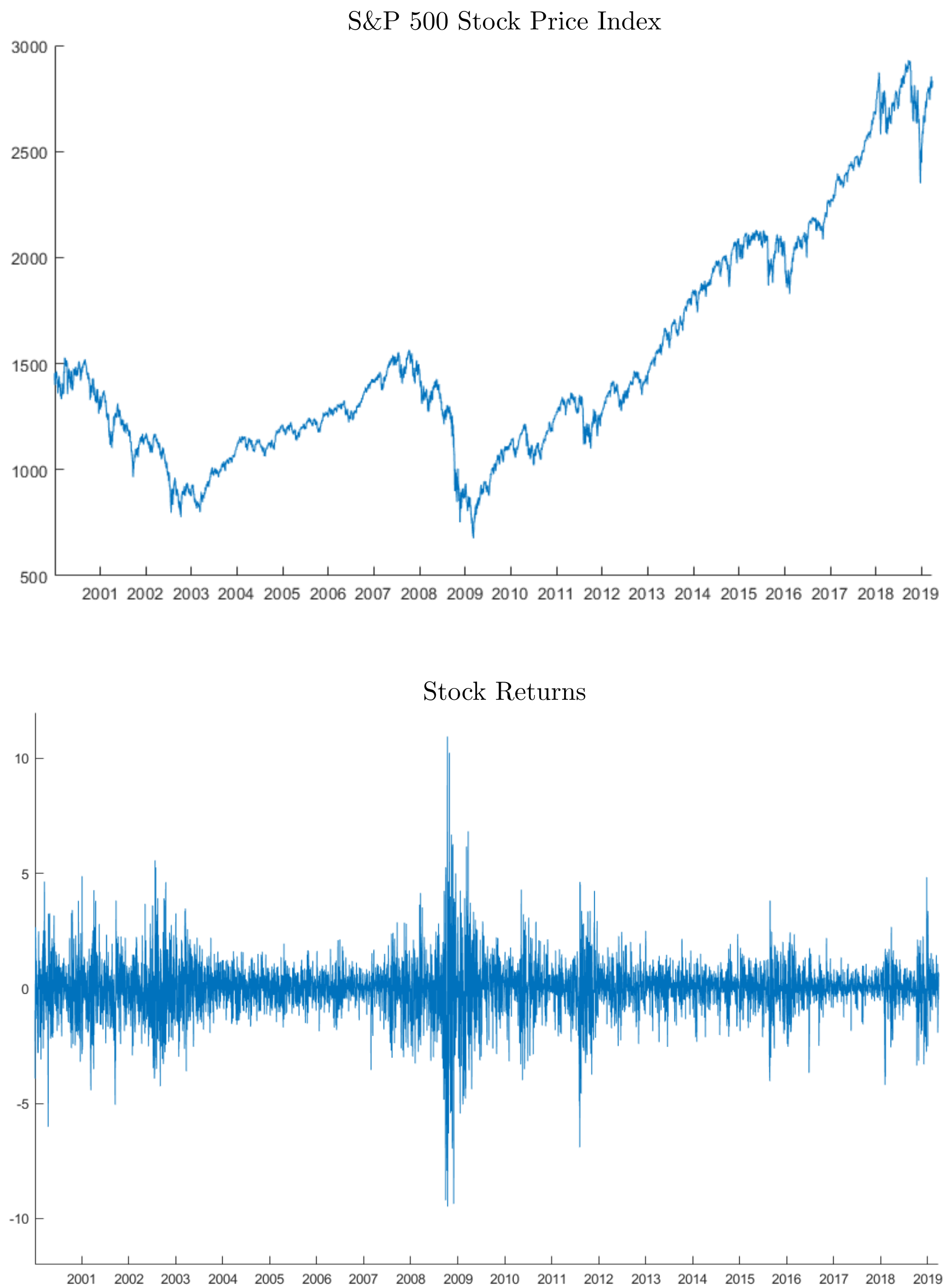

Figure 1.2: The upper panel displays the daily S&P 500 price index for a period ranging from 2000:M1 to 2019:M3 for a total of 4840 observations. The lower panel depicts the corresponding daily log-returns in percentage.

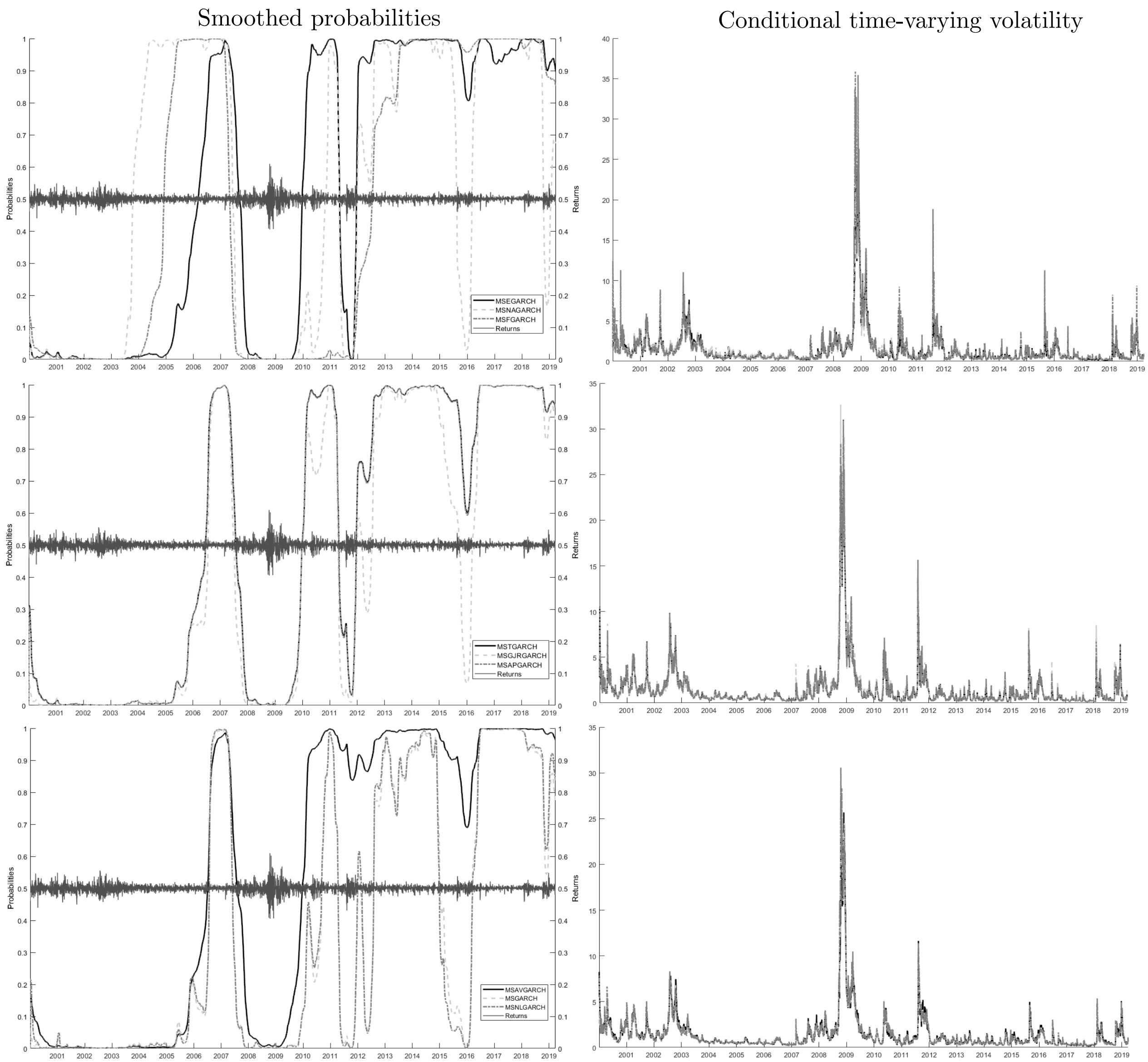

Figure 1.3: Conditional volatilities and estimated smoothed probabilitites of being in the high-mean, low-volatility regime.

1.5.2 Proof of stationarity conditions

Proof of Theorem 1. The volatility equation of the MSGARCH family in (1.6)-(1.7) for $\lambda > 0$ can be rewritten as

$$\sigma^{\lambda}_{t,s_t} = \xi_{s_t} + \left\{ \lambda\, \alpha_{s_t} \left[|z_{t-1,s_t} - \psi_{s_t}| - \gamma_{s_t} \left(z_{t-1,s_t} - \psi_{s_t} \right) \right]^{\hat{\lambda}} + \beta_{s_t} \right\} \overline{\sigma^{\lambda}_{t-1,s_t}}, \tag{1.16}$$

$$\overline{\sigma^{\lambda}_{t-1,s_t}} = \mathbb{E}\left(\sigma^{\lambda}_{t-1,s_{t-1}} \mid \mathcal{I}_{t-1}, s_t \right) = \sum_{j=1}^{k} \mathbb{P}\left(s_{t-1} = j \mid \mathcal{I}_{t-1},\, s_t \right) \sigma^{\lambda}_{t-1,j} \tag{1.17}$$

where $\xi_{s_t} = \lambda\, \omega_{s_t} + 1 - \beta_{s_t}$. Taking expectations in (1.16) conditional on $s_t = i \in \{1, \ldots, k\}$ yields

$$\mathbb{E}\left(\sigma^{\lambda}_{t,i} \mid s_t = i \right) = \xi_i + \left\{ \lambda\, \alpha_i\, \mathbb{E}\left[|z_{t-1,i} - \psi_i| - \gamma_i \left(z_{t-1,i} - \psi_i \right) \right]^{\hat{\lambda}} + \beta_i \right\} \mathbb{E}\left(\overline{\sigma^{\lambda}_{t-1,i}} \mid s_t = i \right) \tag{1.18}$$

Using the law of iterated expectations we have

$$\begin{aligned} \mathbb{E}\left(\overline{\sigma^{\lambda}_{t-1,i}} \mid s_t = i \right) &= \mathbb{E}\left[\mathbb{E}\left(\sigma^{\lambda}_{t-1,s_{t-1}} \mid \mathcal{I}_{t-1}, s_t = i \right) \mid s_t = i \right] = \mathbb{E}\left(\sigma^{\lambda}_{t-1,s_{t-1}} \mid s_t = i \right) \\ &= \mathbb{E}\left[\mathbb{E}\left(\sigma^{\lambda}_{t-1,s_{t-1}} \mid s_{t-1}, s_t = i \right) \mid s_t = i \right] = \mathbb{E}\left[\mathbb{E}\left(\sigma^{\lambda}_{t-1,s_{t-1}} \mid s_{t-1} \right) \mid s_t = i \right] \\ &= \sum_{j=1}^{k} \mathbb{P}(s_{t-1} = j \mid s_t = i)\, \mathbb{E}\left(\sigma^{\lambda}_{t-1,j} \mid s_{t-1} = j \right) \end{aligned} \tag{1.19}$$

where the antepenultimate equality uses the fact that the expected value given the current state does not depend on any future states.

Define the k-vectors $\boldsymbol{\xi} = \left[\xi_1, \ldots, \xi_k \right]^{\top}$, $\boldsymbol{\beta} = \left[\beta_1, \ldots, \beta_k \right]^{\top}$,

$$\boldsymbol{\Delta}_t = \left[\lambda\, \alpha_1 \left[|z_{t,1} - \psi_1| - \gamma_1 \left(z_{t,1} - \psi_1 \right) \right]^{\hat{\lambda}}, \ldots, \lambda\, \alpha_k \left[|z_{t,k} - \psi_k| - \gamma_k \left(z_{t,k} - \psi_k \right) \right]^{\hat{\lambda}} \right]^{\top}, \tag{1.20}$$

$$\mathbf{H}^{(\boldsymbol{\lambda})}_t = \left[\mathbb{E}\left(\sigma^{\lambda}_{t,1} \mid s_t = 1 \right), \ldots, \mathbb{E}\left(\sigma^{\lambda}_{t,k} \mid s_t = k \right) \right]^{\top}. \tag{1.21}$$

Let $\mathbf{Q}$ denote a $k \times k$ matrix with elements

$$\{\mathbf{Q}\}_{i,j} = \mathbb{P}(s_{t-1} = j \mid s_t = i) = \frac{\pi_j}{\pi_i}\, p_{ji}, \qquad i, j = 1, \ldots, k \tag{1.22}$$

and $\mathbf{K}$ be a $k \times k$ matrix given by

$$\mathbf{K} = \mathbb{E}\left[\operatorname{diag}\left(\boldsymbol{\Delta}_t + \boldsymbol{\beta}\right)\right] \tag{1.23}$$

Then, by (1.18)-(1.19), we have

$$\mathbf{H}_t^{(\boldsymbol{\lambda})} = \boldsymbol{\xi} + \mathbf{K}\mathbf{Q}\,\mathbf{H}_{t-1}^{(\boldsymbol{\lambda})} \tag{1.24}$$

which converges as t tends to infinity if and only if $\rho(\mathbf{KQ}) < 1$.

The remainder of the proof follows easily by applying the condition that the existence of the moment of a certain order implies the existence of all moments of a lower order:

- **Case** $\lambda \geq 2$. In this case, a sufficient condition for the existence of $\mathbf{H}_t^{(\mathbf{2})}$ is the existence of $\mathbf{H}_t^{(\boldsymbol{\lambda})}$. As shown above, $\mathbf{H}_t^{(\boldsymbol{\lambda})}$ converges as t tends to infinity if and only if $\rho(\mathbf{KQ}) < 1$.

- **Case** $0 < \lambda \leq 2$. The same reasoning applies to this case. A necessary condition for the existence of $\mathbf{H}_t^{(\mathbf{2})}$ is that of $\mathbf{H}_t^{(\boldsymbol{\lambda})}$. Therefore, we can conclude that $\rho(\mathbf{KQ}) < 1$ is a necessary condition for stationarity of our model when $0 < \lambda \leq 2$. □

Proof of Theorem 2. For $\lambda \geq 2$ we have $\left\lceil \frac{2}{\lambda} \right\rceil = 1$ and the results follows by Theorem 1. Assume $0 < \lambda < 2$. Raising (1.16) to the power $n \in \mathbb{N}$ yields

$$\sigma_{t,i}^{\lambda n} = \sum_{j=0}^{n} \binom{n}{j} \xi_i^j \left\{\lambda\,\alpha_i\,\left[|z_{t-1,i} - \psi_i| - \gamma_i(z_{t-1,i} - \psi_i)\right]^{\hat{\lambda}} + \beta_i\right\}^{n-j} \left[\overline{\sigma_{t-1,i}^{\lambda}}\right]^{n-j} \tag{1.25}$$

Consider the terms $\left[\overline{\sigma_{t-1,i}^{\lambda}}\right]^{n-j}$ for $j = 0, \ldots, n$. Since the function $f(x) = x^n$, $n \in \mathbb{N}$, is convex in $[0, \infty)$, we can apply Jensen's inequality to obtain:

$$\left[\overline{\sigma_{t-1,i}^{\lambda}}\right]^{n-j} = \left[\mathbb{E}\left(\sigma_{t-1,s_{t-1}}^{\lambda} \mid \mathcal{I}_{t-1},\, s_t = i\right)\right]^{n-j} \leq \mathbb{E}\left(\sigma_{t-1,s_{t-1}}^{\lambda(n-j)} \mid \mathcal{I}_{t-1},\, s_t = i\right) = \overline{\sigma_{t-1,i}^{\lambda(n-j)}} \tag{1.26}$$

Substituting (1.26) into (1.25) yields

$$\sigma_{t,i}^{\lambda n} \leq \sum_{j=0}^{n} \binom{n}{j} \xi_i^j \left\{\lambda\,\alpha_i\,\left[|z_{t-1,i} - \psi_i| - \gamma_i(z_{t-1,i} - \psi_i)\right]^{\hat{\lambda}} + \beta_i\right\}^{n-j} \overline{\sigma_{t-1,i}^{\lambda(n-j)}} \tag{1.27}$$

Therefore, the expectation of $\sigma_{t,i}^{\lambda n}$ conditional on $s_t = i \in \{1, \ldots, k\}$ satisfies the following inequality:

$$\mathbb{E}\left(\sigma_{t,i}^{\lambda n} \mid s_t = i\right) \leq \tag{1.28}$$
$$\sum_{j=0}^{n} \binom{n}{j} \xi_i^j \, \mathbb{E}\left\{\lambda\, \alpha_i \, \left[|z_{t-1,i} - \psi_i| - \gamma_i(z_{t-1,i} - \psi_i)\right]^{\hat{\lambda}} + \beta_i\right\}^{n-j} \mathbb{E}\left(\overline{\sigma_{t-1,i}^{\lambda(n-j)}} \mid s_t = i\right)$$

Replacing $\overline{\sigma_{t-1,i}^{\lambda}}$ by $\overline{\sigma_{t-1,i}^{\lambda(n-j)}}$ in (1.19) gives

$$\begin{aligned} \mathbb{E}\left(\overline{\sigma_{t-1,i}^{\lambda(n-j)}} \mid s_t = i\right) &= \sum_{j=1}^{k} \mathbb{P}(s_{t-1} = j \mid s_t = i)\, \mathbb{E}\left(\sigma_{t-1,j}^{\lambda(n-j)} \mid s_{t-1} = j\right) \\ &= \sum_{j=1}^{k} \frac{\pi_j}{\pi_i} p_{ij}\, \mathbb{E}\left(\sigma_{t-1,j}^{\lambda(n-j)} \mid s_{t-1} = j\right) \end{aligned} \tag{1.29}$$

Thus, substituting (1.29) into (1.28) we obtain

$$\mathbb{E}\left(\sigma_{t,i}^{\lambda n} \mid s_t = i\right) \leq \tag{1.30}$$
$$\sum_{j=0}^{n} \binom{n}{j} \xi_i^j \, \mathbb{E}\left\{\lambda\, \alpha_i \, \left[|z_{t-1,s_t} - \psi_i| - \gamma_i(z_{t-1,s_t} - \psi_i)\right]^{\hat{\lambda}} + \beta_i\right\}^{n-j} \sum_{j=1}^{k} \frac{\pi_j}{\pi_i} p_{ij}\, \mathbb{E}\left(\sigma_{t-1,j}^{\lambda(n-j)} \mid s_{t-1} = j\right)$$

Let $\boldsymbol{\xi}^{(n)} = \text{diag}\left[\xi_1^n, \ldots, \xi_k^n\right]$ and let $\mathbf{K}_n$ be a $k \times k$ matrix given by

$$\mathbf{K}_n = \mathbb{E}\left[\text{diag}\left(\boldsymbol{\Delta}_t + \boldsymbol{\beta}\right)\right]^n, \quad n \in \mathbb{N} \tag{1.31}$$

with $\boldsymbol{\Delta}_t$ defined in (1.20). Equation (1.28) can be expressed in matrix form as follows:

$$\mathbf{H}_t^{(\lambda n)} \leq \mathbf{K}_n\, \mathbf{Q}\, \mathbf{H}_{t-1}^{(\lambda n)} + \sum_{j=1}^{n} \binom{n}{j} \boldsymbol{\xi}^{(j)}\, \mathbf{K}_{n-j}\, \mathbf{Q}\, \mathbf{H}_{t-1}^{(\lambda(n-j))} \tag{1.32}$$

where $\mathbf{H}_t^{(\boldsymbol{\lambda n})} = \left[\mathbb{E}\left(\sigma_{t,1}^{\lambda n} \mid s_t = 1\right), \ldots, \mathbb{E}\left(\sigma_{t,k}^{\lambda n} \mid s_t = k\right)\right]^\top$ and $\mathbf{Q}$ is defined in (1.22). In (1.32), the inequality $\leq$ represents an element by element inequality. That is, the $i \in \{1, \ldots, k\}$ element in $\mathbf{H}_t^{(\boldsymbol{\lambda n})}$ is less than or equal the i element of the $k \times 1$ vector resulting in the right hand side of the inequality.

We now show that if $\rho\left(\mathbf{K}_i\, \mathbf{Q}\right) < 1$ for $i = 1, \ldots, n$, the vector of expected values $\mathbf{H}_t^{(\lambda n)}$

converges as t goes to infinity. The convergence follows by induction on n. The base case $n = 1$ is given in the proof of Theorem 1, Equation (1.24). Suppose the result holds for $n-1$. That is, $\lim_{t\to\infty} \mathbf{H}_t^{(\lambda(n-1))}$ exists if $\rho\left(\mathbf{K}_i\,\mathbf{Q}\right) < 1$ for $i = 1,\dots,n-1$. Proceeding recursively in (1.32) we obtain:

$$\mathbf{H}_t^{(\lambda n)} \leq \left(\mathbf{K}_n\,\mathbf{Q}\right)^t\,\mathbf{H}_0^{(\lambda n)} + \sum_{j=1}^{n}\binom{n}{j}\sum_{i=0}^{t-1}\left(\mathbf{K}_n\,\mathbf{Q}\right)^i\,\boldsymbol{\xi}^{(j)}\,\mathbf{K}_{n-j}\,\mathbf{Q}\,\mathbf{H}_{t-i-1}^{(\lambda(n-j))} \tag{1.33}$$

By the induction hypothesis we have that the right-hand side of (1.33) converges if $\rho\left(\mathbf{K}_i\,\mathbf{Q}\right) < 1$ for $i = 1,\dots,n$. The convergence of this side of the inequality implies the convergence of $\mathbf{H}_t^{(\lambda n)}$.

Let $n = \left\lceil \frac{2}{\lambda} \right\rceil$. Since $0 < \lambda < 2$, it follows that $\lambda\,n \geq 2$. Therefore, we can conclude that $\rho\left(\mathbf{K}_i\,\mathbf{Q}\right) < 1$ for $i = 1,\dots,\left\lceil \frac{2}{\lambda} \right\rceil$, is a sufficient condition for the convergence of $\mathbf{H}_t^{(2)}$. □

Proof of Corollary 1. When λ approaches 0, the matrix $\mathbf{K}_n$ approaches $\left[\mathrm{diag}\left(\boldsymbol{\beta}\right)\right]^n$ and the natural number $\left\lceil \frac{2}{\lambda} \right\rceil$ tends to ∞. Therefore, the sufficient condition in Theorem 2 becomes

$$\rho\left(\left[\mathrm{diag}\left(\boldsymbol{\beta}\right)\right]^n\mathbf{Q}\right) < 1, \quad \forall\, n \in \mathbb{N} \tag{1.34}$$

We now show that this condition is equivalent to

$$|\beta_i| \leq 1 \;\; \forall\, i \in \{1,\dots,k\} \;\; \text{where} \;\; \exists\, i \in \{1,\dots,k\} \;\; \text{s.t.} \;\; |\beta_i| < 1 \tag{1.35}$$

<u>(1.34) ⇒ (1.35)</u>. Assume (1.34) holds. Let $\{\eta_{1,n},\dots,\eta_{k,n}\}$ denote the spectrum of $\left[\mathrm{diag}\left(\boldsymbol{\beta}\right)\right]^n\mathbf{Q}$. Since the trace of a matrix is equal to the sum of its eigenvalues, we have

$$\mathrm{trace}\left\{\left[\mathrm{diag}\left(\boldsymbol{\beta}\right)\right]^n\mathbf{Q}\right\} = \sum_{i=1}^{k}\beta_i^n\,p_{ii} = \sum_{i=1}^{k}\eta_{i,n} \leq \sum_{i=1}^{k}|\eta_{i,n}| < k, \quad \forall\, n \in \mathbb{N} \tag{1.36}$$

where the last inequality follows by (1.34). Therefore, $\lim\limits_{n\to\infty}\sum_{i=1}^{k}\beta_i^n p_{ii} \leq k$ which implies $|\beta_i| \leq 1$, $\forall\, i = 1,\dots,k$.

It remains to show that there must exists at least one $i \in \{1,\dots,k\}$ such that $|\beta_i| < 1$. To obtain a contradiction, suppose that $|\beta_i| = 1$, $\forall\, i$. Then, $\left[\mathrm{diag}\left(\boldsymbol{\beta}\right)\right]^n$ becomes the identity matrix of order k for any even natural number n and condition (1.34) becomes $\rho(\mathbf{Q}) < 1$. Note that the matrix $\mathbf{Q}$ and the transition probability matrix $\mathbf{P}$ are similar matrices. Indeed, $\mathbf{Q} = [\text{diag}(\pi_1, \ldots, \pi_k)]^{-1}\mathbf{P}\,\text{diag}(\pi_1, \ldots, \pi_k)$. Therefore, the eigenvalues of $\mathbf{Q}$ are equal to those of $\mathbf{P}$. Since $\mathbf{P}$ is the transition probability matrix of an ergodic Markov chain, unity is an eigenvalue of $\mathbf{P}$ and all other eigenvalues are inside the unit circle (see Hamilton, 1994, pp. 681). Thus, $\rho(\mathbf{Q}) = \rho(\mathbf{P}) = 1$ and we obtain a contradiction.

(1.35) $\Rightarrow$ (1.34). We now prove that (1.35) implies (1.34). Condition (1.35) yields

$$\sum_{j=1}^{k} \left|([\text{diag}(\boldsymbol{\beta})]^n\,\mathbf{Q})_{ij}\right| = |\beta_i^n| \sum_{j=1}^{k} \mathbb{P}(s_{t-1} = j \mid s_t = i) = |\beta_i^n| \leq 1, \quad \forall n \in \mathbb{N} \tag{1.37}$$

with strict inequality for at least one i. In order to prove that (1.34) is satisfied, we use the following result for the spectral radius of an irreducible matrix with absolute row sum bounded by one (see e.g. Lancaster and Tismenetsky, 1985, pp. 377):

Lemma 1.5.1. *Let $\boldsymbol{C} = (c_{ij})_{i,j=1}^k$ be irreducible and satisfy $\sum_{j=1}^k |c_{ij}| \leq 1$, for $j = 1, \ldots, k$, with strict inequality for at least one i. Then, $\rho(\boldsymbol{C}) < 1$.*

If we show that $[\text{diag}(\boldsymbol{\beta})]^n\,\mathbf{Q}$ is irreducible, condition (1.34) holds by Lemma 1.5.1. Note that $[\text{diag}(\boldsymbol{\beta})]^n\,\mathbf{Q} = [\text{diag}(\boldsymbol{\beta})]^n\,[\text{diag}(\pi_1, \ldots, \pi_k)]^{-1}\,\mathbf{P}\,\text{diag}(\pi_1, \ldots, \pi_k)$. Suppose first that none of the parameters β_i is null. Therefore, $[\text{diag}(\boldsymbol{\beta})]^n\,\mathbf{Q}$ can be obtained pre- and post-multiplying $\mathbf{P}$ by diagonal matrices with non null diagonal elements. Then, the non zero entries in $[\text{diag}(\boldsymbol{\beta})]^n\,\mathbf{Q}$ coincide with those in $\mathbf{P}$ and both matrices share directed graph. Since a square matrix is irreducible if and only if its directed graph is strongly connected (see Lancaster and Tismenetsky, 1985, pp. 528-529) and $\mathbf{P}$ is irreducible, we have that $[\text{diag}(\boldsymbol{\beta})]^n\,\mathbf{Q}$ is irreducible $\forall\, n \in \mathbb{N}$.

Suppose now that there exists i such that $\beta_i = 0$, then $\rho([\text{diag}(\boldsymbol{\beta})]^n\,\mathbf{Q}) < 1$ follows easily by applying the following result:

Lemma 1.5.2. *Let $C = (c_{ij})_{i,j=1}^k \in \mathbb{C}^{k\times k}$, $D = (d_{ij})_{i,j=1}^k \in \mathbb{R}^{k\times k}$ and let $|C|$ denote a nonnegative matrix with elements $|c_{ij}|$. If $|C| \leq D$, then $\rho(C) \leq \rho(D)$.*

Since $|[\text{diag}(\boldsymbol{\beta})]^n\,\mathbf{Q}| \leq |[\text{diag}(\boldsymbol{\beta}^*)]^n\,\mathbf{Q}|$ for any $\boldsymbol{\beta}^*$ obtained from $\boldsymbol{\beta}$ by replacing the zeros entries by non zero entries, it follows from Lemma 1.5.2 that $\rho([\text{diag}(\boldsymbol{\beta})]^n\,\mathbf{Q}) \leq \rho(|[\text{diag}(\boldsymbol{\beta}^*)]^n\,\mathbf{Q}|)$.

Take the replaced non zero entries in $\boldsymbol{\beta}^*$ be inside the unit circle. We have seen that $|[\text{diag}(\boldsymbol{\beta}^*)]^n \mathbf{Q}|$ is irreducible and therefore $\rho([\text{diag}(\boldsymbol{\beta})]^n \mathbf{Q}) \leq \rho(|[\text{diag}(\boldsymbol{\beta}^*)]^n \mathbf{Q}|) < 1$ by Lemma 1.5.1. □

1.5.3 Maximum likelihood estimation

In this appendix, we develop the log-likelihood function of our Markov-switching GARCH model in (1.6)-(1.7). Estimation of the model can be done following the quasi-maximum likelihood method.

The log-likelihood function is given by

$$\mathcal{L}(\Theta) = \sum_{t=1}^{T} \log f(r_t \mid \Theta, \mathcal{I}_{t-1}) \tag{1.38}$$

where Θ is the vector of parameters and $f(r_t \mid \Theta, \mathcal{I}_{t-1})$ is the density of r_t given the past observable information in $\mathcal{I}_{t-1} = \{r_{t-1}, r_{t-2}, \ldots\}$. The density can be decomposed as

$$f(r_t \mid \Theta, \mathcal{I}_{t-1}) = \sum_{j=1}^{k} f(r_t \mid \Theta, \mathcal{I}_{t-1}, s_t = j)\, \mathbb{P}(s_t = j \mid \mathcal{I}_{t-1}) \tag{1.39}$$

The probabilities $\mathbb{P}(s_t = j \mid \mathcal{I}_{t-1})$, $j = 1, \ldots, k$ denote the predicted probabilities at time t and can be calculated using the law of total probabilities as follows:

$$\mathbb{P}(s_t = j \mid \mathcal{I}_{t-1}) = \sum_{i=1}^{k} \mathbb{P}(s_t = j \mid \mathcal{I}_{t-1}, s_{t-1} = i)\; \mathbb{P}(s_{t-1} = i \mid \mathcal{I}_{t-1}) \tag{1.40}$$

Since the state variable follows a first-order Markov process with constant transition probabilities, we have that the first term of the summation in (1.40) is constant and equal to the probability of transition from regime i to regime j: $\mathbb{P}(s_t = j \mid \mathcal{I}_{t-1}, s_{t-1} = i) = \mathbb{P}(s_t = j \mid s_{t-1} = i) := p_{ij}$. Furthermore, by the Bayes' rule the probabilities $\mathbb{P}(s_{t-1} = i \mid \mathcal{I}_{t-1})$ for $i = 1, \ldots, k$, referred to as filtered probabilities, follow a first-order recursive structure :

$$\begin{aligned} \mathbb{P}\left(s_{t-1}=i \mid \mathcal{I}_{t-1}\right) &= \mathbb{P}\left(s_{t-1}=i \mid \mathcal{I}_{t-2}, r_{t-1}\right) \\ &= \frac{f\left(r_{t-1} \mid \Theta,\, \mathcal{I}_{t-2},\, s_{t-1}=i\right) \mathbb{P}\left(s_{t-1}=i \mid \mathcal{I}_{t-2}\right)}{f\left(r_{t-1} \mid \Theta,\, \mathcal{I}_{t-2}\right)} \\ &= \frac{f\left(r_{t-1} \mid \Theta,\, \mathcal{I}_{t-2},\, s_{t-1}=i\right) \mathbb{P}\left(s_{t-1}=i \mid \mathcal{I}_{t-2}\right)}{\sum_{j=1}^{k} f(r_{t-1} \mid \mathcal{I}_{t-2}, s_{t-1}=j)\, \mathbb{P}\left(s_{t-1}=j \mid \mathcal{I}_{t-2}\right)} \\ &= \frac{f\left(r_{t-1} \mid \Theta,\, \mathcal{I}_{t-2},\, s_{t-1}=i\right) \sum_{m=1}^{k} p_{mi}\, \mathbb{P}\left(s_{t-2}=m \mid \mathcal{I}_{t-2}\right)}{\sum_{j=1}^{k} f(r_{t-1} \mid \mathcal{I}_{t-2}, s_{t-1}=j) \sum_{m=1}^{k} p_{mj}\, \mathbb{P}\left(s_{t-2}=m \mid \mathcal{I}_{t-2}\right)} \end{aligned} \tag{1.41}$$

Thus, the filtered probabilities can be obtained in a recursive way from its previous values and the transition probabilities.

Moreover, the Klaassen's (2002) probabilities used in the calculation of the time-varying volatility in (1.7) can be calculated from the predicted and filtered probabilities in (1.40)-(1.41) and the transition probabilities as follows:

$$\mathbb{P}\left(s_{t-1}=i \mid s_t=j, \mathcal{I}_{t-1}\right) = \frac{\mathbb{P}\left(s_t=j \mid s_{t-1}=i\right) \mathbb{P}\left(s_{t-1}=i \mid \mathcal{I}_{t-1}\right)}{\mathbb{P}\left(s_t=j \mid \mathcal{I}_{t-1}\right)} \tag{1.42}$$

where we again use the fact that the regime s_t depends on past observations $\mathcal{I}_{t-1}$ only through the value of s_{t-1} and, therefore, $\mathbb{P}\left(s_t=j \mid \mathcal{I}_{t-1}, s_{t-1}=i\right) = \mathbb{P}\left(s_t=j \mid s_{t-1}=i\right)$.

Finally, the smoothed probabilities $\mathbb{P}\left(s_t=i \mid \mathcal{I}_T\right)$, $i = 1, \ldots, k$, represent the inference about the regime the process was at time t based on data from the whole sample period until date T. Kim (1994) develops an algorithm for calculating these probabilities. Let $\boldsymbol{\xi}_{t|t} = \left[\mathbb{P}\left(s_t=1 \mid \mathcal{I}_t\right), \ldots, \mathbb{P}\left(s_t=k \mid \mathcal{I}_t\right)\right]^\top$, $\boldsymbol{\xi}_{t+1|t} = \left[\mathbb{P}\left(s_{t+1}=1 \mid \mathcal{I}_t\right), \ldots, \mathbb{P}\left(s_{t+1}=k \mid \mathcal{I}_t\right)\right]^\top$ and $\boldsymbol{\xi}_{t|T} = \left[\mathbb{P}\left(s_t=1 \mid \mathcal{I}_T\right), \ldots, \mathbb{P}\left(s_t=k \mid \mathcal{I}_T\right)\right]^\top$ denote the vectors of filtered, predicted and smoothed probabilities, respectively. Then, the smoothed probabilities at time t are obtained by backward recursion as follows:

$$\boldsymbol{\xi}_{t|T} = \boldsymbol{\xi}_{t|t} \circ \left\{\mathbf{P}^\top \left(\boldsymbol{\xi}_{t+1|T} \,./\, \boldsymbol{\xi}_{t+1|t}\right)\right\} \tag{1.43}$$

where ./ denotes the element-wise division and $\mathbf{P}$ is the $k \times k$ matrix with the transition probability $\mathbb{P}\left(s_t=j \mid s_{t-1}=i\right)$ in position $p_{ij} = \{\mathbf{P}\}_{ij}$. See Hamilton and Susmel (1994), Chapter 22.4.

Chapter 2

Forecasting Value-at-Risk and Expected Shortfall with Markov-switching GARCH models

Abstract

This paper presents a comparison study of the forecasting performance of a wide set of models in terms of Value-at-Risk (VaR) and Expected Shortfall (ES) predictions. We propose a flexible specification that allows for two regimes and asymmetry in the conditional volatility within regime, and nests the most commonly used Markov-switching GARCH models. Using data for four international stock market indices, we show that asymmetric Markov-switching GARCH models significantly outperform nonparametric, semiparametric, single-regime GARCH and symmetric models. Furthermore, the Model Confidence Set (MCS) procedure of Hansen et al. (2011) identifies our most flexible Markov-switching GARCH specification as the best performing model for VaR and ES predictions. Moreover, according to Diebold and Mariano's (1995) tests of equal predictive ability, the superiority of our model is significant and consistent across the four indices.

2.1 Introduction

Measurement of risk is central to the process of managing risk in financial institutions and corporations. Value-at-Risk (VaR) has been the standard risk measure for the past two decades. VaR gives the worst loss over a target horizon that will not be exceeded with a given level of confidence (Jorion, 2007). Despite its ease of computation and conceptual simplicity, VaR has important deficiencies. First, VaR is not a coherent risk measure as it does not meet the subadditivity axiom (Artzner et al., 1999). Thus, there may be cases in which, contrary to the idea of diversification, the total VaR of a portfolio is larger than the sum of the VaRs of the portfolio components. In addition, VaR measures do not provide any information regarding the loss beyond the estimated VaR level.

After the financial crisis of 2007–2008, the Basel Committee on Banking Supervision (2010) placed more emphasis on Expected Shortfall (ES) as a measure of risk complementing VaR. ES is defined as the expected return on an asset conditional on the return being below the VaR level. In contrast to VaR, ES is a coherent measure. The Basel III Accord, implemented in 2019, proposed a transition from VaR at a 99% confidence level to ES with a confidence level of 97.5%: "ES must be computed on a daily basis for the bank-wide internal models to determine market risk capital requirements."; "In calculating ES, a bank must use a 97.5th percentile, one-tailed confidence level" (Basel Committee on Banking Supervision, 2019).

There are many approaches in the current literature to forecasting VaR and ES. These include non-parametric methods, e.g. historical simulation (using past or in-sample quantiles); semi-parametric approaches, e.g. the dynamic quantile regression CAViaR model of Engle and Manganelli (2004) or recently proposed models for joint estimation of VaR and ES (Patton et al., 2019; Taylor, 2019; Taylor, 2020; Storti and Wang, 2021); and parametric approaches that fully specify model dynamics and distributional assumptions, e.g. generalized autoregressive conditional heteroskedasticity (GARCH) models (see Engle, 1982; Bollerslev, 1986). GARCH models in particular are widely used among both academics and practitioners (see e.g., McNeil and Frey, 2000; Christoffersen et al., 2004; Hansen and Lunde, 2005; Kuester et al., 2006).

Several studies have shown that incorporating structural shifts into the GARCH equation enables improved forecast of the volatility (see e.g., Lamoureux and Lastrapes, 1990; Gray, 1996; Klaassen, 2002; Haas et al., 2004; Marcucci, 2005). Following Hamilton (1989), one popular way to incorporate these structural shifts is through a discrete unobservable variable following a Markov process. See Ardia et al. (2018) and Caporale and Zekokh (2019) for recent studies on Markov-switching GARCH models.

In this paper, we propose a flexible Markov-switching GARCH (MSGARCH) model and evaluate its ability to forecast VaR and ES. Our specification nests several symmetric and asymmetric models and is based on a family of single-regime GARCH models introduced by

Hentschel (1995). The symmetric models include the standard GARCH model of Bollerslev (1986), the absolute value GARCH (AVGARCH) (Schwert, 1989; Taylor, 2008) and the non-linear GARCH (NLGARCH) model of Higgins and Bera (1992). The nested asymmetric models are the exponential GARCH (EGARCH) model of Nelson (1990), the threshold GARCH (TGARCH) model of Zakoian (1994), the GJRGARCH model of Glosten et al. (1993), the nonlinear asymmetric GARCH (NAGARCH) of Engle and Ng (1993) and the asymmetric power GARCH (APGARCH) model of Ding et al. (1993). All these models are extended to allow for regime switches in the dynamics of the volatility. A challenge for MSGARCH models, however, is the path dependence problem. As shown by Gray (1996), given a Markov chain with k different regimes and T observations, the evaluation of the likelihood requires integration over all possible k^T paths. This is infeasible in practice as the number of possible paths grows exponentially with time. To avoid this problem, we follow the collapsing procedure introduced by Gray (1996) and later extended by Klaassen (2002).

We perform a thorough analysis using daily data for four stock indices (the S&P 500, Dow Jones Industrial Average, the NIKKEI 225 and the FTSE 100) to evaluate the forecasting performance of our model and the models nested within it. Nonparametric rolling window models and the semiparametric models proposed by Patton et al. (2019) are also included in our analysis. In total, we compare 26 different models. To the best of the authors' knowledge, the present study is the first to provide a comparison of MSGARCH models with nonparametric or semiparametric models in terms of their ability to forecast VaR and ES. To test the competing models under varying market conditions, we consider a forecasting period spanning the 2008 financial crisis.

Our results are summarized as follows: the Model Confidence Set (MCS) procedure of Hansen et al. (2011) identifies our most flexible MSGARCH specification as the best performing model for VaR and ES forecasting within the set of competing models. Furthermore, according to Diebold and Mariano (1995) tests of equal predictive ability, the superiority of our model is significant and consistent across the four indices.

This paper is structured into three main sections. Section 2.2 introduces the model specifications. The data used in our study and the forecasting evaluation are presented in Section 2.3. Finally, Section 2.4 concludes.

2.2 Methodology

This section presents the specifications of the models used in the out-of-sample forecast evaluation exercise in Section 2.3. Section 2.2.1 introduces our proposed Markov-switching GARCH (MSGARCH) model and the nested models within it. Nonparametric and semiparametric models for VaR and ES are presented in Section 2.2.2.

2.2.1 Risk forecasting with Markov-switching GARCH models

Let $r_t \in \mathbb{R}$ denote the log-return of a financial asset at time t, $\mathcal{I}_t = \{r_{t-i},\, i \geq 0\}$ denote the set of available information and let $s_t \in \{1, \ldots, k\}$ be a first-order ergodic Markov chain representing the (unobserved) regime at time t. Following Hentschel's (1995) nesting of single-regime GARCH models, we propose the following MSGARCH model:

$$
\begin{aligned}
r_t &= \mu_{s_t} + \varepsilon_t \qquad (2.1)\\
\varepsilon_t &= z_{t,s_t}\, \sigma_{t,s_t} \mid s_t,\, \mathcal{I}_{t-1} \sim t_{\nu_{s_t}}(0, \sigma^2_{t,s_t})\\
\frac{\sigma^{\lambda}_{t,s_t} - 1}{\lambda} &= \omega_{s_t} + \alpha_{s_t}\left[|z_{t-1,s_t} - \psi_{s_t}| - \gamma_{s_t}\left(z_{t-1,s_t} - \psi_{s_t}\right)\right]^{\hat{\lambda}} \overline{\sigma^{\lambda}_{t-1,s_t}} + \beta_{s_t}\frac{\overline{\sigma^{\lambda}_{t-1,s_t}} - 1}{\lambda},
\end{aligned}
$$

where $\lambda \geq 0$, $\hat{\lambda} > 0$; $t_{\nu_{s_t}}(0, \sigma^2_{t,s_t})$ is a Student-t distribution with ν_{s_t} degrees of freedom and $\left\{z_{t,s_t} = \frac{r_t - \mu_{s_t}}{\sigma_{t,s_t}}\right\}$ is a sequence of independent and identically distributed (iid) zero-mean unit-variance random variables. To avoid the path-dependence problem, we follow Gray's (1996) and Klaassen's (2002) collapsing procedure and define $\overline{\sigma^{\lambda}_{t-1,s_t}}$ as the expectation of $\sigma^{\lambda}_{t-1,s_{t-1}}$ over the set of states, conditional on past information $\mathcal{I}_{t-1}$ and the current regime s_t:

$$
\overline{\sigma^{\lambda}_{t-1,s_t}} = \sum_{i=1}^{k} \mathbb{P}\left(s_{t-1} = i \mid s_t,\, \mathcal{I}_{t-1}\right)\, \sigma^{\lambda}_{t-1,i}. \qquad (2.2)
$$

When $\lambda \neq 0$ and $\lambda \neq \frac{1}{n}$, $n \in \mathbb{N}$, the parameters ω_{s_t}, α_{s_t}, β_{s_t}, τ_{s_t} and ψ_{s_t} in (2.1) satisfy

$$
\lambda\,\omega + 1 - \beta > 0, \qquad \alpha,\, \beta \geq 0, \qquad |\gamma| \leq 1 \qquad (2.3)
$$

to ensure that the conditional variance takes positive real values[1].

[1] Conditions for stationarity of model (2.1)-(2.2) are provided in Chapter 1.

By appropriately choosing the parameters λ, $\hat{\lambda}$, τ_{s_t} and ψ_{s_t}, model (2.1) nests a wide-range of symmetric and asymmetric MSGARCH models as shown in Table 2.1[2].

[Table 2.1 here]

The VaR and ES at time t at $(1-\alpha)\%$ significance level are calculated as follows

$$\begin{aligned}\mathrm{VaR}_t^{\alpha} &= \mu_t + \sum_{i=1}^{k} \mathbb{P}\left(s_t = i \mid \mathcal{I}_{t-1}\right) \sigma_{t,i} \sqrt{\frac{\nu_i - 2}{\nu_i}}\, t_{\nu_i}^{-1}(\alpha) \\ \mathrm{ES}_t^{\alpha} &= \mu_t + \sum_{i=1}^{k} \mathbb{P}\left(s_t = i \mid \mathcal{I}_{t-1}\right) \sigma_{t,i} \sqrt{\frac{\nu_i - 2}{\nu_i}} \frac{g_{\nu_i}(t_{\nu_i}^{-1}(\alpha))}{\alpha} \left(\frac{\nu_i + (t_{\nu_i}^{-1}(\alpha))^2}{\nu_i - 1}\right)\end{aligned} \tag{2.4}$$

where g_ν denotes the density of the standard t (see e.g., McNeil et al., 2015, pp. 71). In our application to forecast VaR and ES, we use two-regime ($k = 2$) MSGARCH models and single-regime ($k = 1$) GARCH models. The next section presents the remaining models considered in our study.

2.2.2 Existing models for ES and VaR

In this section, we introduce a set of nonparametric and semiparametric models for ES and VaR. The nonparametric models are the simple rolling window estimation of quantiles, known as historical simulation. The semiparametric models include filtered historical simulation and the models recently proposed by Patton et al. (2019).

Nonparametric models for ES and VaR

Historical simulations are the most widely used VaR forecast methods at commercial banks (see e.g., Berkowitz et al., 2011). Let v_t and e_t represent the values of VaR and ES, respectively, at time t. For a given rolling window size, w, a historical simulation model estimates v_t and

[2] Section 1.2.1 in Chapter 1 provides a discussion and description of the asymmetric and symmetric models being nested. An illustration of the asymmetry of the models using the news impact curve of Pagan and Schwert (1990) and Engle and Ng (1993) is also given.

e_t as follows:

$$\begin{aligned} v_t &= \widehat{\text{Quantile}}\{r_i\}_{i=t-w}^{t-1} \\ e_t &= \frac{1}{\alpha\, w} \sum_{i=t-w}^{t-1} r_i \, \mathbb{I}_{\{r_i \leq v_i\}} \end{aligned} \tag{2.5}$$

where $\widehat{\text{Quantile}}\{r_i\}_{i=t-w}^{t-1}$ is the sample quantile of r_i over the period $i \in [\, t-w, t-1\,]$ and $\mathbb{I}_{\{\cdot\}}$ denotes the indicator function. In our forecast evaluation exercise, we select three commonly used rolling window sizes corresponding to six months ($w = 125$), one year ($w = 250$) and two years ($w = 500$) of daily return observations.

The principal advantage of the historical simulation method is that it is easy to implement. However, it is based on the assumption of independent and identically distributed returns. This assumption is unrealistic as it contradicts the well-known volatility clustering property of financial returns. Another inherent problem of historical simulation models is the difficulty in finding the optimal size of the rolling window (Pritsker, 2006).

Semiparametric models for ES and VaR

Semiparametric models impose a parametric structure on the dynamics of VaR and ES but make no assumptions about the conditional distribution of returns. The first semiparametric model we consider is a combination of a parametric GARCH$(1, 1)$ with a nonparametric distribution:

$$\begin{aligned} r_t &= \mu_t + z_t \, \sigma_t \\ \sigma_t^2 &= \omega + \gamma \, \varepsilon_{t-1}^2 + \beta \, \sigma_{t-1}^2 \\ v_t &= \mu_t + \mathrm{F}^{-1}(\alpha) \, \sigma_t \\ e_t &= \mu_t + \mathbb{E}\,[\, z_t \mid z_t \leq \mathrm{F}^{-1}(\alpha) \,] \; \sigma_t \end{aligned} \tag{2.6}$$

where F represents the empirical distribution function (EDF) of $\{z_t\}$. This approach is known as "filtered historical simulation". In what follows, we denote model (2.6) as GARCH-EDF.

We also consider the semiparametric models proposed by Patton et al. (2019). Their work builds on the results in Fissler et al.'s (2016) that show that ES is jointly elicitable with VaR.

That is, there exists a class of loss functions that is consistent for VaR and ES and, therefore, minimizing the expected loss using any of these loss functions returns the true values of VaR and ES. Based on the "generalized autoregressive score" (GAS) framework of Creal et al. (2013), Patton et al. (2019) propose four models for VaR and ES:

- Two-factor GAS model (GAS-2F):

$$\begin{bmatrix} v_t \\ e_t \end{bmatrix} = \mathbf{w} + \mathbf{B} \begin{bmatrix} v_{t-1} \\ e_{t-1} \end{bmatrix} + \mathbf{A} \begin{bmatrix} \lambda_{v,t-1} \\ \lambda_{e,t-1} \end{bmatrix} \tag{2.7}$$

  where $\mathbf{w}$ is a (2×1) vector, $\mathbf{B}$ and $\mathbf{A}$ are (2×2) matrices and

$$\begin{bmatrix} \lambda_{v,t-1} \\ \lambda_{e,t-1} \end{bmatrix} = \begin{bmatrix} -v_{t-1} \left(\mathbb{I}_{\{r_{t-1} \leq v_{t-1}\}} - \alpha \right) \\ -\frac{1}{\alpha} \mathbb{I}_{\{r_{t-1} \leq v_{t-1}\}} \, r_{t-1} - e_{t-1} \end{bmatrix} \tag{2.8}$$

  is the forcing variable of the GAS model.

- One-factor GAS model (GAS-1F):

$$\begin{aligned} v_t &= a \exp\{\kappa_t\}, \\ e_t &= b \exp\{\kappa_t\}, \quad b < a < 0, \\ \kappa_t &= \omega + \beta\, \kappa_{t-1} + \gamma \frac{1}{b \exp\{\kappa_{t-1}\}} \left(\frac{1}{\alpha} \mathbb{I}_{\{r_{t-1} \leq a \exp(\kappa_{t-1})\}} \, r_{t-1} - b \exp(\kappa_{t-1}) \right) \end{aligned} \tag{2.9}$$

- GARCH-FZ:

$$\begin{aligned} v_t &= a\, \sigma_t, \\ e_t &= b\, \sigma_t, \quad b < a < 0, \\ \sigma_t^2 &= \omega + \beta\, \sigma_{t-1}^2 + \gamma\, r_{t-1}^2 \end{aligned} \tag{2.10}$$

- Hybrid (GAS/GARCH) model:

$$
\begin{aligned}
v_t &= a \exp\{\kappa_t\}, \qquad (2.11)\\
e_t &= b \exp\{\kappa_t\}, \quad b < a < 0,\\
\kappa_t &= \omega + \beta\,\kappa_{t-1} + \gamma \frac{1}{b \exp\{\kappa_{t-1}\}} \left(\frac{1}{\alpha} \mathbb{I}_{\{r_{t-1} \leq a \exp(\kappa_{t-1})\}}\, r_{t-1} - b \exp(\kappa_{t-1}) \right) + \delta \log |r_{t-1}|
\end{aligned}
$$

The four models proposed by Patton et al. (2019) are estimated by minimizing the following loss function

$$
L_{FZ0}(r, v, e; \alpha) = -\frac{1}{\alpha e} (v - r)\, \mathbb{I}_{\{r \leq v\}} + \frac{v}{e} + \log(-e) - 1 \qquad (2.12)
$$

See Lazar and Xue (2020) for an application of Patton et al.'s (2019) models using intraday data.

2.3 Data and Out-of-sample Forecasting

We now apply the models introduced in Section 2.2 to forecast and backtest the 99%, 97.5%, 95% and 90% VaR and ES of four international equity indices. We consider single-regime and two-regime Markov-switching versions of the GARCH models introduced in Section 2.2.1. The model in (2.1)-(2.2) is denoted by FGARCH when the number of regimes is one and by MSFGARCH when we allow for two regimes. The nested models, presented in Table 2.1, are also estimated for comparison. In total, we consider three nonparametric models, five semiparametric models, nine single-regime GARCH models and nine Markov-switching GARCH models. Section 2.3.1 describes the data. Section 2.3.2 presents the testing design and the results of the paper.

2.3.1 Data description

We consider daily prices of the S&P 500 index, the Dow Jones Industrial Average, the NIKKEI 225 index of Japanese stocks and the FTSE 100 index of UK stocks.[3] The sample period is

[3]The data can be downloaded from https://realized.oxford-man.ox.ac.uk/

from 3 January 2000 to 31 December 2019, yielding between 4866 and 5039 observations per series (the exact numbers vary due to differences in holidays and market closures). To assess how the forecasting models perform during the 2008 financial crisis, we use data until 31 December 2007 for estimation and reserve the last 12 years for evaluation of VaR and ES forecasts and model comparison.

In Table 2.2, we report summary statistics of the four daily equity returns over the full sample period. Average annualized returns range from 0.661% for the FTSE to 4.691% for the DJIA and annualized standard deviations range from 17.694% to 23.539%. The return series exhibit mild negative skewness and significant kurtosis, which motivates our choice of the t-distribution in model (2.1). The lower two panels of Table 2.2 present the sample VaR and ES at different confidence levels.

[Table 2.2 here]

2.3.2 Backtesting VaR and ES

The most popular procedures for testing the accuracy of VaR forecasts use the "hit sequence" of VaR violations (see Nieto and Ruiz, 2016),

$$\mathrm{Hit}_t^\alpha = \mathbb{I}_{\{r_t \leq \mathrm{VaR}_t^\alpha\}} \tag{2.13}$$

For a correctly specified model, we expect that the observed return values will only be worse than the VaR forecast $100 \times \alpha\%$ of the time. Hence, ideally, the hit variable should have a mean value of α (see Kupiec, 1995). This property is known as Unconditional Coverage (UC). The hit sequence should also be completely unpredictable and, therefore, independently distributed as a Bernouilli random variable, $\mathrm{Hit}_t^\alpha \overset{iid}{\sim} \mathrm{Bernouilli}(\alpha)$. Furthermore, the occurrences of observations outside the interval provided by the VaR forecasts must be homogeneous over the sample and not come in clusters (Christoffersen, 1998). This property is called Conditional Coverage (CC). Christoffersen (1998) introduces a three-step procedure for testing the correct unconditional coverage, independence and conditional coverage. All three tests can be implemented in the likelihood ratio (LR) framework.

Table 2.3 presents the number of rejections of the null of correct unconditional coverage, independence and conditional coverage for each model across the four equity indices. The significance level considered for the tests is 5%. Semiparametric models and MSGARCH models have a lower number of rejections than historical simulation models and single-regime models across the four values of α. Overall, the best models for VaR based on the UC, independence and CC tests are the asymmetric Markov-switching GARCH models where the λ parameter in (2.1) is freely estimated. That is, the MSAPGARCH and MSFGARCH models.

[Table 2.3 here]

However, these tests based on the binary hit variable have been criticized in several studies. For instance, Thiele (2019) has recently shown that the tests have an undesirable power asymmetry, with a greater ability to identify conservative VaR models than those that underestimate risk on average. In the following section, we perform recently proposed backtesting methods to jointly backtest VaR and ES forecasts.

Joint Backtests of VaR and ES

As in Patton et al. (2019), we jointly backtest the VaR and ES forecasts generated by different models using the L_{FZ0} loss function of Fissler et al. (2016) given in (2.12). Columns 2-5 in Table 2.4 to Table 2.7 present the average out of sample losses for $\alpha = 1\%$, $\alpha = 2.5\%$, $\alpha = 5\%$ and $\alpha = 10\%$. We show results for the 26 models and the four equity return series. The smallest loss is highlighted in bold, the second-lowest is highlighted in italics. Our more flexible specification in (2.1)-(2.2) with two-regimes, MSFGARCH model, consistently provides the smallest losses across indices and values of α. In fact, it is only beaten by other asymmetric MSGARCH models for the FTSE index when $\alpha = 1\%$ or $\alpha = 10\%$.

[Table 2.4 - Table 2.7 here]

The ranking of the models based on the losses in Table 2.4 - Table 2.7 are given in Table 2.8. The model with the smallest loss is ranked 1 and the model with the highest loss is ranked 26. Columns 6 and 12 give the average rank across the four series and columns 7 and 13 give the rank for each value of α. As noted above, the MSFGARCH model ranks first across the four indices. On the other hand, nonparametric methods based on rolling windows are at the bottom of the ranking.

[Table 2.8 here]

Figure 2.1 - Figure 2.4 show the VaR and ES forecasts when $\alpha = 5\%$ for the four indices using our more general regime-switching GARCH model in (2.1), the GARCH-FZ model in (2.10) and the rolling window model in (2.5) with a rolling window of 500 observations. The risk measures obtained from the parametric and semiparametric models follow a dynamic pattern mimicking the fluctuations in returns. On the other hand, the nonparametric model yield stepped VaR and ES forecasts that little resembles the movements in returns. It is therefore not surprising that nonparametric models based on rolling windows do not provide accurate VaR and ES forecasts.

[Figure 2.1 - Figure 2.4 here]

While average losses are useful for an initial look at forecast performance, they do not reveal whether the gains are statistically significant. To analyse the relative performance of each model, we use the Diebold–Mariano (DM) test on the loss differences. In this way, we use t-statistics from the DM test to compare the losses of each pair of models, using the L_{FZ0} loss function. Figure 2.5 to Figure 2.8 show the results of the DM tests for the S&P 500, DJIA, NIKKEI 225 and FTSE 100 indices. The numbering of the models used in the figures corresponds to column 1 in Table 2.3. The tests are conducted as “row model minus column model” and so a negative t-statistic indicates that the row model outperforms the column model. Values of the t-statistics smaller than -1.96, indicating losses from the row model significantly smaller than losses from the column model at the 95% confidence level, are depicted in dark green. Values of the t-statistics greater than 1.96, showing significant outperformance of the column model, are indicated in dark red. Light green (red) indicates that the row (column) model performs better than the column (row) model but not significantly at the 95% confidence level.

[Figure 2.5 - Figure 2.8 here]

For the S&P 500, the MSFGARCH model significantly outperforms all competing model when $\alpha = 5\%$ and $\alpha = 10\%$. Although the MSFGARCH model still provides the smallest losses for values of $\alpha = 1\%$ and $\alpha = 2.5\%$, these losses are not statistically lower than those obtained from other regime-switching GARCH models such as the MSGJRGARCH or the MSNAGARCH models. When considering the DJIA and NIKKEI 225 indices, the MSF-GARCH model significantly outperforms for any value of α. Finally, for the FTSE 100 index only the MSTGARCH and MSNAGARCH models significantly outperform the MSFGARCH model for one of the four values of α, $\alpha = 10\%$. We can conclude that, based on Diebold and Mariano's (1995) tests, the MSFGARCH model provides losses that are significantly smaller than those obtained from the competing models.

An alternative forecast comparison is the Model Confidence Set (MCS) approach of Hansen et al. (2011). This test enables one to obtain a set of models for which there is a pre-specified probability that the set contains the best model. It is similar to the DM test for two models but estimates the distribution of the test statistic using a bootstrap procedure. In this study, we consider the 90% confidence level and use two methods: the R method which uses sums of absolute values for calculating the test statistic, and the SQ method which uses the summed squares. Table 2.9 presents the number of models within the MCS test using the block bootstrap with a block length of 12 and 10,000 replications based on the losses generated from the L_{FZ0} loss function. The results from the MCS approach reaffirm the fact that the MSFGARCH model provides a superior forecast accuracy. Indeed, the MSFGARCH model is included in the final set in 15 out of the 16 forecast events. It is followed by the MSTGARCH and MSNAGARCH models which are included only 5 times.

[Table 2.9 here]

To individually evaluate the VaR and ES estimates for the different models, we follow Patton et al. (2019) in using regression frameworks related to those in Engle and Manganelli (2004). Correct forecasts of ES and VaR must satisfy:

$$\mathbb{E}\left[\partial L_{FZ0}(r_t, v_t, e_t; \alpha)/\partial v_t \mid \mathcal{I}_{t-1}\right] = \mathbb{E}\left[\partial L_{FZ0}(r_t, v_t, e_t; \alpha)/\partial e_t \mid \mathcal{I}_{t-1}\right] = 0 \tag{2.14}$$

which is equivalent to $\mathbb{E}\left(\lambda_{v,t} \mid \mathcal{I}_{t-1}\right) = \mathbb{E}\left(\lambda_{e,t} \mid \mathcal{I}_{t-1}\right) = 0$ where $\lambda_{v,t}$ and $\lambda_{e,t}$ are defined in (2.8). Based on this, Patton et al. (2019) propose a linear regression of $\lambda^s_{v,t} = \frac{\lambda_{v,t}}{v_t}$ and $\lambda^s_{e,t} = \frac{\lambda_{e,t}}{e_t}$ on its lagged values and any useful function of past information. Consider the following regressions:

$$\lambda^s_{v,t} = \mathrm{a}_0 + \sum_{i=1}^{l} \mathrm{a}_{t-i}\lambda^s_{v,t-i} + \mathrm{a}_{l+1}v_{t-1} + \epsilon_{v,t} \quad (2.15)$$

$$\lambda^s_{e,t} = \mathrm{b}_0 + \sum_{i=1}^{l} \mathrm{b}_{t-i}\lambda^s_{v,t-i} + \mathrm{b}_{l+1}e_{t-1} + \epsilon_{e,t} \quad (2.16)$$

where the a_i and b_i are the parameters of the regression and $\varepsilon_{v,t}$, $\epsilon_{e,t}$ are the regression residuals. The test then consists of testing the significance of all coefficients in the regressions. Note that (2.15) is just the dynamic quantile regression test of Engle and Manganelli (2004). As in Engle and Manganelli (2004), we choose $l = 4$. Columns 6-13 in Table 2.4 to Table 2.7 present the p-values from the test of the goodness-of-fit of the VaR and ES forecasts for different values of α. Entries greater than 0.10 (indicating no evidence against optimality at the 0.10 level) are in bold and entries between 0.05 and 0.10 are in italics. For $\alpha = 1\%$, no model passes the test across the four equity indices. For $\alpha = 2.5\%$ and $\alpha = 5\%$, most of the semiparametric and MSGARCH models pass the tests for at least two of the indices. For $\alpha = 10\%$, the results are mixed. Only a few single-regime GARCH models and the MSFGARCH model pass the tests for the S&P 500 and DJIA indices; most of the models pass the test for the FTSE index, while only semiparametric and single-regime GARCH models pass the test for the NIKKEI 225 index. Nevertheless, as discussed in Nolde et al. (2017) and Patton et al. (2019), it is difficult to discuss the relative of the models when many different models pass (or fail) a goodness-of-fit test. In this case, comparative backtesting methods such as the DM or MCS tests above are more appropriate approaches.

2.4 Conclusion

Following the Basel III Accord, more attention is placed on Expected Shortfall (ES) as a measure of risk complementing and partly replacing Value-at-Risk (VaR) procedures. In this paper, we propose a flexible Markov-switching GARCH model and evaluate its ability to forecast the VaR and ES of four international stock indices. Namely, the S&P 500, Dow Jones Industrial Average, the NIKKEI 225 and the FTSE 100. We compare the forecasting performance of our general model with a set of models used in the literature including nonparametric, semiparametric, single-regime and Markov-switching GARCH models. To test the competing models under varying market conditions, we consider a forecasting period spanning the 2008 financial crisis.

Using traditional and comparative backtesting procedures, we show that asymmetric MS-GARCH models significantly outperform nonparametric, semiparametric, single-regime and symmetric GARCH models. The Model Confidence Set (MCS) procedure of Hansen et al. (2011) identifies our most flexible specification as the best performing model for VaR and ES across a range of tail probability values used in risk management. Furthermore, based on Diebold and Mariano (1995) tests, the superiority of our model is significant and consistent across the four indices and probabilities level considered in our study.

2.5 Appendix: Tables and Figures

Table 2.1: Nested GARCH models.

Model	λ	$\hat{\lambda}$	γ	ψ	Volatility Specification	Reference
EGARCH	0	1	free	0	$\ln \sigma_t^2 = \omega + \alpha \left(\lvert z_{t-1}\rvert - \mathbb{E}\lvert z_{t-1}\rvert - \gamma z_{t-1}\right) + \beta \ln \sigma_{t-1}^2$	Nelson (1991)
AVGARCH	1	1	0	0	$\sigma_t = \omega + \left[\alpha \lvert z_{t-1}\rvert + \beta\right] \sigma_{t-1}$	Schwert (1989); Taylor (2008)
TGARCH	1	1	$\lvert\gamma\rvert \leq 1$	0	$\sigma_t = \omega + \left[\alpha \left(\lvert z_{t-1}\rvert - \gamma z_{t-1}\right) + \beta\right] \sigma_{t-1}$	Zakoian (1994)
GARCH	2	2	0	0	$\sigma_t^2 = \omega + \left[\alpha z_{t-1}^2 + \beta\right] \sigma_{t-1}^2$	Bollerslev (1986)
GJRGARCH	2	2	free	0	$\sigma_t^2 = \omega + \left[\alpha \left(\lvert z_{t-1}\rvert - \gamma z_{t-1}\right)^2 + \beta\right] \sigma_{t-1}^2$	Glosten et al. (1993)
NAGARCH	2	2	0	free	$\sigma_t^2 = \omega + \left[\alpha \left(z_{t-1} - \psi\right)^2 + \beta\right] \sigma_{t-1}^2$	Engle and Ng (1993)
NLGARCH	free	λ	0	0	$\sigma_t^\lambda = \omega + \left[\alpha \lvert z_{t-1}\rvert^\lambda + \beta\right] \sigma_{t-1}^\lambda$	Higgins and Bera (1992)*
APGARCH	free	λ	$\lvert\gamma\rvert \leq 1$	0	$\sigma_t^\lambda = \omega + \left[\alpha \left(\lvert z_{t-1}\rvert - \gamma z_{t-1}\right)^\lambda + \beta\right] \sigma_{t-1}^\lambda$	Ding et al. (1993)*

*Nested if $\lambda > 0$.

Table 2.2: Summary Statistics.

	S&P 500	DJI	NIKKEI	FTSE
Mean (Annualized)	4.053	4.691	0.865	0.661
Std dev (Annualized)	18.765	17.694	23.539	17.987
Skewness	-0.218	-0.135	-0.432	-0.175
Kurtosis	11.242	11.039	9.448	9.525
99% VaR	-3.416	-3.255	-4.101	-3.241
97.5% VaR	-2.525	-2.358	-3.033	-2.404
95% VaR	-1.876	-1.767	-2.344	-1.779
90% VaR	-1.270	-1.171	-1.669	-1.222
99% ES	-4.820	-4.542	-5.943	-4.520
95% ES	-3.656	-3.434	-4.456	-3.439
97.5% ES	-2.908	-2.734	-3.546	-2.752
90% ES	-2.222	-2.086	-2.764	-2.107

Note: The sample period is from January 2000 to December 2019.

Table 2.3: Rejections based on tests for VaR across the four indices for different values of α.

		$\alpha = 1\%$			$\alpha = 2.5\%$			$\alpha = 5\%$			$\alpha = 10\%$		
		UC	Ind	CC	UC	Ind	CC	UC	Ind	CC	UC	Ind	CC
1	RW-125	2	2	4	**0**	*1*	*1*	**0**	2	2	**0**	3	3
2	RW-250	*1*	3	3	**0**	3	4	**0**	3	3	**0**	*1*	*1*
3	RW-500	**0**	*1*	4	*1*	3	4	**0**	3	3	**0**	3	3
4	GARCH-EDF	2	*1*	2	*1*	**0**	**0**	**0**	**0**	**0**	**0**	*1*	*1*
5	GAS-2F	4	**0**	4	2	**0**	*1*	**0**	**0**	**0**	2	**0**	*1*
6	GAS-1F	*1*	**0**	*1*	**0**	**0**	**0**	**0**	**0**	**0**	*1*	**0**	*1*
7	GARCH-FZ	*1*	**0**	2	*1*	**0**	**0**	**0**	**0**	**0**	**0**	*1*	*1*
8	Hybrid	*1*	**0**	*1*	**0**	**0**	**0**	**0**	**0**	**0**	*1*	**0**	*1*
9	EGARCH	4	**0**	4	4	**0**	4	2	**0**	2	**0**	**0**	**0**
10	AVGARCH	4	2	4	4	**0**	4	4	**0**	3	*1*	*1*	2
11	TGARCH	4	**0**	4	4	**0**	4	2	**0**	2	*1*	**0**	*1*
12	GARCH	3	**0**	3	3	**0**	3	*1*	**0**	*1*	**0**	*1*	*1*
13	GJRGARCH	3	**0**	3	3	**0**	3	*1*	**0**	*1*	**0**	**0**	**0**
14	NAGARCH	3	**0**	3	3	**0**	3	4	**0**	4	**0**	**0**	**0**
15	NLGARCH	3	**0**	4	3	**0**	3	*1*	**0**	*1*	**0**	*1*	*1*
16	APGARCH	4	**0**	3	4	**0**	4	2	**0**	*1*	**0**	**0**	**0**
17	FGARCH	4	**0**	4	4	**0**	3	3	**0**	3	*1*	**0**	*1*
18	MSEGARCH	2	**0**	2	*1*	**0**	*1*	**0**	**0**	**0**	**0**	**0**	**0**
19	MSAVGARCH	**0**	3	3	**0**	**0**	**0**	*1*	**0**	*1*	2	**0**	2
20	MSTGARCH	*1*	**0**	2	*1*	**0**	*1*	**0**	**0**	**0**	*1*	**0**	**0**
21	MSGARCH	**0**	**0**	*1*	**0**	**0**	**0**	*1*	**0**	**0**	3	**0**	2
22	MSGJRGARCH	*1*	*1*	2	*1*	**0**	**0**	**0**	**0**	**0**	2	**0**	2
23	MSNAGARCH	*1*	**0**	*1*	**0**	**0**	**0**	4	**0**	4	*1*	**0**	*1*
24	MSNLGARCH	**0**	**0**	*1*	**0**	**0**	**0**	**0**	**0**	**0**	3	**0**	3
25	MSAPGARCH	*1*	**0**	2	*1*	**0**	*1*	**0**	**0**	**0**	**0**	**0**	**0**
26	MSFGARCH	**0**	**0**	*1*	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**

Note: This table presents the number of model rejections based on unconditional coverage, independence and conditional coverage tests of VaR for the four daily equity return series over the out-of-sample period for 26 different forecasting models. The first three rows (models 1–3) correspond to rolling window historical forecasts, the next five rows correspond to forecasts based on the semiparametric models in Section 2.2.2, the next eight rows (models 9–17) correspond to single-regime GARCH models and the last 8 rows (models 18–28) correspond to the different Markov-switching GARCH models nested in model (2.1).

Table 2.4: Out-of-sample average losses and goodness-of-fit tests ($\alpha = 1\%$)

	Average Loss				GoF p-values VaR				GoF p-values ES			
	S&P	DJIA	NIKKEI	FTSE	S&P	DJIA	NIKKEI	FTSE	S&P	DJIA	NIKKEI	FTSE
RW-125	1.4712	1.3960	1.8247	1.3059	0.0229	**0.1145**	0.0000	0.0015	0.0146	*0.0647*	0.0000	0.0019
RW-250	1.5036	1.4556	1.9135	1.3749	0.0353	*0.0720*	0.0036	0.0001	0.0448	0.0354	0.0010	0.0001
RW-500	1.6125	1.5307	1.9937	1.4609	*0.0673*	0.0163	0.0045	0.0002	0.0315	0.0038	0.0000	0.0000
GARCH-EDF	1.2729	1.1285	1.5454	1.1194	0.0087	**0.3429**	0.0000	0.0000	0.0111	0.0164	0.0000	0.0000
GAS-2F	1.2588	1.1956	1.6131	1.2263	0.0000	0.0184	0.0000	0.0002	0.0000	0.0002	0.0000	0.0018
GAS-1F	1.2204	1.1801	1.6075	1.1784	0.0000	0.0000	0.0000	**0.6653**	0.0000	0.0000	0.0000	**0.8570**
GARCH-FZ	1.2347	1.1466	1.5046	1.0943	0.0022	0.0011	0.0000	0.0000	0.0056	0.0115	0.0000	0.0000
Hybrid	1.2342	1.1798	1.5172	1.1996	0.0000	0.0000	0.0000	**0.1009**	0.0000	0.0000	0.0000	*0.0860*
EGARCH	1.4114	1.2439	1.5027	1.2266	0.0005	0.0156	0.0000	0.0012	0.0002	0.0120	0.0000	0.0006
AVGARCH	1.4487	1.2888	1.9547	1.2718	0.0002	0.0007	0.0000	0.0004	0.0003	0.0008	0.0000	0.0003
TGARCH	1.4107	1.2341	1.8951	1.2756	0.0000	0.0004	0.0000	0.0006	0.0001	0.0002	0.0000	0.0004
GARCH	1.2995	1.1790	1.5821	1.1604	0.0084	0.0116	0.0000	0.0000	0.0077	0.0003	0.0000	0.0000
GJRGARCH	1.2521	1.1320	1.5166	1.1721	0.0000	0.0000	0.0000	0.0033	0.0000	0.0000	0.0000	0.0017
NAGARCH	1.3064	1.1837	1.5046	1.1730	0.0000	*0.0933*	0.0000	0.0000	0.0000	*0.0605*	0.0000	0.0000
NLGARCH	1.2846	1.1785	1.5853	1.1601	0.0117	0.0116	0.0002	0.0000	0.0097	0.0003	0.0029	0.0000
APGARCH	1.2965	1.1983	1.5093	1.1947	0.0047	0.0254	0.0000	0.0018	0.0032	0.0404	0.0001	0.0008
FGARCH	1.3070	1.2062	1.5105	1.1700	0.0002	0.0048	0.0196	0.0004	0.0001	0.0049	0.0105	0.0003
MSEGARCH	1.2340	1.1962	*1.3952*	1.1684	*0.0635*	**0.1423**	0.0084	0.0292	0.0446	*0.0990*	0.0085	0.0159
MSAVGARCH	1.2037	1.1024	1.5570	1.0663	**0.4200**	0.0065	**0.1096**	0.0000	**0.4495**	0.0012	**0.2436**	0.0010
MSTGARCH	1.1775	1.0785	1.4520	**0.9634**	**0.4375**	**0.3378**	**0.5871**	0.0000	**0.3414**	**0.1909**	**0.4400**	0.0000
MSGARCH	1.1837	1.0986	1.5482	1.0710	0.0000	0.0000	0.0178	0.0003	0.0000	0.0000	**0.1130**	0.0001
MSGJRGARCH	1.1565	1.0568	1.4757	1.0845	0.0000	0.0000	0.0116	*0.0511*	0.0000	0.0000	0.0037	0.0396
MSNAGARCH	*1.1430*	*1.0388*	1.4167	1.1076	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
MSNLGARCH	1.1808	1.0979	1.5466	1.0765	0.0000	0.0000	0.0193	0.0000	0.0000	0.0000	*0.0772*	0.0000
MSAPGARCH	1.1814	1.1087	1.4553	*0.9831*	**0.3625**	**0.2552**	**0.5737**	0.0000	**0.2872**	**0.1307**	**0.4192**	0.0000
MSFGARCH	**1.1313**	**0.9392**	**1.3265**	0.9911	*0.0751*	0.0036	0.0001	0.0067	0.0461	0.0021	0.0001	0.0040

Note: The left panel of this table presents the average losses, using the L_{FZ0} loss function, for four daily equity return series, over the out-of-sample period, for $\alpha = 1\%$. The lowest average loss in each column is highlighted in bold, the second-lowest is highlighted in italics. The middle and right panels of this table present p-values from goodness-of-fit tests of the VaR and ES forecasts respectively. Values that are greater than 0.10 (indicating no evidence against optimality at the 0.10 level) are in bold and values between 0.05 and 0.10 are in italics.

Table 2.5: Out-of-sample average losses and goodness-of-fit tests ($\alpha = 2.5\%$)

	Average Loss				GoF p-values VaR				GoF p-values ES			
	S&P	DJIA	NIKKEI	FTSE	S&P	DJIA	NIKKEI	FTSE	S&P	DJIA	NIKKEI	FTSE
RW-125	1.1982	1.1201	1.5068	1.0669	0.0028	0.0011	0.0018	0.0000	0.0064	0.0046	0.0023	0.0000
RW-250	1.2285	1.1590	1.5465	1.1251	0.0400	**0.1319**	0.0167	0.0003	0.0343	*0.0733*	0.0049	0.0000
RW-500	1.3380	1.2690	1.6248	1.2288	0.0000	0.0000	0.0000	0.0000	0.0000	0.0001	0.0001	0.0000
GARCH-EDF	1.0316	0.9410	1.3112	0.9493	**0.1353**	**0.3658**	**0.7650**	**0.2204**	**0.1608**	**0.4859**	**0.8105**	**0.1670**
GAS-2F	1.0312	0.9834	1.3542	1.0038	0.0121	**0.2188**	**0.3687**	0.0016	0.0189	**0.5534**	**0.2024**	*0.0902*
GAS-1F	1.0380	0.9684	1.3553	0.9749	0.0000	**0.2143**	0.0000	**0.6596**	0.0000	**0.6145**	0.0000	**0.8712**
GARCH-FZ	1.0245	0.9259	1.3034	0.9550	*0.0963*	**0.4062**	**0.6781**	*0.0650*	**0.1354**	**0.4773**	**0.6892**	0.0263
Hybrid	1.0299	0.9354	1.3041	0.9606	0.0000	**0.6525**	0.0029	**0.2490**	0.0000	**0.8701**	0.0157	**0.2090**
EGARCH	1.0929	0.9842	1.3005	0.9872	0.0097	*0.0978*	0.0002	0.0156	0.0014	0.0442	0.0010	0.0029
AVGARCH	1.1829	1.0638	1.5607	1.0548	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
TGARCH	1.1477	1.0312	1.5286	1.0447	0.0000	0.0001	0.0000	0.0000	0.0000	0.0001	0.0000	0.0000
GARCH	1.0549	0.9691	1.3344	0.9773	0.0292	**0.1072**	**0.5507**	0.0045	0.0326	*0.0902*	**0.3779**	0.0017
GJRGARCH	1.0234	0.9275	1.3023	0.9668	*0.0807*	*0.0826*	*0.0624*	0.0035	0.0239	*0.0750*	0.0187	0.0007
NAGARCH	1.0398	0.9519	1.2947	0.9618	0.0107	0.0489	*0.0629*	0.0171	0.0067	*0.0626*	0.0176	0.0051
NLGARCH	1.0500	0.9689	1.3345	0.9783	0.0332	**0.1072**	**0.5586**	0.0041	0.0376	*0.0905*	**0.3558**	0.0016
APGARCH	1.0389	0.9642	1.2986	0.9736	0.0339	*0.0636*	**0.1714**	0.0026	0.0079	0.0355	*0.0959*	0.0006
FGARCH	1.0330	0.9647	1.3012	0.9516	0.0029	*0.0885*	0.0281	0.0044	0.0004	0.0274	0.0042	0.0009
MSEGARCH	0.9987	0.9452	1.2366	0.9491	**0.5214**	**0.7694**	**0.9105**	**0.1236**	**0.2081**	**0.3610**	**0.8987**	0.0417
MSAVGARCH	1.0008	0.9204	1.3188	0.9042	**0.2773**	**0.4829**	**0.1750**	**0.7915**	**0.5028**	0.0000	**0.2076**	**0.7798**
MSTGARCH	0.9669	0.8834	1.2487	*0.8682*	**0.4926**	**0.9392**	**0.3348**	0.0340	**0.4145**	**0.8699**	**0.2421**	0.0337
MSGARCH	0.9853	0.9118	1.3130	0.9127	**0.5685**	**0.3204**	**0.7471**	**0.5731**	**0.7067**	**0.4231**	**0.8026**	**0.4839**
MSGJRGARCH	0.9623	0.8747	1.2653	0.9117	**0.5677**	**0.4707**	**0.9718**	**0.1531**	**0.4761**	**0.2617**	**0.8178**	*0.0826*
MSNAGARCH	*0.9533*	*0.8589*	*1.2218*	0.9125	**0.4322**	**0.3893**	0.0030	**0.5765**	**0.3488**	**0.2343**	0.0047	**0.4388**
MSNLGARCH	0.9837	0.9111	1.3119	0.9166	**0.6317**	**0.3194**	**0.6842**	**0.3529**	**0.5461**	**0.4284**	**0.7709**	**0.1564**
MSAPGARCH	0.9691	0.9052	1.2478	0.8765	**0.3651**	**0.5539**	0.0175	**0.1176**	**0.2803**	**0.3557**	0.0352	*0.0743*
MSFGARCH	**0.9342**	**0.8032**	**1.1432**	**0.8436**	**0.4285**	**0.8904**	0.0233	0.0059	**0.2145**	**0.8774**	0.0220	0.0021

Note: The left panel of this table presents the average losses, using the L_{FZ0} loss function, for four daily equity return series, over the out-of-sample period, for $\alpha = 2.5\%$. The lowest average loss in each column is highlighted in bold, the second-lowest is highlighted in italics. The middle and right panels of this table present p-values from goodness-of-fit tests of the VaR and ES forecasts respectively. Values that are greater than 0.10 (indicating no evidence against optimality at the 0.10 level) are in bold and values between 0.05 and 0.10 are in italics.

Table 2.6: Out-of-sample average losses and goodness-of-fit tests ($\alpha = 5\%$)

	Average Loss				GoF p-values VaR				GoF p-values ES			
	S&P	DJIA	NIKKEI	FTSE	S&P	DJIA	NIKKEI	FTSE	S&P	DJIA	NIKKEI	FTSE
RW-125	0.9710	0.8910	1.2752	0.8742	0.0021	0.0145	0.0000	0.0002	0.0114	0.0135	0.0010	0.0000
RW-250	1.0045	0.9442	1.2931	0.9234	0.0001	0.0001	0.0002	0.0002	0.0033	0.0011	0.0029	0.0000
RW-500	1.0953	1.0474	1.3536	0.9845	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
GARCH-EDF	0.8306	0.7601	1.1324	0.7900	*0.0715*	**0.6559**	**0.9774**	**0.3855**	**0.2452**	**0.6312**	**0.9922**	**0.4459**
GAS-2F	0.8622	0.7976	1.1408	0.7950	**0.1292**	**0.2752**	**0.3580**	**0.9376**	**0.4954**	**0.3232**	0.0254	**0.7555**
GAS-1F	0.8489	0.7781	1.1383	0.7892	0.0401	**0.3987**	*0.0519*	**0.8265**	**0.1680**	**0.5085**	0.0319	**0.8882**
GARCH-FZ	0.8332	0.7561	1.1321	0.7896	0.0373	**0.3866**	**0.8995**	**0.2130**	**0.1358**	**0.4209**	**0.9622**	**0.3752**
Hybrid	0.8522	0.7664	1.1145	0.7957	0.0360	**0.1422**	0.0173	**0.5640**	**0.1534**	**0.2555**	0.0114	**0.5082**
EGARCH	0.8456	0.7591	1.1098	0.7941	0.0173	**0.2689**	**0.2287**	0.0364	0.0045	*0.0903*	*0.0733*	0.0050
AVGARCH	0.9480	0.8635	1.2858	0.8650	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
TGARCH	0.9142	0.8362	1.2613	0.8521	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
GARCH	0.8385	0.7705	1.1392	0.8021	**0.2646**	**0.5776**	**0.6612**	*0.0879*	**0.1232**	**0.3691**	**0.5381**	0.0276
GJRGARCH	0.8070	0.7312	1.1130	0.7857	**0.1830**	**0.3938**	**0.8673**	**0.1147**	*0.0921*	**0.3528**	**0.5196**	0.0183
NAGARCH	1.0030	0.9208	1.2649	0.9521	*0.0573*	**0.3745**	**0.1333**	0.0177	*0.0595*	**0.3418**	**0.1481**	0.0334
NLGARCH	0.8367	0.7704	1.1387	0.8019	**0.1371**	**0.5776**	**0.6652**	*0.0695*	**0.1003**	**0.3700**	**0.5126**	0.0271
APGARCH	0.8119	0.7440	1.1070	0.7850	0.0408	**0.3226**	**0.2243**	*0.0594*	0.0156	**0.1451**	**0.1389**	0.0064
FGARCH	1.0020	0.9415	1.2758	0.8159	0.0177	**0.1602**	0.0127	**0.1262**	0.0009	0.0314	0.0040	0.0114
MSEGARCH	0.7906	0.7386	1.0830	0.7617	**0.2883**	**0.6277**	**0.1187**	**0.5481**	**0.1463**	**0.4632**	**0.1628**	**0.1734**
MSAVGARCH	0.8165	0.7539	1.1298	0.7560	0.0024	0.0064	0.0009	**0.8777**	0.0406	0.0146	0.0051	**0.8933**
MSTGARCH	*0.7761*	0.7080	1.0712	*0.7287*	**0.1594**	**0.2094**	0.0286	**0.5045**	**0.5543**	**0.4835**	*0.0556*	**0.2747**
MSGARCH	0.8130	0.7476	1.1285	0.7619	0.0133	0.0086	0.0051	**0.3020**	*0.0673*	0.0227	0.0287	**0.7439**
MSGJRARCH	0.7821	*0.7070*	1.0899	0.7497	**0.2150**	**0.1814**	0.0411	**0.7374**	**0.4447**	**0.5919**	**0.1045**	**0.4299**
MSNAGARCH	0.8053	0.7766	1.1273	0.8732	*0.0508*	**0.1374**	0.0003	**0.3376**	**0.1901**	**0.2590**	0.0018	**0.3472**
MSNLGARCH	0.8129	0.7475	1.1285	0.7645	0.0124	0.0164	0.0055	**0.1524**	*0.0643*	0.0352	0.0292	**0.3295**
MSAPGARCH	0.7772	0.7156	*1.0698*	0.7316	**0.1610**	**0.1340**	0.0013	**0.3450**	**0.4438**	*0.0922*	0.0056	**0.1588**
MSFGARCH	**0.7490**	**0.6611**	**1.0063**	**0.7050**	**0.1077**	**0.5318**	**0.1716**	**0.3055**	**0.4384**	**0.6655**	*0.0703*	0.0364

Note: The left panel of this table presents the average losses, using the L_{FZ0} loss function, for four daily equity return series, over the out-of-sample period, for $\alpha = 5\%$. The lowest average loss in each column is highlighted in bold, the second-lowest is highlighted in italics. The middle and right panels of this table present p-values from goodness-of-fit tests of the VaR and ES forecasts respectively. Values that are greater than 0.10 (indicating no evidence against optimality at the 0.10 level) are in bold and values between 0.05 and 0.10 are in italics.

Table 2.7: Out-of-sample average losses and goodness-of-fit tests ($\alpha = 10\%$)

	Average Loss				GoF p-values VaR				GoF p-values ES			
	S&P	DJIA	NIKKEI	FTSE	S&P	DJIA	NIKKEI	FTSE	S&P	DJIA	NIKKEI	FTSE
RW-125	0.7014	0.6354	1.0094	0.6396	0.0000	0.0000	0.0000	0.0063	0.0014	0.0000	0.0000	0.0004
RW-250	0.7257	0.6686	1.0170	0.6687	0.0000	0.0000	0.0008	0.0359	0.0001	0.0000	0.0005	0.0001
RW-500	0.8019	0.7525	1.0525	0.7257	0.0000	0.0000	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000
GARCH-EDF	0.5910	0.5274	0.9183	0.5858	0.0065	0.0398	**0.4546**	**0.6258**	**0.1564**	**0.1229**	**0.7096**	**0.4086**
GAS-2F	0.5885	0.5179	0.9057	0.5913	0.0099	**0.1493**	**0.2607**	*0.0510*	**0.2487**	**0.2314**	**0.3360**	0.0050
GAS-1F	0.6287	0.5201	0.9085	0.5825	0.0049	0.0060	**0.2833**	**0.8673**	**0.1512**	*0.0542*	**0.5271**	**0.7001**
GARCH-FZ	0.5931	0.5255	0.9197	0.5860	0.0026	0.0119	**0.4191**	**0.7205**	*0.0865*	*0.0967*	**0.4837**	**0.5171**
Hybrid	0.5962	0.5144	0.8974	0.5814	0.0003	0.0116	**0.7274**	**0.4629**	*0.0682*	**0.1265**	**0.8893**	**0.3090**
EGARCH	0.5806	0.5116	0.8944	0.5718	0.0245	0.0186	**0.5081**	**0.2733**	0.0166	0.0275	**0.3000**	*0.0765*
AVGARCH	0.6752	0.6044	1.0021	0.6314	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
TGARCH	0.6464	0.5788	0.9844	0.6161	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
GARCH	0.5915	0.5303	0.9209	0.5905	*0.0874*	*0.0772*	**0.4549**	**0.6666**	**0.2376**	*0.0765*	**0.4701**	**0.2113**
GJRGARCH	0.5642	0.4976	0.9010	0.5670	**0.2980**	*0.0544*	**0.8141**	**0.9261**	**0.4612**	**0.2500**	**0.8931**	**0.3049**
NAGARCH	0.5605	0.4971	0.8924	0.5613	0.0379	*0.0553*	**0.7499**	**0.9299**	*0.0783*	**0.1274**	**0.6771**	**0.4949**
NLGARCH	0.5913	0.5303	0.9195	0.5905	*0.0622*	*0.0772*	**0.3855**	**0.7453**	**0.2292**	*0.0764*	**0.4195**	**0.2909**
APGARCH	0.5631	0.5011	0.8933	0.5648	**0.1148**	0.0484	**0.6619**	**0.2698**	**0.1494**	*0.0815*	**0.5473**	0.0446
FGARCH	0.5576	0.5050	0.8942	0.6382	**0.4428**	**0.3381**	0.0297	**0.5820**	*0.0625*	0.0223	0.0109	**0.2158**
MSEGARCH	0.5498	0.5001	0.8812	0.5503	0.0307	0.0363	0.0180	**0.6344**	**0.2216**	**0.1514**	*0.0667*	**0.3305**
MSAVGARCH	0.5918	0.5340	0.9100	0.5654	0.0000	0.0000	0.0008	**0.6281**	0.0000	0.0000	0.0088	**0.6233**
MSTGARCH	*0.5484*	0.4819	0.8626	**0.5361**	0.0013	0.0243	0.0468	**0.9695**	0.0420	*0.0905*	**0.1249**	**0.7813**
MSGARCH	0.5878	0.5296	0.9111	0.5679	0.0000	0.0000	0.0021	**0.4196**	0.0000	0.0001	0.0246	**0.4801**
MSGJRGARCH	0.5555	0.4874	0.8827	0.5444	0.0003	0.0015	0.0494	**0.5606**	0.0167	0.0452	**0.1449**	**0.8129**
MSNAGARCH	0.5492	*0.4587*	*0.8396*	0.5398	0.0000	0.0000	0.0003	**0.2568**	0.0147	0.0094	0.0028	**0.4659**
MSNLGARCH	0.5878	0.5297	0.9112	0.5658	0.0000	0.0000	0.0012	**0.2083**	0.0001	0.0001	0.0199	**0.2604**
MSAPGARCH	0.5492	0.4882	0.8595	*0.5372*	0.0017	0.0363	0.0276	**0.7342**	0.0414	0.0361	*0.0632*	**0.4378**
MSFGARCH	**0.5189**	**0.4569**	**0.8266**	0.5531	*0.0898*	**0.1451**	0.0342	**0.7399**	**0.1258**	*0.0981*	0.0354	**0.3135**

Note: The left panel of this table presents the average losses, using the L_{FZ0} loss function, for four daily equity return series, over the out-of-sample period, for $\alpha = 10\%$. The lowest average loss in each column is highlighted in bold, the second-lowest is highlighted in italics. The middle and right panels of this table present p-values from goodness-of-fit tests of the VaR and ES forecasts respectively. Values that are greater than 0.10 (indicating no evidence against optimality at the 0.10 level) are in bold and values between 0.05 and 0.10 are in italics.

Table 2.8: Out-of-sample performance rankings.

	$\alpha = 1\%$						$\alpha = 2.5\%$					
	SPX	DJI	N225	FTSE	Average	Rank	SPX	DJI	N225	FTSE	Average	Rank
RW-125	24	24	22	24	23.50	24	24	24	22	24	23.50	23
RW-250	25	25	24	25	24.75	25	25	25	24	25	24.75	25
RW-500	26	26	26	26	26.00	26	26	26	26	26	26.00	26
GARCH-EDF	15	9	14	10	12.00	11	14	12	14	10	12.50	13
GAS-2F	14	17	21	20	18.00	20	13	20	20	21	18.50	19
GAS-1F	9	15	20	17	15.25	16	16	17	21	17	17.75	17
GARCH-FZ	12	11	8	8	9.75	9	11	9	12	12	11.00	10
Hybrid	11	14	13	19	14.25	13	12	11	13	13	12.25	12
EGARCH	22	22	7	21	18.00	21	21	21	9	20	17.75	18
AVGARCH	23	23	25	22	23.25	23	23	23	25	23	23.50	24
TGARCH	21	21	23	23	22.00	22	22	22	23	22	22.25	22
GARCH	18	13	18	12	15.25	17	20	19	18	18	18.75	20
GJRGARCH	13	10	12	15	12.50	12	10	10	11	15	11.50	11
NAGARCH	19	16	9	16	15.00	15	18	14	7	14	13.25	15
NLGARCH	16	12	19	11	14.50	14	19	18	19	19	18.75	21
APGARCH	17	19	10	18	16.00	18	17	15	8	16	14.00	16
FGARCH	20	20	11	14	16.25	19	15	16	10	11	13.00	14
MSEGARCH	10	18	2	13	10.75	10	8	13	3	9	8.25	6
MSAVGARCH	8	7	17	4	9.00	8	9	8	17	4	9.50	9
MSTGARCH	4	4	4	1	3.25	2	4	4	5	2	3.75	3
MSGARCH	7	6	16	5	8.50	7	7	7	16	7	9.25	8
MSGJRGARCH	3	3	6	7	4.75	4	3	3	6	5	4.25	4
MSNAGARCH	2	2	3	9	4.00	3	2	2	2	6	3.00	2
MSNLGARCH	5	5	15	6	7.75	6	6	6	15	8	8.75	7
MSAPGARCH	6	8	5	2	5.25	5	5	5	4	3	4.25	5
MSFGARCH	1	1	1	3	1.50	1	1	1	1	1	1.00	1

	$\alpha = 5\%$						$\alpha = 10\%$					
	SPX	DJI	NIKKEI	FTSE	Average	Rank	SPX	DJI	N225	FTSE	Average	Rank
RW-125	22	22	22	23	22.25	22	24	24	24	24	24.00	24
RW-250	25	25	25	24	24.75	25	25	25	25	25	25.00	25
RW-500	26	26	26	26	26.00	26	26	26	26	26	26.00	26
GARCH-EDF	12	13	15	13	13.25	13	15	16	18	16	16.25	18
GAS-2F	19	19	19	15	18.00	19	14	13	13	20	15.00	15
GAS-1F	17	18	16	11	15.50	16	21	14	14	15	16.00	17
GARCH-FZ	13	11	14	12	12.50	12	19	15	20	17	17.75	19
Hybrid	18	14	9	16	14.25	15	20	12	11	14	14.25	12
EGARCH	16	12	7	14	12.25	11	11	11	10	13	11.25	10
AVGARCH	21	21	24	21	21.75	21	23	23	23	22	22.75	23
TGARCH	20	20	20	20	20.00	20	22	22	22	21	21.75	22
GARCH	15	16	18	18	16.75	18	17	19	21	18	18.75	21
GJRGARCH	7	5	8	10	7.50	6	10	7	12	11	10.00	9
NAGARCH	24	23	21	25	23.25	24	8	6	7	7	7.00	7
NLGARCH	14	15	17	17	15.75	17	16	20	19	19	18.50	20
APGARCH	8	7	6	9	7.50	7	9	9	8	8	8.50	8
FGARCH	23	24	23	19	22.25	23	7	10	9	23	12.25	11
MSEGARCH	5	6	4	6	5.25	5	5	8	5	5	5.75	6
MSAVGARCH	11	10	13	5	9.75	10	18	21	15	9	15.75	16
MSTGARCH	2	3	3	2	2.50	2	2	3	4	1	2.50	2
MSGARCH	10	9	12	7	9.50	9	13	17	16	12	14.50	14
MSGJRGARCH	4	2	5	4	3.75	4	6	4	6	4	5.00	5
MSNAGARCH	6	17	10	22	13.75	14	3	2	2	3	2.50	3
MSNLGARCH	9	8	11	8	9.00	8	12	18	17	10	14.25	13
MSAPGARCH	3	4	2	3	3.00	3	4	5	3	2	3.50	4
MSFGARCH	1	1	1	1	1.00	1	1	1	1	6	2.25	1

Note: This table presents the rankings (with the best-performing model ranked 1 and the worst ranked 26) based on the average losses obtained with the L_{FZ0} loss function.

Table 2.9: The 90% MCS for the R and SQ methods across the four stock indices.

	R method					SQ method				
	$\alpha = 1\%$	$\alpha = 2.5\%$	$\alpha = 5\%$	$\alpha = 10\%$	Total	$\alpha = 1\%$	$\alpha = 2.5\%$	$\alpha = 5\%$	$\alpha = 10\%$	Total
RW-125	0	0	0	0	0	0	0	0	0	0
RW-250	0	0	0	0	0	0	0	0	0	0
RW-500	0	0	0	0	0	0	0	0	0	0
GARCH-EDF	0	0	0	0	0	0	0	0	0	0
GAS-2F	0	0	0	0	0	0	0	0	0	0
GAS-1F	1	0	0	0	1	1	0	0	0	1
GARCH-FZ	0	0	0	0	0	0	0	0	0	0
Hybrid	1	0	0	0	1	1	0	0	0	1
EGARCH	0	0	0	0	0	0	0	0	0	0
AVGARCH	0	0	0	0	0	0	0	0	0	0
TGARCH	0	0	0	0	0	0	0	0	0	0
GARCH	0	0	0	0	0	0	0	0	0	0
GJRGARCH	0	0	0	0	0	0	0	0	0	0
NAGARCH	0	0	0	0	0	0	0	0	0	0
NLGARCH	0	0	0	0	0	0	0	0	0	0
APGARCH	0	0	0	0	0	0	0	0	0	0
FGARCH	0	0	0	0	0	0	0	0	0	0
MSEGARCH	1	0	0	1	2	1	0	0	1	2
MSAVGARCH	1	0	0	0	1	1	0	0	0	1
MSTGARCH	2	1	1	1	5	2	2	0	1	5
MSGARCH	1	0	0	0	1	1	0	0	0	1
MSGJRGARCH	1	1	0	1	3	1	1	0	1	3
MSNAGARCH	1	1	0	3	5	1	1	0	3	5
MSNLGARCH	1	0	0	0	1	1	0	0	0	1
MSAPGARCH	1	0	1	1	3	1	0	0	1	2
MSFGARCH	3	4	4	4	**15**	3	4	4	4	**15**

Note: This table presents the number of indices for which each method is within the MCS at the 95% confidence level based on the L_{FZ0} loss function. The highest value (in bold) means that the model is the most favoured one across the four stock indices and for different probability levels.

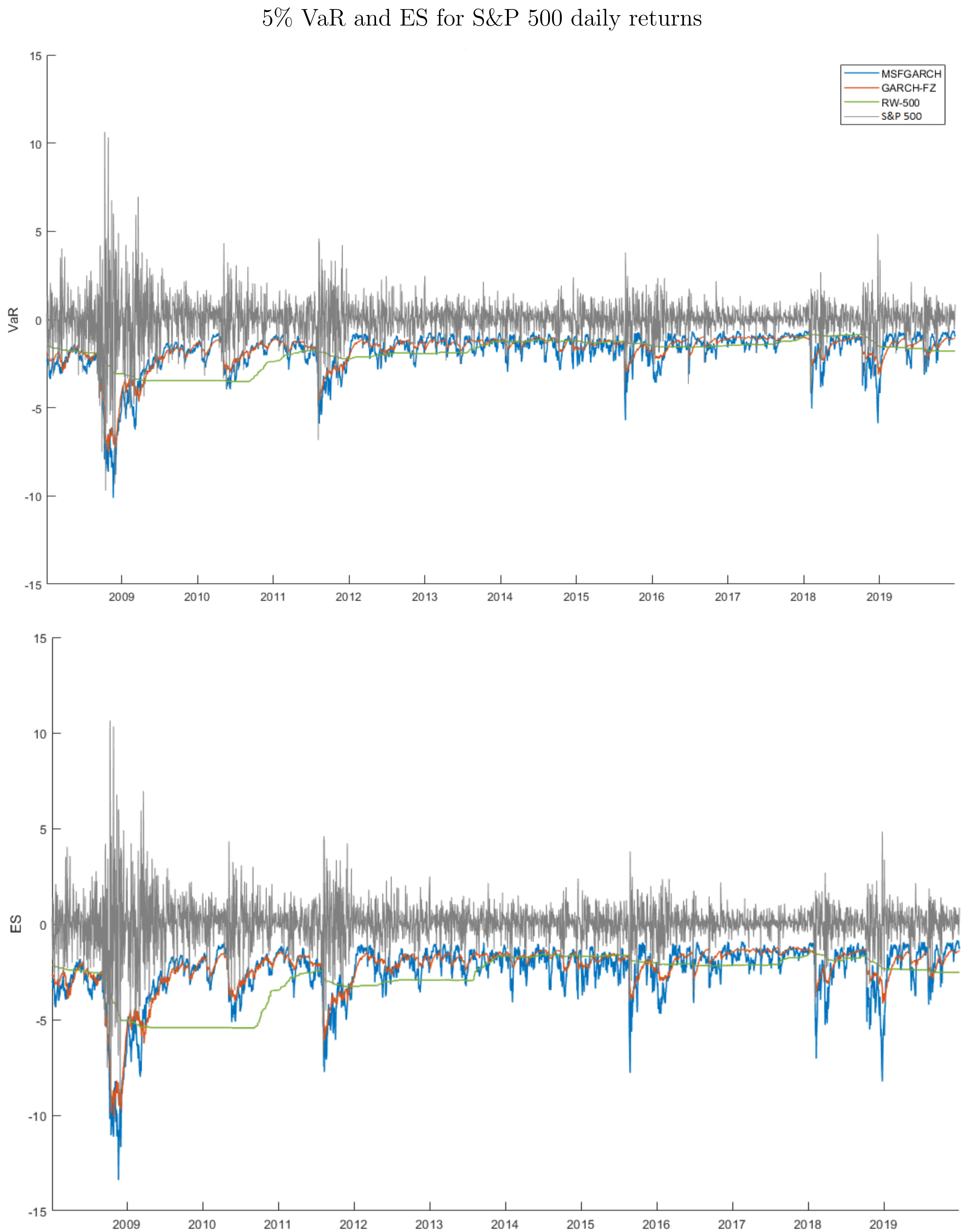

Figure 2.1: Value-at-Risk (VaR) and Expected Shortfall (ES) at the 5% risk level for the S&P 500 index provided by the MSFGARCH model in (2.1), the GARCH-FZ model in (2.10) and the rolling window in (2.5) using 500 observations.

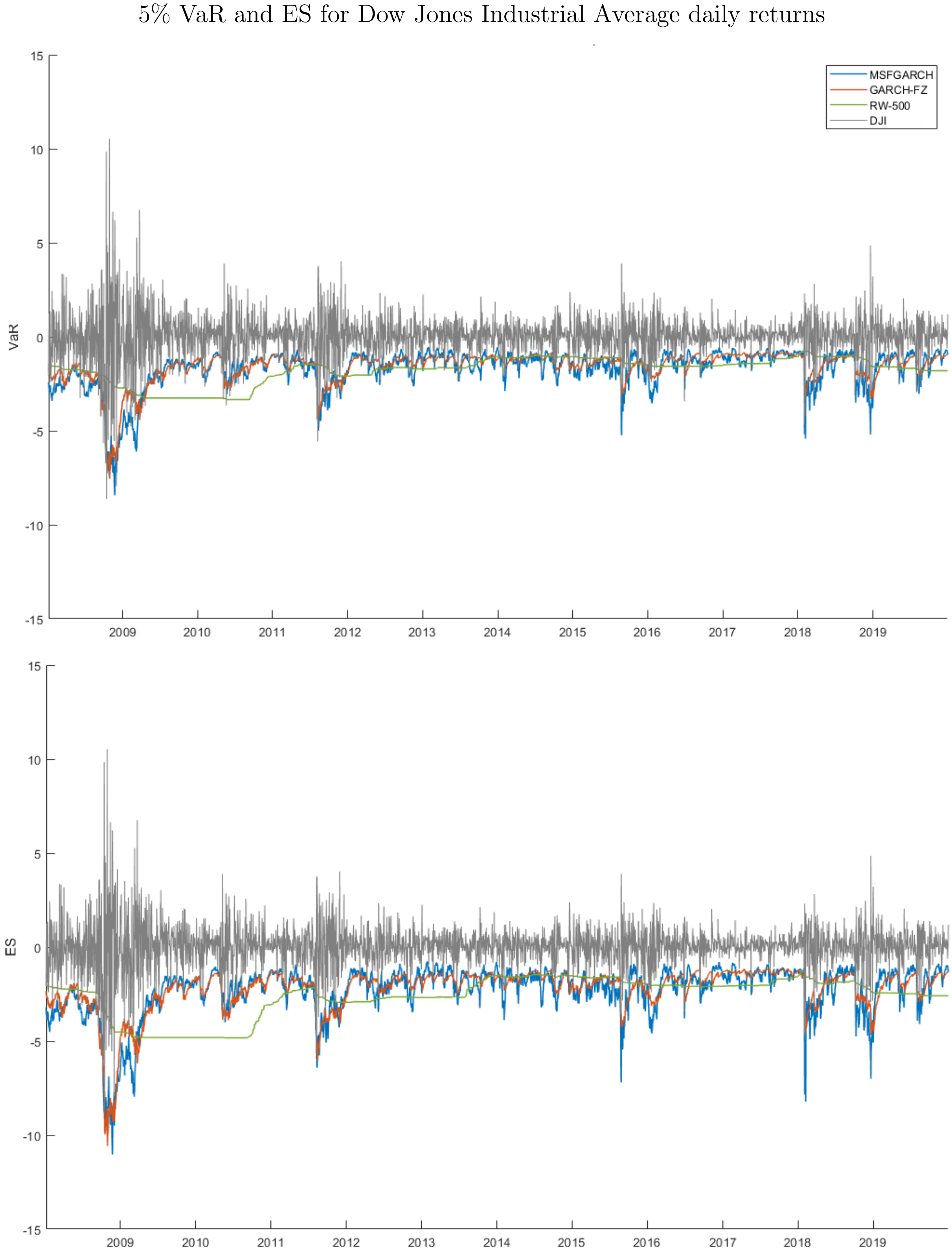

Figure 2.2: Value-at-Risk (VaR) and Expected Shortfall (ES) at the 5% risk level for the S&P 500 index provided by the MSFGARCH model in (2.1), the GARCH-FZ model in (2.10) and the rolling window in (2.5) using 500 observations.

5% VaR and ES for NIKKEI 225 daily returns

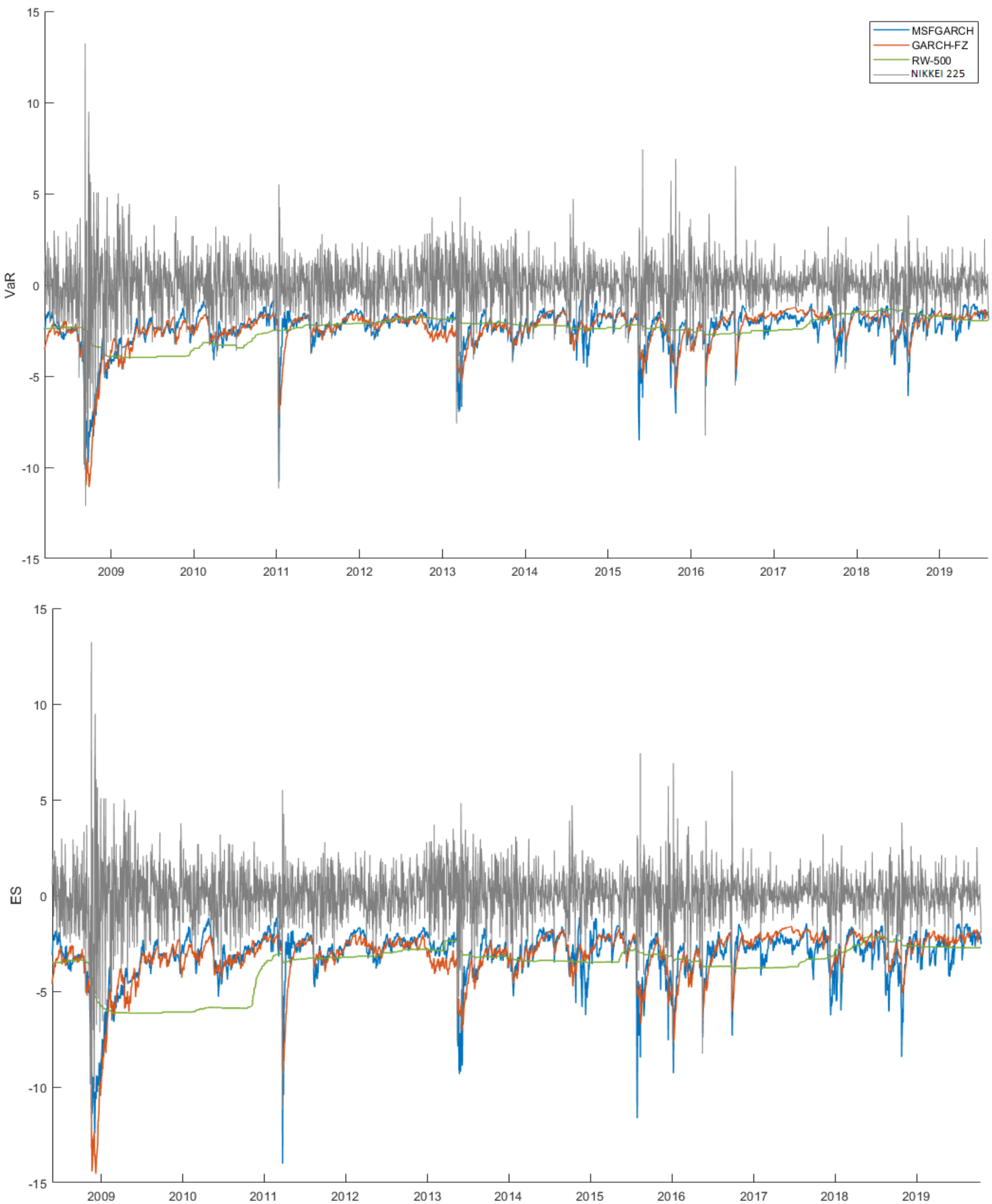

Figure 2.3: Value-at-Risk (VaR) and Expected Shortfall (ES) at the 5% risk level for the S&P 500 index provided by the MSFGARCH model in (2.1), the GARCH-FZ model in (2.10) and the rolling window in (2.5) using 500 observations.

5% VaR and ES for FTSE 100 daily returns

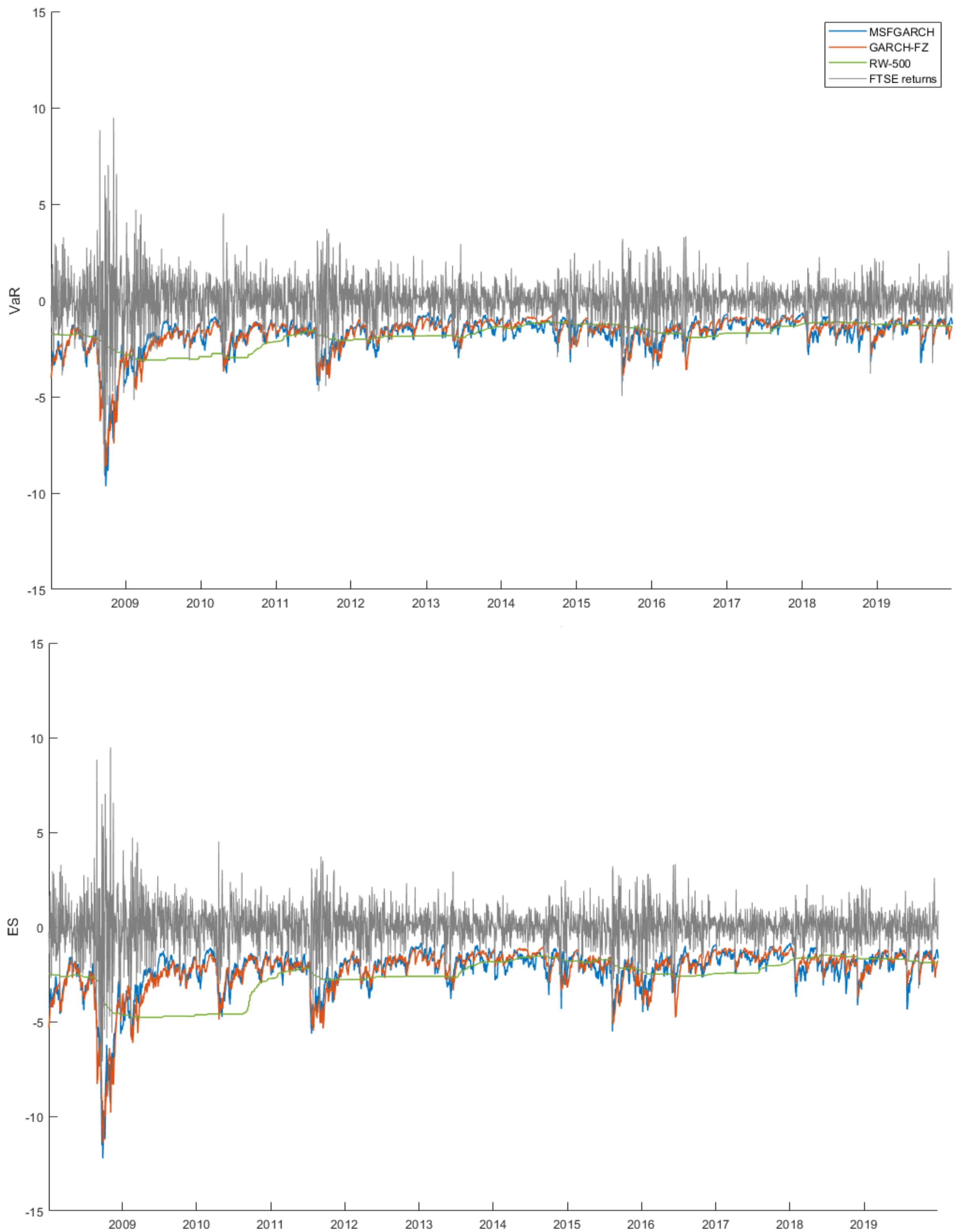

Figure 2.4: Value-at-Risk (VaR) and Expected Shortfall (ES) at the 5% risk level for the S&P 500 index provided by the MSFGARCH model in (2.1), the GARCH-FZ model in (2.10) and the rolling window in (2.5) using 500 observations.

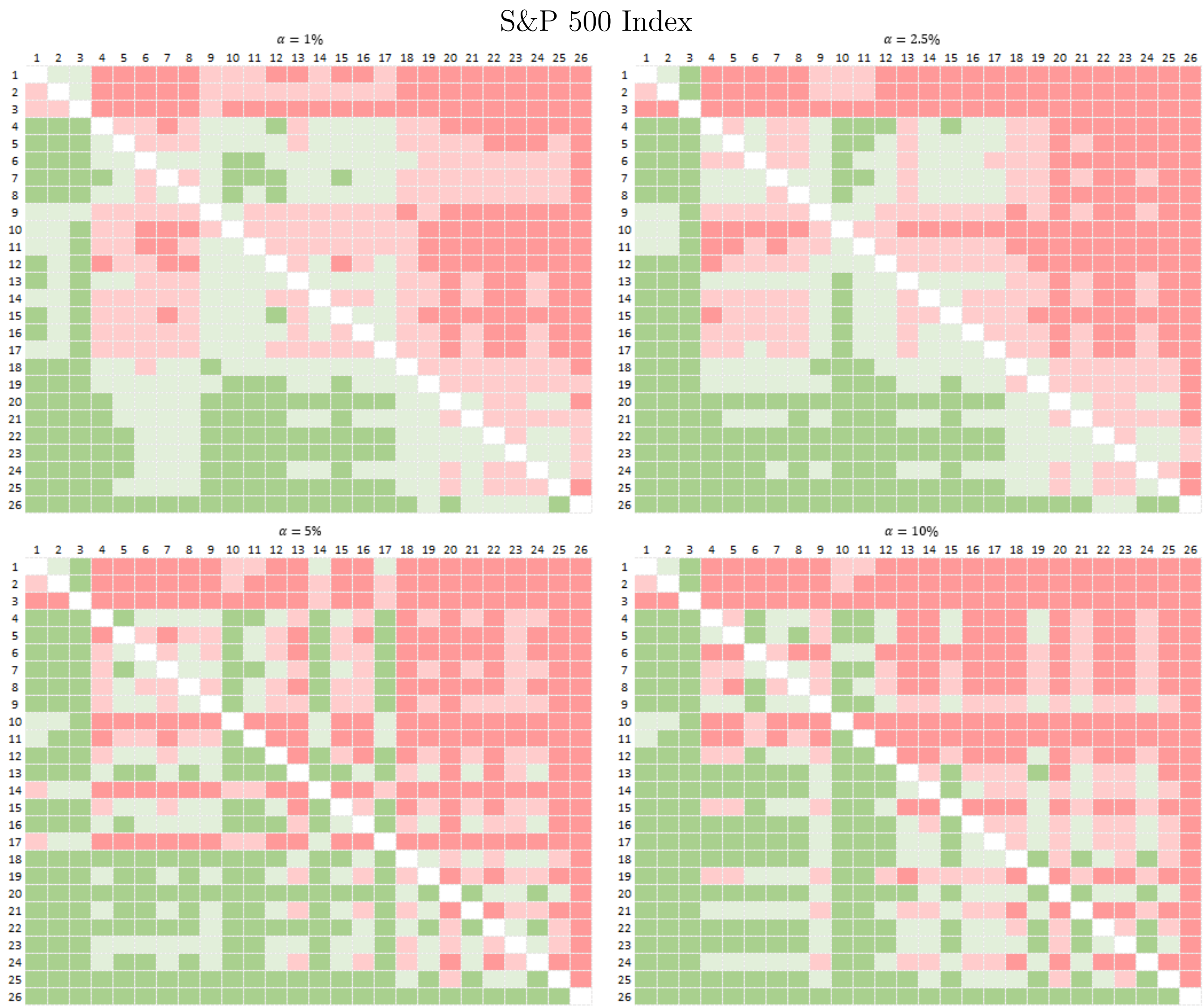

Figure 2.5: Colour map based on the DM test comparing the average losses obtained with the L_{FZ0} loss function over the out-of-sample period for 26 different models for the S&P 500. Dark green blocks indicate that the row model has lower average loss than the column model at the 5% significance level; light-green blocks mean that the row model has lower average loss than the column model, but it is not significantly different from it. Dark red blocks mean that the row model has significantly higher average loss than the column model; light red blocks mean that the row model has greater average loss than the column model, but it is not significantly different from it.

Dow Jones Industrial Average Index

Figure 2.6: Colour map based on the DM test comparing the average losses obtained with the L_{FZ0} loss function over the out-of-sample period for 26 different models for the Dow Jones Industrial Average index. Dark green blocks indicate that the row model has lower average loss than the column model at the 5% significance level; light-green blocks mean that the row model has lower average loss than the column model, but it is not significantly different from it. Dark red blocks mean that the row model has significantly higher average loss than the column model; light red blocks mean that the row model has greater average loss than the column model, but it is not significantly different from it.

NIKKEI 225 Index

Figure 2.7: Colour map based on the DM test comparing the average losses obtained with the L_{FZ0} loss function over the out-of-sample period for 26 different models for the NIKKEI 225 Index. Dark green blocks indicate that the row model has lower average loss than the column model at the 5% significance level; light-green blocks mean that the row model has lower average loss than the column model, but it is not significantly different from it. Dark red blocks mean that the row model has significantly higher average loss than the column model; light red blocks mean that the row model has greater average loss than the column model, but it is not significantly different from it.

FTSE 100 Index

Figure 2.8: Colour map based on the DM test comparing the average losses obtained with the L_{FZ0} loss function over the out-of-sample period for 26 different models for the FTSE 100 Index. Dark green blocks indicate that the row model has lower average loss than the column model at the 5% significance level; light-green blocks mean that the row model has lower average loss than the column model, but it is not significantly different from it. Dark red blocks mean that the row model has significantly higher average loss than the column model; light red blocks mean that the row model has greater average loss than the column model, but it is not significantly different from it.

Chapter 3

Uncertainty and volatility: A Markov-switching GARCH-MIDAS approach

Abstract

In this paper, we examine the relationship between S&P 500 returns volatility and macroeconomic uncertainty. We extend the GARCH-MIDAS model proposed by Engle et al. (2013) to account for regime changes. To the best of the author's knowledge, our specification encompasses all GARCH-MIDAS models proposed in the literature. We show that our model provides more accurate volatility forecasts than the nested GARCH-MIDAS models at forecast horizons of 1 day, 2 weeks, 1 month, 2 months and 3 months. Furthermore, we find that models that incorporate the daily VIX index as an explanatory variable achieve more precise 1 day-ahead forecasts than those that consider the monthly uncertainty measure of Ludvigson et al. (2020). On the other hand, Ludvigson et al.'s (2020) index outperforms for longer forecast horizons. Consequently, our findings suggest that while high-frequency uncertainty indices such as the VIX variable are suitable for short-horizon forecasts, low-frequency uncertainty proxies provide better forecasts at longer horizons ranging from 2 weeks to 3 months.

3.1 Introduction

The field of financial economics contains a vast literature on modelling and forecasting asset price volatility. Risk management, derivative pricing and hedging, portfolio selection, and many other financial activities depend on volatility forecasts. The generalized autoregressive conditional heteroskedasticity (GARCH) class of models has been particularly common in volatility modelling since its introduction by Bollerslev (1986). GARCH models, however, can only incorporate variables sampled at identical frequencies. This is a major drawback since, although researchers and market participants are mainly concerned with daily volatility, most macroeconomic variables containing useful market information are available at a lower frequency (such as monthly frequency).

Fortunately, Ghysels et al. (2004), Ghysels et al. (2005) and Ghysels et al. (2007) propose MIxed Data Sampling (MIDAS) regressions that directly accommodate variables sampled at different frequencies. More recently, Engle et al. (2013) combines GARCH with mixed data frequency introducing the GARCH-MIDAS model. This model has proven to be useful in explaining and predicting variations in volatility by economic variables (see, e.g., Conrad and Loch, 2015; Amado et al., 2019; Conrad and Kleen, 2020; Wang et al., 2020). In GARCH-MIDAS, the conditional variance is decomposed into a short-term daily GARCH component and a long-term component that is driven by macroeconomic explanatory variables observed at a monthly, quarterly or even lower frequency.

In this paper, we extend the GARCH-MIDAS model to account for regime changes. Our model has a specification rich enough to nest all GARCH-MIDAS models already proposed in the literature[1]. Failure to account for regime changes leads to high-volatility persistence in GARCH models (Lamoureux and Lastrapes, 1990). Furthermore, incorporating structural shifts into GARCH have been shown to improve volatility forecasts (Gray, 1996; Klaassen, 2002; Haas et al., 2004; Pan et al., 2017; Ardia et al., 2018). Besides regime changes and mixed-frequency data, our proposed model has the advantage that it accommodates the presence of a "leverage effect", that is the asymmetric response of volatility to positive or negative return innovations. There is strong evidence in the literature to support the asymmetric impact of innovations on volatility. However, to the best of the authors' knowledge, no asymmetric Markov-switching GARCH-MIDAS model has been applied so far in the literature. In this paper, we fill this gap by allowing for different types of asymmetry and functional forms for the conditional volatility.

We apply our model and various nested models to daily returns of the S&P 500 index from January 1990 to November 2020. Our sample period includes not only the Great Recession of 2008 but also the COVID-19 pandemic episode. We aim to provide a better understanding

[1] We are not aware of papers that have proposed a GARCH-MIDAS model that is not included in our specification.

of how the stock market reacts during periods of heightened uncertainty in the economy. As explanatory variables, we use two measures of uncertainty observed at a monthly frequency: the Financial Uncertainty (FU) index of Ludvigson et al. (2020) and the Macroeconomic Uncertainty (MU) index of Jurado et al. (2015). The Financial Uncertainty index contains 147 financial time series including dividend-price ratio and earnings-price ratio, growth rates of aggregate dividends and prices, default and term spreads, yields on corporate bonds of different rating grades, yields on Treasuries and yield spreads, and a broad cross-section of industry, size, book-market, and momentum portfolio equity returns. The Macroeconomic Uncertainty index is calculated using 132 series representing broad categories of macroeconomic time series: real output and income, employment and hours, real retail, manufacturing and trade sales, consumer spending, housing starts, inventories and inventory sales ratios, orders and unfilled orders, compensation and labor costs, capacity utilization measures, price indexes, bond and stock market indexes, and foreign exchange measures. Following the COVID-19 pandemic, the authors report four estimates of uncertainty: total FU, economic FU, total MU and economic MU. "Total" uncertainty includes an estimate of the effect of COVID-19 on uncertainty. The authors include COVID-related information such as the number of increased hospitalization and positive COVID cases[2]. Since we are interested in studying the effect of COVID in the stock market, we mainly focus on "total" uncertainty throughout the paper[3]. We also consider the daily Chicago Board Options Exchange Volatility Index (VIX) as a measure of daily fluctuations in the market. We estimate Markov-switching GARCH-MIDAS models with one, two or three explanatory variables.

Our results yield the following in-sample and out-of-sample conclusions. First, our model provides a significantly better in-sample fit than the competing models. Furthermore, the results suggest that the combination FU-VIX gives a superior characterisation of the data than the combination MU-VIX or models with one explanatory variable. In fact, the MU variable is not found to provide useful information when estimated along with the VIX.

Regarding forecasts of volatility, we are interested in forecast horizons of 1 day, 2 weeks, 1 month, 2 months and 3 months. We show that our model provides more accurate forecasts than

[2] See `https://www.sydneyludvigson.com/` for a description of the series.

[3] As robustness check to our results, we also consider a sample period that excludes the COVID-19 pandemic.

the remaining GARCH-MIDAS models at essentially all horizons. We also find that models with the FU variable outperform for longer forecast horizons of 2 weeks, 1 month, 2 months and 3 months. On the other hand, models that consider the VIX variable achieve more precise 1 day-ahead forecasts than those that incorporate only one of the monthly macroeconomic variables.

This paper is organized as follows. In Section 3.2, we introduce our Markov-switching GARCH-MIDAS model and the model nested within it. Section 3.3 presents the data and the estimation results. Volatility forecasting is performed in Section 3.4. Section 3.5 reports robustness checks. Finally, Section 3.6 concludes.

3.2 Methodology

Let $r_{i,t}$ denote the log-return of a financial asset on day i of month t. We assume the returns follow a GARCH-MIDAS process:

$$\begin{aligned} r_{i,t} &= \mu + \varepsilon_{i,t} \\ \varepsilon_{i,t} &= \sqrt{\tau_t}\, g_{i,t}\, z_{i,t} \end{aligned} \tag{3.1}$$

where τ_t is the so-called long-term volatility component as it evolves according to low-frequency explanatory variables; $g_{i,t}$ is the short-term volatility component and follows a GARCH process; $z_{i,t}$ is an identically and independently distributed standard normal random variable.

Let N_t denote the number of days in month t and let $\mathcal{I}_{i,t}$ denote the information set available up to day $i \in \{1, \ldots, N_t\}$ of month t. We have that

$$\sigma^2_{i,t} = \mathrm{var}\left(\varepsilon_{i,t} \mid \mathcal{I}_{i-1,t}\right) = \tau_t\, g^2_{i,t} \tag{3.2}$$

is the conditional variance of returns.

Following Hentschel's (1995) nesting of single-regime GARCH models, we model the Box-Cox (1964) transformation to the short-term volatility component, $\frac{g^{\lambda}_{i,t}-1}{\lambda}$, as follows:

$$\frac{g^{\lambda}_{i,t}-1}{\lambda} = \omega + \alpha\left(|z_{i-1,t}| - \gamma\, z_{i-1,t}\right)^{\hat{\lambda}} g^{\,\lambda}_{i-1,t} - \psi\,\frac{\varepsilon_{i-1,t}}{\sqrt{\tau_t}} + \beta\,\frac{g^{\lambda}_{i-1,t}-1}{\lambda}, \quad \lambda \geq 0,\ \hat{\lambda} > 0 \tag{3.3}$$

Appendix 3.7.4 shows that, after reparametrization, an equivalent specification of (3.3) has the form:

$$\begin{aligned} \ln g_{i,t}^2 &= \omega + \alpha \left(|z_{i-1,t}| - \mathbb{E}|z_{i-1,t}| \right) - \gamma \, z_{i-1,t} - \psi \, \frac{\varepsilon_{i-1,t}}{\sqrt{\tau_t}} + \beta \ln g_{i-1,t}^{\,2}, && \text{if} \;\; \lambda = 0, \; \hat{\lambda} = 1, \\ g_{i,t}^{\lambda} &= \omega + \alpha \left(|z_{i-1,t}| - \gamma \, z_{i-1,t} \right)^{\hat{\lambda}} g_{i-1,t}^{\,\lambda} - \psi \, \frac{\varepsilon_{i-1,t}}{\sqrt{\tau_t}} + \beta \, g_{i-1,t}^{\,\lambda}, && \text{if} \;\; \lambda > 0, \; \hat{\lambda} > 0 \end{aligned} \tag{3.4}$$

Taking expectations on (3.4), we have that:

$$\begin{aligned} \mathbb{E}\left[\ln g_{i,t}^2 \right] &= \omega / (1 - \beta), && \text{if} \;\; \lambda = 0, \; \hat{\lambda} = 1, \\ \mathbb{E}\left[g_{i,t}^{\lambda} \right] &= \omega / \left(1 - \alpha \, \mathbb{E} \left(|z_{i,t}| - \gamma \, z_{i,t} \right)^{\hat{\lambda}} - \beta \right), && \text{if} \;\; \lambda > 0, \; \hat{\lambda} > 0 \end{aligned} \tag{3.5}$$

The long-term volatility component τ_t is given by

$$\tau_t \;=\; \exp \left\{ m + \theta^X \sum_{k=1}^{K} \varphi_k(w_1^X, w_2^X) \, X_{t-k} \right\} \tag{3.6}$$

where X is a financial or macroeconomic variable that can be observed at a frequency lower than daily (e.g., monthly or weekly). The use of the exponential function in (3.6) ensures that τ_t is nonnegative and the weighting scheme follows the Beta lag structure:

$$\varphi_k(w_1^X, w_2^X) \;=\; \frac{\left[k/(1+K)^{w_1^X - 1} \right] \left[1 - k/(1+K)^{w_2^X - 1} \right]}{\sum_{j=1}^{K} \left[j/(1+K)^{w_1^X - 1} \right] \left[1 - j/(1+K)^{w_2^X - 1} \right]} \tag{3.7}$$

The specification in (3.7) is introduced by Ghysels et al. (2004) and explored further in Ghysels et al. (2007). It is the most common weighting scheme in the context of GARCH-MIDAS regressions (see, e.g., Engle and Ng, 1993; Conrad and Kleen, 2020). Its popularity is due to its flexibility to accommodate various lag structures. If $w_1^X = w_2^X = 1$, we obtain equal weights $\varphi_k(w_1^X, w_2^X) = \frac{1}{K}$. The restriction $w_1^X = 1$, $w_2^X > 1$ guarantees a decaying pattern, i.e., the maximum weight is at the first lag. As w_2^X increases, we obtain a faster decaying pattern. By construction, all the weights $\varphi_k(w_1^X, w_2^X)$ are non-negative and sum to one.

As noted by Pan et al. (2017), the existence of a constant term in both short- and long-term components would lead to identification problems. Therefore, in the following section we substitute the parameter ω in (3.4) by the term $1 - \alpha \, \mathbb{E} \left(|z_{i,t}| - \gamma \, z_{i,t} \right)^{\hat{\lambda}} - \beta$ if $\lambda > 0$, and by

0 when $\lambda = 0$[4]. It follows from (3.5) that, after imposing these restrictions, the unconditional variance of the short-term component becomes 1 when $\lambda = 2$.

The specification of the short-term volatility component in (3.4) is intended to be a nesting framework in which several GARCH specifications are included. Indeed, by appropriately choosing the parameters λ, $\hat{\lambda}$, γ and ψ in (3.4), our model nests a wide-range of symmetric and asymmetric GARCH-MIDAS models as shown in Table 3.1. The parameters γ and ψ introduce asymmetry in the conditional volatility of returns, thus accounting for the well-known "leverage effect". Therefore, the asymmetric models are those that do not require that both the parameters γ and ψ be equal to 0. The specification of our model permits tests of different types of asymmetry and functional forms for the conditional volatility.

[Table 3.1 here]

Several studies have shown that incorporating structural shifts into GARCH models improves volatility forecasting (see e.g., Lamoureux and Lastrapes, 1990; Gray, 1996; Klaassen, 2002; Haas et al., 2004; Pan et al., 2017). We introduce regime changes in our model in the following section.

3.2.1 Markov-switching GARCH-MIDAS models

In this section, we present two types of Markov-switching GARCH-MIDAS models. In Section 3.2.2, we consider switches in the long-term volatility component. In Section 3.2.3, we modify the short-term volatility component in (3.4) to account for regime changes.

3.2.2 Regime-switching in the long-term volatility component

Let $s_{i,t} \in \{1, 2\}$ be a first-order Markov chain representing the (unobserved) regime. In our first specification, we allow the parameters m and θ^X in the long-term component (3.6) to switch between regimes as follows:

$$\tau_t(s_{i,t}) = \exp\left\{ m_{s_{i,t}} + \theta^X_{s_{i,t}} \sum_{k=1}^{K} \varphi_k(w_1^X, w_2^X) X_{t-k} \right\} \tag{3.8}$$

[4] An alternative would be to eliminate the constant term in (3.6). That is, imposing $m = 0$. We consider this specification in Section 3.2.3.

where the MIDAS weighting function, $\varphi_k(w_1^X, w_2^X)$ is given in (3.7).

As previously pointed out, the existence of a constant term in both the long- and short-term volatility components would lead to identification problems. Therefore, we substitute the parameter ω in (3.4) by the term $1 - \alpha\,\mathbb{E}\left(|z_{i,t}| - \gamma\, z_{i,t}\right)^{\hat{\lambda}} - \beta$ if $\lambda > 0$, and by 0 when $\lambda = 0$:

$$\ln g^2_{i,t}\left(s_{i,t}\right) \;=\; \alpha\left(|z_{i-1,t}| - \mathbb{E}|z_{i-1,t}|\right) - \gamma\, z_{i-1,t} - \psi\,\frac{\varepsilon_{i-1,t}}{\sqrt{\overline{\tau_t}}} + \beta \ln \overline{g^2_{i-1,t}}, \tag{3.9}$$

$$g^{\lambda}_{i,t}\left(s_{i,t}\right) \;=\; 1 - \alpha\,\mathbb{E}\left(|z_{i-1,t}| - \gamma\, z_{i,t}\right)^{\hat{\lambda}} - \beta + \alpha\left(|z_{i-1,t}| - \gamma z_{i-1,t}\right)^{\hat{\lambda}}\,\overline{g^{\lambda}_{i-1,t}} - \psi\,\frac{\varepsilon_{i-1,t}}{\sqrt{\overline{\tau_t}}} + \beta\,\overline{g^{\lambda}_{i-1,t}}$$

It is common practice in the context of GARCH-MIDAS models to consider this substitution (see, e.g., Engle et al., 2013; Conrad and Loch, 2015; Conrad and Kleen, 2020). However, to the best of the authors' knowledge, this is the first paper in the Markov-switching GARCH framework that imposes this restriction. For instance, Pan et al. (2017) and Ma et al. (2020) consider a constant term in the short-term volatility component.

In order to avoid the path-dependence problem, we follow Klaassen's (2002)[5] procedure and define

$$\ln \overline{g^2_{i-1,t}} \;=\; \mathbb{E}\left(\ln g^2_{i-1,t} \mid s_{i,t},\, \mathcal{I}_{i-1,t}\right) \;=\; \sum_{j=1}^{2} \mathbb{P}\left(s_{i-1,t} = j \mid s_{i,t},\, \mathcal{I}_{i-1,t}\right) \ln g^2_{i-1,t}(s_{i-1,t} = j) \tag{3.10}$$

$$\overline{g^{\lambda}_{i-1,t}} \;=\; \mathbb{E}\left(g^{\lambda}_{i-1,t} \mid s_{i,t},\, \mathcal{I}_{i-1,t}\right) \;=\; \sum_{j=1}^{2} \mathbb{P}\left(s_{i-1,t} = j \mid s_{i,t},\, \mathcal{I}_{i-1,t}\right) g^{\lambda}_{i-1,t}(s_{i-1,t} = j) \tag{3.11}$$

$$\overline{\tau_t} \;=\; \mathbb{E}\left(\tau_t \mid s_{i,t},\, \mathcal{I}_{i-1,t}\right) \;=\; \sum_{j=1}^{2} \mathbb{P}\left(s_{i-1,t} = j \mid s_{i,t},\, \mathcal{I}_{i-1,t}\right) \tau_t(s_{i-1,t} = j) \tag{3.12}$$

The state variable $s_{i,t}$ is assumed to follow an homogeneous first-order Markov process with constant transition probabilities $p_{jj} = \mathbb{P}\left(s_{i,t} = j \mid s_{i-1,t} = j\right)$, $j = 1, 2$.

3.2.3 Regime-switching in the short-term volatility component

We now consider an alternative Markov-switching model in which the short-term volatility component switches while the long-term component remains constant across regimes. In this specification, the long-term component follows equations (3.6)-(3.7) with $m = 0$ while we

[5] Klaassen (2002) refines the collapsing procedure introduced by Gray (1996). Other collapsing procedures are proposed by Dueker (1997) and Haas et al. (2004).

allow the constant term in (3.4) to switch between regimes:

$$
\begin{aligned}
\ln g_{i,t}^2(s_{i,t}) &= \omega_{s_{i,t}} + \alpha\left(|z_{i-1,t}| - \mathbb{E}|z_{i-1,t}|\right) - \gamma\, z_{i-1,t} - \psi\,\frac{\varepsilon_{i-1,t}}{\sqrt{\tau_t}} + \beta\,\ln\overline{g_{i-1,t}^2},\\
g_{i,t}^{\lambda}(s_{i,t}) &= \omega_{s_{i,t}} + \alpha\left(|z_{i-1,t}| - \gamma\, z_{i-1,t}\right)^{\hat\lambda}\,\overline{g_{i-1,t}^{\lambda}} - \psi\,\frac{\varepsilon_{i-1,t}}{\sqrt{\tau_t}} + \beta\,\overline{g_{i-1,t}^{\lambda}}
\end{aligned}
\tag{3.13}
$$

where $\ln\overline{g_{i-1,t}^2}$ follows (3.10) and $\overline{g_{i-1,t}^{\lambda}}$ follows (3.11).

We could allow for switches not only in the constant term, ω, but also in the rest of the parameters. However, to make our model tractable and parsimonious we consider the parameters α, β, γ and ψ to be constant across regimes. Furthermore, as noted by Pan et al.'s (2017), it is unlikely to find significant differences between regimes in the α, β, γ and ψ parameters.

Pan et al.'s (2017) regime-switching GARCH-MIDAS is nested in our model when $\lambda = \hat\lambda = 2$ and $\gamma = \psi = 0$ (Standard GARCH-MIDAS model in Table 3.1)[6]. Our model also nests the single-regime GJRGARCH-MIDAS model used in Conrad and Kleen's (2020) (see Appendix 3.7.4).

3.3 Data and Estimation Results

3.3.1 Data

In this section, we apply the models introduced in Section 3.2 to daily returns of the S&P 500 calculated as $r_{i,t} = 100\,(\ln p_{i,t} - \ln p_{i-1,t})$. The period under consideration dates from 1^{st} January 1990 to 30^{st} November 2020. As explanatory variable we use the Financial Uncertainty (FU) measure of Ludvigson et al. (2020) and the Macroeconomic Uncertainty (MU) measure of Jurado et al. (2015)[7]. These variables are observed at a monthly frequency. We also consider the Chicago Board Options Exchange Volatility Index (VIX) observed daily[8]. The CBOE Volatility Index (VIX) is a measure of expected price fluctuations in the S&P 500 Index options over the next 30 days. It is available from the year 1990 onwards[9]. Table 3.2

[6] See equation (3) in Pan et al. (2017).
[7] https://www.sydneyludvigson.com/macro-and-financial-uncertainty-indexes
[8] https://www.cboe.com/indices/
[9] We convert the VIX variable to daily levels by dividing it by $\sqrt{252}$.

presents summary statistics of the data and Figure 3.1 plots the macroeconomic variables.

[Table 3.2 and Figure 3.1 here]

As explained in Conrad and Kleen (2020), assuming a large enough lag length K in the long-term volatility component ensures that the data identifies the optimal weighting scheme. We use $K = 36$ for the monthly variables. That is, three years of monthly observations. For the daily VIX variable, we choose 3 lags as is shown by Conrad and Kleen (2020) to be adequate. Furthermore, we find that the restricted Beta weighting scheme, $w_1^X = 1$ in (3.7), is the best choice for our explanatory variables. That is, the optimal weights follow a decaying pattern from the beginning[10,11].

3.3.2 Estimation results with one explanatory variable

In this section, we estimate the models introduced in Section 3.2 for the two monthly macroeconomic variables FU and MU and for the daily volatility index VIX. We estimate two-regime models and their single-regime counterparts.

Switching in the long-term volatility component

Table 3.3, Table 3.4 and Table 3.5 compare goodness of fit statistics of the models introduced in Section 3.2.2 with FU, MU and VIX as explanatory variables, respectively. We refer to our more general model as MSFGARCH-MIDAS when the number of regimes is two and FGARCH-MIDAS when we impose one regime.

[Table 3.3 - 3.5 here]

The second and third columns list, respectively, the number of parameters n and the maximized log-likelihood LL. The Akaike information criterion (AIC) and Bayesian information

[10] We find from our estimation results, reported in Section 3.3.2 to Section 3.3.4, that the models estimate a high value of the weighting parameter w^X, indicating that most of the weight is placed at the first lag. Therefore, our chosen values of K are found to be too large but guarantee that the models identify the optimal weight.

[11] We also estimate the models with the "exponential Almon lag" specification used in Ghysels et al. (2007) and Ghysels et al. (2005). The "exponential Almon lag" yield essentially the same τ dynamics as the Beta weighting scheme. Therefore, we only report the results for the Beta weighting scheme.

criterion (BIC) calculated as AIC $= -2\,(LL - n)$ and BIC $= -2\,LL + n\log(T)$, $T = 7764$ being the sample size, are displayed in the fourth and sixth columns. Irrespective of the macroeconomic variable considered, the AIC and BIC rank the asymmetric regime-switching MSFGARCH-MIDAS, MSEGARCH-MIDAS and MSQGARCH-MIDAS models as the three best specifications. On the other hand, the single-regime symmetric models AVGARCH-MIDAS, GARCH-MIDAS and NLGARCH-MIDAS are categorised as the worst performing models by both criteria.

We further perform likelihood ratio tests for testing the estimation performance of the asymmetric models against their symmetric counterparts. The statistics from these tests are reported in the antepenultimate and penultimate columns of the tables. The null hypothesis of symmetry, corresponding to the model in the first column of the same row, is tested against the alternative that asymmetry is governed by the parameter γ or the parameter ψ. The LRT statistics for these tests are asymptotically distributed as a $\chi^2(1)$ random variable. The statistics far exceed the value of $\chi^2(1)$ at 1%, 5% and 10% levels of confidence, thus rejecting the null of symmetry.

The last column in the tables tests the different nested models in Table 3.1 against the alternative of the most general MSFGARCH-MIDAS model. The LRT statistics for these tests are asymptotically distributed as a χ^2 random variable with a degree of freedoms equal to the difference in the number of parameters. The only regime-switching model that is not rejected at a 1% significance level in favour of the MSFGARCH-MIDAS is the MSQGARCH-MIDAS for the VIX variable. Thus, our more general model provides a superior characterisation of the data when considering the monthly macroeconomic variables and it is only beaten by the more parsimonious MSQGARCH-MIDAS when the daily VIX is the explanatory variable.

A difficulty arises when testing a Markov-switching model versus a single-regime model since the state parameter is unidentified under the null hypothesis (see Davies, 1977). This problem is typically approached by treating the traditional likelihood ratio test (LRT) statistic as a function of the unidentified parameter and obtaining an upper bound for the statistic across all possible values of the parameter (see Davies, 1977; Garcia et al., 1996; *inter alia*). Garcia et al. (1996) follows Davies's (1977) approach and derives an upper bound for the significance level of the LRT when testing a Markov-switching autoregressive model against a single-regime model[12]. Given the LR statistic $LR = 2\left(LR_1 - LR_0\right)$ where LR_0 is the log-likelihood under the null and LR_1 the log-likelihood under the alternative, the upper bound proposed by Garcia et al. (1996) is given as $\mathbb{P}\left(\chi_d^2 > LR\right) + 2\left(\frac{LR}{2}\right)^{\frac{d}{2}} \exp\left(-\frac{LR}{2}\right)\left[\Gamma\left(\frac{d}{2}\right)\right]^{-1}$, where d is the difference in the number of parameters. For the values of the LR statistics corresponding to the null of single-regime models in the last columns of Table 3.3 - 3.5, the term $\exp\left(-\frac{LR}{2}\right)$ is essentially 0 so the upper bound for our tests is approximately equal to $\mathbb{P}\left(\chi_d^2 > LR\right)$, the usual marginal level associated with the traditional LR test[13]. We obtain a LRT statistics that far exceeds the marginal level associated with the traditional LR at any significance level. Therefore, if this bound could be used in the context of GARCH-MIDAS models we would strongly reject the null of a single-regime specification. Unfortunately, an extension of Garcia et al.'s (1996) bound to GARCH-MIDAS is not currently available in the literature. Its derivation constitutes a substantial task for future research. In any case, the large values of the LRT statistics support the regime-switching specification.

In summary, the estimation results show that our proposed MSFGARCH-MIDAS model significantly outperforms existing models in explaining the data and that both asymmetry and regime changes are important features of returns. This is a significant finding of our research since, so far in the literature, the presence of asymmetries in Markov-switching GARCH-MIDAS models has been neglected. The publications either account for regime-changes in symmetric GARCH-MIDAS (for instance, Pan et al., 2017 and Ma et al., 2020 use the standard MSGARCH-MIDAS model) or asymmetries in single-regime GARCH-MIDAS (e.g., Conrad and Kleen, 2020 and Wang et al., 2020 use the single-regime GJRGARCH model). In this sense, our paper is the first to combine regime switches and asymmetries in GARCH-MIDAS models and we have shown that there is strong evidence to consider both aspects of the volatility. Failure to account for either of these characteristics of the volatility would lead to misleading inferences and poor out-of-sample volatility forecasts (as shown below in Section 3.4).

Table 3.6 - 3.8 show the estimated parameters from the Markov-switching MIDAS mod-

[12]The bounds holds assuming that the likelihood function has a single peak.

[13]The smallest LRT statistic for testing single-regime against the MSFGARCH-MIDAS model is given in Table 3.3. It takes a value of 172.75 and corresponds to the null of the single-regime FGARCH-MIDAS model. Being $d = 4$ the difference in the number of parameters, we have $\exp\left(-\frac{172.75}{4}\right) = 1.75 \cdot 10^{-19} \approx 0$.

els[14]. Regime 1 corresponds to the low-volatility regime while regime 2 is a high-volatility regime. Unsurprisingly, the parameters governing asymmetry γ and ψ are highly significant. Regarding the parameters in the long-term volatility component (3.8), the coefficient θ^{FU} of the FU variable in Table 3.6 is significant in both regimes. However, this parameter appears to be constant across regimes. We formally test $\theta_1^{\text{FU}} = \theta_2^{\text{FU}}$ through a Wald test. The p-values from the tests are presented in the last row of the table. All the p-values are far greater than 10%. Thus, there is not enough evidence to suggest that the parameter θ^{FU} switches across regimes. If $\theta_1^{\text{FU}} = \theta_2^{\text{FU}}$, then from (3.8) we have that $\frac{\tau_t(s_{i,t}=1)}{\tau_t(s_{i,t}=2)} = \frac{\exp(m_1)}{\exp(m_2)}$ and the unconditional variance in regime one relative to that in regime 2 is given by $\frac{\overline{\sigma}_1^2}{\overline{\sigma}_2^2} = \frac{\exp(m_1)\,\overline{g}^2}{\exp(m_2)\,\overline{g}^2} = \frac{\exp(m_1)}{\exp(m_2)}$. The penultimate row in the table shows the ratio $\overline{\sigma}_1^2/\overline{\sigma}_2^2$. The three best specifications MSFGARCH-MIDAS, MSEGARCH-MIDAS and MSQGARCH-MIDAS, estimate that the unconditional variance in the low-volatility regime is approximately 35% of the unconditional variance in the high-volatility regime. For the remaining models, the ratio is approximately 25%. Similarly, from Table 3.7 the coefficient of the MU variable θ^{MU} is significant within regime but not significantly different across regimes. For most of the models, the ratio $\overline{\sigma}_1^2/\overline{\sigma}_2^2$ does not deviate a lot from the value obtained when FU was the explanatory variable in the previous table. On the other hand, the coefficient θ^{VIX} in Table 3.8 is highly significant in both regimes for the majority of the models and there is enough evidence to suggest that θ^{VIX} switches across regimes for all of the models. Therefore, it is not possible to obtain the ratio of the unconditional variance in regime 1 over the unconditional variance in regime 2 solely from the estimated parameters.

[Table 3.6 - 3.8 here]

Figures 3.2, 3.3 and 3.4 plot the smoothed probabilities of being in the low-volatility regime for the three best specifications MSFGARCH-MIDAS, MSQGARCH-MIDAS and MSEGARCH-MIDAS with FU, MU and VIX as explanatory variable, respectively. Periods of high-volatility include, among others, the dot-com bubble, around the year 2003, the global financial crisis of 2008 and the recent COVID global pandemic in 2020. The main difference between the probabilities in the three figures is that the models with MU as explanatory variable estimate

[14]Table 3.37 - 3.39 contain the parameter estimates for the single-regime models.

a persistent period in the high-volatility state from 1997 to 2001. On the other hand, the models with the FU or VIX variables estimate the period from the end of 1998 to the beginning of 1999 in the low-volatility regime. This behaviour may be explained by the fact that while the MU index dropped 6.8% from August 1998 to the end of July 1999, the FU (VIX) index decreased 22.8% (62%).

[Figure 3.2 - 3.4 here]

Figure 3.5 - 3.7 show the time-varying conditional volatilities obtained from the MSFGARCH-MIDAS, MSQGARCH-MIDAS and MSEGARCH-MIDAS models with one explanatory variable. The conditional volatilities from the models with monthly long-term component show similar patterns. However, models with the FU (MU) index estimate higher (lower) volatility during the 2008 financial crisis than during the COVID-19 pandemic. This is not surprising given that the FU variable reaches its maximum level during the crisis of 2008 while COVID is the highest uncertainty episode for the MU index. Furthermore, the models with the daily VIX in the long-term component estimate far higher volatility during the first quarter of 2020 than the models with monthly variables. This result may be driven by the fact that while the FU (MU) variable increased its value by about 23.9% (48.8%) from January 2020 to March 2020, the VIX index increased its value by approximately 562% during the same period (see Figure 3.1).

[Figure 3.5 - 3.7 here]

On balance, the monthly macroeconomic variables and the daily VIX appear to provide useful information to explain returns. In the following section, we consider switches in the short-term volatility component and compare the results with those obtained when the regime changes occur in the long-term component.

Switching in the short-term volatility component

Table 3.9 - 3.11 give goodness of fit statistics of the models introduced in Section 3.2.3. The ranking of the models based on the AIC and BIC criteria remains almost unchanged compared to those obtained in Section 3.3.2 (see Table 3.3 - 3.5). Tests for asymmetry in the 8^{th} and 9^{th}

columns again reject the null of symmetry in favour of the asymmetric models. Additionally, based on LRT tests in the last column of the table, the general MSFGARCH-MIDAS model is preferred over the remaining models with the exception of the MSQGARCH-MIDAS model when FU or VIX are the explanatory variables in the long-term component.

[Table 3.9 - 3.11 here]

Table 3.12 - 3.14 provide the estimated parameters from the Markov-switching MIDAS models in Section 3.2.3[15]. As before, regime 1 (2) corresponds to the low-(high-) volatility regime. Again, we find the parameters governing asymmetry γ and ψ highly significant. Furthermore, the coefficient associated with the macroeconomic variables θ^{FU}, θ^{MU} and θ^{VIX} are also significant suggesting again the usefulness of the variables to characterise the returns.

[Table 3.12 - 3.14 here]

The unconditional short-term variance in each of the regimes, $\overline{g}_j^2$ $j = 1, 2$, and $\overline{\sigma}_1^2/\overline{\sigma}_2^2$ are given in the last three rows of the tables. For the models with the FU variable (Table 3.12) the unconditional variance in the low-volatility regime is approximately 30% the unconditional variance in the high-volatility regime. When considering the MU variable, the ratio $\overline{g}_1^2/\overline{g}_2^2$ is around 26% from the asymmetric models while around 17% for the symmetric models. Finally, for the models with the daily VIX in the long-term component the ratio of the unconditional variances is about 42%.

Note that the values of the AIC and BIC information criteria from the models that allow switches in the long-term volatility component (Table 3.3 - 3.5) are smaller than those obtained from the models that allow switches in the short-term volatility component (Table 3.9 - 3.11). Consequently, we obtain a better description of the data when allowing the coefficient associated with the macroeconomic variables to switch across regimes[16]. For this reason, in what follows we use the models introduced in Section 3.2.2. That is, those that allow switches in the long-term component[17].

[15] Table 3.40 - 3.42 present the parameter estimates for the single-regime models.

[16] Other papers such as Pan et al. (2017) only consider switches in the short-term volatility component.

[17] We have also estimated models that allow switches in both long- and short-term components simultaneously. In these models, the short-term volatility component follows equation (3.13) while the long-term volatility component follows (3.8) (with $m = 0$ to avoid identification problems). We do not find significant improvements in terms of fit. Furthermore, the increment in the number of parameters makes these models less tractable than those considered in Section 3.2.2 and Section 3.2.3.

From the previous results, the VIX variable seems to be good at capturing the daily movements in the long-term volatility component. Our subsequent analysis is based on the combination of information in the monthly variables with that in the daily volatility index. With this aim, we consider models that include VIX and one of the monthly variables in the long-term component in Section 3.3.3.

3.3.3 Estimation results with two explanatory variables

We now extend the long-term volatility component in (3.8) to include two explanatory variables. In addition to the monthly variables in Section 3.3.2, we include the VIX variable. We choose to include VIX as one of the explanatory variables since it appears to be a relevant variable in explaining returns. We argue that omitting this variable from the estimation would lead to biased and misleading inference[18]. The short-term volatility component follows equations (3.9)-(3.12) while the long-term volatility component is now given by

$$\tau_{i,t}\left(s_{i,t}\right) = \exp\left\{m_{s_{i,t}} + \theta^{X}_{s_{i,t}} \sum_{k=1}^{36} \varphi_k(1, w^{X}) X_{t-k} + \theta^{\text{VIX}}_{s_{i,t}} \sum_{k=1}^{3} \varphi_k(1, w^{\text{VIX}})\ \text{VIX}_{i-k,t}\right\} \tag{3.14}$$

where X denotes either of the FU or MU variables.

Log-likelihood values, AIC and BIC criteria from the different models with two explanatory variables are given in Table 3.15 and Table 3.16. As in the previous section, the MSQGARCH-MIDAS, MSEGARCH-MIDAS and MSFGARCH-MIDAS are found to provide the best fit irrespective of the monthly variable considered. The symmetric single-regime models again give the worst description of the data and tests for symmetry are rejected in favour of the asymmetric models. When comparing the models with the most general model in the last column of the tables, the MSQGARCH-MIDAS and MSEGARCH-MIDAS are the only models not rejected at a 1% significance level. Furthermore, according to the AIC and BIC information criterion presented in the tables, the combination FU-VIX provides a superior characterisation of the data than MU-VIX.

[Table 3.15 and Table 3.16 here]

[18] We confirm this argument in Section 3.3.4 where we include the two monthly variables and the daily variable simultaneously in the long-term component.

The estimated parameters are presented in Table 3.17 and Table 3.18. For most of the models, the VIX and FU coefficients θ^{VIX} and θ^{FU} are found to be significant in at least one regime and, according to the Wald tests in the last two rows of Table 3.17, the parameters θ^{VIX} and θ^{FU} are significantly different across regimes. On the other hand, from Table 3.18 very few models estimate significant θ^{MU} parameters. Consequently, the variable MU does not seem to provide complementary information to that included in the VIX[19].

[Table 3.17 and Table 3.18 here]

3.3.4 Estimation results with three explanatory variables

We further consider models that include three explanatory variables. The models in this section include the three covariates simultaneously: the two monthly indexes FU and MU and the VIX variable. The short-term volatility component follows equations (3.9)-(3.12) while the long-term volatility component is given by

$$\begin{aligned}\tau_{i,t}\left(s_{i,t}\right) = \exp\Bigg\{ & m_{s_{i,t}} + \theta^{\mathrm{FU}}_{s_{i,t}} \sum_{k=1}^{36} \varphi_k(1, w^{\mathrm{FU}})\mathrm{FU}_{t-k} + \theta^{\mathrm{MU}}_{s_{i,t}} \sum_{k=1}^{36} \varphi_k(1, w^{\mathrm{MU}})\mathrm{MU}_{t-k} \\ & + \theta^{\mathrm{VIX}}_{s_{i,t}} \sum_{k=1}^{3} \varphi_k(1, w^{\mathrm{VIX}})\mathrm{VIX}_{i-k,t} \Bigg\}\end{aligned} \tag{3.15}$$

Table 3.19 contains goodness of fit statistics from the single-regime and two-regime models. As before, the MSQGARCH-MIDAS, MSEGARCH-MIDAS and MSFGARCH-MIDAS yield the best fit while the symmetric single-regime models are at the bottom of the AIC and BIC rankings. The LRT tests in the last three columns of the table again make clear the significance of asymmetry and regime-switching in returns.

[Table 3.19 here]

The estimated parameters are presented in Table 3.20. The coefficients θ^{FU} and θ^{VIX} are significant in at least one regime for most of the models. On the other hand, the coefficient

[19]This behaviour is also evident from the smoothed probabilities and time-varying volatilities depicted in Figure 3.9 and Figure 3.11. The probabilities and volatilities obtained from the combinations MU-VIX are very similar to those obtained from the models with just the daily VIX variable in the long-term component in Figure 3.4 and Figure 3.7.

θ^{MU} is not significant in any of the regimes. The last panel in the table reports likelihood ratio tests (LRT) for the null of only one of the monthly variables with VIX in the long-term component. There is not enough evidence to suggest that the unrestricted model provides a better in-sample fit than the combination FU-VIX. This result reaffirms the conclusions in Section 3.3.3 where we found that MU does not add complementary information to that in the VIX. For this reason, we do not consider models that include both MU and VIX simultaneously as explanatory variables in the out-of-sample exercise in the following section.

[Table 3.20 here]

3.4 Forecasting volatility

In this section, we compare the forecasting ability of the models that allow switches in the long-term volatility component introduced in Section 3.2.2 and estimated in Section 3.3. We evaluate the predictive performance of the models that consider either one of the FU, MU and VIX variables and the models with both VIX and FU as explanatory variables. We attempt to answer the following two questions: (1) Do two-regime GARCH-MIDAS models provide more accurate volatility predictions than single-regime GARCH-MIDAS models? (2) Is accounting for asymmetry in volatility important when forecasting volatility?

To find the answer to these questions, we obtain and evaluate cumulative volatility forecasts for horizons up to 3 months. The out-of-sample period comprises January 2007 to November 2020. Our out-of-sample period covers the 2008 financial crisis and the recent COVID pandemic in order to compare the performance of the models under different market conditions. Estimation is performed on a rolling window basis and the model parameters are updated on a monthly frequency[20].

The forecasts are obtained on the last day N_t of month t. We denote the k-step-ahead variance forecast by $\hat{\sigma}^2_{k,t+1|t}$ with $k \leq N_{t+1}$. That is, $\hat{\sigma}_{k,t+1|t}$ is the volatility forecast on day k of month $t+1$ given the information up to the last day of month t, $\mathcal{I}_{N_t,t}$. For simplicity of

[20]We do not update the value of the parameters daily since the computational cost of the daily updating was high (about two days per model). As shown by Ardia and Hoogerheide (2014), the forecast performance of GARCH models does not change significantly when moving from a daily updating frequency to a higher updating frequency.

notation, we write $\mathcal{I}_t$ instead of $\mathcal{I}_{N_t,t}$. The optimal forecast is given by

$$\hat{\sigma}^2_{k,t+1|t} = \mathbb{E}\left(\sigma^2_{k,t+1} \mid \mathcal{I}_t\right) = \mathbb{E}\left(\tau_{k,t+1}\, g^2_{k,t+1} \mid \mathcal{I}_t\right) \tag{3.16}$$

We approximate $\mathbb{E}\left(\tau_{k,t+1}\, g^2_{k,t+1} \mid \mathcal{I}_t\right)$ in (3.16) by[21]

$$\mathbb{E}\left(\tau_{k,t+1}\, g^2_{k,t+1} \mid \mathcal{I}_t\right) \approx \sum_{j=1}^{2} \mathbb{P}\left(s_{k,t+1} = j \mid \mathcal{I}_t\right) \tau_{k,t+1}\left(s_{k,t+1} = j\right) \left[\mathbb{E}\left(g^{\lambda}_{k,t+1} \mid s_{k,t+1} = j, \mathcal{I}_t\right)\right]^{\frac{2}{\lambda}} \tag{3.17}$$

The probabilities of being in each regime on day k of month $t+1$ given $\mathcal{I}_t$ are obtained by:

$$\mathrm{P}_{s_{k,t+1|t}} = \begin{bmatrix} \mathbb{P}(s_{k,t+1} = 1 \mid \mathcal{I}_t) \\ \mathbb{P}(s_{k,t+1} = 2 \mid \mathcal{I}_t) \end{bmatrix} = \begin{bmatrix} p_{11} & 1 - p_{22} \\ 1 - p_{11} & p_{22} \end{bmatrix}^{k-1} \begin{bmatrix} \mathbb{P}(s_{1,t+1} = 1 \mid \mathcal{I}_t) \\ \mathbb{P}(s_{1,t+1} = 2 \mid \mathcal{I}_t) \end{bmatrix} \tag{3.18}$$

The expectation of the k-step-ahead short-term component in (3.17) is a function of the one-step-ahead short-term component, $g^{\lambda}_{1,t+1|t}$. Let $\triangle = \alpha\, \mathbb{E}\left(|z_{i,t}| - \gamma\, z_{i,t}\right)^{\hat{\lambda}} + \beta$ and consider the following matrices

$$\mathrm{T}_{k,t+1} = \begin{bmatrix} \tau_{k,t+1}\left(s_{k,t+1} = 1\right) \\ \tau_{k,t+1}\left(s_{k,t+1} = 2\right) \end{bmatrix}, \quad \mathrm{H}^{\lambda}_{k,t+1|t} = \begin{bmatrix} \mathbb{E}\left(g^{\lambda}_{k,t+1} \mid s_{k,t+1} = 1,\, \mathcal{I}_t\right) \\ \mathbb{E}\left(g^{\lambda}_{k,t+1} \mid s_{k,t+1} = 2,\, \mathcal{I}_t\right) \end{bmatrix} \tag{3.19}$$

and

$$\Psi = \triangle \begin{bmatrix} \mathbb{P}\left(s_{k-1,t+1} = 1 \mid s_{k,t+1} = 1\right) & \mathbb{P}\left(s_{k-1,t+1} = 2 \mid s_{k,t+1} = 1\right) \\ \mathbb{P}\left(s_{k-1,t+1} = 1 \mid s_{k,t+1} = 2\right) & \mathbb{P}\left(s_{k-1,t+1} = 2 \mid s_{k,t+1} = 2\right) \end{bmatrix}. \tag{3.20}$$

Appendix 3.7.4 shows that $\mathrm{H}^{\lambda}_{k,t+1|t}$ can be obtained in a recursive way as follows

$$\mathrm{H}^{\lambda}_{k,t+1|t} = (1 - \triangle)\, \mathbf{1}_{2\times 1} + \Psi\, \mathrm{H}^{\lambda}_{k-1,t+1|t} = \Psi^{k-1}\, \mathrm{H}^{\lambda}_{1,t+1|t} + \sum_{j=0}^{k-2} \Psi^{j}\, (1 - \triangle)\, \mathbf{1}_{2\times 1} \tag{3.21}$$

[21] For $\lambda = 2$ we have an equality.

where $\mathbf{1}_{2\times 1}$ is a 2×1 vector of ones.

Therefore, we calculate the optimal forecasts as:

$$\hat{\sigma}^2_{k,t+1|t} = \mathrm{P}^{\top}_{s_{k,t+1|t}} \left[\mathrm{T}_{k,t+1} \circ \left(\mathrm{H}^{\lambda}_{k,t+1|t} \right)^{\cdot\frac{2}{\lambda}} \right] \tag{3.22}$$

where $\top$ denotes the transpose operator, $\circ$ denotes the Hadamard product of matrices and $.^{\frac{2}{\lambda}}$ denotes the element-wise exponent.

For forecasting horizons that are beyond one-low frequency period, we have

$$\mathrm{H}^{\lambda}_{k,t+s|t} = \Psi^{N_{1:s}+k-1} \mathrm{H}^{\lambda}_{1,t+1|t} + \sum_{j=0}^{N_{1:s}+k-2} \Psi^{j} \left(1 - \triangle\right) \mathbf{1}_{2\times 1}, \quad s > 1 \tag{3.23}$$

where $N_{1:s} = \sum_{h=2}^{s} N_{t+h-1}$. As in Conrad and Kleen's (2020), we forecast τ_{t+s} by τ_{t+1}. Thus, we obtain

$$\hat{\sigma}^2_{k,t+s|t} = \mathrm{P}^{\top}_{s_{k,t+s|t}} \left[\mathrm{T}_{k,t+1} \circ \left(\mathrm{H}^{\lambda}_{k,t+s|t} \right)^{\cdot\frac{2}{\lambda}} \right] \tag{3.24}$$

Finally, cumulative forecasts are obtained as

$$\hat{\sigma}^2_{1:k,t+s|t} = \sum_{j=1}^{k} \hat{\sigma}^2_{j,t+s|t} \tag{3.25}$$

In Section 3.4.1, we consider cumulative volatility forecasts for horizon up to 3 months. In particular, we consider one day forecasts ($s = 1$, $k = 1$), 2 weeks cumulative forecasts ($s = 1$, $k = 10$), 1 month cumulative forecasts ($s = 1, k = 22$), 2 months cumulative forecasts ($s = 2$, $k = 22$) and 3 months cumulative forecasts ($s = 3$, $k = 22$).

3.4.1 Forecasting performance evaluation

We evaluate the forecasts against the daily realised variances, $RV_{i,t}$, defined as the sum of 5-minute intraday subsampling log-returns on day i of month t plus the squared overnight log-return[22],[23]. To asses the prediction accuracy of each model, we use the QLIKE loss function,

$$\text{QLIKE}\left(RV_{1:k,t+s}, \hat{\sigma}^2_{1:k,t+s|t}\right) = \frac{RV_{1:k,t+s}}{\hat{\sigma}^2_{1:k,t+s|t}} - \ln\left(\frac{RV_{1:k,t+s}}{\hat{\sigma}^2_{1:k,t+s|t}}\right) - 1 \tag{3.26}$$

where $RV_{1:k,t+s} = \sum_{i=1}^{k} RV_{i,t+s}$. As shown by Patton (2011), the QLIKE loss function is robust to noise in the volatility proxy.

Table 3.21 - 3.24 contain the average QLIKE losses for each of the single-regime and two-regime GARCH-MIDAS models and the different forecast horizons. The smallest average loss is highlighted in bold; the second smallest is in italics. The models that consider the VIX variable, both alone and together with FU, give the smallest losses for 1 day-ahead forecasts. On the other hand, the models with only FU as explanatory variable provide the smallest average losses for longer forecast horizons from 2 weeks to 3 months. Moreover, models with the MU index as explanatory variable also achieve better performance than models that consider the VIX or FU-VIX variables for 2 months and 3 months horizons. This finding is in line with the results in Megaritis et al. (2021) which show that the MU variable outperforms the VIX for 3 month and 12 months horizons in predicting stock market volatility and jump tail risk in the post-2007 crisis period. Even so, the MU variable results in higher losses than the FU variable for all horizons.

[Table 3.21 - 3.24 here]

Regarding the individual performance of the models, our more general MSFGARCH-MIDAS model achieves the lowest QLIKE for most of the horizons and economic variables considered. In particular, the performance of our model is remarkable for the FU and FU-VIX variables. The MSAPGARCH-MIDAS model also does a good job in terms of providing small QLIKE losses.

To further evaluate the individual performance of the models, we consider the Model Confidence Set (MCS) approach of Hansen et al. (2011). This approach consists of obtaining

[22] Several papers, such as Pan et al. (2017), use the squared daily returns as the volatility proxy. It has been shown, however, that squared daily returns are unbiased but noisy volatility proxy (see Patton, 2011).

[23] The data can be downloaded from `https://www.oxford-man.ox.ac.uk/`.

a set of models for which there is a pre-specified probability that the set contains the best model. The distribution of the test statistic is estimated using a bootstrap procedure. In this paper, we consider 90% confidence level, 10,000 bootstrap replications and use the range statistic proposed by Hansen et al. (2011). The optimal block-length is obtained following the procedure in Politis and White (2004) and Patton et al. (2009)[24]. The models included in the final set are shaded in blue in the tables. For the FU and FU-VIX variables, the only model included in the final set at all horizons is the MSFGARCH-MIDAS model. The asymmetric MSTGARCH-MIDAS, MSQGARCH-MIDAS and MSAPGARCH-MIDAS are also included in the MCS at almost all horizons for the remaining variables, VIX and MU. On balance, our model outperforms for most of the horizons considered and its superiority is particularly evident for the FU and FU-VIX combination when no other model enters the MCS at all horizons.

In an attempt to obtain a better picture of the performance of the models, we divide the sample into three different subsamples using the method in Conrad and Kleen (2020). The authors consider forecasts to be in the low-/normal-/high-volatility period if the level of the realised variance on the day the forecasts was issued is below the 25% quantile, between the 25% and 75% quantile, or above the 75% quantile of the full-sample realised variances, respectively. We have 1013 observations in the low, 1587 observations in the normal and 828 observations in the high regime. That is, approximately 30% of the sample is in the low-volatility period, 46% of the sample is in the normal-volatility period, and 24% of the sample is in the high-volatility period.

Table 3.25 - 3.28 contain the forecasts results on these subsamples. The MSFGARCH-MIDAS model is included in the three volatility regimes at essentially all horizons. As for the whole out-of-sample period, our model clearly dominates in all regimes for the FU and FU-VIX variables. When considering the VIX or MU variables our model also performs very well especially in the normal- and high-volatility periods and at shorter horizons. In the low-volatility period, other more parsimonious models such as the MSTGARCH-MIDAS, MSAPGARCH-MIDAS or even the symmetric MSAVGARCH-MIDAS and MSNLGARCH-MIDAS models are competitive models.

[24] For implementation of the MCS procedure, we use the MFE Matlab Toolbox by Kevin Sheppard: `https://www.kevinsheppard.com/code/matlab/mfe-toolbox/`.

[Table 3.25 - 3.28 here]

In summary, the most general MSFGARCH-MIDAS outperforms for the FU and FU-VIX variables when evaluating the forecasts across the whole out-of-sample period and in subsamples of low-, normal- and high-volatility regimes. For VIX and MU variables, other more parsimonious models such as the MSTGARCH-MIDAS and MSAPGARCH-MIDAS also perform well, and in the low-regime symmetric regime-switching models achieve good performance. Nevertheless, the MSFGARCH-MIDAS model is included in the final MCS for almost all horizons even in the low-volatility regime.

3.5 Robustness

In this section, we provide robustness of the previous results by considering an alternative out-of-sample period which excludes the COVID-19 pandemic. COVID-19 constitutes a period of high uncertainty particularly during the first quarter of 2020 where uncertainty achieved historical records. We now consider pre-COVID data until December 2019. As in Section 3.4, we apply a rolling forecasting scheme where the window size is determined by the length of the first estimation period ending in December 2006. The models are re-estimated on a monthly basis. We evaluate the forecasting performance of the models during the period January 2007 to December 2019.

Table 3.29 - 3.32 present the average QLIKE losses for each of the GARCH-MIDAS models and the different forecast horizons. Our conclusions remain essentially unaltered. First, our model significantly outperforms at all horizons when considering the FU and FU together with VIX variables. For the MU variable, the results are mixed. Surprisingly, the symmetric single-regime GARCH models, AVGARCH-MIDAS and NLGARCH-MIDAS perform very well at horizons of 2 and 3 months. This result may be driven by the fact that the performance of the single-regime symmetric models substantially deteriorates during periods of high uncertainty which explains the bad performance of the models when the COVID-19 period is included in the out-of-sample period. Nevertheless, asymmetric regime-switching models such as MSFGARCH-MIDAS and MSAPGARCH-MIDAS achieve the best performance at shorter horizons. When VIX is the only explanatory variable, asymmetric regime-switching

models still perform the best. However, more single-regime models are included in the final MCS set than when including the COVID-19 pandemic episode, particularly at longer horizons. This again manifests that periods of heightened uncertainty hinder the performance of single-regime models.

[Table 3.29 - 3.32 here]

Results in low-, normal- and high- volatility periods are presented in Table 3.33 - 3.36. The tables again show the remarkable performance of asymmetric regime-switching models. In particular, our more general MSFGARCH-MIDAS model is the only model included in the final set at virtually all horizon and volatility regimes.

[Table 3.33 - 3.36 here]

Overall, we have shown that our proposed model performs best at essentially all the horizons considered and that this superiority is robust to the sample period under consideration. As we would expect, single-regime models achieve a better performance when excluding the high-uncertainty period during the COVID-19 pandemic. Even so, regime-switching models are superior irrespective of the sample period.

3.6 Conclusion

We introduce a general model that nests several single-regime and regime-switching GARCH-MIDAS models as special cases. We apply this general model and the nested models to daily returns of the S&P 500 index from January 1990 to November 2020. As explanatory variables in the long-term volatility component, we use two measures of uncertainty: the Financial Uncertainty index of Ludvigson et al. (2020) and the Macroeconomic Uncertainty (MU) index of Jurado et al. (2015). Even though both measures of uncertainty provide useful information to explain returns, the FU variable provides a superior description of the data than the other uncertainty index. Furthermore, we combine the information in the monthly variables with the daily Chicago Board Options Exchange Volatility Index (VIX) and find that the MU index does not provide complementary information to the VIX.

We consider two different types of Markov-switching GARCH-MIDAS models. First, we allow regime changes in the long-term volatility component. We then consider switches in the short-term component. Our results show that asymmetries and regime changes are important features of the volatility and that failure to account for either of these properties leads to misleading inferences and biased forecasts. This is a significant finding of our research since, so far in the literature, the presence of asymmetries in Markov-switching GARCH-MIDAS models has been neglected.

In the volatility forecasting evaluation exercise, we show that the models that consider the VIX variable provide the most accurate predictions for 1 day-ahead forecasts. On the other hand, the models with the FU variable outperform for longer forecast horizons of 2 weeks, 1 month, 2 months and 3 months horizons. Moreover, models with the MU index as explanatory variable also achieve better performance than models that consider the VIX variable (alone or with FU) for 2 months and 3 months horizons. This finding is in line with the results in Megaritis et al. (2021) which show that the MU variable outperforms the VIX for 3 months and 12 months horizons in predicting stock market volatility in the post-2007 crisis period. Even so, we find that the FU variable significantly outperforms the MU variable at all horizons considered. Thus, our findings provide a useful indication for selecting the appropriate explanatory variable depending on the forecasts horizon under consideration. While high-frequency uncertainty proxies such as the VIX variable appear to perform well at one-day horizon, low-frequency uncertainty indexes perform better at longer forecast horizons. Furthermore, we find that our proposed model achieves the lowest QLIKE losses at essentially all horizons and is the only model that is almost always included in the Model Confidence Set of Hansen et al. (2011). This suggests that our model is more useful than other more parsimonious GARCH-MIDAS models when forecasting the volatility of returns.

3.7 Appendix

3.7.1 Tables

Table 3.1: Nested GARCH models in the short-term component.

λ	$\hat{\lambda}$	γ	ψ	Model	Reference
0	1	free	0	Exponential GARCH (EGARCH)	Nelson (1991)
1	1	0	0	Absolute Value GARCH (AVGARCH)	Schwert (1989); Taylor (2008)
1	1	$\lvert\gamma\rvert \leq 1$	0	Threshold GARCH (TGARCH)	Zakoian (1994)
2	2	0	0	Standard GARCH (GARCH)	Bollerslev (1986)
2	2	free	0	GJRGARCH	Glosten et al. (1993)
2	2	0	free	Quadratic GARCH (QGARCH) Asymmetric GARCH (AGARCH)	Sentana (1995) Engle (1990)
free	λ	0	0	Nonlinear GARCH (NLGARCH)	Higgins and Bera (1992)*
free	λ	$\lvert\gamma\rvert \leq 1$	0	Asymmetric Power GARCH (APGARCH)	Ding et al. (1993)*

*Nested if $\lambda > 0$.

Table 3.2: Summary statistics of returns and explanatory variables.

	Starts	Frecuency	Mean	Min.	Max.	Std. Dev.	Skew.	Kurt.
S&P 500 returns	1990:M1	Daily	0.0302	-12.7652	10.9572	1.1517	-0.4093	14.3369
VIX	1990M1	Daily	1.2255	0.5758	5.2090	0.5119	0.1388	0.7108
FU index	1987:M1	Monthly	0.8920	0.6293	1.5463	0.1844	0.8373	3.2776
MU index	1987:M1	Monthly	0.6380	0.5286	1.2190	0.1084	2.7023	11.7180

Note: This table presents summary statistics for the log-returns of the S&P 500, the Chicago Board Options Exchange Volatility Index (VIX), the Financial Uncertainty (FU) index of Ludvigson et al. (2020) and the Macroeconomic Uncertainty (MU) index of Jurado et al. (2015). The sample ends in 2020:M11.

Table 3.3: Model comparisons, long-term component switching, FU.

			AIC		BIC		LRT		
Model	n	LL	Value	Rank	Value	Rank	$H_A:\gamma$ free	$H_A:\psi$ free	H_A : MSFGARCH
Panel A : $\lambda = 0$									
EGARCH	7	-10041.68	20097.36	10	20146.06	10			227.22***
MSEGARCH	11	-9940.19	19902.37	2	19978.90	1			24.23***
Panel B: $\lambda = 1$									
AVGARCH	6	-10230.77	20473.55	18	20515.29	18	390.11***		605.40***
TGARCH	7	-10035.72	20085.43	9	20134.13	8			215.29***
MSAVGARCH	10	-10091.13	20202.25	13	20271.83	13	239.33***		326.11***
MSTGARCH	11	-9971.46	19964.93	5	20041.45	5			86.78***
Panel C: $\lambda = 2$									
GARCH	6	-10216.84	20445.68	16	20487.43	16	274.80***	338.49***	577.54***
GJRGARCH	7	-10079.44	20172.88	12	20221.58	12			302.73***
QGARCH	7	-10047.60	20109.19	11	20157.89	11			239.05***
MSGARCH	10	-10099.17	20218.34	15	20287.91	15	201.56***	312.20***	342.19***
MSGJRGARCH	11	-9998.39	20018.78	6	20095.31	6			140.63***
MSQGARCH	11	-9943.07	19908.14	3	19984.67	3			29.99***
Panel D: λ free									
NLGARCH	7	-10216.34	20446.68	17	20495.39	17	363.80***		576.54***
APGARCH	8	-10034.44	20084.88	8	20140.54	9			212.74***
FGARCH	10	-10014.45	20048.90	7	20118.47	7			172.75***
MSNLGARCH	11	-10090.66	20203.32	14	20279.85	14	259.89***		325.18***
MSAPGARCH	12	-9960.72	19945.43	4	20028.92	4			65.28***
MSFGARCH	14	-9928.07	19884.15	1	19981.55	2			

Note: n denotes the number of parameters to be estimated and LL the maximized log-likelihood value of a given model. AIC and BIC are the Akaike information criterion and the Bayesian information criterion, respectively, computed as $\text{AIC} = -2\,(LL - n)$ and $\text{BIC} = -2\,LL + n\log(T)$ where $T = 7764$ is the sample size. The LRTs in the antepenultimate and penultimate columns are the likelihood test statistics for the following tests: H_0 : the symmetric specification corresponding to the model in the first column of the same row, versus H_A : the parameters γ or ψ are freely estimated. Under the null hypothesis, the LRT statistics are asymptotically χ^2-distributed with 1 degrees of freedom. The last column reports the LRT of the problem: H_0 : the specification corresponding to the model in the first column of the same row versus H_A : the most general model in (3.1), (3.8)-(3.12) denoted here MSFGARCH model. *,** and *** denote statistical significance at 1%, 5% and 10% levels, respectively.

Table 3.4: Model comparisons, long-term component switching, MU.

			AIC		BIC		LRT		
Model	n	LL	Value	Rank	Value	Rank	$H_A:\gamma$ free	$H_A:\psi$ free	H_A : MSFGARCH
Panel A : $\lambda = 0$									
EGARCH	7	-10118.77	20251.54	10	20300.24	10			275.84***
MSEGARCH	11	-9998.06	20018.12	3	20094.65	3			34.42***
Panel B: $\lambda = 1$									
AVGARCH	6	-10288.65	20589.31	18	20631.05	18	368.48***		615.62***
TGARCH	7	-10104.42	20222.83	8	20271.53	8			247.14***
MSAVGARCH	10	-10116.53	20253.07	11	20322.64	12	224.28***		271.38***
MSTGARCH	11	-10004.39	20030.79	5	20107.32	4			47.10***
Panel C: $\lambda = 2$									
GARCH	6	-10270.82	20553.64	16	20595.39	16	253.82***	291.36***	579.95***
GJRGARCH	7	-10143.91	20301.83	15	20350.53	15			326.13***
QGARCH	7	-10125.14	20264.28	13	20312.98	11			288.59***
MSGARCH	10	-10123.43	20266.86	14	20336.43	14	181.53***	253.90***	285.17***
MSGJRGARCH	11	-10032.67	20087.33	6	20163.86	6			103.64***
MSQGARCH	11	-9996.48	20014.96	2	20091.49	2			31.27***
Panel D: λ free									
NLGARCH	7	-10270.32	20554.65	17	20603.35	17	331.96***		578.96***
APGARCH	8	-10104.35	20224.69	9	20280.35	9			247.00***
FGARCH	10	-10080.36	20200.71	7	20250.29	7			199.02***
MSNLGARCH	11	-10115.79	20253.58	12	20330.11	13	227.64***		269.89***
MSAPGARCH	12	-10001.97	20027.94	4	20111.42	5			42.25***
MSFGARCH	14	-9980.85	19989.69	1	20087.09	1			

Note: As in Table 3.3.

Table 3.5: Model comparisons, long-term component switching, VIX.

			AIC		BIC		LRT		
Model	n	LL	Value	Rank	Value	Rank	$H_A : \gamma$ free	$H_A : \psi$ free	H_A : MSFGARCH
Panel A : $\lambda = 0$									
EGARCH	7	-9980.21	19974.41	12	20023.11	10			214.44***
MSEGARCH	11	-9879.50	19780.99	3	19857.52	2			13.02***
Panel B: $\lambda = 1$									
AVGARCH	6	-10061.34	20134.69	18	20176.43	17	152.94***		376.71***
TGARCH	7	-9984.87	19983.75	14	20032.45	11			223.77***
MSAVGARCH	10	-9964.14	19948.29	7	20017.86	9	84.89***		182.32***
MSTGARCH	11	-9914.43	19850.86	5	19927.39	5			82.89***
Panel C: $\lambda = 2$									
GARCH	6	-10059.80	20131.60	17	20173.34	16	97.49***	166.60***	373.63***
GJRGARCH	7	-10011.05	20036.11	15	20084.81	15			276.14***
QGARCH	7	-9976.50	19967.00	10	20015.70	8			207.02***
MSGARCH	10	-9973.29	19966.58	9	20036.15	12	69.67***	171.66***	200.61***
MSGJRGARCH	11	-9927.54	19877.08	6	19953.61	6			109.10***
MSQGARCH	11	-9875.64	19773.29	1	19849.82	1			5.32
Panel D: λ free									
NLGARCH	7	-10058.76	20131.52	16	20200.23	18	152.81***		371.55***
APGARCH	8	-9982.36	19980.72	13	20036.38	13			218.75***
FGARCH	10	-9976.62	19973.24	11	20042.82	14			207.27***
MSNLGARCH	11	-9956.11	19934.23	8	20010.76	7	94.00***		166.26***
MSAPGARCH	12	-9909.11	19842.23	4	19925.72	4			72.26***
MSFGARCH	14	-9872.99	19773.97	2	19871.37	3			

Note: As in Table 3.3.

Table 3.6: Estimated MSGARCH-MIDAS models, long-term component switching, FU

	MSEGARCH	MSAVGARCH	MSTGARCH	MSGARCH	MSGJRGARCH	MSQGARCH	MSNLGARCH	MSAPGARCH	MSFGARCH
									0.7415***
λ	0	1	1	2	2	2	1.4574***	0.6334***	(0.2526)
							(0.2641)	(0.0863)	1.9820***
$\hat{\lambda}$	1	1	1	2	2	2			(1.2224)
μ	0.0352***	0.0670***	0.0391**	0.0678***	0.0460***	0.0339***	0.0685***	0.0349***	0.0320***
	(0.0083)	(0.0081)	(0.0188)	(0.0083)	(0.0072)	(0.0086)	(0.0086)	(0.0000)	(0.0084)
α	0.0257*	0.0440***	0.0762***	0.0391***	0.0378***	0.0255***	0.0446***	0.0733***	0.0037
	(0.0150)	(0.0062)	(0.0067)	(0.0059)	(0.0055)	(0.0080)	(0.0028)	(0.0060)	(0.0068)
β	0.9126***	0.9568***	0.8815***	0.9526***	0.8623***	0.8791***	0.9551***	0.8993***	0.9113***
	(0.0133)	(0.0065)	(0.2190)	(0.0079)	(0.0271)	(0.0201)	(0.0062)	(0.0126)	(0.0125)
γ	-0.1929***		0.9999***		0.9996***			0.9999***	-0.8977***
	(0.0145)		(0.2597)		(0.0552)			(0.0002)	(0.3271)
ψ						0.1858***			0.0883***
						(0.0154)			(0.0333)
m_1	-3.9439***	-4.0586***	-3.5495***	-4.3514***	-3.7332***	-3.8415***	-4.4063***	-3.5477***	-3.7679***
	(0.2377)	(0.5697)	(0.5258)	(0.5850)	(0.3326)	(0.2775)	(1.0890)	(0.2925)	(0.2738)
m_2	-2.9008***	-2.6212***	-2.0630	-3.0584***	-2.2404***	-2.8251***	-3.0219***	-2.2898***	-2.6933***
	(0.3523)	(0.7284)	(1.5079)	(0.6489)	(0.5850)	(0.4030)	(1.0872)	(0.6168)	(0.3932)
θ_1^{FU}	3.6803***	4.0146***	3.4815***	4.0775***	3.4512***	3.6699***	4.2490***	3.5370***	3.5655***
	(0.2338)	(0.5775)	(1.0343)	(0.5931)	(0.3130)	(0.2891)	(1.1527)	(0.2627)	(0.2759)
θ_2^{FU}	3.6091***	4.0315***	3.0203	4.2555***	2.9726***	3.6527***	4.3202***	3.3844***	3.4205***
	(0.3820)	(0.7027)	(2.7274)	(0.7012)	(0.4862)	(0.4536)	(1.1796)	(0.6583)	(0.4384)
w^{FU}	404.66	519.70*	104.58	520.05***	505.85	513.85*	506.88	106.21	509.73
	(526.84)	(267.42)	(595.88)	(103.98)	(657.05)	(280.11)	(854.57)	(138.77)	(865.71)
p_{11}	0.9841	0.9803	0.9888	0.9849	0.9757	0.9834	0.9811	0.9878	0.9842
p_{22}	0.9685	0.9326	0.9681	0.9582	0.9582	0.9670	0.9415	0.9547	0.9716
$\overline{\sigma}_1^2/\overline{\sigma}_2^2$	0.3524	0.2375	0.2262	0.2744	0.2248	0.3619	0.2505	0.2843	0.3414
$\theta_1^{FU}=\theta_2^{FU}$	[0.8227]	[0.9647]	[0.7951]	[0.6277]	[0.7446]	[0.9585]	[0.8512]	[0.8156]	[0.6547]

Note: This table shows parameter estimates for the two-regime model in (3.1), (3.8)-(3.12) and the nested models, presented in Table 3.1. The explanatory variable in the long-term component is the Financial Uncertainty (FU) index of Ludvigson et al. (2020). Standard errors are displayed in parenthesis; p-values are displayed as $[\cdot]$. *,** and *** denote statistical significance at 1%, 5% and 10% levels, respectively.

Table 3.7: Estimated MSGARCH-MIDAS models, long-term component switching, MU

	MSEGARCH	MSAVGARCH	MSTGARCH	MSGARCH	MSGJRGARCH	MSQGARCH	MSNLGARCH	MSAPGARCH	MSFGARCH
λ	0	1	1	2	2	2	1.2345***	0.7899***	3.5091***
							(0.2159)	(0.0983)	(0.5408)
$\hat{\lambda}$	1	1	1	2	2	2			0.9620***
									(0.1353)
μ	0.0296***	0.0676***	0.0360***	0.0682***	0.0424***	0.0267***	0.0684***	0.0335***	0.0287***
	(0.0088)	(0.0030)	(0.0096)	(0.0087)	(0.0105)	(0.0071)	(0.0082)	(0.0000)	(0.0089)
α	0.0779***	0.0539***	0.0766***	0.0482***	0.0000	0.0517***	0.0550***	0.0780***	0.3163***
	(0.0158)	(0.0046)	(0.0098)	(0.0035)	(0.0000)	(0.0072)	(0.0043)	(0.0087)	(0.0731)
β	0.9543***	0.9529***	0.8934***	0.9488***	0.9047***	0.8788***	0.9516***	0.9004***	0.7394***
	(0.0148)	(0.0039)	(0.0243)	(0.0034)	(0.0209)	(0.0192)	(0.0042)	(0.0190)	(0.0435)
γ	-0.1643***		0.9999***		0.1491***			0.9999***	0.9999***
	(0.0180)		(0.0899)		(0.0391)			(0.0004)	(0.0034)
ψ						0.1784***			0.0274
						(0.0163)			(0.0187)
m_1	-2.8181***	-2.6585***	-2.5759***	-4.0364***	-2.0619	-2.9243***	-2.7936***	-2.6513***	-2.8094***
	(0.3681)	(0.4028)	(0.4556)	(0.6479)	(0.0000)	(0.2809)	(0.4657)	(0.3939)	(0.6355)
m_2	-1.7165***	-1.3225**	-1.5780***	-2.7284***	-1.1734	-1.8387***	-1.4426**	-1.6525***	-1.7467***
	(0.4415)	(0.6284)	(0.5817)	(0.8006)	(1.5845)	(0.2446)	(0.6040)	(0.4622)	(0.7159)
θ_1^{MU}	3.2713***	3.8236***	3.0521***	4.6121***	2.3238	3.6336***	3.8473***	3.2505***	3.0545***
	(0.4945)	(0.6028)	(0.5892)	(0.9433)	(3.4856)	(0.4313)	(0.6285)	(0.6207)	(0.5468)
θ_2^{MU}	3.4487***	4.4105***	3.4514***	5.4177***	3.0847***	3.9080***	4.4337***	3.6181***	3.6144***
	(0.5521)	(0.8570)	(0.7043)	(1.2418)	(1.0853)	(0.4361)	(0.8138)	(0.6103)	(0.5372)
w^{MU}	369.58***	498.12***	479.21***	505.14**	465.94	503.02	504.75**	454.83	467.14
	(129.51)	(116.33)	(111.12)	(226.81)	(420.89)	(751.99)	(218.93)	(296.62)	(805.75)
p_{11}	0.9902	0.9866	0.9896	0.9885	0.9814	0.9916	0.9866	0.9899	0.9933
p_{22}	0.9843	0.9497	0.9829	0.9582	0.9307	0.9848	0.9500	0.9825	0.9842
$\overline{\sigma}_1^2/\overline{\sigma}_2^2$	0.3323	0.2629	0.3686	0.2704	0.4113	0.3377	0.2590	0.3683	0.3455
$\theta_1^{\text{MU}} = \theta_2^{\text{MU}}$	[0.7399]	[0.3903]	[0.5556]	[0.2169]	[0.7933]	[0.5725]	[0.3290]	[0.5978]	[0.3894]

Note: This table shows parameter estimates for the two-regime model in (3.1), (3.8)-(3.12) and the nested models, presented in Table 3.1. The explanatory variable in the long-term component is the Macroeconomic Uncertainty (MU) index of Jurado et al. (2015). Standard errors are displayed in parenthesis; p-values are displayed as [·]. *,** and *** denote statistical significance at 1%, 5% and 10% levels, respectively.

Table 3.8: Estimated MSGARCH-MIDAS models, long-term component switching, VIX

	MSEGARCH	MSAVGARCH	MSTGARCH	MSGARCH	MSGJRGARCH	MSQGARCH	MSNLGARCH	MSAPGARCH	MSFGARCH
λ	0	1	1	2	2	2			3.0747***
							1.5045**	0.6373***	(0.2040)
$\hat{\lambda}$	1	1	1	2	2	2	(0.7108)	(0.1321)	0.8791***
									(0.0338)
μ	0.0300***	0.0475***	0.0325***	0.0445***	0.0369	0.0285***	0.0457***	0.0321***	0.0279***
	(0.0084)	(0.0130)	(0.0085)	(0.0109)	(0.0972)	(0.0079)	(0.0102)	(0.0000)	(0.0000)
α	-0.0471	0.0190	0.0452***	0.0181*	0.0179	0.0000	0.0208	0.0461***	0.0021***
	(0.0327)	(0.0247)	(0.0050)	(0.0102)	(0.0862)	(0.0002)	(0.0133)	(0.0056)	(0.0000)
β	0.8032***	0.9712***	0.8762***	0.9630***	0.8909	0.8200***	0.9667***	0.8822***	0.8094***
	(0.0748)	(0.0357)	(0.0288)	(0.0152)	(2.0606)	(0.0478)	(0.0125)	(0.0291)	(0.0000)
γ	-0.1846***		0.9999***		0.9974***			0.9999***	0.9890***
	(0.0306)		(0.1684)		(0.3736)			(0.0003)	(0.0081)
ψ						0.1788***			0.2463***
						(0.0226)			(0.0000)
m_1	-2.6909***	-2.7982**	-2.4514***	-2.8495***	-2.6154***	-2.5702***	-2.7871***	-2.4127***	-2.5459***
	(0.1317)	(1.2499)	(0.1295)	(0.4289)	(0.4885)	(0.1308)	(0.6078)	(0.1250)	(0.1028)
m_2	-0.9192***	-0.4327	0.1197	-0.2122	0.0673	-0.7505***	-0.2162	0.0546	-0.6634***
	(0.3324)	(2.2438)	(0.5792)	(1.1042)	(1.3893)	(0.2806)	(1.3698)	(0.2416)	(0.0000)
θ_1^{VIX}	0.1117***	0.1245**	0.1089***	0.1265***	0.1138***	0.1094***	0.1241***	0.1071***	0.1091***
	(0.0059)	(0.0508)	(0.0058)	(0.0175)	(0.0336)	(0.0056)	(0.0268)	(0.0070)	(0.0061)
θ_2^{VIX}	0.0640***	0.0572*	0.0391***	0.0498***	0.0392***	0.0606***	0.0518***	0.0417***	0.0595***
	(0.0084)	(0.0312)	(0.0099)	(0.0147)	(0.0168)	(0.0071)	(0.0163)	(0.0033)	(0.0000)
w^{VIX}	2.1424***	5.4054	3.3429***	5.1081**	3.9998**	2.2599***	5.0227*	3.0827***	2.3687***
	(0.6861)	(5.1170)	(0.7781)	(2.3733)	(1.3021)	(0.5209)	(3.0005)	(0.7147)	(0.0148)
p_{11}	0.9853	0.9852	0.9952	0.9916	0.9912	0.9876	0.9897	0.9938	0.9881
p_{22}	0.9627	0.9326	0.9681	0.9582	0.9326	0.9649	0.9415	0.9547	0.9616
$\overline{\sigma}_1^2/\overline{\sigma}_2^2$	0.1701	0.0939	0.0764	0.0716	0.0684	0.1621	0.0765	0.0848	0.1522
$\theta_1^{\text{VIX}} = \theta_2^{\text{VIX}}$	[0.0000]	[0.0083]	[0.0000]	[0.0000]	[0.0276]	[0.0000]	[0.0001]	[0.0000]	[0.0000]

Note: This table shows parameter estimates for the two-regime model in (3.1), (3.8)-(3.12) and the nested models, presented in Table 3.1. The explanatory variable in the long-term component is the Chicago Board Options Exchange Volatility Index (VIX). Standard errors are displayed in parenthesis; p-values are displayed as $[\cdot]$. *,** and *** denote statistical significance at 1%, 5% and 10% levels, respectively.

Table 3.9: Model comparisons, short-term component switching, FU.

			AIC		BIC		LRT		
Model	n	LL	Value	Rank	Value	Rank	$H_A : \gamma$ free	$H_A : \psi$ free	H_A : MSFGARCH
Panel A : $\lambda = 0$									
EGARCH	7	-10046.80	20107.60	10	20156.30	9			190.64***
MSEGARCH	10	-9971.89	19963.78	3	20033.35	3			40.83***
Panel B: $\lambda = 1$									
AVGARCH	6	-10236.78	20485.56	18	20527.30	18	387.39***		570.60***
TGARCH	7	-10043.09	20100.17	9	20148.87	7			183.22***
MSAVGARCH	9	-10163.99	20345.98	15	20408.59	15	325.05***		425.02***
MSTGARCH	10	-10001.46	20022.92	5	20092.50	5			99.97***
Panel C: $\lambda = 2$									
GARCH	6	-10223.63	20459.26	16	20501.01	16	274.95***	335.87***	544.31***
GJRGARCH	7	-10086.16	20206.32	12	20235.02	12			269.36***
QGARCH	7	-10055.70	20125.39	11	20174.09	11			208.44***
MSGARCH	9	-10151.15	20320.29	14	20382.91	13	225.59***	397.29***	399.34***
MSGJRGARCH	10	-10038.35	20096.71	7	20166.28	10			173.76***
MSQGARCH	10	-9952.50	19925.00	1	19994.57	1			2.05
Panel D: λ free									
NLGARCH	7	-10223.55	20461.11	17	20509.81	17	363.27***		544.15***
APGARCH	8	-10041.92	20099.84	8	20155.50	8			180.89***
FGARCH	10	-10025.83	20071.66	6	20141.24	6			148.71***
MSNLGARCH	10	-10147.44	20314.88	13	20384.45	14	305.82***		391.92***
MSAPGARCH	11	-9994.53	20011.05	4	20087.58	4			86.10***
MSFGARCH	13	-9951.48	19928.95	2	20019.40	2			

Note: n denotes the number of parameters to be estimated and LL the maximized log-likelihood value of a given model. AIC and BIC are the Akaike information criterion and the Bayesian information criterion, respectively, computed as AIC $= -2\,(LL - n)$ and BIC $= -2\,LL + n\log(T)$ where $T = 7764$ is the sample size. The LRTs in the antepenultimate and penultimate columns are the likelihood test statistics for the following tests: H_0 : the symmetric specification corresponding to the model in the first column of the same row, versus H_A : the parameters γ or ψ are freely estimated. Under the null hypothesis, the LRT statistics are asymptotically χ^2-distributed with 1 degrees of freedom. The last column reports the LRT of the problem: H_0 : the specification corresponding to the model in the first column of the same row versus H_A : the most general model introduced in Section 3.2.3, denoted here MSFGARCH model. *,** and *** denote statistical significance at 1%, 5% and 10% levels, respectively.

Table 3.10: Model comparisons, short-term component switching, MU.

			AIC		BIC		LRT		
Model	n	LL	Value	Rank	Value	Rank	$H_A : \gamma$ free	$H_A : \psi$ free	H_A : MSFGARCH
Panel A : $\lambda = 0$									
EGARCH	7	-10123.15	20260.30	10	20309.00	10			261.74***
EGARCH	10	-10026.50	20073.00	3	20142.57	3			68.44***
Panel B: $\lambda = 1$									
AVGARCH	6	-10293.22	20598.43	18	20640.17	18	364.67***		601.87***
TGARCH	7	-10110.88	20235.76	8	20284.47	7			237.20***
AVGARCH	9	-10204.99	20427.97	15	20490.59	15	347.82***		425.41***
TGARCH	10	-10031.08	20082.16	5	20151.73	4			77.60***
Panel C: $\lambda = 2$									
GARCH	6	-10274.16	20560.32	16	20602.06	16	251.74***	285.24***	563.76***
GJRGARCH	7	-10148.29	20310.58	12	20359.28	12			312.02***
QGARCH	7	-10131.54	20277.07	11	20325.78	11			278.51***
GARCH	9	-10197.30	20412.60	13	20475.22	13	268.96***	355.85***	410.04***
GJRGARCH	10	-10062.82	20145.64	6	20215.21	6			141.08***
QGARCH	10	-10019.37	20058.75	2	20128.32	2			54.19***
Panel D: λ free									
NLGARCH	7	-10273.92	20561.84	17	20610.54	17	326.13***		563.28***
APGARCH	8	-10110.86	20237.71	9	20293.37	8			237.15***
FGARCH	10	-10107.39	20234.77	7	20304.34	9			230.21***
NLGARCH	10	-10196.15	20412.30	14	20481.87	14	336.99***		407.74***
APGARCH	11	-10027.66	20077.31	4	20153.84	5			70.75***
FGARCH	13	-9992.28	20010.56	1	20101.00	1			

Note: As in Table 3.9.

Table 3.11: Model comparisons, short-term component switching, VIX.

			AIC		BIC		LRT		
Model	n	LL	Value	Rank	Value	Rank	$H_A : \gamma$ free	$H_A : \psi$ free	H_A : MSFGARCH
Panel A : $\lambda = 0$									
EGARCH	7	-9982.35	19978.70	8	20027.40	7			182.16***
MSEGARCH	10	-9904.36	19828.73	3	19898.30	2			26.19***
Panel B: $\lambda = 1$									
AVGARCH	6	-10069.30	20150.61	18	20192.35	17	160.63***		356.07***
TGARCH	7	-9988.99	19991.98	14	20040.68	8			195.44***
MSAVGARCH	9	-9982.63	19983.26	12	20045.87	11	46.86***		182.72***
MSTGARCH	10	-9959.20	19938.40	5	20007.97	5			135.86***
Panel C: $\lambda = 2$									
GARCH	6	-10066.99	20145.99	16	20207.73	16	104.09***	172.92***	351.45***
GJRGARCH	7	-10014.95	20043.89	15	20092.59	15			247.35***
QGARCH	7	-9980.53	19975.07	7	20023.77	6			178.53***
MSGARCH	9	-9983.24	19984.47	10	20047.09	12	14.99***	182.88***	183.93***
MSGJRGARCH	10	-9975.74	19971.48	6	20041.05	9			168.94***
MSQGARCH	10	-9891.80	19803.60	1	19873.17	1			1.06
Panel D: λ free									
NLGARCH	7	-10066.79	20147.58	17	20196.28	18	160.58***		351.04***
APGARCH	8	-9986.50	19989.00	13	20044.66	10			190.46***
FGARCH	10	-9979.37	19978.74	9	20048.31	13			176.20***
MSNLGARCH	10	-9981.39	19982.79	11	20052.36	14	63.23***		180.25***
MSAPGARCH	11	-9949.78	19921.56	4	19998.09	4			117.02***
MSFGARCH	13	-9891.27	19808.54	2	19898.98	3			

Note: As in Table 3.9.

Table 3.12: Estimated MSGARCH-MIDAS models, short-term component switching, FU

	MSEGARCH	MSAVGARCH	MSTGARCH	MSGARCH	MSGJRGARCH	MSQGARCH	MSNLGARCH	MSAPGARCH	MSFGARCH
λ	0	1	1	2	2	2	5.9967***	0.6417**	1.6086
							(1.2057)	(0.2984)	(1.7190)
$\hat{\lambda}$	1	1	1	2	2	2			0.8027
									(0.7471)
μ	0.0306***	0.0663***	0.0350**	0.0674***	0.0403***	0.0256**	0.0670	0.0324***	0.0289***
	(0.0660)	(0.0150)	(0.0095)	(0.0048)	(0.0089)	(0.0024)	(0.0083)	(0.0009)	(0.0104)
ω_1	-0.4579***	0.0251***	0.0180***	0.0035***	0.0038*	0.0026	0.0000	0.0310***	0.0063
	(0.0094)	(0.0084)	(0.0085)	(0.0083)	(0.0021)	(0.0124)	(0.0000)	(0.0072)	(0.0142)
ω_2	-0.3126***	0.0488***	0.0294***	0.0133***	0.0107**	0.0079***	0.0001	0.0435***	0.0149
	(0.0693)	(0.0062)	(0.0086)	(0.0013)	(0.0054)	(0.0043)	(0.0001)	(0.0113)	(0.0259)
α	-0.0004	0.0592	0.0861***	0.0544***	0.0470***	0.0307	0.0049***	0.0832***	0.0766***
	(0.0253)	(0.0392)	(0.0071)	(0.0251)	(0.0050)	(0.0674)	(0.0057)	(0.0230)	(0.1617)
β	0.8812***	0.7628***	0.8365***	0.7513***	0.7916***	0.8529***	0.5080***	0.8537***	0.8404***
	(0.0147)	(0.0419)	(0.0147)	(0.0389)	(0.0187)	(0.1217)	(0.1057)	(0.0385)	(0.1285)
γ	-0.2083***		0.9999***		0.9992***			0.9999***	0.9999***
	(0.0133)		(0.0852)		(0.0002)			(0.0006)	(0.0019)
ψ						0.0421***			0.0465
						(0.0122)			(0.1016)
θ^{FU}	3.4696***	3.6705***	3.1034***	3.5412***	3.0251***	3.4985***	3.5609***	3.2830***	3.2838***
	(0.2424)	(0.4440)	(0.9948)	(0.3975)	(0.5502)	(0.4517)	(0.3572)	(1.1091)	(0.8875)
w^{FU}	448.11	474.31**	510.89***	509.38*	493.07	503.35	470.96	153.35*	510.06
	(733.54)	(216.13)	(67.28)	(298.74)	(430.85)	(898.71)	(614.86)	(87.79)	(669.52)
p_{11}	0.9761	0.9750	0.9857	0.9709	0.9820	0.9677	0.9754	0.9838	0.9808
p_{22}	0.9627	0.9642	0.9753	0.9591	0.9762	0.9498	0.9661	0.9688	0.9682
$\overline{g}_1^2$	0.0212	0.0175	0.0362	0.0180	0.0334	0.0227	0.0166	0.0317	0.0295
$\overline{g}_2^2$	0.0720	0.0660	0.0964	0.0683	0.0938	0.0675	0.0583	0.0918	0.0864
$\overline{\sigma}_1^2/\overline{\sigma}_2^2$	0.2943	0.2655	0.3751	0.2631	0.3559	0.3357	0.2843	0.3450	0.3409

Note: This table shows parameter estimates for the two-regime model introduced in Section 3.2.3 and the nested models, presented in Table 3.1. The explanatory variable in the long-term component is the Financial Uncertainty (FU) index of Ludvigson et al. (2020). Standard errors are displayed in parenthesis. *,** and *** denote statistical significance at 1%, 5% and 10% levels, respectively.

Table 3.13: Estimated MSGARCH-MIDAS models, short-term component switching, MU

	MSEGARCH	MSAVGARCH	MSTGARCH	MSGARCH	MSGJRGARCH	MSQGARCH	MSNLGARCH	MSAPGARCH	MSFGARCH
λ	0	1	1	2	2	2	1.9261***	0.7928***	3.4269***
							(0.5086)	(0.1154)	(0.6057)
$\hat{\lambda}$	1	1	1	2	2	2			1.1066***
									(0.1639)
μ	0.0260***	0.0666***	0.0322***	0.0664***	0.0357***	0.0228***	0.0663***	0.0322***	0.0248***
	(0.0087	0.0092)	(0.0082)	(0.0084)	(0.0082)	(0.0086)	(0.0085)	(0.0006)	(0.0080)
ω_1	-0.2402***	0.0195***	0.0203***	0.0043*	0.0107**	0.0069***	0.0052*	0.0256***	0.0018
	(0.0476)	(0.0024)	(0.0055)	(0.0023)	(0.0053)	(0.0020)	(0.0029)	(0.0052)	(0.0015)
ω_2	-0.1283***	0.0514	0.0394***	0.0286**	0.0420**	0.0223***	0.0299**	0.0433***	0.0145*
	(0.0333)	(0.0315)	(0.0111)	(0.0123)	(0.0210)	(0.0071)	(0.0147)	(0.0075)	(0.0085)
α	0.0685***	0.0758*	0.0887***	0.0700***	0.0510***	0.0656***	0.0745***	0.0884***	0.3026***
	(0.0222)	(0.0438)	(0.0054)	(0.0259)	(0.0044)	(0.0073)	(0.0223)	(0.0056)	(0.0929)
β	0.9203***	0.8429***	0.8549***	0.8253***	0.8157***	0.8391***	0.8215***	0.8642***	0.7058***
	(0.0140)	(0.0732)	(0.0131)	(0.0468)	(0.0156)	(0.0155)	(0.0491)	(0.0142)	(0.0454)
γ	-0.1869***		0.9999***		0.9985***			0.9999***	0.9997***
	(0.0128		(0.0789)		(0.0019)			(0.0005)	(0.0038)
ψ						0.0657***			0.0074
						(0.0098)			(0.0061)
θ^{MU}	3.5196***	3.7377**	3.1705***	3.5652***	2.0958**	3.2591***	3.5453***	3.3281***	2.5295***
	(0.3922)	(1.7019)	(0.9957)	(0.5429)	(0.8819)	(0.4660)	(0.5164)	(0.4442)	(0.5522)
w^{MU}	296.07**	507.99	9.46	512.91	38.67	8.13	514.44*	530.28***	8.1489
	(118.45)	(790.06)	(7.36)	(642.64)	(113.68)	(5.06)	(302.80)	(154.88)	(5.3021)
p_{11}	0.9904	0.9842	0.9948	0.9834	0.9947	0.9932	0.9854	0.9936	0.9941
p_{22}	0.9838	0.9642	0.9894	0.9591	0.9907	0.9867	0.9661	0.9871	0.9863
$\overline{g}_1^2$	0.0491	0.0407	0.0744	0.0414	0.1300	0.0729	0.0386	0.0672	0.1812
$\overline{g}_2^2$	0.1998	0.2828	0.2805	0.2729	0.5089	0.2344	0.2382	0.2528	0.6150
$\overline{\sigma}_1^2/\overline{\sigma}_2^2$	0.2457	0.1438	0.2652	0.1518	0.2555	0.3110	0.1622	0.2657	0.2946

Note: This table shows parameter estimates for the two-regime model introduced in Section 3.2.3 and the nested models, presented in Table 3.1. The explanatory variable in the long-term component is the Macroeconomic Uncertainty (MU) index of Jurado et al. (2015). Standard errors are displayed in parenthesis. *,** and *** denote statistical significance at 1%, 5% and 10% levels, respectively.

Table 3.14: Estimated MSGARCH-MIDAS models, short-term component switching, VIX

	MSEGARCH	MSAVGARCH	MSTGARCH	MSGARCH	MSGJRGARCH	MSQGARCH	MSNLGARCH	MSAPGARCH	MSFGARCH
λ	0	1	1	2	2	2	1.5693***	0.4176***	2.3709***
							(0.1342)	(0.1553)	(0.2871)
$\hat{\lambda}$	1	1	1	2	2	2			0.0015
									(0.0011)
μ	0.0304***	0.0465***	0.0693***	0.0467***	0.0388***	0.0253***	0.0464***	0.0321***	0.0250***
	(0.0101)	(0.0084)	(0.0224)	(0.0084)	(0.0085)	(0.0085)	(0.0084)	(0.0000)	(0.0088)
ω_1	-0.8394*	0.2548***	0.0324***	0.0654***	0.0150	0.0211***	0.1167***	0.1004**	0.0135**
	(0.5063)	(0.0163)	(0.0093)	(0.0094)	(0.0129)	(0.0041)	(0.0155)	(0.0443)	(0.0055)
ω_2	-0.5590*	0.3947***	0.0473***	0.1586***	0.0405***	0.0501***	0.2312***	0.1185**	0.0345***
	(0.3195)	(0.0276)	(0.0106)	(0.0274)	(0.0112)	(0.0144)	(0.0209)	(0.0501)	(0.0129)
α	-0.1158	0.0000	0.0506***	0.0000	0.0169***	0.0000	0.0000	0.0440***	0.0801
	(0.0830)	(0.0000)	(0.0083)	(0.0000)	(0.0062)	(0.0000)	(0.0000)	(0.0114)	(0.0515)
β	0.6804***	0.0000	0.8034***	0.0000	0.7617***	0.7677***	0.0000	0.8084***	0.7020***
	(0.1809)	(0.0000)	(0.0426)	(0.0000)	(0.1664)	(0.0599)	(0.0000)	(0.0488)	(0.0364)
γ	-0.2027***		0.9999***		0.9975***			0.9999***	0.0328**
	(0.0406))		(0.2196)		(0.0042)			(0.0002)	(0.0166)
ψ						0.0717***			0.0527***
						(0.0071)			(0.0136)
θ^{VIX}	0.1001***	0.1064***	0.0930***	0.1065***	0.0998***	0.0937***	0.1064***	0.0950***	0.0926***
	(0.0063)	(0.0035)	(0.0134)	(0.0034)	(0.0042)	(0.0062)	(0.0036)	(0.0084)	(0.0081)
w^{VIX}	1.8030***	5.3399***	2.9869***	5.2743***	3.5916***	1.6918***	5.3662***	2.7701***	2.1155***
	(0.5763)	(1.2189)	(0.6715)	(1.2776)	(0.7598)	(0.6416)	(1.3289)	(0.6205)	(0.7998)
p_{11}	0.9740	0.9748	0.9794	0.9739	0.9654	0.9742	0.9752	0.9786	0.9799
p_{22}	0.9627	0.9642	0.9695	0.9591	0.9266	0.9489	0.9661	0.9688	0.9625
$\overline{g}_1^2$	0.0723	0.0649	0.0431	0.0654	0.0735	0.0907	0.0647	0.0899	0.0954
$\overline{g}_2^2$	0.1739	0.1558	0.0915	0.1586	0.1979	0.2157	0.1547	0.1994	0.2108
$\overline{\sigma}_1^2/\overline{\sigma}_2^2$	0.4159	0.4165	0.4715	0.4121	0.3716	0.4203	0.4183	0.4511	0.4525

Note: This table shows parameter estimates for the two-regime model introduced in Section 3.2.3 and the nested models, presented in Table 3.1. The explanatory variable in the long-term component is the Chicago Board Options Exchange Volatility Index (VIX). Standard errors are displayed in parenthesis. *,** and *** denote statistical significance at 1%, 5% and 10% levels, respectively.

Table 3.15: Model comparisons, long-term component switching, FU-VIX.

			AIC		BIC		LRT		
Model	n	LL	Value	Rank	Value	Rank	$H_A : \gamma$ free	$H_A : \psi$ free	H_A : MSFGARCH
Panel A : $\lambda = 0$									
EGARCH	9	-9953.92	19925.85	9	19988.46	8			201.82***
MSEGARCH	14	-9856.06	19900.12	3	19837.52	2			6.09
Panel B : $\lambda = 1$									
AVGARCH	8	-10057.74	20131.48	18	20207.14	17	186.78***		409.46***
TGARCH	9	-9964.35	19946.71	13	20009.32	10			222.68***
MSAVGARCH	13	-9954.80	19935.61	11	20026.05	13	95.81***		203.58***
MSTGARCH	14	-9906.90	19841.80	5	19939.20	5			107.77***
Panel C : $\lambda = 2$									
GARCH	8	-10055.04	20126.09	16	20201.75	16	121.99***	207.19***	404.06***
GJRGARCH	9	-9994.05	20006.10	15	20068.71	15			282.07***
QGARCH	9	-9951.45	19920.90	8	19983.52	7			196.87***
MSGARCH	13	-9961.47	19948.95	14	20039.39	14	102.47***	234.63***	216.92***
MSGJRGARCH	14	-9922.06	19872.11	6	19969.52	6			138.09***
MSQGARCH	14	-9855.98	19739.95	1	19837.35	1			5.92
Panel D : λ free									
NLGARCH	9	-10054.21	20126.42	17	20209.04	18	187.54***		402.40***
APGARCH	10	-9960.44	19940.88	12	20010.45	11			214.86***
FGARCH	12	-9947.02	19918.05	7	20001.53	9			188.02***
MSNLGARCH	14	-9950.31	19928.62	10	20026.02	12	109.59***		194.60***
MSAPGARCH	15	-9895.52	19821.03	4	19925.39	4			85.01***
MSFGARCH	17	-9853.01	19900.03	2	19858.30	3			

Note: n denotes the number of parameters to be estimated and LL the maximized log-likelihood value of a given model. AIC and BIC are the Akaike information criterion and the Bayesian information criterion, respectively, computed as AIC $= -2\,(LL - n)$ and BIC $= -2\,LL + n\log(T)$ where $T = 7764$ is the sample size. The LRTs in the antepenultimate and penultimate columns are the likelihood test statistics for the following tests: H_0 : the symmetric specification corresponding to the model in the first column of the same row, versus H_A : the parameters γ or ψ are freely estimated. Under the null hypothesis, the LRT statistics are asymptotically χ^2-distributed with 1 degrees of freedom. The last column reports the LRT of the problem: H_0 : the specification corresponding to the model in the first column of the same row versus H_A : the most general model, denoted here MSFGARCH model. *,** and *** denote statistical significance at 1%, 5% and 10% levels, respectively.

Table 3.16: Model comparisons, long-term component switching, MU-VIX.

			AIC		BIC		LRT		
Model	n	LL	Value	Rank	Value	Rank	$H_A : \gamma$ free	$H_A : \psi$ free	H_A : MSFGARCH
Panel A : $\lambda = 0$									
EGARCH	9	-9975.83	19969.66	12	20032.28	9			217.27***
MSEGARCH	14	-9872.80	19773.59	3	19870.99	2			11.20**
Panel B : $\lambda = 1$									
AVGARCH	8	-10061.33	20138.66	18	20194.31	17	161.04***		388.27***
TGARCH	9	-9980.81	19979.62	14	20042.23	12			227.23***
MSAVGARCH	13	-9956.88	19939.76	9	20030.20	8	108.94***		179.37***
MSTGARCH	14	-9909.67	19847.35	5	19944.75	5			84.96***
Panel C : $\lambda = 2$									
GARCH	8	-10059.71	20135.41	16	20191.07	16	102.04***	175.24***	385.02***
GJRGARCH	9	-10008.69	20035.37	15	20097.99	15			282.98***
QGARCH	9	-9972.09	19962.17	10	20024.79	7			209.78***
MSGARCH	13	-9960.21	19946.41	8	20036.85	11	71.33***	184.62***	186.02***
MSGJRGARCH	14	-9924.54	19877.08	6	19974.48	6			114.69***
MSQGARCH	14	-9867.89	19763.79	1	19861.19	1			1.40
Panel D : λ free									
NLGARCH	9	-10058.71	20135.43	17	20198.05	18	162.28***		383.04***
APGARCH	10	-9977.58	19975.15	13	20044.72	13			220.76***
FGARCH	12	-9971.89	19967.77	11	20051.26	14			209.38***
MSNLGARCH	14	-9953.86	19935.71	7	20033.11	10	108.34***		173.32***
MSAPGARCH	15	-9899.68	19829.37	4	19933.73	4			64.98***
MSFGARCH	17	-9867.19	19768.39	2	19886.66	3			

Note: As in Table 3.15.

Table 3.17: Estimated MSGARCH-MIDAS models, long-term component switching, FU-VIX

	MSEGARCH	MSAVGARCH	MSTGARCH	MSGARCH	MSGJRGARCH	MSQGARCH	MSNLGARCH	MSAPGARCH	MSFGARCH
λ	0	1	1	2	2	2	1.4270***	0.5313**	1.8390***
							(0.4165)	(0.2175)	(0.4520)
$\hat{\lambda}$	1	1	1	2	2	2			1.1836
									(1.0777)
μ	0.0320***	0.0499***	0.0368***	0.0460*	0.0410***	0.0301***	0.0475***	0.0349***	0.0301***
	(0.0079)	(0.0104)	(0.0130)	(0.0246)	(0.0092)	(0.0076)	(0.0091)	(0.0000)	(0.0083)
α	-0.0606	0.0155	0.0500***	0.0010	0.0000	0.0000	0.0144***	0.0481***	0.0000
	(0.0628)	(0.0191)	(0.0057)	(0.0815)	(0.0000)	(0.0000)	(0.0050)	(0.0070)	(0.0000)
β	0.8331***	0.9786***	0.8606***	0.9709***	0.8599***	0.8472***	0.9820***	0.8812***	0.8447***
	0.0611	0.0367	0.0238	0.1648	0.0457	0.0434	0.0049	0.0766	0.0492
γ	-0.1871***		0.9999***		0.0802***			0.9999***	-0.3350
	(0.0338)		(0.1604)		(0.0177)			(0.0004)	(0.3410)
ψ						0.1746***			0.1697***
						(0.0233)			(0.0558)
m_1	-3.0852***	-2.8066***	-2.8942***	-3.5213***	-2.9637***	-2.9385***	-2.8506***	-2.8935***	-2.9753***
	(0.1217)	(0.4205)	(0.2221)	(0.5286)	(0.2219)	(0.1568)	(0.3979)	(0.3035)	(0.1810)
m_2	-2.1713***	-1.4869***	-1.7373***	-1.5397	-1.7647***	-1.9953***	-1.5217***	-1.8020***	-2.0666***
	(0.7024)	(0.4229)	(0.5487)	(2.1479)	(0.6772)	(0.3449)	(0.4116)	(0.5797)	(0.3963)
θ_1^{FU}	0.7630	-0.4119	0.4661	1.0582***	0.2069*	0.6459**	-0.7015	0.7316	0.7182**
	(0.4647)	(2.7802)	(0.3600)	(0.3724)	(0.1229)	(0.2537)	(0.6500)	(0.9187)	(0.3693)
θ_2^{FU}	1.7380***	1.1237	1.6911***	-0.4506	1.6763***	1.7150***	0.9900**	2.0092	1.9194***
	(0.6192)	(1.2092)	(0.5157)	(1.1373)	(0.5351)	(0.4499)	(0.4870)	(1.6820)	(0.5590)
θ_1^{VIX}	0.0936***	0.1406	0.1059***	0.1012***	0.1184***	0.0967***	0.1566***	0.0938**	0.0949***
	(0.0286)	(0.1139)	(0.0129)	(0.0207)	(0.0098)	(0.0103)	(0.0234)	(0.0452)	(0.0143)
θ_2^{VIX}	0.0429***	0.0478*	0.0300***	0.1111	0.0304**	0.0390***	0.0515***	0.0256	0.0356***
	(0.0149)	(0.0253)	(0.0090)	(0.1107)	(0.0137)	(0.0078)	(0.0155)	(0.0442)	(0.0094)
w^{FU}	422.31	518.12	511.59	1.0098	512.26	516.15	500.08	227.49***	499.11
	(787.4272)	(935.1448)	(345.5859)	(2.6951)	(501.9760)	(612.0390)	(316.6408)	(49.3430)	(915.58)
w^{VIX}	2.3357**	6.1138	3.9395***	5.0749**	4.7698***	2.6556***	6.4044***	3.2920**	2.2961***
	(0.9787)	(4.6213)	(0.9582)	(2.2368)	(1.3613)	(0.8524)	(1.9288)	(1.5603	(0.7102)
p_{11}	0.9807	0.9751	0.9881	0.9809	0.9814	0.9848	0.9813	0.9824	0.9792
p_{22}	0.9627	0.9345	0.9698	0.9694	0.9519	0.9695	0.9644	0.9335	0.9475
$\theta_1^{FU} = \theta_2^{FU}$	[0.0126]	[0.3706]	[0.0329]	[0.1777]	[0.0044]	[0.0232]	[0.0068]	[0.1609]	[0.0186]
$\theta_1^{VIX} = \theta_2^{VIX}$	[0.2279]	[0.3124]	[0.0000]	[0.9177]	[0.0000]	[0.0000]	[0.0000]	[0.0000]	[0.0001]

Note: This table shows parameter estimates for the two-regime model with two explanatory variables in the long-term volatility component: the Financial Uncertainty (FU) index of Ludvigson et al. (2020) and the Chicago Board Options Exchange Volatility Index (VIX). Standard errors are displayed in parenthesis; p-values are displayed as [·]. *,** and *** denote statistical significance at 1%, 5% and 10% levels, respectively.

Table 3.18: Estimated MSGARCH-MIDAS models, long-term component switching, MU-VIX

	MSEGARCH	MSAVGARCH	MSTGARCH	MSGARCH	MSGJRGARCH	MSQGARCH	MSNLGARCH	MSAPGARCH	MSFGARCH
λ	0	1	1	2	2	2	1.4801***	0.6026***	2.2779***
							(0.5081)	(0.1101)	(0.7182)
$\hat{\lambda}$	1	1	1	2	2	2			5.9962*
									(3.1598)
μ	0.0290***	0.0473***	0.0343**	0.0453***	0.0381***	0.0278***	0.0431***	0.0323***	0.0278***
	(0.0084)	(0.0111)	(0.0173)	(0.0109)	(0.0086)	(0.0088)	(0.0125)	(0.0000)	(0.0093)
α	-0.0295	0.0204	0.0479*	0.0097	0.0000	0.0013	0.0202**	0.0472***	0.0000
	(0.0294)	(0.0158)	(0.0263)	(0.0097)	(0.0000)	(0.0119)	(0.0083)	(0.0056)	(0.0001)
β	0.8379***	0.9709***	0.8716***	0.9874***	0.8781***	0.8362***	0.9631***	0.8932***	0.8225***
	(0.0634)	(0.0181)	(0.3318)	(0.0202)	(0.0393)	(0.0524)	(0.0200)	(0.0230)	(0.1088)
γ	-0.1781***		0.9999***		0.0772***			0.9999***	-0.6300
	(0.0294)		(0.0693)		(0.0147)			(0.0002)	(2.1780)
ψ						0.1781***			0.2096**
						(0.0269)			(0.0982)
m_1	-2.7504***	-2.5103***	-2.6316***	-2.3469	-2.7085***	-2.6026***	-2.6314***	-2.5064***	-2.6099***
	(0.2951	0.6801)	(0.1448)	(2.1956)	(0.2474)	(0.2693)	(0.4636)	(0.2570)	(0.3201)
m_2	-1.4689***	-0.9082	-0.9743	-0.4180	-0.8867	-1.2850***	0.4438	-1.1105*	-1.3158***
	(0.3630)	(0.9128)	(3.8969)	(2.0677)	(0.6600)	(0.3759)	(2.5264)	(0.6164)	(0.3983)
θ_1^{MU}	0.1466	-0.3938	0.2670	-1.7191	0.0726	0.0760	-0.2588	0.3224	0.0854
	(0.4934)	(0.3581)	(2.7082)	(4.2411)	(0.3499)	(0.3340)	(0.7726)	(0.3261)	(0.6377)
θ_2^{MU}	1.4673*	1.1302	1.7085	-0.6084	1.5195	1.5663*	-1.1269	2.4463*	1.4598*
	(0.8847)	(2.3869)	(6.8837)	(3.7298)	(0.9388)	(0.8679)	(3.0169)	(1.2815)	(0.7720)
θ_1^{VIX}	0.1116***	0.1233***	0.1081	0.1520***	0.1157***	0.1098***	0.1253***	0.1007***	0.1101***
	(0.0073)	(0.0319)	(0.0686)	(0.0209)	(0.0105)	(0.0066)	(0.0105)	(0.0084)	(0.0091)
θ_2^{VIX}	0.0455***	0.0490	0.0287	0.0585***	0.0294***	0.0410***	0.0550***	0.0215*	0.0441***
	(0.0143)	(0.0476)	(0.1781)	(0.0063)	(0.0066)	(0.0140)	(0.0139)	(0.0113)	(0.0122)
w^{MU}	398.74	578.98	490.54	2.5299	409.97	501.22	2.2493	260.41	387.79
	(594.1362)	(628.4954)	(720.2537)	(2.9189)	(348.2923)	(563.7854)	(2.4616)	(164.6635)	(994.70)
w^{VIX}	2.2878***	5.2370	3.5526	5.9009***	4.2409***	2.1970***	5.2319***	3.0059***	2.2019**
	(0.8385)	(4.0233)	(4.1079)	(1.5888)	(1.0170)	(0.7602)	(1.7969)	(0.7363)	(1.0040)
p_{11}	0.9864	0.9868	0.9926	0.9836	0.9894	0.9855	0.9933	0.9906	0.9875
p_{22}	0.9627	0.9307	0.9652	0.9606	0.9436	0.9454	0.9692	0.9265	0.9611
$\theta_1^{\text{MU}} = \theta_2^{\text{MU}}$	[0.2726]	[0.4603]	[0.7333]	[0.3567]	[0.1737]	[0.1789]	[0.7799]	[0.1269]	[0.2407]
$\theta_1^{\text{VIX}} = \theta_2^{\text{VIX}}$	[0.0000]	[0.0003]	[0.4698]	[0.0000]	[0.0000]	[0.0000]	[0.0000]	[0.0000]	[0.0000]

Note: This table shows parameter estimates for the two-regime model with two explanatory variables in the long-term volatility component: the Macroeconomic Uncertainty (MU) index of Jurado et al. (2015) and the Chicago Board Options Exchange Volatility Index (VIX). Standard errors are displayed in parenthesis; p-values are displayed as [·]. *,** and *** denote statistical significance at 1%, 5% and 10% levels, respectively.

Table 3.19: Model comparisons, long-term component switching, FU-MU-VIX.

			AIC		BIC		LRT		
Model	n	LL	Value	Rank	Value	Rank	$H_A: \gamma$ free	$H_A: \psi$ free	H_A : MSFGARCH
Panel A : $\lambda = 0$									
EGARCH	11	-9953.92	19929.84	9	20006.37	8			203.82***
MSEGARCH	17	-9854.51	19743.02	2	19861.29	2			5.00
Panel B : $\lambda = 1$									
AVGARCH	10	-10056.57	20133.14	18	20202.71	17	184.49***		409.12***
TGARCH	11	-9964.33	19950.65	13	20027.18	10			224.63***
MSAVGARCH	16	-9952.25	19936.51	11	20047.82	12	104.02***		200.49***
MSTGARCH	17	-9900.24	19834.49	5	19952.76	5			96.47***
Panel C : $\lambda = 2$									
GARCH	10	-10053.10	20126.20	16	20195.77	16	118.17***	203.33***	402.18***
GJRGARCH	11	-9994.01	20010.03	15	20086.56	15			284.01***
QGARCH	11	-9951.44	19924.87	8	20001.40	7			198.85***
MSGARCH	16	-9968.01	19968.01	14	20079.33	14	98.84***	227.81***	231.99***
MSGJRGARCH	17	-9918.59	19871.17	6	19989.45	6			133.15***
MSQGARCH	17	-9854.10	19742.20	1	19860.48	1			4.18
Panel D : λ free									
NLGARCH	11	-10054.22	20130.45	17	20206.98	18	187.70***		404.43***
APGARCH	12	-9960.37	19944.75	12	20028.24	11			216.73***
FGARCH	14	-9946.97	19921.93	7	20019.33	9			189.91***
MSNLGARCH	17	-9949.05	19932.10	10	20050.38	13	110.95***		194.08***
MSAPGARCH	18	-9893.58	19823.15	4	19948.38	4			83.13***
MSFGARCH	20	-9852.01	19744.02	3	19883.17	3			

Note: As in Table 3.15.

Table 3.20: Estimated MSGARCH-MIDAS models, long-term component switching, FU-MU-VIX

	MSEGARCH	MSAVGARCH	MSTGARCH	MSGARCH	MSGJRGARCH	MSQGARCH	MSNLGARCH	MSAPGARCH	MSFGARCH
λ	0	1	1	2	2	2	1.4875*	0.5570	2.5122***
							(0.8603)	(1.1793)	(0.2957)
$\hat{\lambda}$	1	1	1	2	2	2			0.0642
									(0.1323)
μ	0.0321***	0.0475***	0.0366***	0.0474***	0.0415***	0.0303***	0.0492***	0.0343***	0.0310***
	(0.0081)	(0.0092)	(0.0101)	(0.0178)	(0.0091)	(0.0081)	(0.0110)	(0.0000)	(0.0000)
α	-0.0723*	0.0131***	0.0495	0.0019	0.0199***	0.0000	0.0128	0.0495	0.0159
	(0.0374)	(0.0046)	(0.0061)	(0.0288)	(0.0035)	(0.0000)	(0.0093)	(0.0097)	(0.0326)
β	0.8204***	0.9840***	0.8814***	0.9705***	0.8774***	0.8427***	0.9843***	0.8877***	0.8012***
	(0.0481)	(0.0078)	(0.0526)	(0.0537)	(0.0453)	(0.0417)	(0.0171)	(0.2412)	(0.0566)
γ	-0.1957***		0.9999***		0.9977***			0.9999***	0.9954***
	(0.0242)		(0.2924)		(0.0016)			(0.0027)	(0.0211)
ψ						0.1773***			0.2301***
						(0.0245)			(0.0289)
m_1	-3.0104***	-2.3915***	-2.7370***	-3.0512*	-2.8901***	-2.9629***	-2.5613***	-2.7707	-2.8137***
	(0.1975)	(0.7581)	(0.3437)	(1.7905)	(0.2454)	(0.1953)	(0.9835)	(2.6192)	(0.1826)
m_2	-2.3459***	-0.9952*	-1.7767	-1.4486	-1.7546**	-1.7868***	-1.1438	-2.0303	-2.2342***
	(0.5096)	(0.5741)	(1.1614)	(5.4213)	(0.7791)	(0.5767)	(0.7475)	(2.3425)	(0.7737)
θ_1^{FU}	0.9623**	-0.4978	0.8098	1.4922	0.4903	0.6720**	-0.2295	0.8314	0.7924
	(0.4272)	(0.5597)	(0.7293)	(5.7583)	(0.5712)	(0.3202)	(0.8813)	(3.1491)	(0.4840)
θ_2^{FU}	1.8648***	1.2419**	1.3903	-0.3252	1.5700*	1.8945***	1.4735**	1.6718**	1.8807***
	(0.4036)	(0.5037)	(1.3691)	(8.0829)	(0.8829)	(0.4694)	(0.6100)	(0.8276)	(0.5753)
θ_1^{MU}	-0.3566	-0.8612	-0.2027	-1.3533	-0.2403	-0.0321	-0.9332	-0.1003	-0.3525
	(0.3729)	(1.7074)	(0.5708)	(13.7050)	(0.4870)	(0.0233)	(2.4388)	(3.8928)	(0.4059)
θ_2^{MU}	0.2082	-1.1314	1.4556	0.0187	0.5519	-0.5869	-1.1020	1.2952	0.2261
	(0.7311)	(0.9640)	(1.8975)	(0.7925)	(0.9738)	(1.3211)	(1.3232)	(9.2975)	(1.7181)
θ_1^{VIX}	0.0918***	0.1535***	0.0907***	0.1028	0.1100***	0.0975***	0.1491***	0.0879	0.0946***
	(0.0132)	(0.0322)	(0.0246)	(0.0780)	(0.0169)	(0.0106)	(0.0556)	(0.1189)	(0.0142)
θ_2^{VIX}	0.0420***	0.0521***	0.0152	0.1038*	0.0228*	0.0392***	0.0462***	0.0174	0.0401***
	(0.0118)	(0.0122)	(0.0176)	(0.0530)	(0.0139)	(0.0093)	(0.0100)	(0.1833)	(0.0109)
w^{FU}	438.530	504.097	480.575**	1.0050	505.280	508.042	513.987	609.053	167.331***
	(520.952)	(770.550)	(225.862)	(17.610)	(610.918)	(823.388)	(723.200)	(791.526)	(33.137)
w^{MU}	181.054	5.3070	510.704	1.0238	496.703	4.8182	5.5978	350.856	65.953
	(492.997)	(5.4832)	(800.524)	(32.158)	(995.619)	(4.8921)	(8.732)	(507.948)	(320.668)
w^{VIX}	1.9793***	6.4710***	3.4679***	5.1561***	4.5793***	2.5741***	6.1542***	3.1191	2.1474***
	(0.7190)	(2.0297)	(0.9374)	(2.0002)	(1.1520)	(0.8940)	(1.8133)	(4.3413)	(0.7103)
p_{11}	0.9759	0.9796	0.9892	0.9818	0.9805	0.9823	0.9750	0.9862	0.9800
p_{22}	0.9422	0.9591	0.9353	0.9611	0.9263	0.9624	0.9414	0.9273	0.9487
$\text{LRT}_{H_0:\text{FU-VIX}}$	[0.3765]	[0.1641]	[0.0040]	[0.4249]	[0.0740]	[0.2897]	[0.4711]	[0.2758]	[0.5712]
$\text{LRT}_{H_0:\text{MU-VIX}}$	[0.0000]	[0.0260]	[0.0003]	[0.9655]	[0.0077]	[0.0000]	[0.0222]	[0.0067]	[0.0000]

Note: The explanatory variables in the long-term component are the Financial Uncertainty (FU) index, the Macroeconomic Uncertainty (MU) index and the Chicago Board Options Exchange Volatility Index (VIX). Standard errors are displayed in parenthesis; p-values are displayed as [·]. *,** and *** denote statistical significance at 1%, 5% and 10% levels, respectively.

Table 3.21: QLIKE losses and MCS: full out-of-sample period, FU.

	1d	2w	1m	2m	3m
EGARCH	0.2881	0.2410	0.2663	0.4229	0.5144
AVGARCH	0.3392	0.2620	0.2809	0.4265	0.5091
TGARCH	0.2777	0.2299	0.2630	0.4226	0.5074
GARCH	0.3374	0.2631	0.2846	0.4333	0.5168
GJRGARCH	0.2977	0.2422	0.2711	0.4298	0.5170
QGARCH	0.3104	0.2542	0.2835	0.4313	0.5126
NLGARCH	0.3371	0.2626	0.2840	0.4316	0.5143
APGARCH	0.2787	0.2301	0.2629	0.4211	*0.5054*
FGARCH	0.2898	0.2419	0.2712	*0.4156*	**0.4985**
MSEGARCH	0.2661	0.2184	*0.2458*	0.4212	0.5224
MSAVGARCH	0.3044	0.2475	0.2862	0.4661	0.5746
MSTGARCH	0.2643	0.2270	0.2735	0.4498	0.5392
MSGARCH	0.3110	0.2530	0.2906	0.4784	0.5858
MSGJRGARCH	0.2931	0.2484	0.2863	0.4476	0.5140
MSQGARCH	0.2709	0.2283	0.2669	0.4391	0.5238
MSNLGARCH	0.3146	0.2454	0.2735	0.4658	0.5855
MSAPGARCH	*0.2601*	*0.2175*	0.2572	0.4237	0.5108
MSFGARCH	**0.2519**	**0.2057**	**0.2381**	**0.4146**	0.5111

Note: This table presents forecasts performance at five horizons: 1-day (1d), 2-weeks (2w), 1-month (1m), 2-moths (2m) and 3-months (3m). The table reports the average QLIKE losses in the full out-of-sample period from 2007:M1 to 2020:M11. The explanatory variable in the long-term component is the Financial Uncertainty (FU) index of Ludvigson et al. (2020). The models with the lowest average losses are highlighted in bold; the second lowest average losses are in italics. The table also presents the results from the 90% MCS for each horizon. The models included in the final set are shaded in blue.

Table 3.22: QLIKE losses and MCS: full out-of-sample period, MU.

	1d	2w	1m	2m	3m
EGARCH	0.2966	0.2794	0.3400	0.6309	0.7277
AVGARCH	0.3519	0.2900	0.3547	0.6296	0.7303
TGARCH	0.2837	0.2594	0.3440	0.6158	0.7134
GARCH	0.3522	0.2932	0.3628	0.6791	0.8015
GJRGARCH	0.3078	0.2645	0.3228	0.5918	0.7104
QGARCH	0.3612	0.2859	0.3622	0.5926	0.6642
NLGARCH	0.3485	0.2894	0.3582	0.6615	0.7762
APGARCH	0.2877	0.2649	0.3478	0.5981	0.6860
FGARCH	0.2964	0.2759	0.3604	0.5779	**0.6294**
MSEGARCH	0.2797	0.2599	0.3173	0.5635	0.6789
MSAVGARCH	0.3089	0.2646	0.3270	0.5917	0.7581
MSTGARCH	0.2675	*0.2377*	0.3108	0.5587	0.6466
MSGARCH	0.3157	0.2709	0.3382	0.6246	0.7943
MSGJRGARCH	0.2969	0.2634	0.3362	0.6112	0.7322
MSQGARCH	0.2774	0.2530	0.3180	**0.5516**	*0.6317*
MSNLGARCH	0.3125	0.2687	0.3313	0.5959	0.7218
MSAPGARCH	**0.2640**	**0.2334**	*0.3090*	*0.5555*	0.6458
MSFGARCH	*0.2655*	0.2442	**0.3058**	0.5703	0.6413

Note: As in Table 3.21 except that the explanatory variable in the long-term component is the Macroeconomic Uncertainty (MU) index of Jurado et al. (2015).

Table 3.23: QLIKE losses and MCS: full out-of-sample period, VIX.

	1d	2w	1m	2m	3m
EGARCH	0.2657	0.2574	0.3729	0.7785	0.9397
AVGARCH	0.2805	0.2482	0.3525	0.7598	0.9289
TGARCH	0.2585	0.2435	0.3531	0.7428	0.8972
GARCH	0.2833	0.2515	0.3583	0.7786	0.9545
GJRGARCH	0.2706	0.2472	0.3562	0.7834	0.9585
QGARCH	0.2714	0.2586	0.3668	0.7494	0.9022
NLGARCH	0.2804	0.2487	0.3554	0.7709	0.9429
APGARCH	0.2590	0.2457	0.3563	0.7297	0.8757
FGARCH	0.2767	0.2585	0.3662	0.7346	0.8837
MSEGARCH	0.2491	0.2424	0.3567	0.7667	0.9428
MSAVGARCH	0.2609	0.2541	0.3684	0.7793	0.9639
MSTGARCH	0.2527	0.2487	0.3583	*0.7122*	*0.8577*
MSGARCH	0.2644	0.2601	0.3751	0.7889	0.9753
MSGJRGARCH	0.2606	0.2512	0.3571	0.7248	0.8740
MSQGARCH	0.2537	0.2493	0.3580	0.7221	0.8688
MSNLGARCH	0.2569	0.2506	0.3651	0.7921	0.9972
MSAPGARCH	*0.2445*	**0.2414**	**0.3483**	**0.7008**	**0.8471**
MSFGARCH	**0.2423**	*0.2424*	*0.3509*	0.7301	0.8838

Note: As in Table 3.21 except that the explanatory variable in the long-term component is the Chicago Board Options Exchange Volatility Index (VIX).

Table 3.24: QLIKE losses and MCS: full out-of-sample period, FU-VIX .

	1d	2w	1m	2m	3m
EGARCH	0.2666	0.2419	0.3209	0.6326	0.7805
AVGARCH	0.2861	0.2533	0.3642	0.7936	0.9725
TGARCH	0.2570	0.2283	0.3082	0.6122	0.7516
GARCH	0.2878	0.2545	0.3650	0.7967	0.9778
GJRGARCH	0.2697	0.2395	0.3393	0.7269	0.8896
QGARCH	0.2713	0.2430	0.3195	0.6158	0.7543
NLGARCH	0.2859	0.2528	0.3639	0.7961	0.9765
APGARCH	0.2580	0.2295	0.3074	0.5949	0.7279
FGARCH	0.2712	0.2442	0.3171	0.6012	0.7371
MSEGARCH	0.2526	*0.2260*	*0.2999*	0.6095	0.7641
MSAVGARCH	0.2529	0.2393	0.3560	0.7753	0.9492
MSTGARCH	0.2516	0.2353	0.3131	0.5899	*0.6926*
MSGARCH	0.2672	0.2534	0.3714	0.8231	1.0150
MSGJRGARCH	0.2675	0.2436	0.3256	0.6369	0.7525
MSQGARCH	0.2620	0.2419	0.3122	*0.5828*	**0.6903**
MSNLGARCH	0.2624	0.2449	0.3453	0.7149	0.8654
MSAPGARCH	**0.2471**	0.2272	0.3093	0.5958	0.7210
MSFGARCH	*0.2508*	**0.2258**	**0.2931**	**0.5664**	0.6997

Note: As in Table 3.21 except that the explanatory variables in the long-term component are the Financial Uncertainty (FU) index of Ludvigson et al. (2020) and the Chicago Board Options Exchange Volatility Index (VIX).

Table 3.25: QLIKE losses and MCS: subsamples of low/normal/high levels of realised volatility, FU.

	Low-volatility regime					Normal-volatility regime					High-volatility regime				
	1d	2w	1m	2m	3m	1d	2w	1m	2m	3m	1d	2w	1m	2m	3m
EGARCH	0.3534	0.2635	0.2942	0.4400	0.5671	0.2648	0.2451	0.2775	0.4501	0.5053	0.2393	0.2027	0.2180	0.3610	*0.4672*
AVGARCH	0.4127	0.2799	0.2992	0.4342	**0.5288**	0.2914	0.2518	0.2844	0.4444	*0.4937*	0.3409	0.2629	0.2631	0.3931	0.5145
TGARCH	0.3346	0.2462	0.2872	0.4394	0.5499	0.2559	0.2351	0.2771	0.4497	0.4993	0.2391	0.1987	0.2144	0.3610	0.4711
GARCH	0.4338	0.2926	0.3060	0.4460	0.5593	0.2945	0.2546	0.2930	0.4577	0.5022	0.3048	0.2482	0.2539	0.3825	0.4927
GJRGARCH	0.3815	0.2727	0.2997	0.4494	0.5780	0.2754	0.2468	0.2893	0.4613	0.5069	0.2362	0.1990	0.2102	*0.3569*	**0.4618**
QGARCH	0.3857	0.2962	0.3263	0.4545	0.5618	0.2726	0.2528	0.2926	0.4563	0.5027	0.2476	0.2082	0.2244	0.3659	0.4714
NLGARCH	0.4318	0.2911	0.3050	0.4442	0.5578	0.2939	0.2540	0.2921	0.4552	0.4987	0.3071	0.2490	0.2542	0.3823	0.4909
APGARCH	0.3317	0.2457	0.2883	0.4356	0.5392	0.2551	0.2351	0.2763	0.4479	0.4988	0.2431	0.1983	0.2138	0.3627	0.4768
FGARCH	0.3561	0.2693	0.3025	*0.4280*	*0.5302*	0.2634	0.2445	0.2820	*0.4407*	**0.4909**	0.2439	0.2031	0.2203	0.3630	0.4744
MSEGARCH	0.3342	0.2405	0.2683	**0.4279**	0.5587	0.2414	0.2230	*0.2610*	0.4545	0.5188	*0.2244*	*0.1845*	*0.1980*	0.3600	0.4850
MSAVGARCH	0.3455	0.2368	0.2737	0.4738	0.6468	0.2716	0.2465	0.3030	0.4693	0.5194	0.3163	0.2615	0.2778	0.4618	0.5923
MSTGARCH	0.3112	0.2395	0.2977	0.4743	0.5864	0.2492	0.2337	0.2922	0.4759	0.5316	0.2339	0.1958	0.2142	0.3800	0.4962
MSGARCH	0.3624	0.2486	0.2840	0.5097	0.6871	0.2792	0.2514	0.3070	0.4773	0.5281	0.3106	0.2618	0.2757	0.4540	0.5724
MSGJRGARCH	0.3460	0.2595	0.2974	0.4476	0.5402	0.2718	0.2542	0.3008	0.4580	0.4943	0.2695	0.2205	0.2475	0.4342	0.5198
MSQGARCH	0.3207	0.2549	0.2999	0.4551	0.5496	0.2519	0.2326	0.2808	0.4680	0.5182	0.2264	0.1904	0.2089	0.3745	0.5029
MSNLGARCH	0.3346	*0.2251*	**0.2544**	0.4738	0.6385	0.2542	0.2265	0.2769	0.4731	0.5482	0.4073	0.3070	0.2989	0.4539	0.5921
MSAPGARCH	*0.3088*	0.2272	0.2769	0.4318	0.5303	*0.2407*	*0.2217*	0.2716	0.4457	0.5020	0.2330	0.1931	0.2114	0.3816	0.5038
MSFGARCH	**0.3072**	**0.2210**	*0.2619*	0.4369	0.5563	**0.2292**	**0.2100**	**0.2487**	**0.4364**	0.5039	**0.2161**	**0.1805**	**0.1965**	**0.3554**	0.4694

Note: The table presents average QLIKE losses at five horizons: 1-day (1d), 2-weeks (2w), 1-month (1m), 2-moths (2m) and 3-months (3m) across three different volatility regimes. A forecast falls into the low/normal/high-volatility regime when the level of the realised variance on the day the forecast is obtained is below the 25% quantile, between the 25% and 75% quantile, or above the 75% quantile of the full sample realised variances. The explanatory variable in the long-term component is the Financial Uncertainty (FU) index of Ludvigson et al. (2020). The out-of-sample period ranges from 2007:M1 to 2020:M11. The models with the lowest average losses are highlighted in bold; the second lowest average losses are in italics. The table also presents the results from the 90% MCS for each horizon across regimes. The models included in the final set are shaded in blue.

Table 3.26: QLIKE losses and MCS: subsamples of low/normal/high levels of realised volatility, MU.

	Low-volatility regime					Normal-volatility regime					High-volatility regime				
	1d	2w	1m	2m	3m	1d	2w	1m	2m	3m	1d	2w	1m	2m	3m
EGARCH	0.3399	0.3028	0.3763	0.6943	0.8371	0.2822	0.2871	0.3526	0.6583	0.6852	0.2543	0.2341	0.2766	0.5117	0.6755
AVGARCH	0.3773	0.2557	0.3128	0.6802	0.9505	0.2977	0.2746	0.3774	0.6709	0.6492	0.4223	0.3645	0.3755	0.5049	0.6165
TGARCH	0.3208	0.2631	0.3497	0.6534	0.8888	0.2679	0.2713	0.3801	0.6718	0.6667	0.2553	0.2306	0.2717	0.4693	0.5885
GARCH	0.4004	0.2753	0.3326	0.7521	1.0562	0.3146	0.2902	0.3979	0.7093	0.6985	0.3659	0.3269	0.3470	0.5494	0.6873
GJRGARCH	0.3738	0.2866	0.3441	0.6432	0.8904	0.2906	0.2704	0.3477	0.6296	0.6597	0.2537	0.2290	**0.2551**	0.4651	0.5871
QGARCH	0.4871	0.3194	0.3949	0.6595	0.8844	0.2922	0.2886	0.3932	0.6429	0.6108	0.2664	0.2379	0.2694	**0.4255**	**0.4972**
NLGARCH	0.3896	0.2668	0.3248	0.7278	1.0190	0.3072	0.2835	0.3902	0.6947	0.6800	0.3766	0.3325	0.3514	0.5337	0.6637
APGARCH	0.3239	0.2687	0.3569	0.6515	0.8717	0.2709	0.2762	0.3853	0.6514	0.6327	0.2599	0.2360	*0.2688*	*0.4370*	*0.5612*
FGARCH	0.3368	0.2879	0.3739	0.6090	0.7593	0.2759	0.2847	0.3979	0.6171	**0.5758**	0.2678	0.2402	0.2772	0.4741	0.5734
MSEGARCH	0.2942	0.2541	0.3098	0.5465	0.7768	0.2774	0.2743	0.3344	0.5877	0.6312	0.2558	0.2387	0.2932	0.5393	0.6507
MSAVGARCH	0.3278	0.2370	**0.2911**	**0.5197**	*0.7049*	0.2778	0.2560	0.3315	0.6053	0.6986	0.3433	0.3156	0.3717	0.6650	0.9373
MSTGARCH	*0.2818*	*0.2266*	0.3119	0.5584	0.7246	0.2663	0.2516	0.3304	*0.5787*	0.6140	0.2463	*0.2235*	0.2756	0.5259	0.6138
MSGARCH	0.3435	0.2488	0.3107	0.5853	0.8085	0.2895	0.2655	0.3465	0.6356	0.7467	0.3314	0.3109	0.3671	0.6649	0.8682
MSGJRGARCH	0.3342	0.2712	0.3482	0.6330	0.8622	0.2799	0.2631	0.3514	0.6427	0.6851	0.2828	0.2574	0.3032	0.5374	0.6634
MSQGARCH	0.3081	0.2602	0.3308	0.5477	0.7075	0.2710	0.2591	0.3304	**0.5753**	*0.6059*	0.2435	0.2307	0.2765	0.5098	0.5883
MSNLGARCH	0.3328	0.2401	*0.2969*	*0.5260*	0.7143	0.2796	0.2587	0.3356	0.6124	0.6755	0.3496	0.3235	0.3748	0.6620	0.8197
MSAPGARCH	0.2819	**0.2208**	0.3033	0.5359	0.7225	**0.2608**	**0.2467**	*0.3300*	0.5878	0.6272	**0.2395**	**0.2200**	0.2760	0.5179	0.5875
MSFGARCH	**0.2810**	0.2446	0.2995	0.5853	**0.7034**	*0.2623*	*0.2505*	**0.3201**	0.5852	0.6101	*0.2412*	0.2281	0.2821	0.5225	0.6251

Note: As in Table 3.25 except that the explanatory variable in the long-term component is the Macroeconomic Uncertainty (MU) index of Jurado et al. (2015).

Table 3.27: QLIKE losses and MCS: subsamples of low/normal/high levels of realised volatility, VIX.

	Low-volatility regime					Normal-volatility regime					High-volatility regime				
	1d	2w	1m	2m	3m	1d	2w	1m	2m	3m	1d	2w	1m	2m	3m
EGARCH	0.3072	0.2482	0.3440	0.7900	1.1677	0.2240	0.2426	0.4042	0.8410	0.8466	0.2881	0.3016	0.3626	0.6651	0.8394
AVGARCH	0.3305	0.2284	**0.3164**	0.7967	1.1540	0.2274	0.2270	0.3738	0.7869	0.8110	0.3192	0.3188	0.3695	0.6819	0.8794
TGARCH	0.2985	0.2325	0.3282	0.7526	1.0930	0.2204	0.2320	0.3847	0.7993	0.8104	0.2772	*0.2823*	*0.3362*	**0.6416**	*0.8241*
GARCH	0.3425	0.2380	0.3244	0.8165	1.1969	0.2298	0.2294	0.3833	0.8166	0.8428	0.3136	0.3176	0.3666	0.6794	0.8720
GJRGARCH	0.3230	0.2393	0.3303	0.8104	1.2029	0.2278	0.2330	0.3910	0.8443	0.8633	0.2878	0.2903	**0.3354**	0.6540	0.8419
QGARCH	0.3118	0.2550	0.3414	0.7423	1.0834	0.2273	0.2395	0.3928	0.8062	0.8137	0.2962	0.3049	0.3621	0.6685	0.8503
NLGARCH	0.3365	0.2338	*0.3222*	0.8158	1.1928	0.2289	0.2291	0.3811	0.8025	0.8235	0.3105	0.3115	0.3613	0.6749	0.8660
APGARCH	0.2956	0.2330	0.3290	0.7280	1.0470	0.2192	0.2318	0.3848	0.7852	0.7934	0.2824	0.2900	0.3481	*0.6443*	**0.8238**
FGARCH	0.3271	0.2549	0.3396	0.7256	1.0588	0.2277	0.2387	0.3910	0.7856	0.7917	0.3004	0.3070	0.3655	0.6668	0.8459
MSEGARCH	0.2744	0.2269	0.3276	0.7889	1.1512	*0.2127*	0.2238	0.3758	0.8014	0.8355	0.2814	0.3008	0.3686	0.6926	0.8935
MSAVGARCH	*0.2706*	**0.2166**	0.3232	0.7907	1.0908	0.2167	*0.2233*	**0.3649**	0.7612	0.8286	0.3259	0.3611	0.4412	0.8181	1.0677
MSTGARCH	**0.2697**	0.2313	0.3356	0.7248	*0.9937*	0.2173	0.2286	0.3704	*0.7221*	*0.7512*	0.2944	0.3108	0.3743	0.6952	0.8955
MSGARCH	0.2776	0.2240	0.3299	0.8257	1.1549	0.2223	0.2272	0.3674	0.7496	0.8138	0.3226	0.3715	0.4575	0.8377	1.0653
MSGJRGARCH	0.2882	0.2374	0.3393	0.7518	1.0224	0.2231	0.2304	0.3679	0.7292	0.7631	0.2943	0.3121	0.3700	0.7011	0.9050
MSQGARCH	0.2739	0.2283	0.3255	**0.7059**	**0.9660**	0.2177	0.2268	0.3685	0.7374	0.7689	0.2924	0.3196	0.3880	0.7297	0.9415
MSNLGARCH	0.2731	*0.2211*	0.3281	0.8183	1.1725	0.2168	**0.2228**	*0.3661*	0.7824	0.8570	0.3067	0.3431	0.4199	0.7973	1.0514
MSAPGARCH	0.2715	0.2294	0.3295	*0.7139*	0.9945	0.2143	0.2266	0.3689	**0.7173**	**0.7403**	*0.2619*	0.2850	0.3425	0.6701	0.8713
MSFGARCH	0.2713	0.2379	0.3311	0.7379	1.0425	**0.2121**	0.2295	0.3738	0.7707	0.7943	**0.2570**	**0.2756**	0.3431	0.6608	0.8613

Note: As in Table 3.25 except that the explanatory variable in the long-term component is the Chicago Board Options Exchange Volatility Index (VIX).

Table 3.28: QLIKE losses and MCS: subsamples of low/normal/high levels of realised volatility, FU-VIX.

	Low-volatility regime					Normal-volatility regime					High-volatility regime				
	1d	2w	1m	2m	3m	1d	2w	1m	2m	3m	1d	2w	1m	2m	3m
EGARCH	0.3149	0.2440	0.3117	0.6624	0.9724	0.2287	0.2304	0.3405	0.6687	0.7008	0.2728	0.2659	0.3067	0.5432	0.6983
AVGARCH	0.3331	0.2359	0.3296	0.8753	1.2873	0.2336	0.2314	0.3895	0.8103	0.8266	0.3289	0.3230	0.3726	0.6818	0.8671
TGARCH	0.3001	0.2247	0.2964	0.6380	0.9190	0.2243	0.2205	0.3307	0.6422	0.6732	0.2610	0.2508	*0.2908*	*0.5390*	*0.6972*
GARCH	0.3430	0.2437	0.3347	0.8847	1.3118	0.2356	0.2328	0.3934	0.8190	0.8354	0.3212	0.3164	0.3627	0.6666	0.8421
GJRGARCH	0.3250	0.2395	0.3218	0.8046	1.1982	0.2348	0.2321	0.3791	0.7657	0.7759	0.2679	0.2595	0.2980	0.5764	0.7299
QGARCH	0.3228	0.2511	0.3127	0.6326	0.9142	0.2314	0.2290	0.3372	0.6482	0.6763	0.2735	0.2651	0.3063	0.5491	0.7082
NLGARCH	0.3373	0.2395	0.3324	0.8835	1.3064	0.2351	0.2327	0.3928	0.8172	0.8336	0.3215	0.3148	0.3620	0.6694	0.8469
APGARCH	0.2975	0.2249	0.2955	0.6058	0.8594	0.2227	0.2194	0.3257	0.6247	0.6576	0.2678	0.2562	0.2976	0.5396	0.7018
FGARCH	0.3221	0.2525	0.3116	0.6120	0.8789	0.2314	0.2314	0.3340	0.6332	0.6652	0.2759	0.2641	0.3036	0.5418	0.7016
MSEGARCH	0.2951	0.2249	**0.2879**	0.6437	0.9457	*0.2154*	**0.2164**	*0.3173*	0.6310	0.6792	0.2649	0.2509	0.2927	0.5422	0.7045
MSAVGARCH	0.2905	**0.2203**	0.3377	0.8955	1.2802	**0.2143**	0.2288	0.3828	0.7792	0.8093	0.2806	0.2883	0.3401	0.6400	0.8125
MSTGARCH	**0.2817**	0.2227	0.2950	*0.5947*	*0.8096*	0.2196	0.2256	0.3288	0.5908	**0.5958**	0.2736	0.2685	0.3117	0.5939	0.7351
MSGARCH	0.3131	0.2383	0.3470	0.9188	1.3196	0.2259	0.2380	0.3954	0.8370	0.8770	0.2906	0.3087	0.3695	0.7002	0.9067
MSGJRGARCH	0.3021	0.2312	0.3030	0.6724	0.9367	0.2311	0.2324	0.3450	0.6348	0.6381	0.2945	0.2825	0.3242	0.6114	0.7463
MSQGARCH	0.2885	0.2325	0.3002	0.6004	**0.7996**	0.2300	0.2324	0.3265	**0.5807**	*0.6028*	0.2874	0.2742	0.3099	0.5800	0.7245
MSNLGARCH	0.2995	0.2238	0.3226	0.8162	1.1681	0.2202	0.2261	0.3569	0.6899	0.7051	0.2974	0.3114	0.3624	0.6565	0.8025
MSAPGARCH	*0.2871*	*0.2217*	0.2988	0.6045	0.8343	0.2154	0.2195	0.3280	0.6167	0.6492	*0.2545*	*0.2489*	0.2955	0.5596	0.7200
MSFGARCH	0.2930	0.2333	*0.2925*	**0.5907**	0.8438	0.2158	*0.2166*	**0.3093**	*0.5872*	0.6248	**0.2508**	**0.2363**	**0.2732**	**0.5110**	**0.6670**

Note: As in Table 3.25 except that the explanatory variables in the long-term component are the Financial Uncertainty (FU) index of Ludvigson et al. (2020) and the Chicago Board Options Exchange Volatility Index (VIX).

Table 3.29: QLIKE losses and MCS: pre-COVID period, FU.

	1d	2w	1m	2m	3m
EGARCH	0.2815	0.2290	0.2384	0.3563	**0.4300**
AVGARCH	0.3348	0.2497	0.2524	0.3738	0.4476
TGARCH	0.2718	0.2182	0.2349	0.3645	0.4378
GARCH	0.3351	0.2519	0.2524	0.3659	0.4413
GJRGARCH	0.2937	0.2308	0.2397	0.3591	0.4349
QGARCH	0.2964	0.2431	0.2576	0.3749	0.4435
NLGARCH	0.3348	0.2513	0.2518	0.3647	0.4391
APGARCH	0.2715	0.2182	0.2358	0.3675	0.4405
FGARCH	0.2828	0.2308	0.2454	0.3646	0.4366
MSEGARCH	0.2621	0.2094	*0.2213*	0.3607	0.4444
MSAVGARCH	0.2975	0.2276	0.2361	0.3516	0.4369
MSTGARCH	0.2613	0.2149	0.2407	0.3872	0.4737
MSGARCH	0.3052	0.2337	0.2421	0.3539	0.4372
MSGJRGARCH	0.2891	0.2392	0.2612	0.3880	0.4523
MSQGARCH	0.2634	0.2184	0.2403	0.3853	0.4637
MSNLGARCH	0.3063	0.2260	0.2301	*0.3514*	0.4433
MSAPGARCH	*0.2567*	*0.2065*	0.2300	0.3687	0.4473
MSFGARCH	**0.2464**	**0.1972**	**0.2135**	**0.3471**	*0.4312*

Note: This table presents forecasts performance at five horizons: 1-day (1d), 2-weeks (2w), 1-month (1m), 2-moths (2m) and 3-months (3m). The table reports the average QLIKE losses in the full out-of-sample period from 2007:M1 to 2019:M12. The explanatory variable in the long-term component is the Financial Uncertainty (FU) index of Ludvigson et al. (2020). The models with the lowest average losses are highlighted in bold; the second lowest average losses are in italics. The table also presents the results from the 90% MCS for each horizon. The models included in the final set are shaded in blue.

Table 3.30: QLIKE losses and MCS: pre-COVID period, MU.

	1d	2w	1m	2m	3m
EGARCH	0.2824	0.2588	0.2834	0.4289	0.5267
AVGARCH	0.3345	0.2497	0.2538	**0.3822**	**0.4709**
TGARCH	0.2726	0.2296	0.2562	0.4213	0.5194
GARCH	0.3374	0.2602	0.2675	0.3975	0.4891
GJRGARCH	0.2978	0.2449	0.2645	0.4187	0.5079
QGARCH	0.3349	0.2513	0.2728	0.4081	*0.4769*
NLGARCH	0.3336	0.2548	0.2617	*0.3914*	0.4815
APGARCH	0.2740	0.2304	0.2532	0.3998	0.4888
FGARCH	0.2797	0.2370	0.2597	0.4063	0.4824
MSEGARCH	0.2633	0.2350	0.2594	0.4269	0.5055
MSAVGARCH	0.2901	0.2331	0.2650	0.4534	0.5952
MSTGARCH	0.2577	*0.2174*	0.2582	0.4341	0.5010
MSGARCH	0.2996	0.2416	0.2763	0.4726	0.6061
MSGJRGARCH	0.2876	0.2456	0.2886	0.4780	0.5746
MSQGARCH	0.2656	0.2305	0.2681	0.4514	0.5135
MSNLGARCH	0.2944	0.2374	0.2693	0.4583	0.5535
MSAPGARCH	*0.2537*	**0.2116**	*0.2527*	0.4314	0.5056
MSFGARCH	**0.2509**	0.2181	**0.2493**	0.4258	0.5044

Note: As in Table 3.29 except that the explanatory variable in the long-term component is the Macroeconomic Uncertainty (MU) index of Jurado et al. (2015).

Table 3.31: QLIKE losses and MCS: pre-COVID period, VIX.

	1d	2w	1m	2m	3m
EGARCH	0.2573	0.2276	0.2655	0.4776	0.5874
AVGARCH	0.2723	0.2224	0.2554	0.4634	0.5750
TGARCH	0.2509	0.2172	0.2550	0.4642	0.5718
GARCH	0.2766	0.2270	0.2597	0.4710	0.5868
GJRGARCH	0.2638	0.2226	0.2568	0.4740	0.5914
QGARCH	0.2618	0.2313	0.2677	0.4771	0.5838
NLGARCH	0.2735	0.2240	0.2566	0.4627	0.5743
APGARCH	0.2506	0.2182	0.2574	0.4646	0.5682
FGARCH	0.2679	0.2310	0.2670	0.4687	0.5723
MSEGARCH	0.2420	*0.2115*	**0.2519**	0.4782	0.6075
MSAVGARCH	0.2410	0.2121	0.2607	0.4881	0.6142
MSTGARCH	0.2457	0.2215	0.2650	*0.4563*	0.5627
MSGARCH	0.2446	0.2175	0.2642	0.4779	0.5955
MSGJRGARCH	0.2519	0.2242	0.2643	0.4569	*0.5620*
MSQGARCH	0.2471	0.2219	0.2664	0.4772	0.5882
MSNLGARCH	0.2374	**0.2091**	0.2565	0.4871	0.6267
MSAPGARCH	*0.2372*	0.2132	0.2542	**0.4495**	**0.5549**
MSFGARCH	**0.2348**	0.2133	*0.2521*	0.4661	0.5792

Note: As in Table 3.29 except that the explanatory variable in the long-term component is the Chicago Board Options Exchange Volatility Index (VIX).

Table 3.32: QLIKE losses and MCS: pre-COVID period, FU-VIX .

	1d	2w	1m	2m	3m
EGARCH	0.2599	0.2210	0.2441	0.4082	0.5094
AVGARCH	0.2779	0.2243	0.2529	0.4435	0.5531
TGARCH	0.2503	0.2081	0.2343	0.4000	0.4972
GARCH	0.2808	0.2269	0.2539	0.4421	0.5536
GJRGARCH	0.2631	0.2146	0.2375	0.4064	0.5116
QGARCH	0.2634	0.2229	0.2459	0.4069	0.5039
NLGARCH	0.2788	0.2251	0.2528	0.4423	0.5531
APGARCH	0.2506	0.2092	0.2364	0.4026	0.4984
FGARCH	0.2643	0.2249	0.2467	0.4070	0.5041
MSEGARCH	0.2489	**0.2072**	**0.2279**	0.4016	0.5120
MSAVGARCH	0.2478	0.2128	0.2521	0.4609	0.5861
MSTGARCH	0.2461	0.2183	0.2504	0.4034	*0.4775*
MSGARCH	0.2623	0.2272	0.2664	0.4790	0.6016
MSGJRGARCH	0.2625	0.2257	0.2533	*0.3988*	**0.4759**
MSQGARCH	0.2580	0.2278	0.2579	0.4111	0.4953
MSNLGARCH	0.2574	0.2227	0.2580	0.4302	0.5244
MSAPGARCH	**0.2427**	0.2089	0.2435	0.4142	0.5033
MSFGARCH	*0.2437*	*0.2075*	*0.2283*	**0.3869**	0.4825

Note: As in Table 3.29 except that the explanatory variables in the long-term component are the Financial Uncertainty (FU) index of Ludvigson et al. (2020) and the Chicago Board Options Exchange Volatility Index (VIX).

Table 3.33: QLIKE losses and MCS: subsamples of low/normal/high levels of realised volatility, pre-COVID period, FU.

	Low-volatility regime					Normal-volatility regime					High-volatility regime				
	1d	2w	1m	2m	3m	1d	2w	1m	2m	3m	1d	2w	1m	2m	3m
EGARCH	0.3374	0.2499	0.2670	0.3373	*0.3701*	0.2640	0.2375	0.2391	0.3646	0.4505	0.2300	0.1758	0.1867	0.3501	0.4624
AVGARCH	0.3960	0.2631	0.2694	0.3564	0.3934	0.2924	0.2467	0.2508	0.3777	0.4556	0.3267	0.2255	0.2150	0.3695	0.4973
TGARCH	0.3195	0.2302	0.2571	0.3512	0.3893	0.2561	0.2271	0.2380	0.3749	0.4548	0.2291	0.1747	0.1842	0.3447	0.4631
GARCH	0.4177	0.2761	0.2747	0.3503	0.3848	0.2970	0.2503	0.2527	0.3722	0.4539	0.2923	0.2136	0.2078	0.3579	0.4846
GJRGARCH	0.3646	0.2560	0.2678	0.3481	0.3850	0.2765	0.2392	0.2458	0.3707	0.4533	0.2241	0.1718	0.1772	**0.3363**	**0.4594**
QGARCH	0.3707	0.2804	0.2967	0.3700	0.3996	0.2718	0.2453	0.2569	0.3812	0.4579	0.2367	0.1799	0.1918	0.3531	0.4684
NLGARCH	0.4156	0.2745	0.2737	0.3494	0.3842	0.2963	0.2495	0.2518	0.3703	0.4506	0.2946	0.2145	0.2081	0.3577	0.4827
APGARCH	0.3162	0.2294	0.2583	0.3536	0.3916	0.2555	0.2275	0.2390	0.3786	0.4580	0.2329	0.1742	0.1834	0.3458	0.4653
FGARCH	0.3399	0.2536	0.2725	0.3494	0.3870	0.2637	0.2364	0.2455	0.3731	0.4518	0.2332	0.1788	0.1923	0.3486	0.4666
MSEGARCH	0.3195	0.2254	0.2393	0.3354	0.3740	0.2392	0.2161	*0.2256*	0.3749	0.4685	*0.2162*	*0.1640*	*0.1738*	0.3512	0.4825
MSAVGARCH	0.3412	0.2224	0.2367	*0.3227*	0.3723	0.2697	0.2365	0.2443	*0.3474*	*0.4372*	0.2888	0.2095	0.2088	0.3814	0.5137
MSTGARCH	0.3007	0.2229	0.2633	0.3880	0.4423	0.2476	0.2237	0.2454	0.3970	0.4914	0.2264	0.1769	0.1868	0.3496	0.4772
MSGARCH	0.3595	0.2355	0.2488	0.3343	0.3833	0.2789	0.2433	0.2499	**0.3468**	**0.4367**	0.2816	0.2065	0.2088	0.3795	0.5026
MSGJRGARCH	0.3389	0.2462	0.2687	0.3648	0.4098	0.2676	0.2475	0.2665	0.3901	0.4572	0.2552	0.2027	0.2236	0.3936	0.4936
MSQGARCH	0.3114	0.2391	0.2681	0.3728	0.4089	0.2484	0.2246	0.2422	0.3957	0.4793	0.2197	0.1682	0.1820	0.3631	0.4994
MSNLGARCH	0.3299	0.2129	**0.2224**	**0.3054**	**0.3487**	0.2514	0.2173	0.2270	0.3563	0.4629	0.3748	0.2510	0.2343	0.3855	0.5190
MSAPGARCH	*0.2979*	*0.2118*	0.2471	0.3502	0.3925	*0.2392*	*0.2135*	0.2328	0.3782	0.4629	0.2266	0.1754	0.1865	0.3557	0.4831
MSFGARCH	**0.2932**	**0.2074**	*0.2350*	0.3344	0.3725	**0.2267**	**0.2026**	**0.2122**	0.3498	0.4517	**0.2078**	**0.1599**	**0.1693**	*0.3390*	*0.4621*

Note: The table presents average QLIKE losses at five horizons: 1-day (1d), 2-weeks (2w), 1-month (1m), 2-moths (2m) and 3-months (3m) across three different volatility regimes. A forecast falls into the low/normal/high-volatility regime when the level of the realised variance on the day the forecast is obtained is below the 25% quantile, between the 25% and 75% quantile, or above the 75% quantile of the full sample realised variances. The out-of-sample period ranges from 2007:M1 to 2019:M12. The explanatory variable in the long-term component is the Financial Uncertainty (FU) index of Ludvigson et al. (2020). The models with the lowest average losses are highlighted in bold; the second lowest average losses are in italics. The table also presents the results from the 90% MCS for each horizon across regimes. The models included in the final set are shaded in blue.

Table 3.34: QLIKE losses and MCS: subsamples of low/normal/high levels of realised volatility, pre-COVID period, MU.

	Low-volatility regime					Normal-volatility regime					High-volatility regime				
	1d	2w	1m	2m	3m	1d	2w	1m	2m	3m	1d	2w	1m	2m	3m
EGARCH	0.3363	0.2927	0.3469	0.3827	0.4107	0.2721	0.2729	0.2775	0.4465	0.5414	0.2350	0.1939	0.2214	0.4523	0.6373
AVGARCH	0.3722	0.2357	**0.2565**	**0.3380**	**0.3820**	0.2915	0.2578	0.2596	**0.3957**	0.4791	0.3663	0.2483	0.2361	0.4039	0.5617
TGARCH	0.3172	0.2405	0.2888	0.3892	0.4536	0.2599	0.2410	0.2549	0.4393	0.5249	0.2421	0.1963	0.2201	0.4227	0.5877
GARCH	0.3976	0.2559	0.2757	0.3519	0.3885	0.3056	0.2727	0.2763	0.4069	0.4933	0.3205	0.2399	0.2387	0.4299	0.6015
GJRGARCH	0.3731	0.2745	0.3081	0.3948	0.4513	0.2816	0.2520	0.2649	0.4375	0.5168	0.2364	0.1960	0.2110	0.4095	0.5587
QGARCH	0.5011	0.3021	0.3400	0.4266	0.4780	0.2803	0.2527	0.2671	0.4158	*0.4736*	0.2471	0.1924	*0.2052*	**0.3688**	**0.4821**
NLGARCH	0.3859	0.2470	0.2678	*0.3463*	*0.3841*	0.2997	0.2666	0.2699	*0.4022*	0.4868	0.3302	0.2399	0.2360	0.4206	0.5881
APGARCH	0.3204	0.2442	0.2931	0.3911	0.4491	0.2625	0.2445	0.2550	0.4168	0.4918	0.2394	0.1887	**0.2024**	*0.3739*	0.5306
FGARCH	0.3337	0.2610	0.3038	0.4026	0.4590	0.2651	0.2440	*0.2536*	0.4069	**0.4709**	0.2430	0.1966	0.2164	0.4003	0.5324
MSEGARCH	0.2965	0.2442	0.2784	0.3722	0.4320	0.2602	0.2522	0.2605	0.4379	0.5080	0.2280	0.1929	0.2357	0.4711	0.5889
MSAVGARCH	0.3247	0.2239	*0.2575*	0.3474	0.4026	0.2644	0.2358	0.2627	0.4651	0.5921	0.2880	0.2322	0.2705	0.5509	0.8316
MSTGARCH	*0.2820*	*0.2132*	0.2781	0.3929	0.4308	0.2560	0.2340	0.2592	0.4392	0.5104	0.2278	0.1902	0.2315	0.4702	0.5669
MSGARCH	0.3428	0.2358	0.2761	0.3845	0.4391	0.2775	0.2466	0.2748	0.4791	0.6218	0.2823	0.2340	0.2747	0.5623	0.7758
MSGJRGARCH	0.3363	0.2626	0.3198	0.4450	0.5227	0.2692	0.2493	0.2851	0.4929	0.5762	0.2596	0.2167	0.2559	0.4859	0.6338
MSQGARCH	0.3121	0.2524	0.3066	0.4361	0.4932	0.2590	0.2409	0.2699	0.4643	0.5224	*0.2213*	0.1864	0.2186	0.4401	*0.5208*
MSNLGARCH	0.3304	0.2267	0.2629	0.3534	0.4054	0.2666	0.2391	0.2671	0.4729	0.5652	0.2954	0.2402	0.2739	0.5490	0.7084
MSAPGARCH	**0.2820**	**0.2059**	0.2659	0.3734	0.4423	*0.2509*	**0.2281**	0.2555	0.4489	0.5269	0.2216	*0.1862*	0.2301	0.4633	0.5404
MSFGARCH	0.2825	0.2306	0.2675	0.3763	0.4479	**0.2475**	*0.2288*	**0.2492**	0.4412	0.5123	**0.2178**	**0.1835**	0.2255	0.4474	0.5566

Note: As in Table 3.33 except that the explanatory variable in the long-term component is the Macroeconomic Uncertainty (MU) index of Jurado et al. (2015).

Table 3.35: QLIKE losses and MCS: subsamples of low/normal/high levels of realised volatility, pre-COVID period, VIX.

	Low-volatility regime					Normal-volatility regime					High-volatility regime				
	1d	2w	1m	2m	3m	1d	2w	1m	2m	3m	1d	2w	1m	2m	3m
EGARCH	0.3058	0.2242	0.2683	0.3709	0.4019	0.2192	0.2134	0.2492	0.5029	0.6089	0.2711	0.2642	0.2993	0.5608	0.7681
AVGARCH	0.3264	0.2082	*0.2491*	0.3800	0.4066	0.2237	0.2061	0.2382	0.4644	0.5704	0.2967	0.2739	0.3002	0.5640	0.7852
TGARCH	0.2955	0.2091	0.2563	0.3626	0.3891	0.2168	0.2060	0.2417	0.4876	0.5902	0.2602	0.2517	0.2828	**0.5430**	*0.7550*
GARCH	0.3399	0.2177	0.2548	0.3834	0.4146	0.2260	0.2076	0.2419	0.4786	0.5914	0.2947	0.2796	0.3053	0.5649	0.7839
GJRGARCH	0.3207	0.2174	0.2574	0.3764	0.4108	0.2239	0.2081	0.2440	0.4987	0.6126	0.2696	0.2610	0.2861	0.5478	0.7669
QGARCH	0.3111	0.2328	0.2713	0.3643	0.3924	0.2216	0.2123	0.2485	0.4999	0.5997	0.2793	0.2708	0.3048	0.5711	0.7824
NLGARCH	0.3335	0.2133	0.2527	0.3827	0.4103	0.2250	0.2069	0.2394	0.4642	0.5715	0.2913	0.2734	0.2998	0.5590	0.7762
APGARCH	0.2925	0.2092	0.2574	0.3590	0.3831	0.2154	0.2056	0.2423	0.4881	0.5866	0.2654	0.2565	0.2898	*0.5472*	**0.7547**
FGARCH	0.3274	0.2326	0.2692	*0.3565*	0.3832	0.2223	0.2107	0.2465	0.4860	0.5824	0.2840	0.2723	0.3076	0.5714	0.7792
MSEGARCH	0.2748	0.2008	**0.2483**	0.3791	0.4327	0.2083	**0.1972**	**0.2328**	0.4892	0.6134	0.2672	0.2578	0.2987	0.5792	0.8054
MSAVGARCH	**0.2704**	**0.1929**	0.2505	0.3831	0.4111	0.2077	0.2006	0.2443	0.4892	0.6120	0.2671	0.2611	0.3087	0.6123	0.8614
MSTGARCH	0.2732	0.2080	0.2628	0.3621	*0.3796*	0.2118	0.2056	0.2429	*0.4529*	0.5588	0.2763	0.2729	0.3151	0.5760	0.7893
MSGARCH	0.2787	0.1990	0.2528	0.3789	0.4135	0.2128	0.2038	0.2422	0.4618	0.5761	0.2635	0.2708	0.3253	0.6282	0.8506
MSGJRGARCH	0.2877	0.2147	0.2674	0.3732	0.3821	0.2198	0.2106	0.2445	**0.4490**	*0.5548*	0.2697	0.2667	0.3041	0.5735	0.7911
MSQGARCH	0.2773	0.2034	0.2523	0.3565	**0.3787**	0.2132	0.2046	0.2448	0.4801	0.5869	0.2753	0.2819	0.3295	0.6167	0.8416
MSNLGARCH	0.2738	*0.1967*	0.2517	0.3806	0.4483	*0.2075*	*0.1997*	0.2431	0.4971	0.6217	0.2492	0.2464	0.2927	0.5962	0.8499
MSAPGARCH	0.2733	0.2037	0.2537	**0.3538**	0.3848	0.2100	0.2042	0.2426	0.4555	**0.5517**	*0.2444*	*0.2461*	*0.2811*	0.5527	0.7645
MSFGARCH	*0.2729*	0.2150	0.2582	0.3669	0.3950	**0.2075**	0.2040	*0.2380*	0.4866	0.5951	**0.2411**	**0.2343**	**0.2765**	0.5474	0.7691

Note: As in Table 3.33 except that the explanatory variable in the long-term component is the Chicago Board Options Exchange Volatility Index (VIX).

Table 3.36: QLIKE losses and MCS: subsamples of low/normal/high levels of realised volatility, pre-COVID period, FU-VIX.

	Low-volatility regime					Normal-volatility regime					High-volatility regime				
	1d	2w	1m	2m	3m	1d	2w	1m	2m	3m	1d	2w	1m	2m	3m
EGARCH	0.3092	0.2267	0.2580	0.3530	0.3876	0.2260	0.2120	0.2315	0.4172	0.5211	0.2598	0.2326	0.2518	0.4561	0.6327
AVGARCH	0.3302	0.2128	0.2514	0.3788	0.4069	0.2300	0.2074	0.2338	0.4329	0.5396	0.3052	0.2752	0.2971	0.5445	0.7539
TGARCH	0.2935	0.2059	*0.2423*	0.3446	0.3759	0.2224	0.2028	0.2252	0.4061	0.5044	0.2463	*0.2209*	0.2410	*0.4519*	*0.6286*
GARCH	0.3410	0.2206	0.2554	0.3833	0.4178	0.2318	0.2083	0.2346	0.4333	0.5441	0.3013	0.2757	0.2957	0.5328	0.7343
GJRGARCH	0.3201	0.2154	0.2456	0.3588	0.3981	0.2314	0.2075	0.2291	0.4097	0.5172	0.2512	0.2291	0.2461	0.4574	0.6367
QGARCH	0.3182	0.2341	0.2607	0.3473	0.3788	0.2284	0.2112	0.2322	0.4143	0.5112	0.2598	0.2323	0.2530	0.4616	0.6399
NLGARCH	0.3349	0.2161	0.2532	0.3828	0.4137	0.2313	0.2080	0.2341	0.4325	0.5429	0.3013	0.2738	0.2947	0.5359	0.7394
APGARCH	0.2907	0.2064	0.2439	0.3416	0.3706	0.2209	0.2024	0.2258	0.4107	0.5068	0.2535	0.2246	0.2452	0.4564	0.6352
FGARCH	0.3174	0.2359	0.2615	0.3464	0.3787	0.2290	0.2140	0.2334	0.4152	0.5121	0.2631	0.2330	0.2528	0.4610	0.6388
MSEGARCH	0.2917	0.2078	**0.2348**	0.3545	0.3996	0.2138	**0.2001**	**0.2168**	0.3995	0.5130	0.2586	0.2212	*0.2407*	0.4608	0.6447
MSAVGARCH	0.2891	**0.1967**	0.2592	0.4579	0.5195	**0.2104**	0.2060	0.2387	0.4340	0.5612	0.2655	0.2481	0.2733	0.5181	0.7137
MSTGARCH	**0.2822**	0.2063	0.2440	**0.3348**	**0.3593**	0.2149	0.2098	0.2377	0.3898	*0.4562*	0.2568	0.2489	0.2807	0.5058	0.6601
MSGARCH	0.3137	0.2150	0.2677	0.4185	0.4470	0.2219	0.2154	0.2492	0.4674	0.5909	0.2756	0.2683	0.3034	0.5771	0.8071
MSGJRGARCH	0.3042	0.2137	0.2440	*0.3351*	*0.3597*	0.2263	0.2158	0.2398	**0.3796**	**0.4539**	0.2765	0.2595	0.2891	0.5073	0.6572
MSQGARCH	0.2903	0.2176	0.2549	0.3555	0.3927	0.2264	0.2187	0.2458	0.3969	0.4762	0.2760	0.2572	0.2818	0.4988	0.6548
MSNLGARCH	0.3020	0.2046	0.2551	0.4013	0.4555	0.2144	0.2076	0.2371	0.3896	0.4746	0.2827	0.2767	0.3065	0.5442	0.7025
MSAPGARCH	*0.2841*	*0.2045*	0.2484	0.3503	0.3674	*0.2125*	0.2029	0.2338	0.4159	0.5067	*0.2449*	0.2258	0.2547	0.4835	0.6593
MSFGARCH	0.2916	0.2183	0.2456	0.3458	0.3817	0.2138	*0.2016*	*0.2201*	*0.3893*	0.4848	**0.2396**	**0.2056**	**0.2208**	**0.4274**	**0.5989**

Note: As in Table 3.33 except that the explanatory variables in the long-term component are the Financial Uncertainty (FU) index of Ludvigson et al. (2020) and the Chicago Board Options Exchange Volatility Index (VIX).

3.7.2 Figures

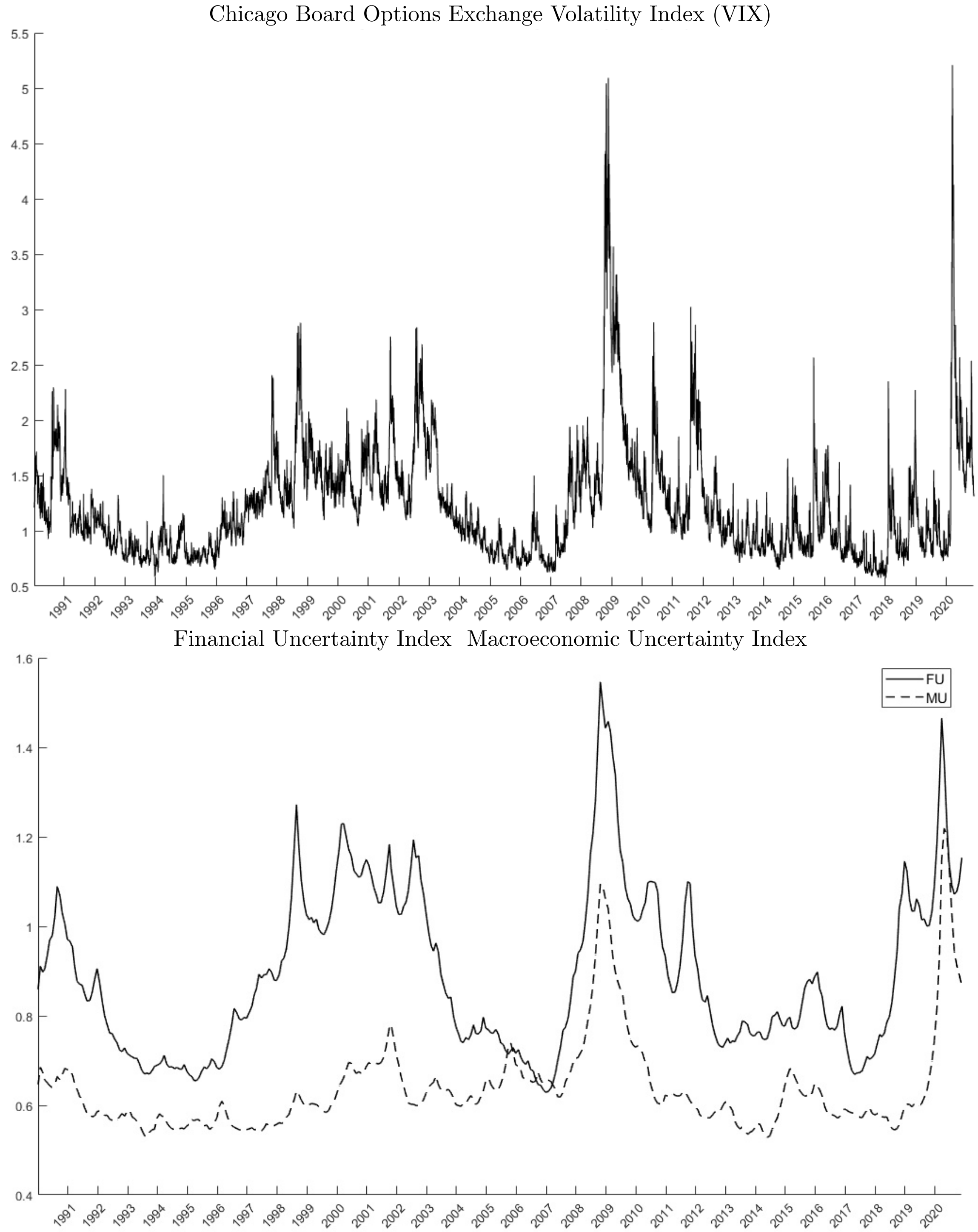

Figure 3.1: Macroeconomic variables from 1990:M1 to 2020:M11. The top panel plots the daily Chicago Board Options Exchange Volatility Index (VIX). The monthly Financial Uncertainty (FU) index of Ludvigson et al. (2020) and the Macroeconomic Uncertainty (MU) index of Jurado et al. (2015) are depicted in the bottom panel.

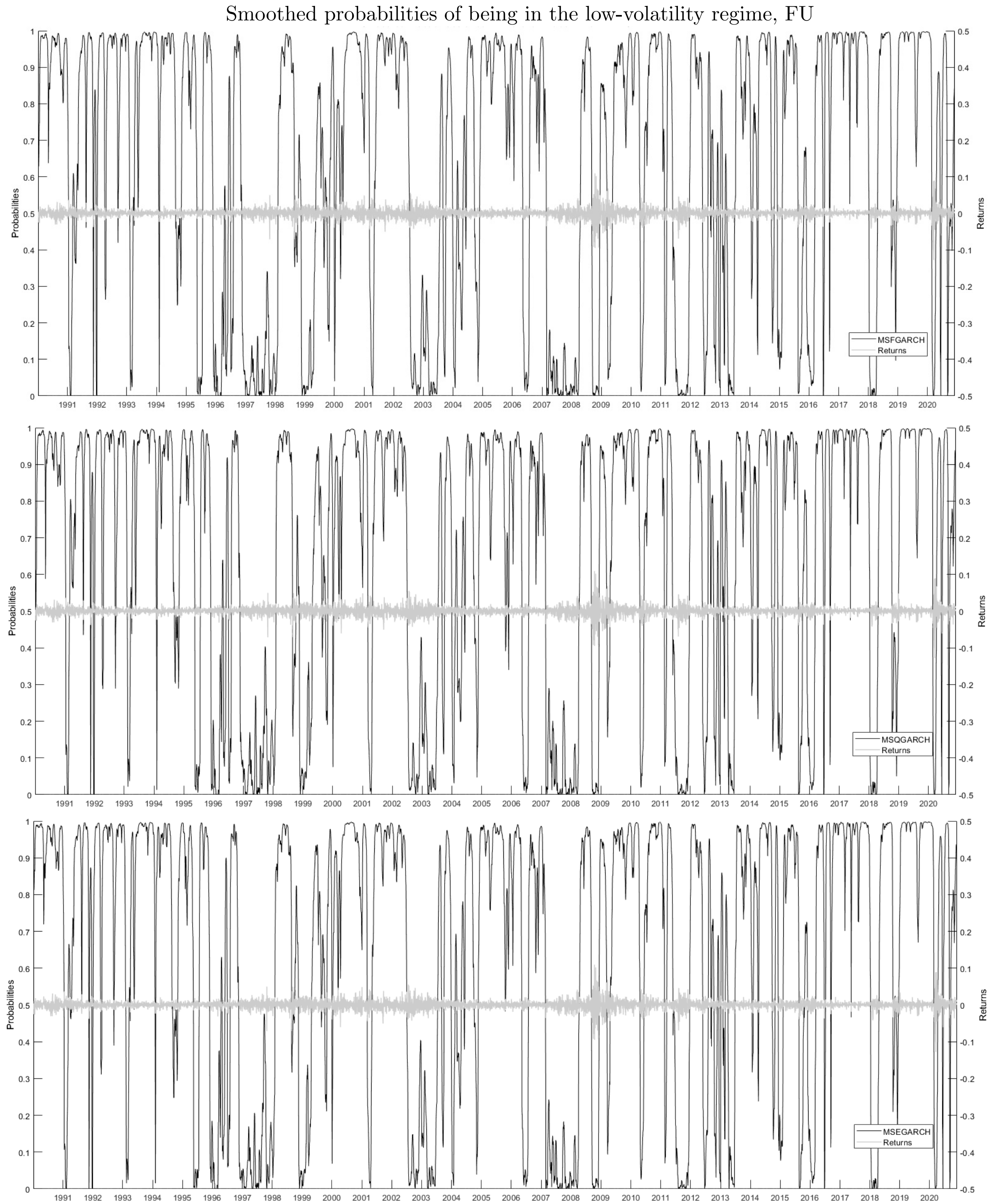

Figure 3.2: Estimated smoothed probabilities of being in the low-volatility regime when the Financial Uncertainty (FU) index of Ludvigson et al. (2020) is the explanatory variable in the long-term component.

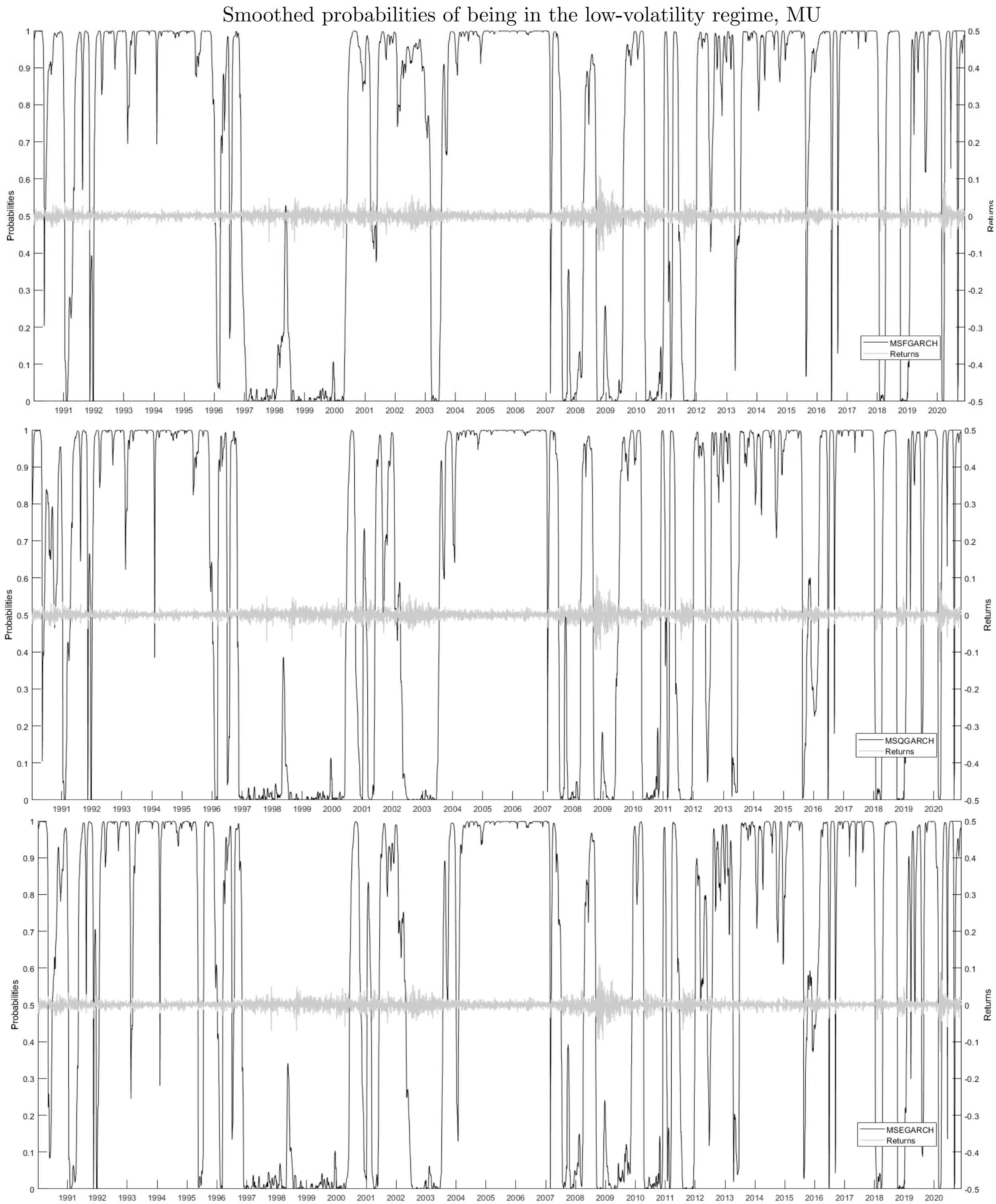

Figure 3.3: Estimated smoothed probabilities of being in the low-volatility regime when the Macroeconomic Uncertainty (MU) index of Jurado et al. (2015) is the explanatory variable in the long-term component.

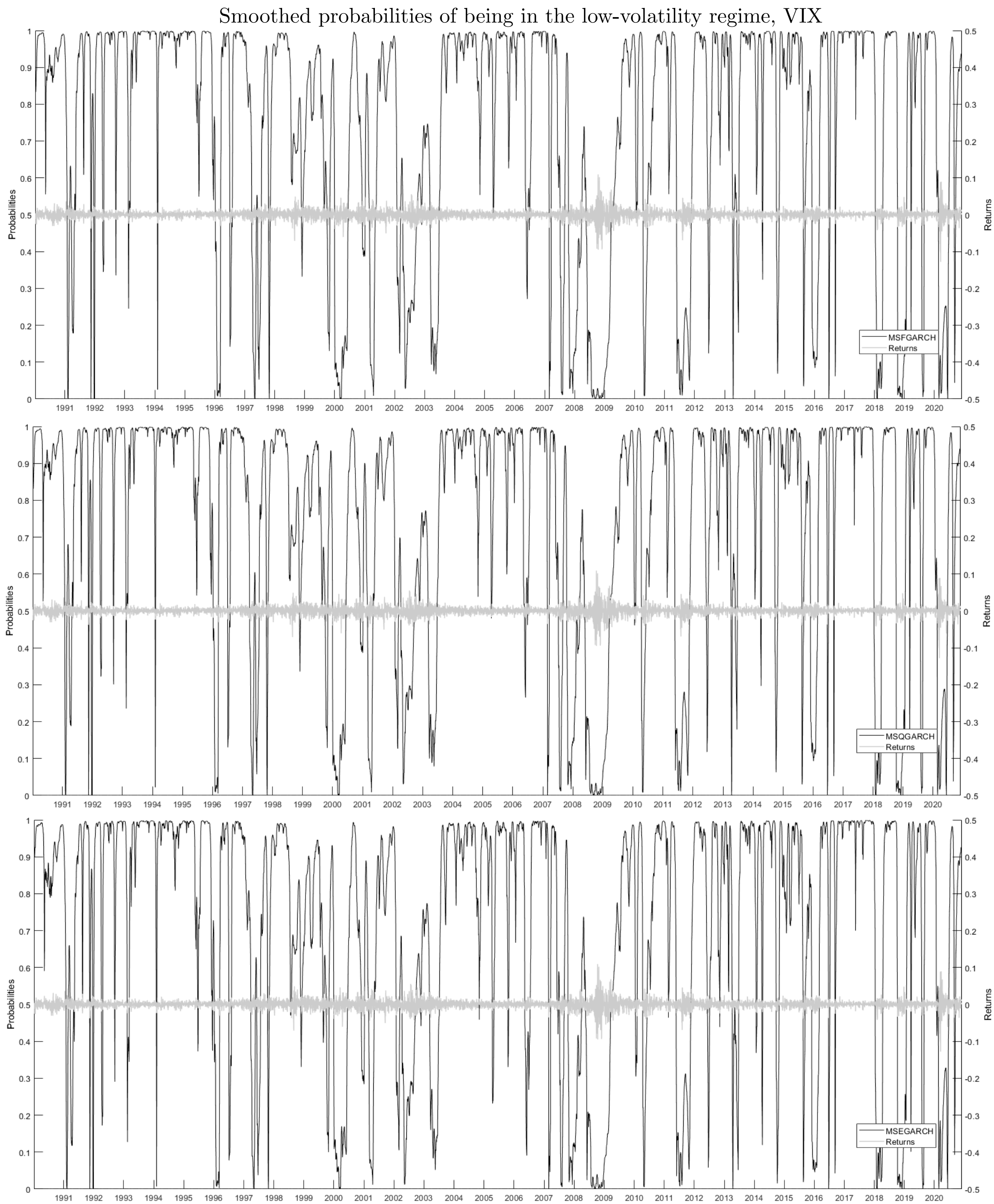

Figure 3.4: Estimated smoothed probabilities of being in the low-volatility regime when the Chicago Board Options Exchange Volatility Index (VIX) is the explanatory variable in the long-term component.

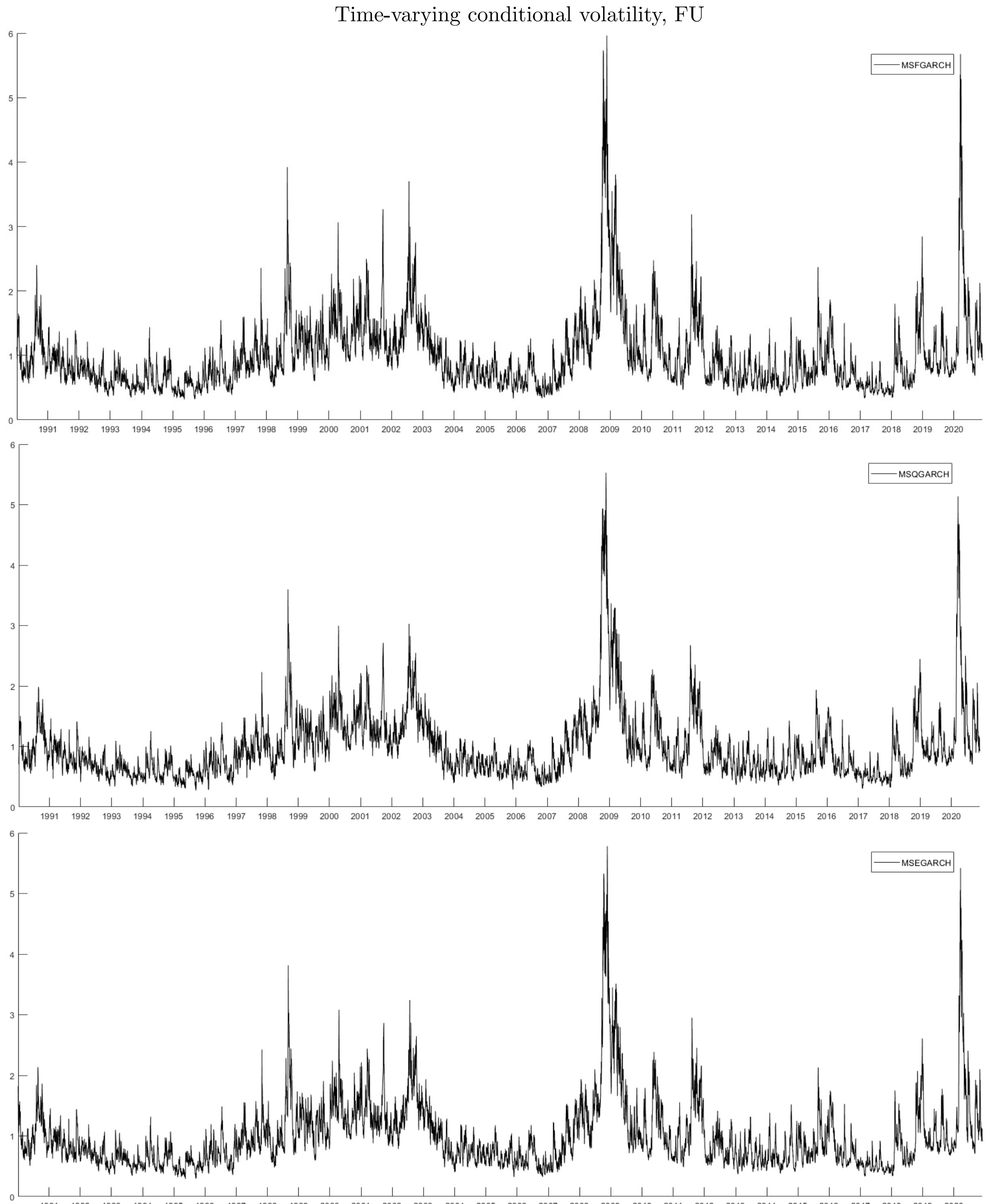

Figure 3.5: Estimated time-varying conditional volatilities when the Financial Uncertainty (FU) index of Ludvigson et al. (2020) is the explanatory variables in the long-term component.

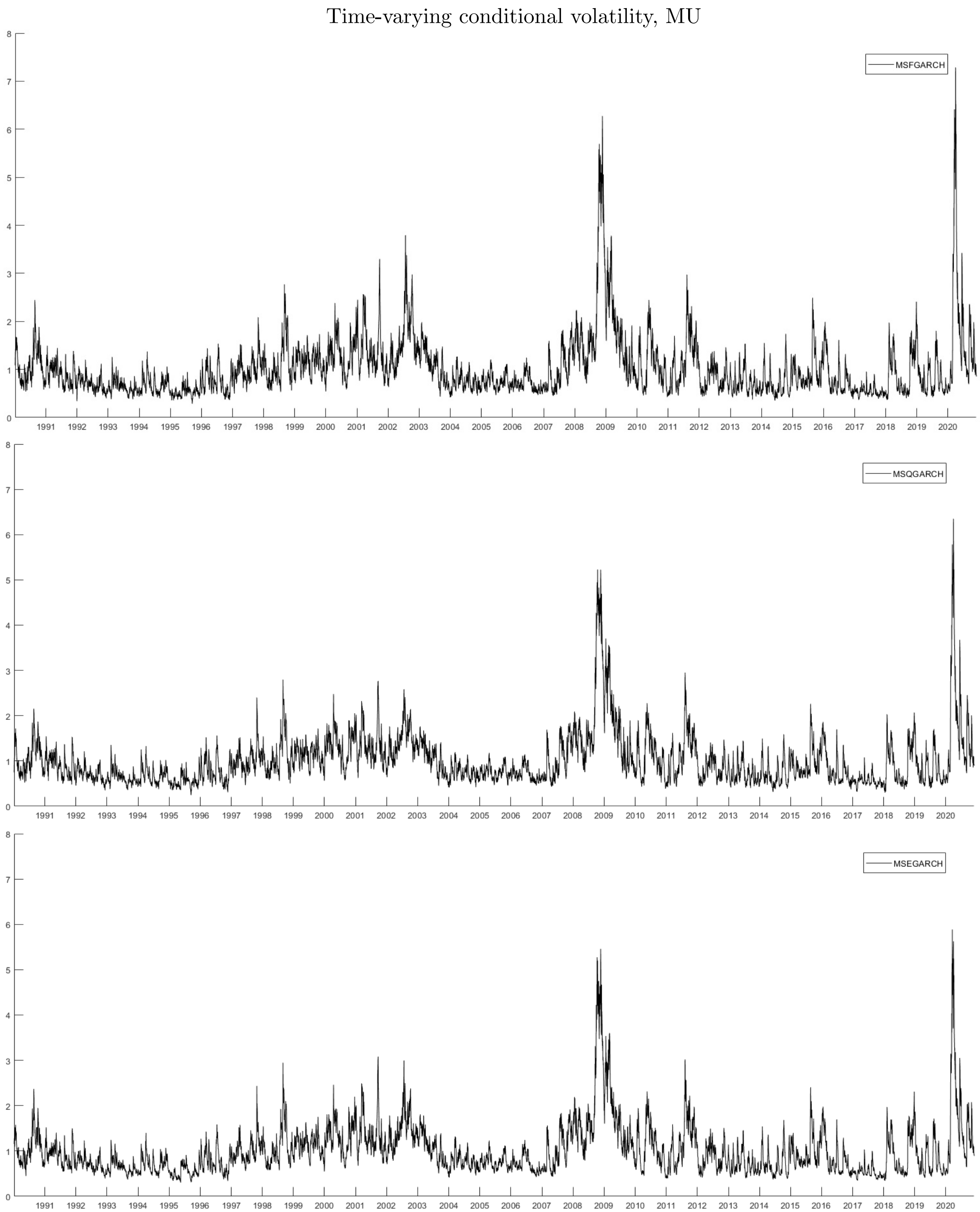

Figure 3.6: Estimated time-varying conditional volatilities when the Macroeconomic Uncertainty (MU) index of Jurado et al. (2015) is the explanatory variables in the long-term component.

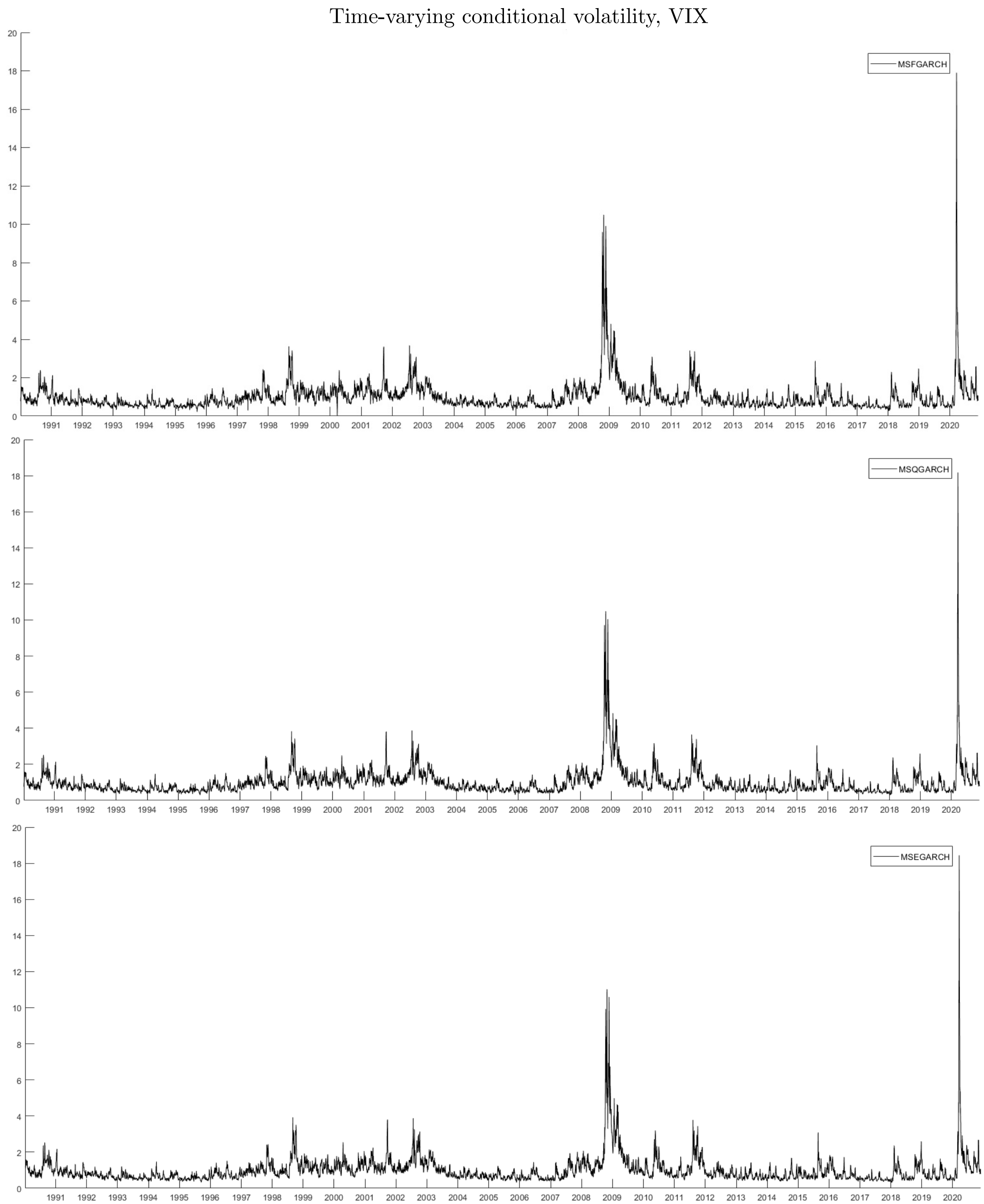

Figure 3.7: Estimated time-varying conditional volatilities when the Chicago Board Options Exchange Volatility Index (VIX) is the explanatory variables in the long-term component.

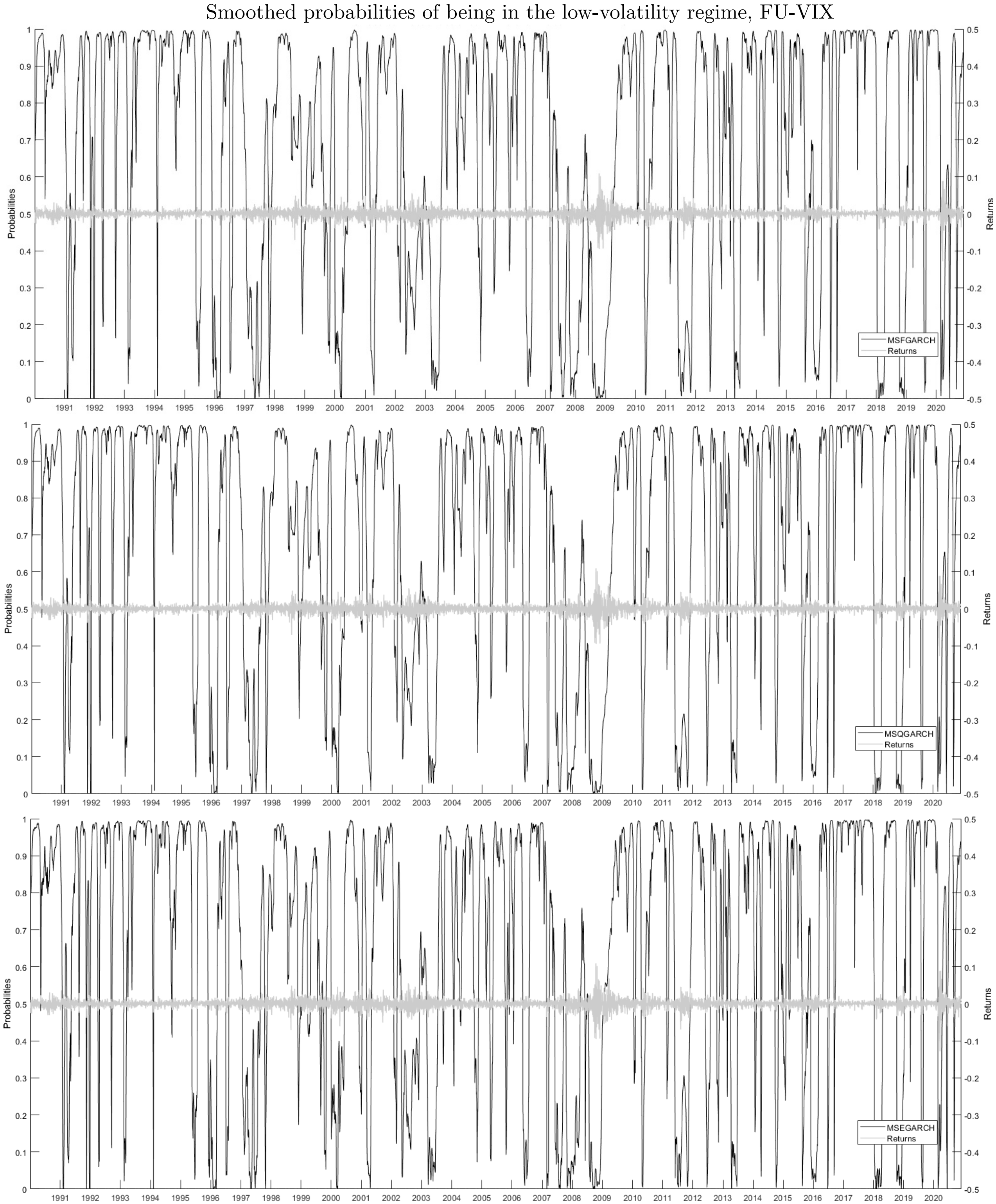

Figure 3.8: Estimated smoothed probabilities of being in the low-volatility regime when the Financial Uncertainty (FU) index of Ludvigson et al. (2020) and the Chicago Board Options Exchange Volatility Index (VIX) are the explanatory variables in the long-term component.

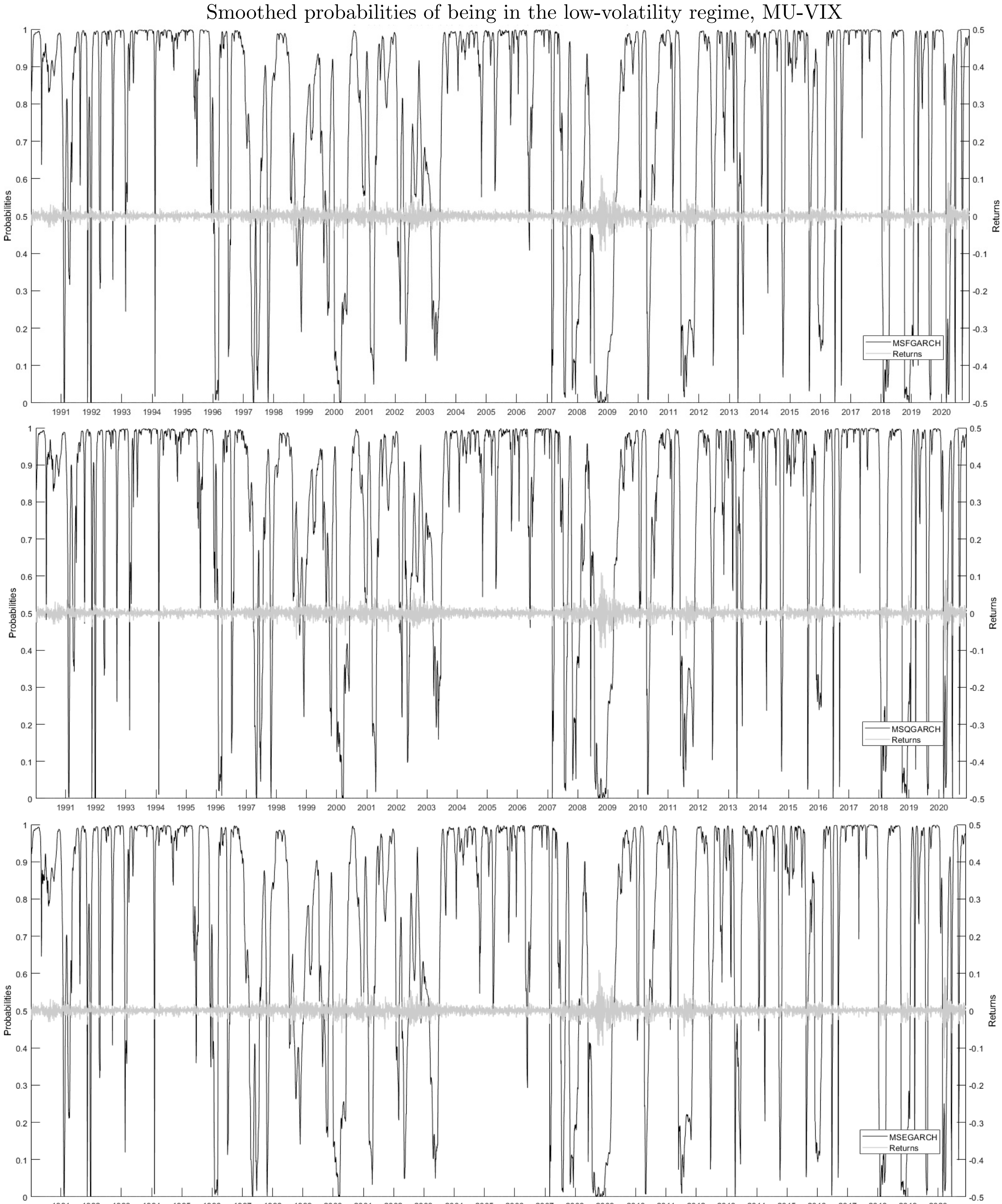

Figure 3.9: Estimated smoothed probabilities of being in the low-volatility regime when the Macroeconomic Uncertainty (MU) index of Jurado et al. (2015) and the Chicago Board Options Exchange Volatility Index (VIX) are the explanatory variables in the long-term component.

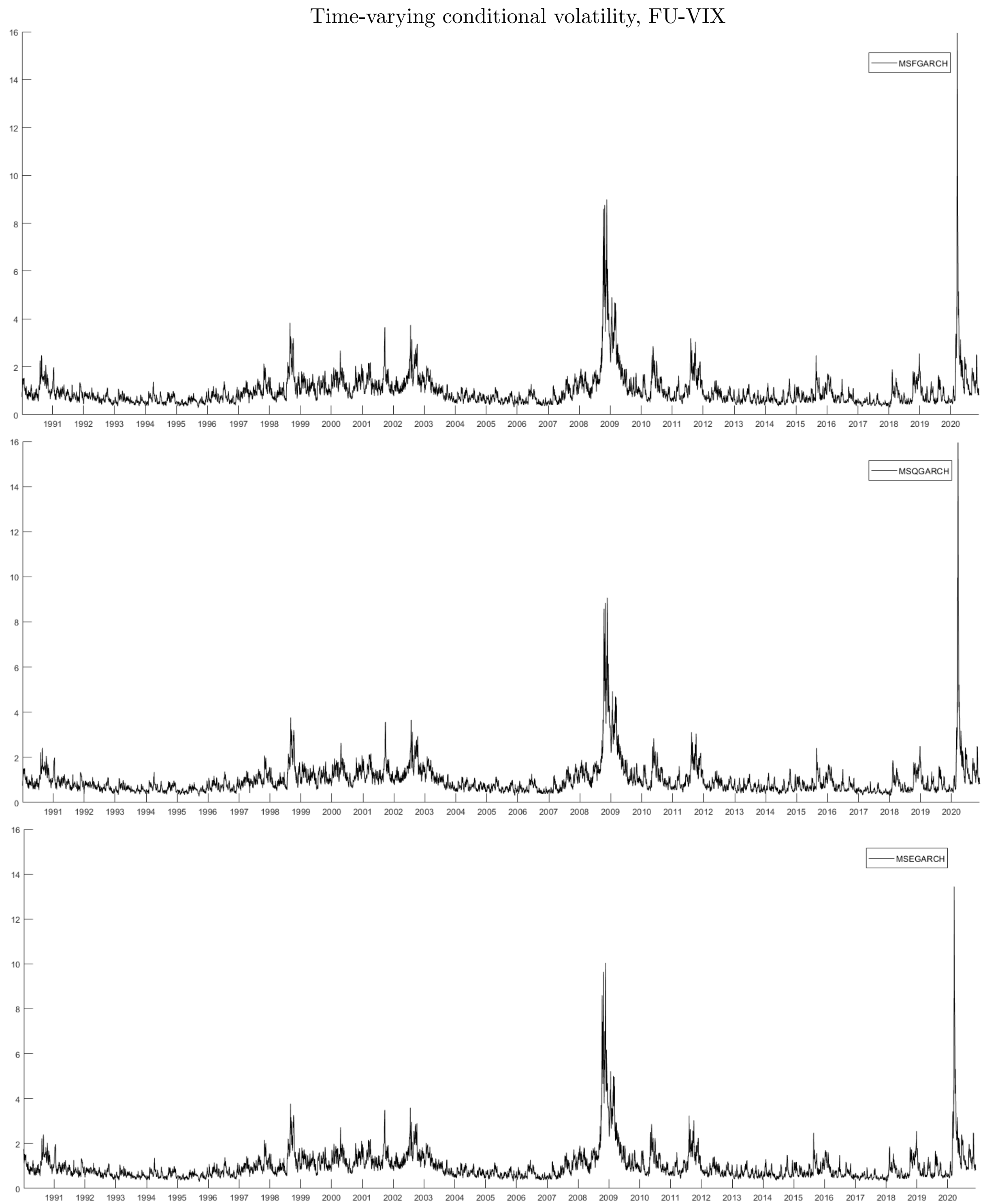

Figure 3.10: Estimated time-varying conditional volatilities when the Financial Uncertainty (FU) index of Ludvigson et al. (2020) and the Chicago Board Options Exchange Volatility Index (VIX) are the explanatory variables in the long-term component.

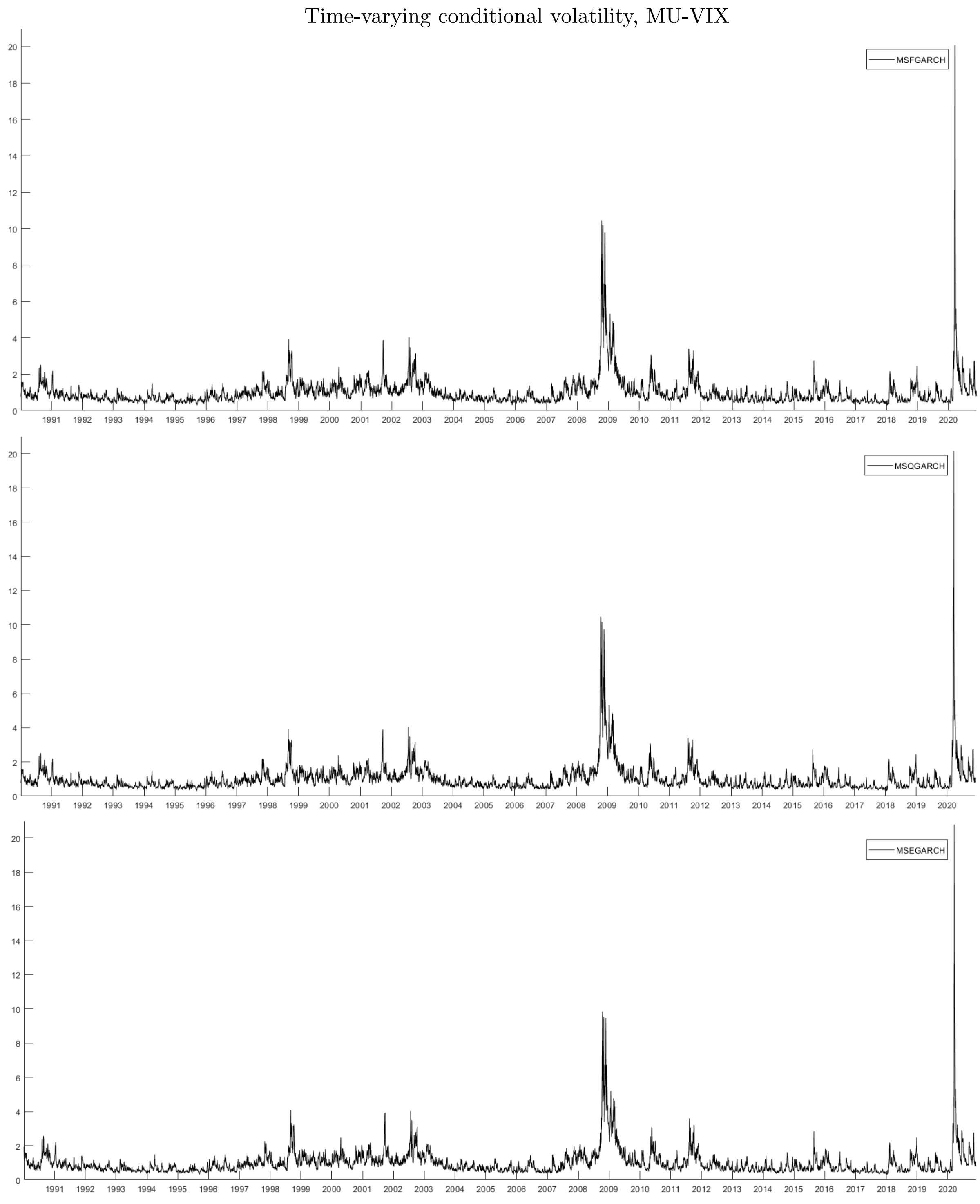

Figure 3.11: Estimated time-varying conditional volatilities when the Macroeconomic Uncertainty (MU) index of Jurado et al. (2015) and the Chicago Board Options Exchange Volatility Index (VIX) are the explanatory variables in the long-term component.

3.7.3 Additional Tables

Table 3.37: Estimated GARCH-MIDAS models, long-term component, FU

	EGARCH	AVGARCH	TGARCH	GARCH	GJRGARCH	QGARCH	NLGARCH	APGARCH	FGARCH
λ	0	1	1	2	2	2	1.8121*** (0.0005)	0.8745*** (0.0001)	0.4967*** (0.0005)
$\hat{\lambda}$	1	1	1	2	2	2			1.6079*** (0.0014)
μ	0.0308*** (0.0000)	0.0629*** (0.0000)	0.0302*** (0.0000)	0.0588*** (0.0000)	0.0326*** (0.0000)	0.0263*** (0.0000)	0.0594*** (0.0000)	0.0292*** (0.0000)	0.0251*** (0.0000)
α	0.1227*** (0.0000)	0.1039*** (0.0001)	0.0834*** (0.0000)	0.0942*** (0.0000)	0.0000*** (0.0000)	0.0720*** (0.0000)	0.0986*** (0.0000)	0.0852*** (0.0000)	0.0164*** (0.0000)
β	0.9315*** (0.0000)	0.8617*** (0.0001)	0.8689*** (0.0000)	0.8498*** (0.0001)	0.8423*** (0.0000)	0.8489*** (0.0000)	0.8521*** (0.0001)	0.8733*** (0.0000)	0.9259*** (0.0000)
γ	0.1570*** (0.0000)		0.9999*** (0.0001)		0.1807*** (0.0001)			0.9999*** (0.0000)	-0.5998*** (0.0007)
ψ						0.1654*** (0.0000)			0.0672*** (0.0001)
m	-3.2898*** (0.0013)	-3.1930*** (0.0013)	-3.0650*** (0.0015)	-3.2503*** (0.0016)	-3.1606*** (0.0018)	-3.2317*** (0.0014)	-3.2393*** (0.0016)	-3.0544*** (0.0015)	-3.0493*** (0.0017)
θ^{FU}	3.4707*** (0.0015)	3.5787*** (0.0017)	3.3416*** (0.0017)	3.5296*** (0.0020)	3.3751*** (0.0020)	3.4914*** (0.0016)	3.5346*** (0.0020)	3.3435*** (0.0016)	3.3176*** (0.0018)
w^{FU}	416.66*** (0.8329)	534.45*** (0.3219)	474.97*** (0.3535)	502.09*** (0.0166)	452.04*** (0.3181)	502.15*** (0.0415)	552.41*** (0.2915)	500.44*** (0.7240)	500.88*** (0.2278)

Note: This table shows parameter estimates for the single-regime model in (3.1), (3.8)-(3.12) and the nested models, presented in Table 3.1. The explanatory variable in the long-term component is the Financial Uncertainty (FU) index of Ludvigson et al. (2020).

Table 3.38: Estimated GARCH-MIDAS models, long-term component, MU

	EGARCH	AVGARCH	TGARCH	GARCH	GJRGARCH	QGARCH	NLGARCH	APGARCH	FGARCH
λ	0	1	1	2	2	2	1.8287*** (0.0011)	0.9701*** (0.0001)	0.6085*** (0.0005)
$\hat{\lambda}$	1	1	1	2	2	2			1.7037*** (0.0010)
μ	0.0280*** (0.0000)	0.0645*** (0.0000)	0.0243*** (0.0000)	0.0598*** (0.0000)	0.0291*** (0.0000)	0.0210*** (0.0000)	0.0603*** (0.0000)	0.0240*** (0.0001)	0.0223*** (0.0001)
α	0.1503*** (0.0000)	0.1141*** (0.0000)	0.0857*** (0.0000)	0.1076*** (0.0000)	0.0060*** (0.0000)	0.0889*** (0.0000)	0.1113*** (0.0000)	0.0865*** (0.0000)	0.0237*** (0.0000)
β	0.9633*** (0.0000)	0.8832*** (0.0000)	0.8982*** (0.0000)	0.8696*** (0.0000)	0.8771*** (0.0001)	0.8736*** (0.0000)	0.8718*** (0.0000)	0.8986*** (0.0000)	0.9451*** (0.0000)
γ	0.1356*** (0.0000)		0.9283*** (0.0000)		0.1722*** (0.0001)			0.9272*** (0.0000)	-0.4963*** (0.0002)
ψ						0.1340*** (0.0000)			0.0805*** (0.0001)
m	-2.4465*** (0.0056)	-2.0891*** (0.0051)	-2.1679*** (0.0064)	-2.0289*** (0.0051)	-2.2223*** (0.0118)	-2.4994*** (0.0073)	-2.0389*** (0.0052)	-2.1678*** (0.0063)	-2.2360*** (0.0087)
θ^{MU}	3.7146*** (0.0092)	3.7017*** (0.0084)	3.5429*** (0.0103)	3.3427*** (0.0083)	3.4082*** (0.0185)	3.9612*** (0.0120)	3.3892*** (0.0087)	3.5535*** (0.0100)	3.6504*** (0.0136)
w^{MU}	23.5083*** (0.3499)	501.5850*** (0.0910)	20.1394*** (0.3872)	502.6379*** (0.6055)	15.6314*** (0.4903)	14.9320*** (0.2596)	506.2740*** (110.3022)	20.6051*** (0.3617)	19.4422*** (0.4427)

Note: This table shows parameter estimates for the single-regime model in (3.1), (3.8)-(3.12) and the nested models, presented in Table 3.1. The explanatory variable in the long-term component is the Macroeconomic Uncertainty (MU) index of Jurado et al. (2015).

Table 3.39: Estimated GARCH-MIDAS models, long-term component, VIX

	EGARCH	AVGARCH	TGARCH	GARCH	GJRGARCH	QGARCH	NLGARCH	APGARCH	FGARCH
λ	0	1	1	2	2	2	1.5917*** (0.0012)	0.7443*** (0.0001)	2.2980*** (0.0008)
$\hat{\lambda}$	1	1	1	2	2	2			0.9767*** (0.0002)
μ	0.0186*** (0.0000)	0.0343*** (0.0000)	0.0225*** (0.0000)	0.0334*** (0.0000)	0.0244*** (0.0000)	0.0156*** (0.0000)	0.0339*** (0.0000)	0.0203*** (0.0001)	0.0181*** (0.0000)
α	0.0644*** (0.0000)	0.0406*** (0.0000)	0.0526*** (0.0000)	0.0382*** (0.0000)	0.0000*** (0.0000)	0.0352*** (0.0000)	0.0419*** (0.0000)	0.0552*** (0.0000)	0.1004*** (0.0000)
β	0.8949*** (0.0000)	0.9221*** (0.0000)	0.8688*** (0.0000)	0.9021*** (0.0000)	0.8537*** (0.0000)	0.8497*** (0.0000)	0.9100*** (0.0000)	0.8733*** (0.0000)	0.8246*** (0.0000)
γ	0.1227*** (0.0000)		0.9999*** (0.0004)		0.0933*** (0.0000)			0.9999*** (0.0000)	0.9968*** (0.0009)
ψ						0.1329*** (0.0000)			0.0562*** (0.0001)
m	-2.1154*** (0.0002)	-2.2043*** (0.0002)	-2.0365*** (0.0002)	-2.2434*** (0.0001)	-2.1639*** (0.0001)	-2.0914*** (0.0002)	-2.2253*** (0.0002)	-2.0155*** (0.0004)	-2.0296*** (0.0002)
θ^{VIX}	0.0964*** (0.0000)	0.1045*** (0.0000)	0.0944*** (0.0000)	0.1036*** (0.0000)	0.0979*** (0.0000)	0.0969*** (0.0000)	0.1037*** (0.0000)	0.0944*** (0.0000)	0.0957*** (0.0000)
w^{VIX}	3.0153*** (0.0008)	5.6346*** (0.0006)	3.3301*** (0.0000)	5.2318*** (0.0009)	3.7157*** (0.0007)	2.9706*** (0.0011)	5.3542*** (0.0004)	3.2612*** (0.0004)	3.1007*** (0.0009)

Note: This table shows parameter estimates for the single-regime model in (3.1), (3.8)-(3.12) and the nested models, presented in Table 3.1. The explanatory variable in the long-term component is the Chicago Board Options Exchange Volatility Index (VIX).

Table 3.40: Estimated GARCH-MIDAS models, short-term component, FU

	EGARCH	AVGARCH	TGARCH	GARCH	GJRGARCH	QGARCH	NLGARCH	APGARCH	FGARCH
λ	0	1	1	2	2	2	1.7498*** (0.0006)	0.8806*** (0.0001)	0.5714*** (0.0004)
$\hat{\lambda}$	1	1	1	2	2	2			1.5401 (0.0011)
μ	0.0300*** (0.0000)	0.0637*** (0.0000)	0.0294*** (0.0000)	0.0598*** (0.0000)	0.0322*** (0.0000)	0.0253*** (0.0000)	0.0606*** (0.0000)	0.0282*** (0.0000)	0.0242*** (0.0000)
ω	-0.2291*** (0.0001)	0.0128*** (0.0000)	0.0151*** (0.0000)	0.0028*** (0.0000)	0.0033*** (0.0000)	0.0037*** (0.0000)	0.0041*** (0.0000)	0.0181*** (0.0000)	0.0264*** (0.0000)
α	0.1259*** (0.0000)	0.1086*** (0.0000)	0.0856*** (0.0000)	0.1004*** (0.0000)	0.0471*** (0.0000)	0.0768*** (0.0000)	0.1064*** (0.0000)	0.0872*** (0.0000)	0.0221*** (0.0000)
β	0.9294*** (0.0000)	0.8535*** (0.0001)	0.8644*** (0.0000)	0.8390*** (0.0000)	0.8358*** (0.0000)	0.8401*** (0.0000)	0.8419*** (0.0000)	0.8688*** (0.0000)	0.9166*** (0.0000)
γ	0.1590*** (0.0000)		0.9999*** (0.0001)		0.9989*** (0.0000)			0.9999*** (0.0000)	-0.5629*** (0.0004)
ψ						0.0357*** (0.0000)			0.1488*** (0.0000)
θ^{FU}	3.4199*** (0.0015)	3.4454*** (0.0016)	3.2510*** (0.0017)	3.3276*** (0.0019)	3.2467*** (0.0019)	3.3773*** (0.0016)	3.3401*** (0.0019)	3.2586*** (0.0016)	3.2193*** (0.0018)
w^{FU}	340.65*** (0.8535)	508.46*** (0.2083)	502.51*** (1.2559)	505.82*** (0.5762)	482.92*** (0.5336)	500.83*** (0.6613)	508.74*** (0.2024)	494.02*** (0.3784)	501.04*** (0.4927)

Note: This table shows parameter estimates for the single-regime model introduced in Section 3.2.3 and the nested models, presented in Table 3.1. The explanatory variable in the long-term component is the Financial Uncertainty (FU) index of Ludvigson et al. (2020).

Table 3.41: Estimated GARCH-MIDAS models, short-term component, MU

	EGARCH	AVGARCH	TGARCH	GARCH	GJRGARCH	QGARCH	NLGARCH	APGARCH	FGARCH
λ	0	1	1	2	2	2	1.8445*** (0.0005)	0.9813*** (0.0002)	1.8196*** (0.0036)
$\hat{\lambda}$	1	1	1	2	2	2			1.0072 (0.0004)
μ	0.0261*** (0.0000)	0.0646*** (0.0000)	0.0231*** (0.0000)	0.0600*** (0.0000)	0.0289*** (0.0000)	0.0199*** (0.0000)	0.0605*** (0.0000)	0.0229*** (0.0000)	0.0244*** (0.0000)
ω	-0.0825*** (0.0002)	0.0113*** (0.0000)	0.0130*** (0.0000)	0.0045*** (0.0000)	0.0045*** (0.0000)	0.0041*** (0.0000)	0.0053*** (0.0000)	0.0133*** (0.0000)	0.0053*** (0.0000)
α	0.1525*** (0.0000)	0.1167*** (0.0000)	0.0879*** (0.0000)	0.1093*** (0.0000)	0.0648*** (0.0000)	0.0917*** (0.0000)	0.1131*** (0.0000)	0.0884*** (0.0000)	0.1781*** (0.0003)
β	0.9628*** (0.0000)	0.8808*** (0.0000)	0.8959*** (0.0000)	0.8679*** (0.0000)	0.8749*** (0.0000)	0.8693*** (0.0000)	0.8697*** (0.0000)	0.8961*** (0.0000)	0.8359*** (0.0002)
γ	0.1363*** (0.0000)		0.9135*** (0.0000)		0.6704 (0.0001)			0.9132*** (0.0000)	0.9999*** (0.0000)
ψ						0.0437*** (0.0001)			-0.0089*** (0.0000)
θ^{MU}	3.3737*** (0.0079)	3.0666*** (0.0071)	3.1622*** (0.0083)	2.7026*** (0.0064)	2.9656*** (0.0102)	3.5913*** (0.0081)	2.6817*** (0.0004)	3.1702*** (0.0084)	3.0267*** (0.0091)
w^{MU}	24.8865*** (0.3097)	500.0225*** (0.0678	20.8562*** (0.2588)	484.7510*** (2.7561)	18.6724*** (0.3323)	16.4678*** (0.1786)	500.3641*** (285.4460)	21.1368*** (0.2448)	21.1070*** (0.3443)

Note: This table shows parameter estimates for the single-regime model introduced in Section 3.2.3 and the nested models, presented in Table 3.1. The explanatory variable in the long-term component is the Macroeconomic Uncertainty (MU) index of Jurado et al. (2015).

Table 3.42: Estimated GARCH-MIDAS models, short-term component, VIX

	EGARCH	AVGARCH	TGARCH	GARCH	GJRGARCH	QGARCH	NLGARCH	APGARCH	FGARCH
λ	0	1	1	2	2	2	1.5572*** (0.0010)	1.3528*** 0.7476*** (0.0000)	(0.0001)
$\hat{\lambda}$	1	1	1	2	2	2			1.7690*** (0.0013)
μ	0.0184*** (0.0000)	0.0358*** (0.0000)	0.0226*** (0.0000)	0.0349*** (0.0000)	0.0247*** (0.0000)	0.0158*** (0.0000)	0.0356*** (0.0000)	0.0204*** (0.0001)	0.0157*** (0.0000)
ω	-0.2291*** (0.0001)	0.0219*** (0.0000)	0.0347*** (0.0000)	0.0088*** (0.0000)	0.0127*** (0.0000)	0.0154*** (0.0000)	0.0134*** (0.0000)	0.0454*** (0.0000)	0.0288*** (0.0000)
α	0.0657*** (0.0000)	0.0492*** (0.0000)	0.0549*** (0.0000)	0.0459*** (0.0000)	0.0252*** (0.0000)	0.0378*** (0.0000)	0.0511*** (0.0000)	0.0573*** (0.0000)	0.0265*** (0.0000)
β	0.8915*** (0.0000)	0.8953*** (0.0000)	0.8602*** (0.0000)	0.8728*** (0.0000)	0.8410*** (0.0000)	0.8381*** (0.0000)	0.8812*** (0.0000)	0.8652*** (0.0000)	0.8574*** (0.0001)
γ	0.1248*** (0.0000)		0.9999*** (0.0004)		0.9940*** (0.0000)			0.9999*** (0.0000)	-0.4064*** (0.0002)
ψ						0.0490*** (0.0000)			0.0858*** (0.0000)
θ^{VIX}	0.0962*** (0.0000)	0.1033*** (0.0000)	0.0941*** (0.0000)	0.1024*** (0.0000)	0.0973*** (0.0000)	0.0965*** (0.0000)	0.1025*** (0.0000)	0.0941*** (0.0000)	0.0953*** (0.0000)
w^{VIX}	2.9667*** (0.0008)	5.5043*** (0.0008)	3.2290*** (0.0001)	5.0849*** (0.0009)	3.6076*** (0.0007)	2.8403*** (0.0012)	5.2112*** (0.0005)	3.1616*** (0.0004)	2.7464*** (0.0006)

Note: This table shows parameter estimates for the single-regime model introduced in Section 3.2.3 and the nested models, presented in Table 3.1. The explanatory variable in the long-term component is the Chicago Board Options Exchange Volatility Index (VIX).

Table 3.43: Estimated GARCH-MIDAS models, long-term component, FU-VIX

	EGARCH	AVGARCH	TGARCH	GARCH	GJRGARCH	QGARCH	NLGARCH	APGARCH	FGARCH
λ	0	1	1	2	2	2			0.9758***
							1.7289***	0.7021***	(0.0006)
$\hat{\lambda}$	1	1	1	2	2	2	(0.1441)	(0.0000)	1.8861***
									(0.0019)
μ	0.0232***	0.0360***	0.0275***	0.0357***	0.0280***	0.0206***	0.0359***	0.0247***	0.0206***
	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0126)	(0.0001)	(0.0000)
α	0.0608***	0.0417***	0.0584***	0.0395***	0.0000***	0.0354***	0.0415***	0.0606***	0.0143***
	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0222)	(0.0000)	(0.0000)
β	0.8944***	0.9142***	0.8611***	0.8932***	0.8459***	0.8515***	0.9006***	0.8678***	0.8821***
	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0539)	(0.0000)	(0.0000)
γ	0.1374***		0.9999***		0.1040***			0.9999***	-0.4777***
	(0.0000)		(0.0001)		(0.0000)			(0.0000)	(0.0003)
ψ						0.1434***			0.0954***
						(0.0000)			(0.0001)
m	-2.7767***	-2.5354***	-2.7023***	-2.6354***	-2.7741***	-2.7376***	-2.6241***	-2.6701***	-2.7338***
	(0.0011)	(0.0012)	(0.0012)	(0.0013)	(0.0012)	(0.0011)	(0.0708)	(0.0012)	(0.0012)
θ^{FU}	1.2876***	0.5649***	1.2864***	0.6920***	1.1736***	1.2681***	0.7000***	1.2783***	1.3629***
	(0.0018)	(0.0024)	(0.0021)	(0.0024)	(0.0022)	(0.0018)	(0.1965)	(0.0020)	(0.0017)
θ^{VIX}	0.0707***	0.0952***	0.0696***	0.0919***	0.0755***	0.0718***	0.0916***	0.0694***	0.0674***
	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0098)	(0.0000)	(0.0000)
w^{FU}	344.37***	502.85***	515.98***	500.50***	537.68***	499.53***	502.49***	506.86***	501.20***
	(0.8701)	(1.3287)	(2.6327)	(0.1162)	(1.1628)	(0.0156)	(225.81)	(1.2020)	(1.1177)
w^{VIX}	3.0933***	5.9704***	3.4586***	5.5619***	3.9186***	3.1593***	500.5424***	3.3671***	2.8283***
	(0.0008)	(0.0008)	(0.0000)	(0.0003)	(0.0003)	(0.0009)	(551.6)	(0.0003)	(0.0009)

Note: This table shows parameter estimates for the single-regime model with two explanatory variables in the long-term volatility component: the Financial Uncertainty (FU) index of Ludvigson et al. (2020) and the Chicago Board Options Exchange Volatility Index (VIX).

Table 3.44: Estimated GARCH-MIDAS models, long-term component, MU-VIX

	EGARCH	AVGARCH	TGARCH	GARCH	GJRGARCH	QGARCH	NLGARCH	APGARCH	FGARCH
λ	0	1	1	2	2	2	1.5960***	0.7171***	2.1212***
							(0.0006)	(0.0002)	(0.0013)
$\hat{\lambda}$	1	1	1	2	2	2			0.9016***
									(0.0005)
μ	0.0192***	0.0344***	0.0234***	0.0335***	0.0248***	0.0161***	0.0341***	0.0209***	0.0188***
	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0001)	(0.0003)	(0.0000)
α	0.0660***	0.0407***	0.0543***	0.0384***	0.0000***	0.0363***	0.0421***	0.0566***	0.1051***
	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
β	0.8990***	0.9218***	0.8705***	0.9018***	0.8548***	0.8551***	0.9096***	0.8762***	0.8313***
	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
γ	0.1249***		0.9999***		0.0967***			0.9999***	0.9921***
	(0.0000)		(0.0002)		(0.0000)			(0.0000)	(0.0000)
ψ						0.1342***			0.0508***
						(0.0000)			(0.0000)
m	-2.4106***	-2.2289***	-2.3614***	-2.3446***	-2.4102***	-2.3820***	-2.2681***	-2.3559***	-2.3257***
	(0.0016)	(0.0002)	(0.0021)	(0.0004)	(0.0021)	(0.0018)	(0.0036)	(0.0027)	(0.0018)
θ^{MU}	0.6455***	0.0486***	0.7149***	0.1782***	0.5391***	0.6507***	0.0857***	0.7529***	0.6807***
	(0.0032)	(0.0007)	(0.0042)	(0.0002)	(0.0044)	(0.0036)	(0.0077)	(0.0042)	(0.0037)
θ^{VIX}	0.0904***	0.1042***	0.0879***	0.1030***	0.0929***	0.0907***	0.1031***	0.0875***	0.0891***
	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0001)	(0.0000)	(0.0000)
w^{MU}	319.23***	500.23***	501.45***	1.4194***	337.52***	499.40***	500.52***	502.07***	500.62***
	(12.2276)	(3.0569)	(2.3833)	(0.0984)	(2.3530)	(1.3241)	(14.9688)	(15.0157)	(0.9705)
w^{VIX}	3.0438***	5.6491***	3.3720***	5.2475***	3.7477***	3.0398***	5.3719***	3.3059***	3.1569***
	(0.0015)	(0.0009)	(0.0000)	(0.0043)	(0.0004)	(0.0007)	(0.0038)	(0.0007)	(0.0006)

Note: This table shows parameter estimates for the single-regime model with two explanatory variables in the long-term volatility component: the Macroeconomic Uncertainty (MU) index of Jurado et al. (2015) and the Chicago Board Options Exchange Volatility Index (VIX).

Table 3.45: Estimated GARCH-MIDAS models, long-term component, FU-MU-VIX

	EGARCH	AVGARCH	TGARCH	GARCH	GJRGARCH	QGARCH	NLGARCH	APGARCH	FGARCH
λ	0	1	1	2	2	2	1.7420***	0.6999***	0.9757***
							(0.0008)	(0.0001)	(0.0006)
$\hat{\lambda}$	1	1	1	2	2	2			1.8856***
									(0.0033)
μ	0.0232***	0.0356***	0.0273***	0.0354***	0.0280***	0.0206***	0.0357***	0.0244***	0.0205***
	(0.0000)	(0.0001)	(0.0002)	(0.0001)	(0.0000)	(0.0001)	(0.0000)	(0.0000)	(0.0000)
α	0.0608***	0.0409***	0.0585***	0.0387***	0.0000***	0.0355***	0.0408***	0.0607***	0.0143***
	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
β	0.8945***	0.9146***	0.8613***	0.8934***	0.8458***	0.8517***	0.9017***	0.8683***	0.8825***
	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
γ	0.1374***		0.9999***		0.1037***			0.9999***	-0.4829
	(0.0000)		(0.0003)		(0.0000)			(0.0000)	(0.0003)
ψ						0.1434***			0.0957***
						(0.0000)			(0.0001)
m	-2.7823***	-2.3598***	-2.7209***	-2.4851***	-2.7531***	-2.7489***	-2.5159***	-2.6992***	-2.7579***
	(0.0010)	(0.0042)	(0.0019)	(0.0046)	(0.0004)	(0.0026)	(0.0021)	(0.0016)	(0.0014)
θ^{FU}	1.2818***	0.7163***	1.2659***	0.8379***	1.1964***	1.2563***	0.8220***	1.2462***	1.3383***
	(0.0019)	(0.0004)	(0.0100)	(0.0000)	(0.0040)	(0.0012)	(0.0012)	(0.0013)	(0.0016)
θ^{MU}	0.0186***	-0.5075***	0.0639***	-0.4607***	-0.0708***	0.0384***	-0.3585***	0.1006***	0.0814***
	(0.0004)	(0.0076)	(0.0137)	(0.0089)	(0.0056)	(0.0038)	(0.0033)	(0.0019)	(0.0007)
θ^{VIX}	0.0706***	0.0957***	0.0694***	0.0925***	0.0757***	0.0717***	0.0921***	0.0691***	0.0671***
	(0.0000)	(0.0000)	(0.0001)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)
w^{FU}	379.56***	507.98***	734.50	505.72***	468.68***	502.11***	161.90***	879.56	506.05***
	(8.2093)	(30.1051)	(503.49)	(11.6832)	(1.5770)	(36.7224)	(0.1813)	(560.95)	(30.232)
w^{MU}	228.308***	11.0856***	526.11	14.12***	311.096***	502.56***	43.486***	46.2464***	502.11***
	(1.7871)	(0.1894)	(675.04)	(0.4846)	(0.6380)	(46.2031)	(0.9149)	(2.5181)	(22.7349)
w^{VIX}	3.0922***	5.9295***	3.4592***	5.5287***	3.9199***	3.1609***	18.9269***	3.3746***	2.8295***
	(0.0011)	(0.0017)	(0.0108)	(0.0043)	(0.0002)	(0.0006)	(0.0051)	(0.0010)	(0.0007)
$\text{LRT}_{H_0:\text{FU-VIX}}$	[0.9998]	[0.3104]	[0.9802]	[0.3296]	[0.9608]	[0.9900]	[0.4404]	[0.9324]	[0.9512]
$\text{LRT}_{H_0:\text{MU-VIX}}$	[0.0000]	[0.0086]	[0.0000]	[0.0013]	[0.0000]	[0.0000]	[0.0112]	[0.0000]	[0.0000]

Note: This table shows parameter estimates for the single-regime model with three explanatory variables in the long-term volatility component: the Financial Uncertainty (FU) index of Ludvigson et al. (2020), the Macroeconomic Uncertainty (MU) index of Jurado et al. (2015) and the Chicago Board Options Exchange Volatility Index (VIX).

3.7.4 Proofs

Equivalent specification of model (3.3). In this appendix we show that specification (3.3) is equivalent to:

$$\begin{aligned} \ln g^2_{i,t} &= \omega + \alpha\left(|z_{i-1,t}| - \mathbb{E}|z_{i-1,t}|\right) - \gamma\, z_{i-1,t} - \psi\,\frac{\varepsilon_{i-1,t}}{\sqrt{\tau_t}} + \beta\,\ln g^{\,2}_{i-1,t}, && \text{if } \lambda = 0,\ \hat{\lambda} = 1, \\ g^{\lambda}_{i,t} &= \omega + \alpha\left(|z_{i-1,t}| - \gamma\, z_{i-1,t}\right)^{\hat{\lambda}} g^{\,\lambda}_{i-1,t} - \psi\,\frac{\varepsilon_{i-1,t}}{\sqrt{\tau_t}} + \beta\, g^{\,\lambda}_{i-1,t}, && \text{if } \lambda > 0,\ \hat{\lambda} > 0 \end{aligned}$$

Equation (3.3) can be written as:

$$\frac{g^{\lambda}_{i,t} - 1}{\lambda} = \omega^* + \alpha^*\left(|z_{i-1,t}| - \gamma^*\, z_{i-1,t}\right)^{\hat{\lambda}} g^{\,\lambda}_{i-1,t} - \psi^*\,\frac{\varepsilon_{i-1,t}}{\sqrt{\tau_t}} + \beta\,\frac{g^{\lambda}_{i-1,t} - 1}{\lambda} \tag{3.27}$$

- Case $\lambda = 0,\ \hat{\lambda} = 1$. In this case, $\lim_{\lambda\to 0^+} \frac{g^{\lambda}_{i,t}-1}{\lambda} = \ln g_{i,t}$ and (3.27) is given by

$$\ln g_{i,t} = \omega^* + \alpha^*\left(|z_{i-1,t}| - \gamma^*\, z_{i-1,t}\right) - \psi^*\,\frac{\varepsilon_{i-1,t}}{\sqrt{\tau_t}} + \beta\,\ln g_{i-1,t} \tag{3.28}$$

  Multiplying by 2 on both sides and subtracting and adding the constant mean of $|z_{i,t}|$, (3.28) becomes

$$\ln g^2_{i,t} = \omega + \alpha\left(|z_{i-1,t}| - \mathbb{E}\,|z_{i-1,t}|\right) - \gamma\, z_{i-1,t} - \psi\,\frac{\varepsilon_{i-1,t}}{\sqrt{\tau_t}} + \beta\,\ln g^2_{i-1,t} \tag{3.29}$$

  where $\alpha = 2\,\alpha^*$, $\omega = 2\omega^* + \alpha\,\mathbb{E}|z_{i-1,t}|$, $\gamma = \alpha\,\gamma^*$ and $\psi = 2\,\psi^*$.

- Case $\lambda > 0,\ \hat{\lambda} > 0$. It follows that (3.27) is equivalent to:

$$g^{\lambda}_{i,t} = \omega + \alpha\left(|z_{i-1,t}| - \gamma\, z_{i-1,t}\right)^{\hat{\lambda}} g^{\,\lambda}_{i-1,t} - \psi\,\frac{\varepsilon_{i-1,t}}{\sqrt{\tau_t}} + \beta\, g^{\lambda}_{i-1,t} \tag{3.30}$$

  where $\omega = \lambda\omega^* + 1 - \beta$, $\alpha = \lambda\alpha^*$, $\gamma = \gamma^*$ and $\psi = \lambda\psi^*$. □

Nesting of Conrad and Kleen's (2020) model. Assume one regime and impose $\lambda = \hat{\lambda} = 2$ and $\psi = 0$ in (3.4):

$$\begin{aligned} g_{i,t}^2 &= \omega + \alpha \left(|z_{i-1,t}| - \gamma\, z_{i-1,t}\right)^2 g_{i-1,t}^2 + \beta\, g_{i-1,t}^2 \\ &= \omega + \alpha \left[(1+\gamma^2)\, z_{i-1,t}^2 - 2\,\gamma\, |z_{i-1,t}|\; z_{i-1,t}\right] g_{i-1,t}^2 + \beta\, g_{i-1,t}^2 \\ &= \omega + \alpha \left[(1-\gamma)^2 + 4\,\gamma\, \mathbb{I}_{\{\varepsilon_{i-1,t}<0\}}\right] \frac{\varepsilon_{i-1,t}^2}{\tau_t} + \beta\, g_{i-1,t}^2 \end{aligned} \tag{3.31}$$

The last step in (3.31) follows since

$$(1+\gamma^2)\, z_{i-1,t}^2 - 2\,\gamma\, |z_{i-1,t}|\; z_{i-1,t} = \left\{ \begin{array}{ll} (1-\gamma)^2\; z_{i-1,t}^2, & \text{if } z_{i-1,t} \geq 0 \\ (1+\gamma)^2\; z_{i-1,t}^2, & \text{if } z_{i-1,t} < 0 \end{array} \right\} \tag{3.32}$$

Therefore, equation (3.31) is equivalent to:

$$g_{i,t}^2 = \omega + \left(\alpha^* + \gamma^*\, \mathbb{I}_{\{\varepsilon_{i-1,t}<0\}}\right) \frac{\varepsilon_{i-1,t}^2}{\tau_t} + \beta\, g_{i-1,t}^2 \tag{3.33}$$

where $\alpha^* = \alpha\,(1-\gamma)^2$ and $\gamma^* = 4\,\alpha\,\gamma$. This is the model proposed by Conrad and Kleen (2020)[25]. □

Recursive form of the vector of conditional expectations, $H^{\lambda}_{k,t+1|t}$. In this appendix, we show that the vector of conditional expectations,

$$\mathrm{H}^{\lambda}_{k,t+1|t} = \begin{bmatrix} \mathbb{E}\left(g^{\lambda}_{k,t+1} \mid s_{k,t+1} = 1,\, \mathcal{I}_t\right) \\ \mathbb{E}\left(g^{\lambda}_{k,t+1} \mid s_{k,t+1} = 2,\, \mathcal{I}_t\right) \end{bmatrix} \tag{3.34}$$

follows the recursive vector form in (3.21).

Let $\triangle = \alpha\, \mathbb{E}\left(|z_{i,t}| - \gamma\, z_{i,t}\right)^{\hat{\lambda}} + \beta$. From equation (3.9) and (3.11), we have

$$\mathbb{E}\left(g^{\lambda}_{k,t+1} \mid s_{k,t+1} = j, \mathcal{I}_t\right) = (1-\triangle) + \triangle\, \mathbb{E}\left[\mathbb{E}\left(g^{\lambda}_{k-1,t+1} \mid s_{k,t+1} = j, \mathcal{I}_{k-1,t+1}\right) \mid s_{k,t+1} = j, \mathcal{I}_t\right] \tag{3.35}$$

[25] Note that the authors include a constant parameter in the long-term component $\tau_{i,t}$ and not in the short-term component where they impose $w = 1 - \alpha - \gamma/2 - \beta$. See equation (3) in Conrad and Kleen (2020). We consider this specification in 3.2.2. Note also the different notation used in their paper - the authors use $\sqrt{g_{i,t}}$ to denote the short-term volatility component while we use $g_{i,t}$.

The conditional expectation over the conditional short-term in (3.35) can be obtained by

$$\begin{aligned}
&\mathbb{E}\left[\,\mathbb{E}(g^{\lambda}_{k-1,t+1} \mid s_{k,t+1} = j, \mathcal{I}_{k-1,t+1}) \mid s_{k,t+1} = j, \mathcal{I}_t\,\right] = \qquad (3.36)\\
&\quad = \mathbb{E}\left(g^{\lambda}_{k-1,t+1} \mid s_{k,t+1} = j, \mathcal{I}_t\right)\\
&\quad = \mathbb{E}\left[\sum_{q=1}^{2} \mathbb{P}\left(s_{k-1,t+1} = q \mid s_{k,t+1} = j,\, \mathcal{I}_t\right)\, g^{\lambda}_{k-1,t+1} \mid s_{k,t+1} = j,\, \mathcal{I}_t\right]\\
&\quad = \sum_{q=1}^{2} \int_{\mathcal{I}_t} \mathbb{P}\left(s_{k-1,t+1} = q \mid s_{k,t+1} = j,\, \mathcal{I}_t\right) \mathbb{P}\left(\mathcal{I}_t \mid s_{k,t+1} = j\right)\, g^{\lambda}_{k-1,t+1}\, d\mathcal{I}_t\\
&\quad = \sum_{q=1}^{2} \int_{\mathcal{I}_t} \mathbb{P}\left(\mathcal{I}_t \mid s_{k-1,t+1} = q,\, s_{k,t+1} = j\right) \mathbb{P}\left(s_{k-1,t+1} = q \mid s_{k,t+1} = j\right)\; g^{\lambda}_{k-1,t+1}\, d\mathcal{I}_t\\
&\quad = \sum_{q=1}^{2} \mathbb{P}\left(s_{k-1,t+1} = q \mid s_{k,t+1} = j\right) \mathbb{E}\left(g^{\lambda}_{k-1,t+1} \mid s_{k-1,t+1} = q,\, \mathcal{I}_t\right)
\end{aligned}$$

The last equality follows since the expected value given the current state does not depend on any future states. Therefore, $\mathbb{E}\left(g^{\lambda}_{k-1,t+1} \mid s_{k-1,t+1} = q,\, s_{k,t+1} = j,\, \mathcal{I}_t\right) = \mathbb{E}\left(g^{\lambda}_{k-1,t+1} \mid s_{k-1,t+1} = q,\, \mathcal{I}_t\right)$.

Let

$$\Psi \;=\; \triangle \begin{bmatrix} \mathbb{P}\left(s_{k-1,t+1} = 1 \mid s_{k,t+1} = 1\right) & \mathbb{P}\left(s_{k-1,t+1} = 2 \mid s_{k,t+1} = 1\right) \\ \mathbb{P}\left(s_{k-1,t+1} = 1 \mid s_{k,t+1} = 2\right) & \mathbb{P}\left(s_{k-1,t+1} = 2 \mid s_{k,t+1} = 2\right) \end{bmatrix}$$

From (3.35)-(3.36), we have

$$\mathrm{H}^{\lambda}_{k,t+1|t} \;=\; (1-\triangle)\; \mathbf{1}_{2\times 1} + \;\Psi\, \mathrm{H}^{\lambda}_{k-1,t+1|t} \;=\; \Psi^{k-1}\, \mathrm{H}^{\lambda}_{1,t+1|t} + \sum_{j=0}^{k-2} \Psi^{j}\; (1-\triangle)\; \mathbf{1}_{2\times 1}$$

and the proof is complete. □

Conclusion

The three main chapters of this dissertation focus on modelling and forecasting stock market volatility with Markov-switching GARCH models (MSGARCH). In Chapter 1, we present a flexible MSGARCH that nests a wide range of popular single-regime and regime-switching GARCH models. In the theoretical part of the chapter, we derive necessary and sufficient conditions for asymptotic stationarity of the model. In an empirical application, we apply the nested two-regime MSGARCH models and their single-regime counterparts to daily returns of the S&P 500 Index. We show that Markov-switching models accounting for an asymmetric impact of returns on volatility significantly outperform single-regime models and symmetric models both in explaining volatility and forecasting Value-at-Risk (VaR).

Chapter 2 further considers the risk management implications of modelling volatility with Markov-switching GARCH models. With this aim, we present a comparative study of a wide set of models including single-regime and two-regime GARCH models, nonparametric models and semiparametric models in terms of forecasting Value-at-Risk (VaR) and Expected Shortfall (ES). Using daily data for four stock indices, the S&P 500, Dow Jones Industrial Average, the NIKKEI 225 and the FTSE 100, we show that Markov-switching GARCH models, in particular our more flexible specification, significantly outperform all other models in the set of competing models.

In Chapter 3, we focus on forecasting volatility with financial and macroeconomic uncertainty. Following the seminal paper by Engle et al. (2013), we combine our proposed MS-GARCH models with the MIxed Data Sampling (MIDAS) approach of Ghysels et al. (2004). This combination allows the inclusion of macroeconomic variables observed at a monthly, quarterly or even lower frequency into the specification of daily volatility. We consider the monthly Financial Uncertainty (FU) index of Jurado et al. (2015), the monthly Macroeconomic Uncertainty (MU) index of Ludvigson et al. (2020) and the daily Chicago Board Options Exchange Volatility Index (VIX) as explanatory variables. We show that the inclusion of macroeconomic and financial uncertainty indices when modelling S&P 500 returns significantly improves volatility predictions. In particular, the daily VIX index provides more accurate 1 day-ahead volatility forecasts. On the other hand, the monthly MU and FU indices provide more accurate forecasts at longer horizon of 2 weeks, 1 month, 2 months and 3 months. Thus, our findings are useful for selecting the appropriate explanatory variable based on the required forecasts horizon: while high-frequency indices are more appropriate for short-horizon forecasts, low-frequency indices are more suitable for longer forecasts horizons. Furthermore, we show that our proposed MSGARCH-MIDAS model outperforms all other single-regime and regime-switching models nested within it. This superiority is consistent across forecasts horizons and robust to the sample period under consideration.

Throughout this thesis, we have shown that there is strong evidence of two regimes in equity returns and that failure to account for regime changes in the dynamics of the volatility leads to biased inference and misleading out-of-sample forecasts. As a direction for further research, we see the implications of considering more than two regimes in our model. For instance, we could allow for more than two regimes to study how the stock market reacted during the COVID-19 pandemic to macroeconomic policies and health policies implemented during the crisis (i.e., Incubation, Outbreak, Fear and Rebound phases). Capelle-Blancard and Desroziers (2020) divide their sample from January 2020 to April 2020 into sub-samples to account for these four phases. We argue that incorporating different regimes in the model itself would provide a better understanding of how the return volatility is affected during the crisis. We leave this for future research.

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