An efficient reduced-order method for stochastic eigenvalue analysis

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Abstract

This paper presents an efficient numerical algorithm to compute eigenvalues of stochastic problems. The proposed method represents stochastic eigenvectors by a sum of the products of unknown random variables and deterministic vectors. Stochastic eigenproblems are thus decoupled into deterministic and stochastic analyses. Deterministic vectors are computed efficiently via a few number of deterministic eigenvalue problems. Corresponding random variables and stochastic eigenvalues are solved by a reduced-order stochastic eigenvalue problem that is built by deterministic vectors. The computational effort and storage of the proposed algorithm increase slightly as the stochastic dimension increases. It can solve high-dimensional stochastic problems with low computational effort, thus the proposed method avoids the curse of dimensionality with great success. Numerical examples compared to existing methods are given to demonstrate the good accuracy and high efficiency of the proposed method.

Keywords: Structural stochastic eigenvalues, Stochastic finite element method, Reduced-order equations, High-dimensional problems, Curse of dimensionality

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1. Introduction

Developments in numerical techniques and computing hardware have made it possible to solve high-resolution models in various computational physics problems. The considerable influence of uncertainties on system behavior has led to the development of dedicated numerical methods for uncertainty analysis. Predicting uncertainty propagation on the physical models has become an essential part of the analysis and design of practical engineering systems.

As an important part of structural dynamics analysis, the eigenvalue problem has received extensive attention and is well understood for deterministic problems [1, 2]. Its extension, known as the stochastic eigenvalue problem, is introduced to consider influence of uncertainties on system dynamics analysis. There are several kinds of methods for solving stochastic eigenvalue problems. The first kind of method is Monte Carlo simulation (MCS) [3]. As a most powerful tool for stochastic analysis, MCS can be applied to almost all kinds of stochastic problems. MCS is very easy to implement by use of the already existing deterministic solvers but has poor convergence. A large number of deterministic eigenvalue problems are solved in order to compute high-accuracy stochastic eigenvalues, which are computationally expensive, especially for large-scale problems. Several improvements, e.g. quasi-MCS and multilevel MCS [4], are used to save computational costs of MCS. The second kind of method is perturbation methods [5]. The perturbation approach expands the random variables around their mean values to first/second order by use of Taylor series. It is easy to implement and has high computational efficiency. However, being limited to small variability and only giving statistical moments make the method unsuitable for complex stochastic problems. Several improvements of perturbation methods are given in references [6, 7]. The third kind of method is the Polynomial chaos-based (PC) method [8]. In this kind of method, the stochastic matrices, the stochastic eigenvectors and the stochastic eigenvalues are projected onto a stochastic space spanned by (generalized) PC basis. Stochastic Galerkin method is then used to transform the original stochastic eigenvalue problem into a nonlinear system of coupled deterministic equations whose size is much larger than that of the original problem. The Newton-Raphson method is adopted to solve the nonlinear system. PC-based methods have rigorous mathematical foundation and good convergence. It is general-purpose and can be applied

to both real- and complex-valued stochastic eigenvalue problems. However, the computational efficiency of the PC-based method needs to be further improved. Several improvements [7, 9] are proposed to reduce computational costs. Other extensions of PC-based methods are to solve stochastic eigenvalue problems by combing PC expansion and deterministic numerical techniques, e.g. power method [10], inverse power method [11, 12, 13], subspace iteration [14, 15, 16]. Other methods are also proposed to solve stochastic eigenvalue problems. The stochastic collocation method [17] is to approximate stochastic eigenvalues and eigenvectors via an interpolation approach. It is non-intrusive and only uses deterministic solvers to solve a set of random samples of the solutions. The methods in references [18, 19] transform the stochastic eigenvalue problem into initial equations with a pseudo time parameter, which is then solved by the PC-based method. The homotopy approach is proposed in the reference [20], which expands stochastic eigenvalues and eigenvectors by an infinite multivariate series and adopts homotopy analysis to compute expanded coefficients. Stochastic eigenvalues and eigenvectors are solved by combining PC expansion and dedicatedly iterative algorithms for low-rank approximations in references [21, 22].

We mention another method, known as reduced-order method (ROM). In this method, stochastic eigenvalues and eigenvectors are solved by a reduced-order (normally small-scale) eigenvalue problem, which is obtained by projection subspace of the large-scale problem. Solutions of the reduced-order eigenvalue problem are very close to the exact solutions if the subspace is similar to the space of stochastic eigenvectors of the original problem [1]. The key of this kind of method is to construct a good projection approximation subspace. Several methods are proposed for this purpose, e.g. perturbation-based subspace [23], optimization-based subspace [24, 25], stochastic Krylov subspace [26] and subspace of mean matrices [27]. Another point that needs to pay more attention is the stochastic dimension, which has significant influence on the computational accuracy and efficiency of numerical methods for stochastic eigenvalue problems. There are usually a large number of uncertain parameters in many applications, e.g. the input is approximated by random fields with a large number of random variables, which leads to the curse of dimensionality in high-dimensional stochastic spaces. MCS and its extensions [4] can overcome curse of dimensionality but a large number of deterministic simulations are needed. Stochastic collocation method [17] needs to construct high-dimensional interpolation formulas of the solutions based

on a certain amount of deterministic simulations. The computational effort of PC-based methods increases dramatically as the number of stochastic dimensions and/or the order of PC basis increase, which is prohibitively expensive. Sparse PC method is an available tool to reduce the computational effort [28].

We also highlight another kind of method called polynomial dimension decomposition (PDD) [29]. Similar to PC-based methods, orthogonal polynomial bases are used in the PDD method to approximate stochastic responses. The PDD method develops a dimensional hierarchy of the stochastic response, which alleviates the curse of dimensionality suffered from PC-based methods. The application of the PDD method in stochastic eigenvalue problems is given in references [30, 31], which allows lower-variate approximations of stochastic eigenvalues and lower-dimensional numerical integration for the statistical moments. To avoid the difficulty that the polynomial basis does not work well in capturing strongly local variations of solutions, e.g. non-smooth and discontinuous stochastic solutions, a spline chaos expansion [32] is also proposed, which represents stochastic solutions by using orthogonal B-spline bases. Combining and generalizing ideas of the dimension decomposition and the spline chaos expansion, the spline dimensional decomposition method is developed to solve the high-dimensional and dynamical stochastic problems [33, 34], which avoids the curse of dimensionality to a great extent. The above methods have become powerful methods in uncertainty quantification and have great potential for very complex stochastic problems.

Although a lot of excellent work have been discussed above, the development of numerical methods for stochastic eigenvalue problems is still an attractive topic, especially robust, efficient and accurate methods for solving large-scale, high-dimensional stochastic eigenvalue problems. In this paper, we focus on developing efficient numerical algorithms to compute first several maximum/minimum stochastic eigenvalues of problems with high stochastic dimensions and large scale. The stochastic eigenvectors are firstly approximated by summing a set of products of random variables and deterministic vectors. The deterministic vectors (also considered as a set of reduced basis) are computed via a few number of decoupled deterministic eigenvalue problems and a dedicated iterative algorithm, where the deterministic eigenvalue problems are obtained by combining an approximation of the stochastic eigenvector and stochastic Galerkin method.

A reduced-order stochastic eigenvalue problem based on the obtained reduced basis is used to solve stochastic eigenvalues of the original problem and random coefficients of the reduced basis. All stochastic analysis, including solving reduced basis and solving the reduced-order stochastic eigenvalue problem, are implemented by non-intrusive sampling methods, which has less computational effort and is almost independent of the stochastic dimension. The curse of dimensionality induced by the high-dimensional stochastic problem is thus avoided, which is demonstrated by using a numerical example with up to a hundred stochastic dimensions. Also, compared to PC and PDD methods, the non-intrusiveness of the proposed method is highlighted, which does not require dedicated approximation structures of the solutions in stochastic spaces. The proposed method is easy to implement and existing solvers can be embedded into solving procedure. It combines the fast convergence of intrusive methods and the weak dimensionality dependency of non-intrusive methods. Another advantage is that the proposed method obtains sample representations of stochastic eigenvalues and semi-explicit representations of stochastic eigenvectors, which provides a pathway to describe the probability density function of the quantity of interest and can be readily applied to structural stochastic dynamical analysis [35, 36, 37].

The paper is organized as follows: Section 2 gives the basic setting of the stochastic eigenvalue problem and a brief description of the PC method for solving stochastic eigenvalue problems. In Section 3, we propose a new reduced-order method to solve stochastic eigenvalue problems, including the construction of stochastic eigenvectors, the solution of reduced basis, the applicability to high-dimensional stochastic problems and the proofs of convergence and optimal approximation. The algorithm implementation of the proposed method is elaborated in Section 4. Following that, several numerical examples of low- and high-dimensional cases are given to demonstrate the performance of the proposed method in Section 5, and conclusions and discussions follow in Section 6.

2. Stochastic eigenvalue problems

In this paper, let $(\Theta, \Sigma, \mathcal{P})$ be a complete probability space, where Θ denotes the space of elementary events, Σ is the σ-algebra defined on Θ and $\mathcal{P}$ is the probability measure. We consider

the following stochastic eigenvalue equation,

$$K(\theta)\, u(\theta) = \lambda(\theta)\, u(\theta), \tag{1}$$

where $\theta \in \Theta$, $K(\theta) \in \mathbb{R}^{n\times n}$ is the real symmetric and positive definite stochastic stiffness matrix related to stochastic properties of the physical models, $\lambda(\theta) \in \mathbb{R}$ and $u(\theta) \in \mathbb{R}^n$ are stochastic eigenvalues and stochastic eigenvectors, respectively. The orthonormal condition

$$u_i^T(\theta)\, u_j(\theta) = \delta_{ij},\ \theta \in \Theta \tag{2}$$

is met for the eigenvectors $u_i(\theta)$ and $u_j(\theta)$, where δ_{ij} is the Kronecker delta function. In a general setting, the matrix $K(\theta)$ depends on a finite set of real valued random variables reduced from the infinite-dimensional probability space. When the inputs are random fields represented with a set of random variables, the number of random variables may be large, which possibly induces the so called curse of dimensionality.

2.1. Polynomial Chaos expansion for solving stochastic eigenvalue problems

Polynomial Chaos expansion is a powerful tool for stochastic analysis and has been applied to many kinds of stochastic problems. A PC-based method [8] is proposed in order to solve Eq. (1). In this method, $K(\theta)$, $\lambda(\theta)$ and $u(\theta)$ are represented by PC basis in the form,

$$K(\theta) = \sum_{i=0}^{l-1} \psi_i(\theta)\, K_i,\ u_m(\theta) = \sum_{i=0}^{p-1} \psi_i(\theta)\, u_m^{(i)},\ \lambda_m(\theta) = \sum_{i=0}^{p-1} \psi_i(\theta)\, \lambda_m^{(i)}, \tag{3}$$

where $\{K_i\}_{i=0}^{l-1} \in \mathbb{R}^{n\times n}$ are deterministic matrices, $\left\{u_m^{(i)}\right\}_{i=0}^{p-1} \in \mathbb{R}^n$ are deterministic vectors and $\left\{\lambda_m^{(i)}\right\}_{i=0}^{p-1} \in \mathbb{R}$ are expanded coefficients. All of them are unknown and need to be computed by a coupled system of equations. Moreover, the orthonormal condition Eq. (2) is written as

$$u_k^T(\theta)\, u_m(\theta) = \sum_{i=0}^{p-1}\sum_{j=0}^{p-1} \psi_i(\theta)\, \psi_j(\theta)\, u_k^{(i)T} u_m^{(j)} = \delta_{km}. \tag{4}$$

Substituting the expansion Eq. (3) into Eq. (1) and applying stochastic Galerkin method [38, 39] to Eq. (1) and (4) yield a deterministic system of equations

$$\sum_{i=0}^{l-1}\sum_{j=0}^{p-1} c_{ijq} K_i u^{(j)} = \sum_{i=0}^{p-1}\sum_{j=0}^{p-1} c_{ijq} \lambda_i u^{(j)} \tag{5}$$

and the equation of the orthonormal condition

$$\sum_{i=0}^{p-1}\sum_{j=0}^{p-1} c_{ijq}\boldsymbol{u}_k^{(i)T}\boldsymbol{u}_m^{(j)} = \delta_{km}\delta_{q0}, \tag{6}$$

where the coefficients is given by $c_{ijq} = \mathbb{E}\left\{\psi_i(\theta)\psi_j(\theta)\psi_q(\theta)\right\}$, $q = 0, \cdots, p-1$. Eq. (5) and (6) can be rewritten as a compact system of nonlinear equations,

$$\sum_{i=0}^{p-1}\sum_{j=0}^{p-1}\left(\mathbf{B}_i\boldsymbol{\Gamma}_i - \lambda_i\boldsymbol{\Gamma}_i\right)\mathbf{u} = 0,\ \mathbf{B}_i = 0,\ i > l \tag{7}$$

and

$$\mathbf{u}_k^T\boldsymbol{\Gamma}_q\mathbf{u}_m = \delta_{km}\delta_{q0}, \tag{8}$$

where the matrices $\mathbf{B}_i$ and $\boldsymbol{\Gamma}_i$ are given by

$$\mathbf{B}_i = \begin{bmatrix} K_i & & 0 \\ & \ddots & \\ 0 & & K_i \end{bmatrix} \in \mathbb{R}^{np\times np},\ \boldsymbol{\Gamma}_i = \left[c_{ijq}I_n\right]_{j,q=0}^{p-1} \in \mathbb{R}^{np\times np}, \tag{9}$$

$I_n \in \mathbb{R}^{n\times n}$ is the identity matrix.

The above PC-based method gives a powerful tool to solve stochastic eigenvalue problems. It has a rigorous mathematical foundation and is of high accuracy. The method can provide probability density descriptions for eigenvalues and eigenvectors instead of moments. However, it needs to be further improved in terms of computational efficiency. On one hand, it requires to solve the augmented nonlinear system Eq. (7) for each eigenpair $\{\lambda(\theta), u(\theta)\}$. On the other hand, the size of Eq. (7) is prohibitively large as the stochastic dimension, the order of PC basis and the degree of freedom of the physical model increase, whose solution is time-consuming. We remark that the dimension decomposition method, including polynomial dimension decomposition [29] and the spline dimension decomposition[34], can be used to reduce the computational cost and capture high-accuracy stochastic solutions. Similar to the PC method, the method adopts orthogonal polynomial/spline bases to approximate stochastic solutions and develops a hierarchical decomposition to overcome some difficulties arising in PC methods, e.g. the curse of dimensionality, capturing discontinuous stochastic solutions. An exhaustively comparative study of the two methods of solving stochastic eigenvalue problems can be found in the reference [31]. In this paper,

we will explore a non-intrusive method instead of intrusive solution approximations in stochastic spaces.

3. A reduced-order method for stochastic eigenvalue analysis

In this section, we develop an efficient method to solve the stochastic eigenproblem Eq. (1). A new expansion similar to the PC expansion is used to approximate stochastic eigenvectors, but the random coefficients and deterministic vectors are not known a priori. A numerical algorithm is proposed to compute corresponding deterministic vectors and a reduced-order eigenequation is then used to solve stochastic eigenvalues of the original eigenproblem.

3.1. Reduced-order stochastic eigenvalue equation

Similar to the expansion Eq. (3), we decompose the stochastic eigenvector into the deterministic and stochastic spaces and consider the expansion of the stochastic eigenvector $u_m(\theta)$ in a form

$$u_m(\theta) = \sum_{i=1}^{k} \phi_m^{(i)}(\theta)\, d_i = D\phi_m(\theta) \in \mathbb{R}^n, \tag{10}$$

where the deterministic vector $d_i \in \mathbb{R}^n$, the random variable $\phi_m^{(i)}(\theta) \in \mathbb{R}$, the deterministic matrix $D = [d_i]_{i=1}^k \in \mathbb{R}^{n\times k}$, the random vector $\phi_m(\theta) = \left[\phi_m^{(i)}(\theta)\right]_{i=1}^k \in \mathbb{R}^k$. All of them are not known a priori and need to be solved. Moreover, we let the orthonormal condition $D^T D = I_k$ hold, i.e. the vectors $\{d_i\}_{i=1}^k$ are orthogonal $d_i^T d_j = \delta_{ij}$. The original eigenproblem Eq. (1) thus becomes as

$$K(\theta)\, D\phi_m(\theta) = \lambda_m(\theta)\, D\phi_m(\theta). \tag{11}$$

It is noted that Eq. (11) is insoluble since both the matrix D and the random vector $\phi_m(\theta)$ are unknown. If one of them has been known, the other is readily computable. In this way, an available way is to fix one of them and then to solve the other. Inspired by the classical subspace iteration method [1, 2] for solving deterministic eigenvalue problems and reduced basis methods, we assume the matrix D has been known and then solve the unknown random vector $\phi_m(\theta)$. By use of the matrix D, a reduced-order eigenvalue problem can be obtained,

$$\widetilde{K}_k(\theta)\, \phi_m(\theta) = \lambda_m(\theta)\, \phi_m(\theta), \tag{12}$$

where the reduced-order stochastic matrix $\widetilde{K}_k(\theta)$ is given by $\widetilde{K}_k(\theta) = D^T K(\theta) D \in \mathbb{R}^{k\times k}$ and $\{\phi_m(\theta)\}_m$ meets the orthonormal condition $\phi_i^T(\theta)\phi_m(\theta) = \delta_{im}$ almost everywhere (a.e.) since they are eigenvectors of the eigenequation (12). In this way, we recall the the orthonormal condition Eq. (2) of the original eigenproblem,

$$u_i^T(\theta) u_j(\theta) = \phi_i^T(\theta) D^T D \phi_j(\theta) = \delta_{ij} \tag{13}$$

holds naturally, thus extra equations (like Eq. (4)) are not needed to impose the orthonormal condition. The size of the reduced-order problem Eq. (12) is k, which is much lower than the size of the original eigenproblem Eq. (1). In our experience, a small number k normally achieves a good approximation of the stochastic eigenvectors of the original eigenproblem. Several methods can be used to solve Eq. (12) efficiently and accurately, e.g. the Monte Carlo simulation and the PC-based method described in Section 2.1. In order to enable the proposed method to solve high-dimensional stochastic eigenproblems, we adopt a non-intrusive sampling method to solve Eq. (12) in this paper, which is easy to implement and has high accuracy and high efficiency thanking to the small size k. By Eq. (12), the randomness of the matrix $K(\theta)$ is transferred to the reduced-order matrix $\widetilde{K}_k(\theta)$. Thus the input random variables are propagated through a reduced-order eigenvalue system to the stochastic eigenvalues of both reduced-order and original eigenequations and the stochastic eigenvectors of the reduced-order eigenequations. However, compared to the PC method, the proposed method cannot provide an explicit representation between the input random variables and the stochastic eigenvalues/eigenvectors. To avoid this point, an available method is to represent the obtained $\lambda_m(\theta)$ and $\phi_m(\theta)$ using the PC basis.

The key part of the proposed method is to determine the matrix $D = [d_i]_{i=1}^k$. For this purpose, we solve the vector d_i one by one by considering the following approximation of the stochastic eigenvector $u(\theta)$,

$$\min_{d\in\mathbb{R}^n} \|u(\theta) - d\|^2, \tag{14}$$

where $\|\cdot\|^2$ is defined as $\|u(\theta)\|^2 = \mathbb{E}\left\{u(\theta)^T u(\theta)\right\}$ and $d \in \mathbb{R}^n$ is an unknown deterministic vector. From a vector approximation point of view, we approximate the stochastic eigenvector $u(\theta)$ by using the deterministic vector d. The approximation accuracy is very low due to the loss of randomness in the vector d, thus it is wrong to some extent. To illustrate the reasonableness of

the approximation, we consider Eq. (14) from a subspace point of view. In Section 3.5, we will show that the stochastic eigenvalues of Eq. (12) converge to the stochastic eigenvalues of Eq. (1) if the stochastic vector $u(\theta)$ is nearly in the subspace obtained by deterministic vectors $\{d_i\}_i$. In this sense, Eq. (14) is used to construct a subspace that nearly includes the stochastic vector $u(\theta)$. In practice, the stochastic eigenvector $u(\theta)$ is not known a priori, thus the vector d cannot be computed directly by using Eq. (14). To avoid this difficulty, we substitute the stochastic vector $u(\theta) = d$ into the eigenproblem Eq. (1) and thus obtain the following stochastic residual

$$\mathcal{R}(\theta) = [K(\theta) - \lambda(\theta) I_n] d \in \mathbb{R}^n, \tag{15}$$

only the random variable $\lambda(\theta)$ and the deterministic vector d in which are unknown. Thus the problem is to find $\lambda(\theta)$ and d to minimize $\|\mathcal{R}(\theta)\|^2$,

$$\min_{\lambda(\theta)\in\mathbb{R},\ d\in\mathbb{R}^n} \|\mathcal{R}(\theta)\|^2 = \min_{\lambda(\theta)\in\mathbb{R},\ d\in\mathbb{R}^n} \|K(\theta) d - \lambda(\theta) d\|^2. \tag{16}$$

Remark 1. *From a vector approximation point of view, a better approximation of the stochastic eigenvector $u(\theta)$ is given as follows*

$$\min_{\varphi(\theta)\in\mathbb{R},\ d\in\mathbb{R}^n} \|u(\theta) - \varphi(\theta) d\|^2, \tag{17}$$

where $\varphi(\theta) \in \mathbb{R}$ is a unknown random variable. Eq. (17) *can be considered as a kind of rank-1 random singular value decomposition (SVD) of $u(\theta)$. Specifically, giving the sample representation $u(\boldsymbol{\theta}) \in \mathbb{R}^{n\times n_s}$ (n_s is the number of random samples) of $u(\theta)$ we have the following rank-1 SVD,*

$$u(\boldsymbol{\theta}) \approx d\varphi(\boldsymbol{\theta}),\ \varphi(\boldsymbol{\theta}) \in \mathbb{R}^{1\times n_s}. \tag{18}$$

In the sense of sample representation, Eq. (17) *is the optimal rank-1 approximation [40] of $u(\theta)$. But it is not easy to solve the couple $\{\varphi(\theta), d\}$ since both of them are unknown. The solution of Eq.* (17) *needs further study and we only consider the approximation Eq.* (14) *in this paper.*

We now focus on computing the random variable $\lambda(\theta)$ and the deterministic vector d in Eq. (16). To this end, we develop an alternating minimization iteration, the idea of which is to fix one of $\lambda(\theta)$ and d to solve the other and then to update the fixed one according to the solution.

Specifically, for a known random variable $\lambda(\theta)$ (or a given initial value), we apply the stochastic Galerkin method [38, 39] to the stochastic residual Eq. (15) and transform it as

$$\mathbb{E}\{\lambda(\theta)\mathcal{R}(\theta)\} = 0, \tag{19}$$

which is used to solve the vector d and can be rewritten as a compact form,

$$K^* d = \lambda^* d, \tag{20}$$

where the deterministic matrix $K^* = \mathbb{E}\{\lambda(\theta) K(\theta)\} \in \mathbb{R}^{n\times n}$ and $\lambda^* = \mathbb{E}\left\{\lambda^2(\theta)\right\} \in \mathbb{R}$. Eq. (20) is a classically deterministic eigenvalue equation, which can be solved by use of existing methods, e.g. power method, Lanczos method and QR method [2]. The details will be discussed in the next subsection. After solving the vector d by Eq. (20), the random variable $\lambda(\theta)$ is updated by the Galerkin procedure,

$$d^T \mathcal{R}(\theta) = 0, \tag{21}$$

equivalently,

$$\lambda(\theta) = \frac{d^T K(\theta) d}{d^T d} \in \mathbb{R}, \tag{22}$$

which can be simplified as $\lambda(\theta) = d^T K(\theta) d$ by considering the normalization $d^T d = 1$.

There are stochastic computations involved in Eq. (20) and (22), i,e. the expectation $\mathbb{E}\{\lambda(\theta) K(\theta)\}$ and the deterministic vector-stochastic matrix multiplication $d^T K(\theta) d$. A common method is to approximate $K(\theta)$ and $\lambda(\theta)$ by use of an Eq. (3)-like expansion. In this method, the size of the equation for solving expanded coefficients of $\lambda(\theta)$ is $(m+p)!/(m!p!)$, where $(\cdot)!$ represents the factorial operator, m and p are the number of random variables and the order of PC basis. It increases dramatically as the stochastic dimension increases, for instance, the size is about 1×10^3 when $m = 10$, $p = 4$ and 4.6×10^6 when $m = 100$, $p = 4$. An available method to avoid the difficulty is sparse PC expansion [28]. In order to overcome the dependence on stochastic dimensions, we adopt a non-intrusive sampling method,

$$\mathbb{E}\{\lambda(\theta) K(\theta)\} = \mathbb{E}\{\lambda(\boldsymbol{\theta}) K(\boldsymbol{\theta})\},\ \lambda(\boldsymbol{\theta}) = d^T K(\boldsymbol{\theta}) d \in \mathbb{R}^{n_s}, \tag{23}$$

where $\lambda(\boldsymbol{\theta}) \in \mathbb{R}^{n_s}$ and $K(\boldsymbol{\theta}) \in \mathbb{R}^{n\times n\times n_s}$ represent samples of the random variable $\lambda(\theta)$ and the matrix $K(\theta)$, respectively. It is noted that Eq. (23) has low computational effort even for very high

stochastic dimensions, the applicability of which to high-dimensional problems will be discussed in Section 3.3.

It is seen from Eq. (20) and (22) that $d^T K(\theta) d \neq 0$ needs to be hold. In other words, the proposed method does not work on the case $\lambda(\theta) = 0$. If there are zero eigenvalues in the problem, we can adopt the frequency-shifting strategy [2] to move the eigenvalues away from zero. We remark that although the proposed method is derived from real symmetric and positive definite stochastic matrices, it can be extended to more general stochastic matrices, e.g. the non-symmetric complex stochastic matrices, which is simply illustrated in Example 5.1.3. Also, nonlinear stochastic eigenvalue problems arise in many practical problems [31, 33]. The current version of the proposed method cannot solve nonlinear stochastic eigenvalue problems well. The proposed method is possible to be extended to the nonlinear eigenvalue problems by combining the idea in this paper and deterministic nonlinear eigenvalue methods [41, 42], which is out of the scope of this paper and an exhaustive study of which will be presented in following-up studies. In addition, the proposed method does not require the correlation of input random variables. As shown in Eq. (23), we adopt a non-intrusive way to perform stochastic computations. After generating samples of correlated/independent input random variables, all computations of the proposed method are the same. Thus the proposed method can be applied to both correlated and independent input random variables.

3.2. Solution of deterministic eigenvectors

In this section, we simply discuss the method for solving the deterministic eigenproblem Eq. (20). For each vector d, the single vector iteration method is enough for our purpose. In this paper, we adopt the power iteration for the explanation of our method. For complex and large-scale problems in practice, other numerical methods can be found in the reference [2] and they can be readily used as an alternative to the power method in this paper.

By adopting the power iteration to compute the maximum eigenvalue of Eq. (20), a new solution $d^{(j)}$ is computed based on a known approximation $d^{(j-1)}$,

$$d^{(j)} = K^* d^{(j-1)}, \tag{24}$$

where the deterministic matrix is inherited from Eq. (20) and d is the deterministic vector to be solved. The iteration Eq. (24) is stopped until d converges.

Although Eq. (24) is only used to solve a single vector d, it can be readily extended to solve a set of vectors $d_1, \cdots, d_k$. To illustrate this point, assuming that the first $k-1$ vectors $d_1, \cdots, d_{k-1}$ have been known, we calculate the k-th vector d_k. Eq. (24) is still adopted

$$d_k^{(j)} = K^* d_k^{(j-1)}, \tag{25}$$

which is stopped until the vector d_k converges. It is noted that d_k is the eigenvector of the different matrix $K^* = \mathbb{E}\{\lambda_k(\theta) K(\theta)\}$ since $\lambda_k(\theta)$ are different random variables that vary with k. To speed up the computation and to avoid the overlapping eigenmodes, we let the vector $d_k^{(j)}$ orthogonal to the already obtained vectors $d_1, \cdots, d_{k-1}$. Here we utilize Gram-Schmidt orthonormalization

$$d_k^{(j)} = d_k^{(j)} - \sum_{i=1}^{k-1} \frac{d_k^{(j)T} d_i}{d_i^T d_i} d_i, \; d_k^{(j)} = d_k^{(j)} \Big/ \left(d_k^{(j)T} d_k^{(j)} \right), \tag{26}$$

which needs to hold along whole iterative process of Eq. (25) until d_k converges. The iterations Eq. (25) and (26) are very similar to the classical power method for solving the eigenvectors of a deterministic matrix, but the randomness is embedded into the matrix K^* in this paper, which allows the subspace obtained by $[d_1, \cdots, d_k]$ to be a good approximation of the space of the first few stochastic eigenvectors. In the same way, we can compute new vector $d_{k+1}, d_{k+2}, \cdots$ by using Eq. (25) until the specified number of items is calculated. Similarly, the inverse power method can be used to compute deterministic vectors $\{d_i\}_i$ that are used to approximate the first several minimum stochastic eigenvectors.

We remark that it is suggested to only use the proposed method to compute the first few maximum and minimum stochastic eigenvalues and eigenvectors, although it can be used to solve more eigenvalues and eigenvectors. Much more reduced basis $\{d_i\}_i$ are required when a large number of stochastic eigenvalues and stochastic eigenvectors are considered. Although the calculation of deterministic vectors $\{d_i\}_i$ is readily implemented and can be accelerated and improved by other methods, e.g. the Lanczos method and the QR method, the size of the reduced-order eigenequation (12) increases as the number of reduced basis $\{d_i\}_i$. More computational effort is needed for the solution of Eq. (12). Hence, although the proposed method can be applied to calculate a

large number of eigenvalues and eigenvectors, it only speeds up the computation to a lesser extent. An extreme case is that when we consider all eigenvalues and eigenvectors of the matrix $K(\theta)$, the size of reduced-order matrix D is the same as the stochastic matrix $K(\theta)$ and the size of the reduced-order Eq. (12) is the same as the original problem, which makes the "reduced-order" no sense.

3.3. High-dimensional stochastic eigenvalue problems

In this subsection, we show that the proposed method can be applied to high-dimensional stochastic eigenvalue problems without any modification and extra computational effort. We assume that the stochastic matrix $K(\theta)$ can be represented in a series expansion form

$$K(\theta) = \sum_{j=1}^{r} \xi_j(\theta) K_j, \tag{27}$$

where $\left\{\xi_j(\theta)\right\}_{j=1}^{r}$ are random variables described by probability distributions, random samples or PC approximation, $\left\{K_j\right\}_{j=1}^{r} \in \mathbb{R}^{n\times n}$ are deterministic matrices. High-dimensional cases are induced by a large value of r. For non-separated stochastic matrices, Eq. (27) can be obtained by PC expansion in Eq. (3), or a third-order tensor of random samples $K(\boldsymbol{\theta}) \in \mathbb{R}^{n\times n\times n_s}$ is generated by using non-intrusive methods.

We introduce the following sample matrix of random variables $\left\{\xi_j(\theta)\right\}_{j=1}^{r}$,

$$\xi(\boldsymbol{\theta}) = \begin{bmatrix} \xi_1\left(\theta^{(1)}\right) & \cdots & \xi_r\left(\theta^{(1)}\right) \\ \vdots & \ddots & \vdots \\ \xi_1\left(\theta^{(n_s)}\right) & \cdots & \xi_r\left(\theta^{(n_s)}\right) \end{bmatrix} \in \mathbb{R}^{n_s\times r} \tag{28}$$

and the sample vector of $\lambda(\theta)$ is $\lambda(\boldsymbol{\theta}) = \left[\lambda\left(\theta^{(1)}\right), \cdots, \lambda\left(\theta^{(n_s)}\right)\right]^T \in \mathbb{R}^{n_s}$. Thus stochastic computations in Eq. (23) are reformulated as

$$\mathbb{E}\left\{\lambda(\boldsymbol{\theta}) K(\boldsymbol{\theta})\right\} = \sum_{j=1}^{r} \mathbb{E}\left\{\lambda(\boldsymbol{\theta}) \xi_j(\boldsymbol{\theta})\right\} K_j,\ \lambda(\theta) = \sum_{j=1}^{r} \xi_j(\theta) d^T K_j d = \xi(\boldsymbol{\theta}) \mathbf{c} \in \mathbb{R}^{n_s}, \tag{29}$$

where the coefficient vector $\mathbf{c} = \left[d^T K_1 d, \cdots, d^T K_r d\right]^T \in \mathbb{R}^r$ and the expectation $\mathbb{E}\left\{\lambda(\boldsymbol{\theta}) \xi_j(\boldsymbol{\theta})\right\}$ are calculated efficiently in a non-intrusive way

$$\left[\mathbb{E}\left\{\lambda(\boldsymbol{\theta}) \xi_j(\boldsymbol{\theta})\right\}\right]_{j=1}^{r} = \frac{1}{n_s} \lambda(\boldsymbol{\theta})^T \xi(\boldsymbol{\theta}) \in \mathbb{R}^r. \tag{30}$$

In this way, we use the same method for solving both low- and high-dimensional stochastic problems and do not need to design dedicated algorithms for high-dimensional cases. The computational effort increases slightly as the dimension increases since only extra memories for storing $\xi(\boldsymbol{\theta}) \in \mathbb{R}^{n_s \times r}$ and $\left\{K_j\right\}_{j=1}^{r}$ are needed.

Remark 2. *Combining Eq.* (20) *and* (22) *we have*

$$\mathbb{E}\left\{\left[d^T K(\theta) d\right] K(\theta)\right\} d = \lambda^* d \tag{31}$$

or the separated form

$$\left(\sum_{j=1}^{r} g_j(d) K_j\right) d = \lambda^* d, \tag{32}$$

where the scalar function $g_j(d) = \sum_{i=1}^{r} \mathbb{E}\left\{\xi_i(\theta)\xi_j(\theta)\right\}\left(d^T K_i d\right) \in \mathbb{R}$ *and* $\mathbb{E}\left\{\xi_i(\theta)\xi_j(\theta)\right\}$ *are given by*

$$\left[\mathbb{E}\left\{\xi_i(\boldsymbol{\theta})\xi_j(\boldsymbol{\theta})\right\}\right]_{i,j=1}^{r} = \frac{1}{n_s}\xi(\boldsymbol{\theta})^T \xi(\boldsymbol{\theta}) \in \mathbb{R}^{r\times r}. \tag{33}$$

Eq. (31) *and* (32) *are deterministic nonlinear eigenvalue problems. The vector* d *can be considered as the eigenvector of the matrix obtained by the combination of* $\left\{K_j\right\}_{j=1}^{r}$*. In this paper, we solve stochastic eigenproblems by iteratively solving linear eigenvalue problems. Compared to the proposed method, only one stochastic computation Eq.* (33) *is required for solving Eq.* (32) *and the vector* d *is solved only by deterministic problems. However, nonlinear eigenvalue problems are beyond the scope of this article and can be found in references [41, 42].*

3.4. Maximum and minimum stochastic eigenvalues

The proposed method utilizes a two-step strategy to compute eigenvalues of the original eigenproblem Eq. (1), that is, the first step is to solve deterministic eigenvalue problems to generate a set of reduced basis and the second step is to solve a reduced-order stochastic eigenvalue problem. In this section we discuss computing original maximum and minimum stochastic eigenvalues by combining two-step eigenvalue analysis.

The maximum and minimum eigenvalues of Eq. (1) are given by

$$\lambda_{\max}(\theta) = \max_{u(\theta)\neq 0} \frac{u^T(\theta) K(\theta) u(\theta)}{u^T(\theta) u(\theta)},\ \lambda_{\min}(\theta) = \min_{u(\theta)\neq 0} \frac{u^T(\theta) K(\theta) u(\theta)}{u^T(\theta) u(\theta)}. \tag{34}$$

In the first step, maximum/minimum stochastic eigenvectors $u(\theta)$ are approximated by the vector d that is solved by the minimization Eq. (16),

$$\max_{d\neq 0} \frac{d^T K^* d}{d^T d} \to d_{\max}, \ \min_{d\neq 0} \frac{d^T K^* d}{d^T d} \to d_{\min}, \tag{35}$$

which indicates that $d_{\max}/d_{\min}$ are the maximum/minimum eigenvectors of K^*. Further, maximum/minimum stochastic eigenvectors of the original eigenproblem are solved by the eigenvalues problem Eq. (12) in the second step,

$$\phi_{\max}(\theta) = \max_{\phi(\theta)\neq 0} \frac{\phi^T(\theta)\widetilde{K}_k(\theta)\phi(\theta)}{\phi^T(\theta)\phi(\theta)}, \ \phi_{\min}(\theta) = \min_{\phi(\theta)\neq 0} \frac{\phi^T(\theta)\widetilde{K}_k(\theta)\phi(\theta)}{\phi^T(\theta)\phi(\theta)}. \tag{36}$$

which indicates that $\phi_{\max}(\theta)/\phi_{\min}(\theta)$ are the maximum/minimum eigenvectors of the reduced-order stochastic eigenvalue problem.

Thus the maximum/minimum stochastic eigenvalues and eigenvectors of the original eigenproblems are obtained by two-step maximum/minimum eigenvalue problems,

$$u_{\max,i}(\theta) = D_{\max}\phi_{\max,i}(\theta), \ u_{\min,i}(\theta) = D_{\min}\phi_{\min,i}(\theta), \ i = 1, 2, \cdots \tag{37}$$

where $D_{\max} = [d_{\max,1}, \cdots, d_{\max,k}] \in \mathbb{R}^{n\times k}$, $\phi_{\max,i}(\theta) \in \mathbb{R}^k$ and $D_{\min} = [d_{\min,1}, \cdots, d_{\min,k}] \in \mathbb{R}^{n\times k}$, $\phi_{\min,i}(\theta) \in \mathbb{R}^k$ are the first several maximum/minimum eigenvectors obtained by Eq. (20) and (12).

3.5. Convergence analysis of stochastic eigenvalues

In this section, we extend the analysis of deterministic eigenvalue problems to stochastic cases. We will demonstrate that if an eigenvector of the matrix $K(\theta)$ is nearly in the subspace $\mathcal{K}(D_k)$ consisting of the reduced-order matrix D_k (we denote the matrix D in Eq. (10) as D_k in this section), the corresponding stochastic eigenvalue of the reduced-order eigenproblem Eq. (12) converges to the stochastic eigenvalue of the full-order eigenproblem Eq. (1).

We adopt some concepts from the perturbation theory of eigenspaces [2, 43]. The acute angle between a vector v and a subspace $\mathcal{K}$ is defined as the smallest acute angle between v and all vectors $w \in \mathcal{K}$

$$\angle(v, \mathcal{K}) = \min_{w\in\mathcal{K}} \angle(v, w), \tag{38}$$

where $\angle(v, w)$ is defined as the acute angle between the nonzero vectors v and w and

$$\cos\angle(v, w) = \frac{|(v, w)|}{\|v\|_2\|w\|_2},\ 0 \le \angle(v, w) \le \frac{\pi}{2}. \tag{39}$$

In this way, recalling Eq. (1) and (12), the angle between the eigenvector $u(\theta)$ of the full-order eigenproblem and the subspace $\mathcal{K}(D_k)$ is given by

$$\alpha(\theta) = \angle(u(\theta), \mathcal{K}(D_k)). \tag{40}$$

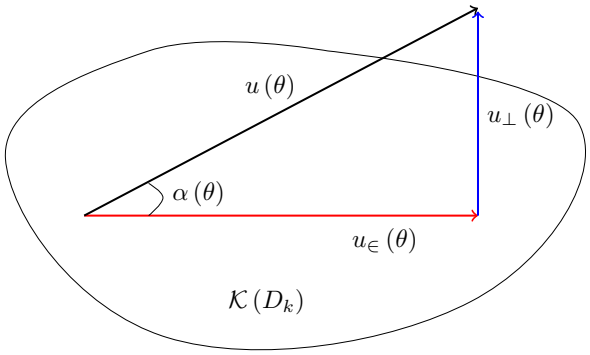


Figure 1: Decomposition of the eigenvector $u(\theta)$.

As illustrated in Fig. 1, we decompose the eigenvector $u(\theta)$ into two parts

$$u(\theta) = u_\in(\theta) + u_\perp(\theta) \qquad \text{a.e.}, \tag{41}$$

where the components are

$$u_\in(\theta) = D_k D_k^T u(\theta) \in \mathcal{K}(D_k),\ u_\perp(\theta) = \left(I - D_k D_k^T\right) u(\theta) \perp \mathcal{K}(D_k). \tag{42}$$

According to Eq. (38) and (39), the relationships between the L^2 norms of $u_\in(\theta)$, $u_\perp(\theta)$ and the angle $\alpha(\theta)$ are given by

$$\|u_\in(\theta)\|_2 = \cos\alpha(\theta),\ \|u_\perp(\theta)\|_2 = \sin\alpha(\theta). \tag{43}$$

Considering $K(\theta)u(\theta)$ by Eq. (41) we have

$$K(\theta)u(\theta) = K(\theta)u_\in(\theta) + K(\theta)u_\perp(\theta) = K(\theta)D_k D_k^T u(\theta) + K(\theta)u_\perp(\theta), \tag{44}$$

Substituting which into Eq. (1) and multiplying by D_k^T from left yields

$$D_k^T K(\theta) u(\theta) = \widetilde{K}_k(\theta) D_k^T u(\theta) + D_k^T K(\theta) u_\perp(\theta) = \lambda(\theta) D_k^T u(\theta), \tag{45}$$

which is equivalent to

$$\left(\lambda(\theta) I - \widetilde{K}_k(\theta)\right) D_k^T u(\theta) = D_k^T K(\theta) u_\perp(\theta) \qquad \text{a.e..} \tag{46}$$

Further, the eigendecomposition of the matrix $\widetilde{K}_k(\theta)$ is

$$\widetilde{K}_k(\theta) = \widetilde{U}(\theta) \widetilde{\Lambda}(\theta) \widetilde{U}(\theta)^T \qquad \text{a.e.,} \tag{47}$$

where $\widetilde{\Lambda}(\theta) \in \mathbb{R}^{k\times k}$ is a diagonal matrix consisting of the eigenvalues $\widetilde{\lambda}(\theta)$ of the reduced-order matrix $\widetilde{K}_k(\theta)$. Substituting Eq. (47) into Eq. (46) we have

$$\widetilde{U}(\theta)\left(\lambda(\theta) I - \widetilde{\Lambda}(\theta)\right) \widetilde{U}(\theta)^T D_k^T u(\theta) = D_k^T K(\theta) u_\perp(\theta), \tag{48}$$

multiplying which by $\widetilde{U}(\theta)^T$ from left we have

$$\left(\lambda(\theta) I - \widetilde{\Lambda}(\theta)\right) \widetilde{U}(\theta)^T D_k^T u(\theta) = \widetilde{U}(\theta)^T D_k^T K(\theta) u_\perp(\theta) \qquad \text{a.e..} \tag{49}$$

Thus the following inequality holds

$$\min_{\lambda(\theta)\in \mathrm{eig}(K(\theta))} \left|\lambda(\theta) - \widetilde{\lambda}(\theta)\right| \left\|\widetilde{U}(\theta)^T D_k^T u(\theta)\right\|_2 \le \left\|\left(\lambda(\theta) I - \widetilde{\Lambda}(\theta)\right) \widetilde{U}(\theta)^T D_k^T u(\theta)\right\|_2 = \left\|\widetilde{U}(\theta)^T D_k^T K(\theta) u_\perp(\theta)\right\|_2, \tag{50}$$

equivalently,

$$\min_{\lambda(\theta)\in \mathrm{eig}(K(\theta))} \left|\widetilde{\lambda}(\theta) - \lambda(\theta)\right| \le \frac{\left\|\widetilde{U}(\theta)^T D_k^T K(\theta) u_\perp(\theta)\right\|_2}{\left\|\widetilde{U}(\theta)^T \left(D_k^T D_k\right) D_k^T u(\theta)\right\|_2} \le \frac{\|K(\theta)\|_2 \|u_\perp(\theta)\|_2}{\|u_\epsilon(\theta)\|_2} = \|K(\theta)\|_2 \tan\alpha(\theta) \qquad \text{a.e..} \tag{51}$$

The right side of Eq. (51) only depends on the values $\|K(\theta)\|_2$ and $\tan\alpha(\theta)$. The value $\|K(\theta)\|_2$ is fixed for a given matrix $K(\theta)$. Hence we can conclude $\left|\widetilde{\lambda}(\theta) - \lambda(\theta)\right| \to 0$ as the angle $\alpha(\theta) \to 0$. In other words, if the eigenvector $u(\theta)$ of the matrix $K(\theta)$ is nearly in $\mathcal{K}(D_k)$, the stochastic eigenvalue $\widetilde{\lambda}(\theta)$ of the reduced-order matrix $\widetilde{K}_k(\theta)$ converges to the stochastic eigenvalue $\lambda(\theta)$ of the full-order matrix $K(\theta)$. It is noted that we need to make sure that the subspace $\mathcal{K}(D_k)$ is "good"

enough such that the target eigenvector is nearly in it. A simple way for this purpose is to increase the dimension of the subspace. Furthermore, only symmetrically positive definite matrix $K(\theta)$ is considered in this paper. Eq. (51)-like bound estimations for more generally deterministic matrices can be found in the references [43, 44] and we can extend these theories to the corresponding stochastic cases.

3.6. Optimal approximation

In this section, we will demonstrate that the stochastic eigenvalues and eigenvectors of the reduced-order matrix $\widetilde{K}_k(\theta)$ are considered optimal approximations to the stochastic eigenvalues and eigenvectors of the full-order matrix $K(\theta)$ from the given subspace $\mathcal{K}(D_k)$. For this purpose, we will show that

$$\min_{Q(\theta)\in\mathbb{R}^{k\times k}} \|K(\theta) D_k - D_k Q(\theta)\|_2 \tag{52}$$

reaches its minimum value when $Q(\theta) = D_k^T K(\theta) D_k$ (i.e. the reduced-order stochastic matrix in Eq. (12)).

Let $\mathbf{D} = [D_k, D_r] \in \mathbb{R}^{n\times n}$ be an orthogonal matrix, where the reduced-order matrix $D_k \in \mathbb{R}^{n\times k}$, the supplementary matrix $D_r \in \mathbb{R}^{n\times r}$, $r = n - k$. Thus we have

$$\widetilde{K}(\theta) = \mathbf{D}^T K(\theta)\mathbf{D} = [D_k, D_r]^T K(\theta)[D_k, D_r] = \begin{bmatrix} D_k^T K(\theta) D_k & D_k^T K(\theta) D_r \\ D_r^T K(\theta) D_k & D_r^T K(\theta) D_r \end{bmatrix} = \begin{bmatrix} \widetilde{K}_{kk}(\theta) & \widetilde{K}_{kr}(\theta) \\ \widetilde{K}_{rk}(\theta) & \widetilde{K}_{rr}(\theta) \end{bmatrix}. \tag{53}$$

To proof Eq. (52), we let

$$Q(\theta) = \widetilde{K}_{kk}(\theta) + S(\theta), \ \forall S(\theta) \in \mathbb{R}^{k\times k}. \tag{54}$$

Hence we have

$$\begin{aligned}
&[K(\theta) D_k - D_k Q(\theta)]^T [K(\theta) D_k - D_k Q(\theta)] \\
=&\left[K(\theta) D_k - D_k\left(\widetilde{K}_{kk}(\theta) + S(\theta)\right)\right]^T \left[K(\theta) D_k - D_k\left(\widetilde{K}_{kk}(\theta) + S(\theta)\right)\right] \\
=&\left[K(\theta) D_k - D_k\widetilde{K}_{kk}(\theta)\right]^T \left[K(\theta) D_k - D_k\widetilde{K}_{kk}(\theta)\right] - \left[K(\theta) D_k - D_k\widetilde{K}_{kk}(\theta)\right]^T (D_k S(\theta)) \\
&\quad - (D_k S(\theta))^T \left[K(\theta) D_k - D_k\widetilde{K}_{kk}(\theta)\right] + (D_k S(\theta))^T (D_k S(\theta))
\end{aligned}$$

$$
\begin{aligned}
=&\left[K(\theta)D_k - D_k\widetilde{K}_{kk}(\theta)\right]^T\left[K(\theta)D_k - D_k\widetilde{K}_{kk}(\theta)\right] - \left(D_k^T K(\theta)^T D_k - \widetilde{K}_{kk}(\theta)^T\right)S(\theta) \\
&- S(\theta)^T\left(D_k^T K(\theta)D_k - \widetilde{K}_{kk}(\theta)\right) + S(\theta)^T S(\theta) \\
=&\left[K(\theta)D_k - D_k\widetilde{K}_{kk}(\theta)\right]^T\left[K(\theta)D_k - D_k\widetilde{K}_{kk}(\theta)\right] + S(\theta)^T S(\theta) \qquad \text{a.e.,}
\end{aligned} \tag{55}
$$

which reaches the minimum when $S(\theta) = 0$, i.e. $Q(\theta) = \widetilde{K}_{kk}(\theta)$, thus Eq. (52) is proved. Furthermore, the minimum value is given by

$$
\begin{aligned}
&\left\|K(\theta)D_k - D_k\widetilde{K}_{kk}(\theta)\right\|_2 \\
=&\left\|[D_k, D_r]\,[D_k, D_r]^T K(\theta)D_k - D_k\widetilde{K}_{kk}(\theta)\right\|_2 \\
=&\left\|\left[D_k\widetilde{K}_{kk}(\theta) + D_r\widetilde{K}_{rk}(\theta)\right] - D_k\widetilde{K}_{kk}(\theta)\right\|_2 \\
=&\left\|\widetilde{K}_{rk}(\theta)\right\|_2 \qquad \text{a.e..}
\end{aligned} \tag{56}
$$

4. Algorithm implementation

The proposed method for solving the stochastic eigenvalue problem Eq. (1) is summarized in Algorithm 1, which includes two parts in turn. The first part is from step 1 to 14, which is to compute the reduced-order matrix D and includes a triple-loop iteration. The innermost loop, which is from step 4 to 9, is used to solve the vector $d_k^{(j)}$ from a given random variable $\lambda_k^{(j)}(\theta)$, where the subscript k represents the k-th reduced basis, the superscript j represents the j-th iteration $\left\{\lambda_k^{(j)}(\theta), d_k^{(j)}\right\}$ and the superscript q only locally works on the power iteration for deterministic eigenproblems. The maximum/minimum eigenvector $d_k^{(j)}$ is solved by power/inverse power methods in step 6 and the orthonormalization is processed in step 7. The convergence error in step 8 is defined as

$$
\varepsilon_{d,k,j,q} = \frac{\left\|d_k^{(j,q)} - d_k^{(j,q-1)}\right\|_2}{\left\|d_k^{(j,q-1)}\right\|_2} = \left\|d_k^{(j,q)} - d_k^{(j,q-1)}\right\|_2, \tag{57}
$$

which measures the convergence of the eigenvector $d_k^{(j)}$ of the deterministic eigenproblem Eq. (20). The middle loop from step 2 to 12 corresponds to computing the k-th couple $\{\lambda_k(\theta), d_k\}$. The random variable $\lambda_k(\theta)$ is initialized by n_s random samples in step 2 and is updated in step 10 based

on the vector $d_k^{(j)}$ obtained by the innermost loop. The convergence error in step 11 is defined as

$$\varepsilon_{l,k,j} = \frac{\left\| d_k^{(j)} - d_k^{(j-1)} \right\|_2}{\left\| d_k^{(j-1)} \right\|_2} = \left\| d_k^{(j)} - d_k^{(j-1)} \right\|_2, \tag{58}$$

which measures the difference between $d_k^{(j)}$ and $d_k^{(j-1)}$. The iteration is stopped when $d_k^{(j)}$ is almost the same as $d_k^{(j-1)}$. The outermost loop from step 1 to 14 is used to compute the deterministic vector d_k one after another, where $k_{\max}$ is the specified number of retained terms. For the second part, the stochastic eigenvalues of both reduced-order and original eigenequations and the stochastic eigenvectors of the reduced-order eigenequation (12) are solved in step 15. The stochastic eigenvectors of the original eigenproblem Eq. (1) are then calculated in step 16 by combining reduced basis and stochastic eigenvectors of the reduced-order eigenproblem. Although only the standard eigenvalue problem is considered, Algorithm 1 can be readily extended to generalized eigenvalue

Algorithm 1 Algorithm for solving stochastic eigenvalue problems

1: **while** $k \leq k_{\max}$ **do**
2: Initialize $\lambda_k^{(0)}(\theta) = \left\{ \lambda_k^{(0)}\left(\theta^{(i)}\right) \right\}_{i=1}^{n_s} \in \mathbb{R}^{n_s}$
3: **while** $\varepsilon_{l,k,j} > \varepsilon_l$ **do**
4: Initialize $d_k^{(j,0)} \in \mathbb{R}^n$
5: **while** $\varepsilon_{d,k,j,q} > \varepsilon_d$ **do**
6: Compute $d_k^{(j,q)}$ by Eq. (25) (inverse power method for the minimum eigenvectors)
7: Orthogonalize $d_k^{(j,q)} \perp d_i,\ i = 1, \cdots, k-1$ and normalize $\left\| d_k^{(j,q)} \right\|_2 = 1$
8: Compute the iterative error $\varepsilon_{d,k,j,q}$
9: $q \leftarrow q + 1$
10: Compute $\lambda_k^{(j)}(\theta)$ by Eq. (23)
11: Compute the iterative error $\varepsilon_{l,k,j}$
12: $j \leftarrow j + 1$
13: Update the matrix $D = [D, d_k] \in \mathbb{R}^{n \times k}$
14: $k \leftarrow k + 1$
15: Compute eigenpairs $\{\lambda_m(\theta), \phi_m(\theta)\}_m$ by Eq. (12)
16: Compute the m-th stochastic eigenvector $u_m(\theta) = D\phi_m(\theta)$

equations, which will be demonstrated in numerical examples.

5. Numerical examples

Numerical implementations of the proposed method are illustrated with the aid of five examples. For all considered examples, convergence errors in step 3 and 5 of Algorithm 1 are set as $\varepsilon_l = \varepsilon_d = 1 \times 10^{-4}$. 1×10^4 random samples are adopted in step 2 of Algorithm 1. Reference solutions are computed by adopting 1×10^4 MCS. All examples are tested on a laptop (dual-core, Intel Core i7, 2.40GHz).

5.1. Eigenvalues of stochastic matrices

In this example, we test the proposed method using separated and non-separated stochastic matrices and compare the performance of the proposed method, MCS and PC method.

5.1.1. Separated stochastic matrix

In this case, we consider the stochastic eigenvalue problem Eq. (1) and compute stochastic eigenvalues of the following stochastic matrix,

$$K(\theta) = \sum_{i=1}^{r} \xi_i(\theta) K_i \in \mathbb{R}^{n\times n}, \tag{59}$$

where $\{\xi_i(\theta)\}_{i=1}^r$ are mutually independent uniform random variables on $[1, 2]$ and deterministic matrices $\{K_i\}_{i=1}^r \in \mathbb{R}^{n\times n}$ are given by

$$K_i = U_i^T D_i U_i \in \mathbb{R}^{n\times n}. \tag{60}$$

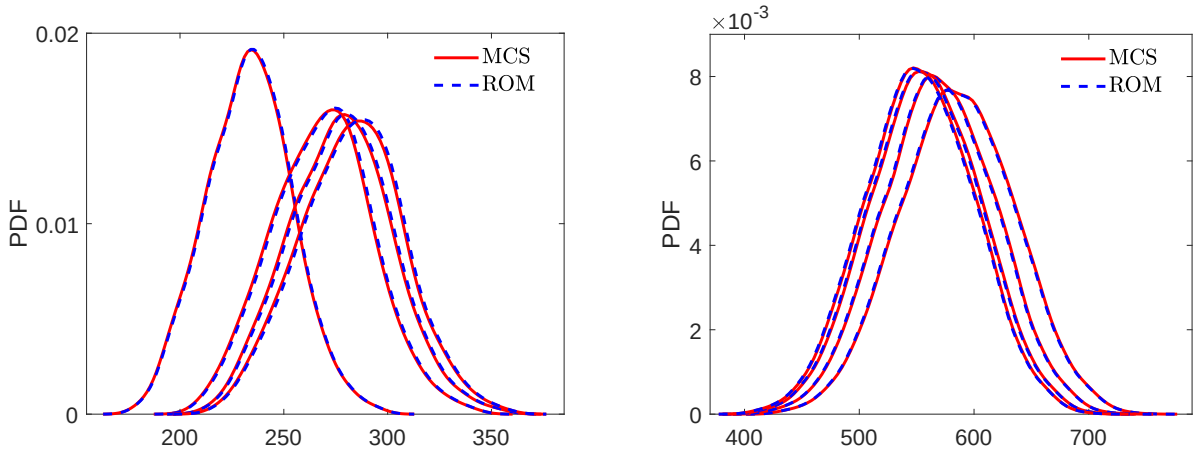
We set the matrix size $n = 50$, the stochastic dimension $r = 5$ and the retained number of Algorithm 1 $k_{\max} = 10$ in this case. Deterministic matrices D_i and U_i are a sample realization of stochastic matrices $D_i(\theta)$ and $U_i(\theta)$, where the diagonal matrix $D_i(\theta) = \mathrm{diag}([\eta_{i1}(\theta), \cdots, \eta_{in}(\theta)]) \in \mathbb{R}^{n\times n}$, $\left\{\eta_{ij}(\theta)\right\}_{i=1,\cdots,r,j=1,\cdots,n}$ are mutually independent uniform random variables on $[10, 100]$ and $U_i(\theta) \in \mathbb{R}^{n\times n}$ is a matrix formed by the orthonormal basis of the stochastic matrix $[\varsigma_{ikl}(\theta)]_{k,l=1}^n \in \mathbb{R}^{n\times n}$, $\{\varsigma_{ikl}(\theta)\}_{i=1,\cdots,r;\ k,l=1,\cdots,n}$ are mutually independent uniform random variables on $[0, 1]$.

Fig. 2 shows probability density functions (PDFs) of first four minimum and maximum stochastic eigenvalues obtained by the proposed ROM and MCS. For both minimum and maximum

stochastic eigenvalues, PDFs obtained by ROM have good agreements with MCS, which demonstrate good accuracy of the proposed ROM. It is seen from Fig. 2b that the proposed method works well even for highly close eigenvalues, which allows the proposed method to deal with the problems with close eigenmodes. It is noted that the rank order of each sample realization of stochastic eigenvalues is fixed, for instance, $\lambda_1\left(\theta^{(i)}\right) > \lambda_2\left(\theta^{(i)}\right)$, $i = 1, \cdots, n_s$ must hold if we consider $\lambda_1(\theta)$ and $\lambda_2(\theta)$ as the largest and second largest stochastic eigenvalues, which means that there is strong correlations between stochastic eigenvalues. We compute the correlations of eight stochastic eigenvalues (including the first four minimum eigenvalues shown in Fig. 2a and the first four maximum eigenvalues shown in Fig. 2b) and the Pearson correlation coefficient

$$\rho_{ij} = \frac{\mathbb{E}\left\{\lambda_i(\theta)\,\lambda_j(\theta)\right\} - \mathbb{E}\left\{\lambda_i(\theta)\right\}\mathbb{E}\left\{\lambda_j(\theta)\right\}}{\sqrt{\mathbb{E}\left\{\lambda_i^2(\theta)\right\} - \left[\mathbb{E}\left\{\lambda_i(\theta)\right\}\right]^2}\sqrt{\mathbb{E}\left\{\lambda_j^2(\theta)\right\} - \left[\mathbb{E}\left\{\lambda_j(\theta)\right\}\right]^2}} \tag{61}$$

is adopted. As listed in Tab. 1, minimum stochastic eigenvalues are strongly correlative to maximum stochastic eigenvalues, which verifies the correlations between stochastic eigenvalues. The correlation decreases if the values of two stochastic eigenvalues are far away.

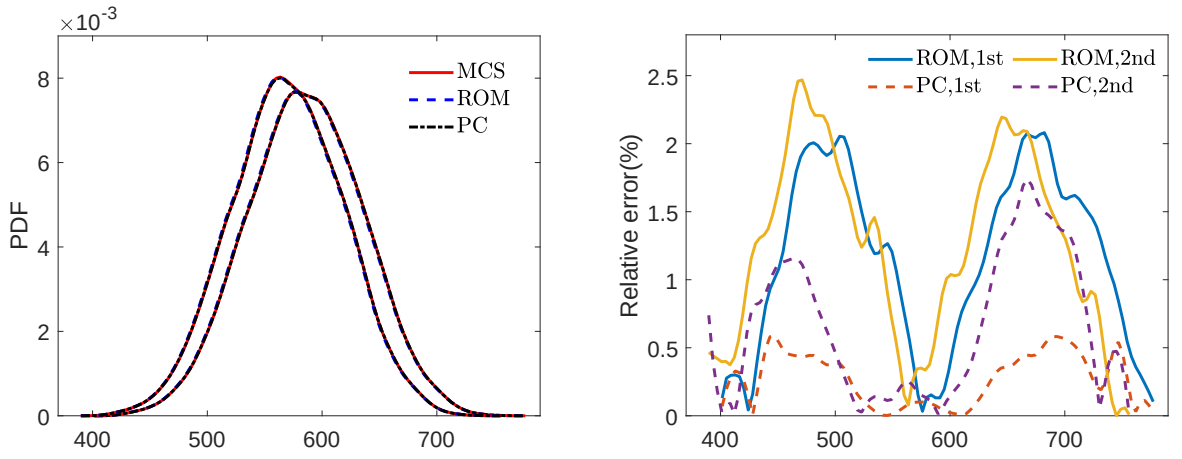


(a) PDFs of first four minimum eigenvalues.

(b) PDFs of first four maximum eigenvalues.

Figure 2: PDFs of first four minimum and maximum eigenvalues.

As a comparison, we solve the problem by use of the PC method described in Section 2.1. Two-order Legendre PC basis of five uniform random variables are adopted. The number of PC basis is 21 and the size of the derived nonlinear system of equations is 1050. It is seen from Fig. 3a



(a) Comparisions of PDFs.

(b) Errors of PDFs of ROM and PC relative to MCS.

Figure 3: Comparisons of PDFs of first two maximum eigenvalues obtained by MCS, ROM and PC method and corresponding relative errors.

that both ROM and PC method are in very good accordance with MCS. Relative errors (defined by $\left|\frac{\text{PDF}-\text{PDF}_{\text{MCS}}}{\text{PDF}_{\text{MCS}}}\right| \times 100\%$) depicted in Fig. 3b indicate both of their errors are small enough, but PC method has a bit better accuracy than ROM.

We test computational efficiencies of the proposed method, MCS and PC method. Tab. 2 shows computational times of minimum and maximum eigenvalues obtained by ROM, MCS and PC methods. The computational costs of ROM are obviously less than MCS for both minimum and maximum eigenvalues, which demonstrate the high efficiency of the proposed ROM. Compared to ROM, the PC method needs more effort since the size of the derived nonlinear system of equations is larger than that of the original problem and Newton-Raphson is used to solve the nonlinear system. This difficulty will be more pronounced for large-scale and high-dimensional stochastic problems. More efficient methods are necessary to reduce the computational effort of the PC method. Total computational times of ROM consist of the cost for computing the matrix D and the cost for solving the reduced-order stochastic eigenvalue problem and the former is normally much higher than the latter. It is the opposite in this example since only a small matrix size is tested.

5.1.2. *Non-separated stochastic matrix*

In this case, we consider a non-separated stochastic matrix

$$K_{ij}(\theta) = \exp\left(-\frac{|x_i - x_j|}{l_x(\theta)}\right) \in \mathbb{R}^{n\times n},\ x_i, x_j \in [0,1], \tag{62}$$

which is discretized with $n = 100$ degrees of freedom, where $l_x(\theta)$ is the uniform random variable on $[0.5, 1]$.

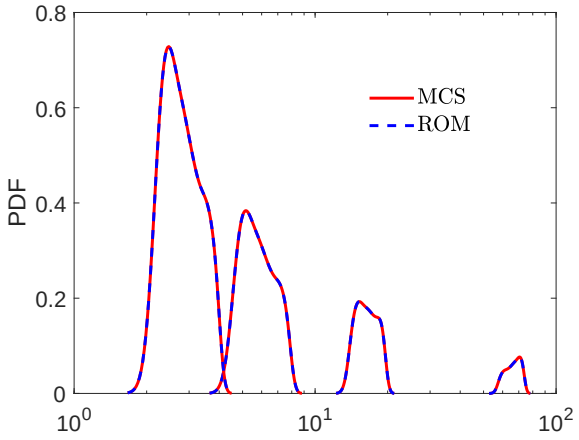


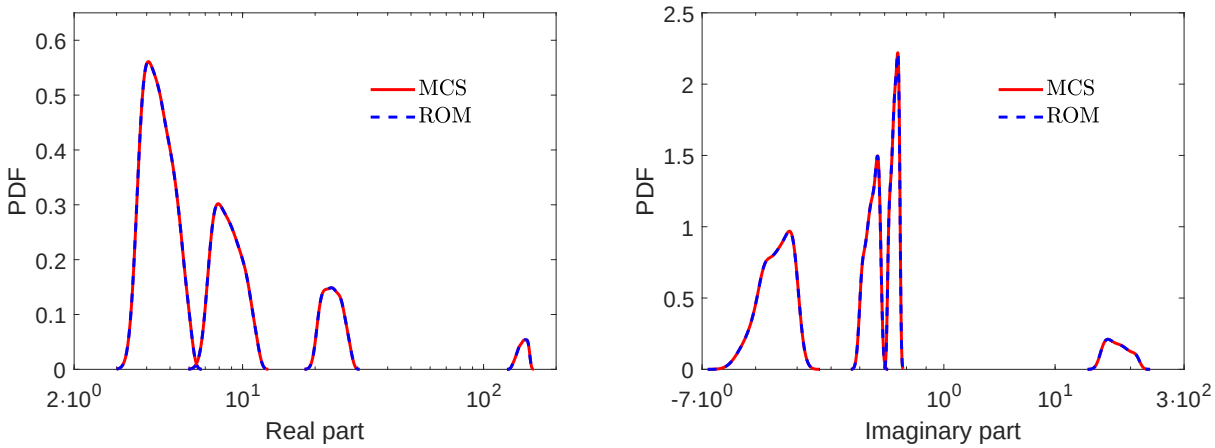
Figure 4: PDFs of first four maximum eigenvalues.

Expansion methods are available to approximate $K_{ij}(\theta)$, for instance, Eq. (3)-like can be obtained by adopting PC method to expand $K(\theta)$. It is noted that the proposed method can be applied to non-separated stochastic matrices without any modification. In this case, we use $n_s = 1 \times 10^4$ random samples $K(\boldsymbol{\theta}) \in \mathbb{R}^{n\times n\times n_s}$ instead of the approximation of $K(\theta)$ and set $k_{\max} = 10$. PDFs of first four maximum eigenvalues compared to MCS are shown in Fig. 4, which indicates our method has good accuracy for non-separated stochastic matrices. The computational time are 103.19s for MCS and 21.08s for ROM, including 13.72s for computing the matrix D and 7.36s for solving the reduced-order eigenvalue problem. Compared to separated cases, more effort are needed to compute $\mathbb{E}\{\lambda(\boldsymbol{\theta})K(\boldsymbol{\theta})\} \in \mathbb{R}^{n\times n}$ and $d^T K(\boldsymbol{\theta}) d \in \mathbb{R}^{n_s}$ for non-separated stochastic matrices.

5.1.3. *Non-symmetric complex stochastic matrix*

In this case, we consider a non-symmetric complex stochastic matrix

$$K_{ij}(\theta) = \exp\left(-\frac{|x_i - x_j|}{l_{x,1}(\theta)}\right) + \exp\left(-\frac{\sqrt{x_i - x_j}}{l_{x,2}(\theta)}\right) \in \mathbb{C}^{n\times n},\ x_i, x_j \in [0,1], \tag{63}$$



(a) PDFs of the real parts of first four maximum eigenvalues.

(b) PDFs of the imaginary parts of first four maximum eigenvalues.

Figure 5: PDFs of first four maximum eigenvalues.

which is discretized with $n = 100$ degrees of freedom, where $l_{x,1}(\theta)$ and $l_{x,2}(\theta)$ are uniform random variables on $[0.5, 1]$ and $[1, 1.5]$, respectively, and their Pearson correlation coefficient is 0.5.

In this case, we adopt $n_s = 1 \times 10^4$ random samples $K(\boldsymbol{\theta}) \in \mathbb{R}^{n \times n \times n_s}$ to describe the stochastic matrix and set $k_{\max} = 10$. The complex stochastic eigenvalue has the form $\lambda(\theta) = \lambda_{\text{real}}(\theta) + i_\lambda \cdot \lambda_{\text{imag}}(\theta)$, where $i_\lambda = \sqrt{-1}$ is the imaginary unit, $\lambda_{\text{real}}(\theta)$ and $\lambda_{\text{imag}}(\theta)$ are real-part and imaginary-part random variables. PDFs of first four maximum eigenvalues compared to MCS are shown in Fig. 5, where PDFs of the first four real-part random variables $\left\{\lambda_{j,\text{real}}(\theta)\right\}_{j=1}^{4}$ are depicted in Fig. 5a and PDFs of the first four imaginary-part random variables $\left\{\lambda_{j,\text{imag}}(\theta)\right\}_{j=1}^{4}$ are depicted in Fig. 5b. The PDFs of both real-part and imaginary-part random variables have good agreements with that of MCS, which demonstrates the proposed method still works well in this case. The computational times are 143.96s for MCS and 31.27s for ROM, including 27.17s for computing the matrix D and 4.10s for solving the reduced-order eigenvalue problem. The proposed method saves a lot of costs compared with MCS.

5.2. Stochastic vibration modes of a membrane

This example is to calculate the stochastic vibration modes of a membrane, which requires the solution of the following eigenvalue partial differential equation,

$$-\nabla \cdot \left[c\left(x, y, \theta\right) \nabla u\left(x, y, \theta\right) \right] = \lambda\left(\theta\right) u\left(x, y, \theta\right) \tag{64}$$

defined on the domain shown in Fig. 6a and $u\left(x, y, \theta\right) = 0$ holds on all boundaries (including inner and outer boundaries). The finite element mesh is depicted in Fig. 6b, including $n_p = 1157$ nodes $n_e = 1874$ triangular elements. The coefficient $c\left(x, y, \theta\right)$ is a Gaussian random field with mean function $c_0(x, y) = 2$ and covariance function

$$C_{cc}\left(x_1, y_1; x_2, y_2\right) = \sigma_c^2 \exp\left(-\frac{|x_1 - x_2|}{l_x} - \frac{|y_1 - y_2|}{l_y} \right), \tag{65}$$

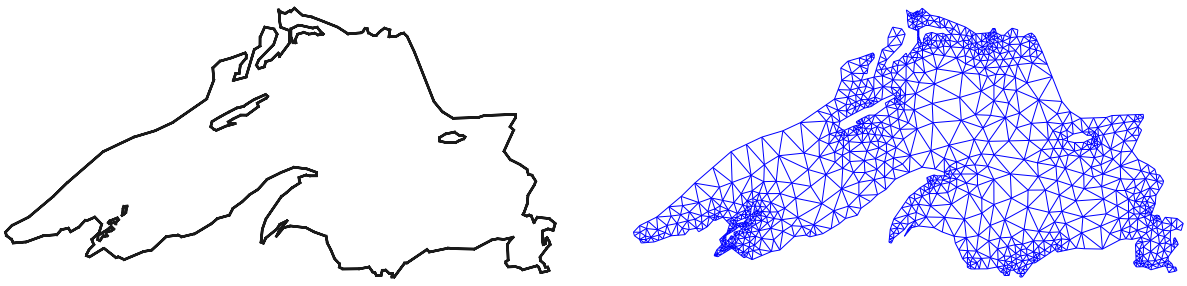
where the variance $\sigma_c = 0.1$ and correlation lengths $l_i = \max\left(i\right) - \min\left(i\right)$, $i = x, y$.

By use of Karhunen–Loève expansion [45, 46], the random field $c\left(\theta, x, y\right)$ is approximated as

$$c\left(x, y, \theta\right) = \sum_{j=0}^{r} \xi_j\left(\theta\right) \sqrt{\kappa_j} c_j\left(x, y\right), \tag{66}$$

where $\xi_0\left(\theta\right) \equiv 1$, $\kappa_0 \equiv 1$, r is the truncated number, $\left\{\xi_j\left(\theta\right)\right\}_{j=1}^{r}$ are mutually independent standard Gaussian random variables and $\left\{ \sqrt{\kappa_j}, c_j\left(x, y\right) \right\}_{j=1}^{r}$ are solved by the following homogeneous Fredholm integral equation of the second kind,

$$\int_{\Omega} C_{cc}\left(x_1, y_1; x_2, y_2\right) c_j\left(x_1, y_1\right) dx_1 dy_1 = \kappa_j c_j\left(x_2, y_2\right). \tag{67}$$



(a) Geometry. (b) Finite element mesh.

Figure 6: Model of the membrane and its finite element mesh.

In order to ensure the well-posedness of Eq. (64), we need to keep $\min_{x,y\in\Omega}\left(c\left(x,y,\theta\right)\right) > 0,\ \forall\theta\in\Theta$ in the practical numerical implementation. For this purpose, the sample realization $\theta^{(i)}$ such that $\min_{x,y\in\Omega}\left(c\left(x,y,\theta^{(i)}\right)\right) < 1\times10^{-3}$ will be dropped out. Thus $c\left(x,y,\theta\right)$ is considered as a truncated Gaussian random field in the numerical processing. In this example, we truncate Eq. (66) at $r = 10$ and the first six $\left\{c_j\left(x,y\right)\right\}_{j=1}^{6}$ obtained by Eq. (67) are depicted in Fig. 7.

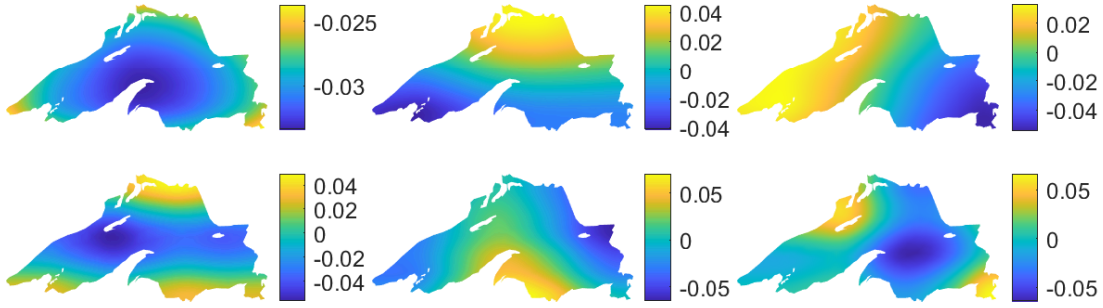


Figure 7: First six eigenvectors $\left\{c_j\left(x,y\right)\right\}_{j=1}^{6}$.

Considering the weak form of Eq. (64) we have

$$\lambda\left(\theta\right)\int_{\Omega}u\left(x,y,\theta\right)v\left(x,y\right)dxdy = \int_{\Omega}c\left(x,y,\theta\right)\nabla u\left(x,y,\theta\right)\nabla v\left(x,y\right)dxdy \tag{68}$$

$$= \sum_{j=0}^{r}\xi_j\left(\theta\right)\sqrt{\kappa_j}\int_{\Omega}c_j\left(x,y\right)\nabla u\left(x,y,\theta\right)\nabla v\left(x,y\right)dxdy, \tag{69}$$

where $v(x,y)$ is the test function. A stochastic eigenequation is thus generated as

$$K\left(\theta\right)u\left(\theta\right) = \lambda\left(\theta\right)Mu\left(\theta\right), \tag{70}$$

where $K\left(\theta\right) = \sum_{j=0}^{r}\xi_j\left(\theta\right)K_j \in \mathbb{R}^{n_p\times n_p}$, matrices $\left\{K_j\right\}_{j=0}^{r}$ and M are computed by using shape functions $\{\varpi_k\left(x,y\right)\}$ of triangular elements,

$$K_{j,kl} = \sqrt{\kappa_j}\int_{\Omega}c_j\left(x,y\right)\nabla\varpi_k\left(x,y\right)\nabla\varpi_l\left(x,y\right)dxdy, \tag{71}$$

$$M_{kl} = \int_{\Omega}\varpi_k\left(x,y\right)\varpi_l\left(x,y\right)dxdy,\ k,l = 1,\cdots,n. \tag{72}$$

From the perspective of practical engineering applications, it is more concerned about the first several minimum eigenvalues since they are related to structural frequencies and natural vibration

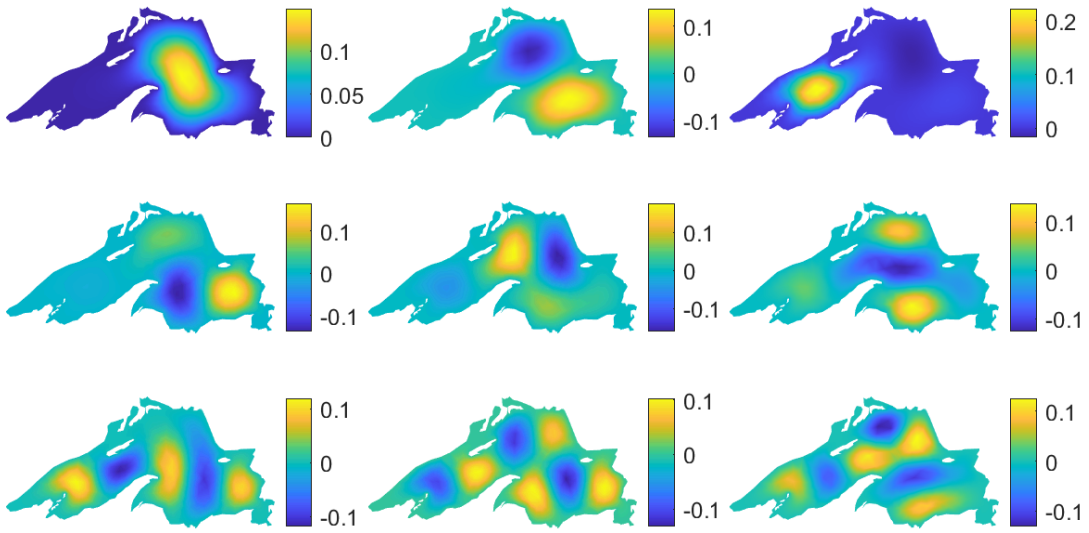
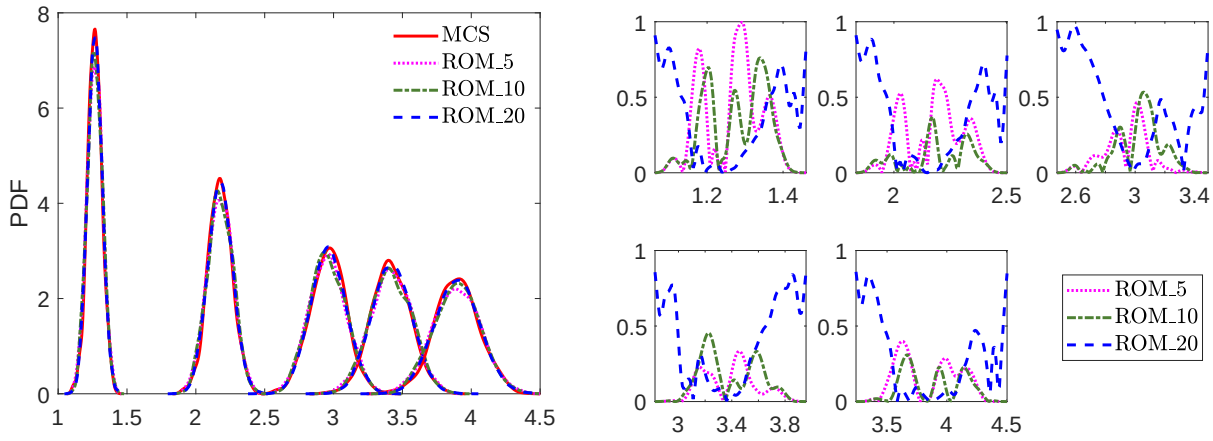


Figure 8: First nine reduced basis $\{d_i\}_{i=1}^{9}$.

modes. We only compute the first five minimum eigenvalues in this example. By use of the proposed ROM, the first nine vectors $\{d_i\}_{i=1}^{9}$ of the matrix D are shown in Fig. 8. It is seen that the mode of d_i becomes less important as the number i increases. In other words, we can use fewer $\{d_i\}$ to approximate stochastic eigenvectors in this example. Different retained terms $k_{\max} = 5, 10, 20$ are thus tested. It is seen from Fig. 9a that all PDFs of three cases are good enough to approximate the reference solutions and they have similar approximation errors (seen from Fig. 9b), which demonstrate five retained terms $\{d_i\}_{i=1}^{5}$ are enough for this example. However, we cannot determine the retained term $k_{\max}$ a priori for practical problems. The preselection of $k_{\max}$ as small as possible is still an open problem for our method and needs further study. Also, the computational time 18.73s (including 9.97s for computing the matrix D and 8.76s for solving reduced-order eigenvalue problem) of ROM is much lower than 4.03×10^3s for 1×10^4 MCS, which verifies the high efficiency of the proposed ROM again.

5.3. Stochastic eigenvalue analysis for a single part of robotic arm

This example considers a linear elastic robotic arm shown in Fig. (10a), stochastic eigenvalue analysis of only single part of which is proceed. The finite mesh is depicted in Fig. (10b), including $n_p = 2062$ nodes and $n_e = 6729$ tetrahedron elements. Material properties of the arm are Poisson's



(a) PDFs of first five minimum eigenvalues from MCS and 5, 10, 20 reduced basis.

(b) Errors of PDFs of 5-, 10-, 20-term ROM relative to MCS.

Figure 9: PDFs of first five maximum eigenvalues obtained by different numbers of reduced basis and corresponding relative errors.

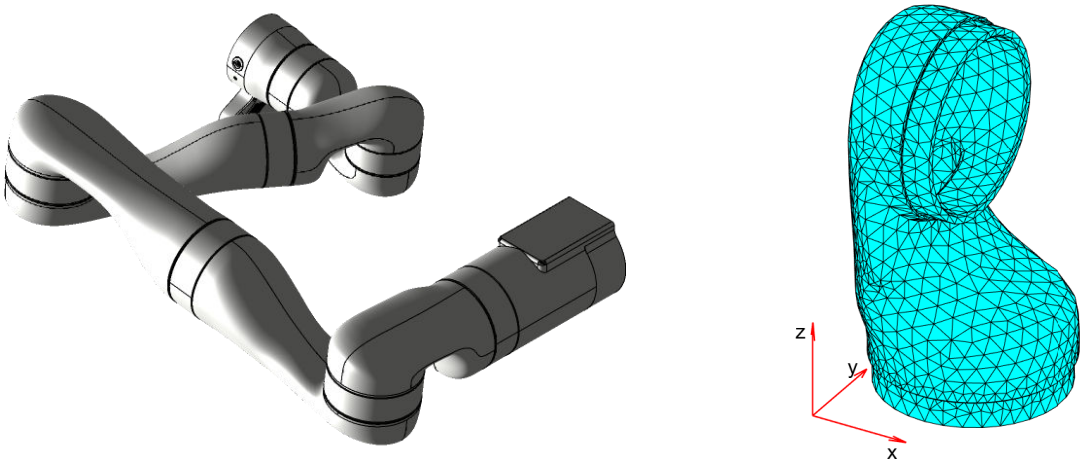
ratio $\nu = 0.30$ and mass density $\rho = 2000\text{kg/m}^3$. The Young's modulus $E(x, y, \theta)$ is considered as a Gaussian random field with mean function $E_0(x, y) = 1.50 \times 10^{11}\text{Pa}$ and covariance function

$$C_{EE}(x_1, y_1, z_1; x_2, y_2, z_2) = \sigma_E^2 \exp\left(-\frac{|x_1 - x_2|}{l_x} - \frac{|y_1 - y_2|}{l_y} - \frac{|z_1 - z_2|}{l_z}\right), \tag{73}$$

where the variance $\sigma_E = 1.50 \times 10^{10}$ and correlation lengths $l_i = \max(i) - \min(i)$, $i = x, y, z$.

Similar to Example 5.2, we expand the random filed $E(x, y, \theta)$ by use of Eq. (66) and obtain a same stochastic eigenequation as Eq. (70). Similar to the random field $c(x, y, \theta)$ considered in Eq. (66), the sample realization $\theta^{(i)}$ such that $\min_{x,y,z\in\Omega}\left(E\left(x, y, z, \theta^{(i)}\right)\right) < 1 \times 10^{-3}$ is discarded to ensure that the Young's modulus is positive. The retained number is set as $k_{\max} = 20$ in this case and the first six minimum eigenvalues are computed. A low-dimensional case $r = 10$ is firstly considered. It is seen from Fig. 11 that PDFs of the first six minimum eigenvalues are still in very good accordance with MCS.

The computational time of the case $r = 10$ is 73.12s (seen from Tab. 3) and corresponding MCS cost is 1.52×10^4s. For large-scale problems, ROM is more efficient since it only solves a few number of large-scale deterministic eigenequations. To verify the validity of the proposed method for high-dimensional stochastic problems, different stochastic dimensions $r = 10, 30, 60, 100$ are



(a) Model. (b) Finite element mesh.

Figure 10: Arm model and its finite element mesh.

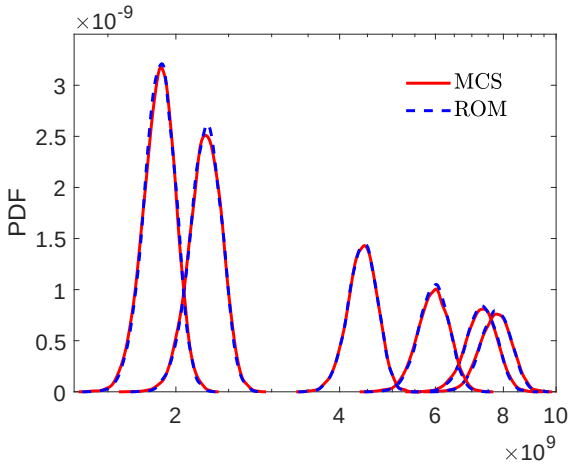


Figure 11: PDFs of first six minimum eigenvalues.

tested and their computational times are listed in Tab. 3, which indicate that the proposed ROM is efficient even for a stochastic dimension up to 100. As the stochastic dimension increases, computational times for computing the matrix D increase since extra effort and storage are needed to a large number of matrices $\left\{K_j\right\}_{j=1}^{r}$. Computational times for solving reduced-order stochastic eigenequations are almost changeless since the size k_{max} is chosen to be fixed.

6. Conclusions

This paper proposes an efficient reduced-order algorithm for solving stochastic eigenvalue problems and certifies its accuracy and efficiency with the aid of several numerical examples. By constructing an approximation of stochastic eigenvectors and developing a dedicated iterative algorithm, solutions of reduced basis are transformed into a few number of deterministic eigenproblems. Existing solvers can be readily incorporated into the computational procedure. Based on the obtained reduced basis, the original eigenequation is transformed into a reduced-order eigenvalue problem, whose solution is solved by use of a non-intrusive sampling method. The proposed method has low computational effort even for very high-dimensional stochastic problems. The curse of dimensionality is thus avoided with great success, which has been illustrated by the numerical example of up to 100 dimensions. In these senses, the proposed method is particularly appropriate for large-scale and high-dimensional stochastic eigenvalue analysis of practical interests.

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Table 1: Correlations between stochastic eigenvalues.

Eigenvalues	$\lambda_1^{\mathrm{MinEv}}(\theta)$	$\lambda_2^{\mathrm{MinEv}}(\theta)$	$\lambda_3^{\mathrm{MinEv}}(\theta)$	$\lambda_4^{\mathrm{MinEv}}(\theta)$	$\lambda_1^{\mathrm{MaxEv}}(\theta)$	$\lambda_2^{\mathrm{MaxEv}}(\theta)$	$\lambda_3^{\mathrm{MaxEv}}(\theta)$	$\lambda_4^{\mathrm{MaxEv}}(\theta)$
$\lambda_1^{\mathrm{MinEv}}(\theta)$	**1.0000**							
$\lambda_2^{\mathrm{MinEv}}(\theta)$	0.9755	**1.0000**						
$\lambda_3^{\mathrm{MinEv}}(\theta)$	0.9973	0.9825	**1.0000**				**sym.**	
$\lambda_4^{\mathrm{MinEv}}(\theta)$	0.9952	0.9819	0.9939	**1.0000**				
$\lambda_1^{\mathrm{MaxEv}}(\theta)$	0.9969	0.9749	0.9937	0.9916	**1.0000**			
$\lambda_2^{\mathrm{MaxEv}}(\theta)$	0.9935	0.9750	0.9891	0.9894	0.9975	**1.0000**		
$\lambda_3^{\mathrm{MaxEv}}(\theta)$	0.9954	0.9816	0.9935	0.9947	0.9984	0.9975	**1.0000**	
$\lambda_4^{\mathrm{MaxEv}}(\theta)$	0.9955	0.9804	0.9932	0.9947	0.9977	0.9983	0.9994	**1.0000**

$\lambda_i^{\mathrm{MinEv}}(\theta)$: the i-th minimum eigenvalues, $\lambda_i^{\mathrm{MaxEv}}(\theta)$: the i-th maximum eigenvalues.

Table 2: Computational costs of minimum and maximum eigenvalues.

	MinEv	MinEv (MC)	MaxEv	MaxEv (MC)	MaxEv (PC)
Solving costs (D)	1.70		1.37		
Solving costs (ROM)	6.98		6.62		
Total costs (seconds)	8.68	91.05	8.99	93.40	46.28

MinEv: minimum eigenvalues, MaxEv: maximum eigenvalues.

Table 3: Computational costs of different stochastic dimensions r.

Dimensions	10	30	60	100
Solving costs (D)	63.21	101.63	225.97	251.94
Solving costs (ROM)	9.91	8.09	9.80	10.59
Total costs (seconds)	73.12	109.72	235.27	262.53