AN ALGORITHM BASED ON AN ITERATIVE OPTIMAL STOPPING METHOD FOR FELLER PROCESSES WITH APPLICATIONS TO IMPULSE CONTROL, PERTURBATION, AND POSSIBLY ZERO RANDOM DISCOUNT PROBLEMS

SUHANG DAI

Institute for Financial and Actuarial Mathematics, Department of Mathematical Sciences, University of Liverpool, L69 7ZL, United Kingdom

OLIVIER MENOUKEU-PAMEN

African Institute for Mathematical Sciences, Ghana
University of Ghana, Ghana
Institute for Financial and Actuarial Mathematics, Department of Mathematical Sciences, University of Liverpool, L69 7ZL, United Kingdom

Abstract. In this paper we present an iterative optimal stopping method for general optimal stopping problems for Feller processes. We show using an approximating scheme that the value function of an optimal stopping problem for some general operator is the unique viscosity solution to an Hamilton-Jacobi-Bellman equation (see for example Theorem 2.3 and Theorem 2.4). We apply our results to study impulse control problems for Feller-Markov processes and derive explicit solutions in the case of one dimensional regular Feller diffusion. We also use our result to study optimal stopping problems for both regime switching and semi-Markov processes and characterise their value functions as the limit of iterative optimal stopping problems (see Corollary 4.2 and Proposition 4.3). Finally, we examine optimal stopping problems for random (possibly zero) discount.

Optimal stopping; Feller process; Viscosity solutions; Hamilton-Jacobi-Bellman equation; Iterative optimal stopping.

E-mail addresses: `sgsdai@liverpool.ac.uk, menoukeu@liverpool.ac.uk`.

2010 *Mathematics Subject Classification.* Primary 60G40; 60J25; 47D07, Secondary 60J35.

This paper is part of a Ph.D. thesis at the University of Liverpool, defended in 2018 by Suhang Dai under the supervision of Olivier Menoukeu Pamen. This paper takes up in particular Chapter 4 dedicated to iterative methods of optimal stopping control with the same applications (see [1]).

The project on which this publication is based has been carried out with funding provided by the Alexander von Humboldt Foundation, under the programme financed by the German Federal Ministry of Education and Research entitled German Research Chair No 01DG15010.

1. Introduction

The iterative method for impulse control problems was first introduced in [2], assuming that the state process is given by a diffusion process. The idea is to reduce the quasi-variational inequality to a sequence of variational inequality.

Similar results can also be found in [3, 4, 5, 6]. When the state process is a Feller process the author in [7] studied the regularity of the value function of impulse control problems using the iterative optimal method. On of the motivation of this paper come from [1], where the author studies an optimal stopping problem for a normal Markov process $X := (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, X(t), \theta_t, \boldsymbol{P}^x)$ on the state space $(E, \mathcal{E})$, where $(\Omega, \mathcal{F})$ is a measurable space, $\{\mathcal{F}_t\}_{t\geq 0}$ is a right continuous and completed filtration, $\{X(t)\}_{t\geq 0}$ is a *càdlàg* stochastic process, $\{\theta_t\}_{t\geq 0}$ is the shift operator and $\boldsymbol{P}^x$ denotes the probability measure on $(\Omega, \mathcal{F})$ for $x \in \mathsf{E}$.

More precisely, for a Feller process $\{X(t)\}_{t\geq 0}$, the problem is as follows: find $\tau^* \in \mathcal{T}$ such that

$$V(x) := \sup_{\tau\in\mathcal{T}} J_x(\tau) := \sup_{\tau\in\mathcal{T}} \boldsymbol{E}^x \Big[\int_0^\tau e^{-as} f(X^x(s))\, \mathrm{d}s + e^{-a\tau} g(X^x(\tau)) \Big] = J_x(\tau^*), \tag{1.1}$$

for each $x \in \mathsf{E}$ (a locally compact, separable metric space with metric ρ) and $\mathcal{T}$ is the family of all $\{\mathcal{F}_t\}_{t\geq 0}$-stopping times. Here f is a running benefit function, g is a terminal reward function, $a > 0$ is a constant discount factor and X is a Feller process starting at x at $t = 0$. The above value function is characterised as the unique viscosity solution to

$$\min\{aw - \mathcal{A}w - f, w - g\} = 0,$$

where $\mathcal{A}$ is a generator derived from some semigroup. Noting that most of the impulse control problems can be reduced to iterative optimal stopping problems, we extend the results in [1, Chapter 3] (see for example Theorem 2.4).

This work extends the setting of [1, Chapter 3] to include more general bequest functions and terminal rewards. We also consider processes constructed by perturbations (see Section 4.2) and optimal stopping problems without discount (see Section 5.2.1). The value functions to the above problems satisfy Hamilton-Jacobi-Bellman (HJB) equations and we show that they are unique viscosity solutions to these HJB equations. The proof is based on iterated stopping arguments (see for example Theorem 2.3 and Theorem 2.4). The main difference between our method and the traditional one developed in [8] is that our generator is given by semigroup whereas in [8], the generator is an elliptic operator. An advantage of the proposed approach is that it enables to solve the HJB equation in more abstract cases. More precisely, we establish the existence of the viscosity solution to the equation

$$\min\{aw - \mathcal{A}w - \mathsf{F}w, w - \mathsf{G}w\} = 0, \tag{1.2}$$

where F and G are abstract operators on $\mathcal{C}_b(\mathsf{E})$. Similar formulation can be found in [9, Chapter 2.] in which the author presents impulse control problems for deterministic processes. See also [4, Chapter 8], where the authors study impulse control problems for jump diffusion, that is, the operator G is defined by

$$\mathsf{G}u(x) := \sup_{y\in\mathsf{E}} (u(y) + K(x, y)), \tag{1.3}$$

with $K : \mathsf{E} \times \mathsf{E} \to \mathbb{R}$, a function satisfying some conditions. We consider a class of stochastic impulse control problems, where the controlled process is a one dimensional regular Feller Process. We list some conditions ensuring that the value function is the unique viscosity solution to the HJB equation associated to the optimal stopping problem. We also give sufficient conditions under which the problem can be solved explicitly (see Assumption 3.1). In this situation, the value function is also given in terms of characteristic functions (see equation (3.5).) This result extends those in [10] (Brownian motion case) and [11] (case of geometric Brownian motion) to Feller diffusions.

In fact, iterative optimal stopping methods have recently been discussed in literature. For instance, [12] analysed the properties of the solution of a finite time optimal stopping (American) option pricing problem under regime switching by iterative optimal stopping method. A similar approach was also used in [13]. In [14], the authors studied iterated optimal stopping for jump diffusion processes. In this work, we suggest an unifying method by incorporating perturbations into Feller processes. (See Section 4)

Added to this, our approach enables us to explore optimal stopping problems without discount (see Section 5.2.1). The zero discount is typical to finite time optimal stopping. However, in case of the infinite horizon optimal stopping problems without discount, one needs more conditions to ensure that the value function is finite. Moreover, when the discount rate is zero, there is a limited number of available work based on Feller semigroup. Here, we employ an iterative optimal stopping approach to transform our problem. To the best of our knowledge, there has not been any work in this direction using the iterated optimal stopping approach. Let us mention for example the interesting work [15] in which the authors characterise the value function of an optimal stopping problem with zero discount as a viscosity solution to an HJB equation. As compared to [15], we do not need non-uniform ergodic property of the controlled process in this paper.

The results obtained here can be applied to study optimal stopping and impulse control problems for bounded and continuous benefit functions f (see for example Sections 3,4 and 5). Let us observe however that we cannot handle the case of unbounded benefit and bequest function f and g in this work. One way of overcoming this is to extend our operators to weighted spaces (see for example [16]). There is a wide range of optimal stopping and impulse control problems for unbounded f and g.

The authors in [17] proved existence of optimal controls for a general stochastic impulse control problem. For that purpose, they characterise the value function as the pointwise minimum of a set of superharmonic functions. They also describe this value function as the unique continuous viscosity solution of the quasi-variational inequalities (QVIs), and as the limit of a sequence of iterated optimal stopping problems. The author in [18] characterises the solution of impulse control problems in terms of superharmonic functions. Assuming that the process X is a general Markov process, it is shown that the value function of an impulse control problem is the minimal function in a convex set of superharmonic functions. The works [19, 20, 21] study both impulse and optimal stopping problems for diffusion processes. It is worth mentioning that the author in [20] derived a new mathematical characterisation of the value function in the continuation region as a linear function in some transformed space. Special feature of this work includes the fact that one does not have to guess optimal strategies using a verification lemma. Note that our setting does not cover the one in the above mentioned papers since their underlying function are not globally bounded. However, except in the work [18], all the other papers assume a diffusion process.

The rest of the paper is organised as follows: In Section 2, we formulate the control problem and derive the main results. In Section 3,we study an impulse control problem and derive explicit solutions in the case of one dimensional regular Feller diffusion. Then, we are able to reduce the regime switching optimal stopping problem to an iterative optimal stopping problem without regime switching, reduce the optimal stopping problem for semi-Markov process to an iterative optimal stopping problems for two dimensional deterministic process (see Section 4). Finally, we study an optimal stopping problem of random discount which can be zero in Section 5.

We will use the following notations in this paper:

- $B(\mathsf{E})$ is the space of all bounded Borel measurable functions on E;
- $\mathcal{C}(\mathsf{E})$ is the space of all continuous functions on E;
- $\mathcal{C}_c(\mathsf{E}) := \{w \in \mathcal{C}(\mathsf{E}); w \text{ has compact support}\}$;
- $\mathcal{C}_0(\mathsf{E}) := \{w \in \mathcal{C}(\mathsf{E}); w \text{ vanishes at infinity}\}$;
- $\mathcal{C}_*(\mathsf{E}) := \{w \in \mathcal{C}(\mathsf{E}); w \text{ converges at infinity}\}$;
- $\mathcal{C}_b(\mathsf{E}) := \mathcal{C}(\mathsf{E}) \cap B(\mathsf{E})$;
- $USC(\mathsf{E})$ (*respectively*, $LSC(\mathsf{E})$) denotes the space Borel-measurable upper (*respectively*, lower) semicontinuous function on E.

2. Problem formulation and Main theorems

In this section, we present the optimal control problem and give and prove the main results. We denote by $\|\cdot\|_\infty$ the supremum norm that is for any $w \in B(\mathsf{E})$, $\|f\|_\infty := \sup_{x\in\mathsf{E}} |f(x)|$.

Definition 2.1. A Feller process is a stochastic process $\{X(t)\}_{t\geq 0}$ such that the operator

$$\mathcal{P}_t w(x) := \boldsymbol{E}^x[w(X(t))|X(0) = x], \text{ for } t \in [0,\infty), x \in \mathsf{E}$$

satisfies

(i) $\mathcal{P}_{t+s} = \mathcal{P}_t \circ \mathcal{P}_s$, for all $t, s \geq 0$; $\mathcal{P}_0 = \mathcal{I}$, where $\mathcal{I}$ is the identity operator.
(ii) For each $t \geq 0$, if $w \in \mathcal{C}_0(\mathsf{E})$, $0 \leq w \leq 1$, then, $0 \leq \mathcal{P}_t w \leq 1$.
(iii) (*Feller Property*) $\mathcal{P}_t : \mathcal{C}_0(\mathsf{E}) \to \mathcal{C}_0(\mathsf{E})$ for all $t \geq 0$.
(iv) (*Strong Continuous Property*) $\lim_{t\to 0^+} \|\mathcal{P}_t w - w\|_\infty = 0$ for $w \in \mathcal{C}_0(\mathsf{E})$.

We will denote by $\{X^x(t)\}_{t\geq 0} = \{X(t)\}_{t\geq 0}$ the process starting at x at time $t = 0$.

Definition 2.2. An *infinitesimal generator* of a Feller semigroup $\{\mathcal{P}_t\}_{t\geq 0}$ or a Feller process $\{X(t)\}_{t\geq 0}$ is a linear operator $(\mathcal{L}, D(\mathcal{L}))$, with $\mathcal{L} : D(\mathcal{L}) \subseteq \mathcal{C}_0(\mathsf{E}) \to \mathcal{C}_0(\mathsf{E})$ defined by

$$\mathcal{L}w := \lim_{t\to 0^+} \frac{\mathcal{P}_t w - w}{t} \text{ for } w \in D(\mathcal{L}), \tag{2.1}$$

where the domain $D(\mathcal{L}) := \{w \in \mathcal{C}_0(\mathsf{E}); \text{such that the limit in (2.1) exists in } \mathcal{C}_0(\mathsf{E})\}$.

Definition 2.3. A *resolvent* $\{\mathcal{R}_\lambda\}_{\lambda>0}$ is defined by

$$\mathcal{R}_\lambda w(x) := \int_0^\infty e^{-\lambda t} \mathcal{P}_t w(x)\,\mathrm{d}t \text{ for } x \in \mathsf{E} \text{ and } w \in \mathcal{C}_0(\mathsf{E}).$$

Set $\mathsf{F}u = f$ and $\mathsf{G} = g$ and define the infinitesimal generator of a Feller process by:

$$\begin{aligned} D(\mathcal{A}) &:= \{u \in \mathcal{C}_*(\mathsf{E}); u - u(\partial) \in D(\mathcal{G})\}, \\ \mathcal{A}u &:= \mathcal{G}(u - u(\partial)), \end{aligned} \tag{2.2}$$

where $(\mathcal{G}, D(\mathcal{G}))$ is the core of Feller process X. Define

$$\mathcal{C}_*(\mathsf{E}) := \{w \in \mathcal{C}(\mathsf{E}); w \text{ converges at the infinity of } \mathsf{E}\}.$$

All stopping times are taken in $\tau \in \mathcal{T}$. From now on we write $\sup_\tau$ instead of $\sup_{\tau\in\mathcal{T}}$. Let us recall the subsequent results from [1, Chapter 3]

Theorem 2.1. *Suppose $f, g \in \mathcal{C}_b(\mathsf{E})$ and $a > 0$. Let V be the value function defined by*

$$V(x) := \sup_{\tau} \boldsymbol{E}^x \Big[\int_0^{\tau} e^{-as} f(X(s)) \mathrm{d}s + e^{-a\tau} g(X(\tau)) \Big].$$

Then V is the unique viscosity solution $w \in \mathcal{C}_b(\mathsf{E})$ associated with $(\mathcal{A}, D(\mathcal{A}))$ to

$$\min\{aw - \mathcal{A}w - f, w - g\} = 0, \tag{2.3}$$

with $(\mathcal{A}, D(\mathcal{A}))$ given by (2.2).

Theorem 2.2. *Suppose $a > 0$ and $f, g \in \mathcal{C}_b(\mathsf{E})$. Let $w_1 \in USC(\mathsf{E})$ and $w_2 \in LSC(\mathsf{E})$ be the viscosity subsolution and supersolution to (2.3), respectively. If w_1 and w_2 are bounded from above and below, respectively, then, $w_1 \leq w_2$.*

Now, we formulate the problem we wish to solve. Defining the operator $\mathcal{T}_{\mathsf{F},\mathsf{G}}$ by:

$$\mathcal{T}_{\mathsf{F},\mathsf{G}} w(x) := \sup_{\tau} \boldsymbol{E}^x \Big[\int_0^{\tau} e^{-as} \mathsf{F}w(X(s)) \mathrm{d}s + e^{-a\tau} \mathsf{G}w(X(\tau)) \Big],$$

where X is a Feller process with state space E and $a > 0$ is the constant discount rate, $\mathsf{F} : B(\mathsf{E}) \to B(\mathsf{E})$ and $\mathsf{G} : B(\mathsf{E}) \to B(\mathsf{E})$. Note that the stopping time τ could be infinite as considered in Section 5.2. In this paper, we consider the following dynamic programming equation

$$w = \mathcal{T}_{\mathsf{F},\mathsf{G}} w. \tag{2.4}$$

Note that the above problem can be thought of as an impulse control problem (see Section 3), when G is of the form (1.3).

We aim at showing that under certain conditions, the solution to (2.4) is the unique viscosity solution of the following Hamilton-Jacobi-Bellman (HJB) equation

$$\min\{aw - \mathcal{A}w - \mathsf{F}w, w - \mathsf{G}w\} = 0. \tag{2.5}$$

Below we give the definition of viscosity subsolution and supersolution (compare with [1, Definition 3.1]).

Definition 2.4. A function $w \in USC(\mathsf{E})$ (*respectively*, $w \in LSC(\mathsf{E})$) is a *viscosity subsolution* (*respectively*, *supersolution*) associated with $(\mathcal{A}, D(\mathcal{A}))$ to (2.5) if for all $\phi \in D(\mathcal{A})$ such that $\phi - w$ has a global minimum (*respectively*, maximum) at $x_0 \in \mathsf{E}$ with $\phi(x_0) = w(x_0)$,

$$\min\{a\phi(x_0) - \mathcal{A}\phi(x_0) - \mathsf{F}w(x_0), \phi(x_0) - \mathsf{G}w(x_0)\} \leq (\geq) 0.$$

Furthermore, $w \in \mathcal{C}(\mathsf{E})$ is a *viscosity solution* associated with $(\mathcal{A}, D(\mathcal{A}))$ to (2.15) if it is both a viscosity supersolution and a viscosity subsolution.

Now, we present the main results of this paper.

2.1. Solutions to $w = \mathcal{T}_{\mathsf{F},\mathsf{G}} w$.

In this subsection, we show that there exists a unique solution to (2.4).

Definition 2.5. Let $\mathcal{Z}$ be an operator.

(i) $\mathcal{Z}$ is *monotonic* if for any $u_1 \geq u_2$, $\mathcal{Z}u_1 \geq \mathcal{Z}u_2$.
(ii) $\mathcal{Z}$ is *convex* if for $u_1, u_2 \in \mathcal{C}_b(\mathsf{E})$ and $0 \leq p \leq 1$, we have $\mathcal{Z}(pu_1 + (1-p)u_2) \leq p\mathcal{Z}u_1 + (1-p)\mathcal{Z}u_2$.

We make the following standard assumptions on the operators of F and G.

Assumption 2.1.

(i) $\mathsf{F} : \mathcal{C}_b(\mathsf{E}) \to \mathcal{C}_b(\mathsf{E})$ *and* $\mathsf{G} : \mathcal{C}_b(\mathsf{E}) \to \mathcal{C}_b(\mathsf{E})$.
(ii) The operators F *and* G *are monotonic and convex.*

As a direct consequence of the above assumption, we have the following result.

Lemma 2.1. *Suppose Assumption 2.1 holds. Then,*
(i) $\mathcal{T}_{\mathsf{F},\mathsf{G}} : \mathcal{C}_b(\mathsf{E}) \to \mathcal{C}_b(\mathsf{E})$,
(ii) $\mathcal{T}_{\mathsf{F},\mathsf{G}}$ *is monotonic and convex.*

Proof. *(i)* Let $u \in \mathcal{C}_b(\mathsf{E})$, $f_u := \mathsf{F}u$ and $g_u := \mathsf{G}u$. By Assumption 2.1 *(i)*, $f_u, g_u \in \mathcal{C}_b(\mathsf{E})$. Therefore, using [1, Theorem 3.3], the value function of the optimal stopping problem is in $\mathcal{C}_b(\mathsf{E})$.

(ii) Using the fact that the operators F and G are also monotonic and convex, we conclude that the operator $\mathcal{T}_{\mathsf{F},\mathsf{G}}$ is monotonic and convex. Indeed

$$\begin{aligned}
&\alpha\mathcal{T}_{\mathsf{F},\mathsf{G}}(w(x)) + (1-\alpha)\mathcal{T}_{\mathsf{F},\mathsf{G}}(u(x)) \\
&= \alpha \sup_\tau \boldsymbol{E}^x\Big[\int_0^\tau e^{-as}\mathsf{F}w(X(s))\mathrm{d}s + e^{-a\tau}\mathsf{G}w(X(\tau))\Big] \\
&\quad + (1-\alpha)\sup_\tau \boldsymbol{E}^x\Big[\int_0^\tau e^{-as}\mathsf{F}u(X(s))\mathrm{d}s + e^{-a\tau}\mathsf{G}u(X(\tau))\Big] \\
&= \sup_\tau \boldsymbol{E}^x\Big[\int_0^\tau \alpha e^{-as}\mathsf{F}w(X(s))\mathrm{d}s + \alpha e^{-a\tau}\mathsf{G}w(X(\tau))\Big] \\
&\quad + \sup_\tau \boldsymbol{E}^x\Big[\int_0^\tau (1-\alpha)e^{-as}\mathsf{F}u(X(s))\mathrm{d}s + (1-\alpha)e^{-a\tau}\mathsf{G}u(X(\tau))\Big] \\
&\geq \sup_\tau \boldsymbol{E}^x\Big[\int_0^\tau e^{-as}(\alpha\mathsf{F}w(X(s)) + (1-\alpha)\mathsf{F}u(X(s)))\mathrm{d}s + e^{-a\tau}(\alpha\mathsf{G}w(X(\tau)) + (1-\alpha)\mathsf{G}u(X(\tau)))\Big] \\
&\geq \sup_\tau \boldsymbol{E}^x\Big[\int_0^\tau e^{-as}\mathsf{F}(\alpha w + (1-\alpha)u)(X(s))\mathrm{d}s + e^{-a\tau}\mathsf{G}(\alpha w + (1-\alpha)u)(X(\tau))\Big] \\
&= \mathcal{T}_{\mathsf{F},\mathsf{G}}(\alpha w + (1-\alpha)u)(x),
\end{aligned}$$

where in the last inequality we have used the convexity of F and G. The proof of the monotonicity follows similarly.

□

We also make the following assumption.

Assumption 2.2.
(i) There exists a positive constant $\kappa > 0$ *and* $w_+ \in \mathcal{C}_b(\mathsf{E})$ *such that*

$$w_+(x) - \kappa \geq \mathcal{T}_{\mathsf{F},\mathsf{G}}w_+(x) \text{ for } x \in \mathsf{E}. \tag{2.6}$$

(ii) There exists $p_1, p_2 \in \mathbb{R}$ *satisfying* $0 \leq p_1 \leq a$, $0 \leq p_2 \leq 1$ *and* $\min\{p_1/a, p_2\} < 1$ *such that*

$$\mathsf{F}(u + C) - \mathsf{F}u \leq p_1 C \text{ and } \mathsf{G}(u + C) - \mathsf{G}u \leq p_2 C$$

for all $u \in \mathcal{C}_b(\mathsf{E})$ *and constant function* $C > 0$.

Remark 2.1. Let us mention that Assumption 2.1 is necessary to obtain uniqueness of a continuous solution to $w = \mathcal{T}_{\mathsf{F},\mathsf{G}}w$, whereas Assumption 2.2 provides the upper and lower bounds to that solution. We will see this in more detail in what follows.

The next lemma will be needed in the proof to our results.

Lemma 2.2. *Suppose Assumption 2.2 holds.*

(i) Let $\kappa > 0$ and $w_+ \in \mathcal{C}_b(\mathsf{E})$ satisfying (2.6). Then, for any constant function $C > 0$, we have

$$(w_+ + C)(x) - \kappa \geq \mathcal{T}_{\mathsf{F},\mathsf{G}}(w_+ + c)(x) \text{ for } x \in \mathsf{E}.$$

(ii) There exists a function $w_0 \in \mathcal{C}_*(\mathsf{E})$ such that

$$w_0 \leq \mathcal{T}_{\mathsf{F},\mathsf{G}} w_0. \tag{2.7}$$

Proof. *(i)* For any $C > 0$, using Assumption 2.2 *(ii)*, we have

$$\begin{aligned}
\mathcal{T}_{\mathsf{F},\mathsf{G}}(w_+ + C)(x) &= \sup_{\tau} \boldsymbol{E}^x \Big[\int_0^{\tau} e^{-as} \mathsf{F}(w_+ + C)(X(s)) \mathrm{d}s + e^{-a\tau} \mathsf{G}(w_+ + c)(X(\tau)) \Big] \\
&\leq \sup_{\tau} \boldsymbol{E}^x \Big[\int_0^{\tau} e^{-as} \big(\mathsf{F} w_+(X(s)) + aC \big) \mathrm{d}s + e^{-a\tau} \big(\mathsf{G} w_+(X(\tau)) + C \big) \Big] \\
&= (\mathcal{T}_{\mathsf{F},\mathsf{G}} w_+ + C)(x) \\
&\leq (w_+ + C)(x) - \kappa,
\end{aligned}$$

where the last inequality is from Assumption 2.2 *(i)*.

(ii) We assume that $p_1/a < 1$. The case $p_2 < 1$ can be proved similarly. Let $\phi_0(x) := 0$ for all $x \in \mathsf{E}$. Let M be a constant such that

$$M \geq \|\mathsf{F}\phi_0\|_\infty / (a - p_1).$$

Define a constant function ϕ by $\phi(x) := -M$ for all $x \in \mathsf{E}$. Then $a\phi - \mathsf{F}\phi \leq 0$. In fact, by Assumption 2.2 *(ii)*, $\mathsf{F}\phi_0 - \mathsf{F}\phi \leq p_1 M$ and thus $-\mathsf{F}\phi \leq p_1 M - \mathsf{F}\phi_0$. Hence, $a\phi - \mathsf{F}\phi \leq (a - p_1)\phi - \mathsf{F}\phi_0 \leq 0$.

Since $\mathcal{A}\phi \geq 0$, by the positive maximum principle, we have $a\phi - \mathcal{A}\phi - \mathsf{F}\phi \leq 0$ and thus

$$\min\{a\phi - \mathcal{A}\phi - \mathsf{F}\phi, \phi - \mathsf{G}\phi\} \leq 0. \tag{2.8}$$

Therefore, ϕ is a viscosity subsolution to (2.8). On the other hand, since $\mathcal{T}_{\mathsf{F},\mathsf{G}}\phi$ is the value function for the optimal stopping problem, $\mathcal{T}_{\mathsf{F},\mathsf{G}}\phi$ and thus ϕ are the viscosity solutions to (2.8) (see [1, Theorem 3.26]). By the comparison principle (see [1, Theorem 3.27]), we have $\mathcal{T}_{\mathsf{F},\mathsf{G}}\phi \geq \phi$. Choose $w_0 = \phi$. □

Theorem 2.3. *Suppose Assumption 2.1 and Assumption 2.2 hold. Then there exists a unique solution $w \in \mathcal{C}_b(\mathsf{E})$ to*

$$w = \mathcal{T}_{\mathsf{F},\mathsf{G}} w. \tag{2.9}$$

Proof. Using Lemma 2.2 *(ii)*, there exists $w_0 \in \mathcal{C}_b(\mathsf{E})$ such that

$$\mathcal{T}_{\mathsf{F},\mathsf{G}} w_0 \geq w_0.$$

Define $w_{n+1} := \mathcal{T}_{\mathsf{F},\mathsf{G}} w_n$ for $n \in \mathbb{N}$. By Assumption 2.2 *(i)*, there exists $\kappa > 0$, $w_+ \in \mathcal{C}_b(\mathsf{E})$ such that

$$w_+ - \kappa \geq \mathcal{T}_{\mathsf{F},\mathsf{G}} w_+.$$

Since $w_0 \in \mathcal{C}_b(\mathsf{E})$, we have $w_1 = \mathcal{T}_{\mathsf{F},\mathsf{G}} w_0 \in \mathcal{C}_b(\mathsf{E})$. There exists $c_0 > 0$ such that $w_1 < c_0$. Choose c large enough and define $w_+^* := w_+ + c \geq w_1$. Then, by Lemma 2.2 *(i)* , we have

$$w_+^* - \kappa \geq \mathcal{T}_{\mathsf{F},\mathsf{G}} w_+^*. \tag{2.10}$$

Thus, we obtain

$$0 \le w_1 - w_0 \le w_+^* - w_0.$$

Now, we want to prove that there exists $0 \le \gamma < 1$ such that

$$w_{n+1} - w_n \le \gamma^n (w_+^* - w_n) \text{ for all } n \in \mathbb{N}. \tag{2.11}$$

We prove this by induction. (2.11) holds when $n = 0$, assume that (2.11) holds for all $n \le m$ where m is some positive integer. We want to prove that

$$w_{m+2} - w_{m+1} \le \gamma^{m+1} (w_+^* - w_{m+1}). \tag{2.12}$$

Since $\mathcal{T}_{\mathsf{F},\mathsf{G}}$ is monotonic by Lemma 2.1 and $w_1 = \mathcal{T}_{\mathsf{F},\mathsf{G}} w_0 \ge w_0$, it follows that the sequence $\{w_n\}_{n\in\mathbb{N}}$ is increasing. Using (2.12), we have

$$w_{m+1} \le \gamma^m w_+^* + (1-\gamma^m) w_n.$$

Thus by monotonicity and convexity of $\mathcal{T}_{\mathsf{F},\mathsf{G}}$, we have

$$\begin{aligned}
\mathcal{T}_{\mathsf{F},\mathsf{G}} w_{m+1} &\le \mathcal{T}_{\mathsf{F},\mathsf{G}} (\gamma^m w_+^* + (1-\gamma^m) w_m) \\
&\le \gamma^m \mathcal{T}_{\mathsf{F},\mathsf{G}} w_+^* + (1-\gamma^m) \mathcal{T}_{\mathsf{F},\mathsf{G}} w_m \\
&\le \gamma^m (w_+^* - \kappa) + (1-\gamma^m) w_{m+1} \\
&= w_{m+1} + \gamma^m (w_+^* - w_{m+1} - \kappa)
\end{aligned}$$

Let $x \in \mathsf{E}$, then we have

$$\begin{aligned}
\mathcal{T}_{\mathsf{F},\mathsf{G}} w_{m+1}(x) &= w_{m+1}(x) + \gamma^m \frac{(w_+^* - w_{m+1})(x) - \kappa}{(w_+^* - w_{m+1})(x)} (w_+^* - w_{m+1})(x) \\
&= w_{m+1}(x) + \gamma^m \big(1 - \frac{\kappa}{(w_+^* - w_{m+1})(x)}\big)(w_+^* - w_{m+1})(x) \\
&\le w_{m+1}(x) + \gamma^m \big(1 - \frac{\kappa}{\|w_+^* - w_0\|_\infty}\big)(w_+^* - w_{m+1})(x),
\end{aligned}$$

where the last inequality is from the fact that $w_+^* \ge w_m \ge w_0$. Choosing

$$\gamma = \max\Big\{0, 1 - \frac{\kappa}{\|w_+^* - w_0\|_\infty}\Big\},$$

we get $(w_{n+1} - w_n)(x) \le \gamma^n \|w_+^* - w_n\|_\infty \le \gamma^n \|w_+^* - w_0\|_\infty$ for all $n \in \mathbb{N}$. Therefore, $\{w_n\}_{n\in N}$ is a Cauchy sequence in $(\mathcal{C}_b(\mathsf{E}), \|\cdot\|_\infty)$ and there exists $w_\infty \in \mathcal{C}_b(\mathsf{E})$ such that $\{w_n\}_{n\in N}$ uniformly converges to w_∞ and satisfies $\mathcal{T}_{\mathsf{F},\mathsf{G}} w_\infty = w_\infty$. The existence of the solution to (2.9) is proved.

For the uniqueness, we only need to prove that $w_\infty \in \mathcal{C}_b(\mathsf{E})$ is the unique solution to (2.9). This can be shown using the comparison principle as shown below. □

The following corollary derived from Theorem 2.3 gives the convergence rate of the iterative optimal stopping scheme.

Corollary 2.1. *Suppose Assumption 2.1 and Assumption 2.2 hold. Consider the following numerical algorithm*

$$w_{m+1} = \mathcal{T}_{\mathsf{F},\mathsf{G}} w_m$$

and starting from w_0 *that satisfies (2.7). Then*

$$\lim_{m\to\infty} \|w_m - w\|_\infty \leq C \lim_{m\to\infty} \gamma^m,$$

where C *is some strictly positive constant and*

$$\gamma = \max\Big\{0, 1 - \frac{\kappa}{\|w_+^* - w_0\|_\infty}\Big\}.$$

Using similar arguments as in the above theorem, we derive the subsequent comparison principle.

Proposition 2.1. *(Comparison Principle) Suppose Assumption 2.1 and Assumption 2.2 hold. Let* w *be the solution to* (2.9). *If* $u \geq (\leq)\mathcal{T}_{\mathsf{F},\mathsf{G}} u$*, then* $u \geq (\leq) w$.

Proof. Assume there exists $v_+ \in \mathcal{C}_b(\mathsf{E})$ satisfying $v_+ \geq \mathcal{T}_{\mathsf{F},\mathsf{G}} v_+$. Let us prove that $v_+ \geq w_\infty$. Assume by contradiction that there exists some x_0 such that $v_+(x_0) < w_\infty(x_0)$. Then, since $w_+^* \geq w_\infty$, there exists $0 < \gamma \leq 1$ such that

$$w_\infty(x_0) - v_+(x_0) = \gamma(w_+^*(x_0) - v_+(x_0)). \tag{2.13}$$

Since w_∞ satisfies $w_\infty = \mathcal{T}_{\mathsf{F},\mathsf{G}} w_\infty$ and $\mathcal{T}_{\mathsf{F},\mathsf{G}}$ is convex, we have

$$\begin{aligned} w_\infty(x_0) = \mathcal{T}_{\mathsf{F},\mathsf{G}} w_\infty(x_0) &= \mathcal{T}_{\mathsf{F},\mathsf{G}}(\gamma w_+^* + (1-\gamma) v_+)(x_0) \\ &\leq \gamma \mathcal{T}_{\mathsf{F},\mathsf{G}} w_+^*(x_0) + (1-\gamma)\mathcal{T}_{\mathsf{F},\mathsf{G}} v_+(x_0) \\ &\leq \gamma(w_+^*(x_0) - \kappa) + (1-\gamma) v_+(x_0), \end{aligned}$$

where the last inequality follows from (2.10) and $v_+ \geq \mathcal{T}_{\mathsf{F},\mathsf{G}} v_+$. Therefore, there exists $\kappa > 0$

$$w_\infty(x_0) - v_+(x_0) \leq \gamma(w_+^*(x_0) - v_+(x_0) - \kappa).$$

Since $\gamma > 0$, this contradicts (2.13). Then, $v_+ \geq \mathcal{T}_{\mathsf{F},\mathsf{G}} v_+$ implies $v_+ \geq w_\infty$.

On the other hand, assume there exists $v_- \in \mathcal{C}_b(\mathsf{E})$ satisfying $v_- \leq \mathcal{T}_{\mathsf{F},\mathsf{G}} v_-$. To prove $v_- \leq w_\infty$, assume there exists some x_0 such that $w_+^*(x_0) \geq v_+(x_0) > w_\infty(x_0)$. Then, similarly, there exists $0 < \gamma \leq 1$ such that

$$v_-(x_0) - w_\infty(x_0) = \gamma(w_+^*(x_0) - w_\infty(x_0)). \tag{2.14}$$

Then, since $v_- \leq \mathcal{T}_{\mathsf{F},\mathsf{G}} v_-$ we have

$$v_-(x_0) - w_\infty(x_0) < \gamma(w_+^*(x_0) - w_\infty(x_0) - \kappa).$$

This contradicts (2.14). Therefore, $v_- \leq \mathcal{T}_{\mathsf{F},\mathsf{G}} v_-$ implies $v_- \leq w_\infty$.

Thus if v is a solution to (2.9) then $v = w_\infty$. From the above computations, $v \geq (\leq)\mathcal{T}_{\mathsf{F},\mathsf{G}} v$ implies that $v \geq (\leq) w_\infty$ and the result follows. □

2.2. **Viscosity Solution.** In this subsection, we show that under Assumptions 2.1 and 2.2, the solution to (2.9) is the unique viscosity solution of the Hamilton-Jacobi-Bellman (HJB) equation given by

$$\min\{aw - \mathcal{A}w - \mathsf{F}w, w - \mathsf{G}w\} = 0.$$

Theorem 2.4. *Suppose Assumption 2.1 and Assumption 2.2 hold. Then there exists a unique viscosity solution* $w \in \mathcal{C}_b(\mathsf{E})$ *to*

$$\min\{aw - \mathcal{A}w - \mathsf{F}w, w - \mathsf{G}w\} = 0. \tag{2.15}$$

In addition, this solution is a solution to (2.9).

Proof. By Theorem 2.1, a function $w \in \mathcal{C}_b(\mathsf{E})$ is a solution to $\mathcal{T}_{\mathsf{F},\mathsf{G}}w = w$ if and only if w is a viscosity solution to (2.15). Since there exists a unique solution to $\mathcal{T}_{\mathsf{F},\mathsf{G}}w = w$ by Theorem 2.3, this completes the proof. □

Proposition 2.2. *(Comparison Principle) Suppose that Assumption 2.1 and Assumption 2.2 holds. Let $w_1 \in \mathcal{C}_b(\mathsf{E})$ and $w_2 \in \mathcal{C}_b(\mathsf{E})$ be a viscosity subsolution and supersolution to* (2.15)*. Then, $w_1 \le w_2$.*

Proof. Using Theorem 2.2, we know that if w_1 (*respectively*, w_2) is a viscosity subsolution (*respectively*, supersolution), then $w_1 \le \mathcal{T}_{\mathsf{F},\mathsf{G}}w_1$ (*respectively*, $w_2 \ge \mathcal{T}_{\mathsf{F},\mathsf{G}}w_2$). Therefore, using Proposition 2.1, we have that $w_1 \le w_\infty \le w_2$. □

Based on Proposition 2.2, we provide a sufficient condition for Assumption 2.2 *(i)* to hold.

Corollary 2.2. *Assume there exists a positive constant $\kappa > 0$ and a viscosity supersolution $w_+ \in \mathcal{C}_b(\mathsf{E})$ to*

$$\min\{aw_+ - \mathcal{A}w_+ - \mathsf{F}_\kappa w_+, w_+ - \mathsf{G}_\kappa w_+\} = 0,$$

where $\mathsf{F}_\kappa w_+ := \mathsf{F}w_+ + a\kappa$ and $\mathsf{G}_\kappa w_+ := \mathsf{G}w_+ + \kappa$. Then, Assumption 2.2 (i) holds.

Proof. Since w_+ is the viscosity supersolution, by Proposition 2.2, we have

$$\begin{aligned} w_+(x) &\ge \mathcal{T}_{\mathsf{F}_\kappa,\mathsf{G}_\kappa}w_+(x), \\ &= \sup_\tau \boldsymbol{E}^x\Big[\int_0^\tau e^{-as}(\mathsf{F}w_+(X(s)) + a\kappa)\mathrm{d}s + e^{-as\tau}(\mathsf{G}w_+(X(\tau)) + \kappa)\Big] \\ &= \kappa + \mathcal{T}_{\mathsf{F},\mathsf{G}}w_+(x). \end{aligned}$$

Then, the proof is finished. □

3. Application 1: Impulse control problems

In this section, we show the link between the value function of some impulse control problems and the unique viscosity solution to some HJB equations. Such relationship has been studied before (see for example [4, 6, 7] and [22] for general Markov processes). We extend the above results in two directions. First, we characterise the value function of an impulse control for Feller processes as a viscosity solution to an HJB equations; second, we relax the assumption of the performance functional (see Assumption 3.1 *(iii)*). The latter assumption is a sufficient condition to obtain Assumption 2.2 *(i)*. Such assumption can for example be found in [6]. Note however that [6] studies impulse control problem for jump diffusions and use an approach different to the iterative approach for general Feller processes.

Consider a general Feller Markov process and let us introduce the following impulse control problem studied in [7]. Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, \theta_t, X(t), \boldsymbol{P}^x)$ be a Markov process. Define $\Omega_\infty := (\Omega)^{\times\infty}$ and $\mathcal{F}^n_t := \mathcal{F}^{\times n}_t$ for $n \in \mathbb{N}$. The shift operator is defined by $\theta^n_t\omega(s) := (\theta_t\omega_1(s), \theta_t\omega_2(s), \ldots, \theta_t\omega_n(s))$ for $\omega = (\omega_1, \omega_2, \ldots) \in \Omega_\infty$. A sequence of $\pi := \{\tau_i, \xi_i\}_{i\in\mathbb{N}}$ is called an *admissible control* strategy if

(1) τ_i is a $\mathcal{F}^n_t \times \{\emptyset, \Omega\}^{\times\infty}$-measurable stopping time, $\tau_i \le \tau_{i+1}$ and $\lim_{n\to\infty}\tau_n = \infty$.
(2) ξ_i is $\mathcal{F}_{\tau_i} \times \{\emptyset, \Omega\}^{\times\infty}$-measurable.

The trajectory of the controlled process $\{X^\pi(t)\}_{t\ge0}$ is defined by using coordinates $X_t(\omega) = X_t(\omega_n)$ for $t \in [\tau_n, \tau_n + 1)$ and $\omega = (\omega_1, \omega_2, \ldots) \in \Omega_\infty$. The process X^π shifts to a new state ξ_n at τ_n and it generates a new probability measure $\boldsymbol{P}^{\pi,x}$ (see for example [7, Section 5] for more information). The impulse control problem consists in finding the optimal admissible strategy π that maximizes

$$J(x,\pi) := \boldsymbol{E}^{\pi,x}\Big[\int_0^\infty e^{-as} f(X^\pi(s))ds + \sum_{i=1}^\infty e^{-a\tau_i} K(X^\pi(\tau_i^-), X^\pi(\tau_i))\Big],$$

where $f : \mathsf{E} \to \mathbb{R}$ is a continuous bounded function and $K : \mathsf{E}\times\mathsf{E} \to \mathbb{R}$ is the reward obtained at ith impulse control. The value function of the above problem is defined by

$$V(x) := \sup_\pi J(x,\pi). \tag{3.1}$$

The notion of viscosity solution is often used to solve the variational inequality associated with the value function for such an impulse control problem (see for example [6, 22, 23]).

3.1. **Main results.** In this section we derive the main results. It is worth mentioning that, the value function can be characterized by the viscosity solution to

$$\min\{aw - \mathcal{A}w - f, g - \mathcal{M}w\} = 0,$$

with

$$\mathcal{M}u(x) := sup_{y\in\mathsf{E}}(u(y) + K(x,y)).$$

In order to solve the problem (3.1), we make the following assumption which guarantees that Assumption 2.1 and Assumption 2.2 are satisfied.

Assumption 3.1.

(i) $\mathcal{M} : \mathcal{C}_b(\mathsf{E}) \to \mathcal{C}_b(\mathsf{E})$.

(ii) There exists a constant $C > 0$ *such that*

$$K(x,y) + K(y,z) \le K(x,z) - C \text{ for all } x,y,z \in \mathsf{E}.$$

(iii) Fix the constant $C > 0$ *from (ii). There exists a function* $u \in \mathcal{C}_b(\mathsf{E}) \cap \mathcal{R}_a(\mathcal{C}_b(\mathsf{E}))$*, a point* $z_0 \in \mathsf{E}$ *and a constant* $\kappa > 0$ *such that for all* $x \in \mathsf{E}$*,*

$$0 \le u(x) - K_{z_0}(x) \le C - \kappa$$

where $K_{z_0}(x) := K(x, z_0)$.

Remark 3.1. Assumption 3.1 *(i)* and *(ii)* are common in the literature of impulse control problems. In general, when studying a general impulse control problem, most papers (see for example [22, 23]) use the following stronger assumption in the place of Assumption 3.1 *(iii)*, namely: $K(x,y) < -C$ for all $x,y \in \mathsf{E}$. However, the preceding assumption failed to be satisfied in some interesting applications in finance (see Remark 3.2 *(i)*).

Proposition 3.1. *Suppose Assumption 3.1 holds and* $f \in \mathcal{C}_b(\mathsf{E})$.

(i) There exists a unique viscosity solution $w \in \mathcal{C}_b(\mathsf{E})$ *to*

$$\min\{aw - \mathcal{A}w - f, w - \mathcal{M}w(x)\} = 0. \tag{3.2}$$

(ii) Additionally, suppose the value function $V \in \mathcal{C}_b(\mathsf{E})$ *defined by* (3.1) *satisfies the following dynamic programming equation*

$$w(x) = \mathcal{T}_{f,\mathcal{M}}w := \sup_\tau \boldsymbol{E}^x\Big[\int_0^\tau e^{-as} f(X(s))ds + e^{-a\tau}\mathcal{M}w(X(\tau))\Big]. \tag{3.3}$$

Then, $V = w$*, where* w *is the unique viscosity solution to* (3.2).

Proof. We simply need to check that Assumption 2.1 and Assumption 2.2 are satisfied and the result will follow from Theorem 2.3.

(i) Let $\mathsf{F}u := f$ and $\mathsf{G}u := \mathcal{M}u$ for all $u \in \mathcal{C}_b(\mathsf{E})$. Then, Assumption 2.1 follows from Assumption 3.1 *(i)* and the convexity and monotonicity properties of G can be proved as in [22]. Additionally, the convexity and monotonicity property of F follows from the fact that $f \in \mathcal{C}_b(\mathsf{E})$. Furthermore, since $\mathcal{M}(u+c) = \mathcal{M}u + c$ for any $u \in \mathcal{C}_b(\mathsf{E})$ and constant function c, we only need to verify Assumption 2.2 *(i)*. Now fix $z_0 \in \mathsf{E}$, using Assumption 3.1 *(ii)*, there exists a constant $C > 0$ such that

$$K(x,y) + K(y,z_0) \leq K(x,z_0) - C.$$

Define $\mathcal{R}_a(\mathcal{C}_b(\mathsf{E})) := \{u \in \mathcal{C}_b(\mathsf{E});\ \text{there exists } v \in \mathcal{C}_b(\mathsf{E}) \text{ such that } u = (a - \mathcal{A})v\}$. Then, there exists $u \in \mathcal{C}_b(\mathsf{E}) \cap \mathcal{R}_a(\mathcal{C}_b(\mathsf{E}))$ such that for any $x \in \mathsf{E}$,

$$\begin{aligned} & u(x) - \sup_{y\in\mathsf{E}}(u(y) + K(x,y)) \\ \geq & u(x) - \sup_{y\in\mathsf{E}}(u(y) + K(x,z_0) - K(y,z_0)) + C \\ \geq & u(x) - K(x,z_0) - \sup_{y\in\mathsf{E}}(u(y) - K(y,z_0)) + C \\ \geq & 0 - (C - \kappa) + C \geq \kappa. \end{aligned}$$

Here the second inequality is from Assumption 3.1 *(iii)*. Hence, $u - \mathcal{M}u \geq \kappa$.

Furthermore, since $u \in \mathcal{R}_a(\mathcal{C}_b(\mathsf{E}))$, there exists $h \in \mathcal{C}_b(\mathsf{E})$ such that $h = (a - \mathcal{A})u$. Define $u^* := u + (\|h\|_\infty + \|f\|_\infty)/a + \kappa$. We have $(a - \mathcal{A})u^* = h + (\|h\|_\infty + \|f\|_\infty) + a\kappa \geq f + a\kappa$. Additionally, since $u - \mathcal{M}u \geq \kappa$ implies $u^* - \mathcal{M}u^* \geq \kappa$, u^* satisfies

$$\min\{au^* - \mathcal{A}u^* - f - a\kappa, u^* - \mathcal{M}u^* - \kappa\} = 0.$$

Then, by Corollary 2.2, Assumption 2.2 *(ii)* is shown.

(ii) The proof of the claim follows by applying Theorem 2.1. □

Proposition 3.2. *Suppose Assumption 3.1 holds. Let* $w_0 := \mathcal{R}_a f$ *and* $w_{n+1} := \mathcal{T}_{f,\mathcal{M}} w_n$, *where* $\mathcal{T}_{f,\mathcal{M}}$ *is defined by* (3.3). *Then, there exists a function* $w \in \mathcal{C}_b(\mathsf{E})$ *such that the sequence of functions* $\{w_n\}_{n\in\mathbb{N}}$ *converges to* w *uniformly as* $n \to \infty$. *Additionally,* w *is the solution to* (3.3)

Proof. Since w_0 is the subsolution to

$$\min\{aw - \mathcal{A}w - f, w - \mathcal{M}w\} = 0,$$

then $w_0 \leq \mathcal{T}_{f,\mathcal{M}} w_0$. The claim follows from Theorem 2.3. □

In the next section, we use results from Proposition 3.1 and 3.2 to examine an impulse control problem for a one-dimensional regular Feller diffusion.

3.2. **Examples.** Let X be a one-dimensional regular diffusion with state space $\mathsf{E} = [L, R] \subseteq \mathbb{R}$. We show that some specific impulse control problems can be considered under the optimal stopping framework, assuming that the premium function K in the impulse control problem defined by (3.1) takes the following form

$$K(x,y) = \begin{cases} k_1(x) - k_1(y) - c_1 & \text{for } x > y, \\ k_2(x) - k_2(y) - c_2 & \text{for } x \leq y, \end{cases}$$

where k_1, k_2 are functions and c_1, c_2 are constants. We are interested in the following impulse control problem:

$$\begin{aligned} V^{(k_{1,2})}(x) &:= \sup_\pi \boldsymbol{E}^x\Big[\sum_{i=1}^\infty e^{-a\tau_i} K(X(\tau_i^-), X(\tau_i))\Big] \\ &= \sup_\pi \boldsymbol{E}^x\Big[\sum_{i=1}^\infty e^{-a\tau_i} \mathbf{1}_{X^\pi(\tau_i^-) > X^\pi(\tau_i)}((k_1(X^\pi(\tau_i^-)) - k_1(X^\pi(\tau_i)) - c_1) \\ &\quad + \sum_{i=1}^\infty e^{-a\tau_i} \mathbf{1}_{X^\pi(\tau_i^-) \le X^\pi(\tau_i)}(k_2(X^\pi(\tau_i^-)) - k_2(X^\pi(\tau_i))) - c_2)\Big]. \end{aligned} \tag{3.4}$$

The above problem often appears in actuarial science and is refereed to as dividend and investment with different proportional costs and fixed costs. More particularly, we consider the subsequent forms:

(1) Linear form: $k_1^{(l)}(x) = \beta_1 x$ and $k_2^{(l)}(x) = \beta_2 x$.
(2) Exponential form: $k_1^{(e)}(x) = \beta_1 e^x$ and $k_2^{(e)}(x) = \beta_2 e^x$.
(3) Quadratic form: $k_1^{(q)}(x) = \beta_1 x^2 + \gamma_1 x$ and $k_2^{(q)}(x) = \beta_2 x^2 + \gamma_2 x$.

Here $\beta_1 \le \beta_2$ and $\gamma_1 \le \gamma_2$. In finance, the first one can be used to study dividends and investment problems (compatre with the premium in [10] in which the author studied and impulse control for diffusions with fixed and proportional cost). The second form can be applied to exponential of diffusion processes. The last situation can be found in [11] and represents the impulse control problem with quadratic costs. Here, although the functions k_1 and k_2 are not bounded in any of the aforementioned cases, we suppose their value functions can be obtained from the convergence of the value function with $k_1 \wedge L$ and $k_2 \wedge L$, as $L \to \infty$. More specifically, the above choice of K and value function can be found in some practical examples as shown below (see for instance [11, 24, 25]).

Remark 3.2.

(i) *Dividend and injection with fixed cost.* A popular example pertains to optimal dividend in financial and actuarial mathematics. Let $\mathsf{E} \subseteq \mathbb{R}$ be a compact subset, $k(x) = x$ and $c(x) = c_0 > 0$ so that $K(x, y) = x - y - c_0$. The associated optimal stopping control problem can be seen as an optimal proportional dividend and capital injection problem with fixed cost k.

(ii) *Dividend and injection for exponential Lévy process.* Suppose the process Y is an exponential Lévy process, i.e. $Y(t) = e^{X(t)}$, where X is a Lévy process. Denote by V_Y the value function to the impulse control problem of Y for dividend and injection given by (1). Now, let $k(x) = e^x$, $c(x) = c_0$ so that $K(x, y) = e^x - e^y - c_0$. Its value function is defined as V_X. Then, we can easily get $V_X(x) = V_Y(e^x)$ for $x \in \mathbb{R}$.

In order to solve (3.4), we follow the idea in [26] combined with the approach introduced in [1, Chapter 3]. Here we assume that X is a regular Feller diffusion, i.e., $\boldsymbol{P}^x\big[\tau_y < \infty\big] > 0$ for all $x \in \mathsf{E}$, where $\tau_y := \inf\{t > 0; X(t) = y\}$. Let $x_0 \in (L, R)$ and define the function

$$\psi_1(x) := \begin{cases} \boldsymbol{E}^x\big[e^{-a\tau_z}\big] & \text{for } x \le z \\ 1/\boldsymbol{E}^z\big[e^{-a\tau_x}\big] & \text{for } x \ge z \end{cases} \text{ and } \psi_2(x) := \begin{cases} 1/\boldsymbol{E}^z\big[e^{-a\tau_x}\big] & \text{for } x \le z \\ \boldsymbol{E}^x\big[e^{-a\tau_z}\big] & \text{for } x \ge z \end{cases}. \tag{3.5}$$

Here, we separate the points in state space E into two regions:

$$\mathfrak{C} := \{x \in [L,R]; V(x) > \mathcal{M}V(x)\},$$
$$\mathfrak{D} := \{x \in [L,R]; V(x) = \mathcal{M}V(x)\}.$$

For simplicity, we only consider the continuation region $\mathfrak{C}$ which is connected and we distinguish three different cases:

Case I $\mathfrak{C} = (l,r)$,
Case II $\mathfrak{C} = (L,r)$ or $[L,r)$,
Case III $\mathfrak{C} = (l,R)$ or $(l,R]$,

where $L < l < r < R$. Since Case II and Case III are similar, we only consider Case I and Case II. We will characterize the value function $V^{(k_{1,2})}$ defined by (3.4).

Case I: Let $\mathfrak{C} = (l,r)$. In this case, when the process reaches l or r, we exercise the impulse strategy which alters the state of the process from l or r to some point inside (l,r). We have the following verification result.

Proposition 3.3. *Assume $k_2 - k_1$ is an increasing function, and there exist 4 constants (l,r,p_1,p_2) such that the functions*

$$u(x) := p_1\psi_1(x) + p_2\psi_2(x)$$
$$u_i(x) := u(x) - k_i(x) \text{ for } i = 1,2,$$

satisfy

(i) u_1 has a local minimum at l and u_2 has a local minimum at r,
(ii) $u_1(r) = \sup_{y\in[l,r]} u_1(y) - c_1$ and $u_2(l) = \sup_{y\in[l,r]} u_2(y) - c_2$.

Define

$$w_{p_1,p_2,l,r}(x) := \begin{cases} k_2(x) + u(l) - k_2(l) & \text{for } x \in [L,l), \\ u(x) & \text{for } x \in [l,r], \\ k_1(x) + u(r) - k_1(r) & \text{for } x \in (r,R]. \end{cases}$$

Then the value function satisfies $V^{(k_{1,2})} \geq w_{p_1,p_2,l,r}$, where $V^{(k_{1,2})}$ is defined by (3.4).

Furthermore, suppose

(iii) $u_1(y) - u_1(x) \leq c_1$ and $u_2(x) - u_2(y) \leq c_2$ for any $l \leq x < y \leq r$.
(iv) k_2 is a viscosity supersolution to $aw(x) - \mathcal{A}w(x) - a(k_2(l) - u(l)) = 0$ for $x \in [L,l)$ and k_1 is a viscosity supersolution to $aw(x) - \mathcal{A}w(x) - a(k_1(r) - u(r)) = 0$ for $x \in (r,R]$.

Then, the equality holds, i.e., $V^{(k_{1,2})}(x) = w_{p_1,p_2,l,r}(x)$ for $x \in [L,R]$.

Remark 3.3. Let O be an open subset of E. Here a function is a *viscosity supersolution (respectively, subsolution)* to

$$aw - \mathcal{A}w - f = 0 \text{ for } x \in O$$

for all $\phi \in D(\mathcal{A})$ such that $\phi - w$ has a global minimum (*respectively*, maximum) at $x_0 \in O$ with $\phi(x_0) = w(x_0)$,

$$a\phi(x_0) - \mathcal{A}\phi(x_0) - \mathsf{F}w(x_0) \leq (\geq) 0.$$

Proof. We know from Proposition 3.1 that the value function $V^{k_{1,2}}$ is the unique viscosity solution to

$$\min\{aw - \mathcal{A}w, w - \mathcal{M}w\} = 0 \text{ for } x \in [L,R]. \tag{3.6}$$

We only need to show that the function $w_{p_1,p_2,l,r}$ is also a viscosity solution to (3.6).

(1) Let $x \in (l,r)$. Since $w_{p_1,p_2,l,r}(x) = u(x)$ for $x \in (l,r)$, $w_{p_1,p_2,l,r}$ is a viscosity solution to

$$aw(x) - \mathcal{A}w(x) = 0 \text{ for } x \in (l,r).$$

It follows from the definition of $w_{p_1,p_2,l,r}$, conditions *(ii)* and *(iii)* that $w_{p_1,p_2,l,r}(x) \geq \mathcal{M}w_{p_1,p_2,l,r}(x)$ and thus (3.6) is satisfied.

(2) Let $x \in (l,r)^c$. For $x = l, r$, condition *(i)* implies that $w_{p_1,p_2,l,r}$ is a viscosity supersolution to $aw(x) - \mathcal{A}w(x) = 0$. On the other hand, ondition *(iv)* implies $w_{p_1,p_2,l,r}$ is a viscosity supersolution to $aw(x) - \mathcal{A}w(x) = 0$ for $x = [l,r]^c$. Then, to verify (3.6), we only need to show $w_{p_1,p_2,l,r}(x) = \mathcal{M}w_{p_1,p_2,l,r}(x)$. This follows from condition *(ii)* and the fact that $k_2 - k_1$ is an increasing function. □

Case II: We consider two cases: $\mathfrak{C} = (L,r)$ and $\mathfrak{C} = [L,r)$, respectively. For $\mathfrak{C} = (L,r)$, when the process reaches the boundary L or r, the impulse strategy is exercised in the same way as described above. However, when $\mathfrak{C} = [L,r)$, the impulse strategy is applied when the process reaches r only. We will show that a similar conclusion holds as in the above case.

Proposition 3.4. *Let $L < R$. Assume that $k_1 - k_2$ is an increasing function, and there exist 3 constants (r, p_1, p_2) such that the functions*

$$u(x) := p_1\psi_1(x) + p_2\psi_2(x)$$
$$u_i(x) := u(x) - k_i(x) \text{ for } i = 1, 2,$$

satisfy

(i) $p_2 \geq 0$. *Furthermore, if* $p_2 > 0$, *we suppose* $u_2(L) = \sup_{y\in[L,r]} u_2(y) - c_2$ *and* $\psi_2(L) < \infty$.

(ii) u_1 *has a local minimum at* r.

(iii) $u_1(r) = \sup_{y\in[L,r]} u_1(y) - c_1$.

Define

$$w_{p_1,p_2,r}(x) := \begin{cases} u(x) & \text{for } x \in [L,r], \\ k_1(x) + u(r) - k_1(r) & \text{for } x \in (r,R]. \end{cases}$$

Then the value function satisfies $V^{k_{1,2}} \geq w_{p_1,p_2,r}$.

In addition, suppose that

(iv) $u_1(y) - u_1(x) \leq c_1$ *and* $u_2(x) - u_2(y) \leq c_2$ *for any* $L \leq x < y \leq r$.

(v) k_1 *is a viscosity supersolution to* $aw(x) - \mathcal{A}w(x) - a(k_1(r) - u(r)) = 0$ *for* $x \in (r,R]$.

Then, the equality holds, i.e., $V^{k_{1,2}}(x) = w_{p_1,p_2,r}(x)$

Proof. Similiar to the proof of Proposition 3.3. □

3.3. **Explicit Solutions.** In this section, we illustrate the above result by studying an impulse control problem for an absorbing Feller diffusion on $[0,\infty)$. In this case, we obtain an explicit solution to problem (3.9) below. Similar problem was solved in [10].

Example 3.1. *(Absorbing Feller diffusion on $[0,\infty)$) An absorbing Feller process is a diffusion process with absorbing boundary whose generator is given by*

$$\begin{aligned} D(\mathcal{A}) &:= \{u \in \mathcal{C}_0([a,\infty)) \cap \mathcal{C}^2([a,\infty)); \frac{1}{2}\sigma^2 D_{xx}u(0) + \mu D_x u(0) = 0\}, \\ \mathcal{A}u(x) &:= \frac{1}{2}\sigma^2 D_{xx}u(x) + \mu D_x u(x). \end{aligned} \tag{3.7}$$

In this case, ψ_1 and ψ_2 as in (3.5) are reduced to

$$\psi_1(x) = e^{l_1 x} - e^{l_2 x} \text{ and } \psi_2(x) = e^{l_2 x}, \tag{3.8}$$

where $l_1 = \frac{-\mu+\sqrt{\mu^2+2a\sigma^2}}{\sigma^2}$ and $l_2 = \frac{-\mu-\sqrt{\mu^2+2a\sigma^2}}{\sigma^2}$.

Recall that we are interested in the following impulse control problem.

$$\begin{aligned} V(x) := \sup_{\pi} \boldsymbol{E}^x \Big[&\sum_{i=1}^{\infty} e^{-a\tau_i} \mathbf{1}_{X^\pi(\tau_i^-) > X^\pi(\tau_i)} ((k_1(X^\pi(\tau_i^-)) - k_1(X^\pi(\tau_i)) - c_1) \\ &+ \sum_{i=1}^{\infty} e^{-a\tau_i} \mathbf{1}_{X^\pi(\tau_i^-) \leq X^\pi(\tau_i)} (k_2(X^\pi(\tau_i^-)) - k_2(X^\pi(\tau_i))) - c_2) \Big], \end{aligned} \tag{3.9}$$

where X is a one-dimensional Brownian diffusion. Such a problem was solved in [10] for $k_1(x) = \beta_1 x$ and $k_2(x) = \beta_2 x$. Note that in their work, they analysed a combined impulse and stochastic control problem. Here, we only focus on the impulse control problem with linear form. Furthermore, the impulse problem with function k_1 and k_2 of the exponential and quadratic type (see for example Remark 3.2) can be also solved similarly.

In the sequel, we illustrate a linear case for an absorbing diffusion when $\mathfrak{C} = [L, r)$.

Corollary 3.1. *Let X be an absorbing Brownian motion whose generator is given by (3.7) and the value function V is defined by (3.9), with $k_1 = \beta_1 x$ and $k_2 = \beta_2 x$ and $\beta_2 > \beta_1 > 0$. Let ψ_1 and ψ_2 be given by (3.8).*

Case I For $\mu \leq 0$, assume there exists $c \in \mathbb{R}$ and $x^* \in (0, \infty)$ such that

- *(i) $c\psi_1 - \beta_1 x$ has a local minimum at x^* .*
- *(ii) $c_1 = c\psi_1(x^*) - \beta_1 x^*$.*
- *(iii) $c\psi_1(x) - \beta_1 x$ is decreasing in $[L, x^*]$.*

Then, the value function is

$$V(x) = \begin{cases} c\psi_1(x) & \text{for } x \in [0, x_*], \\ k_1(x) - k_1(x^*) + c\psi_1(x) & \text{for } x \in (x_*, \infty). \end{cases}$$

Case II For $\mu > 0$, assume there exist (p_1, p_2, x^*) such that

- *(i) $p_1 \in \mathbb{R}$, $p_2 > 0$.*
- *(ii) $p_1\psi_1(x) + p_2\psi_2(x) - k_1(x)$ has a local minimum at x^*.*
- *(iii) $p_1\psi_1(x^*) + p_2\psi_2(x^*) - \beta_1 x^* = \max_{x\in[0,x^*]}\{p_1\psi_1(x) + p_2\psi_2(x) - \beta_1 x\} - c_1 = p_1\psi_1(x_r) + p_2\psi_2(x_r) - \beta_1 x_r$ where $x_r \in [0, x^*]$. $p_1\psi_1(x) + p_2\psi_2(x) - \beta_1 x$ is increasing in $[0, x_r]$ and decreasing in $[x_r, x^*]$.*
- *(iv) $p_1\psi_1(0) + p_2\psi_2(0) = \max_{x\in[0,x^*]}\{p_1\psi_1(x) + p_2\psi_2(x) - \beta_2 x\} - c_2 = p_1\psi_1(x_r) + p_2\psi_2(x_l) - k_2 x_l$, where $x_l \in [0, x^*]$. Additionally, $p_1\psi_1(x) + p_2\psi_2(x) - \beta_2 x$ is increasing in $[0, x_l]$ and decreasing in $[x_l, x^*]$.*

Then, the value function is given by

$$V(x) = \begin{cases} p_1\psi_1(x) + p_2\psi_2(x) & \text{for } x \in [0, x^*], \\ k_1(x) - k_1(x^*) + p_1\psi_1(x^*) + p_2\psi_2(x^*) & \text{for } x \in (x^*, \infty). \end{cases}$$

Proof. Theses results can be proved using Proposition 3.4. Case I is from the case $[L, r)$ and Case II is from the case (L, r), when $L = 0, r = x^*$. □

The numerical results give an idea of what we could obtain from the above result. Set the parameter values to be $\beta_1 = 0.9$, $\beta_2 = 1.5$, $c_1 = 2$ and $c_2 = 4$.

Case I: Set $\mathfrak{C} = [0, x^*)$, $\mu = -1, \sigma = 1$. *In addition, one can show that* $\psi_1(x) = e^{2.0488x} - e^{-0.0488x}$, $\psi_2(x) = e^{-0.0488x}$, *as well as deriving* $c = 0.0017$ *and* $x^* = 2.71$. *Based on these values, we plotted the function* u_1 *in the figure.*

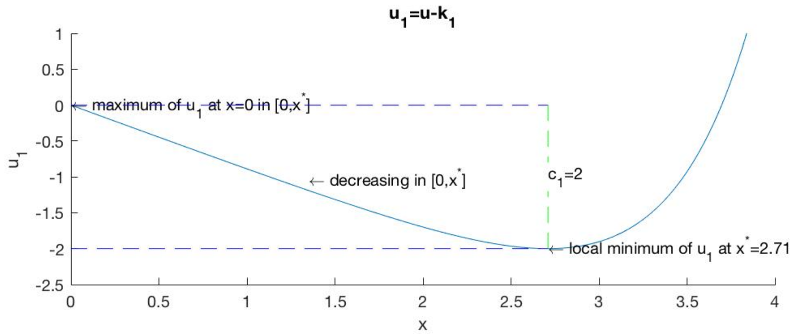

Figure 1. This graph sketches $u_1(x) = c\psi_1(x) - \beta_1 x$. Since ψ_1 is convex, it has a minimum at $x^* = 2.71$. Additionally, we can see it decreases from 0 to $x^* = 2.71$ and then increases. Hence, the maximum of u_1 in $[0, x^*]$ is at $x = 0$. Thus Corollary 3.1 Case I *(ii)* is satisfied under $u_1(0) - u_1(x^*) = c_1$.

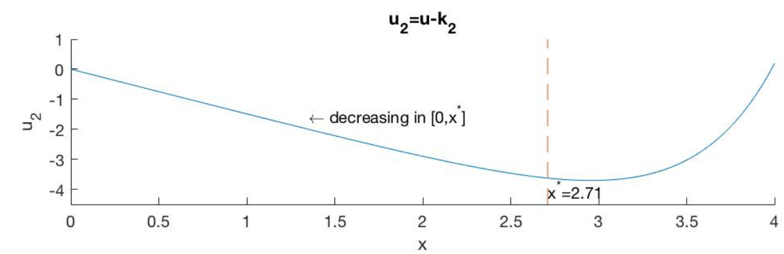

Figure 2. This graph sketches $u_2(x) = c\psi_1(x) - \beta_2 x$. Since u_2 is decreasing from 0 to x^* and $\beta_2 \geq \beta_1$, then u_2 is decreasing in $[0, x^*]$. Thus, since $c_2 > 0$, we have that $u_2(x) - u_2(y) \leq c_2$ for any $x > y$.

The above suggests that the optimal strategy is as follows: an impulse from $x^* = 2.71$ *to* 0 *when the process reaches* $x^* = 2.71$.

Case II: Here $\mathfrak{C} = (l, r)$. *In this case we assume that the parameters have the same value except for* $\mu = 1$. *Consequently,* $\psi_1(x) = e^{0.0488x} - e^{-2.0488x}$ *and* $\psi_2(x) = e^{-2.0488x}$. *Furthermore, one has* $p_1 = 10.01$, $p_2 = 4.33$ *and* $r = 12$.

Again intuitively, the desired strategy is to carry out an impulse from $r = 12$ *to* $x_r = 1.62$ *when the process reaches* $r = 12$. *Furthermore, it suggests to exercise an impulse from* 0 *to* $x_l = 1.16$ *when the process reaches* 0.

4. Application 2: Perturbation and Its Applications

In the above section, we mainly discussed the specific case for the impulse control problem whose operator is given by $\mathsf{G}u = \sup_{y \in \mathsf{E}}(u(y) + K(x, y))$. In this section, we analyse a series of problems assuming the operator F given by perturbation. Note that, the construction of the Feller semigroup based on the perturbation relies on Hille-Yosida theorem. This allows

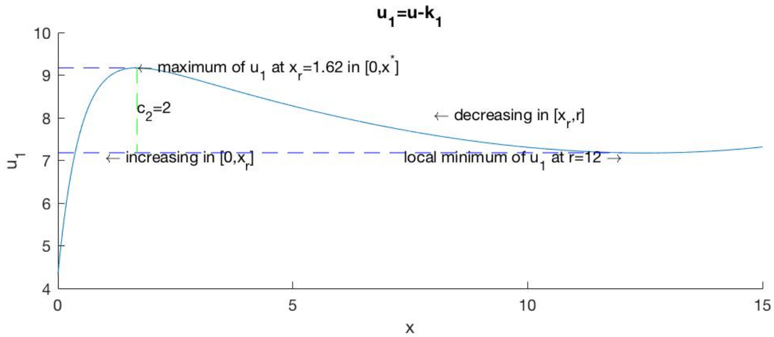

FIGURE 3. This graph sketches $u_1(x) = p_1\psi_1(x) + p_2\psi_2(x) - \beta_1 x$. As it can be seen, u_1 has a local maximum at $x_r = 1.62$ and a local minimum at $r = 12$. Additionally, u_1 is increasing in $[0, x_r]$ and is decreasing in $[x_r, x_*]$. The condition we need to impose here is that $u_1(x_r) - u_1(r) = c_2$.

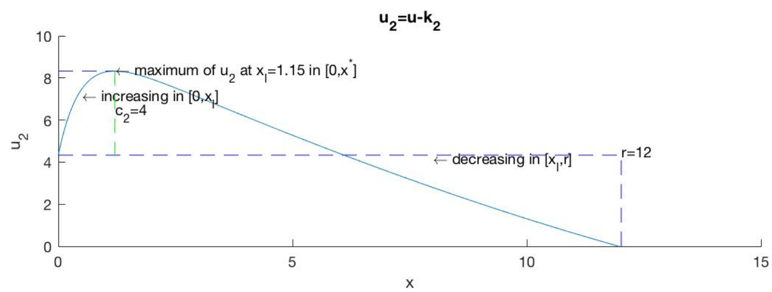

FIGURE 4. This graph sketches $u_2(x) = p_1\psi_1(x) + p_2\psi_2(x) - \beta_1 x$. Again, u_1 has a local maximum at $x_l = 1.15$. It entitles that u_1 is increasing in $[0, x_r]$ and is decreasing in $[x_r, x_*]$. We need to impose the condition $u_1(x_r) - u_1(0) = c_2$.

us to verify the existence of Feller process with jumps (see for example [27, Section 4.3.] and [28, Corollary 9.51.]).

Let b be a non-negative function in $\mathcal{C}_b(\mathsf{E})$, λ be a non-negative constant and $\mathcal{B}$ be a linear operator on $\mathcal{C}_b(\mathsf{E})$. Then we can define the perturbation operator $\mathcal{A}_{pb} : B(\mathsf{E}) \to B(\mathsf{E})$ by

$$\mathcal{A}_{pb} w(x) := b(x)\mathcal{B}w(x) - \lambda b(x) w(x) \text{ for } x \in \mathsf{E}, w \in B(\mathsf{E}).$$

To construct the process with perturbation, we make the following assumptions on the operator $\mathcal{B}$.

Assumption 4.1.

(i) $\mathcal{B}$ *is a linear operator and* $\mathcal{B} : \mathcal{C}_b(\mathsf{E}) \to \mathcal{C}_b(\mathsf{E})$.
(ii) $\mathcal{B}$ *is positive and bounded with* $\lambda \geq \|\mathcal{B}\|_\infty$.

We start with the following lemma (see [28, Corollary 9.51.]).

Lemma 4.1. *Suppose Assumption 4.1 holds and* $\mathcal{B} : \mathcal{C}_0(\mathsf{E}) \to \mathcal{C}_0(\mathsf{E})$. *Let* $(\mathcal{A}_0, D(\mathcal{A}_0))$ *be the generator of some Feller process. Then,* $(\mathcal{A}_0 + \mathcal{A}_{pb}, D(\mathcal{A}_0))$ *is also the generator of some Feller semigroup.*

Proof. One can check the positive maximum property of $\mathcal{A}_{pb}$ and $\mathcal{A}_{pb} : \mathcal{C}_0(\mathsf{E}) \to \mathcal{C}_0(\mathsf{E})$. □

Now let $(\mathcal{A}_0, D(\mathcal{A}_0))$ be a Feller semigroup and X be a Feller process with the infinitesimal generators $(\mathcal{A}_0 + \mathcal{A}_{pb}, D(\mathcal{A}_0))$. We are interested in the optimal stopping problem

$$V(x) := \sup_{\tau} \boldsymbol{E}^x \Big[\int_0^\tau e^{-as} f(X(s)) \mathrm{d}s + e^{-a\tau} g(X(\tau)) \Big]. \tag{4.1}$$

4.1. **Main results.** The following results, derived from Theorem 2.4 characterize the value function V given by (4.1) in the viscosity sense.

Proposition 4.1. *Suppose Assumption 4.1 holds and* $(\mathcal{A}_0, D(\mathcal{A}_0))$ *is a generator of a Feller process. If* X *is a Feller process with infinitesimal generator* $(\mathcal{A} + \mathcal{A}_{pb}, D(\mathcal{A}))$*, then the value function defined by* (4.1) *is the unique viscosity solution* $w \in \mathcal{C}_b(\mathsf{E})$ *to*

$$\min\{aw - \mathcal{A}_0 w - (\mathcal{A}_{pb} w + f), w - g\} = 0. \tag{4.2}$$

Proof. We first show that there exists a unique viscosity solution to (4.2). After transforming (4.2) using $\mathcal{A}_{pb} u := b\mathcal{B}u - \lambda b u$, its viscosity solution is equivalent to that of

$$\min\{(a + \lambda b)w - \mathcal{A}_0 w - (b\mathcal{B}w + f), w - g\} = 0.$$

Since $a + \lambda b \in \mathcal{C}_b(\mathsf{E})$ and $a + \lambda b > 0$, this is further equivalent to the viscosity solution to

$$\min\Big\{ w - \frac{1}{a + \lambda b}\mathcal{A}_0 w - \frac{b\mathcal{B}w + f}{a + \lambda b}, w - g \Big\} = 0. \tag{4.3}$$

We use Theorem 2.4 to show that there exists a viscosity solution to (4.3). Define F and G by

$$\mathsf{F}u := \frac{b\mathcal{B}u + f}{a + \lambda b} \text{ and } \mathsf{G}u := g.$$

We only need to verify that Assumption 2.1 and Assumption 2.2 are satisfied.

(i) Since $\mathcal{B}$ is defined from $\mathcal{C}_b(\mathsf{E})$ to itself and $b \in \mathcal{C}_b(\mathsf{E})$, we have that F is defined from $C_b(\mathsf{E})$ to itself.

(ii) The monotonic property of F in Assumption 2.1 follows from the fact that $\mathcal{B}$ is positive, $a + \lambda b > 0$ and $b \geq 0$.

(iii) The convexity of F in Assumption 2.1 follows from the linearity of $\mathcal{B}$, that is,

$$\begin{aligned}\mathsf{F}(pu_1 + (1-p)u_2) &= \frac{b\mathcal{B}(pu_1 + (1-p)u_2) + pf + (1-p)f}{a + \lambda b} \\ &= p\mathsf{F}u_1(x) + (1-p)\mathsf{F}u_2(x).\end{aligned}$$

(iv) Let $\kappa > 0$ be a constant and $w_+ := \max\{\frac{\|f\|_\infty}{a} + (a + \lambda\|b\|_\infty)\kappa, \|g\|_\infty + \kappa\}$ be a constant function. Then,

$$\begin{aligned}&\min\Big\{ w_+ - \frac{1}{a + \lambda b}\mathcal{A}_0 w_+ - \frac{b\mathcal{B}w_+ + f}{a + \lambda b} - \kappa, w_+ - g - \kappa \Big\} \\ =& \min\Big\{ \frac{aw_+}{a + \lambda} - \frac{\mathcal{A}_0 w_+ + b\mathcal{B}w_+ + \lambda w_+}{a + \lambda b} - \frac{f}{a + \lambda b} - \kappa, w_+ - g - \kappa \Big\} \\ =& \min\Big\{ \frac{aw_+ - f}{a + \lambda b} - \kappa, w_+ - g - \kappa \Big\} \geq 0.\end{aligned}$$

Hence, using Lemma 2.2, we have Assumption 2.2 *(i)*.

(v) Assumption 2.2 *(ii)* is true, since $\mathsf{F}(w + C) - \mathsf{F}w = \frac{b\mathcal{B}C}{a+\lambda} \leq \frac{C\lambda b}{a+\lambda b} \leq 1$.

The verification for G is straightforward and is omitted here. Thus, we can conclude that there exists a unique viscosity solution to (4.2) by Theorem 2.4 (or equivalently (4.3)).

Next, we prove that the value function V defined by (4.1) is the unique viscosity solution to (4.2). Since X is a Feller process, the value function V defined by (4.1) is the unique viscosity solution to

$$\min\{aw - \mathcal{A}w - f, w - g\} = 0. \tag{4.4}$$

where $\mathcal{A} := \mathcal{A}_0 + \mathcal{A}_{pb}$. By the uniqueness of the viscosity solution to (4.4), we only need to prove that the viscosity solution w to (4.2) is also aviscosity solution to (4.4). Let w be the viscosity solution to (4.3) (that is, a viscosity solution to (4.2)). Assume $\phi \in D(\mathcal{A}_0)$ satisfies $\phi - w$ has a global minimum at $x_0 \in \mathsf{E}$ such that $\phi(x_0) = w(x_0)$. Since w is a viscosity subsolution to (4.3), we have

$$\min\Big\{\phi(x_0) - \frac{1}{a+\lambda b}\mathcal{A}_0\phi(x_0) - \mathsf{F}w(x_0), \phi(x_0) - g(x_0)\Big\} \le 0.$$

In addition, since $\phi \ge w$ and F is increasing, we have $\mathsf{F}\phi \ge \mathsf{F}w$ and then

$$\min\Big\{\phi(x_0) - \frac{1}{a+\lambda b}\mathcal{A}_0\phi(x_0) - \mathsf{F}\phi(x_0), \phi(x_0) - g(x_0)\Big\} \le 0.$$

This is the same as

$$\min\{a\phi(x_0) - \mathcal{A}\phi(x_0) - f(x_0), \phi(x_0) - g(x_0)\} \le 0.$$

Therefore, w is also a viscosity subsolution to (4.2). The case of the viscosity supersolution can be proved similarly. □

Next, we construct a numerical scheme to derive the value function.

Proposition 4.2. *Suppose that assumptions in Proposition 4.1 are in force. Let $v_0 \in \mathcal{C}_b(\mathsf{E})$ be a viscosity subsolution to*

$$\min\{aw - \mathcal{A}_0 w - (\mathsf{F}_{pb}w + f), w - g\} = 0.$$

Let v_n be the viscosity solution to

$$\min\{aw - \mathcal{A}_0 w - \mathsf{F}_{pb}(v_{n-1} + f), w - g\} = 0,$$

or equivalently,

$$v_n(x) := \sup_\tau \boldsymbol{E}^x\Big[\int_0^\tau e^{-s}\frac{b(Y(s))\mathcal{B}v_{n-1}(Y(s)) + f(Y(s))}{a+\lambda b(Y(s))}\mathrm{d}s + e^{-\tau}g(Y(s))\Big],$$

where Y is a Feller process with the infinitesimal generator $(\frac{\mathcal{A}_0}{a+\lambda b}, D(\mathcal{A}_0))$. Then v_n converges uniformly to the viscosity solution $w \in \mathcal{C}_b(\mathsf{E})$ to (4.2).

Proof. Since we have proved the value function is the viscosity solution to (4.4). Then, we can transform our problem by an iterative optimal stopping method. □

Next, we present three examples that satisfy Assumption 4.1 : jump processes, regime switching Feller processes and semi-Markov processes. We recall that the iterative optimal stopping method was also used in [13] for regime switching and [14] for pricing of the American option for jump processes. Results obtained from our method are consistent with theirs.

4.2. **Examples.**

4.2.1. *Compound Poisson operator.* Here, we only consider the simple case introduced in [14]. The authors in [14] study an optimal stopping problem for American options pricing. Its value function is defined by

$$V^{(c)}(x) := \sup_{\tau} \boldsymbol{E}^x \big[e^{-a\tau}(K - e^{X(\tau)})^+ \big],$$

where X is a jump diffusion, i.e.,

$$X(t) = (\mu - \frac{1}{\sigma^2})t + \sigma W(t) + \sum_{n=1}^{N(t)} S_n,$$

with $W(t)$ a standard Brownian motion, $N(t)$ is a Poisson process with intensity $\lambda_0 > 0$ and $\{S_n\}_{n\in\mathbb{N}}$ a sequence of independent and identical random variables. Here, X is a Lévy process with infinitesimal generator

$$\begin{aligned} D(\mathcal{A}) &:= \mathcal{C}_*^2(\mathbb{R}) \\ \mathcal{A}u(x) &:= (\mu - \frac{1}{\sigma^2})D_x u(x) + \frac{1}{2}\sigma^2 D_{xx}u(x) + \int_{\mathbb{R}} (u(x+y) - u(x))\lambda F(\mathrm{d}y), \end{aligned}$$

where F is the distribution of S_n.

In this way, we can decompose the infinitesimal generator $(\mathcal{A}, D(\mathcal{A}))$ by

$$\begin{aligned} \mathcal{A}_0 u(x) &:= (\mu - \frac{1}{\sigma^2})D_x u(x) + \frac{1}{2}\sigma^2 D_{xx}u(x) \text{ for } u \in D(\mathcal{A}_0) := \mathcal{C}_*^2(\mathbb{R}), \\ \mathsf{F}_{bp}u(x) &:= \int_{\mathbb{R}} (u(x+y) - u(x))\alpha(x, \mathrm{d}y), \end{aligned}$$

where $\alpha(x, \mathrm{d}y) := \lambda F(\mathrm{d}y)$ such that $(\mathcal{A}, D(\mathcal{A})) = (\mathcal{A}_0 + \mathsf{F}_{bp}, D(\mathcal{A}_0)$. Then, using Proposition 4.2, we obtain the value function $V^{(c)}$ as follows.

Corollary 4.1. *Let $v_0(x) := (K - e^x)^+$. Define*

$$v_n(x) := \sup_{\tau} \boldsymbol{E}^x \Big[\int_0^{\tau} e^{-s} \frac{\lambda_0}{a + \lambda_0} \big(\int_{\mathbb{R}} v_{n-1}(Y(s) + y) F(\mathrm{d}y) \big) \mathrm{d}s + e^{-\tau}(K - e^{Y(\tau)})^+ \Big],$$

where Y is a diffusion defined by $Y(t) = \frac{(\mu - \frac{1}{\sigma^2})}{a + \lambda_0} t + \frac{\sigma}{a + \lambda_0} W(t)$. Then, the sequence of functions $\{v_n\}_{n\in\mathbb{N}}$ converges to the value function $V^{(c)}$ uniformly.

Remark 4.1. Notice that results computed by the proposed iterative optimal stopping method in Corollary 4.1 coincides with those in [14, Section 3]. Similar problem was solved in [29] when X is a Lévy process. It is worth mentioning that our setting does not allow us to tackle infinite activity Lévy processes since the partial integro differential operator in this case is not a bounded.

4.2.2. *Regime Switching Process.* This example is an extension from [13], where regime switching diffusion processes were studied. We generalize the underlying processes to regime switching Feller processes by adding a perturbation operator. Here, $\mathcal{S} := \{1, 2, \ldots, N\}$ is a finite discrete space, where N is a positive integer. Let $(\mathcal{A}_i, D(\mathcal{A}_i))$ be the infinitesimal generators of some Feller semigroups on $\mathcal{C}_0(\mathsf{E})$. Then, define the operator $(\mathcal{A}_0^{(r)}, D(\mathcal{A}_0^{(r)}))$ as follows:

$$\begin{aligned} D(\mathcal{A}_0^{(r)}) &:= \{u \in \mathcal{C}_0(S \times \mathsf{E}); u(i, \cdot) \in D(\mathcal{G}_i)\}, \\ \mathcal{A}_0^{(r)} u(i, x) &:= \mathcal{A}_i u_i(x) \text{ for } i \in S \text{ and } x \in \mathsf{E}, \end{aligned}$$

where $u_i(x) := u(i,x)$. By Hille-Yosida theorem, the above generator is the infinitesimal generator of some Feller semigroup. Additionally, let us introduce a bounded operator

$$\mathcal{A}_{pb}^{(r)} u(i,x) := \sum_{j\in N} q_{ij}(x)(u(j,x) - u(i,x)),$$

where $q_{ij} \in \mathcal{C}_b(\mathsf{E})$ and $q_{ij} \geq 0$. Since $\mathcal{A}_{pb}^{(r)}$ satisfies the positive maximum principle and $\mathcal{A}_{pb}^{(r)} : \mathcal{C}_0(\mathsf{E}) \to \mathcal{C}_0(\mathsf{E})$, the operator $((\mathcal{A}_0^{(r)} + \mathcal{A}_{pb}^{(r)}, D(\mathcal{A}^{(r)})))$ is the infinitesimal generator of some Feller semigroup.

Then, there exists a corresponding Feller process $(I(s), X(s))$ with state space $\mathcal{S}\times\mathsf{E}$ whose infinitesimal generator is $(\mathcal{A}_0^{(r)} + \mathcal{A}_{pb}^{(r)}, D(\mathcal{A}^{(r)}))$. Therefore, our interest lies in the optimal stopping problem of the Feller process $(I(s), X(s))$:

$$V^{(r)}(i,x) := \sup_\tau \boldsymbol{E}^{i,x}\Big[\int_0^\tau e^{-as} f(I(s),X(s))\mathrm{d}s + e^{-a\tau} g(I(\tau),X(\tau))\Big]. \tag{4.5}$$

We can once more characterise the above value function using the iterative optimal stopping method below.

Corollary 4.2. *Let $v_0(i,x) := g(i,x)$. Define*

$$v_n(i,x) = \sup_\tau \boldsymbol{E}^{i,x}\Big[\int_0^\tau e^{-s}\big(\frac{f(i,Y^{(i)}(s))}{a+\sum_{j\in N} q_{ij}(x)} - \sum_{j\in N}\frac{q_{ij}(x)v_{n-1}(j,x)}{a+\sum_{j\in N} q_{ij}(x)}\big)\mathrm{d}s + e^{-\tau} g(i,Y^{(i)}(\tau))\Big]$$

for $n \geq 1$, where $Y^{(i)}$ is a process with the generator $(\frac{1}{a+\sum_{j\in N} q_{ij}(x)}\mathcal{A}_i, D(\mathcal{A}_i))$. Then, the value function v_n converges to the value function $V^{(r)}$ defined by (4.5) *uniformly.*

Proof. The proof follows from Proposition 4.2. □

Remark 4.2. The above result generalised the one in [13] in which the state process is given by regime switching diffusions only.

4.2.3. *Semi-Markov process.* Finally, we study an application to optimal stopping problems for semi-Markov processes. To the best of our knowledge, this problem has not been solved using viscosity methods in literature. Let us illustrate this by the following example. Consider a risk process

$$X(t) := X(0) + t - \sum_{n=1}^{N^{(s)}(t)} S_n,$$

where $N^{(s)}(t)$ is a renewal process with inter-arrival time $\{T_n\}_{n\in\mathbb{N}}$ having the distribution law F_T, and $\{S_n\}_{n\in\mathbb{N}}$ is a sequence of i.i.d random variables with a distribution function F. Let $\xi(t)$ be the time from the last jump and $Y := \{\xi(t), X(t)\}_{t\geq 0}$ be a Markov process. Then, its infinitesimal generator $(\mathcal{A}, D(\mathcal{A}))$ is

$$D(\mathcal{A}) := \{u \in \mathcal{C}_0([0,\infty]\times\mathbb{R}); u \text{ has first order derivative and } \frac{\partial}{\partial\xi}u(\infty,x) = 0\},$$

$$\mathcal{A}u(\xi,x) := \frac{\partial}{\partial\xi}u(\xi,x) + \frac{\partial}{\partial x}u(\xi,x) + s(\xi)\int_{\mathbb{R}}(u(0,x+\zeta) - u(\xi,x))\mathrm{d}F(\zeta),$$

where the function s is the hazard function of the distribution F_T.

Then, we decompose the generator $\mathcal{A}$ as

$$\mathcal{A}_0 u(\xi, x) := \frac{\partial}{\partial \xi} u(\xi, x) + \frac{\partial}{\partial x} u(\xi, x),$$
$$\mathcal{A}_{pb} u(\xi, x) := s(\xi) \int_{\mathbb{R}} (u(0, x+\zeta) - u(\xi, x)) \mathrm{d}F(\zeta).$$

Here, we show numerical approximation results deduced from the iterative optimal stopping method. We consider the following optimal stopping problem

$$V(x) := \sup_{\tau} \boldsymbol{E}^x \big[e^{-a\tau} g(X(\tau)) \big].$$

Proposition 4.3. *Assume $g \in \mathcal{C}_b(\mathbb{R})$ and $s \in \mathcal{C}_b(\mathbb{R})$.*

(i) The value function $V(x) = w(0, x)$ for $x \in \mathbb{R}$, where w is the unique viscosity solution $w \in \mathcal{C}_b([0, \infty] \times \mathbb{R})$ to

$$\min\{aw - \mathcal{A}_0 w - \mathcal{A}_{pb} w, w - \bar{g}\} = 0. \tag{4.6}$$

where $\bar{g}(\xi, z) := g(x)$ for $y \in [0, \infty]$ and $z \in \mathbb{R}$.

(ii) Let $v_0 = \bar{g}$. Define v_n as the viscosity solution in $w \in \mathcal{C}_b([0, \infty] \times \mathbb{R})$ to

$$\min\{aw - \mathcal{A}_0 w - \mathcal{A}_{pb} v_{n-1}, w - \bar{g}\} = 0,$$

or equivalently,

$$v_n(y, z) := \sup_{\tau} \boldsymbol{E}^{(y,z)} \Big[- \int_0^{\tau} e^{-\int_0^s (a + s(Y(l)))\mathrm{d}l} s(Y(s)) \int_{\mathbb{R}} v_{n-1}(0, Z(s) + \zeta) \mathrm{d}F(\zeta) \mathrm{d}s$$
$$+ e^{-\int_0^{\tau} (a + s(Y(l)))\mathrm{d}l} \bar{g}(Y(\tau), Z(\tau)) \Big],$$

where $\{Y(t), Z(t)\}_{t \geq 0}$ is a Feller process with generator $(\mathcal{A}_0, D(\mathcal{A}_0))$. Then, $v_n(0, \cdot)$ converges to the value function V uniformly.

Proof. The proof follows from Proposition 4.2. □

Remark 4.3. Based on Proposition 4.3 *(ii)*, the optimal stopping problem for semi-Markov process can also be solved by constructing an iterative optimal stopping problem for two-dimensional deterministic processes.

Specifically, let T_n be a mixture exponential distribution and S_n be an exponential distribution, i.e.,

$$F_T(x) := 1 - \beta e^{-\lambda_1 x} - (1 - \beta) e^{-\lambda_2 x},$$
$$F_S(x) := 1 - e^{-\gamma x},$$

where $\beta \in [0, 1]$ is the weight, $\lambda_1, \lambda_2, \gamma$ are three positive parameters. Then, the force rate of the inter-arrival time is

$$s_{\beta}(y) = \frac{\beta \lambda_1 e^{-\lambda_1 y} + (1 - \beta) \lambda_2 e^{-\lambda_2 y}}{\beta e^{-\lambda_1 y} + (1 - \beta) e^{-\lambda_2 y}}.$$

Consider the subsequent optimal stopping problem

$$V(x) := \sup_{\tau} \boldsymbol{E}^x \big[e^{-a\tau} (X(\tau)) \vee 0) \wedge L \big].$$

The value function V can be described as the viscosity solution to the following equation

$$\min\{aw(\xi,x) - \mathcal{A}_0 w(\xi,x) - s_\beta(\xi)\int_{\mathbb{R}^+}(u(0,x-\zeta)-u(\xi,x))\lambda e^{-\lambda\zeta}\mathrm{d}\zeta,$$
$$w(\xi,x) - (x \vee 0)\wedge L\} = 0.$$

We derive a numerical solution for such problem. Here, $\bar{g}$ in (4.6) is given by $\bar{g}(y,z) := (y \vee 0) \wedge c$. We solve the value function numerically using the iterative optimal stopping method. As a consequence, we sketch both the value function and exercise boundaries under different scenarios based on various choices of β. Assume that $\lambda_1 = 1$, $\lambda_2 = 3$, $\gamma = 1$, discount rate $a = 0.25$ and $L = 2$ The rate β can take values between 0 and 1.

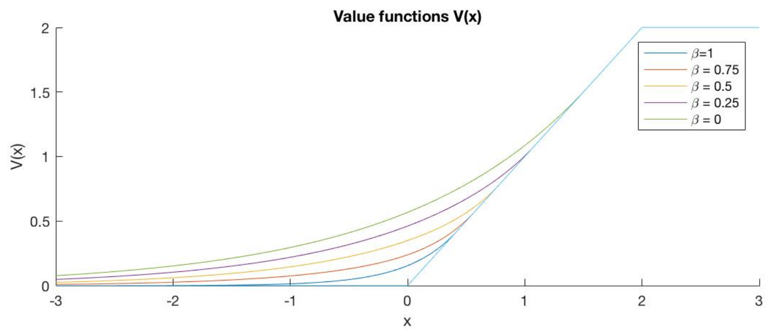

FIGURE 5. Since the hazard rate of F_T is an increasing function of β, then the frequency of the negative jumps increases. Besides, since the payoff g function is an increasing function, intuitively speaking, the value function $V^{(\beta)}$ increases with β as shown in the figure.

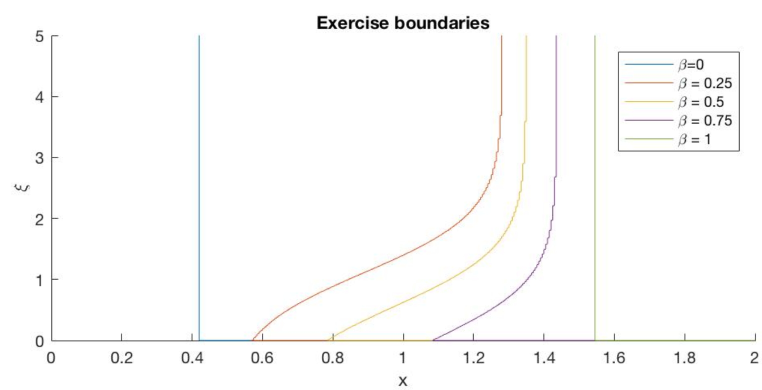

FIGURE 6. Each line represents the boundary of an exercise. We should stop when $\{\xi(t), X(t)\}_{t\geq 0}$ hit the left side of the line. We can see that for each $\beta \in (0,1)$, when the time from the last jump ξ continues to grow, we will stop at rising levels of the state x based on process X. However, when $\beta = 0$ or $\beta = 1$, since the process X is Markov, the optimal stopping strategy does not depend on the time ξ.

5. Application 3: Non-negative Random discount

In the previous sections, the discount rate a is a positive constant. The aim of this section is to relax the assumption on the discount rate, assuming that it is a random variable.

5.1. Main results.

We start by studying the properties of the value function

$$V(x) := \sup_{\tau} \boldsymbol{E}^x\Big[\int_0^{\tau} e^{-\int_0^s r(X(s))\mathrm{d}s} f(X(s))\mathrm{d}s + e^{-\int_0^{\tau} r(X(s))\mathrm{d}s} g(X(\tau))\Big], \tag{5.1}$$

where $r \in \mathcal{C}_b(\mathsf{E})$ is a random non-negative discount rate and $f, g \in \mathcal{C}_b(\mathsf{E})$. It is worth mentioning that the discount rate r could be zero. For example, the work [15] considers an optimal stopping problem for non-uniformly ergodic Feller-Markov processes. The authors proved the continuity of the value function V and its characterisation in the viscosity sense, that is, they showed that V is a viscosity solution to

$$\min\{rw - \mathcal{A}w - f, w - g\} = 0. \tag{5.2}$$

Note, however, that they did not prove the uniqueness of the viscosity solution. Here, we provide the proof of the uniqueness of the viscosity solution as a consequence of Theorem 2.4. The ergodic property (see [15]) of the Feller process is not necessary in our proof for the uniqueness. Instead, we make the following assumption.

Assumption 5.1. *There exist $\kappa > 0$ and $w_+ \in \mathcal{C}_b(\mathsf{E})$ such that w_+ is a viscosity supersolution to*

$$rw - \mathcal{A}w - f - \kappa = 0. \tag{5.3}$$

This is a reasonable assumption for common problems encountered in literature. For instance, suppose that r is a continuous bounded function and $\inf_{x\in\mathsf{E}} r(x) = a > 0$. For this case, we can choose $w_+ = \frac{\|f\|_\infty + 1}{a}$ and $\kappa = 1$ so that

$$rw_+ - \mathcal{A}w_+ - f - \kappa \geq \|f\|_\infty + 1 - f - 1 \geq 0.$$

Then, Assumption 5.1 is satisfied. In particular, if r is a constant function, it reduces to the results discussed in the above sections.

There have been extensive works under the previous setting. Hence, we would like to devote more attention to the case when $\inf_{x\in\mathsf{E}} r(x) = 0$. The next result gives existence and uniqueness of the viscosity solution to (5.2).

Proposition 5.1. *Suppose that Assumption 5.1 holds. Let $r \in \mathcal{C}_b(\mathsf{E})$ and $r \geq 0$, $f, g \in \mathcal{C}_b(\mathsf{E})$. There exists a unique viscosity solution to*

$$\min\{rw - \mathcal{A}w - f, w - g\} = 0. \tag{5.4}$$

Proof. Let us first observe that the viscosity solution to (5.4) is equivalent to the viscosity solution to

$$\min\{(1+r)w - \mathcal{A}w - (w+f), w - g\} = 0. \tag{5.5}$$

Since $r \in \mathcal{C}_b(\mathsf{E})$ and $r \geq 0$, it follows that the viscosity solution to (5.5) is equivalent to the viscosity solution associated with $(\frac{1}{1+r}\mathcal{A}, D(\mathcal{A}))$ to

$$\min\Big\{w - \frac{1}{1+r}\mathcal{A}w - \mathsf{F}w, w - \mathsf{G}w\Big\} = 0, \tag{5.6}$$

where $\mathsf{F}w := \frac{w+f}{1+r}$ and $\mathsf{G}u := g$. Then, we only need to verify all the conditions of Assumption 2.1 and Assumption 2.2. The properties of G are obvious and we only prove the properties of F as follows.

(i) We know that $r \in \mathcal{C}_b(\mathsf{E})$ and $r \geq 0$ thus $\frac{1}{1+r} \in \mathcal{C}_b(\mathsf{E})$. Since $f \in \mathcal{C}_b(\mathsf{E})$, then Assumption 2.1 *(i)* holds. Let $u_1, u_2 \in \mathcal{C}_b(\mathsf{E})$ and $0 \leq p \leq 1$, we have

$$\begin{aligned} p\mathsf{F}u_1 + (1-p)\mathsf{F}u_2 =& p\frac{u_1+f}{1+r} + (1-p)\frac{u_2+f}{1+r} \\ =& \frac{pu_1 + (1-p)u_2 + f}{1+r} \\ =& \mathsf{F}(pu_1 + (1-p)u_2). \end{aligned}$$

Thus, the operator F is convex. Additionally, if $u_1 \geq u_2$, $\mathsf{F}u_1 = \frac{u_1+f}{1+r} \geq \frac{u_2+f}{1+r} = \mathsf{F}u_2$. Therefore, Assumption 2.1 holds.

(ii) Using Assumption 5.1, let w_+ be the viscosity supersolution to $rw - \mathcal{A} - f = 0$. Define $w_+^* := w_+ + \|w_+\|_\infty + \|g\|_\infty$. Then, w_+^* is a viscosity supersolution to

$$\min\{rw_+^* - \mathcal{A} - f, w - g\} = 0,$$

which is equivalent to

$$\min\left\{w - \frac{1}{1+r}\mathcal{A}w - \mathsf{F}w, w - g\right\} = 0.$$

Therefore, by Corollary 2.2, Assumption 2.2 *(i)* holds. Let $C > 0$, $p_1 = 1$ and $u \in \mathcal{C}_b(\mathsf{E})$. Since $\frac{1}{1+r} \leq 1$, we get $\mathsf{F}(u+C) - \mathsf{F}u = \frac{C}{1+r} \leq p_1 C$. Hence, Assumption 2.2 holds.

Here Assumption 2.2 *(i)* follows from (5.3). Thus, there exists a unique viscosity solution to (5.6), which is equivalent to the viscosity solution to (5.4). By [15, Theorem 1.1], the value function function is the viscosity solution to (5.4). Whence, by Theorem 2.4, the proof is completed. □

To the best of our knowledge there are no general results on the uniqueness of the viscosity solution when the value function is given by (5.1) and X is a Feller process. However, under appropriate assumptions, it is possible to show using Proposition 5.1 that there always exists a unique viscosity solution to (5.3).

Let us look into more details with several examples satisfying Assumption 5.1. As emphasized above, we have to assume that the value function is the underlying viscosity solution, yet such assumption is justifiable in most cases.

5.2. **Examples.**

5.2.1. *Non-uniformly ergodic Markov process.* Similarly as in [15, Section 2.2], the authors introduced a zero potential function

$$q(x) = \lim_{T\to\infty} \boldsymbol{E}^x\Big[\int_0^T (f(X(s)) - \mu(f))\mathrm{d}s\Big],$$

where μ is an invariate measure of the process X and $\mu(f)$ is a negative constant depending on f . By [15, Lemma 2.2], the process $Z(t) = \int_0^t (f(X(s)) - \mu(f)) + q(X(t))$ is a martingale.

Additionally, in this example, we assume that q is a bounded function and $\mu(f) < 0$. Let κ such that $0 < \kappa < -\mu(f)$, then q is a viscosity supersolution to

$$-\mathcal{A}w - f + \kappa = 0.$$

The zero potential function q is not necessarily bounded from above if E is not compact. Thus, the value function in [15] is only continuous but not bounded.

Corollary 5.1. *Assume that the conditions of [15, Theorem 1.1] are in force and q is bounded and $\mu(f) < 0$. Then, the value function V defined by*

$$V(x) := \sup_{\tau} \boldsymbol{E}^x \Big[\int_0^{\tau} f(X(s)) \mathrm{d}s + g(X(\tau)) \Big],$$

is a continuous and bounded function. Additionally, the value function is the unique viscosity solution to

$$\min\{-\mathcal{A}w - f, w - g\} = 0.$$

Proof. As mentioned before, there exists $\kappa < -\mu(f)$ such that the zeros potential function $q(x)$ is the viscosity supersolution to

$$-\mathcal{A}w - f + \kappa = 0.$$

If Assumption 5.1 is satisfied and the claim follows from Proposition 5.1. □

On the other hand, we should mention that we do not need the ergodicity of $(\mathcal{G}, D(\mathcal{G}))$ to show there exists a unique viscosity solution to (5.4). For example, if there exists $C_0 < 0$ such that $f \le C_0$, (5.3) in Corollary 5.1 holds for any Feller process.

5.2.2. *Optimal stopping with random costs of observation.* In this example, we are interested in the following problem

$$V(x) := \sup_{\tau} \boldsymbol{E}^x \Big[\int_0^{\tau} f(X(s)) \mathrm{d}s + g(X(\tau)) \Big], \tag{5.7}$$

where X is a Feller process which does not necessarily satisfy the ergodic property. We have the next result.

Corollary 5.2. *Suppose that there exists a constant $c > 0$ such that $f \le -c$. Then, the value function V defined by* (5.7) *is the unique viscosity solution to*

$$\min\{-\mathcal{A}w - f, w - g\} = 0.$$

Proof. Similar to the proof of Corollary 5.1. Choose $w_+ := 0$ and $\kappa = \frac{c}{2}$ and then Assumption 5.1 holds and the result follows. □

In particular, let $f = -c$ be a constant function and $g \in \mathcal{C}_b(\mathsf{E})$. The value function of the optimal stopping time problem defined by

$$V(x) := \sup_{\tau} \boldsymbol{E}^x \big[-c\tau + g(X(\tau)) \big]$$

can be characterized by the viscosity solution to

$$\min\{c - \mathcal{A}w, w - g\} = 0.$$

5.2.3. *Finite time horizon optimal stopping problems.* A finite time horizon optimal stopping problem is also a popular topic in previous literature. However, compared to infinite time horizon problems, such problems often do not include the discount cost, i.e., $a = 0$. Consequently, in this part, we will study the finite time horizon optimal stopping problems using Proposition 4.1 and obtain some direct results. Consider a process $(D, X) = \{D(t), X(t)\}_{t\geq 0}$ on $\mathsf{E} := \mathbb{R}^+ \times \mathbb{R}^n$ with infinitesimal generator

$$D(\mathcal{A}^{(time)}) := \{u \in \mathcal{C}_*(\mathsf{E}); \frac{\partial}{\partial t}u(t,x) \in \mathcal{C}_0(\mathsf{E}), u_t \in D(\mathcal{A}) \text{ for } t \in \mathbb{R}^+\},$$
$$\mathcal{A}^{(time)}u(t,x) := \frac{\partial}{\partial t}u(t,x) + b(t)\mathcal{A}u_t(x),$$

where $u_t(x) := u(t,x)$. By [30], (D, Y) is a Feller process if $(\mathcal{A}, D(\mathcal{A})$ is the generator of the Feller semigroup. Additionally, let $T > 0$, we are interested in the following finite time horizon optimal stopping problem

$$V(d,x) := \boldsymbol{E}^{(d,x)}\Big[\int_0^{\tau\wedge T} f(D(s), X(s))\mathrm{d}s + g(D(\tau\wedge T), X(\tau\wedge T))\Big]. \tag{5.8}$$

Remark 5.1. In general, the above optimal stopping problems are commonly studied for the time inhomogeneous diffusions, whose operator $\mathcal{A}^{(time)}$ are of a parabolic type (see for example [6]). Here, we extend past results using Proposition 5.1 and we do not restrict our operator to be of parabolic type. However, we have to assume that $f(T,\cdot) = g(T,\cdot) = 0$.

First, define the operator $(\mathcal{A}^{(time)}_{[0,T]}, D(\mathcal{A}^{(time)}_{[0,T]}))$ by

$$D(\mathcal{A}^{(time)}_{[0,T]}) := \{u \in \mathcal{C}_b([0,T]\times\mathbb{R}^n); \text{there exists a continue extension } u_* \in D(\mathcal{A}^{time})\},$$
$$\mathcal{A}^{(time)}_{[0,T]}u(t,x) := \frac{\partial}{\partial t}u(t,x) + b(t)\mathcal{A}u_t(x).$$

Then, variational characterization of the value function is shown in the following corollary.

Corollary 5.3. *Assume that $f, g \in \mathcal{C}_b([0,T]\times\mathbb{R}^n)$, $f(T,x) = 0$ and $g(T,x) = 0$ for all $x \in \mathbb{R}^n$. Then, the value function V defined by (5.8) is in $\mathcal{C}_b([0,T]\times\mathbb{R}^n)$. Moreover, the value function V is the unique viscosity solution $w \in \mathcal{C}_b([0,T)\times\mathbb{R}^n)$ to*

$$\min\{-\mathcal{A}^{(time)}_{[0,T]}w - f, w - g\} = 0,$$

with the boundary condition $w(T,\cdot) = 0$.

Proof. Define the continuous extensions of the functions f and g by

$$\tilde{f}(t,x) := \begin{cases} f(t,x) & \text{for } x \in [0,T)\times\mathbb{R}^n \\ T-t & \text{for } x \in [T,T+1)\times\mathbb{R}^n \\ -1 & \text{for } x \in [T+1,\infty)\times\mathbb{R}^n \end{cases}$$
$$\tilde{g}(t,x) := \begin{cases} g(t,x) & \text{for } x \in [0,T)\times\mathbb{R}^n \\ 0 & \text{for } x \in [T,\infty)\times\mathbb{R}^n. \end{cases}$$

Due to the fact that $f(T,\cdot) = g(T,\cdot) = 0$, $\tilde{f}$ and $\tilde{g}$ are continuous functions. Define

$$\tilde{V}(x) := \sup_{\tau}\boldsymbol{E}^x\Big[\int_0^{\tau}\tilde{f}(s,X(s))\mathrm{d}s + \tilde{g}(\tau,X(\tau))\Big].$$

Since $\tau \wedge T$ is also a F_t stopping time, we have $V(t,x) \leq \tilde{V}(x)$. On the other hand, for any $\varepsilon > 0$, there exists a stopping time $\tilde{\tau}$ satisfying

$$\begin{aligned}\tilde{V}(x) - \varepsilon &\leq \boldsymbol{E}^x\Big[\int_0^{\tilde{\tau}} \tilde{f}(s, X(s))\mathrm{d}s + \tilde{g}(\tau, X(\tilde{\tau}))\Big] \\ &= \boldsymbol{E}^x\Big[\int_0^{\tilde{\tau}\wedge T} \tilde{f}(s, X(s))\mathrm{d}s + \mathbf{1}_{\tilde{\tau}>T}\int_{\tilde{\tau}\wedge T}^{\tilde{\tau}} \tilde{f}(s, X(s))\mathrm{d}s + \tilde{g}(\tilde{\tau}, X(\tilde{\tau}))\Big] \\ &\leq \boldsymbol{E}^x\Big[\int_0^{\tilde{\tau}\wedge T} \tilde{f}(s, X(s))\mathrm{d}s + \tilde{g}(\tilde{\tau}\wedge T, X(\tilde{\tau}\wedge T))\Big],\end{aligned}$$

where the last inequality comes from $f(t,x) \leq 0$ and $g(t,x) = 0$ for $t \geq T$. Additionally, as $\varepsilon \to 0$, $\tilde{V}(x) \geq V(t,x)$. Therefore, the value function $\tilde{V}$ is equal to V. Since $\tilde{f}, \tilde{g} \in \mathcal{C}_b(\mathbb{R}^+ \times \mathbb{R}^n)$, the value function $\tilde{V}$ in $\mathcal{C}_b(\mathsf{E})$ is a viscosity solution to

$$\min\{-\mathcal{A}w - \tilde{f}, w - \tilde{g}\} = 0. \tag{5.9}$$

Now, let us prove that the viscosity solution to (5.9) is unique. Define

$$u(t) := \begin{cases} -(\|f\|_\infty + 1) & \text{for } t \in [0, T+1) \\ -(\|f\|_\infty + 1) + (\|f\|_\infty + 1)(T + 2 - t) & \text{for } t \in [T+1, T+2) \\ 0. \end{cases}$$

Define $w_+(t,x) := \int_0^t u(s)\mathrm{d}s$ such that $\mathcal{A}^{(time)}w_+(t,x) = \frac{\partial w_+(t,x)}{\partial t} = u(t)$. As a result, we have

$$-\mathcal{A}^{(time)}w(t,x) - f(t,x) = -u(t) + f(t,x) \geq 1.$$

Hence, Assumption 5.1 holds. Then, by Proposition 5.1, there exists a unique viscosity solution to (5.9). Furthermore, since $\tilde{V}(t,x) = 0$ for $t \geq T$, the value function v can be characterized by the viscosity solution to

$$\min\{-\mathcal{A}_0^{(time)}w - f, w - g\} = 0$$

with boundary condition $w(T,x) = 0$. □

Conclusion

In this paper, we have suggested an iterative optimal stopping method for general optimal stopping problems for Feller processes. More precisely, we use an approximating scheme to show that the value function is the unique viscosity solution to an Hamilton-Jacobi-Bellman equation. Unlinke in the traditional litterature, in which the generator is given by a partial differential operator, we assume in this work that the generator is given by a generator of some semigroup. One of the advantages of our method is that it provides a unified framework, and enables to solve the HJB equations in more abtract cases. We can then apply our technique to solve optimal stopping and impulse control problems with the state process not only only given by a diffusion process, but also by compound poisson processes, semi-Markov processes, etc. We can also tackle the case of infinite horizon optimal stopping problem with zero discount.

Note however that if our method allows us to tackle problems with more general bequest functions and terminal rewards, we can still at the moment not handle the case in which these functions are unbounded. Nevertheless, we believe that this can be overcomed by extending our operators in some weighted spaces (see for example [16]).

Thus our methodology gives other perspectives of research, namely the study of optimal stopping and impulse control problems with integrable bequest functions and terminal rewards.

Another interesting study would be to addressed the problem of optimal stopping for negative discount rate as studies in [31]. This often arises in the stock loan problem. In fact, when the loan interest rate is higher than the risk-free rate, the problem reduces to the valuation of an American call option with a negative discount rate (see for example [32]).

References

[1] S. Dai, *Viscosity of optimal stopping problems for Feller processes and their applications*, PhD thesis, University of Liverpool, 2018. `https://livrepository.liverpool.ac.uk/3022740/`

[2] A. Bensoussan, *Impulse control and quasi-variational inequalities*, Gauthier-Villars, 1984.

[3] B. Øksendal, A. Sulem, *Optimal consumption and portfolio with both fixed and proportional transaction costs*, J. Control Optim., 40 (6) (2002) 1765-1790.

[4] B. Øksendal, A. Sulem, *Applied Stochastic Control of Jump Diffusions*, second ed., Springer, Berlin Heidelberg, 2007.

[5] I. Hdhiri, M. Karouf, *Risk sensitive impulse control of non-markovian processes*, Math. Methods Oper. Res., 74 (1) (2011) 1-20.

[6] R. C. Seydel, *Existence and uniqueness of viscosity solutions for QVI associated with impulse control of jump-diffusions*, Stoch. Process. Their Appl., 119 (10) (2009) 3719-3748.

[7] M. Robin, *Contrôle impulsionnel des processus de Markov*, PhD thesis, Université Paris Dauphine-Paris X, 1978 (in French). `https://tel.archives-ouvertes.fr/tel-00735779`

[8] M. G. Crandall, H. Ishii, P.-L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Am. Math. Soc., 27 (1) (1992) 1-67.

[9] J. Zabczyk, *Mathematical control theory: an introduction*, Springer Science & Business Media, 2009.

[10] H. Guan, Z. Liang, *Viscosity solution and impulse control of the diffusion model with reinsurance and fixed transaction costs*, Insur. Math. Econ., 54 (2014) 109-122.

[11] M. Ohnishi, M. Tsujimura, *An impulse control of a geometric brownian motion with quadratic costs*, Eur. J. Oper. Res., 168 (2) (2006) 311-321.

[12] H. Le, H. Wang, *A finite time horizon optimal stopping problem with regime switching*, J. Control Optim., 48 (8) (2010) 5193-5213.

[13] J. Babbin, P. A. Forsyth, G. Labahn, *A comparison of iterated optimal stopping and local policy iteration for american options under regime switching*, J. Sci. Comput., 58 (2) (2014) 409-430.

[14] E. Bayraktar, H. Xing, *Pricing American options for jump diffusions by iterating optimal stopping problems for diffusions*, Math. Methods Oper. Res., 70 (3) (2009) 505-525.

[15] J. Palczewski , L. Stettner, *Infinite horizon stopping problems with (nearly) total reward criteria*, Stoch. Process. Their Appl., 124 (12) (2014) 3887-3920.

[16] K.-J. Engel, R. Nagel, *One-parameter semigroups for linear evolution equations*, volume 194. Springer Science & Business Media, 1999.

[17] C. Belak, S. Christensen, F. T. Seifried, *A general verification result for stochastic impulse control problems* J. Control Optim., 55 (2) (2017) 627-649.

[18] S. Christensen, *On the solution of general impulse control problems using superharmonic functions*, Stoch. Process. Their Appl., 124 (1) (2014) 709-729.
[19] L. H. R Alvarez, J. Lempa, *On the optimal stochastic impulse control of linear diffusions*, J. Control Optim., 47 (2) (2008) 703-732.
[20] M. Egami, *A direct solution method for stochastic impulse control problems of one- dimensional diffusions*, J. Control Optim., 47 (3) (2008) 1191-1218.
[21] A. Irle, V. Paulsen, *Solving problems of optimal stopping with linear costs of observations* Seq. Anal., 23 (3) (2004) 297-316.
[22] X. Guo, G. Wu, *Smooth fit principle for impulse control of multidimensional diffusion processes*, J. Control Optim., 48 (2) (2009) 594-617.
[23] M. H. A. Davis, X. Guo, G. Wu, *Impulse control of multidimensional jump diffusions*, J. Control Optim., 48 (8) (2010) 5276–5293.
[24] R. Korn, Y. Melnyk, F. T. Seifried, *Stochastic impulse control with regime-switching dynamics*, Eur. J. Oper. Res., 260 (3) (2017)1024-1042.
[25] D. Yao, H. Yang, R. Wang, *Optimal dividend and capital injection problem in the dual model with proportional and fixed transaction costs*, Eur. J. Oper. Res., 211 (3) (2011) 568–576.
[26] M. Beibel, H. R. Lerche, *Optimal stopping of regular diffusions under random dis- counting*, Theory Probab. Its Appl., 45 (4) (2001) 547-557.
[27] B. Böttcher, R. L. Schilling, J. Wang, *Lévy-Type Processes: Construction, Approximation and Sample Path Properties,* Lévy Matters III. Lecture Notes in Mathematics, 2099, 2013.
[28] K. Taira, *Semigroups, boundary value problems and Markov processes*, second ed., Springer, 2004.
[29] E. Mordecki, *Optimal stopping and perpetual options for Lévy processes*, Finance Stoch., 6 (4) (2002) 473–493.
[30] A. Mijatovic, M. Pistorius, *On additive time-changes of feller processes*, In Progress in Analysis and Its Applications: Proceedings of the 7th International ISAAC Congress (13-18 July 2009), London, UK, pages 431–437, 2010.
[31] Z. Palmowski, J. L. Pérez, K. Yamazaki, *Double continuation regions for American options under Poisson exercise opportunities*, Math. Financ., 31 (2) (2021) 722-771.
[32] J. Xia, X. Y. Zhou, Stock loans. Math. Financ., 17 (2) (2007) 307-317.