Non-stationary response determination of nonlinear systems subjected to combined deterministic and evolutionary stochastic excitations

Renjie Han[a], Vasileios C. Fragkoulis[b], Fan Kong[c,*], Michael Beer[b,d,e], Yongbo Peng[f]

[a] *College of Civil Engineering, Tongji University, 1239 Siping Road, Shanghai, 200092, China*
[b] *Institute for Risk and Reliability, Leibniz Universität Hannover, Callinstr. 34, Hannover, 30167, Germany*
[c] *College of Civil Engineering, Hefei University of Technology, 193 Tunxi Road, Hefei, 230009, China*
[d] *Institute of Risk and Uncertainty, University of Liverpool, Peach Street, Liverpool, L69 7ZF, UK*
[e] *International Joint Research Center for Resilient Infrastructure & International Joint Research Center for Engineering Reliability and Stochastic Mechanics, Tongji University, Shanghai, 200092, China*
[f] *State Key Laboratory of Disaster Reduction in Civil Engineering & Shanghai Institute of Disaster Prevention and Relief, Tongji University, Shanghai, 200092, China*

Abstract

A semi-analytical method is proposed for determining the response of a lightly damped single-degree-of-freedom nonlinear system subjected to combined deterministic and non-stationary stochastic excitations. This is attained by combining the stochastic averaging and statistical linearization methodologies. Specifically, first, the system response is decomposed into two components, namely the deterministic and the stochastic parts. This

*Corresponding author.
Email addresses: `HanRenjie@whut.edu.cn` (Renjie Han), `fragkoulis@irz.uni-hannover.de` (Vasileios C. Fragkoulis), `kongfan@hfut.edu.cn` (Fan Kong), `beer@irz.uni-hannover.de` (Michael Beer), `pengyongbo@tongji.edu.cn` (Yongbo Peng)

leads to a set of coupled differential sub-equations governing, respectively, the deterministic and the stochastic component of the response. Next, aiming at solving the set of differential sub-equations, an additional expression is derived by applying the statistical linearization methodology, followed by the application of a stochastic averaging step to the stochastic sub-equations. Therefore, an equivalent time-varying linear system is defined for the original nonlinear system. The stochastic averaging method is then applied to the linearized system for reducing its order, and thus, its complexity from a solution perspective. In this regard, an additional equation is derived, which connects the deterministic and stochastic components of the response. The latter and the deterministic differential sub-equations are solved simultaneously for determining the system response. A single-degree-of-freedom nonlinear system exhibiting quadratic and cubic nonlinear stiffness is considered for assessing the reliability of the proposed technique. The obtained results are compared with pertinent Monte-Carlo simulation estimates.

Keywords: Nonlinear system, Evolutionary stochastic process, Statistical linearization, Stochastic averaging, Combined excitation

1. Introduction

The development of methodologies and techniques for efficiently treating the uncertainty in engineering systems constitutes a critical aspect of the research pertaining to random vibration of structural and mechanical systems. In the field of stochastic dynamics of structural systems, in particular, this relates to developing solution frameworks to account for the nonlinear and hysteretic behavior of the system, as well as for the realistic modeling of the applied excitations (e.g., [1, 2]).

In this regard, a number of different tools exhibiting a varying degree of efficiency have been developed over the last years for determining the stochastic response, and assessing the risk and reliability of complex engineering systems. These include, for instance, the development of solution frameworks based on the concept of the Wiener path integral (e.g., [3, 4, 5]), or on the probability density evolution method (e.g., [6, 7, 8]), as well as methods based on utilizing compressive sampling tools for attaining a sparse representation of the stochastic system response (e.g., [9, 5]), and various harmonic wavelet-based tools for joint time-frequency stochastic response analysis (e.g., [10, 11, 12, 13]). Nevertheless, it can be argued that two of the most versatile tools for treating nonlinear systems are the statistical linearization method (e.g., [14, 15, 16]) and the stochastic averaging method (e.g., [17, 18, 19]). Through a plethora of engineering applications, these classical methodologies of random vibration theory have proven to be efficient, as well as straightforward in their application tools for determining the stochastic response of nonlinear systems. Clearly, both methods are readily applied to systems exhibiting a wide-range of nonlinearities and hysteretic behaviors, systems endowed with fractional derivative elements and subjected to both stationary and non-stationary stochastic excitations (e.g., [20, 18, 21, 19, 22]).

Further, in terms of engineering applications, it can be argued that there is an increasing interest in developing efficient solution schemes to determine the response of nonlinear systems subjected to combined deterministic and stochastic excitations. Such systems are naturally met, for instance, in energy harvesting applications (e.g., [23, 24]), as well as in the modeling of slender structures (e.g., [25, 26]) and aircraft rotation mechanisms (e.g., [27]). In this regard, various solution schemes have been proposed to determine the nonlinear system response, such as these applied to systems endowed with fractional derivative elements (e.g., [28, 29]) and systems exhibiting singular parameter matrices (e.g., [30]). In addition, extensive literature exists on the first-passage problem of systems subjected to combined deterministic and stochastic excitations (e.g., [31, 32, 33, 34]); a detailed discussion on the topic is found in [35]. However, most of the literature for nonlinear systems subjected to combined excitation pertains to the determination of the system stationary response, whereas to the best of the authors' knowledge very few results are available for assessing the non-stationary behavior of such systems (e.g., [36]).

In general, the rigorous mathematical modeling of environmental excitations necessitates the use of the theory of non-stationary stochastic processes. This is due to that the statistics and frequency content of most environmental loads are efficiently modeled as time-varying processes. Taking this into account and further considering the reliability assessment of systems subjected to combined deterministic and stochastic excitations, in this paper, a semi-analytical technique is proposed for obtaining the response of single-degree-of-freedom (SDOF) systems subjected to combined deterministic and non-stationary stochastic excitations described by evolutionary power spectrum forms. This is attained by decomposing the system response into two parts, namely the deterministic and stochastic components, and constructing a coupled set of differential deterministic/stochastic sub-equations to be solved for determining the two components. For the solution of the coupled set of equations, an additional expression is derived by resorting to a statistical linearization- and averaging-based framework. Specifically, first, the nonlinear stochastic system is replaced by an equivalent linear one, where the time-dependent damping and stiffness coefficients (time-dependent equivalent elements) of the latter are readily found by closed-form expressions obtained via a mean-square minimization criterion. The equivalent elements depend on the form of the considered nonlinearity, the deterministic component, as well as the time-varying variance of the stochastic component of the system response. Hence, considering the case of a lightly damped system, a stochastic averaging step is used for efficiently treating the linearized system by reducing its order. This leads to the formulation of the Fokker-Planck equation governing the evolution in time of the amplitude of the stochastic response component. Finally, utilizing a solution of the Rayleigh form for the Fokker-Planck equation which is proposed in [37], yields a deterministic first-order nonlinear differential equation to be solved simultaneously with the deterministic sub-equations for determining both time-varying deterministic response component and the stochastic response variance. The present semi-analytical technique can be construed as a substantial extension of the pioneering work by Spanos and co-workers [38], as well as the subsequent development by Kong and co-workers [36], to account for systems subjected to deterministic and non-stationary stochastic excitation with non-separable power spectral density. The efficiency and reliability of the proposed technique are demonstrated in the example section by considering an SDOF nonlinear system exhibiting quadratic and cubic nonlinear stiffness. Various levels of nonlinearity strength are considered in conjunction with different non-stationary stochastic excitation forms. The obtained results are compared to Monte-Carlo simulation (MCS) estimates.

2. Mathematical formulation

2.1. Single-degree-of-freedom systems subjected to combined deterministic and evolutionary stochastic excitations

The governing equation of motion of an SDOF nonlinear system subject to combined periodic and non-stationary excitation is given by

$$\ddot{x} + 2\xi\omega_{\mathrm{n}}\dot{x} + g(t, x, \dot{x}, \ddot{x}) = F(t) + Q(t), \tag{1}$$

where x, $\dot{x}$ and $\ddot{x}$ denote, respectively, the displacement, velocity and acceleration of the system. ξ, ω_{n} are the damping ratio and the natural frequency of the corresponding linear system, and $g(t, x, \dot{x}, \ddot{x})$ represents the nonlinear restoring force depending on x, $\dot{x}$, and $\ddot{x}$. Further, $F(t)$ is the deterministic excitation defined herein by the modulated harmonic function

$$F(t) = F_0 \exp(-\mu_0 |t - t_0|) \cos(\omega_0 t), \tag{2}$$

where F_0 and ω_0 are, respectively, the amplitude and frequency of the harmonic excitation, the parameter μ_0 is related to the decaying rate of $F(t)$, whereas t_0 represents the time instant when $F(t)$ attains its peak value; $Q(t)$ is a zero-mean non-stationary Gaussian stochastic excitation possessing a broad spectrum. In the ensuing analysis, $Q(t)$ is considered as either a modulated non-stationary white/colored noise with a separable power spectral density (PSD), or a non-stationary noise with non-separable PSD.

Next, considering that the system in Eq. (1) is subject to a combined excitation, the corresponding response is decomposed into a combination of a deterministic and a stochastic component ([38]). That is

$$x(t) = \mu_x(t) + \hat{x}(t), \tag{3}$$

where $\mu_x(t)$ is the deterministic component of the response and, without loss of generality, $\hat{x}(t)$ is the corresponding zero-mean stochastic component. In passing, note that the stochastic response is not a zero-mean process in general. Nevertheless, it can be considered as such by defining a new process as the difference between the non-zero-mean response and its mean; a detailed discussion is found in [14]. Further, it is also noted that for the case of a lightly damped system, $\hat{x}(t)$ is considered as a Gaussian process. Next, differentiating twice with respect to time on both sides of Eq. (3) yields

$$\dot{x} = \dot{\mu}_x + \dot{\hat{x}} \tag{4}$$

and

$$\ddot{x} = \ddot{\mu}_x + \ddot{\hat{x}}, \tag{5}$$

and substituting Eqs. (3-5) into the equation of motion Eq. (1) yields

$$(\ddot{\mu}_x + \ddot{\hat{x}}) + 2\xi\omega_{\mathrm{n}}(\dot{\mu}_x + \dot{\hat{x}}) + g(t, \mu_x + \hat{x}, \dot{\mu}_x + \dot{\hat{x}}, \ddot{\mu}_x + \ddot{\hat{x}}) = Q(t) + F(t). \tag{6}$$

Further, ensemble averaging on both sides of Eq. (6), the zero-mean response component of the linear terms is eliminated, and thus, the deterministic sub-equation is derived

$$\ddot{\mu}_x + 2\xi\omega_{\mathrm{n}}\dot{\mu}_x + \mathrm{E}\left[g(t, \mu_x + \hat{x}, \dot{\mu}_x + \dot{\hat{x}}, \ddot{\mu}_x + \ddot{\hat{x}})\right] = F(t), \tag{7}$$

where $\mathrm{E}\left[\cdot\right]$ denotes the mathematical expectation operator. Then, it is assumed for simplicity that the nonlinear function $g\left(\cdot\right)$ in Eq. (1) depends only on x and $\dot{x}$ (i.e., $g\left(x, \dot{x}\right)$), which readily encloses the nonlinear behavior modeling of diverse components in civil and/or mechanical engineering applications. Moreover, adopting the Gaussian assumption for the stochastic response $\hat{x}$ (e.g., [14]), the mean of the nonlinear term in Eq. (6) is written as a function of μ_x, $\dot{\mu}_x$, $c(t)$ and $\dot{c}(t)$, where the time-dependent coefficient $c(t)$ can be construed as the non-stationary response variance of the system (e.g., [37, 39, 40, 22]), to be determined in section 2.2; that is

$$\mathrm{E}\left[g(t, \mu_x + \hat{x}, \dot{\mu}_x + \dot{\hat{x}})\right] = \gamma\left(\mu_x, \dot{\mu}_x, c(t), \dot{c}(t)\right). \tag{8}$$

Substituting Eq. (8) into Eq. (7) leads to a coupled state-space equation

$$\begin{cases} \dot{q}_1 = q_2 \\ \dot{q}_2 = F(t) - 2\xi\omega_{\mathrm{n}}q_2 - \gamma\left(q_1, q_2, c(t), \dot{c}(t)\right) \end{cases}, \tag{9}$$

where $q_1 = \mu_x$ and $q_2 = \dot{\mu}_x$ for simplicity. Clearly, additional expressions connecting the deterministic and stochastic components of the response are required for solving the coupled state-space equation defined by Eq. (9). These supplementary equations are derived in section 2.2 by resorting to a combination of the statistical linearization and stochastic averaging methodologies.

2.2. Statistical linearization- and stochastic averaging-based framework for determining the response stochastic component

In this section, a treatment based on the statistical linearization (e.g., [14, 40]) and stochastic averaging methodologies (e.g., [17]) is proposed for deriving an additional set of equations, which are required for solving the coupled system defined in Eq. (9).

In this regard, subtracting Eq. (7) from Eq. (1), and also taking into account Eq. (8), leads to the stochastic sub-equation

$$\ddot{\hat{x}} + 2\xi\omega_{\mathrm{n}}\dot{\hat{x}} + h(t, \mu_x, \hat{x}, c(t), \dot{\mu}_x, \dot{\hat{x}}, \dot{c}(t)) = Q(t), \tag{10}$$

where

$$h(t, \mu_x, \hat{x}, c(t), \dot{\mu}_x, \dot{\hat{x}}, \dot{c}(t)) = g(t, \mu_x + \hat{x}, \dot{\mu}_x + \dot{\hat{x}}) - \gamma(\mu_x, \dot{\mu}_x, c(t), \dot{c}(t)). \tag{11}$$

Next, for the application of the statistical linearization method, first, an equivalent time-varying linear system for the oscillator in Eq. (10) is defined in the form

$$\ddot{\hat{x}} + \beta_{\mathrm{eq}}(t)\dot{\hat{x}} + \omega_{\mathrm{eq}}^2(t)\hat{x} = Q(t), \tag{12}$$

where $\beta_{\mathrm{eq}}(t)$ and $\omega_{\mathrm{eq}}^2(t)$ denote the time-dependent equivalent linear elements. For the determination of the equivalent elements in Eq. (12), first, the error between the original system and its linear equivalent is formed. Then, considering a mean-square minimization criterion and also adopting the Gaussian response assumption (e.g., [14]), the equivalent linear elements are given by

$$\beta_{\mathrm{eq}}(t) = 2\xi\omega_{\mathrm{n}} + \mathrm{E}\left[\frac{\partial h}{\partial \dot{\hat{x}}}\right] = \lambda_1\left(\mu_x, \dot{\mu}_x, c(t), \dot{c}(t)\right) \tag{13}$$

and

$$\omega_{\mathrm{eq}}^2(t) = \mathrm{E}\left[\frac{\partial h}{\partial \hat{x}}\right] = \lambda_2\left(\mu_x, \dot{\mu}_x, c(t), \dot{c}(t)\right), \tag{14}$$

where h denotes the nonlinear function in Eq. (11). Clearly, the time-dependent equivalent elements defined in Eqs. (13) and (14) depend on μ_x, $\dot{\mu}_x$, $c(t)$ and $\dot{c}(t)$. A detailed derivation of Eqs. (12-14) is found in [14, 41]. Next, a stochastic averaging-based framework is proposed to reduce the order of the equivalent linear system in Eq. (12), and thus, treat the linearized system in a more computational efficient manner.

In this regard, considering that the system in Eq. (1) is lightly damped, or, equivalently, $\beta_{\mathrm{eq}}(t)$ attains small values, it can be argued that the system response follows a pseudo-harmonic behavior. That is

$$\hat{x}\left(t\right) = a(t)\cos\phi(t) \tag{15}$$

and

$$\dot{\hat{x}}\left(t\right) = -\omega_{\mathrm{eq}}(t)a(t)\sin\phi(t), \tag{16}$$

where $\phi = \phi(t) = t\omega_{\mathrm{eq}}(t) + \theta(t)$, with $a = a(t)$ denoting the time-dependent amplitude of the pseudo-harmonic response, given by

$$a^2\left(t\right) = \hat{x}^2(t) + \frac{\dot{\hat{x}}^2(t)}{\omega_{\mathrm{eq}}^2(t)}, \tag{17}$$

and $\theta = \theta(t)$ the corresponding phase. The latter are considered to be slowly varying functions with respect to time, and approximately constant over one cycle of oscillation (e.g., [14, 22]). Then, substituting Eqs. (15) and (16) into Eq. (12) yields ([17])

$$\dot{a}(t) = -\beta_{\text{eq}}(t)a(t)\sin^2\phi(t) - \frac{Q(t)}{\omega_{\text{eq}}(t)}\sin\phi(t) \tag{18}$$

and

$$\dot{\theta}(t) = -\beta_{\text{eq}}(t)\sin\phi(t)\cos\phi(t) - \frac{Q(t)}{a(t)\omega_{\text{eq}}(t)}\cos\phi(t). \tag{19}$$

Next, the standard stochastic averaging method (e.g., [17]) is applied to decouple the system response amplitude and phase given by Eqs. (18) and (19), respectively. In this regard, considering that the stochastic excitation $Q(t)$ in Eq. (1) possesses a wide-band spectrum, it can be proved that ([39])

$$-Q(t)\sin\phi(t) \approx \frac{\pi S(\omega_{\text{eq}}(t), t)}{2\omega_{\text{eq}}(t)a(t)} + \left(\pi S(\omega_{\text{eq}}(t), t)\right)^{1/2}\eta(t), \tag{20}$$

where $\eta(t)$ denotes a delta correlated zero-mean process of intensity one (e.g., [37, 40, 22]). Further, substituting Eq. (20) into Eq. (18) and deterministically averaging leads to the first-order differential equation for the response amplitude

$$\dot{a}(t) = -\frac{1}{2}\beta_{\text{eq}}(t)a(t) + \frac{\pi S\left(\omega_{\text{eq}}(t), t\right)}{2a(t)\omega_{\text{eq}}^2(t)} - \frac{\left(\pi S\left(\omega_{\text{eq}}(t), t\right)\right)^{1/2}}{\omega_{\text{eq}}^2(t)}\eta(t). \tag{21}$$

The corresponding Fokker–Planck partial differential equation governing the evolution in time of the response amplitude probability density function

(PDF) takes the form (e.g., [37, 1, 22])

$$
\begin{aligned}
\frac{\partial p(a,t)}{\partial t} = & - \frac{\partial}{\partial a}\left\{\left(-\frac{1}{2}\beta_{\mathrm{eq}}(t)a(t) + \frac{\pi S\left(\omega_{\mathrm{eq}}(t),t\right)}{2a(t)\omega_{\mathrm{eq}}^2(t)}\right)p(a,t)\right\} \\
& + \frac{\pi S\left(\omega_{\mathrm{eq}}(t),t\right)}{2\omega_{\mathrm{eq}}^2(t)}\frac{\partial^2 p(a,t)}{\partial a^2}. \qquad (22)
\end{aligned}
$$

Next, considering that a solution of the Fokker-Planck equation is already available for the case of a linear oscillator subject to stationary stochastic excitation (e.g., [17, 18]), following closely [37], for the non-stationary case the solution to Eq. (22) is sought in the Rayleigh form

$$
p(a,t) = \frac{a}{c(t)}\exp\left(-\frac{a^2}{2c(t)}\right). \qquad (23)
$$

In Eq. (23), $c(t)$ denotes the time-dependent variance of the response process $\hat{x}(t)$ to be determined; the interested reader is referred to [37, 39, 40, 18] for a detailed derivation of Eqs. (18-23) and further discussion on the topic. Moreover, it is noted that the closed-form expression in Eq. (23) for determining the non-stationary response amplitude PDF for nonlinear oscillators, has been recently generalized in [22] to account for nonlinear oscillators endowed with fractional derivative elements. Finally, substituting Eq. (23) into the Fokker-Planck equation Eq. (22) and manipulating, yields ([40])

$$
\dot{c}(t) = -\beta_{\mathrm{eq}}(c(t))c(t) + \frac{\pi S\left(\omega_{\mathrm{eq}}(c(t)),t\right)}{\omega_{\mathrm{eq}}^2(c(t))}. \qquad (24)
$$

Clearly, the deterministic first-order nonlinear differential equation defined in Eq. (24) greatly facilitates the determination of the non-stationary response amplitude PDF of Eq. (23), since its solution can be readily found by any standard numerical scheme. Further, taking into account the system nonlinearity in Eq. (11) it can be argued that the time-dependent equivalent linear elements $\beta_{eq}(t)$ and $\omega_{eq}(t)$ depend, in essence, on $c(t)$, μ_x and $\dot{\mu}_x$. Therefore, Eqs. (13) and (14) are written in the form

$$\beta_{\text{eq}}(c(t)) = \lambda_1 \left(\mu_x, \dot{\mu}_x, c(t), \dot{c}(t)\right) \tag{25}$$

and

$$\omega_{\text{eq}}^2(c(t)) = \lambda_2 \left(\mu_x, \dot{\mu}_x, c(t), \dot{c}(t)\right), \tag{26}$$

which, in conjunction with Eq. (24), constitute the additional set of differential equations to be solved for determining the deterministic and stochastic response of the system in Eq. (1). In this regard, combining Eqs. (24-26) with Eq. (9), and further defining

$$\mathbf{q} = \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}^{\text{T}} = \begin{bmatrix} \mu_x & \dot{\mu}_x & c(t) \end{bmatrix}^{\text{T}}, \tag{27}$$

leads to the nonlinear system

$$\dot{q}_1 = q_2, \tag{28a}$$

$$\dot{q}_2 = F(t) - 2\xi\omega_{\text{n}} q_2 - \gamma\left(q_1, q_3, q_2, \dot{q}_3\right), \tag{28b}$$

$$\dot{q}_3 = -\beta_{\text{eq}}(c(t)) q_3 + \frac{\pi S\left(\omega_{\text{eq}}(c(t)), t\right)}{\omega_{\text{eq}}^2(c(t))}, \tag{28c}$$

which can be solved by utilizing the Runge-Kutta method.

2.3. Mechanization of the semi-analytical technique

To further elucidate the theoretical developments in section 2.2, the mechanization of the semi-analytical technique is concisely described in this section. It involves the following steps:

1. Utilize Eq. (8) for determining the expectation of the nonlinear function $\gamma\left(q_1, q_3, q_2, \dot{q}_3\right)$ in Eq. (28b).
2. Determine the time-dependent equivalent linear elements $\beta_{\mathrm{eq}}(c(t))$ and $\omega_{\mathrm{eq}}(c(t))$ of Eqs. (25-26), and substitute the corresponding expressions into Eq. (28c).
3. Solve the system defined in Eqs. (28a-28c) simultaneously to obtain the deterministic response and the variance of the stochastic response. This is attained by applying any standard numerical scheme, such as the Runge-Kutta method.

3. Numerical Examples

In this section, an SDOF nonlinear system exhibiting quadratic and cubic nonlinear stiffness is considered for assessing the reliability of the proposed technique. The system governing equation of motion is given by

$$\ddot{x} + 2\xi\omega_{\mathrm{n}}\dot{x} + \omega_n^2 x\left(1 + \varepsilon_1 x + \varepsilon_2 x^2\right) = Q\left(t\right) + F\left(t\right), \tag{29}$$

where the coefficients ε_1, $\varepsilon_2 > 0$ denote the intensity of the quadratic and cubic nonlinearity, respectively, and $F(t)$ and $Q(t)$ are the deterministic and non-stationary stochastic excitations.

Decomposing the system response into a deterministic and a stochastic component, Eq. (29) takes the form

$$\begin{aligned}(\ddot{\mu}_x + \ddot{\hat{x}}) + 2\xi\omega_{\mathrm{n}}\left(\dot{\mu}_x + \dot{x}\right) + \omega_n^2\left(\mu_x + \hat{x}\right)\left(1 + \varepsilon_1\left(\mu_x + \hat{x}\right) + \varepsilon_2(\mu_x + \hat{x})^2\right)& \\ = Q\left(t\right) + F\left(t\right).&\end{aligned} \tag{30}$$

Then, taking into account Eq. (6), the nonlinear restoring force is given by

$$\mathrm{g}(\mu_x + \hat{x}, \dot{\mu}_x + \dot{\hat{x}}) = \omega_{\mathrm{n}}^2\left(\mu_x + \hat{x}\right)\left(1 + \varepsilon_1\left(\mu_x + \hat{x}\right) + \varepsilon_2(\mu_x + \hat{x})^2\right), \tag{31}$$

and thus, ensemble averaging, Eq. (8) yields

$$\gamma\left(\mu_x, c(t)\right) = \omega_n^2\left(\mu_x + \varepsilon_1\left(c(t) + \mu_x^2\right) + \varepsilon_2\left(3\mu_x c(t) + \mu_x^3\right)\right), \tag{32}$$

whereas, Eq. (11) is written as

$$\begin{aligned}h\left(t, \mu_x, \hat{x}, c(t)\right) = {}& \omega_{\mathrm{n}}^2\left(\mu_x + \hat{x}\right)\left(1 + \varepsilon_1\left(\mu_x + \hat{x}\right) + \varepsilon_2(\mu_x + \hat{x})^2\right) \\ & - \omega_{\mathrm{n}}^2\left(\mu_x + \varepsilon_1\left(c(t) + \mu_x^2\right) + \varepsilon_2\left(3\mu_x c(t) + \mu_x^3\right)\right).\end{aligned} \tag{33}$$

In this regard, the time-dependent equivalent linear elements in Eqs. (25) and (26) become

$$\beta_{\mathrm{eq}}(c(t)) = 2\xi\omega_{\mathrm{n}} \tag{34}$$

and

$$\omega_{\mathrm{eq}}^2(c(t)) = \omega_{\mathrm{n}}^2\left(1 + 2\varepsilon_1\mu_x + 3\varepsilon_2\left(c(t) + \mu_x^2\right)\right), \tag{35}$$

respectively, where the non-stationary response variance $c(t)$ is determined by the deterministic first-order nonlinear differential equation Eq. (24). Finally, substituting Eq. (32) and Eqs. (34-35) into Eqs. (28a-28c) yields the system of equations

$$\dot{q}_1 = q_2, \tag{36a}$$

$$\dot{q}_2 = -2\xi\omega_{\mathrm{n}}q_2 - \omega_{\mathrm{n}}^2\left(q_1 + \varepsilon_1\left(q_1^2 + q_3\right) + \varepsilon_2\left(3q_1q_3 + q_1^3\right)\right) + F(t), \tag{36b}$$

$$\dot{q}_3 = -2\xi\omega_{\mathrm{n}}q_3 + \frac{\pi S\left(\sqrt{\omega_{\mathrm{n}}^2\left[1 + 3\varepsilon\left(q_3 + q_1^2\right)\right]}, t\right)}{\omega_{\mathrm{n}}^2\left[1 + 3\varepsilon\left(q_3 + q_1^2\right)\right]}. \tag{36c}$$

Clearly, the solution of the system defined in Eqs. (36a-36c), and thus, the determination of the response vector $\mathbf{q}$ in Eq. (27), relies on the specific form of the evolutionary power spectrum $S(\omega, t)$. In the ensuing analysis, the reliability of the proposed method is assessed by considering four different cases of non-stationary stochastic excitation loads, each one described by a different power spectrum.

3.1. Systems subjected to combined deterministic excitation and modulated white noise

In this section, the non-stationary stochastic excitation of the system in Eq. (29) is modeled as a modulated white noise excitation described by the power spectrum

$$S\left(\omega, t\right) = S_0\rho^2(t). \tag{37}$$

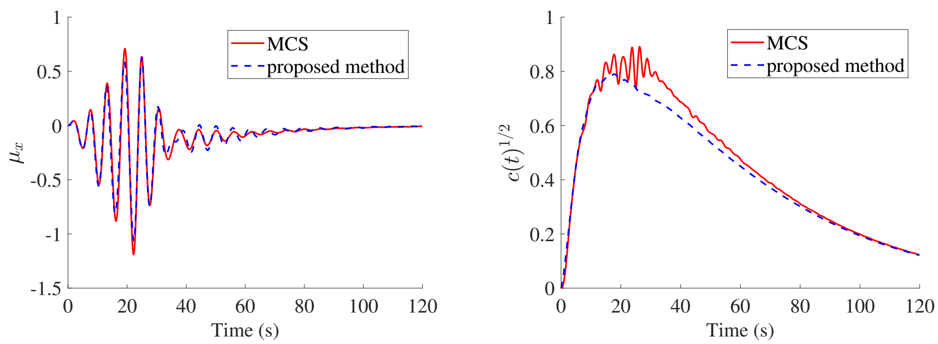
In Eq. (37), S_0 denotes the constant spectral density of the white noise and $\rho(t)$ is the deterministic time-modulating function

$$\rho\left(t\right) = H\left(\exp\left(\mu_1 t\right) - \exp\left(\mu_2 t\right)\right), \tag{38}$$

where H is the amplitude of the modulating function, and the parameters μ_1 and μ_2 account for the modulation speed.

The following set of parameters are considered for the numerical implementation: $\xi = 0.05$, $\omega_{\mathrm{n}} = 1$, $\varepsilon_1 = \varepsilon_2 = \varepsilon = 0.5$ for the system; $F_0 = 1$, $\omega_0 = 1$, $\mu_0 = 0.15$, $t_0 = 20$ for the deterministic excitation; and $S_0 = 0.1$, $H = 1$, $\mu_1 = 0.025$, $\mu_2 = 0.25$ for the random excitation. Further, the obtained results are compared with MCS data ($10,000$ samples). Specifically, the samples of the stochastic excitation used in the MCS estimates are generated by multiplying the modulating function $\rho(t)$ in Eq. (38) by relevant white noise samples synthesized using the spectral representation method [42].

The deterministic component of the response displacement and the standard deviation of the stochastic component of the response displacement obtained by the proposed method are shown in Figs. 1(a) and 1(b), respectively; MCS data are also included for comparison. Clearly, the deterministic response displacement component obtained by the proposed method agrees well with the MCS data. The method is not only capable of capturing the salient characteristics of the modulated harmonic wave, such as the pulses and the time-varying amplitudes, but also agrees with the fact that the harmonic reference line is not exactly zero, due to the quadratic nonlinear factor in Eq. (29). Further, Fig. 1(b) demonstrates a satisfactory agreement between the standard deviation of the stochastic response displacement obtained by the proposed method and the MCS data. However, it is noted that since the employed stochastic averaging step has eliminated the harmonic characteristic of the standard deviation, the proposed method does not capture the harmonic-like oscillations visible between 20s and 30s of the MCS data.



(a) (b)

Figure 1: (a) Time history of the deterministic response displacement component; and (b) standard deviation of the stochastic response displacement component of the SDOF nonlinear system in Eq. (29) subjected to combined harmonic excitation and non-stationary modulated white noise excitation described by the PSD in Eq. (37). MCS data (10,000 samples) are included for comparison.

3.2. Systems subjected to combined deterministic excitation and modulated colored noise

In this section, it is assumed that the system non-stationary excitation is described by the modulated Kanai-Tajimi power spectrum

$$S(\omega, t) = S_0 \rho^2(t) \frac{1 + 4\xi_{\mathrm{g}}^2 \left(\omega/\omega_{\mathrm{g}}\right)^2}{\left(1 - \left(\omega/\omega_{\mathrm{g}}\right)^2\right)^2 + 4\xi_{\mathrm{g}}^2 \left(\omega/\omega_{\mathrm{g}}\right)^2}, \tag{39}$$

where ξ_{g} and ω_{g} are the damping ratio and natural period of the site; S_0 represents the constant spectral density of the noise, whereas the deterministic time-modulating function $\rho(t)$ is defined in Eq. (38). For the numerical implementation, the colored noise parameter values are selected as $\omega_{\mathrm{g}} = 1$,

$\xi_g = 0.4$, whereas all other system parameter values are the same as in section 3.1.

In this regard, applying the herein proposed method, the time history of the deterministic response displacement component, as well as the standard deviation of the stochastic response displacement are shown in Figs. 2(a) and 2(b), where MCS data (10,000 samples) are also included for comparison. Similar to the excitation case considered in section 3.1, it is clear that the proposed method is in extremely good agreement with the relevant MCS data.

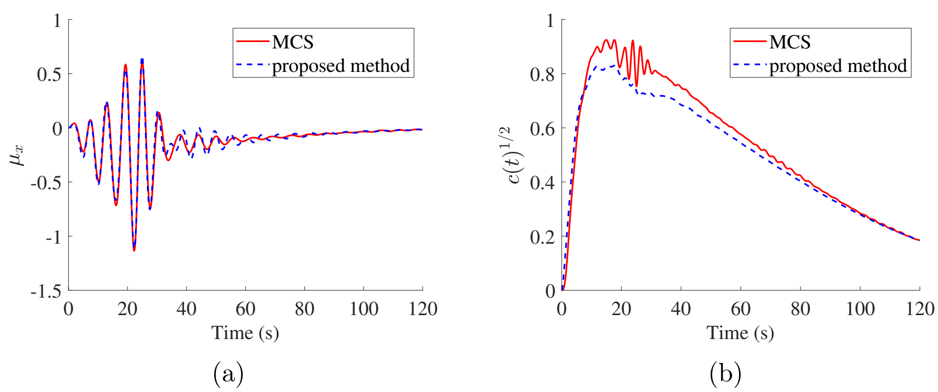


Figure 2: (a) Time history of the deterministic response displacement component; and (b) standard deviation of the stochastic response displacement component of the SDOF nonlinear system in Eq. (29) subjected to combined harmonic excitation and non-stationary modulated colored noise excitation described by the evolutionary power spectrum in Eq. (39). MCS data (10,000 samples) are included for comparison.

3.3. Systems subjected to combined deterministic excitation and evolutionary stochastic excitation

In this section, the proposed approach is utilized to determine the response of the SDOF nonlinear oscillator in Eq. (29) subjected to deterministic and stochastic excitation described by two different non-separable power spectrum forms.

In this regard, first, the evolutionary Clough-Penzien model is considered

$$S(\omega,t)=S_0\rho^2(t)\frac{\omega_{\mathrm{g}}^4(t)+4\xi_{\mathrm{g}}^2(t)\omega_{\mathrm{g}}^2(t)\omega^2}{\left(\omega^2-\omega_{\mathrm{g}}^2(t)\right)^2+4\xi_{\mathrm{g}}^2(t)\omega_{\mathrm{g}}^2(t)\omega^2} \times\frac{\omega^4}{\left(\omega^2-\omega_{\mathrm{f}}^2(t)\right)^2+4\xi_{\mathrm{f}}^2(t)\omega_{\mathrm{f}}^2(t)\omega^2}, \tag{40}$$

where

$$\omega_{\mathrm{g}}(t)=\omega_{\mathrm{g}}-c\frac{t}{T}, \tag{41a}$$

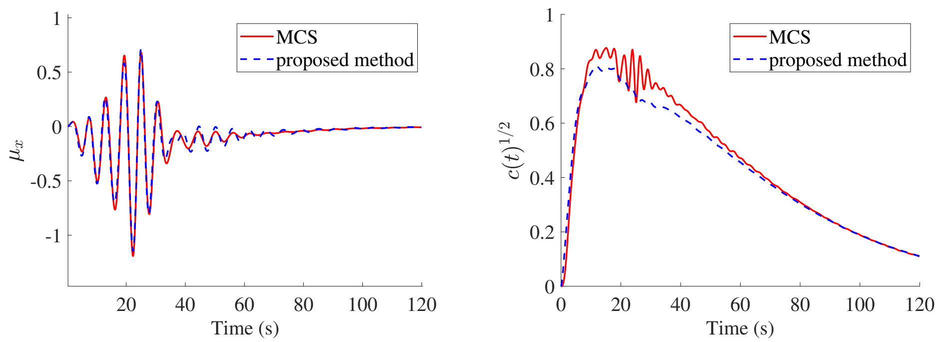
$$\xi_{\mathrm{g}}(t)=\xi_{\mathrm{g}}+d\frac{t}{T}, \tag{41b}$$

$$\omega_{\mathrm{f}}(t)=0.1\omega_{\mathrm{g}}(t), \tag{41c}$$

$$\xi_{\mathrm{f}}(t)=\xi_{\mathrm{g}}(t). \tag{41d}$$

In Eqs. (41a-41d), T denotes the time duration of the applied white noise excitation process, c and d are the parameters of the evolutionary Clough-Penzien model denoting the varying rate of the site parameters with respect to time ([43]), $\rho(t)$ is the modulating function defined in Eq. (38), and S_0 denotes the constant spectral density. For the numerical implementation, the parameter values of the evolutionary Clough-Penzien model are $c = 0.4$ and $d = 0.25$, whereas the other parameters of the system have the same values as the parameters in section 3.2.

The time history of the deterministic response displacement component and the standard deviation of the stochastic response displacement component obtained by the proposed method are shown in Figs. 3(a)-3(b), respectively, where relevant MCS data are also shown for comparison. Obviously, the proposed method is in good agreement with the MCS data.



(a) (b)

Figure 3: (a) Time history of the deterministic response displacement component; and (b) standard deviation of the stochastic response displacement component of the SDOF nonlinear system in Eq. (29) subjected to combined harmonic excitation and non-separable non-stationary stochastic excitation described by the evolutionary power spectrum in Eq. (40). MCS data (10,000 samples) are included for comparison.

The applicability of the proposed method to systems with different levels of nonlinearity is further investigated in the following. For convenience in the comparisons, the time-average modulus (TAM) of the deterministic response is defined as

$$P = \frac{1}{T}\int_0^T \sqrt{\mu_x^2(t)}\mathrm{d}t, \tag{42}$$

whereas

$$\bar{\sigma} = \frac{1}{T}\int_0^T \sqrt{c(t)}\mathrm{d}t \tag{43}$$

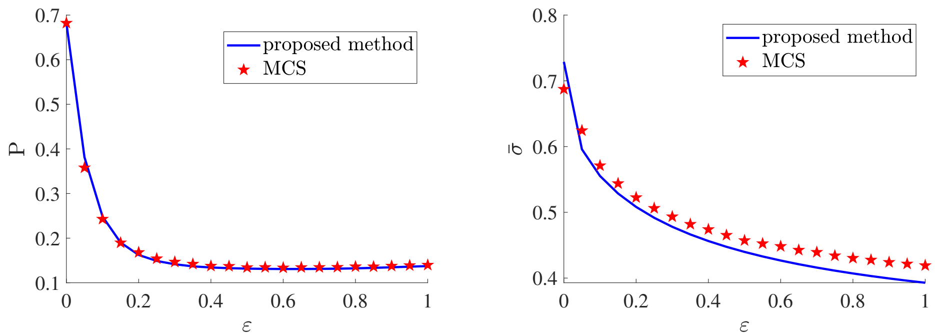
denotes the time-averaged standard deviation (TASTD) of the non-stationary stochastic response component. The considered parameter values in this example remain the same as these used in the typical parameters setting in sections 3.1 and 3.2, apart from the values of the nonlinearity strength $\varepsilon_1 = \varepsilon_2 = \varepsilon$ which vary between 0 and 1. Finally, relevant MCS estimates (10, 000 samples) are provided in the ensuing analysis for comparison.

In this regard, the variation of the TAM of the deterministic response and the TASTD of the stochastic response versus a varying intensity of the nonlinear strength are shown in Figs. 4(a) and 4(b), respectively. Fig. 4(a) clearly shows that the TAM derived by the proposed method is in perfect agreement with the MCS estimate. In Fig. 4(b), on the contrary, although the error in determining the TASTD by the proposed method slightly increases with increasing the intensity of the nonlinearity, as compared to the MCS estimate, yet the margin of the error remains relatively small. This aspect demonstrates the capacity of the proposed method to efficiently capture the TASTD of the non-stationary stochastic response component, even for the case of highly nonlinear systems. Further, it is seen that although the TASTD decreases with increasing the nonlinearity magnitude, the same action affects in a different way the bahavior of the TAM, which decreases dramatically at the beginning but then remains almost the same.

Next, it is assumed that the evolutionary power spectrum that corresponds to the stochastic excitation component applied to the nonlinear oscillator in Eq. (29) is of the non-separable kind (e.g., [39, 22])

$$S(\omega, t) = S_0 \left(\frac{\omega}{5\pi}\right)^2 t^2 \rho^2(t) \exp\left(-\left(\frac{\omega}{5\pi}\right)^2 t\right), \tag{44}$$

where $\rho(t)$ denotes the time-modulating function defined in Eq. (38). In the ensuing analysis, the parameter values of the system and the excitation are the same with these used in section 3.1.

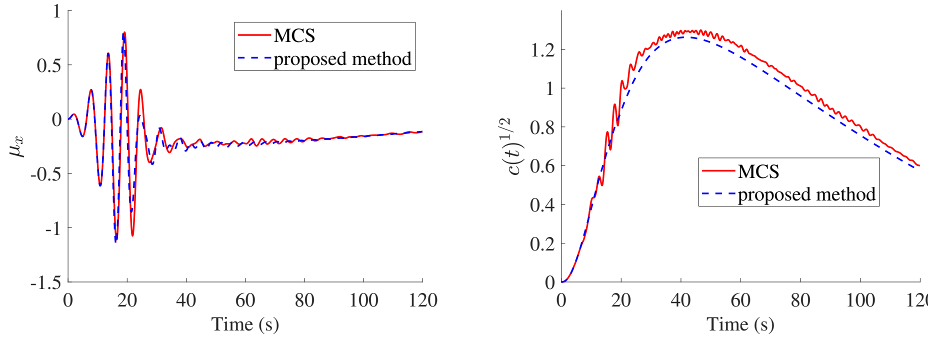


(a) (b)

Figure 4: (a) TAM of the deterministic response displacement component; and (b) TASTD of the stochastic response displacement component versus nonlinearity magnitude of the SDOF nonlinear system in Eq. (29), subjected to combined harmonic excitation and non-separable non-stationary stochastic excitation described by the evolutionary power spectrum in Eq. (40). MCS data (10, 000 samples) are included for comparison.

In this regard, the time history of the deterministic response displacement component obtained by the proposed method is shown in Figs. 5(a), while a relevant MCS estimate (10, 000 samples) is also included in the same figure for comparison. Further, Fig. 5(b) shows the standard deviation of the stochastic response displacement component and the corresponding MCS estimate. It is readily seen that the results obtained by the proposed method agree well with the MCS estimates, even for the case of non-stationary stochastic excitation described by the non-separable power spectrum in Eq. (44). Further, comparing Fig. 3(a) with Fig. 5(a) and Fig. 3(b) with Fig. 5(b), it is clear that, the application of different kind of stochastic excitation spectra to the same system, not only affects the stochastic response displacement component, but also alters the deterministic component of the response.

Finally, similar to the case of the evolutionary Clough-Penzien model, the variation of the TAM of the deterministic response and the TASTD of the

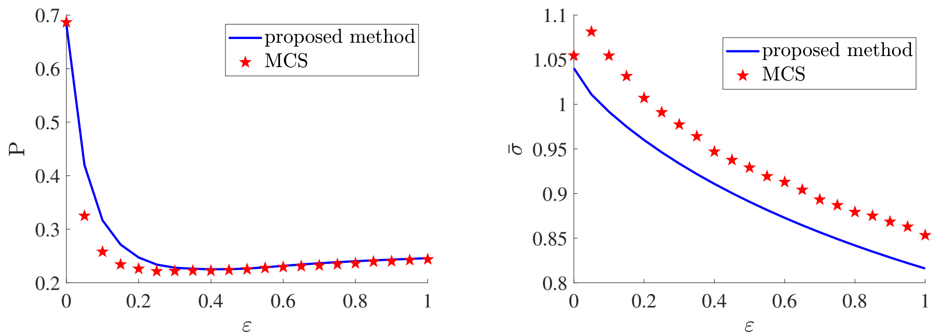


(a) (b)

Figure 5: (a) Time history of the deterministic response displacement component; and (b) standard deviation of the stochastic response displacement component of the SDOF nonlinear system in Eq. (29) subjected to combined harmonic excitation and non-separable non-stationary stochastic excitation described by the evolutionary power spectrum in Eq. (44). MCS data (10,000 samples) are included for comparison.

stochastic response, versus varying nonlinear strength is shown in Figs. 6(a) and 6(b), and analogous conclusions are drawn. Specifically, the TAM obtained by the proposed method is in satisfactory agreement with the MCS estimate, as it is seen in Fig. 6(a). In Fig. 6(b), in spite the increasing error between the TASTD and the MCS estimate with increasing the nonlinearity strength, the margin of the error is acceptable. Further, increasing the nonlinearity magnitude results in decreasing the TASTD, whereas the TAM first decreases dramatically over the anterior part of the considered nonlinearity magnitude, and then slightly increases over the posterior part of it.

Overall, it can be argued that the herein proposed method exhibits satisfactory accuracy for the considered SDOF systems subjected to non-stationary stochastic excitation described by modulated noise and non-separable noise. In addition, the method can be readily applied to systems exhibiting strong nonlinearity.



(a) (b)

Figure 6: (a) TAM of the deterministic response displacement component; and (b) TASTD of the stochastic response displacement component versus nonlinearity magnitude of the SDOF nonlinear system in Eq. (29), subjected to combined harmonic excitation and non-separable non-stationary stochastic excitation described by the evolutionary power spectrum in Eq. (44). MCS data (10, 000 samples) are included for comparison.

4. Concluding remarks

In this paper, a semi-analytical technique has been developed for determining the stochastic response of single-degree-of-freedom nonlinear systems subjected to combined deterministic and non-stationary stochastic excitations. This has been attained by resorting to the combination of the statistical linearization and stochastic averaging methods. Based on the nature of the excitation, the response of the system has been decomposed into a deterministic component and a zero-mean stochastic component. These have been treated separately. Specifically, first, a coupled set of differential sub-equations governing the deterministic component of the response have been derived. Next, an additional equation has been constructed by applying the statistical linearization and stochastic averaging methodologies. In this regard, applying the statistical linearization has led to a linearized equivalent system. Further, adopting a mean-square minimization criterion, standard closed-form expressions have been derived for determining the time-varying equivalent damping and stiffness coefficients of the linearized system. It has been shown that the latter not only depend on the nonlinearity and the deterministic response component, but also on the unknown time-dependent variance of the stochastic response. Therefore, considering that the system is lightly damped, a stochastic averaging step has been applied for reducing the order of the linearized system, and thus, the computational effort required for its solution. This has been achieved by assuming that the zero-mean stochastic response follows a pseudo-harmonic behavior characterized by slowly varying over one cycle of oscillation response amplitude and phase. Subsequently, a first-order differential equation for the amplitude of the stochastic response has been constructed, and a solution for the corresponding Fokker-Planck equation has led to a first-order nonlinear differential equation for determining the time-dependent stochastic response variance. The latter, in conjunction with the expressions for the equivalent damping and stiffness elements, as well as the coupled set of differential sub-equations have led to the computation of the system response.

A single-degree-of-freedom nonlinear system exhibiting quadratic and cubic nonlinear stiffness, and subjected to deterministic and non-stationary stochastic excitations has been considered in the numerical example section. The efficiency of the proposed technique has been demonstrated by considering different levels of nonlinearity and various forms of non-stationary stochastic excitation. These include the case of modulated white and colored noise processes, as well as non-stationary stochastic excitation described by evolutionary power spectrum forms of the non-separable kind. The latter have demonstrated the applicability of the technique to arbitrary excitation evolutionary power spectrum forms. Pertinent Monte-Carlo simulation data have been used to assess the accuracy of the proposed technique.

CRediT authorship contribution statement

R. Han: Writing- original draft, investigation, visualization. **V.C. Fragkoulis**: Writing- original draft, Writing- review & editing, methodology, funding acquisition, validation. **F. Kong**: Conceptualization, funding acquisition, project administration. **M. Beer**: Resources, supervision. **Y. Peng**: Resources, supervision.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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