E-Companion

Can We Work More Safely and Healthily with Robot Partners? A Human-Friendly Robot-Human Coordinated Order Fulfillment Scheme

Online Appendix A. Proofs

A.1. Proof of Theorem 4.1

Using Eq. (8) (the text), we have

$$\gamma_k = \frac{Y_k}{Q_{k-1} + \mu_k X_k} \quad , \ \forall k \qquad \text{(A1)}$$
$$\Rightarrow Y_k = \gamma_k (Q_{k-1} + \mu_k X_k)$$

Input Eq. (A1) into Eq. (9) (the text), and then, we have

$$\delta_k = \frac{X_k}{A_k + \nu_k Y_k}$$
$$\Rightarrow X_k = \delta_k \left(A_k + \nu_k \gamma_k (Q_{k-1} + \mu_k X_k) \right) \quad , \ \forall k$$
$$\Rightarrow X_k (1 - \delta_k \mu_k \nu_k \gamma_k) = \delta_k (A_k + \nu_k \gamma_k Q_{k-1})$$
$$\Rightarrow \hat{X}_k = \frac{\delta_k (A_k + \nu_k \gamma_k Q_{k-1})}{1 - \mathbf{\Omega}_k} = \frac{1}{1 - \mathbf{\Omega}_k} \left(\delta_k A_k + \frac{\mathbf{\Omega}_k Q_{k-1}}{\mu_k} \right)$$

(A2)

Input Eq. (A2) into Eq. (A1), and then, we have

$$Y_k = \gamma_k (Q_{k-1} + \mu_k X_k)$$
$$\Rightarrow \hat{Y}_k = \gamma_k \left(Q_{k-1} + \frac{\delta_k \mu_k A_k + \mathbf{\Omega}_k Q_{k-1}}{1 - \mathbf{\Omega}_k} \right) = \frac{1}{1 - \mathbf{\Omega}_k} \left(\frac{\mathbf{\Omega}_k A_k}{\nu_k} + \gamma_k Q_{k-1} \right) \quad , \ \forall k$$

(A3)

Thus, Theorem 4.1 is proven.

A.2. Proof of Corollary 4.1

Given Theorem 4.1, we have the estimates $\hat{X}_k$ (Eq. (A2)) and $\hat{Y}_k$ (Eq. (A3). Using Eqs. (A2) and Eq. (1) (the text), we have $\hat{Q}_k$

$$\hat{Q}_k = Q_{k-1} + \hat{\hat{X}}_k = Q_{k-1} + \frac{\mu_k}{1-\mathbf{\Omega}_k}\left(\delta_k A_k + \frac{\mathbf{\Omega}_k Q_{k-1}}{\mu_k}\right) = \frac{1}{1-\mathbf{\Omega}_k}\left(\delta_k \mu_k A_k + Q_{k-1}\right) \quad , \ \forall k \qquad \text{(A4)}$$

Using Eqs. (3) (text), (A3) and (A4), we can derive the estimate $\hat{\lambda}_k$ as

$$\begin{aligned}\hat{\lambda}_k &= \beta_0 + \beta_1 \hat{Y}_k + \beta_2 \hat{Q}_k \\ &= \beta_0 + \beta_1 \frac{\gamma_k}{1-\mathbf{\Omega}_k}\left(\delta_k \mu_k A_k + Q_{k-1}\right) + \beta_2 \frac{1}{1-\mathbf{\Omega}_k}\left(\delta_k \mu_k A_k + Q_{k-1}\right) \quad , \ \forall k \qquad \text{(A5)} \\ &= \beta_0 + (\beta_1 \gamma_k + \beta_2)\left(\frac{\delta_k \mu_k A_k + Q_{k-1}}{1-\mathbf{\Omega}_k}\right)\end{aligned}$$

Based on Eqs. (A5) and (2) (text), we have the estimate ($\Delta\hat{\rho}_k$) of the fatigue increment in each time interval k ($\forall k$) as

$$\Delta\hat{\rho}_k = 1 - e^{-\hat{\lambda}_k} = 1 - e^{-\left(\beta_0 + (\beta_1\gamma_k + \beta_2)\left(\frac{\delta_k \mu_k A_k + Q_{k-1}}{1-\mathbf{\Omega}_k}\right)\right)} \quad , \ \forall k \qquad \text{(A6)}$$

A.3. Proof of Corollary 5.1

Given Eq. (15) (the text) and the data regarding A_k and Y_k collected through time interval k, the optimal estimate ($\hat{\mathbf{\Phi}}_{k+1}$) of $\mathbf{\Phi}_{k+1}$ is

$$\hat{\mathbf{\Phi}}_{k+1} = \hat{\mathbf{F}}_k + \hat{\mathbf{W}}_k \qquad \text{(A7)}$$

where $\hat{\mathbf{F}}_k$ is the linear minimum mean square (LMMS) estimate of $\mathbf{F}_k$. If $\hat{\mathbf{W}}_k$ is the LMMS estimate of $\mathbf{W}_k$ using the data regarding A_k and Y_k collected through time interval k, we, then, have

$$\hat{\mathbf{W}}_k = E\left[(\mathbf{W}_k - \mathbf{m_W})(\mathbf{Y}_k - \mathbf{m_Y})^{\mathrm{T}}\right]\left\{E\left[(\mathbf{Y}_k - \mathbf{m_Y})(\mathbf{Y}_k - \mathbf{m_Y})^{\mathrm{T}}\right]\right\}^{-1}(\mathbf{Y}_k - \mathbf{m_Y}) + \mathbf{m_W} \qquad \text{(A8)}$$

where $\mathbf{m_W}$ is the mean of $\mathbf{W}_k$; and $\mathbf{m_Y}$ is the mean of $\mathbf{Y}_k$. Since $\mathbf{W}_k - \mathbf{m_W}$ and $\mathbf{Y}_k - \mathbf{m_Y}$ are orthogonal, we have $(\mathbf{W}_k - \mathbf{m_W})(\mathbf{Y}_k - \mathbf{m_Y})^{\mathrm{T}} = \mathbf{0}$. Moreover, $\mathbf{m_W} = \mathbf{0}$ as $\mathbf{W}_k$ is a white noise vector.

Thus, we can derive $\hat{\mathbf{W}}_k = \mathbf{0}$ from Eq. (A8) such that $\hat{\mathbf{\Phi}}_{k+1} = \hat{\mathbf{F}}_k$ by Eq. (A7). Thus, Corollary 5.1 is proven.

A.4. Proof of Corollary 5.2

Given Eqs. (15) and (26) of Corollary 5.1 (the text), we have $\mathbf{\Phi}_{k+1}$ and $\hat{\mathbf{\Phi}}_{k+1|k}$ given, respectively, by

$$\mathbf{\Phi}_{k+1} = \mathbf{F}_k + \mathbf{W}_k \tag{A9}$$

$$\hat{\mathbf{\Phi}}_{k+1|k} = \hat{\mathbf{F}}_k \tag{A10}$$

Then, we have the state prediction error $\Delta\mathbf{\Phi}_{k+1|k}$ given by

$$\begin{aligned}\Delta\mathbf{\Phi}_{k+1|k} &\equiv \mathbf{\Phi}_{k+1} - \hat{\mathbf{\Phi}}_{k+1|k} \\ &= \left(\mathbf{F}_k - \hat{\mathbf{F}}_k\right) + \mathbf{W}_k \\ &= \left(\hat{\mathbf{F}}'_k \Delta\mathbf{\Phi}_{k|k}\right) + \mathbf{W}_k \qquad , \textit{where } \Delta\mathbf{\Phi}_{k|k} \equiv \mathbf{\Phi}_k - \hat{\mathbf{\Phi}}_k\end{aligned} \tag{A11}$$

Using Eq. (A11), the prior prediction ($\hat{\mathbf{P}}_{k+1|k}$) for the covariance matrix of state estimation error $\mathbf{P}_{k+1}$ is, thus, given by

$$\begin{aligned}\hat{\mathbf{P}}_{k+1|k} &\equiv E[\Delta\mathbf{\Phi}_{k+1|k}\Delta\mathbf{\Phi}^{\mathrm{T}}_{k+1|k}] \\ &= E\left[\left[\left(\hat{\mathbf{F}}'_k\Delta\mathbf{\Phi}_{k|k}\right) + \mathbf{W}_k\right]\left[\left(\hat{\mathbf{F}}'_k\Delta\mathbf{\Phi}_{k|k}\right) + \mathbf{W}_k\right]^{\mathrm{T}}\right] \\ &= \hat{\mathbf{F}}'_k E[\Delta\mathbf{\Phi}_{k|k}\Delta\mathbf{\Phi}^{\mathrm{T}}_{k|k}]\hat{\mathbf{F}}'^{\mathrm{T}}_k + \hat{\mathbf{F}}'_k E[\Delta\mathbf{\Phi}_{k|k}\mathbf{W}_k{}^{\mathrm{T}}] + E[\mathbf{W}_k\Delta\mathbf{\Phi}^{\mathrm{T}}_{k|k}]\hat{\mathbf{F}}'^{\mathrm{T}}_k + E[\mathbf{W}_k\mathbf{W}_k{}^{\mathrm{T}}]\end{aligned} \tag{A12}$$

Since $\Delta\mathbf{\Phi}_{k|k}$ and $\mathbf{W}_k$ are uncorrelated, $E[\Delta\mathbf{\Phi}_{k|k}\mathbf{W}_k{}^{\mathrm{T}}] = E[\mathbf{W}_k\Delta\mathbf{\Phi}^{\mathrm{T}}_{k|k}] = \mathbf{0}$. Thus, Eq. (A12) can be rewritten as

$$\begin{aligned}\hat{\mathbf{P}}_{k+1|k} &= \hat{\mathbf{F}}'_k E[\Delta\mathbf{\Phi}_{k|k}\Delta\mathbf{\Phi}^{\mathrm{T}}_{k|k}]\hat{\mathbf{F}}'^{\mathrm{T}}_k + E[\mathbf{W}_k\mathbf{W}_k{}^{\mathrm{T}}] \\ &= \hat{\mathbf{F}}'_k\hat{\mathbf{P}}_{k|k}\hat{\mathbf{F}}'^{\mathrm{T}}_k + \mathbf{R}_k\end{aligned} \tag{A13}$$

Thus, Corollary 5.2 is proven.

A.5. Proof of Corollary 5.3

It is noted that the measurement residual $\Delta\mathbf{y}_{k+1|k}$ in time interval $k+1$ and the measurement residual vector $\Delta\mathbf{Y}_k$ collected through time interval k are orthogonal. Moreover, $\hat{\mathbf{\Phi}}_{k+1|k}$ can be treated as the LMMS estimate of the state vector ($\mathbf{\Phi}_{k+1}$) based on $\Delta\mathbf{Y}_k$ (by Corollary 5.1). Utilizing the properties of LMMS estimation in the aspect of incorporation of orthogonal data, the LMMS estimate ($\hat{\mathbf{\Phi}}_{k+1}$) of $\mathbf{\Phi}_{k+1}$ based on $\Delta\mathbf{Y}_k$ and $\Delta\mathbf{y}_{k+1|k}$ are the sum of the two individual estimates. That is

$$\hat{\mathbf{\Phi}}_{k+1} = \hat{\mathbf{\Phi}}_{k+1|k} + E[\mathbf{\Phi}_{k+1} \mid \Delta\mathbf{y}_{k+1|k}] \qquad ,\forall k \tag{A14}$$

where $E[\mathbf{\Phi}_{k+1} \mid \Delta\mathbf{y}_{k+1|k}]$ is defined as the best estimate of $\mathbf{\Phi}_{k+1}$ based on $\Delta\mathbf{y}_{k+1|k}$. Utilizing the LMMS estimation, $E[\mathbf{\Phi}_{k+1} \mid \Delta\mathbf{y}_{k+1|k}]$ can be written as

$$\begin{aligned} &E[\mathbf{\Phi}_{k+1} \mid \Delta\mathbf{y}_{k+1|k}] \\ &\quad = E[(\mathbf{\Phi}_{k+1} - \mathbf{m}_{\mathbf{\Phi}})(\Delta\mathbf{y}_{k+1|k} - \mathbf{m}_{\Delta\mathbf{y}})^{\mathrm{T}}] \cdot \{E[(\Delta\mathbf{y}_{k+1|k} - \mathbf{m}_{\Delta\mathbf{y}})(\Delta\mathbf{y}_{k+1|k} - \mathbf{m}_{\Delta\mathbf{y}})^{\mathrm{T}}]\}^{-1} \cdot (\Delta\mathbf{y}_{k+1|k} - \mathbf{m}_{\Delta\mathbf{y}}) + \mathbf{m}_{\mathbf{\Phi}} \\ &\quad = E[(\mathbf{\Phi}_{k+1} - \mathbf{m}_{\mathbf{\Phi}})\Delta\mathbf{y}^{\mathrm{T}}_{k+1|k}] \cdot \{E[\Delta\mathbf{y}_{k+1|k}\Delta\mathbf{y}^{\mathrm{T}}_{k+1|k}]\}^{-1} \cdot \Delta\mathbf{y}_{k+1|k} + \mathbf{m}_{\mathbf{\Phi}} \\ &\quad = \left(E[\mathbf{\Phi}_{k+1}\Delta\mathbf{y}^{\mathrm{T}}_{k+1|k}] - \mathbf{m}_{\mathbf{\Phi}}E[\Delta\mathbf{y}^{\mathrm{T}}_{k+1|k}]\right) \cdot \{E[\Delta\mathbf{y}_{k+1|k}\Delta\mathbf{y}^{\mathrm{T}}_{k+1|k}]\}^{-1} \cdot \Delta\mathbf{y}_{k+1|k} + \mathbf{m}_{\mathbf{\Phi}} \qquad ,\forall k \\ &\quad = E[\mathbf{\Phi}_{k+1}\Delta\mathbf{y}^{\mathrm{T}}_{k+1|k}] \cdot \{E[\Delta\mathbf{y}_{k+1|k}\Delta\mathbf{y}^{\mathrm{T}}_{k+1|k}]\}^{-1} \cdot \Delta\mathbf{y}_{k+1|k} + \mathbf{m}_{\mathbf{\Phi}} \end{aligned} \tag{A15}$$

where $\mathbf{m}_{\mathbf{\Phi}}$ and $\mathbf{m}_{\Delta\mathbf{y}}$ are the means of $\mathbf{\Phi}_{k+1}$ and $\Delta\mathbf{y}_{k+1|k}$, respectively; and $\mathbf{m}_{\Delta\mathbf{y}} = \mathbf{0}$. Let the Kalman gain $\mathbf{G}_{k+1}$ be defined as

$$\mathbf{G}_{k+1} \equiv E[\mathbf{\Phi}_{k+1}\Delta\mathbf{y}^{\mathrm{T}}_{k+1|k}] \cdot \{E[\Delta\mathbf{y}_{k+1|k}\Delta\mathbf{y}^{\mathrm{T}}_{k+1|k}]\}^{-1} \quad , \ \forall k \tag{A16}$$

Using Eqs. (A15) and (A16), Eq. (A14) can, then, be rewritten as

$$\hat{\mathbf{\Phi}}_{k+1} = \hat{\mathbf{\Phi}}_{k+1|k} + \mathbf{G}_{k+1} \cdot \Delta\mathbf{y}_{k+1|k} + \mathbf{m}_{\mathbf{\Phi}} \qquad , \ \forall k \tag{A17}$$

Thus, Corollary 5.3 is proven.

A.6. Proof of Corollary 5.4

Given Corollary 5.3 and Eq. (19), we have $\Delta\mathbf{y}_{k+1|k}$ given by

$$\begin{aligned}\Delta\mathbf{y}_{k+1|k} &= \mathbf{Y}_{k+1} - \hat{\mathbf{Y}}_{k+1|k} \\ &= \mathbf{H}_{k+1} + \boldsymbol{\varepsilon}_{k+1} - \hat{\mathbf{H}}_{k+1|k} \qquad , \quad \forall k \\ &= \hat{\mathbf{H}}'_{k+1}\Delta\boldsymbol{\Phi}_{k+1|k} + \boldsymbol{\varepsilon}_{k+1}\end{aligned} \tag{A18}$$

Using Eq. (A18), we can have $E[\Delta\mathbf{y}_{k+1|k}\Delta\mathbf{y}^{\mathrm{T}}_{k+1|k}]$ (shown in Eq. (A16)) given by

$$\begin{aligned}E[\Delta\mathbf{y}_{k+1|k}\Delta\mathbf{y}^{\mathrm{T}}_{k+1|k}] &= E\left[\left(\hat{\mathbf{H}}'_{k+1}\Delta\boldsymbol{\Phi}_{k+1|k} + \boldsymbol{\varepsilon}_{k+1}\right)\left(\hat{\mathbf{H}}'_{k+1}\Delta\boldsymbol{\Phi}_{k+1|k} + \boldsymbol{\varepsilon}_{k+1}\right)^{\mathrm{T}}\right] \qquad ,\forall k \\ &= \hat{\mathbf{H}}'_{k+1}E[\Delta\boldsymbol{\Phi}_{k+1|k}\Delta\boldsymbol{\Phi}^{\mathrm{T}}_{k+1|k}]\hat{\mathbf{H}}'^{\mathrm{T}}_{k+1|k} + \hat{\mathbf{H}}'_{k+1}E[\Delta\boldsymbol{\Phi}_{k+1|k}\boldsymbol{\varepsilon}^{\mathrm{T}}_{k+1}] + E[\boldsymbol{\varepsilon}_{k+1}\Delta\boldsymbol{\Phi}^{\mathrm{T}}_{k+1|k}]\hat{\mathbf{H}}'^{\mathrm{T}}_{k+1} + E[\boldsymbol{\varepsilon}_{k+1}\boldsymbol{\varepsilon}^{\mathrm{T}}_{k+1}]\end{aligned} \tag{A19}$$

Since $\Delta\boldsymbol{\Phi}_{k+1|k}$ and $\boldsymbol{\varepsilon}_{k+1}$ are uncorrelated, $E[\Delta\boldsymbol{\Phi}_{k+1|k}\boldsymbol{\varepsilon}^{\mathrm{T}}_{k+1}] = E[\boldsymbol{\varepsilon}_{k+1}\Delta\boldsymbol{\Phi}^{\mathrm{T}}_{k+1|k}] = \mathbf{0}$ in Eq. (A19). Using the definition of the prior prediction ($\hat{\mathbf{P}}_{k+1|k}$) for the covariance matrix of state estimation error, we have $\hat{\mathbf{P}}_{k+1|k}$ given by

$$\hat{\mathbf{P}}_{k+1|k} \equiv E[\Delta\boldsymbol{\Phi}_{k+1|k}\Delta\boldsymbol{\Phi}^{\mathrm{T}}_{k+1|k}] \qquad , \quad \forall k \tag{A20}$$

Then, Eq. (A19) can be rewritten as

$$E[\Delta\mathbf{y}_{k+1|k}\Delta\mathbf{y}^{\mathrm{T}}_{k+1|k}] = \hat{\mathbf{H}}'_{k+1}\hat{\mathbf{P}}_{k+1|k}\hat{\mathbf{H}}'^{\mathrm{T}}_{k+1} + \boldsymbol{\Lambda}_{k+1} \quad , \quad \forall k \tag{A21}$$

where $\boldsymbol{\Lambda}_{k+1} \equiv E[\boldsymbol{\varepsilon}_k\boldsymbol{\varepsilon}_k{}^{\mathrm{T}}]$.

Similarly, utilizing the definition of the state prediction error, we have $E[\boldsymbol{\Phi}_{k+1}\Delta\mathbf{y}_{k+1|k}]$ (shown in Eq. (A16)) given by

$$\begin{aligned}E[\boldsymbol{\Phi}_{k+1}\Delta\mathbf{y}^{\mathrm{T}}_{k+1|k}] &= E[(\hat{\boldsymbol{\Phi}}_{k+1|k} + \Delta\boldsymbol{\Phi}_{k+1|k})\Delta\mathbf{y}^{\mathrm{T}}_{k+1|k}] \qquad , \quad \forall k \\ &= E[\hat{\boldsymbol{\Phi}}_{k+1|k}\Delta\mathbf{y}^{\mathrm{T}}_{k+1|k}] + E[\Delta\boldsymbol{\Phi}_{k+1|k}\Delta\mathbf{y}^{\mathrm{T}}_{k+1|k}]\end{aligned} \tag{A22}$$

Since $\hat{\boldsymbol{\Phi}}_{k+1|k}$ and $\Delta\mathbf{y}^{\mathrm{T}}_{k+1|k}$ are uncorrelated with each other; and moreover, $\hat{\boldsymbol{\Phi}}_{k+1|k}$ and $\Delta\boldsymbol{\Phi}_{k+1|k}$ are orthogonal, we have $E[\hat{\boldsymbol{\Phi}}_{k+1|k}\Delta\mathbf{y}^{\mathrm{T}}_{k+1|k}] = \mathbf{0}$. Accordingly, Eq. (A22) can be rewritten as

$$E[\boldsymbol{\Phi}_{k+1}\Delta\mathbf{y}^{\mathrm{T}}_{k+1|k}] = E[\Delta\boldsymbol{\Phi}_{k+1|k}\Delta\mathbf{y}^{\mathrm{T}}_{k+1|k}] \qquad , \quad \forall k \tag{A23}$$

Input Eq. (A18) into (A23), we have

$$\begin{aligned} E[\mathbf{\Phi}_{k+1}\Delta\mathbf{y}^{\mathrm{T}}_{k+1|k}] &= E[\Delta\mathbf{\Phi}_{k+1|k}\Delta\mathbf{y}^{\mathrm{T}}_{k+1|k}] \\ &= E[\Delta\mathbf{\Phi}_{k+1|k}(\hat{\mathbf{H}}'_{k+1}\Delta\mathbf{\Phi}_{k+1|k}+\boldsymbol{\varepsilon}_{k+1})^{\mathrm{T}}] \quad , \ \forall k \\ &= E[(\Delta\mathbf{\Phi}_{k+1|k}\Delta\mathbf{\Phi}^{\mathrm{T}}_{k+1|k}\hat{\mathbf{H}}'^{\mathrm{T}}_{k+1})+\Delta\mathbf{\Phi}_{k+1|k}\boldsymbol{\varepsilon}^{\mathrm{T}}_{k+1}] \end{aligned} \tag{A24}$$

But $\Delta\mathbf{\Phi}_{k+1|k}$ and $\boldsymbol{\varepsilon}^{\mathrm{T}}_{k+1}$ are uncorrelated, indicating $E[\Delta\mathbf{\Phi}_{k+1|k}\boldsymbol{\varepsilon}^{\mathrm{T}}_{k+1}]=\mathbf{0}$, and thus, Eq. (A24) can be rewritten as

$$E[\mathbf{\Phi}_{k+1}\Delta\mathbf{y}^{\mathrm{T}}_{k+1|k}] = E[(\Delta\mathbf{\Phi}_{k+1|k}\Delta\mathbf{\Phi}^{\mathrm{T}}_{k+1|k}\mathbf{H}'^{\mathrm{T}}_{k+1})] = \hat{\mathbf{P}}_{k+1|k}\hat{\mathbf{H}}'^{\mathrm{T}}_{k+1} \quad , \ \forall k \tag{A25}$$

By Eqs. (A16), (A21), and (A25), we, therefore, have the Kalman gain $\mathbf{G}_{k+1}$ given by

$$\mathbf{G}_{k+1} = \hat{\mathbf{P}}_{k+1|k}\hat{\mathbf{H}}'^{\mathrm{T}}_{k+1}(\hat{\mathbf{H}}'_{k+1}\hat{\mathbf{P}}_{k+1|k}\hat{\mathbf{H}}'^{\mathrm{T}}_{k+1}+\mathbf{\Lambda}_{k+1})^{-1} \quad , \ \forall k \tag{A25}$$

where $\mathbf{\Lambda}_{k+1} \equiv E[\boldsymbol{\varepsilon}_k\boldsymbol{\varepsilon}_k{}^{\mathrm{T}}]$. Thus, Corollary 5.4 is proven.

A.7. Proof of Corollary 5.5

First, let the state estimation error vector $\Delta\mathbf{\Phi}_{k+1|k+1}$ be defined as $\Delta\mathbf{\Phi}_{k+1|k+1} \equiv \mathbf{\Phi}_{k+1} - \hat{\mathbf{\Phi}}_{k+1}$.

Using Eq. (A17), $\Delta\mathbf{\Phi}_{k+1|k+1}$, then, becomes

$$\begin{aligned} \Delta\mathbf{\Phi}_{k+1|k+1} &\equiv \mathbf{\Phi}_{k+1} - \hat{\mathbf{\Phi}}_{k+1} \\ &= \mathbf{\Phi}_{k+1} - (\hat{\mathbf{\Phi}}_{k+1|k} + \mathbf{G}_{k+1}\cdot\Delta\mathbf{y}_{k+1|k} + \mathbf{m}_{\mathbf{\Phi}}) \\ &= \Delta\mathbf{\Phi}_{k+1|k} - \mathbf{G}_{k+1}\Delta\mathbf{y}_{k+1|k} - \mathbf{m}_{\mathbf{\Phi}} \\ &= \Delta\mathbf{\Phi}_{k+1|k} - \mathbf{G}_{k+1}(\hat{\mathbf{H}}'_{k+1}\Delta\mathbf{\Phi}_{k+1|k}+\boldsymbol{\varepsilon}_{k+1}) - \mathbf{m}_{\mathbf{\Phi}} \quad \text{(by Eq.(A18))} \qquad , \ \forall k \\ &= \Delta\mathbf{\Phi}_{k+1|k} - \mathbf{G}_{k+1}\hat{\mathbf{H}}'_{k+1}\Delta\mathbf{\Phi}_{k+1|k} - \mathbf{G}_{k+1}\boldsymbol{\varepsilon}_{k+1} - \mathbf{m}_{\mathbf{\Phi}} \\ &= (\mathbf{I} - \mathbf{G}_{k+1}\hat{\mathbf{H}}'_{k+1})\Delta\mathbf{\Phi}_{k+1|k} - \mathbf{G}_{k+1}\boldsymbol{\varepsilon}_{k+1} - \mathbf{m}_{\mathbf{\Phi}} \end{aligned} \tag{A26}$$

Therefore, using Eq. (A26) we have $\hat{\mathbf{P}}_{k+1|k+1}$ given by

$$
\begin{aligned}
\hat{\mathbf{P}}_{k+1|k+1} &\equiv E[\Delta\mathbf{\Phi}_{k+1|k+1}\Delta\mathbf{\Phi}_{k+1|k+1}^{\mathrm{T}}] \\
&= E\left[\left((\mathbf{I}-\mathbf{G}_{k+1}\hat{\mathbf{H}}'_{k+1})\Delta\mathbf{\Phi}_{k+1|k}-\mathbf{G}_{k+1}\boldsymbol{\varepsilon}_{k+1}-\mathbf{m}_{\mathbf{\Phi}}\right)\left((\mathbf{I}-\mathbf{G}_{k+1}\hat{\mathbf{H}}'_{k+1})\Delta\mathbf{\Phi}_{k+1|k}-\mathbf{G}_{k+1}\boldsymbol{\varepsilon}_{k+1}-\mathbf{m}_{\mathbf{\Phi}}\right)^{\mathrm{T}}\right] \\
&= (\mathbf{I}-\mathbf{G}_{k+1}\hat{\mathbf{H}}'_{k+1})E[\Delta\mathbf{\Phi}_{k+1|k}\Delta\mathbf{\Phi}_{k+1|k}^{\mathrm{T}}](\mathbf{I}-\mathbf{G}_{k+1}\hat{\mathbf{H}}'_{k+1})^{\mathrm{T}}-(\mathbf{I}-\mathbf{G}_{k+1}\hat{\mathbf{H}}'_{k+1})E[\Delta\mathbf{\Phi}_{k+1|k}\boldsymbol{\varepsilon}_{k+1}^{\mathrm{T}}]\mathbf{G}_{k+1}^{\mathrm{T}} \\
&\quad -(\mathbf{I}-\mathbf{G}_{k+1}\hat{\mathbf{H}}'_{k+1})E[\Delta\mathbf{\Phi}_{k+1|k}]\mathbf{m}_{\mathbf{\Phi}}^{\mathrm{T}} \\
&\quad -\mathbf{G}_{k+1}E[\boldsymbol{\varepsilon}_{k+1}\Delta\mathbf{\Phi}_{k+1|k}^{\mathrm{T}}](\mathbf{I}-\mathbf{G}_{k+1}\hat{\mathbf{H}}'_{k+1})^{\mathrm{T}}+\mathbf{G}_{k+1}E[\boldsymbol{\varepsilon}_{k+1}\boldsymbol{\varepsilon}_{k+1}^{\mathrm{T}}]\mathbf{G}_{k+1}^{\mathrm{T}}+\mathbf{G}_{k+1}E[\boldsymbol{\varepsilon}_{k+1}]\mathbf{m}_{\mathbf{\Phi}}^{\mathrm{T}} \\
&\quad -\mathbf{m}_{\mathbf{\Phi}}E[\Delta\mathbf{\Phi}_{k+1|k}^{\mathrm{T}}](\mathbf{I}-\mathbf{G}_{k+1}\hat{\mathbf{H}}'_{k+1})^{\mathrm{T}}+\mathbf{m}_{\mathbf{\Phi}}E[\boldsymbol{\varepsilon}_{k+1}]\mathbf{G}_{k+1}^{\mathrm{T}}+\mathbf{m}_{\mathbf{\Phi}}\mathbf{m}_{\mathbf{\Phi}}^{\mathrm{T}} \qquad ,\forall k
\end{aligned} \tag{A27}
$$

Since $\Delta\mathbf{\Phi}_{k+1|k}$ and $\boldsymbol{\varepsilon}_{k+1}$ are uncorrelated; and moreover, $E[\Delta\mathbf{\Phi}_{k+1|k}]=E[\boldsymbol{\varepsilon}_{k+1}]=\mathbf{0}$, Eq. (A27) can be rewritten as

$$
\begin{aligned}
\hat{\mathbf{P}}_{k+1|k+1} &= (\mathbf{I}-\mathbf{G}_{k+1}\hat{\mathbf{H}}'_{k+1})E[\Delta\mathbf{\Phi}_{k+1|k}\Delta\mathbf{\Phi}_{k+1|k}^{\mathrm{T}}](\mathbf{I}-\mathbf{G}_{k+1}\hat{\mathbf{H}}'_{k+1})^{\mathrm{T}}+\mathbf{G}_{k+1}E[\boldsymbol{\varepsilon}_{k+1}\boldsymbol{\varepsilon}_{k+1}^{\mathrm{T}}]\mathbf{G}_{k+1}^{\mathrm{T}}+\mathbf{m}_{\mathbf{\Phi}}\mathbf{m}_{\mathbf{\Phi}}^{\mathrm{T}} \\
&= (\mathbf{I}-\mathbf{G}_{k+1}\hat{\mathbf{H}}'_{k+1})\hat{\mathbf{P}}_{k+1|k}(\mathbf{I}-\mathbf{G}_{k+1}\hat{\mathbf{H}}'_{k+1})^{\mathrm{T}}+\mathbf{G}_{k+1}\mathbf{\Lambda}_{k+1}\mathbf{G}_{k+1}^{\mathrm{T}}+\mathbf{m}_{\mathbf{\Phi}}\mathbf{m}_{\mathbf{\Phi}}^{\mathrm{T}} \qquad , \ \forall k \\
&= (\mathbf{I}-\mathbf{G}_{k+1}\hat{\mathbf{H}}'_{k+1})\hat{\mathbf{P}}_{k+1|k}-\hat{\mathbf{P}}_{k+1|k}\hat{\mathbf{H}}'^{\mathrm{T}}_{k+1}\mathbf{G}_{k+1}^{\mathrm{T}}+\mathbf{G}_{k+1}(\hat{\mathbf{H}}'_{k+1}\hat{\mathbf{P}}_{k+1|k}\hat{\mathbf{H}}'^{\mathrm{T}}_{k+1}+\mathbf{\Lambda}_{k+1})\mathbf{G}_{k+1}^{\mathrm{T}}+\mathbf{m}_{\mathbf{\Phi}}\mathbf{m}_{\mathbf{\Phi}}^{\mathrm{T}}
\end{aligned} \tag{A28}
$$

But from Eq. (A25), we have $\mathbf{G}_{k+1}=\hat{\mathbf{P}}_{k+1|k}\hat{\mathbf{H}}'^{\mathrm{T}}_{k+1}(\hat{\mathbf{H}}'_{k+1}\hat{\mathbf{P}}_{k+1|k}\hat{\mathbf{H}}'^{\mathrm{T}}_{k+1}+\mathbf{\Lambda}_{k+1})^{-1}$. Let Eq. (A25) be inputted into Eq. (A28), and then, we have

$$
\hat{\mathbf{P}}_{k+1|k+1}=(\mathbf{I}-\mathbf{G}_{k+1}\hat{\mathbf{H}}'_{k+1})\hat{\mathbf{P}}_{k+1|k}+\mathbf{m}_{\mathbf{\Phi}}\mathbf{m}_{\mathbf{\Phi}}^{\mathrm{T}} \qquad , \ \forall k \tag{A29}
$$

Thus, Corollary 5.5 is proven.