# PARTIAL COMPACTIFICATION OF STABILITY MANIFOLDS BY MASSLESS SEMISTABLE OBJECTS 

NATHAN BROOMHEAD, DAVID PAUKSZTELLO, DAVID PLOOG, JON WOOLF

Dedicated to Mike Prest on the occasion of his seventieth birthday.


#### Abstract

We introduce two partial compactifications of the space of Bridgeland stability conditions of a triangulated category. First we consider lax stability conditions where semistable objects are allowed to have mass zero but still have a phase. The subcategory of massless objects is thick and there is an induced classical stability on the quotient category. We study deformations of lax stability conditions. Second we consider the space arising by identifying lax stability conditions which are deformation-equivalent with fixed charge. This second space is stratified by stability spaces of Verdier quotients of the triangulated category by thick subcategories of massless objects. We illustrate our results through examples in which the Grothendieck group has rank 2. For these, our partial compactification can be explicitly described and related to the wall-and-chamber structure of the stability space.


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Ginzburg algebra $C=D^{b}\left(\Gamma_{2} A_{2}\right)$


Bound path algebra $C=D^{b}\left(\Lambda_{1,2,0}\right)$

Examples of quotient stability spaces $\operatorname{Stab}^{Q}(\mathrm{C})^{*}$ up to $\mathbb{C}$ action; see Figure 2.

[^0]
## Glossary

## Slicings.

Slice(C), the set of locally finite slicings on C; page 7
$P \in \operatorname{Slice}(\mathrm{C})$ is adapted to thick $\mathrm{N} \subset \mathrm{C}$ if $P$ restricts to N and if $P(I) \cap \mathrm{N} \subset P(I)$ are Serre subcategories for all $I=[\varphi, \varphi+1)$ and $I=(\varphi, \varphi+1]$; page 10
$P \in \operatorname{Slice}(\mathrm{C})$ is well-adapted to thick $\mathrm{N} \subset \mathrm{C}$ if it is adapted and the quotient slicing $P_{\mathrm{C} / \mathrm{N}}$ is locally finite; page 11

## Charges.

$v: K(\mathrm{C}) \rightarrow \Lambda$, a surjective homomorphism onto a lattice with fixed inner product; page 8
$\operatorname{Hom}\left(\Lambda / \Lambda_{\mathrm{N}}, \mathbb{C}\right) \subset \operatorname{Hom}(\Lambda, \mathbb{C})$, the set of charge maps $\Lambda \rightarrow \mathbb{C}$ vanishing on $\Lambda_{N}$
$\operatorname{Hom}\left(\Lambda_{N}, \mathbb{C}\right) \subset \operatorname{Hom}(\Lambda, \mathbb{C})$, charges on $\Lambda_{N}$, subset via the inner product; page 8

## The spaces.

$\operatorname{Stab}(\mathrm{C}) \subseteq \operatorname{Slice}(\mathrm{C}) \times \operatorname{Hom}(\Lambda, \mathrm{C})$, the set of stability conditions (i.e. supported pre-stability conditions) on C whose charge map factors as $Z: K(\mathrm{C}) \xrightarrow{v} \Lambda \rightarrow \mathbb{C}$; page 15
$\mathcal{Z}: \operatorname{Stab}(\mathrm{C}) \rightarrow \operatorname{Hom}(\Lambda, \mathbb{C})$, the charge map (also for larger spaces); pages 16 and 21
$\operatorname{Stab}^{l}(\mathrm{C}) \subseteq \operatorname{Slice}(\mathrm{C}) \times \operatorname{Hom}(\Lambda, \mathbb{C})$, the set of lax stability conditions (semistable objects can have mass 0 , with the support condition); page 21
$\operatorname{Stab}^{L}(\mathrm{C})=\operatorname{Stab}^{l}(\mathrm{C}) \cap \overline{\operatorname{Stab}(\mathrm{C})}$, adding the closure condition; page 21
$\operatorname{Stab}^{Q}(\mathrm{C})=\operatorname{Stab}^{L}(\mathrm{C}) / \sim$, the space of quotient stability conditions, where two lax stability conditions are equivalent if they have the same charge and lie in the same connected component of the corresponding fibre of $\operatorname{Stab}^{L}(\mathrm{C}) \rightarrow \operatorname{Hom}(\Lambda, \mathbb{C})$; page 39

$\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})=\left\{\sigma \in \operatorname{Stab}^{L}(\mathrm{C}) \mid \mathrm{N}_{\sigma}=\mathrm{N}\right\}$, the subset with massless subcategory N ; page 21 $\operatorname{Stab}^{L S}(\mathrm{C}, \mathrm{N})=\left\{\sigma \in \overline{\operatorname{Stab}(\mathrm{C})} \mid \mathrm{N}_{\sigma}=\mathrm{N}, \mu_{\mathrm{N}}(\sigma) \in \operatorname{Stab}(\mathrm{C} / \mathrm{N})\right\} ;$ page 21
Open subsets.
$B_{\varepsilon}(\sigma)=\left\{(Q, W): d(P, Q)<\varepsilon\right.$ and $\left.\|W-Z\|_{\sigma}<\sin (\pi \varepsilon)\right\} ;$ page 22
$V_{\varepsilon}(\sigma)=\left\{(Q, W) \in B_{\varepsilon}(\sigma):\left\|W_{\mathrm{N}}\right\|_{\sigma}<\sin (\pi \varepsilon)\right\}$ for $\sigma \in \operatorname{Stab}^{L}(\mathrm{~N}) \mathrm{C}$; page 36
Open neighbourhoods: $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N}) \subseteq \mathrm{U}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{N}) \subseteq \mathrm{V}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{N}) \subseteq \mathrm{B}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{N}) \subseteq \operatorname{Stab}^{L}(\mathrm{C})$ :
$\mathrm{B}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{N})=\bigcup_{\sigma \in \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})} B_{\varepsilon}(\sigma) \cap \operatorname{Stab}^{L}(\mathrm{C})$; page 27
$\mathrm{V}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{N})=\bigcup_{\sigma \in \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})} V_{\varepsilon}(\sigma) \cap \operatorname{Stab}^{L}(\mathrm{C})$; page 36
$\mathrm{U}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{N})=\left\{\tau \in \mathrm{V}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{N}): \tau \in B_{\varepsilon}\left(\Phi_{\mathrm{N}}(\tau)\right)\right\} ;$ page 37

## The maps.

$\mu_{\mathrm{N}}: \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N}) \rightarrow \operatorname{Stab}(\mathrm{C} / \mathrm{N})$, the map sending a lax stability condition $(P, Z)$ with massless subcategory N to the massive stability condition $\mu_{\mathrm{N}}(P, Z)=\left(P_{\mathrm{C} / \mathrm{N}}, Z\right)$ on the quotient.
It extends to a continuous map $\mu_{\mathrm{N}}: \overline{\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})} \rightarrow \operatorname{Stab}^{L}(\mathrm{C} / \mathrm{N})$; page 24
$\rho_{\mathrm{N}}: \mathrm{B}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{N}) \rightarrow \operatorname{Stab}^{L}(\mathrm{~N})$, the restriction map $\rho_{\mathrm{N}}(P, Z)=\left(P_{\mathrm{N}}, Z_{\mathrm{N}}\right)=\left(P \cap \mathrm{~N},\left.Z\right|_{\Lambda_{\mathrm{N}}}\right)$; page 23 $\Phi_{\mathrm{N}}: \mathrm{V}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{N}) \rightarrow \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$, a deformation retraction; page 36

Support propagation. Support propagates from a component $\Sigma \subset \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ (page 32) if

$$
\underset{\varepsilon>0}{\exists} \underset{\sigma \in \Sigma}{\forall}\left\{\tau=(Q, W) \in B_{\varepsilon}(\sigma) \left\lvert\, \begin{array}{l}
\left.\rho_{\mathrm{N}}(\tau) \in \operatorname{Stab}^{L}(\mathrm{~N})\right\} \text { and } \\
\left\|Z-\left(W-W_{\mathrm{N}}\right)\right\|_{\sigma}<\sin (\pi \varepsilon)
\end{array}\right.\right\} \subset \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{~N}) .
$$

## 1. Introduction

The space of stability conditions on a non-zero triangulated category is always non-compact when non-empty: the mass of an object may tend to zero or infinity, and the phases of objects may tend to infinity. We construct a partial compactification in which we add boundary strata where the masses of objects in certain thick subcategories vanish. The points of these boundary strata can be interpreted as stability conditions on quotient categories.

We have in mind applications to the study of the topology of stability spaces and of their wall-and-chamber structure, and also to the construction of new stability conditions. Firstly, the partial compactification is always contractible, and so maybe a useful stepping stone in establishing the conjectured contractibility of stability spaces. Secondly, under a suitable technical condition, a neighbourhood of each boundary stratum has a simple product structure. This provides new information about the boundary of the stability space. Thirdly, there is a close connection between boundary strata and walls which we hope will prove useful in understanding the wall-and-chamber structure. Roughly, the stratum where a stable object's mass vanishes is the end point of the walls where that object is a destabilising subobject or quotient. And fourthly, the key technical ingredient in the local model is a deformation result for stability conditions with massless objects. Under suitable conditions, this allows one to construct stability conditions from ones on a thick subcategory and on the quotient by it. This is reminiscent of the tilting process by which stability conditions are constructed on complex algebraic surfaces and 3 -folds by perturbing the charge of a 'very-weak stability condition'. The key extra data required to perform this tilting is a Bogomolov-Gieseker type inequality. We require a stronger condition because we impose the extra requirement that the deformation is continuous in the slicing metric.

Description of results. Let C be a triangulated category and $v: K(\mathrm{C}) \rightarrow \Lambda$ a surjective homomorphism from its Grothendieck group onto a finite rank lattice. Write Slice(C) for the space of locally finite slicings of $C$. The space of stability conditions $\mathrm{Stab}(\mathrm{C})$ is the subspace of Slice $(\mathrm{C}) \times \operatorname{Hom}(\Lambda, \mathbb{C})$ consisting of pairs $(P, Z)$ with $Z(c) \in \mathbb{R}_{>0} e^{i \pi \varphi}$ whenever $0 \neq c \in P(\varphi)$ is semistable of phase $\varphi$, and satisfying the support property $\inf M_{(P, Z)}>0$ where

$$
M_{(P, Z)}=\left\{\frac{|Z(c)|}{\|v(c)\|}: 0 \neq c \in \mathrm{C} \text { stable }\right\}
$$

is the (normalised) mass distribution. The support property is usually stated in terms of semistable objects, but it is equivalent to consider only stable objects and this turns out to be crucial when one allows semistable objects with zero mass. For simplicity in the introduction we assume that all stability spaces are connected.

A pair $(P, Z)$ in the boundary of $\operatorname{Stab}(\mathrm{C})$ in $\operatorname{Slice}(\mathrm{C}) \times \operatorname{Hom}(\Lambda, \mathbb{C})$ satisfies the conditions
(1) $Z(c) \in \mathbb{R}_{\geq 0} e^{i \pi \varphi}$ whenever $0 \neq c \in P(\varphi)$ and
(2) $\inf M_{(P, Z)}=0$.

We call such a pair $(P, Z)$ a lax pre-stability condition and refer to $c \in P(\varphi)$ with $Z(c)=0$ as a massless semistable object. We say the pair is a lax stability condition if it satisfies the modified support property $\inf \left(M_{(P, Z)}-\{0\}\right)>0$, i.e. if zero is an isolated point of the normalised mass distribution. Intuitively, this forces a separation between massive and massless objects which leads to a 'de-coupling' of the massive and massless parts of the theory. This manifests geometrically in local product descriptions near such points in the boundary. The space of lax stability conditions $\operatorname{Stab}^{L}(\mathrm{C})$ is the subset of points in the closure of $\mathrm{Stab}(\mathrm{C})$ satisfying this modified support property.

An example may help to understand the restrictions we impose on boundary points - see also $\S 12.5$. The stability space $\operatorname{Stab}(X)$ of a strictly positive genus smooth complex projective curve $X$ is isomorphic to $\mathbb{H} \times \mathbb{C}$, where $\mathbb{H}$ is the strict upper half-plane in $\mathbb{C}$. The boundary points $\mathbb{Q} \times \mathbb{C}$ where the masses of line bundles vanish do not appear in $\operatorname{Stab}^{L}(X)$ because the slicings do not converge as we approach them. Nor do the points $(\mathbb{R}-\mathbb{Q}) \times \mathbb{C}$ appear because no
objects become massless at these points, and therefore since $\inf M_{(P, Z)}=0$ we must also have $\inf \left(M_{(P, Z)}-\{0\}\right)=0$. This example shows that any partial compactification of the stability space including points where our support property fails is likely to have rather complicated local geometry. It would be pleasant to have a partial compactification including the points $\mathbb{Q} \times \mathbb{C}$, but the topology on it would have to allow for the slicings to vary discontinuously. The techniques we use rely on the convergence of slicings so prohibit consideration of such boundary points. In contrast, in the genus zero case $X=\mathbb{P}^{1}$ there is one boundary stratum in $\operatorname{Stab}^{L}(X)$ corresponding to the vanishing mass of each line bundle $\mathcal{O}(n)$ for $n \in \mathbb{Z}$.

Returning to the general situation, the full subcategory N of massless objects in a lax stability condition $(P, Z)$ is thick, and there is an induced stability condition on the quotient $\mathrm{C} / \mathrm{N}$ with the same charge $Z$ and for which the semistable objects of phase $\varphi$ are those in the isomorphism closure of $P(\varphi)$ in $\mathrm{C} / \mathrm{N}$. More precisely, the induced stability condition has charge $Z$ considered as an element of the subspace $\operatorname{Hom}\left(\Lambda / \Lambda_{N}, \mathbb{C}\right) \subset \operatorname{Hom}(\Lambda, \mathbb{C})$ where $\Lambda_{N}$ is the saturation of the subgroup $\{v(c): c \in \mathrm{~N}\}$ of $\Lambda$. It satisfies the support property with respect to the homomorphism $K(\mathrm{C} / \mathrm{N}) \rightarrow \Lambda / \Lambda_{\mathrm{N}}$ induced from $v$.

When $\Lambda_{N}$ has rank one, $\operatorname{Stab}^{L}(\mathrm{C})$ contains a real-codimension one boundary stratum homeomorphic to $\operatorname{Stab}(\mathrm{C} / \mathrm{N}) \times \mathbb{R}$. Up to shift, the massless objects in N are all semistable with common phase, and this phase is recorded by the factor $\mathbb{R}$. There is a local product description at each point of this boundary stratum. The normal factor is homeomorphic to

$$
\operatorname{Stab}^{L}(\mathbb{N}) \cong \operatorname{Stab}(\mathbb{N}) \cup \operatorname{Stab}^{L}(\mathbb{N}, \mathrm{~N}) \cong \mathbb{C} \cup(-\infty+i \mathbb{R})
$$

where $\operatorname{Stab}^{L}(\mathrm{~N}, \mathrm{~N})$ is the unique boundary stratum in $\operatorname{Stab}^{L}(\mathrm{~N})$ at which all objects in N are massless. In summary, $\operatorname{Stab}(\mathrm{C})$ is the interior of a manifold with boundary each of whose boundary components is homeomorphic to $\operatorname{Stab}(\mathrm{C} / \mathrm{N}) \times \mathbb{R}$ for a some thick subcategory N of C .

Under additional assumptions this picture generalises to higher rank $\Lambda_{\mathrm{N}}$, allowing us to describe higher codimension strata in the boundary of Stab(C). Namely if 'support propagates' along a boundary stratum where objects in N are massless then a neighbourhood of that stratum is homeomorphic to a neighbourhood of

$$
\operatorname{Stab}(\mathrm{C} / \mathrm{N}) \times \operatorname{Stab}^{L}(\mathrm{~N}, \mathrm{~N}) \subset \operatorname{Stab}(\mathrm{C} / \mathrm{N}) \times \operatorname{Stab}^{L}(\mathrm{~N})
$$

This 'support propagation' condition is satisfied by points in the boundary of any 'finite type' component, but we do not know whether it holds more generally.
The fibres of the charge map $\operatorname{Stab}^{L}(\mathrm{C}) \rightarrow \operatorname{Hom}(\Lambda, \mathbb{C}):(P, Z) \mapsto Z$ are not discrete, because the phases of massless objects may vary whilst the charge remains constant. The massless subcategory N and the induced stability condition on $\operatorname{Stab}(\mathrm{C} / \mathrm{N})$ are locally constant on the fibres. We define the space of quotient stability conditions $\operatorname{Stab}^{Q}(\mathrm{C})$ to be the topological quotient space of $\operatorname{Stab}^{L}$ (C) by the equivalence relation identifying points in the same component of a fibre. Each point of $\operatorname{Stab}^{Q}(\mathrm{C})$ can be interpreted as a stability condition on the quotient $\mathrm{C} / \mathrm{N}$ by the massless subcategory N . The space of quotient stability conditions contains $\operatorname{Stab}(\mathrm{C})$ as a subspace, and has a complex codimension one boundary stratum $\operatorname{Stab}(\mathrm{C} / \mathrm{N})$ for each boundary component $\operatorname{Stab}(\mathrm{C} / \mathrm{N}) \times \mathbb{R}$ in $\operatorname{Stab}^{L}(\mathrm{C})$ arising from a rank one massless subcategory N .

When support propagates from all boundary strata in $\operatorname{Stab}^{L}(\mathrm{C})$ we can say more. The space of quotient stability conditions decomposes as a union

$$
\operatorname{Stab}^{Q}(\mathrm{C})=\bigcup_{i \in I} \operatorname{Stab}\left(\mathrm{C} / \mathrm{N}_{i}\right)
$$

of stability spaces of quotients of $C$. The mass of each object $c \in C$ extends to a continuous function $m_{\bullet}(c): \operatorname{Stab}^{Q}(\mathrm{C}) \rightarrow \mathbb{R}_{\geq 0}$ and $\operatorname{Stab}(\mathrm{C} / \mathrm{N})$ is the subset where this mass vanishes if and only if $c \in \mathrm{~N}$. It follows that each stratum is locally closed with closure

$$
\overline{\operatorname{Stab}\left(\mathrm{C} / \mathrm{N}_{i}\right)}=\bigcup_{i \leq j} \operatorname{Stab}\left(\mathrm{C} / \mathrm{N}_{j}\right)
$$

where the indexing set $I$ is partially-ordered by the inclusion of thick subcategories. (We do not have a general characterisation of which thick subcategories $\mathrm{N}_{i}$ appear.) Moreover,
under the strong propagation assumption, the space of quotient stability conditions has a local product structure: an open neighbourhood of $\operatorname{Stab}(C / N)$ in $\operatorname{Stab}^{Q}(\mathrm{C})$ is homeomorphic to a neighbourhood of the 'central' fibre of the second projection

$$
\operatorname{Stab}(\mathrm{C} / \mathbf{N}) \times \operatorname{Stab}^{Q}(\mathrm{~N}) \rightarrow \operatorname{Stab}^{Q}(\mathrm{~N})
$$

that is the fibre over the point stratum of $\operatorname{Stab}^{Q}(\mathrm{~N})$ where all objects of N are massless. Intuitively, the massive and massless parts of the theory 'de-couple' and can be treated independently of each other.

The charge extends to a continuous map $\operatorname{Stab}^{Q}(\mathrm{C}) \rightarrow \operatorname{Hom}(\Lambda, C)$ given on a stratum by

$$
\operatorname{Stab}(C / N) \rightarrow \operatorname{Hom}\left(\Lambda / \Lambda_{N}, \mathbb{C}\right) \hookrightarrow \operatorname{Hom}(\Lambda, \mathbb{C})
$$

i.e. the composite of the charge map for the stability space of the quotient and the natural inclusion. The fibres are discrete and the restriction to each stratum is a local homeomorphism. This allows us to view $\operatorname{Stab}^{Q}(\mathrm{C})$ as a 'stratified branched cover' of $\operatorname{Hom}(\Lambda, \mathbb{C})$. However, care is required since some fibres may be empty, corresponding to the existence of 'forbidden' massless subcategories. Moreover, some points may be infinitely ramified. For example, when $\Lambda_{N}$ has rank one $\operatorname{Stab}^{Q}(\mathrm{~N}) \cong \mathbb{C} \cup\{-\infty\}$ with charge map the extension of the exponential $\mathbb{C} \rightarrow \mathbb{C}^{*}$ by $\exp (-\infty)=0$ to create a branched cover infinitely ramified over the origin. Indeed, the local product description shows that this is the typical situation along a codimension one stratum.

There are several other recent constructions of (partial) compactifications of stability spaces. Bolognese [5] constructs a metric completion, at least when the charge map is a covering map, which seems to be closely related to $\operatorname{Stab}^{Q}(\mathrm{C})$. Its points can also be interpreted as stability conditions on quotients of $C$ by thick subcategories of massless objects. The difference is that she uses a notion of 'limiting support' for Cauchy sequences of stability conditions, and it is not immediately obvious how this relates to the notion of support we use to define lax stability conditions - see §13.1.

Bapat, Deopurkar and Licata [1] take a very different approach. They consider, by analogy with the Thurston compactification of Teichmüller space, the closure of the image of

$$
\operatorname{Stab}(\mathrm{C}) / \mathbb{C} \rightarrow \mathbb{R P}^{\mathrm{C}}: \sigma \mapsto\left[m_{\sigma}(c): c \in \mathrm{C}\right]
$$

and conjecture that under 'mild conditions' on $C$ this is a (real) manifold with boundary whose interior is $\operatorname{Stab}(\mathrm{C}) / \mathbb{C}$. The above map extends continuously to $\operatorname{Stab}^{Q}(\mathrm{C})^{*} / \mathbb{C}$ allowing us to compare the two spaces, which we do in two simple examples in $\S 13.2$. Under appropriate conditions it seems reasonable to hope that this extension is a homeomorphism between the interiors, and has dense image in the boundary.

Discrepancies between classical and lax stability conditions. In many ways lax stability conditions behave much as classical stability conditions do. However, in some respects lax stability conditions have weaker categorical and analytical properties than classical stability conditions. We highlight the differences. Let $\sigma=(P, Z)$ be a pair consisting of a slicing $P$ on $C$ and a charge $Z \in \operatorname{Hom}(\Lambda, \mathbb{C})$.
(1) The definitional distinction is that the mass of a non-zero semistable object $c \in P(\varphi)$ is classically required to be positive but can be zero if $\sigma$ is lax.
(2) The support property for a stability condition $\sigma$ implies that the slicing $P$ is locally finite. By contrast, the support condition for a lax stability condition $\sigma$ does not imply local finiteness of the slicing (which we therefore impose as a separate condition).
(3) The slices $P(\varphi)$ are always abelian length categories if $\sigma$ is a classical stability condition. If $\sigma$ is a lax stability condition then we only know that $P(\varphi)$ is a quasi-abelian length category; in particular, we don't know whether the Jordan-Hölder property holds.
(4) A classical stability condition $\sigma$ induces a norm $\|-\|_{\sigma}$ on $\operatorname{Hom}(\Lambda, \mathbb{C})$ whereas a lax stability condition induces a semi-norm.
(5) If $\sigma$ is a classical stability condition then any sufficiently close element of Slice(C) $\times$ $\operatorname{Hom}(\Lambda, \mathbb{C})$ is again a stability condition. By contrast, we do not know if elements near

|  |  | Stability conditions | lax stability conditions |
| :--- | :--- | :---: | :---: |
| $(1)$ | Masses are | positive; | non-negative. |
| $(2)$ | Slicings are locally-finite | automatically; | by additional condition. |
| $(3)$ | Slices $P(\varphi)$ are | abelian categories; | quasi-abelian categories. |
| $(4)$ | $\\|-\\|_{\sigma}$ is a | norm; | semi-norm. |
| $(5)$ | Small deformations are | stability conditions; | lax pre-stability conditions. |
| $(6)$ | Geometric structure: | complex manifold; | stratified space. |

Table 1. Summary of discrepancies between ordinary and lax stability conditions
a lax stability condition have the support property; in general they may only be lax pre-stability conditions.
(6) Finally, the space of classical stability conditions $\mathrm{Stab}(\mathrm{C})$ forms a complex manifold modelled on $\operatorname{Hom}(\Lambda, \mathbb{C})$. The space of lax stability conditions $\operatorname{Stab}^{L}(\mathrm{C})$ is a stratified space under good conditions.

Structure of the article. Section 2 fixes notation. Section 3 discusses the relationship between slicings on $C$ and on a thick subcategory $N$ and the quotient $C / N$. This is a key ingredient of the deformation results in Section 6. In Section 4 we define lax stability conditions and prove that the massless subcategory of a lax stability condition is thick and that there is a naturally induced classical stability condition on the quotient.

Section 5 contains the first results on the space of lax stability conditions, including the continuity of masses and phases and the 'semi-continuity' of massless subcategories. Section 6 is the technical heart of the paper. In Section 6.1 we extend Bridgeland's result on lifting charge deformations to the context of classical stability conditions. In Section 6.2 we discuss the extent to which the support property 'propagates' when we deform a lax stability condition.

In Section 7 the deformation results are applied to describe the local topology of the space of lax stability conditions. In Section 8 we finally define the space $\operatorname{Stab}^{Q}(\mathrm{C})$ of quotient stability conditions. The results about the local structure of $\operatorname{Stab}^{L}(\mathrm{C})$ descend to corresponding results about the local structure of $\operatorname{Stab}^{Q}(\mathrm{C})$.

Section 9 examines support propagation in the simplest case in which the saturation $\Lambda_{N}$ of the image of $K(\mathrm{~N}) \rightarrow K(\mathrm{C})$ has rank one. In this case the massless subcategory N is generated by a set of stable objects with common phase. Support propagates from the corresponding boundary stratum in $\operatorname{Stab}^{L}(\mathrm{C})$ and the stratum is homeomorphic to $\operatorname{Stab}(\mathrm{C} / \mathrm{N}) \times \mathbb{R}$ and has real codimension one. The corresponding stratum in $\operatorname{Stab}^{Q}(\mathrm{C})$ has complex codimension one.

In Section 10 we consider the partial compactification of a 'finite type component' $\operatorname{Stab}(\mathrm{C})$. This case is much simpler than the general one because $\operatorname{Stab}^{L}(C)=\overline{\operatorname{Stab}(C)}$. We show that the massless subcategories occurring in a finite type component of $\mathrm{Stab}(\mathrm{C})$ are precisely those generated by subsets of simple objects in the heart of a stability condition in the component. We also show that support propagates from each boundary stratum so that the strong forms of our local structure results apply, in particular each stratum in $\operatorname{Stab}^{L}(\mathrm{C})$ and in $\operatorname{Stab}^{Q}(\mathrm{C})$ has a product neighbourhood.

The universal cover $G$ of $\mathrm{GL}_{2}^{+}(\mathbb{R})$ acts on $\operatorname{Stab}(\mathrm{C})$. In Section 11 we describe the closure of the $G$-orbit of a stability condition $\sigma$ in $\operatorname{Stab}^{Q}(\mathrm{C})$ in terms of the phase diagram of $\sigma$, i.e. the set of 'occupied' phases for which there is a non-zero semistable object. This is a key ingredient of Section 12 in which we illustrate our results in some simple two-dimensional examples. In each case we are able to identify $\operatorname{Stab}(\mathrm{C}) / \mathbb{C}$ holomorphically as either $\mathbb{C}$ or the the Poincare disk, and also identify its wall-and-chamber structure. The walls are given by smooth analytic curves with endpoints on the boundary where the destabilising subobject and quotient respectively become massless.

Finally, in Section 13 we compare our approach to those of Bolognese [5] and Bapat, Deopurkar and Licata [1].

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## 2. Notation and preliminaries

2.1. Quasi-abelian categories. It has been known since Bridgeland's original article [7] that quasi-abelian categories are important in the theory of stability conditions. In this text they appear even more prominently because slices of lax stability conditions are in general not abelian categories (as with stability conditions) but only quasi-abelian; see slicing $P_{t}$ in Example 3.14.

Recall, e.g. from [31], that a quasi-abelian category is an additive category with kernels and cokernels and such that the pullback of a strict epimorphism is a strict epimorphism, and the pushout of a strict monomorphism is a strict monomorphism. Here a strict morphism is one for which the canonical morphism from its coimage to its image is an isomorphism. A length quasi-abelian category is one which is both artinian and noetherian.
Example 2.1 ([10, Examples 3.5 and 6.9$])$. Let $\mathcal{E}$ be the full additive subcategory of finitedimensional $\mathbf{k}$-vector spaces generated by $\mathbf{k}^{2}$ and $\mathbf{k}^{3}$. In this example, any non-zero map $\mathbf{k}^{2} \rightarrow \mathbf{k}^{3}$ has kernel and cokernel 0 . Therefore the coimage $\mathbf{k}^{2}$ and image $\mathbf{k}^{3}$ are not isomorphic and the morphism is not strict. In particular, $\mathbf{k}^{2}$ and $\mathbf{k}^{3}$ are simple objects of $\mathcal{E}$, and $\mathcal{E}$ is a length quasi-abelian category. However, the Jordan-Hölder property fails: $\mathbf{k}^{6}=\mathbf{k}^{2} \oplus \mathbf{k}^{2} \oplus \mathbf{k}^{2}=\mathbf{k}^{3} \oplus \mathbf{k}^{3}$ has two Jordan-Hölder filtrations with non-isomorphic factors and of different lengths.
2.2. Slicings. Let C be a triangulated category with shift functor $c \mapsto c[1]$. A slicing $P$ on C is a collection of full additive subcategories $P(\varphi)$ for each $\varphi \in \mathbb{R}$ such that
(1) $P(\varphi+1)=P(\varphi)[1]$ for all $\varphi \in \mathbb{R}$;
(2) $\operatorname{Hom}_{c}\left(c, c^{\prime}\right)=0$ whenever $c \in P(\varphi)$ and $c^{\prime} \in P\left(\varphi^{\prime}\right)$ with $\varphi>\varphi^{\prime}$;
(3) each $0 \neq c \in \mathrm{C}$ admits a finite filtration i.e. a finite sequence of morphisms

with cones $a_{i} \in P\left(\varphi_{i}\right)$ where $\varphi_{1}>\varphi_{2}>\cdots>\varphi_{n}$.
The objects in $P(\varphi)$ are called semistable of phase $\varphi$, the filtration is the Harder-Narasimhan filtration (henceforth abbreviated to HN filtration) of $c$, and the objects $a_{i}$ are called the semistable factors of $c$. The filtration, in particular the semistable factors, are determined uniquely up to isomorphism when they exist. The maximal and minimal phases of $0 \neq c \in \mathrm{C}$ are $\varphi^{+}(c)=\varphi_{1}$ and $\varphi^{-}(c)=\varphi_{n}$, respectively.

For any slicing $P$ and interval $I \subset \mathbb{R}$ let $P(I)$ denote the full subcategory of C on those objects whose semistable factors with respect to the slicing have phases in $I$. When $I=(a, b]$ we omit the outer brackets and simply write write $P(a, b]$ and so on. The category $P(I)$ is quasi-abelian when $I$ has length strictly less than one. A stable object is a simple semistable object, that is a semistable object of some phase $\varphi$ with no proper strict subobjects in the quasi-abelian category $P(\varphi)$.

The slicing $P$ is locally finite if there is some $\varepsilon>0$ such that $P(\varphi-\varepsilon, \varphi+\varepsilon)$ is a length quasi-abelian category for each $\varphi \in \mathbb{R}$. In particular, for a locally finite slicing each slice $P(\varphi)$ is a quasi-abelian length category. It follows that each semistable object has a finite composition series whose factors are stable objects. However, we do not know in general that the set of these
stable factors, nor the multiplicity with which each occurs, are well-defined - see [14] for a discussion of when a quasi-abelian category satisfies the Jordan-Hölder Theorem. Let Slice(C) denote the space of locally finite slicings on C . This has a metric

$$
d(P, Q)=\sup _{0 \neq c \in \mathrm{C}} \max \left\{\left|\varphi_{P}^{-}(c)-\varphi_{Q}^{-}(c)\right|,\left|\varphi_{P}^{+}(c)-\varphi_{Q}^{+}(c)\right|\right\} .
$$

For any slicing $P$ and $\varphi \in \mathbb{R}$ the inclusions of $P(-\infty, \varphi)$ and $P(-\infty, \varphi]$ into C have respective left adjoints $H_{P}^{<\varphi}$ and $H_{P}^{\leq \varphi}$. Dually, the inclusions of $P(\varphi, \infty)$ and $P[\varphi, \infty)$ into C have respective right adjoints $H_{P}^{>\varphi}$ and $H_{P}^{>\varphi}$. We use the notation

$$
H_{P}^{(\varphi, \psi)}=H_{P}^{>\varphi} \circ H_{P}^{<\psi}=H_{P}^{<\psi} \circ H_{P}^{>\varphi},
$$

and similarly for semi-closed and closed intervals. We also use the shorthand $H_{P}^{\varphi}=H_{P}^{[\varphi, \varphi]}$. When $I$ is an interval of strict length one, that is $I$ is either $(\varphi, \varphi+1]$ or $[\varphi, \varphi+1)$ for some $\varphi \in \mathbb{R}$, the subcategory $P(I)$ is the heart of a bounded t-structure on C and $H_{P}^{I}: \mathrm{C} \rightarrow P(I)$ is the associated cohomological functor taking triangles in C to long exact sequences in $P(I)$.
Remark 2.2. The right adjoint to the inclusion $P(0, \infty) \hookrightarrow \mathrm{C}$ is the truncation below associated to the bounded t -structure on C with heart $P(0,1]$, i.e. it is the functor classically denoted $\tau^{\leq 0}$. This unfortunate clash of notation arises because the factors in a HNfiltration are ordered by decreasing phase. To avoid confusion we use the notation $H_{P}^{>0}$ instead.

For $c \in \mathrm{C}$ the $P$-semistable factors of $H_{P}^{(\varphi, \psi)}(c)$ are precisely the $P$-semistable factors of $c$ with phases in the interval $(\varphi, \psi)$, and $H_{P}^{\varphi}(c)$ is the $P$-semistable factor with phase $\varphi$, or zero if no such factor exists. If $Q$ is another slicing with $Q(-\infty, \varphi) \subset P(-\infty, \psi)$ then

$$
H_{Q}^{<\varphi}=H_{Q}^{<\varphi} H_{P}^{<\psi}
$$

and similarly for closed intervals, and the dual cases. We omit the subscript $P$ from the notation when the slicing is understood from the context.
2.3. Charges. Fix a finite rank lattice $\Lambda$ and a surjective homomorphism $v: K(\mathrm{C}) \rightarrow \Lambda$. For $Z \in \operatorname{Hom}(\Lambda, \mathbb{C})$ and $c \in \mathrm{C}$ we abuse notation by writing $Z(c)$ for $Z(v([c]))$.

Suppose N is a thick subcategory of C . The Verdier quotient $\mathrm{C} / \mathrm{N}$ is a triangulated category with the same objects as $C$ and a morphism $c^{\prime} \rightarrow c$ in $\mathrm{C} / \mathrm{N}$ given by a roof $c^{\prime} \leftarrow c^{\prime \prime} \rightarrow c$ where the cone of the first morphism $c^{\prime} \leftarrow c^{\prime \prime}$ is in N ; see, for example, [17, 20].

The homomorphism $K(\mathrm{~N}) \rightarrow K(\mathrm{C})$ induced from the inclusion $\mathrm{N} \rightarrow \mathrm{C}$ need not be injective, but its cokernel is $K(\mathrm{C} / \mathrm{N})$. Let $\Lambda_{\mathrm{N}} \subset \Lambda$ be the minimal primitive sublattice containing the image of $K(\mathrm{~N}) \rightarrow K(\mathrm{C}) \rightarrow \Lambda$, so that $\Lambda / \Lambda_{\mathrm{N}}$ is again a lattice. Let $v_{\mathrm{N}}: K(\mathrm{~N}) \rightarrow \Lambda_{\mathrm{N}}$ and $v_{\mathrm{C} / \mathrm{N}}: K(\mathrm{C} / \mathrm{N}) \rightarrow \Lambda / \Lambda_{\mathrm{N}}$ denote the induced homomorphisms. The map $v_{\mathrm{N}}$ may not be surjective but it always has finite index.

We also fix an inner product $\langle\cdot, \cdot\rangle$ on $\Lambda \otimes \mathbb{R}$ and denote the associated norm by $\|\cdot\|$. This norm restricts to a norm on $\Lambda_{N} \otimes \mathbb{R}$ and also induces a norm on the quotient

$$
\Lambda / \Lambda_{N} \otimes \mathbb{R} \cong(\Lambda \otimes \mathbb{R}) /\left(\Lambda_{N} \otimes \mathbb{R}\right)
$$

defined by $\left\|\lambda+\Lambda_{N} \otimes \mathbb{R}\right\|=\inf \left\{\|\lambda+\alpha\|: \alpha \in \Lambda_{N} \otimes \mathbb{R}\right\}$. Alternatively, this is given by identifying $\Lambda / \Lambda_{N} \otimes \mathbb{R}$ with the orthogonal complement of $\Lambda_{N} \otimes \mathbb{R}$ and taking the restriction of $\|\cdot\|$.

The orthogonal projection $\Lambda \otimes \mathbb{R} \rightarrow \Lambda_{N} \otimes \mathbb{R}$ induces a splitting $\operatorname{Hom}\left(\Lambda_{N}, \mathbb{C}\right) \hookrightarrow \operatorname{Hom}(\Lambda, \mathbb{C})$ and we use this to identify $\operatorname{Hom}\left(\Lambda_{\mathrm{N}}, \mathbb{C}\right)$ with its image in $\operatorname{Hom}(\Lambda, \mathbb{C})$.
2.4. Spaces of stability conditions. We work with stability conditions on C whose charges factor through $v: K(\mathrm{C}) \rightarrow \Lambda$ and satisfy the support condition. (See $\S 4.1$ for the definition.) We denote the space of these by $\operatorname{Stab}(\mathrm{C})$, leaving the lattice $\Lambda$ implicit.

Stability conditions on thick subcategories N of C , and on the quotients $\mathrm{C} / \mathrm{N}$ by these play a prominent role. The charges of these are always understood to factor through $v_{\mathrm{N}}$ and $v_{\mathrm{C} / \mathrm{N}}$ respectively. We denote the respective spaces of stability conditions by $\operatorname{Stab}(N)$ and $\operatorname{Stab}(\mathrm{C} / \mathrm{N})$, again omitting the lattices from the notation.

## 3. Restriction, Descent and glueing of Slicings

Let $\mathrm{N} \subset \mathrm{C}$ be a thick subcategory, and $\mathrm{C} / \mathrm{N}$ the quotient triangulated category. We investigate the relationship between slicings of C and slicings of N and $\mathrm{C} / \mathrm{N}$. In this section we do not assume that slicings are locally finite unless stated otherwise.

Definition 3.1. A slicing $P$ of C is compatible with a pair ( $P_{\mathrm{N}}, P_{\mathrm{C} / \mathrm{N}}$ ) of slicings of N and $\mathrm{C} / \mathrm{N}$ if there are inclusions of objects $P_{\mathrm{N}}(\varphi) \subset P(\varphi) \subset P_{\mathrm{C} / \mathrm{N}}(\varphi)$ for each $\varphi \in \mathbb{R}$.

Proposition 3.2. There is at most one pair ( $P_{\mathrm{N}}, P_{\mathrm{C} / \mathrm{N}}$ ) compatible with each slicing $P$. When it exists, $P_{\mathrm{N}}(\varphi)=P(\varphi) \cap \mathrm{N}$ and $P_{\mathrm{C} / \mathrm{N}}(\varphi)$ is the isomorphism closure of $P(\varphi)$ in $\mathrm{C} / \mathrm{N}$.

Conversely, there is at most one slicing $P$ compatible with each pair $\left(P_{\mathrm{N}}, P_{\mathrm{C} / \mathrm{N}}\right)$. When it exists $c \in P(\varphi)$ if and only if $c \in P_{\mathrm{C} / \mathrm{N}}(\varphi)$ and

$$
\begin{equation*}
\operatorname{Hom}_{\mathcal{C}}(b, c)=0=\operatorname{Hom}_{\mathcal{C}}(c, d) \tag{1}
\end{equation*}
$$

for all $b \in P_{\mathrm{N}}(\psi)$ with $\psi>\varphi$ and all $d \in P_{\mathrm{N}}\left(\psi^{\prime}\right)$ with $\psi^{\prime}<\varphi$.
Proof. Suppose the pair $\left(P_{\mathrm{N}}, P_{\mathrm{C} / \mathrm{N}}\right)$ is compatible with $P$. Then $P_{\mathrm{N}}(\varphi) \subset P(\varphi)$ so that the HNfiltration of any $c \in \mathrm{~N}$ with $P_{\mathrm{N}}$-semistable factors is also the HNfiltration with $P$-semistable factors. The uniqueness of HNfiltrations implies that $P(\varphi) \cap \mathrm{N} \subset P_{\mathrm{N}}(\varphi)$, hence that $P_{\mathrm{N}}(\varphi)=$ $P(\varphi) \cap \mathrm{N}$ for each $\varphi \in \mathbb{R}$.

Denote by $P(\varphi)_{\mathrm{C} / \mathrm{N}}$ the closure of $P(\varphi)$ in $\mathrm{C} / \mathrm{N}$ under isomorphisms. We claim $P_{\mathrm{C} / \mathrm{N}}(\varphi)=$ $P(\varphi)_{\mathrm{C} / \mathrm{N}}$. Since $P(\varphi) \subset P_{\mathrm{C} / \mathrm{N}}(\varphi)$ it is clear that $P(\varphi)_{\mathrm{C} / \mathrm{N}} \subset P_{\mathrm{C} / \mathrm{N}}(\varphi)$. Moreover, again by uniqueness, the HNfiltration of any $c \in \mathrm{C}$ with $P$-semistable factors descends to the HNfiltration of $c$ with $P_{\mathrm{C} / \mathrm{N}^{-}}$-semistable factors if we simply ignore any factors in N . Thus if $c \in P_{\mathrm{C} / \mathrm{N}}(\varphi)$ it has a HNfiltration in C with all factors in N except for one factor, say $c^{\prime}$, in $P(\varphi)$. Thus $c \cong c^{\prime}$ in $\mathrm{C} / \mathrm{N}$ and $P_{\mathrm{C} / \mathrm{N}}(\varphi) \subset P(\varphi)_{\mathrm{C} / \mathrm{N}}$ establishing the claim.

In the other direction, if $P$ is compatible with $\left(P_{\mathrm{N}}, P_{\mathrm{C} / \mathrm{N}}\right)$ then by definition $P(\varphi) \subset P_{\mathrm{C} / \mathrm{N}}(\varphi)$. We saw above that any $c \in P_{\mathrm{C} / \mathrm{N}}(\varphi)$ has a HNfiltration in C with all factors in N apart from a single factor in $P(\varphi)$. Therefore $H_{P}^{>\varphi}(c)$ and $H_{P}^{<\varphi}(c)$ are in N and so vanish precisely when (1) holds. Thus $c \in P(\varphi)$ if and only if $c \in P_{\mathrm{C} / \mathrm{N}}(\varphi)$ and (1) holds. This shows that $P$, when it exists, is uniquely determined by the pair ( $P_{\mathrm{N}}, P_{\mathrm{C} / \mathrm{N}}$ ).
Lemma 3.3. If $P$ is compatible with a pair $\left(P_{\mathrm{N}}, P_{\mathrm{C} / \mathrm{N}}\right)$ of locally finite slicings then $P$ is also locally finite.

Proof. Let $I \subset \mathbb{R}$ be an interval such that both $P_{\mathrm{N}}(I)$ and $P_{\mathrm{C} / \mathrm{N}}(I)$ are quasi-abelian length categories. Let $a_{0} \hookrightarrow a_{1} \hookrightarrow \cdots \hookrightarrow a$ be an increasing sequence of strict subobjects of $a$ in $P(I)$. This can be considered as an increasing sequence of strict subobjects in $P_{\mathrm{C} / \mathrm{N}}(I)$ via the quotient functor $\mathrm{C} \rightarrow \mathrm{C} / \mathrm{N}$ since the strict exact structures on $P_{\mathrm{N}}(I)$ and $P_{\mathrm{C} / \mathrm{N}}(I)$ are induced by the triangulated structures on C and $\mathrm{C} / \mathrm{N}$, respectively, and the quotient functor is exact. Since $P_{\mathrm{C} / \mathrm{N}}(I)$ is length this chain stabilises, i.e. there is some $n \in \mathbb{N}$ such that $a_{n} \cong a_{n+1} \cong \ldots \cong a$ in $P_{\mathrm{C} / \mathrm{N}}(I)$. Pushing the sequence of strict monomorphisms out along $a_{n+1} \rightarrow a_{n+1} / a_{n}$ we obtain a diagram

whose bottom row is an increasing sequence of strict subobjects of $a / a_{n}$ in $P_{\mathrm{N}}(I)$. Since the latter is length this bottom row also stabilises. It follows that the original sequence stabilises so that $P(I)$ is noetherian. The proof that it is artinian is dual.

In the next sections we discuss the more subtle question of when compatible slicings exist.
3.1. Restriction and descent. Let $\mathrm{N} \subset \mathrm{C}$ be a thick subcategory. We say that a slicing $P$ of C restricts to N if the HNfactors of each $c \in \mathrm{~N}$ lie in N . In that case the full subcategories $P(\varphi) \cap \mathrm{N}$ define a slicing $P_{\mathrm{N}}(\varphi)$ of N . The question of when $P$ descends to a compatible slicing of $\mathrm{C} / \mathrm{N}$ is more involved.
Definition 3.4. A slicing $P$ of C is adapted to a thick subcategory N if it restricts to N and $P(I) \cap \mathrm{N}$ is a Serre subcategory of $P(I)$ for each strict length one interval $I \subset \mathbb{R}$.
Lemma 3.5. Let $P$ be a slicing of $C$ adapted to the thick subcategory $N$. Fix $c \in P(\varphi)$ and $b \in \mathrm{C}$. Then the following two conditions are equivalent:
(1) $b \cong c$ in $\mathrm{C} / \mathrm{N}$
(2) the only semistable factor of $b$ not in N is a factor $b_{0} \in P(\varphi)$ with $b_{0} \cong c$ in $\mathrm{C} / \mathrm{N}$.

Proof. Clearly if $b$ has a unique semistable factor $b_{0} \in P(\varphi)$ not in N then $b \cong b_{0}$ in $\mathrm{C} / \mathrm{N}$, and thus if $b_{0} \cong c$ in $\mathrm{C} / \mathrm{N}$ then also $b \cong c$ in $\mathrm{C} / \mathrm{N}$.

For the other direction, it is enough to show that, given $c \in \mathrm{C}$ with $H^{<\varphi}(c)$ and $H^{>\varphi}(c)$ in N and a morphism in $\operatorname{Hom}_{\mathrm{C}}(c, b)$ or $\operatorname{Hom}_{\mathrm{C}}(b, c)$ with cone in N , then $H^{<\varphi}(b)$ and $H^{>\varphi}(b)$ are also in N . The cases are similar, and we only consider the first in which there is an exact triangle $d[-1] \rightarrow c \rightarrow b \rightarrow d$ with $d \in \mathrm{~N}$. Applying the cohomological functor $H^{(\varphi, \varphi+1]}$ yields a long exact sequence

$$
\cdots \rightarrow H^{[\varphi, \varphi+1]}(c) \rightarrow H^{(\varphi, \varphi+1]}(b) \rightarrow H^{(\varphi, \varphi+1]}(d) \rightarrow \cdots .
$$

The assumptions on $c$ and $d$ imply that the first and third terms are in $P(\varphi, \varphi+1] \cap \mathrm{N}$. Since this is a Serre subcategory of $P(\varphi, \varphi+1]$ so too is the middle term. For the same reason $H^{(\varphi+n, \varphi+n+1]}(b) \in \mathrm{N}$ for all $n \in \mathbb{N}$, which implies $H^{>\varphi}(b) \in \mathrm{N}$. To show that $H^{<\varphi}(b) \in \mathrm{N}$ one proceeds similarly using the cohomological functor $H^{[\varphi-1, \varphi)}$.
Proposition 3.6. Let $P$ be a slicing of $C$. Then the following conditions are equivalent:
(1) $P$ is adapted to $N$.
(2) There is a pair $\left(P_{\mathrm{N}}, P_{\mathrm{C} / \mathrm{N}}\right)$ of slicings of N and $\mathrm{C} / \mathrm{N}$ compatible with $P$.

Proof. If there is a compatible pair ( $P_{\mathrm{N}}, P_{\mathrm{C} / \mathrm{N}}$ ) then we saw above that $P_{\mathrm{N}}(\varphi)=P(\varphi) \cap \mathrm{N}$ so that $P$ restricts to N . Moreover, the quotient functor $\mathrm{C} \rightarrow \mathrm{C} / \mathrm{N}$ restricts to an exact functor $P(\varphi, \varphi+1] \rightarrow P_{\mathrm{C} / \mathrm{N}}(\varphi, \varphi+1]$ between abelian categories with $P(\varphi, \varphi+1] \cap \mathrm{N}$ as kernel. Hence the latter is a Serre subcategory for each $\varphi \in \mathbb{R}$. The argument showing $P[\varphi, \varphi+1) \cap \mathrm{N}$ is a Serre subcategory is similar. Thus $P$ is adapted to N .

Now suppose that $P$ is adapted to N . The subcategories $P_{\mathrm{N}}(\varphi):=P(\varphi) \cap \mathrm{N}$ define a slicing of N. We must show that $P$ also descends to a slicing of $\mathrm{C} / \mathrm{N}$. Proposition 3.2 shows that we must define $P_{\mathrm{C} / \mathrm{N}}(\varphi)$ to be the closure of $P(\varphi)$ under isomorphisms in $\mathrm{C} / \mathrm{N}$. By construction $P_{\mathrm{C} / \mathrm{N}}(\varphi)$ is a full additive subcategory of $\mathrm{C} / \mathrm{N}$ for each $\varphi \in \mathbb{R}$, satisfying $P_{\mathrm{C} / \mathrm{N}}(\varphi+1)=P_{\mathrm{C} / \mathrm{N}}(\varphi)[1]$. Moreover, ignoring any factors in N , the image in $\mathrm{C} / \mathrm{N}$ of the HNfiltration of $0 \neq c \in \mathrm{C}$ with respect to the slicing $P$ provides a finite filtration with factors in these subcategories and with strictly decreasing phases. Therefore to show that $P_{\mathrm{C} / \mathrm{N}}$ is a slicing we must show that there are no non-zero morphisms in $\mathrm{C} / \mathrm{N}$ from $c \in P_{\mathrm{C} / \mathrm{N}}(\varphi)$ to $c^{\prime} \in P_{\mathrm{C} / \mathrm{N}}\left(\varphi^{\prime}\right)$ when $\varphi>\varphi^{\prime}$.

It is enough to show that $\operatorname{Hom}_{\mathrm{C} / \mathrm{N}}\left(c, c^{\prime}\right)=0$ for $c \in P(\varphi)$ and $c^{\prime} \in P\left(\varphi^{\prime}\right)$. A morphism in $\operatorname{Hom}_{\mathrm{C} / \mathrm{N}}\left(c, c^{\prime}\right)$ is represented by a 'roof' $c \leftarrow b \rightarrow c^{\prime}$ in C where the cone on the left hand morphism is in N. By Lemma 3.5, $H^{>\varphi}(b)$ and $H^{<\varphi}(b)$ are in N and moreover, $\operatorname{Hom}_{\mathcal{C}}\left(H^{>\varphi}(b), c\right)=0=\operatorname{Hom}_{\mathcal{C}}\left(H^{>\varphi}(b), c^{\prime}\right)$ because $c \in P(\varphi)$ and $c^{\prime} \in P\left(\varphi^{\prime}\right)$ for $\varphi^{\prime}<\varphi$. Thus we can construct dashed arrows to form a commutative diagram

in which the morphisms $b \rightarrow H^{\leq \varphi}(b)$ and $H^{\varphi}(b) \rightarrow H^{\leq \varphi}(b)$ are the canonical ones. These are isomorphisms in $\mathrm{C} / \mathrm{N}$. Therefore the bottom 'roof' $c \leftarrow H^{\varphi}(b) \rightarrow c^{\prime}$ represents the same morphism on $\mathrm{C} / \mathrm{N}$. Since $\operatorname{Hom}_{\mathrm{C}}\left(H^{\varphi}(b), c^{\prime}\right)=0$ we deduce $\operatorname{Hom}_{\mathrm{C} / \mathrm{N}}\left(c, c^{\prime}\right)=0$ as required.
Corollary 3.7. Let $P$ and $Q$ be slicings on C that are adapted to N , and let $P_{\mathrm{N}}, Q_{\mathrm{N}} \in \operatorname{Slice}(\mathrm{N})$, $P_{\mathrm{C} / \mathrm{N}}, Q_{\mathrm{C} / \mathrm{N}} \in \operatorname{Slice}(\mathrm{C} / \mathrm{N})$ be the induced slicings. Then

$$
d\left(P_{\mathrm{N}}, Q_{\mathrm{N}}\right) \leq d(P, Q) \quad \text { and } \quad d\left(P_{\mathrm{C} / \mathrm{N}}, Q_{\mathrm{C} / \mathrm{N}}\right) \leq d(P, Q)
$$

Proof. The HNfiltrations of $c \in \mathrm{~N}$ with respect to $P$ and to $P_{\mathrm{N}}$ coincide. The first statement follows since the supremum defining $d\left(P_{\mathrm{N}}, Q_{\mathrm{N}}\right)$ is taken over a subset of that defining $d(P, Q)$. The second statement follows because

$$
\begin{aligned}
d(P, Q)<\varepsilon & \Longleftrightarrow P(\varphi) \subset Q(\varphi-\varepsilon, \varphi+\varepsilon) & & \forall \varphi \in \mathbb{R} \\
& \Longleftrightarrow P(\varphi) \subset Q_{\mathrm{C} / \mathrm{N}}(\varphi-\varepsilon, \varphi+\varepsilon) & & \forall \varphi \in \mathbb{R} \\
& \Longleftrightarrow P_{\mathrm{C} / \mathrm{N}}(\varphi) \subset Q_{\mathrm{C} / \mathrm{N}}(\varphi-\varepsilon, \varphi+\varepsilon) & & \forall \varphi \in \mathbb{R} \\
& \Longleftrightarrow d\left(P_{\mathrm{C} / \mathrm{N}}, Q_{\mathrm{C} / \mathrm{N}}\right)<\varepsilon & &
\end{aligned}
$$

where we have used the fact that $P_{\mathrm{C} / \mathrm{N}}(\varphi)$ is the isomorphism closure of $P(\varphi)$ in $\mathrm{C} / \mathrm{N}$, and similarly for $Q_{\mathrm{C} / \mathrm{N}}(\varphi)$.

If $P$ is a locally finite slicing adapted to a thick subcategory N then the restriction $P_{\mathrm{N}}$ is clearly locally finite. However, we do not know whether the slicing on the quotient $P_{\mathrm{C} / \mathrm{N}}$ is locally finite when $P$ is so. Therefore we introduce the following enhancement of Definition 3.4.

Definition 3.8. A locally finite slicing $P$ of C is well-adapted to a thick subcategory N if it is adapted to it and the quotient slicing $P_{\mathrm{C} / \mathrm{N}}$ is locally finite.
Corollary 3.9. Let $P$ be a slicing of $C$. Then the following conditions are equivalent:
(1) $P$ is locally finite and well-adapted to N .
(2) There is a pair $\left(P_{\mathrm{N}}, P_{\mathrm{C} / \mathrm{N}}\right)$ of locally finite slicings of N and $\mathrm{C} / \mathrm{N}$ compatible with $P$.

Proof. If $P$ is locally finite and well-adapted to N then the restricted slicing $P_{\mathrm{N}}$ and quotient slicing $P_{\mathrm{C} / \mathrm{N}}$ exist and are locally finite. The slicing $P$ is compatible with them.

Conversely, if $P$ is compatible with the pair ( $P_{\mathrm{N}}, P_{\mathrm{C} / \mathrm{N}}$ ) of locally finite slicings then $P$ is locally finite by Lemma 3.3, it is adapted to N by Proposition 3.6 and indeed is well-adapted since $P_{\mathrm{C} / \mathrm{N}}$ is locally finite by assumption.
3.2. Glueing. Let $\mathrm{N} \subset \mathrm{C}$ be a thick subcategory. In this subsection we establish a criterion for when slicings of N and $\mathrm{C} / \mathrm{N}$ can be glued to a compatible slicing on C . An important consequence is that the set of pairs of slicings which glue to a locally finite slicing is open.

Proposition 3.10. Let $\left(Q_{\mathrm{N}}, Q_{\mathrm{C} / \mathrm{N}}\right)$ be a pair of slicings of N and $\mathrm{C} / \mathrm{N}$. Then the following conditions are equivalent:
(1) There is a compatible locally finite slicing $Q$ of C .
(2) There is a locally finite slicing $P$ of C compatible with a pair ( $P_{\mathrm{N}}, P_{\mathrm{C} / \mathrm{N}}$ ) such that $d\left(P_{\mathrm{N}}, Q_{\mathrm{N}}\right)<\varepsilon$ and $d\left(P_{\mathrm{C} / \mathrm{N}}, Q_{\mathrm{C} / \mathrm{N}}\right)<\varepsilon$ where $\varepsilon>0$ is sufficiently small that the categories $P(\varphi-2 \varepsilon, \varphi+2 \varepsilon)$ are length for each $\varphi \in \mathbb{R}$.
One direction is trivial, if there is a compatible locally finite slicing $Q$ then we set $P=Q$ and are done. The proof of the other direction is rather long, so we break it down into a number of results. We retain the notation and assumptions of the statement throughout this section. By Proposition 3.2 there is at most one choice for the slicing: for $\varphi \in \mathbb{R}$ the full subcategory $Q(\varphi)$ must be defined by $c \in Q(\varphi) \Longleftrightarrow c \in Q_{\mathrm{C} / \mathrm{N}}(\varphi)$ and, for any $b \in Q_{\mathrm{N}}(\psi)$,
(i) $\psi>\varphi$ implies $\operatorname{Hom}_{\mathrm{C}}(b, c)=0$ and
(ii) $\psi<\varphi$ implies $\operatorname{Hom}_{\mathrm{C}}(c, b)=0$.

That is, $Q(\varphi)=Q_{\mathrm{C} / \mathrm{N}}(\varphi) \cap Q_{\mathrm{N}}(>\varphi)^{\perp} \cap{ }^{\perp} Q_{\mathrm{N}}(<\varphi)$. Clearly $Q(\varphi+1)=Q(\varphi)[1]$ for all $\varphi \in \mathbb{R}$. Moreover, $Q(\varphi) \cap \mathrm{N}=Q_{\mathrm{N}}(\varphi)$ and $Q(\varphi) \subset Q_{\mathrm{C} / \mathrm{N}}(\varphi)$. As usual, we extend the notation to intervals $I \subset \mathbb{R}$ by defining $Q(I)$ to be the extension-closure of the $Q(\varphi)$ for $\varphi \in I$. By definition $Q(I) \subset Q_{\mathrm{C} / \mathrm{N}}(I)$ for any interval $I$.
Lemma 3.11. For each $\varphi \in \mathbb{R}$ we have $Q(\varphi) \subset P(\varphi-\varepsilon, \varphi+\varepsilon)$.
Proof. Suppose $c \in Q(\varphi)$. Then as $Q(\varphi) \subset Q_{\mathrm{C} / \mathrm{N}}(\varphi) \subset P_{\mathrm{C} / \mathrm{N}}(\varphi-\varepsilon, \varphi+\varepsilon)$ we know that $H_{\bar{P}}^{>\varphi+\varepsilon}(c), H_{P}^{\leq \varphi-\varepsilon}(c) \in \mathrm{N}$. Thus $H_{P}^{>\varphi+\varepsilon}(c) \in Q_{\mathrm{N}}(>\varphi)$ and $H_{P}^{\leq \varphi-\varepsilon}(c) \in Q_{\mathrm{N}}(<\varphi)$ and so by the definition of $Q(\varphi)$ we have

$$
\operatorname{Hom}_{\mathcal{C}}\left(H_{P}^{\geq \varphi+\varepsilon}(c), c\right)=0=\operatorname{Hom}_{\mathcal{C}}\left(c, H_{\bar{P}}^{\leq \varphi-\varepsilon}(c)\right) .
$$

It follows that $H_{\bar{P}}^{>\varphi+\varepsilon}(c)=0=H_{\bar{P}}^{<\varphi-\varepsilon}(c)$ so that $c \in P(\varphi-\varepsilon, \varphi+\varepsilon)$ as claimed.
Lemma 3.12. If $c \in Q(\varphi)$ and $c^{\prime} \in Q\left(\varphi^{\prime}\right)$ with $\varphi>\varphi^{\prime}$ then $\operatorname{Hom}_{\mathcal{C}}\left(c, c^{\prime}\right)=0$.
Proof. Suppose $\gamma \in \operatorname{Hom}_{\mathcal{C}}\left(c, c^{\prime}\right)$. Since $c \in Q_{\mathrm{C} / \mathrm{N}}(\varphi)$ and $c^{\prime} \in Q_{\mathrm{C} / \mathrm{N}}\left(\varphi^{\prime}\right)$ the morphism $\gamma$ vanishes in $\mathrm{C} / \mathrm{N}$. Hence it must factor through some $b \in \mathrm{~N}$. We therefore have a diagram

in which the upper triangle commutes and the lower triangle is exact. Condition (ii) implies that $\operatorname{Hom}_{\mathrm{C}}\left(c, H^{<\varphi}(b)\right)=0$, and hence that there is a dashed morphism making the left hand triangle commute. Condition (i) and the condition $\varphi^{\prime}<\varphi$ imply that $\operatorname{Hom}_{\mathrm{C}}\left(H \geq \varphi(b), c^{\prime}\right)=0$. Hence $\gamma=0$ as claimed.

It remains to check that each $c \in \mathrm{C}$ has a HNfiltration with respect to $Q$. We do so by induction on the length of the HNfiltration of $c$ in $\mathrm{C} / \mathrm{N}$ with respect to $Q_{\mathrm{C} / \mathrm{N}}$. The next result provides the base case.

Lemma 3.13. Suppose $c \in Q_{\mathrm{C} / \mathrm{N}}(\varphi)$. Then $c$, considered as an object of C , has a HNfiltration with respect to $Q$ with all factors in N except for a single factor in $Q(\varphi)$.
Proof. Let $0<\varepsilon<\frac{1}{2}$ and $\mathrm{A}=P(\varphi-\varepsilon, \varphi+\varepsilon)$ and to begin with, assume $c \in \mathrm{~A}$.
We first show that $c \in Q(\varphi)$ if and only if $\operatorname{Hom}_{\mathcal{C}}(b, c)=0=\operatorname{Hom}_{\mathcal{C}}(c, d)$ for all $b \in \mathrm{~A} \cap Q_{\mathrm{N}}(>\varphi)$ and $d \in \mathrm{~A} \cap Q_{\mathrm{N}}(<\varphi)$. One direction is clear: when $c \in Q(\varphi)$ the vanishing conditions follow immediately from the definition of $Q$. For the other direction suppose $b \in Q_{\mathrm{N}}(>\varphi)$. There is an exact triangle

$$
H_{P}^{\geq \varphi+\varepsilon}(b) \rightarrow b \rightarrow H_{P}^{<\varphi+\varepsilon}(b) \rightarrow H_{P}^{\geq \varphi+\varepsilon}(b)[1] .
$$

Note that $\operatorname{Hom}_{\mathcal{C}}(H \geq \varphi+\varepsilon(b), c)=0$ because $c \in \mathrm{~A} \subset P(<\varphi+\varepsilon)$. Therefore any morphism from $b$ to $c$ factors through $h:=H_{P}^{<\varphi+\varepsilon}(b)$. Since $d\left(P_{\mathrm{N}}, Q_{\mathrm{N}}\right)<\varepsilon$ we have

$$
H_{P}^{\geq \varphi+\varepsilon}(b)[1] \in P_{\mathrm{N}}(\geq \varphi+\varepsilon+1) \subset Q_{\mathrm{N}}(\geq \varphi+1) \subset Q_{\mathrm{N}}(>\varphi) .
$$

Since $Q_{\mathrm{N}}(>\varphi)$ is extension-closed the above triangle shows that $h \in Q_{\mathrm{N}}(>\varphi)$. Moreover, $h \in \mathrm{~A}$ because $b \in Q_{\mathrm{N}}(>\varphi) \subset P_{\mathrm{N}}(>\varphi-\varepsilon)$. Therefore $\operatorname{Hom}_{\mathrm{C}}(h, c)=0$ by assumption, and hence $\operatorname{Hom}_{\mathrm{C}}(b, c)=0$ too. A dual argument shows that $\operatorname{Hom}_{\mathrm{C}}(c, d)=0$ for all $d \in Q_{\mathrm{N}}(<\varphi)$. Hence $c \in Q(\varphi)$ as claimed.

We now use this criterion to construct a HNfiltration for $c \in \mathrm{~A}$ with all factors in N except for a single factor in $Q(\varphi)$ isomorphic to $c$ in $\mathrm{C} / \mathrm{N}$. Let $b$ be a maximal strict subobject of $c$ in the subcategory $\mathrm{A} \cap Q_{\mathrm{N}}(>\varphi)$ of A . We can always find such a $b$ (possibly zero) because A is a
quasi-abelian length category. Then $c^{\prime}=c / b$ has no non-zero strict subobjects in $\mathrm{A} \cap Q_{\mathrm{N}}(>\varphi)$ because if $b^{\prime} \hookrightarrow c^{\prime}$ is such a strict subobject we can pullback to obtain a commutative diagram

whose rows are strict short exact sequences and whose vertical morphisms are strict monomorphisms. In particular $b^{\prime \prime}$ is a strict subobject of $c$ in $\mathrm{A} \cap Q_{\mathrm{N}}(>\varphi)$ and by maximality of $b$ we deduce that $b \cong b^{\prime \prime}$ and therefore that $b^{\prime}=0$. By assumption $b$ has a HNfiltration with respect to $Q_{\mathrm{N}}$. Using this we can construct a finite filtration of $c$ in A whose quotients are a sequence of $Q_{\mathrm{N}}$-semistable objects of strictly decreasing phase in $\mathrm{A} \cap Q_{\mathrm{N}}(>\varphi)$, except for the final quotient $c^{\prime}$ which has no non-zero strict subobjects in $\mathrm{A} \cap Q_{\mathrm{N}}(>\varphi)$.

A dual argument constructs a finite filtration of this final quotient $c^{\prime}$ whose first term $c^{\prime \prime}$ has no non-zero strict quotients in $\mathrm{A} \cap Q_{\mathrm{N}}(<\varphi)$ and whose other quotients form a sequence of $Q_{\mathrm{N}}$-semistable objects of strictly decreasing phase in $\mathrm{A} \cap Q_{\mathrm{N}}(<\varphi)$. It follows that $c^{\prime \prime}$ cannot have any non-zero strict subobjects in $\mathrm{A} \cap Q_{\mathrm{N}}(>\varphi)$ either, for any such would lift to a subobject of $c^{\prime}$. To summarise we have constructed a strict subquotient $c^{\prime \prime}$ of $c$ in A such that
(1) $c^{\prime \prime} \cong c$ in $\mathrm{C} / \mathrm{N}$, in particular $c^{\prime \prime} \in Q_{\mathrm{C} / \mathrm{N}}(\varphi)$;
(2) $c^{\prime \prime}$ has no non-zero strict subobjects in $\mathrm{A} \cap Q_{\mathrm{N}}(>\varphi)$;
(3) $c^{\prime \prime}$ has no non-zero strict quotients in $\mathrm{A} \cap Q_{\mathrm{N}}(<\varphi)$.

Recalling that the image of any morphism in the quasi-abelian category A is a strict subobject of the target, and dually that the coimage is a strict quotient of the source, we conclude that $\operatorname{Hom}_{\mathrm{C}}\left(b, c^{\prime \prime}\right)=0=\operatorname{Hom}_{\mathrm{C}}\left(c^{\prime \prime}, d\right)$ for all $b \in \mathrm{~A} \cap Q_{\mathrm{N}}(>\varphi)$ and $d \in \mathrm{~A} \cap Q_{\mathrm{N}}(<\varphi)$. Hence $c^{\prime \prime} \in Q(\varphi)$ and concatenating the filtrations of $b$ and of $c^{\prime}$ using iterated applications of the octahedral axiom yields the desired HNfiltration.

Now consider the general case, i.e. remove the assumption that $c \in A$. By the first part $c_{0}=H_{P}^{(\varphi-\varepsilon, \varphi+\varepsilon)}(c)$ has a HNfiltration with respect to $Q$. Noting $H_{P}^{>\varphi+\varepsilon} H_{P}^{>\varphi-\varepsilon}(c)=H_{P}^{>\varphi+\varepsilon}(c)$ and applying the octahedral axiom, there is a commutative diagram

whose rows and columns extend to exact triangles. Moreover $H_{\bar{P}}^{>\varphi+\varepsilon}(c) \in \mathrm{N}$ because

$$
c \in Q_{\mathrm{C} / \mathrm{N}}(\varphi) \subset P_{\mathrm{C} / \mathrm{N}}(\varphi-\varepsilon, \varphi+\varepsilon) .
$$

Indeed $H_{P}^{>\varphi+\varepsilon}(c) \in Q_{\mathrm{N}}(>\varphi)$ because $d\left(P_{\mathrm{N}}, Q_{\mathrm{N}}\right)<\varepsilon$. Considering the top row, and recalling that $H_{Q}^{>\varphi}\left(c_{0}\right) \in \mathrm{N}$ too, shows that $c_{1} \in Q_{\mathrm{N}}(>\varphi)$. Therefore, by considering the middle column, we can construct a HNfiltration for $c$ with $Q$-semistable factors by concatenating the filtrations of $c_{1}$ and of $H_{\bar{Q}}^{\leq \varphi}\left(c_{0}\right)$.

Proof of Proposition 3.10. Suppose $c \in \mathrm{C}$ has a HNfiltration of length $k \in \mathbb{N}$ in $\mathrm{C} / \mathrm{N}$ with $Q_{\mathrm{C} / \mathrm{N}^{-}}$ semistable factors. We show that $c$ has a HNfiltration in C with $Q$-semistable factors. If $k=0$ then $c \in \mathrm{~N}$ and we simply take the HNfiltration with respect to $Q_{\mathrm{N}}$. If $k=1$ then the result holds by Lemma 3.13. Therefore we assume that $k>1$ and that the result holds for any object with a strictly shorter HNfiltration in $\mathbf{C} / \mathrm{N}$. Choose a representative $b \in Q_{\mathrm{C} / \mathrm{N}}(\psi)$ for the highest phase factor of $c$ so that there is an exact triangle $b \rightarrow c \rightarrow d \rightarrow b[1]$ in C . By induction we
may assume both $b$ and $d$ have HNfiltrations with $Q$-semistable factors. In particular there is an exact triangle

$$
H_{\bar{Q}}^{\geq \psi}(b) \rightarrow b \rightarrow H_{Q}^{<\psi}(b) \rightarrow H_{\bar{Q}}^{\geq \psi}(b)[1]
$$

in C. Since $H_{Q}^{\geq \psi}(b) \in Q(\geq \psi) \subset Q_{\mathrm{C} / \mathrm{N}}(\geq \psi)$ and $H_{Q}^{<\psi}(b) \in Q(<\psi) \subset Q_{\mathrm{C} / \mathrm{N}}(<\psi)$ we deduce that $H_{Q}^{<\psi}(b) \in \mathrm{N}$ and that $H_{\bar{Q}}^{>\psi}(b) \rightarrow b$ is an isomorphism in $\mathrm{C} / \mathrm{N}$. Therefore we may assume $b \in Q(\geq \psi)$. Having done so we argue similarly with $d$. There is an exact triangle

$$
H_{\bar{Q}}^{\geq \psi}(d) \rightarrow d \rightarrow H_{Q}^{<\psi}(d) \rightarrow H_{\bar{Q}}^{\geq \psi}(d)[1]
$$

where now $H_{\bar{Q}}{ }^{\psi}(d) \in \mathrm{N}$ and $d \rightarrow H_{Q}^{<\psi}(d)$ is an isomorphism in $\mathrm{C} / \mathrm{N}$. Hence there is a commutative diagram

whose rows and columns extend to exact triangles. By considering the top row we see that $b^{\prime} \in Q(\geq \psi)$. Thus the middle column shows that we may assume, by judicious choice of representatives, that $b \in Q(\geq \psi)$ and $d \in Q(<\psi)$. Having done so, we obtain a $Q$ HNfiltration for $c$ by concatenating those of $b$ and $d$.

Clearly $d(P, Q)<\varepsilon$ because $Q(\varphi) \subset P(\varphi-\varepsilon, \varphi+\varepsilon)$ for all $\varphi \in \mathbb{R}$ by Lemma 3.11. Finally $Q$ is locally finite because $Q(\varphi-\varepsilon, \varphi+\varepsilon) \subset P(\varphi-2 \varepsilon, \varphi+2 \varepsilon)$ and the latter is length.
Example 3.14. Let $C=D^{b}\left(\mathbb{P}^{1}\right)$ be the bounded derived category of coherent sheaves on the projective line $\mathbb{P}^{1}$. All complexes in C decompose into direct sums of their cohomologies (this holds for any smooth curve) and, moreover, all coherent sheaves decompose into direct sums of line bundles $\mathcal{O}(n)$ and torsion sheaves; the latter have the skyscraper sheaves $\mathcal{O}_{x}$ for $x \in \mathbb{P}^{1}$ as their minimal non-zero subsheaves.

For a slicing in C , up to shift and direct sums, various $\mathcal{O}(n)$ and $\mathcal{O}_{x}$ occur as cones of HNfiltrations and, conversely, the decomposition properties of C imply that HNfiltrations exist trivially for any family of subcategories $P(\varphi)$ with $P(\varphi+1)=P(\varphi)[1]$ and Hom-vanishing $\operatorname{Hom}(P(>\varphi), P(\varphi))=0$ and such that all $\mathcal{O}_{x}$ and $\mathcal{O}(n)$ are in the heart $P(0,1]$, again up to shift. Thus the following assignments for $\varphi \in(0,1]$ give slicings on C :

$$
\begin{aligned}
P_{t}(\varphi) & =\left\{\begin{array}{ll}
\langle\mathcal{O}(n): n \in \mathbb{Z}\rangle & \varphi=\frac{1}{2} \\
\left\langle\mathcal{O}_{x}: x \in \mathbb{P}^{1}\right\rangle & \varphi=1 \\
0 & \text { else; }
\end{array}, \quad P_{g}(\varphi)= \begin{cases}\langle\mathcal{O}(n): n \in \mathbb{Z}\rangle & \varphi=\frac{1}{\pi} \arg (-n+i) \in(0,1) \\
\left\langle\mathcal{O}_{x}: x \in \mathbb{P}^{1}\right\rangle & \varphi=1 \\
0 & \text { else } ;\end{cases} \right. \\
P_{b}(\varphi) & = \begin{cases}\langle\mathcal{O}\rangle & \varphi=\frac{1}{2} \\
\left\langle\mathcal{O}_{x}, \mathcal{O}(n), \mathcal{O}(-n)[1]: x \in \mathbb{P}^{1}, n \in \mathbb{N}_{>0}\right\rangle & \varphi=1, \\
0 & \text { else; }\end{cases}
\end{aligned} \quad P_{c}(\varphi)= \begin{cases}\operatorname{coh}\left(\mathbb{P}^{1}\right) & \varphi=1 \\
0 & \text { else. }\end{cases}
$$

$P_{t}$ separates torsion sheaves and line bundles into two slices. $P_{g}$ is the geometric slicing induced by the classical slope of coherent sheaves $\mu(A)=\operatorname{deg}(A) / \mathrm{rk}(A)$; see Example 5.4. The slicing $P_{b}$ occurs in the boundary of the stability space; see Example 6.3 and Subsection 12.9.
These three slicings are locally finite. Note that the slice $P_{t}\left(\frac{1}{2}\right)$ contains the infinite chain $\cdots \rightarrow \mathcal{O}(-1) \rightarrow \mathcal{O} \rightarrow \mathcal{O}(1) \rightarrow \cdots$. Nonetheless $P_{t}$ is locally finite because any non-zero morphism $\mathcal{O}(n) \rightarrow \mathcal{O}(m)$ for $n<m$ is not strict in $P_{t}\left(\frac{1}{2}\right)$ since it has image $\mathcal{O}(n)$ and coimage $\mathcal{O}(m)$. In particular, each $\mathcal{O}(n)$ is simple in the quasi-abelian (but not abelian) category $P_{t}\left(\frac{1}{2}\right)$.

In $P_{c}$, a single slice contains the whole heart; it is not locally finite and used in Example 4.9. See [18] for the classification of stability conditions and bounded $t$-structures on C.

Example 3.15. We continue the above example by considering some thick subcategories of $\mathrm{C}=$ $\mathrm{D}^{b}\left(\mathbb{P}^{1}\right)$ : first the subcategory $\mathrm{N}=\mathrm{N}_{\mathcal{O}}=\operatorname{thick}(\mathcal{O})$ generated by the trivial line bundle. Because every line bundle is an exceptional object, $\mathrm{N} \cong \mathrm{D}^{b}(\mathbf{k})$ and N is an admissible subcategory, i.e. the inclusion $\mathrm{N} \hookrightarrow \mathrm{C}$ has both adjoints. In particular, there is a canonical equivalence $\mathrm{C} / \mathrm{N} \cong \mathrm{N}^{\perp} \cong \operatorname{thick}(\mathcal{O}(-1))$.

Next, for any point $x \in \mathbb{P}^{1}$, let $\mathrm{N}_{x}$ be the subcategory generated by the skyscraper sheaf $\mathcal{O}_{x}$. Moreover, let T be the subcategory of all torsion objects in C. Both subcategories are thick and neither is admissible. We have $\mathrm{N}_{x} \subsetneq \mathrm{~T}$ and $\mathrm{C} / \mathrm{T} \cong \mathrm{N}_{\mathcal{O}}$. The quotient $\mathrm{C} / \mathrm{N}_{x}$ is not Hom-finite: the objects $\mathcal{O}$ and $\mathcal{O}(-1)$ are isomorphic in the quotient but the morphisms $y: \mathcal{O}(-1) \rightarrow \mathcal{O}$ for $y \neq x$ induce non-zero elements of $\operatorname{End}_{\mathrm{C}^{/} \mathrm{N}_{x}}(\mathcal{O})$.

## 4. Lax stability conditions and quotient categories

4.1. Stability conditions. Let C be a triangulated category and $v: K(\mathrm{C}) \rightarrow \Lambda$ a surjective homomorphism from the Grothendieck group to a finite rank lattice. Let Slice(C) be the space of locally finite slicings. A pre-stability condition on C is a pair $(P, Z) \in \operatorname{Slice}(\mathrm{C}) \times \operatorname{Hom}(\Lambda, \mathbb{C})$ such that $0 \neq c \in P(\varphi)$ implies $Z(c)=m(c) \exp (i \pi \varphi)$ for some $m(c) \in \mathbb{R}_{>0}$. An object $c \in P(\varphi)$ is said to be semistable of phase $\varphi$ and $m(c)$ is its mass. The mass of any $c \in \mathrm{C}$ is defined to be

$$
m(c):=\sum_{i=1}^{k} m\left(c_{i}\right)
$$

where $c_{i} \in P\left(\varphi_{i}\right)$ for $i=1, \ldots, k$ are the semistable factors of $c$ with respect to the slicing $P$. The maximal and minimal phases of $0 \neq c \in \mathrm{C}$ are $\varphi^{+}(c)=\varphi_{1}$ and $\varphi^{-}(c)=\varphi_{k}$ respectively. A stability condition is a pre-stability condition $\sigma=(P, Z)$ for which there exists $K>0$ such that

$$
m(c)=|Z(v(c))| \geq \frac{1}{K}\|v(c)\|
$$

for all semistable $c \in \mathrm{C}$. This latter condition is referred to as the support property [27, §2.1]; it is independent of the choice of norm because $\operatorname{dim}(\Lambda \otimes \mathbb{R})<\infty$. The support property has three important consequences. First, and most obviously, it implies that the infimal mass

$$
\mu_{\sigma}=\inf \{m(c): 0 \neq c \in \mathrm{C}\} \geq \frac{1}{K} \inf \{\|\lambda\|: 0 \neq \lambda \in \Lambda\}
$$

is strictly positive. Second, it implies that the slicing $P$ is locally finite, in fact that $P(I)$ is a length category for any interval $I \subset \mathbb{R}$ of length $|I|<1$. This is because if $c \in P(I)$ has a composition series in $P(I)$ with $n$ non-zero factors then elementary trigonometry shows that

$$
|Z(c)| \geq n \mu_{\sigma} \cos \left(\frac{\pi}{2}|I|\right)
$$

so that the length $n$ of any composition series of $c$ in $P(I)$ is bounded above. Third, it implies that the generalised norm

$$
U \mapsto\|U\|_{\sigma}=\sup \left\{\frac{|U(c)|}{|Z(c)|}: 0 \neq c \in P(\varphi), \varphi \in \mathbb{R}\right\}
$$

defined in [6] is actually a norm on $\operatorname{Hom}(\Lambda, \mathbb{C})$ because $\|U\|_{\sigma} \leq K\|U\|$ where

$$
\|U\|=\sup \{|U(\lambda)|: \lambda \in \Lambda \otimes \mathbb{R},\|\lambda\|=1\}
$$

denotes the operator norm. In fact the support property is equivalent to $\|\cdot\|_{\sigma}$ being a norm, see [2, Appendix B].

The central result in the theory of stability conditions is the following deformation theorem.
Theorem 4.1 ( $[6$, Theorem 7.1 and Lemma 6.2$]$ ). Let $\sigma=(P, Z)$ be a pre-stability condition. Then for any $0<\varepsilon<1 / 8$ and $W \in \operatorname{Hom}(\Lambda, \mathbb{C})$ with $\|W-Z\|_{\sigma}<\sin (\pi \varepsilon)$ there is a unique pre-stability condition $\tau=(Q, W)$ with $d(P, Q)<\varepsilon$. Moreover, if $\sigma$ is a stability condition then so is $\tau$.

Let $\operatorname{Stab}(\mathrm{C})$ be the set of stability conditions and $\mathcal{Z}: \operatorname{Stab}(C) \rightarrow \operatorname{Hom}(\Lambda, C)$ the second projection. We refer to this as the charge map. The above deformation theorem shows that the charge map is a local homeomorphism, and therefore that $\operatorname{Stab}(\mathrm{C})$ can be given the structure of a, possibly empty, complex manifold of dimension $\mathrm{rk}(\Lambda)$.
4.2. Lax stability conditions. We start with a modified version of Bridgeland's notion of stability condition without the condition that the masses of non-zero objects have to be positive, and with a concomitantly modified support condition.

Definition 4.2 (Lax pre-stability condition). A lax pre-stability condition on C is a pair $(P, Z) \in$ Slice $(\mathbb{C}) \times \operatorname{Hom}(\Lambda, \mathbb{C})$ such that $0 \neq c \in P(\varphi)$ implies $Z(c)=m(c) \exp (i \pi \varphi)$ for some $m(c) \in$ $\mathbb{R}_{\geq 0}$.
As in the classical case we refer to $m(c)$ as the mass of a semistable object $c \in P(\varphi)$. The mass of any $c \in \mathrm{C}$ is again defined as $m(c):=m\left(c_{1}\right)+\cdots+m\left(c_{k}\right)$ where $c_{1} \in P\left(\varphi_{1}\right), \ldots, c_{k} \in P\left(\varphi_{k}\right)$ are the semistable factors of $c$ with respect to the slicing $P$. We define the maximal and minimal phases of $0 \neq c \in \mathrm{C}$ to be $\varphi^{+}(c)=\varphi_{1}$ and $\varphi^{-}(c)=\varphi_{k}$ as before.

Definition 4.3. An object $c \in \mathrm{C}$ is called massive if $m(c)>0$, and massless if $m(c)=0$. Note that $0 \in \mathrm{C}$ has no semistable factors, so that $m(0)=0$, i.e. 0 is always a massless object.

The massless subcategory N of a lax pre-stability condition $\sigma$ is the full subcategory on the massless objects. When $\mathrm{N}=0$, i.e. the mass of every non-zero semistable object is strictly positive, $\sigma$ is a pre-stability condition; sometimes for emphasis we say it is classical.
Proposition 4.4. The massless subcategory N of a lax pre-stability condition $\sigma=(P, Z)$ is a thick subcategory of C to which the slicing $P$ is adapted.

Proof. We start with the second claim. Clearly every semistable factor of a massless object is massless since the mass of an object is the sum of the masses of its semistable factors. Moreover, for any interval $I$ of the form $(\varphi, \varphi+1]$ or $[\varphi, \varphi+1)$ the full subcategory $P(I)$ is the heart of a t-structure, and hence abelian. The intersection $P(I) \cap \mathrm{N}$ is a Serre subcategory because $m(c)=0 \Longleftrightarrow Z(c)=0$ for $c \in P(I)$. Therefore $P$ is adapted to N .

It is clear that N is closed under shifts, so we need only show it is closed under extensions and direct summands. Suppose that $b \in \mathrm{C}$ sits in a triangle $a \rightarrow b \rightarrow c \rightarrow a[1]$ with $a, c \in \mathrm{~N}$. Taking cohomology with respect to the t-structure with heart $P(0,1]$ we obtain a long exact sequence $\cdots \rightarrow H^{i} a \rightarrow H^{i} b \rightarrow H^{i} c \rightarrow \cdots$ of objects of $P(0,1]$. By assumption $m(a)=0=m(c)$, so that $m\left(H^{i} a\right)=0=m\left(H^{i} c\right)$ for each $i \in \mathbb{Z}$ since the HNfiltration of an object $x$ of C is a refinement of the decomposition of $x$ into its cohomology with respect to the heart $P(0,1]$. It follows from the fact that $\mathrm{N} \cap P(0,1]$ is a Serre subcategory that $m\left(H^{i} b\right)=0$ too. Hence $m(b)=\sum_{i \in \mathbb{Z}} m\left(H^{i} b\right)=0$, and N is extension-closed.

Since the set of semistable factors of $a \oplus b$ is the union of the sets of semistable factors of $a$ and $b$, we obtain that N is thick from the following chain of equivalences:

$$
a \oplus b \in \mathrm{~N} \Longleftrightarrow m(a \oplus b)=0 \Longleftrightarrow m(a)+m(b)=0 \Longleftrightarrow m(a)=0=m(b) \Longleftrightarrow a, b \in \mathrm{~N} .
$$

Lemma 4.5. Suppose $\sigma=(P, Z)$ is a lax pre-stability condition. Then $\mathrm{N}=\operatorname{triang}(S)$ is the triangulated closure of the set $S$ of stable massless objects in the heart $P(0,1]$.

Proof. Evidently triang $(S) \subset \mathrm{N}$. To see the other inclusion consider the HNfiltration of a massless object $c \in \mathbf{N}$. There are finitely many semistable factors. Each factor is massless, and has a finite composition series with stable massless objects. Up to shift each of these stable objects has phase in $(0,1]$. Therefore $c \in \operatorname{triang}(S)$ and $\mathrm{N} \subset \operatorname{triang}(S)$.

Remark 4.6. When $\sigma=(P, Z)$ is a classical pre-stability condition the slices $P(\varphi)$ are abelian categories [6, Lemma 5.2]. However, this is not necessarily the case when $\sigma$ is lax since the argument relies on the charge of each semistable object being non-zero - see Example 4.17. Nevertheless, each slice $P(\varphi)$ is a quasi-abelian length category, and so each semistable object $c$ has at least one finite composition series with stable factors. Indeed, every composition series
of $c$ must have finite length, but the lengths need not be the same, and the multi-sets of stable factors need not be unique up to isomorphism.

Definition 4.7 (Lax stability condition). A lax stability condition is a lax pre-stablity condition $\sigma=(P, Z)$ for which there exists $K>0$ such that

$$
m(s)=|Z(v(s))| \geq \frac{1}{K}\|v(s)\|
$$

for all massive stable objects $s \in \mathrm{C}$. We refer to this as the support condition for a lax prestability condition.

When all non-zero objects are massive this support condition coincides with the usual one in §4.1. Since every massive stable object is semistable it is clear that the usual support condition implies the above one. In the other direction, for a semistable object $c$ the inequality $m(c) \geq\|v(c)\| / K$ follows by applying the above condition to the stable factors of $c$, each of which is massive.
It is clear that massless objects must be excluded from any analogue of the support property for lax pre-stability conditions. The reason it is important to consider only stable massive objects is to avoid support failing simply because there is a massive semistable object $b$ and a non-zero massless semistable object $c$ of the same phase. In that situation the sums $b \oplus c^{n}$ for $n \in \mathbb{N}$ are semistable with fixed mass $m\left(b \oplus c^{n}\right)=m(b)$ but with $\left\|v\left(b \oplus c^{n}\right)\right\| \rightarrow \infty$ as $n \rightarrow \infty$.

The support condition guarantees a 'mass gap' between the massive and massless objects of a lax stability condition.

Lemma 4.8. There is a uniform lower bound on the mass of massive objects of a lax stability condition.

Proof. Suppose $c \in C$ is massive. Then

$$
\begin{aligned}
m(c) & \geq \inf \{m(s): \text { massive stable } s \in P(\varphi), \varphi \in \mathbb{R}\} \\
& \geq \inf \left\{\frac{\|v(s)\|}{K}: \text { massive stable } s \in P(\varphi), \varphi \in \mathbb{R}\right\} \\
& \geq \frac{\inf \{\|\lambda\|: 0 \neq \lambda \in \Lambda\}}{K} .
\end{aligned}
$$

where the last term is strictly positive because $\Lambda$ has finite rank.
However, the presence of massless objects means that, unlike in the classical setting, the support condition does not imply that the slicing is locally finite.

Example 4.9. Let $\mathrm{C}=\mathrm{D}^{b}\left(\mathbb{P}^{1}\right)$ and $\sigma=(P, Z)$ with charge $Z=0$ and the slicing $P=P_{c}$ of Example 3.14, i.e. $P(1)=\operatorname{coh}\left(\mathbb{P}^{1}\right)$. Then $\sigma$ satisfies the support property since every non-zero object is massless. However, it is not locally finite since $P(1-\varepsilon, 1+\varepsilon)=P(1)$ is not length for any $0<\varepsilon<1 / 2$.
4.3. Semi-norms and support. A lax pre-stability condition $\sigma=(P, Z)$ defines a (generalised) semi-norm

$$
W \mapsto\|W\|_{\sigma}=\sup \left\{\frac{|W(s)|}{|Z(s)|}: \text { massive stable } s \in P(\varphi), \varphi \in \mathbb{R}\right\}
$$

on $\operatorname{Hom}(\Lambda, \mathbb{C})$. By convention we set $\sup (\varnothing)=0$ so that $\|W\|_{\sigma}=0$ for all $W \in \operatorname{Hom}(\Lambda, \mathbb{C})$ when $\sigma$ has no massive objects. The adjective 'generalised' refers to the fact that we allow $\|W\|_{\sigma}=\infty$ if the supremum does not exist, see Example 6.4 below. This is only a semi-norm because it is possible for a non-zero charge $W$ to vanish on all massive stable objects, so that $\|W\|_{\sigma}=0$. When $\sigma$ is classical this is the usual (generalised) norm because if $c$ is semistable then

$$
|W(c)| \leq \sum_{s \in S}|W(s)| \leq\|W\|_{\sigma} \sum_{s \in S}|Z(s)|=\|W\|_{\sigma}|Z(c)|
$$

where $S$ is the multi-set of stable factors of $c$, all of which are necessarily massive since $\sigma$ is classical.

Definition 4.10 (Full stability condition). A lax pre-stability condition $\sigma$ is full if the seminorm $\|\cdot\|_{\sigma}$ is bounded on the unit ball in $\operatorname{Hom}(\Lambda, \mathbb{C})$, i.e. if there exists $K>0$ such that

$$
\|W\|_{\sigma} \leq K\|W\|
$$

for all $W \in \operatorname{Hom}(\Lambda, \mathbb{C})$, where $\|W\|=\sup \{|W(\lambda)|: \lambda \in \Lambda \otimes \mathbb{R},\|\lambda\|=1\}$ is the operator norm. This is independent of the norm on $\Lambda \otimes \mathbb{R}$, and reduces to the usual notion [7] when $\sigma$ is classical.

The next result is a simple extension of [2, Proposition B.4] and [3, Lemma 11.4], following [27, §2.1], to the case of lax stability conditions.

Proposition 4.11. For a lax pre-stability condition $\sigma=(P, Z)$ with charge factoring through $v: K(\mathrm{C}) \rightarrow \Lambda$ the following are equivalent:
(1) $\sigma$ is a lax stability condition;
(2) $\sigma$ is full;
(3) there exists a quadratic form $\Delta$ on $\Lambda \otimes \mathbb{R}$ such that
(a) $\Delta(v(s)) \geq 0$ for each massive stable $s \in \mathrm{C}$;
(b) $\Delta$ is negative definite on $\operatorname{ker} Z \subset \Lambda \otimes \mathbb{R}$.

Proof. (1) $\Longrightarrow(2),(3)$. Suppose $\sigma$ satisfies the support property. Then for $W \in \operatorname{Hom}(\Lambda, \mathbb{C})$ we have

$$
\begin{aligned}
\|W\|_{\sigma} & =\sup \left\{\frac{|W(s)|}{|Z(s)|}: \text { massive stable } s \in P(\varphi), \varphi \in \mathbb{R}\right\} \\
& \leq K \sup \left\{\frac{|W(s)|}{\|v(s)\|}: \text { massive stable } s \in P(\varphi), \varphi \in \mathbb{R}\right\} \\
& \leq K \sup \left\{\frac{|W(\lambda)|}{\|\lambda\|}: 0 \neq \lambda \in \Lambda\right\}=K\|W\|
\end{aligned}
$$

so that $\sigma$ is full. Moreover, the quadratic form $\Delta(\lambda)=K^{2}|Z(\lambda)|^{2}-\|\lambda\|^{2}$ on $\Lambda \otimes \mathbb{R}$ satisfies the properties of the third condition in the statement.
$(2) \Longrightarrow(1)$. Now suppose $\sigma$ is full. Assume, for a contradiction, that $\sigma$ does not satisfy the support property. Then there is a sequence $\left(s_{n}\right)$ of massive stable objects for which

$$
m_{\sigma}\left(s_{n}\right)=\left|Z\left(s_{n}\right)\right|<\frac{\left\|v\left(s_{n}\right)\right\|}{n} .
$$

One can choose $W_{n} \in \operatorname{Hom}(\Lambda, \mathbb{C})$ with $\left\|W_{n}\right\|=1$ and $\left|W_{n}\left(s_{n}\right)\right|=\left\|v\left(s_{n}\right)\right\|$. But then

$$
\left\|W_{n}\right\|_{\sigma} \geq \frac{\left|W_{n}\left(s_{n}\right)\right|}{\left|Z\left(s_{n}\right)\right|}>n \frac{\left|W_{n}\left(s_{n}\right)\right|}{\| v\left(s_{n}\right)| |}=n
$$

so that $\|\cdot\|_{\sigma}$ is not bounded on the (compact) unit ball. This contradicts the fact that $\sigma$ is full, so $\sigma$ does not satisfy support after all.
$(3) \Longrightarrow$ (1). Finally, suppose that $\Delta$ is a quadratic form with $\Delta(v(s)) \geq 0$ for every massive stable object $s \in \mathrm{C}$ and whose restriction to $\operatorname{ker} Z$ is negative definite. In particular if $\Delta(\lambda)>0$ then $\lambda \notin \operatorname{ker} Z$ so $|Z(\lambda)|^{2}>0$. Therefore, because the unit ball is compact, there exists $K>0$ with

$$
\lambda \mapsto K^{2}|Z(\lambda)|^{2}-\Delta(\lambda)
$$

a positive definite form on $\Lambda \otimes \mathbb{R}$. If $\|\cdot\|$ is the induced norm then

$$
K^{2}|Z(s)|^{2}=\|v(s)\|^{2}+\Delta(v(s)) \geq\|v(s)\|^{2}
$$

for each massive stable object $s \in \mathrm{C}$. Therefore $\sigma$ satisfies support.

Example 4.12. Let $\mathrm{C}=\mathrm{D}^{b}\left(\mathbb{P}^{1}\right)$. Its Grothendieck group is $\Lambda=K(\mathrm{C})=K\left(\mathbb{P}^{1}\right) \cong \mathbb{Z}^{2}$ using the basis $[\mathcal{O}]$ (structure sheaf) and $\left[\mathcal{O}_{x}\right]$ (skyscraper sheaves). The inner product is chosen so that this basis is orthonormal. Let $\sigma=(P, Z)$ be the lax pre-stability condition defined by the charge $Z=0$ and the slicing $P=P_{t}$ from Example 3.14, i.e. $P(1)=\left\langle\mathcal{O}_{x}: x \in \mathbb{P}^{1}\right\rangle$ and $P(1 / 2)=\langle\mathcal{O}(n): n \in \mathbb{Z}\rangle$. Every object is massless so $\sigma$ trivially satisfies the support property and so is a lax stability condition.

Let $\tau=(P, W)$ be the lax pre-stability condition with the same slicing as $\sigma$ but charge $W\left(\mathcal{O}_{x}\right)=0, W(\mathcal{O})=i$. Since there are no massive $\sigma$-stable objects, $\|W\|_{\sigma}=0$. The massless subcategory of $\tau$ is thick $\left(\mathcal{O}_{x}: x \in \mathbb{P}^{1}\right)$ and the massive stable objects are, up to shifts, the line bundles $\mathcal{O}(n)$ for $n \in \mathbb{Z}$. Therefore

$$
\|U\|_{\tau}=\sup \left\{\frac{|U(b)|}{|W(b)|}: b \text { massive } \tau \text {-stable }\right\}=\sup \{|U(\mathcal{O}(n))|: n \in \mathbb{Z}\}
$$

which is infinite for example when $U\left(\mathcal{O}_{x}\right)=1$ and $U(\mathcal{O})=0$. Thus $\tau$ does not satisfy support and is not a lax stability condition.
4.4. Stability conditions on quotients. A lax pre-stability condition $\sigma$ on C with massless subcategory N induces a pre-stability condition $\mu_{\mathrm{N}}(\sigma)$ on the quotient $\mathrm{C} / \mathrm{N}$. We think of this as the 'massive part' of $\sigma$, and refer to it as the associated pre-stability condition on the quotient. We say 'stability condition on the quotient' to distinguish these from the 'quotient stability conditions' of Section 8.
Proposition 4.13. Let $\sigma=(P, Z)$ be a lax pre-stability condition with massless subcategory N . Put $\mu_{\mathrm{N}}(\sigma)=\left(P_{\mathrm{C} / \mathrm{N}}, Z\right) \in \operatorname{Slice}(\mathrm{C} / \mathrm{N}) \times \operatorname{Hom}\left(\Lambda / \Lambda_{\mathrm{N}}, \mathbb{C}\right)$.
(1) If the slicing $P$ is well-adapted to N then $\mu_{\mathrm{N}}(\sigma)$ is a pre-stability condition on $\mathrm{C} / \mathrm{N}$.
(2) If $\sigma$ is a lax stability condition then $\mu_{\mathrm{N}}(\sigma)$ is a stability condition on $\mathrm{C} / \mathrm{N}$.

Proof. (1) By Proposition 4.4 and Proposition 3.6, $P$ is adapted to N and so is compatible with a pair of slicings ( $P_{\mathrm{N}}, P_{\mathrm{C} / \mathrm{N}}$ ) on the massless subcategory N and the quotient $\mathrm{C} / \mathrm{N}$. If it is well-adapted then $P_{\mathrm{C} / \mathrm{N}}$ is locally finite. The charge $Z$ lies in the subspace $\operatorname{Hom}\left(\Lambda / \Lambda_{\mathrm{N}}, \mathbb{C}\right)$ and is compatible with the slicing $P_{\mathrm{C} / \mathrm{N}}$. There are no massless objects in $\mathrm{C} / \mathrm{N}$. Therefore, $\mu_{\mathrm{N}}(\sigma)=\left(P_{\mathrm{C} / \mathrm{N}}, Z\right)$ is a pre-stability condition on $\mathrm{C} / \mathrm{N}$.
(2) Let $\sigma$ be a lax stability condition. The support property implies that the slicing $P_{\mathrm{C} / \mathrm{N}}$ is locally finite because there are no massless objects in $\mathrm{C} / \mathrm{N}$. Hence $P$ is well-adapted to N and $\mu_{\mathrm{N}}(\sigma)$ is a lax pre-stability condition by (1). We claim that $\mu_{\mathrm{N}}(\sigma)$ satisfies the support property. By Lemma 3.5, the HNfiltration of a $\mu_{\mathrm{N}}(\sigma)$-semistable object $b$ has a unique massive $\sigma$-semistable factor $c$. Let $S$ be a multi-set of massive stable factors of $c$. Then there is some constant $K>0$ with

$$
m_{\mu_{N}(\sigma)}(b)=m_{\sigma}(c)=\sum_{s \in S} m_{\sigma}(s) \geq \frac{1}{K} \sum_{s \in S}\|v(s)\| \geq \frac{1}{K}\left\|\sum_{s \in S} v(s)\right\|=\frac{1}{K}\left\|v_{\mathrm{C} / \mathrm{N}}(b)\right\|
$$

using the restricted norm on $\left(\Lambda / \Lambda_{N}\right) \otimes \mathbb{R}$ for last term. Hence $\mu_{\mathrm{N}}(\sigma)$ satisfies the support property.

Remark 4.14. We can generalise Proposition 4.13 slightly. The same argument shows that if $\mathrm{M} \subset \mathrm{N}$ is a thick subcategory of the massless subcategory of $\sigma=(P, Z)$ to which the slicing is well-adapted, then
(1) there is a lax pre-stability condition $\mu_{\mathrm{M}}(\sigma)=\left(P_{\mathrm{C} / \mathrm{M}}, Z\right)$ on $\mathrm{C} / \mathrm{M}$;
(2) $\mu_{M}(\sigma)$ is a lax stability condition when $\sigma$ is a lax stability condition.

The last point follows from the same calculation as before (or the lemma below), but note that we now need the extra local-finiteness assumption as this no longer follows from the support property when there are massless objects.

Given a lax pre-stability condition $\sigma$ with massless subcategory N then by the proposition there is an induced pre-stability condition $\mu_{\mathrm{N}}(\sigma)$ on the quotient. Moreover, $\sigma$ restricts to a
fully lax pre-stability condition $\rho_{\mathrm{N}}(\sigma)$ on N by restricting the slicing of $\sigma$ to N and assigning the zero charge map. This latter construction is taken up again in Corollary 5.8.

Lemma 4.15. Suppose $\sigma=(P, Z)$ is a lax pre-stability condition with massless subcategory N and that the slicing is well-adapted to a thick subcategory $\mathrm{M} \subset \mathrm{N}$. Then $\|W\|_{\mu_{\mathrm{M}}(\sigma)} \leq\|W\|_{\sigma}$ for all $W \in \operatorname{Hom}\left(\Lambda / \Lambda_{\mathrm{M}}, \mathbb{C}\right)$ with equality when $\mathrm{M}=\mathrm{N}$.
Proof. Suppose $c$ is a massive $\mu_{\mathrm{M}}(\sigma)$-stable object of phase $\varphi$. Then by Lemma 3.13 its HNfiltration with respect to $\sigma$ has a single massive semistable factor $b$ also of phase $\varphi$, with all other factors in M. Furthermore, any composition series of $b$ in $P(\varphi)$ must have exactly one massive stable factor, $a$ say, with all other factors in M since otherwise $c$ would fail to be $\mu_{\mathrm{M}}(\sigma)$-stable. Therefore $a \cong c$ in $\mathrm{C} / \mathrm{M}$ so that $W(a)=W(c)$. It follows that

$$
\begin{aligned}
\|W\|_{\mu_{\mathrm{M}}(\sigma)} & =\sup \left\{\frac{|W(c)|}{|Z(c)|}: c \text { massive } \mu_{\mathrm{M}}(\sigma) \text {-stable }\right\} \\
& \leq \sup \left\{\frac{|W(a)|}{|Z(a)|}: a \text { massive } \sigma \text {-stable }\right\}=\|W\|_{\sigma} .
\end{aligned}
$$

Now suppose that $\mathrm{M}=\mathrm{N}$ and $b \in P(\varphi)$ is a massive stable object. Let $S$ be a multi-set of $\mu_{\mathrm{N}}(\sigma)$-stable factors of $b$ considered as an object of $P_{\mathrm{C} / \mathrm{N}}(\varphi)$. Then

$$
|W(b)| \leq \sum_{s \in S}|W(s)| \leq\|W\|_{\mu_{N}(\sigma)} \sum_{s \in S}|Z(s)|=\left|\left|W \|_{\mu_{N}(\sigma)}\right| Z(b)\right|
$$

by the triangle inequality, the definition of the semi-norm $\|\cdot\|_{\mu_{N}(\sigma)}$, the fact that $W(s)=0$ when $s \in \mathrm{~N}$, and the fact that all $s \in S$ have phase $\varphi$. Rearranging, $\|W\|_{\sigma} \leq\|W\|_{\mu_{\mathrm{N}}(\sigma)}$.

The support property satisfied by a lax stability condition is stronger than that for the induced stability condition on the quotient. The following technical lemma will be useful later in Lemma 9.5. The case $\mathrm{M}=\mathrm{N}$ provides a criterion for distinguishing a lax pre-stability condition which satisfies support on the quotient $\mathrm{C} / \mathrm{N}$ from a genuine lax stability condition.

Lemma 4.16. Suppose $\sigma$ is a lax pre-stability condition with massless subcategory N. Let M be a thick subcategory of N such that each object of M has HNfiltration in M , and assume that $\mu_{\mathrm{M}}(\sigma)$ is a lax stability condition. Then any sequence $\left(b_{n}\right)$ of massive $\sigma$-stable objects with $m_{\sigma}\left(b_{n}\right) /\left\|v\left(b_{n}\right)\right\| \rightarrow 0$ as $n \rightarrow \infty$ contains a subsequence $\left(c_{n}\right)$ with

$$
\lim _{n \rightarrow \infty}\left(\frac{v\left(c_{n}\right)}{\left\|v\left(c_{n}\right)\right\|}\right)=\lambda \in \Lambda_{\mathrm{M}} \otimes \mathbb{R} .
$$

Moreover, $\sigma$ is a lax stability condition if and only if there is no such sequence $\left(c_{n}\right)$.
Proof. Suppose there is such a sequence $\left(c_{n}\right)$. Then

$$
\frac{m_{\sigma}\left(c_{n}\right)}{\left\|v\left(c_{n}\right)\right\|}=\left|Z\left(\frac{v\left(c_{n}\right)}{\left\|v\left(c_{n}\right)\right\|}\right)\right| \rightarrow|Z(\lambda)|=0
$$

since $\Lambda_{\mathrm{M}} \otimes \mathbb{R} \subset \Lambda_{\mathrm{N}} \otimes \mathbb{R} \subset \operatorname{ker}(Z)$. Therefore $\sigma$ does not satisfy the support property.
Conversely, suppose that $\sigma$ does not satisfy the support property. Then there is a sequence $\left(b_{n}\right)$ of massive stable objects with $m_{\sigma}\left(b_{n}\right) /\left\|v\left(b_{n}\right)\right\| \rightarrow 0$. Let $v_{\mathrm{C} / \mathrm{M}}: K(\mathrm{C}) \rightarrow \Lambda / \Lambda_{\mathrm{M}}$ be the composite of $v$ and the quotient by the primitive sublattice $\Lambda_{\mathrm{M}}$. Note that $Z\left(v_{\mathrm{C} / \mathrm{M}}\left(b_{n}\right)\right) \neq 0$ because $b_{n}$ is massive and that $m_{\sigma}\left(b_{n}\right) /\left\|v_{\mathrm{C} / \mathrm{M}}\left(b_{n}\right)\right\|$ is bounded below because $\mu_{\mathrm{M}}(\sigma)$ satisfies the support property by assumption. Therefore writing

$$
\frac{m_{\sigma}\left(b_{n}\right)}{\left\|v\left(b_{n}\right)\right\|}=\frac{m_{\sigma}\left(b_{n}\right)}{\left\|v_{\mathrm{C} / \mathrm{M}}\left(b_{n}\right)\right\|} \cdot \frac{\left\|v_{\mathrm{C} / \mathrm{M}}\left(b_{n}\right)\right\|}{\left\|v\left(b_{n}\right)\right\|}
$$

we deduce that $v_{\mathrm{C} / \mathrm{M}}\left(b_{n}\right) /\left\|v\left(b_{n}\right)\right\| \rightarrow 0$. Passing to a subsequence $\left(c_{n}\right)$ such that the unit vectors $v\left(c_{n}\right) /\left\|v\left(c_{n}\right)\right\|$ converge we have

$$
\lim _{n \rightarrow \infty}\left(\frac{v\left(c_{n}\right)}{\left\|v\left(c_{n}\right)\right\|}\right)=\lim _{n \rightarrow \infty}\left(\frac{v\left(c_{n}\right)-v_{\mathrm{C} / \mathrm{M}}\left(c_{n}\right)}{\left\|v\left(c_{n}\right)\right\|}\right) \in \Lambda_{\mathrm{M}} \otimes \mathbb{R}
$$

where we consider $v_{\mathrm{C} / \mathrm{M}}\left(c_{n}\right) \in \Lambda \otimes \mathbb{R}$ via the orthogonal splitting.
Example 4.17. Let $\tau$ be the lax pre-stability condition on $\mathrm{C}=\mathrm{D}^{b}\left(\mathbb{P}^{1}\right)$ with massless subcategory $\mathrm{N}=\operatorname{thick}\left(\mathcal{O}_{x}: x \in \mathbb{P}^{1}\right)$ defined in Example 4.12. The quotient $\mathrm{C} / \mathrm{N} \simeq \operatorname{thick}_{\mathrm{C} / \mathrm{N}}(\mathcal{O})$ is generated by a single object with charge $i$ and phase $1 / 2$ in the stability condition on the quotient $\mu_{\mathrm{N}}(\tau)$. It follows that $\mu_{\mathrm{N}}(\tau)$ satisfies the support property so that $\tau$ is a lax pre-stability condition which satisfies support on the quotient $\mathrm{C} / \mathrm{N}$. However, the sequence $\mathcal{O}(n)$ of massive stable objects shows that $\tau$ does not satisfy the support property so that $\tau$ is not a lax stability condition.

Recall that pre-stability conditions $\sigma=(P, Z)$ and $\tau=(Q, Z)$ with the same charge and with $d(P, Q)<1$ are equal [6, Lemma 6.4]. There is an analogue for lax pre-stability conditions; the only difference is that the slicing on the massless objects is not determined by the charge so we must fix this too.

Corollary 4.18. If $\sigma=(P, Z)$ and $\tau=(Q, Z)$ are lax pre-stability conditions with the same charge $Z$, the same massless subcategory N , the same massless slicing $P_{\mathrm{N}}=Q_{\mathrm{N}}$ and $d(P, Q)<1$, then $\sigma=\tau$.

Proof. The induced pre-stability conditions $\mu_{\mathrm{N}}(\sigma)$ and $\mu_{\mathrm{N}}(\tau)$ have the same charge and the distance between their slicings is $d\left(P_{\mathrm{C} / \mathrm{N}}, Q_{\mathrm{C} / \mathrm{N}}\right) \leq d(P, Q)<1$. Hence $P_{\mathrm{C} / \mathrm{N}}=Q_{\mathrm{C} / \mathrm{N}}$ by [ 6 , Lemma 6.4]. Since $P_{\mathrm{N}}=Q_{\mathrm{N}}$, and the glued slicing is unique, we deduce that $P=Q$.

## 5. The space of lax stability conditions

Fix a finite rank lattice $\Lambda$ with a surjective homomorphism $v: K(\mathrm{C}) \rightarrow \Lambda$ and a norm $\|\cdot\|$ on $\Lambda \otimes \mathbb{R}$. Let $\operatorname{Stab}(\mathrm{C})$ be the space of stability conditions whose charges factor through $v$. Recall $[6, \S 6]$ that this has the subspace topology from the inclusion $\operatorname{Stab}(\mathrm{C}) \subset \operatorname{Slice}(\mathrm{C}) \times \operatorname{Hom}(\Lambda, \mathbb{C})$ where the right hand side has the topology from the metric

$$
d((Q, W),(P, Z))=\max \{d(P, Q),\|W-Z\|\}
$$

arising from the metric on Slice $(C)$ and the operator norm on $\operatorname{Hom}(\Lambda, \mathbb{C})$.
Definition 5.1. Let $\operatorname{Stab}^{l}(\mathrm{C})$ be the subset of lax stability conditions in Slice $(\mathrm{C}) \times \operatorname{Hom}(\Lambda, \mathbb{C})$ equipped with the subspace topology. Let

$$
\operatorname{Stab}^{L}(\mathrm{C})=\operatorname{Stab}^{l}(\mathrm{C}) \cap \overline{\operatorname{Stab}(\mathrm{C})}
$$

be the subspace of lax stability conditions in the boundary of $\operatorname{Stab}(\mathrm{C})$. Since this will be the principal object we study we refer to it as the space of lax stability conditions. We also introduce the larger space $\operatorname{Stab}^{L S}(\mathrm{C})$ of lax pre-stability conditions which satisfy support on the quotient of C by the massless subcategory and which lie in the closure of $\operatorname{Stab}(\mathrm{C})$.

For a thick subcategory N of C , let $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N}) \subset \operatorname{Stab}^{L}(\mathrm{C})$ and $\operatorname{Stab}^{L S}(\mathrm{C}, \mathrm{N}) \subset \operatorname{Stab}^{L S}(\mathrm{C})$ denote the subsets where the massless subcategory is N , respectively. For a thick subcategory N of C , the subset $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N}) \subset \operatorname{Stab}^{L}(\mathrm{C})$ will be called a stratum of $\mathrm{Stab}^{L}(\mathrm{C})$. For each space, the charge map $\mathcal{Z}$ is the second projection, e.g. $\mathcal{Z}: \operatorname{Stab}^{L}(\mathrm{C}) \rightarrow \operatorname{Hom}(\Lambda, \mathbb{C})$.

The subspace $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ for $\mathrm{N}=0$ is just the space of (classical) stability conditions: $\operatorname{Stab}^{L}(\mathrm{C}, 0)=\operatorname{Stab}(\mathrm{C})$. In the other extreme case $\mathrm{N}=\mathrm{C}$ of lax stability conditions with zero charge map, the subspace $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{C})$ is homeomorphic to the closure in Slice(C) of the set of slicings of stability conditions in $\operatorname{Stab}(\mathrm{C})$.

By Proposition $4.13(2)$, there is a map $\mu_{\mathrm{N}}: \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N}) \rightarrow \operatorname{Stab}(\mathrm{C} / \mathrm{N})$. Note that $\operatorname{Stab}^{L}(\mathrm{C}) \subset$ $\operatorname{Stab}^{L S}(\mathrm{C})$ and $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N}) \subset \operatorname{Stab}^{L S}(\mathrm{C}, \mathrm{N})$ because the support property for $\sigma$ implies the support property for $\mu_{\mathrm{N}}(\sigma)$, but not vice versa in general.
Remark 5.2. For a lax pre-stability condition $\sigma=(P, Z)$ to be in $\operatorname{Stab}^{L}(\mathrm{C})$, it needs to have two properties: the support condition and the closure condition, i.e. $\sigma \in \overline{\operatorname{Stab}(C)}$. Of these two, the support condition is the main issue. Indeed, Corollary 6.5 and Lemma 6.6 imply that if
$\sigma$ has the support property and the restricted slicing on the massless subcategory N is in the closure of the set of slicings occuring in $\operatorname{Stab}(\mathrm{N})$ then $\sigma \in \overline{\operatorname{Stab}(\mathrm{C})}$. Theorem 6.17 refines this result by showing that under the above assumptions all sufficiently small deformations of $\sigma$ are also in $\overline{\operatorname{Stab}(C)}$.

One way to find an example of a lax stability condition not in $\overline{\operatorname{Stab}(C)}$ would be to find one whose massless subcategory N had empty stability space. However, since the thick subcategory N always comes with the slicing restricted from $P$, so in particular carries bounded t-structures, the obstruction to $\operatorname{Stab}(N) \neq \varnothing$ could only be the absence of compatible stability functions on the heart. We do not know if such examples exist.

Lemma 5.3. Suppose $\sigma=(P, Z)$ is a lax pre-stability condition in $\overline{\operatorname{Stab}(\mathrm{C})}$. Then $P(\varphi-\varepsilon, \varphi+\varepsilon)$ is length for any $0<\varepsilon<1 / 2$.

Proof. Fix $0<\varepsilon<1 / 2$ and $n>0$ with $\varepsilon+1 / n<1 / 2$. Since $\sigma \in \overline{\operatorname{Stab}(\mathrm{C})}$ there is $\tau=(Q, W) \in$ $\operatorname{Stab}(\mathrm{C})$ with $d(P, Q)<1 / n$. Then, for all $\varphi \in \mathbb{R}$,

$$
P(\varphi-\varepsilon, \varphi+\varepsilon) \subset Q\left(\varphi-\varepsilon-\frac{1}{n}, \varphi+\varepsilon+\frac{1}{n}\right)
$$

Since the latter is length, so is the former because the strict exact structure is inherited from the triangulated structure on $C$.
5.1. Semi-norm neighbourhoods. For $\varepsilon>0$ and a lax pre-stability condition $\sigma=(P, Z)$ define a subset of $\operatorname{Slice}(\mathbb{C}) \times \operatorname{Hom}(\Lambda, \mathbb{C})$ by

$$
B_{\varepsilon}(\sigma)=\left\{(Q, W): d(P, Q)<\varepsilon \text { and }\|W-Z\|_{\sigma}<\sin (\pi \varepsilon)\right\}
$$

Proposition 4.11 implies that $B_{\varepsilon}(\sigma)$ is open precisely when $\sigma$ is a lax stability condition. In this case it contains all sufficiently small metric balls about $\sigma$, but it need not be contained within any such metric ball because $\|\cdot\|_{\sigma}$ is only a semi-norm. If $\sigma$ does not satisfy the support property then $B_{\varepsilon}(\sigma)$ need not even contain any metric ball about $\sigma$. It is also important to note that the condition $\|W-Z\|_{\sigma}<\sin (\pi \varepsilon)$ is asymmetric in $W$ and $Z$ because $Z$ is the charge of $\sigma$. This asymmetry is illustrated in the example below.

Example 5.4. Let $\sigma_{g}=\left(P_{g}, Z_{g}\right)$ the classical geometric stability condition on $\mathrm{C}=\mathrm{D}^{b}\left(\mathbb{P}^{1}\right)$ with charge $Z_{g}=-\operatorname{deg}+i \cdot \mathrm{rk}$, and slicing $P_{g}$ from Example 3.14, i.e. $P_{g}(1)=\left\langle\mathcal{O}_{x}: x \in \mathbb{P}^{1}\right\rangle$ and $P_{g}(\varphi)=\langle\mathcal{O}(n)\rangle$ for $\varphi=\frac{1}{\pi} \arg (-n+i) \in(0,1)$. And let $\sigma_{d}=\left(P_{d}, Z_{d}\right):=\left(P_{g}, 0\right)$ be the lax stability condition with the same slicing but zero charge so that all objects are massless.

Now $d\left(P_{g}, P_{d}\right)=0$ and $\left\|Z_{g}-Z_{d}\right\|_{\sigma_{g}}=\left\|Z_{g}\right\|_{\sigma_{g}}=1$ and $\left\|Z_{d}-Z_{g}\right\|_{\sigma_{d}}=\left\|Z_{g}\right\|_{\sigma_{d}}=0$ because there are no massive $\sigma_{d}$-stable objects. Thus $\sigma_{g} \in B_{\varepsilon}\left(\sigma_{d}\right)$ but $\sigma_{d} \notin B_{\varepsilon}\left(\sigma_{g}\right)$ for any $\varepsilon>0$.

For classical $\sigma$ the intersections of the $B_{\varepsilon}(\sigma)$ with $\operatorname{Stab}(\mathrm{C})$ form a basis for the topology, see [6, §6]. The semi-norm neighbourhoods $B_{\varepsilon}(\sigma)$ are similarly useful for studying the topology of $\operatorname{Stab}^{L}(\mathrm{C})$. Clearly, if $\sigma \in \operatorname{Stab}^{L}(\mathrm{C})$ then $B_{\varepsilon}(\sigma) \cap \operatorname{Stab}(\mathrm{C}) \neq \varnothing$ for any $\varepsilon>0$. The next result is a partial converse.

Lemma 5.5. Suppose $\sigma$ is a lax pre-stability condition such that $B_{\varepsilon}(\sigma) \cap \operatorname{Stab}(\mathrm{C}) \neq \varnothing$. Then $\sigma$ is a lax stability condition.

Proof. Let $\sigma=(P, Z)$ and suppose $\tau=(Q, W) \in B_{\varepsilon}(\sigma) \cap \operatorname{Stab}(\mathrm{C})$. Suppose $c$ is a massive $\sigma$-stable object, and let $S$ be the set of its $\tau$-semistable factors. Then

$$
\begin{aligned}
m_{\sigma}(c)=|Z(c)| & >\frac{1}{1+\sin (\pi \varepsilon)}|W(c)|
\end{aligned}
$$

where we have used successively the norm bound $\|W-Z\|_{\sigma}<\sin (\pi \varepsilon)$ and the triangle inequality, the fact that $d(P, Q)<\varepsilon$, the support property for $\tau$, and the triangle inequality for the norm on $\Lambda \otimes \mathbb{R}$.

The HNfactors of a massless object are, by definition, massless. In fact, this property persists in an open neighbourhood in the following sense.

Lemma 5.6. Let $\sigma=(P, Z)$ be a lax pre-stability condition with massless subcategory N . If $d(P, Q)<1 / 8$ then $Q$ restricts to a slicing $Q_{\mathrm{N}}$ on N . In particular this applies to the slicing of any lax pre-stability condition in $B_{\varepsilon}(\sigma)$ for $0<\varepsilon<1 / 8$.

Proof. Let $d(P, Q)<\varepsilon<1 / 8$. We must show that the $Q$-semistable factors of any $c \in \mathrm{~N}$ lie in N . Suppose $b \in Q(\varphi)$ is a $Q$-semistable factor of $c \in \mathrm{~N}$. Then $b$ is also a $Q$-semistable factor of $c^{\prime}=H_{P}^{(\varphi-\varepsilon, \varphi+\varepsilon)}(c)$ because $Q(\varphi) \subset P(\varphi-\varepsilon, \varphi+\varepsilon)$ so that

$$
H_{Q}^{\varphi}\left(c^{\prime}\right)=H_{Q}^{\varphi} H_{P}^{(\varphi-\varepsilon, \varphi+\varepsilon)}(c)=H_{Q}^{\varphi}(c)=b
$$

Moreover, $c^{\prime} \in \mathrm{N}$ too, since each of its $P$-semistable factors lies in N . In particular, $Z\left(c^{\prime}\right)=0$.
Let $b_{1}, \ldots, b_{m}$ be the $Q$-semistable factors of $c^{\prime}$, so that $b=b_{i}$ for some $1 \leq i \leq m$. Since $c^{\prime} \in P(\varphi-\varepsilon, \varphi+\varepsilon) \subset Q(\varphi-2 \varepsilon, \varphi+2 \varepsilon)$ we have

$$
b_{1}, \ldots, b_{m} \in Q(\varphi-2 \varepsilon, \varphi+2 \varepsilon) \subset P(\varphi-3 \varepsilon, \varphi+3 \varepsilon) .
$$

Now let $b_{i j}$ for $j=1, \ldots, n_{i}$ be the $P$-semistable factors of $b_{i}$ for $1 \leq i \leq m$. Since $6 \varepsilon<1$ the equation

$$
\sum_{i, j} Z\left(b_{i j}\right)=\sum_{i} Z\left(b_{i}\right)=Z\left(c^{\prime}\right)=0
$$

implies that $Z\left(b_{i j}\right)=0$, and hence that $b_{i j} \in \mathrm{~N}$, for each $1 \leq i \leq m$ and $1 \leq j \leq n_{i}$. In particular, all $P$-semistable factors of $b$ lie in N , so that $b \in \mathrm{~N}$ as claimed.

Recall the constructions of restriction and quotient stability conditions:

$$
\begin{array}{lll}
\rho_{\mathrm{N}}: & \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{~N}) \rightarrow \operatorname{Stab}^{L}(\mathrm{~N}), & \sigma=(P, Z) \mapsto \rho_{\mathrm{N}}(\sigma)=\left(P_{\mathrm{N}}, 0\right) ; \\
\mu_{\mathrm{N}}: & \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{~N}) \rightarrow \operatorname{Stab}(\mathrm{C} / \mathrm{N}), & \sigma=(P, Z) \mapsto \mu_{\mathrm{N}}(\sigma)=\left(P_{\mathrm{C} / \mathrm{N}}, Z\right) ;
\end{array}
$$

for the latter, see Proposition 4.13. The next lemma says that these maps are contractions and hence, in particular, continuous.

Lemma 5.7. Let $\sigma$ and $\tau$ be lax pre-stability conditions on C such that their slicings are adapted to a thick subcategory N of C . Then

$$
d\left(\mu_{\mathrm{N}}(\sigma), \mu_{\mathrm{N}}(\tau)\right) \leq d(\sigma, \tau) \quad \text { and } \quad d\left(\rho_{\mathrm{N}}(\sigma), \rho_{\mathrm{N}}(\tau)\right) \leq d(\sigma, \tau)
$$

Proof. Writing $\sigma=(P, Z)$ and $\tau=(Q, W)$, Corollary 3.7 gives $d\left(P_{\mathrm{N}}, Q_{\mathrm{N}}\right) \leq d(P, Q)$ and $d\left(P_{\mathrm{C} / \mathrm{N}}, Q_{\mathrm{C} / \mathrm{N}}\right) \leq d(P, Q)$. For any $U \in \operatorname{Hom}(\Lambda, \mathbb{C})$ let $U_{\mathrm{N}}=\left.U\right|_{\Lambda_{N}} \in \operatorname{Hom}\left(\Lambda_{\mathrm{N}}, \mathbb{C}\right)$ be the restriction. Using the embedding $\operatorname{Hom}\left(\Lambda_{\mathrm{N}}, \mathbb{C}\right) \hookrightarrow \operatorname{Hom}(\Lambda, \mathbb{C})$ arising from the inner product on $\Lambda \otimes \mathbb{R}$, see $\S 2.3$, we may also consider $U_{\mathrm{N}} \in \operatorname{Hom}(\Lambda, \mathbb{C})$. Write (in this proof only) $U_{\mathrm{N}}^{\prime}=U-U_{\mathrm{N}}$ for the component in the orthogonal complement $\operatorname{Hom}\left(\Lambda / \Lambda_{N}, \mathbb{C}\right)$ to $\operatorname{Hom}\left(\Lambda_{N}, \mathbb{C}\right)$. Because $U=U_{N}+U_{N}^{\prime}$ is an orthogonal decomposition: $\|U\|^{2}=\left\|U_{N}\right\|^{2}+\left\|U_{N}^{\prime}\right\|^{2}$. Applied to $U:=Z-W$, this shows that $\left\|Z_{\mathrm{N}}-W_{\mathrm{N}}\right\| \leq\|Z-W\|$ and $\left\|Z_{\mathrm{N}}^{\prime}-W_{\mathrm{N}}^{\prime}\right\| \leq\|Z-W\|$. The claimed inequalities follow because $\rho_{\mathrm{N}}(\sigma)=\left(P_{\mathrm{N}}, Z_{\mathrm{N}}\right)$ and $\mu_{\mathrm{N}}(\sigma)=\left(P_{\mathrm{C} / \mathrm{N}}, Z_{\mathrm{N}}^{\prime}\right)$, and likewise for $\tau$.

If $\sigma=(P, Z) \in \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ then its restriction to the massless subcategory produces a lax stability condition $\rho_{\mathrm{N}}(\sigma)=(P \cap \mathrm{~N}, 0)$ in which all objects are massless. By Lemma 5.6, this construction extends to nearby lax stability conditions.
Corollary 5.8. Given $\sigma=(P, Z) \in \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ and $0<\varepsilon<1 / 16$ there is a continuous map

$$
\begin{aligned}
\rho_{\mathrm{N}}:\left\{\tau=(Q, W) \in \operatorname{Stab}^{L}(\mathrm{C}): d(P, Q)<\varepsilon\right\} & \rightarrow \operatorname{Stab}^{L}(\mathrm{~N}) \\
\tau=(Q, W) & \mapsto \rho_{\mathrm{N}}(\tau)=\left(Q \cap \mathrm{~N}, W_{\mathrm{N}}\right),
\end{aligned}
$$

in particular this restricts to a map $\rho_{\mathrm{N}}: B_{\varepsilon}(\sigma) \cap \operatorname{Stab}^{L}(\mathrm{C}) \rightarrow \operatorname{Stab}^{L}(\mathrm{~N})$.

Proof. The restriction, $\rho_{\mathrm{N}}(\tau)$, is a lax pre-stability condition by Lemma 5.6. It also satisfies the support property because $Q_{\mathrm{N}}(\varphi) \subset Q(\varphi)$ and so a $\rho_{\mathrm{N}}(\tau)$-semistable object $s$ is also $\tau$-semistable, hence $v(s)=v_{\mathrm{N}}(s) \in \Lambda_{\mathrm{N}}$. In particular, the restricted charge map $W_{\mathrm{N}}$ satisfies

$$
m_{\rho_{\mathrm{N}}(\tau)}(s)=\left|W_{\mathrm{N}}\left(v_{\mathrm{N}}(s)\right)\right|=|W(v(s))| \geq \frac{1}{K}\|v(s)\|=\frac{1}{K}\left\|v_{\mathrm{N}}(s)\right\| .
$$

Next, we check that $\rho_{\mathrm{N}}(\tau) \in \overline{\operatorname{Stab}(\mathbb{N})}$. Since $\tau \in \operatorname{Stab}^{L}(\mathrm{C}) \subseteq \overline{\operatorname{Stab}(\mathrm{C})}$, for each $\varepsilon>0$ there exists $\sigma_{\varepsilon}=\left(P_{\varepsilon}, Z_{\varepsilon}\right) \in \operatorname{Stab}(\mathrm{C})$ with $d\left(\tau, \sigma_{\varepsilon}\right)<\varepsilon$ in the metric on Slice $(\mathrm{C}) \times \operatorname{Hom}(\Lambda, \mathbb{C})$. As $d\left(P, P_{\varepsilon}\right) \leq d(P, Q)+d\left(Q, P_{\varepsilon}\right)<\varepsilon+\varepsilon=2 \varepsilon$, the slicing $P_{\varepsilon}$ restricts to N by Lemma 5.6. Hence, $d\left(\rho_{\mathrm{N}}(\tau), \rho_{\mathrm{N}}\left(\sigma_{\varepsilon}\right)\right)<\varepsilon$ in the metric on Slice $(\mathrm{N}) \times \operatorname{Hom}\left(\Lambda_{\mathrm{N}}, \mathbb{C}\right)$. It follows that $\rho_{\mathrm{N}}(\tau) \in \overline{\operatorname{Stab}(\mathbb{N})}$. Finally, the map $\rho_{\mathrm{N}}$ is continuous by Lemma 5.7.

The next result shows that the massless subcategory of a lax pre-stability condition varies semi-continuously.
Lemma 5.9. Suppose $\sigma$ is a lax pre-stability condition with massless subcategory N . If $\tau=$ $(Q, W) \in B_{\varepsilon}(\sigma)$ for $0<\varepsilon<1 / 8$, then the massless subcategories $\mathrm{N}_{\tau} \subset \mathrm{N}$ are nested. Moreover, the inclusion is equality if and only if $W \in \operatorname{Hom}\left(\Lambda / \Lambda_{N}, \mathbb{C}\right) \subset \operatorname{Hom}(\Lambda, \mathbb{C})$.

Proof. Let $\sigma=(P, Z)$. For $\sigma$-stable $c \in \mathrm{C}$ there is an inequality

$$
\begin{equation*}
(1-\sin (\pi \varepsilon))|Z(c)| \leq|W(c)| . \tag{2}
\end{equation*}
$$

This is evident if $c$ is massless, and follows from the definition of the norm and the (reverse) triangle inequality if it is massive.

Now suppose $b \in Q(\varphi)$ is $\tau$-semistable. Let $S$ be a (finite) multi-set of $\sigma$-stable factors of $b$. Since $\tau \in B_{\varepsilon}(\sigma)$ we know that $S \subset P(\varphi-\varepsilon, \varphi+\varepsilon) \subset Q(\varphi-2 \varepsilon, \varphi+2 \varepsilon)$. Hence, using elementary trigonometry and the inequality (2) we have

$$
|W(b)| \geq \cos (2 \pi \varepsilon) \sum_{s \in S}|W(s)| \geq(1-\sin (\pi \varepsilon)) \cos (2 \pi \varepsilon) \sum_{s \in S}|Z(s)| .
$$

Therefore $m_{\tau}(b) \geq(1-\sin (\pi \varepsilon)) \cos (2 \pi \varepsilon) m_{\sigma}(b)$. In particular, if $b \in \mathrm{~N}_{\tau}$ then $b \in \mathrm{~N}$.
For the equality statement, clearly, $W \in \operatorname{Hom}\left(\Lambda / \Lambda_{N}, \mathbb{C}\right)$ if and only if $W(c)=0$ for all $c \in \mathrm{~N}$. In particular, if $\mathrm{N}_{\tau}=\mathrm{N}$ then $W \in \operatorname{Hom}\left(\Lambda / \Lambda_{\mathrm{N}}, \mathbb{C}\right)$. Conversely, if $W \in \operatorname{Hom}\left(\Lambda / \Lambda_{\mathrm{N}}, \mathbb{C}\right)$ then, by Lemma 5.6, the $\tau$-semistable factors of any $c \in \mathrm{~N}$ are also in N which implies that $m_{\tau}(c)=0$ because $|W(c)| \leq \sum_{s \in S}|W(s)| \leq\left|\left|W \|_{\sigma} \sum_{s \in S}\right| Z(s)\right|=0$, where $S$ is the multi-set of $\tau$-semistable factors of $c$. Hence $\mathrm{N}_{\tau} \supset \mathrm{N}$ and so we have equality as claimed.

We next show that the quotient stability map extends to the closure of $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$.
Proposition 5.10. The map $\mu_{\mathrm{N}}: \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N}) \rightarrow \operatorname{Stab}(\mathrm{C} / \mathrm{N})$ extends to a continuous map

$$
\mu_{\mathrm{N}}: \overline{\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{~N})} \rightarrow \operatorname{Stab}^{L}(\mathrm{C} / \mathrm{N})
$$

Moreover, $\mu_{\mathrm{N}}(\sigma) \in \operatorname{Stab}(\mathrm{C} / \mathrm{N}) \Longleftrightarrow \sigma \in \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$.
Proof. By Proposition 4.13, the assignment $\sigma=(P, Z) \mapsto\left(P_{\mathrm{C} / \mathrm{N}}, Z\right)=\mu_{\mathrm{N}}(\sigma)$ defines a map $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N}) \rightarrow \mathrm{Stab}(\mathrm{C} / \mathrm{N})$. It is continuous by Lemma 5.7.

If $\sigma$ is in the boundary of $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ in $\operatorname{Stab}^{L}(\mathrm{C})$ then it has massless subcategory $\mathrm{N}_{\sigma} \supset \mathrm{N}$ by Lemma 5.9. We claim that the slicing $P$ is well-adapted to N .

Suppose $c \in \mathbf{N}$ has a $\sigma$-semistable factor $b \notin \mathbf{N}$. For sufficiently close $\tau=(Q, W) \in$ $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ the HNfiltration of $c$ with respect to $\tau$ will be the concatenation of the filtrations of its $\sigma$-semistable factors. Since $b \notin \mathrm{~N}$ its filtration must contain at least one factor not in N . This contradicts Proposition 4.4 which says that $Q$ is adapted to N . Therefore $P$ restricts to N after all.

Now suppose that $b \in \mathrm{~N} \cap P(I)$ and that $0 \rightarrow a \rightarrow b \rightarrow c \rightarrow 0$ is a short exact sequence in the abelian category $P(I)$. Then for sufficiently close $\tau=(Q, W)$ in $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ and suitable strict length one interval $J$ it is also a short exact sequence in $Q(J)$. Therefore $a, c \in \mathrm{~N}$ because
$Q$ is adapted to N by Proposition 4.4. Since $\mathrm{N} \cap P(I)$ is clearly extension-closed it is therefore a Serre subcategory of $P(I)$. Thus $P$ is adapted to N .

By Proposition 3.6 there is a slicing $P_{\mathrm{C} / \mathrm{N}}$ on the quotient. Since $d\left(P_{\mathrm{C} / \mathrm{N}}, Q_{\mathrm{C} / \mathrm{N}}\right) \leq d(P, Q)$ and $\sigma \in \overline{\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})}$ this slicing is the limit of slicings appearing in $\operatorname{Stab}(\mathrm{C} / \mathrm{N})$. Thus $P_{\mathrm{C} / \mathrm{N}}$ is locally finite by Lemma 5.3. Therefore $P$ is well-adapted to N as claimed.

It follows that $\mu_{\mathrm{N}}(\sigma)$ is well-defined by Remark 4.14, is in the boundary of $\operatorname{Stab}(\mathrm{C} / \mathrm{N})$, inherits the support property from $\sigma$ and has massless subcategory $\mathrm{N}_{\sigma} / \mathrm{N}$. Hence it is in $\operatorname{Stab}^{L}(\mathrm{C} / \mathrm{N})$ as claimed.

Finally we relate the semi-norms associated to nearby lax pre-stability conditions. This is the (weaker) analogue of the fact that the norms associated to stability conditions in the same component of $\operatorname{Stab}(\mathrm{C})$ are equivalent, cf. [6, Lemma 6.2].
Lemma 5.11. Let $\sigma$ be a lax pre-stability condition with massless subcategory N and let $\tau \in$ $B_{\varepsilon}(\sigma)$. Then for any $U \in \operatorname{Hom}\left(\Lambda / \Lambda_{\mathrm{N}}, \mathbb{C}\right)$

$$
\|U\|_{\tau} \leq \frac{\|U\|_{\sigma}}{(1-\sin (\pi \varepsilon)) \cos (2 \pi \varepsilon)}
$$

Proof. Let $\sigma=(P, Z)$ and $\tau=(Q, W)$. Suppose $b \in Q(\varphi)$ is a massive $\tau$-stable object and $S$ is a multi-set of its $\sigma$-stable factors. Note that $|U(s)| \leq\|U\|_{\sigma}|Z(s)|$ for each $s \in S$ by definition of $\|\cdot\|_{\sigma}$ when $s$ is massive, and trivially when $s$ is massless since then $U(s)=0$. Therefore, using (2) and the fact that $S \subset P(\varphi-\varepsilon, \varphi+\varepsilon) \subset Q(\varphi-2 \varepsilon, \varphi+2 \varepsilon)$, we have

$$
|U(b)| \leq \sum_{s \in S}|U(s)| \leq\|U\|_{\sigma} \sum_{s \in S}|Z(s)| \leq \frac{\|U\|_{\sigma}}{1-\sin (\pi \varepsilon)} \sum_{s \in S}|W(s)| \leq \frac{\|U\|_{\sigma}}{1-\sin (\pi \varepsilon)} \cdot \frac{|W(b)|}{\cos (2 \pi \varepsilon)} .
$$

Dividing by $|W(b)|$ and taking the supremum over all massive $\tau$-stable $b \in \mathrm{C}$ gives the result.

### 5.2. Continuity of masses and phases.

Proposition 5.12. For each $0 \neq c \in \mathrm{C}$ the functions $\sigma \mapsto m_{\sigma}(c)$ and $\sigma \mapsto \varphi_{\sigma}^{ \pm}(c)$ are continuous on $\operatorname{Stab}^{L}(\mathrm{C})$.
Proof. Fix $0 \neq c \in \mathrm{C}$. The result is immediate for the minimal and maximal phases $\varphi_{\sigma}^{ \pm}(c)$. To show that the mass is continuous consider $\sigma=(P, Z) \in \operatorname{Stab}^{L}(\mathrm{C})$. For sufficiently small $\varepsilon>0$ and $\tau=(Q, W)$ with $d(P, Q)<\varepsilon$ the HNfiltration of $c$ with respect to $\tau$ is the concatenation of the filtrations of the $\sigma$-semistable factors $\left\{c_{i}\right\}$ of $c$. Hence

$$
m_{\tau}(c)-m_{\sigma}(c)=\sum_{i}\left(m_{\tau}\left(c_{i}\right)-m_{\sigma}\left(c_{i}\right)\right) .
$$

Therefore it suffices to consider the case in which $c$ is $\sigma$-semistable. Assume $c \in P(\varphi)$ and let $S$ be a multi-set of $\tau$-stable factors of $c$. Since $d(P, Q)<\varepsilon$ and $c \in P(\varphi)$ we have $S \subset Q(\varphi-\varepsilon, \varphi+\varepsilon)$. By the triangle inequality and elementary trigonometry

$$
|W(c)| \leq \sum_{s \in S}|W(s)| \leq \frac{|W(c)|}{\cos (2 \pi \varepsilon)}
$$

and therefore

$$
\left|m_{\tau}(c)-m_{\sigma}(c)\right|=\left|\sum_{s \in S}\right| W(s)|-|Z(c)|| \leq \max \left\{|Z(c)|-|W(c)|, \frac{|W(c)|}{\cos (2 \pi \varepsilon)}-|Z(c)|\right\} .
$$

Applying the triangle inequality to each term on the right-hand side and the operator norm bounds $|W(c)-Z(c)| \leq\|W-Z\| \cdot\|v(c)\|$ and $|Z(c)| \leq\|Z\| \cdot\|v(c)\|$ we obtain

$$
\left|m_{\tau}(c)-m_{\sigma}(c)\right| \leq \max \left\{\|W-Z\|, \frac{\|W-Z\|+(1-\cos (2 \pi \varepsilon))\|Z\|}{\cos (2 \pi \varepsilon)}\right\}\|v(c)\| .
$$

Requiring $\|W-Z\|<\varepsilon$, in addition to $d(P, Q)<\varepsilon$, we see the bound can be made arbitrarily small by reducing $\varepsilon$. The result follows.

Proposition 5.13. For any $c \in C$ the function $\sigma \mapsto m_{\sigma}(c)$ is locally constant on the fibres of the charge map $\mathcal{Z}: \operatorname{Stab}^{L}(\mathrm{C}) \rightarrow \operatorname{Hom}(\Lambda, \mathbb{C})$. Moreover, the set of phases of the massive semistable factors of $c$ is also locally constant on the fibres of $\mathcal{Z}$.

Proof. Fix $c \in \mathrm{C}$ and $\sigma \in \operatorname{Stab}^{L}(\mathrm{C})$. Then for $\tau$ sufficiently close to $\sigma$ the HNfiltration of $c$ with respect to $\tau$ is the concatenation of the filtrations of the $\sigma$-semistable factors of $c$. Suppose $c_{i}$ is one of these $\sigma$-semistable factors. The charges of the $\tau$-semistable factors of $c_{i}$ lie in a cone of angle $2 \pi \varepsilon$ in $\mathbb{C}$, centred on the phase of $c_{i}$. If $\mathcal{Z}(\tau)=\mathcal{Z}(\sigma)$ then all but one of the $\tau$-semistable factors of $c_{i}$ must be massless because otherwise the massive semistable factors would already destabilise $c_{i}$ with respect to $\sigma$. The unique massive factor must have the same charge, in particular the same mass, as $c_{i}$. It follows that $m_{\tau}(c)=m_{\sigma}(c)$, and also that the sets of phases of the massive factors of $c$ with respect to $\sigma$ and $\tau$ are the same.

Corollary 5.14. The subcategory N of massless objects, and the stability condition on the quotient $\mu_{\mathrm{N}}(\sigma)$ are locally constant on the fibres of the charge map $\mathcal{Z}: \operatorname{Stab}^{L}(\mathrm{C}) \rightarrow \operatorname{Hom}(\Lambda, \mathbb{C})$.

Proof. The massless subcategory N and the semistable objects of the stability condition on the quotient $\mu_{\mathrm{N}}(\sigma)$ are locally constant on the fibres of the projection by Proposition 5.13. By construction the charge of $\mu_{\mathrm{N}}(\sigma)$ is constant.
5.3. Group actions. Let $\operatorname{Aut}_{\Lambda}(\mathrm{C})$ be the subgroup of auto-equivalences $\alpha: \mathrm{C} \rightarrow \mathrm{C}$ which descend (necessarily uniquely) to an isomorphism $[\alpha]: \Lambda \rightarrow \Lambda$ with $v \circ \alpha=[\alpha] \circ v$. Then $\operatorname{Aut}_{\Lambda}(\mathrm{C})$ acts smoothly on the left of $\operatorname{Stab}(\mathrm{C})$ via

$$
(P, Z) \mapsto\left(\alpha \circ P, Z \circ[\alpha]^{-1}\right) .
$$

There is also a smooth right action of the universal cover $G$ of the orientation-preserving component $\mathrm{GL}_{2}^{+}(\mathbb{R})$. An element $g \in G$ corresponds to a pair $\left(T_{g}, \theta_{g}\right)$ where $T_{g}$ is the projection of $g$ to $\mathrm{GL}_{2}^{+}(\mathbb{R})$ under the covering map and $\theta_{g}: \mathbb{R} \rightarrow \mathbb{R}$ is an increasing map with $\theta_{g}(t+1)=\theta_{g}(t)+1$ which induces the same map as $T_{g}$ on the circle $\mathbb{R} / 2 \mathbb{Z}=\left(\mathbb{R}^{2}-\{0\}\right) / \mathbb{R}_{>0}$. The element acts by

$$
\begin{equation*}
(P, Z) \mapsto\left(P \circ \theta_{g}, T_{g}^{-1} \circ Z\right) \tag{3}
\end{equation*}
$$

where we think of the central charge as taking values in $\mathbb{R}^{2}$. This action preserves the semistable and stable objects and the HNfiltrations of all objects. The subgroup consisting of pairs with $T$ conformal is isomorphic to $\mathbb{C}$ with $w \in \mathbb{C}$ acting via

$$
(P, Z) \mapsto(P(\varphi+\operatorname{Re} w), \exp (-i \pi w) Z))
$$

i.e. by rotating the phases and rescaling the masses of semistable objects. The $\mathbb{C}$ action is free provided $\mathbb{C} \neq 0$. Clearly the charge map $\mathcal{Z}: \operatorname{Stab}(C) \rightarrow \operatorname{Hom}(\Lambda, \mathbb{C})$ is equivariant with respect to these actions and the evident actions on $\operatorname{Hom}(\Lambda, \mathbb{C})$.

The group actions preserve the semi-norms $\|\cdot\|_{\sigma}$ for $\sigma \in \operatorname{Stab}^{L}(\mathrm{C})$ in the sense that

$$
\begin{equation*}
\|\alpha \cdot U \cdot w\|_{\alpha \cdot \sigma \cdot w}=\|U\|_{\sigma} \tag{4}
\end{equation*}
$$

for any automorphism $\alpha \in \operatorname{Aut}_{\Lambda}(\mathrm{C})$, element $w \in \mathbb{C}$ and charge $U \in \operatorname{Hom}(\Lambda, \mathbb{C})$. They also preserve the semi-norm neighbourhoods: $\alpha \cdot B_{\varepsilon}(\sigma) \cdot w=B_{\varepsilon}(\alpha \cdot \sigma \cdot w)$ for any $\alpha \in \operatorname{Aut}_{\Lambda}(\mathrm{C})$ and $w \in \mathbb{C}$.

Lemma 5.15. The actions of $\operatorname{Aut}_{\Lambda}(\mathrm{C})$ and of $G$ on $\operatorname{Stab}(\mathrm{C})$ extend uniquely to continuous actions on $\operatorname{Stab}^{L}(\mathrm{C})$ so that the charge map is equivariant. Elements of $G$ preserve $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ and each $\alpha \in \operatorname{Aut}_{\Lambda}(\mathrm{C})$ maps $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ to $\operatorname{Stab}^{L}(\mathrm{C}, \alpha(\mathrm{N}))$. The map $\mu_{\mathrm{N}}: \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N}) \rightarrow$ $\operatorname{Stab}(\mathrm{C} / \mathrm{N})$ is $G$-equivariant and such that


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commutes for each $\alpha \in \operatorname{Aut}_{\Lambda}(\mathrm{C})$. In particular $\mu_{\mathrm{N}}$ is equivariant for the subgroup of $\operatorname{Aut}_{\Lambda}(\mathrm{C})$ preserving N .

Proof. The actions of $\operatorname{Aut}_{\Lambda}(\mathrm{C})$ and $G$ extend to continuous actions on $\operatorname{Slice}(\mathrm{C}) \times \operatorname{Hom}(\Lambda, \mathbb{C})$ which preserve the subsets of lax and classical stability conditions. Hence they preserve $\operatorname{Stab}^{L}$ (C). The equivariance of the charge map and the properties of the $\mu_{\mathrm{N}}$ are easy to verify.
5.4. Neighbourhoods of strata. Fix $0<\varepsilon<1 / 8$ and consider the open neighbourhood

$$
\mathrm{B}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{~N})=\operatorname{Stab}^{L}(\mathrm{C}) \cap \bigcup_{\sigma \in \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{~N})} B_{\varepsilon}(\sigma)
$$

of $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ in $\operatorname{Stab}^{L}(\mathrm{C})$. Intuitively this is the subset where objects of N , and only those, are close to massless. These neighbourhoods are $\mathbb{C}$-invariant and compatible with the action of automorphisms in the sense that $\alpha \cdot \mathrm{B}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{N})=\mathrm{B}_{\varepsilon}^{L}(\mathrm{C}, \alpha(\mathrm{N}))$. Note that $\mathrm{B}_{\varepsilon}^{L}(\mathrm{C}, 0)=\mathrm{Stab}(\mathrm{C})$, and that $\mathrm{B}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{C})=\operatorname{Stab}^{L}(\mathrm{C})$ because the closure of the $\mathbb{C}$-orbit of any $\sigma \in \operatorname{Stab}^{L}(\mathrm{C})$ contains a lax stability condition with massless subcategory C .

For $\sigma \in \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$, Corollary 5.8 shows that restriction $(Q, W) \mapsto\left(Q \cap \mathrm{~N},\left.W\right|_{\Lambda_{\mathrm{N}}}\right)$ gives a continuous map $\rho_{\mathrm{N}}: B_{\varepsilon}(\sigma) \cap \operatorname{Stab}^{L}(\mathrm{C}) \rightarrow \operatorname{Stab}^{L}(\mathrm{~N})$. It clearly extends to $\mathrm{B}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{N})$ so that there is a commutative diagram

of continuous maps which are equivariant for the right action of $\mathbb{C}$ and for the left action of the subgroup of automorphisms preserving N . More generally, $\alpha \cdot \rho_{\mathrm{N}}(\sigma)=\rho_{\alpha(\mathrm{N})}(\alpha \cdot \sigma)$ as elements of $\operatorname{Stab}^{L}(\alpha(\mathrm{~N}))$ for any $\alpha \in \operatorname{Aut}_{\Lambda}(\mathrm{C})$.

## 6. Deforming lax stability conditions

The technical heart of the theory of stability conditions is Theorem 4.1 which governs their deformation. We cannot expect such a simple result for lax stability conditions, but it turns out that it is still possible to deform them in a reasonable way. The heuristic is that the massive and massless parts of a lax stability condition deform independently.
6.1. Tangential, normal and fibrewise deformations. For a lax stability condition $\sigma=$ $(P, Z) \in \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ with massless subcategory N , the base of the charge map $\mathcal{Z}: \operatorname{Stab}^{L}(\mathrm{C}) \rightarrow$ $\operatorname{Hom}(\Lambda, \mathbb{C})$ decomposes as

$$
\operatorname{Hom}\left(\Lambda_{N}, \mathbb{C}\right) \oplus \operatorname{Hom}\left(\Lambda / \Lambda_{\mathrm{N}}, \mathbb{C}\right)
$$

with $Z \in \operatorname{Hom}\left(\Lambda / \Lambda_{\mathrm{N}}, \mathbb{C}\right)$. Here, as elsewhere, we consider $\operatorname{Hom}\left(\Lambda_{N}, \mathbb{C}\right)$ as a subspace of $\operatorname{Hom}(\Lambda, \mathbb{C})$ using the splitting arising from the inner product on $\Lambda \otimes \mathbb{R}$ - see §2.3. It is geometrically appealing to distinguish three (not mutually exclusive) cases of deformation:
(1) A tangential deformation of $\sigma$ is given by varying the charge in $\operatorname{Hom}\left(\Lambda / \Lambda_{N}, \mathbb{C}\right)$. Such a deformation fixes the massless subcategory and hence stays inside $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$.
(2) A normal deformation of $\sigma$ is given by varying the charge in $\operatorname{Hom}\left(\Lambda_{\mathrm{N}}, \mathbb{C}\right)$. Such a deformation moves out of $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ into $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{M})$ for some thick subcategory M of N . We think of this as deforming in a normal slice to the stratum.
(3) A fibrewise deformation of $\sigma$ takes place when the charge function is fixed, i.e. only the slicing is deformed. In the classical setting, the charge map $\operatorname{Stab}(\mathrm{C}) \rightarrow \operatorname{Hom}(\Lambda, \mathbb{C})$ has discrete fibres, so there are no non-trivial fibrewise deformations. However for the lax stability condition $\sigma$ we can potentially vary the slicing on N in a continuous way, as this is not controlled by the charge function.

We begin with a variant of Theorem 4.1 which shows that a lax stability condition with massless subcategory N can be freely deformed in the normal direction with respect to a nearby stability condition on N .

Theorem 6.1. Suppose $\sigma=(P, Z)$ is a lax pre-stability condition on C with massless subcategory N . Then there is some $0<\varepsilon_{0}<1 / 8$ such that for any $0<\varepsilon<\varepsilon_{0}$ and pre-stability condition $\tau_{\mathrm{N}}=\left(Q_{\mathrm{N}}, W_{\mathrm{N}}\right)$ on N with

- $\left\|W_{\mathrm{N}}\right\|_{\sigma}<\sin (\pi \varepsilon)$
- and $d\left(P_{\mathrm{N}}, Q_{\mathrm{N}}\right)<\varepsilon$
there is a unique pre-stability condition $\tau=(Q, W)$ on C with
- charge $W=Z+W_{\mathrm{N}}$,
- restricted slicing $Q \cap \mathrm{~N}=Q_{\mathrm{N}}$
- and $d(P, Q)<\varepsilon$.

If $\sigma \in \operatorname{Stab}^{l}(\mathrm{C})$ and $\tau_{\mathrm{N}} \in \operatorname{Stab}(\mathrm{N})$ then $\tau \in \operatorname{Stab}(\mathrm{C})$.
Remark 6.2. This is a normal deformation at least whenever $\tau_{\mathrm{N}}$ is a stability condition — the charge $Z$ is changed by $W_{N} \in \operatorname{Hom}\left(\Lambda_{N}, \mathbb{C}\right)$, and since the deformation $\tau$ is in $\operatorname{Stab}(\mathbb{C})$ which has discrete fibres over $\operatorname{Hom}(\Lambda, C)$, the slicing of $\tau$ cannot be deformed in the fibre. The construction defines a continuous map

$$
\begin{equation*}
\delta_{\mathrm{N}}: \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{~N}) \times_{\text {Slice }(\mathrm{N})} \operatorname{Stab}(\mathrm{N}) \rightarrow \operatorname{Stab}(\mathrm{C}) \tag{5}
\end{equation*}
$$

where the fibre product denotes the set of pairs $\left(\sigma, \tau_{\mathrm{N}}\right) \in \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N}) \times \operatorname{Stab}(\mathrm{N})$ whose slicings agree on N and the dashed arrow indicates that the map is only defined on the open subset where the charge $W_{\mathrm{N}}$ of $\tau_{\mathrm{N}}$ satisfies $\left\|W_{\mathrm{N}}\right\|_{\sigma}<\sin (\pi \varepsilon)$ for some suitably small $\varepsilon>0$.

The continuity of the charge $Z+W_{\mathrm{N}}$ of $\delta_{\mathrm{N}}\left(\sigma, \tau_{\mathrm{N}}\right)$ is evident; the continuity of the slicing follows from the fact that $\operatorname{Stab}(\mathrm{C})$ is locally homeomorphic to $\operatorname{Hom}(\Lambda, \mathbb{C})$ and Corollary 4.18.

Proof. Choose $0<\varepsilon_{0}<1 / 8$ sufficiently small so that $P\left(\varphi-4 \varepsilon_{0}, \varphi+4 \varepsilon_{0}\right)$ is a length category for all $\varphi \in \mathbb{R}$ and fix $0<\varepsilon<\varepsilon_{0}$. Recall that $P(s, t)$ is a thin subcategory, see [6, Definition 7.2], if $0<t-s<1-2 \varepsilon$, and that this implies that it is quasi-abelian. The charge $W=Z+W_{N}$ defines a skewed stability function, see [6, Definition 4.4], on any thin subcategory $P(s, t)$. That is, $W: K(P(s, t)) \rightarrow \mathbb{C}$ is a group homomorphism taking every non-zero object into a rotated copy of the strict half-plane $\mathbb{H} \cup \mathbb{R}_{<0}$. To see why, suppose that $c \in P(\varphi)$ for some $\varphi \in(s, t)$. Let $A$ be a finite multi-set of stable factors of $c$ in the quasi-abelian length category $P(\varphi)$. If, on the one hand, $a \in A$ is a massless stable object in N then $a \in Q_{\mathrm{N}}(\varphi-\varepsilon, \varphi+\varepsilon)$ and so $W(a)=\left(Z+W_{\mathrm{N}}\right)(a)=W_{\mathrm{N}}(a)$ is non-zero and therefore one can assign the phase $\frac{1}{\pi} \arg W_{\mathrm{N}}(a) \in(s-\varepsilon, t+\varepsilon)$ to $a$. On the other hand, if $a \notin \mathrm{~N}$ is a massive stable factor then

$$
|W(a)-Z(a)|<\sin (\pi \varepsilon)|Z(a)|
$$

because $\|W-Z\|_{\sigma}=\left\|W_{N}\right\|_{\sigma}<\sin (\pi \varepsilon)$. Therefore $W(a) \neq 0$ and differs in phase from $Z(a)$ by less than $\varepsilon$ and again the phase of $a$ with respect to $W$ lies in $(s-\varepsilon, t+\varepsilon)$. Since $W(c)=\sum_{a \in A} W(a)$ we conclude that $W(c) \neq 0$ too and the phase of $c, \frac{1}{\pi} \arg W(c) \in(s-\varepsilon, t+\varepsilon)$.

The remainder of the proof follows that of Theorem 4.1 in [6, §7] verbatim. This is possible because, after the above initial step of showing that $W$ defines a skewed stability function on each thin subcategory, the charge $Z$ and the masses of objects with respect to $\sigma$ play no role in the proof, one only uses the locally finite slicing $P$. Therefore the same argument goes through even though $\sigma$ is lax, and we can construct a unique pre-stability condition $\tau=(Q, W)$ with $d(P, Q)<\varepsilon$.

By Lemma 5.6 the slicing $Q$ restricts to a slicing $Q \cap \mathbf{N}$ on $\mathbf{N}$ with $d\left(P_{\mathbf{N}}, Q \cap \mathbf{N}\right)<\varepsilon$. Since $\sigma$ has massless subcategory N we know $Z \in \operatorname{Hom}\left(\Lambda / \Lambda_{\mathrm{N}}, \mathbb{C}\right)$ so that $\left.W\right|_{\Lambda_{N}}=W_{\mathrm{N}}$. Therefore $\tau=(Q, W)$ restricts to a pre-stability condition on N with charge $W_{\mathrm{N}}$ and slicing within distance $2 \varepsilon$ of $Q_{\mathrm{N}}$. By Corollary 4.18 it follows that the restriction is $\tau_{\mathrm{N}}$, i.e. that $Q \cap \mathrm{~N}=Q_{\mathrm{N}}$ as claimed.

Finally, we must verify that $\tau$ satisfies the support property when $\sigma$ and $\tau_{\mathrm{N}}$ do. We may assume that $\sigma$ and $\tau_{\mathrm{N}}$ satisfy $K$-support for the same constant $K$. Suppose $b \in Q(\varphi)$ and let
$S$ be a multi-set of its $\sigma$-stable factors. Since $d(P, Q)<\varepsilon$ these factors lie in $P(\varphi-\varepsilon, \varphi+\varepsilon) \subset$ $Q(\varphi-2 \varepsilon, \varphi+2 \varepsilon)$. Therefore

$$
m_{\tau}(b)=|W(b)| \geq \cos (2 \pi \varepsilon) \sum_{s \in S}|W(s)| .
$$

We consider massive factors in $S-\mathrm{N}$ and massless ones in $S \cap \mathrm{~N}$ separately. For a massive $\sigma$-stable factor $s$ the triangle inequality in $\mathbb{C}$, the norm estimate $\|W-Z\|_{\sigma}<\sin (\pi \varepsilon)$, and support for $\sigma$ imply that

$$
|W(s)| \geq|Z(s)|-|Z(s)-W(s)|>(1-\sin (\pi \varepsilon))|Z(s)| \geq \frac{1-\sin (\pi \varepsilon)}{K}\|v(s)\| .
$$

For a massless factor $s$ in $S \cap \mathrm{~N}$ consider the set $T$ of its $\tau$-semistable factors. Since $d(P, Q)<\varepsilon$ these lie in $Q(\psi-\varepsilon, \psi+\varepsilon)$ where $s \in P(\psi)$. Moreover, since the slicing $Q$ restricts to the slicing $Q_{\mathrm{N}}$ on N we know that each $t \in T$ is a $\tau_{\mathrm{N}}$-semistable object in N , and in particular that $W(t)=Z(t)+W_{\mathrm{N}}(t)=W_{\mathrm{N}}(t)$. Together with support for $\tau_{\mathrm{N}}$ and the triangle inequality for the norm $\|\cdot\|$ these observations yield the inequality

$$
|W(s)| \geq \cos (2 \pi \varepsilon) \sum_{t \in T}|W(t)|=\cos (2 \pi \varepsilon) \sum_{t \in T}\left|W_{\mathrm{N}}(t)\right| \geq \frac{\cos (2 \pi \varepsilon)}{K} \sum_{t \in T}\|v(t)\| \geq \frac{\cos (2 \pi \varepsilon)}{K}\|v(s)\| .
$$

Combining these estimates for the massive and massless factors of $b$, and using the triangle inequality for the norm again, gives

$$
m_{\tau}(b) \geq \frac{1}{L} \sum_{s \in S}\|v(s)\| \geq \frac{1}{L}\|v(b)\|
$$

where $L=K \max \{1 /(1-\sin (\pi \varepsilon)), 1 / \cos (\pi \varepsilon)\}$.
Hence $\tau$ satisfies the support property and so is a stability condition in $\operatorname{Stab}(\mathrm{C})$.
Example 6.3. Let $\mathrm{C}=\mathrm{D}^{b}\left(\mathbb{P}^{1}\right)$ and $\Lambda=K\left(\mathbb{P}^{1}\right) \cong \mathbb{Z}^{2}$ with basis $[\mathcal{O}],\left[\mathcal{O}_{x}\right]$. The inner product is chosen so that this basis is orthonormal. Let $\sigma=(P, Z)$ be the lax stability condition with charge $Z(\mathcal{O})=0, Z\left(\mathcal{O}_{x}\right)=-1$ and slicing $P=P_{b}$ from Example 3.14, i.e. $P(1)=$ $\left\langle\mathcal{O}_{x}, \mathcal{O}(n), \mathcal{O}(-n)[1]: x \in \mathbb{P}^{1}, n \in \mathbb{N}_{>0}\right\rangle$ and $P(1 / 2)=\langle\mathcal{O}\rangle$. The massless subcategory $\mathrm{N}=$ thick $(\mathcal{O})$ and the massive stable objects are, up to shifts, the skyscrapers $\mathcal{O}_{x}$ for $x \in \mathbb{P}^{1}$, the line bundle $\mathcal{O}(1)$ and the shifted line bundle $\mathcal{O}(-1)[1]$. This lax stability condition can be deformed to a classical one using the previous result. Let $\tau_{\mathrm{N}}=\left(Q_{\mathrm{N}}, W_{\mathrm{N}}\right)$ where $Q_{\mathrm{N}}=P_{\mathrm{N}}$ is the restricted slicing with $Q_{\mathrm{N}}(1 / 2)=\langle\mathcal{O}\rangle$ and $W_{\mathrm{N}}(\mathcal{O})=r i$ for some $r>0$. Considered as a charge in $\operatorname{Hom}(\Lambda, \mathbb{C})$ via the orthogonal splitting we also have $W_{\mathrm{N}}\left(\mathcal{O}_{x}\right)=0$. Therefore

$$
\left\|W_{\mathrm{N}}\right\|_{\sigma}=\sup \left\{\frac{\left|W_{\mathrm{N}}(c)\right|}{|Z(c)|}: c \text { massive } \sigma \text {-stable }\right\}=\sup \left\{\frac{r}{n}: n \neq 0\right\}=r
$$

and the conditions of Theorem 6.1 are satisfied. The deformed stability condition $\tau=(Q, W)$ has charge $W\left(\mathcal{O}_{x}\right)=-1, W(\mathcal{O}(n))=-n+r i$ and heart $Q(0,1]=\operatorname{coh}\left(\mathbb{P}^{1}\right)$. Note that $d(P, Q)=$ $\arctan (r)$ so that the slicing converges to $P$ as $r \rightarrow 0$.
Example 6.4. In contrast there are lax pre-stability conditions which cannot be deformed to classical ones. Again on $\mathrm{C}=\mathrm{D}^{b}\left(\mathbb{P}^{1}\right)$, let $\tau=(Q, W)$ be defined by the charge $W\left(\mathcal{O}_{x}\right)=0$, $W(\mathcal{O})=i$ and slicing $Q=P_{t}$ from Example 3.14, i.e. $Q(1 / 2)=\langle\mathcal{O}(n): n \in \mathbb{Z}\rangle$ and $Q(1)=$ $\left\langle\mathcal{O}_{x}: x \in \mathbb{P}^{1}\right\rangle$. The massless subcategory $\mathrm{N}=\operatorname{thick}\left(\mathcal{O}_{x}: x \in \mathbb{P}^{1}\right)$ and the massive stable objects are, up to shifts, the line bundles $\mathcal{O}(n)$ for $n \in \mathbb{Z}$. Let $\tau_{\mathrm{N}}=\left(Q_{\mathrm{N}}, W_{\mathrm{N}}\right)$ where $W_{\mathrm{N}}\left(\mathcal{O}_{x}\right)=w$ for some $0 \neq w \in \mathbb{C}$ and $Q_{N}$ is a compatible slicing. Considering $W_{\mathrm{N}}$ as a charge in $\operatorname{Hom}(\Lambda, \mathbb{C})$ via the orthogonal splitting we also have $W_{\mathrm{N}}(\mathcal{O})=0$. Therefore

$$
\left\|W_{\mathrm{N}}\right\|_{\tau}=\sup \left\{\frac{\left|W_{\mathrm{N}}(c)\right|}{|W(c)|}: c \text { massive } \tau \text {-stable }\right\}=\sup \{|n \| w|: n \in \mathbb{Z}\}=\infty .
$$

Thus the conditions of Theorem 6.1 are not satisfied. Indeed we have already seen in Examples 4.12 and 4.17 that $\tau$ is not a lax stability condition, so not in $\operatorname{Stab}^{L}$ (C), because it does not satisfy the support property as well as not being in the closure of $\operatorname{Stab}(\mathrm{C})$.

Theorem 6.1 leads to the following inductive criterion for recognising when a lax stability condition is in $\mathrm{Stab}^{L}(\mathrm{C})$.

Corollary 6.5. Suppose $\sigma \in \operatorname{Stab}^{l}(\mathrm{C})$ is a lax stability condition with massless subcategory N . Then $\sigma \in \operatorname{Stab}^{L}(\mathrm{C}) \Longleftrightarrow \rho_{\mathrm{N}}(\sigma) \in \operatorname{Stab}^{L}(\mathrm{~N})$.
Proof. If $\sigma=(P, Z) \in \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ then $\rho_{\mathrm{N}}(\sigma) \in \operatorname{Stab}^{L}(\mathrm{~N})$ by Lemma 5.6.
Conversely, if $\rho_{\mathrm{N}}(\sigma)=\left(P_{\mathrm{N}}, 0\right) \in \operatorname{Stab}^{L}(\mathrm{~N})$ then we can choose a sequence of stability conditions $\left(Q_{n}, W_{n}\right) \in \operatorname{Stab}(\mathbb{N})$ converging to $\left(P_{\mathrm{N}}, 0\right)$ in the sense that $d\left(P_{\mathrm{N}}, Q_{n}\right) \rightarrow 0$ and $W_{n} \rightarrow 0$ in the operator norm on $\operatorname{Hom}\left(\Lambda_{\mathrm{N}}, \mathbb{C}\right)$. It follows that $W_{n} \rightarrow 0$ in the operator norm on $\operatorname{Hom}(\Lambda, \mathbb{C})$, where as usual we consider $W_{n} \in \operatorname{Hom}(\Lambda, \mathbb{C})$ via the fixed splitting $\operatorname{Hom}\left(\Lambda_{N}, \mathbb{C}\right) \hookrightarrow \operatorname{Hom}(\Lambda, \mathbb{C})$. Since $\sigma$ satisfies the support property this implies $\left\|W_{n}\right\|_{\sigma} \rightarrow 0$. Therefore we can apply Theorem 6.1 to lift this sequence uniquely to a sequence of stability conditions $\left(P_{n}, Z+W_{n}\right) \in \operatorname{Stab}(\mathrm{C})$ converging to $(P, Z)=\sigma$. Hence $\sigma \in \operatorname{Stab}^{L}(\mathrm{C})$ as claimed.

The above criterion is tautological, and useless, when every object is massless i.e. when $\mathrm{N}=\mathrm{C}$. In that case we have the following result.

Lemma 6.6. Suppose $P$ is a slicing. Then

$$
(P, 0) \in \operatorname{Stab}^{L}(\mathrm{C}) \Longleftrightarrow P \in \overline{\{Q:(Q, W) \in \operatorname{Stab}(\mathrm{C})\}}
$$

Proof. Suppose $(P, 0) \in \operatorname{Stab}^{L}(\mathrm{C})$. Then there is a sequence $\left(P_{n}, Z_{n}\right)$ of stability conditions in $\operatorname{Stab}(\mathrm{C})$ converging to it, in particular with $P_{n} \rightarrow P$.

Conversely, suppose $P_{n} \rightarrow P$ and there exists a charge $Z_{n}$ such that $\left(P_{n}, Z_{n}\right) \in \operatorname{Stab}(\mathrm{C})$ for each $n \in \mathbb{N}$. Then $P$ is locally finite by Lemma 5.3. Moreover, $\left(P_{n}, Z_{n} / n\left\|Z_{n}\right\|\right) \in \operatorname{Stab}(\mathrm{C})$ and converges to $(P, 0)$ in $\operatorname{Slice}(\mathrm{C}) \times \operatorname{Hom}(\Lambda, \mathbb{C})$ as $n \rightarrow \infty$. So $(P, 0) \in \overline{\operatorname{Stab}(\mathrm{C})}$ and since the support property is automatic when all objects are massless in fact $(P, 0) \in \operatorname{Stab}^{L}(\mathrm{C})$.
Example 6.7. Suppose $\mathrm{C}=\mathrm{D}^{b}\left(\mathbf{k} A_{2}\right)$, where $A_{2}$ is the quiver $1 \longrightarrow 2$. Define a sequence of stability conditions $\tau_{n}=\left(Q_{n}, W_{n}\right)$ via $W_{n}\left(S_{1}\right)=-1 / n$ and $W_{n}\left(S_{2}\right)=i / e^{n}$, where $S_{1}$ and $S_{2}$ are the simple modules at 1 and 2 in the standard heart, respectively. Note that the indecomposable projective module $P_{1}$ in $\bmod \left(\mathbf{k} A_{2}\right)$ is also $\tau_{n}$-semistable. The limit slicing $Q=\lim _{n \rightarrow \infty} Q_{n}$ is given by $Q(1 / 2)=\operatorname{add}\left(S_{2}\right)$ and $Q(1)=\operatorname{add}\left(S_{1} \oplus P_{1}\right)$. Note that $Q$ is not a slicing for any (pre-)stability condition on $\mathrm{D}^{b}\left(\mathbf{k} A_{2}\right)$ because the slice $Q(1)$ is not abelian.

We now refine Theorem 6.1 by considering deformations of a lax pre-stability condition with massless subcategory N to one with massless subcategory $\mathrm{M} \subset \mathrm{N}$. We saw in the proof of Proposition 5.10 that if $\sigma=(P, Z)$ is in the closure of $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{M})$ then $P$ is well-adapted to M. Therefore it is natural to impose this condition. The result is weaker than Theorem 6.1 in that even if we start with lax stability conditions on C and N , we only show that the resulting deformation is a lax pre-stability condition which satisfies support on the quotient $C / M$ but not necessarily on C ; the set of such is denoted $\mathrm{Stab}^{l s}(\mathrm{C}, \mathrm{M})$ below. In the notation of Remark 6.2, we get a continuous map

$$
\begin{equation*}
\delta_{\mathrm{N}, \mathrm{M}}: \operatorname{Stab}^{l}(\mathrm{C}, \mathrm{~N}) \times_{\text {Slice }(\mathrm{N})} \operatorname{Stab}^{l}(\mathrm{~N}, \mathrm{M}) \longrightarrow \operatorname{Stab}^{l s}(\mathrm{C}, \mathrm{M}) \tag{6}
\end{equation*}
$$

where the domain is the set of pairs $\left(\sigma, \tau_{\mathrm{N}}\right)$ whose slicings agree on N , and are well-adapted to the subcategory M . It is not clear that this is an open subset of the fibre product. It is a natural question when the image of this map is within lax stability conditions; this is discussed in Subsection 6.2.

Proposition 6.8. Suppose $\sigma=(P, Z)$ is a lax pre-stability condition on C with massless subcategory N such that $P$ is well-adapted to the thick subcategory M of N . Then there is some $0<\varepsilon_{0}<1 / 8$ such that for any $0<\varepsilon<\varepsilon_{0}$ and lax pre-stability condition $\tau_{\mathrm{N}}=\left(Q_{\mathrm{N}}, W_{\mathrm{N}}\right)$ on N with

- massless subcategory M ,
- $\left\|W_{\mathrm{N}}\right\|_{\sigma}<\sin (\pi \varepsilon)$
- and $d\left(P_{\mathrm{N}}, Q_{\mathrm{N}}\right)<\varepsilon$
there is a unique lax pre-stability condition $\tau=(Q, W)$ on C with
- massless subcategory M ,
- $W=Z+W_{\mathrm{N}}$,
- restricted slicing $Q \cap \mathrm{~N}=Q_{\mathrm{N}}$
- and $d(P, Q)<\varepsilon$.

If in addition $\sigma \in \operatorname{Stab}^{l}(\mathrm{C})$ and $\tau_{\mathrm{N}} \in \operatorname{Stab}^{l}(\mathrm{~N})$ then $\tau \in \operatorname{Stab}^{l s}(\mathrm{C})$.
Remark 6.9. We do not know if a version of the last statement holds that includes the closure property, i.e. whether $\sigma \in \operatorname{Stab}^{L}(\mathrm{C})$ and $\tau_{\mathrm{N}} \in \operatorname{Stab}^{L}(\mathrm{~N})$ implies $\tau \in \operatorname{Stab}^{L S}(\mathrm{C})$.

Proof. For $\mathrm{M}=0$ this is Theorem 6.1. When $\mathrm{M} \neq 0$ the strategy is to reduce to this case by taking the quotient by M and then lifting back up to C using Proposition 3.10.

We first observe that $Q_{\mathrm{N}}$ is well-adapted to M . It is adapted to M by Proposition 4.4, so we just need to see that $Q_{\mathrm{N} / \mathrm{M}}$ is locally finite. The restricted slicing $P_{\mathrm{C} / \mathrm{M}}(\varphi) \cap \mathrm{N} / \mathrm{M}=P_{\mathrm{N} / \mathrm{M}}(\varphi)$ for each $\varphi \in \mathbb{R}$ because the proof of Proposition 3.2 shows that each is the full subcategory of $\mathrm{N} / \mathrm{M}$ consisting of those objects having an HNfiltration with one factor in $P(\varphi) \cap \mathrm{N}$ and all others in M . Hence, since $d\left(P_{\mathrm{N} / \mathrm{M}}, Q_{\mathrm{N} / \mathrm{M}}\right) \leq d\left(P_{\mathrm{N}}, Q_{\mathrm{N}}\right)<\varepsilon$ by Corollary 3.7, and $P_{\mathrm{N} / \mathrm{M}}$ is a locally finite slicing, so is $Q_{\mathrm{N} / \mathrm{M}}$. Now, by Proposition $4.13, \tau_{\mathrm{N}}$ induces a pre-stability condition $\mu_{\mathrm{M}}\left(\tau_{\mathrm{N}}\right)=\left(Q_{\mathrm{N} / \mathrm{M}}, W_{\mathrm{N}}\right)$ on $\mathrm{N} / \mathrm{M}$. Moreover, since $P$ is well-adapted to M, Remark 4.14 shows there is a lax pre-stability condition $\mu_{\mathrm{M}}(\sigma)=\left(P_{\mathrm{C} / \mathrm{M}}, Z\right)$ on $\mathrm{C} / \mathrm{M}$ with massless subcategory $\mathrm{N} / \mathrm{M}$. We verify that these satisfy the conditions of Theorem 6.1.

The charge $W_{N}$ is in $\operatorname{Hom}\left(\Lambda_{N} / \Lambda_{M}, \mathbb{C}\right)$ and the splitting $\operatorname{Hom}\left(\Lambda_{N}, \mathbb{C}\right) \hookrightarrow \operatorname{Hom}(\Lambda, \mathbb{C})$ restricts to one $\operatorname{Hom}\left(\Lambda_{N} / \Lambda_{\mathrm{M}}, \mathbb{C}\right) \hookrightarrow \operatorname{Hom}\left(\Lambda / \Lambda_{\mathrm{M}}, \mathbb{C}\right)$. We use this to consider $W_{\mathrm{N}}$ as an element of $\operatorname{Hom}\left(\Lambda / \Lambda_{\mathrm{M}}, \mathbb{C}\right)$. With this identification $\left\|W_{\mathrm{N}}\right\|_{\mu_{\mathrm{M}}(\sigma)} \leq\left\|W_{\mathrm{N}}\right\|_{\sigma}<\sin (\pi \varepsilon)$ by Lemma 4.15. Since the restricted slicing $P_{\mathrm{C} / \mathrm{M}}(\varphi) \cap \mathrm{N} / \mathrm{M}=P_{\mathrm{N} / \mathrm{M}}(\varphi)$ for each $\varphi \in \mathbb{R}$ and $d\left(P_{\mathrm{N} / \mathrm{M}}, Q_{\mathrm{N} / \mathrm{M}}\right) \leq$ $d\left(P_{\mathrm{N}}, Q_{\mathrm{N}}\right)<\varepsilon$ the conditions of Theorem 6.1 are satisfied. Applying that result we construct a pre-stability condition $\left(Q_{\mathrm{C} / \mathrm{M}}, W_{\mathrm{C} / \mathrm{M}}\right)$ on $\mathrm{C} / \mathrm{M}$ where $W_{\mathrm{C} / \mathrm{M}}=Z+W_{\mathrm{N}}$ and $d\left(P_{\mathrm{C} / \mathrm{M}}, Q_{\mathrm{C} / \mathrm{M}}\right)<\varepsilon$.

Since $d\left(P_{\mathrm{M}}, Q_{\mathrm{M}}\right) \leq d\left(P_{\mathrm{N}}, Q_{\mathrm{N}}\right)<\varepsilon$ we can use Proposition 3.10 to glue $Q_{\mathrm{M}}$ and $Q_{\mathrm{C} / \mathrm{M}}$ to a locally finite slicing $Q$ with $d(P, Q)<\varepsilon$ by Lemma 3.11. By Lemma 5.6 this slicing $Q$ restricts to N . It follows from the construction that the restriction is the slicing glued from $Q_{\mathrm{M}}$ and $Q_{\mathrm{N} / \mathrm{M}}$, which by uniqueness is $Q_{\mathrm{N}}$. Thus we have constructed a lax pre-stability condition $\tau=(Q, W)$ with $d(P, Q)<\varepsilon$, massless subcategory M , charge $W=Z+W_{\mathrm{N}}$ and slicing $Q$ restricting to $Q_{\mathrm{N}}$ on N. Corollary 4.18 implies that $\tau$ is unique with these properties.

Finally, when $\sigma$ and $\tau_{N}$ are lax stability conditions then $\mu_{M}(\sigma)$ is a lax stability condition and $\mu_{\mathrm{M}}\left(\tau_{\mathrm{N}}\right)$ a stability condition. Therefore by the last part of Theorem 6.1 the pre-stability condition on the quotient $\mu_{\mathrm{M}}(\tau)=\left(Q_{\mathrm{C} / \mathrm{M}}, Z+W_{\mathrm{N}}\right)$ is in $\operatorname{Stab}(\mathrm{C} / \mathrm{M})$. Hence, $\tau \in \operatorname{Stab}^{l s}(\mathrm{C})$ as claimed.

The deformations described in this result are in general neither purely normal nor fibrewise, but a mixture of the two: the charge $W$ is changed by $W_{N} \in \operatorname{Hom}\left(\Lambda_{N}, \mathbb{C}\right)$, so for $\mathrm{M} \neq \mathrm{N}$, a component of the deformation occurs in the normal direction, but for $M \neq 0$ the slicing on $N$ is not fully determined by $W_{\mathrm{N}}$, and the choice of $Q_{\mathrm{N}}$ determines the fibrewise deformation. When $\mathrm{M}=\mathrm{N}$ the deformation is purely fibrewise since the charge, indeed the associated pre-stability condition on the quotient, is fixed and only the massless slicing is deformed.

Corollary 6.10. Suppose $\sigma=(P, Z)$ is a lax pre-stability condition on $C$ with massless subcategory N . Then there is some $\varepsilon_{0}>0$ such that for any $0<\varepsilon<\varepsilon_{0}$ and $Q_{\mathrm{N}} \in \operatorname{Slice}(\mathrm{N})$ with $d\left(P_{\mathrm{N}}, Q_{\mathrm{N}}\right)<\varepsilon$ there is a unique lax pre-stability condition $\tau=(Q, Z)$ on C with

- massless subcategory N
- massless slicing $Q_{\mathrm{N}}$
- and $d(P, Q)<\varepsilon$.

If $\sigma$ is a lax stability condition then we may choose $\varepsilon_{0}=1 / 4$.

Proof. This is the special case $\mathrm{M}=\mathrm{N}$ of Proposition 6.8, i.e. $\tau_{\mathrm{N}}=\left(Q_{\mathrm{N}}, 0\right)$ except for the last statement. If $\sigma$ is a lax stability condition then $P(I)$ is length for any interval $I$ of length strictly less than one by Lemma 5.3. Therefore we may choose $\varepsilon_{0}=1 / 4$.

Finally, we consider tangential deformations where the charge is varied in $\operatorname{Hom}\left(\Lambda / \Lambda_{\mathrm{N}}, \mathbb{C}\right)$ and the massless slicing remains fixed.
Proposition 6.11. Suppose $\sigma=(P, Z)$ is a lax pre-stability condition on C with massless subcategory N . Then for any $0<\varepsilon<1 / 8$ and $W \in \operatorname{Hom}\left(\Lambda / \Lambda_{N}, \mathbb{C}\right)$ satisfying $\|W-Z\|_{\sigma}<$ $\sin (\pi \varepsilon)$ there is a unique lax pre-stability condition $\tau=(Q, W)$ with $d(P, Q)<\varepsilon$ and massless slicing $Q_{\mathrm{N}}=P_{\mathrm{N}}$. If in addition $\sigma \in \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ and $\tau \in \operatorname{Stab}^{l}(\mathrm{C})$ then $\tau \in \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$.
Proof. By Lemma 4.15, $\|W-Z\|_{\mu_{\mathrm{N}}(\sigma)}=\|W-Z\|_{\sigma}<\sin (\pi \varepsilon)$ where $\mu_{\mathrm{N}}(\sigma)=\left(P_{\mathrm{C} / \mathrm{N}}, Z\right)$ is the induced stability condition in $\operatorname{Stab}(\mathrm{C} / \mathrm{N})$. Therefore, Theorem 4.1 allows us to construct a stability condition $\left(Q_{\mathrm{C} / \mathrm{N}}, W\right)$ in $\operatorname{Stab}(\mathrm{C} / \mathrm{N})$ with $d\left(P_{\mathrm{C} / \mathrm{N}}, Q_{\mathrm{C} / \mathrm{N}}\right)<\varepsilon$.

By Proposition 3.10 the slicings $P_{\mathrm{N}}$ and $Q_{\mathrm{C} / \mathrm{N}}$ can be glued to a locally finite slicing $Q$ on C. Since $Q(\varphi) \subseteq Q_{\mathrm{C} / \mathrm{N}}(\varphi)$ for all $\varphi \in \mathbb{R}$, we have thus constructed a lax pre-stability condition $\tau=(Q, W)$ with massless subcategory N , restricted slicing $Q_{\mathrm{N}}=P_{\mathrm{N}}$, and such that $d(P, Q)<\varepsilon$. Uniqueness follows from Corollary 4.18. Since $P_{\mathrm{N}}=Q_{\mathrm{N}}$ the last statement follows from Corollary 6.5 .
6.2. Support propagation. In the classical case support propagates in components of the stability space: all nearby deformations of a stability condition are also stability conditions, and not just pre-stability conditions. We now discuss the extent to which this remains true for lax stability conditions.
Definition 6.12. We say that support propagates from

- a lax stability condition $\sigma \in \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ if there is $\varepsilon>0$ such that any $\tau=(Q, W) \in$ $B_{\varepsilon}(\sigma)$ with $\left\|Z-\left(W-W_{\mathrm{N}}\right)\right\|_{\sigma}<\sin (\pi \varepsilon)$ and $\rho_{\mathrm{N}}(\tau) \in \operatorname{Stab}^{L}(\mathrm{~N})$ is in $\operatorname{Stab}^{L}(\mathrm{C})$. See Figure 1 for a schematic illustration of the charge conditions.
- a component $\Sigma$ of $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ if there is an $\varepsilon>0$ such that support propagates from all $\sigma \in \Sigma$ with respect to $\varepsilon$.
- $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ if it propagates from all components.

This condition on the lax stability condition $\sigma$ means that nearby lax pre-stability conditions $\tau$ have the support property provided $\left\|Z-\left(W-W_{\mathrm{N}}\right)\right\|_{\sigma}<\sin (\pi \varepsilon)$ and $\rho_{\mathrm{N}}(\tau) \in \operatorname{Stab}^{L}(\mathrm{~N})$. The last condition is necessary because it is implied by $\tau \in B_{\varepsilon}(\sigma) \cap \operatorname{Stab}^{L}(\mathrm{C})$. We illustrate this in the degenerate example below.
Example 6.13. Consider the classical geometric stability condition $\sigma_{g}=\left(P_{g}, Z_{g}\right)$ on $\mathrm{C}=$ $\mathrm{D}^{b}\left(\mathbb{P}^{1}\right)$ from Examples 3.14 and 5.4. By sending all charges to zero uniformly, we obtain a lax stability condition $\sigma=\left(P_{g}, 0\right)$ with massless subcategory $\mathrm{N}=\mathrm{C}$. Consider the lax stability condition $\tau=(Q, 0)$ given by $Q(\varphi)=P_{g}(\varphi)$ for $\varphi \notin \frac{1}{2} \mathbb{Z}$ and $Q\left(\frac{1}{2}+\varepsilon\right)=P_{g}\left(\frac{1}{2}\right)=\langle\mathcal{O}\rangle$ for sufficiently small $\varepsilon>0$. Since we have not changed the phase of any other slices, $Q$ cannot be compatible with any linear charge. Hence, $\tau \in B_{\varepsilon}(\sigma)$ is a lax stability condition with the same charge as $\sigma$ but $\tau \notin \operatorname{Stab}^{L}(\mathrm{~N})=\operatorname{Stab}^{L}(\mathrm{C})$.
Example 6.14. Support propagates in the following cases:
(1) For $\operatorname{Stab}(C)$, i.e. classical stability conditions, by Theorem 4.1.
(2) For $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{C})$, i.e. for lax stability conditions $(P, 0)$ with massless subcategory $\mathrm{N}=\mathrm{C}$. The propagation condition is tautological in this case because $\|\cdot\|_{\sigma}=0$ and $\rho_{\mathrm{N}}=\mathrm{id}$.
(3) For $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ where $\mathrm{rk}\left(\Lambda_{\mathrm{N}}\right)=1$. This result is a combination of Theorem 6.1 (normal deformations) and Theorem 9.4 (tangential deformations). The condition that $\rho_{\mathrm{N}}(\tau) \in$ $\operatorname{Stab}^{L}(\mathrm{~N})$ is necessary in this case.
(4) For any component of $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ in the boundary of a finite type component $\operatorname{Stab}(\mathrm{C})$, i.e. a component in which every heart is a length abelian category, by Corollary 10.7. In this case we take $\Lambda=K(\mathrm{C})$.

$$
\|W-Z\|_{\sigma}<\sin (\pi \varepsilon)
$$



Figure 1. The definition of support propagation from $\sigma=(P, Z) \in \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ imposes two conditions on the charge of $\tau=(Q, W)$. These are illustrated schematically above. We require two conditions because $\|\cdot\|_{\sigma}$ is only a semi-norm. When $Z-W$ is in the subspace $\operatorname{Hom}\left(\Lambda / \Lambda_{N}, \mathbb{C}\right)$ on which it restricts to a norm the conditions coincide, and agree with the condition $\|W-Z\|_{\mu_{N}(\sigma)}<\sin (\pi \varepsilon)$ for the quotient stability condition $\mu_{\mathrm{N}}(\sigma)$. In particular when $\mathrm{N}=0$ they reduce to the usual condition in the classical case. At the other extreme, when $\mathrm{N}=\mathrm{C}$, the conditions are vacuous because $\|\cdot\|_{\sigma}=0$. Another special case which we use frequently is when $Z-W \in \operatorname{Hom}\left(\Lambda_{\mathrm{N}}, \mathbb{C}\right)$ in which case they again reduce to $\|W-Z\|_{\sigma}<\sin (\pi \varepsilon)$.

We do not know of any examples in which support does not propagate. However, it is entirely possible that such examples exist.

The next result is a refinement of Corollary 6.5.
Corollary 6.15. Suppose support propagates from $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$. Let $\sigma$ be a lax stability condition with massless subcategory N and $\mathrm{M} \subset \mathrm{N}$ be a thick subcategory. Then

$$
\sigma \in \overline{\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{M})} \Longleftrightarrow \rho_{\mathrm{N}}(\sigma) \in \overline{\operatorname{Stab}^{L}(\mathrm{~N}, \mathrm{M})}
$$

Proof. If $\sigma=(P, Z) \in \overline{\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{M})}$ then the continuity of $\rho_{\mathrm{N}}$ implies $\rho_{\mathrm{N}}(\sigma) \in \overline{\operatorname{Stab}^{L}(\mathrm{~N}, \mathrm{M})}$.
Now suppose that $\rho_{\mathrm{N}}(\sigma)=\left(P_{\mathrm{N}}, 0\right)$ is in the closure of $\operatorname{Stab}^{L}(\mathrm{~N}, \mathrm{M})$. We always have $\overline{\operatorname{Stab}^{L}(\mathrm{~N}, \mathrm{M})} \subseteq \operatorname{Stab}^{l}(\mathrm{~N}) \cap \overline{\operatorname{Stab}(\mathrm{N})}=\operatorname{Stab}^{L}(\mathrm{~N})$. Thus $\sigma \in \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ by Corollary 6.5. Next, $P_{\mathrm{N}}$ is well-adapted to M by the proof of Proposition 5.10, i.e. $P_{\mathrm{N} / \mathrm{M}}$ is locally finite and $P_{\mathrm{N}}$ is adapted to M . Thus $P$, being well-adapted to N by Proposition 4.13, is also adapted to M and moreover $P_{\mathrm{C} / \mathrm{N}}$ is locally finite. As $P_{\mathrm{C} / \mathrm{M}}$ is compatible with the pair $\left(P_{\mathrm{N} / \mathrm{M}}, P_{\mathrm{C} / \mathrm{N}}\right)$, we conclude from Lemma 3.3 that $P_{\mathrm{C} / \mathrm{M}}$ is locally finite. Hence $P$ is actually well-adapted to M.
Thus we can apply Proposition 6.8 to $\sigma$ and a sequence of lax stability conditions in $\operatorname{Stab}^{L}(\mathrm{~N}, \mathrm{M})$ converging to $\rho_{\mathrm{N}}(\sigma)$ to construct a sequence of lax pre-stability conditions with massless subcategory M which satisfy support on the quotient $\mathrm{C} / \mathrm{M}$ and which converge to $\sigma$. We conclude by noting that support propagation implies this is eventually a sequence of lax stability conditions in $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{M})$.

Remark 6.16. If we do not assume support propagates then we can only conclude that $\sigma$ is in $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ and is in the limit in $\operatorname{Slice}(\mathrm{C}) \times \operatorname{Hom}(\Lambda, \mathbb{C})$ of a sequence of lax pre-stability conditions with massless subcategory $M$ which satisfy support on the quotient $C / M$.

Definition 6.12 asks for two properties of lax pre-stability conditions occurring as small deformations of $\sigma \in \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ : firstly the support property and secondly the property of being in the closure of $\operatorname{Stab}(\mathrm{C})$. The next result shows that the latter is automatic. Thus the terminology 'support propagation' is justified.
Theorem 6.17. Suppose $\sigma=(P, Z) \in \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$. Then for any sufficiently small $\varepsilon>0$ the set of lax pre-stability conditions $\tau=(Q, W) \in B_{\varepsilon}(\sigma)$ with $\left\|Z-\left(W-W_{\mathrm{N}}\right)\right\|_{\sigma}<\sin (\pi \varepsilon)$ and $\rho_{\mathrm{N}}(\tau) \in \operatorname{Stab}^{L}(\mathrm{~N})$ is contained in $\overline{\operatorname{Stab}(\mathrm{C})}$.

Proof. Let $\sigma=(P, Z) \in \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ so that $\rho_{\mathrm{N}}(\sigma)=\left(P_{\mathrm{N}}, Z_{\mathrm{N}}=0\right)$. Since $\sigma$ satisfies the support property there is some $K>0$ for which $\|v(c)\| \leq K|Z(c)|$ whenever $c$ is a massive stable object. Recall from Proposition 4.11 that this implies that $\|\cdot\|_{\sigma} \leq K\|\cdot\|$ on $\operatorname{Hom}(\Lambda, \mathbb{C})$.

Let $\tau=(Q, W) \in B_{\varepsilon}(\sigma)$ be a lax pre-stability condition with $\rho_{\mathrm{N}}(\tau) \in \operatorname{Stab}^{L}(\mathrm{~N})$ and $\| Z-$ $\left(W-W_{\mathrm{N}}\right) \|_{\sigma}<\sin (\pi \varepsilon)$. Thus $d\left(P_{\mathrm{N}}, Q_{\mathrm{N}}\right) \leq d(P, Q)<\varepsilon$ by Corollary 3.7 and

$$
\left\|W_{\mathrm{N}}\right\|_{\sigma} \leq\|Z-W\|_{\sigma}+\left\|Z-\left(W-W_{\mathrm{N}}\right)\right\|_{\sigma}<2 \sin (\pi \varepsilon)
$$

by the triangle inequality. To construct $\tau^{\prime} \in \operatorname{Stab}(\mathrm{C})$ suitably close to $\tau$ we proceed in a sequence of steps. We first deform $\sigma$ to $\sigma^{\prime} \in \operatorname{Stab}(\mathrm{C})$ using Theorem 6.1. We then apply Bridgeland's deformation theorem, Theorem 4.1, to deform the charge of $\sigma^{\prime}$ to obtain a stability condition $\tau^{\prime}$ whose charge is closer to that of $\tau$. Finally, we check that $\tau^{\prime}$ is indeed suitably close to $\tau$.

Step 1: Deforming $\sigma$ to $\sigma^{\prime} \in \operatorname{Stab}(\mathrm{C})$.
Since $\rho_{\mathrm{N}}(\tau) \in \operatorname{Stab}^{L}(\mathrm{~N}) \subset \overline{\operatorname{Stab}(\mathbf{N})}$, for any $\varepsilon>0$ there is $\tau_{\mathrm{N}}^{\prime}=\left(Q_{\mathrm{N}}^{\prime}, W_{\mathrm{N}}+W_{\mathrm{N}}^{\prime}\right) \in \operatorname{Stab}(\mathrm{N})$ such that $d\left(Q_{\mathrm{N}}, Q_{\mathrm{N}}^{\prime}\right)<\varepsilon$ and $\left\|\left(W_{\mathrm{N}}+W_{\mathrm{N}}^{\prime}\right)-W_{\mathrm{N}}\right\|=\left\|W_{\mathrm{N}}^{\prime}\right\|<\sin (\pi \varepsilon) / K$. The latter inequality implies $\left\|W_{N}^{\prime}\right\|_{\sigma} \leq \sin (\pi \varepsilon)$ and then the triangle inequality gives

$$
d\left(P_{\mathrm{N}}, Q_{\mathrm{N}}^{\prime}\right)<2 \varepsilon \quad \text { and } \quad\left\|W_{\mathrm{N}}+W_{\mathrm{N}}^{\prime}\right\|_{\sigma}<3 \sin (\pi \varepsilon) .
$$

Set $\delta:=\max \{2 \varepsilon, \arcsin (3 \sin (\pi \varepsilon)) / \pi\}$ and note that $\delta \rightarrow 0$ as $\varepsilon \rightarrow 0$. In particular we can choose $\varepsilon>0$ sufficiently small that $\delta<1 / 8$. By Theorem 6.1 there is a unique stability condition $\sigma^{\prime}=\left(P^{\prime}, Z^{\prime}\right) \in B_{\delta}(\sigma)$ with charge $Z^{\prime}=Z+W_{\mathrm{N}}+W_{\mathrm{N}}^{\prime}$ and restricted slicing $P^{\prime} \cap \mathrm{N}=Q_{\mathrm{N}}^{\prime}$.
Step 2: Deform the charge of $\sigma^{\prime}$ to get $\tau^{\prime} \in \operatorname{Stab}(\mathrm{C})$.
Set $U:=W-Z-W_{\mathrm{N}} \in \operatorname{Hom}\left(\Lambda / \Lambda_{\mathrm{N}}, \mathbb{C}\right)$. Then by Lemma 5.11

$$
\|U\|_{\sigma^{\prime}} \leq \frac{\|U\|_{\sigma}}{C} \leq \frac{\|W-Z\|_{\sigma}+\left\|W_{N}^{\prime}\right\|_{\sigma}}{C} \leq \frac{2 \sin (\pi \varepsilon)}{C}<\sin (3 \pi \varepsilon)
$$

where $C:=(1-\sin (\pi \delta)) \cos (2 \pi \delta) \rightarrow 1$ as $\varepsilon \rightarrow 0$ so that the final inequality holds for sufficiently small $\varepsilon>0$. Possibly further shrinking $\varepsilon$ we can apply Theorem 4.1 to obtain a stability condition $\tau^{\prime}=\left(Q^{\prime}, W^{\prime}\right) \in B_{3 \varepsilon}\left(\sigma^{\prime}\right)$ with $W^{\prime}=Z^{\prime}+U=W+W_{\mathrm{N}}^{\prime}$.

Step 3: $\tau^{\prime}$ is close to $\tau$.
By construction $\left\|W^{\prime}-W\right\|=\left\|W_{N}^{\prime}\right\|<\sin (\pi \varepsilon)$. Therefore it suffices to show that $d\left(Q^{\prime}, Q\right)<\varepsilon$. Consider a $\tau^{\prime}$-semistable object $b \in Q^{\prime}(\varphi)$. Using $d(P, Q)<\varepsilon, d\left(P, P^{\prime}\right)<\delta$ and $d\left(P^{\prime}, Q^{\prime}\right)<3 \varepsilon$ gives $d\left(Q, Q^{\prime}\right)<\delta+4 \varepsilon$ and hence

$$
Q^{\prime}(\varphi) \subset Q(\varphi-\delta-4 \varepsilon, \varphi+\delta+4 \varepsilon) \subset P(\varphi-\delta-5 \varepsilon, \varphi+\delta+5 \varepsilon)=: \mathrm{A} .
$$

For sufficiently small $\varepsilon>0$ Lemma 5.3 ensures that A is a quasi-abelian length category. The simple objects in A are the $\sigma$-stable objects. The HNfiltration of $b$ with respect to $\tau$ is obtained by grouping the simple factors in some composition series of $b$ in A into a sequence of $\tau$-semistable objects of strictly decreasing phase. Therefore it suffices to show that the phases of these simple factors with respect to $\tau$ and $\tau^{\prime}$ differ by less than $\varepsilon$.

If $a \in \mathrm{~A}$ is a massless simple object then there is nothing to prove because $d\left(Q_{\mathrm{N}}^{\prime}, Q_{\mathbf{N}}\right)<\varepsilon$. If $a \in \mathrm{~A}$ is massive then it suffices to show that $\left|W_{\mathrm{N}}^{\prime}(a)\right| /|W(a)|$ can be made arbitrarily small, uniformly for all such $a$, by choosing $\left\|W_{N}^{\prime}\right\|$ sufficiently small. We do so by making a series of estimates.

The operator norm estimate and the support property for $\sigma$ provide a bound

$$
\left|W_{\mathrm{N}}^{\prime}(a)\right| \leq\left\|W_{\mathrm{N}}^{\prime}\right\| \cdot\|v(a)\| \leq K\left\|W_{\mathrm{N}}^{\prime}\right\| \cdot|Z(a)|
$$

for some constant $K>0$. Moreover, since $Z \in \operatorname{Hom}\left(\Lambda / \Lambda_{\mathrm{N}}, \mathbb{C}\right)$ and $d(P, Q)<\varepsilon$ we have

$$
|Z(a)| \leq \sum_{t \in T}|Z(t)| \leq \sum_{t \in T}| | Z \|_{\tau}|W(t)| \leq \frac{\|Z\|_{\tau}}{\cos (2 \pi \varepsilon)}|W(a)|
$$

where $T$ is a set of $\tau$-stable factors of the object $a$. Finally,

$$
\|Z\|_{\tau} \leq \frac{\|Z\|_{\sigma}}{(1-\sin (\pi \varepsilon)) \cos (2 \pi \varepsilon)}=\frac{1}{(1-\sin (\pi \varepsilon)) \cos (2 \pi \varepsilon)}
$$

by Lemma 5.11 because $Z \in \operatorname{Hom}\left(\Lambda / \Lambda_{\mathrm{N}}, \mathbb{C}\right)$. Combining these estimates, we have the bound

$$
\frac{\left|W_{N}^{\prime}(a)\right|}{|W(a)|} \leq \frac{K| | W_{N}^{\prime} \|}{(1-\sin (\pi \varepsilon)) \cos (2 \pi \varepsilon)^{2}}
$$

uniformly in $a$. The result follows because we can make the right hand side smaller than any given $\varepsilon>0$ by rescaling $W_{N}^{\prime}$.
Proposition 6.18. Suppose support propagates from $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$. Then the map $\delta_{\mathrm{N}}$ defined in Remark 6.2 extends to a continuous map

$$
\delta_{\mathrm{N}}: \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{~N}) \times \text { Slice }(\mathrm{N}) \operatorname{Stab}^{L}(\mathrm{~N}) \longrightarrow \operatorname{Stab}^{L}(\mathrm{C})
$$

defined on the open subset of pairs $\left(\sigma, \tau_{\mathrm{N}}\right)=\left((P, Z),\left(P \cap \mathrm{~N}, W_{\mathrm{N}}\right)\right)$ with $\left\|W_{\mathrm{N}}\right\|_{\sigma}<\sin (\pi \varepsilon)$ for some suitably small $\varepsilon>0$.

Proof. Let $\mathrm{M} \subset \mathrm{N}$ be the massless subcategory of $\tau_{\mathrm{N}}$. Then $P_{\mathrm{N}}=P \cap \mathrm{~N}$ is adapted to M by Proposition 4.4. It follows that the $P$-semistable factors of an object $c \in \mathrm{M}$ are the $P_{\mathrm{N}^{-}}$ semistable factors, thus are in M . Moreover, for any strict length one interval $I$ the intersection $P(I) \cap \mathrm{M}=P_{\mathrm{N}}(I) \cap \mathrm{M}$ is a Serre subcategory of $P_{\mathrm{N}}(I)$, which is in turn a Serre subcategory of $P(I)$. Hence, $P$ is adapted to M . Therefore, by Proposition 3.6, there is an induced slicing $P_{\mathrm{C} / \mathrm{M}}$ on $\mathrm{C} / \mathrm{M}$. This induced slicing is compatible with $P_{\mathrm{C} / \mathrm{N}}$ and $P_{\mathrm{N} / \mathrm{M}}$, both of which are locally finite because they are respectively the slicings of the classical stability conditions $\mu_{\mathrm{N}}(\sigma)$ and $\mu_{\mathrm{M}}\left(\tau_{\mathrm{N}}\right)$. Hence $P_{\mathrm{C} / \mathrm{M}}$ is locally finite by Lemma 3.3 so that $P$ is well-adapted to M .

Thus Proposition 6.8 lets us define $\delta_{\mathrm{N}}\left(\sigma, \tau_{\mathrm{N}}\right)$ as the unique lax pre-stability condition $(Q, W=$ $\left.Z+W_{\mathrm{N}}\right)$ with $d(P, Q)<\varepsilon$ and restricted slicing $Q \cap \mathrm{~N}=P_{\mathrm{N}}$. The assumption that support propagates implies $\delta_{\mathrm{N}}\left(\sigma, \tau_{\mathrm{N}}\right) \in \operatorname{Stab}^{L}(\mathrm{C})$ so that $\delta_{\mathrm{N}}$ is well-defined as a map of sets.

The restriction of $\delta_{\mathrm{N}}$ to (an open subset of) $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N}) \times_{\text {Slice( } \mathrm{N})} \operatorname{Stab}(\mathrm{N})$ is continuous by Remark 6.2. The proof of Theorem 6.17 shows that the extension is continuous on each normal slice $\{\sigma\} \times_{\text {Slice }(\mathbb{N})} \operatorname{Stab}^{L}(\mathrm{~N})$. Together these facts imply the extension is continuous.

## 7. The topology of the space of lax stability conditions

We apply the deformation results of $\S 6$ to study the topology of the space of lax stability conditions. Our starting point is the following consequence of support propagation.

Proposition 7.1. Suppose that support propagates from $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$. Then the following maps are homeomorphisms onto unions of components:
(1) $\mu_{\mathrm{N}} \times \rho_{\mathrm{N}}: \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N}) \rightarrow \operatorname{Stab}(\mathrm{C} / \mathrm{N}) \times \operatorname{Stab}^{L}(\mathrm{~N}, \mathrm{~N})$;
(2) $\mu_{\mathrm{N}} \times$ id: $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N}) \times{ }_{\text {Slice( }}$ ) $\operatorname{Stab}^{L}(\mathrm{~N}) \rightarrow \operatorname{Stab}(\mathrm{C} / \mathrm{N}) \times \operatorname{Stab}^{L}(\mathrm{~N})$.

Here $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N}) \times_{\text {Slice }(\mathbb{N})} \operatorname{Stab}^{L}(\mathrm{~N})$ denotes the set of pairs $(P, Z)$ in $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ and $\left(Q_{\mathrm{N}}, W_{\mathrm{N}}\right)$ in $\operatorname{Stab}^{L}(\mathrm{~N})$ such that the restricted slicing $P_{\mathrm{N}}=Q_{\mathrm{N}}$.

Proof. (1) The quotient map $\mu_{\mathrm{N}}: \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N}) \rightarrow \operatorname{Stab}(\mathrm{C} / \mathrm{N})$ is well-defined by Proposition 4.13 and continuous by Proposition 5.10. The restriction map $\rho_{\mathrm{N}}: \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N}) \rightarrow \operatorname{Stab}^{L}(\mathrm{~N}, \mathrm{~N})$ is continuous by Corollary 5.8. The product $\mu_{\mathrm{N}} \times \rho_{\mathrm{N}}$ is injective: given $\sigma=(P, Z) \in \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$, the slicings $P_{\mathrm{N}}$ and $P_{\mathrm{C} / \mathrm{N}}$ determine $P$ uniquely by Propositions 3.2, 3.6 and 4.4; the charge $Z$ is determined by its factorisation through $\Lambda / \Lambda_{N}$.

By Corollary 6.10, $\sigma$ can be deformed along nearby slicings in $\operatorname{Stab}^{L}(\mathrm{~N}, \mathrm{~N}) \subseteq \operatorname{Slice}(\mathrm{N})$ to a unique lax pre-stability condition which, by the assumption that support propagates from $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$, is in fact in $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$. Proposition 6.11 is the analogous statement about tangential deformations, i.e. deforming charges in $\operatorname{Hom}\left(\Lambda / \Lambda_{N}, \mathbb{C}\right)$ or, equivalently, deforming the stability condition on the quotient $\mathrm{C} / \mathrm{N}$. Therefore, there is an open metric ball around $\left(\mu_{\mathrm{N}}(\sigma), \rho_{\mathrm{N}}(\sigma)\right)$ in the image of $\mu_{\mathrm{N}} \times \rho_{\mathrm{N}}$. Thus the map in (1) is open. The statement about surjectivity on connected components follows.
(2) As in the first part, the map $\mu_{\mathrm{N}} \times$ id is injective because the slicing of $\sigma$ in $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ can be reconstructed uniquely from the slicing of $\mu_{\mathrm{N}}(\sigma)$ and the slicing on N . It is surjective because given $\left(\sigma_{\mathrm{C} / \mathrm{N}}, \sigma_{\mathrm{N}}\right)$ in a component meeting the image of $\mu_{\mathrm{N}} \times$ id we can apply the first part to construct $\sigma \in \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ with $\mu_{\mathrm{N}}(\sigma)=\sigma_{\mathrm{C} / \mathrm{N}}$ and restricted slicing that of $\sigma_{\mathrm{N}}$. Then $\left(\sigma, \sigma_{\mathrm{N}}\right)$ is the required lift of $\left(\sigma_{\mathrm{C} / \mathrm{N}}, \sigma_{\mathrm{N}}\right)$. The inverse is continuous because the reconstruction of the slicing on C from those on $\mathrm{C} / \mathrm{N}$ and N is continuous in the slicing metric; see Lemma 3.11.

We show in Corollary 7.4 that the product description of $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ extends to a deformation retract neighbourhood, and we describe how pairs of strata fit together in Proposition 7.8.
7.1. Neighbourhoods of strata. Our aim is to construct a deformation retract neighbourhood of $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$. The construction depends on the splitting $\operatorname{Hom}\left(\Lambda_{\mathrm{N}}, \mathbb{C}\right) \hookrightarrow \operatorname{Hom}(\Lambda, \mathbb{C})$ induced from the inner product on $\Lambda \otimes \mathbb{R}$, and therefore we need a neighbourhood which is adapted to this. We define one as follows. Fix $0<\varepsilon<1 / 8$ and for $\sigma \in \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ set

$$
V_{\varepsilon}(\sigma)=\left\{\tau=(Q, W) \in B_{\varepsilon}(\sigma):\left\|W_{\mathrm{N}}\right\|_{\sigma}<\sin (\pi \varepsilon)\right\}
$$

where $W_{N}$ is the restriction of $W$ to $\Lambda_{N}$ considered as an element of $\operatorname{Hom}(\Lambda, \mathbb{C})$ via the splitting. Thus $V_{\varepsilon}(\sigma)$ is open in $\operatorname{Slice}(\mathrm{C}) \times \operatorname{Hom}(\Lambda, \mathbb{C})$. Then define an open subset of $\operatorname{Stab}^{L}(\mathrm{C})$ by

$$
\mathrm{V}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{~N})=\operatorname{Stab}^{L}(\mathrm{C}) \cap \bigcup_{\sigma \in \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{~N})} V_{\varepsilon}(\sigma)
$$

These smaller neighbourhoods have the same good properties as the $\mathrm{B}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{N})$, namely they are $\mathbb{C}$-invariant and satisfy $\alpha \cdot \mathrm{V}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{N})=\mathrm{V}_{\varepsilon}^{L}(\mathrm{C}, \alpha(\mathrm{N}))$ for automorphisms $\alpha \in \operatorname{Aut}_{\Lambda}(\mathrm{C})$. Furthermore, $\mathrm{V}_{\varepsilon}^{L}(\mathrm{C}, 0)=\operatorname{Stab}(\mathrm{C})$ and $\mathrm{V}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{C})=\operatorname{Stab}^{L}(\mathrm{C})$ because the closure of the $\mathbb{C}$-orbit of any $\sigma \in \operatorname{Stab}^{L}(\mathrm{C})$ contains a lax stability condition with massless subcategory C .

Lemma 7.2. Assume that support propagates from $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$. Then, for sufficiently small $\varepsilon>0$, the identity on $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ extends to a continuous map

$$
\Phi_{\mathrm{N}}: \mathrm{V}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{~N}) \longrightarrow \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{~N})
$$

mapping $(Q, W)$ to the unique element of $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ with charge $W-W_{\mathrm{N}}$, massless slicing $Q_{\mathrm{N}}$ and slicing within distance $5 \varepsilon$ of $Q$. Moreover, $\Phi_{\mathrm{N}}$ is $\mathbb{C}$-equivariant and satisfies $\alpha \cdot \Phi_{\mathrm{N}}(\tau)=$ $\Phi_{\alpha(\mathrm{N})}(\alpha \cdot \tau)$ for each automorphism $\alpha \in \operatorname{Aut}_{\Lambda}(\mathrm{C})$ and $\tau \in \mathrm{V}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{N})$.

Remark 7.3. The map $\Phi_{N}$ is induced by the projection $\pi_{N}$ of charges onto their $\Lambda / \Lambda_{N}$ component, $\pi_{\mathrm{N}}(W)=W-W_{\mathrm{N}}$ :


Proof. Given $\tau=(Q, W) \in \mathrm{V}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{N})$ choose $\sigma=(P, Z) \in \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ such that $\tau \in B_{\varepsilon}(\sigma)$ and $\left\|W_{\mathrm{N}}\right\|_{\sigma}<\sin (\pi \varepsilon)$ where $\rho_{\mathrm{N}}(\tau)=\left(Q_{\mathrm{N}}, W_{\mathrm{N}}\right)$ and the slicing restricts by Lemma 5.6. Then $W-W_{\mathrm{N}} \in \operatorname{Hom}\left(\Lambda / \Lambda_{\mathrm{N}}, \mathbb{C}\right)$ and

$$
\left\|Z-\left(W-W_{\mathrm{N}}\right)\right\|_{\sigma} \leq\|Z-W\|_{\sigma}+\left\|W_{\mathrm{N}}\right\|_{\sigma}<2 \sin (\pi \varepsilon) \leq \sin (3 \pi \varepsilon)
$$

where we assume $\varepsilon \leq 1 / 6$ which enables the final trigonometric inequality.
Hence, for sufficiently small $\varepsilon>0$, we can apply Proposition 6.11 to construct a lax prestability condition $\sigma^{\prime}=\left(P^{\prime}, W-W_{\mathrm{N}}\right)$ with restricted slicing $P_{\mathrm{N}}^{\prime}=P_{\mathrm{N}}$ and $d\left(P, P^{\prime}\right)<3 \varepsilon$. Therefore $\sigma^{\prime} \in B_{3 \varepsilon}(\sigma)$, the restriction $\rho_{\mathrm{N}}\left(\sigma^{\prime}\right)=\rho_{\mathrm{N}}(\sigma)$, and $\left\|Z-\left(W-W_{\mathrm{N}}\right)\right\|_{\sigma}<\sin (3 \pi \varepsilon)$. Thus we can apply the support propagation assumption to conclude that $\sigma^{\prime}$ is in $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ when $\varepsilon$ is sufficiently small.

Since $\tau \in B_{\varepsilon}(\sigma)$ we know that $d(P, Q)<\varepsilon$. Hence $d\left(P_{\mathrm{N}}, Q_{\mathrm{N}}\right)<\varepsilon$ too. Therefore we can deform the massless slicing of $\sigma^{\prime}$ using Corollary 6.10 to obtain a lax pre-stability condition $\Phi_{\mathrm{N}}(\tau)=\left(Q^{\prime}, W-W_{\mathrm{N}}\right)$ with massless slicing $Q_{\mathrm{N}}^{\prime}=Q_{\mathrm{N}}$ and $d\left(P^{\prime}, Q^{\prime}\right)<\varepsilon$, from which it follows that $d\left(Q, Q^{\prime}\right)<5 \varepsilon$. It follows from Corollary 4.18 that $\Phi_{\mathrm{N}}(\tau)$ is well-defined and independent of the particular choice of $\sigma$ used to construct it. For sufficiently small $\varepsilon>0$ the support propagation assumption implies that $\Phi_{\mathrm{N}}(\tau)$ is in $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$.

By construction $\Phi_{\mathrm{N}}$ is continuous in the charge and in the restricted slicing on $N$. Since $\rho_{\mathrm{N}} \Phi_{\mathrm{N}}(\tau)=\left(Q_{\mathrm{N}}, 0\right)$, in order to see that $\Phi_{\mathrm{N}}: \mathrm{V}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{N}) \rightarrow \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ is continuous, it is sufficient to show that $\mu_{\mathrm{N}} \Phi_{\mathrm{N}}(\tau)$ varies continuously in $\tau$. If $\tau^{\prime}$ is sufficiently close to $\tau$ then, since the slicing distance between $\tau$ and $\Phi_{\mathrm{N}}(\tau)$ is less than $5 \varepsilon$, the slicing distance between $\Phi_{\mathrm{N}}(\tau)$ and $\Phi_{\mathrm{N}}\left(\tau^{\prime}\right)$ can be made strictly less than 1 . Therefore the slicing distance between $\mu_{\mathrm{N}} \Phi_{\mathrm{N}}(\tau)$ and $\mu_{\mathrm{N}} \Phi_{\mathrm{N}}\left(\tau^{\prime}\right)$ is also strictly less than 1 . Deformations of classical stability conditions with slicing within distance 1 are uniquely determined by the deformation of the charge, i.e. if the charge deforms continuously then so does the stability condition. Since $\mu_{N} \Phi_{N}(\tau)$ and $\mu_{\mathrm{N}} \Phi_{\mathrm{N}}\left(\tau^{\prime}\right)$ are classical stability conditions whose slicings are within distance 1 of each other whose charges are also close, it follows that $\mu_{\mathrm{N}} \Phi_{\mathrm{N}}(\tau)$ varies continuously with $\tau$.

Moreover $\Phi_{\mathrm{N}}$ restricts to the identity on $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$. The characterisation of $\Phi_{\mathrm{N}}(\tau)$ in the statement follows immediately from the above construction, as does the compatibility with the actions of $\mathbb{C}$ and of $\operatorname{Aut}_{\Lambda}(\mathrm{C})$.

Using the map $\Phi_{\mathrm{N}}$ we can, for sufficiently small $\varepsilon>0$, define a smaller, and even better behaved, neighbourhood of $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ namely

$$
\mathrm{U}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{~N})=\left\{\tau \in \mathrm{V}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{~N}): \tau \in B_{\varepsilon}\left(\Phi_{\mathrm{N}}(\tau)\right)\right\}
$$

The compatibility of $\Phi_{N}$ with the actions of $\mathbb{C}$ and $\operatorname{Aut}_{\Lambda}(\mathrm{C})$ ensures that these neighbourhoods are $\mathbb{C}$-invariant and that $\mathrm{U}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{N})$ is mapped to $\mathrm{U}_{\varepsilon}^{L}(\mathrm{C}, \alpha(\mathrm{N}))$ by $\alpha \in \operatorname{Aut}_{\Lambda}(\mathrm{C})$. In the two extreme cases of $N=0$ and $C$ the extra condition is vacuous, and we have equalities

$$
\begin{aligned}
\mathrm{U}_{\varepsilon}^{L}(\mathrm{C}, 0) & =\mathrm{V}_{\varepsilon}^{L}(\mathrm{C}, 0)=\mathrm{B}_{\varepsilon}^{L}(\mathrm{C}, 0)=\operatorname{Stab}(\mathrm{C}) \\
\mathrm{U}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{C}) & =\mathrm{V}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{C})=\mathrm{B}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{C})=\operatorname{Stab}^{L}(\mathrm{C}) .
\end{aligned}
$$

Corollary 7.4. Suppose support propagates from $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$. Then, for sufficiently small $\varepsilon>0$ the map

$$
\Phi_{\mathrm{N}} \times \rho_{\mathrm{N}}: \mathrm{U}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{~N}) \longrightarrow \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{~N}) \times_{\text {Slice( } \mathrm{N})} \operatorname{Stab}^{L}(\mathrm{~N})
$$

is a homeomorphism onto the set of pairs $\left(\sigma, \tau_{\mathrm{N}}\right)$ with $\left\|W_{\mathrm{N}}\right\|_{\sigma}<\sin (\pi \varepsilon)$ where $\tau_{\mathrm{N}}=\left(Q_{\mathrm{N}}, W_{\mathrm{N}}\right)$.
Proof. The map $\Phi_{\mathrm{N}} \times \rho_{\mathrm{N}}$ is continuous and, by definition of $\mathrm{U}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{N})$, has image in the subset

$$
\left\{\left(\sigma, \tau_{\mathrm{N}}=\left(Q_{\mathrm{N}}, W_{\mathrm{N}}\right)\right):\left\|W_{\mathrm{N}}\right\|_{\sigma}<\sin (\pi \varepsilon)\right\}
$$

The assumption that support propagates means that the map $\delta_{\mathrm{N}}$ defined in Proposition 6.18 exists. We claim it is inverse to $\Phi_{\mathrm{N}} \times \rho_{\mathrm{N}}$.

Let $\tau=\delta_{\mathrm{N}}\left(\sigma, \tau_{\mathrm{N}}\right)$, where $\sigma=(P, Z)$. By construction $\rho_{\mathrm{N}}(\tau)=\tau_{\mathrm{N}}$. Moreover $\tau \in \mathrm{V}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{N})$ so that $\Phi_{\mathrm{N}}(\tau)$ is well-defined. Then $\Phi_{\mathrm{N}}(\tau)$ has charge $Z$, massless subcategory N and massless slicing $Q_{\mathrm{N}}$. The slicing of $\Phi_{\mathrm{N}}(\tau)$ is within distance $5 \varepsilon$ of $P$ so that $\Phi_{\mathrm{N}}(\tau)=\sigma$ by Corollary 4.18
provided that $5 \varepsilon<1$. We conclude that $\tau \in B_{\varepsilon}\left(\Phi_{\mathrm{N}}(\tau)\right)$ so that $\tau \in \mathrm{U}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{N})$ and $\left(\Phi_{\mathrm{N}} \times \rho_{\mathrm{N}}\right) \circ \delta_{\mathrm{N}}$ is the identity.
Now set $\tau^{\prime}=\delta_{\mathrm{N}}\left(\Phi_{\mathrm{N}}(\tau), \rho_{\mathrm{N}}(\tau)\right)$. This is a lax stability condition with the same charge, massless subcategory and massless slicing as $\tau$. Moreover, its slicing is within distance $5 \varepsilon$ of that of $\tau$. Thus $\tau^{\prime}=\tau$ by Corollary 4.18 if again $5 \varepsilon<1$ and $\delta_{N} \circ\left(\Phi_{N} \times \rho_{N}\right)$ is the identity too.

Clearly $\left\{\left(\sigma,\left(Q_{\mathrm{N}}, W_{\mathrm{N}}\right)\right):\left\|W_{\mathrm{N}}\right\|_{\sigma}<\sin (\pi \varepsilon)\right\}$ is an open neighbourhood of $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ in $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N}) \times \times_{\text {Slice }(\mathbb{N})} \operatorname{Stab}^{L}(\mathrm{~N})$, and therefore $\mathrm{U}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{N})$ is an open neighbourhood of $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$. Indeed it is a deformation retract neighbourhood.

Definition 7.5. Suppose support propagates from $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ so that $\Phi_{N} \times \rho_{N}$ is a homeomorphism on $\mathrm{U}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{N})$ for some sufficiently small $\varepsilon>0$. For $t \in[0,1]$ let

$$
\Phi_{\mathrm{N}, t}: \mathrm{U}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{~N}) \rightarrow \mathrm{U}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{~N})
$$

be the map corresponding to $\left(\sigma,\left(Q_{\mathrm{N}}, W_{\mathrm{N}}\right)\right) \mapsto\left(\sigma,\left(Q_{\mathrm{N}}, t W_{\mathrm{N}}\right)\right)$ under $\Phi_{\mathrm{N}} \times \rho_{\mathrm{N}}$. Note that $\Phi_{\mathrm{N}, 0}=$ $\Phi_{\mathrm{N}}$ and $\Phi_{\mathrm{N}, 1}=\mathrm{id}$.

When $\mathrm{N}=\mathrm{C}$ the retraction $\Phi_{\mathrm{N}, t}$ is defined on the entire space of lax stability conditions because $\mathrm{U}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{C})=\operatorname{Stab}^{L}(\mathrm{C})$. In this case $\Phi_{\mathrm{N}, t}(Q, W)=(Q, t W)=(Q, W) \cdot i \log (t) / \pi$ is obtained by dilating the masses of objects using the right action of $i \mathbb{R} \subset \mathbb{C}$ defined in $\S 5.3$. More generally

commutes for each $t \in[0,1]$ where the bottom map is given by mass dilation in $\operatorname{Stab}^{L}(\mathrm{~N})$. It is also clear that the maps $\Phi_{N, t}$ preserve the intersections $U_{\varepsilon}^{L}(C, N) \cap \overline{\operatorname{Stab}^{L}(C, M)}$ for thick $M \subset N$.
7.2. Pairs of strata. We now consider how the subsets $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{M})$ and $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ fit together when $\mathrm{M} \subset \mathrm{N}$ are thick subcategories. The first observation is that the frontier condition fails: that is in general

$$
\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{~N}) \cap \overline{\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{M})} \neq \varnothing \nRightarrow \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{~N}) \subset \overline{\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{M})} .
$$

Example 7.6. Let $C=D^{b}\left(\mathbb{P}^{1}\right)$ and consider the pair $M=\operatorname{thick}(\mathcal{O}) \subset D^{b}\left(\mathbb{P}^{1}\right)=N$. The above intersection contains the slicings $P$ occurring in $\operatorname{Stab}(\mathrm{C})$ for which $\mathcal{O}$ is semistable, so is nonempty. However, it is not the whole of $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{C})$ because it does not contain any slicings for which $\mathcal{O}$ is unstable, such as those in which the only two semistable objects are $\mathcal{O}(1)$ and $\mathcal{O}(2)$ and their shifts.
Nevertheless the situation is quite well-behaved. When we restrict to $\overline{\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{M})}$ the maps $\Phi_{\mathrm{N}}$ and $\rho_{\mathrm{N}}$ are compatible with passing to the quotient via $\mu_{\mathrm{M}}$. Effectively then we can reduce to the case $\mathrm{M}=0$.

Lemma 7.7. Suppose $\mathrm{M} \subset \mathrm{N}$ are thick subcategories such that support propagates from both $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ and $\operatorname{Stab}^{L}(\mathrm{C} / \mathrm{M}, \mathrm{N} / \mathrm{M})$. Then for sufficiently small $\varepsilon>0$ the map $\mu_{\mathrm{M}}$ restricts to a map

$$
\mathrm{U}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{~N}) \cap \overline{\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{M})} \longrightarrow \mathrm{U}_{\varepsilon}^{L}(\mathrm{C} / \mathrm{M}, \mathrm{~N} / \mathrm{M}) .
$$

Proof. The support propagation assumptions ensure that the maps $\Phi_{\mathrm{N}, t}$ for $t \in[0,1]$ and $\Phi_{\mathrm{N} / \mathrm{M}}$ are defined.
Let $\tau=(Q, W) \in \mathrm{U}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{N}) \cap \overline{\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{M})}$ and $\rho_{\mathrm{N}}(\tau)=\left(Q_{\mathrm{N}}, W_{\mathrm{N}}\right)$. Then $\Phi_{\mathrm{N}}(\tau)$ is also in $\overline{\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{M})}$ because $\Phi_{\mathrm{N}, t}$ preserves the intersection $\mathrm{U}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{N}) \cap \overline{\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{M})}$, and $\mu_{\mathrm{M}}\left(\Phi_{\mathrm{N}}(\tau)\right)$ is
well-defined and lies in $\operatorname{Stab}^{L}(C / M, N / M)$; see Remark 4.14. Then

$$
\left\|W_{\mathrm{N}}\right\|_{\mu_{\mathrm{M}}\left(\Phi_{\mathrm{M}}(\tau)\right)} \leq\left\|W_{\mathrm{N}}\right\|_{\Phi_{\mathrm{M}}(\tau)}<\sin (\pi \varepsilon)
$$

by Lemma 4.15 and, by Lemma 5.7, the distance between the slicings of $\mu_{\mathrm{M}}\left(\Phi_{\mathrm{N}}(\tau)\right)$ and $\mu_{\mathrm{M}}(\tau)$ is less than that between those of $\Phi_{\mathrm{N}}(\tau)$ and $\tau$, which in turn is less than $\varepsilon$ because $\tau \in B_{\varepsilon}\left(\Phi_{\mathrm{N}}(\tau)\right)$. Therefore, we also have $\mu_{\mathrm{M}}(\tau) \in B_{\varepsilon}\left(\mu_{\mathrm{M}}\left(\Phi_{\mathrm{M}}(\tau)\right)\right)$. This shows that $\mu_{\mathrm{M}}(\tau)$ is in $\mathrm{V}_{\varepsilon}^{L}(\mathrm{C} / \mathrm{M}, \mathrm{N} / \mathrm{M})$ so that $\Phi_{\mathrm{N} / \mathrm{M}}\left(\mu_{\mathrm{M}}(\tau)\right)$ is well-defined. Applying Corollary 4.18 then shows that

$$
\Phi_{\mathrm{N} / \mathrm{M}}\left(\mu_{\mathrm{M}}(\tau)\right)=\mu_{\mathrm{M}}\left(\Phi_{\mathrm{N}}(\tau)\right)
$$

because they have the same charge $W-W_{\mathrm{N}}$, the same massless subcategory $\mathrm{N} / \mathrm{M}$ and massless slicing, and their slicings are within distance one of each other for sufficiently small $\varepsilon>0$. Therefore, $\mu_{\mathrm{M}}(\tau) \in B_{\varepsilon}\left(\Phi_{\mathrm{N} / \mathrm{M}}\left(\mu_{\mathrm{M}}(\tau)\right)\right)$ and so is in $\mathrm{U}_{\varepsilon}^{L}(\mathrm{C} / \mathrm{M}, \mathrm{N} / \mathrm{M})$ as claimed.
Proposition 7.8. Suppose $\mathrm{M} \subset \mathrm{N}$ are thick subcategories such that support propagates from both $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ and $\operatorname{Stab}^{L}(\mathrm{C} / \mathrm{M}, \mathrm{N} / \mathrm{M})$. Then for sufficiently small $\varepsilon>0$ there is a commutative diagram


Proof. The support propagation assumptions and the previous lemma ensure the maps in the central square are well-defined, and that the square commutes. Consider the left hand square and choose

$$
\sigma=(P, Z) \in \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{~N}) \cap \overline{\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{M})} .
$$

Recall $\mu_{\mathrm{N}}(\sigma)=\left(P_{\mathrm{C} / \mathrm{N}}, Z\right)$ from Proposition 4.13 where $P_{\mathrm{C} / \mathrm{N}}(\varphi)$ is the isomorphism closure of $P(\varphi)$ in $\mathrm{C} / \mathrm{N}$. Similarly, $\mu_{\mathrm{M}}(\sigma)=\left(P_{\mathrm{C} / \mathrm{M}}, Z\right)$ where $P_{\mathrm{C} / \mathrm{M}}(\varphi)$ is the isomorphism closure of $P(\varphi)$ in $\mathrm{C} / \mathrm{M}$. Applying $\mu_{\mathrm{N} / \mathrm{M}}$ the charge remains the same, namely $Z$, and the category of semistable objects of phase $\varphi$ becomes the isomorphism closure of $P_{C / M}(\varphi)$ in $(C / M) /(N / M) \simeq C / N$ which is, as before, the isomorphism closure of $P(\varphi)$. Hence $\mu_{\mathrm{N}}=\mu_{\mathrm{N} / \mathrm{M}} \circ \mu_{\mathrm{M}}$.

Now consider the right hand square. Choose

$$
\sigma=(P, Z) \in \mathrm{U}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{~N}) \cap \overline{\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{M})} .
$$

Both $\mu_{\mathrm{M}} \circ \rho_{\mathrm{N}}(\sigma)$ and $\rho_{\mathrm{N} / \mathrm{M}} \circ \mu_{\mathrm{M}}(\sigma)$ have charge $\left.Z\right|_{\Lambda_{\mathrm{N}}}$. On the one hand, the category of semistable objects of $\mu_{\mathrm{M}} \circ \rho_{\mathrm{N}}(\sigma)$ with phase $\varphi$ is the isomorphism closure of $P(\varphi) \cap \mathrm{N}$ in $\mathrm{N} / \mathrm{M}$. On the other hand, that of $\rho_{\mathrm{N} / \mathrm{M}} \circ \mu_{\mathrm{M}}(\sigma)$ is the isomorphism closure of $P(\varphi)$ in $\mathrm{C} / \mathrm{M}$ intersected with $\mathrm{N} / \mathrm{M}$. This clearly contains the former and hence is the same as the former since nested slicings are equal. Therefore $\mu_{\mathrm{M}} \circ \rho_{\mathrm{N}}=\rho_{\mathrm{N} / \mathrm{M}} \circ \mu_{\mathrm{M}}$.

## 8. The space of quotient stability conditions

8.1. Definition. Define an equivalence relation on the points of $\operatorname{Stab}^{L}(\mathrm{C})$ by $\sigma \sim \tau$ if they have the same charge and lie in the same connected component of the corresponding fibre of $\operatorname{Stab}^{L}(\mathrm{C}) \rightarrow \operatorname{Hom}(\Lambda, \mathbb{C})$. We refer to an equivalence class as a quotient stability condition on C and denote the class of $\sigma$ by $[\sigma]$. If $\sigma \sim \tau$ then by Corollary 5.14 they have the same massless subcategory, $\mathbf{N}$ say, and induce the same stability condition $\mu_{\mathrm{N}}(\sigma)=\mu_{\mathrm{N}}(\tau)$ in $\operatorname{Stab}(\mathrm{C} / \mathrm{N})$.
Remark 8.1. If support propagates from $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ then

$$
\mu_{\mathrm{N}} \times \rho_{\mathrm{N}}: \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{~N}) \rightarrow \operatorname{Stab}^{L}(\mathrm{C} / \mathrm{N}) \times \operatorname{Stab}^{L}(\mathrm{~N}, \mathrm{~N})
$$

is a homeomorphism onto a union of components by Proposition 7.1. In this case $\sigma$ and $\tau$ are in the same component of the fibre of the charge map precisely when $\rho_{\mathrm{N}}(\sigma)$ and $\rho_{\mathrm{N}}(\tau)$ are in the same component of $\operatorname{Stab}^{L}(\mathrm{~N}, \mathrm{~N})$.

Conjecturally, non-empty stability spaces are contractible, in particular connected. If this is the case, and support propagates from all strata $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$, then a quotient stability condition is specified by a choice of massless subcategory N and stability condition in $\operatorname{Stab}(\mathrm{C} / \mathrm{N})$. Which thick subcategories N arise as massless categories remains a subtle question.

The space of quotient stability conditions is defined to be $\operatorname{Stab}^{Q}(\mathrm{C})=\operatorname{Stab}^{L}(\mathrm{C}) / \sim$ equipped with the quotient topology. By definition the charge map factors through the quotient, and we denote this factorisation also by $\mathcal{Z}$.

The theme of this section is that $\operatorname{Stab}^{Q}(\mathrm{C})$ is a stratified space in a reasonable way. The first step is to specify a decomposition. Let $\operatorname{Stab}^{Q}(\mathrm{C}, \mathrm{N})=\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N}) / \sim$ be the subspace of quotient stability conditions with massless subcategory N. Evidently

$$
\operatorname{Stab}^{Q}(\mathrm{C})=\bigsqcup_{\text {thick } N \subset C} \operatorname{Stab}^{Q}(\mathrm{C}, \mathrm{~N})
$$

is a disjoint union of these subsets. The strata are the connected components of these pieces.
The maximal and minimal dimensional strata are easy to identify. The fibres of the charge map on $\operatorname{Stab}(\mathrm{C})$ are discrete so there are homeomorphisms

$$
\operatorname{Stab}(C) \cong \operatorname{Stab}^{L}(\mathrm{C}, 0) \cong \operatorname{Stab}^{Q}(\mathrm{C}, 0)
$$

Thus the usual space of stability conditions embeds continuously in $\operatorname{Stab}^{Q}(\mathrm{C}, 0)$ as a union of strata. At the other extreme $\operatorname{Stab}^{Q}(\mathrm{C}, \mathrm{C})=\pi_{0}\left(\mathcal{Z}^{-1}(0)\right)$ is the set of components of the fibre $\mathcal{Z}^{-1}(0)=\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{C})$, with the discrete topology. These are the 0 -dimensional strata. If $\operatorname{Stab}(\mathrm{C})$ is connected there is a unique such stratum by Lemma 6.6.
8.2. The stratification of the space of quotient stability conditions. We show that, under suitable technical assumptions, $\operatorname{Stab}^{Q}(\mathrm{C})$ is a stratified space. By this we mean that the strata $\operatorname{Stab}^{Q}(\mathrm{C}, \mathrm{N})$ are locally closed subspaces satisfying the frontier condition, i.e. the closure of each such stratum is a union of strata.

For $\sigma \in \operatorname{Stab}^{Q}(\mathrm{C}, \mathrm{N})$ and $c \in \mathrm{C}$ the mass $m_{\sigma}(c)$ is the mass of $c$ in the corresponding stability condition on $\mathrm{C} / \mathrm{N}$. For massive $c \in \mathrm{C}$ we define $\tilde{\varphi}_{\sigma}^{ \pm}(c)$ to be the minimal and maximal phases of the HNfactors of $c$ in $\mathrm{C} / \mathrm{N}$.

Lemma 8.2. For each $c \in C$ the mass $m_{\bullet}(c): \operatorname{Stab}^{Q}(C) \rightarrow \mathbb{R}_{\geq 0}$ is continuous. The phases $\tilde{\varphi}_{\bullet}^{ \pm}(c): m_{\bullet}(c)^{-1}\left(\mathbb{R}_{>0}\right) \rightarrow \mathbb{R}$ are well-defined and $\tilde{\varphi}_{\bullet}^{-}(c)$ and $\tilde{\varphi}_{\bullet}^{+}(c)$ are respectively upper and lower semi-continuous.

Proof. Recall that the mass of $c$ is continuous on $\operatorname{Stab}^{L}(\mathrm{C})$ and is constant on equivalence classes, since it depends only on the massive HNfactors. The continuity of masses on $\mathrm{Stab}^{Q}$ (C) follows immediately. The statements for phases follow similarly using the fact that the minimal and maximal phases are respectively bounded above and below by the minimal and maximal phases of the massive factors (which are constant on equivalence classes).

It follows immediately from the continuity of masses that the subsets $\operatorname{Stab}^{Q}(\mathrm{C}, \mathrm{N})$ and hence also their connected components, i.e. the strata, are locally closed. The following consequence of Proposition 7.1 identifies the strata as components of spaces of stability conditions on various quotient categories.
Corollary 8.3. Suppose that support propagates from $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$. Then the factorisation of $\mu_{\mathrm{N}}: \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N}) \rightarrow \operatorname{Stab}(\mathrm{C} / \mathrm{N})$ through the quotient map induces a homeomorphism between the component of $\sigma \in \operatorname{Stab}^{Q}(\mathrm{C}, \mathrm{N})$ and the component of $\mu_{\mathrm{N}}(\sigma) \in \operatorname{Stab}(\mathrm{C} / \mathrm{N})$. Thus each stratum of $\operatorname{Stab}^{Q}(\mathrm{C})$ can be given the structure of a complex manifold in such a way that the restriction of the charge map is a local homeomorphism to $\operatorname{Hom}\left(\Lambda / \Lambda_{N}, \mathbb{C}\right)$.

We begin by describing the equivalence relation in a neighbourhood of $\operatorname{Stab}^{L}$ (C,N).

Lemma 8.4. Suppose support propagates from both $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ and $\operatorname{Stab}^{L}(\mathrm{C} / \mathrm{M}, \mathrm{N} / \mathrm{M})$ for each thick $\mathrm{M} \subset \mathrm{N}$. Let $\sigma, \tau \in \mathrm{U}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{N})$ for some sufficiently small $\varepsilon>0$. Then $\sigma \sim \tau$ if and only if $\mu_{\mathrm{N}}\left(\Phi_{\mathrm{N}}(\sigma)\right)=\mu_{\mathrm{N}}\left(\Phi_{\mathrm{N}}(\tau)\right)$ and $\rho_{\mathrm{N}}(\sigma) \sim \rho_{\mathrm{N}}(\tau)$ as elements of $\operatorname{Stab}^{L}(\mathrm{~N})$.
Proof. Suppose $\sigma \sim \tau$. Then $\rho_{\mathrm{N}}(\sigma) \sim \rho_{\mathrm{N}}(\tau)$ as elements of $\operatorname{Stab}^{L}(\mathbb{N})$. Moreover, $\sigma$ and $\tau$ have the same massless subcategory, $M$ say, so that $\sigma, \tau \in \mathrm{U}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{N}) \cap \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{M})$. Since we also have $\mu_{\mathrm{M}}(\sigma)=\mu_{\mathrm{M}}(\tau)$ Proposition 7.8 implies that $\mu_{\mathrm{N}}\left(\Phi_{\mathrm{N}}(\sigma)\right)=\mu_{\mathrm{N}}\left(\Phi_{\mathrm{N}}(\tau)\right)$.

Conversely, suppose $\mu_{\mathrm{N}}\left(\Phi_{\mathrm{N}}(\sigma)\right)=\mu_{\mathrm{N}}\left(\Phi_{\mathrm{N}}(\tau)\right)$ and $\rho_{\mathrm{N}}(\sigma) \sim \rho_{\mathrm{N}}(\tau)$. Then $\sigma$ and $\tau$ have the same massless subcategory, namely the common massless subcategory M of $\rho_{\mathrm{N}}(\sigma)$ and $\rho_{\mathrm{N}}(\tau)$. Because $\Phi_{\mathrm{N}}(\sigma)$ and $\Phi_{\mathrm{N}}(\tau)$ have the same charge, as do $\rho_{\mathrm{N}}(\sigma)$ and $\rho_{\mathrm{N}}(\tau)$, we get that $\sigma$ and $\tau$ have the same charge. Next, $\rho_{\mathrm{M}}\left(\rho_{\mathrm{N}}(\sigma)\right)$ and $\rho_{\mathrm{M}}\left(\rho_{\mathrm{N}}(\tau)\right)$ are in the same component of $\operatorname{Stab}^{L}(\mathrm{M}, \mathrm{M})$, since $\rho_{\mathrm{N}}(\sigma) \sim \rho_{\mathrm{N}}(\tau)$. Then $\sigma \sim \tau$ by Remark 8.1, as $\rho_{\mathrm{M}} \circ \rho_{\mathrm{N}}=\rho_{\mathrm{M}}$.
Lemma 8.5. Suppose support propagates from both $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ and $\operatorname{Stab}^{L}(\mathrm{C} / \mathrm{M}, \mathrm{N} / \mathrm{M})$ for each thick $\mathrm{M} \subset \mathrm{N}$. Then, for sufficiently small $\varepsilon>0$, the open neighbourhood $\mathrm{U}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{N})$ is a union of equivalence classes.
Proof. Suppose $\tau \in \mathrm{U}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{N})$. By definition, the class $[\tau]$ is connected in the subspace topology from $\operatorname{Stab}^{L}(\mathrm{C})$. Its intersection with $\mathrm{U}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{N})$ is an open subset of $[\tau]$, so it suffices to show this intersection is also closed in $[\tau]$. By Corollary 7.4, $\mathrm{U}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{N})$ is homeomorphic via $\Phi_{\mathrm{N}} \times \rho_{\mathrm{N}}$ to

$$
\left\{\left(\sigma, \tau_{\mathrm{N}}=\left(Q_{\mathrm{N}}, W_{\mathrm{N}}\right)\right):\left\|W_{\mathrm{N}}\right\|_{\sigma}<\sin (\pi \varepsilon)\right\} \subset \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{~N}) \times_{\text {Slice(N) }} \operatorname{Stab}^{L}(\mathrm{~N})
$$

And by the previous result $\tau^{\prime} \in[\tau] \cap \mathrm{U}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{N})$ if and only if $\mu_{\mathrm{N}}\left(\Phi_{\mathrm{N}}\left(\tau^{\prime}\right)\right)=\mu_{\mathrm{N}}\left(\Phi_{\mathrm{N}}(\tau)\right)$ and $\rho_{\mathrm{N}}\left(\tau^{\prime}\right) \sim \rho_{\mathrm{N}}(\tau)$. The set of such $\tau^{\prime}$ is closed in the above fibre product, therefore also in $\mathrm{U}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{N})$ and in $[\tau]$.

These lemmas show that $\mathrm{U}_{\varepsilon}^{Q}(\mathrm{C}, \mathrm{N})=\mathrm{U}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{N}) / \sim$ is an open neighbourhood of $\operatorname{Stab}^{Q}(\mathrm{C}, \mathrm{N})$ in $\operatorname{Stab}^{Q}(\mathrm{C})$ and allow us to describe the stratification within this neighbourhood.

Corollary 8.6. Suppose that support propagates from $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$. Then for sufficiently small $\varepsilon>0$ the map

$$
\mu_{\mathrm{N}} \circ \Phi_{\mathrm{N}} \times\left[\rho_{\mathrm{N}}\right]: \mathrm{U}_{\varepsilon}^{Q}(\mathrm{C}, \mathrm{~N}) \rightarrow \operatorname{Stab}(\mathrm{C} / \mathrm{N}) \times \operatorname{Stab}^{Q}(\mathrm{~N}), \quad[\sigma] \mapsto\left(\mu_{\mathrm{N}}\left(\Phi_{\mathrm{N}}(\sigma)\right),\left[\rho_{\mathrm{N}}(\sigma)\right]\right)
$$

is a homeomorphism onto an open subset. Moreover, if for each thick $\mathrm{M} \subset \mathrm{N}$ support also propagates from $\operatorname{Stab}^{L}(\mathrm{C} / \mathrm{M}, \mathrm{N} / \mathrm{M})$ then the homeomorphism above restricts to a biholomorphism from $\mathrm{U}_{\varepsilon}^{Q}(\mathrm{C}, \mathrm{N}) \cap \operatorname{Stab}^{Q}(\mathrm{C}, \mathrm{M})$ onto (an open subset of) a union of components of $\operatorname{Stab}(\mathrm{C} / \mathrm{N}) \times$ $\operatorname{Stab}^{Q}(\mathrm{~N}, \mathrm{M})$. In particular, locally we have

$$
\overline{\operatorname{Stab}^{Q}(\mathrm{C}, \mathrm{M})} \cong \overline{\operatorname{Stab}(\mathrm{C} / \mathrm{N}) \times \operatorname{Stab}^{Q}(\mathrm{~N}, \mathrm{M})} \cong \operatorname{Stab}(\mathrm{C} / \mathrm{N}) \times \overline{\operatorname{Stab}^{Q}(\mathrm{~N}, \mathrm{M})}
$$

Proof. The existence of the homeomorphism, and the fact that it is stratum-preserving, follow from Corollary 7.4 and Lemma 8.4. It restricts to a holomorphic isomorphism because it is compatible with the charge maps.
Theorem 8.7. Suppose that support propagates from both $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ and $\operatorname{Stab}^{L}(\mathrm{C} / \mathrm{M}, \mathrm{N} / \mathrm{M})$ for each thick $\mathrm{M} \subset \mathrm{N}$. Then the space of quotient stability conditions is stratified by complex manifolds. More precisely,

$$
\operatorname{Stab}^{Q}(\mathrm{C})=\bigsqcup_{\text {thick } N \subset C} \operatorname{Stab}^{Q}(\mathrm{C}, \mathrm{~N})
$$

decomposes into a disjoint union of locally closed subsets. Each $\operatorname{Stab}^{Q}(\mathrm{C}, \mathrm{N})$ is a complex manifold of dimension $\operatorname{rk}\left(\Lambda / \Lambda_{N}\right)$; we refer to its connected components as strata. This decomposition satisfies the frontier condition: the closure of each stratum is a union of strata.

Proof. We have already seen that there is such a decomposition into locally closed subsets $\operatorname{Stab}^{Q}(\mathrm{C}, \mathrm{N})$ and, by Corollary $8.3, \operatorname{Stab}^{Q}(\mathrm{C}, \mathrm{N})$ is a complex manifold locally homeomorphic to $\operatorname{Hom}\left(\Lambda / \Lambda_{\mathrm{N}}, \mathbb{C}\right)$.

For the last part, consider $[\sigma] \in \operatorname{Stab}^{Q}(\mathbf{C}, \mathbf{N})$. Note that $\left[\rho_{\mathrm{N}}(\sigma)\right] \in \operatorname{Stab}^{Q}(\mathbf{N}, \mathrm{~N})$ which is a discrete set of points. If $\left[\rho_{\mathrm{N}}(\sigma)\right] \in \overline{\operatorname{Stab}^{Q}(\mathrm{~N}, \mathrm{M})}$ then $[\sigma] \in \overline{\operatorname{Stab}^{Q}(\mathrm{C}, \mathrm{M})}$ by the last part of Corollary 8.6. Since $\operatorname{Stab}^{Q}(N, N)$ is discrete and we have a local product description, an open neighbourhood of $[\sigma]$ in $\operatorname{Stab}^{Q}(\mathrm{C}, \mathrm{N})$ is contained in

$$
\operatorname{Stab}(\mathrm{C} / \mathrm{N}) \times\left\{\left[\rho_{\mathrm{N}}(\sigma)\right]\right\} \subset \operatorname{Stab}(\mathrm{C} / \mathrm{N}) \times \overline{\operatorname{Stab}^{Q}(\mathrm{~N}, \mathrm{M})}
$$

Therefore by the formula at the end of Corollary 8.6 the connected component of $[\sigma]$ in $\operatorname{Stab}^{Q}(\mathrm{C}, \mathrm{N})$ is in the closure of $\operatorname{Stab}^{Q}(\mathrm{C}, \mathrm{M})$.

Corollary 8.8. Suppose support propagates from $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$. Let $\Phi_{\mathrm{N}, t}$ be the deformation retraction of Definition 7.5. Then

$$
\left[\Phi_{\mathrm{N}, t}\right]: \mathrm{U}_{\varepsilon}^{Q}(\mathrm{C}, \mathrm{~N}) \rightarrow \mathrm{U}_{\varepsilon}^{Q}(\mathrm{C}, \mathrm{~N}), \quad[\sigma] \mapsto\left[\Phi_{\mathrm{N}, t}(\sigma)\right]
$$

is an almost-stratum-preserving deformation retraction of $\mathrm{U}_{\varepsilon}^{Q}(\mathrm{C}, \mathrm{N})$ onto $\operatorname{Stab}^{Q}(\mathrm{C}, \mathrm{N})$. By this we mean that the track of each point under the deformation retraction remains in the same stratum until the last point, at which it may enter a higher codimension stratum.

Proof. Suppose $\sigma, \tau \in \mathrm{U}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{N})$ and $\sigma \sim \tau$. Then $\Phi_{\mathrm{N}, t}(\sigma) \sim \Phi_{\mathrm{N}, t}(\tau)$ by Lemma 8.4, because $\Phi_{\mathrm{N}}\left(\Phi_{\mathrm{N}, t}(\sigma)\right)=\Phi_{\mathrm{N}}(\sigma)$ and $\rho_{\mathrm{N}}\left(\Phi_{\mathrm{N}, t}(\sigma)\right)=\rho_{\mathrm{N}}(\sigma) \cdot i \log (t) / \pi$. Thus $\Phi_{\mathrm{N}, t}$ descends to the quotient. Since $\Phi_{\mathrm{N}, t}(\sigma)$ has the same massless subcategory as $\sigma$ for $t \in(0,1]$ the resulting deformation retraction is almost-stratum-preserving, i.e. its tracks remain in the same stratum until the last moment, when they move into the deeper stratum $\operatorname{Stab}^{Q}(\mathrm{C}, \mathrm{N})$.
8.3. Group actions. To complete our description of the space of quotient stability conditions, we note that the actions of $\operatorname{Aut}_{\Lambda}(\mathrm{C})$ and $G$ descend to it, and that they respect the stratification and local models.

Corollary 8.9. The actions of $\operatorname{Aut}(\mathrm{C})$ and of $G$ on $\mathrm{Stab}(\mathrm{C})$ extend uniquely to continuous actions on $\mathrm{Stab}^{Q}(\mathrm{C})$. The maps in the diagram

are equivariant with respect to these actions.
The actions respect the stratification: elements of $G$ preserve $\operatorname{Stab}^{Q}(\mathrm{C}, \mathrm{N})$ whereas the action of an automorphism $\alpha \in \operatorname{Aut}_{\Lambda}(\mathrm{C})$ takes $\operatorname{Stab}^{Q}(\mathrm{C}, \mathrm{N})$ to $\operatorname{Stab}^{Q}(\mathrm{C}, \alpha(\mathrm{N}))$. Moreover, the neighbourhood $\mathrm{U}_{\varepsilon}^{Q}(\mathrm{C}, \mathrm{N})$ is $\mathbb{C}$-invariant and $\alpha \in \operatorname{Aut}_{\Lambda}(\mathrm{C})$ maps it to $\mathrm{U}_{\varepsilon}^{Q}(\mathrm{C}, \alpha(\mathrm{N}))$.

When support propagates from $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ the actions are compatible with the local model in Corollary 8.6 in that

commutes, and that the horizontal maps are $\mathbb{C}$-equivariant with respect to the evident diagonal action of $\mathbb{C}$ on the right hand terms.

Proof. It is easy to check that the actions preserve the equivalence relation and so descend to the quotient. Lemma 5.15 implies they respect the stratification as stated. The final part follows from the properties of the maps $\mu_{\mathrm{N}}, \rho_{\mathrm{N}}$ and $\Phi_{\mathrm{N}}$ - see Lemma 5.15, Section 5.4, and Lemma 7.2 respectively.

## 9. Codimension one strata

We investigate the case in which the massless subcategory is non-zero but 'as small as possible'. More precisely, throughout this section we fix a thick subcategory N for which the saturation $\Lambda_{N}$ of the image of $K(\mathrm{~N}) \rightarrow K(\mathrm{C}) \rightarrow \Lambda$ is a rank one lattice. We abuse terminology by saying that the massless subcategory N has rank one. We assume that $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ is non-empty. For simplicity of notation we also assume that it is connected; otherwise we consider each component separately. The main result of this section, Theorem 9.4, is that $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ is a component of $\operatorname{Stab}(\mathrm{C} / \mathrm{N}) \times \mathbb{R}$. In particular it is a real codimension one boundary stratum in $\operatorname{Stab}^{L}(\mathrm{C})$.
9.1. Objects and phases. When $\operatorname{rk}\left(\Lambda_{\mathrm{N}}\right)=1$ the massless subcategory N has a simple form. In particular, up to shift, all the massless objects must have the same phase. This remains true even for nearby stability conditions.

Lemma 9.1. Suppose $\sigma \in \overline{\operatorname{Stab}(C)}$ is a lax pre-stability condition with rank one massless subcategory N . Then $\mathrm{N}=\operatorname{triang}(S)$ is the triangulated closure of a set $S$ of stable objects of the same phase.

Proof. Write $\sigma=(P, Z)$. Suppose $0 \neq c \in \mathrm{~N}$ is a $\sigma$-semistable object with phase $0<\varphi(c) \leq 1$. Given $0<\varepsilon<1 / 8$ we can choose $\tau=(Q, W) \in \operatorname{Stab}(\mathrm{C})$ with $d(P, Q)<\varepsilon$. By Lemma 5.6 all $\tau$-semistable factors of $c$ are in N . Since they have phases in $(\varphi(c)-\varepsilon, \varphi(c)+\varepsilon)$ and $m_{\tau}(c)>0$ we deduce that $\left.W\right|_{\Lambda_{N}} \neq 0$. Therefore we can choose a generator $\lambda \in \Lambda_{N}$ with $W(\lambda) \in \mathbb{H} \cup \mathbb{R}_{<0}$. Let $\psi \in(0,1]$ be the phase of $W(\lambda)$. Then $|\varphi(c)-\psi|<\varepsilon$ and if $c^{\prime} \in N$ is another $\sigma$-semistable object with phase $0<\varphi\left(c^{\prime}\right) \leq 1$ we have

$$
\left|\varphi(c)-\varphi\left(c^{\prime}\right)\right| \leq|\varphi(c)-\psi|+\left|\psi-\varphi\left(c^{\prime}\right)\right|<2 \varepsilon .
$$

It follows that $\varphi(c)=\varphi\left(c^{\prime}\right)=\varphi$, say, for all $\sigma$-semistable objects in N , i.e. $\mathrm{N} \cap P(0,1] \subset P(\varphi)$.
Since $\sigma \in \overline{\operatorname{Stab}(\mathrm{C})}$ its slicing $P$ is locally finite so that $P(\psi)$ is a quasi-abelian length category. The full subcategory $\mathrm{N} \cap P(0,1]$ is closed under extensions, strict subobjects and strict quotients. Therefore $\mathrm{N} \cap P(0,1]=\langle S\rangle$ is the extension-closure of a subset $S$ of simple objects of $P(\varphi)$. Hence $\mathrm{N}=\operatorname{triang}(S)$.

Corollary 9.2. Fix $0<\varepsilon<1 / 8$. There is a set $S$ of objects with $\mathrm{N}=\operatorname{triang}(S)$ and such that for each $\sigma$ in the open neighbourhood $\mathrm{B}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{N})$ of $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ all objects in $S$ are stable of the same phase.

Proof. Let $\sigma=(P, Z) \in \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$. By the previous lemma there is $0<\varphi \leq 1$ and a subset $S$ of simple objects in $P(\varphi)$ with $\mathrm{N}=\operatorname{triang}(S)$. Suppose $\tau=(Q, W) \in B_{\varepsilon}(\sigma) \cap \operatorname{Stab}^{L}(\mathrm{C})$. Lemma 5.6 implies that

$$
\langle S\rangle \subset Q(\varphi-\varepsilon, \varphi+\varepsilon) \cap \mathrm{N} \subset P(\varphi-2 \varepsilon, \varphi+2 \varepsilon) \cap \mathrm{N} .
$$

Since $P(0,1] \cap \mathrm{N}=\langle S\rangle$ and $0<\varepsilon<1 / 8$ the right-hand side above is $\langle S\rangle[k]$ for $k=0$ or $\pm 1$. Since it contains $\langle S\rangle$ we must have $k=0$ so that the above containments are equalities. Therefore all $\tau$-semistable factors of each $s \in S$ are also in $\langle S\rangle$, and since each object $s \in S$ is simple in $\langle S\rangle$ this implies that each $s \in S$ is $\tau$-semistable. It follows as in the proof of the previous lemma that all $s \in S$ have the same phase, so that in fact $\langle S\rangle=Q(\psi) \cap \mathrm{N}$ for some $\psi \in(\varphi-\varepsilon, \varphi+\varepsilon)$. Since each $s \in S$ is simple, we deduce that each $s \in S$ is actually $\tau$-stable. The result now follows from the definition

$$
\mathrm{B}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{~N})=\bigcup_{\sigma \in \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{~N})} B_{\varepsilon}(\sigma) \cap \operatorname{Stab}^{L}(\mathrm{C})
$$

of the neighbourhood and the fact that we have assumed $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ is connected.
9.2. The neighbourhood of a stratum. The neighbourhood of the stratum $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ has a simple form when $\operatorname{rk}\left(\Lambda_{N}\right)=1$ : it is a fibration over $\mathbb{C}$.

Recall that $\operatorname{Stab}(\mathrm{N})$ is the space of stability conditions on N whose charges factor through $\Lambda_{N} \cong \mathbb{Z}$. The space of non-zero charges is $\operatorname{Hom}\left(\Lambda_{N}, \mathbb{C}\right)-\{0\} \cong \mathbb{C}^{*}$. As $\mathbb{C}$ acts freely on $\operatorname{Stab}(\mathbb{N})$ we deduce that each component is homeomorphic to the universal cover $\mathbb{C}$, with $\mathbb{C}$ acting freely and transitively. The image of the continuous restriction map

$$
\rho_{\mathrm{N}}: \mathrm{B}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{~N}) \rightarrow \operatorname{Stab}^{L}(\mathrm{~N}), \quad(P, Z) \mapsto\left(P \cap \mathrm{~N},\left.Z\right|_{\Lambda_{\mathrm{N}}}\right)
$$

consists of lax stability conditions in which the simple objects of the generating set $S$ are stable and have a common phase.

Lemma 9.3. Fix $0<\varepsilon<1 / 8$. The map $\rho_{\mathrm{N}}$ restricts to a holomorphic fibration from $\mathrm{B}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{N}) \cap$ $\mathrm{Stab}(\mathrm{C})$ onto the component of $\mathrm{Stab}(\mathrm{N})$ consisting of stability conditions with heart a shift of $\langle S\rangle$.

Proof. If $\sigma=(P, Z) \in \mathrm{B}_{\varepsilon}^{L}(\mathrm{C}, \mathrm{N}) \cap \operatorname{Stab}(\mathrm{C})$ then clearly $\rho_{\mathrm{N}}(\sigma) \in \operatorname{Stab}(\mathrm{N})$. Moreover the objects of $S$ are stable with respect to $\rho_{\mathrm{N}}(\sigma)$ and by Corollary 9.2 have a common phase. It follows that the heart $P(0,1] \cap \mathrm{N}$ is a shift of the abelian length category $\langle S\rangle$. The $\mathbb{C}$-equivariance of $\rho_{\mathrm{N}}$ implies that the restriction is surjective onto this component. Indeed it is a holomorphic submersion whose fibres are biholomorphic to one another, and therefore it is a holomorphic fibration.

The only massless subcategory appearing in the boundary of this component is N itself. A lax stability condition is determined by a common phase $\varphi \in \mathbb{R}$ for the objects in $S$. Thus we have a commuting diagram

where we define $\exp (-\infty+i r)=0$ for $r \in \mathbb{R}$. In the next section we show that the restriction of $\rho_{\mathrm{N}}$ to $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ is a fibration over $-\infty+i \mathbb{R}$ with fibre $\operatorname{Stab}(\mathrm{C} / \mathrm{N})$.
9.3. The boundary stratum. In this section we show that the boundary stratum $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ is (a component of) the product $\operatorname{Stab}(\mathrm{C} / \mathrm{N}) \times \mathbb{R}$ of the space of stability conditions on the quotient category $\mathrm{C} / \mathrm{N}$ and a factor $\mathbb{R}$ recording the common phase of the massless objects. Recall that, for simplicity of notation only, we assume that $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ is connected.

We first observe that there are inclusions

$$
\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{~N}) \subset \operatorname{Stab}^{L S}(\mathrm{C}, \mathrm{~N}) \subset \operatorname{Stab}(\mathrm{C} / \mathrm{N}) \times \mathbb{R}
$$

where we abuse notation by identifying $\sigma \in \operatorname{Stab}^{L S}(\mathrm{C}, \mathrm{N})$ with its image under the continuous inclusion $\mu_{\mathrm{N}} \times \rho_{\mathrm{N}}$, and where $\operatorname{Stab}^{L S}(\mathrm{C}, \mathrm{N})$ denotes the set of lax pre-stability conditions $\sigma$ in the closure of $\operatorname{Stab}(\mathrm{C})$ with massless subcategory N and such that $\mu_{\mathrm{N}}(\sigma)$ is a stability condition on $\mathrm{C} / \mathrm{N}$; see Definition 5.1.

Theorem 9.4. Support propagates from $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$. Therefore, $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ is a component of $\operatorname{Stab}(\mathrm{C} / \mathrm{N}) \times \mathbb{R}$ and $\operatorname{Stab}^{Q}(\mathrm{C}, \mathrm{N})$ is a component of $\operatorname{Stab}(\mathrm{C} / \mathrm{N})$.
Proof. Let $\sigma=(P, Z) \in \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$. We show that support propagates from $\sigma$ in two steps:
(1) for $\tau$ sufficiently close to $\sigma$, we have $\tau \in \operatorname{Stab}^{L S}(\mathrm{C}, \mathrm{N})$; and,
(2) any $\tau \in \operatorname{Stab}^{l s}(\mathrm{C})$ with massless subcategory N satisfies the support property.

Step 1: Choose $\varepsilon>0$ sufficiently small that we can apply Theorem 6.1 and Theorem 6.17. Let $\tau=(Q, W) \in B_{\varepsilon}(\sigma)$. If $\left.W\right|_{\Lambda_{N}} \neq 0$ then $\tau$ is a (classical) pre-stability condition in $B_{\varepsilon}(\sigma)$ and is
therefore actually a stability condition by the argument at the end of the proof of Theorem 6.1. Therefore we may assume that $W \in \operatorname{Hom}\left(\Lambda / \Lambda_{\mathrm{N}}, \mathbb{C}\right)$. Then by Theorem 6.17

$$
\begin{equation*}
\left\{\tau=(Q, W) \in B_{\varepsilon}(\sigma): W \in \operatorname{Hom}\left(\Lambda / \Lambda_{\mathrm{N}}, \mathbb{C}\right)\right\} \subset \operatorname{Stab}^{L S}(\mathrm{C}, \mathrm{~N}) \tag{8}
\end{equation*}
$$

because $\mu_{\mathrm{N}}(\sigma)$ is a stability condition and so therefore is any sufficiently small deformation of it. (The condition that $\rho_{\mathrm{N}}(\tau)$ is in $\operatorname{Stab}^{L}(\mathbb{N})$ is automatic because $\operatorname{Stab}^{L}(\mathrm{~N}, \mathrm{~N}) \cong \mathbb{R}$.)
Step 2: It remains to show that any $\tau \in \operatorname{Stab}^{L S}(\mathrm{C}, \mathrm{N})$ satisfies the support property, for which we need the following lemma, whose proof we defer.
Lemma 9.5. Suppose $\sigma=(P, Z) \in \operatorname{Stab}^{L S}(\mathrm{C}, \mathrm{N})$. Let $S \subset P(\varphi)$ be a set of stable objects such that $\mathrm{N}=\operatorname{triang}(S)$. Then, up to shifts, any sequence $\left(b_{n}\right)$ of massive stable objects with $m\left(b_{n}\right) /\left\|v\left(b_{n}\right)\right\| \rightarrow 0$ contains a subsequence $\left(c_{n}\right)$ whose phases converge and such that

$$
\lim _{n \rightarrow \infty}\left(\frac{v\left(c_{n}\right)}{\left\|v\left(c_{n}\right)\right\|}\right)=\frac{v(s)}{\|v(s)\|}
$$

for any $s \in S$. Moreover, $\lim _{n \rightarrow \infty} \varphi\left(c_{n}\right) \in \varphi+2 \mathbb{Z}$ for any such subsequence.
Let $S \subset Q(\varphi)$ be a set of $\tau$-stable objects such that $\mathrm{N}=\operatorname{triang}(S)$. By Lemma 9.5 any sequence $\left(b_{n}\right)$ of massive stable objects with $m\left(b_{n}\right) /\left\|v\left(b_{n}\right)\right\| \rightarrow 0$ has, up to shifts, a subsequence $\left(c_{n}\right)$ with $\varphi\left(c_{n}\right) \rightarrow \varphi$. Without loss of generality we may assume $\varphi\left(c_{n}\right) \leq \varphi$ for all $n \in \mathbb{N}$; the other case is similar. Since $c_{n}$ is stable and not in $S$ this implies that $\operatorname{Hom}\left(s, c_{n}\right)=0$ for all $s \in S$.

Perturbing the common phase of the massless objects slightly using Corollary 6.10, we can find $\tau^{\prime}=\left(Q^{\prime}, W\right)$ also in $\operatorname{Stab}^{L S}(\mathrm{C}, \mathrm{N})$ with $\mu_{\mathrm{N}}\left(\tau^{\prime}\right)=\mu_{\mathrm{N}}(\tau)$ and $S \subset Q^{\prime}\left(\varphi^{\prime}\right)$ where $\varphi^{\prime}>\varphi$. The $c_{n}$ remain massive and stable for $\tau^{\prime}$ by Proposition 3.2 because

$$
c_{n} \in Q_{\mathrm{C} / \mathrm{N}}\left(\varphi\left(c_{n}\right)\right)=Q_{\mathrm{C} / \mathrm{N}}^{\prime}\left(\varphi\left(c_{n}\right)\right)
$$

where $\varphi\left(c_{n}\right)<\varphi^{\prime}$ and $\operatorname{Hom}\left(s, c_{n}\right)=0$ for all $s \in S$. However now $\varphi^{\prime}\left(c_{n}\right)=\varphi\left(c_{n}\right) \rightarrow \varphi<\varphi^{\prime}$ contradicting Lemma 9.5. We conclude that there is no sequence $\left(b_{n}\right)$ of massive stable objects with $m\left(b_{n}\right) /\left\|v\left(b_{n}\right)\right\| \rightarrow 0$. Therefore $\tau$ satisfies the support property and so is in $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$.

The final statement on the space of quotient stability conditions follows from Proposition 7.1 and the definition of $\operatorname{Stab}^{Q}(\mathrm{C}, \mathrm{N})$ because $\operatorname{Stab}^{L}(\mathrm{~N}, \mathrm{~N}) \cong \mathbb{R}$.
Proof of Lemma 9.5. If $\sigma$ satisfies the support property then there is no sequence $\left(b_{n}\right)$ with $m\left(b_{n}\right) /\left\|v\left(b_{n}\right)\right\| \rightarrow 0$ and the result is vacuously true. Therefore we assume $\sigma$ does not satisfy the support property i.e. there is a sequence $\left(b_{n}\right)$ of massive stable objects with $m\left(b_{n}\right) /\left\|v\left(b_{n}\right)\right\| \rightarrow 0$.

Since $\Lambda_{\mathrm{N}}$ has rank one and by Corollary 9.2 the objects in $S$ are stable with a common phase in nearby classical stability conditions, there is $\lambda \in \Lambda_{N} \otimes \mathbb{R}$ with $\|\lambda\|=1$ such that $v(s) \in \mathbb{R}_{>0} \cdot \lambda$ for all $s \in S$. Lemma 4.16 tells us that there is a subsequence $\left(c_{n}\right)$ such that

$$
\lim _{n \rightarrow \infty}\left(\frac{v\left(c_{n}\right)}{\left\|v\left(c_{n}\right)\right\|}\right) \in \Lambda_{\mathrm{N}} \otimes \mathbb{R} .
$$

By replacing $c_{n}$ with $c_{n}[1]$ if necessary we can ensure that the limit is $\lambda=v(s) /\|v(s)\|$. Further shifting by an even integer if necessary we may assume that $\varphi\left(c_{n}\right) \in[\varphi-1, \varphi+1]$ for each $n \in \mathbb{N}$. Therefore, we can pass to a subsequence $\left(c_{n}\right)$ whose phases converge. It remains to show that the limit is the common phase $\varphi$ of the objects $s \in S$.

Suppose that $\left(\sigma_{m}\right)$ is a sequence in $\operatorname{Stab}(\mathrm{C})$ tending to $\sigma$. By the argument in the proof of Corollary 9.2 , for sufficiently large $m \in \mathbb{N}$, the objects in $S$ are $\sigma_{m}$-stable with common phase $\varphi_{m}(s)$ for all $s \in S$. For similar reasons, $c_{n}$ is massive for each $\sigma_{m}$ and its charge has a well-defined phase $\varphi_{m}\left(c_{n}\right)$ in the interval $\left(\varphi\left(c_{n}\right)-1 / 8, \varphi\left(c_{n}\right)+1 / 8\right)$. Since $d\left(P_{m}, P\right) \rightarrow 0$ and $\varphi\left(c_{n}\right) \rightarrow \varphi^{\prime}$, say, for any $\varepsilon>0$ we can find $M, N \in \mathbb{N}$ such that

$$
\left|\varphi_{m}\left(c_{n}\right)-\varphi^{\prime}\right| \leq\left|\varphi_{m}\left(c_{n}\right)-\varphi\left(c_{n}\right)\right|+\left|\varphi\left(c_{n}\right)-\varphi^{\prime}\right|<\varepsilon
$$

whenever $m \geq M$ and $n \geq N$. Therefore we can switch the limits to compute

$$
\varphi^{\prime}=\lim _{n \rightarrow \infty} \varphi\left(c_{n}\right)=\lim _{n \rightarrow \infty} \lim _{m \rightarrow \infty} \varphi_{m}\left(c_{n}\right)=\lim _{m \rightarrow \infty} \lim _{n \rightarrow \infty} \varphi_{m}\left(c_{n}\right)=\lim _{m \rightarrow \infty} \varphi_{m}(s)=\varphi
$$

where the penultimate step follows because the phase $\varphi_{m}$ only depends on the normalised vector in $\Lambda \otimes \mathbb{R}$ and $v\left(c_{n}\right) /\left\|v\left(c_{n}\right)\right\| \rightarrow v(s) /\|v(s)\|$.
Remark 9.6. In Step 2 of the proof of Theorem 9.4 we established the support property for all elements of $\operatorname{Stab}^{L S}(\mathrm{C}, \mathrm{N})$. This special situation in the codimension one case is not obvious in general: Example 6.4 gives an example of a lax pre-stability condition which satisfies support on the quotient of $\mathrm{C} / \mathrm{N}$ but does not satisfy the stronger support property required to be a lax stability condition. However, this is not quite a counter-example because it is not in the closure of $\operatorname{Stab}(\mathrm{C})$ since the slicings do not converge as we approach it.

## 10. Finite type components

Let C be a triangulated category with $K(\mathrm{C}) \cong \mathbb{Z}^{n}$ and set $v=\mathrm{id}: K(\mathrm{C}) \rightarrow \Lambda$. For simplicity of notation we assume that $\operatorname{Stab}(\mathrm{C})$ is connected; otherwise consider a component. Say that a stability condition $\sigma=(P, Z)$ has discrete phase distribution if $\{\varphi \in \mathbb{R}: P(\varphi) \neq 0\}$ is a discrete subset of $\mathbb{R}$, and that it has algebraic heart if $P(0,1]$ is a finite length abelian category with finitely many simple objects. Finally, we say that $\operatorname{Stab}(\mathrm{C})$ has finite type if each stability condition has an algebraic heart admitting only finitely many torsion pairs [30].

Lemma 10.1. Assume that $K(\mathrm{C}) \cong \mathbb{Z}^{n}$ is free abelian. The following are equivalent:
(1) Each $\sigma \in \operatorname{Stab}(\mathrm{C})$ has algebraic heart.
(2) $\operatorname{Stab}(\mathrm{C})$ has finite type.
(3) Stab(C) has dimension $n$ and each stability condition has discrete phase distribution.

Proof. (1) $\Longrightarrow(2)$ : Suppose each $\sigma \in \operatorname{Stab}(C)$ has algebraic heart. Then the set of hearts of stability conditions in $\operatorname{Stab}(\mathrm{C})$ is closed under HRS-tilting at simple objects of these hearts (see, for example, [19, 33] for the definition), because a stability condition with such a heart is given by freely assigning a charge in the upper half plane to each simple object of the heart.

Moreover, by [30, Corollary 3.27] the poset of strata is pure of length $n=\operatorname{rk} K(\mathrm{C})$, i.e. each maximal chain in the poset has length $n$. This implies that each heart has only finitely many torsion pairs because the poset of torsion pairs in the heart has a uniform bound on the valency of each element and on the length of each chain. Therefore $\operatorname{Stab}(\mathrm{C})$ has finite type.
$(2) \Longrightarrow(3)$ : Now suppose $\operatorname{Stab}(C)$ has finite type. The phase distribution of each $\sigma \in$ $\operatorname{Stab}(\mathrm{C})$ must be discrete, for otherwise rotating phases would yield an infinite sequence of tilts, and thus infinitely many torsion pairs in the heart of $\sigma$.
$(3) \Longrightarrow(1)$ : Finally, suppose each $\sigma \in \operatorname{Stab}(C)$ has discrete phase distribution. Then for each $\sigma$ there exists $\varepsilon>0$ such that $P_{\sigma}(0, \varepsilon)=0$. Hence by [30, Lemma 3.1] the heart of $\sigma$ is algebraic.

In general, we do not know which thick subcategories N of C can appear as massless subcategories. If N occurs as a massless subcategory then, by Corollary 7.4, both $\operatorname{Stab}(\mathrm{C} / \mathrm{N})$ and $\operatorname{Stab}(N)$ are non-empty. However this is not sufficient, as can be already seen for $\mathrm{D}^{b}\left(\mathbb{P}^{1}\right)$.
Example 10.2. Let $\mathrm{C}=\mathrm{D}^{b}\left(\mathbb{P}^{1}\right)$ with $\Lambda=K(\mathrm{C}) \cong \mathbb{Z}^{2}$ and let $\mathrm{N}=\operatorname{thick}\left(\mathcal{O}_{x}: x \in \mathbb{P}^{1}\right)$. Note $\Lambda_{N}=\operatorname{im}(K(\mathbb{N}) \rightarrow K(\mathrm{C}) \rightarrow \Lambda) \cong \mathbb{Z}$ even though $K(\mathbb{N}) \cong \mathbb{Z}^{\mathbb{P}^{1}}$. With these lattices, $\operatorname{Stab}(\mathbb{N}) \cong \mathbb{C}$ and $\operatorname{Stab}(\mathrm{C} / \mathrm{N}) \cong \mathbb{C}$ are non-empty. However, it follows from Proposition 12.6 that N does not occur as a massless subcategory because the skyscrapers $\mathcal{O}_{x}$ are not simple objects in any algebraic heart.

For a finite type component of a stability space the situation is much simpler: we can classify the massless subcategories completely, and show that support propagates from each boundary stratum.

Corollary 10.3. Suppose $\operatorname{Stab}(\mathrm{C})$ has finite type and $\sigma=(P, Z) \in \overline{\operatorname{Stab}(\mathrm{C})}$. Then for any $\varphi \in \mathbb{R}$ the full subcategory $P(\varphi, \varphi+1]$ is the heart of some classical stability condition in $\operatorname{Stab}(\mathrm{C})$, and so, in particular, is algebraic. Moreover, $\sigma$ has discrete phase distribution.

Proof. Rotating phases by the $\mathbb{C}$ action it suffices to prove the first part for $\varphi=0$. Choose $\tau=(Q, W)$ in $\operatorname{Stab}(\mathrm{C})$ with $d(P, Q)<1 / 2$ so that the hearts $P(0,1]$ and $Q(0,1]$ are both contained in $Q(-1 / 2,3 / 2]$. Then $P(0,1]$ is obtained by first tilting from $Q(0,1]$ to $Q(-1 / 2,1 / 2$ ] and then tilting from that to $P(0,1]$. Since the set of hearts of stability conditions in the finitetype component $\operatorname{Stab}(\mathrm{C})$ is closed under tilting, $P(0,1]$ is the heart of some stability condition in $\operatorname{Stab}(\mathrm{C})$. Therefore $P(0,1]$ is algebraic.

Since $P(\varphi, \varphi+1]$ is algebraic there exists $\varepsilon>0$, depending on $\varphi$, for which $P(\varphi, \varphi+\varepsilon)=0$. It follows that $\sigma$ has discrete phase distribution.
Corollary 10.4. Suppose $\operatorname{Stab}(\mathrm{C})$ has finite type. Then N is the massless subcategory of some $\sigma \in \operatorname{Stab}^{L}(\mathrm{C})$ if and only if N is the triangulated closure of a finite subset of simple objects in the heart of some classical stability condition $\tau \in \operatorname{Stab}(\mathrm{C})$.
Proof. Suppose N is generated by a finite subset $I$ of simple objects in the heart $Q(0,1]$ of some classical $\tau=(Q, W) \in \operatorname{Stab}(\mathrm{C})$. By deforming the charge we may assume that

$$
W(s)= \begin{cases}-r & \Longleftrightarrow s \in I, \\ -1 & \Longleftrightarrow s \notin I,\end{cases}
$$

for some $r \in \mathbb{R}_{>0}$. Clearly the charges converge as $r \rightarrow 0$. Moreover, since $Q(0,1]=Q(1)$ for all $r$ the slicings also converge as $r \rightarrow 0$. In the limit as $r \rightarrow 0$ we obtain a lax stability condition $\sigma$ with massless subcategory N and the same slicing $Q$. Therefore the only stable objects of $\sigma$ are the simple objects of $Q(0,1]$. Since there are finitely many of these, $\sigma$ satisfies the support property and therefore lies in $\operatorname{Stab}^{L}(\mathrm{C})$.
Now suppose $\sigma=(P, Z) \in \operatorname{Stab}^{L}(\mathrm{C})$. By Lemma 4.5 the massless subcategory N is the triangulated closure of the set of stable massless objects in $P(0,1]$. Since $P(0,1]$ is algebraic each stable massless object has a composition series whose factors are massless simple objects, each of which is of course also stable. Therefore $\mathrm{N}=\operatorname{triang}(S)$ is the triangulated closure of the set $S$ of massless simple objects in $P(0,1]$. The result follows because $P(0,1]$ is the heart of some classical stability condition in $\operatorname{Stab}(\mathrm{C})$ by Corollary 10.3.
Remark 10.5. Corollary 10.4 tells us that massless subcategories need not be admissible. An example is the derived-discrete algebra $A=\Lambda(2,1,0)$; see [4] for notation and [26, §8] for a detailed description of $\mathrm{D}^{b}(A)$. One of the two simple modules $S$ is a 2 -spherical object. Therefore, its thick hull $\mathrm{N}:=\operatorname{thick}(S)$ occurs as a massless subcategory. On the other hand, the duality property $\operatorname{Hom}(S,-)=\operatorname{Hom}(-, S[2])^{*}$ implies that an adjoint of the inclusion $\mathrm{N} \hookrightarrow \mathrm{D}^{b}(A)$ would lead to a splitting $\mathrm{D}^{b}(A) \cong \mathrm{N} \oplus \mathrm{M}$ but $\mathrm{D}^{b}(A)$ is indecomposable.
Proposition 10.6. If $\operatorname{Stab}(\mathrm{C})$ has finite type, then $\operatorname{Stab}^{L}(\mathrm{C})=\overline{\operatorname{Stab}(\mathrm{C})}$.
Proof. Suppose $\sigma=(P, Z) \in \overline{\operatorname{Stab}(C)}$ does not satisfy the support property. Then we can find a sequence $\left(c_{n}\right)$ of massive stable objects with $m_{\sigma}\left(c_{n}\right) /\left\|v\left(c_{n}\right)\right\| \rightarrow 0$. Shifting the objects if necessary we may assume they all have phase in $(0,1]$ and hence, by passing to a subsequence, that the phases $\varphi\left(c_{n}\right) \rightarrow \varphi$ as $n \rightarrow \infty$. Corollary 10.3 states that $\sigma$ has a discrete phase distribution so, again passing to a subsequence if required, we may assume that $\varphi\left(c_{n}\right)=\varphi$ for all $n \in \mathbb{N}$. Thus the $c_{n}$ are simple objects in $P(\varphi)$, and therefore also simple in $P(\varphi-1, \varphi]$. There are only finitely many iso-classes of such simple objects because $P(\varphi-1, \varphi]$ is algebraic by Corollary 10.3. So, passing to a subsequence for a final time, we may assume that $\left(c_{n}\right)$ is a constant sequence. Since the $c_{n}$ are massive this contradicts the fact that $m_{\sigma}\left(c_{n}\right) /\left\|v\left(c_{n}\right)\right\| \rightarrow 0$. We conclude that no such sequence exists, i.e. that $\sigma$ satisfies support after all.
Corollary 10.7. If $\operatorname{Stab}(\mathrm{C})$ has finite type, then support propagates from $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$.
Proof. This follows immediately from the fact that $\operatorname{Stab}^{L}(\mathrm{C})=\overline{\operatorname{Stab}(C)}$ and Theorem 6.17 which show that deformations of an element of $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ remain in the closure of $\operatorname{Stab}(\mathrm{C})$.
Corollary 10.8. Suppose $\operatorname{Stab}(C)$ has finite type and that $\sigma \in \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$. Then $\mu_{\mathrm{N}}(\sigma)$ is in a finite type component of $\operatorname{Stab}(\mathrm{C} / \mathrm{N})$.

Proof. By Corollary 10.7, the entire component of $\mu_{\mathrm{N}}(\sigma)$ in $\operatorname{Stab}(\mathrm{C} / \mathrm{N})$ is in the image of $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ under $\mu_{\mathrm{N}}$. The slicing of each lax stability condition $\sigma=(P, Z)$ in $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ has discrete phase distribution by Corollary 10.3. Since $P_{\mathrm{C} / \mathrm{N}}(\varphi)=0$ when $P(\varphi)=0$ the slicing $P_{\mathrm{C} / \mathrm{N}}$ of $\mu_{\mathrm{N}}(\sigma)$ also has discrete phase distribution. Thus the component of $\mu_{\mathrm{N}}(\sigma)$ has finite type by Lemma 10.1.

Remark 10.9. We have thus shown that the conditions for the structure results of $\S 7$ and $\S 8$ are satisfied. If $\operatorname{Stab}(\mathrm{C})$ is a finite type component then the space $\operatorname{Stab}^{L}(\mathrm{C})$ of lax stability conditions is a union of components of $\operatorname{Stab}(\mathrm{C} / \mathrm{N}) \times \operatorname{Stab}^{L}(\mathrm{~N}, \mathrm{~N})$, where N is the triangulated closure of a finite set of simple objects in the heart of a stability condition in Stab(C). Each component of $\operatorname{Stab}(\mathrm{C} / \mathrm{N})$ which appears has finite type. The space $\mathrm{Stab}^{Q}(\mathrm{C})$ of quotient stability conditions is therefore stratified by these finite type components.

## 11. Closures of $G$-orbits

Recall that the the universal cover $G$ of $\mathrm{GL}_{2}^{+}(\mathbb{R})$ acts on stability spaces and our partial compactifications. In this section we describe the closures of $G$-orbits. The phase diagrams $\Phi_{\sigma}=\{\varphi+\mathbb{Z}: P(\varphi) \neq 0\} \subset \mathbb{R} / \mathbb{Z}$ of stability conditions $\sigma=(P, Z)$ in the same orbit are related by orientation-preserving diffeomorphisms of the circle $\mathbb{R} / \mathbb{Z}$. The structure of the closure of the orbit in both $\operatorname{Stab}(\mathrm{C})$ and in $\operatorname{Stab}^{Q}(\mathrm{C})^{*}$ can be described in terms of the phase diagram. Here $\operatorname{Stab}^{Q}(\mathrm{C})^{*}$ is the space obtained by removing the strata where all objects are massless from $\operatorname{Stab}^{Q}(\mathrm{C})$. This is more convenient to consider because the action of $\mathbb{C}$ on $\operatorname{Stab}^{Q}(\mathrm{C})^{*}$ is free and in fact we will describe the closure of $(\sigma \cdot G) / \mathbb{C}$ in $\operatorname{Stab}(\mathrm{C}) / \mathbb{C}$ and in $\operatorname{Stab}^{Q}(\mathrm{C})^{*} / \mathbb{C}$.

Fix a stability condition $\sigma=(P, Z)$. If there is only one point in the phase diagram $\Phi_{\sigma}$ then there is a non-trivial stabiliser and $(\sigma \cdot G) / \mathbb{C}$ is a point. In this case the orbit is closed in $\operatorname{Stab}(\mathrm{C}) / \mathbb{C}$ and also in $\operatorname{Stab}^{Q}(\mathrm{C})^{*} / \mathbb{C}$. Henceforth, we assume that $\Phi_{\sigma}$ consists of at least two points. In particular the image of the charge is the whole of $\mathbb{C}$ so that the $G$-orbit through $\sigma$ is free and $(\sigma \cdot G) / \mathbb{C}$ is biholomorphic to the Poincaré disk $\mathbb{D}$ because $G \cong \mathbb{C} \times \mathbb{H}$. We show that the closures of the orbit in $\operatorname{Stab}(\mathrm{C}) / \mathbb{C}$ and in $\operatorname{Stab}^{Q}(\mathrm{C})^{*} / \mathbb{C}$ are homeomorphic to subsets of the closed disk, with appropriate topologies. Our constructions require an explicit identification of the Poincaré disk and the strict upper half-plane; we choose $f: \mathbb{D} \rightarrow \mathbb{H}$ given by

$$
f(w)=i \frac{1+w}{1-w}
$$

Note that $f$ extends to a homeomorphism from the closure of the disk to $\mathbb{H} \cup \mathbb{R} \cup\{\infty\}$. For each $w \in \overline{\mathbb{D}}$ we choose a charge $Z_{w}=M_{w} \circ Z$ where $M_{w} \in \operatorname{End}_{\mathbb{R}}(\mathbb{C})$ satisfies

$$
M_{w}(1)=1 \quad \text { and } \quad M_{w}(i)=f(w) \quad(w \neq 1)
$$

and $M_{1}(1)=0, M_{1}(i)=-1$. Note that $M_{w}(i) \in \mathbb{H}$ when $w \in \mathbb{D}$ so that there is a unique compatible slicing $P_{w}$ with $P_{w}(0,1]=P(0,1]$. Mapping $w$ to the $\mathbb{C}$-orbit of $\sigma_{w}=\left(P_{w}, Z_{w}\right)$ gives an explicit identification $\mathbb{D} \cong(\sigma \cdot G) / \mathbb{C}$. The reason for this particular choice is that

$$
\begin{aligned}
P(\varphi) \subset \operatorname{ker} Z_{w} & \Longleftrightarrow M_{w}\left(e^{i \pi \varphi}\right)=0 \\
& \Longleftrightarrow M_{w}(1) \cos (\pi \varphi)+f(w) \sin (\pi \varphi)=0 \\
& \Longleftrightarrow(w=1 \text { and } \varphi=0) \text { or }\left(w=e^{2 \pi i \varphi} \text { and } \varphi \neq 0\right) \\
& \Longleftrightarrow w=e^{2 \pi i \varphi}
\end{aligned}
$$

so that points on $\partial \mathbb{D}$ correspond to charges for which semistable objects of a certain phase have vanishing mass in a natural way. For $w=e^{2 \pi i \varphi} \in \partial \mathbb{D}$ and $c \in P\left(\varphi^{\prime}\right)$ the sine rule yields

$$
Z_{w}(c)=\left\{\begin{array}{cl}
|Z(c)| \frac{\sin \pi\left(\varphi-\varphi^{\prime}\right)}{\sin \pi \varphi} & \text { if } w \neq 1  \tag{9}\\
-|Z(c)| \sin \pi \varphi^{\prime} & \text { if } w=1 \\
48 &
\end{array}\right.
$$

When $P(\varphi)=0$ there is a unique choice of slicing $P_{w}$ for $w=e^{2 \pi i \varphi}$ which is compatible with $Z_{w}$ and with $P(0,1] \subset P_{w}[0,1]$, namely

$$
P_{w}(1)=P(\varphi, \varphi+1]=P[\varphi, \varphi+1)
$$

and all slices with phase in $(0,1)$ are zero. One can verify that $\sigma_{w}=\left(P_{w}, Z_{w}\right)$ is a pre-stability condition.

Lemma 11.1. Suppose $w=e^{2 \pi i \varphi} \in \partial \mathbb{D}$ and $P(\varphi)=0$. Then $\sigma_{w}=\left(P_{w}, Z_{w}\right)$ is a stability condition if and only if $\varphi$ is not an accumulation point of the phase diagram $\Phi_{\sigma}$.

Proof. Suppose $\varphi \neq 0$ so that $M_{w}(1)=1$. If $\varphi$ is an accumulation point of $\Phi_{\sigma}$ and $\varepsilon>0$ then using the first equation in (9) shows that one can choose $\varphi^{\prime} \in \Phi_{\sigma}$ sufficiently close to $\varphi$ that

$$
\left|Z_{w}(c)\right| \leq \frac{\varepsilon}{\|Z\|}|Z(c)| \leq \varepsilon\|v(c)\|
$$

for any $c \in P\left(\varphi^{\prime}\right)$. Hence the support property fails since $c$ remains semistable for $\sigma_{w}$.
Conversely, if $\varphi$ is not an accumulation point then there is $L>0$ such that

$$
\left|Z_{w}(c)\right| \geq L|Z(c)| \geq K L\|v(c)\|
$$

for all $c \in P\left(\varphi^{\prime}\right)$ where $\varphi^{\prime} \in(0,1]$, and $K$ is a support constant for $\sigma$. Therefore the same inequality holds for all $c \in P_{w}(1)=P(\varphi, \varphi+1]$ so that $\sigma_{w}$ satisfies the support property as claimed.

The case $\varphi=0$ is similar but uses the second equation in (9).
When $P(\varphi) \neq 0$ there is a one-parameter family of compatible slicings $P_{w}^{\psi}$ for $w=e^{2 \pi i \varphi}$ compatible with $Z_{w}$ and with $P(0,1] \subset P_{w}[0,1]$. They differ by the choice of phase for the massless objects in $P(\varphi)$. Namely, for each $\psi \in[0,1]$ there is a unique such slicing $P_{w}^{\psi}$ with

$$
P_{w}^{\psi}(1) \supset P(\varphi, \varphi+1) \quad \text { and } \quad P_{w}^{\psi}(\psi) \supset P(\varphi)
$$

and all other slices with phase in $(0,1]$ zero. One can verify that $\sigma_{w}^{\psi}=\left(P_{w}^{\psi}, Z_{w}\right)$ is a lax prestability condition. We now give criteria for when it is in the closure of the orbit, and when it satisfies the support property.

Lemma 11.2. Suppose $w=e^{2 \pi i \varphi} \in \partial \mathbb{D}$ and $P(\varphi) \neq 0$. The lax pre-stability condition $\sigma_{w}^{\psi}$ is in the closure of the orbit $\sigma \cdot G$ in $\operatorname{Slice}(\mathrm{C}) \times \operatorname{Hom}(\Lambda, \mathbb{C})$ if and only if one of the following conditions holds
(1) $\varphi$ is an isolated point of $\Phi_{\sigma}$ and $\psi \in[0,1]$;
(2) $\varphi$ is an accumulation point of $\Phi_{\sigma}$ such that $P(\varphi-\varepsilon, \varphi)=0$ for some $\varepsilon>0$ and $\psi=1$;
(3) $\varphi$ is an accumulation point of $\Phi_{\sigma}$ such that $P(\varphi, \varphi+\varepsilon)=0$ for some $\varepsilon>0$ and $\psi=0$.

In particular, the slicing $P_{w}^{\psi}$ is locally-finite in each of the above cases.
Proof. If there is a sequence in $\Phi_{\sigma}$ tending to $\varphi$ from below then the slicings of a sequence in $\sigma \cdot G$ can only converge to $P_{w}^{\psi}$ if $\psi=0$. Similarly, if there is a sequence in $\Phi_{\sigma}$ tending to $\varphi$ from above then the slicings of a sequence in $\sigma \cdot G$ can only converge to $P_{w}^{\psi}$ if $\psi=1$. It follows that if $\sigma_{w}^{\psi}$ is in the closure then we are in one of the three cases in the statement. In each of those cases one can construct a sequence $\left(w_{n}\right)$ in $\mathbb{D}$ converging to $w \in \partial \mathbb{D}$ and with $\sigma_{w_{n}} \rightarrow \sigma_{w}^{\psi}$ by controlling the limiting phase of objects in $P(\varphi)$ appropriately.
Lemma 11.3. Suppose $w=e^{2 \pi i \varphi} \in \partial \mathbb{D}$ and $P(\varphi) \neq 0$ and moreover that the lax pre-stability condition $\sigma_{w}^{\psi}$ is in the closure of the orbit $\sigma \cdot G$. Then it satisfies the support property if, and only if, there is some $\varepsilon>0$ such that either
(1) $P(\varphi-\varepsilon, \varphi)=0$ and no simple object in $P[\varphi, \varphi+1)$ lies in $P(\varphi, \varphi+\varepsilon)$; or,
(2) $P(\varphi, \varphi+\varepsilon)=0$ and no simple object in $P(\varphi, \varphi+1]$ lies in $P(\varphi+1-\varepsilon, \varphi+1)$.

In particular, if $\varphi$ is isolated both conditions are satisfied and $\sigma_{w}^{\psi}$ satisfies the support property.

Proof. Under the assumption $P(\varphi-\varepsilon, \varphi)=0$ the heart $P[\varphi, \varphi+1)$ is length so that it makes sense to speak of simple objects, and similarly for the second case. We only need to check the support property for massive stable objects. Each such is also $\sigma$ stable. If $\varphi$ is an isolated point of $\Phi_{\sigma}$ then any massive $\sigma_{w}^{\psi}$ stable object lies in a slice $P\left(\varphi^{\prime}\right)$ whose phase $\varphi^{\prime}$ is bounded away from $\varphi$. Again the support property follows as in the proof of Lemma 11.1.

If $\varphi$ is an accumulation point of $\Phi_{\sigma}$ with $P(\varphi-\varepsilon, \varphi)=0$ then $\psi=1$ by Lemma 11.2. It is enough to consider massive stable objects in $P_{w}^{\psi}(1)=P[\varphi, \varphi+1)$. These are the simple objects of this heart which are not in $P(\varphi)$. Therefore each such lies in $P\left(\varphi^{\prime}\right)$ for some $\varphi^{\prime} \in(\varphi, \varphi+1)$. The result then follows as in the proof of Lemma 11.1. The other case is similar.
Remark 11.4. If C has an algebraic heart then any length heart is algebraic, i.e. has finitely many iso-classes of simple objects. In this situation the conditions simplify to $P(\varphi-\varepsilon, \varphi)=0$ or $P(\varphi, \varphi+\varepsilon)=0$, i.e. to $\varphi$ being isolated or at worst a one-sided accumulation point.
Definition 11.5. We define subsets of the closure of the Poincaré disk by

$$
\begin{aligned}
\mathbb{D}_{\sigma} & =\mathbb{D} \cup\left\{e^{2 \pi i \varphi}: \varphi \notin \overline{\Phi_{\sigma}}\right\} \\
\text { and } \mathbb{D}_{\sigma}^{Q} & =\mathbb{D} \cup\left\{e^{2 \pi i \varphi}: \varphi \text { satisfies either (1) or (2) of Lemma } 11.3 \text { for some } \varepsilon>0\right\} .
\end{aligned}
$$

Note that $\mathbb{D}_{\sigma} \subset \mathbb{D}_{\sigma}^{Q}$ because (1) and (2) are satisfied when $\varphi \notin \overline{\Phi_{\sigma}}$. For each $w \in \mathbb{D}_{\sigma}^{Q}$ we have defined a quotient stability condition $\bar{\sigma}_{w} \in \operatorname{Stab}^{Q}(\mathrm{C})^{*}$, namely $\bar{\sigma}_{w}=\left[\sigma_{w}\right]$ for $w \in \mathbb{D}_{\sigma}$ and $\bar{\sigma}_{w}=\left[\sigma_{w}^{\psi}\right]$ for $w \in \mathbb{D}_{\sigma}^{Q}-\mathbb{D}_{\sigma}$.
Proposition 11.6. Let $\overline{\sigma \cdot G} / \mathbb{C}$ denote the closure of the orbit of $\sigma$ in $\operatorname{Stab}^{Q}(\mathrm{C})^{*} / \mathbb{C}$. Then

$$
\mathbb{D}_{\sigma}^{Q} \rightarrow \overline{\sigma \cdot G} / \mathbb{C} \text { given by } \quad w \mapsto \bar{\sigma}_{w} \cdot \mathbb{C}
$$

is a bijection, and restricts to a bijection between $\mathbb{D}_{\sigma}$ and the closure of the orbit of $\sigma$ in $\operatorname{Stab}(\mathrm{C}) / \mathbb{C}$.
Proof. By construction, and Lemma 11.2, $\bar{\sigma}_{w}$ is in $\overline{\sigma \cdot G}$ so the map is well-defined. Since the image of the charge $Z$ is the whole of $\mathbb{C}$ we can find $\lambda, \mu \in \Lambda \otimes \mathbb{R}$ with $Z(\lambda)=i$ and $Z(\mu)=1$. Then the assignment

$$
(Q, W) \mapsto f^{-1}\left(\frac{W(\lambda)}{W(\mu)}\right)
$$

descends to a map $\overline{\sigma \cdot G} / \mathbb{C} \rightarrow \overline{\mathbb{D}}$ taking $\bar{\sigma}_{w}$ to $w$. We claim this is the inverse. Certainly, if $w$ is the image of $(Q, W)$ then $W$ is in the same $\mathbb{C}$ orbit as $Z_{w}$. It follows that the image of this map is precisely $\mathbb{D}_{\sigma}^{Q}$, because we have shown that there are no quotient stability conditions in $\overline{\sigma \cdot G}$ with charge $Z_{w}$ for $w \notin \mathbb{D}_{\sigma}^{Q}$. Moreover, there is a unique quotient stability condition in $\overline{\sigma \cdot G}$ with charge $Z_{w}$, which establishes the claim.
Corollary 11.7. The orbit $\sigma \cdot G$ is closed in $\operatorname{Stab}(\mathrm{C})$ and in $\operatorname{Stab}^{Q}(\mathrm{C})^{*}$ if and only if the phase diagram $\Phi_{\sigma}$ is dense.
Proof. This follows immediately from the above result since $\mathbb{D}_{\sigma}^{Q}=\mathbb{D}$ when $\Phi_{\sigma}$ is dense. Alternatively, note that the Bridgeland metric induces the standard hyperbolic metric on the quotient $(\sigma \cdot G) / \mathbb{C} \cong \mathbb{D}$ up to a factor [34, Proposition 4.1]. Since the hyperbolic metric is complete, the orbit is closed.

Finally we describe the topology on $\mathbb{D}_{\sigma}^{Q}$ for which the bijection in Proposition 11.6 is a homeomorphism. This is the topology in which a sequence $\left(w_{n}\right)$ converges to $w$ if and only if the charges $Z_{w_{n}}$ converge in $\operatorname{Hom}(\Lambda, \mathbb{C}) / \mathbb{R}_{>0}$ and the slicings $P_{w_{n}}$ converge in Slice $(\mathrm{C})$. Since $d\left(P_{w}, P_{w^{\prime}}\right) \leq 1$ for $w, w^{\prime} \in \mathbb{D}_{\sigma}^{Q}$ the convergence of charges implies uniform convergence of the phases of massive objects. This occurs whenever $w_{n} \rightarrow w$ in the subspace topology from $\mathbb{C}$. To ensure that the phase of the massless objects in $P(\varphi)$ converges we have to refine this topology in a neighbourhood of each $w \in \mathbb{D}_{\sigma}^{Q}-\mathbb{D}_{\sigma}$ so that convergence also implies that

$$
\lim _{n \rightarrow \infty} \frac{w_{n}-w}{\left\lvert\, \begin{array}{c}
w_{n}-w \mid \\
50
\end{array}\right.}
$$

is a well-defined unit tangent vector in $T_{w} \mathbb{C}$. In other words, the required topology arises from the real blow-up $\beta: \widetilde{\mathbb{C}} \rightarrow \mathbb{C}$ at the points in $\mathbb{D}_{\sigma}^{Q}-\mathbb{D}_{\sigma}$. More precisely, let $\widetilde{\mathbb{D}}_{\sigma}^{Q}$ be the subspace of $\beta^{-1}\left(\mathbb{D}_{\sigma}^{Q}\right)$ consisting of $\beta^{-1}\left(\mathbb{D}_{\sigma}\right)$ and those points $\left(w=e^{2 \pi i \varphi}, v\right)$ in the exceptional divisors where the unit tangent vector $v \in T_{w} \mathbb{C}$ is such that $\varphi$ and

$$
\psi=\frac{1}{\pi} \arg d_{w} f(v)
$$

satisfy one of the conditions (1), (2) or (3) in Lemma 11.2. The above discussion leads to the following description of orbit closures.
Corollary 11.8. Equip $\mathbb{D}_{\sigma}^{Q}$ with the quotient topology from $\beta: \widetilde{\mathbb{D}}_{\sigma}^{Q} \rightarrow \mathbb{D}_{\sigma}^{Q}$. Then

$$
\mathbb{D}_{\sigma}^{Q} \rightarrow \overline{\sigma \cdot G} / \mathbb{C}: w \mapsto \bar{\sigma}_{w} \cdot \mathbb{C}
$$

is a homeomorphism, and restricts to a homeomorphism between $\mathbb{D}_{\sigma}$ and the closure of the orbit of $\sigma$ in $\operatorname{Stab}(\mathrm{C}) / \mathbb{C}$.
Remark 11.9. In fact, although we have not filled in all the details, the subspace $\widetilde{\mathbb{D}}_{\sigma}^{Q}$ is homeomorphic to the closure of $(\sigma \cdot G) / \mathbb{C}$ in $\operatorname{Stab}^{L}(\mathrm{C})^{*} / \mathbb{C}$.

## 12. Two-dimensional stability spaces

We illustrate our results in the simplest non-trivial case in which the stability space is a 2 dimensional complex manifold. In this context there is a very close relationship between the boundary strata we add and the wall-and-chamber structure of the stability space.
12.1. Walls and chambers. Suppose that $\Lambda$ is a rank two lattice, and moreover that if $C$ contains a length heart then that heart has two iso-classes of simple objects and $\Lambda=K(\mathrm{C}) \cong \mathbb{Z}^{2}$. This assumption is reasonable because otherwise the stability space of C is naturally higherdimensional. Since $\operatorname{rk}(\Lambda)=2$ the quotient $\operatorname{Stab}(\mathrm{C}) / \mathbb{C}$ is a non-compact Riemann surface and

$$
\operatorname{Stab}(\mathrm{C}) \cong \operatorname{Stab}(\mathrm{C}) / \mathbb{C} \times \mathbb{C}
$$

as a complex manifold because all holomorphic bundles on a non-compact Riemann surface are holomorphically trivial [16, Theorem 30.4]. Therefore it suffices to describe $\operatorname{Stab}(\mathrm{C}) / \mathbb{C}$.

The action of $\mathbb{C}$ preserves the set of semistable objects, so the wall-and-chamber structure of the stability space descends to a partition of $\operatorname{Stab}(\mathrm{C}) / \mathbb{C}$ into open chambers and real codimension one walls between them. The charge map descends to a holomorphic map $\operatorname{Stab}(\mathrm{C}) / \mathbb{C} \rightarrow \mathbb{P}(\operatorname{Hom}(\Lambda, \mathbb{C})) \cong \mathbb{C P}^{1}$ which by abuse of notation we also denote $\mathcal{Z}$. The equatorial copy of $\mathbb{R} \mathbb{P}^{1}$ consists of the charges with rank one image.

The action of the universal cover $G$ of $\mathrm{GL}_{2}^{+}(\mathbb{R})$ also preserves the sets of semistable objects, so each chamber and each wall is a union of orbits. The image of a free orbit in $\operatorname{Stab}(\mathrm{C}) / \mathbb{C}$ is a copy of $G / \mathbb{C} \cong \mathbb{H}$ biholomorphic to its image under the charge map. This image is either the Southern or Northern hemisphere in $\mathbb{C P}^{1}$. We refer to these free orbits as cells.

The image of a non-free orbit in $\operatorname{Stab}(\mathrm{C}) / \mathbb{C}$ is a point. The heart of a stability condition in such an orbit is length, and so by our assumption has two isoclasses of simple objects, say $s$ and $t$. There is a one-parameter family consisting of images of orbits through stability conditions for which $s$ and $t$ are semistable of the same phase and where the ratio $m(s) / m(t)$ varies in $\mathbb{R}_{>0}$. This describes a real analytic curve in $\operatorname{Stab}(\mathrm{C}) / \mathbb{C}$ which we refer to as a cell-wall. Each cell-wall is isomorphic to its image in $\mathbb{P H o m}(\Lambda, \mathrm{C})$ which is an arc in the equator from the point $Z(s)=0$ to $Z(t)=0$. The following lemma is immediate.
Lemma 12.1. Suppose $C$ admits an algebraic heart with two simple objects. Then there is a bijection between the set of cell-walls and the set of algebraic hearts (up to shift).

If there are non-split extensions between the simple objects $s$ and $t$ then a cell-wall is a genuine wall in the stability space along which these extensions are strictly semistable. All walls are of this form; in particular no two walls intersect. If there are no non-split extensions, for instance if the heart is semisimple, then the cell-wall lies in the same chamber as the two neighbouring cells.

Lemma 12.2. Each chamber of $\operatorname{Stab}(\mathrm{C}) / \mathbb{C}$ is a linear chain of cells, separated by cell-walls. If it is a doubly-infinite chain then it is biholomorphic to $\mathbb{C}$, otherwise it is biholomorphic to $\mathbb{H}$.

Proof. Let $C$ be a cell. If it has no cell-walls in its closure then it is a chamber and we are done. Let $W$ be a cell-wall in the closure, and let $s$ and $t$ be simple objects in a corresponding algebraic heart. Their phases agree on $W$ and we may assume $\varphi(t)-1<\varphi(s)<\varphi(t)$ in $C$.

Suppose $W^{\prime}$ is another cell-wall in the closure of $C$ which is not an actual wall. Let $s^{\prime}$ and $t^{\prime}$ be simple objects in a corresponding algebraic heart. Since $s$ and $t$ are semistable on $W^{\prime}$, some shifts $s[m]$ and $t[n]$ lie in this heart. Indeed, since we assume $\varphi(s)<\varphi(t)$ in $C$, we may apply a shift so that the heart contains $s[1]$ and $t$. Since these are indecomposable and there are no non-split extensions between $s^{\prime}$ and $t^{\prime}$ we deduce, after swapping $s^{\prime}$ and $t^{\prime}$ if necessary, that $s[1]$ and $t$ are respectively self-extensions of $s^{\prime}$ and $t^{\prime}$. However, the classes of $s[1]$ and $t$ are primitive in $\Lambda=K(\mathrm{C})$ so in fact $s^{\prime}=s[1]$ and $t^{\prime}=t$. It follows that the closure of $C$ in $\operatorname{Stab}(\mathrm{C}) / \mathbb{C}$ is precisely $W \cup C \cup W^{\prime}$.

Arguing inductively we conclude that the chamber is either a linear chain of cells as claimed or a cycle. The possibility that it is a cycle is easily excluded since the phase difference between $s$ and $t$ is well-defined and monotonic as we move along a chain of cells.

For the final part note that the chamber is an open subset of the universal cover $\mathbb{C}$ of $\mathbb{P H o m}(\Lambda, \mathrm{C})-\{Z(s)=0, Z(t)=0\}$. If the chain of cells is doubly-infinite then it is the entirety of $\mathbb{C}$. Otherwise it is a proper simply-connected open subset of $\mathbb{C}$ and is therefore biholomorphic to $\mathbb{H}$ by the Riemann Mapping Theorem.
12.2. The Speiser graph. The topology and geometry of $\operatorname{Stab}(\mathbb{C}) / \mathbb{C}$ is encoded combinatorially in what we call the Speiser graph. This is the dual graph $\operatorname{Sp}(\mathrm{C})$ to the cell and cell-wall decomposition, i.e. it has a vertex for each cell and an edge between two vertices when they have a common cell-wall in their closures. (More properly the Speiser graph is defined when $\operatorname{Stab}(\mathrm{C}) / \mathbb{C} \rightarrow \mathbb{P} \operatorname{Hom}(\Lambda, \mathrm{C})$ has a finite set $S$ of singular values. It is the preimage in $\operatorname{Stab}(\mathrm{C}) / \mathbb{C}$ of the dual graph in $\mathbb{P H o m}(\Lambda, \mathrm{C})-S$ to a Jordan curve passing through the singular values in $S$ in cyclic order, see for example [12, §2]. However, this definition is too restrictive for our setting where there may be infinitely many singular values.) Since the cells and cell-walls are contractible the Speiser graph encodes the homotopy type of $\operatorname{Stab}(\mathbb{C}) / \mathbb{C}$. Indeed we can embed the Speiser graph in $\operatorname{Stab}(\mathrm{C}) / \mathbb{C}$ so that each vertex is in the corresponding cell and each edge is a smooth curve crossing the corresponding cell-wall once transversely (and not crossing any other cell-wall). Then $\operatorname{Stab}(\mathrm{C}) / \mathbb{C}$ deformation retracts onto the embedded Speiser graph.

Conjecture 12.3. Each component of the Speiser graph $\mathrm{Sp}(\mathrm{C})$ is a tree and therefore each component of $\operatorname{Stab}(\mathrm{C}) / \mathbb{C}$ is contractible.

When the Speiser graph is a tree the Uniformisation Theorem, see for example [16, Theorem 27.9], tells us that $\operatorname{Stab}(\mathrm{C}) / \mathbb{C}$ is biholomorphic to either $\mathbb{C}$ (parabolic type) or to the Poincaré disk $\mathbb{D}$ (hyperbolic type). Either case may occur, for instance if $C$ is the bounded derived category of representations of the quiver with two vertices and no arrows then $\operatorname{Stab}(\mathrm{C}) / \mathbb{C} \cong \mathbb{C}$ whereas if $\mathrm{C}=\mathrm{D}^{b}(X)$ is the bounded derived category of coherent sheaves on a smooth complex projective curve $X$ of strictly positive genus then $\operatorname{Stab}(\mathrm{C}) / \mathbb{C} \cong \mathbb{D}[6,28]$. The type is parabolic if the Brownian motion on $\operatorname{Stab}(\mathrm{C}) / \mathbb{C}$ is recurrent and hyperbolic if it is transient [21, 22]. The idea behind the following conjecture is that the Brownian motion can be combinatorially modelled by the random walk on the Speiser graph with suitable transition probabilities on each edge (provided at least that we are not in the trivial case in which the Speiser graph has a single vertex and no edges). See [12] for a nice discussion.

Conjecture 12.4. Assume the Speiser graph contains at least one edge. Equip the charge space $\mathbb{P} \operatorname{Hom}(\Lambda, \mathbb{C}) \cong \mathbb{C P} \mathbb{P}^{1}$ with the constant curvature Riemannian metric in which the equator (consisting of charges with rank one image) has length one. Assign each edge in the Speiser graph a transition probability given by the length of the image in charge space of the corresponding cell-wall. Then $\operatorname{Stab}(\mathbb{C}) / \mathbb{C}$ is parabolic if the resulting random walk is recurrent and hyperbolic if it is transient.

The following criterion, proved by comparing with the random walk on the standard 2 dimensional lattice, is useful for establishing recurrence and shows that it does not depend delicately on the transition probabilities.

Theorem 12.5 ( $[13, \S 2.4])$. Suppose $\Gamma$ is an infinite connected graph, with a uniform upper bound on the valency of its vertices. Assign transition probabilities in $[\varepsilon, 1-\varepsilon]$, where $\varepsilon>0$, to each edge. Then the resulting random walk is recurrent if the vertices of $\Gamma$ can be embedded in $\mathbb{R}^{2}$ with a uniform lower bound on the distance between any two vertices, and a uniform upper bound on the length of each edge.
12.3. The space of quotient stability conditions. We remove the boundary strata of $\operatorname{Stab}^{L}(\mathrm{C})$ where all objects are massless, and denote the resulting space by $\operatorname{Stab}^{L}(\mathrm{C})^{*}$. Therefore $r k\left(\Lambda_{N}\right)=1$ when the massless subcategory $N \neq 0$. In this case $\operatorname{Stab}^{L}(C, N)$ is a union of components each homeomorphic to $\mathbb{C} \times \mathbb{R}$ by Theorem 9.4. The action of $\mathbb{C}$ on $\operatorname{Stab}^{L}(\mathrm{C})^{*}$ is free and the quotient is a union of the open subset $\operatorname{Stab}(\mathrm{C}) / \mathbb{C}$ and a copy of $\mathbb{R}$ for each boundary component in $\operatorname{Stab}^{L}(\mathrm{C})^{*}$. Passing to $\operatorname{Stab}^{Q}(\mathrm{C})^{*}$ replaces each of these copies of $\mathbb{R}$ by a point.

Recall, we abuse notation by using $\mathcal{Z}$ to denote the map $\operatorname{Stab}^{L}(\mathrm{C})^{*} / \mathbb{C} \rightarrow \mathbb{P}(\operatorname{Hom}(\Lambda, \mathbb{C})) \cong$ $\mathbb{C P}^{1}$ induced from the charge map on the space of lax stability conditions. This maps points of $\operatorname{Stab}^{Q}(\mathbb{C}, \mathrm{~N}) / \mathbb{C}$ where N is massless to the point $\operatorname{Hom}\left(\Lambda / \Lambda_{\mathrm{N}}, \mathbb{C}\right)$. By Corollary 8.6 such a point has a punctured neighbourhood in $\mathrm{Stab}^{Q}(\mathrm{C})^{*} / \mathbb{C}$ isomorphic as a complex manifold to the universal cover of a punctured disk centred at $\operatorname{Hom}\left(\Lambda / \Lambda_{N}, \mathbb{C}\right)$. The partial compactification $\operatorname{Stab}^{Q}(\mathrm{C})^{*} / \mathbb{C}$ is obtained locally by adding a point to the universal cover over the puncture. Such a point is called a logarithmic singularity of $\mathcal{Z}$. It follows that $\operatorname{Stab}^{Q}(\mathrm{C})^{*} / \mathbb{C}$ embeds in the Mazurkiewicz completion which is the union of $\operatorname{Stab}(\mathrm{C}) / \mathbb{C}$ with all logarithmic singularities of $\mathcal{Z}$, see for example [15]. We do not claim that this embedding is a homeomorphism, i.e. that every logarithmic singularity occurs as a boundary point in $\operatorname{Stab}^{Q}(\mathrm{C})^{*} / \mathbb{C}$.

The space $\operatorname{Stab}^{L}(\mathrm{C})^{*} / \mathbb{C}$ is recovered by performing a real blow-up at each of the boundary points, resolving them to copies of $\mathbb{R}$. Note that $\operatorname{Stab}^{L}(\mathrm{C})^{*} / \mathbb{C}$ is not a Riemann surface with boundary because each boundary stratum has a holomorphic function which vanishes along it, namely the charge of (any) object which becomes massless on that stratum.

The next result classifies the stable massless objects.
Proposition 12.6. An object is massless and stable at some point in $\operatorname{Stab}^{Q}(\mathrm{C})^{*}$ precisely if it is simple in some algebraic heart of C .

Proof. Suppose $s$ is simple in an algebraic heart H . Let $t$ be the other simple object. Define stability conditions $\left(P, Z_{n}\right)$ for $n \in \mathbb{N}$ with slicing $P(1)=P(0,1]=\mathrm{H}$ and charge $Z_{n}(s)=-1 / n$ and $Z_{n}(t)=-1$. The limit $\sigma=\lim _{n \rightarrow \infty}\left(P, Z_{n}\right)$ is a lax stability condition in $\operatorname{Stab}^{L}(\mathrm{C})^{*}$ for which $s$ is massless and stable. (By construction the limit is in the closure of $\operatorname{Stab}(\mathrm{C})$ with locally-finite slicing $P$, and the support property follows immediately because, up to shift, $t$ is the only massive stable object.)

Now suppose $s$ is a massless stable object at some point in $\operatorname{Stab}^{Q}(\mathrm{C})^{*}$. Since $\operatorname{rk}(\Lambda)=2$ each free $G$-orbit in $\operatorname{Stab}(\mathrm{C})$ is open, and $\operatorname{Stab}^{Q}(\mathrm{C})^{*}$ is the union of their closures. Hence $s$ is massless and stable at some point in the closure of a free orbit $\sigma \cdot G$ where $\sigma=(P, Z) \in \operatorname{Stab}(\mathrm{C})$. Then by Remark 11.4 and Proposition 11.6 the object $s$ is stable in some slice $P(\varphi)$ such that $P(\varphi-\varepsilon, \varphi)=0$ or $P(\varphi, \varphi+\varepsilon)=0$ for sufficiently small $\varepsilon>0$ (or both). In the first case $P[\varphi, \varphi+1)$ is a length heart in which $s$ is simple and in the second $P(\varphi-1, \varphi]$.
Corollary 12.7. The boundary points of $\operatorname{Stab}^{Q}(\mathrm{C})^{*} / \mathbb{C}$ are in bijection with iso-classes of massless stable objects up to shift.
Proof. We must show that there is a unique boundary point of $\operatorname{Stab}^{Q}(\mathrm{C})^{*} / \mathbb{C}$ at which a given stable object is massless. Suppose $\sigma \in \operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ where $\mathrm{N}=\operatorname{triang}_{\mathrm{C}}(s)$ is generated by the stable massless object $s$. By Proposition 12.6, $s$ is simple in some algebraic heart and our standing assumption $K(\mathrm{C}) \cong \mathbb{Z}^{2}$ implies $K(\mathrm{C} / \mathrm{N}) \cong \mathbb{Z}$ and $\operatorname{Stab}(\mathrm{C} / \mathrm{N}) \cong \mathbb{C}$. Thus the induced
stability condition $\sigma_{\mathrm{C} / \mathrm{N}}=\left(P_{\mathrm{C} / \mathrm{N}}, Z_{\mathrm{C} / \mathrm{N}}=Z\right)$ on the quotient is fixed up to the action of $\mathbb{C}$. By Proposition 3.2 the slicing $P$ is uniquely determined by $P_{\mathrm{C} / \mathrm{N}}$ and $P_{\mathrm{N}}$, i.e. by the slicing of the quotient and the phase of $s$. However, by Theorem 9.4 the component of $\sigma$ in $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$ is isomorphic to $\operatorname{Stab}(\mathrm{C} / \mathrm{N}) \times \mathbb{R} \cong \mathbb{C} \times \mathbb{R}$ so that all possible such $\sigma$ occur in the same component of $\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{N})$. Hence they all determine the same point of $\operatorname{Stab}^{Q}(\mathrm{C})^{*} / \mathbb{C}$ as claimed.

Recall that cell-walls in $\operatorname{Stab}(\mathrm{C}) / \mathbb{C}$ correspond to algebraic hearts (up to shift) and that the masses of the two simple objects vanish at the respective ends of the cell-wall. Therefore the cell-wall starts and ends at the boundary points where these simple objects become massless.
12.4. The exchange graph. The exchange graph $\mathrm{EG}(\mathrm{C})$ has one vertex for each algebraic heart of a stability condition in $\operatorname{Stab}(\mathrm{C})$ and an edge whenever two hearts are related by a simple HRS tilt; see $[19,33]$. Each vertex of $\mathrm{EG}(\mathrm{C})$ is 4 -valent because we can tilt left or right at each of the two simple objects of the corresponding heart. The shift acts freely on EG(C) and we denote the quotient, the projective exchange graph, by $\mathrm{EG}(\mathrm{C}) / \mathbb{Z}$. This quotient has one vertex for each algebraic heart up to shift, i.e. one vertex for each cell-wall. If there is an edge between two vertices then they share a common end point in the boundary of $\operatorname{Stab}^{Q}(\mathrm{C})^{*} / \mathbb{C}$. The massless stable object at this boundary point is (up to shift) the common simple object of the hearts corresponding to the two vertices.

The exchange graph can be embedded into $\operatorname{Stab}^{Q}(\mathrm{C})^{*} / \mathbb{C}$ by placing a vertex on each cellwall and embedding edges as smooth curves in the unique cell containing the two cell-walls corresponding to its vertices in its closure.
12.5. Dense phase case. Suppose that $\operatorname{Stab}(\mathrm{C})$ contains a stability condition $\sigma$ whose phase diagram $\Phi_{\sigma}$ is dense. Then the $G$-orbit of $\sigma$ is free and by Corollary 11.7 closed. Since the orbit is also open it is an entire connected component, consisting of a single chamber with no walls. This is the case in which the Speiser graph is trivial. Every stability condition in this component has dense phases and therefore no stability condition in this component has an algebraic heart. Our theory adds no boundary points to this component.

This situation occurs for the space $\operatorname{Stab}(X)$ of numerical stability conditions on a smooth complex projective curve $X$ of genus $g>0$, see $[6,28]$ respectively for the elliptic curve and higher genus cases. It also occurs for the space $\operatorname{Stab}(Q)$ of stability conditions on the bounded derived category of finite-dimensional representations of a 2 -vertex quiver $Q$ with an oriented cycle, see [11, Remark 3.33] for the existence of a dense-phase stability condition.
12.6. Non-dense phase case. Now suppose that there is at least one $\sigma$ in $\operatorname{Stab}(\mathrm{C})$ with nondense phases. By the above, every stability condition in the component of $\sigma$ has non-dense phases, and therefore lies in the $\mathbb{C}$ orbit of a stability condition with length heart. In this situation we assume $\Lambda=K(\mathrm{C}) \cong \mathbb{Z}^{2}$.

Using the description of the closure of the orbit $(\sigma \cdot G) / \mathbb{C}$ in Proposition 11.6 and Corollary 11.8 , we can describe the component of $\sigma$ in $\operatorname{Stab}^{Q}(\mathrm{C})^{*} / \mathbb{C}$. First we determine the chamber containing $\sigma$ and its walls. The latter correspond to non-trivial algebraic hearts whose simple objects are stable in the chamber. Then we find the stable objects in the adjacent chambers and repeat. In the examples we consider there are only finitely many chambers up to the action of $\operatorname{Aut}_{\Lambda}(\mathrm{C})$ so this is an effective strategy.

It is easy to find the cell decompositions of chambers (since each chamber is a linear chain of cells) and thence construct the Speiser graph. In our examples this is always a tree, so $\operatorname{Stab}(\mathrm{C})$ is contractible. The type of $\operatorname{Stab}(\mathrm{C}) / \mathbb{C}$, either parabolic or hyperbolic, is known from previous results and we confirm Conjecture 12.4 in these cases by showing that the random walk on the Speiser graph is respectively recurrent or transient. In the former case $\operatorname{Aut}_{\Lambda}(\mathrm{C})$ acts by rotations and translations of $\mathbb{C}$, and in the latter by rotations, translations and ideal rotations of $\mathbb{D}$.
12.7. The $A_{2}$ quiver. Let $Q$ be the $A_{2}$ quiver with two vertices and one arrow. In this section we consider several stability spaces associated with algebras associated with the $A_{2}$ quiver, namely the classical $A_{2}$ path algebra, and the the associated 2-Calabi-Yau Ginzburg algebra.
12.7.1. Classic $A_{2}$. First we consider the bounded derived category $\mathrm{D}^{b}\left(A_{2}\right)$ of finite-dimensional representations of the classic $A_{2}$. Its stability space $\operatorname{Stab}\left(A_{2}\right) \cong \mathbb{C}^{2}$ was first described by King [24], see [8] for further references.

The standard heart of $\mathrm{D}^{b}\left(A_{2}\right)$ has two exceptional simple objects $s$ and $t$ with one non-split extension $0 \rightarrow s \rightarrow e \rightarrow t \rightarrow 0$ between them. It is easy to construct a stability condition in which $s, e$ and $t$ are the only stable objects up to shift. Its phase diagram has three isolated phases. By Corollary 11.8 the cell containing this stability condition has three cell-walls, and three boundary points where respectively $s, e$ and $t$ become massless. The object $e$ destabilises as we cross the cell-wall along which $s$ and $t$ have the same phase, and we enter a chamber in which only $s$ and $t$ are stable. Since $e$ is the unique indecomposable extension between any shifts of $s$ and $t$, this chamber is a chain of cells indexed by $\mathbb{N}$. Similar considerations apply to the other two walls of the initial cell. Thus the stability space has four chambers, one in which $s, e$ and $t$ are all stable and three in which pairs of them are stable. The Speiser graph has one central vertex with three infinite linear graphs attached. The random walk on this is recurrent by Theorem 12.5 in agreement with the fact that $\operatorname{Stab}(\mathrm{C}) / \mathbb{C} \cong \mathbb{C}$ is parabolic. The category $\mathrm{D}^{b}\left(A_{2}\right)$ is fractional Calabi-Yau; the Serre functor $S$ satisfies $S^{3}=[1]$. This acts by rotation on $\operatorname{Stab}^{Q}(\mathrm{C})^{*} / \mathbb{C}$ preserving the central chamber and cyclically permuting the other three chambers, and also the three boundary points. See Figure 2 for an illustration.
12.7.2. 2-Calabi-Yau $A_{2}$. Now consider the 2-Calabi-Yau category $\mathrm{D}^{b}\left(\Gamma_{2} A_{2}\right)$, where $\Gamma_{2} A_{2}$ is the Ginzburg dg algebra of the $A_{2}$ quiver; see [23, §7.2], for example, for details of the construction.

The stability space $\operatorname{Stab}\left(\Gamma_{2} A_{2}\right) / \mathbb{C} \cong \mathbb{D}$ is the universal cover of the thrice punctured Riemann sphere and was first described in [32]. See [8] for a detailed discussion and further references, and also $[25,30]$ for more general discussions of the stability spaces of the Ginzburg dg-algebras associated to Dynkin quivers.

The standard heart of $\mathrm{D}^{b}\left(\Gamma_{2} A_{2}\right)$ has two 2-spherical simple objects $s$ and $t$, with one non-split extension $0 \rightarrow s \rightarrow e \rightarrow t \rightarrow 0$ and $0 \rightarrow t \rightarrow f \rightarrow s \rightarrow 0$ in each direction. Each 2-spherical object in $\mathrm{D}^{b}\left(\Gamma_{2} A_{2}\right)$ generates a twist automorphism. For example, applying the twist $\mathrm{TW}_{s}$ about $s$ to the triangle $s \rightarrow e \rightarrow t \rightarrow s[1]$ yields the triangle $s[-1] \rightarrow t \rightarrow f \rightarrow s$, and then applying $\mathrm{TW}_{t}$ yields the rotation $e[-1] \rightarrow t[-1] \rightarrow s \rightarrow e$ of the original triangle. In particular $e$ and $f$ are also 2 -spherical. The subgroup of $\operatorname{Aut}\left(\mathrm{D}^{b}\left(\Gamma_{2} A_{2}\right)\right)$ generated by $\mathrm{TW}_{s}$ and $\mathrm{TW}_{t}$ is isomorphic to the Artin-Tits braid group $B_{3}$ of the $A_{2}$ quiver, i.e. the braid group on three strands. The centre $Z\left(B_{3}\right)$ is generated by a single automorphism which acts as the Serre functor $S=[2]$. Let $\mathcal{S}$ be the set of equivalence classes of spherical objects in $\mathrm{D}^{b}\left(\Gamma_{2} A_{2}\right)$ up to isomorphism and shift. We abuse notation by using the same notation for spherical objects and their classes in $\mathcal{S}$; this is harmless since the twists $\mathrm{TW}_{s}=\mathrm{TW}_{s[1]}$ agree. The quotient $B_{3} / Z\left(B_{3}\right) \cong \mathrm{PSL}_{2}(\mathbb{Z})$ acts on $\mathcal{S}$ and the stabiliser of $s$ is the infinite cyclic subgroup generated by $\mathrm{Tw}_{s}$. From the above examples $t$ is in the orbit of $s$ (indeed the action on $\mathcal{S}$ is transitive although we do not need this).

Now consider the stability space $\operatorname{Stab}\left(\Gamma_{2} A_{2}\right)$. As in the $\mathrm{D}^{b}\left(A_{2}\right)$ case one can easily construct a stability condition in which, up to shift, the stable objects are the two simple objects $s$ and $t$ of the standard heart and one, $e$ say, of the two extensions between them. The phase diagram has three isolated phases so the corresponding cell in $\operatorname{Stab}\left(\Gamma_{2} A_{2}\right) / \mathbb{C}$ has three cell-walls. As before $e$ destabilises as we cross the wall where $s$ and $t$ have the same phase, but now the other extension $f$ becomes stable on the far side of the wall. Thus we enter the chamber obtained by applying $\mathrm{TW}_{s}$ to the initial one. Similar considerations apply to the other walls of the initial chamber. Therefore $\mathrm{PSL}_{2}(\mathbb{Z})$ acts transitively on the chambers in $\operatorname{Stab}\left(\Gamma_{2} A_{2}\right) / \mathbb{C}$, each of which is a single cell bounded by three walls. There are three stable 2-spherical objects in each chamber whose respective masses vanish at the three boundary points of the chamber. The action of $\mathrm{PSL}_{2}(\mathbb{Z})$ on chambers is free because no pair of distinct spherical objects, a fortiori no triple, is fixed.

The action on walls is also free and it quickly follows from the examples of twist actions that it is transitive.

In conformity with Corollary 10.4 and Corollary 12.7, the only massless stable objects are the simple objects of the hearts of stability conditions (all of which are algebraic; see e.g. [30]) and there is one boundary point in $\operatorname{Stab}^{Q}\left(\Gamma_{2} A_{2}\right)^{*} / \mathbb{C}$ for each (up to shift and isomorphism). The Speiser graph is the Cayley graph of $\mathrm{PSL}_{2}(\mathbb{Z})$ with respect to the generating set consisting of the images of $\mathrm{TW}_{s}, \mathrm{TW}_{e}$ and $\mathrm{TW}_{t}$. It is an infinite trivalent tree as expected from Conjecture 12.3 and the random walk on it is transient as expected from Conjecture 12.4. The twist rw $s$ acts by a hyperbolic isometry fixing the boundary point at which $s$ is massless. Therefore, $\mathrm{Tw}_{s}$ acts either by an ideal rotation about that point or a translation, since, locally at the fixed point the action universally covers the action on $\mathbb{P H o m}(\Lambda, \mathbb{C})$. This is given by the matrix

$$
\left(\begin{array}{cc}
-1 & 0 \\
1 & 1
\end{array}\right)
$$

with respect to the basis $\{[s],[t]\}$ of $\Lambda=K\left(\mathrm{D}^{b}\left(\Gamma_{2} A_{2}\right)\right)$. Since the eigenvalues are $\pm 1$ the twist acts by an ideal rotation. Up to isometry, the three boundary points where $s, e$ and $t$ are massless can be chosen arbitrarily on $\partial \mathbb{D}$ and this fixes the remaining boundary points of $\operatorname{Stab}^{Q}\left(\Gamma_{2} A_{2}\right)^{*} / \mathbb{C}$ uniquely. They form a dense subset of $\partial \mathbb{D}$. See Figure 2 for an illustration.
12.8. A discrete derived category. Let $Q=\Lambda_{2,1,0}$ be the bound quiver with two vertices, one arrow in each direction, and the zero relation given by the composite of these arrows. The (principal component of the) stability space $\operatorname{Stab}\left(\Lambda_{2,1,0}\right) \cong \mathbb{C}^{2}$ was first described in [33], see also $[9,30]$ for proofs that the stability space is connected and generalisations to other discrete derived categories.

Let $s$ be the simple representation at the vertex with no relation, and $t=t_{0}$ the other simple representation. The object $s$ is 2 -spherical and $t_{0}$ is exceptional. Since $\operatorname{TW}_{s}(s)=s[-1]$ the twist $\mathrm{TW}_{s}$ generates an infinite cyclic subgroup of automorphisms. Set $t_{n}=\mathrm{TW}_{s}^{n}\left(t_{0}\right)$. There are unique non-split extensions $0 \rightarrow s \rightarrow t_{-1} \rightarrow t_{0} \rightarrow 0$ and $0 \rightarrow t_{0} \rightarrow t_{1} \rightarrow s \rightarrow 0$. In particular there is a stability condition in which $s, t_{-1}$ and $t_{0}$ are the only stable objects up to shift. This lives in a chamber with three walls, and three boundary points at which these objects are respectively massless. Crossing the wall where $t_{-1}$ destabilises we enter a chamber in which $t_{1}$ is stable. As in the previous example this chamber is the image of the initial one under the action of the twist $\mathrm{TW}_{s}$, and similarly crossing the wall where $t_{0}$ destabilises we enter the image of the initial chamber under $\mathrm{TW}_{s}^{-1}$. However, if we cross the wall where the spherical object $s$ destabilises then we enter a chamber in which only $t_{0}$ and $t_{1}$ are stable. Unlike the previous chambers which consist of a single cell, this is the union of a sequence of cells, and cell-walls upon which the phases of $t_{0}$ and $t_{1}[n]$ agree for $n \in \mathbb{N}$. In summary, there is one free orbit of chambers with three stable objects (one spherical and two exceptional) and one free orbit of chambers with two stable objects (both exceptional) under the action generated by $\mathrm{Tw}_{s}$. As expected, there is one boundary point in $\operatorname{Stab}^{Q}\left(\Lambda_{2,1,0}\right)^{*} / \mathbb{C}$ at which each of $s$ and $\left\{t_{n}: n \in \mathbb{Z}\right\}$ is massless. Clearly $\mathrm{Tw}_{s}$ fixes the former and acts freely and transitively on the latter. Its square $\mathrm{TW}_{s}^{2}$ acts trivially on the Grothendieck group so the images of the boundary points labelled by the $t_{n}$ map to one of two points in charge space according to whether $n$ is even or odd.

The Speiser graph has vertices $\mathbb{Z} \times \mathbb{N}$ with edges $(m, 0)$ to $(m+1,0)$ and $(m, n)$ to $(m, n+1)$ for $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. This is a tree as predicted by Conjecture 12.3. Moreover, the random walk on the Speiser graph is recurrent by Theorem 12.5, as predicted by Conjecture 12.4. The twist $\mathrm{TW}_{s}$ acts by an isometry without fixed points on $\operatorname{Stab}\left(\Lambda_{2,1,0}\right) / \mathbb{C} \cong \mathbb{C}$ and therefore acts by a translation of $\mathbb{C}$. This is illustrated in Figure 2.
12.9. The projective line. The stability space $\operatorname{Stab}\left(\mathbb{P}^{1}\right) / \mathbb{C} \cong \mathbb{C}$ was first described in [29], see also [28]. The 'classical' stability condition has heart the coherent sheaves, with stable objects the skyscrapers $\mathcal{O}_{x}$ for $x \in \mathbb{P}^{1}$ and the line bundles $\mathcal{O}(k)$ for $k \in \mathbb{Z}$. The cell containing it has a sequence of cell-walls indexed by $\mathbb{Z}$ separated by boundary points at which $\mathcal{O}(k)$ is massless for $k \in \mathbb{Z}$. An example for a lax stability condition at the boundary is given by the slicing


Figure 2. Illustrations of $\operatorname{Stab}^{Q}(\mathrm{C})^{*} / \mathbb{C}$. In each case this is a contractible non-compact Riemann surface. We depict it as a disk, which is holomorphically accurate in the hyperbolic case (Ginzburg algebra $\Gamma_{2} A_{2}$, shown orange). In the parabolic cases (purple) it is only topologically accurate, but has the advantage that we can more easily visualise the partial compactification. This is obtained by adding logarithmic singularities on the boundary circle at which certain objects become massless. These points are labelled by the corresponding stable massless object. Chambers are indicated by the shading. The set of stable objects in a chamber (up to shift) is the set of massless stable objects at its boundary vertices. Walls are shown as arcs of circles spanning a pair of points on the boundary. The objects labelling these two boundary points have equal phase on the wall; they are the two simple objects of the corresponding algebraic heart. The subset of stable objects which become strictly semistable on a wall is therefore the set of massless stable objects labelling the other vertices of the chamber. The Speiser graph $\operatorname{Sp}(C)$ is shown in red and the quotient $\operatorname{EG}(\mathrm{C}) / \mathbb{Z}$ of the exchange graph by the shift in blue. To avoid clutter cell-walls which are not walls are omitted; these can be inferred from the Speiser and exchange graphs.
$P_{b}$ of Example 3.14 with charge map $Z(\mathcal{O})=0$ and $Z(\mathcal{O}(1))=i$. These boundary points accumulate on the boundary at a point where the charge of the skyscrapers vanishes. However, Proposition 12.6 shows this point is not in $\operatorname{Stab}^{Q}\left(\mathbb{P}^{1}\right)^{*} / \mathbb{C}$ because the skyscrapers are not simple in any algebraic heart and so cannot become massless (see also Example 4.17). We indicate this omitted point by a white dot in Figure 2 and label it by $\mathcal{O}_{x}$ to indicate that the skyscrapers are stable in the adjacent chamber.

Crossing the wall spanning the points where $\mathcal{O}(k)$ and $\mathcal{O}(k+1)$ are massless the skyscrapers and all other line bundles destabilise and we enter a chamber in which the only stable objects are $\mathcal{O}(k)$ and $\mathcal{O}(k+1)$. This chamber is the union of a sequence of cells separated by walls on which the phases of $\mathcal{O}(k+1)$ and $\mathcal{O}(k)[n]$ for $n>0$ agree.

The Speiser graph is the union of $\mathbb{Z}$ copies of the graph with vertices $\mathbb{N}$ and edges from $n$ to $n+1$ joined at the 0 vertices. This is a tree as expected from Conjecture 12.3. The central vertex has infinite valence so Theorem 12.5 does not apply. Nevertheless the random walk is recurrent as expected from Conjecture 12.4.

The infinite cyclic group generated by the automorphism $-\otimes \mathcal{O}(1)$ preserves the chamber containing the 'classical' stability condition (but does not fix any stability condition in this chamber) and acts freely and transitively on the chambers in which only two objects are stable. It also acts freely and transitively on the boundary points. It follows that it acts by a translation on $\operatorname{Stab}\left(\mathbb{P}^{1}\right) / \mathbb{C} \cong \mathbb{C}$. See Figure 2 for an illustration.
Superficially, this closely resembles the previous example. However, there are several important (and inter-related) differences. In this case there is a chamber bounded by a countably infinite family of walls; there are stable objects whose mass does not vanish; the images of the boundary points accumulate in charge space.

## 13. Comparisons with other constructions

We compare our partial compactification of the stability space with two alternative approaches, namely Bolognese's 'local compactification' [5] and Bapat, Deopurkar and Licata's 'Thurston compactification' [1].
13.1. Bolognese's 'local compactification'. In [5] Bolognese constructs an alternative 'local compactification' of $\operatorname{Stab}(\mathrm{C})$ using a metric completion of $\operatorname{Stab}(\mathrm{C})$. In order to do so she assumes that $\mathcal{Z}: \operatorname{Stab}(\mathrm{C}) \rightarrow \operatorname{Hom}(\Lambda, \mathbb{C})$ is a cover of the complement of a locally finite union $\Delta \subset \operatorname{Hom}(\Lambda, \mathbb{C})$ of submanifolds. She fixes an inner product on the underlying real space of $\operatorname{Hom}(\Lambda, \mathbb{C})$ and gives $\operatorname{Stab}(\mathrm{C})$ the geodesic metric $d_{B}$ induced from the pullback of the associated metric. Since $\operatorname{Stab}(C)$ is locally homeomorphic to $\operatorname{Hom}(\Lambda, \mathbb{C})$ with its norm topology this metric induces the usual topology on $\operatorname{Stab}(C)$. Her local compactification $\widehat{\operatorname{Stab}}(\mathrm{C})$ is the subspace of the metric completion consisting of equivalence classes of Cauchy sequences satisfying the limiting support property below. As a topological space $\widehat{\operatorname{Stab}}(\mathrm{C})$ is independent of the choice of inner product [5, Lemma 3.6].

Definition 13.1 (Limiting support property [5, Definition 4.3]). A Cauchy sequence ( $\sigma_{n}$ ) in the metric $d_{B}$ on $\operatorname{Stab}(\mathrm{C})$ has the limiting support property if $\liminf _{n \rightarrow \infty} C_{n}=C>0$ where for each $n \in \mathbb{N}$ the constant $C_{n}$ is the infimum of those $K>0$ such that $\left|Z_{n}(c)\right|>K\|v(c)\|$ for every $c \in P_{n}(\varphi)$ with $\lim _{n \rightarrow \infty} Z_{n}(c) \neq 0$. (The set of such constants $K$ is non-empty because each $\sigma_{n}$ is in $\operatorname{Stab}(\mathrm{C})$ and so satisfies the support property.) This property is well-defined on equivalence classes of Cauchy sequences by [5, Lemma 4.4].
In fact Bolognese shows that one can construct $\widehat{\text { Stab }}(\mathrm{C})$ using only $\mathcal{Z}$-local Cauchy sequences, that is Cauchy sequences $\left(\sigma_{n}\right)$ for $d_{B}$ which eventually lie in an open subset $U \subset \operatorname{Stab}(\mathrm{C})$ homeomorphic to its image via $\mathcal{Z}$. More precisely, she shows that any Cauchy sequence is equivalent to a $\mathcal{Z}$-local one, and that if two $\mathcal{Z}$-local Cauchy sequences are equivalent then they are $\mathcal{Z}$-local with respect to the same open $U$ [5, Theorem 3.7]. Moreover, each $\mathcal{Z}$-local Cauchy sequence determines a thick subcategory of objects which become massless in the limit, and a well-defined stability condition on the quotient category [5, Propositions 4.2 and Theorem 6.1]. Finally, $\mathcal{Z}$-local Cauchy sequences are equivalent precisely when they determine the same massless subcategory and stability condition on the quotient [5, Theorem 6.2].

Lemma 13.2. Suppose $\left(\sigma_{n}\right)$ is a $\mathcal{Z}$-local Cauchy sequence for $d_{B}$ on $\operatorname{Stab}(\mathrm{C})$. Then $\left(\sigma_{n}\right)$ converges to a lax pre-stability condition $\sigma$ in the product metric on $\operatorname{Slice}(\mathrm{C}) \times \operatorname{Hom}(\Lambda, \mathbb{C})$.

Proof. By construction the sequence $\left(Z_{n}\right)$ of charges converges in norm in $\operatorname{Hom}(\Lambda, \mathbb{C})$, say $Z_{n} \rightarrow Z$ as $n \rightarrow \infty$. The full subcategories $P(\varphi):=\left\{c \in \mathrm{C}: \varphi_{\sigma_{n}}^{ \pm}(c) \rightarrow \varphi\right\}$ define a slicing [5, Proposition 5.3]. Bolognese notes in the proof of [5, Proposition 5.2] that the sequence ( $P_{n}$ ) of slicings is Cauchy in the slicing metric on Slice(C). It follows that $P_{n} \rightarrow P$ in Slice(C). For, if this were not the case, then, after passing to a subsequence, we may assume there is some $\varepsilon>0$ with $d\left(P, P_{n}\right) \geq \varepsilon$ for all $n \in \mathbb{N}$. In other words there is a sequence $\left(\varphi_{n}\right)$ of phases and objects $c_{n} \in P\left(\varphi_{n}\right)$ such that $c_{n} \notin P_{n}\left(\varphi_{n}-\varepsilon, \varphi_{n}+\varepsilon\right)$. However, we know that for each $n \in \mathbb{N}$ there is some $N$ with $c_{n} \in P_{m}\left(\varphi_{n}-\varepsilon / 2, \varphi_{n}+\varepsilon / 2\right)$ whenever $m \geq N$. Choosing $N$ sufficiently large that we also have $d\left(P_{n}, P_{m}\right)<\varepsilon / 2$ for all $m, n \geq N$ leads to a contradiction.

Therefore $\left(\sigma_{n}\right)$ converges to $(P, Z)$ in the product metric on $\operatorname{Slice}(\mathbb{C}) \times \operatorname{Hom}(\Lambda, \mathbb{C})$. It is easy to confirm that $\sigma=(P, Z)$ is a lax pre-stability condition, i.e. that $Z(c) \in \mathbb{R}_{\geq 0} e^{\pi i \varphi}$ whenever $c \in P(\varphi)$.

Let $\operatorname{Stab}^{B}(\mathrm{C})$ be the subset of the closure of $\operatorname{Stab}(\mathrm{C})$ in $\operatorname{Slice}(\mathrm{C}) \times \operatorname{Hom}(\Lambda, \mathbb{C})$ consisting of the limits of $\mathcal{Z}$-local Cauchy sequences for $d_{B}$ which satisfy the limiting support property. In order to compare Bolognese's local compactification with our constructions, we need to be able to compare this with $\operatorname{Stab}^{L}(\mathrm{C})$. Unfortunately the relationship is not obvious since our support property is phrased in terms of massive stable objects in some $P(\varphi)$ and Bolognese's limiting support property is phrased in terms of semistable objects in some $P_{n}(\varphi)$ which remain massive in the limit. Since the HNfiltration of a $\sigma_{n}$-semistable object with respect to $\sigma$ may contain massless objects, and similarly the other way round, there is no direct argument relating the two notions of support.

If $\operatorname{Stab}^{B}(\mathrm{C})=\operatorname{Stab}^{L}(\mathrm{C})$ then there should be a homeomorphism $\widehat{\operatorname{Stab}}(\mathrm{C}) \cong \operatorname{Stab}^{Q}(\mathrm{C})$. For instance, this is so for the example of the $A_{1}$ quiver computed in Bolognese's paper. More generally an inclusion in either direction should extend to a map between $\widehat{\operatorname{Stab}}(\mathrm{C})$ and $\operatorname{Stab}^{Q}(\mathrm{C})$ in the corresponding direction.
13.2. Bapat, Deopurkar and Licata's 'Thurston compactification'. In [1], the authors take a different approach to compactifying the stability space, more precisely the quotient $\operatorname{Stab}(C) / \mathbb{C}$, of a k-linear triangulated category C. By analogy with Thurston's compactification of Teichmüller space, they consider the map

$$
m: \operatorname{Stab}(\mathrm{C}) / \mathbb{C} \rightarrow \mathbb{P}\left(\mathbb{R}^{\mathrm{C}}\right): \sigma \cdot \mathbb{C} \mapsto\left[m_{\sigma}(c): 0 \neq c \in \mathrm{C}\right] .
$$

When $\mathbb{P}\left(\mathbb{R}^{C}\right)$ has the topology induced from the product topology on $\mathbb{R}^{C}$ this map is continuous, for instance by Proposition 5.12. Automorphisms of $C$ act on $\mathbb{P}\left(\mathbb{R}^{C}\right)$ by pre-composing a realvalued function on the objects of C with the inverse automorphism. The map $m$ is equivariant for $\operatorname{Aut}_{\Lambda}(\mathrm{C})$ because $m_{\alpha \cdot \sigma(c)}=m_{\sigma}\left(\alpha^{-1} c\right)$.

Under appropriate conditions on C, Bapat, Deopurkar and Licata conjecture that $m$ is a homeomorphism onto its image $M(\mathrm{C})$, and that the closure $\overline{M(\mathrm{C})}$ is a manifold with boundary and interior $M(\mathrm{C})$. Moreover, motivated by the description of boundary points of Thurston's compactification as functionals given by unsigned intersections with closed curves, they conjecture that there is a suitable class $\mathcal{S}$ of objects such that the functionals

$$
\overline{\operatorname{hom}}(s):=\left[\sum_{n \in \mathbb{Z}} \operatorname{dim}_{\mathbf{k}} \operatorname{Hom}_{\mathrm{C}}(s, c[n]) \mid 0 \neq c \in \mathrm{C}\right] \quad(s \in \mathcal{S})
$$

form a dense subset of the boundary $\partial M(\mathrm{C})$. These conjectures hold for the 2-Calabi-Yau category associated to the $A_{2}$ quiver [1, $\left.\S 5\right]$. More generally, when $C$ is the 2-Calabi-Yau category associated to any connected quiver, they show $m$ is injective, that the closure of the image is compact, and that the functionals of all 2 -spherical objects lie in the boundary.

Let $\operatorname{Stab}^{L}(\mathrm{C})^{*}=\operatorname{Stab}^{L}(\mathrm{C})-\operatorname{Stab}^{L}(\mathrm{C}, \mathrm{C})$ be the space of lax stability conditions with the stratum where all objects are massless deleted, and define $\operatorname{Stab}^{Q}(\mathrm{C})^{*}$ similarly. We can extend $m$ to a map from $\operatorname{Stab}^{L}(\mathrm{C})^{*} / \mathbb{C}$ defined in the same way; the extension is also continuous by

Proposition 5.12. This extension factorises continuously through the quotient

because the masses depend only on the associated quotient stability condition. (We abuse notation by denoting the extension and the factorisation by $m$.) By construction the images of both $\operatorname{Stab}^{L}(\mathrm{C})^{*} / \mathbb{C}$ and $\operatorname{Stab}^{Q}(\mathrm{C})^{*} / \mathbb{C}$ are contained within $\overline{M(\mathrm{C})}$. At least when C is the 2-Calabi-Yau category associated to a connected quiver, Bapat, Deopurkar and Licata expect the boundary points in the image of $\operatorname{Stab}^{Q}(\mathrm{C})^{*} / \mathbb{C}$ to be dense in the boundary [1, Remark 4.9]. We show that this is true for the $A_{2}$ quiver. Before discussing that example we consider the non Calabi-Yau case. We slightly modify the construction by considering the map

$$
\operatorname{Stab}^{Q}(\mathrm{C})^{*} / \mathbb{C} \rightarrow \mathbb{P}\left(\mathbb{R}^{\mathcal{S}}\right): \sigma \mapsto\left[m_{\sigma}(s): s \in \mathcal{S}\right]
$$

where $\mathcal{S}$ is a suitable class of objects with the property that the masses of objects in $\mathcal{S}$ uniquely determine the masses of all objects.

Example 13.3. Recall the description of $\operatorname{Stab}\left(A_{2}\right) / \mathbb{C}$ in $\S 12.7$. In each stability condition in $\operatorname{Stab}\left(A_{2}\right)$ the semistable objects are, up to shift, a subset of at least two of $\{s, e, t\}$ where $s$ and $t$ are the two simple representations and $e$ the extension between them. It follows that the masses of objects in $\mathcal{S}=\{s, e, t\}$ determine the masses of all objects. This remains true for lax stability conditions in $\operatorname{Stab}^{L}\left(A_{2}\right)$. Therefore it suffices to consider the map

$$
\operatorname{Stab}\left(A_{2}\right) / \mathbb{C} \rightarrow \mathbb{P}\left(\mathbb{R}^{3}\right): \sigma \mapsto\left[m_{\sigma}(s): m_{\sigma}(e): m_{\sigma}(t)\right]
$$

The image is cut out by the inequalities $x_{0}, x_{1}, x_{2}>0$ (the masses are strictly positive) together with the cyclic permutations of the inequality $x_{0}-x_{1}+x_{2} \leq 0$ (the mass of an extension is bounded by the sum of the masses of its factors). If we normalise so that $x_{0}+x_{1}+x_{2}=1$ then the image can be viewed as the shaded triangle, with vertices omitted, in the 2 -simplex in Figure 3. In particular we see that the map is not injective because when, for example, $e$ is unstable the masses of $s$ and $t$ do not suffice to determine their phases. The three boundary points in $\operatorname{Stab}^{Q}\left(A_{2}\right)^{*} / \mathbb{C}$ where the masses of $s, e$ and $t$ respectively vanish are mapped to the three omitted vertices. So in this example, $\operatorname{Stab}^{Q}\left(A_{2}\right)^{*} / \mathbb{C}$ surjects onto Bapat, Deopurkar and Licata's compactification, and the boundaries coincide. The boundary points correspond precisely to the functionals $\overline{\text { hom }}(c)$ for $c \in \mathcal{S}$.

Example 13.4. Let $\mathcal{S}$ be the set of equivalence classes of spherical objects in $\mathrm{D}^{b}\left(\Gamma_{2} A_{2}\right)$ up to isomorphism and shift. Then by [1, Proposition 5.7] the map

$$
m: \operatorname{Stab}\left(\Gamma_{2} A_{2}\right) / \mathbb{C} \rightarrow \mathbb{P}\left(\mathbb{R}^{\mathcal{S}}\right): \sigma \mapsto\left[m_{\sigma}(s): s \in \mathcal{S}\right]
$$

is a homeomorphism onto its image which we denote $M\left(\Gamma_{2} A_{2}\right)$. Moreover, $M\left(\Gamma_{2} A_{2}\right)$ is homeomorphic to an open disk by [1, Proposition 5.20]. After choosing an element $s \in \mathcal{S}$ to map to $[1: 0]$, the action of the Artin-Tits braids group induces a bijection $\mathcal{S} \cong \mathbb{P}\left(\mathbb{Z}^{2}\right)$. Using this identification the map

$$
\mathcal{S} \rightarrow \mathbb{P}\left(\mathbb{R}^{\mathcal{S}}\right): s \mapsto\left[\operatorname{hom}(s)\left(s^{\prime}\right): s^{\prime} \in \mathcal{S}\right]
$$

extends uniquely to a homeomorphism from $\mathbb{P}\left(\mathbb{R}^{2}\right)$ onto the boundary of $M\left(\Gamma_{2} A_{2}\right)$ in $\mathbb{P}\left(\mathbb{R}^{S}\right)$ by [1, Propositions 5.13 and 5.18]. The closure $\overline{M\left(\Gamma_{2} A_{2}\right)}$ is homeomorphic to a closed disk. The functional $\overline{\mathrm{hom}}(s)$ is the unique fixed point of the spherical twist $\mathrm{TW}_{s}$.

Now consider the extension $m$ : $\operatorname{Stab}^{Q}\left(\Gamma_{2} A_{2}\right)^{*} / \mathbb{C} \rightarrow \overline{M\left(\Gamma_{2} A_{2}\right)}$. Recall from $\S 12.7$ and Figure 2 that the partial compactification $\operatorname{Stab}^{Q}\left(\Gamma_{2} A_{2}\right)^{*} / \mathbb{C}$ contains one boundary point for each $s \in \mathcal{S}$ at which the objects in the class $s$ become massless. This boundary point is fixed by $\mathrm{TW}_{s}$ because $\mathrm{TW}_{s}$ acts by a shift on $s$. The equivariance of $m$ implies that this point is mapped to


Figure 3. The image of $\operatorname{Stab}\left(A_{2}\right) / \mathbb{C} \rightarrow \Delta^{2}: \sigma \mapsto\left(\lambda m_{\sigma}(s), \lambda m_{\sigma}(e), \lambda m_{\sigma}(t)\right)$ where $\lambda=m_{\sigma}(s)+m_{\sigma}(e)+m_{\sigma}(t)$ is shaded red. The chamber of the stability space in which $s, e$ and $t$ are stable is mapped homeomorphically to the interior. The chamber in which only $s$ and $t$ are stable, together with its bounding wall, are projected down onto the edge $x_{0}-x_{1}+x_{2}=0$, and similarly for the other two chambers. The three boundary points in $\operatorname{Stab}^{Q}\left(A_{2}\right)^{*} / \mathbb{C}$ where the masses of $s, e$ and $t$ respectively vanish are mapped to the three black vertices.
$\overline{\operatorname{hom}}(s)$. (At first sight this looks odd because $\sum_{n \in \mathbb{Z}} \operatorname{Hom}(s, s[n])=2$ is non-zero. However we are working in an infinite-dimensional projective space and [1, Proposition 4.5] states that

$$
\lim _{n \rightarrow \infty} \frac{m_{\sigma}\left(\mathrm{TW}_{s}^{n}\left(s^{\prime}\right)\right)}{n}=m_{\sigma}(s) \overline{\operatorname{hom}}(s)\left(s^{\prime}\right)
$$

for any $s^{\prime} \in \mathcal{S}$ and stability condition $\sigma$ in which $s$ is stable. Since $\mathrm{TW}_{s}$ fixes $s$ up to a shift, this implies that $\overline{\mathrm{hom}}(s)(s)=0$ as expected.) We conclude that

$$
m: \operatorname{Stab}^{Q}\left(\Gamma_{2} A_{2}\right)^{*} / \mathbb{C} \rightarrow \overline{M\left(\Gamma_{2} A_{2}\right)}
$$

is a continuous embedding, restricting to a homeomorphism between the interiors, and whose image is dense in the boundary. This accords with the expectations of [1, Remark 4.9], and provides a modular interpretation of the boundary points $\overline{\mathrm{hom}}(s)$ as quotient stability conditions.

## 14. Open questions

We end with four open questions. (1) and (2) relate to the properties of composition series in quasi-abelian categories. This technical issue plays a role because the slices $P(\varphi)$ of a lax stability condition are quasi-abelian but, in contrast to the situation for classical stability conditions, need not be abelian. (3) and (4) relate to support properties for lax stability conditions. There are various different notions in the literature, and it would be good to understand which are equivalent, and to what extent each has the crucial propagation property enjoyed by support for classical stability conditions.
(1) Are there examples where the Jordan-Hölder property fails for the slices $P(\varphi)$ of a lax stability condition? If there are then, whilst the HNfiltrations of objects are unique, their refinements to filtrations with stable factors would not be. Presumably this would have implications for the wall-and-chamber structure.
(2) Let $P$ be a locally finite slicing on C and $\mathrm{N} \subset \mathrm{C}$ a thick subcategory such that $P$ descends to $\mathrm{C} / \mathrm{N}$. Is the slicing $P_{\mathrm{C} / \mathrm{N}}$ on the quotient also locally finite? What if $P$ is the slicing of a lax pre-stability condition with massless subcategory N? (If $P$ is the slicing of a lax stability condition then the support property guarantees that $P_{\mathrm{C} / \mathrm{N}}$ is locally finite, see Proposition 4.13.)
(3) Are there examples where support propagation fails? See Definition 6.12 and the discussion on page 33. This is crucial for the range of applicability of our results.
(4) What is the relationship between the support property for a lax pre-stability condition $\sigma$ we use and Bolognese's notion of limiting support for a sequence of stability conditions converging to $\sigma$ ? Understanding this is key to clarifying the relationship between our partial compactification and Bolognese's, see §13.1.

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Contact: nathan.broomhead@plymouth.ac.uk, d.pauksztello@lancaster.ac.uk, david.ploog@uis.no, jonathan.woolf@liverpool.ac.uk


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