Four loop anomalous dimension of non-singlet quark currents in the RI′ scheme

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Abstract. We construct the five loop anomalous dimensions of the basic fields in Quantum Chromodynamics in a linear covariant gauge in the modified Regularization Invariant (RI′) scheme. Using these core results we also compute the four loop Green's function where the quark mass operator and vector current are separately inserted in a quark 2-point function. These are necessary for measurements of the same quantities on the lattice. The Green's functions are provided in both the modified Minimal Subtraction ($\overline{\text{MS}}$) and RI′ schemes. All the results are available for a general Lie group.

1 Introduction.

Using lattice gauge theory to study the non-perturbative regime of Quantum Chromodynamics (QCD) has advanced our understanding of quark masses as well as providing accurate theoretical predictions of masses such as that of the proton and neutron together with the spectrum of mesons and other baryons. Central to lattice gauge theory analyses is the use of supercomputers to numerically evaluate Green's functions, for instance, in the low energy region where the mechanisms behind colour and quark confinement operate. Measurements of such Green's functions are carried out on a regular discretized spacetime with the separation distance acting as a regulator that when reduced in magnitude requires a careful extrapolation to continuum spacetime. This has to be implemented in such a way that errors are under control and not significant. Such a task is not straightforward and while this problem is generally very well understood now, having independent information on any Green's function of interest would provide an important quality control tool. Indeed it was recognized in [1, 2], for instance, and emphasised in [3] that the high loop order perturbative evaluation of Green's functions could provide useful information, albeit at high energy. In other words a series of low energy lattice measurements could be extrapolated to larger energies which should therefore overlap with a similar extension of the perturbative result to lower energies. Such a programme has proved successful over the years in contributing to the understanding and improvement of lattice errors. It has also benefitted from the intense use of high performance computing (HPC) facilities devoted to lattice computations as well as the development of techniques to evaluate Feynman graphs to higher loop order using new algorithms written in efficient symbolic manipulation languages. These have equally gained from similar improvements in computing power.

While this summarizes the overall vision of the earlier ideas of [1, 2, 3] there are ongoing technical and field theory issues to be overcome. On the lattice side evaluating Green's functions can be financially expensive especially when they involve operators with a spacetime derivative. This can be circumvented to a degree by using a specific renormalization scheme developed in [1, 2] and named the modified regularization invariant (RI$'$) scheme. It is a minor variation of the related RI scheme but RI$'$ is more popularly used now since it does not require taking a spacetime derivative on the lattice. By contrast the scheme that is used in virtually all perturbative continuum calculations is the modified minimal subtraction ($\overline{\text{MS}}$) scheme, [4, 5]. It is usually coupled with a regularization that is dimensional where the spacetime dimension is analytically continued to $d = 4 - 2\epsilon$ with the small parameter ϵ acting as the regularizing parameter. The main reason why the $\overline{\text{MS}}$ scheme is used together with dimensional regularization is that in this configuration one can evaluate massless Feynman integrals and thence renormalize Green's functions to a very high loop order. A case in point is the five loop renormalization of QCD, [6, 7, 8, 9, 10]. While the lattice measurements of Green's functions are carried out in the RI$'$ scheme the parallel continuum calculations are always in $\overline{\text{MS}}$. Therefore before matching comparisons and subsequent error analyses can be determined the dependence of the Green's function over all energy scales has to be found in the same scheme. For practical reasons this is much easier to carry out in the continuum since the RI$'$ scheme can be defined in that case, [1, 2, 11, 12, 13], allowing for the renormalization of QCD. However, with the move to improve precision on the lattice low energy side there is now a need to progress loop calculations beyond the present orders that are available for Green's functions that are of immediate importance.

This is the purpose of the article. We will renormalize the QCD Lagrangian at four loops in the RI$'$ scheme and by using properties of the renormalization group we will determine the renormalization group functions of the fields at *five* loops. The RI$'$ β-function is already trivially available since the scheme is defined in such a way that the coupling constant renormalization is carried out in an $\overline{\text{MS}}$ way. In other words at the subtraction point of a vertex function only

the poles with respect to ϵ are absorbed into the coupling renormalization constant. Therefore the coefficients of each term of the five loop RI$'$ scheme β-function are formally equivalent to those of the $\overline{\text{MS}}$ one. The reason why the field anomalous dimensions are important in the RI$'$ scheme is that they are required for renormalizing Green's functions with operator insertions, [1, 2]. In particular recent lattice studies [14, 15, 16] have, for instance, measured quark bilinear operators inserted in a quark 2-point function at zero momentum in the RI$'$ scheme. So as such Green's functions are of current interest we have also renormalized the quark mass operator and vector current to four loops and will provide the analytic and numerical values of the respective Green's functions to the same order. For example the former will be useful for lattice studies that relate to quark mass determination. The article therefore extends the three loop results of [11, 12, 13]. One corollary is that we will provide the five loop RI$'$ quark mass dimension for a general colour group in a linear covariant gauge. Although lattice measurements are invariably in the Landau gauge the continuum calculations are carried out for a non-zero gauge parameter partly as it can be used for cross-checks on the evaluation of the Feynman graphs. However it is also important to have such data for other problems. For instance, the study of the conformal window in gauge theories has been an important pursuit in recent years given the potential for it to give insight into four dimensional conformal field theories, as well as the connection with beyond the Standard Model ideas. The conformal window is governed by the existence of the Banks-Zaks fixed point [17, 18]. Such a fixed point has associated critical exponents that can be estimated perturbatively. As they are renormalization group invariants they have to be scheme independent although in a truncation of their perturbative expansion this would only be true approximately. So having the RI$'$ scheme renormalization group functions to a new level of precision will be important for extending the results of [19, 20, 21] which is currently in progress, [22]. As an example the measurement of the quark mass anomalous dimension exponent has been looked at on the lattice in [23, 24]. Having the five loop result in another scheme could assist with improving the continuum estimate.

The article is organized as follows. We discuss the method for extracting the four loop RI$'$ renormalization constants in Section 2 as well as how we use properties of the renormalization group to deduce the five loop anomalous dimensions of the fields and covariant gauge parameter in the same scheme. The corresponding results are contained in Section 3 together with the related conversion functions. Sections 4 and 5 respectively focus on the four loop renormalization of the quark mass operator and vector current. They contain the value of the Green's function under consideration in both the $\overline{\text{MS}}$ and RI$'$ schemes. We summarize our conclusions in Section 6 while Appendix A records expressions for a general Lie group.

2 Background.

By way of orientation we recall the bare QCD Lagrangian with a linear covariant gauge fixing is

$$L \;=\; -\; \frac{1}{4} G^a_{\text{o}\,\mu\nu} G^{a\,\mu\nu}_{\text{o}} \;-\; \frac{1}{2\alpha_{\text{o}}} (\partial^\mu A^a_{\text{o}\,\mu})^2 \;-\; \bar{c}^a_{\text{o}} \partial_\mu D^\mu_{\text{o}} c^a_{\text{o}} \;+\; i\bar{\psi}^{iI}_{\text{o}} \slashed{D}_{\text{o}} \psi^{iI}_{\text{o}} \tag{2.1}$$

where the subscript $_{\text{o}}$ denotes a bare field or parameter. The renormalization constants are introduced in the canonical way through a rescaling of the bare entities

$$A^{a\,\mu}_{\text{o}} \;=\; \sqrt{Z_A}\, A^{a\,\mu} \;\;,\;\; c^a_{\text{o}} \;=\; \sqrt{Z_c}\, c^a \;\;,\;\; \psi_{\text{o}} \;=\; \sqrt{Z_\psi}\, \psi \;\;,\;\; g_{\text{o}} \;=\; \mu^\epsilon Z_g\, g \;\;,\;\; \alpha_{\text{o}} \;=\; Z^{-1}_\alpha Z_A\, \alpha \tag{2.2}$$

where g is the coupling constant, μ is the arbitrary mass scale associated with the dimensionally regularized version of the action. It is these renormalization constants that are determined with respect to a scheme, and in particular the RI$'$ scheme, that we will be interested in here. We

recall that the RI$'$ scheme was introduced in connection with the renormalization of operators of interest to lattice field theory but can be equally applied to the renormalization of the Lagrangian in the continuum, [1, 2]. This was carried out in [11, 12, 13] to three loops previously. Briefly the prescription defining the RI$'$ scheme is that the renormalization constants of the fields are fixed by ensuring that at the subtraction point each 2-point function has no $O(a)$ correction where $a = g^2/(16\pi^2)$. While this is similar to the momentum subtraction scheme given in [25, 26] it differs in the prescription for defining the coupling renormalization constant. In that case Z_g is determined using the same criterion as that of the $\overline{\text{MS}}$ scheme. Consequently the RI$'$ scheme β-function is formally the same as that of the $\overline{\text{MS}}$ one in terms of the coefficients of the polynomial in a but that variable is regarded as an RI$'$ scheme coupling constant rather than an $\overline{\text{MS}}$ one.

In [13] the three loop renormalization constants were deduced from applying the MINCER algorithm, [27, 28], to the evaluation of the Feynman graphs comprising the gluon, ghost and quark 2-point functions. While QCD had been renormalized at four loops, [29, 30, 31, 32, 33], this was achieved via a variety of different techniques. More recently though the renormalization of QCD has progressed to the five loop level, [6, 7, 8, 9, 10, 33, 34, 35]. Underlying the computations carried out by one group was the extension of MINCER to its four loop successor FORCER, [36, 37], together with infrared rearrangement, [38, 39]. One consequence of the use of FORCER was the provision of each of the 2-point functions to four loops in terms of bare parameters to particularly high order in the ϵ expansion, [40]. This data provides the core starting point for implementing the renormalization of (2.1) in the RI$'$ scheme. Therefore we have applied the RI$'$ prescription using the symbolic manipulation language FORM, [41, 42], and extracted the four loop values of Z_A, Z_α, Z_c and Z_ψ using the method of [43]. These are converted into the associated renormalization group functions via the relations

$$
\begin{aligned}
\gamma_\phi^{\text{RI}'}(a, \alpha) &= \beta^{\text{RI}'}(a)\frac{\partial \ln Z_\phi^{\text{RI}'}}{\partial a} + \alpha\gamma_\alpha^{\text{RI}'}(a, \alpha)\frac{\partial \ln Z_\phi^{\text{RI}'}}{\partial \alpha} \\
\gamma_\alpha^{\text{RI}'}(a, \alpha) &= \left[\beta^{\text{RI}'}(a)\frac{\partial \ln Z_\alpha^{\text{RI}'}}{\partial a} - \gamma_A^{\text{RI}'}(a, \alpha)\right]\left[1 - \alpha\frac{\partial \ln Z_\alpha^{\text{RI}'}}{\partial \alpha}\right]^{-1}
\end{aligned}
\tag{2.3}
$$

where ϕ represents either gluon, ghost or quark with respective labels A, c and ψ for future reference. The relation for the gauge parameter α is for a covariant gauge fixing parameter in general but in the case of the linear gauge fixing used here Z_α is unity in any scheme leading to the relation

$$
\gamma_A^{\text{RI}'}(a, \alpha) = -\gamma_\alpha^{\text{RI}'}(a, \alpha)
\tag{2.4}
$$

which is true to all orders. We use the convention that the variables of the renormalization group functions are those in the scheme indicated in the superscript unless they are labelled explicitly. Also the argument of the RI$'$ β-function does not involve α since the vertex renormalization is carried out according to the $\overline{\text{MS}}$ prescription.

We delay presenting the explicit four loop expressions for the simple reason that it is possible to determine the five loop renormalization group functions in the RI$'$ scheme. This is achieved from knowledge of several quantities which are the five loop $\overline{\text{MS}}$ renormalization group functions from [9, 10, 33, 34, 35] and the full four loop renormalization constants in the RI$'$ scheme itself. By the latter we mean that the finite part of the renormalization constants is absolutely crucial to extending our four loop RI$'$ scheme expressions to the next loop order. These allow us to construct conversion functions which provide the second key ingredient of the relevant formalism. From the properties of the renormalization group the anomalous dimensions in separate schemes are related by

$$
\gamma_\phi^{\text{RI}'}\left(a_{\text{RI}'}, \alpha_{\text{RI}'}\right) = \left[\gamma_\phi^{\overline{\text{MS}}}\left(a_{\overline{\text{MS}}}, \alpha_{\overline{\text{MS}}}\right) + \beta^{\overline{\text{MS}}}\left(a_{\overline{\text{MS}}}\right)\frac{\partial}{\partial a_{\overline{\text{MS}}}}\ln C_\phi\left(a_{\overline{\text{MS}}}, \alpha_{\overline{\text{MS}}}\right)\right.
$$

4

$$+ \ \alpha_{\overline{\text{MS}}} \gamma_\alpha^{\overline{\text{MS}}}\left(a_{\overline{\text{MS}}}, \alpha_{\overline{\text{MS}}}\right) \frac{\partial}{\partial \alpha_{\overline{\text{MS}}}} \ln C_\phi\left(a_{\overline{\text{MS}}}, \alpha_{\overline{\text{MS}}}\right)\Bigg]_{\overline{\text{MS}} \to \text{RI}'} . \quad (2.5)$$

We have labelled the variables according to the scheme they correspond to in order to avoid ambiguity. The mapping indicated via the restriction on the right hand side means that the expression inside the square brackets is evaluated in terms of $\overline{\text{MS}}$ variables. However since the left hand side is the RI' scheme anomalous dimension the $\overline{\text{MS}}$ variables have to be mapped to their RI' counterparts. While

$$a_{\text{RI}'} \ = \ a_{\overline{\text{MS}}} \quad (2.6)$$

the relation between the gauge parameters is more involved. We have determined this to the requisite order and note, for brevity, that the $SU(3)$ relation is

$$
\begin{aligned}
\alpha_{\text{RI}'}\Big|^{SU(3)} \ = \ & \left[1 + \left[\frac{10}{9}N_f - \frac{97}{12} - \frac{3}{2}\alpha - \frac{3}{4}\alpha^2\right]a\right. \\
& + \left[27\zeta_3 - \frac{7143}{32} - \frac{9}{16}\alpha^3 - \frac{5}{3}N_f\alpha - \frac{5}{3}N_f\alpha^2 + \frac{4}{3}\zeta_3 N_f + \frac{9}{16}\alpha^4 + \frac{95}{16}\alpha^2 \right. \\
& \left. + \frac{463}{32}\alpha + \frac{2473}{72}N_f - 18\zeta_3\alpha\right]a^2 \\
& + \left[\frac{63225}{64}\zeta_5 - \frac{10221367}{1152} - \frac{61105}{972}N_f^2 - \frac{36213}{32}\zeta_3\alpha - \frac{21763}{108}\zeta_3 N_f - \frac{11417}{288}N_f\alpha^2\right. \\
& - \frac{8439}{128}\alpha^3 - \frac{3049}{48}N_f\alpha - \frac{2320}{9}\zeta_5 N_f - \frac{1449}{32}\zeta_3\alpha^2 - \frac{315}{64}\zeta_5\alpha^4 - \frac{261}{16}\alpha^4 \\
& - \frac{45}{8}\zeta_5\alpha^3 - \frac{33}{2}\zeta_4 N_f - \frac{27}{64}\alpha^6 - \frac{25}{27}N_f^2\alpha^2 - \frac{16}{3}\zeta_3 N_f^2\alpha - \frac{11}{4}\zeta_3 N_f\alpha^2 - \frac{5}{4}N_f\alpha^3 \\
& + \frac{4}{3}N_f^2\alpha + \frac{15}{8}N_f\alpha^4 + \frac{27}{16}\alpha^5 + \frac{32}{27}\zeta_3 N_f^2 + \frac{81}{8}\zeta_4\alpha + \frac{81}{32}\zeta_4\alpha^2 + \frac{117}{32}\zeta_3\alpha^4 \\
& + \frac{243}{32}\zeta_4 + \frac{1341}{32}\zeta_3\alpha^3 + \frac{3105}{8}\zeta_5\alpha + \frac{3465}{32}\zeta_5\alpha^2 + \frac{13941}{8}\zeta_3 + \frac{30835}{384}\alpha^2 \\
& \left. + \frac{39423}{128}\alpha + \frac{4939411}{2592}N_f + 117\zeta_3 N_f\alpha\right]a^3 \\
& + \left[\frac{230899397}{2048}\zeta_3 + \frac{246393489}{1024}\zeta_5 - \frac{1174691657}{2304} - \frac{277127487}{2048}\zeta_7\right. \\
& - \frac{273998929}{31104}N_f^2 - \frac{70979391}{4096}\zeta_7\alpha - \frac{56971325}{2592}\zeta_3 N_f - \frac{54117607}{1024}\zeta_3\alpha \\
& - \frac{18363505}{288}\zeta_5 N_f - \frac{13026679}{6912}N_f\alpha^2 - \frac{12195539}{4608}N_f\alpha - \frac{2566431}{2048}\zeta_7\alpha^2 \\
& - \frac{1624581}{512}\zeta_3\alpha^2 - \frac{1199043}{2048}\alpha^4 - \frac{1100549}{512}\alpha^3 - \frac{992391}{1024}\zeta_5\alpha^3 - \frac{851175}{1024}\zeta_6\alpha \\
& - \frac{327213}{512}\zeta_3^2\alpha - \frac{305965}{576}\zeta_4 N_f - \frac{226449}{512}\zeta_3^2\alpha^2 - \frac{216815}{96}\zeta_5 N_f\alpha \\
& - \frac{204525}{32}\zeta_6 - \frac{137781}{1024}\zeta_7\alpha^4 - \frac{56025}{512}\zeta_3^2\alpha^3 - \frac{45465}{256}\zeta_5\alpha^4 - \frac{23247}{1024}\zeta_4\alpha^3 \\
& - \frac{19755}{512}\zeta_5\alpha^5 - \frac{17631}{512}\zeta_3\alpha^5 - \frac{13041}{2048}\zeta_4\alpha^4 - \frac{6071}{96}N_f\alpha^3 - \frac{5977}{216}\zeta_4 N_f^2 \\
& - \frac{4873}{9}\zeta_3 N_f^2\alpha - \frac{3267}{512}\alpha^5 - \frac{1117}{8}\zeta_3 N_f\alpha^3 - \frac{880}{27}\zeta_5 N_f^3 - \frac{567}{256}\alpha^7 - \frac{351}{64}\zeta_3\alpha^6 \\
& - \frac{175}{16}\zeta_5 N_f\alpha^4 - \frac{128}{3}\zeta_3^2 N_f^2 - \frac{97}{36}N_f^2\alpha^3 - \frac{40}{27}N_f^3\alpha - \frac{15}{8}N_f\alpha^6 - \frac{7}{2}\zeta_3 N_f^2\alpha^2 \\
& - \frac{5}{2}\zeta_5 N_f^2\alpha^2 + \frac{9}{4}\zeta_4 N_f\alpha + \frac{9}{4}\zeta_4 N_f^2\alpha + \frac{9}{4}\zeta_3^2 N_f\alpha^3 + \frac{23}{2}\zeta_3 N_f\alpha^4 + \frac{25}{12}N_f^2\alpha^4 \\
& + \frac{45}{4}\zeta_3^2 N_f\alpha^2 + \frac{45}{8}N_f\alpha^5 + \frac{81}{256}\alpha^8 + \frac{115}{2}\zeta_5 N_f^2\alpha + \frac{153}{4}\zeta_4 N_f\alpha^2 + \frac{160}{27}\zeta_3 N_f^3\alpha
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{321}{4}\zeta_3^2 N_f \alpha + \frac{595}{32}\zeta_5 N_f \alpha^3 + \frac{945}{128}\zeta_5 \alpha^6 + \frac{1501}{243}\zeta_3 N_f^3 + \frac{1683}{8}\zeta_3^2 N_f \\
&+ \frac{1765}{128}N_f \alpha^4 + \frac{1863}{256}\zeta_3^2 \alpha^5 + \frac{3069}{128}\alpha^6 + \frac{4725}{1024}\zeta_6 \alpha^4 + \frac{10647}{4096}\zeta_7 \alpha^5 \\
&+ \frac{16775}{16}\zeta_6 N_f + \frac{19575}{512}\zeta_4 \alpha^2 + \frac{19737}{512}\zeta_3^2 \alpha^4 + \frac{22275}{1024}\zeta_6 \alpha^2 + \frac{24975}{1024}\zeta_6 \alpha^3 \\
&+ \frac{26901}{32}\zeta_7 N_f \alpha + \frac{40279}{96}\zeta_5 N_f \alpha^2 + \frac{51203}{1296}N_f^2 \alpha^2 + \frac{87701}{576}\zeta_3 N_f \alpha^2 \\
&+ \frac{132615}{512}\zeta_7 \alpha^3 + \frac{197811}{32}\zeta_3^2 + \frac{285995}{1728}N_f^2 \alpha + \frac{303233}{1944}\zeta_3 N_f^2 \\
&+ \frac{423043}{162}\zeta_5 N_f^2 + \frac{488271}{2048}\zeta_3 \alpha^4 + \frac{636039}{1024}\zeta_4 \alpha + \frac{1312185}{512}\alpha^2 \\
&+ \frac{2029821}{1024}\zeta_3 \alpha^3 + \frac{2785621}{288}\zeta_3 N_f \alpha + \frac{2910357}{2048}\zeta_4 + \frac{4167895}{34992}N_f^3 \\
&+ \frac{4295115}{1024}\zeta_5 \alpha^2 + \frac{25666081}{864}\zeta_7 N_f + \frac{32185629}{1024}\zeta_5 \alpha + \frac{128069033}{18432}\alpha \\
&+ \left. \frac{541249745}{3888}N_f + \zeta_4 N_f^3 + 8\zeta_3 N_f^2 \alpha^3 \right] a^4 \; + \; O(a^5) \Bigg] \alpha
\end{aligned}
\tag{2.7}
$$

which extends the previous loop order expression given in [13]. In (2.7) ζ_n is the Riemann zeta function. The full expression for an arbitrary group is available in the data file associated with the arXiv version of this article. To effect the mapping indicated on the right side of (2.5) one has to invert the relation (2.7). The conversion functions in (2.5) are defined by

$$
C_\phi(a, \alpha) \;=\; \frac{Z_\phi^{\mathrm{RI}'}}{Z_\phi^{\overline{\mathrm{MS}}}}
\tag{2.8}
$$

where we use the $\overline{\mathrm{MS}}$ scheme parameters as the argument variables. While the renormalization constants are divergent, $C_\phi(a, \alpha)$ is ϵ finite. This follows by recalling that one has to convert the RI$'$ variables of the numerator using (2.6) and (2.7).

3   Field anomalous dimensions.

This section records the field anomalous dimensions at five loops given the avenue provided by the renormalization group construction (2.5). First, we note that the conversion functions are only needed to four loops since in (2.5) their derivative with respect to either variable is each multiplied by a renormalization group function. The combination of both will be $O(a_{\overline{\mathrm{MS}}}^5)$ which is the order we are interested in. In this section we will provide results for the $SU(3)$ colour group for brevity and therefore note that in the Landau gauge we have

$$
\begin{aligned}
C_A(a, 0)\Big|^{SU(3)} \;=\; & 1 + \left[\frac{97}{12} - \frac{10}{9}N_f \right] a + \left[\frac{100}{81}N_f^2 + \frac{83105}{288} - \frac{11299}{216}N_f - \frac{4}{3}\zeta_3 N_f - 27\zeta_3 \right] a^2 \\
& + \left[\frac{16}{9}\zeta_3 N_f^2 - \frac{8228977}{2592}N_f - \frac{63225}{64}\zeta_5 - \frac{17433}{8}\zeta_3 - \frac{1000}{729}N_f^3 - \frac{243}{32}\zeta_4 \right. \\
& \left. + \frac{33}{2}\zeta_4 N_f + \frac{2320}{9}\zeta_5 N_f + \frac{25915}{108}\zeta_3 N_f + \frac{164395}{972}N_f^2 + \frac{44961125}{3456} \right] a^3 \\
& + \left[\frac{62302764631}{82944} - \frac{1758762815}{7776}N_f - \frac{324121925}{2048}\zeta_3 - \frac{262747689}{1024}\zeta_5 \right. \\
& - \frac{25666081}{864}\zeta_7 N_f - \frac{15059695}{34992}N_f^3 - \frac{3161781}{2048}\zeta_4 - \frac{515843}{162}\zeta_5 N_f^2
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{362555}{648}\zeta_3 N_f^2 - \frac{174483}{32}\zeta_3^2 - \frac{16775}{16}\zeta_6 N_f - \frac{1943}{216}\zeta_4 N_f^2 \\
&- \frac{1107}{8}\zeta_3^2 N_f - \frac{229}{27}\zeta_3 N_f^3 + \frac{400}{9}\zeta_3^2 N_f^2 + \frac{880}{27}\zeta_5 N_f^3 \\
&+ \frac{10000}{6561}N_f^4 + \frac{204525}{32}\zeta_6 + \frac{469333}{576}\zeta_4 N_f + \frac{27270475}{864}\zeta_3 N_f \\
&+ \frac{60587905}{864}\zeta_5 N_f + \frac{190787741}{10368}N_f^2 + \frac{277127487}{2048}\zeta_7 - \zeta_4 N_f^3 \Big] a^4 \\
&+ O(a^5) \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad (3.1)
\end{aligned}
$$

$$
\begin{aligned}
C_c(a,0)\Big|^{SU(3)} &= 1 + 3a + \left[\frac{5829}{64} - \frac{135}{16}\zeta_3 - \frac{95}{16}N_f\right]a^2 \\
&+ \Big[\frac{5161}{648}N_f^2 - \frac{198001}{432}N_f - \frac{40449}{64}\zeta_3 - \frac{1755}{32}\zeta_5 - \frac{33}{4}\zeta_4 N_f + \frac{2}{3}\zeta_3 N_f^2 \\
&+ \frac{59}{2}\zeta_3 N_f + \frac{243}{64}\zeta_4 + \frac{1082353}{288}\Big]a^3 \\
&+ \Big[\frac{15567976783}{73728} - \frac{151911987}{4096}\zeta_3 - \frac{79190001}{2048}\zeta_5 - \frac{39621021}{1024}N_f - \frac{204525}{64}\zeta_6 \\
&- \frac{150979}{7776}N_f^3 - \frac{52025}{128}\zeta_4 N_f - \frac{11907}{16}\zeta_7 N_f - \frac{699}{8}\zeta_5 N_f^2 - \frac{425}{48}\zeta_3 N_f^2 \\
&- \frac{5}{18}\zeta_3 N_f^3 + \frac{1}{2}\zeta_4 N_f^3 + \frac{165}{16}\zeta_4 N_f^2 + \frac{337}{4}\zeta_3^2 N_f + \frac{16775}{32}\zeta_6 N_f \\
&+ \frac{250085}{64}\zeta_5 N_f + \frac{1149471}{512}\zeta_3^2 + \frac{2625583}{768}\zeta_3 N_f + \frac{5819877}{4096}\zeta_4 \\
&+ \frac{8093153}{4608}N_f^2 + \frac{100880073}{8192}\zeta_7\Big]a^4 \; + \; O(a^5) \quad\quad\quad\quad\quad (3.2)
\end{aligned}
$$

and

$$
\begin{aligned}
C_\psi(a,0)\Big|^{SU(3)} &= 1 + \left[12\zeta_3 - \frac{359}{9} + \frac{7}{3}N_f\right]a^2 \\
&+ \Big[\frac{24722}{81}N_f - \frac{439543}{162} - \frac{1570}{243}N_f^2 - \frac{1165}{3}\zeta_5 - \frac{440}{9}\zeta_3 N_f \\
&+ \frac{79}{4}\zeta_4 + \frac{8009}{6}\zeta_3\Big]a^3 \\
&+ \Big[\frac{21391}{1458}N_f^3 - \frac{356864009}{5184}\zeta_5 - \frac{146722043}{864} - \frac{29889697}{5184}\zeta_3^2 - \frac{1294381}{108}\zeta_3 N_f \\
&- \frac{1276817}{972}N_f^2 - \frac{440}{9}\zeta_5 N_f^2 - \frac{20}{3}\zeta_4 N_f^2 + \frac{8}{27}\zeta_3 N_f^3 + \frac{100}{3}\zeta_6 N_f + \frac{2291}{72}\zeta_4 N_f \\
&+ \frac{5704}{27}\zeta_3 N_f^2 + \frac{565939}{864}\zeta_4 + \frac{1673051}{324}\zeta_5 N_f + \frac{3807625}{10368}\zeta_6 + \frac{6747755}{288}\zeta_7 \\
&+ \frac{55476671}{1944}N_f + \frac{317781451}{2592}\zeta_3 - 1029\zeta_7 N_f - 24\zeta_3^2 N_f\Big]a^4 \; + \; O(a^5) \;\; (3.3)
\end{aligned}
$$

with the expressions for non-zero α and an arbitrary Lie group available in the article's data file. With these it is therefore a straightforward exercise to deduce

$$
\begin{aligned}
\gamma_A^{RI'}(a,0)\Big|^{SU(3)} &= \left[\frac{2}{3}N_f - \frac{13}{2}\right]a + \left[\frac{250}{9}N_f - \frac{3727}{24} - \frac{20}{27}N_f^2\right]a^2 \\
&+ \Big[\frac{200}{243}N_f^3 + \frac{5210}{3}N_f + \frac{9747}{16}\zeta_3 - \frac{2127823}{288} - \frac{1681}{18}N_f^2 - \frac{119}{3}\zeta_3 N_f
\end{aligned}
$$

$$
\begin{aligned}
& \left. - \frac{16}{9}\zeta_3 N_f^2 \right] a^3 \\
& + \left[\frac{373823}{1458} N_f^3 - \frac{3011547563}{6912} - \frac{6816713}{648} N_f^2 - \frac{2897113}{216} \zeta_3 N_f \right. \\
& \quad - \frac{845275}{96} \zeta_5 N_f - \frac{2000}{2187} N_f^4 + \frac{88}{27} \zeta_3 N_f^3 + \frac{4640}{9} \zeta_5 N_f^2 + \frac{60427}{162} \zeta_3 N_f^2 \\
& \quad \left. + \frac{1431945}{64} \zeta_5 + \frac{18987543}{256} \zeta_3 + \frac{221198219}{1728} N_f \right] a^4 \\
& + \left[\frac{1791101885}{34992} N_f^3 - \frac{5296290721381}{165888} - \frac{88396975485}{16384} \zeta_7 \right. \\
& \quad - \frac{28725816895}{23328} N_f^2 - \frac{4380999739}{2592} \zeta_3 N_f - \frac{4144018255}{1152} \zeta_5 N_f \\
& \quad + \frac{11001160807}{6912} \zeta_7 N_f + \frac{13433378467}{2048} \zeta_3 + \frac{42793732125}{4096} \zeta_5 \\
& \quad + \frac{51527836699}{4608} N_f - \frac{25666081}{324} \zeta_7 N_f^2 - \frac{18057505}{26244} N_f^4 - \frac{942553}{64} \zeta_3^2 N_f \\
& \quad - \frac{761560}{81} \zeta_5 N_f^3 - \frac{34749}{4} \zeta_4 - \frac{16744}{81} \zeta_3 N_f^3 - \frac{13771}{54} \zeta_4 N_f^2 \\
& \quad - \frac{13456}{9} \zeta_3^2 N_f^2 - \frac{4120}{243} \zeta_3 N_f^4 + \frac{1618}{81} \zeta_4 N_f^3 + \frac{3136}{27} \zeta_3^2 N_f^3 + \frac{7040}{81} \zeta_5 N_f^4 \\
& \quad + \frac{20000}{19683} N_f^5 + \frac{53227}{36} \zeta_4 N_f + \frac{137558185}{432} \zeta_5 N_f^2 + \frac{231090011}{2592} \zeta_3 N_f^2 \\
& \quad \left. + \frac{589719519}{2048} \zeta_3^2 \right] a^5 \; + \; O(a^6)
\end{aligned}
\tag{3.4}
$$

$$
\begin{aligned}
\left. \gamma_c^{\mathrm{RI}'}(a,0) \right|^{SU(3)} = \; & - \frac{9}{4} a + \left[\frac{13}{4} N_f - \frac{813}{16} \right] a^2 \\
& + \left[\frac{21}{4} \zeta_3 N_f + \frac{14909}{48} N_f + \frac{5697}{32} \zeta_3 - \frac{157303}{64} - \frac{125}{18} N_f^2 \right] a^3 \\
& + \left[\frac{2705}{162} N_f^3 - \frac{219384137}{1536} - \frac{288155}{216} N_f^2 - \frac{132749}{96} \zeta_3 N_f - \frac{15175}{16} \zeta_5 N_f \right. \\
& \quad \left. + \frac{19}{2} \zeta_3 N_f^2 + \frac{221535}{32} \zeta_5 + \frac{9207729}{512} \zeta_3 + \frac{30925009}{1152} N_f \right] a^4 \\
& + \left[\frac{44742833}{7776} N_f^3 - \frac{373954139515}{36864} - \frac{26906792787}{32768} \zeta_7 + \frac{6421603437}{4096} \zeta_3 \right. \\
& \quad + \frac{18444521475}{8192} \zeta_5 + \frac{69770872099}{27648} N_f - \frac{463728159}{4096} \zeta_3^2 - \frac{280840685}{768} \zeta_5 N_f \\
& \quad - \frac{90397253}{384} \zeta_3 N_f - \frac{28603559}{144} N_f^2 - \frac{37306}{729} N_f^4 - \frac{24057}{8} \zeta_4 - \frac{3969}{2} \zeta_7 N_f^2 \\
& \quad - \frac{809}{12} \zeta_4 N_f^2 - \frac{437}{2} \zeta_3^2 N_f^2 - \frac{415}{3} \zeta_5 N_f^3 - \frac{172}{3} \zeta_3 N_f^3 + \frac{1627}{8} \zeta_4 N_f \\
& \quad \left. + \frac{498713}{64} \zeta_3 N_f^2 + \frac{720477}{128} \zeta_3^2 N_f + \frac{4411505}{288} \zeta_5 N_f^2 + \frac{318680873}{3072} \zeta_7 N_f \right] a^5 \\
& + O(a^6)
\end{aligned}
\tag{3.5}
$$

and

$$
\begin{aligned}
\left. \gamma_\psi^{\mathrm{RI}'}(a,0) \right|^{SU(3)} = \; & \left[\frac{67}{3} - \frac{4}{3} N_f \right] a^2 + \left[\frac{104}{27} N_f^2 + \frac{52321}{36} + 16 \zeta_3 N_f - \frac{4472}{27} N_f - \frac{607}{2} \zeta_3 \right] a^3 \\
& + \left[\frac{75355}{81} N_f^2 - \frac{15631129}{324} \zeta_3 - \frac{1537880}{81} N_f - \frac{1000}{81} N_f^3 + \frac{121558}{27} \zeta_3 N_f \right.
\end{aligned}
$$

$$+ \frac{8966278}{81} + \frac{15846715}{1296}\zeta_5 - 830\zeta_5 N_f - 80\zeta_3 N_f^2 \Big] a^4$$

$$+ \Big[\frac{86560}{2187} N_f^4 - \frac{61222404563}{10368}\zeta_3 - \frac{26588447977}{27648}\zeta_7 - \frac{23362550497}{11664}N_f$$

$$+ \frac{2727697601}{10368}\zeta_3^2 + \frac{2947769225}{972}\zeta_5 + \frac{10680056713}{11664}\zeta_3 N_f$$

$$+ \frac{136110640603}{15552} - \frac{1254469405}{2916}\zeta_5 N_f - \frac{9978134}{243}\zeta_3 N_f^2$$

$$- \frac{9708268}{2187} N_f^3 - \frac{645017}{54}\zeta_3^2 N_f - \frac{6811}{2}\zeta_7 N_f^2 - \frac{3520}{27}\zeta_5 N_f^3 - \frac{704}{9}\zeta_3^2 N_f^2$$

$$+ \frac{43264}{81}\zeta_3 N_f^3 + \frac{3815110}{243}\zeta_5 N_f^2 + \frac{4429579}{36}\zeta_7 N_f + \frac{222450779}{1458}N_f^2 \Big] a^5$$

$$+ O(a^6) \tag{3.6}$$

at five loops. We have checked that the previous three loop results of [11, 12, 13] are reproduced since the four loop renormalization constants were derived from the FORCER data provided in [40]. These relations now together with the five loop RI$'$ β-function complete the derivation of the RI$'$ scheme renormalization group functions to that order. More importantly we have determined the RI$'$ renormalization constant for the quark itself. This is one of our main goals since it is central to the renormalization of quark bilinear operators in the RI$'$ scheme that are of importance in lattice matching analyses.

4 Quark mass operator.

We now turn to the determination of the Green's functions with zero momentum operator insertions which also extend the results provided in [13]. Unlike the derivation of the five loop anomalous dimensions of the fields in the RI$'$ scheme of the previous section no expression is available for the Green's function as a function of the bare parameters that is measured on the lattice involving the quark mass operator. Therefore we have computed the four loop corrections explicitly for the present analysis. First, to be specific the Green's function of the quark mass operator we will consider is

$$G_{\bar{\psi}\psi}(p) = \langle \psi(p) \, [\bar{\psi}\psi](0) \, \bar{\psi}(-p) \rangle \,. \tag{4.1}$$

As $G_{\bar{\psi}\psi}(p)$ involves a Lorentz scalar operator it has only one form factor which can be accessed by taking the spinor trace, normalized by the trace over the unit matrix. This produces Feynman integrals that are Lorentz scalars which means we can apply the four loop automatic Feynman graph computation programme FORCER, [36, 37], to it. Previously, the three loop MINCER programme was used in [13]. Here we have computed the 5728 four loop graphs contributing to (4.1) as well as the 1, 13 and 244 diagrams respectively at one, two and three loops. The lower loop graphs were calculated with the FORCER algorithm both for consistency and as a check on previous work. In addition to using FORCER to evaluate the large number of graphs, that are generated electronically using QGRAF [44], we used the FORM `color.h` module [41] since it automatically determines the colour group factor associated with each graph prior to integration. The module is based on [45]. Consequently we have evaluated

$$\Sigma_{\bar{\psi}\psi}^{(1)}(p) = \frac{1}{4}\mathrm{tr}\Big[G_{\bar{\psi}\psi}(p) \Big] \tag{4.2}$$

where tr is the Lorentz spinor trace and found

$$\Sigma_{\bar{\psi}\psi}^{(1)\,\overline{\mathrm{MS}}}(p)\Big|_{\alpha=0} = 1 + 4C_F a$$

$$+ \left[\frac{1531}{24} C_F C_A + 13 C_F^2 - \frac{52}{3} N_f T_F C_F - 21 \zeta_3 C_F C_A + 12 \zeta_3 C_F^2 \right] a^2$$

$$+ \left[\frac{4315565}{3888} C_F C_A^2 + \frac{3005}{18} C_F^2 C_A + \frac{916}{3} C_F^3 - \frac{131048}{243} N_f T_F C_F C_A \right.$$

$$- \frac{4699}{18} N_f T_F C_F^2 + \frac{12224}{243} N_f^2 T_F^2 C_F + \frac{405}{4} \zeta_5 C_F C_A^2 - 40 \zeta_5 C_F^2 C_A$$

$$- 120 \zeta_5 C_F^3 - \frac{69}{16} \zeta_4 C_F C_A^2 + 6 \zeta_4 C_F^2 C_A - 24 \zeta_4 N_f T_F C_F C_A$$

$$+ 24 \zeta_4 N_f T_F C_F^2 - \frac{20305}{36} \zeta_3 C_F C_A^2 + \frac{626}{3} \zeta_3 C_F^2 C_A + 38 \zeta_3 C_F^3$$

$$\left. + \frac{772}{9} \zeta_3 N_f T_F C_F C_A + \frac{176}{3} \zeta_3 N_f T_F C_F^2 + \frac{32}{9} \zeta_3 N_f^2 T_F^2 C_F \right] a^3$$

$$+ \left[\frac{2200233199}{93312} C_F C_A^3 - \frac{759}{2} \frac{d_F^{abcd} d_A^{abcd}}{N_F} - \frac{41391917}{31104} C_F^2 C_A^2 \right.$$

$$+ \frac{1095131}{72} C_F^3 C_A + \frac{2033}{12} C_F^4 + 808 N_f \frac{d_F^{abcd} d_F^{abcd}}{N_F}$$

$$- \frac{66598003}{3888} N_f T_F C_F C_A^2 - \frac{33045443}{3888} N_f T_F C_F^2 C_A - \frac{305695}{72} N_f T_F C_F^3$$

$$+ \frac{7064407}{1944} N_f^2 T_F^2 C_F C_A + \frac{1017025}{486} N_f^2 T_F^2 C_F^2 - \frac{161152}{729} N_f^3 T_F^3 C_F$$

$$+ \frac{43295}{16} \zeta_7 \frac{d_F^{abcd} d_A^{abcd}}{N_F} - \frac{2227141}{768} \zeta_7 C_F C_A^3 + \frac{365813}{32} \zeta_7 C_F^2 C_A^2$$

$$- \frac{43351}{2} \zeta_7 C_F^3 C_A + 15078 \zeta_7 C_F^4 + 1764 \zeta_7 N_f \frac{d_F^{abcd} d_F^{abcd}}{N_F}$$

$$- \frac{2499}{4} \zeta_7 N_f T_F C_F C_A^2 + 1764 \zeta_7 N_f T_F C_F^2 C_A - \frac{4275}{64} \zeta_6 \frac{d_F^{abcd} d_A^{abcd}}{N_F}$$

$$- \frac{91125}{256} \zeta_6 C_F C_A^3 + \frac{6125}{8} \zeta_6 C_F^2 C_A^2 - 900 \zeta_6 C_F^3 C_A + 400 \zeta_6 C_F^4$$

$$+ 200 \zeta_6 N_f T_F C_F C_A^2 + 150 \zeta_6 N_f T_F C_F^2 C_A - 300 \zeta_6 N_f T_F C_F^3$$

$$+ \frac{7075}{32} \zeta_5 \frac{d_F^{abcd} d_A^{abcd}}{N_F} + \frac{2673791}{1152} \zeta_5 C_F C_A^3 + 203 \zeta_5 C_F^2 C_A^2 - 2225 \zeta_5 C_F^3 C_A$$

$$- 5950 \zeta_5 C_F^4 - 1440 \zeta_5 N_f \frac{d_F^{abcd} d_F^{abcd}}{N_F} - \frac{5933}{72} \zeta_5 N_f T_F C_F C_A^2$$

$$- 170 \zeta_5 N_f T_F C_F^2 C_A + 360 \zeta_5 N_f T_F C_F^3 - \frac{1688}{9} \zeta_5 N_f^2 T_F^2 C_F C_A$$

$$+ 112 \zeta_5 N_f^2 T_F^2 C_F^2 + \frac{9099}{64} \zeta_4 \frac{d_F^{abcd} d_A^{abcd}}{N_F} + \frac{34871}{1536} \zeta_4 C_F C_A^3$$

$$- \frac{2451}{16} \zeta_4 C_F^2 C_A^2 + \frac{921}{2} \zeta_4 C_F^3 C_A - 276 \zeta_4 C_F^4 - 180 \zeta_4 N_f \frac{d_F^{abcd} d_F^{abcd}}{N_F}$$

$$- \frac{5377}{16} \zeta_4 N_f T_F C_F C_A^2 + 42 \zeta_4 N_f T_F C_F^2 C_A + 279 \zeta_4 N_f T_F C_F^3$$

$$+ 72 \zeta_4 N_f^2 T_F^2 C_F C_A - 72 \zeta_4 N_f^2 T_F^2 C_F^2 + \frac{16}{3} \zeta_4 N_f^3 T_F^3 C_F$$

$$+ \frac{21183}{16} \zeta_3 \frac{d_F^{abcd} d_A^{abcd}}{N_F} - \frac{24776933}{1728} \zeta_3 C_F C_A^3 - \frac{1457}{8} \zeta_3 C_F^2 C_A^2$$

$$+ 9049 \zeta_3 C_F^3 C_A - 4652 \zeta_3 C_F^4 - 552 \zeta_3 N_f \frac{d_F^{abcd} d_F^{abcd}}{N_F}$$

$$
\begin{aligned}
&+ \frac{432649}{72}\zeta_3 N_f T_F C_F C_A^2 + \frac{3155}{3}\zeta_3 N_f T_F C_F^2 C_A + 1226\zeta_3 N_f T_F C_F^3 \\
&- \frac{556}{3}\zeta_3 N_f^2 T_F^2 C_F C_A - \frac{2920}{3}\zeta_3 N_f^2 T_F^2 C_F^2 - \frac{128}{27}\zeta_3 N_f^3 T_F^3 C_F \\
&- \frac{74031}{32}\zeta_3^2 \frac{d_F^{abcd} d_A^{abcd}}{N_F} + \frac{101263}{64}\zeta_3^2 C_F C_A^3 - \frac{11359}{4}\zeta_3^2 C_F^2 C_A^2 \\
&+ 3320\zeta_3^2 C_F^3 C_A - 1472\zeta_3^2 C_F^4 - 192\zeta_3^2 N_f \frac{d_F^{abcd} d_F^{abcd}}{N_F} \\
&+ 384\zeta_3^2 N_f T_F C_F C_A^2 - 812\zeta_3^2 N_f T_F C_F^2 C_A + 200\zeta_3^2 N_f T_F C_F^3 \Big] a^4 \\
&+ O(a^5)
\end{aligned}
\tag{4.3}
$$

in the Landau gauge in the $\overline{\text{MS}}$ scheme for an arbitrary colour group where N_F is the dimension of the fundamental representation. The result for an arbitrary gauge is included in the associated data file. Aside from the usual colour factors T_F, C_F and C_A the rank 4 fully symmetric tensor d_R^{abcd} appears for both the fundamental F and adjoint A representations. Their properties are given in [45] and are defined by

$$
d_R^{abcd} = \frac{1}{6}\text{Tr}\left(T^a T^{(b} T^c T^{d)}\right)
\tag{4.4}
$$

for the group generators T^a in representation R. The trace Tr is over the colour spinor indices. In determining (4.3) we have verified the $\overline{\text{MS}}$ four loop quark mass anomalous dimension found in [31, 32, 46] is reproduced as a check on our procedure. For practical applications to lattice matching we note that (4.3) numerically evaluates to

$$
\begin{aligned}
\Sigma_{\bar\psi\psi}^{(1)\,\overline{\text{MS}}}(p)\Big|_{\alpha=0}^{SU(3)} &= 1 + 5.333333a + [202.948878 - 11.555556 N_f]a^2 \\
&\quad + [18.192836 N_f^2 - 1070.591116 N_f + 8966.208391]a^3 \\
&\quad + [439203.615244 - 81491.908465 N_f + 3721.663435 N_f^2 - 36.830872 N_f^3]a^4 \\
&\quad + O(a^5)
\end{aligned}
\tag{4.5}
$$

for $SU(3)$. As the four loop correction is important in assisting error analysis for lattice matching it is worthwhile estimating the effect of the new term. If we take the value of the strong coupling $\alpha_s = g^2/(4\pi)$ to be $\alpha_s = 0.12$ then at successive loop orders (4.5) gives 1.05092958, 1.06627508, 1.07142857 and 1.07331808 respectively. The difference between the two and three loop values is around 0.5% but between three and four loops this drops to around 0.2%. This rough comparison suggests that the four loop expression could refine matching errors for quark mass measurements.

Having established one of our main results we turn now to the other scheme of interest which is RI$'$. We recall the condition to renormalize the quark mass operator in that scheme is

$$
\frac{1}{4}\lim_{\epsilon\to 0}\text{tr}\left[Z_{\bar\psi\psi}^{\text{RI}'} Z_\psi^{\text{RI}'}\langle\psi(p)[\bar\psi\psi](0)\bar\psi(-p)\rangle\right]\Bigg|_{p^2=\mu^2} = 1
\tag{4.6}
$$

which runs parallel to that for the quark wave function renormalization. Applying this to the value we obtained for the Green's function where the parameters are bare allows us to find $Z_{\bar\psi\psi}^{\text{RI}'}$ at four loops. Using the renormalization group procedure of the previous section means we can deduce the four loop contribution to the quark mass conversion that was given in [11, 12, 13] at lower order. For brevity we note the expression in the Landau gauge for $SU(3)$ is

$$
C_m(a,0)\Big|^{SU(3)} = 1 - \frac{16}{3}a + \left[\frac{83}{9}N_f + \frac{152}{3}\zeta_3 - \frac{3779}{18}\right]a^2
$$

$$
+ \left[\frac{217390}{243} N_f - \frac{3115807}{324} - \frac{7514}{729} N_f^2 - \frac{4720}{27} \zeta_3 N_f - \frac{2960}{9} \zeta_5 - \frac{32}{27} \zeta_3 N_f^2 \right.
$$
$$
\left. + \frac{80}{3} \zeta_4 N_f + \frac{195809}{54} \zeta_3 \right] a^3
$$
$$
+ \left[\frac{96979}{4374} N_f^3 - \frac{744609145}{1296} - \frac{52383125}{17496} N_f^2 - \frac{3837631}{1728} \zeta_7 - \frac{2017309}{81} \zeta_3 N_f \right.
$$
$$
- \frac{843077}{54} \zeta_3^2 - \frac{359855}{81} \zeta_5 N_f - \frac{16960}{9} \zeta_4 - \frac{11500}{9} \zeta_6 N_f - \frac{8776}{27} \zeta_3^2 N_f
$$
$$
- \frac{343}{2} \zeta_7 N_f - \frac{100}{3} \zeta_4 N_f^2 - \frac{8}{9} \zeta_4 N_f^3 + \frac{40}{81} \zeta_3 N_f^3 + \frac{560}{3} \zeta_5 N_f^2 + \frac{11542}{9} \zeta_4 N_f
$$
$$
\left. + \frac{33964}{81} \zeta_3 N_f^2 + \frac{9369745}{432} \zeta_5 + \frac{86284171}{324} \zeta_3 + \frac{247516535}{2916} N_f + 5500 \zeta_6 \right] a^4
$$
$$
+ O(a^5) \tag{4.7}
$$

where the full colour group and gauge dependence for this and all the other results in this section are provided in the associated data file. Similar to the last section knowing the $\overline{\text{MS}}$ five loop quark mass anomalous dimension, [35], means we can deduce the five loop RI' scheme counterpart which is

$$
\gamma_m^{\text{RI}'}(a,0)\Big|^{SU(3)} = -4a + \left[\frac{52}{9} N_f - 126 \right] a^2
$$
$$
+ \left[\frac{17588}{27} N_f + \frac{3344}{3} \zeta_3 - \frac{20174}{3} - \frac{856}{81} N_f^2 - \frac{128}{9} \zeta_3 N_f \right] a^3
$$
$$
+ \left[\frac{16024}{729} N_f^3 - \frac{141825253}{324} - \frac{611152}{243} N_f^2 - \frac{298241}{27} \zeta_3 N_f - \frac{6160}{3} \zeta_5 \right.
$$
$$
\left. - \frac{4160}{3} \zeta_5 N_f + \frac{5984}{27} \zeta_3 N_f^2 + \frac{3519059}{54} N_f + \frac{7230017}{54} \zeta_3 \right] a^4
$$
$$
+ \left[\frac{3457170034}{243} \zeta_3 - \frac{32529575441}{972} - \frac{492112429}{243} \zeta_3 N_f - \frac{314416490}{729} N_f^2 \right.
$$
$$
- \frac{220508981}{432} \zeta_7 - \frac{214516115}{486} \zeta_5 N_f - \frac{22686158}{27} \zeta_3^2 - \frac{380896}{6561} N_f^4
$$
$$
- \frac{289936}{243} \zeta_3 N_f^3 - \frac{68992}{81} \zeta_3^2 N_f^2 - \frac{8320}{27} \zeta_5 N_f^3 - \frac{3236}{27} \zeta_4 N_f^2
$$
$$
+ \frac{1372}{3} \zeta_7 N_f^2 + \frac{3254}{9} \zeta_4 N_f + \frac{3172244}{81} \zeta_3^2 N_f + \frac{5666800}{243} \zeta_5 N_f^2
$$
$$
+ \frac{14071511}{1458} N_f^3 + \frac{21021016}{243} \zeta_3 N_f^2 + \frac{42627823}{648} \zeta_7 N_f + \frac{1646302180}{243} N_f
$$
$$
\left. + \frac{1681793075}{972} \zeta_5 - 5346 \zeta_4 \right] a^5 + O(a^6) \tag{4.8}
$$

for the same configuration. Finally we note

$$
\Sigma_{\bar{\psi}\psi}^{(1)\ \text{RI}'}(p) = 1 \tag{4.9}
$$

follows trivially from the definition of the RI' scheme.

5 Vector current.

We now turn to the other operator of interest which is the flavour non-singlet vector current $\bar{\psi}\gamma^\mu\psi$ and, in a similar procedure to that of the previous section, determine the analogous

Green's function to (4.1) which is

$$G^\mu_{\bar\psi\gamma^\mu\psi}(p) \;=\; \langle\psi(p)\,[\bar\psi\gamma^\mu\psi](0)\,\bar\psi(-p)\rangle \;=\; \Sigma^{(1)}_{\bar\psi\gamma^\mu\psi}(p)\gamma^\mu \;+\; \Sigma^{(2)}_{\bar\psi\gamma^\mu\psi}(p)\frac{p^\mu\slashed{p}}{p^2}\;. \tag{5.1}$$

As the operator is clearly a Lorentz vector its zero momentum insertion in the quark 2-point function has to decompose into a basis of vectors depending on the external momenta and γ^μ by Lorentz symmetry. This is reflected in the two form factors $\Sigma^{(i)}_{\bar\psi\gamma^\mu\psi}(p)$ of (5.1) which are Lorentz scalars. To evaluate them to four loops using FORCER, which can only be applied to scalar Feynman integrals, we project out the form factors via the relations

$$\Sigma^{(1)}_{\bar\psi\gamma^\mu\psi}(p) \;=\; \frac{1}{4(d-1)}\left[\mathrm{tr}\left(\gamma_\mu G^\mu_{\bar\psi\gamma^\mu\psi}(p)\right) \;-\; \mathrm{tr}\left(\frac{p_\mu\slashed{p}}{p^2}G^\mu_{\bar\psi\gamma^\mu\psi}(p)\right)\right]$$

$$\Sigma^{(2)}_{\bar\psi\gamma^\mu\psi}(p) \;=\; -\frac{1}{4(d-1)}\left[\mathrm{tr}\left(\gamma_\mu G^\mu_{\bar\psi\gamma^\mu\psi}(p)\right) \;-\; d\,\mathrm{tr}\left(\frac{p_\mu\slashed{p}}{p^2}G^\mu_{\bar\psi\gamma^\mu\psi}(p)\right)\right] \tag{5.2}$$

in the same way as [13]. One property of the non-singlet vector current is that since it is a physical operator that relates to charge conservation its anomalous dimension is zero. In other words the renormalization constant of the current itself is unity in *all* renormalization schemes meaning $\Sigma^{(i)}_{\bar\psi\gamma^\mu\psi}(p)$ are finite for $i=1$ and 2 despite individual graphs being divergent. Although this provides a check on the computation we have carried out a more stringent check in that the Green's function has to satisfy the Slavnov-Taylor identity underlying the charge current conservation. In practical terms this means that $\Sigma^{(1)}_{\bar\psi\gamma^\mu\psi}(p)$ must be equivalent to the quark 2-point function after its renormalization in the same scheme. We have verified explicitly that this is indeed the case in the $\overline{\text{MS}}$ scheme. For the RI' scheme agreement follows trivially as a result of the Slavnov-Taylor identity and the RI' scheme condition defining Z_ψ which is, [1, 2],

$$\lim_{\epsilon\to 0}\left[Z^{\text{RI}'}_\psi \Sigma_\psi(p)\right]\Bigg|_{p^2=\mu^2} \;=\; \slashed{p} \tag{5.3}$$

and produces

$$\Sigma^{(1)\,\text{RI}'}_{\bar\psi\gamma^\mu\psi}(p) \;=\; 1 \;+\; O(a^5) \tag{5.4}$$

for all α and colour groups to the order we have calculated to. By contrast the $\overline{\text{MS}}$ expression is

$$\begin{aligned}
\Sigma^{(1)\,\overline{\text{MS}}}_{\bar\psi\gamma^\mu\psi}(p)\Bigg|_{\alpha=0} \;=\;& 1 + \left[\frac{41}{4}C_A C_F - \frac{7}{2}C_F T_F N_f - \frac{5}{8}C_F^2 - 3\zeta_3 C_A C_F\right]a^2 \\
&+ \left[\frac{1570}{81}C_F T_F^2 N_f^2 + \frac{159257}{648}C_A^2 C_F - \frac{11887}{81}C_A C_F T_F N_f - \frac{3139}{24}\zeta_3 C_A^2 C_F\right.\\
&\quad - \frac{997}{24}C_A C_F^2 - \frac{79}{6}C_F^2 T_F N_f - \frac{73}{12}C_F^3 - \frac{69}{16}\zeta_4 C_A^2 C_F + \frac{52}{3}\zeta_3 C_A C_F T_F N_f \\
&\quad + \frac{165}{4}\zeta_5 C_A^2 C_F - 20\zeta_5 C_A C_F^2 + 6\zeta_4 C_A C_F^2 + 16\zeta_3 C_F^2 T_F N_f \\
&\quad \left.+ 44\zeta_3 C_A C_F^2\right]a^3 \\
&+ \left[400\zeta_6 C_F^4 - \frac{5660951}{1296}C_A^2 C_F T_F N_f - \frac{1278281}{864}C_A^2 C_F^2 - \frac{1073621}{288}\zeta_3 C_A^3 C_F\right.\\
&\quad - \frac{447197}{432}C_A C_F^2 T_F N_f - \frac{264467}{384}\zeta_7 C_A^3 C_F - \frac{63769}{72}\zeta_5 C_A^2 C_F T_F N_f \\
&\quad - \frac{42945}{128}C_F^4 - \frac{21391}{243}C_F T_F^3 N_f^3 - \frac{20725}{256}\zeta_6 C_A^3 C_F - \frac{18627}{512}\zeta_4 C_A^3 C_F \\
&\quad - \frac{10185}{2}\zeta_7 C_A C_F^3 - \frac{4697}{8}\zeta_7\frac{d_F^{abcd}d_A^{abcd}}{N_F} - \frac{4275}{64}\zeta_6\frac{d_F^{abcd}d_A^{abcd}}{N_F} - \frac{3889}{2}\zeta_3 C_F^4
\end{aligned}$$

$$- \frac{2263}{3}\zeta_3 C_A^2 C_F^2 - \frac{1263}{16}\zeta_4 C_A^2 C_F^2 - \frac{1225}{3}\zeta_5 C_A C_F^3 - \frac{796}{3}\zeta_3 C_F^2 T_F^2 N_f^2$$

$$- \frac{455}{2}\frac{d_F^{abcd} d_A^{abcd}}{N_F} - \frac{441}{2}\zeta_7 C_A^2 C_F T_F N_f - \frac{351}{4}\zeta_3^2 C_A^2 C_F^2$$

$$- \frac{280}{3}\zeta_3 C_A C_F T_F^2 N_f^2 - \frac{16}{9}\zeta_3 C_F T_F^3 N_f^3 - \frac{9}{16}\zeta_4 C_A^2 C_F T_F N_f$$

$$+ \frac{20}{3}\zeta_5 C_F^3 T_F N_f + \frac{440}{9}\zeta_5 C_A C_F T_F^2 N_f^2 + \frac{613}{16}\zeta_3 \frac{d_F^{abcd} d_A^{abcd}}{N_F}$$

$$+ \frac{779}{6}\zeta_3 C_A C_F^2 T_F N_f + \frac{1669}{8}\zeta_5 C_A^2 C_F^2 + \frac{3339}{64}\zeta_4 \frac{d_F^{abcd} d_A^{abcd}}{N_F}$$

$$+ \frac{3925}{8}\zeta_6 C_A^2 C_F^2 + \frac{4929}{32}\zeta_3^2 \frac{d_F^{abcd} d_A^{abcd}}{N_F} + \frac{10795}{64}\zeta_3^2 C_A^3 C_F$$

$$+ \frac{12415}{32}\zeta_5 \frac{d_F^{abcd} d_A^{abcd}}{N_F} + \frac{21775}{144} C_F^3 T_F N_f + \frac{27895}{16}\zeta_7 C_A^2 C_F^2$$

$$+ \frac{32359}{8}\zeta_3 C_A C_F^3 + \frac{46643}{24}\zeta_3 C_A^2 C_F T_F N_f + \frac{79531}{216} C_F^2 T_F^2 N_f^2$$

$$+ \frac{124783}{108} C_A C_F T_F^2 N_f^2 + \frac{278975}{288} C_A C_F^3 + \frac{2256031}{1152}\zeta_5 C_A^3 C_F$$

$$+ \frac{164005363}{31104} C_A^3 C_F - 1240\zeta_5 N_f \frac{d_F^{abcd} d_F^{abcd}}{N_F} - 970\zeta_5 C_F^4 - 900\zeta_6 C_A C_F^3$$

$$- 384\zeta_3^2 N_f \frac{d_F^{abcd} d_F^{abcd}}{N_F} - 160\zeta_3^2 C_A C_F^2 T_F N_f - 150\zeta_4 C_F^4 - 103\zeta_3 C_F^3 T_F N_f$$

$$- 80\zeta_3^2 C_F^4 - 50\zeta_6 C_A^2 C_F T_F N_f - 24\zeta_4 C_F^3 T_F N_f - 12\zeta_4 C_F^2 T_F^2 N_f^2$$

$$+ 12\zeta_4 C_A C_F T_F^2 N_f^2 + 84\zeta_3^2 C_A^2 C_F T_F N_f + 100\zeta_6 C_A C_F^2 T_F N_f$$

$$+ 118\zeta_5 C_A C_F^2 T_F N_f + 156\zeta_3^2 C_A C_F^3 + 318\zeta_4 C_A C_F^3 + 474 N_f \frac{d_F^{abcd} d_F^{abcd}}{N_F}$$

$$+ 882\zeta_7 C_A C_F^2 T_F N_f + 1124\zeta_3 N_f \frac{d_F^{abcd} d_F^{abcd}}{N_F} + 3528\zeta_7 C_F^4 \bigg] a^4$$

$$+ O(a^5) \tag{5.5}$$

in the Landau gauge. For completeness and to assist with the extraction of the relevant part of the Green's function for lattice matching we have

$$\Sigma^{(2)\,\overline{\text{MS}}}_{\bar\psi \gamma^\mu \psi}(p)\bigg|_{\alpha=0} = \bigg[3C_F^2 + 4C_F T_F N_f - \frac{25}{2} C_A C_F \bigg] a^2$$

$$+ \bigg[\frac{242}{3} C_A C_F^2 + \frac{245}{4}\zeta_3 C_A^2 C_F + \frac{1528}{9} C_A C_F T_F N_f - \frac{19979}{72} C_A^2 C_F$$

$$- \frac{208}{9} C_F T_F^2 N_f^2 - \frac{28}{3} C_F^2 T_F N_f - 24\zeta_3 C_A C_F^2 - 16\zeta_3 C_A C_F T_F N_f$$

$$- 3C_F^3 \bigg] a^3$$

$$+ \bigg[\frac{2039}{8} C_F^4 - \frac{761141}{108} C_A^3 C_F - \frac{32773}{48} C_A C_F^3 - \frac{15830}{9} C_A C_F T_F^2 N_f^2$$

$$- \frac{10375}{16}\zeta_5 C_A^3 C_F - \frac{4514}{3}\zeta_3 C_A^2 C_F T_F N_f - \frac{2186}{9} C_F^2 T_F^2 N_f^2$$

$$- \frac{1113}{4}\zeta_3 \frac{d_F^{abcd} d_A^{abcd}}{N_F} - \frac{1019}{2}\zeta_3 C_A^2 C_F^2 - \frac{367}{3} C_F^3 T_F N_f$$

$$+ \frac{224}{3}\zeta_3 C_A C_F T_F^2 N_f^2 + \frac{855}{4}\zeta_5 \frac{d_F^{abcd} d_A^{abcd}}{N_F} + \frac{4000}{27} C_F T_F^3 N_f^3$$

$$
\begin{aligned}
&+ \frac{7031}{36} C_A C_F^2 T_F N_f + \frac{56764}{9} C_A^2 C_F T_F N_f + \frac{154157}{72} C_A^2 C_F^2 \\
&+ \frac{307915}{96} \zeta_3 C_A^3 C_F - 1705 \zeta_3 C_A C_F^3 - 1280 \zeta_5 C_F^4 - 1130 \zeta_5 C_A^2 C_F^2 \\
&- 480 \zeta_5 C_A C_F^2 T_F N_f - 256 N_f \frac{d_F^{abcd} d_F^{abcd}}{N_F} - 60 \zeta_3 C_A C_F^2 T_F N_f \\
&+ 128 \zeta_3 C_F^3 T_F N_f + 134 \frac{d_F^{abcd} d_A^{abcd}}{N_F} + 192 \zeta_3 C_F^2 T_F^2 N_f^2 \\
&\left. + 490 \zeta_5 C_A^2 C_F T_F N_f + 800 \zeta_3 C_F^4 + 2880 \zeta_5 C_A C_F^3 \right] a^4 \; + \; O(a^5) \; . \; (5.6)
\end{aligned}
$$

We also record

$$
\begin{aligned}
\left. \Sigma_{\bar{\psi}\gamma^\mu\psi}^{(2)\,\mathrm{RI}'}(p) \right|_{\alpha=0} =\ & \left[3 C_F^2 + 4 C_F T_F N_f - \frac{25}{2} C_A C_F \right] a^2 \\
& + \left[\frac{242}{3} C_A C_F^2 + \frac{245}{4} \zeta_3 C_A^2 C_F + \frac{1528}{9} C_A C_F T_F N_f - \frac{19979}{72} C_A^2 C_F \right. \\
& \quad - \frac{208}{9} C_F T_F^2 N_f^2 - \frac{28}{3} C_F^2 T_F N_f - 24 \zeta_3 C_A C_F^2 - 16 \zeta_3 C_A C_F T_F N_f \\
& \quad \left. - 3 C_F^3 \right] a^3 \\
& + \left[\frac{1027}{4} C_F^4 - \frac{761141}{108} C_A^3 C_F - \frac{15830}{9} C_A C_F T_F^2 N_f^2 - \frac{10375}{16} \zeta_5 C_A^3 C_F \right. \\
& \quad - \frac{4514}{3} \zeta_3 C_A^2 C_F T_F N_f - \frac{2164}{3} C_A C_F^3 - \frac{2060}{9} C_F^2 T_F^2 N_f^2 \\
& \quad - \frac{1113}{4} \zeta_3 \frac{d_F^{abcd} d_A^{abcd}}{N_F} - \frac{328}{3} C_F^3 T_F N_f + \frac{224}{3} \zeta_3 C_A C_F T_F^2 N_f^2 \\
& \quad + \frac{855}{4} \zeta_5 \frac{d_F^{abcd} d_A^{abcd}}{N_F} + \frac{995}{9} C_A C_F^2 T_F N_f + \frac{4000}{27} C_F T_F^3 N_f^3 \\
& \quad + \frac{56764}{9} C_A^2 C_F T_F N_f + \frac{81691}{36} C_A^2 C_F^2 + \frac{307915}{96} \zeta_3 C_A^3 C_F \\
& \quad - 1696 \zeta_3 C_A C_F^3 - 1280 \zeta_5 C_F^4 - 1130 \zeta_5 C_A^2 C_F^2 - 547 \zeta_3 C_A^2 C_F^2 \\
& \quad - 480 \zeta_5 C_A C_F^2 T_F N_f - 256 N_f \frac{d_F^{abcd} d_F^{abcd}}{N_F} - 48 \zeta_3 C_A C_F^2 T_F N_f \\
& \quad + 128 \zeta_3 C_F^3 T_F N_f + 134 \frac{d_F^{abcd} d_A^{abcd}}{N_F} + 192 \zeta_3 C_F^2 T_F^2 N_f^2 \\
& \quad \left. + 490 \zeta_5 C_A^2 C_F T_F N_f + 800 \zeta_3 C_F^4 + 2880 \zeta_5 C_A C_F^3 \right] a^4 \; + \; O(a^5) \quad (5.7)
\end{aligned}
$$

to complete the analysis in both schemes. We note that $\left. \Sigma_{\bar{\psi}\gamma^\mu\psi}^{(2)\,\mathrm{RI}'}(p) \right|_{\alpha=0}$ differs from its $\overline{\mathrm{MS}}$ counterpart at four loops. The difference between the four loop coefficients in each expression is related to the product of the coefficient of the two loop Landau gauge terms of $Z_\psi^{\mathrm{RI}'}$ and $\left. \Sigma_{\bar{\psi}\gamma^\mu\psi}^{(1)\,\overline{\mathrm{MS}}}(p) \right|_{\alpha=0}$. Full expressions for $\Sigma_{\bar{\psi}\gamma^\mu\psi}^{(i)}(p)$ in both schemes for an arbitrary linear covariant gauge are included in the associated data file. Finally we record the numerical values are

$$
\begin{aligned}
\left. \Sigma_{\bar{\psi}\gamma^\mu\psi}^{(1)\,\overline{\mathrm{MS}}}(p) \right|_{\alpha=0}^{SU(3)} =\ & 1 + [25.464206 - 2.333333 N_f] a^2 \\
& + [6.460905 N_f^2 - 246.442650 N_f + 1489.980500] a^3 \\
& + [78097.273944 - 18600.055317 N_f + 1123.006264 N_f^2 - 15.027633 N_f^3] a^4 \\
& + O(a^5)
\end{aligned}
$$

$$\Sigma^{(2)\,\overline{\mathrm{MS}}}_{\bar{\psi}\gamma^\mu\psi}(p)\Big|^{SU(3)}_{\alpha=0} = [2.666667N_f - 44.666667]a^2$$

$$+ [292.793438N_f - 7.703704N_f^2 - 2177.073682]a^3$$
$$+ [24.691358N_f^3 - 1674.510402N_f^2 + 29042.096003N_f - 131900.241538]a^4$$
$$+ O(a^5)$$

$$\Sigma^{(2)\,\mathrm{RI}'}_{\bar{\psi}\gamma^\mu\psi}(p)\Big|^{SU(3)}_{\alpha=0} = [2.666667N_f - 44.666667]a^2$$

$$+ [292.793438N_f - 7.703704N_f^2 - 2177.073682]a^3$$
$$+ [24.691358N_f^3 - 1668.288179N_f^2 + 28869.969231N_f - 130762.840335]a^4$$
$$+ O(a^5) \tag{5.8}$$

when the colour group is $SU(3)$. To gauge the effect the four loop term has we have again evaluated the two $\overline{\mathrm{MS}}$ scheme expressions at $\alpha_s = 0.12$ when $N_f = 3$. For $\Sigma^{(1)\,\overline{\mathrm{MS}}}_{\bar{\psi}\gamma^\mu\psi}(p)\Big|^{SU(3)}_{\alpha=0}$ we note that the value at the three successive loop orders are 1.001684, 1.002388 and 1.002654 as there is no one loop term while the respective values are -0.003344, -0.004535 and -0.005027 for $\Sigma^{(2)\,\overline{\mathrm{MS}}}_{\bar{\psi}\gamma^\mu\psi}(p)\Big|^{SU(3)}_{\alpha=0}$. For $\Sigma^{(2)\,\mathrm{RI}'}_{\bar{\psi}\gamma^\mu\psi}(p)\Big|^{SU(3)}_{\alpha=0}$ the two and three loop values are the same as the $\overline{\mathrm{MS}}$ ones while the four loop value is -0.005021. Clearly there is a small discrepancy between the three and four loop $\overline{\mathrm{MS}}$ values for channel 1 which suggests that there is a degree of convergence.

6 Discussion.

We have now extended the renormalization group functions of QCD in the RI' scheme to five loops. In addition the 2-point functions of the three fields are also available to four loops in the same scheme. However the results of more immediate use provided in this article are the determination of the four loop Green's functions where the quark mass operator and separately the vector current are inserted at zero momentum in a quark 2-point function in both the $\overline{\mathrm{MS}}$ and RI' schemes. These are important for the wider and ongoing lattice gauge theory programme of measuring quark masses more accurately. Our observation is that the four loop corrections of these operator Green's functions are not significantly different from their three loop values at a particular reference point. This should in principle allow for a better understanding of errors in extrapolating and matching lattice data to high energy for the exceptional momentum configuration considered here. We recall that the corresponding non-exceptional point symmetric point renormalization was carried out to three loops in [47] in the Landau gauge. While this equated to the loop order achieved in [13], various analyses such as that of [48] used those results to evaluate operator renormalization constants on the lattice. For instance, [48] confirmed the expected behaviour of the renormalization constants over a wide range of momenta down to infrared scales. At a particular point it turned out that the exceptional momentum case began to deviate from expectations. As the three loop RI' perturbative renormalization was used for that it would be interesting to see whether the four loop information of this study improves the behaviour in the infrared and if so then how well does it compare with the symmetric point measurements based on both the two, [13], and three loop Landau gauge data, [47].

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A   Expressions for a general Lie group.

As the renormalization group functions for a general colour group are of interest, for example with respect to the Casimir location and dependence on the number of colours, we record the various anomalous dimensions in the Landau gauge for the RI$'$ scheme. First the expressions for the field anomalous dimensions are

$$
\begin{aligned}
\gamma_A^{\mathrm{RI}'}(a,0) &= \left[\frac{4}{3}T_F N_f - \frac{13}{6}C_A\right]a + \left[4C_F T_F N_f - \frac{3727}{216}C_A^2 - \frac{80}{27}T_F^2 N_f^2 + \frac{452}{27}C_A T_F N_f\right]a^2 \\
&+ \left[\frac{26054}{81}C_A^2 T_F N_f - \frac{2127823}{7776}C_A^3 - \frac{290}{3}C_A T_F^2 N_f^2 - \frac{280}{3}\zeta_3 C_A C_F T_F N_f \right.\\
&\qquad - \frac{188}{3}C_F T_F^2 N_f^2 - \frac{64}{3}\zeta_3 C_A T_F^2 N_f^2 + \frac{98}{3}\zeta_3 C_A^2 T_F N_f + \frac{128}{3}\zeta_3 C_F T_F^2 N_f^2 \\
&\qquad \left.+ \frac{291}{2}C_A C_F T_F N_f + \frac{361}{16}\zeta_3 C_A^3 + \frac{1600}{243}T_F^3 N_f^3 - 2C_F^2 T_F N_f\right]a^3 \\
&+ \left[192\zeta_3 C_A T_F^3 N_f^3 - \frac{3012081353}{559872}C_A^4 - \frac{4608833}{1458}C_A^2 T_F^2 N_f^2 \right.\\
&\qquad - \frac{821905}{243}C_A C_F T_F^2 N_f^2 - \frac{32000}{2187}T_F^4 N_f^4 - \frac{29905}{48}\zeta_5 C_A^3 T_F N_f \\
&\qquad - \frac{21920}{9}\zeta_3 C_A^2 C_F T_F N_f - \frac{10185}{64}\zeta_5 \frac{d_A^{abcd}d_A^{abcd}}{N_A} - \frac{8788}{9}\zeta_3 C_A C_F^2 T_F N_f \\
&\qquad - \frac{7498}{9}\zeta_3 C_A^2 T_F^2 N_f^2 - \frac{4160}{27}C_A C_F^2 T_F N_f - \frac{3712}{9}\zeta_3 C_F T_F^3 N_f^3 \\
&\qquad - \frac{1760}{27}C_F^2 T_F^2 N_f^2 - \frac{989}{12}\zeta_3 \frac{d_A^{abcd}d_A^{abcd}}{N_A} - \frac{512}{3}\zeta_3 N_f^2 \frac{d_F^{abcd}d_F^{abcd}}{N_A} \\
&\qquad - \frac{512}{9}N_f \frac{d_F^{abcd}d_A^{abcd}}{N_A} + \frac{640}{3}\zeta_5 C_A^2 T_F^2 N_f^2 + \frac{659}{144}\frac{d_A^{abcd}d_A^{abcd}}{N_A} + \frac{704}{9}N_f^2 \frac{d_F^{abcd}d_F^{abcd}}{N_A} \\
&\qquad + \frac{1376}{3}\zeta_3 N_f \frac{d_F^{abcd}d_A^{abcd}}{N_A} + \frac{5408}{9}\zeta_3 C_F^2 T_F^2 N_f^2 + \frac{5968}{3}\zeta_3 C_A C_F T_F^2 N_f^2 \\
&\qquad + \frac{8999}{36}\zeta_3 C_A^3 T_F N_f + \frac{134632}{243}C_F T_F^3 N_f^3 + \frac{475205}{1536}\zeta_5 C_A^4 + \frac{716429}{768}\zeta_3 C_A^4 \\
&\qquad + \frac{956764}{2187}C_A T_F^3 N_f^3 + \frac{2130347}{486}C_A^2 C_F T_F N_f + \frac{529941473}{69984}C_A^3 T_F N_f \\
&\qquad - 760\zeta_5 C_A^2 C_F T_F N_f - 640\zeta_5 C_F^2 T_F^2 N_f^2 - 46C_F^3 T_F N_f \\
&\qquad \left.+ 120\zeta_5 N_f \frac{d_F^{abcd}d_A^{abcd}}{N_A} + 320\zeta_5 C_A C_F T_F^2 N_f^2 + 1520\zeta_5 C_A C_F^2 T_F N_f\right]a^4 \\
&+ \left[\frac{28974184619}{995328}\zeta_3 C_A^5 + \frac{4892001875}{110592}\zeta_5 C_A^5 - \frac{5298101107831}{40310784}C_A^5 \right.\\
&\qquad - \frac{18075016135}{157464}C_A^3 T_F^2 N_f^2 - \frac{5939172295}{294912}\zeta_7 C_A^5 + \frac{1091693262941}{5038848}C_A^4 T_F N_f \\
&\qquad - \frac{790040875}{10368}\zeta_5 C_A^4 T_F N_f - \frac{674419927}{62208}\zeta_3 C_A^4 T_F N_f \\
&\qquad - \frac{221064985}{1458}C_A^2 C_F T_F^2 N_f^2 - \frac{199659583}{110592}\zeta_3^2 C_A^5
\end{aligned}
$$

$$- \frac{140495179}{13824}\zeta_3 C_A \frac{d_A^{abcd}d_A^{abcd}}{N_A} - \frac{121750363}{12288}\zeta_7 C_A \frac{d_A^{abcd}d_A^{abcd}}{N_A}$$

$$- \frac{68513455}{1296}\zeta_3 C_A^3 C_F T_F N_f - \frac{32911972}{19683} C_A T_F^4 N_f^4 - \frac{15005047}{648}\zeta_3 C_A^3 T_F^2 N_f^2$$

$$- \frac{4571425}{768}\zeta_5 C_A \frac{d_A^{abcd}d_A^{abcd}}{N_A} - \frac{3031633}{54}\zeta_3 C_A^2 C_F^2 T_F N_f - \frac{963825}{16}\zeta_5 C_A^3 C_F T_F N_f$$

$$- \frac{951901}{324} C_A C_F^2 T_F^2 N_f^2 - \frac{677390}{81} C_A^2 C_F^2 T_F N_f - \frac{591424}{27}\zeta_3 C_A C_F T_F^3 N_f^3$$

$$- \frac{453040}{9}\zeta_5 C_A C_F^2 T_F^2 N_f^2 - \frac{442757}{36}\zeta_7 C_A^3 C_F T_F N_f$$

$$- \frac{429088}{27}\zeta_3 T_F N_f^2 \frac{d_F^{abcd}d_F^{abcd}}{N_A} - \frac{385745}{72}\zeta_3^2 T_F N_f \frac{d_A^{abcd}d_A^{abcd}}{N_A}$$

$$- \frac{368000}{27}\zeta_5 C_A C_F T_F^3 N_f^3 - \frac{338560}{243}\zeta_3 C_A T_F^4 N_f^4$$

$$- \frac{258632}{27}\zeta_3 C_A N_f^2 \frac{d_F^{abcd}d_F^{abcd}}{N_A} - \frac{227363}{9}\zeta_3^2 C_A N_f \frac{d_F^{abcd}d_A^{abcd}}{N_A}$$

$$- \frac{121352}{27} C_F T_F^4 N_f^4 - \frac{100040}{27} C_A N_f \frac{d_F^{abcd}d_A^{abcd}}{N_A} - \frac{88480}{3}\zeta_7 C_A C_F^3 T_F N_f$$

$$- \frac{58339}{576}\zeta_3^2 C_A^4 T_F N_f - \frac{52960}{9}\zeta_5 T_F N_f^2 \frac{d_F^{abcd}d_A^{abcd}}{N_A} - \frac{40480}{9}\zeta_5 C_A^2 T_F^3 N_f^3$$

$$- \frac{35840}{3}\zeta_7 C_A C_F^2 T_F^2 N_f^2 - \frac{20224}{9}\zeta_3^2 C_A N_f^2 \frac{d_F^{abcd}d_F^{abcd}}{N_A}$$

$$- \frac{17344}{9} T_F N_f^3 \frac{d_F^{abcd}d_F^{abcd}}{N_A} - \frac{8069}{2} C_A C_F^3 T_F N_f - \frac{7928}{9}\zeta_3^2 C_A^3 T_F^2 N_f^2$$

$$- \frac{7877}{72} T_F N_f \frac{d_A^{abcd}d_A^{abcd}}{N_A} - \frac{6400}{3}\zeta_5 T_F N_f^3 \frac{d_F^{abcd}d_F^{abcd}}{N_A} - \frac{5120}{3}\zeta_3 C_F N_f^2 \frac{d_F^{abcd}d_F^{abcd}}{N_A}$$

$$- \frac{4160}{3} C_F N_f^2 \frac{d_F^{abcd}d_F^{abcd}}{N_A} - \frac{3580}{3}\zeta_5 C_F N_f \frac{d_F^{abcd}d_A^{abcd}}{N_A} - \frac{1588}{3}\zeta_3 C_A C_F^3 T_F N_f$$

$$- \frac{1544}{3}\zeta_3 C_F N_f \frac{d_F^{abcd}d_A^{abcd}}{N_A} - \frac{1274}{3}\zeta_7 C_A N_f \frac{d_F^{abcd}d_A^{abcd}}{N_A} - \frac{832}{3}\zeta_4 T_F N_f^2 \frac{d_F^{abcd}d_A^{abcd}}{N_A}$$

$$- \frac{707}{18}\zeta_4 C_A^4 T_F N_f - \frac{692}{9}\zeta_4 C_A^2 C_F T_F^2 N_f^2 - \frac{572}{3}\zeta_4 C_A \frac{d_A^{abcd}d_A^{abcd}}{N_A}$$

$$- \frac{416}{3}\zeta_4 C_A N_f^2 \frac{d_F^{abcd}d_F^{abcd}}{N_A} - \frac{392}{3}\zeta_7 T_F N_f^2 \frac{d_F^{abcd}d_A^{abcd}}{N_A} - \frac{352}{9}\zeta_4 C_F^2 T_F^3 N_f^3$$

$$- \frac{286}{9}\zeta_4 C_A^2 C_F^2 T_F N_f + \frac{22}{9}\zeta_4 C_A^3 T_F^2 N_f^2 + \frac{112}{9}\zeta_4 C_A^2 T_F^3 N_f^3 + \frac{143}{36}\zeta_4 C_A^5$$

$$+ \frac{224}{9}\zeta_4 C_A C_F T_F^3 N_f^3 + \frac{256}{3}\zeta_4 T_F N_f^3 \frac{d_F^{abcd}d_F^{abcd}}{N_A} + \frac{352}{3}\zeta_4 T_F N_f \frac{d_A^{abcd}d_A^{abcd}}{N_A}$$

$$+ \frac{533}{9}\zeta_4 C_A^3 C_F T_F N_f + \frac{748}{9}\zeta_4 C_A C_F^2 T_F^2 N_f^2 + \frac{944}{3} C_F N_f \frac{d_F^{abcd}d_A^{abcd}}{N_A}$$

$$+ \frac{1352}{3}\zeta_4 C_A N_f \frac{d_F^{abcd}d_A^{abcd}}{N_A} + \frac{4157}{6} C_F^4 T_F N_f + \frac{6784}{3}\zeta_3^2 C_A^2 C_F T_F^2 N_f^2$$

$$+ \frac{8192}{9}\zeta_3^2 T_F N_f^3 \frac{d_F^{abcd}d_F^{abcd}}{N_A} + \frac{10240}{81}\zeta_5 C_A T_F^4 N_f^4 + \frac{10526}{9} C_F^3 T_F^2 N_f^2$$

$$+ \frac{12800}{3}\zeta_5 C_F N_f^2 \frac{d_F^{abcd}d_F^{abcd}}{N_A} + \frac{12896}{9} T_F N_f^2 \frac{d_F^{abcd}d_A^{abcd}}{N_A} + \frac{15872}{27}\zeta_3^2 C_A^2 T_F^3 N_f^3$$

$$
\begin{aligned}
&+ \frac{17665}{3}\zeta_3^2 C_A^3 C_F T_F N_f + \frac{20480}{27}\zeta_5 C_F T_F^4 N_f^4 + \frac{35680}{9}\zeta_5 C_A N_f^2 \frac{d_F^{abcd} d_F^{abcd}}{N_A} \\
&+ \frac{35840}{3}\zeta_7 C_F^3 T_F^2 N_f^2 + \frac{44800}{9}\zeta_7 C_A^2 C_F T_F^2 N_f^2 + \frac{82240}{27}\zeta_3 T_F N_f^3 \frac{d_F^{abcd} d_F^{abcd}}{N_A} \\
&+ \frac{86464}{9}\zeta_3^2 T_F N_f^2 \frac{d_F^{abcd} d_A^{abcd}}{N_A} + \frac{88480}{3}\zeta_7 C_A^2 C_F^2 T_F N_f \\
&+ \frac{99775}{9}\zeta_5 C_A N_f \frac{d_F^{abcd} d_A^{abcd}}{N_A} + \frac{102400}{9}\zeta_5 C_F^2 T_F^3 N_f^3 + \frac{131467}{24}\zeta_7 T_F N_f \frac{d_A^{abcd} d_A^{abcd}}{N_A} \\
&+ \frac{163784}{27} C_A N_f^2 \frac{d_F^{abcd} d_F^{abcd}}{N_A} + \frac{223186}{81} C_F^2 T_F^3 N_f^3 + \frac{224195}{144}\zeta_5 T_F N_f \frac{d_A^{abcd} d_A^{abcd}}{N_A} \\
&+ \frac{237440}{81}\zeta_3 C_F T_F^4 N_f^4 + \frac{483430}{9}\zeta_5 C_A^2 C_F^2 T_F N_f + \frac{522002}{9}\zeta_3 C_A^2 C_F T_F^2 N_f^2 \\
&+ \frac{640000}{19683} T_F^5 N_f^5 + \frac{745015}{3456} C_A \frac{d_A^{abcd} d_A^{abcd}}{N_A} + \frac{1098749}{27}\zeta_3 C_A N_f \frac{d_F^{abcd} d_A^{abcd}}{N_A} \\
&+ \frac{1488560}{27}\zeta_5 C_A^2 C_F T_F^2 N_f^2 + \frac{1615204}{27}\zeta_3 C_A C_F^2 T_F^2 N_f^2 + \frac{2995112}{243}\zeta_3 C_A^2 T_F^3 N_f^3 \\
&+ \frac{3683755}{108}\zeta_5 C_A^3 T_F^2 N_f^2 + \frac{3835049}{864}\zeta_3 T_F N_f \frac{d_A^{abcd} d_A^{abcd}}{N_A} + \frac{35251481}{729} C_A C_F T_F^3 N_f^3 \\
&+ \frac{46576943}{1152}\zeta_7 C_A^4 T_F N_f + \frac{66141673}{4608}\zeta_3^2 C_A \frac{d_A^{abcd} d_A^{abcd}}{N_A} + \frac{925396565}{39366} C_A^2 T_F^3 N_f^3 \\
&+ \frac{1619234591}{11664} C_A^3 C_F T_F N_f - 14400\zeta_3 C_F^2 T_F^3 N_f^3 - 13920\zeta_5 C_F^3 T_F^2 N_f^2 \\
&- 12628\zeta_7 C_A^3 T_F^2 N_f^2 - 4864\zeta_3^2 C_A C_F^2 T_F^2 N_f^2 - 2048\zeta_3^2 C_A C_F T_F^3 N_f^3 \\
&- 476\zeta_7 C_F N_f \frac{d_F^{abcd} d_A^{abcd}}{N_A} - 288\zeta_3^2 C_F N_f \frac{d_F^{abcd} d_A^{abcd}}{N_A} + 128\zeta_3 C_F^4 T_F N_f \\
&+ 1280\zeta_3 C_F^3 T_F^2 N_f^2 + 2048\zeta_3^2 C_F^2 T_F^3 N_f^3 + 28300\zeta_5 C_A C_F^3 T_F N_f \Big] a^5 \\
&+ O(a^6)
\end{aligned}
\tag{A.1}
$$

$$
\begin{aligned}
\gamma_c^{\mathrm{RI}'}(a,0) =\ & -\frac{3}{4} C_A a + \left[\frac{13}{6} C_A T_F N_f - \frac{271}{48} C_A^2\right] a^2 \\
&+ \left[\frac{13}{2}\zeta_3 C_A^2 T_F N_f - \frac{157303}{1728} C_A^3 - \frac{250}{27} C_A T_F^2 N_f^2 + \frac{61}{4} C_A C_F T_F N_f + \frac{211}{32}\zeta_3 C_A^3 \right. \\
&\quad \left. + \frac{13445}{216} C_A^2 T_F N_f - 12\zeta_3 C_A C_F T_F N_f\right] a^3 \\
&+ \left[\frac{2825}{48}\zeta_3 C_A^3 T_F N_f - \frac{219440027}{124416} C_A^4 - \frac{82325}{162} C_A^2 T_F^2 N_f^2 - \frac{10295}{54} C_A C_F T_F^2 N_f^2 \right. \\
&\quad - \frac{675}{2}\zeta_3 C_A^2 C_F T_F N_f - \frac{609}{8}\zeta_3 \frac{d_A^{abcd} d_A^{abcd}}{N_A} - \frac{505}{8}\zeta_5 C_A^3 T_F N_f - \frac{295}{12} C_A C_F^2 T_F N_f \\
&\quad - \frac{158}{3}\zeta_3 C_A^2 T_F^2 N_f^2 + \frac{69}{32}\frac{d_A^{abcd} d_A^{abcd}}{N_A} + \frac{10185}{128}\zeta_5 \frac{d_A^{abcd} d_A^{abcd}}{N_A} + \frac{10820}{243} C_A T_F^3 N_f^3 \\
&\quad + \frac{70545}{1024}\zeta_5 C_A^4 + \frac{123715}{216} C_A^2 C_F T_F N_f + \frac{365387}{1536}\zeta_3 C_A^4 + \frac{9013883}{5184} C_A^3 T_F N_f \\
&\quad - 74\zeta_3 C_A C_F^2 T_F N_f - 60\zeta_5 N_f \frac{d_F^{abcd} d_A^{abcd}}{N_A} - 60\zeta_5 C_A^2 C_F T_F N_f \\
&\quad \left. + 48\zeta_3 N_f \frac{d_F^{abcd} d_A^{abcd}}{N_A} + 120\zeta_5 C_A C_F^2 T_F N_f + 128\zeta_3 C_A C_F T_F^2 N_f^2\right] a^4
\end{aligned}
$$

$$
\begin{aligned}
+ \Bigg[&\frac{60381388927}{1119744} C_A^4 T_F N_f - \frac{374313393145}{8957952} C_A^5 - \frac{695857939}{196608} \zeta_7 C_A^5 \\
& - \frac{68955821}{2916} C_A^3 T_F^2 N_f^2 - \frac{17068475}{2304} \zeta_5 C_A^4 T_F N_f - \frac{8364239}{648} C_A^2 C_F T_F^2 N_f^2 \\
& - \frac{7523471}{1024} \zeta_3 C_A \frac{d_A^{abcd} d_A^{abcd}}{N_A} - \frac{3068531}{8192} \zeta_3^2 C_A^5 - \frac{2164553}{1536} \zeta_3 C_A^4 T_F N_f \\
& - \frac{596896}{2187} C_A T_F^4 N_f^4 - \frac{448897}{1024} \zeta_3^2 C_A \frac{d_A^{abcd} d_A^{abcd}}{N_A} - \frac{328191}{32} \zeta_3 C_A^3 C_F T_F N_f \\
& - \frac{317605}{144} \zeta_3 C_A^3 T_F^2 N_f^2 - \frac{172253}{144} C_A^2 C_F^2 T_F N_f - \frac{164135}{96} \zeta_5 T_F N_f \frac{d_A^{abcd} d_A^{abcd}}{N_A} \\
& - \frac{77889}{64} \zeta_7 T_F N_f \frac{d_A^{abcd} d_A^{abcd}}{N_A} - \frac{42109}{384} \zeta_3^2 C_A^4 T_F N_f - \frac{40805}{6} \zeta_5 C_A N_f \frac{d_F^{abcd} d_A^{abcd}}{N_A} \\
& - \frac{33635}{8} \zeta_5 C_A^3 C_F T_F N_f - \frac{16681}{24} \zeta_7 C_A^3 C_F T_F N_f - \frac{12808}{3} \zeta_3 C_A^2 C_F^2 T_F N_f \\
& - \frac{6560}{3} \zeta_5 C_A C_F^2 T_F^2 N_f^2 - \frac{5887}{24} C_A C_F^2 T_F^2 N_f^2 - \frac{5504}{3} \zeta_3 T_F N_f^2 \frac{d_F^{abcd} d_A^{abcd}}{N_A} \\
& - \frac{3641}{48} T_F N_f \frac{d_A^{abcd} d_A^{abcd}}{N_A} - \frac{1333}{2} \zeta_3^2 C_A N_f \frac{d_F^{abcd} d_A^{abcd}}{N_A} - \frac{1292}{9} C_A N_f \frac{d_F^{abcd} d_A^{abcd}}{N_A} \\
& - \frac{680}{9} \zeta_5 C_A^2 T_F^3 N_f^3 - \frac{422}{3} \zeta_3^2 C_A^3 T_F^2 N_f^2 - \frac{320}{3} \zeta_5 C_A C_F T_F^3 N_f^3 - \frac{51}{4} \zeta_4 C_A^4 T_F N_f \\
& + \frac{8}{3} C_F N_f \frac{d_F^{abcd} d_A^{abcd}}{N_A} + \frac{11}{8} \zeta_4 C_A^5 + \frac{41}{2} \zeta_4 C_A^3 C_F T_F N_f + \frac{1093}{2} \zeta_3^2 C_A^3 C_F T_F N_f \\
& + \frac{1988}{9} C_A N_f^2 \frac{d_F^{abcd} d_F^{abcd}}{N_A} + \frac{3655}{16} \zeta_3^2 T_F N_f \frac{d_A^{abcd} d_A^{abcd}}{N_A} + \frac{6320}{3} \zeta_5 T_F N_f^2 \frac{d_F^{abcd} d_A^{abcd}}{N_A} \\
& + \frac{13504}{27} \zeta_3 C_A^2 T_F^3 N_f^3 + \frac{14605}{3} \zeta_5 C_A^2 C_F^2 T_F N_f + \frac{40651}{6} \zeta_3 C_A^2 C_F T_F^2 N_f^2 \\
& + \frac{42707}{6} \zeta_3 C_A N_f \frac{d_F^{abcd} d_A^{abcd}}{N_A} + \frac{123395}{72} \zeta_5 C_A^3 T_F^2 N_f^2 + \frac{147841}{768} C_A \frac{d_A^{abcd} d_A^{abcd}}{N_A} \\
& + \frac{152269}{64} \zeta_3 T_F N_f \frac{d_A^{abcd} d_A^{abcd}}{N_A} + \frac{313051}{162} C_A C_F T_F^3 N_f^3 + \frac{2230991}{768} \zeta_7 C_A^4 T_F N_f \\
& + \frac{6298537}{8192} \zeta_7 C_A \frac{d_A^{abcd} d_A^{abcd}}{N_A} + \frac{11508905}{1536} \zeta_5 C_A \frac{d_A^{abcd} d_A^{abcd}}{N_A} \\
& + \frac{16814735}{864} C_A^3 C_F T_F N_f + \frac{37229609}{8748} C_A^2 T_F^3 N_f^3 + \frac{189347125}{24576} \zeta_5 C_A^5 \\
& + \frac{1765579181}{221184} \zeta_3 C_A^5 - 1680 \zeta_7 C_A C_F^3 T_F N_f - 1240 \zeta_3 C_A C_F T_F^3 N_f^3 \\
& - 470 \zeta_5 C_F N_f \frac{d_F^{abcd} d_A^{abcd}}{N_A} - 384 \zeta_3^2 C_A C_F^2 T_F^2 N_f^2 - 316 \zeta_3 C_A N_f^2 \frac{d_F^{abcd} d_F^{abcd}}{N_A} \\
& - 294 \zeta_7 C_A^3 T_F^2 N_f^2 - 128 \zeta_3^2 C_A N_f^2 \frac{d_F^{abcd} d_F^{abcd}}{N_A} - 66 \zeta_4 C_A \frac{d_A^{abcd} d_A^{abcd}}{N_A} \\
& - 48 \zeta_4 C_A N_f^2 \frac{d_F^{abcd} d_F^{abcd}}{N_A} - 25 C_A C_F^3 T_F N_f - 14 \zeta_4 C_A^2 C_F T_F^2 N_f^2 \\
& - 11 \zeta_4 C_A^2 C_F^2 T_F N_f - 7 \zeta_4 C_A^3 T_F^2 N_f^2 + 22 \zeta_4 C_A C_F^2 T_F^2 N_f^2 \\
& + 32 \zeta_3^2 T_F N_f^2 \frac{d_F^{abcd} d_A^{abcd}}{N_A} + 44 \zeta_3 C_F N_f \frac{d_F^{abcd} d_A^{abcd}}{N_A} + 78 \zeta_3 C_A C_F^3 T_F N_f \\
& + 144 \zeta_3^2 C_F N_f \frac{d_F^{abcd} d_A^{abcd}}{N_A} + 156 \zeta_4 C_A N_f \frac{d_F^{abcd} d_A^{abcd}}{N_A} + 238 \zeta_7 C_F N_f \frac{d_F^{abcd} d_A^{abcd}}{N_A}
\end{aligned}
$$

$$+ 240\zeta_5 C_A N_f^2 \frac{d_F^{abcd} d_F^{abcd}}{N_A} + 392\zeta_7 C_A N_f \frac{d_F^{abcd} d_A^{abcd}}{N_A} + 416\zeta_3^2 C_A^2 C_F T_F^2 N_f^2$$

$$+ 1470\zeta_5 C_A C_F^3 T_F N_f + 1680\zeta_7 C_A^2 C_F^2 T_F N_f + 1880\zeta_5 C_A^2 C_F T_F^2 N_f^2$$

$$+ 2448\zeta_3 C_A C_F^2 T_F^2 N_f^2 \Big] a^5 \ + \ O(a^6) \tag{A.2}$$

and

$$
\begin{aligned}
\gamma_\psi^{\mathrm{RI}'}(a,0) \ = \ & \left[\frac{25}{4} C_A C_F - \frac{3}{2} C_F^2 - 2 C_F T_F N_f \right] a^2 \\
& + \Big[8\zeta_3 C_A C_F T_F N_f + 12\zeta_3 C_A C_F^2 - \frac{764}{9} C_A C_F T_F N_f - \frac{245}{8}\zeta_3 C_A^2 C_F - \frac{121}{3} C_A C_F^2 \\
& \quad + \frac{3}{2} C_F^3 + \frac{14}{3} C_F^2 T_F N_f + \frac{104}{9} C_F T_F^2 N_f^2 + \frac{19979}{144} C_A^2 C_F \Big] a^3 \\
& + \Big[848\zeta_3 C_A C_F^3 - \frac{307915}{192}\zeta_3 C_A^3 C_F - \frac{81691}{72} C_A^2 C_F^2 - \frac{28382}{9} C_A^2 C_F T_F N_f \\
& \quad - \frac{2000}{27} C_F T_F^3 N_f^3 - \frac{1027}{8} C_F^4 - \frac{995}{18} C_A C_F^2 T_F N_f - \frac{855}{8}\zeta_5 \frac{d_F^{abcd} d_A^{abcd}}{N_F} \\
& \quad - \frac{112}{3}\zeta_3 C_A C_F T_F^2 N_f^2 + \frac{164}{3} C_F^3 T_F N_f + \frac{547}{2}\zeta_3 C_A^2 C_F^2 + \frac{1030}{9} C_F^2 T_F^2 N_f^2 \\
& \quad + \frac{1082}{3} C_A C_F^3 + \frac{1113}{8}\zeta_3 \frac{d_F^{abcd} d_A^{abcd}}{N_F} + \frac{2257}{3}\zeta_3 C_A^2 C_F T_F N_f + \frac{7915}{9} C_A C_F T_F^2 N_f^2 \\
& \quad + \frac{10375}{32}\zeta_5 C_A^3 C_F + \frac{761141}{216} C_A^3 C_F - 1440\zeta_5 C_A C_F^3 - 400\zeta_3 C_F^4 \\
& \quad - 245\zeta_5 C_A^2 C_F T_F N_f - 96\zeta_3 C_F^2 T_F^2 N_f^2 - 67 \frac{d_F^{abcd} d_A^{abcd}}{N_F} - 64\zeta_3 C_F^3 T_F N_f \\
& \quad + 24\zeta_3 C_A C_F^2 T_F N_f + 128 N_f \frac{d_F^{abcd} d_F^{abcd}}{N_F} + 240\zeta_5 C_A C_F^2 T_F N_f + 565\zeta_5 C_A^2 C_F^2 \\
& \quad + 640\zeta_5 C_F^4 \Big] a^4 \\
& + \Big[\frac{17300466253}{186624} C_A^4 C_F - \frac{2511034225}{23328} C_A^3 C_F T_F N_f - \frac{685190527}{36864}\zeta_7 C_A^4 C_F \\
& \quad - \frac{135324833}{3888} C_A^3 C_F^2 - \frac{121549691}{2304}\zeta_3 C_A^4 C_F - \frac{5756552}{729} C_A C_F T_F^3 N_f^3 \\
& \quad - \frac{3366343}{432} C_A^2 C_F^2 T_F N_f - \frac{2984107}{16}\zeta_7 C_A^2 C_F^3 - \frac{2272097}{48}\zeta_3 C_A^3 C_F^2 \\
& \quad - \frac{1669235}{96}\zeta_3 C_A \frac{d_F^{abcd} d_A^{abcd}}{N_F} - \frac{1458845}{384}\zeta_5 C_F \frac{d_A^{abcd} d_A^{abcd}}{N_A} - \frac{787558}{9}\zeta_3 C_A C_F^4 \\
& \quad - \frac{781985}{36}\zeta_5 C_A^2 C_F^3 - \frac{654439}{144} C_A \frac{d_F^{abcd} d_A^{abcd}}{N_F} - \frac{543935}{24}\zeta_5 C_A^3 C_F T_F N_f \\
& \quad - \frac{536720}{243} C_F^2 T_F^3 N_f^3 - \frac{377000}{9}\zeta_5 C_A C_F^4 - \frac{238324}{9}\zeta_3 C_A C_F^3 T_F N_f \\
& \quad - \frac{193093}{18} C_A C_F^4 - \frac{125447}{8}\zeta_7 C_F \frac{d_F^{abcd} d_A^{abcd}}{N_F} - \frac{110513}{24}\zeta_3^2 C_A^3 C_F^2 \\
& \quad - \frac{99622}{9}\zeta_3 C_A^2 C_F T_F^2 N_f^2 - \frac{91085}{9}\zeta_5 C_A^2 C_F^2 T_F N_f - \frac{69727}{18} C_A C_F^3 T_F N_f \\
& \quad - \frac{51928}{9}\zeta_3 C_A C_F^2 T_F^2 N_f^2 - \frac{41909}{16}\zeta_7 C_A^3 C_F T_F N_f - \frac{30112}{9} T_F N_f^2 \frac{d_F^{abcd} d_F^{abcd}}{N_F} \\
& \quad - \frac{21080}{3}\zeta_5 T_F N_f \frac{d_F^{abcd} d_A^{abcd}}{N_F} - \frac{20234}{27} C_F^3 T_F^2 N_f^2 - \frac{9206}{3}\zeta_3^2 C_A^2 C_F^2 T_F N_f
\end{aligned}
$$

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\begin{aligned}
& - \frac{7040}{27}\zeta_5 C_A C_F T_F^3 N_f^3 - \frac{5984}{3}C_F N_f \frac{d_F^{abcd}d_F^{abcd}}{N_F} - \frac{5600}{3}\zeta_5 C_A C_F^2 T_F^2 N_f^2 \\
& - \frac{1985}{24}C_F \frac{d_A^{abcd}d_A^{abcd}}{N_A} - \frac{1472}{3}\zeta_3^2 C_A^2 C_F T_F^2 N_f^2 - \frac{320}{9}\zeta_5 C_F^4 T_F N_f \\
& + \frac{113}{6}C_F \frac{d_F^{abcd}d_A^{abcd}}{N_F} + \frac{1911}{2}\zeta_7 C_A^2 C_F T_F^2 N_f^2 + \frac{2560}{9}\zeta_5 C_F^3 T_F^2 N_f^2 \\
& + \frac{2816}{3}\zeta_3^2 C_A C_F^2 T_F^2 N_f^2 + \frac{3577}{64}\zeta_3^2 C_F \frac{d_A^{abcd}d_A^{abcd}}{N_A} + \frac{3584}{9}\zeta_3 C_A C_F T_F^3 N_f^3 \\
& + \frac{3616}{9}\zeta_3 C_F^3 T_F^2 N_f^2 + \frac{4041}{4}\zeta_3^2 C_A^3 C_F T_F N_f + \frac{4977}{8}C_F^5 + \frac{13568}{9}\zeta_3 C_F^2 T_F^3 N_f^3 \\
& + \frac{17120}{9}T_F N_f \frac{d_F^{abcd}d_A^{abcd}}{N_F} + \frac{21760}{3}\zeta_5 T_F N_f^2 \frac{d_F^{abcd}d_F^{abcd}}{N_F} + \frac{22673}{2}\zeta_3^2 C_A^2 C_F^3 \\
& + \frac{23632}{3}\zeta_7 C_A N_f \frac{d_F^{abcd}d_F^{abcd}}{N_F} + \frac{46165}{6}\zeta_7 T_F N_f \frac{d_F^{abcd}d_A^{abcd}}{N_F} + \frac{48101}{18}C_F^4 T_F N_f \\
& + \frac{51943}{8}\zeta_3^2 C_A \frac{d_F^{abcd}d_A^{abcd}}{N_F} + \frac{52550}{9}\zeta_5 C_A^2 C_F T_F^2 N_f^2 + \frac{53116}{3}\zeta_7 C_A^2 C_F^2 T_F N_f \\
& + \frac{76432}{3}\zeta_3 C_A N_f \frac{d_F^{abcd}d_F^{abcd}}{N_F} + \frac{96032}{9}C_A N_f \frac{d_F^{abcd}d_F^{abcd}}{N_F} + \frac{115832}{9}\zeta_3 C_F^4 T_F N_f \\
& + \frac{135731}{192}\zeta_3 C_F \frac{d_A^{abcd}d_A^{abcd}}{N_A} + \frac{153221}{24}\zeta_3 C_A^2 C_F^2 T_F N_f + \frac{198800}{9}\zeta_5 C_A C_F^3 T_F N_f \\
& + \frac{346240}{729}C_F T_F^4 N_f^4 + \frac{781753}{192}\zeta_7 C_F \frac{d_A^{abcd}d_A^{abcd}}{N_A} + \frac{1020625}{384}\zeta_3^2 C_A^4 C_F \\
& + \frac{1097000}{81}C_A C_F^2 T_F^2 N_f^2 + \frac{1155685}{96}\zeta_5 C_A \frac{d_F^{abcd}d_A^{abcd}}{N_F} + \frac{2432579}{18}\zeta_3 C_A^2 C_F^3 \\
& + \frac{2435545}{768}\zeta_7 C_A \frac{d_F^{abcd}d_A^{abcd}}{N_F} + \frac{5525989}{216}C_A^2 C_F^3 + \frac{6642965}{288}\zeta_5 C_A^3 C_F^2 \\
& + \frac{13984033}{192}\zeta_7 C_A^3 C_F^2 + \frac{14900669}{288}\zeta_3 C_A^3 C_F T_F N_f + \frac{21901003}{486}C_A^2 C_F T_F^2 N_f^2 \\
& + \frac{686761615}{27648}\zeta_5 C_A^4 C_F - 47628\zeta_7 C_F^5 - 34400\zeta_5 C_A N_f \frac{d_F^{abcd}d_F^{abcd}}{N_F} \\
& - 18816\zeta_7 C_F^4 T_F N_f - 12096\zeta_3 C_F N_f \frac{d_F^{abcd}d_F^{abcd}}{N_F} - 9384\zeta_3^2 C_A C_F^4 \\
& - 8680\zeta_7 C_F N_f \frac{d_F^{abcd}d_F^{abcd}}{N_F} - 7952\zeta_3^2 C_A N_f \frac{d_F^{abcd}d_F^{abcd}}{N_F} - 5760\zeta_3 T_F N_f^2 \frac{d_F^{abcd}d_F^{abcd}}{N_F} \\
& - 4884\zeta_3^2 C_F \frac{d_F^{abcd}d_A^{abcd}}{N_F} - 4704\zeta_7 C_A C_F^2 T_F^2 N_f^2 - 1027\zeta_3 T_F N_f \frac{d_F^{abcd}d_A^{abcd}}{N_F} \\
& - 896\zeta_3^2 C_A C_F^3 T_F N_f - 776\zeta_3^2 T_F N_f \frac{d_F^{abcd}d_A^{abcd}}{N_F} + 768\zeta_3^2 C_F^4 T_F N_f \\
& + 1015\zeta_5 C_F \frac{d_F^{abcd}d_A^{abcd}}{N_F} + 2048\zeta_3^2 T_F N_f^2 \frac{d_F^{abcd}d_F^{abcd}}{N_F} + 2496\zeta_3^2 C_F^5 \\
& + 3648\zeta_3^2 C_F N_f \frac{d_F^{abcd}d_F^{abcd}}{N_F} + 13216\zeta_7 C_A C_F^3 T_F N_f + 16000\zeta_3 C_F^5 \\
& + 17554\zeta_3 C_F \frac{d_F^{abcd}d_A^{abcd}}{N_F} + 18080\zeta_5 C_F N_f \frac{d_F^{abcd}d_F^{abcd}}{N_F} + 22600\zeta_5 C_F^5 \\
& + 175721\zeta_7 C_A C_F^4 \Big] a^5 \; + \; O(a^6)
\end{aligned}
\tag{A.3}
$$

where N_A is the dimension of the adjoint representation. Similarly the quark mass anomalous dimension is

$$
\begin{aligned}
\gamma_m^{\mathrm{RI}'}(a,0) \;=\; & -3C_F a + \left[\frac{26}{3} C_F T_F N_f - \frac{185}{6} C_A C_F - \frac{3}{2} C_F^2 \right] a^2 \\
& + \left[\frac{7870}{27} C_A C_F T_F N_f - \frac{29357}{54} C_A^2 C_F - \frac{856}{27} C_F T_F^2 N_f^2 - \frac{129}{2} C_F^3 - 88\zeta_3 C_A C_F^2 \right. \\
& \left. - 16\zeta_3 C_F^2 T_F N_f - 9 C_A C_F^2 + 77 C_F^2 T_F N_f + 132\zeta_3 C_A^2 C_F \right] a^3 \\
& + \left[\frac{1261}{8} C_F^4 - \frac{46284559}{3888} C_A^3 C_F - \frac{172912}{81} C_A C_F T_F^2 N_f^2 - \frac{23104}{27} C_F^2 T_F^2 N_f^2 \right. \\
& - \frac{13102}{3} C_A C_F^3 - \frac{10987}{6} \zeta_3 C_A^2 C_F T_F N_f - \frac{8566}{3} \zeta_3 C_A^2 C_F^2 + \frac{224}{3} \zeta_3 C_A C_F T_F^2 N_f^2 \\
& + \frac{520}{3} \zeta_3 C_A C_F^2 T_F N_f + \frac{992}{3} \zeta_3 C_F^2 T_F^2 N_f^2 + \frac{3760}{3} C_F^3 T_F N_f + \frac{32048}{243} C_F T_F^3 N_f^3 \\
& + \frac{95881}{72} C_A^2 C_F^2 + \frac{120385}{24} \zeta_3 C_A^3 C_F + \frac{145717}{54} C_A C_F^2 T_F N_f \\
& + \frac{3050747}{324} C_A^2 C_F T_F N_f - 496\zeta_3 C_F^3 T_F N_f - 240\zeta_3 \frac{d_F^{abcd} d_A^{abcd}}{N_F} - 220\zeta_5 C_A^2 C_F^2 \\
& - 220\zeta_5 C_A^3 C_F - 206\zeta_3 C_A C_F^3 - 160\zeta_5 C_A C_F^2 T_F N_f - 160\zeta_5 C_A^2 C_F T_F N_f \\
& - 64 N_f \frac{d_F^{abcd} d_F^{abcd}}{N_F} + 32 \frac{d_F^{abcd} d_A^{abcd}}{N_F} + 336\zeta_3 C_F^4 + 480\zeta_3 N_f \frac{d_F^{abcd} d_F^{abcd}}{N_F} \\
& \left. + 1320\zeta_5 C_A C_F^3 \right] a^4 \\
& + \left[\frac{11}{2} \zeta_4 C_A^4 C_F - \frac{11072937943}{34992} C_A^4 C_F - \frac{84487076}{729} C_A^2 C_F T_F^2 N_f^2 \right. \\
& - \frac{53524919}{216} C_A^2 C_F^3 - \frac{1523584}{2187} C_F T_F^4 N_f^4 - \frac{1445111}{27} \zeta_3 C_A^3 C_F^2 \\
& - \frac{1441292}{27} C_A C_F^2 T_F^2 N_f^2 - \frac{1186031}{8} \zeta_7 C_A^3 C_F^2 - \frac{1151695}{54} C_F^3 T_F^2 N_f^2 \\
& - \frac{655760}{9} \zeta_3 C_A^2 C_F^3 - \frac{627715}{18} \zeta_5 C_A^3 C_F^2 - \frac{469717}{4} \zeta_3 C_A^3 C_F T_F N_f \\
& - \frac{256531}{12} \zeta_3^2 C_A^4 C_F - \frac{255248}{3} \zeta_7 C_A C_F^3 T_F N_f - \frac{217060}{9} \zeta_5 C_A^3 C_F T_F N_f \\
& - \frac{200500}{9} \zeta_5 C_A C_F^3 T_F N_f - \frac{198121}{4} \zeta_7 C_A \frac{d_F^{abcd} d_A^{abcd}}{N_F} - \frac{150304}{3} \zeta_3^2 C_A^2 C_F^3 \\
& - \frac{110719}{6} \zeta_3 C_A \frac{d_F^{abcd} d_A^{abcd}}{N_F} - \frac{106016}{27} \zeta_3 C_F^2 T_F^3 N_f^3 - \frac{93317}{48} \zeta_7 C_A^3 C_F T_F N_f \\
& - \frac{69472}{9} C_A N_f \frac{d_F^{abcd} d_F^{abcd}}{N_F} - \frac{66344}{3} \zeta_3 C_F^4 T_F N_f - \frac{50995}{8} C_F^5 \\
& - \frac{41505}{2} \zeta_5 C_A \frac{d_F^{abcd} d_A^{abcd}}{N_F} - \frac{39320}{3} \zeta_5 C_F^4 T_F N_f - \frac{21760}{27} \zeta_5 C_A C_F T_F^3 N_f^3 \\
& - \frac{17312}{27} \zeta_3 C_A C_F T_F^3 N_f^3 - \frac{16160}{9} \zeta_5 C_A C_F^2 T_F^2 N_f^2 - \frac{10336}{9} T_F N_f \frac{d_F^{abcd} d_A^{abcd}}{N_F} \\
& - \frac{7804}{3} \zeta_3 C_F \frac{d_A^{abcd} d_A^{abcd}}{N_A} - \frac{7040}{9} \zeta_5 C_F^3 T_F^2 N_f^2 - \frac{6190}{3} \zeta_5 T_F N_f \frac{d_F^{abcd} d_A^{abcd}}{N_F} \\
& - \frac{2902}{9} C_F \frac{d_A^{abcd} d_A^{abcd}}{N_A} - \frac{2048}{3} \zeta_3 C_F N_f^2 \frac{d_F^{abcd} d_F^{abcd}}{N_A} - \frac{2048}{9} C_F N_f \frac{d_F^{abcd} d_A^{abcd}}{N_A} \\
& + \frac{512}{3} \zeta_3^2 C_F^3 T_F^2 N_f^2 + \frac{512}{3} \zeta_3^2 C_A^2 C_F T_F^2 N_f^2 + \frac{1280}{3} \zeta_5 C_F^2 T_F^3 N_f^3
\end{aligned}
$$

$$+ \frac{2031}{2} C_F^4 T_F N_f + \frac{2816}{9} C_F N_f^2 \frac{d_F^{abcd} d_F^{abcd}}{N_A} + \frac{6656}{3} \zeta_3 C_F N_f \frac{d_F^{abcd} d_A^{abcd}}{N_A}$$

$$+ \frac{9241}{3} \zeta_3^2 C_A^3 C_F T_F N_f + \frac{15640}{3} \zeta_5 C_F \frac{d_A^{abcd} d_A^{abcd}}{N_A} + \frac{22928}{9} T_F N_f^2 \frac{d_F^{abcd} d_F^{abcd}}{N_F}$$

$$+ \frac{26200}{3} \zeta_5 C_A^2 C_F T_F^2 N_f^2 + \frac{35777}{9} \zeta_3 C_A C_F^3 T_F N_f + \frac{37916}{3} \zeta_3 C_A C_F^2 T_F^2 N_f^2$$

$$+ \frac{40576}{3} \zeta_3^2 C_A C_F^3 T_F N_f + \frac{44480}{3} \zeta_5 C_A N_f \frac{d_F^{abcd} d_F^{abcd}}{N_F} + \frac{54889}{2} C_A C_F^4$$

$$+ \frac{76349}{2} \zeta_7 C_A^2 C_F^2 T_F N_f + \frac{90464}{3} \zeta_3 C_A N_f \frac{d_F^{abcd} d_F^{abcd}}{N_F} + \frac{96848}{9} C_A \frac{d_F^{abcd} d_A^{abcd}}{N_F}$$

$$+ \frac{102650}{9} \zeta_5 C_A^2 C_F^2 T_F N_f + \frac{107296}{9} \zeta_3 C_F^3 T_F^2 N_f^2 + \frac{163414}{3} \zeta_3 C_A C_F^4$$

$$+ \frac{168400}{3} \zeta_5 C_A C_F^4 + \frac{176281}{18} \zeta_3 C_A^2 C_F^2 T_F N_f + \frac{190282}{9} \zeta_3 C_A^2 C_F T_F^2 N_f^2$$

$$+ \frac{380960}{27} \zeta_5 C_A^4 C_F + \frac{552820}{9} \zeta_5 C_A^2 C_F^3 + \frac{757372}{3} \zeta_7 C_A^2 C_F^3$$

$$+ \frac{1878562}{243} C_F^2 T_F^3 N_f^3 + \frac{2389877}{18} C_A C_F^3 T_F N_f + \frac{5635399}{192} \zeta_7 C_A^4 C_F$$

$$+ \frac{30274283}{648} C_A^2 C_F^2 T_F N_f + \frac{34700285}{2187} C_A C_F T_F^3 N_f^3 + \frac{72189403}{432} \zeta_3 C_A^4 C_F$$

$$+ \frac{231306035}{1944} C_A^3 C_F^2 + \frac{1441558963}{4374} C_A^3 C_F T_F N_f - 169400 \zeta_7 C_A C_F^4$$

$$- 25872 \zeta_7 C_A N_f \frac{d_F^{abcd} d_F^{abcd}}{N_F} - 11840 \zeta_5 C_F N_f \frac{d_F^{abcd} d_F^{abcd}}{N_F}$$

$$- 11752 \zeta_3^2 T_F N_f \frac{d_F^{abcd} d_A^{abcd}}{N_F} - 11376 \zeta_3 T_F N_f^2 \frac{d_F^{abcd} d_F^{abcd}}{N_F} - 7808 \zeta_3^2 C_F^4 T_F N_f$$

$$- 6768 C_F \frac{d_F^{abcd} d_A^{abcd}}{N_F} - 4448 \zeta_3^2 C_A^2 C_F^2 T_F N_f - 4224 \zeta_3^2 C_A N_f \frac{d_F^{abcd} d_F^{abcd}}{N_F}$$

$$- 3872 \zeta_3^2 C_F \frac{d_A^{abcd} d_A^{abcd}}{N_A} - 3840 \zeta_3 C_F \frac{d_F^{abcd} d_A^{abcd}}{N_F} - 2156 \zeta_7 C_A^2 C_F T_F^2 N_f^2$$

$$- 2080 \zeta_5 C_F^5 - 1152 \zeta_3^2 C_A C_F^2 T_F^2 N_f^2 - 848 \zeta_3 C_F^5 - 320 \zeta_5 T_F N_f^2 \frac{d_F^{abcd} d_F^{abcd}}{N_F}$$

$$- 264 \zeta_4 C_F \frac{d_A^{abcd} d_A^{abcd}}{N_A} - 192 \zeta_4 T_F N_f^2 \frac{d_F^{abcd} d_F^{abcd}}{N_F} - 56 \zeta_4 C_A C_F^2 T_F^2 N_f^2$$

$$- 51 \zeta_4 C_A^3 C_F T_F N_f - 44 \zeta_4 C_A C_F^3 T_F N_f - 28 \zeta_4 C_A^2 C_F T_F^2 N_f^2$$

$$+ 82 \zeta_4 C_A^2 C_F^2 T_F N_f + 88 \zeta_4 C_F^3 T_F^2 N_f^2 + 352 C_F N_f \frac{d_F^{abcd} d_F^{abcd}}{N_F}$$

$$+ 624 \zeta_4 T_F N_f \frac{d_F^{abcd} d_A^{abcd}}{N_F} + 1024 \zeta_3^2 T_F N_f^2 \frac{d_F^{abcd} d_F^{abcd}}{N_F} + 1232 \zeta_7 C_F \frac{d_A^{abcd} d_A^{abcd}}{N_A}$$

$$+ 4704 \zeta_7 C_A C_F^2 T_F^2 N_f^2 + 4928 \zeta_3 C_F N_f \frac{d_F^{abcd} d_F^{abcd}}{N_F} + 9408 \zeta_7 T_F N_f^2 \frac{d_F^{abcd} d_F^{abcd}}{N_F}$$

$$+ 11830 \zeta_3 T_F N_f \frac{d_F^{abcd} d_A^{abcd}}{N_F} + 17563 \zeta_7 T_F N_f \frac{d_F^{abcd} d_A^{abcd}}{N_F} + 21472 \zeta_3^2 C_A C_F^4$$

$$+ 25440 \zeta_5 C_F \frac{d_F^{abcd} d_A^{abcd}}{N_F} + 40062 \zeta_3^2 C_A \frac{d_F^{abcd} d_A^{abcd}}{N_F} + 44176 \zeta_3^2 C_A^3 C_F^2$$

$$+ 54880 \zeta_7 C_F^4 T_F N_f \Big] a^5 \ + \ O(a^6) \ . \tag{A.4}$$

One ingredient that was necessary in applying (2.5) to find the above expressions was the relation between the gauge parameter in the $\overline{\text{MS}}$ and RI$'$ schemes. Therefore we record that

$$
\begin{aligned}
\alpha_{\text{RI}'} \;=\; & \left[1 + \left[\frac{20}{9}T_F N_f - \frac{97}{36}C_A - \frac{1}{2}C_A\alpha - \frac{1}{4}C_A\alpha^2\right]a \right.\\
& + \left[\frac{1}{16}C_A^2\alpha^4 - \frac{2381}{96}C_A^2 - \frac{10}{9}C_A T_F N_f\alpha - \frac{10}{9}C_A T_F N_f\alpha^2 - \frac{1}{16}C_A^2\alpha^3 + \frac{55}{3}C_F T_F N_f \right.\\
& + \frac{59}{4}C_A T_F N_f + \frac{95}{144}C_A^2\alpha^2 + \frac{463}{288}C_A^2\alpha - 16\zeta_3 C_F T_F N_f - 2\zeta_3 C_A^2\alpha + 3\zeta_3 C_A^2 \\
& \left. + 8\zeta_3 C_A T_F N_f\right]a^2 \\
& + \left[\frac{1}{16}C_A^3\alpha^5 - \frac{10221367}{31104}C_A^3 - \frac{12071}{288}\zeta_3 C_A^3\alpha - \frac{10499}{243}C_A T_F^2 N_f^2 - \frac{7402}{81}C_F T_F^2 N_f^2 \right.\\
& - \frac{6137}{1296}C_A^2 T_F N_f\alpha^2 - \frac{2813}{1152}C_A^3\alpha^3 - \frac{1492}{9}\zeta_3 C_A C_F T_F N_f - \frac{296}{3}\zeta_3 C_F^2 T_F N_f \\
& - \frac{286}{9}C_F^2 T_F N_f - \frac{256}{9}\zeta_3 C_A T_F^2 N_f^2 - \frac{241}{24}C_A^2 T_F N_f\alpha - \frac{161}{96}\zeta_3 C_A^3\alpha^2 \\
& - \frac{160}{3}\zeta_5 C_A^2 T_F N_f - \frac{100}{81}C_A T_F^2 N_f^2\alpha^2 - \frac{64}{9}\zeta_3 C_A T_F^2 N_f^2\alpha - \frac{55}{6}C_A C_F T_F N_f\alpha \\
& - \frac{55}{6}C_A C_F T_F N_f\alpha^2 - \frac{35}{192}\zeta_5 C_A^3\alpha^4 - \frac{29}{48}C_A^3\alpha^4 - \frac{25}{6}\zeta_3 C_A^2 T_F N_f\alpha^2 - \frac{5}{18}C_A^2 T_F N_f\alpha^3 \\
& - \frac{5}{24}\zeta_5 C_A^3\alpha^3 - \frac{1}{64}C_A^3\alpha^6 + \frac{3}{8}\zeta_4 C_A^3\alpha + \frac{3}{32}\zeta_4 C_A^3\alpha^2 + \frac{5}{12}C_A^2 T_F N_f\alpha^4 + \frac{9}{32}\zeta_4 C_A^3 \\
& + \frac{13}{96}\zeta_3 C_A^3\alpha^4 + \frac{16}{9}C_A T_F^2 N_f^2\alpha + \frac{115}{8}\zeta_5 C_A^3\alpha + \frac{149}{96}\zeta_3 C_A^3\alpha^3 + \frac{202}{9}\zeta_3 C_A^2 T_F N_f\alpha \\
& + \frac{385}{96}\zeta_5 C_A^3\alpha^2 + \frac{608}{9}\zeta_3 C_F T_F^2 N_f^2 + \frac{871}{18}\zeta_3 C_A^2 T_F N_f + \frac{1549}{24}\zeta_3 C_A^3 + \frac{7025}{192}\zeta_5 C_A^3 \\
& + \frac{13141}{1152}C_A^3\alpha + \frac{30835}{10368}C_A^3\alpha^2 + \frac{96809}{324}C_A C_F T_F N_f + \frac{1154561}{3888}C_A^2 T_F N_f \\
& - 80\zeta_5 C_A C_F T_F N_f - 9\zeta_4 C_A^2 T_F N_f + 8\zeta_3 C_A C_F T_F N_f\alpha + 8\zeta_3 C_A C_F T_F N_f\alpha^2 \\
& \left. + 12\zeta_4 C_A C_F T_F N_f + 160\zeta_5 C_F^2 T_F N_f\right]a^3 \\
& + \left[\frac{1}{6}\zeta_3^2 C_A^3 T_F N_f\alpha^3 - \frac{4698915983}{746496}C_A^4 - \frac{51101257}{82944}\zeta_3 C_A^4\alpha - \frac{44177531}{23328}C_A^2 T_F^2 N_f^2 \right.\\
& - \frac{21636697}{186624}C_A^3 T_F N_f\alpha - \frac{13241219}{2916}C_A C_F T_F^2 N_f^2 - \frac{9266411}{6144}\zeta_7 C_A^4 - \frac{7806659}{36864}\zeta_7 C_A^4\alpha \\
& - \frac{2360029}{18432}\zeta_3^2 C_A^4 - \frac{1650287}{432}\zeta_5 C_A^3 T_F N_f - \frac{1100549}{41472}C_A^4\alpha^3 - \frac{576533}{648}C_A C_F^2 T_F N_f \\
& - \frac{515119}{55296}\zeta_5 C_A^4\alpha^3 - \frac{485489}{2592}C_A^2 C_F T_F N_f\alpha - \frac{324935}{162}\zeta_3 C_A^2 C_F T_F N_f \\
& - \frac{311113}{768}\zeta_3\frac{d_A^{abcd}d_A^{abcd}}{N_A} - \frac{310055}{110592}\zeta_5 C_A^4\alpha^4 - \frac{298517}{3456}C_A^3 T_F N_f\alpha^2 - \frac{238325}{4096}\zeta_6 C_A^4 \\
& - \frac{232351}{1296}\zeta_3 C_A^3 T_F N_f - \frac{226975}{512}\zeta_5\frac{d_A^{abcd}d_A^{abcd}}{N_A} - \frac{193639}{4608}\zeta_3 C_A^4\alpha^2 \\
& - \frac{192157}{54}\zeta_3 C_A C_F^2 T_F N_f - \frac{156817}{162}\zeta_3 C_A^2 T_F^2 N_f^2 - \frac{147553}{6144}\zeta_7 C_A^4\alpha^2 \\
& - \frac{99757}{128}\zeta_7\frac{d_A^{abcd}d_A^{abcd}}{N_A} - \frac{82181}{648}C_A^2 C_F T_F N_f\alpha^2 - \frac{51175}{6144}\zeta_6 C_A^4\alpha \\
& - \frac{50925}{512}\zeta_6\frac{d_A^{abcd}d_A^{abcd}}{N_A} - \frac{36166}{9}\zeta_5 C_A^2 C_F T_F N_f - \frac{33515}{192}\zeta_3\frac{d_A^{abcd}d_A^{abcd}}{N_A}\alpha \\
& - \frac{25888}{81}\zeta_3 C_F T_F^3 N_f^3 - \frac{22703}{144}\zeta_5 C_A^3 T_F N_f\alpha - \frac{17449}{128}\zeta_3^2\frac{d_A^{abcd}d_A^{abcd}}{N_A}\alpha
\end{aligned}
$$

$$- \frac{16000}{9}\zeta_5 C_F^2 T_F^2 N_f^2 - \frac{14803}{2048}C_A^4\alpha^4 - \frac{8369}{288}\zeta_3 C_A^3 T_F N_f\alpha^2 - \frac{7505}{162}C_F^2 T_F^2 N_f^2$$

$$- \frac{5452}{27}\zeta_3 C_A^2 T_F^2 N_f^2\alpha - \frac{5350}{3}\zeta_3^2 N_f\frac{d_F^{abcd}d_A^{abcd}}{N_A} - \frac{5195}{128}\zeta_5\frac{d_A^{abcd}d_A^{abcd}}{N_A}\alpha^2$$

$$- \frac{4751}{1296}C_A^3 T_F N_f\alpha^3 - \frac{4565}{9216}\zeta_5 C_A^4\alpha^5 - \frac{4544}{27}N_f\frac{d_F^{abcd}d_A^{abcd}}{N_A} - \frac{3997}{384}\zeta_7\frac{d_A^{abcd}d_A^{abcd}}{N_A}\alpha$$

$$- \frac{3255}{256}\zeta_5\frac{d_A^{abcd}d_A^{abcd}}{N_A}\alpha^3 - \frac{2800}{3}\zeta_7 C_A^2 C_F T_F N_f - \frac{2767}{32}\zeta_4 C_A^3 T_F N_f$$

$$- \frac{2375}{256}\zeta_6\frac{d_A^{abcd}d_A^{abcd}}{N_A}\alpha - \frac{1501}{64}\zeta_3^2\frac{d_A^{abcd}d_A^{abcd}}{N_A}\alpha^2 - \frac{1280}{9}\zeta_5 C_F T_F^3 N_f^3$$

$$- \frac{1213}{108}\zeta_3 C_A^3 T_F N_f\alpha^3 - \frac{1195}{768}\zeta_3\frac{d_A^{abcd}d_A^{abcd}}{N_A}\alpha^4 - \frac{989}{32}\zeta_4\frac{d_A^{abcd}d_A^{abcd}}{N_A} - \frac{985}{3072}\zeta_3^2 C_A^4\alpha^3$$

$$- \frac{875}{1728}C_A^3 T_F N_f\alpha^4 - \frac{784}{9}\zeta_3 C_A C_F T_F^2 N_f^2\alpha - \frac{704}{9}\zeta_3^2 C_A^2 T_F^2 N_f^2 - \frac{640}{27}\zeta_5 C_A T_F^3 N_f^3$$

$$- \frac{633}{128}\zeta_3^2\frac{d_A^{abcd}d_A^{abcd}}{N_A}\alpha^3 - \frac{567}{128}\zeta_7\frac{d_A^{abcd}d_A^{abcd}}{N_A}\alpha^4 - \frac{557}{18}\zeta_3^2 C_A^3 T_F N_f$$

$$- \frac{512}{3}\zeta_3^2 N_f^2\frac{d_F^{abcd}d_F^{abcd}}{N_A} - \frac{487}{384}\frac{d_A^{abcd}d_A^{abcd}}{N_A}\alpha^2 - \frac{385}{96}\frac{d_A^{abcd}d_A^{abcd}}{N_A}\alpha - \frac{320}{81}C_A T_F^3 N_f^3\alpha$$

$$- \frac{189}{256}\zeta_7 C_A^4\alpha^4 - \frac{181}{2048}\zeta_4 C_A^4\alpha^4 - \frac{175}{216}\zeta_5 C_A^3 T_F N_f\alpha^4 - \frac{175}{512}\zeta_6 C_A^4\alpha^2 - \frac{157}{384}\zeta_3 C_A^4\alpha^5$$

$$- \frac{147}{2}\zeta_7 N_f\frac{d_F^{abcd}d_A^{abcd}}{N_A}\alpha - \frac{147}{256}\zeta_3^2 C_A^4\alpha^2 - \frac{121}{1536}C_A^4\alpha^5 - \frac{97}{81}C_A^2 T_F^2 N_f^2\alpha^3$$

$$- \frac{91}{192}\zeta_7\frac{d_A^{abcd}d_A^{abcd}}{N_A}\alpha^5 - \frac{73}{256}\zeta_4 C_A^4\alpha^3 - \frac{55}{24}C_A^2 C_F T_F N_f\alpha^3 - \frac{41}{6}\zeta_4 C_A^2 T_F^2 N_f^2$$

$$- \frac{31}{3}C_F^3 T_F N_f - \frac{13}{192}\zeta_3 C_A^4\alpha^6 - \frac{10}{9}\zeta_5 C_A^2 T_F^2 N_f^2\alpha^2 - \frac{7}{256}C_A^4\alpha^7 - \frac{5}{36}C_A^3 T_F N_f\alpha^6$$

$$- \frac{5}{64}\zeta_3\frac{d_A^{abcd}d_A^{abcd}}{N_A}\alpha^5 - \frac{3}{2}\zeta_4 C_A^2 C_F T_F N_f\alpha + \frac{1}{256}C_A^4\alpha^8 + \frac{3}{64}\zeta_4\frac{d_A^{abcd}d_A^{abcd}}{N_A}\alpha^4$$

$$+ \frac{3}{128}\zeta_4\frac{d_A^{abcd}d_A^{abcd}}{N_A}\alpha^3 + \frac{5}{6}\zeta_4 C_A^3 T_F N_f\alpha + \frac{5}{6}\zeta_3^2 C_A^3 T_F N_f\alpha^2 + \frac{5}{12}C_A^3 T_F N_f\alpha^5$$

$$+ \frac{8}{3}\zeta_4 C_A T_F^3 N_f^3 + \frac{9}{8}\zeta_4\frac{d_A^{abcd}d_A^{abcd}}{N_A}\alpha^2 + \frac{15}{64}\zeta_3^2\frac{d_A^{abcd}d_A^{abcd}}{N_A}\alpha^5 + \frac{21}{512}\zeta_3^2 C_A^4\alpha^5$$

$$+ \frac{25}{27}C_A^2 T_F^2 N_f^2\alpha^4 + \frac{25}{512}\zeta_6\frac{d_A^{abcd}d_A^{abcd}}{N_A}\alpha^4 + \frac{29}{6}\zeta_3^2 C_A^3 T_F N_f\alpha + \frac{32}{9}\zeta_3 C_A^2 T_F^2 N_f^2\alpha^3$$

$$+ \frac{35}{384}\zeta_5\frac{d_A^{abcd}d_A^{abcd}}{N_A}\alpha^5 + \frac{35}{384}\zeta_5 C_A^4\alpha^6 + \frac{41}{6}\zeta_4 C_A^3 T_F N_f\alpha^2 + \frac{49}{2}\zeta_7 N_f\frac{d_F^{abcd}d_A^{abcd}}{N_A}$$

$$+ \frac{50}{9}\zeta_3 C_A^2 T_F^2 N_f^2\alpha^2 + \frac{55}{16}C_A^2 C_F T_F N_f\alpha^4 + \frac{59}{27}\zeta_3 C_A^3 T_F N_f\alpha^4 + \frac{88}{3}\zeta_4 C_F^2 T_F^2 N_f^2$$

$$+ \frac{143}{9}C_A C_F^2 T_F N_f\alpha + \frac{143}{9}C_A C_F^2 T_F N_f\alpha^2 + \frac{148}{3}\zeta_3 C_A C_F^2 T_F N_f\alpha$$

$$+ \frac{148}{3}\zeta_3 C_A C_F^2 T_F N_f\alpha^2 + \frac{227}{36}C_A^2 T_F^2 N_f^2\alpha^2 + \frac{230}{9}\zeta_5 C_A^2 T_F^2 N_f^2\alpha + \frac{245}{6}\zeta_4 C_A C_F^2 T_F N_f$$

$$+ \frac{275}{4}\zeta_6 C_A^3 T_F N_f + \frac{275}{256}\zeta_6\frac{d_A^{abcd}d_A^{abcd}}{N_A}\alpha^3 + \frac{283}{64}\zeta_3\frac{d_A^{abcd}d_A^{abcd}}{N_A}\alpha^3 + \frac{329}{256}\zeta_3^2\frac{d_A^{abcd}d_A^{abcd}}{N_A}\alpha^4$$

$$+ \frac{341}{1152}C_A^4\alpha^6 + \frac{365}{1536}\zeta_4 C_A^4\alpha^2 + \frac{375}{128}\zeta_6\frac{d_A^{abcd}d_A^{abcd}}{N_A}\alpha^2 + \frac{413}{6}\zeta_3 C_A^2 C_F T_F N_f\alpha^2$$

$$+ \frac{475}{6144}\zeta_6 C_A^4\alpha^3 + \frac{539}{8}\zeta_7 C_A^3 T_F N_f\alpha + \frac{575}{12288}\zeta_6 C_A^4\alpha^4 + \frac{595}{432}\zeta_5 C_A^3 T_F N_f\alpha^3$$

$$
\begin{aligned}
&+ \frac{992}{3}\zeta_3 N_f \frac{d_F^{abcd} d_A^{abcd}}{N_A}\alpha + \frac{1279}{6144}\zeta_3^2 C_A^4 \alpha^4 + \frac{1280}{81}\zeta_3 C_A T_F^3 N_f^3 \alpha \\[4pt]
&+ \frac{1313}{96}\zeta_3 \frac{d_A^{abcd} d_A^{abcd}}{N_A}\alpha^2 + \frac{1347}{128}\zeta_4 \frac{d_A^{abcd} d_A^{abcd}}{N_A}\alpha + \frac{1397}{48}\zeta_5 C_A^3 T_F N_f \alpha^2 \\[4pt]
&+ \frac{1625}{9}\zeta_3 C_A^2 C_F T_F N_f \alpha + \frac{2051}{81}C_A C_F T_F^2 N_f^2 \alpha^2 + \frac{2625}{64}\zeta_7 \frac{d_A^{abcd} d_A^{abcd}}{N_A}\alpha^2 \\[4pt]
&+ \frac{2737}{384}\zeta_7 \frac{d_A^{abcd} d_A^{abcd}}{N_A}\alpha^3 + \frac{3319}{3456}\frac{d_A^{abcd} d_A^{abcd}}{N_A} + \frac{4280}{27}\zeta_3 C_A T_F^3 N_f^3 + \frac{4505}{1536}\zeta_5 \frac{d_A^{abcd} d_A^{abcd}}{N_A}\alpha^4 \\[4pt]
&+ \frac{4781}{81}C_A C_F T_F^2 N_f^2 \alpha + \frac{4823}{36864}\zeta_7 C_A^4 \alpha^5 + \frac{6008}{3}\zeta_5 C_A C_F T_F^2 N_f^2 + \frac{6896}{27}N_f^2 \frac{d_F^{abcd} d_F^{abcd}}{N_A} \\[4pt]
&+ \frac{8411}{1536}\zeta_4 C_A^4 \alpha + \frac{9471}{4}\zeta_7 C_A^3 T_F N_f + \frac{15785}{9216}\zeta_7 C_A^4 \alpha^3 + \frac{16586}{27}\zeta_5 C_A^2 T_F^2 N_f^2 \\[4pt]
&+ \frac{29210}{9}\zeta_5 C_A C_F^2 T_F N_f + \frac{49117}{2048}\zeta_4 C_A^4 + \frac{49696}{27}\zeta_3 C_F^2 T_F^2 N_f^2 + \frac{63007}{3072}\zeta_3^2 C_A^4 \alpha \\[4pt]
&+ \frac{69395}{1152}\zeta_5 C_A^4 \alpha^2 + \frac{90341}{27648}\zeta_3 C_A^4 \alpha^4 + \frac{113743}{1458}C_A T_F^3 N_f^3 + \frac{124210}{81}\zeta_3 C_A C_F T_F^2 N_f^2 \\[4pt]
&+ \frac{260717}{432}\zeta_3 C_A^3 T_F N_f \alpha + \frac{322195}{768}\zeta_5 \frac{d_A^{abcd} d_A^{abcd}}{N_A}\alpha + \frac{393026}{729}C_F T_F^3 N_f^3 \\[4pt]
&+ \frac{552001}{11664}C_A^2 T_F^2 N_f^2 \alpha + \frac{651137}{27648}\zeta_3 C_A^4 \alpha^3 + \frac{753337}{768}\zeta_3^2 \frac{d_A^{abcd} d_A^{abcd}}{N_A} + \frac{882095}{27648}C_A^4 \alpha^2 \\[4pt]
&+ \frac{1847129}{6144}\zeta_5 C_A^4 \alpha + \frac{20572427}{2916}C_A^2 C_F T_F N_f + \frac{42950657}{5832}C_A^3 T_F N_f \\[4pt]
&+ \frac{122449741}{82944}\zeta_3 C_A^4 + \frac{129316433}{1492992}C_A^4 \alpha + \frac{338738527}{110592}\zeta_5 C_A^4 - 2240\zeta_7 C_F^3 T_F N_f \\[4pt]
&- 340\zeta_5 N_f \frac{d_F^{abcd} d_A^{abcd}}{N_A}\alpha - 336\zeta_3 N_f^2 \frac{d_F^{abcd} d_F^{abcd}}{N_A} - 256\zeta_3^2 C_F^2 T_F^2 N_f^2 \\[4pt]
&- 150\zeta_6 C_A C_F^2 T_F N_f - 80\zeta_5 C_A C_F^2 T_F N_f \alpha - 80\zeta_5 C_A C_F^2 T_F N_f \alpha^2 \\[4pt]
&- 64\zeta_4 N_f^2 \frac{d_F^{abcd} d_F^{abcd}}{N_A} - 60\zeta_3^2 C_A C_F^2 T_F N_f - 22\zeta_4 C_A C_F T_F^2 N_f^2 \\[4pt]
&- 16\zeta_3 C_A C_F T_F^2 N_f^2 \alpha^2 - 9\zeta_4 C_A^2 C_F T_F N_f \alpha^2 - 3\zeta_3 C_A^2 C_F T_F N_f \alpha^4 \\[4pt]
&+ \zeta_4 C_A^2 T_F^2 N_f^2 \alpha + 2\zeta_3 C_A^2 C_F T_F N_f \alpha^3 + 16\zeta_3^2 N_f \frac{d_F^{abcd} d_A^{abcd}}{N_A}\alpha \\[4pt]
&+ 40\zeta_5 C_A^2 C_F T_F N_f \alpha^2 + 61\zeta_4 C_A^2 C_F T_F N_f + 67\zeta_5 C_A^2 C_F T_F N_f \alpha \\[4pt]
&+ 75\zeta_6 N_f \frac{d_F^{abcd} d_A^{abcd}}{N_A} + 75\zeta_6 C_A^2 C_F T_F N_f + 104\zeta_3 C_F^3 T_F N_f + 172\zeta_4 N_f \frac{d_F^{abcd} d_A^{abcd}}{N_A} \\[4pt]
&+ 256\zeta_3^2 C_A C_F T_F^2 N_f^2 + 320\zeta_5 N_f^2 \frac{d_F^{abcd} d_F^{abcd}}{N_A} + 410\zeta_3^2 C_A^2 C_F T_F N_f \\[4pt]
&+ 1005\zeta_5 N_f \frac{d_F^{abcd} d_A^{abcd}}{N_A} + 1960\zeta_5 C_F^3 T_F N_f + 1964\zeta_3 N_f \frac{d_F^{abcd} d_A^{abcd}}{N_A} \\[4pt]
&+ 2240\zeta_7 C_A C_F^2 T_F N_f \Big] a^4 ~+~ O(a^5) \Big] \alpha \tag{A.5}
\end{aligned}
$$

is the full mapping in an arbitrary gauge.

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