

# Five loop minimal MOM scheme field and quark mass anomalous dimensions in QCD

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**Abstract.** We determine the anomalous dimensions of the gluon, Faddeev-Popov ghost and quark in the minimal MOM scheme to five loops for a general colour group when Quantum Chromodynamics is fixed in a linear covariant gauge. The quark mass anomalous dimension is also constructed in the same scheme.

# 1 Introduction.

High order loop calculations in perturbative quantum field theory are carried out with respect to a renormalization scheme. Invariably the main scheme of choice is the modified minimal subtraction ( $\overline{\text{MS}}$ ) scheme, [1, 2]. It is defined by the prescription that at the subtraction point of a divergent  $n$ -point function in a renormalizable theory the singularities of the Laurent series in the regularization parameter are absorbed into the renormalization constant for that Green's function, [1]. In addition a specific finite part, which is  $4\pi e^{-\gamma}$  where  $\gamma$  is the Euler-Mascheroni constant, is also removed, [2]. The main benefit of the  $\overline{\text{MS}}$  scheme is that high order loop calculations can be pushed to extremely high order analytically. Several impressive examples that illustrate this in recent years are the *five* loop  $\beta$ -function of Quantum Chromodynamics (QCD), [3, 4, 5, 6, 7], and the renormalization of scalar  $\phi^4$  theory at *seven* loops, [8]. While this level of precision for QCD has been crucial for phenomenology studies, the  $\overline{\text{MS}}$  scheme is in one sense a universal scheme. This is because in that scheme the choice for the momentum configuration of the divergent  $n$ -point function where the renormalization constants are defined is virtually free. The only places where one has to be careful are configurations which are termed exceptional where techniques such as infrared rearrangement, [9, 10, 11, 12], have to be employed. In other words the  $\overline{\text{MS}}$  scheme is not tied to any kinematic property of the  $n$ -point function. By contrast kinematic schemes are connected to non-exceptional momentum configurations and information concerning specific properties of the subtraction point are reflected in the structure of the renormalization constants. A well-known set of such schemes are those provided by Celmaster and Gonsalves in [13, 14] for the renormalization of QCD. In the three different momentum subtraction (MOM) schemes, each based on one of the 3-point vertices in the QCD Lagrangian, the respective vertex functions are evaluated at the fully symmetric point. This is where the square of each of the external momenta are all equal to each other. Then the renormalization prescription is that at this symmetric configuration the renormalization constant is chosen so that there are no  $O(a)$  corrections where  $a = g^2/(16\pi^2)$  and  $g$  is the gauge coupling constant. While this scheme has a phenomenological origin other schemes have been introduced to accommodate certain issues.

One such scheme is the minimal momentum subtraction (mMOM) scheme that was introduced in [15] to exploit and extend a particular fundamental property of QCD that was originally observed by Taylor in [16]. More specifically it was proved in [16] that the gluon-ghost vertex function is finite to all orders in the Landau gauge. One consequence is that the  $\overline{\text{MS}}$  QCD  $\beta$ -function can therefore be deduced from the Landau gauge values of the gluon and ghost anomalous dimensions, [17]. However to ease the numerical and financial aspects of making measurements of lattice regularized quantities in QCD the mMOM scheme was developed in such a way that the non-renormalization of the gluon-ghost vertex was maintained for an arbitrary linear covariant gauge, [15]. Although initially defined for lattice regularization of QCD it has a continuum spacetime analogue which was given in [15]. In particular the mMOM  $\beta$ -function was computed to four loops, [15], with the field anomalous dimensions together with the quark mass anomalous dimension following later in [18]. These renormalization group functions were required for studying the conformal window properties of QCD and the associated critical exponents at the Banks-Zaks fixed point that was discovered in [19, 20]. One property of a fixed point of the  $\beta$ -function is that the critical exponents, derived from evaluating the anomalous dimensions at the fixed point, are renormalization group invariants. In other words they are scheme independent. Therefore the four loop mMOM anomalous dimensions were needed to study the convergence of critical exponent estimates [21, 22]. Given the continued interest in such conformal window studies for gauge theories, [21, 22, 23, 24], and theories beyond the Standard Model coupled with the extension of QCD renormalization group functions to five loops,

the aim of this article is to provide the mMOM field and quark mass anomalous dimensions to the same order. In [25] only the five loop mMOM  $\beta$ -function was presented and then used to determine the  $R$  ratio in that scheme to a new loop order. Other phenomenological applications of the mMOM scheme were discussed in [26, 27]. However, using data provided in [25] we have been able to determine the mMOM field and quark mass renormalization constants to four loops. Knowledge of these will then allow us to deduce their *five* loop mMOM anomalous dimensions from a particular property of the renormalization group. In addition the anomalous dimensions are also needed for a parallel study of the fixed points of QCD, including the Banks-Zaks one, in mMOM, [28]. That article examines the fixed point structure in a variety schemes, such as the MOM ones of [13, 14], as well as QCD fixed in both linear and nonlinear covariant gauges. The mMOM aspect of that work relies importantly on the separate five loop results provided here.

The article is organized as follows. We review the basics of the mMOM scheme in Section 2 as well as outlining the renormalization group formalism we use to extract the five loop mMOM renormalization group functions. The full results are recorded in Section 3 and concluding remarks are provided in Section 4.

## 2 Background.

The defining criterion for the mMOM scheme is that the non-renormalization of the ghost-gluon vertex in the Landau gauge observed by Taylor in [16] is preserved for a non-zero linear covariant gauge parameter. In practical terms this translates into a condition between the relevant renormalization constants associated with that vertex which is, [15],

$$Z_g^{\overline{\text{MS}}} \sqrt{Z_A^{\overline{\text{MS}}} Z_c^{\overline{\text{MS}}}} = Z_g^{\text{mMOM}} \sqrt{Z_A^{\text{mMOM}}} Z_c^{\text{mMOM}} \quad (2.1)$$

where  $Z_g$ ,  $Z_A$  and  $Z_c$  are the respective renormalization constants for the gauge coupling constant  $g$  together with the gluon and the Faddeev-Popov ghost fields. Each is labelled by the scheme in which the basic variables  $a$  and the covariant gauge parameter  $\alpha$  are in. The  $\overline{\text{MS}}$  renormalization constants have been determined to five loops [3, 4, 5, 6, 7, 17, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38] over many years. For the mMOM scheme the gluon and ghost renormalization constants are defined in the same way as their counterparts in the MOM scheme of [13, 14]. As one determines the mMOM field renormalization constants order by order in the loop expansion then  $Z_g^{\text{mMOM}}$  is deduced iteratively via (2.1). This procedure was applied in [15] to determine the four loop mMOM  $\beta$ -function with the anomalous dimensions of the gluon, ghost and quark as well as that of the quark mass provided in [18] to the same loop order. In practical terms to find the four loop mMOM results the 2-point functions of the fields were computed to three loops using the MINCER package, [39, 40]. At a technical level the 2-point functions were computed as a Laurent expansion in the dimensional regularization parameter  $\epsilon$  where  $d = 4 - 2\epsilon$ . Such an expansion is necessary to ensure the correct finite part emerges as one applies the mMOM renormalization conditions to find the renormalization constants. Once the three loop mMOM renormalization constants have been determined then the four loop anomalous dimensions can be deduced from a property of the renormalization group equation.

To illustrate this we recall the basic relation of the field renormalization constants with their associated anomalous dimension is

$$\gamma_\phi(a, \alpha) = \beta(a, \alpha) \frac{\partial \ln Z_\phi}{\partial a} + \alpha \gamma_\alpha(a, \alpha) \frac{\partial \ln Z_\phi}{\partial \alpha} \quad (2.2)$$

where  $\phi \in \{A, c, \psi\}$  noting that

$$\gamma_\alpha(a, \alpha) = \left[ \beta(a, \alpha) \frac{\partial \ln Z_\alpha}{\partial a} - \gamma_A(a, \alpha) \right] \left[ 1 - \alpha \frac{\partial \ln Z_\alpha}{\partial \alpha} \right]^{-1} \quad (2.3)$$

for the renormalization of the gauge parameter in general. In a *linear* covariant gauge fixing  $Z_\alpha$  is unity whence the latter relation for the anomalous dimension of  $\alpha$  is trivially related to  $\gamma_A(a, \alpha)$ . Our convention for the relation between the renormalized and bare, denoted by  $\circ$ , parameter is

$$\alpha_\circ = Z_\alpha^{-1} Z_A \alpha. \quad (2.4)$$

We have taken a general position and included  $\alpha$  as an argument of the  $\beta$ -function. In the  $\overline{\text{MS}}$  scheme the  $\beta$ -function is independent of the covariant gauge parameter, [1]. With (2.2) and defining the conversion functions for the two schemes of interest by

$$C_\phi(a, \alpha) = \frac{Z_\phi^{\text{mMOM}}}{Z_\phi^{\overline{\text{MS}}}} \quad (2.5)$$

where the variables of the conversion functions will always be in the  $\overline{\text{MS}}$  scheme, then it is straightforward to relate the anomalous dimensions in one scheme with those in the other. This leads to

$$\begin{aligned} \gamma_\phi^{\text{mMOM}}(a_{\text{mMOM}}, \alpha_{\text{mMOM}}) &= \left[ \gamma_\phi^{\overline{\text{MS}}}(a_{\overline{\text{MS}}}, \alpha_{\overline{\text{MS}}}) + \beta^{\overline{\text{MS}}}(a_{\overline{\text{MS}}}) \frac{\partial}{\partial a_{\overline{\text{MS}}}} \ln C_\phi(a_{\overline{\text{MS}}}, \alpha_{\overline{\text{MS}}}) \right. \\ &\quad \left. + \alpha_{\overline{\text{MS}}} \gamma_\alpha^{\overline{\text{MS}}}(a_{\overline{\text{MS}}}, \alpha_{\overline{\text{MS}}}) \frac{\partial}{\partial \alpha_{\overline{\text{MS}}}} \ln C_\phi(a_{\overline{\text{MS}}}, \alpha_{\overline{\text{MS}}}) \right] \Big|_{\overline{\text{MS}} \rightarrow \text{mMOM}} \end{aligned} \quad (2.6)$$

where we label the variables and functions by their scheme. The final stage requires one to map the  $\overline{\text{MS}}$  variables on the right side to their mMOM partners to ensure the expression is a function of the correct variables. This is achieved by recalling that the relations between the coupling constant and gauge parameter in each scheme are given by

$$g_{\text{mMOM}} = \frac{Z_g^{\overline{\text{MS}}}}{Z_g^{\text{mMOM}}} g_{\overline{\text{MS}}} \quad , \quad \alpha_{\text{mMOM}} = \frac{Z_A^{\text{mMOM}}}{Z_A^{\overline{\text{MS}}}} \alpha_{\overline{\text{MS}}}. \quad (2.7)$$

The explicit expressions can be deduced once the respective renormalization constants are available at the required order. The key property of (2.2) that allows us to determine the five loop mMOM anomalous dimensions is that the mMOM renormalization constants are only needed to *four* loops which can be seen by examining the  $a$  dependence of each term in (2.6). Therefore all that remains in this exercise is to determine the renormalization constants.

This can be achieved relatively straightforwardly from the data presented in [25]. In [25] the expressions for the 2-point functions as well as the 3-point vertices, where there was a nullified external momentum on one of the fields, were presented as a function of the bare parameters. Moreover the Laurent expansion in powers of  $\epsilon$  was given to four loops. This was determined via the FORCER algorithm, [41, 42], and formed the basis of calculating the five loop  $\overline{\text{MS}}$  renormalization group functions of QCD. Since the expressions are available in terms of bare parameters there is sufficient data for the four loop mMOM renormalization constants to be found through the scheme prescription described earlier. In extracting these and converting them to the renormalization group functions in the mMOM scheme using FORM, [43, 44], we have verified the four loop expressions given in [18]. We have carried out the same procedure for the quark mass anomalous dimension. In this instance we made use of the four loop renormalization

of the quark mass operator provided in [45]. The operator was renormalized in that article in the RI' scheme which is a scheme underpinning the matching of lattice field theory results to continuum perturbation theory. In particular the operator was inserted in a quark 2-point function at zero momentum. While the  $\overline{\text{MS}}$  mass dimension has been known to five loops in the  $\overline{\text{MS}}$  scheme for several years now [36, 37, 38, 46, 47, 48, 49, 50], the finite part of the Green's function was not available at four loops in terms of bare parameters to high enough order in the  $\epsilon$  expansion to implement renormalization in schemes whose prescriptions require finite parts to be absorbed into the renormalization constants. As this is the case for our mMOM analysis we have used the results of [45], where the quark mass operator was inserted at zero momentum in a quark 2-point function, to determine the four loop quark mass renormalization constant in the mMOM scheme.

### 3 Results.

Having established the formalism and strategy to extract the five loop mMOM renormalization group functions we devote this section to recording the various expressions. In evaluating Green's functions to higher loop order it is well-known that the number of Feynman graphs increases. As a consequence the resulting expressions become larger in QCD due to additional colour group factors being introduced through new graph topologies such as diagrams that involve so called light-by-light structures. Therefore we illustrate the essence of our results by recording the relevant expressions for the  $SU(3)$  colour group in the Landau gauge. The full gauge dependence for  $SU(3)$  is given in the Appendix while we provide the expressions for an arbitrary colour group in a data file associated with the arXiv version of this paper. First, we record the four conversion functions are

$$\begin{aligned}
C_A(a, 0)|^{SU(3)} &= 1 + \left[ \frac{97}{12} - \frac{10}{9} N_f \right] a + \left[ \frac{83105}{288} - \frac{11299}{216} N_f - \frac{4}{3} \zeta_3 N_f + \frac{100}{81} N_f^2 - 27 \zeta_3 \right] a^2 \\
&+ \left[ \frac{164395}{972} N_f^2 - \frac{8228977}{2592} N_f - \frac{63225}{64} \zeta_5 - \frac{17433}{8} \zeta_3 - \frac{1000}{729} N_f^3 - \frac{243}{32} \zeta_4 \right. \\
&\quad \left. + \frac{16}{9} \zeta_3 N_f^2 + \frac{33}{2} \zeta_4 N_f + \frac{2320}{9} \zeta_5 N_f + \frac{25915}{108} \zeta_3 N_f + \frac{44961125}{3456} \right] a^3 \\
&+ \left[ \frac{62302764631}{82944} - \frac{1758762815}{7776} N_f - \frac{324121925}{2048} \zeta_3 - \frac{262747689}{1024} \zeta_5 \right. \\
&\quad - \frac{25666081}{864} \zeta_7 N_f - \frac{15059695}{34992} N_f^3 - \frac{3161781}{2048} \zeta_4 - \frac{515843}{162} \zeta_5 N_f^2 \\
&\quad - \frac{362555}{648} \zeta_3 N_f^2 - \frac{174483}{32} \zeta_3^2 - \frac{16775}{16} \zeta_6 N_f - \frac{1943}{216} \zeta_4 N_f^2 - \frac{1107}{8} \zeta_3^2 N_f \\
&\quad - \frac{229}{27} \zeta_3 N_f^3 + \frac{400}{9} \zeta_3^2 N_f^2 + \frac{880}{27} \zeta_5 N_f^3 + \frac{10000}{6561} N_f^4 + \frac{204525}{32} \zeta_6 \\
&\quad + \frac{469333}{576} \zeta_4 N_f + \frac{27270475}{864} \zeta_3 N_f + \frac{60587905}{864} \zeta_5 N_f + \frac{190787741}{10368} N_f^2 \\
&\quad \left. + \frac{277127487}{2048} \zeta_7 - \zeta_4 N_f^3 \right] a^4 + O(a^5) \tag{3.1}
\end{aligned}$$

$$\begin{aligned}
C_c(a, 0)|^{SU(3)} &= 1 + 3a + \left[ \frac{5829}{64} - \frac{135}{16} \zeta_3 - \frac{95}{16} N_f \right] a^2 \\
&+ \left[ \frac{5161}{648} N_f^2 - \frac{198001}{432} N_f - \frac{40449}{64} \zeta_3 - \frac{1755}{32} \zeta_5 - \frac{33}{4} \zeta_4 N_f + \frac{2}{3} \zeta_3 N_f^2 \right. \\
&\quad \left. + \frac{59}{2} \zeta_3 N_f + \frac{243}{64} \zeta_4 + \frac{1082353}{288} \right] a^3
\end{aligned}$$

$$\begin{aligned}
& + \left[ \frac{15567976783}{73728} - \frac{151911987}{4096} \zeta_3 - \frac{79190001}{2048} \zeta_5 - \frac{39621021}{1024} N_f \right. \\
& - \frac{204525}{64} \zeta_6 - \frac{150979}{7776} N_f^3 - \frac{52025}{128} \zeta_4 N_f - \frac{11907}{16} \zeta_7 N_f - \frac{699}{8} \zeta_5 N_f^2 \\
& - \frac{425}{48} \zeta_3 N_f^2 - \frac{5}{18} \zeta_3 N_f^3 + \frac{1}{2} \zeta_4 N_f^3 + \frac{165}{16} \zeta_4 N_f^2 + \frac{337}{4} \zeta_3^2 N_f + \frac{16775}{32} \zeta_6 N_f \\
& + \frac{250085}{64} \zeta_5 N_f + \frac{1149471}{512} \zeta_3^2 + \frac{2625583}{768} \zeta_3 N_f + \frac{5819877}{4096} \zeta_4 + \frac{8093153}{4608} N_f^2 \\
& \left. + \frac{100880073}{8192} \zeta_7 \right] a^4 + O(a^5) \tag{3.2}
\end{aligned}$$

and

$$\begin{aligned}
C_\psi(a, 0)|^{SU(3)} & = 1 + \left[ \frac{7}{3} N_f + 12 \zeta_3 - \frac{359}{9} \right] a^2 \\
& + \left[ \frac{24722}{81} N_f - \frac{439543}{162} - \frac{1570}{243} N_f^2 - \frac{1165}{3} \zeta_5 - \frac{440}{9} \zeta_3 N_f + \frac{79}{4} \zeta_4 \right. \\
& \left. + \frac{8009}{6} \zeta_3 \right] a^3 \\
& + \left[ \frac{21391}{1458} N_f^3 - \frac{356864009}{5184} \zeta_5 - \frac{146722043}{864} - \frac{29889697}{5184} \zeta_3^2 - \frac{1294381}{108} \zeta_3 N_f \right. \\
& - \frac{1276817}{972} N_f^2 - \frac{440}{9} \zeta_5 N_f^2 - \frac{20}{3} \zeta_4 N_f^2 + \frac{8}{27} \zeta_3 N_f^3 + \frac{100}{3} \zeta_6 N_f \\
& + \frac{2291}{72} \zeta_4 N_f + \frac{5704}{27} \zeta_3 N_f^2 + \frac{565939}{864} \zeta_4 + \frac{1673051}{324} \zeta_5 N_f + \frac{3807625}{10368} \zeta_6 \\
& \left. + \frac{6747755}{288} \zeta_7 + \frac{55476671}{1944} N_f + \frac{317781451}{2592} \zeta_3 - 1029 \zeta_7 N_f - 24 \zeta_3^2 N_f \right] a^4 \\
& + O(a^5) \tag{3.3}
\end{aligned}$$

for the fields and

$$\begin{aligned}
C_m(a, 0)|^{SU(3)} & = 1 - \frac{16}{3} a + \left[ \frac{83}{9} N_f + \frac{152}{3} \zeta_3 - \frac{3779}{18} \right] a^2 \\
& + \left[ \frac{217390}{243} N_f - \frac{3115807}{324} - \frac{7514}{729} N_f^2 - \frac{4720}{27} \zeta_3 N_f - \frac{2960}{9} \zeta_5 - \frac{32}{27} \zeta_3 N_f^2 \right. \\
& \left. + \frac{80}{3} \zeta_4 N_f + \frac{195809}{54} \zeta_3 \right] a^3 \\
& + \left[ \frac{40}{81} \zeta_3 N_f^3 - \frac{744609145}{1296} - \frac{52383125}{17496} N_f^2 - \frac{3837631}{1728} \zeta_7 - \frac{2017309}{81} \zeta_3 N_f \right. \\
& - \frac{843077}{54} \zeta_3^2 - \frac{359855}{81} \zeta_5 N_f - \frac{16960}{9} \zeta_4 - \frac{11500}{9} \zeta_6 N_f - \frac{8776}{27} \zeta_3^2 N_f \\
& - \frac{343}{2} \zeta_7 N_f - \frac{100}{3} \zeta_4 N_f^2 - \frac{8}{9} \zeta_4 N_f^3 + \frac{560}{3} \zeta_5 N_f^2 + \frac{11542}{9} \zeta_4 N_f \\
& + \frac{33964}{81} \zeta_3 N_f^2 + \frac{96979}{4374} N_f^3 + \frac{9369745}{432} \zeta_5 + \frac{86284171}{324} \zeta_3 + \frac{247516535}{2916} N_f \\
& \left. + 5500 \zeta_6 \right] a^4 + O(a^5) \tag{3.4}
\end{aligned}$$

for the quark mass renormalization where  $\zeta_n$  is the Riemann zeta function and  $N_f$  is the number of quark flavours. One minor check on the derivation of these expressions is that they are finite with respect to  $\epsilon$  given that they derive from the ratio of four loop renormalization constants.

The conversion functions coupled with the five loop  $\overline{\text{MS}}$  anomalous dimensions derived in [5, 7, 36, 37, 38] for an arbitrary colour group mean that we can extract the corresponding

mMOM expressions to the same loop order via (2.6). Again restricting to the case of Landau gauge and the  $SU(3)$  we arrive at our main results which are

$$\begin{aligned}
\gamma_A^{\text{mMOM}}(a, 0) \Big|^{SU(3)} &= \left[ \frac{2}{3} N_f - \frac{13}{2} \right] a + \left[ \frac{67}{6} N_f - \frac{255}{4} \right] a^2 \\
&+ \left[ \frac{11227}{24} N_f - \frac{8637}{4} - \frac{719}{54} N_f^2 - \frac{229}{12} \zeta_3 N_f - \frac{8}{9} \zeta_3 N_f^2 + 324 \zeta_3 \right] a^3 \\
&+ \left[ \frac{5549393}{192} N_f - \frac{27189875}{256} - \frac{1118977}{648} N_f^2 - \frac{921265}{144} \zeta_5 N_f \right. \\
&\quad \left. - \frac{889231}{144} \zeta_3 N_f - \frac{16}{9} \zeta_3 N_f^3 + \frac{665}{27} N_f^3 + \frac{5143}{27} \zeta_3 N_f^2 + \frac{9280}{27} \zeta_5 N_f^2 \right. \\
&\quad \left. + \frac{1950705}{128} \zeta_5 + \frac{7740879}{256} \zeta_3 \right] a^4 \\
&+ \left[ \frac{1760}{27} \zeta_5 N_f^4 - \frac{71363464263}{16384} \zeta_7 - \frac{16520894997}{2048} - \frac{16359945025}{6912} \zeta_5 N_f \right. \\
&\quad \left. - \frac{8539017539}{20736} \zeta_3 N_f + \frac{3150668061}{2048} \zeta_3 + \frac{23633674547}{18432} \zeta_7 N_f \right. \\
&\quad \left. + \frac{29623505625}{4096} \zeta_5 + \frac{327291152977}{124416} N_f - \frac{613783375}{2592} N_f^2 \right. \\
&\quad \left. - \frac{25237429}{432} \zeta_7 N_f^2 - \frac{979887}{64} \zeta_3^2 N_f - \frac{292855}{54} \zeta_5 N_f^3 - \frac{51647}{36} \zeta_3^2 N_f^2 \right. \\
&\quad \left. - \frac{1861}{27} N_f^4 - \frac{304}{27} \zeta_3 N_f^4 + \frac{2240}{27} \zeta_3^2 N_f^3 + \frac{129545}{162} \zeta_3 N_f^3 \right. \\
&\quad \left. + \frac{2420705}{324} N_f^3 + \frac{13148441}{2592} \zeta_3 N_f^2 + \frac{246191965}{1296} \zeta_5 N_f^2 \right. \\
&\quad \left. + \frac{671260095}{2048} \zeta_3^2 \right] a^5 + O(a^6) \tag{3.5}
\end{aligned}$$

$$\begin{aligned}
\gamma_c^{\text{mMOM}}(a, 0) \Big|^{SU(3)} &= -\frac{9}{4} a + \left[ \frac{3}{4} N_f - \frac{153}{8} \right] a^2 \\
&+ \left[ \frac{1401}{16} N_f - \frac{11691}{16} - \frac{5}{2} N_f^2 + \frac{9}{4} \zeta_3 N_f + \frac{1269}{16} \zeta_3 \right] a^3 \\
&+ \left[ \frac{5}{2} N_f^3 - \frac{16985133}{512} - \frac{16059}{32} \zeta_3 N_f - \frac{13115}{48} N_f^2 - \frac{5895}{16} \zeta_5 N_f \right. \\
&\quad \left. + \frac{739053}{128} N_f + \frac{1140075}{256} \zeta_5 + \frac{2264193}{512} \zeta_3 + 26 \zeta_3 N_f^2 \right] a^4 \\
&+ \left[ \frac{9758887695}{8192} \zeta_5 - \frac{15114361941}{32768} \zeta_7 - \frac{9464865363}{4096} - \frac{34433025}{256} \zeta_5 N_f \right. \\
&\quad \left. - \frac{12230645}{288} N_f^2 - \frac{86231}{2} \zeta_3 N_f - \frac{3969}{2} \zeta_7 N_f^2 - \frac{219}{2} \zeta_3^2 N_f^2 \right. \\
&\quad \left. + \frac{15353}{12} N_f^3 + \frac{477395}{192} \zeta_3 N_f^2 + \frac{550645}{96} \zeta_5 N_f^2 + \frac{814149}{128} \zeta_3^2 N_f \right. \\
&\quad \left. + \frac{34354859}{1024} \zeta_7 N_f + \frac{1106092719}{4096} \zeta_3 + \frac{1670942731}{3072} N_f \right. \\
&\quad \left. - \frac{424982943}{4096} \zeta_3^2 - 65 \zeta_5 N_f^3 - 14 N_f^4 + \zeta_3 N_f^3 \right] a^5 + O(a^6) \tag{3.6}
\end{aligned}$$

$$\begin{aligned}
\gamma_\psi^{\text{mMOM}}(a, 0) \Big|^{SU(3)} &= \left[ \frac{67}{3} - \frac{4}{3} N_f \right] a^2 + \left[ \frac{29675}{36} - \frac{706}{9} N_f - \frac{607}{2} \zeta_3 + \frac{8}{9} N_f^2 + 16 \zeta_3 N_f \right] a^3 \\
&+ \left[ \frac{31003343}{648} - \frac{21683117}{648} \zeta_3 - \frac{2393555}{324} N_f - \frac{272}{9} \zeta_3 N_f^2 - \frac{40}{9} N_f^3 \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{2861}{9} N_f^2 + \frac{74440}{27} \zeta_3 N_f + \frac{15846715}{1296} \zeta_5 - 830 \zeta_5 N_f \Big] a^4 \\
& + \left[ \frac{2313514793}{10368} \zeta_3^2 - \frac{94958116621}{31104} \zeta_3 - \frac{26588447977}{27648} \zeta_7 \right. \\
& + \frac{14723323093}{5184} + \frac{18607183745}{7776} \zeta_5 - \frac{667846415}{1944} \zeta_5 N_f \\
& - \frac{251804567}{432} N_f - \frac{4726621}{243} \zeta_3 N_f^2 - \frac{596849}{54} \zeta_3^2 N_f - \frac{80606}{81} N_f^3 \\
& - \frac{6811}{2} \zeta_7 N_f^2 - \frac{3520}{27} \zeta_5 N_f^3 - \frac{128}{9} \zeta_3^2 N_f^2 + \frac{160}{27} N_f^4 + \frac{24064}{81} \zeta_3 N_f^3 \\
& + \frac{1063237}{27} N_f^2 + \frac{3085750}{243} \zeta_5 N_f^2 + \frac{4429579}{36} \zeta_7 N_f \\
& \left. + \frac{1748707267}{3888} \zeta_3 N_f \right] a^5 + O(a^6) \tag{3.7}
\end{aligned}$$

and

$$\begin{aligned}
\gamma_m^{\text{mMOM}}(a, 0) \Big|^{SU(3)} & = -4a + \left[ \frac{4}{3} N_f - \frac{209}{3} \right] a^2 \\
& + \left[ \frac{5635}{6} \zeta_3 - \frac{95383}{36} - \frac{176}{9} \zeta_3 N_f - \frac{8}{3} N_f^2 + \frac{4742}{27} N_f \right] a^3 \\
& + \left[ \frac{8}{3} N_f^3 - \frac{182707879}{1296} - \frac{309295}{48} \zeta_5 - \frac{159817}{27} \zeta_3 N_f - \frac{13651}{27} N_f^2 \right. \\
& \left. - \frac{3200}{9} \zeta_5 N_f + \frac{1552}{9} \zeta_3 N_f^2 + \frac{5246557}{324} N_f + \frac{15752321}{216} \zeta_3 \right] a^4 \\
& + \left[ \frac{3576071485}{27648} \zeta_7 - \frac{75504232175}{7776} + \frac{9610932889}{5832} N_f + \frac{17917034005}{31104} \zeta_5 \right. \\
& + \frac{187324052147}{31104} \zeta_3 - \frac{310328447}{432} \zeta_3^2 - \frac{257106335}{324} \zeta_3 N_f \\
& - \frac{180251015}{1944} \zeta_5 N_f - \frac{22459484}{243} N_f^2 - \frac{4778536}{81} \zeta_7 N_f - \frac{60928}{81} \zeta_3^2 N_f^2 \\
& - \frac{28096}{81} \zeta_3 N_f^3 - \frac{1600}{9} \zeta_5 N_f^3 - \frac{352}{27} N_f^4 + \frac{1372}{3} \zeta_7 N_f^2 + \frac{464038}{243} N_f^3 \\
& \left. + \frac{948548}{27} \zeta_3 N_f^2 + \frac{1850845}{243} \zeta_5 N_f^2 + \frac{6570181}{162} \zeta_3^2 N_f \right] a^5 \\
& + O(a^6) \tag{3.8}
\end{aligned}$$

where the variables are in the scheme indicated by the label on the renormalization group function. While we have recorded the Landau gauge expressions for the field and quark mass anomalous dimensions an independent check on our formalism and results is that we have reproduced the five loop mMOM  $\beta$ -function for non-zero  $\alpha$  and a general colour group. Although the Landau gauge result was provided in [25] the full gauge dependent result was given in the associated data file of the arXiv version of that publication. We have reproduced this precisely for all  $\alpha$  which means we have used the correct mapping of the mMOM values of  $a$  and  $\alpha$  to their  $\overline{\text{MS}}$  counterparts, and their inverses, in applying the equation analogous to that of (2.6) for the  $\beta$ -function. The same data file contains the coupling constant map, which we have independently verified, but that for the gauge parameter was not available. For completeness we record the  $SU(3)$  version for that mapping in the Appendix for  $\alpha \neq 0$  in our conventions. The full colour group expression is available in the arXiv data file associated with this article.



## 4 Discussion.

To summarize we have derived the renormalization group functions for the gluon, ghost and quark in QCD at five loops in the mMOM scheme as well as the quark mass anomalous dimension. With the five loop mMOM  $\beta$ -function of [25] the set of core QCD renormalization group functions in that scheme is now complete since the running of the linear covariant gauge parameter is given by  $-\gamma_A^{\text{mMOM}}(a, \alpha)$ . Although the  $\beta$ -function is ordinarily of primary importance since it governs the running of the gauge coupling constant, with applications to phenomenology [25], the anomalous dimensions are important for conformal window studies, [21, 22, 23, 24]. These can now be extended to higher order in relation to the known Banks-Zaks infrared fixed point in the mMOM scheme. While the conformal window has been studied at length in the case of the  $SU(3)$  group due to the connection with the strong force of the Standard Model, our results for an arbitrary colour group mean that the properties of gauge groups can be explored. Moreover, such investigations are not limited to the case where quarks are in the fundamental representation since the properties of the conformal window can be explored for other representations that will have applications for beyond the Standard Model physics. For instance there have been several lattice studies of some of these issues that have relied on the perturbative mMOM renormalization group functions, [51, 52]. In the former article the conformal window was examined in  $SU(2)$  when there are three Majorana fermions in the adjoint representation with the aim of understanding the scaling dimensions of bound states in the theory. By contrast in [52] the mMOM scheme results were also of importance in exploring ideas concerning lepton compositeness in the Standard Model. So five loop mMOM expressions should assist with improving the precision of such analyses.

**Acknowledgements.** This work was carried out with the support of the STFC Consolidated Grant ST/T000988/1 (JAG) and an EPSRC Studentship EP/R513271/1 (RHM). For the purpose of open access, the authors have applied a Creative Commons Attribution (CC-BY) licence to any Author Accepted Manuscript version arising. The data representing the main results here are accessible in electronic form from the arXiv ancillary directory associated with the article.

## A Gauge parameter dependent expressions.

We devote the Appendix to providing the full gauge dependence of the  $SU(3)$  anomalous dimensions so that the degree of the polynomial coefficients in  $\alpha$  at each loop order is manifest. We have

$$\begin{aligned}
\gamma_A^{\text{mMOM}}(a, \alpha) \Big|^{SU(3)} &= \left[ \frac{2}{3}N_f + \frac{3}{2}\alpha - \frac{13}{2} \right] a \\
&+ \left[ \frac{67}{6}N_f - \frac{255}{4} - \frac{9}{4}\alpha^3 + \frac{51}{8}\alpha + \frac{51}{8}\alpha^2 - N_f\alpha - N_f\alpha^2 \right] a^2 \\
&+ \left[ \frac{11227}{24}N_f - \frac{8637}{4} - \frac{1161}{16}\zeta_3\alpha - \frac{719}{54}N_f^2 - \frac{495}{32}\alpha^4 - \frac{237}{8}N_f\alpha^2 \right. \\
&\quad \left. - \frac{229}{12}\zeta_3N_f - \frac{189}{16}\alpha^3 - \frac{139}{8}N_f\alpha - \frac{9}{2}\zeta_3N_f\alpha - \frac{9}{4}N_f\alpha^3 - \frac{8}{9}\zeta_3N_f^2 \right. \\
&\quad \left. + \frac{3}{8}N_f\alpha^4 + \frac{9}{4}\zeta_3N_f\alpha^2 + \frac{81}{16}\zeta_3\alpha^3 + \frac{153}{16}\alpha + \frac{5283}{32}\alpha^2 - 54\zeta_3\alpha^2 \right. \\
&\quad \left. + 324\zeta_3 \right] a^3
\end{aligned}$$

$$\begin{aligned}
& + \left[ \frac{5549393}{192} N_f - \frac{27189875}{256} - \frac{1815015}{128} \zeta_3 \alpha - \frac{1118977}{648} N_f^2 \right. \\
& - \frac{921265}{144} \zeta_5 N_f - \frac{889231}{144} \zeta_3 N_f - \frac{231705}{64} \zeta_3 \alpha^2 - \frac{98423}{64} N_f \alpha^2 \\
& - \frac{65295}{128} \alpha^4 - \frac{50291}{32} N_f \alpha - \frac{20727}{256} \alpha^3 - \frac{11745}{128} \alpha^5 - \frac{3579}{32} N_f \alpha^3 \\
& - \frac{1485}{8} \zeta_5 N_f \alpha - \frac{945}{32} \zeta_5 \alpha^5 - \frac{495}{64} \zeta_5 \alpha^4 - \frac{315}{8} \zeta_5 N_f \alpha^2 - \frac{105}{16} \zeta_5 N_f \alpha^4 \\
& - \frac{16}{3} \zeta_3 N_f^2 \alpha^2 - \frac{16}{9} \zeta_3 N_f^3 - \frac{15}{8} \zeta_3 N_f \alpha^4 - \frac{8}{9} N_f^3 \alpha + \frac{9}{4} N_f \alpha^5 \\
& + \frac{15}{8} \zeta_5 N_f \alpha^3 + \frac{27}{4} \zeta_3 \alpha^5 + \frac{32}{9} \zeta_3 N_f^3 \alpha + \frac{123}{4} \zeta_3 N_f \alpha^3 + \frac{417}{32} N_f \alpha^4 \\
& + \frac{637}{8} N_f^2 \alpha + \frac{665}{27} N_f^3 + \frac{1375}{36} N_f^2 \alpha^2 + \frac{5143}{27} \zeta_3 N_f^2 + \frac{6013}{16} \zeta_3 N_f \alpha^2 \\
& + \frac{9280}{27} \zeta_5 N_f^2 + \frac{18939}{8} \zeta_3 N_f \alpha + \frac{19215}{128} \zeta_3 \alpha^3 + \frac{23175}{128} \zeta_5 \alpha^3 \\
& + \frac{52821}{256} \zeta_3 \alpha^4 + \frac{336285}{128} \zeta_5 \alpha^2 + \frac{753165}{128} \zeta_5 \alpha + \frac{1028403}{256} \alpha \\
& \left. + \frac{1386747}{256} \alpha^2 + \frac{1950705}{128} \zeta_5 + \frac{7740879}{256} \zeta_3 - 170 \zeta_3 N_f^2 \alpha \right] a^4 \\
& + \left[ \frac{1760}{27} \zeta_5 N_f^4 - \frac{71363464263}{16384} \zeta_7 - \frac{16520894997}{2048} \right. \\
& - \frac{16359945025}{6912} \zeta_5 N_f - \frac{8999200665}{16384} \zeta_7 \alpha - \frac{8539017539}{20736} \zeta_3 N_f \\
& + \frac{2743735509}{4096} \zeta_5 \alpha + \frac{3150668061}{2048} \zeta_3 + \frac{3614875345}{18432} \alpha \\
& + \frac{23633674547}{18432} \zeta_7 N_f + \frac{29623505625}{4096} \zeta_5 + \frac{327291152977}{124416} N_f \\
& - \frac{1470309201}{2048} \zeta_3 \alpha - \frac{613783375}{2592} N_f^2 - \frac{498113341}{4608} N_f \alpha^2 \\
& - \frac{453256775}{4608} N_f \alpha - \frac{89434611}{2048} \zeta_3^2 \alpha - \frac{35716707}{2048} \zeta_5 \alpha^3 \\
& - \frac{34041915}{2048} \alpha^4 - \frac{32200263}{4096} \zeta_5 \alpha^4 - \frac{32074623}{512} \zeta_7 \alpha^2 \\
& - \frac{26500527}{2048} \alpha^3 - \frac{25237429}{432} \zeta_7 N_f^2 - \frac{24649187}{384} \zeta_5 N_f \alpha \\
& - \frac{14459067}{16384} \zeta_7 \alpha^4 - \frac{11789955}{64} \zeta_3 \alpha^2 - \frac{11441745}{16384} \zeta_7 \alpha^5 \\
& - \frac{5644053}{1024} \zeta_3^2 \alpha^2 - \frac{4574791}{288} \zeta_3 N_f^2 \alpha - \frac{3830409}{1024} \zeta_3^2 \alpha^3 - \frac{2526579}{512} \alpha^5 \\
& - \frac{1410255}{4096} \zeta_5 \alpha^5 - \frac{1348137}{2048} \zeta_3^2 \alpha^4 - \frac{1058511}{512} N_f \alpha^3 - \frac{979887}{64} \zeta_3^2 N_f \\
& - \frac{839241}{256} \zeta_4 \alpha - \frac{746433}{128} \zeta_3 N_f \alpha^3 - \frac{520425}{1024} \zeta_5 \alpha^6 - \frac{407025}{512} \zeta_6 \alpha^2 \\
& - \frac{318969}{1024} \zeta_7 N_f \alpha^3 - \frac{292855}{54} \zeta_5 N_f^3 - \frac{192423}{256} \zeta_3 N_f \alpha^4 \\
& - \frac{186725}{144} \zeta_5 N_f^2 \alpha^2 - \frac{151875}{256} \zeta_6 N_f \alpha - \frac{112077}{2048} \zeta_7 N_f \alpha^4 \\
& - \frac{100115}{144} \zeta_3 N_f^2 \alpha^2 - \frac{95607}{1024} \zeta_3 \alpha^6 - \frac{76275}{512} \zeta_6 \alpha^4 - \frac{54405}{128} \alpha^6 \\
& \left. - \frac{51647}{36} \zeta_3^2 N_f^2 - \frac{50787}{512} \zeta_4 \alpha^3 - \frac{41993}{243} N_f^3 \alpha - \frac{14175}{256} \zeta_6 N_f \alpha^2 \right] a^4
\end{aligned}$$

$$\begin{aligned}
& -\frac{10725}{128}\zeta_5 N_f \alpha^5 - \frac{10125}{256}\zeta_6 N_f \alpha^3 - \frac{8967}{8}\zeta_7 N_f^2 \alpha - \frac{7569}{128}\zeta_3 N_f \alpha^5 \\
& -\frac{4131}{512}\zeta_4 \alpha^5 - \frac{2081}{24}N_f^2 \alpha^3 - \frac{1861}{27}N_f^4 - \frac{1549}{18}N_f^3 \alpha^2 \\
& -\frac{1431}{128}\zeta_4 N_f \alpha^2 - \frac{1131}{16}\zeta_4 N_f^2 \alpha - \frac{459}{128}\zeta_4 N_f \alpha^4 - \frac{304}{27}\zeta_3 N_f^4 \\
& -\frac{260}{3}\zeta_5 N_f^3 \alpha - \frac{32}{3}\zeta_3 N_f^3 \alpha^3 - \frac{21}{4}\zeta_3^2 N_f^2 \alpha^2 - \frac{15}{2}\zeta_5 N_f^2 \alpha^3 + \frac{5}{3}\zeta_5 N_f^3 \alpha^2 \\
& +\frac{8}{3}N_f^3 \alpha^3 + \frac{9}{64}\zeta_3 N_f \alpha^6 + \frac{63}{16}\zeta_4 N_f^2 \alpha^2 + \frac{315}{64}\zeta_5 N_f \alpha^6 + \frac{477}{64}N_f \alpha^6 \\
& +\frac{621}{64}\zeta_3^2 N_f \alpha^5 + \frac{1031}{2}\zeta_3 N_f^2 \alpha^3 + \frac{1053}{64}\zeta_4 N_f \alpha^3 + \frac{1575}{256}\zeta_6 N_f \alpha^4 \\
& +\frac{2240}{27}\zeta_3^2 N_f^3 + \frac{2961}{4}\zeta_4 N_f \alpha + \frac{3159}{64}\zeta_3^2 N_f \alpha^3 + \frac{3549}{1024}\zeta_7 N_f \alpha^5 \\
& +\frac{3610}{9}\zeta_3 N_f^3 \alpha + \frac{7065}{128}\zeta_3^2 N_f \alpha^4 + \frac{9387}{2}\zeta_3^2 N_f \alpha + \frac{14175}{1024}\zeta_6 \alpha^5 \\
& +\frac{19695}{16}N_f \alpha^4 + \frac{19791}{128}N_f \alpha^5 + \frac{27945}{512}\zeta_3^2 \alpha^6 + \frac{36525}{256}\zeta_5 N_f \alpha^4 \\
& +\frac{44145}{512}\zeta_4 \alpha^4 + \frac{51237}{128}\zeta_3^2 N_f \alpha^2 + \frac{66825}{256}\zeta_6 \alpha^3 + \frac{102619}{36}\zeta_5 N_f^2 \alpha \\
& +\frac{114615}{512}\zeta_4 \alpha^2 + \frac{129545}{162}\zeta_3 N_f^3 + \frac{159705}{8192}\zeta_7 \alpha^6 + \frac{316347}{256}\zeta_5 N_f \alpha^3 \\
& +\frac{609525}{2048}\zeta_3^2 \alpha^5 + \frac{2420705}{324}N_f^3 + \frac{2646567}{2048}\zeta_3 \alpha^5 + \frac{2928303}{1024}\zeta_7 N_f \alpha^2 \\
& +\frac{5923125}{1024}\zeta_6 \alpha + \frac{10840253}{1728}N_f^2 \alpha^2 + \frac{12012619}{384}\zeta_3 N_f \alpha^2 \\
& +\frac{13011521}{768}\zeta_5 N_f \alpha^2 + \frac{13148441}{2592}\zeta_3 N_f^2 + \frac{18387049}{96}\zeta_3 N_f \alpha \\
& +\frac{26611431}{512}\zeta_7 N_f \alpha + \frac{27183969}{2048}\zeta_3 \alpha^4 + \frac{28589639}{3456}N_f^2 \alpha \\
& +\frac{36371421}{1024}\zeta_3 \alpha^3 + \frac{47616093}{8192}\zeta_7 \alpha^3 + \frac{177532569}{2048}\zeta_5 \alpha^2 \\
& +\frac{246191965}{1296}\zeta_5 N_f^2 + \frac{671260095}{2048}\zeta_3^2 + \frac{1011473605}{3072}\alpha^2 \\
& -48\zeta_3^2 N_f^2 \alpha - 41N_f^2 \alpha^4 - 5\zeta_3 N_f^3 \alpha^2 + 3\zeta_4 N_f^3 \alpha + 15\zeta_3 N_f^2 \alpha^4] a^5 \\
& + O(a^6) \tag{A.1}
\end{aligned}$$

$$\begin{aligned}
\gamma_c^{\text{mMOM}}(a, \alpha) \Big|^{SU(3)} &= \left[ \frac{3}{4}\alpha - \frac{9}{4} \right] a + \left[ \frac{3}{4}N_f + \frac{27}{16}\alpha^2 - \frac{153}{8} - \frac{27}{16}\alpha \right] a^2 \\
&+ \left[ \frac{1401}{16}N_f - \frac{11691}{16} - \frac{405}{64}\alpha^3 - \frac{81}{16}\zeta_3 \alpha^2 - \frac{57}{16}N_f \alpha - \frac{9}{8}N_f \alpha^2 - \frac{5}{2}N_f^2 \right. \\
&\quad + \frac{9}{4}\zeta_3 N_f + \frac{9}{4}\zeta_3 N_f \alpha + \frac{189}{32}\zeta_3 \alpha + \frac{243}{8}\alpha^2 + \frac{243}{32}\zeta_3 \alpha^3 + \frac{981}{64}\alpha \\
&\quad \left. + \frac{1269}{16}\zeta_3 \right] a^3 \\
&+ \left[ \frac{5}{2}N_f^3 - \frac{16985133}{512} - \frac{182601}{256}\zeta_3 \alpha - \frac{158895}{256}\zeta_5 \alpha^2 - \frac{84699}{512}\alpha^3 \right. \\
&\quad - \frac{38205}{256}\zeta_5 \alpha^3 - \frac{24183}{128}N_f \alpha^2 - \frac{18117}{128}\alpha^4 - \frac{16059}{32}\zeta_3 N_f \\
&\quad \left. - \frac{15693}{64}N_f \alpha - \frac{13115}{48}N_f^2 - \frac{5895}{16}\zeta_5 N_f - \frac{585}{8}\zeta_5 N_f \alpha - \frac{459}{32}\zeta_3 N_f \alpha^2 \right] a^4
\end{aligned}$$

$$\begin{aligned}
& -\frac{27}{8}\zeta_3 N_f \alpha^3 + \frac{9}{8} N_f \alpha^3 + \frac{27}{64} N_f \alpha^4 + \frac{45}{8} N_f^2 \alpha^2 + \frac{45}{8} \zeta_3 N_f \alpha + \frac{49}{8} N_f^2 \alpha \\
& + \frac{315}{16} \zeta_5 N_f \alpha^2 + \frac{729}{128} \zeta_3 \alpha^2 + \frac{2565}{64} \zeta_5 \alpha^4 + \frac{31131}{512} \zeta_3 \alpha^4 + \frac{42417}{256} \zeta_3 \alpha^3 \\
& + \frac{94365}{256} \zeta_5 \alpha + \frac{658287}{512} \alpha^2 + \frac{739053}{128} N_f + \frac{801801}{512} \alpha + \frac{1140075}{256} \zeta_5 \\
& + \frac{2264193}{512} \zeta_3 + 6\zeta_3 N_f^2 \alpha + 26\zeta_3 N_f^2 \Big] a^4 \\
& + \left[ \frac{9758887695}{8192} \zeta_5 - \frac{15114361941}{32768} \zeta_7 - \frac{9464865363}{4096} - \frac{458876853}{4096} \zeta_3 \alpha \right. \\
& - \frac{226707579}{4096} \zeta_5 \alpha^2 - \frac{149725879}{9216} N_f \alpha^2 - \frac{107008841}{3072} N_f \alpha \\
& - \frac{40486293}{4096} \alpha^3 - \frac{36922473}{16384} \zeta_7 \alpha^3 - \frac{34433025}{256} \zeta_5 N_f \\
& - \frac{24205473}{4096} \alpha^4 - \frac{23064993}{32768} \zeta_7 \alpha^4 - \frac{14639049}{4096} \zeta_5 \alpha^3 \\
& - \frac{12230645}{288} N_f^2 - \frac{6663195}{2048} \zeta_7 N_f \alpha - \frac{1478493}{1024} \alpha^5 - \frac{581013}{512} \zeta_4 \alpha \\
& - \frac{492075}{1024} \zeta_6 \alpha^2 - \frac{397467}{4096} \zeta_3^2 \alpha^4 - \frac{390789}{512} \zeta_7 N_f \alpha^2 - \frac{144171}{256} \zeta_3^2 N_f \alpha \\
& - \frac{109683}{256} \zeta_3 N_f \alpha^3 - \frac{86231}{2} \zeta_3 N_f - \frac{66825}{1024} \zeta_6 \alpha^4 - \frac{15795}{1024} \zeta_4 \alpha^3 \\
& - \frac{6111}{128} \zeta_3 N_f \alpha^2 - \frac{4725}{128} \zeta_5 N_f \alpha^4 - \frac{4131}{1024} \zeta_4 \alpha^5 - \frac{3969}{2} \zeta_7 N_f^2 \\
& - \frac{3645}{8} \zeta_5 N_f^2 \alpha - \frac{3013}{64} \zeta_3 N_f^2 \alpha - \frac{435}{16} \zeta_5 N_f^2 \alpha^2 - \frac{405}{128} \zeta_4 N_f \alpha^3 \\
& - \frac{297}{8} \zeta_4 N_f \alpha^2 - \frac{223}{8} N_f^3 \alpha - \frac{219}{2} \zeta_3^2 N_f^2 - \frac{153}{4} \zeta_3 N_f^2 \alpha^2 - \frac{135}{32} N_f^2 \alpha^4 \\
& - \frac{81}{8} \zeta_4 N_f^2 \alpha - \frac{45}{4} \zeta_3^2 N_f^2 \alpha^2 - \frac{33}{2} \zeta_3^2 N_f^2 \alpha - \frac{27}{2} \zeta_3 N_f^2 \alpha^3 - \frac{15}{2} N_f^3 \alpha^2 \\
& + \frac{27}{4} \zeta_3 N_f \alpha^4 + \frac{27}{8} \zeta_4 N_f^2 \alpha^2 + \frac{81}{64} \zeta_3 N_f \alpha^5 + \frac{117}{16} N_f^2 \alpha^3 + \frac{297}{256} N_f \alpha^5 \\
& + \frac{11259}{256} \zeta_3^2 N_f \alpha^3 + \frac{14175}{2048} \zeta_6 \alpha^5 + \frac{15353}{12} N_f^3 + \frac{17307}{128} N_f \alpha^4 \\
& + \frac{17901}{128} \zeta_4 N_f \alpha + \frac{18225}{256} \zeta_6 \alpha^3 + \frac{29403}{128} \zeta_3^2 N_f \alpha^2 + \frac{38637}{1024} \zeta_4 \alpha^4 \\
& + \frac{46935}{512} \zeta_5 N_f \alpha^3 + \frac{198531}{1024} \zeta_4 \alpha^2 + \frac{220563}{4096} \zeta_3^2 \alpha^5 + \frac{304689}{1024} N_f \alpha^3 \\
& + \frac{318969}{2048} \zeta_7 N_f \alpha^3 + \frac{452709}{2048} \zeta_3^2 \alpha^2 + \frac{477395}{192} \zeta_3 N_f^2 + \frac{550645}{96} \zeta_5 N_f^2 \\
& + \frac{683829}{256} \zeta_5 N_f \alpha + \frac{687307}{864} N_f^2 \alpha^2 + \frac{814149}{128} \zeta_3^2 N_f + \frac{832923}{4096} \zeta_3 \alpha^5 \\
& + \frac{1582659}{2048} \zeta_3^2 \alpha^3 + \frac{1814049}{512} \zeta_5 N_f \alpha^2 + \frac{2473413}{256} \zeta_3 N_f \alpha \\
& + \frac{4100625}{2048} \zeta_6 \alpha + \frac{4414969}{2304} N_f^2 \alpha + \frac{4890807}{8192} \zeta_5 \alpha^4 + \frac{5021595}{8192} \zeta_5 \alpha^5 \\
& + \frac{5543559}{32768} \zeta_7 \alpha^5 + \frac{8992593}{2048} \zeta_3 \alpha^2 + \frac{13481559}{4096} \zeta_3^2 \alpha + \frac{13990239}{4096} \zeta_3 \alpha^4 \\
& + \frac{17310645}{2048} \zeta_3 \alpha^3 + \frac{34354859}{1024} \zeta_7 N_f + \frac{59929767}{8192} \zeta_5 \alpha \\
& + \frac{300532491}{16384} \zeta_7 \alpha^2 + \frac{523935217}{6144} \alpha^2 + \frac{694222603}{4096} \alpha + \frac{1106092719}{4096} \zeta_3
\end{aligned}$$

$$\begin{aligned}
& + \frac{1670942731}{3072} N_f + \frac{1894706667}{32768} \zeta_7 \alpha - \frac{424982943}{4096} \zeta_3^2 - 65 \zeta_5 N_f^3 \\
& - 14 N_f^4 + \zeta_3 N_f^3 + \zeta_3 N_f^3 \alpha + 15 \zeta_5 N_f^3 \alpha \Big] a^5 + O(a^6) \tag{A.2}
\end{aligned}$$

and

$$\begin{aligned}
\gamma_\psi^{\text{mMOM}}(a, \alpha) \Big|^{SU(3)} &= \frac{4}{3} \alpha a + \left[ 3\alpha^2 + 6\alpha - \frac{4}{3} N_f + \frac{67}{3} \right] a^2 \\
&+ \left[ \frac{29675}{36} - \frac{706}{9} N_f - \frac{607}{2} \zeta_3 - \frac{309}{2} \zeta_3 \alpha - \frac{45}{2} \zeta_3 \alpha^2 - \frac{8}{3} N_f \alpha + \frac{8}{9} N_f^2 \right. \\
&\quad \left. + \frac{9}{2} \zeta_3 \alpha^3 + \frac{49}{12} \alpha^3 + \frac{107}{4} \alpha^2 + \frac{427}{4} \alpha + 2 N_f \alpha^2 + 8 \zeta_3 N_f \alpha + 16 \zeta_3 N_f \right] a^3 \\
&+ \left[ \frac{31003343}{648} - \frac{21683117}{648} \zeta_3 - \frac{2393555}{324} N_f - \frac{51365}{48} \alpha^2 - \frac{17972}{3} \zeta_3 \alpha \right. \\
&\quad - \frac{13087}{54} N_f \alpha - \frac{2517}{8} \zeta_3 \alpha^2 - \frac{1841}{72} \alpha^4 - \frac{1640}{9} \zeta_5 N_f \alpha - \frac{272}{9} \zeta_3 N_f^2 \\
&\quad - \frac{135}{4} \zeta_5 \alpha^3 - \frac{68}{3} \zeta_3 N_f \alpha^2 - \frac{40}{9} N_f^3 - \frac{3}{4} N_f \alpha^4 + \frac{23}{2} N_f \alpha^3 + \frac{188}{27} N_f^2 \alpha \\
&\quad + \frac{235}{16} \zeta_5 \alpha^4 + \frac{341}{48} \alpha^3 + \frac{539}{9} \zeta_3 \alpha^4 + \frac{635}{4} N_f \alpha^2 + \frac{2381}{12} \zeta_3 \alpha^3 \\
&\quad + \frac{2861}{9} N_f^2 + \frac{3812}{9} \zeta_3 N_f \alpha + \frac{4465}{8} \zeta_5 \alpha^2 + \frac{34175}{12} \zeta_5 \alpha + \frac{74440}{27} \zeta_3 N_f \\
&\quad + \frac{194005}{72} \alpha + \frac{15846715}{1296} \zeta_5 - 830 \zeta_5 N_f - 12 \zeta_3 N_f \alpha^3 - 10 \zeta_5 N_f \alpha^2 \\
&\quad \left. - 2 N_f^2 \alpha^2 \right] a^4 \\
&+ \left[ \frac{2313514793}{10368} \zeta_3^2 - \frac{94958116621}{31104} \zeta_3 - \frac{26588447977}{27648} \zeta_7 \right. \\
&\quad + \frac{14723323093}{5184} + \frac{18607183745}{7776} \zeta_5 - \frac{667846415}{1944} \zeta_5 N_f \\
&\quad - \frac{512366237}{3456} \zeta_3 \alpha - \frac{409534595}{3456} \alpha^2 - \frac{251804567}{432} N_f \\
&\quad - \frac{100358063}{3456} \zeta_5 \alpha^2 - \frac{82530427}{768} \zeta_7 \alpha - \frac{35557175}{432} \zeta_5 \alpha \\
&\quad - \frac{10141489}{1296} \zeta_3 N_f \alpha^2 - \frac{4726621}{243} \zeta_3 N_f^2 - \frac{1141313}{144} \alpha^3 - \frac{841917}{128} \zeta_5 \alpha^3 \\
&\quad - \frac{596849}{54} \zeta_3^2 N_f - \frac{489041}{36} N_f \alpha - \frac{455879}{1728} \alpha^5 - \frac{223663}{243} N_f^2 \alpha^2 \\
&\quad - \frac{151875}{128} \zeta_6 \alpha^2 - \frac{137523}{128} \zeta_5 \alpha^4 - \frac{98477}{72} \zeta_3 N_f \alpha^3 - \frac{80606}{81} N_f^3 \\
&\quad - \frac{72515}{12} \zeta_3^2 N_f \alpha - \frac{44073}{128} \zeta_3^2 \alpha^4 - \frac{37779}{1024} \zeta_7 \alpha^4 - \frac{21651}{16} \zeta_3^2 \alpha^3 \\
&\quad - \frac{14175}{128} \zeta_6 \alpha^3 - \frac{11996}{27} \zeta_3 N_f^2 \alpha - \frac{10125}{128} \zeta_6 \alpha^4 - \frac{6811}{2} \zeta_7 N_f^2 \\
&\quad - \frac{3520}{27} \zeta_5 N_f^3 - \frac{1617}{32} \zeta_7 N_f \alpha^3 - \frac{663}{8} \zeta_4 N_f \alpha^2 - \frac{459}{64} \zeta_4 \alpha^5 - \frac{435}{8} N_f \alpha^4 \\
&\quad - \frac{190}{3} \zeta_5 N_f^2 \alpha^2 - \frac{139}{6} N_f^2 \alpha^3 - \frac{129}{4} \zeta_3^2 N_f \alpha^2 - \frac{128}{9} \zeta_3^2 N_f^2 - \frac{57}{8} N_f \alpha^5 \\
&\quad - \frac{45}{8} \zeta_4 N_f \alpha^3 + \frac{3}{2} N_f^2 \alpha^4 + \frac{9}{2} \zeta_3 N_f \alpha^5 + \frac{40}{3} N_f^3 \alpha^2 + \frac{128}{27} \zeta_3 N_f^3 \alpha \\
&\quad \left. + \frac{160}{3} \zeta_5 N_f^3 \alpha + \frac{160}{27} N_f^4 + \frac{225}{8} \zeta_5 N_f \alpha^4 + \frac{413}{24} \zeta_7 N_f \alpha^2 + \frac{729}{16} \zeta_4 \alpha^4 \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{1575}{128} \zeta_6 \alpha^5 + \frac{2403}{128} \zeta_3^2 \alpha^5 + \frac{3455}{8} \zeta_5 N_f \alpha^3 + \frac{6993}{64} \zeta_4 \alpha^3 \\
& + \frac{8083}{81} \zeta_3 N_f^2 \alpha^2 + \frac{14931}{256} \zeta_7 \alpha^5 + \frac{19490}{27} N_f^2 \alpha + \frac{21519}{32} \zeta_4 \alpha^2 \\
& + \frac{24064}{81} \zeta_3 N_f^3 + \frac{28951}{4} \zeta_3^2 \alpha^2 + \frac{33825}{128} \zeta_5 \alpha^5 + \frac{82327}{24} \zeta_5 N_f \alpha^2 \\
& + \frac{105389}{384} \zeta_3 \alpha^5 + \frac{123931}{96} N_f \alpha^3 + \frac{292055}{1152} \alpha^4 + \frac{414431}{128} \zeta_3 \alpha^4 \\
& + \frac{442155}{128} \zeta_7 \alpha^3 + \frac{641081}{96} \zeta_7 N_f \alpha + \frac{817061}{1536} \zeta_7 \alpha^2 + \frac{1030331}{72} \zeta_3 \alpha^3 \\
& + \frac{1063237}{27} N_f^2 + \frac{1295257}{72} \zeta_3 N_f \alpha + \frac{2129515}{72} \zeta_5 N_f \alpha + \frac{3085750}{243} \zeta_5 N_f^2 \\
& + \frac{4429579}{36} \zeta_7 N_f + \frac{13408341}{128} \zeta_3^2 \alpha + \frac{31261961}{432} \zeta_3 \alpha^2 \\
& + \frac{52889729}{2592} N_f \alpha^2 + \frac{138867799}{1728} \alpha + \frac{1748707267}{3888} \zeta_3 N_f - 2475 \zeta_5 N_f^2 \alpha \\
& - 41 \zeta_3 N_f \alpha^4 - 8 N_f^3 \alpha - 8 \zeta_3^2 N_f^2 \alpha^2 + 6 \zeta_4 N_f^2 \alpha^2 + 24 \zeta_3^2 N_f^2 \alpha \\
& + 128 \zeta_3^2 N_f \alpha^3 \Big] a^5 + O(a^6) \tag{A.3}
\end{aligned}$$

for the three fields of the Lagrangian. The mMOM scheme quark mass anomalous dimension is

$$\begin{aligned}
\gamma_m^{\text{mMOM}}(a, \alpha) \Big|^{SU(3)} &= -4a + \left[ \alpha^2 - \frac{209}{3} + \frac{4}{3} N_f \right] a^2 \\
&+ \left[ \frac{5635}{6} \zeta_3 - \frac{95383}{36} - \frac{176}{9} \zeta_3 N_f - \frac{27}{2} \zeta_3 \alpha^2 - \frac{8}{3} N_f^2 + \frac{23}{12} \alpha^3 + \frac{47}{4} \right. \\
&\quad \left. + \frac{459}{4} \alpha^2 + \frac{4742}{27} N_f - 2 N_f \alpha^2 + 2 N_f \alpha + 99 \zeta_3 \alpha \right] a^3 \\
&+ \left[ \frac{8}{3} N_f^3 - \frac{182707879}{1296} - \frac{309295}{48} \zeta_5 - \frac{159817}{27} \zeta_3 N_f - \frac{47377}{18} \zeta_3 \alpha^2 \right. \\
&\quad - \frac{13651}{27} N_f^2 - \frac{7965}{4} \zeta_5 \alpha - \frac{6821}{144} \alpha^4 - \frac{4202}{9} \zeta_3 N_f \alpha - \frac{3200}{9} \zeta_5 N_f \\
&\quad - \frac{3007}{12} \zeta_3 \alpha^3 - \frac{2413}{6} N_f \alpha^2 - \frac{1575}{8} \zeta_5 \alpha^2 - \frac{176}{27} N_f^2 \alpha - \frac{63}{8} \zeta_3 \alpha^4 \\
&\quad - \frac{45}{4} \zeta_5 \alpha^3 - \frac{21}{2} N_f \alpha^3 + \frac{3}{4} N_f \alpha^4 + \frac{315}{16} \zeta_5 \alpha^4 + \frac{608}{27} \zeta_3 N_f^2 \alpha \\
&\quad + \frac{1019}{108} N_f \alpha + \frac{1265}{27} \zeta_3 N_f \alpha^2 + \frac{1552}{9} \zeta_3 N_f^2 + \frac{10571}{24} \alpha^3 + \frac{12541}{4} \zeta_3 \alpha \\
&\quad \left. + \frac{333899}{48} \alpha^2 + \frac{480463}{144} \alpha + \frac{5246557}{324} N_f + \frac{15752321}{216} \zeta_3 + 6 N_f^2 \alpha^2 \right] a^4 \\
&+ \left[ \frac{3576071485}{27648} \zeta_7 - \frac{75504232175}{7776} + \frac{9610932889}{5832} N_f + \frac{17917034005}{31104} \zeta_5 \right. \\
&\quad + \frac{187324052147}{31104} \zeta_3 - \frac{448693433}{1728} \zeta_3 \alpha^2 - \frac{310328447}{432} \zeta_3^2 \\
&\quad - \frac{257106335}{324} \zeta_3 N_f - \frac{180251015}{1944} \zeta_5 N_f - \frac{135482155}{2592} N_f \alpha^2 \\
&\quad - \frac{51199769}{384} \zeta_5 \alpha - \frac{22459484}{243} N_f^2 - \frac{13215491}{576} \zeta_3 \alpha^3 - \frac{9488839}{288} N_f \alpha \\
&\quad - \frac{6906043}{1152} \zeta_3 \alpha - \frac{4778536}{81} \zeta_7 N_f - \frac{1261649}{16} \zeta_3^2 \alpha - \frac{540175}{144} \alpha^4 \\
&\quad \left. - \frac{421721}{864} \alpha^5 - \frac{363001}{36} \zeta_3 N_f \alpha - \frac{126619}{48} \zeta_5 \alpha^2 - \frac{75565}{48} N_f \alpha^3 \right]
\end{aligned}$$

$$\begin{aligned}
& -\frac{64557}{32}\zeta_4\alpha - \frac{60928}{81}\zeta_3^2 N_f^2 - \frac{47287}{81}\zeta_3 N_f^2 \alpha^2 - \frac{28096}{81}\zeta_3 N_f^3 \\
& -\frac{26173}{8}\zeta_7 N_f \alpha - \frac{20979}{64}\zeta_4 \alpha^2 - \frac{17307}{64}\zeta_3^2 \alpha^4 - \frac{10647}{1024}\zeta_7 \alpha^5 \\
& -\frac{4725}{128}\zeta_6 \alpha^4 - \frac{2187}{16}\zeta_4 \alpha^3 - \frac{1863}{64}\zeta_3^2 \alpha^5 - \frac{1600}{9}\zeta_5 N_f^3 - \frac{830}{3}\zeta_5 N_f^2 \alpha \\
& -\frac{617}{18}\zeta_3 N_f \alpha^4 - \frac{603}{2}\zeta_3^2 N_f \alpha^2 - \frac{352}{27}N_f^4 - \frac{255}{8}\zeta_5 N_f \alpha^3 - \frac{175}{16}\zeta_7 N_f \alpha^2 \\
& -\frac{152}{3}\zeta_3 N_f^2 \alpha^3 - \frac{128}{27}\zeta_3 N_f^3 \alpha - \frac{105}{8}\zeta_5 N_f \alpha^4 - \frac{10}{3}\zeta_5 N_f^2 \alpha^2 - \frac{9}{2}N_f^2 \alpha^4 \\
& +\frac{27}{4}N_f \alpha^5 + \frac{124}{9}N_f^3 \alpha + \frac{135}{8}\zeta_4 N_f \alpha^2 + \frac{223}{6}N_f^2 \alpha^3 + \frac{1037}{4}N_f \alpha^4 \\
& +\frac{1372}{3}\zeta_7 N_f^2 + \frac{1377}{64}\zeta_4 \alpha^4 + \frac{1989}{8}\zeta_4 N_f \alpha + \frac{7639}{9}\zeta_3 N_f^2 \alpha \\
& +\frac{10435}{6}\zeta_3^2 N_f \alpha + \frac{13915}{384}\zeta_3 \alpha^5 + \frac{15425}{8}\zeta_5 N_f \alpha^2 + \frac{21784}{81}N_f^2 \alpha \\
& +\frac{27735}{128}\zeta_5 \alpha^5 + \frac{30375}{128}\zeta_6 \alpha^3 + \frac{42525}{128}\zeta_6 \alpha^2 + \frac{46027}{36}\zeta_3 N_f \alpha^3 \\
& +\frac{64065}{128}\zeta_5 \alpha^4 + \frac{92475}{64}\zeta_3^2 \alpha^3 + \frac{95931}{32}\zeta_5 \alpha^3 + \frac{123604}{9}\zeta_5 N_f \alpha \\
& +\frac{253453}{162}N_f^2 \alpha^2 + \frac{316953}{512}\zeta_7 \alpha^2 + \frac{336231}{1024}\zeta_7 \alpha^4 + \frac{439047}{512}\zeta_7 \alpha^3 \\
& +\frac{455039}{384}\zeta_3 \alpha^4 + \frac{455625}{128}\zeta_6 \alpha + \frac{464038}{243}N_f^3 + \frac{466231}{64}\zeta_3^2 \alpha^2 \\
& +\frac{948548}{27}\zeta_3 N_f^2 + \frac{1850845}{243}\zeta_5 N_f^2 + \frac{6570181}{162}\zeta_3^2 N_f + \frac{6589163}{324}\zeta_3 N_f \alpha^2 \\
& +\frac{31369981}{1152}\alpha^3 + \frac{61494153}{1024}\zeta_7 \alpha + \frac{71828867}{144}\alpha^2 + \frac{1414667209}{3456}\alpha \\
& -18\zeta_4 N_f^2 \alpha - 9\zeta_3^2 N_f \alpha^3 - 8N_f^3 \alpha^2 \Big] a^5 + O(a^6). \tag{A.4}
\end{aligned}$$

The respective conversion functions that these are derived from are

$$\begin{aligned}
C_A(a, \alpha)|^{SU(3)} &= 1 + \left[ \frac{3}{2}\alpha + \frac{3}{4}\alpha^2 + \frac{97}{12} - \frac{10}{9}N_f \right] a \\
&+ \left[ \frac{83105}{288} - \frac{11299}{216}N_f - \frac{5}{3}N_f \alpha - \frac{4}{3}\zeta_3 N_f + \frac{45}{16}\alpha^3 + \frac{100}{81}N_f^2 + \frac{135}{16}\alpha^2 \right. \\
&\quad \left. + \frac{313}{32}\alpha - 27\zeta_3 + 18\zeta_3 \alpha \right] a^2 \\
&+ \left[ \frac{164395}{972}N_f^2 - \frac{8228977}{2592}N_f - \frac{63225}{64}\zeta_5 - \frac{17433}{8}\zeta_3 - \frac{8821}{144}N_f \alpha \right. \\
&\quad - \frac{3465}{32}\zeta_5 \alpha^2 - \frac{3105}{8}\zeta_5 \alpha - \frac{1000}{729}N_f^3 - \frac{477}{32}\zeta_3 \alpha^3 - \frac{469}{32}N_f \alpha^2 - \frac{243}{32}\zeta_4 \\
&\quad - \frac{117}{32}\zeta_3 \alpha^4 - \frac{81}{8}\zeta_4 \alpha - \frac{81}{32}\zeta_4 \alpha^2 + \frac{3}{4}\zeta_3 N_f \alpha^2 + \frac{14}{27}N_f^2 \alpha + \frac{16}{3}\zeta_3 N_f^2 \alpha \\
&\quad + \frac{16}{9}\zeta_3 N_f^2 + \frac{33}{2}\zeta_4 N_f + \frac{45}{8}\zeta_5 \alpha^3 + \frac{315}{64}\zeta_5 \alpha^4 + \frac{1197}{64}\alpha^4 \\
&\quad + \frac{1881}{32}\zeta_3 \alpha^2 + \frac{2320}{9}\zeta_5 N_f + \frac{11961}{128}\alpha^3 + \frac{25915}{108}\zeta_3 N_f + \frac{40539}{128}\alpha^2 \\
&\quad \left. + \frac{42933}{32}\zeta_3 \alpha + \frac{161965}{384}\alpha + \frac{44961125}{3456} - 161\zeta_3 N_f \alpha \right] a^3 \\
&+ \left[ \frac{62302764631}{82944} - \frac{1758762815}{7776}N_f - \frac{324121925}{2048}\zeta_3 - \frac{262747689}{1024}\zeta_5 \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{62931331}{13824}N_f\alpha - \frac{41645709}{1024}\zeta_5\alpha - \frac{25666081}{864}\zeta_7N_f - \frac{15059695}{34992}N_f^3 \\
& -\frac{8797395}{1024}\zeta_5\alpha^2 - \frac{4480985}{288}\zeta_3N_f\alpha - \frac{3319667}{2304}N_f\alpha^2 - \frac{3161781}{2048}\zeta_4 \\
& -\frac{826983}{1024}\zeta_4\alpha - \frac{515843}{162}\zeta_5N_f^2 - \frac{487965}{1024}\zeta_3\alpha^3 - \frac{468495}{2048}\zeta_3\alpha^4 \\
& -\frac{362555}{648}\zeta_3N_f^2 - \frac{174483}{32}\zeta_3^2 - \frac{170451}{512}\zeta_3^2\alpha - \frac{132615}{512}\zeta_7\alpha^3 \\
& -\frac{61911}{512}\zeta_4\alpha^2 - \frac{26901}{32}\zeta_7N_f\alpha - \frac{24975}{1024}\zeta_6\alpha^3 - \frac{23925}{64}\zeta_3N_f\alpha^2 \\
& -\frac{22275}{1024}\zeta_6\alpha^2 - \frac{19737}{512}\zeta_3^2\alpha^4 - \frac{16775}{16}\zeta_6N_f - \frac{14985}{512}\zeta_3\alpha^5 - \frac{10647}{4096}\zeta_7\alpha^5 \\
& -\frac{9621}{64}N_f\alpha^3 - \frac{4725}{1024}\zeta_6\alpha^4 - \frac{1943}{216}\zeta_4N_f^2 - \frac{1863}{256}\zeta_3^2\alpha^5 - \frac{1107}{8}\zeta_3^2N_f \\
& -\frac{995}{32}\zeta_5N_f\alpha^3 - \frac{513}{4}\zeta_3^2N_f\alpha - \frac{229}{27}\zeta_3N_f^3 - \frac{160}{9}\zeta_3N_f^3\alpha - \frac{115}{2}\zeta_5N_f^2\alpha \\
& -\frac{81}{1024}\zeta_4\alpha^3 - \frac{63}{8}\zeta_4N_f\alpha^2 - \frac{45}{4}\zeta_3^2N_f\alpha^2 - \frac{9}{4}\zeta_4N_f^2\alpha - \frac{9}{4}\zeta_3^2N_f\alpha^3 \\
& +\frac{5}{2}\zeta_5N_f^2\alpha^2 + \frac{83}{6}\zeta_3N_f^2\alpha^2 + \frac{279}{4}\zeta_4N_f\alpha + \frac{400}{9}\zeta_3^2N_f^2 + \frac{580}{243}N_f^3\alpha \\
& +\frac{880}{27}\zeta_5N_f^3 + \frac{2879}{3}\zeta_3N_f^2\alpha + \frac{5265}{2048}\zeta_4\alpha^4 + \frac{6647}{32}\zeta_5N_f\alpha^2 + \frac{6781}{432}N_f^2\alpha^2 \\
& +\frac{10000}{6561}N_f^4 + \frac{28575}{256}\zeta_5\alpha^4 + \frac{31635}{512}\zeta_5\alpha^5 + \frac{37719}{256}\alpha^5 + \frac{56025}{512}\zeta_3^2\alpha^3 \\
& +\frac{137781}{1024}\zeta_7\alpha^4 + \frac{156711}{1024}\zeta_5\alpha^3 + \frac{204525}{32}\zeta_6 + \frac{373855}{96}\zeta_5N_f\alpha \\
& +\frac{392337}{512}\zeta_3^2\alpha^2 + \frac{469333}{576}\zeta_4N_f + \frac{651031}{5184}N_f^2\alpha + \frac{851175}{1024}\zeta_6\alpha \\
& +\frac{2124351}{2048}\alpha^4 + \frac{2297709}{512}\alpha^3 + \frac{2371053}{512}\zeta_3\alpha^2 + \frac{2566431}{2048}\zeta_7\alpha^2 \\
& +\frac{27270475}{864}\zeta_3N_f + \frac{49026385}{3072}\alpha^2 + \frac{60587905}{864}\zeta_5N_f + \frac{70979391}{4096}\zeta_7\alpha \\
& +\frac{78128919}{1024}\zeta_3\alpha + \frac{190787741}{10368}N_f^2 + \frac{277127487}{2048}\zeta_7 + \frac{457175255}{18432}\alpha \\
& -\zeta_4N_f^3 + 25\zeta_3N_f\alpha^3]a^4 + O(a^5) \tag{A.5}
\end{aligned}$$

$$\begin{aligned}
C_c(a, \alpha)|^{SU(3)} &= 1 + 3a + \left[ \frac{5829}{64} - \frac{135}{16}\zeta_3 - \frac{95}{16}N_f - \frac{63}{64}\alpha - \frac{27}{16}\zeta_3\alpha^2 + \frac{27}{8}\alpha^2 + \frac{27}{8}\zeta_3\alpha \right] a^2 \\
&+ \left[ \frac{5161}{648}N_f^2 - \frac{198001}{432}N_f - \frac{40449}{64}\zeta_3 - \frac{3159}{64}\zeta_3\alpha^2 - \frac{1755}{32}\zeta_5 - \frac{1755}{32}\zeta_5\alpha \right. \\
&\quad - \frac{621}{64}\zeta_3\alpha^3 - \frac{195}{64}\alpha - \frac{135}{32}\zeta_5\alpha^3 - \frac{39}{4}\zeta_3N_f\alpha - \frac{33}{4}\zeta_4N_f + \frac{2}{3}\zeta_3N_f^2 \\
&\quad + \frac{59}{2}\zeta_3N_f + \frac{81}{16}\zeta_4\alpha + \frac{81}{64}\zeta_4\alpha^2 + \frac{199}{32}N_f\alpha + \frac{243}{64}\zeta_4 + \frac{945}{32}\zeta_5\alpha^2 \\
&\quad \left. + \frac{3249}{128}\alpha^3 + \frac{11421}{128}\alpha^2 + \frac{14157}{64}\zeta_3\alpha + \frac{1082353}{288} \right] a^3 \\
&+ \left[ \frac{15567976783}{73728} - \frac{151911987}{4096}\zeta_3 - \frac{79190001}{2048}\zeta_5 - \frac{39621021}{1024}N_f \right. \\
&\quad - \frac{7341417}{2048}\zeta_5\alpha - \frac{3869019}{4096}\zeta_7\alpha - \frac{3508811}{2304}\alpha - \frac{1282797}{2048}\zeta_3\alpha^3 \\
&\quad \left. - \frac{1172367}{2048}\zeta_7\alpha^2 - \frac{455517}{1024}\zeta_3^2\alpha - \frac{317277}{4096}\zeta_3\alpha^4 - \frac{289845}{128}\zeta_3\alpha^2 \right]
\end{aligned}$$



$$\begin{aligned}
& -\frac{214893}{8192}\zeta_7\alpha^4 - \frac{204525}{64}\zeta_6 - \frac{173743}{10368}N_f^2\alpha - \frac{158493}{128}\zeta_3N_f\alpha - \frac{150979}{7776}N_f^3 \\
& - \frac{92113}{512}N_f\alpha^2 - \frac{83943}{2048}\zeta_5\alpha^3 - \frac{52025}{128}\zeta_4N_f - \frac{36045}{1024}\zeta_5\alpha^4 - \frac{27135}{2048}\zeta_4\alpha^3 \\
& - \frac{25947}{1024}\zeta_3^2\alpha^3 - \frac{11907}{16}\zeta_7N_f - \frac{4725}{2048}\zeta_6\alpha^4 - \frac{699}{8}\zeta_5N_f^2 - \frac{633}{16}\zeta_5N_f\alpha^2 \\
& - \frac{425}{48}\zeta_3N_f^2 - \frac{135}{16}\zeta_3^2N_f\alpha^2 - \frac{45}{4}\zeta_3^2N_f\alpha - \frac{9}{8}\zeta_4N_f^2\alpha - \frac{5}{18}\zeta_3N_f^3 + \frac{1}{2}\zeta_4N_f^3 \\
& + \frac{15}{2}\zeta_5N_f^2\alpha + \frac{81}{64}\zeta_4N_f\alpha^2 + \frac{117}{64}\zeta_4N_f\alpha + \frac{153}{8}\zeta_3N_f^2\alpha + \frac{165}{16}\zeta_4N_f^2 \\
& + \frac{337}{4}\zeta_3^2N_f + \frac{1161}{1024}\zeta_3^2\alpha^4 + \frac{6033}{64}\zeta_5N_f\alpha + \frac{9639}{4096}\zeta_4\alpha^4 + \frac{16281}{1024}\zeta_4\alpha^2 \\
& + \frac{16775}{32}\zeta_6N_f + \frac{27459}{256}\zeta_3N_f\alpha^2 + \frac{60075}{2048}\zeta_6\alpha + \frac{85725}{2048}\zeta_6\alpha^3 + \frac{107325}{2048}\zeta_6\alpha^2 \\
& + \frac{130977}{1024}\zeta_3^2\alpha^2 + \frac{250085}{64}\zeta_5N_f + \frac{257823}{2048}\zeta_4\alpha + \frac{956907}{4096}\zeta_7\alpha^3 + \frac{1007289}{4096}\alpha^4 \\
& + \frac{1149471}{512}\zeta_3^2 + \frac{1279017}{1024}\alpha^3 + \frac{1780139}{3456}N_f\alpha + \frac{2625583}{768}\zeta_3N_f \\
& + \frac{3540987}{2048}\zeta_5\alpha^2 + \frac{5819877}{4096}\zeta_4 + \frac{8093153}{4608}N_f^2 + \frac{29075517}{2048}\zeta_3\alpha \\
& + \left. \frac{35695263}{8192}\alpha^2 + \frac{100880073}{8192}\zeta_7 \right] a^4 + O(a^5) \tag{A.6}
\end{aligned}$$

and

$$\begin{aligned}
C_\psi(a, \alpha)^{SU(3)} &= 1 - \frac{4}{3}\alpha a + \left[ \frac{7}{3}N_f + 12\zeta_3 + 12\zeta_3\alpha - \frac{359}{9} - \frac{49}{18}\alpha^2 - 26\alpha \right] a^2 \\
& + \left[ \frac{24722}{81}N_f - \frac{439543}{162} - \frac{322351}{432}\alpha - \frac{4139}{48}\alpha^2 - \frac{1570}{243}N_f^2 - \frac{1165}{3}\zeta_5 \right. \\
& - \frac{929}{54}\alpha^3 - \frac{440}{9}\zeta_3N_f - \frac{410}{3}\zeta_5\alpha - \frac{9}{2}\zeta_4\alpha - \frac{9}{4}\zeta_4\alpha^2 + \frac{20}{81}\zeta_3\alpha^3 + \frac{53}{2}\zeta_3\alpha^2 \\
& \left. + \frac{79}{4}\zeta_4 + \frac{1436}{3}\zeta_3\alpha + \frac{1547}{36}N_f\alpha + \frac{8009}{6}\zeta_3 - 16\zeta_3N_f\alpha - 15\zeta_5\alpha^2 \right] a^3 \\
& + \left[ \frac{21391}{1458}N_f^3 - \frac{2065961635}{62208}\alpha - \frac{356864009}{5184}\zeta_5 - \frac{146722043}{864} - \frac{74862851}{20736}\alpha^2 \right. \\
& - \frac{29889697}{5184}\zeta_3^2 - \frac{1294381}{108}\zeta_3N_f - \frac{1276817}{972}N_f^2 - \frac{326689}{2592}\alpha^4 \\
& - \frac{240893}{288}\zeta_5\alpha^2 - \frac{171487}{144}\zeta_5\alpha - \frac{106555}{128}\alpha^3 - \frac{94613}{1458}N_f^2\alpha - \frac{85687}{192}\zeta_4\alpha \\
& - \frac{11771}{27}\zeta_5N_f\alpha - \frac{9179}{4}\zeta_3^2\alpha - \frac{4851}{64}\zeta_7\alpha^3 - \frac{1913}{108}\zeta_3N_f\alpha^2 - \frac{1437}{16}\zeta_4\alpha^2 \\
& - \frac{757}{16}\zeta_5\alpha^3 - \frac{685}{64}\zeta_5\alpha^4 - \frac{501}{64}\zeta_4\alpha^3 - \frac{440}{9}\zeta_5N_f^2 - \frac{275}{128}\zeta_6\alpha^4 - \frac{20}{3}\zeta_4N_f^2 \\
& + \frac{3}{4}\zeta_4\alpha^4 + \frac{4}{3}\zeta_3^2N_f\alpha + \frac{8}{27}\zeta_3N_f^3 + \frac{21}{8}\zeta_4N_f\alpha^2 + \frac{39}{4}\zeta_3^2\alpha^3 + \frac{76}{9}\zeta_3N_f^2\alpha \\
& + \frac{80}{3}\zeta_5N_f^2\alpha + \frac{100}{3}\zeta_6N_f + \frac{149}{12}\zeta_5N_f\alpha^2 + \frac{155}{64}\zeta_3^2\alpha^4 + \frac{245}{4}\zeta_4N_f\alpha \\
& + \frac{413}{32}\zeta_7\alpha^2 + \frac{2291}{72}\zeta_4N_f + \frac{3725}{64}\zeta_6\alpha^2 + \frac{4651}{32}\zeta_3^2\alpha^2 + \frac{5704}{27}\zeta_3N_f^2 \\
& + \frac{14801}{72}\zeta_3\alpha^3 + \frac{32173}{3888}\zeta_3\alpha^4 + \frac{119995}{864}N_f\alpha^2 + \frac{387653}{192}\zeta_7\alpha + \frac{565939}{864}\zeta_4 \\
& \left. + \frac{1132909}{864}\zeta_3\alpha^2 + \frac{1460149}{72}\zeta_3\alpha + \frac{1673051}{324}\zeta_5N_f + \frac{3807625}{10368}\zeta_6 \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{6747755}{288}\zeta_7 + \frac{55476671}{1944}N_f + \frac{59611205}{15552}N_f\alpha + \frac{317781451}{2592}\zeta_3 \\
& - 1552\zeta_3N_f\alpha - 1029\zeta_7N_f - 24\zeta_3^2N_f - 6\zeta_3^2N_f\alpha^2 - 2\zeta_4N_f^2\alpha + 575\zeta_6\alpha \Big] a^4 \\
& + O(a^5) \tag{A.7}
\end{aligned}$$

for the field conversion functions while we have

$$\begin{aligned}
C_m(a, \alpha)|^{SU(3)} &= 1 - \left[ \frac{16}{3} + \frac{4}{3}\alpha \right] a + \left[ \frac{83}{9}N_f + \frac{152}{3}\zeta_3 - \frac{3779}{18} - \frac{62}{9}\alpha - \frac{11}{9}\alpha^2 \right] a^2 \\
& + \left[ \frac{217390}{243}N_f - \frac{3115807}{324} - \frac{36235}{216}\alpha - \frac{12695}{216}\alpha^2 - \frac{7514}{729}N_f^2 - \frac{4720}{27}\zeta_3N_f \right. \\
& - \frac{2960}{9}\zeta_5 - \frac{1291}{108}\alpha^3 - \frac{761}{9}\zeta_3\alpha - \frac{32}{27}\zeta_3N_f^2 - \frac{8}{9}\zeta_3N_f\alpha + \frac{80}{3}\zeta_4N_f \\
& \left. + \frac{211}{18}\zeta_3\alpha^2 + \frac{815}{54}N_f\alpha + \frac{195809}{54}\zeta_3 \right] a^3 \\
& + \left[ \frac{40}{81}\zeta_3N_f^3 - \frac{744609145}{1296} - \frac{52383125}{17496}N_f^2 - \frac{26904209}{3888}\alpha - \frac{4767911}{648}\zeta_3\alpha \right. \\
& - \frac{3837631}{1728}\zeta_7 - \frac{3053677}{1296}\alpha^2 - \frac{2017309}{81}\zeta_3N_f - \frac{843077}{54}\zeta_3^2 - \frac{711149}{1296}\alpha^3 \\
& - \frac{359855}{81}\zeta_5N_f - \frac{115433}{1296}\alpha^4 - \frac{77737}{2187}N_f^2\alpha - \frac{16960}{9}\zeta_4 - \frac{11500}{9}\zeta_6N_f \\
& - \frac{8776}{27}\zeta_3^2N_f - \frac{3973}{12}\zeta_3^2\alpha - \frac{3155}{8}\zeta_5\alpha^2 - \frac{2072}{81}\zeta_3N_f\alpha^2 - \frac{343}{2}\zeta_7N_f \\
& - \frac{122}{9}\zeta_4N_f\alpha - \frac{100}{3}\zeta_4N_f^2 - \frac{91}{2}\zeta_7\alpha - \frac{81}{8}\zeta_4\alpha - \frac{80}{27}\zeta_3N_f^2\alpha - \frac{55}{16}\zeta_5\alpha^4 \\
& - \frac{27}{2}\zeta_4\alpha^2 - \frac{27}{4}\zeta_3^2\alpha^2 - \frac{27}{8}\zeta_4\alpha^3 - \frac{8}{9}\zeta_4N_f^3 + \frac{28}{9}\zeta_3\alpha^4 + \frac{160}{3}\zeta_5N_f\alpha \\
& + \frac{525}{64}\zeta_7\alpha^2 + \frac{560}{3}\zeta_5N_f^2 + \frac{11542}{9}\zeta_4N_f + \frac{15317}{216}\zeta_3\alpha^3 + \frac{31595}{36}\zeta_3\alpha^2 \\
& + \frac{32579}{324}N_f\alpha^2 + \frac{33964}{81}\zeta_3N_f^2 + \frac{34204}{81}\zeta_3N_f\alpha + \frac{59275}{216}\zeta_5\alpha + \frac{96979}{4374}N_f^3 \\
& + \frac{3764537}{2916}N_f\alpha + \frac{9369745}{432}\zeta_5 + \frac{86284171}{324}\zeta_3 + \frac{247516535}{2916}N_f - 45\zeta_5\alpha^3 \\
& \left. + 10\zeta_5N_f\alpha^2 + 5500\zeta_6 \right] a^4 + O(a^5) \tag{A.8}
\end{aligned}$$

for the quark mass conversion function. Finally the  $SU(3)$  gauge parameter mapping is

$$\begin{aligned}
\alpha_{\text{mMOM}}|^{SU(3)} &= \alpha + \left[ \frac{10}{9}N_f\alpha - \frac{97}{12}\alpha - \frac{3}{2}\alpha^2 - \frac{3}{4}\alpha^3 \right] a \\
& + \left[ \frac{4}{3}\zeta_3N_f\alpha - \frac{7143}{32}\alpha - \frac{9}{16}\alpha^4 - \frac{5}{3}N_f\alpha^2 - \frac{5}{3}N_f\alpha^3 + \frac{9}{16}\alpha^5 + \frac{95}{16}\alpha^3 + \frac{463}{32}\alpha^2 \right. \\
& \left. + \frac{2473}{72}N_f\alpha - 18\zeta_3\alpha^2 + 27\zeta_3\alpha \right] a^2 \\
& + \left[ \frac{15}{8}N_f\alpha^5 - \frac{10221367}{1152}\alpha - \frac{61105}{972}N_f^2\alpha - \frac{36213}{32}\zeta_3\alpha^2 - \frac{21763}{108}\zeta_3N_f\alpha \right. \\
& - \frac{11417}{288}N_f\alpha^3 - \frac{8439}{128}\alpha^4 - \frac{3049}{48}N_f\alpha^2 - \frac{2320}{9}\zeta_5N_f\alpha - \frac{1449}{32}\zeta_3\alpha^3 \\
& - \frac{315}{64}\zeta_5\alpha^5 - \frac{261}{16}\alpha^5 - \frac{45}{8}\zeta_5\alpha^4 - \frac{33}{2}\zeta_4N_f\alpha - \frac{27}{64}\alpha^7 - \frac{25}{27}N_f^2\alpha^3 \\
& - \frac{16}{3}\zeta_3N_f^2\alpha^2 - \frac{11}{4}\zeta_3N_f\alpha^3 - \frac{5}{4}N_f\alpha^4 + \frac{4}{3}N_f^2\alpha^2 + \frac{27}{16}\alpha^6 + \frac{32}{27}\zeta_3N_f^2\alpha \\
& \left. + \frac{81}{8}\zeta_4\alpha^2 + \frac{81}{32}\zeta_4\alpha^3 + \frac{117}{32}\zeta_3\alpha^5 + \frac{243}{32}\zeta_4\alpha + \frac{1341}{32}\zeta_3\alpha^4 + \frac{3105}{8}\zeta_5\alpha^2 \right] a^3
\end{aligned}$$

$$\begin{aligned}
& + \frac{3465}{32} \zeta_5 \alpha^3 + \frac{13941}{8} \zeta_3 \alpha + \frac{30835}{384} \alpha^3 + \frac{39423}{128} \alpha^2 + \frac{63225}{64} \zeta_5 \alpha \\
& + \frac{4939411}{2592} N_f \alpha + 117 \zeta_3 N_f \alpha^2 \Big] a^3 \\
& + \left[ \frac{9}{4} \zeta_4 N_f \alpha^2 - \frac{1174691657}{2304} \alpha - \frac{277127487}{2048} \zeta_7 \alpha - \frac{273998929}{31104} N_f^2 \alpha \right. \\
& - \frac{70979391}{4096} \zeta_7 \alpha^2 - \frac{56971325}{2592} \zeta_3 N_f \alpha - \frac{54117607}{1024} \zeta_3 \alpha^2 \\
& - \frac{18363505}{288} \zeta_5 N_f \alpha - \frac{13026679}{6912} N_f \alpha^3 - \frac{12195539}{4608} N_f \alpha^2 - \frac{2566431}{2048} \zeta_7 \alpha^3 \\
& - \frac{1624581}{512} \zeta_3 \alpha^3 - \frac{1199043}{2048} \alpha^5 - \frac{1100549}{512} \alpha^4 - \frac{992391}{1024} \zeta_5 \alpha^4 \\
& - \frac{851175}{1024} \zeta_6 \alpha^2 - \frac{327213}{512} \zeta_3^2 \alpha^2 - \frac{305965}{576} \zeta_4 N_f \alpha - \frac{226449}{512} \zeta_3^2 \alpha^3 \\
& - \frac{216815}{96} \zeta_5 N_f \alpha^2 - \frac{204525}{32} \zeta_6 \alpha - \frac{137781}{1024} \zeta_7 \alpha^5 - \frac{56025}{512} \zeta_3^2 \alpha^4 \\
& - \frac{45465}{256} \zeta_5 \alpha^5 - \frac{23247}{1024} \zeta_4 \alpha^4 - \frac{19755}{512} \zeta_5 \alpha^6 - \frac{17631}{512} \zeta_3 \alpha^6 - \frac{13041}{2048} \zeta_4 \alpha^5 \\
& - \frac{6071}{96} N_f \alpha^4 - \frac{5977}{216} \zeta_4 N_f^2 \alpha - \frac{4873}{9} \zeta_3 N_f^2 \alpha^2 - \frac{3267}{512} \alpha^6 - \frac{1117}{8} \zeta_3 N_f \alpha^4 \\
& - \frac{880}{27} \zeta_5 N_f^3 \alpha - \frac{567}{256} \alpha^8 - \frac{351}{64} \zeta_3 \alpha^7 - \frac{175}{16} \zeta_5 N_f \alpha^5 - \frac{128}{3} \zeta_3^2 N_f^2 \alpha \\
& - \frac{97}{36} N_f^2 \alpha^4 - \frac{40}{27} N_f^3 \alpha^2 - \frac{15}{8} N_f \alpha^7 - \frac{7}{2} \zeta_3 N_f^2 \alpha^3 - \frac{5}{2} \zeta_5 N_f^2 \alpha^3 + \frac{9}{4} \zeta_4 N_f^2 \alpha^2 \\
& + \frac{9}{4} \zeta_3^2 N_f \alpha^4 + \frac{23}{2} \zeta_3 N_f \alpha^5 + \frac{25}{12} N_f^2 \alpha^5 + \frac{45}{4} \zeta_3^2 N_f \alpha^3 + \frac{45}{8} N_f \alpha^6 + \frac{81}{256} \alpha^9 \\
& + \frac{115}{2} \zeta_5 N_f^2 \alpha^2 + \frac{153}{4} \zeta_4 N_f \alpha^3 + \frac{160}{27} \zeta_3 N_f^3 \alpha^2 + \frac{321}{4} \zeta_3^2 N_f \alpha^2 + \frac{595}{32} \zeta_5 N_f \alpha^4 \\
& + \frac{945}{128} \zeta_5 \alpha^7 + \frac{1501}{243} \zeta_3 N_f^3 \alpha + \frac{1683}{8} \zeta_3^2 N_f \alpha + \frac{1765}{128} N_f \alpha^5 + \frac{1863}{256} \zeta_3^2 \alpha^6 \\
& + \frac{3069}{128} \alpha^7 + \frac{4725}{1024} \zeta_6 \alpha^5 + \frac{10647}{4096} \zeta_7 \alpha^6 + \frac{16775}{16} \zeta_6 N_f \alpha + \frac{19575}{512} \zeta_4 \alpha^3 \\
& + \frac{19737}{512} \zeta_3^2 \alpha^5 + \frac{22275}{1024} \zeta_6 \alpha^3 + \frac{24975}{1024} \zeta_6 \alpha^4 + \frac{26901}{32} \zeta_7 N_f \alpha^2 \\
& + \frac{40279}{96} \zeta_5 N_f \alpha^3 + \frac{51203}{1296} N_f^2 \alpha^3 + \frac{87701}{576} \zeta_3 N_f \alpha^3 + \frac{132615}{512} \zeta_7 \alpha^4 \\
& + \frac{197811}{32} \zeta_3^2 \alpha + \frac{285995}{1728} N_f^2 \alpha^2 + \frac{303233}{1944} \zeta_3 N_f^2 \alpha + \frac{423043}{162} \zeta_5 N_f^2 \alpha \\
& + \frac{488271}{2048} \zeta_3 \alpha^5 + \frac{636039}{1024} \zeta_4 \alpha^2 + \frac{1312185}{512} \alpha^3 + \frac{2029821}{1024} \zeta_3 \alpha^4 \\
& + \frac{2785621}{288} \zeta_3 N_f \alpha^2 + \frac{2910357}{2048} \zeta_4 \alpha + \frac{4167895}{34992} N_f^3 \alpha + \frac{4295115}{1024} \zeta_5 \alpha^3 \\
& + \frac{25666081}{864} \zeta_7 N_f \alpha + \frac{32185629}{1024} \zeta_5 \alpha^2 + \frac{128069033}{18432} \alpha^2 + \frac{230899397}{2048} \zeta_3 \alpha \\
& + \frac{246393489}{1024} \zeta_5 \alpha + \frac{541249745}{3888} N_f \alpha + \zeta_4 N_f^3 \alpha + 8 \zeta_3 N_f^2 \alpha^4 \Big] a^4 \\
& + O(a^5) . \tag{A.9}
\end{aligned}$$

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