Hereditarily Indecomposable Julia Continua of Transcendental Entire Functions

Thesis submitted in accordance with the requirements of the University of Liverpool for the degree of Doctor in Philosophy by

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ABSTRACT

This thesis studies the topology of the Julia set of transcendental entire functions of disjoint type. It is known that the Julia set of such entire functions may contain topological objects which could be considered “pathological”. In this sense, we may ask how pathological the Julia set could become. Here, we prove the existence of a transcendental entire function of disjoint type for which the connected components of its Julia set together with infinity are *pseudo-arcs*. Furthermore, the disjoint-type entire function can be chosen to have finite lower order of growth.

To Fátima

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CONTENTS

CHAPTER

ONE

INTRODUCTION

The present work belongs to the field of one-dimensional *holomorphic dynamics*, which is the study of the behaviour under iteration of holomorphic self-maps either in the complex plane $\mathbb{C}$, or in the Riemann sphere $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$. This research was originated by the pioneering work of Pierre Fatou and Gaston Julia in the early 1920s. During their studies, they showed that either $\mathbb{C}$ or $\hat{\mathbb{C}}$ may be divided into two totally invariant sets: the set of stability, currently known as the *Fatou set* –open, possibly empty– and its complement, called the *Julia set* –closed, non-empty, perfect– which is the set where the iterates exhibit chaotic behaviour. More precisely, the Fatou set $\mathcal{F}(f)$ of f is defined as the set of points $z \in \mathbb{C}$ (respectively $z \in \hat{\mathbb{C}}$) such that $\{f^n\}_{n\in\mathbb{N}}$ form a normal family in a neighbourhood of z, whereas the Julia set is $\mathcal{J}(f) := \mathbb{C} \setminus \mathcal{F}(f)$ (respectively $\mathcal{J}(f) = \hat{\mathbb{C}} \setminus \mathcal{F}(f)$).

Following a few decades when research output had been scarce, in the early 1980s the field underwent a resurgence. Some ground-breaking results in rational dynamics were attained, such as Sullivan's work [Sul85], showing that there are no wandering components in the Fatou set of a rational map, and Douady and Hubbard's work on the Mandelbrot set. In transcendental dynamics, substantial contributions were made by Baker, Misiurewicz and Devaney.

It is understood that chaotic dynamics often leads to a complicated topology. For even one of the simplest families of holomorphic functions, for instance, the *quadratic family* $f(z) = z^2 + c$, for $c \in \mathbb{C}$, it is known that the Julia set may have a noticeably intricate topological structure. Then, our main motivation is to understand this type of dynamics in the transcendental setting making use of topological tools. In particular, we will make use of *continuum theory*, see §2.2 for an introduction on this topic.

The Julia set of a polynomial can be totally disconnected; that is, its connected components are points. Thus, a natural question is whether the Julia set of transcendental entire functions always contains non-trivial continua, i.e., non-empty compact connected metric spaces. This question was positively answered by Baker in 1975 [Bak75, Corollary 1]. Therefore, a second question is the following, what type of continua can arise as the Julia set of a transcendental entire function? In 1981, Misiurewicz [Mis81] proved that the Julia set of e^z is the whole plane $\mathbb{C}$, which topologically does not tell us much. Nonetheless, researchers started to study the *exponential family*, i.e., $E_\lambda \colon \mathbb{C} \to \mathbb{C}$, $z \mapsto \lambda e^z$ for $\lambda \in \mathbb{C} \setminus \{0\}$, for different parameters. In 1984, Devaney and Krych [DK84] studied the map E_λ with $\lambda \in (0, 1/e)$, and showed that its Julia set $\mathcal{J}(E_\lambda)$ is a Cantor set of curves. Moreover, in 1993, Devaney [Dev93] proved that the Julia set of E_λ with $\lambda > 1/e$ contains invariant indecomposable continua (see Figure 1.1), which are defined as follows. A continuum is *indecomposable* if it cannot be written as the union of two of its proper subcontinua (see Definition 2.2.7).

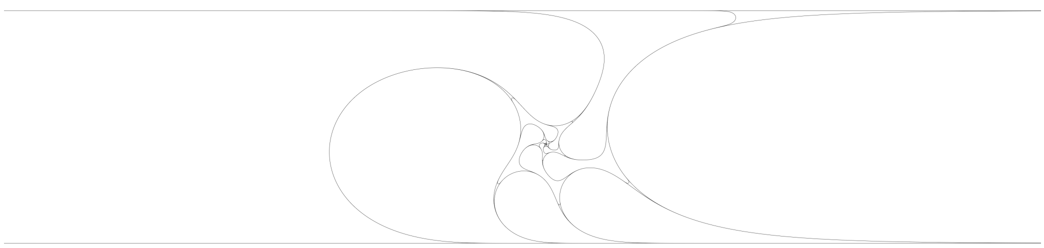


Figure 1.1: An illustration of an indecomposable continuum of $\mathcal{J}(E_\lambda)$ with $\lambda > 1/e$.
Picture, from [Rem10], provided by Rempe.

Therefore, the topological dynamics on the Julia set is extremely rich, even in the case of one of the simplest transcendental entire functions. In spite of extensive investigations on the exponential family, which still continue to date, some questions regarding the dynamics of this family remain open [SRG15]. Part of this significant research can be found in [DT86, EL92, AO93, Rem03].

In the field of dynamical systems, it is common to first study *hyperbolic systems*, since their understanding is usually a key step in building a more general theory. In holomorphic dynamics, a polynomial p is said to be *hyperbolic* if it is expanding with respect to a suitable conformal metric defined on a neighbourhood of its compact Julia set $\mathcal{J}(p)$. By [DHL84, Theorem 1, page 21], it is equivalent to say that every critical value of p belongs to the basin of an attracting periodic cycle (see Definition 2.4.2, which is stated for transcendental maps, but it also holds for polynomials). For transcendental entire functions, infinity is an essential singularity and therefore their Julia sets are no longer compact. However, with some slight modification on the notion of expansion, i.e. the hyperbolic metric is defined in a punctured neighbourhood of infinity, it is possible to have an analogous definition and characterization as in the polynomial case (see [RGS17, Theorem and Definition 1.3]).

Let us now focus on *hyperbolic transcendental dynamics*. Let f be a transcendental entire function. The set of *singular values* $S(f)$ of f consists of the closure of the union of the critical and asymptotic values of f (see §2.4). The function f is *hyperbolic* if $S(f)$ is bounded and every singular value tends to an attracting periodic cycle of f under iteration (see Definition 2.4.1).

We now take an example and briefly discuss its Julia set. The function $f\colon z \mapsto \frac{4\pi}{3} \cdot (1 - \cos(z))$ is hyperbolic (see Figure 1.2). It has a unique superattracting fixed point at $z = 0$, which is in fact the only attracting cycle of f. Its Julia set is locally connected [BFRG15, Corollary 1.9]. This implies that this Julia set is topologically a Sierpiński carpet (see Figure 2.1(c) for an illustration of the Sierpiński carpet). In addition, it is known that the Sierpiński carpet contains a homeomorphic copy of any one (topological) dimensional planar continuum

[Why57].

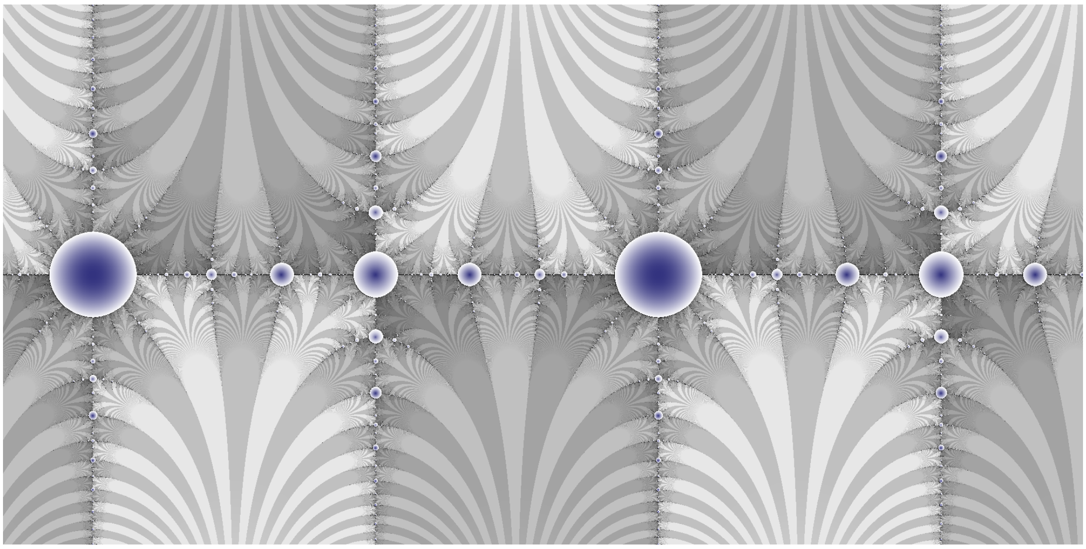


Figure 1.2: Illustration of the dynamical plane of the function $\frac{4\pi}{3} \cdot (1 - \cos(z))$. The blue region represents the Fatou set, while is drawn in grey the Julia set.
Picture provided by Rempe.

So, clearly in the class of hyperbolic entire functions we can still find an abundance of continua. Let us now explore a certain subclass of hyperbolic entire functions, that is, we will now consider the class of *disjoint-type entire functions*.

Definition 1.0.1. (Disjoint-type entire function).
A transcendental entire function $f\colon \mathbb{C} \to \mathbb{C}$ is of *disjoint type* if there is a bounded Jordan domain D such that $S(f) \subset D$ and $f(\overline{D}) \subset D$.

Equivalently, f is hyperbolic with connected Fatou set (see [BK07, Lemma 3.1] and [Rem16, Definition 1.1]). Even though this class has the simplest combinatorial structure, their properties allow us to build a better understanding of their dynamics. Furthermore, these maps belong to the well-studied *Eremenko–Lyubich class* $\mathcal{B}$, which consists of all transcendental entire functions with bounded singular set. Conversely, for any map $f \in \mathcal{B}$, there exists a sufficiently small positive $\lambda \in \mathbb{R}$ such that the function

$$f_\lambda \colon z \mapsto \lambda f(z)$$

is of disjoint type. If $f \in \mathcal{B}$ and additionally is hyperbolic, thus by [Rem09, Theorem 5.2], the topological dynamics of f on $\mathcal{J}(f)$ can be described using the topological dynamics of λf on $\mathcal{J}(\lambda f)$. Further discussion on the importance of this class can be found in [Rem16].

It is known that for a disjoint-type entire function f, its Julia set $\mathcal{J}(f)$ has uncountably many connected components, each of which is closed and unbounded. This motivates the introduction of the following definition.

Definition 1.0.2. (Julia continuum).
Let f be a transcendental entire function of disjoint type and let C be a component of the Julia set $\mathcal{J}(f)$. We say that

$$\hat{C} := C \cup \{\infty\}$$

is a *Julia continuum* of f.

Since C is a connected and unbounded set, when we add infinity, we obtain a compact, connected set, and hence we have a continuum.

Let us now explore some examples. Consider the map $F_\lambda \colon z \mapsto \lambda \sin(z)$, with $\lambda \in (0, 1)$ (see Figure 1.3). Note that F_λ is of disjoint type. Indeed, it has two singular values $\pm\lambda$ (which are the critical values), it has a fixed point at $z_0 = 0$ and all real starting values tend to z_0. Thus $F_\lambda(z)$ is hyperbolic, and its Fatou set consists only of the immediate basin of attraction of $z_0 = 0$. Hence it is of disjoint type.

In 1926, Fatou [Fat26] studied this map and showed that $\mathcal{J}(F_\lambda)$ consists of infinitely many curves which tend to infinity under iteration. This example shows that the Julia continua of $F_\lambda(z)$ are arcs, which are the simplest example of non-degenerate continua.

As a second example of a Julia continuum, in [RRRS11, Theorem 8.4] the authors showed the following:

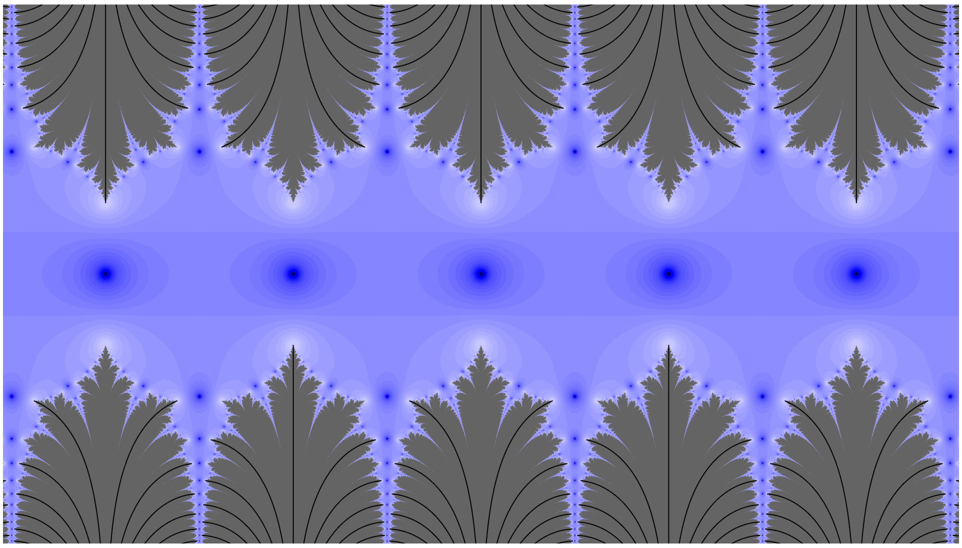


Figure 1.3: Illustration of the dynamical plane of the function $F_\lambda(z)$. The Fatou set is coloured in blue, while the Julia set is in grey.

Picture provided by Rempe.

Theorem 1.0.3. *There exists a transcendental entire function of disjoint type for which every Julia continuum contains no arcs.*

This example may highlight how complicated the Julia continua could become. So due to the complexity of this example and the existence of the aforementioned indecomposable continua in certain Julia sets, we can ask whether the Julia continua may be hereditarily indecomposable, which is defined as follows. A continuum is said to be *hereditarily indecomposable* provided that each of its subcontinua is indecomposable.

The notion of hereditarily indecomposable continua has been studied since 1920. In §1.1, we give a brief outline of the history and comments regarding this set. A famous example of a hereditarily indecomposable continuum is *the pseudo-arc* (see §2.2). So, in particular we may ask whether a Julia continuum of a transcendental entire function of disjoint type may be a pseudo-arc. This question was positively answered by Rempe [Rem16, Theorem 1.5]. More specifically, he proved a more general result which is that any continuum that is homeomorphic to an inverse limit of a self-map of the interval $[0, 1]$ fixing the origin can be realised as an invariant Julia continuum [Rem16, Theorem 2.7]. Furthermore, he proved the following.

Theorem 1.0.4. *There exists a transcendental entire function f of disjoint type such that every Julia continuum of f is a pseudo-arc.*

The goal of this thesis is to give a simpler construction and more direct proof of Theorem 1.0.4. This will be achieved by following some ideas of a classical result of Henderson [Hen64]. Moreover, due to the nature of our construction, we will obtain control of the *lower order of growth* of the function, which is defined as follows.

Definition 1.0.5. For a transcendental entire function f, the *lower order of growth* of f is

$$\liminf_{r\to\infty} \frac{\log\log M(r,f)}{\log r},$$

where $M(r,f) = \max\{|f(z)|\colon |z| = r\}$.

We furthermore obtain the following:

Theorem 1.0.6. *There exists a transcendental entire function f of disjoint type such that every Julia continuum of f is a pseudo-arc. Furthermore, f can be chosen to have finite lower order of growth.*

We note that the functions constructed in [Rem16] have infinite lower order of growth. We now describe the strategy of our construction, and remark on some similarities and differences between our construction and the ones from [RRRS11, Theorem 1.1] and [Rem16].

From Definition 1.0.1, recall that D is a bounded Jordan domain, we have that $f\colon \mathcal{V} \to \mathbb{C}\setminus D$ is a covering map, where $\mathcal{V} := f^{-1}(\mathbb{C}\setminus D)$. The connected components of $\mathcal{V}$ are called *tracts* of f; each such tract V of $\mathcal{V}$ is simply connected. Then $f\colon V \to \mathbb{C}\setminus D$ is a universal covering. So the general strategy from [RRRS11, Rem16] and this thesis starts by constructing a suitable simply connected domain T with $\overline{T} \subseteq \mathbb{H}$ and a conformal isomorphism,

$$F\colon T \to \mathbb{H},$$

where T is disjoint from all its $2\pi i$-translates and $\mathbb{H}$ denotes the right half plane. Then the map F is constructed so that

$$f\colon \exp(T) \to \{z \in \mathbb{C}\colon |z| > 1\}$$

defined by $f(\exp(z)) = \exp(F(z))$ is the universal covering map. For such maps, we can define the Julia set by

$$\mathcal{J}(f) := \{z \in \mathbb{C}\colon f^n(z) \in \mathcal{V} \text{ for all } n \in \mathbb{N} \text{ and } |f^n(z)| \geq 1\}.$$

Then, using an approximation result (see Theorem 3.1.7), we can obtain a disjoint-type entire function g such that $\mathcal{J}(g)$ is homeomorphic to the Julia set $\mathcal{J}(f)$ of f.

Now, in [RRRS11, Theorem 1.1], the tract T is constructed by adding a long sequence of "wiggles". Roughly speaking, if we take two real parts $r < R$ then the tract increases up to R then it turns back to reach r and then it grows again. More specifically, T traverses (r, R) three times (see Figure 1.4). The proof of Theorem 1.0.4 in [Rem16] uses a much more complicated construction than this one; we refer to [Rem16, Figure 9].

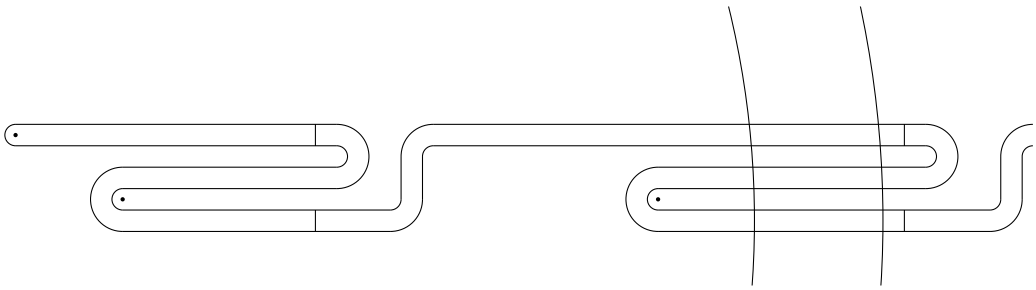


Figure 1.4: The tract used in [RRRS11, Theorem 1.1].
Figure provided by Rempe.

In contrast, our proof of Theorem 1.0.4 will use the same type of tracts as in Figure 1.4. Naturally, the location of the wiggles will have to be chosen very carefully.

1.1 Historical comments on the pseudo-arc

Knaster [Kna22], in his PhD dissertation, constructed the first hereditarily indecomposable continuum in 1922, giving a positive answer to a question posed by himself and Kuratowski in 1921.

Previously, in 1920, Knaster and Kuratowski also asked whether a simple closed curve is the only planar *homogeneous continuum*, meaning that for every two points of the space there exists a homeomorphism from the space onto itself which maps one of the points to the other. In 1924, Mazurkiewicz showed that the circle is the only homogeneous locally connected planar continuum.

On the other hand, in 1921, Mazurkiewicz asked whether the arc is the only finite-dimensional, non-degenerate *hereditarily equivalent continuum* (meaning that it is homeomorphic to each of its non-degenerate subcontinua). This question was formally answered by Moise [Moi48] in 1948, who constructed another hereditarily equivalent continuum, which he called the *pseudo-arc*, due to this property in common with the arc. More precisely, his construction was a family of topologically equivalent continua in the plane having the property of being hereditarily indecomposable and homeomorphic to each of their non-degenerate subcontinua.

In the same year, 1948, Bing [Bin48], inspired by Moise's continuum, also constructed "another" hereditarily indecomposable continuum, which he proved to be a homogeneous continuum. This result was very surprising, intriguing and non-intuitive among the community. Since Bing's construction was made using chainable continua; we will give a brief introduction on this topic in §2.2. Furthermore, because of the complexity and technicality of his proof it was not accepted immediately. In fact, in 1953 Kapuano [Kap53] claimed that the pseudo-arc is not homogeneous. Later a flaw was found in his work, so he published an attempt to correct it. The community seemed to favour Bing's and Moise's work over Kapuano's. However, Esenin-Volpin, a reviewer of Referativnyi Zhurnal wrote in 1955: "in the light of this, the problem of Knaster and Kuratowski remains open".

Due to this intriguing situation, Knaster in 1955 asked two of his students A. Lelek and M. Rochowski, to review Bing's and Kapuano's work and verify which was the correct argument. They did this hard work and concluded that Bing's proof was correct. Their work was handwritten in 60 detailed pages in Polish, and it was never published [Cha98].

Lastly, in 1951, Bing [Bin51a] proved that all hereditarily indecomposable arc-like continua are homeomorphic. Therefore, Knaster, Moise and Bing's constructions are all homeomorphic, and we call any such continuum a pseudo-arc.

1.2 Structure of the thesis

Chapter 2 establishes the terminology and collects some background results that we shall require throughout our work. In particular, we give an introduction to continuum theory focusing on hereditarily indecomposable continua. Then we introduce some tools from hyperbolic geometry needed to make estimates in the right half plane. Later, we give a short background on holomorphic dynamics. In the last section, we present some results on logarithmic coordinates. In Chapter 3, we introduce the class of conformal isomorphism in which our construction takes place. We also give sufficient conditions on a function in this class to ensure that the Julia continuum is a pseudo-arc. In Chapter 4, we introduce a subclass with certain geometric properties that allows us to have convergence in our tracts in the sense of the Carathéodory kernel theorem. Chapter 5 is devoted to the proofs of Theorem 1.0.4 and Theorem 1.0.6.

CHAPTER

TWO

PRELIMINARIES

In this chapter, we present some results and notation which are used throughout the thesis. In §2.1, we introduce general notation. §2.2 presents a general introduction to continuum theory, particularly the concepts of arc-like and hereditarily indecomposable continua and some of their properties. §2.3 collects some basic results on hyperbolic geometry. Next, in §2.4, we include general basic results on holomorphic dynamics. We end in §2.5 by reviewing the logarithmic change of coordinates, which is an important tool for studying functions in the class $\mathcal{B}$ of transcendental entire functions.

2.1 Notation

Throughout the thesis, $\mathbb{C}$ denotes the complex plane, $\hat{\mathbb{C}}$ denotes the Riemann sphere, and

$$\mathbb{H} = \{z \in \mathbb{C}\colon \operatorname{Re} z > 0\}$$

is the right half plane. A disc of radius $r > 0$ around a point $z \in \mathbb{C}$ is defined by

$$D_r(z) := \{w \in \mathbb{C}\colon |z - w| < r\}.$$

The unit disc centred at 0 is abbreviated as $\mathbb{D}$ and $S^1 = \partial\mathbb{D}$.

For two non-empty sets $A, B \subset \mathbb{C}$, the *Euclidean distance* between A and B is denoted by

$$\operatorname{dist}(A, B) := \inf\{|z - w| \colon z \in A,\ w \in B\}.$$

The *diameter* of A is given by

$$\operatorname{diam}(A) = \sup\{|z - w| \colon z, w \in A\}.$$

If A is a subset of $\mathbb{C}$, $\operatorname{int}(A)$, $\overline{A}$ and ∂A denote the interior, the closure and the boundary of A in $\mathbb{C}$, respectively. The cardinality of a set A will be denoted by $\#A$.

Lef $f \colon \mathbb{C} \to \mathbb{C}$ be a transcendental entire function. For each $n \geq 1$, we denote by f^n the n-th iterate of f, that is,

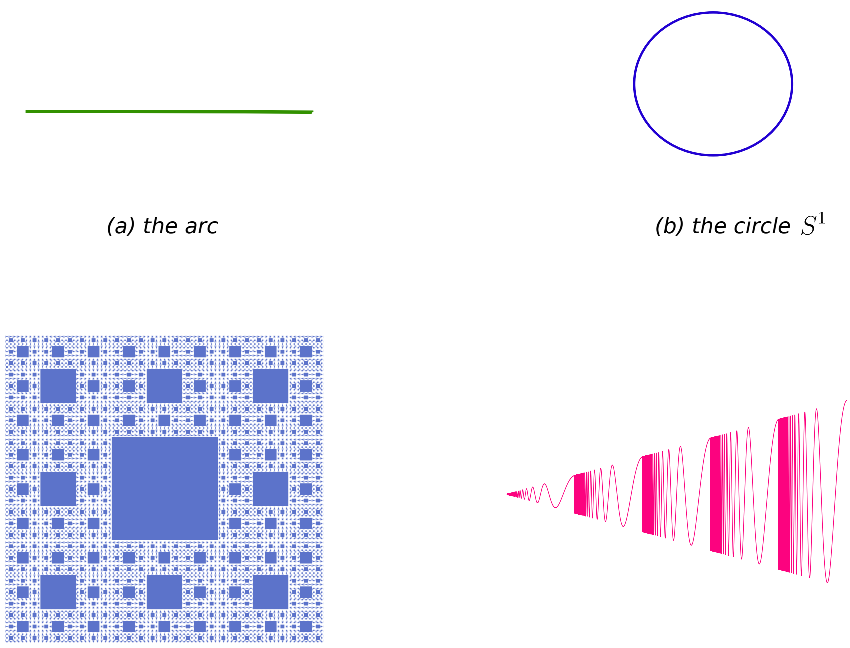
$$f^n := \underbrace{f \circ \cdots \circ f}_{n \text{ times}}.$$

The *Fatou set* of f, denoted by $\mathcal{F}(f)$, is the largest open set where the family $\{f^n\}_{n\in\mathbb{N}}$ is normal. The *Julia set* of f, denoted by $\mathcal{J}(f)$, is the complement in $\mathbb{C}$ of $\mathcal{F}(f)$. Finally, we conclude any proof of a claim with the symbol $\triangle$, and the rest of the proofs with the symbol $\square$.

2.2 Introduction to continuum theory

In this section, we give a brief introduction to continuum theory. In particular, we focus on results regarding heriditarily indecomposable continua, as well as the notion of arc-like continua. For general texts in continuum theory, we refer to [NJ92, Mac05].

Definition 2.2.1. (Continuum).
A *continuum* X is a non-empty compact connected metric space. We say that a continuum is *non-degenerate* if it contains more than one point.



(a) the arc

(b) the circle S^1

(c) the Sierpiński carpet

(d) an arc-like continuum

Figure 2.1: Some examples of continua. In particular, Figures (a) and (d) are arc-like continua, while (b) and (c) are not.

Before we introduce the *arc-like* notion, let us first start with the following definition.

Definition 2.2.2. (Chainable continuum).
Let X be a continuum. A *chain* $\mathcal{U}$ in X is a finite sequence, $U_1, \ldots, U_n$ of open subsets of X such that for all $i, j \in \{1, \ldots, n\}$,

$$U_i \cap U_j \neq \emptyset \text{ if and only if } |i - j| \leq 1.$$

Each U_j is called a *link* of $\mathcal{U}$; *the mesh* of $\mathcal{U}$ is

$$\text{mesh}(\mathcal{U}) := \max\{\text{diam}(U_j) \colon 1 \leq j \leq n\}.$$

Given $\varepsilon > 0$, $\mathcal{U}$ is said to be an ε-*chain* if $\text{mesh}(\mathcal{U}) < \varepsilon$. We say that X is *chainable* if there exists an ε-chain covering X for each $\varepsilon > 0$.

Now we present a few examples of chainable continua. Let us start with the arc $[0, 1]$ which is clearly chainable (see Figure 2.2).

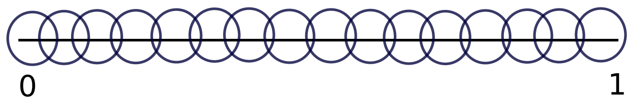


Figure 2.2: The arc $[0, 1]$. The intersection of the discs shown with the interval $X = [0, 1]$ form a chain covering X.

Let $Y = \{0\} \times [-1, 1] \cup \left\{\left(x, \sin\left(\frac{1}{x}\right)\right) \in \mathbb{R}^2 \colon x \in \left(0, \frac{2}{\pi}\right]\right\}$. Y is called either the *topologist's sine curve* or the $\sin(1/x)$-*continuum*. Note that Y is also chainable (see Figure 2.3).

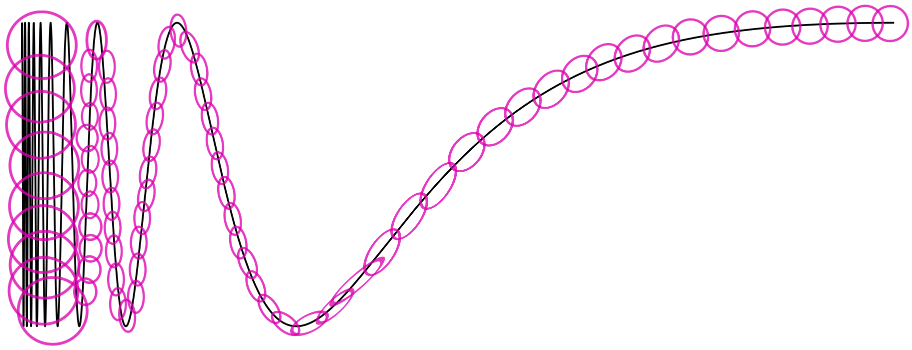


Figure 2.3: The $\sin(1/x)$-continuum. The intersection of the discs with Y forms a chain covering Y.

Chains (say in $\mathbb{C}$) can also be used to construct chainable continua. Suppose that we define a nested sequence of chains whose links are topological discs in the plane, and whose mesh tends to 0. Then the union of the closures of the links of each chain is a continuum, and their intersection is again a continuum by the following elementary result.

Theorem 2.2.3. ([NJ92, Theorem 1.8]).
Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of non-degenerate continua such that $X_i \supset X_{i+1}$ for

each $i \in \{1, 2, \ldots\}$*, and let*

$$X = \bigcap_{i=1}^{\infty} X_i.$$

Then, X *is a continuum.*

An important class of one-dimensional continua is given by arc-like continua (see Figure 2.1 (a) and (d)). There are several different equivalent definitions. We now introduce the following notion.

Definition 2.2.4. (Arc-like continuum).
A continuum X is said to be *arc-like* if for every $\varepsilon > 0$, there exists a continuous surjective function $g \colon X \to [0, 1]$ such that

$$\operatorname{diam}(g^{-1}(t)) < \varepsilon$$

for all $t \in [0, 1]$. Such g is called an ε*-map*.

Observation 2.2.5. *Let* $\varepsilon > 0$ *and let* $g \colon X \to [0, 1]$ *be an* ε*-map. There exists* $\delta > 0$ *with the following property. If* $U \subset [0, 1]$ *and* $\operatorname{diam}(U) < \delta$*, then* $\operatorname{diam}(g^{-1}(U)) < \varepsilon$*.*

Proof. Note that $\operatorname{diam}(U) = \operatorname{diam}(\overline{U})$, then the statement will be proved for a closed set. Let $\varepsilon > 0$ and suppose, by the way of contradiction, that for every $n \in \mathbb{N}$ there is a closed subset B_n of Y such that

$$\operatorname{diam}(B_n) < \frac{1}{n} \quad \text{and} \quad \operatorname{diam}(g^{-1}(B_n)) \geq \varepsilon.$$

Since X is compact, we can get a subsequence $\{B_{n_k}\}_{k=1}^{\infty}$ of $\{B_n\}_{n=1}^{\infty}$ such that $\{g^{-1}(B_{n_k})\}_{k=1}^{\infty}$ converges to a closed subset K of X. Observe that $\operatorname{diam}(K) \geq \varepsilon$. On the other hand, without loss of generality, we can assume that $\{B_{n_k}\}_{k=1}^{\infty}$ converges to a closed subset B of Y. Since $\lim_{n\to\infty} \operatorname{diam}(B_n) = 0$, we then obtain that $\operatorname{diam}(B) = 0$. This implies that there is $y \in Y$ such that $B = \{y\}$. Therefore, we have obtained that $g(K) = \{y\}$ by continuity, and thus

$$\operatorname{diam}(g^{-1}(y)) \geq \operatorname{diam}(K) \geq \varepsilon,$$

which is a contradiction to the fact that g is a ε-map. Hence, the claim is proved. □

As mentioned above, there are some equivalent definitions of arc-like. The following result tells us that both classes stated before are equivalent.

Theorem 2.2.6. ([NJ92, Theorem 12.11]).
A continuum X is arc-like if and only if it is chainable.

Remark. Note that the proof that an arc-like continuum is chainable follows from the definition and Observation 2.2.5.

Indeed, let $\varepsilon > 0$ and let $g\colon X \to [0,1]$ be an ε-map. By Observation 2.2.5, there is $\delta > 0$ such that for $U \subset [0,1]$ with $\mathrm{diam}(U) < \delta$, we have that $\mathrm{diam}(g^{-1}(U)) < \varepsilon$. Let $n \in \mathbb{N}$ such that $\frac{1}{n} < \frac{\delta}{2}$. Then, we get that $\{g^{-1}([0,\frac{2}{n})), g^{-1}((\frac{1}{n},\frac{3}{n})), g^{-1}((\frac{2}{n},\frac{4}{n})), \ldots, g^{-1}((\frac{n-3}{n},\frac{n-1}{n})), g^{-1}((\frac{n-2}{n},1])\}$ is an ε-chain of X covering X, and thus X is chainable as claimed.

We now introduce the notion of hereditarily indecomposable continua and give a few examples, as well as some of their properties.

Definition 2.2.7. (Indecomposable continuum).
A continuum X is said to be *decomposable* if there are two proper subcontinua A, B of X such that $X = A \cup B$. Otherwise, X is *indecomposable*.

Definition 2.2.8. (Hereditarily indecomposable continuum).
We say X is *hereditarily decomposable* (*indecomposable*) if each non-degenerate subcontinuum of X is decomposable (indecomposable).

Before we present some examples of these concepts, we state the following result which gives us a useful and basic property of a decomposable continuum.

Proposition 2.2.9. *A continuum X is decomposable if and only if X contains a proper subcontinuum with non-empty interior.*

Proof. Suppose X is decomposable, then there are two proper subcontinua A and B of X such that $X = A \cup B$. Note that $X \setminus A$ is non-empty and contained in B. Hence, $\mathrm{Int}(B) \neq \emptyset$.

Now, assume A is a proper subcontinuum of X such that $\text{Int}(A) \neq \emptyset$. If $X \setminus A$ is connected, this implies that $\overline{X \setminus A}$ is a continuum. Therefore, $X = A \cup (\overline{X \setminus A})$. Hence, X is decomposable.

Suppose $X \setminus A$ is not connected. Then, there exist two non-empty disjoint open sets U, V of X such that

$$X \setminus A = U \cup V.$$

Then $X \setminus (A \cup U) = V$; this implies that $A \cup U$ is closed. Therefore $A \cup U$ is compact. We now prove that $A \cup U$ is connected. Otherwise, by way of contradiction, there exist two non-empty disjoint closed subsets W_1, W_2 of X such that $A \cup U = W_1 \cup W_2$. Since A is connected, we can assume that $A \subset W_1$. This implies that $W_2 \subset U$, then $W_2 \cap \overline{V} = \emptyset$. Thus,

$$X = W_2 \cup (W_1 \cup \overline{V}),$$

which is a contradiction, because W_2 and $(W_1 \cup \overline{V})$ are disjoint closed subsets of X. Hence, $A \cup U$ is connected. This means that, $A \cup U$ is a subcontinuum of X. Likewise, $A \cup V$ is a subcontinuum of X. Therefore $X = (A \cup U) \cup (A \cup V)$. Hence, X is decomposable. □

Observe that examples in Figures 2.2, 2.3 are decomposable, in fact, they are hereditarily decomposable. In the case of the arc, example in Figure 2.2, note that all non-degenerate subcontinua are arcs, hence the continuum is hereditarily decomposable. In the case of figure 2.3, $X := \sin(1/x)$-continuum, we have two types of non-degenerate subcontinua. Some subcontinua are arcs and others are homeomorphic to X. Therefore X is hereditarily decomposable as well.

The *Knaster buckethandle*, $\mathcal{K}$, is an example of an indecomposable continuum (see Figure 2.4). It can be constructed as follows. Let us consider the Cantor set, $\mathcal{C}$, on the x-axis from 0 to 1, now the continuum consists of

1. Every semi-circle in $\mathbb{R}^2$ in the upper half plane, with center at the point $(\frac{1}{2}, 0)$ and passing through a point of $\mathcal{C}$.

2. Every semi-circle in $\mathbb{R}^2$ in the lower half plane, which has, for some $n \in \mathbb{N}$, the center at the point $(\frac{5}{2\cdot 3^n}, 0)$ and passes through a point of the Cantor set lying in the interval $\left[\frac{2}{3^n}, \frac{1}{3^{n-1}}\right]$.

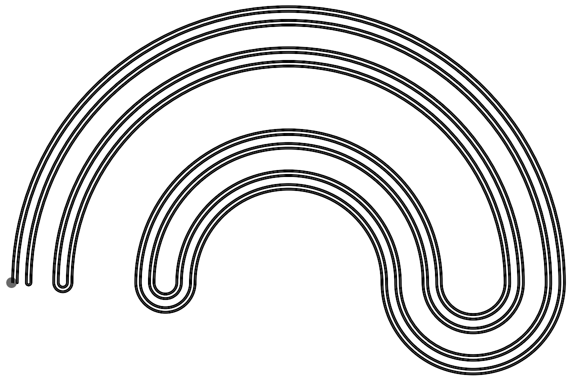


Figure 2.4: Knaster buckethandle

Figure provided by Rempe.

Before we give a brief explanation of why $\mathcal{K}$ is indecomposable, first we say that a *composant*, for some point p, of a continuum X is the set of all of points $x \in X$ such that there is a proper subcontinuum $Y \subset X$ containing p and x. It turns out that the only proper subcontinua of $\mathcal{K}$ are arcs, therefore the composants are equal to the arc components. Furthermore, every arc in $\mathcal{K}$ has empty interior. Hence $\mathcal{K}$ is indecomposable by Proposition 2.2.9. There are other techniques to prove that $\mathcal{K}$ is indecomposable, these may be found in [Kur68, Remark of Theorem 8] and [NJ92, Section 2.9] (this approach uses inverse limits).

We have presented examples of decomposable and indecomposable continua. As we explained before, the examples in Figures 2.2 and 2.3 are also hereditarily decomposable continua. As stated in the introduction, due to Bing's characterization, we use the following definition.

Definition 2.2.10. (Hereditarily indecomposable continuum).
A *pseudo-arc* is a hereditarily indecomposable arc-like continuum.

Since the pseudo-arc is arc-like, a common technique to construct it is by embedding chainable continua via Theorem 2.2.3. We now briefly explain this construction. We do not attempt to give all details, we just want to emphasize the complexity of this continuum and vaguely give an idea of its construction. We present Bing's description [Bin48].

A chain $\mathcal{V}$ is a *refinement* of a chain $\mathcal{U}$ if each link of $\mathcal{V}$ is contained in a link of $\mathcal{U}$. We say that a chain $\mathcal{V} = \{V_1, V_2, \ldots, V_m\}$ is *crooked* in $\mathcal{U} = \{U_1, \ldots, U_n\}$ if $\mathcal{V}$ is a refinement of $\mathcal{U}$ and any subchain of $\mathcal{V}$ passing through at least 4 consecutive links of $\mathcal{U}$ must follow a pattern like a 'z', that is, suppose we take 4 links on $\mathcal{U} = \{U_1, \ldots, U_4\}$, then a subchain in $\mathcal{V}$ has to cross up to U_3 then comes back to U_2 and then end in U_4 (see Figure 2.5).

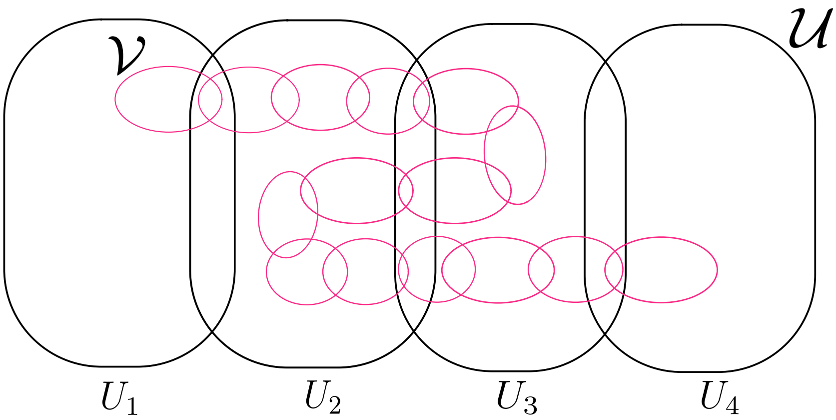


Figure 2.5: Illustration of the definition of crookedness.

More precisely, $\mathcal{V} = \{V_1, V_2, \ldots, V_n\}$ is crooked in $\mathcal{U} = \{U_1, \ldots, U_m\}$ if $\mathcal{V}$ is a refinement of $\mathcal{U}$. Further, if $k, l \in \{1, \ldots, m\}$ with $k + 3 \leq l$ and $i, j \in \{1, \ldots, n\}$ such that $V_i \subseteq U_k$ and $V_j \subseteq U_l$, then there exist $r, s \in \{1, \ldots, n\}$ such that $V_r \subseteq U_{l-1}$ and $V_s \subseteq U_{k+1}$ satisfying either $i < r < s < j$ or $j < s < r < i$; see Figure 2.6 for an ilustration with 5 links in $\mathcal{U}$.

It is possible to construct a recursive sequence of open connected chains $\mathcal{U}_1, \mathcal{U}_2, \ldots$ satisfying:

1. $\mathcal{U}_{n+1}$ is crooked in $\mathcal{U}_n$.

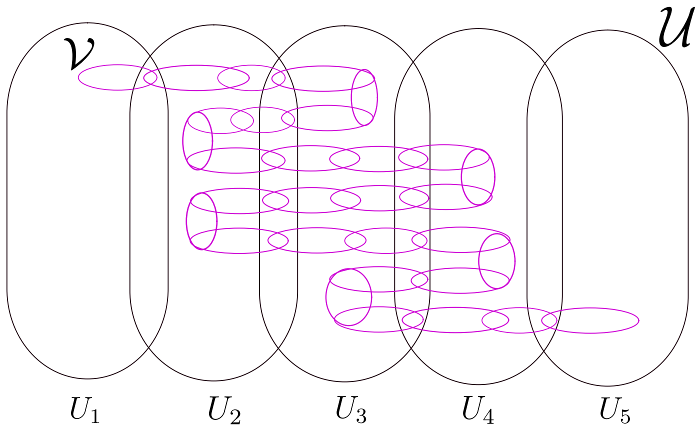


Figure 2.6: Illustration of crookedness with 5 links.

2. $\text{diam}(U_i) \leq 1/2^n$ for every $U_i \in \mathcal{U}_n$.

3. The first element of $\mathcal{U}_{n+1}$ is contained in the first element of $\mathcal{U}_n$, and the last element of $\mathcal{U}_{n+1}$ is contained in the last element of $\mathcal{U}_n$.

Then, $X := \bigcap_j (\overline{\bigcup \mathcal{U}_j})$ is an arc-like continuum, and it follows from the definition of crookedness that it is hereditarily indecomposable. So X is a pseudo-arc.

For a more detailed introduction to these particular topics we refer to [Bin51a, Bin51b] and to [Lew99] for a survey and further results in heriditarily indecomposable, arc-like continua.

2.3 Hyperbolic metric

A powerful tool commonly used in complex dynamics to obtain estimates in simply connected domains is the hyperbolic distance. Here we give some results about the hyperbolic geometry of plane domains which we shall require. A more detailed discussion on this topic can be found in [BM07] and [KL07], for instance.

A model of the hyperbolic plane, commonly known as the Poincaré disc model, is the unit disk $\mathbb{D}$ with the hyperbolic metric given by

$$\rho_{\mathbb{D}}(z)|dz| = \frac{2|dz|}{1-|z|^2},$$

where $\rho_{\mathbb{D}}\colon \mathbb{D} \to (0,\infty)$, $\rho_{\mathbb{D}}(z)|dz|$ is called a conformal metric and $\rho_{\mathbb{D}}(z) = \frac{2}{1-|z|^2}$ denotes the density of the hyperbolic metric.

For a simply connected domain $X \neq \mathbb{C}$, *the Riemann mapping theorem* [BM07, Theorem 6.1] states that there exists a conformal isomorphism from X onto $\mathbb{D}$. This allows us to transfer the hyperbolic metric of $\mathbb{D}$ to any simply connected domain X. That is, we can define the hyperbolic metric of X as follows.

Definition 2.3.1. Let $f\colon X \to \mathbb{D}$ be a conformal map, mapping X onto $\mathbb{D}$, where X is a simply connected domain. Then the *density of the hyperbolic metric* of X is defined by

$$\rho_X(z) = \rho_{\mathbb{D}}(f(z))|f'(z)|.$$

Remark. ρ_X is independent of the choice of the conformal map f. Let us justify this, suppose $g\colon X \to \mathbb{D}$ is a conformal map. Then $g = M \circ f$, where M is a Möbius self-map of $\mathbb{D}$. Since the hyperbolic metric $\rho_{\mathbb{D}}|dz|$ is invariant under M, i.e., $\rho_{\mathbb{D}}(z) = \rho_{\mathbb{D}}(M(z))|M'(z)|$, hence we obtain

$$\begin{aligned}
\rho_{\mathbb{D}}(g(z))|g'(z)| &= \rho_{\mathbb{D}}(M(f(z)))|M'(f(z))||f'(z)| \\
&= \frac{\rho_{\mathbb{D}}(f(z))}{|M'(f(z))|}|M'(f(z))||f'(z)| \\
&= \rho_{\mathbb{D}}(f(z))|f'(z)|.
\end{aligned}$$

As an example of Definition 2.3.1, we compute the hyperbolic metric of the right half plane $\mathbb{H}$.

Example 2.3.2. (Hyperbolic metric of $\mathbb{H}$).
To compute the hyperbolic metric of $\mathbb{H}$, let $\phi\colon \mathbb{H} \to \mathbb{D}$ be a conformal map defined

by $\phi(z) = \frac{1-z}{1+z}$. Then, we have

$$\begin{aligned} \rho_{\mathbb{H}}(z) &= \rho_{\mathbb{D}}(\phi(z))|\phi'(z)| \\ &= \frac{2|\phi'(z)|}{1-|\phi(z)|^2} \\ &= \frac{4|1+z|^2}{(|1+z|^2-|1-z|^2)|1+z|^2} \\ &= \frac{4}{(1+\operatorname{Re} z)^2-(\operatorname{Im} z)^2-(1-\operatorname{Re} z)^2+(\operatorname{Im} z)^2} \\ &= \frac{1}{\operatorname{Re} z}. \end{aligned} \tag{2.1}$$

Let us now introduce the notion of covering maps.

Definition 2.3.3. (Covering and universal covering maps).
Let $X, Y \subseteq \mathbb{C}$ be domains.

* A local homeomorphism $\varphi \colon X \to Y$ is called a *covering* if each $y \in Y$ has a connected neighbourhood V such that every connected component of $\varphi^{-1}(V)$ is mapped by φ homeomorphically onto V.

* A covering map $\varphi \colon X \to Y$ is called *universal* if X is simply connected.

A domain $U \subset \mathbb{C}$ is called *hyperbolic* if the complement of U contains at least two points. A generalization of the Riemann mapping theorem is *the Planar uniformization theorem* [BM07, Theorem 10.2], which allows us to define the hyperbolic metric also on a multiply connected domain, that is, if U is a hyperbolic domain then there exists a holomorphic covering map $\varphi \colon \mathbb{D} \to U$.

The following result shows us that the hyperbolic metric of $\mathbb{D}$ can be transferred to any hyperbolic domain. A proof of this result can be found in [BM07, Theorem 10.3].

Theorem 2.3.4. *Let U be a hyperbolic domain. Let $\varphi \colon \mathbb{D} \to U$ be a holomorphic universal covering. Then there is a unique metric $\rho_U(z)|dz|$ such that*

$$\rho_{\mathbb{D}}(z) = \rho_U(\varphi(z))|\varphi'(z)|.$$

Given $z, w \in U$ we define the *hyperbolic length* of a piecewise smooth curve $\gamma \subset U$ by

$$\ell_U(\gamma) = \int_\gamma \rho_U(z)|dz|$$

and we also define the *hyperbolic distance* between two points $z, w \in U$ by

$$d_U(z,w) = \inf \ell_U(\gamma),$$

where the infimum is taken over all piecewise smooth curves γ connecting z and w in U.

Given a holomorphic function $f\colon U \to V$ between hyperbolic domains, we denote the *hyperbolic derivative* of f with respect to the metrics in U and V by

$$||\operatorname{D} f(z)||_U^V := |f'(z)|\frac{\rho_V(f(z))}{\rho_U(z)}.$$

If $U \subset V$, then let $\iota\colon U \to V$ be the inclusion map, and we define

$$\begin{aligned} ||\operatorname{D} f(z)||_V^V &= \frac{1}{||\operatorname{D} \iota||_U^V}||\operatorname{D} f(z)||_U^V \\ &= |f'(z)|\frac{\rho_U(z)}{\rho_V(z)}\frac{\rho_V(f(z))}{\rho_U(z)} \\ &= |f'(z)|\frac{\rho_V(f(z))}{\rho_V(z)}. \end{aligned}$$

and we abbreviate $||\operatorname{D} f(z)||_V := ||\operatorname{D} f(z)||_V^V$.

We will often make use of the following result, known as Pick's theorem, which provides us properties of the hyperbolic metric when a domain is mapped into another. A proof of this can be found in [BM07, Theorem 10.5] or [Mil11, Theorem 2.11].

Theorem 2.3.5. (Pick's theorem).
Let U and V be hyperbolic domains, and let $f\colon U \to V$ be a holomorphic function. Then the following holds,

1. *f does not increase the hyperbolic metric, i.e., for all $z \in V$,*

$$|f'(z)|\rho_V(f(z)) \leq \rho_U(z);$$

or, equivalently, $||\,\mathrm{D}\, f(z)||_U^V \leq 1$.

2. *For any $z \in U$, $||\,\mathrm{D}\, f(z)||_U^V = 1$ if and only if f is a covering map. In this case, f is a local isometry.*

3. *If $U \subsetneq V$, then $\rho_U(z) > \rho_V(z)$ for every $z \in U$.*

There are few cases where we can compute the hyperbolic metric. The following example will be frequently considered throughout this work. So let us compute the hyperbolic metric of the strip $\tilde{S}$ of height 2π.

Example 2.3.6. (Hyperbolic metric of $\tilde{S}$).
The hyperbolic metric of $\tilde{S} \coloneqq \{z \in \mathbb{C} \colon |\operatorname{Im}(z)| < \pi\}$ can be computed by using the holomorphic covering map $\psi\colon \tilde{S} \to \mathbb{H}$ given by $\psi(z) = e^{z/2}$. Then, by Pick's theorem 2.3.5 we obtain that

$$\begin{aligned} 1 &= \frac{\rho_{\mathbb{H}}(\psi(z))}{\rho_{\tilde{S}}(z)}|\psi'(z)| \\ \rho_{\tilde{S}}(z) &= \rho_{\mathbb{H}}(\psi(z))|\psi'(z)| \\ &= \frac{|e^{z/2}|}{2\operatorname{Re}(e^{z/2})}. \end{aligned}$$

Now, set $z = x + iy$, so by rewriting $e^{z/2} = e^{x/2} \cdot e^{iy/2}$, we have

$$\rho_{\tilde{S}} = \frac{e^{x/2}}{2e^{x/2} \cdot \cos(y/2)} = \frac{1}{2\cos(\operatorname{Im} z/2)}. \tag{2.2}$$

It will be important to have estimates on the hyperbolic metric of a domain. The following standard estimate on the hyperbolic density in a simply connected domain is obtained from the Schwarz lemma and Koebe's 1/4–theorem. See [BM07, Theorems 8.2 and 8.6] or [Mil11, Corollary A.8] for more details.

Theorem 2.3.7. (Standard estimate on the hyperbolic density).

Let $U \subset \mathbb{C}$ be a simply connected domain. Then, for $z \in U$,

$$\frac{1}{2\,\mathrm{dist}(z, \partial U)} \leq \rho_U(z) \leq \frac{2}{\mathrm{dist}(z, \partial U)}. \tag{2.3}$$

2.4 Background on holomorphic dynamics

This section deals with the most essential definitions and results in holomorphic dynamics. For general texts in holomorphic dynamics we refer to [Mil11, CG13], and for an introduction on the iteration of meromorphic and transcendental entire functions we particularly refer to [Ber93, Sch10]. Additionally, since the main functions considered in this thesis belongs to the Eremenko–Lyubich class, we refer to [Six18] for a survey thereof.

Fix a transcendental entire function $f\colon \mathbb{C} \to \mathbb{C}$. Recall that f^n denotes the n-th iteration of f, for each $n \geq 0$.

Definition 2.4.1. (Periodic points and cycles)
A point $z \in \mathbb{C}$ is said to be *periodic* if there exists $n \geq 1$ such that $f^n(z) = z$. The smallest n satisfying this property is called the *period* of z. A periodic point of period one is called a *fixed point*. If z is a periodic point, then

$$\{z, f(z), \dots, f^{n-1}(z)\}$$

is called the *periodic cycle* of z. A point $z \in \mathbb{C}$ is *preperiodic* if $f^n(z)$ is a periodic point for some $n \geq 1$, and we say that z is *strictly preperiodic* if it is preperiodic but it is not periodic.

A significant part of the local behaviour near these points is determined by the *multiplier*. The multiplier of a point z of period n is defined by $\mu(z) := (f^n)'(z)$. Furthermore, periodic points can be classified in terms of the multiplier as follows.

1. A periodic point is *attracting* if $0 \leq |\mu(z)| < 1$.

2. A periodic point is *repelling* if $|\mu(z)| > 1$.

3. A periodic point is *indifferent* if $|\mu(z)| = 1$.

An attracting point z is called *superattracting* if $\mu(z) = 0$. Since the multiplier of an indifferent periodic point is of the form $e^{2\pi i a}$ where $0 \leq a < 1$. We can classify the point as a *rationally indifferent point* if a is rational and as an *irrationally indifferent point* otherwise.

Definition 2.4.2. (Immediate attracting basin).
Let $B := \{z, f(z), \ldots, f^{p-1}(z)\}$ be a periodic cycle of period p which contains an attracting point z_0. The *basin of attraction* $\mathcal{A}(B)$ of B is the open set $\mathcal{A}(B) := \left\{z \in \mathbb{C}\colon f^{np}(z) \to w, \text{ for } w \in B\right\}$. We say that the *immediate attracting basin* is the union of connected components of $\mathcal{A}(B)$ containing the points of B.

We now introduce the concept of singular values.

Definition 2.4.3. (Singular values)
Let f be a transcendental entire function. A point $c \in \mathbb{C}$ is called a *critical value* of f if there exists a point $z \in \mathbb{C}$ such that $f'(z) = 0$ and $f(z) = c$. A point $w \in \mathbb{C}$ is called an *asymptotic value* of f if there exits a curve $\gamma\colon (0, \infty) \to \mathbb{C}$ such that $\gamma(t) \to \infty$ as $t \to \infty$ but $f(\gamma(t)) \to w$ as $t \to \infty$. Denote $CV(f)$ and $AV(f)$ as the set of all critical and of all asymptotic values, respectively. The *singular set* of f, $S(f)$, is,

$$S(f) := \overline{CV(f) \cup AV(f)}.$$

Its elements are called *singular values*. Equivalently, $S(f)$ is the smallest closed set such that

$$f : \mathbb{C} \setminus f^{-1}(S(f)) \to \mathbb{C} \setminus S(f)$$

is a covering map.

Recall from Chapter 1, that the *Fatou set* of f is the set of all points that have a neighbourhood in which $\{f^n\}_{n\in\mathbb{N}}$ forms a normal family, and its complement is the *Julia set* $\mathcal{J}(f) := \mathbb{C} \setminus \mathcal{F}(f)$. The following result gathers some of the fundamental properties of the Fatou and Julia sets of a transcendental entire function. A proof of this may be found in [Ber93, Lemmas 1, 2 and 3 and Theorem 3].

We say that a set A is *completely invariant* if $z \in A$ if and only if $f(z) \in A$. A set X is *perfect* if it is closed, non-empty and has no isolated points.

Theorem 2.4.4. (Properties of transcendental entire functions).
Let f be a transcendental entire function. Then:

(i) $\mathcal{F}(f) = \mathcal{F}(f^n)$ *and* $\mathcal{J}(f) = \mathcal{J}(f^n)$ *for all* $n \geq 1$.

(ii) $\mathcal{F}(f)$ *and* $\mathcal{J}(f)$ *are completely invariant.*

(iii) Either $\mathcal{J}(f) = \mathbb{C}$ *or* $\operatorname{Int}(\mathcal{J}(f)) = \emptyset$.

(iv) $\mathcal{J}(f)$ *is perfect.*

Let us turn back our attention to *singular values* since they play an important role in the dynamics of an entire function. Let us recall that the *Eremenko–Lyubich class* is defined as follows,

$$\mathcal{B} := \{f : \mathbb{C} \to \mathbb{C} \, \text{transcendental entire} : S(f) \;\text{ is bounded}\}.$$

We also introduce the *escaping set* of a transcendental entire function,

$$I(f) := \left\{z \in \mathbb{C} \colon \lim_{n \to \infty} f^n(z) = \infty\right\}.$$

Observe that $I(f^n) = I(f)$ for all $n \geq 1$. Eremenko [Ere89] started the first systematic study of the escaping set for transcendental entire functions. Part of this study showed the following properties. The last statement of this result comes from [EL92].

Theorem 2.4.5. (Properties of the escaping set).
If f is a transcendental entire function, then

$$\mathcal{J}(f) = \partial I(f) \quad \text{and} \quad \mathcal{J}(f) \cap I(f) \neq \emptyset.$$

Furthermore, if $f \in \mathcal{B}$, *then* $\mathcal{J}(f) = \overline{I(f)}$.

Additionally, as already mentioned in the introduction, for a map $f \in \mathcal{B}$, it is possible to obtain a disjoint-type entire function. In order to prove this, we first say that two functions $f, g \in \mathcal{B}$ are *quasiconformal equivalent* (*near infinity*) if there exist quasiconformal maps $\varphi, \psi \colon \mathbb{C} \to \mathbb{C}$ such that

$$\psi(f(z)) = g(\varphi(z))$$

for all $z \in \mathbb{C}$ (whenever $|f(z)|$ or $g(\varphi(z))$ is large enough).

Observation 2.4.6. *Let $f \in \mathcal{B}$. Then for all $\lambda \in \mathbb{C} \setminus \{0\}$ with sufficiently small modulus, the function $g(z) := \lambda f(z)$ is of disjoint type.*

Proof. First, note that f and g are quasiconformal equivalent by taking ψ as the identity map and $\varphi(z) = z/\lambda$. Now, since $f \in \mathcal{B}$, we can choose $R > 0$ such that $\{S(f), f(0), 0\} \subset D_R(0)$ For sufficiently small λ, $\lambda f(D_R(0)) \subset D_R(0)$. This implies that for any such λ, $g(D_R(0)) \subset D_R(0)$. Furthermore, it is not difficult to see that $S(g) = \lambda S(f) \subset D_R(0)$. Hence, g is of disjoint type. □

Lastly, note that by Theorem 2.4.4(ii), if U is a connected component of $\mathcal{F}(f)$, then $f^n(U)$ is contained in a component of $\mathcal{F}(f)$. Thus, this allows us to classify these components under their orbits.

Definition 2.4.7. (Fatou components).
Let U be a connected component of $\mathcal{F}(f)$. We say that U is *periodic* if, there exists $n \geq 1$ such that $f^n(U) \subseteq U$. We call U *preperiodic* if $f^n(U)$ is periodic for some $n > 1$. A component which is preperiodic but not periodic is called *strictly preperiodic*. If U is not preperiodic, then U is called a *wandering domain*.

We now state the classification of periodic Fatou components for transcendental entire functions, which is essentially due to Cremer [Cre32] and Fatou [Fat19]. For detailed information on the history, we refer to [Ber93].

Theorem 2.4.8. (Classification of periodic Fatou components).
Let f be a transcendental entire function and let U be a periodic Fatou component of period p. Then one of the following holds:

(a) U contains an attracting periodic point z_0. Then U is the immediate attracting basin of z_0.

(b) ∂U *contains a periodic point* z_0 *of period* p *and* $f^{np}(z) \to z_0$ *as* $n \to \infty$ *for all* $z \in U$*. Then* $(f^p)'(z_0) = 1$ *and we call* U *the immediate parabolic basin or Leau domain.*

(c) *There is an analytic homeomorphism* $\psi\colon U \to \mathbb{D}$ *such that* $\psi(f^p(\psi^{-1}(z))) = e^{2\pi i a}z$ *for some* $a \in \mathbb{R} \setminus \mathbb{Q}$*. In this case,* U *is called a Siegel disc.*

(d) $f^{np}(z) \to \infty$ *as* $n \to \infty$ *for* $z \in U$*. In this case,* U *is called a Baker domain.*

Remark. Recall from Definition 3.1.1 that f is of disjoint type if there exists a bounded Jordan domain D, such that $S(f) \subset D$ and $f(\overline{D}) \subset D$. First, let us note that Baker domains lie in $I(f)$, then it follows from Theorem 2.4.5 that f has no Baker domains. Additionally, it can be deduced from [Ber93, Theorem 7] that f has no parabolic basin or Siegel discs.

Now, if U is a Fatou component of f, then by Montel's theorem $D \subset U$. Since any cycle of an immediate attracting basin contains at least one singular value, then U is the unique immediate attracting basin of f, by [Ber93, Theorem 7].

2.5 Logarithmic coordinates

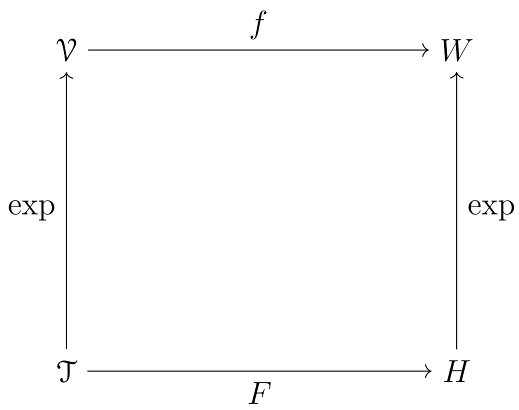
A common technique for studying the dynamics in the Eremenko-Lyubich class $\mathcal{B}$ is the *logarithmic change of coordinates*. This was first used in dynamics in [EL92]. For a more detailed overview of this technique we refer to [RRRS11, Section 2], [Rem16, Section 3] or [Six18, Section 5].

Definition 2.5.1. (The logarithmic transform).
For a transcendental entire function $f \in \mathcal{B}$, fix a disk $D_R := D_R(0) \subset \mathbb{C}$ with $R > 0$, such that $S(f) \subseteq D_R$ and additionally $f(0) \in D_R$. Consider $W := \mathbb{C} \setminus \overline{D_R}$ and define

$$\mathcal{V} := f^{-1}(W).$$

The connected components V of $\mathcal{V}$ are called the *tracts* of f. By construction, the restriction of f to each connected component V of $\mathcal{V}$, $f\colon V \to W$ is a universal covering.

Let $H := \exp^{-1}(W)$ and let $\mathcal{T} := \exp^{-1}(\mathcal{V})$. Since f is a universal covering on every connected component of $\mathcal{V}$ and $0 \notin W \supset \mathcal{V}$, then we can *lift* to a map $F\colon \mathcal{T} \to H$ such that the following diagram commutes:



We call F a *logarithmic transform* of f, and it can be chosen to be $2\pi i$-periodic. Furthermore, we call each connected component of $\mathcal{T}$ a *logarithmic tract* of F.

By construction, we have the following properties.

Definition 2.5.2. (Properties of a logarithmic transform).

(a) H is a $2\pi i$-periodic unbounded Jordan domain which contains a right half plane.

(b) $\mathcal{T} \neq \emptyset$ is $2\pi i$-periodic, and the real part of points in $\mathcal{T}$ is bounded from below, but unbounded from above.

(c) Each tract $T \in \mathcal{T}$ is an unbounded Jordan domain that is disjoint from all its $2\pi i$-translates. The restriction $F|_T\colon T \to H$ is a conformal isomorphism whose continuous extension to the closure of T in $\hat{\mathbb{C}}$ satisfies $F(\infty) = \infty$, and we denote the inverse of F_T by F_T^{-1}.

(d) The components of $\mathcal{T}$ have pairwise disjoint closures and accumulate only at ∞; i.e., if $(z_n)_{n\in\mathbb{N}}$ is a sequence of points of $\mathcal{T}$, all belonging to different components of $\mathcal{T}$, then $z_n \to \infty$ as $n \to \infty$.

Remark 2.5.3. The *class* $\mathcal{B}_{\log}$ is defined to be those functions

$$F \colon \mathcal{T} \to H$$

satisfying properties (a)–(d), independently of whether they arise from an entire function $f \in \mathcal{B}$ or not. We also denote $\mathcal{B}_{\log}^{p}$ the class of functions $F \in \mathcal{B}_{\log}$ such that F is $2\pi i$-periodic. See [Rem16, Definition 3.3].

By the Carathéodory–Torhorst's theorem [Pom92, Theorem 2.1], for every $T \in \mathcal{T}$ the function $F|_T$ in (c) can be continuously extended to the boundary of T. Furthermore, since T is a Jordan domain, this extension is a homeomorphism. In particular, $F|_T$ extends continuously to a homeomorphism between $\overline{T}$ and $\overline{H}$, and combining this with (d) implies that F extends continuously to $\overline{\mathcal{T}}$. Then we denote the *Julia set* of F

$$\mathcal{J}(F) := \{z \in \overline{\mathcal{T}} \colon F^n(z) \in \overline{\mathcal{T}} \text{ for all } n \geq 0\},$$

to be the set of points whose iterates are always contained in $\overline{\mathcal{T}}$. We say that a logarithmic transform F is of *disjoint type* if the boundaries of the tracts of F do not intersect the boundary of H, that is $\overline{\mathcal{T}} \subset H$. If $F \in \mathcal{B}_{\log}$ is of disjoint type, in particular, the Julia set can be defined by

$$\mathcal{J}(F) = \bigcap_{n \geq 0} F^{-n}(\overline{\mathcal{T}}). \tag{2.4}$$

Remark 2.5.4. If an entire function $f \in \mathcal{B}$ has a logarithmic transform F which is of disjoint type, then f itself is of disjoint type, and so

$$\exp(\mathcal{J}(F)) = \mathcal{J}(f). \tag{2.5}$$

The notion of *external address* for functions in the class $\mathcal{B}_{\log}^{p}$ provides us a partition of $\mathcal{J}(F)$, that is, this concept allows to assign symbolic dynamics as follows.

Definition 2.5.5. (External addresses).
Let $F \in \mathcal{B}_{\log}^{p}$ be of disjoint type. An (*infinite*) *external address* is a sequence

$\underline{s} = T_0T_1T_2\dots$ of tracts of F. The external address $\underline{s}$ is *bounded* if it contains only finitely many different tracts.
For each external address $\underline{s}$, we then denote

$$\begin{aligned}\mathcal{J}_{\underline{s}}(F) &:= \{z \in \mathcal{J}(F)\colon F^n(z) \in T_n \text{ for all } n \geq 0\}\\ &= \{z \in \mathcal{J}(F)\colon F^n(z) \in \overline{T_n} \text{ for all } n \geq 0\},\\ &\text{and}\\ \widehat{\mathcal{J}}_{\underline{s}}(F) &:= \mathcal{J}_{\underline{s}}(F) \cup \{\infty\}.\end{aligned}$$

The external address $\underline{s}$ is said to be *admissible* if $\mathcal{J}_{\underline{s}}(F)$ is non-empty. Then, in this case $\widehat{\mathcal{J}}_{\underline{s}}(F)$ is called a *Julia continuum* of F.
For $n \geq 0$, we denote

$$F^n_{\underline{s}} := F|_{T_{n-1}} \circ \dots \circ F|_{T_0} \qquad \text{and} \qquad F^{-n}_{\underline{s}} := (F^n_{\underline{s}})^{-1}.$$

Remark 2.5.6. In the case where $F \in \mathcal{B}^p_{\log}$ is of disjoint type, we can write $\widehat{\mathcal{J}}_{\underline{s}}(F)$ as a nested intersection of continua, that is,

$$\widehat{\mathcal{J}}_{\underline{s}}(F) = \bigcap_{n=0}^{\infty}(F^{-n}_{\underline{s}}(\overline{T}_n) \cup \{\infty\}) \tag{2.6}$$

Therefore, $\widehat{\mathcal{J}}_{\underline{s}}(F)$ is a continuum by Theorem 2.2.3. Moreover, note that $\underline{s}^1 \neq \underline{s}^2$, implies that for some $n \geq 0$, the sets $\widehat{\mathcal{J}}_{\underline{s}^1}(F)$ and $\widehat{\mathcal{J}}_{\underline{s}^2}(F)$ belong to different connected components of $F^{-n}(\mathcal{T})$ (recall that $\mathcal{T}$ denotes the domain of F, see Remark 2.5.3). Hence, every connected component of $\mathcal{J}(F)$ is contained in a single Julia continuum.

From the above remark, we can consider connected components of the Julia set of F which remain in the same single Julia continuum. So we conclude this section with the following concept.

Definition 2.5.7. (Invariant Julia continuum in class $\mathcal{B}^p_{\log}$).
Let $F \in \mathcal{B}^p_{\log}$ and denote $\underline{s} := T_0T_0\dots$. An *invariant Julia continuum* is given by

$$\widehat{\mathcal{J}}_{\underline{s}}(F) = \mathcal{J}_{\underline{s}}(F) \cup \{\infty\} \tag{2.7}$$

where $\mathcal{J}_{\underline{s}}(F) := \{z \in \mathcal{J}(F) \colon F^n(z) \in \overline{T}_0 \text{ for all } n \geq 0\}$.

CHAPTER

THREE

DISJOINT-TYPE MODELS AND REALISATION OF PSEUDO-ARCS

Recall that given a disjoint-type entire function f, we say that a *Julia continuum* of f is a connected component of its Julia set, $\mathcal{J}(f)$, together with infinity.

In this chapter, we show how a *pseudo-arc* can arise as a Julia continuum of a transcendental entire function f. In order to construct such a function f, we will make use of a technique introduced by Bishop [Bis15] and results of Rempe [Rem09]. That is, we shall first construct a suitable "disjoint-type model" F (see Definition 3.1.1) with specific geometric conditions, then approximate this map by an entire function (see Theorem 3.1.7 for a precise statement).

In the first section, we introduce a class $\mathcal{H}$ of "model functions" and show that any function in this class can be approximated by a transcendental entire function in a way that preserves the topology of Julia continua. In the following section, inspired by a result of Henderson (compare [Hen64, Lemma 1]), we give a sufficient condition for the Julia continuum of a function $F \in \mathcal{H}$ to be a pseudo-arc.

3.1 Disjoint-type models

We start by introducing a suitable class $\mathcal{H}$ of conformal isomorphisms, which will be the model functions in our construction. Then, we look at the basic properties of this class.

Definition 3.1.1. (The class $\mathcal{H}$).
Let $\mathcal{T}$ denote the set of all simply connected domains

$$T \subset S := \{x + iy \colon x > 4, |y| < \pi\}$$

with $5 \in T$, and the following properties:

(i) T is unbounded,

(ii) ∂T is locally connected,

(iii) there is only one access to infinity in T; i.e., any two curves in T connecting the same finite endpoint to ∞ are homotopic.

If $T \in \mathcal{T}$, then there exists a unique conformal isomorphism $F \colon T \to \mathbb{H}$ with $F(5) = 5$ and such that F extends continuously to ∞ with $F(\infty) = \infty$. The family $\mathcal{H}$ consists of all such functions F.

Remark 3.1.2. We now justify the existence of this function. By the Riemann mapping theorem, there is a conformal isomorphism $F \colon T \to \mathbb{H}$ satisfying $F(5) = 5$. This map is unique up to post-compositions with Möbius transformations of $\mathbb{H}$ fixing 5.

By the Carathéodory–Torhorst theorem [Pom92, Theorem 2.1], $F^{-1} \colon \mathbb{H} \to T$ extends continuously to $\overline{\mathbb{H}} \cup \{\infty\}$. Then (iii) implies that there exists exactly one point $\varsigma \in \partial\mathbb{H} \cup \{\infty\}$ such that $F^{-1}(\varsigma) = \infty$. By post-composing F with a Möbius transformation, we may assume that $\varsigma = \infty$, then this makes F unique. Since $\overline{\mathbb{H}} \cup \{\infty\}$ is compact, then F extends continuously to ∞.

The following result provides uniform expansion for functions in the class $\mathcal{H}$.

Lemma 3.1.3. (Expansion of F).
Let $F \in \mathcal{H}$*. Then* $|F'(z)| \geq \frac{\operatorname{Re} F(z)}{2}$ *for all* $z \in T$*. In particular, if* $\operatorname{Re} F(z) \geq 4$*, then*

$$|F'(z)| \geq 2. \tag{3.1}$$

Proof. Note that $T \subseteq S \subseteq \tilde{S} := \{x + iy \colon |y| < \pi\}$. Then by Pick's theorem 2.3.5, $\rho_T \geq \rho_{\tilde{S}}$, which implies

$$\rho_T(z) \geq \rho_{\tilde{S}}(z) = \frac{1}{2\cos(\operatorname{Im} z/2)} \geq \frac{1}{2} \qquad \text{for all } \ z \in T. \tag{3.2}$$

See Example 2.3.6 for the explicit expression of $\rho_{\tilde{S}}(z)$. On the other hand, F is a conformal isomorphism, and hence a local isometry between T and $\mathbb{H}$ with their respective hyperbolic metrics. Thus, again by Pick's theorem 2.3.5, we have

$$1 = ||DF(z)||_T^{\mathbb{H}} = |F'(z)| \cdot \frac{\rho_{\mathbb{H}}(F(z))}{\rho_T(z)}.$$

Since $\rho_{\mathbb{H}}(F(z)) = \frac{1}{\operatorname{Re} F(z)}$ (see Example 2.3.2), we get

$$|F'(z)| = \frac{\rho_T(z)}{\rho_{\mathbb{H}}(F(z))} \geq \frac{\operatorname{Re} F(z)}{2}.$$

Hence, the claim holds. □

Definition 3.1.4. (Julia continuum of F).
Let $F \in \mathcal{H}$. We denote the *Julia set* of F by

$$\begin{aligned} \mathcal{J}(F) &:= \{z \in T \colon F^n(z) \in T \ \text{ for all } \ n \geq 0\} = \bigcap_{n \geq 0} F^{-n}(T) \\ &= \bigcap_{n \geq 0} F^{-n}(\overline{T}). \end{aligned}$$

We also denote $\widehat{\mathcal{J}}(F) = \mathcal{J}(F) \cup \{\infty\}$, which will be called the *Julia continuum* of F.

Remark. In the definition of the Julia set of F, note that the last equality follows from the fact that $F^{-(n+1)}(\overline{T}) \subset F^{-n}(T) \subset F^{-n}(\overline{T})$ for $n \geq 0$.

Additionally, observe that $F^{-n}(\overline{T})\cup\{\infty\}$ is compact and connected for each $n \geq 0$. Therefore, $\widehat{\mathcal{J}}(F)$ is a continuum by Theorem 2.2.3.

As a consequence of the expansion property (3.1) of F, two different points in the Julia continuum must separate under iteration eventually.

Lemma 3.1.5. (Expansion along orbits).
Let $F \in \mathcal{H}$ and $n \geq 0$. Suppose $z, w \in F^{-n}(T)$ and $\min\{\operatorname{Re} F^n(z), \operatorname{Re} F^n(w)\} \geq 4$. Then

$$|F^n(z) - F^n(w)| \geq 2^n \cdot |z - w|. \tag{3.3}$$

Proof. For $n = 1$, let γ be a straight line joining $F(z)$ and $F(w)$, i.e., $\gamma\colon [0,1] \to \mathbb{H}$, defined by $\gamma(t) = t \cdot F(z) + (1-t) \cdot F(w)$.
Since $\{x + iy\colon x \geq 4\}$ is convex, then $\operatorname{Re} v \geq 4$, for all $v \in \gamma$. Consider the curve $\tilde{\gamma} := F^{-1} \circ \gamma$, and note that $\tilde{\gamma}$ joins z and w. By (3.1), we have that $|(F^{-1})'(v)| \leq \frac{1}{2}$. Consequently,

$$\begin{aligned} |z - w| \leq \ell(\tilde{\gamma}) &= \int_0^1 |\tilde{\gamma}'(t)| dt = \int_0^1 |(F^{-1})'(\gamma(t))||\gamma'(t)| dt \\ &\leq \frac{1}{2} \int_0^1 |\gamma'(t)| dt = \frac{1}{2} |F(z) - F(w)|. \end{aligned}$$

Thus, (3.3) is proved for $n = 1$. Since $z, w \in F^{-k}(T)$, for each $k \geq 0$, the points $F^k(z), F^k(w)$ belong to T. Hence, we can inductively apply the same argument as before and (3.3) follows. □

Observation 3.1.6. (Periodic extension of F).
Let $F \in \mathcal{H}$ and denote $\mathbb{H}_1 := \{a + ib\colon a > 1\}$. We define $\tilde{F}\colon \tilde{T} \to \mathbb{H}_1$ where $\tilde{T} = F^{-1}(\mathbb{H}_1)$. We also define

$$\hat{\mathfrak{T}} = \bigcup_{m \in \mathbb{Z}} \{z + 2\pi i m\colon z \in \tilde{T}\},$$

and $\hat{F}\colon \hat{\mathfrak{T}} \to \mathbb{H}_1$ by $\hat{F}(z + 2\pi i m) := \tilde{F}(z)$ for all $z \in \tilde{T}$, $m \in \mathbb{Z}$. Then, $\hat{F} \in \mathcal{B}^p_{\log}$ in the sense of Remark 2.5.3. Further, $\hat{F}$ will be referred to as the periodic extension *of F.*

Moreover, $\widehat{\mathfrak{J}}(F)$ is an invariant Julia continuum of $\hat{F}$ in the sense of Definition 2.5.7.

Proof. First, since $5 \in T$, we have $\tilde{T} \neq \emptyset$, and thus $\hat{\mathfrak{T}} \neq \emptyset$. By construction, $\tilde{T}$ is an unbounded Jordan domain,; this implies that every component $\hat{T}$ of $\hat{\mathfrak{T}}$ is an unbounded Jordan domain. Observe that $\hat{\mathfrak{T}}$ and $\mathbb{H}$ are $2\pi i$-periodic by definition. Furthermore, since $\tilde{T} \subset T$, then for any connected component $\hat{T} \in \hat{\mathfrak{T}}$, we have that

$$\hat{F}|_{\hat{T}} \colon \hat{T} \to \mathbb{H}_1$$

is a conformal isomorphism. Lastly, let $(w_n)_{n\in\mathbb{N}}$ be a sequence of points of $\hat{\mathfrak{T}}$ all belonging to different components, so we can write $w_n = z_n + 2\pi i m_n$, then for every $n \in \mathbb{N}$, w_n is in a different component of $\hat{\mathfrak{T}}$. Therefore, $w_n \to \infty$ as $n \to \infty$. □

The following result will allow us to deduce Theorems 1.0.4 and 1.0.6 from corresponding results for functions in $\mathcal{H}$. It states that we can pass from a conformal isomorphism $F \in \mathcal{H}$ to an entire function $g \in \mathcal{B}$ such that both functions have the same topological structure on their Julia sets. This result follows from [Rem16, Theorem 3.5] which is a consequence of [Bis15, Theorem 1.2]. Still, for completeness, we include a version adapted to our setting.

Theorem 3.1.7. (Realisation of disjoint-type models).
Suppose $F \in \mathcal{H}$. Then there exists a disjoint-type function $g \in \mathcal{B}$ such that every Julia continuum of g is homeomorphic to a Julia continuum of the periodic extension $\hat{F}$ of F.
In addition, the function g is uniformly bounded on $\mathbb{C} \setminus \exp(T \setminus F^{-1}(\operatorname{Re} z \leq \rho))$ for $\rho > 0$.

Proof. Let $\tilde{T} = F^{-1}(\mathbb{H}_1)$, and let $\tilde{F} \colon \tilde{T} \to \mathbb{H}_1$. Now, consider the periodic extension $\hat{F}$ of $\tilde{F}$, then $\hat{F} \in \mathcal{B}_{\log}^p$ by Observation 3.1.6.
Now, we define Θ given by $\Theta(\exp z) = \exp \tilde{F}(z)$ for all $z \in \tilde{T}$. Additionally, let V be the image under the exponential of $\tilde{T}$, that is $V = \exp(\tilde{T}) = \exp(\hat{\mathfrak{T}})$. Furthermore, $\hat{F}$ also satisfies

$$\Theta(\exp z) = \exp \hat{F}(z) \quad \text{for all} \quad z \in \hat{\mathfrak{T}}.$$

In particular, note that $\tilde{T}$ maps one to one to V under exp. That is, V has one preimage under the exponential map on each of the logarithmic tracts $\{\tilde{T} + 2\pi i\mathbb{Z}\}$, see Figure 3.1.

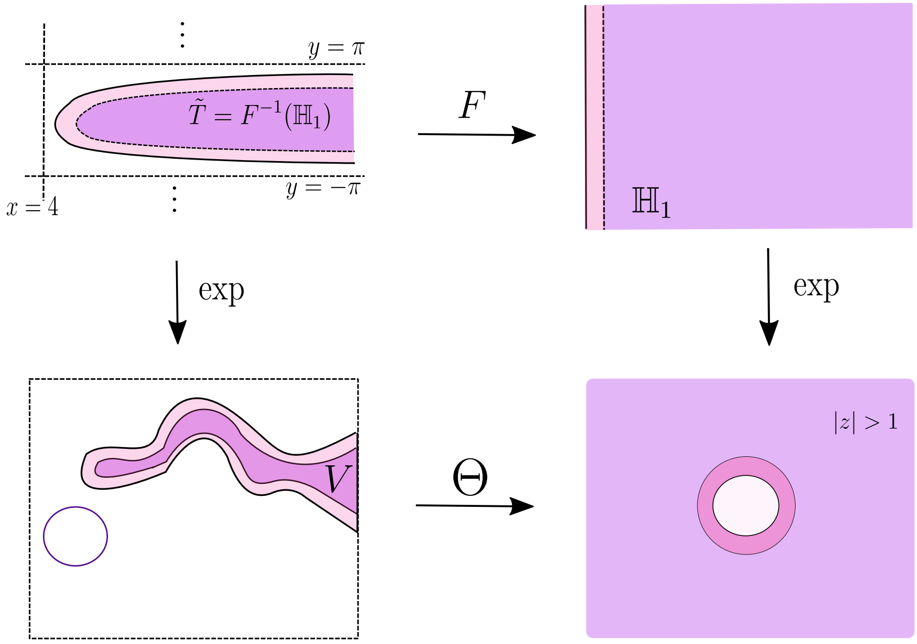


Figure 3.1: Illustration of the proof

The pair (V, Θ) is a model in the sense of [Bis15, page 2]. Following the terminology from the aforementioned paper, for $\rho > 0$ we define

$$V(\rho) := \{z \in V : \operatorname{Re}\Theta(z) > e^{\rho}\},$$
$$V(\delta, \rho) := \{z \in V : e^{\delta} < \operatorname{Re}\Theta(z) < e^{\rho}\}.$$

So by [Bis15, Theorem 1.1], there is a function $f \in \mathcal{B}$ and a quasiconformal homeomorphism $\psi\colon \mathbb{C} \to \mathbb{C}$ such that

$$\Theta = f \circ \psi \quad \text{on } V(2). \tag{3.4}$$

Furthermore, the quasiconformal dilatation of the map ψ is supported on the set $V(\delta, \rho)$, and f is bounded on $\mathbb{C} \setminus \psi(V(2))$. By Observation 2.4.6 we get that

$g := \lambda f$ is of disjoint type for sufficiently small λ. So, if we take an affine map $\phi\colon \mathbb{C} \to \mathbb{C}$ given by $\phi(z) = \lambda z$, then from (3.4) we obtain

$$\phi(\Theta(z)) = g(\psi(z))$$

whenever $\exp(z) \in \psi(V(2)) = T \setminus F^{-1}(\operatorname{Re} z \leq 2)$. Thus, by the above and [Rem09, Theorem 3.5], g and Θ are quasiconformal conjugate on a neighborhood of their Julia sets. More precisely, there is a quasiconformal map $\vartheta\colon \mathbb{C} \to \mathbb{C}$ such that

$$\vartheta \circ \Theta = g \circ \vartheta$$

on an open set that contains both $\mathcal{J}(g)$ and $\mathcal{J}(\Theta)$. So, now let $\hat{C} = C \cup \{\infty\}$ be a Julia continuum of $\hat{F}$, then, by the above, we have

$$\vartheta(\phi(\exp(C))) \cup \{\infty\}$$

is a Julia continuum of g. Therefore, $\vartheta(\phi(\exp \mathcal{J}(\hat{F}))) \cup \{\infty\}$ is the Julia continua of $\widehat{\mathcal{J}}(g)$. Hence every Julia continuum of g is homeomorphic to a Julia continuum of $\hat{F}$. Since f is bounded and $g = \lambda f$ is of disjoint type, thus g is uniformly bounded on $\mathbb{C} \setminus \psi(V(2))$. □

3.2 Sufficient conditions for Julia continua to be pseudo-arcs

In this section, we develop some geometric conditions on a function $F \in \mathcal{H}$ that ensure that $\widehat{\mathcal{J}}(F)$ is a pseudo-arc. We start by introducing a function of one real variable associated to a function $F \in \mathcal{H}$; this function will encapsulate the essential mapping behaviour of F.

Definition 3.2.1. (One-dimensional projection).
Let $F \in \mathcal{H}$. The *one-dimensional projection* of F is the map $\varphi\colon [4, \infty] \to [4, \infty]$ given by

$$\varphi(t) := \operatorname{Re} F^{-1}(t).$$

Lemma 3.2.2. (Properties of φ).
Let $F \in \mathcal{H}$. Then the one-dimensional projection φ has the following properties:

(i) $|\varphi(t) - \varphi(s)| \leq \frac{|t-s|}{2}$ *whenever* $t, s \geq 4$.

(ii) $\varphi(t) < t$ *whenever* $t > 5$, *and furthermore, if* $t \in [4, 5]$ *then* $\varphi(t) < 6$.

(iii) $\varphi(t) \leq 5 + 2(\log t - \log 5)$ *for all* $t \geq 5$.

(iv) $\varphi(t) < t - 1$ *for* $t \geq 7$ *and* $\varphi(t) < \frac{t}{2}$ *for* $t \geq 15$.

Proof. Let $t, s \geq 4$ be such that $t \geq s$, then by Lemma 3.1.3 $|(F^{-1})'(x)| \leq \frac{1}{2}$ for all $x \in [s, t]$. Then we obtain

$$\begin{aligned} |\varphi(t) - \varphi(s)| &= |\operatorname{Re} F^{-1}(t) - \operatorname{Re} F^{-1}(s)| \\ &\leq |F^{-1}(t) - F^{-1}(s)| \\ &\leq \frac{|t-s|}{2}. \end{aligned}$$

Hence (i) holds.

Recall that $F(5) = 5$ by definition, and since F is a conformal isomorphism, then we have $F^{-1}(5) = 5$. Now, if $t \geq 5$ and by applying (i), we then obtain $|\varphi(t) - 5| \leq \frac{|t-5|}{2}$. So, in particular, $\varphi(t) - 5 \leq \frac{t-5}{2}$, which implies $\varphi(t) \leq \frac{t+5}{2}$. Hence $\varphi(t) < t$.

Now, if $|t-5| \leq 1$ and by the previous part, we have $|\varphi(t)-5| \leq (t-5)/2 \leq 1/2$. Therefore, $\varphi(t) < 6$. This proves (ii).

Since $T \subset \tilde{S} = \{x + iy \colon |y| < \pi\}$, then by (2.2) we have that $\rho_T(z) \geq 1/2$ for all $z \in T$. Let $t \geq 5$, then we obtain

$$|F^{-1}(t) - 5| \leq 2 \operatorname{dist}_T(F^{-1}(t), 5) = 2 \operatorname{dist}_{\mathbb{H}}(t, 5) = 2(\log t - \log 5);$$

this implies that

$$\varphi(t) \leq 5 + |F^{-1}(t) - 5| \leq 5 + 2(\log t - \log 5).$$

Hence, (iii) is satisfied.

Let us set $g(t) = 5 + 2(\log t - \log 5)$ for $t \geq 5$. Observe that $g(5) = 5$, and furthermore $g(t) < t$ for all $t > 5$. Thus, by (iii), $\varphi(t) < g(t) < t$. By this inequality, it is enough to show that $g(t) < t - 1$ for $t \geq 7$. First, note that the functions $g(t)$ and $t - 1$ are monotonically increasing real functions. Then, $5 + 2(\log t - \log 5) < t - 1$, which implies that $\log(t^2/25) < t - 6$, then $t^2/25 < \exp(t - 6)$. Therefore, for $t \geq 7$ we get that $g(t) < t - 1$, and hence $\varphi(t) < t - 1$.

Analogously, we can show that $\varphi(t) < t/2$ for $t \geq 15$. That is, we now want to check that $g(t) < t/2$ for $t \geq 15$. This implies that $5 + 2(\log t - \log 5) < t/2$, then $\log(t^2/25) < (t - 10)/2$, so we obtain $t^2/25 < \exp((t - 10)/2)$. Note that if $t = 15$, then the inequality $t^2/25 < \exp(t - 10/2)$ holds. Since again both $g(t)$ and $t/2$ are monotonically increasing real functions, $\varphi(t) < g(t) < t/2$ for $t \geq 15$, and thus (iv) is proved. □

The following result will tell us about the relationship between $F^{-1}(z)$ and $\varphi(\operatorname{Re} z)$ for points z which are not necessarily on the real line.

Lemma 3.2.3. (Relationship between $F^{-1}(z)$ and $\varphi(\operatorname{Re} z)$).
Let $F \in \mathcal{H}$ and let $z \in \mathbb{H}$ be such that $\operatorname{Re} z \geq 4$ and $|\operatorname{Im} z| \leq \pi$. Then

$$|\operatorname{Re} F^{-1}(z) - \varphi(\operatorname{Re} z)| \leq 2.$$

Proof. By assumption and Example 2.3.2, we have

$$\operatorname{dist}_{\mathbb{H}}(z, \operatorname{Re} z) \leq \frac{|\operatorname{Im} z|}{|\operatorname{Re} z|} \leq \frac{\pi}{4} < 1.$$

Recall that $\rho_T(w) \geq 1/2$ for all $w \in T$ (see equation (2.2)). Then we have

$$|\operatorname{Re} F^{-1}(z) - \varphi(\operatorname{Re} z)| \leq |F^{-1}(z) - F^{-1}(\operatorname{Re} z)| \leq 2 \operatorname{dist}_T(F^{-1}(z), F^{-1}(\operatorname{Re} z)).$$

Since F is a conformal isomorphism, and thus by Pick's theorem 2.3.5, F is a local isometry between T and $\mathbb{H}$, we get

$$2 \operatorname{dist}_T(F^{-1}(z), F^{-1}(\operatorname{Re} z)) = 2 \operatorname{dist}_{\mathbb{H}}(z, \operatorname{Re} z) \leq 2.$$

Hence, $|\operatorname{Re} F^{-1}(z) - \varphi(\operatorname{Re} z)| \leq 2$ as claimed. □

Next, in the following result, we see the relation between the iterate on the one-dimensional projection and the one-dimensional projection of the iterate for a given point in the tract T.

Lemma 3.2.4. (Relationship between $\varphi^{n-m}(\operatorname{Re} F^n(z))$ and $\operatorname{Re} F^m(z)$).
Let $F \in \mathcal{H}$, let $n \in \mathbb{N}$ and let $z_0 \in \mathbb{H}$ be such that $\operatorname{Re} z_0 \geq 4$. Set $z_m = F^m(z_0)$ for $m \geq 0$. If $0 \leq m \leq n$ and $|\operatorname{Im} z_m| \leq \pi$ for $m = 0, \ldots, n$, then

$$|\varphi^{n-m}(\operatorname{Re} z_n) - \operatorname{Re} z_m| \leq 4. \tag{3.5}$$

Proof. Fix z_0 and $n \in \mathbb{N}$, we prove (3.5) by induction over $n - m$. Observe that the case $n = m$ is trivial. Set $r_m := \varphi^{n-m}(\operatorname{Re} z_n)$, for $m = 0, \ldots, n$. Now suppose that (3.5) holds for $m \in \{1, \ldots, n\}$, and our goal is to prove it for $m - 1$. We have

$$|r_{m-1} - \varphi(\operatorname{Re} z_m)| = |\varphi(r_m) - \varphi(\operatorname{Re} z_m)| \leq 2 \tag{3.6}$$

by the inductive hypothesis and Lemma 3.2.2(i). Further, we have that $|\operatorname{Im} z_m| \leq \pi$ by assumption. Then by applying Lemma 3.2.3, we obtain

$$|\varphi(\operatorname{Re} z_m) - \operatorname{Re} z_{m-1}| \leq 2. \tag{3.7}$$

Therefore, by combining (3.6) and (3.7), we have

$$|r_{m-1} - \operatorname{Re} z_{m-1}| \leq |r_{m-1} - \varphi(\operatorname{Re} z_m)| + |\varphi(\operatorname{Re} z_m) - \operatorname{Re} z_{m-1}| \leq 4.$$

Hence, the claim holds for $m - 1$ and the proof is complete. □

Definition 3.2.5. (Quadruple).
A *quadruple* is an increasing four-tuple of real numbers contained in $[9, \infty)$. We denote such a quadruple as $Q = [A, B, C, D]$. Since $A < B < C < D$, we also define the *size* of a quadruple as follows:

$$|Q| = \min(A - 5, B - A, C - B, D - C).$$

Additionally, when we refer to the interval of the quadruple, it will be written as $[A, D]$.

We now introduce the following definition which will be a key property in order to obtain a pseudo-arc.

Definition 3.2.6. (Interval mapped crookedly).
Let $F \in \mathcal{H}$, let J be an interval and let $Q = [A, B, C, D]$ be a quadruple. We say that J is *mapped crookedly* over Q by φ^k for $k \geq 1$ if $\varphi^k(J) \supset [A, D]$, and furthermore the convex hull of $J \cap \varphi^{-k}(B)$ intersects the convex hull of $J \cap \varphi^{-k}(C)$ (see Figure 3.2).

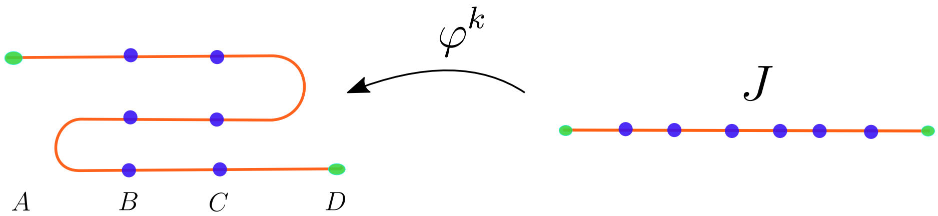


Figure 3.2: Illustration of J mapped crookedly over Q by φ^k.

Remark. The second condition means that either there are two points of $J \cap \varphi^{-k}(B)$ surrounding a point of $J \cap \varphi^{-k}(C)$, or the other way around.

We are now ready to state and prove the main result of this chapter which gives sufficient conditions for a Julia continuum to be a pseudo-arc.

Theorem 3.2.7. (Realisation of pseudo-arcs).
Let $F \in \mathcal{H}$. Suppose there is a constant $K > 0$ such that the following property holds for all integer quadruples $Q = [A, B, C, D]$ with $|Q| \geq K$.

There exists a number $k_0 = k_0(Q) \in \mathbb{N}$, such that for every $k \geq k_0$ and every compact interval $J \subset [6, \infty)$ with $\varphi^k(J) \supseteq [A, D]$, we have that J is mapped crookedly over Q by φ^k.

Then $\widehat{\mathcal{J}}(F)$ is a pseudo-arc.

Proof. Let $\hat{F}$ be the periodic extension of F. Note that $\widehat{\mathfrak{J}}(F)$ is an invariant Julia continuum of $\hat{F}$ by Observation 3.1.6.

Suppose, by way of contradiction, that $\widehat{\mathfrak{J}}(F)$ is not a pseudo-arc. Note that every Julia continuum of $\hat{F}$ is arc-like by [Rem16, Proposition 7.6] and, in particular, $\widehat{\mathfrak{J}}(\hat{F})$ is arc-like as well; thus $\widehat{\mathfrak{J}}(F)$ is also arc-like . Hence, by definition, $\widehat{\mathfrak{J}}(F)$ is not hereditarily indecomposable. This implies that, there is a decomposable subcontinuum $\mathcal{W}$ of $\widehat{\mathfrak{J}}(F)$. Then, $\mathcal{W}$ can be written as the union of $\mathcal{C}^0$ and $\mathcal{C}^1$, each of which is a proper subcontinuum of $\mathcal{W}$. In particular, neither $\mathcal{C}^0 \nsubseteq \mathcal{C}^1$ nor $\mathcal{C}^1 \nsubseteq \mathcal{C}^0$. For $j \in \{0, 1\}$, we denote

$$\mathcal{C}_n^j := F^n(\mathcal{C}^j) \qquad \text{and} \qquad \mathcal{W}_n := \mathcal{C}_n^0 \cup \mathcal{C}_n^1.$$

By assumption, $\mathcal{C}^0, \mathcal{C}^1 \subset \widehat{\mathfrak{J}}(F)$. Then we have $\mathcal{C}_n^j \subset T$ for $j \in \{0, 1\}$. Let z^0, z^1 be two points such that $z^j \in \mathcal{C}^j \setminus (\mathcal{C}^{1-j} \cup \{\infty\})$ for $j \in \{0, 1\}$. Let us now choose a positive number ε such that $\varepsilon < \min\{\text{dist}(z^0, \mathcal{C}^1), \text{dist}(z^1, \mathcal{C}^0)\}$, and note that $\varepsilon > 0$ because $\mathcal{C}^0, \mathcal{C}^1$ are proper subcontinua. This implies that

$$\overline{D_\varepsilon(z^j)} \cap \mathcal{C}^{1-j} = \emptyset \tag{3.8}$$

for $j \in \{0, 1\}$. Now, we fix n sufficiently large such that $2^n > \frac{2\pi+9+3K}{\varepsilon}$ and define $z_n^j = F^n(z^j)$. First, note that by Lemma 3.1.5, we obtain

$$\overline{D_{2^n\varepsilon}(z_n^j)} \cap \mathcal{C}_n^{1-j} = \emptyset. \tag{3.9}$$

Furthermore, the disc in (3.9) contains a straight line segment of length larger than 4π, and thus it separates the strip S in two parts one to the left and one to the right of z_n^j.

Let $j \in \{0, 1\}$ be such that $\operatorname{Re} z_n^j < \operatorname{Re} z_n^{1-j}$. We define an integer quadruple $Q = [A, B, C, D]$ as follows:

$$A := \lceil \operatorname{Re} z_n^j + 4 + K \rceil; \quad B := A + K; \quad D := \lfloor \operatorname{Re} z_n^{1-j} - 4 \rfloor; \quad C := D - K.$$

First, by the choice of n, we have

$$\operatorname{Re} z_n^j + 9 + 3K \le \operatorname{Re} F^n(z), \tag{3.10}$$

for all $z \in \mathcal{C}^{1-j}$; this means that $\mathcal{C}_n^{1-j}$ is to the right of $\operatorname{Re} z_n^j + 9 + 3K$. In particular, we have that

$$\operatorname{Re} z_n^j + 9 + 3K \le \operatorname{Re} z_n^{1-j}$$

and thus $|Q| \ge K$. Moreover, from (3.10), we also have that

$$B + 4 < \operatorname{Re} z_n^j + 9 + 3K \le \operatorname{Re} w,$$

for every $w \in \mathcal{C}_n^{1-j}$. Similarly, $\mathcal{C}_n^j$ is to the left of $\operatorname{Re} z_n^{1-j} - 9 - 3K$, and in particular, all of points in $\mathcal{C}_n^j$ have real part less than $C - 4$.

Let $k_0 = k_0(Q) \in \mathbb{N}$ be as in the hypothesis, and we define $k = n + k_0$. By Lemma 3.2.4 and the choice of A and D, we get

$$\varphi^{k_0}(\operatorname{Re} z_k^j) \le \operatorname{Re} z_n^j + 4 \le A \quad \text{and} \quad \varphi^{k_0}(\operatorname{Re} z_k^{1-j}) \ge \operatorname{Re} z_n^{1-j} - 4 \ge D.$$

Then, if J is the interval bounded by $\operatorname{Re} z_k^j$ and $\operatorname{Re} z_k^{1-j}$, this implies that $\varphi^{k_0}(J) \supset [A, D]$. By assumption, J is mapped crookedly over Q by φ^{k_0}, that is, there is a point of $\varphi^{-k_0}(B) \cap J$ between two points of $\varphi^{-k_0}(C) \cap J$, or vice versa.

More precisely, let us assume the following; there are $\xi_1, \xi_2, \xi_3 \in J$ such that

$$\operatorname{Re} z_k^j \le \xi_1 < \xi_2 < \xi_3 \le \operatorname{Re} z_k^{1-j},$$

$\varphi^{k_0}(\xi_1) = \varphi^{k_0}(\xi_3) = B$ and $\varphi^{k_0}(\xi_2) = C$ (the opposite case is analogous).

Since $z_k^j, z_k^{1-j} \in \mathcal{W}_k$, and $\mathcal{W}_k$ is connected, there are $\tilde{\xi}_1, \tilde{\xi}_3 \in \mathcal{W}_k$ such that $\operatorname{Re} \tilde{\xi}_1 = \xi_1$ and $\operatorname{Re} \tilde{\xi}_3 = \xi_3$ (see Figure 3.3). Now, let z_1 and z_3 be their preimages under $F^{k_0} \colon \mathcal{W}_n \to \mathcal{W}_k$, respectively. By Lemma 3.2.4, we have

$$\operatorname{Re} z_1, \operatorname{Re} z_3 \le B + 4.$$

Thus, $z_1, z_3 \in \mathcal{C}_n^j$, and hence $\tilde{\xi}_1, \tilde{\xi}_3 \in \mathcal{C}_k^j$. Since $\mathcal{C}_k^j$ is a continuum, there is $\tilde{\xi}_2 \in \mathcal{C}_k^j$

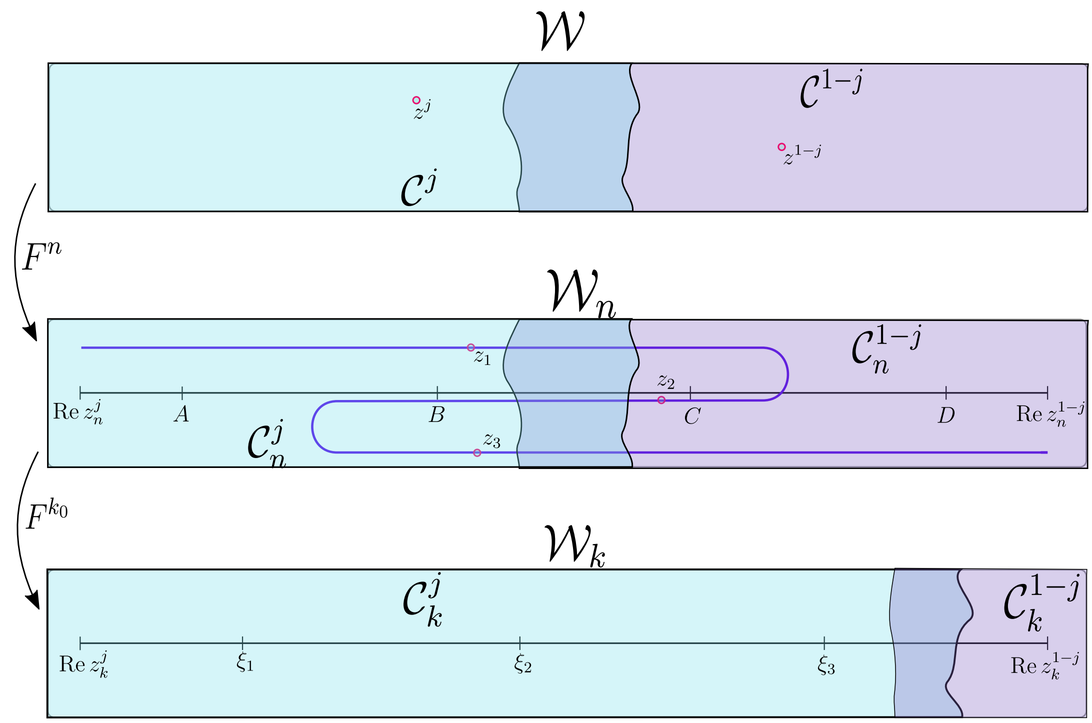


Figure 3.3: Illustration of the proof. Note that in the second level we have that $\varphi^{k_0}(J) \supset [A, D]$ and J is mapped crookedly by assumption.

such that $\operatorname{Re} \tilde{\xi}_2 = \xi_2$. However, the point $z_2 \in \mathcal{C}_n^j$ with $F^{k_0}(z_2) = \tilde{\xi}_2$ satisfies

$$\operatorname{Re} z_2 \geq C - 4,$$

by Lemma 3.2.4. Thus, this gives us the desired contradiction, since $\mathcal{C}_n^j$ is to the left of $C-4$ and so z_2 cannot both be in the continuum $\mathcal{C}_n^j$ and have $\operatorname{Re} z_2 \geq C-4$. Therefore, any subcontinuum $\mathcal{W}$ of $\widehat{\mathcal{J}}(F)$ is indecomposable. Hence $\widehat{\mathcal{J}}(F)$ is hereditarily indecomposable. □

CHAPTER

FOUR

GEOMETRY AND CONVERGENCE ON THE TRACTS

The first section of this chapter discusses the shape and geometric properties of the tracts used in the main construction. We start by introducing a suitable subclass $\mathcal{H}_\nu \subset \mathcal{H}$ of conformal isomorphisms. This subclass will be defined by those tracts with a geometrical property called *Euclidean bounded decorations*, then we establish some conditions on functions in $\mathcal{H}$ to be in $\mathcal{H}_\nu$. Additionally, we introduce another subclass $\mathcal{K}$ of $\mathcal{H}$ that consists of functions $F \in \mathcal{H}$ whose tracts T have a specific form, tending to ∞ through a sequence of "wiggles"(see Definition 4.1.5). Then we show some properties of this subclass. In particular, we prove that the class $\mathcal{K}$ is contained in the class of $\mathcal{H}_\nu$ for a universal constant ν.

In the following section, we focus on the class $\mathcal{K}$ and show that this class has key properties of the one-dimensional projection φ of F which will be preserved under approximation. We start by defining that functions in the class $\mathcal{K}$ are (N, Γ)-*close*. Next, we show that this definition gives us convergence in the sense of the Carathéodory kernel theorem. Finally, we conclude with an approximation result for functions that are (N, Γ)-close.

4.1 Wiggles and a geometric condition

Definition 4.1.1. (Euclidean bounded decorations and class $\mathcal{H}_\nu$).
Let $F\colon T \to \mathbb{H}$ be in the class $\mathcal{H}$. We say that F has *Euclidean bounded decorations* if there is a constant $\nu > 0$ such that

$$\operatorname{diam}\big(F^{-1}\big(\{\zeta \in \mathbb{H}\colon |\zeta| = R\}\big)\big) \leq \nu$$

for all $R \geq 0$.

We denote by $\mathcal{H}_\nu$ the class of all functions $F \in \mathcal{H}$ that satisfy the Euclidean bounded decorations condition for the constant ν.

Remark. For $R > 0$, the set $F^{-1}(\zeta \in \mathbb{H}\colon |\zeta| = R)$ is a hyperbolic geodesic of T that is perpendicular to $F^{-1}((0,\infty))$. We call these geodesics *"vertical" geodesics*.

In order to verify the bounded decoration property, we will make use of the following result which is used to prove the Gehring–Hayman theorem [Pom92, Theorem 4.20].

Lemma 4.1.2. ([Pom92, Lemma 4.21]).
There is a universal constant $K > 0$ with the following property.

Let $f\colon \mathbb{D} \to \mathbb{C}$ be a conformal map. Let $-1 \leq z_1 < z_2 \leq 1$ and assume that $d_\mathbb{D}(z_1, z_2) \geq 1$, where $d_\mathbb{D}$ is the hyperbolic metric of $\mathbb{D}$. For $j \in \{1, 2\}$, let L_j be the hyperbolic line through z_j orthogonal to $\mathbb{R}$ if $|z_j| < 1$, and let $L_j = \{z_j\}$ otherwise. Then, for every curve γ from L_1 to L_2,

$$\operatorname{diam}(f(S)) \leq K \cdot \operatorname{diam}(f(\gamma)),$$

where $S = (z_1, z_2)$.

In the following result we estimate the Euclidean diameter (denoted by diam) of hyperbolic geodesics in $\mathbb{H}$ under F^{-1}.

Proposition 4.1.3. (Euclidean bounded decorations for translated points).
Let $C > 0$. Then there exists a constant $\nu > 0$ with the following property.

Suppose $F \in \mathcal{H}$ and $\theta \in \mathbb{R}$. Suppose that for every $z \in \partial\mathbb{H}$, there is a curve $\alpha_z \subseteq \mathbb{H}$ connecting z to $i\theta + [0,\infty)$ such that

$$\operatorname{diam}\big(F^{-1}(\alpha_z)\big) \leq C.$$

Then, $\operatorname{diam}(\{F^{-1}(z)\colon |z - i\theta| = R\}) \leq \nu$ *for all* $R \geq 0$.

Proof. Let $F \in \mathcal{H}$ and let $R \geq 0$. We consider the curves in $\mathbb{H}$ perpendicular to $i\theta + [0,\infty)$, that is,

$$\beta^+ := \{\zeta \in \mathbb{H}\colon |\zeta - i\theta| = R,\ \operatorname{Im}\zeta > \theta\},$$
$$\beta^- := \{\zeta \in \mathbb{H}\colon |\zeta - i\theta| = R,\ \operatorname{Im}\zeta < \theta\}.$$

Let $z \in \partial\mathbb{H}$ be such that $|z - i\theta| = R$. If $\operatorname{Im} z \geq \theta$, then there is a curve α_{z^+} connecting z to $i\theta + [0,\infty)$ satisfying that

$$\operatorname{diam}\big(F^{-1}(\alpha_{z^+})\big) \leq C$$

by assumption. Likewise, if $\operatorname{Im} z < \theta$, then there is a curve α_{z^-} having the same property (see Figure 4.1).

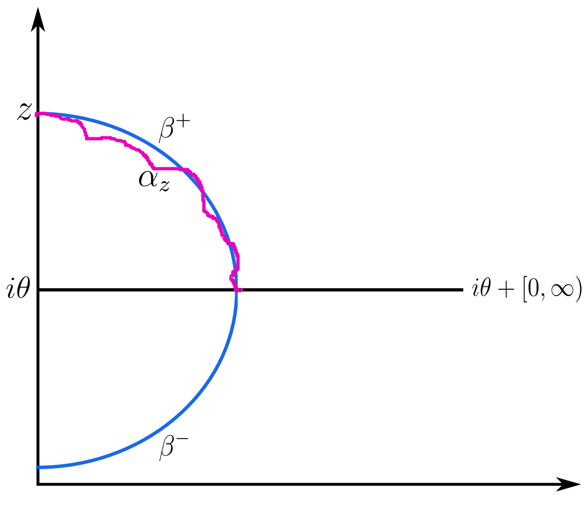


Figure 4.1: Illustration of the setting, in blue are the semi-circles β^+ and β^- and $\alpha_z^+ = \alpha_z$ is in pink.

By Lemma 4.1.2, there is a universal constant K such that

$$\operatorname{diam}(F^{-1}(\beta^+)) \leq K \cdot \operatorname{diam}(F^{-1}(\alpha))$$

for any curve α connecting $i(R+\theta)$ to $i\theta + [0,\infty)$. Then, take α to be the curve α_{z^+}. Thus, we have

$$\operatorname{diam}(F^{-1}(\beta^+)) \leq K \cdot C.$$

The case for β^- is analogous. Hence, we set $\nu := 2 \cdot K \cdot C$ and the claim follows. □

In the following result we set conditions on the class $\mathcal{H}$ to belong to the class $\mathcal{H}_{\nu_0}$ for some $\nu_0 > 0$.

Proposition 4.1.4. (Condition on functions in $\mathcal{H}$ to be in $\mathcal{H}_{\nu_0}$).
For every $C_1 > 0$, there exists a constant $\nu_0 > 0$ satisfying the following.

Let $F \in \mathcal{H}$, and suppose that there is $\theta \in \mathbb{R}$ with the following properties.

(i) There is a curve β connecting 5 to $i\theta$ in $\mathbb{H}$ such that

$$\operatorname{diam}(F^{-1}(\beta)) \leq C_1.$$

(ii) For every $z \in \partial\mathbb{H}$, there is a curve γ_z connecting z to $i\theta + [0,\infty)$ such that

$$\operatorname{diam}(F^{-1}(\gamma_z)) \leq C_1.$$

Then $F \in \mathcal{H}_{\nu_0}$.

Proof. Let $F \in \mathcal{H}$, and take $\theta \in \mathbb{R}$ as in the statement of the proposition. The idea of the proof is to apply first Proposition 4.1.3 to the hypothesis of the statement to see that every point in the boundary of $\mathbb{H}$ can be connected to the real line by a curve whose image under F^{-1} has Euclidean diameter at most C_1. Then, we obtain the conclusion from the same previous proposition for hyperbolic geodesics perpendicular to $[0,\infty)$, since we have connected any point in $\partial\mathbb{H}$ with the real line; that is, the function F has Euclidean bounded decorations and we are done. So this will be achieved by the following claim.

Claim 1. There exists $C_2 > 0$, depending only on C_1, such that every $w \in \partial\mathbb{H}$ can be connected to $[0,\infty)$ by a curve α_w such that

$$\operatorname{diam}(F^{-1}(\alpha_w)) \leq C_2.$$

Proof of Claim. Let us suppose that $\theta > 0$; the case $\theta < 0$ is analogous.

Let $w \in \partial\mathbb{H}$; note that if $w \in i(-\infty, 0]$, then by property (ii) of the assumption, the curve γ_w must intersect $[0,\infty)$, since γ_w connects w to $i\theta + [0,\infty)$. We can then take α_w to be the sub-curve of γ_w which ends at the real line and we are done.

We now assume that $w \in i[0,\infty)$. First, let us note that by Proposition 4.1.3, and using the second assumption (ii) of the hypothesis, there is a constant $\nu > 0$, depending only on C_1, such that any geodesic $\tilde{\alpha}$ in $\mathbb{H}$ perpendicular to $i\theta + [0,\infty)$ satisfies that

$$\operatorname{diam}(F^{-1}(\tilde{\alpha})) \leq \nu.$$

Let $\tilde{\alpha}$ be the geodesic in $\mathbb{H}$ perpendicular to $i\theta + (0,\infty)$ containing w. Now, we consider two sub-cases. If $\operatorname{Im} w \geq 2\theta$, this implies that

$$\tilde{\alpha} \cap [0,\infty) \neq \emptyset$$

and we let α_w be a sub-curve of $\tilde{\alpha}$ connecting w to $[0,\infty)$.

Otherwise, the geodesic $\tilde{\alpha}$ separates $i\theta$ from $[0,\infty)$, and hence it must intersect β. So, we can take a point $p \in \beta \cap \tilde{\alpha}$. Then consider the sub-curve $\tilde{\alpha}_1$ of $\tilde{\alpha}$ connecting w to p, together with the sub-curve $\tilde{\beta}$ of β connecting p to 5, and set

$$\alpha_w := \tilde{\alpha}_1 \cup \tilde{\beta},$$

(see Figure 4.2). So by construction, we have

$$\operatorname{diam}(F^{-1}(\alpha_w)) \leq C_1 + \nu.$$

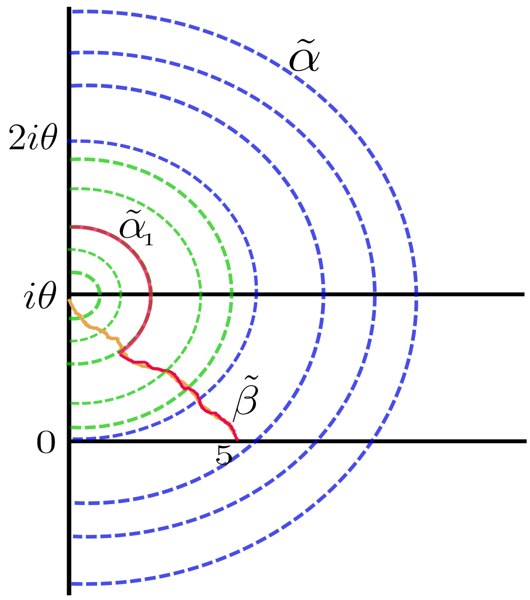


Figure 4.2: Illustration of the geodesics $\tilde{\alpha}$, the geodesics in blue represent those with $\operatorname{Im} z \geq 2\theta$, the ones in green otherwise. The red curve is the sub-curve α_w when $0 < \operatorname{Im} w < 2\theta$.

In either case, we have shown that $\operatorname{diam}(F^{-1}(\alpha_w)) \leq C_1 + \nu =: C_2$. Hence, the proof of the Claim 1 is complete. △

This completes the proof of Proposition 4.1.4. □

In view of Theorem 3.2.7, our goal is to construct a function $F \in \mathcal{H}$ which satisfies its hypothesis. In order to achieve this, we begin with the notion of *wiggles*. In a rough sense, we start with the strip S and we inductively add a sequence of wiggles to the tract over increasing real parts. So, let us start with the following definition.

Definition 4.1.5. (Wiggles and class $\mathcal{K}$).
Let $\{r_j\}_{j=0}^N$ and $\{R_j\}_{j=0}^N$ be sequences of real numbers, with $N \leq \infty$, such that

$$r_0 > 6, \quad r_j > R_{j-1} + 1 \quad \text{and} \quad R_j > r_j + 2,$$

for all $j < N$, and by convention $R_{-1} = 5$. Define a tract T as follows (see Figure 4.3):

$$T := \{x + iy : x > 4, \ |y| < \pi\} \setminus \bigcup_{0 \le j < N} \left(\left\{x + iy : x = r_j, \ -\pi < y \le \frac{\pi}{3}\right\} \cup \left\{x + iy : r_j \le x \le R_j - 1, \ y = \frac{\pi}{3}\right\} \cup \left\{x + iy : x = R_j, \ \frac{-\pi}{3} \le y < \pi\right\} \cup \left\{x + iy : r_j + 1 \le x \le R_j, \ y = \frac{-\pi}{3}\right\} \right),$$

and we say that T has N wiggles. It will be also written as F has N wiggles.

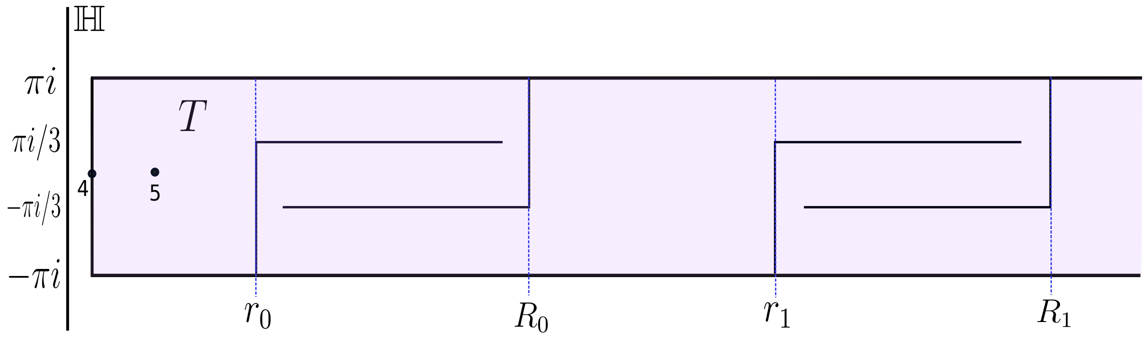


Figure 4.3: Illustration of the tract T of a function $F \in \mathcal{K}$.

In addition, we denote by $\mathcal{K}$ the set of functions $F \in \mathcal{H}$ such that the tract of F is of the above form. For $F \in \mathcal{K}$, we write $r_j(F)$, $R_j(F)$ and $N(F)$ for the numbers r_j, R_j and N appearing in the definition of T.

Notation 4.1.6. Given different elements $F, F_j, \tilde{F} \in \mathcal{K}$, their one-dimensional projections will be always denoted by φ, φ_j, $\tilde{\varphi}$ respectively.

Next, we introduce the notion of wiggles as a set.

Definition 4.1.7. Let $F \in \mathcal{K}$ have N wiggles. We say that a *wiggle* (set) is denoted as follows:

$$W_j(F) := \left\{\zeta \in T : r_j < \operatorname{Re}\zeta < R_j \text{ and } \operatorname{Im}\zeta < \frac{\pi}{3}\right\} \tag{4.1}$$

for all $0 \le j < N$.

In the following result we apply Proposition 4.1.4 to show that the class $\mathcal{K}$ satisfies the Euclidean bounded decorations condition.

Proposition 4.1.8. (Euclidean bounded decorations in $\mathcal{K}$).
There exists a universal constant $\nu_0 > 0$ *such that* $\mathcal{K} \subset \mathcal{H}_{\nu_0}$.

Proof. Let $F \in \mathcal{K}$ and let T be the tract of F. First, let us observe that the set $\partial T \setminus \{4\}$ has two connectected components which are written as follows:

$$\begin{aligned}\partial T^+ :=& \{4 + iy \mid 0 < y \leq \pi\} \cup \{x + i\pi \mid x \geq 4\} \\ & \bigcup_{0 \leq j < N} (\{R_j + iy \mid -\pi/3 \leq y \leq \pi\} \cup \{x - i\pi/3 \mid r_j + 1 \leq x \leq R_j\}),\end{aligned}$$

$$\begin{aligned}\partial T^- :=& \{4 + iy \mid -\pi \leq y < 0\} \cup \{x - i\pi \mid x \geq 4\} \\ & \bigcup_{0 \leq j < N} (\{r_j + iy \mid -\pi \leq y \leq \pi/3\} \cup \{x + i\pi/3 \mid r_j \leq x \leq R_j - 1\}).\end{aligned}$$

These components will be referred as the *upper* and *lower* boundaries of T respectively. The sets ∂T^+ and ∂T^- are semi-open, connected curves. Moreover, those curves do not intersect. This implies that there is a unique point $\theta_0 \in \mathbb{R}$ such that $F^{-1}(i\theta_0) = 4$.

Recall that the idea of the proof is to apply Proposition 4.1.4, that is, we want to show that there are certain curves β and γ_z in $\mathbb{H}$ such that $\operatorname{diam} F^{-1}(\beta)$ and $\operatorname{diam} F^{-1}(\gamma_z)$ for $z \in \partial\mathbb{H}$ are small. However, in this case we do the opposite, we construct curves with small diameter in T, then we define β and γ_z as the images under F of these curves and verify the hypotheses of the aforementioned proposition.

So, note that the curve $\beta := F([4, 5])$ connects $i\theta_0$ to $F(5) = 5$ in $\mathbb{H}$. On the other hand, by the shape of the tract T, we have that every $w \in T$ can be connected to the boundaries ∂T^+ and ∂T^- by either a straight line segment of length at most 2π, or a horizontal segment of length at most 1 (at the bends of the wiggles). Observe that at least one of these segments intersect the geodesic $\alpha := F^{-1}((i\theta_0 + (0, \infty)))$. In particular, for $z' \in \partial T^+$ there is a segment $\gamma'_{z'}$ connecting z' with ∂T^- of length at most 2π. Thus, $\gamma'_{z'}$ intersects α. Now, let $z = F(z') \in \partial\mathbb{H}$, and $\gamma_z := F(\gamma'_{z'})$. So, we set $C := 2\pi + 1$ and therefore the hypotheses hold. Hence $F \in \mathcal{H}_{\nu_0}$ and the proof is complete. □

We shall now obtain uniform bounds for the growth of functions in the class $\mathcal{K}$ (depending on the sequences $(r_j(F))$ and $(R_j(F))$).

Proposition 4.1.9. (Growth of functions in class $\mathcal{K}$).
There exits a constant $\Lambda > 0$ such that the following holds:
Suppose $F \in \mathcal{K}$, and $z \in T$ with $\operatorname{Re} F(z) \geq 4$. If z belongs neither to the middle nor the bottom of any wiggle, i.e., if $z \notin W_j(F)$ for all $0 \leq j < N$ (see Definition 4.1.7), then

$$\frac{\operatorname{Re} z}{\Lambda} \leq \log |F(z)| \leq \Lambda \operatorname{Re} z.$$

If $z \in W_j(F)$, then

$$\frac{R_j}{\Lambda} \leq \log |F(z)| \leq \Lambda R_j.$$

Proof. First, note that there is an arc α in T connecting 5 and ∞, and it remains at distance at least $1/2$ from ∂T, which is defined as follows, (see Figure 4.4).

$$\begin{aligned}\alpha := \left[5, r_0 - \frac{1}{2}\right] \cup \bigcup_{0 \leq j \leq N(F)} \Bigg(&\left(r_j - \frac{1}{2} + i(0, 2\pi/3]\right) \cup \left(\left(r_j - \frac{1}{2}, R_j - \frac{1}{2}\right] + 2\pi i/3\right) \\ &\cup \left(R_j - \frac{1}{2} + i[0, 2\pi/3)\right) \cup \left[r_j + \frac{1}{2}, R_j - \frac{1}{2}\right) \\ &\cup \left(r_j + \frac{1}{2} + i[-2\pi/3, 0)\right) \cup \left(\left(r_j + \frac{1}{2}, R_j + \frac{1}{2}\right] - 2\pi i/3\right) \\ &\cup \left(R_j + \frac{1}{2} + i(-2\pi/3, 0]\right) \cup \left[R_j + \frac{1}{2}, r_{j+1} - \frac{1}{2}\right] \Bigg).\end{aligned}$$

Here we use the convention that $r_{N(F)+1} = \infty$, if $N(F) < \infty$.

Let $w \in \alpha$. Denote by α_w the part of the arc α connecting 5 with w. Now, we want to estimate the Euclidean length $\ell(\alpha_w)$.
Then, let us first assume that $w \notin W_j(F)$ for every j. Let $j < N(F)$ be maximal such that $\operatorname{Re} w > R_j$. Further, observe that $R_j \geq 3j + 8$ by construction (if no

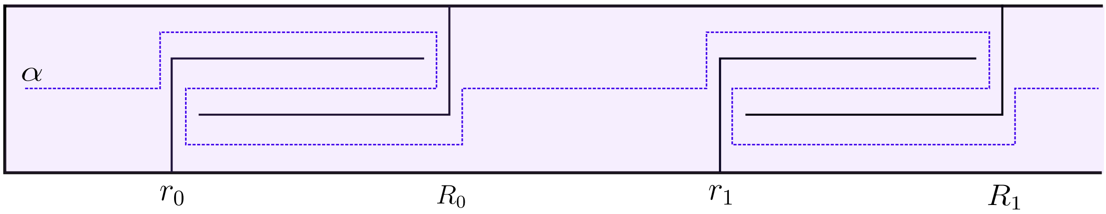


Figure 4.4: The arc α stays at distance at least $\frac{1}{2}$ from ∂T.

such j exists, take $j = -1$). We then have

$$\ell(\alpha_w) \leq \operatorname{Re} w - 5 + \pi + \sum_{0 \leq k \leq j} 2(\pi + R_k - r_k - 1)$$
$$\leq 3 \operatorname{Re} w + (2j+1)\pi < 3 \operatorname{Re} w + R_j \pi < (3+\pi) \operatorname{Re} w.$$

In the other case where $w \in W_j(F)$, we have $\alpha_w \subseteq \alpha_{\tilde{w}}$, where $\tilde{w} := R_j - 2\pi i/3$, and thus

$$\ell(\alpha_w) \leq \ell(\alpha_{\tilde{w}}) \leq (3+\pi) \cdot \operatorname{Re} \tilde{w} = (3+\pi) \cdot R_j.$$

On the other hand, note that any curve connecting 5 with $w \in T$ must have Euclidean length at least $\operatorname{Re} w - 5$. Further, if $w \in W_j(F)$, then such a curve must connect 5 with some point at real part $R_j - 1$, and thus the length should be at least $R_j - 6$.
Now, let $z \in T$ be as in the statement of the proposition, and let

$$\gamma := F^{-1}\big(\{u \in \mathbb{H} \colon |u| = |F(z)|\}\big)$$

be the vertical geodesic containing z, and set $\tilde{z} := F^{-1}(|F(z)|)$, then $\tilde{z} \in \gamma$. By Proposition 4.1.8, the Euclidean diameter of γ is uniformly bounded by ν_0. Define

$$R := \begin{cases} \operatorname{Re} z & \text{if } z \notin W_j(F) \text{ for all } j, \\ R_j & \text{if } z \in W_j(F). \end{cases}$$

For $u \in \alpha$, it follows from the standard estimate (2.3) that $\rho_T(u) \leq \frac{2}{\operatorname{dist}(u, \partial T)} \leq 4$. Further, by (2.2) we have $\rho_T(u) \geq 1/2$.

If $|F(z)| \geq 5$, then by the above, γ separates 5 from ∞, so it intersects α. Then

$$\frac{1}{2}(R - \nu_0 - 5) \leq \operatorname{dist}_T(5, \gamma) \leq 4(3 + \pi) \cdot (R + \nu_0) \leq 4(3 + \pi + \nu_0) \cdot R. \quad (4.2)$$

Since F is a local isometry between T and $\mathbb{H}$, we have

$$\operatorname{dist}_T(5, \gamma) = \operatorname{dist}_T(5, \tilde{z}) = \operatorname{dist}_{\mathbb{H}}(5, |F(z)|) = \log \frac{|F(z)|}{5}.$$

Then, the claim follows when $R > 2(\nu_0 + 5)$. If $4 \leq |F(z)| \leq 5$, then $\operatorname{dist}_T(5, \gamma) \leq \log(5/4)$, so the inequality also holds in this case.

In order to obtain a precise Λ, we have to ensure that (4.2) holds for smaller R. Recall that $R \geq 4$ by assumption, then $4 \leq R \leq 2(\nu_0 + 5)$. Further, $\operatorname{Re} F(z) \geq 4$, so, in particular, we have $|F(z)| \geq 4$. Then, by the above, we have

$$|F(z)| \geq 4 \cdot \frac{R}{2(\nu_0 + 5)}.$$

On the other inequality, set $\Lambda' := 4(3 + \pi + \nu_0)$, this implies that

$$|F(z)| \leq \Lambda' \cdot 2(\nu_0 + 5) \leq \Lambda' \cdot 2(\nu_0 + 5) \cdot \frac{R}{4}.$$

Therefore, if we take $\Lambda := \frac{\Lambda'}{2} \cdot (\nu_0 + 5)$, the claim follows and the proof is complete. □

Remark. The key fact in Proposition 4.1.9 is that Λ is independent of F (and z), so that we have uniform bound on the growth of a function $F \in \mathcal{K}$.

4.2 Convergence on the tracts

As we mentioned before, the position of the wiggles (r_n, R_n) in our example will be determined inductively. An important property is that if $F \in \mathcal{K}$ has N wiggles, and the next wiggle (r_N, R_N) is chosen sufficiently far to the right, then the inverse of the next function $\tilde{F}$ will be close to F^{-1}, at least up to the $(N+1)$-th wiggle. More precisely, let us first introduce the following notion. Recall that $N(F)$ is the number of wiggles of the tract T of F.

Definition 4.2.1. (Functions that are (N, Γ)-close).
Let $F \in \mathcal{K}$ have $N < \infty$ wiggles and let Γ be a positive number. We say that $\tilde{F} \colon \tilde{T} \to \mathbb{H}$ is (N, Γ)-*close* to F if the following hold:

(i) $\tilde{F}$ has more wiggles than F, that is, $N(\tilde{F}) > N$,

(ii) $r_n(\tilde{F}) = r_n(F)$ and $R_n(\tilde{F}) = R_n(F)$ for $0 \le n < N$,

(iii) $r_N(\tilde{F}) \ge \Gamma$.

Remark 4.2.2. From the above definition, observe that (ii) can be rewritten as follows:

$$T \cap \{\operatorname{Re} z < \Gamma\} = \tilde{T} \cap \{\operatorname{Re} z < \Gamma\}.$$

As already mentioned, the constant Λ from Proposition 4.1.9 does not depend on the function $F \in \mathcal{K}$. Thus, as a consequence of the aforementioned result, we have the following property for functions in $\mathcal{K}$ that are sufficiently close.

Proposition 4.2.3. (Points to the left of Γ).
Let $F \in \mathcal{K}$ have N wiggles. Let $K \ge 4$ and let $n_0 \ge 0$. There exists $\Gamma > R_{N-1}(F)$ with the following property.

Suppose $\tilde{F} \in \mathcal{K}$ is (N, Γ)-close to F, $x \in [4, \infty)$ and, $\tilde{\varphi}^n(x) \le K$ for some $n \le n_0$, where $\tilde{\varphi}$ is the one-dimensional projection of $\tilde{F}$. Then $x \le \Gamma$.

Proof. Without loss of generality, we assume that $K \ge R_{N-1}(F)$. Let Λ be the constant from Proposition 4.1.9 and define

$$\Psi(t) := \exp(\Lambda \cdot t) \quad \text{for } t > 0.$$

Let us now set $\Gamma := \Psi^{n_0}(K)$, where Ψ^{n_0} denotes the n_0-th iterate of Ψ. Let $\tilde{F} \in \mathcal{K}$, let n and x be as in the statement of the proposition, and we additionally set $t_k := \tilde{\varphi}^{n-k}(x)$.

Claim 1. $t_k \le \Psi^k(K)$, for $k = 0, \dots, n$.

Proof of Claim. The proof will be given by induction over k. Observe that $t_0 = \tilde{\varphi}^n(x) \le K$ by assumption, thus the claim is proved for $k = 0$. Now, assume the claim holds for t_k, $k < n$. Set $z := \tilde{F}^{-1}(t_{k+1})$, then $\operatorname{Re} z = \operatorname{Re} \tilde{F}^{-1}(t_{k+1}) = \tilde{\varphi}(t_{k+1}) = t_k$. In particular, by the induction hypothesis, we get

$$\operatorname{Re} z \le \Psi^k(K) \le \Gamma.$$

This implies that z is either not in a wiggle, or it is in one of the first N wiggles of $\tilde{T}$. So, by Proposition 4.1.9, we have

$$\log|\tilde{F}(z)| \le \Lambda \cdot \max(\operatorname{Re} z, R_{N-1}(F)) \le \Lambda \cdot \Psi^k(K),$$

and this implies that, $t_{k+1} = \tilde{F}(z) \le \exp(\Lambda \cdot \Psi^k(K)) = \Psi^{k+1}(K)$. Hence, the claim holds for $k + 1$. △

Since $x = t_n$ for $k = n$, this implies that

$$x \le \Psi^n(K) \le \Psi^{n_0}(K) \le \Gamma$$

by Claim 1. Hence, the claim holds and the proof is complete. □

Before we show that Definition 4.2.1 is in the sense of the *Carathéodory kernel*, we first state the following definition and the Carathéodory kernel theorem. Then we introduce the notion of harmonic measure.

Definition 4.2.4. (Carathéodory kernel convergence, [Pom92, Section 1.4]). Let $w_0 \in \mathbb{C}$. Suppose U_n is a sequence of simply connected domains satisfying $w_0 \in U_n$ for all $n \in \mathbb{N}$. We say that

$$U_n \to U \quad \text{as } n \to \infty \qquad \text{with respect to } w_0$$

in the sense of *kernel convergence*, and we write $(U_n, w_0) \to (U, w_0)$, if:

1. Either $U = \{w_0\}$, or $U \neq \mathbb{C}$ is a domain containing w_0 with the following property: For all compact sets $K \subset U$, there exists $N \in \mathbb{N}$ such that for all $n \ge N$, $K \subset U_n$.

2. For all $w \in \partial U$, there exists a sequence $(w_n)_{n\in\mathbb{N}}$ in ∂U_n such that $w_n \to w$ as $n \to \infty$.

Theorem 4.2.5. (Carathéodory kernel theorem, [Pom92, Theorem 1.8]).
Let U_n be a sequence of simply connected domains and let $f_n\colon \mathbb{D} \to U_n$ be a sequence of conformal maps with $f_n(0) = w_0$ and $f_n'(0) > 0$. If $U = \{w_0\}$ let $f(z) \equiv w_0$, otherwise let $f\colon \mathbb{D} \to U$ be the conformal map with $f(0) = w_0$ and $f'(0) > 0$. Then the following are equivalent:

1. *f_n converges locally uniformly to f in $\mathbb{D}$ as $n \to \infty$.*

2. *$(U_n, w_0) \to (U, w_0)$ as $n \to \infty$.*

As mentioned above, we now define the harmonic measure and some of its properties that we might need for following result. For further details in this topic we refer to [GM05].

Definition 4.2.6. Let $\Omega \subseteq \mathbb{C}$ be a simply connected domain whose boundary consists of finitely many Jordan curves. Let $E \subseteq \partial\Omega$ be such that E', the boundary of E with respect to $\partial\Omega$, is null with respect to Ω. We set

$$U(z_0) := \begin{cases} 1 & \text{if } z_0 \in E, \\ 0 & \text{if } z_0 \in \partial\Omega \setminus E. \end{cases}$$

Then U is continuous at each point of $\partial\Omega \setminus E'$, and thus there is a unique function $\omega(z_0, E, \Omega)$ that is harmonic and bounded in Ω and such that

$$\lim_{z\to z_0} \omega(z, E, \Omega) = U(z_0), \quad \text{for } z_0 \in \partial\Omega \setminus E'.$$

The function $\omega(z, E, \Omega)$ is called the *harmonic measure* of E with respect to Ω at z. The harmonic measure can also be seen as the probability of a random walk starting at z hitting the boundary of Ω in the set E.

Remark 4.2.7. By the maximum principle, the function ω takes values between 0 and 1. Therefore, $0 < \omega(z, E, \Omega) < 1$.

We now show that the (N, Γ)-closeness notion is in the sense of Carathéodory convergence.

Lemma 4.2.8. (Carathéodory convergence).
Let $F \in \mathcal{K}$. Suppose that $(F_j)_{j=1}^{\infty}$ is a sequence of functions in $\mathcal{K}$ such that $F_j\colon T_j \to \mathbb{H}$ is (N, Γ_j)-close to $F\colon T \to \mathbb{H}$ with $\Gamma_j \to \infty$ as $j \to \infty$. Then $(T_j, 5) \to (T, 5)$ as $j \to \infty$ in the sense of the Carathéodory kernel.

Proof. Let K be a compact and connected set such that $5 \in K$. We have to show that the following are equivalent:

(a) $K \subset T$,

(b) $K \subset T_j$ for all but at most finitely many j.

By condition (ii) in the Definition 4.2.1, we have that

$$T_j \subset T \qquad \text{and} \qquad T \cap \{z \mid \operatorname{Re} z < \Gamma_j\} \subset T_j,$$

for all $j \in \mathbb{N}$. So, if j is sufficiently large such that $\Gamma_j > \max_{z \in K} \operatorname{Re} z$, then $K \subset T_j$ if and only if $K \subset T$. Additionally, note that any point in the boundary of T will eventually be in the boundary of T_j. Hence, the convergence holds. □

The following result gives us locally uniform convergence on the inverse functions.

Proposition 4.2.9. (Convergence of the inverse functions).
Let $\nu > 0$ and let $F \in \mathcal{H}_\nu$. Suppose that $(F_j)_{j=1}^{\infty}$ is a sequence of functions in $\mathcal{H}_\nu$ such that $(T_j, 5) \to (T, 5)$ as $j \to \infty$ in the sense of the Carathéodory kernel convergence. Then F_j^{-1} converges locally uniformly to F^{-1} as $j \to \infty$.

Proof. Let $\psi_j : \mathbb{D} \to T_j$ be a conformal isomorphism normalised such that $\psi_j(0) = 5$ and $\psi_j'(0) > 0$ for every $j \in \mathbb{N}$. Let us also define $\psi\colon \mathbb{D} \to T$ as a conformal isomorphism having the same properties as ψ_j, that is, $\psi(0) = 5$ and $\psi'(0) > 0$.

By assumption, $(T_j, 5) \to (T, 5)$ in the sense of kernel convergence, then $\psi_j \to \psi$ locally uniformly by the Carathéodory kernel theorem 4.2.5. Recall that ψ extends continuously to $\partial\mathbb{D}$ by the Carathéodory–Torhorst theorem [Pom92, Theorem 2.1], taking the value ∞ at exactly one point $\psi^{-1}(\infty)$ of $\partial\mathbb{D}$.

Now, let us consider the conformal isomorphisms $M_j\colon \mathbb{H} \to \mathbb{D}$ and $M\colon \mathbb{H} \to \mathbb{D}$ defined by $M_j := \psi_j^{-1} \circ F_j^{-1}$ and $M := \psi^{-1} \circ F^{-1}$ respectively. This implies that $F_j^{-1} = \psi_j \circ M_j$ and $F^{-1} = \psi \circ M$ (see Figure 4.5 for an illustration of the setting).

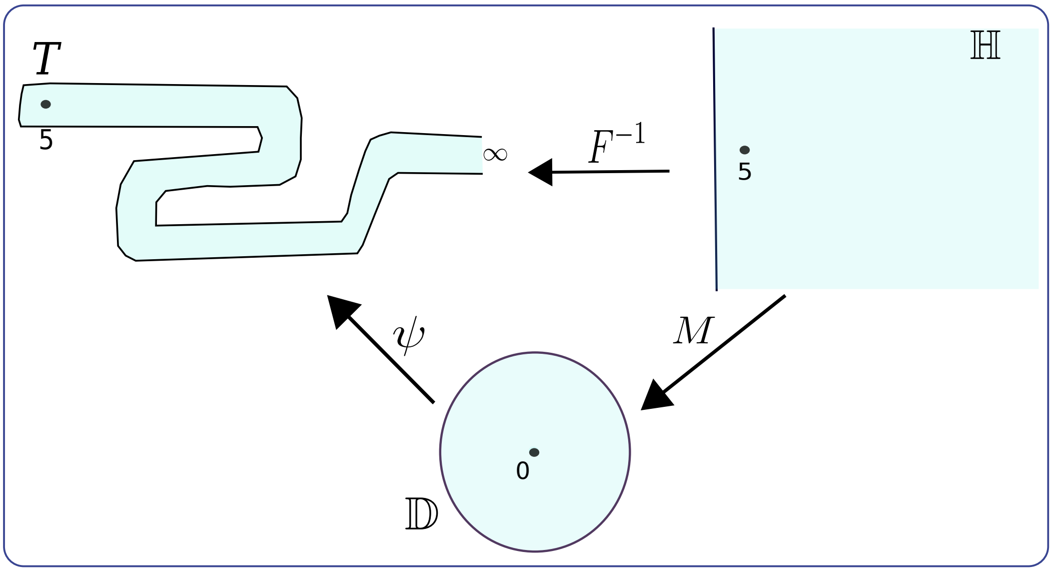


Figure 4.5: Illustration definition of M in terms of F^{-1} and ψ^{-1}.

Note that M_j is a sequence of Möbius transformations with $M_j(5) = 0$ and $M_j(\infty) = \psi_j^{-1}(\infty)$, since $F_j(\infty) = \infty$ (see Definition 3.1.1). Analogously, the map M satisfies that $M(5) = 0$ and $M(\infty) = \psi^{-1}(\infty)$. This implies that it will be sufficient to show that $M_j(\infty) \to M(\infty)$, that is $(\psi_j^{-1} \circ F_j^{-1})(\infty) \to (\psi^{-1} \circ F^{-1})(\infty)$ as $j \to \infty$. Therefore, it remains to prove the following claim.

Claim 1. $\psi_j^{-1}(\infty) \to \psi^{-1}(\infty)$.

Proof of Claim. Recall that $\tilde{S} = \{x + iy \mid -\pi < y < \pi\}$. Let $\varepsilon > 0$ and let $R > 0$ be chosen sufficiently large that the set $\{\operatorname{Re} u \geq R\}$ in $\partial\tilde{S}$ has harmonic measure at most $(\varepsilon/2)/2\pi$ seen from 5. Since $T_j \subset \tilde{S}$, then the set $\{\operatorname{Re} v \geq R\}$ in ∂T_j has also harmonic measure at most $(\varepsilon/2)/2\pi$ seen from 5, independently of j.

Now, we choose a point $w \in T$ such that $\operatorname{Re} w > R + \nu$ and

$$|\psi^{-1}(w) - \psi^{-1}(\infty)| < \varepsilon/4. \tag{4.3}$$

For sufficiently large j, we also have

$$|\psi_j^{-1}(w) - \psi^{-1}(w)| < \varepsilon/4. \tag{4.4}$$

Let α_j be the geodesic of $\mathbb{D}$ passing through $\psi_j^{-1}(w)$ which is perpendicular to the radius connecting 0 and $\psi_j^{-1}(\infty)$. Then $\psi_j(\alpha_j)$ is the vertical geodesic of T_j through w, and hence has diameter at most ν (see Figure 4.6).

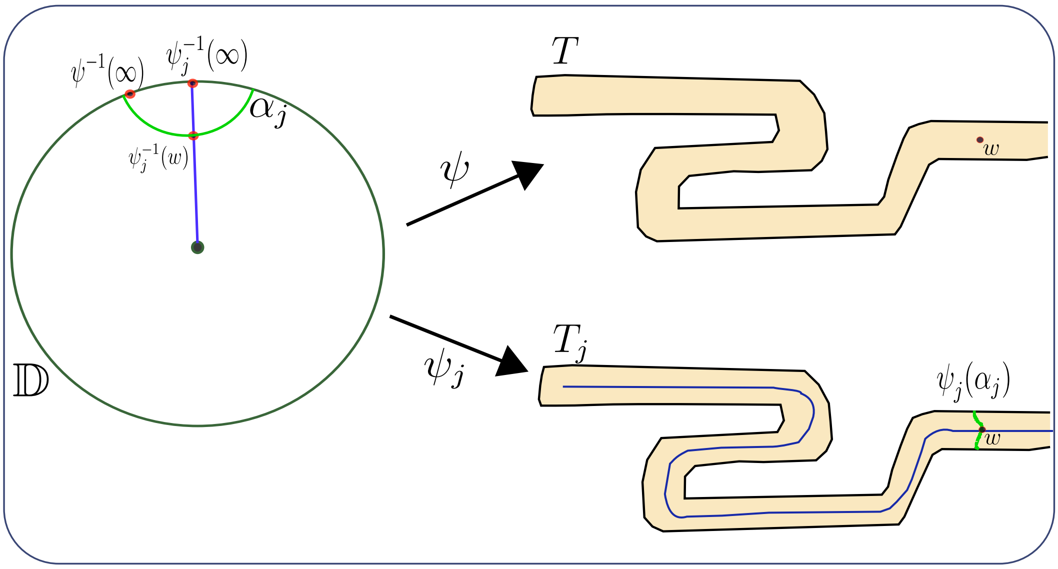


Figure 4.6: Illustration of the choice of the geodesics α_j, in the disc $\mathbb{D}$, and $\psi_j(\alpha_j)$ in T_j, passing through $\psi_j^{-1}(w)$ and w respectively.

By the choice of w, the real part of points in $\psi_j(\alpha_j)$ is greater than R and the arc of $\partial\mathbb{D}$ that is separated by α_j from 0 has harmonic measure at most $(\varepsilon/2)/2\pi$. This implies that this arc has length at most $\varepsilon/2$, and so

$$|\psi_j^{-1}(w) - \psi_j^{-1}(\infty)| < \varepsilon/2. \tag{4.5}$$

Therefore, by (4.3), (4.4) and (4.5), we have shown that

$$|\psi_j^{-1}(\infty) - \psi^{-1}(\infty)| \leq |\psi_j^{-1}(w) - \psi_j^{-1}(\infty)| + |\psi_j^{-1}(w) - \psi^{-1}(w)| + |\psi^{-1}(w) - \psi^{-1}(\infty)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon$$

Since ε was arbitrary, the proof is complete. △

Thus, Proposition 4.2.9 is proved. □

Recall that φ and $\tilde{\varphi}$ are the one-dimensional projections of F and $\tilde{F}$, respectively (see Notation 4.1.6). So we are now ready to state and prove the main approximation result of this section. Note that Proposition 4.2.9 shows that $\tilde{\varphi}$ converges to φ whenever $\tilde{F}$ is (N, Γ)-close to F. In the following result we show that $\tilde{\varphi}^n$ converges to φ^n.

Proposition 4.2.10. (Approximation by functions that are (N, Γ)-close). *Suppose $F \in \mathcal{K}$ has $N < \infty$ wiggles. Let $\varepsilon > 0$ and $\tau \geq 5$. Then there exists a number $\Gamma > R_{N-1}(F)$ with the following property.*

Assume $\tilde{F} \in \mathcal{K}$ is such that $\tilde{F}$ is (N, Γ)-close to F. If $4 \leq t \leq 2R_N(\tilde{F})$ and $\min(\varphi^n(t), \tilde{\varphi}^n(t)) \leq \tau$ for some $n \geq 0$, then

$$|\varphi^n(t) - \tilde{\varphi}^n(t)| \leq \varepsilon. \tag{4.6}$$

Proof. The proof will be given in two steps. First, we prove (4.6) when t belongs to a fixed interval of the real line, then the remaining case.

By Lemma 4.2.8 and Proposition 4.2.9, $\tilde{F}^{-1} \to F^{-1}$ converges locally uniformly on $\mathbb{H}$. So, in particular, $\tilde{\varphi}_j$ converges uniformly to φ on compact subsets of $(0, \infty)$.

Claim 1. Let $\tilde{\tau} > 4$ and let $\delta > 0$ be arbitrary. Then there is $\Gamma(\tilde{\tau}, \delta) > R_{N-1}(F)$ ($\Gamma(\tilde{\tau}, \delta)$ means that Γ depends on both parameters $\tilde{\tau}$ and δ), with the following property. Suppose $\tilde{F} \in \mathcal{K}$ is (N, Γ)-close to F. If $t_1, t_2 \in [4, \tilde{\tau}]$ with $|t_1 - t_2| \leq \delta$, then

$$|\varphi^n(t_1) - \tilde{\varphi}^n(t_2)| \leq \delta \tag{4.7}$$

for all $n \geq 0$.

Proof of Claim. The proof will be given by induction over n. First, without loss of generality, we may assume that $\tilde{\tau} \geq 6$. Then, by Lemma 3.2.2, we have $\varphi([4, \tilde{\tau}]) \cup \tilde{\varphi}([4, \tilde{\tau}]) \subset [4, \tilde{\tau}]$. Now, by Lemma 4.2.8 and Proposition 4.2.9, there is $\Gamma(\tilde{\tau}, \delta)$ such that under the hypothesis of the claim, we have,

$$|\varphi(t) - \tilde{\varphi}(t)| \leq \frac{\delta}{2} \qquad \text{for all } \ t \in [4, \tilde{\tau}]. \tag{4.8}$$

Then for the case $n = 1$, using the contraction property of the map φ and (4.8),

$$|\varphi(t_1) - \tilde{\varphi}(t_2)| \leq |\varphi(t_1) - \varphi(t_2)| + |\varphi(t_2) - \tilde{\varphi}(t_2)| \leq \frac{|t_1 - t_2|}{2} + \frac{\delta}{2} \leq \delta.$$

Thus, (4.7) is proved for $n = 1$. Now assume that the claim holds for n. Then $\tilde{t}_1 := \varphi^n(t_1)$ and $\tilde{t}_2 := \tilde{\varphi}^n(t_2)$ also satisfy the hypothesis of the claim. Applying the base case to $\tilde{t}_1$ and $\tilde{t}_2$, we then obtain the conclusion for $n + 1$. Hence the claim is proved. △

Now, let us assume that $\tilde{F}$ is defined as in the hypothesis for Γ sufficiently large (to be stated below). Let us set $R_{N-1} := R_{N-1}(F) = R_{N-1}(\tilde{F})$. Further, let Λ be the universal constant from Proposition 4.1.9. For $t \geq \exp(\Lambda \cdot R_{N-1})$, we consider the point $z := F^{-1}(t)$; i.e.,

$$F(z) = t \geq \exp(\Lambda \cdot R_{N-1}).$$

Thus,

$$\log |F(z)| \geq \Lambda \cdot R_{N-1}.$$

By Proposition 4.1.9, this means that z belongs neither to the bottom nor to the middle of any wiggle of T. Since $\varphi(t) = \operatorname{Re} z$, Proposition 4.1.9 gives

$$\frac{\varphi(t)}{\Lambda} \leq \log t \leq \Lambda \cdot \varphi(t). \tag{4.9}$$

Moreover, if additionally $t \leq \exp(R_N(\tilde{F})/\Lambda)$, then by a similar argument which also applies to $\tilde{F}$, we obtain the following inequality. Set $w := \tilde{F}^{-1}(t)$, so

$$\Lambda \cdot R_{N-1} \leq \log|\tilde{F}(w)| \leq \frac{R_N(\tilde{F})}{\Lambda}.$$

Again, this means that w does not belong to the bottom two thirds of any wiggle of $\tilde{T}$, and thus the same estimate holds for $\tilde{\varphi}$. That is,

$$\frac{\tilde{\varphi}(t)}{\Lambda} \leq \log t \leq \Lambda \cdot \tilde{\varphi}(t). \tag{4.10}$$

So, in particular, both values $\varphi(t)$ and $\tilde{\varphi}(t)$ from the inequalities (4.9) and (4.10) are comparable up to a multiplicative error of at most Λ^2. In fact, we have the following.

Claim 2. Set $\hat{\Lambda} := \max(6, 4\Lambda^2/3)$. Suppose that $t \leq \exp(R_N(\tilde{F})/\Lambda)$ and further that $n \geq 0$ satisfies $\max(\varphi^n(t), \tilde{\varphi}^n(t)) \geq \Lambda^2 \cdot R_{N-1}$. Then

$$\frac{1}{\hat{\Lambda}} \leq \frac{\varphi^n(t)}{\tilde{\varphi}^n(t)} \leq \hat{\Lambda}. \tag{4.11}$$

Proof of Claim. The proof will be given by induction. Let t be as in the statement of the claim. Observe that the case $n = 0$ is trivial. Now suppose that the equation (4.11) holds for n. Let us first assume that $\varphi^{n+1}(t) \geq \Lambda^2 \cdot R_{N-1}$. Then, it follows from the induction hypothesis that

$$\text{dist}_{\mathbb{H}}(\varphi^n(t), \tilde{\varphi}^n(t)) \leq \log \hat{\Lambda}.$$

Recall that $\rho_T(u) \geq 1/2$ for all $u \in T$ (see Example 2.3.6). Therefore, we have obtained

$$\begin{aligned} |\tilde{\varphi}(\varphi^n(t)) - \tilde{\varphi}^{n+1}(t)| &\leq |\tilde{F}^{-1}(\varphi^n(t)) - \tilde{F}^{-1}(\tilde{\varphi}^n(t))| \\ &\leq 2\,\text{dist}_T(\tilde{F}^{-1}(\varphi^n(t)), \tilde{F}^{-1}(\tilde{\varphi}^n(t))) \\ &= 2\,\text{dist}_{\mathbb{H}}(\varphi^n(t), \tilde{\varphi}^n(t)) \leq 2\log\hat{\Lambda} < \hat{\Lambda}, \end{aligned} \tag{4.12}$$

and the last inequality holds from the fact that $\hat{\Lambda} \geq 6$. Additionally, the assumption on $\varphi^{n+1}(t)$ implies that $\varphi^n(t) \geq \exp(\Lambda \cdot R_{N-1})$; i.e.,

$$\begin{aligned}\varphi^n(t) = |F(F^{-1}(\varphi^n(t)))| &\geq \exp\left(\frac{\operatorname{Re} F^{-1}(\varphi^n(t))}{\Lambda}\right) \\ &= \exp\left(\frac{\varphi^{n+1}(t)}{\Lambda}\right) \\ &\geq \exp(\Lambda \cdot R_{N-1}).\end{aligned}$$

Also, recall that $\varphi(t) < t$ by Lemma 3.2.2. Therefore, we further have $\varphi^n(t) \leq \exp(R_N(\tilde{F})/\Lambda)$ by assumption. Thus, (4.9) and (4.10) apply to $\varphi^n(t)$, then we obtain

$$\frac{\varphi^{n+1}(t)}{\Lambda^2} \leq \tilde{\varphi}(\varphi^n(t)) \leq \Lambda^2 \cdot \varphi^{n+1}(t). \tag{4.13}$$

So if $\tilde{\varphi}^{n+1}(t) \geq \frac{1}{4}\hat{\Lambda}$, and by combining (4.12) and (4.13) we have

$$\varphi^{n+1}(t) \leq \Lambda^2 \tilde{\varphi}(\varphi^n(t)) \leq \Lambda^2(\tilde{\varphi}^{n+1}(t) + \hat{\Lambda}) \leq (\Lambda^2 + \hat{\Lambda}/4)\tilde{\varphi}^{n+1}(t) \leq \hat{\Lambda} \cdot \tilde{\varphi}^{n+1}(t).$$

For the opposite inequality, first note that $\hat{\Lambda} \leq 6\Lambda^2$, then $\frac{1}{4}\hat{\Lambda} \leq \frac{3}{2}\Lambda^2 < \Lambda^2 \cdot R_{N-1}$. Now, if $\tilde{\varphi}^{n+1}(t) \leq \frac{1}{4}\hat{\Lambda}$, thus we get

$$\varphi^{n+1}(t) \geq \frac{1}{\Lambda^2}\tilde{\varphi}(\varphi^n(t)) \geq \frac{1}{\Lambda^2}(\tilde{\varphi}^{n+1}(t) - \hat{\Lambda}) \geq \tilde{\varphi}^{n+1}(t)\left(\frac{1}{\Lambda^2} - \frac{1}{3\hat{\Lambda}}\right) \geq \frac{\tilde{\varphi}^{n+1}(t)}{\hat{\Lambda}}.$$

Therefore, the case for $n+1$ has been proved, and hence the claim holds when $\varphi^n(t) \geq \Lambda^2 \cdot R_{N-1}$. The case for $\tilde{\varphi}^n(t) \geq \Lambda^2 \cdot R_{N-1}$ is completely analogous. △

Now that we have shown that there is at most a multiplicative error $\hat{\Lambda}$ between the pull-backs under φ^n and $\tilde{\varphi}^n$, we are ready to complete the proof. Let $\varepsilon > 0$, we may assume that

$$\tau \geq \max\left(\frac{8\hat{\Lambda}}{\varepsilon}, \Lambda^2 \cdot R_{N-1}\right). \tag{4.14}$$

Further, let us define $\tilde{\tau} := \hat{\Lambda} \cdot \exp(\Lambda \cdot \exp(\Lambda \cdot \tau))$. First, it follows from (4.9) that

$$\varphi^2(t) > \tau \qquad \text{whenever} \qquad t > \tilde{\tau}/\hat{\Lambda}. \tag{4.15}$$

Additionally, if we assume that Γ is larger than $\tilde{\tau}$, then the same inequality holds for $\tilde{\varphi}$, that is

$$\tilde{\varphi}^2(t) > \tau \qquad \text{whenever} \qquad t > \tilde{\tau}/\hat{\Lambda}. \tag{4.16}$$

Now we choose

$$\Gamma \geq \max(\Gamma(\tilde{\tau}, \delta), \tilde{\tau}),$$

where $\Gamma(\tilde{\tau}, \delta)$ is as in Claim 1 with $\delta = \min(\hat{\Lambda}, \varepsilon/2)$. Note that $\tilde{\tau}$ depends only on τ and R_{N-1}, and therefore Γ also depends only on F, ε and τ, as required.

Let us now consider t as in the statement of the proposition. First, it will be enough to show the statement when n is minimal. Indeed, if (4.7) holds for some n_0 with $\min(\varphi^{n_0}(t), \tilde{\varphi}^{n_0}(t)) \leq \tau$, then it holds for all subsequent n by Claim 1; recall that $\Gamma \geq \Gamma(\tau, \varepsilon)$.

Moreover, the claim for $n \leq 2$ holds by the choice of $\tilde{\tau}$ and Claim 1. So let us now take $n \geq 2$, it follows from (4.15) and (4.16) that

$$\varphi^j(t) > \tau \quad \text{and} \quad \tilde{\varphi}^j(t) > \tau$$

for $j < n$, and furthermore either $\varphi^{n-2}(t) \leq \tilde{\tau}/\hat{\Lambda}$ or $\tilde{\varphi}^{n-2}(t) \leq \tilde{\tau}/\hat{\Lambda}$ by assumption. On the other hand, it follows from Claim 2 that $\varphi^{n-2}(t)$ and $\tilde{\varphi}^{n-2}(t)$ are comparable up to a factor at $\hat{\Lambda}$. So, in particular, both values, $\varphi^{n-2}(t)$ and $\tilde{\varphi}^{n-2}(t)$, are less that $\tilde{\tau}$, and so are their images.

Now, we will proceed with a similar argument as in the proof of Claim 2. As in (4.12),

$$|\varphi^{n-1}(t) - \varphi(\tilde{\varphi}^{n-2}(t))| \leq 2\log\hat{\Lambda} < \hat{\Lambda}. \tag{4.17}$$

Additionally, by the above we know that $|\tilde{\varphi}^{n-2}(t)| \leq \tilde{\tau}$ and since $\Gamma \geq \Gamma(\tilde{\tau}, \delta)$, thus the hypotheses of Claim 1 hold, then we have

$$|\varphi(\tilde{\varphi}^{n-2}(t)) - \tilde{\varphi}^{n-1}(t)| \leq \delta. \tag{4.18}$$

Consequently, by combining (4.17) and (4.18), we get

$$|\varphi^{n-1}(t) - \tilde{\varphi}^{n-1}(t)| \leq \hat{\Lambda} + \delta \leq 2\hat{\Lambda}. \tag{4.19}$$

Again, recall that $\rho_T(u) \geq 1/2$ for all $u \in T$ (see (2.2)). So, in particular, by (4.14), we also have

$$\begin{aligned} |\varphi^n(t) - \varphi(\tilde{\varphi}^{n-1}(t))| &\leq |F^{-1}(\varphi^{n-1}(t)) - F^{-1}(\tilde{\varphi}^{n-1}(t))| \\ &\leq 2\,\mathrm{dist}_T(F^{-1}(\varphi^{n-1}(t)), F^{-1}(\tilde{\varphi}^{n-1}(t))) \\ &= 2\,\mathrm{dist}_{\mathbb{H}}(\varphi^{n-1}(t), \tilde{\varphi}^{n-1}(t)). \end{aligned}$$

Since $\{\varphi^{n-1}(t), \tilde{\varphi}^{n-1}(t)\} > \tau$ and together with (4.19). In particular, we have

$$2\,\mathrm{dist}_{\mathbb{H}}(\varphi^{n-1}(t), \tilde{\varphi}^{n-1}(t)) \leq \frac{2\hat{\Lambda}}{\tau} \leq \frac{\varepsilon}{2}.$$

Therefore, by proceeding as above, we have obtained

$$|\varphi^n(t) - \tilde{\varphi}^n(t)| \leq |\varphi^n(t) - \varphi(\tilde{\varphi}^{n-1}(t))| + |\varphi(\tilde{\varphi}^{n-1}(t)) - \tilde{\varphi}^n(t)| \leq \varepsilon/2 + \delta \leq \varepsilon.$$

Hence the proof of Proposition 4.2.10 is complete. □

CHAPTER

FIVE

CONSTRUCTION OF PSEUDO-ARC JULIA CONTINUA

The aim of this chapter is to construct a map $F \in \mathcal{H}$ which satisfies the hypotheses of Theorem 3.2.7 and give the proofs of Theorem 1.0.4 and Theorem 1.0.6. To do so, we adapt some ideas of Henderson [Hen64] to our context. He constructs a self-map of the interval whose inverse limit is the pseudo-arc, starting with $x \to x^2$ and introducing successive "notches" in the graph. We adjust his technique to our setting, replacing maps of the interval by conformal maps of the strip $S = \{x + iy \colon x \geq 4,\ |y| < \pi\}$ into itself.

We start by defining the set $\mathcal{U}_n(\varphi, I)$ (see Definition 5.1.3), where φ is the one-dimensional projection of a function $F \in \mathcal{H}$. This set records all of the intervals which are *mapped minimally* (see Definition 5.1.1), for a given interval I, function φ and iteration step n. Then for each of those intervals, we add inductively a wiggle to our tract as in Definition 4.1.5. This is an analogue of the notches in Henderson's proof. In this way, we obtain a function in $\mathcal{H}$ satisfying the hypotheses of Theorem 3.2.7; the existence of the desired entire function follows from Theorem 3.1.7.

5.1 The set $\mathcal{U}_n(\varphi, I)$ and its properties

5.1.1 Basic properties of $\mathcal{U}_n(\varphi, I)$

We begin by introducing the set $\mathcal{U}_n(\varphi, I)$ and study its properties. This set is adapted from [Hen64]. Throughout this section, let $F \in \mathcal{H}$, and let φ be its one-dimensional projection, as in Definition 3.2.1.

Definition 5.1.1. (Mapping minimally).
Let $I \subset [6, \infty)$ be a closed interval, and let $n \geq 0$. We say that a closed interval $J \subset [4, \infty)$ is *mapped minimally* over I by φ^n if $\varphi^n(J) = I$, and there is no smaller subinterval of J satisfying this property.

In the following result, we give enough conditions of mapping minimally an interval J' to an interval I under φ^n.

Lemma 5.1.2. (Condition of mapping minimally).
Let $I \subset [6, \infty)$ be a closed interval and let $n \geq 0$. Suppose $J \subset [4, \infty)$ is an interval such that $\varphi^n(J) \supseteq I$. Then there exists an interval $J' \subseteq J$ that maps minimally over I by φ^n; in particular, $\varphi^n(J') = I$.

Proof. Define $\chi = \{[\mu, \nu] \subseteq J \colon \varphi^n([\mu, \nu]) \supseteq I\}$. Note that χ is partially ordered by set inclusion. We first show that χ has a minimal element, then we prove that this element has the desired property.

Claim 1. $(\chi, \subseteq)$ has a minimal element.

Proof of Claim. Observe that χ is non-empty, because $J \in \chi$. Let C be a chain in χ, i.e., a totally ordered subset of χ. We want to show that C has a lower bound. Consider $L = \bigcap C$; observe that L is contained in every element of C. Furthermore, L is a nested intersection of compact intervals; hence L is a lower bound for C. Moreover, note that $L \in \chi$. To see this, let $x \in I$, then $(\varphi^n)^{-1}(x) \cap C$ is non-empty and compact for all $C \in \chi$. So,

$$(\varphi^n)^{-1}(x) \cap L = \bigcap_C \big((\varphi^n)^{-1}(x) \cap C\big) \neq \emptyset.$$

Then, by Zorn's lemma, χ has a minimal element. △

Claim 2. Suppose $J' = [a, d]$ is a minimal element of χ. Then $\varphi^n(J') = I$.

Proof of Claim. We shall prove the claim by contrapositive. So suppose that $[a, d] \in \chi$ and $I \subsetneq \widehat{I} := \varphi^n([a, d])$. We then consider the following cases:

- Case 1. $I \subseteq \operatorname{Int} \widehat{I}$.

Set $\varepsilon := \operatorname{dist}(I, \partial\widehat{I})$, then by uniform continuity, there is $\delta > 0$ such that

$$|\varphi^n(x) - \varphi^n(y)| \leq \frac{\varepsilon}{2}$$

whenever $x, y \in [a, d]$ and $|x - y| < \delta$. Since $[a, d] \in \chi$, there are $x', y' \in [a, d]$ such that $\widehat{I} = [\varphi^n(x'), \varphi^n(y')]$. Now set

$$\tilde{J} = [a + \delta, d - \delta].$$

Then there is $\tilde{x} \in \tilde{J}$ such that $|x' - \tilde{x}| \leq \delta$ and then $\varphi^n(\tilde{x})$ is to the left of I. Likewise, $\tilde{J}$ contains a point $\tilde{y}$, where $\varphi^n(\tilde{y})$ is to the right of I. Since $\varphi^n(\tilde{J})$ is connected, hence $\varphi^n(\tilde{J}) \supseteq I$, which implies $\tilde{J}$ is an element of χ, and so J' is not minimal.

- Case 2. I and $\widehat{I}$ have an endpoint in common.

Let α be a common point, then there is $a' \in [a, d]$ such that $\varphi^n(a') = \alpha$. Let β be the other endpoint of $\widehat{I}$, and set $\varepsilon = \operatorname{dist}(\beta, I)$. Define $\delta > 0$ as in case 1. Let

$$\tilde{J} := [a_0, d_0]$$

be the smallest closed interval containing $[a + \delta, d - \delta]$ and a'. This is a proper subinterval of J. Analogously to case 1, $\varphi^n(\tilde{J}) \supseteq I$, therefore $\tilde{J}$ is an element of χ, so J is not minimal. △

Therefore, we have shown that any minimal element of χ has the desired property, and the proof is complete. □

Recall that a quadruple $Q = [A, B, C, D]$ is an increasing four-tuple of real numbers, that is, $A < B < C < D$ (see Definition 3.2.5 for further details).

Definition 5.1.3. (The set $\mathcal{U}_n(\varphi, I)$).
Let $I = [A, D] \subset [6, \infty)$ be a closed interval and let $Q = [A, B, C, D]$ be a

quadruple. For $n \geq 0$, we define

$$\mathcal{U}_n(\varphi, I) := \{J \subset [4, \infty) \colon J \text{ is mapped minimally over } I \text{ by } \varphi^n\}.$$

In this case, if $J \in \mathcal{U}_n(\varphi, I)$, we also say that J is mapped minimally *over* Q by φ^n. Therefore, we also write $\mathcal{U}_n(\varphi, I) := \mathcal{U}_n(\varphi, Q)$. Further, we define

$$\widehat{\mathcal{U}}_n(\varphi, Q) := \{J \in \mathcal{U}_n(\varphi, Q) \colon J \text{ is mapped crookedly over } Q \text{ by } \varphi^n\}.$$

We refer to Definition 3.2.6 for the definition of crookedness.

Remark 5.1.4. With this terminology, the hypotheses of Theorem 3.2.7 can be rephrased as follows: Let $F \in \mathcal{H}$. For all integer quadruple Q, there exists $k_0 \in \mathbb{N}$ with the following property. If

$$\mathcal{U}_k(\varphi, Q) = \widehat{\mathcal{U}}_k(\varphi, Q) \text{ for all } k \geq k_0,$$

then $\widehat{\mathcal{J}}(F)$ is a pseudo-arc. Further, in this case, it will be sufficient to take the constant K as $K = 1$.

Definition 5.1.5. (Interval to the left).
Let $R > 0$ and let I be a closed interval. We say that I is (*strictly*) *to the left* of R if, $(x < R)$ $x \leq R$ for every $x \in I$. Analogously, I is (*strictly*) *to the right* of R for $(x > R)$ $x \geq R$ for all $x \in I$.

Now, let us start with two simple properties, which are consequences of Lemma 3.2.2. For the remainder of this subsection, fix a closed interval $I = [A, D] \subseteq [6, \infty)$.

Observation 5.1.6. *If $J \in \mathcal{U}_n(\varphi, I)$, then $J \subset [6, \infty)$.*

Proof. By Lemma 3.2.2(ii), we have $\varphi([4, 6)) \subset [4, 6)$. On the other hand, if $J \in \mathcal{U}_n(\varphi, I)$ for some $n \geq 0$, we have $\varphi^n(J) = I$, by definition. But $I \subset [6, \infty)$, and therefore $J \cap [4, 6) = \emptyset$. Hence the claim holds. □

Observation 5.1.7. *No interval $J \in \mathcal{U}_n(\varphi, I)$ contains an interval of $\mathcal{U}_m(\varphi, I)$ for $n \neq m$.*

Proof. Observe that $\varphi(t) < t$ for $t \geq 5$ by Lemma 3.2.2(ii). If $J \in \mathcal{U}_n(\varphi, I)$, this implies that $\varphi^n(J) = I$ by Lemma 5.1.2. Therefore, $\varphi^m(J)$ is strictly the left of D, if $m > n$, and it will be strictly to the right of A, if $m < n$. Thus J does not map over I under φ^m for $n \neq m$. □

The next observation shows us that the endpoints of an interval J map to A and D under φ^n.

Observation 5.1.8. *If $J = [a, d] \in \mathcal{U}_n(\varphi, I)$, then $\{\varphi^n(a), \varphi^n(d)\} = \{A, D\}$.*

Proof. Note that if $J \in \mathcal{U}_n(\varphi, I)$, then $\varphi^{-n}(A) \cap J \neq \emptyset$ and $\varphi^{-n}(D) \cap J \neq \emptyset$. Further, any sub-interval of J is bounded by an element of $\varphi^{-n}(A)$ and $\varphi^{-n}(D)$ which also maps over I by Lemma 5.1.2. Hence the only elements of J that can map over A and D are the endpoints and the proof is complete. □

The following result shows that different elements of the same $\mathcal{U}_n(\varphi, I)$ can only intersect at their endpoints.

Lemma 5.1.9. (Pairwise disjoint elements of $\mathcal{U}_n(\varphi, I)$).
For every $n \in \mathbb{N}$, the elements of $\mathcal{U}_n(\varphi, I)$ have pairwise disjoint interiors.

Proof. Suppose, by way of contradiction, that there are two intervals $J_1 = [a_1, d_1]$, $J_2 = [a_2, d_2] \in \mathcal{U}_n(\varphi, I)$ such that $J_1 \cap J_2$ contains more than one point.

Without loss of generality, we can assume $a_1 < a_2$, then $a_2 < d_1$. Since J_1 is mapping minimally over I under φ^n, this implies that

$$\{\varphi^n(a_1), \varphi^n(d_1)\} = \{A, D\}$$

by Observation 5.1.8. Moreover, since $J_2 \in \mathcal{U}_n(\varphi, I)$, then again by Observation 5.1.8, we also have either $\varphi^n(a_2) = A$ or $\varphi^n(a_2) = D$. So, either

$$\{\varphi^n(a_1), \varphi^n(a_2)\} = \{A, D\}, \quad \text{or}$$
$$\{\varphi^n(d_1), \varphi^n(a_2)\} = \{A, D\}.$$

Now, note that $[a_1, a_2]$ is a sub-interval of J_1 that maps minimally over I by φ^n, and hence a contradiction of $J_1 \in \mathcal{U}_n(\varphi, I)$.
Likewise, we have that $[a_2, d_1]$ is a subinterval of J_1 and J_2 that maps minimally over I by φ^n. These contradictions complete the proof. □

Lemma 5.1.10. *Each* $\mathfrak{U}_n(\varphi, I)$ *is finite.*

Proof. Let $n \geq 0$, since $\varphi(t) \to \infty$ as $t \to \infty$, and thus $\varphi^n(t) \to \infty$ as $t \to \infty$, this implies that $\varphi^{-n}(I)$ is bounded. By the condition of minimality, we have

$$\varphi^{-n}(I) \supset \bigcup_{J \in \mathfrak{U}_n(\varphi,I)} J.$$

Set $\varepsilon := \operatorname{diam}(I)$. It follows from Observation 5.1.8 that for every $J = [a, d] \in \mathfrak{U}_n(\varphi, I)$, we have $|\varphi^n(a) - \varphi^n(d)| = \varepsilon$. By Lemma 3.2.2(i), we have

$$|\varphi^n(s) - \varphi^n(t)| \leq \frac{|s-t|}{2^n} \quad \text{for all } \ s, t \in J;$$

in particular,

$$\varepsilon = |\varphi^n(a) - \varphi^n(d)| \leq \frac{|a-d|}{2^n}. \tag{5.1}$$

Set $\delta := 2^n \varepsilon > 0$. Then it follows from (5.1) that $\delta \leq |J|$ for all elements $J \in \mathfrak{U}_n(\varphi, I)$.

Since the interior of the elements of $\mathfrak{U}_n(\varphi, I)$ are pairwise disjoint by Lemma 5.1.9, it follows that

$$\operatorname{diam}(\varphi^{-n}(I)) \geq \operatorname{diam}\left(\bigcup_{J \in \mathfrak{U}_n(\varphi,I)} J\right) \geq \sum_{J \in \mathfrak{U}_n(\varphi,I)} |J| \geq \delta \cdot \#\mathfrak{U}_n(\varphi, I).$$

Hence,

$$\#\mathfrak{U}_n(\varphi, I) \leq \frac{\operatorname{diam}(\varphi^{-n}(I))}{\delta} < \infty$$

as claimed, and the proof is complete. □

The following lemma shows that, for $k \leq n$, the map φ^k induces a map $\mathfrak{U}_n(\varphi, I) \to \mathfrak{U}_{n-k}(\varphi, I)$.

Lemma 5.1.11. (Properties of $\mathfrak{U}_{n-k}(\varphi, I)$).
Let $n \geq 0$ *be a positive integer. Then the following properties hold:*

(i) If $J \in \mathcal{U}_n(\varphi, I)$, then $\varphi^k(J) \in \mathcal{U}_{n-k}(\varphi, I)$ and $J \in \mathcal{U}_k(\varphi, \varphi^k(J))$ for $k = 0, \ldots, n$.

(ii) In particular, for $k \leq n$,

$$\mathcal{U}_n(\varphi, I) = \bigcup_{J \in \mathcal{U}_{n-k}(\varphi, I)} \mathcal{U}_k(\varphi, J).$$

Proof. To prove (i), let $J \in \mathcal{U}_n(\varphi, I)$ and consider $J_k := \varphi^k(J)$. Then, we have

$$\varphi^{n-k}(J_k) = \varphi^{n-k}(\varphi^k(J)) = \varphi^n(J) = I.$$

Now, let us show that $J \in \mathcal{U}_k(\varphi, J_k)$. By definition, it is enough to show that there is no proper sub-interval of J mapping minimally over J_k by φ^k. We proceed by contradiction. Suppose there is $\tilde{J} \subsetneq J$ such that $\varphi^k(\tilde{J}) = J_k$. This implies $\varphi^{n-k}(\varphi^k(\tilde{J})) = \varphi^{n-k}(J_k)$, and thus $\varphi^n(\tilde{J}) = I$. Therefore, $J \notin \mathcal{U}_n(\varphi, I)$, which is a contradiction. Hence the claim holds.

Similarly, we show that $J_k \in \mathcal{U}_{n-k}(\varphi)$. Suppose, by way of contradiction, that there is $\tilde{J}_k \subsetneq J_k$ such that $\varphi^{n-k}(\tilde{J}_k) = I$ and $\tilde{J}_k$ is minimal having this property. Since $\tilde{J}_k \subset J_k = \varphi^k(J)$, this implies, by Lemma 5.1.2, that there is a sub-interval $\tilde{J} \subset J$ such that $\varphi^k(\tilde{J}) = \tilde{J}_k \subsetneq J_k$. Then we have obtained $\varphi^n(\tilde{J}) = \varphi^{n-k}(\varphi^k(\tilde{J})) = I$. Thus, $J \notin \mathcal{U}_n(\varphi, I)$ which is a contradiction. Then, the claim follows.

From (i), we have

$$\mathcal{U}_n(\varphi, I) \subset \bigcup_{J \in \mathcal{U}_{n-k}(\varphi, I)} \mathcal{U}_k(\varphi, J).$$

For the converse inclusion, let $J \in \mathcal{U}_{n-k}(\varphi, I)$ and $L \in \mathcal{U}_k(\varphi, J)$. This implies that $\varphi^k(L) = J$ and hence $\varphi^n(L) = \varphi^{n-k}(J) = I$. So L contains a sub-interval $\tilde{L}$ such that $\tilde{L} \in \mathcal{U}_n(\varphi, I)$ by Lemma 5.1.2. Then $\varphi^k(\tilde{L}) \subset \varphi^k(L) = J$, but by (i) we also have that $\varphi^k(\tilde{L}) \in \mathcal{U}_{n-k}(\varphi, I)$. Therefore, we must have that $L = \tilde{L}$ as required. Hence (ii) holds, and the proof is complete. $\square$

The following result shows that, if an interval in $\mathcal{U}_{n-k}(\varphi, Q)$ is mapped crookedly over a quadruple Q, then this is also true for the corresponding interval in $\mathcal{U}_n(\varphi, Q)$. Recall that the set $\widehat{\mathcal{U}}_n(\varphi, Q)$ is given as

$$\widehat{\mathcal{U}}_n(\varphi, Q) = \{J \in \mathcal{U}_n(\varphi, Q) \colon J \text{ is mapped crookedly over } Q \text{ by } \varphi^n\},$$

we refer to Definition 5.1.3 for all details.

Lemma 5.1.12. (A condition to map crookedly in $\mathcal{U}_n(\varphi, Q)$).
Let $Q = [A, B, C, D]$ be a quadruple. Suppose $J \in \mathcal{U}_n(\varphi, Q)$ and $\varphi^k(J) \in \widehat{\mathcal{U}}_{n-k}(\varphi, Q)$ for some $k \leq n$. Then $J \in \widehat{\mathcal{U}}_n(\varphi, Q)$.

Proof. Let $J \in \mathcal{U}_n(\varphi, I)$, and suppose $\varphi^k(J)$ is mapped crookedly over Q. To fix our ideas, say the preimages $t_B < \tilde{t}_B$ of B under φ^{n-k} surround a preimage t_C of C under φ^{n-k}, in particular $t_C \in [t_B, \tilde{t}_B]$.

By Lemma 5.1.2, there is a sub-interval $\tilde{J}$ of J that maps minimally over $[t_B, \tilde{t}_B]$ by φ^k. In particular, $\tilde{J}$ contains a preimage of t_C under φ^k which is an element of $\varphi^{-n}(C) \cap J$. Further, it follows from Observation 5.1.8 that $\tilde{J}$ is bounded by $\varphi^k(\{t_B, \tilde{t}_B\})$. Note that both preimages of t_B and $\tilde{t}_B$ under φ^k are elements of $\varphi^{-n}(B) \cap J$. Hence, J is mapped crookedly by φ^n, and the proof is complete. □

We conclude this subsection with the following result which is an immediately consequence of Lemma 5.1.11 and Lemma 5.1.12.

Corollary 5.1.13. *Let $Q = [A, B, C, D]$ be a quadruple. If $\mathcal{U}_n(\varphi, Q) = \widehat{\mathcal{U}}_n(\varphi, Q)$, then $\mathcal{U}_k(\varphi, Q) = \widehat{\mathcal{U}}_k(\varphi, Q)$ for all $k \geq n$.*

5.1.2 Approximation on the set $\mathcal{U}_n(\varphi, I)$

In this section, we obtain further properties of the set $\mathcal{U}_n(\varphi, I)$, however these will focus on functions in the class $\mathcal{K}$. In particular, we wish to describe how these sets change when we move from a function with N wiggles to a nearby function with $N + 1$ wiggles. Let us first start with the following result which tells us that the number of intervals in $\mathcal{U}_n(\varphi, I)$ increases only through wiggles.

Recall that $(r_j, R_j)_{j=0}^{N}$, with $N \leq \infty$, is a sequence of pairs of real numbers which determines the position of the wiggles; we refer to Definition 4.1.5 for the reader's convenience.

Proposition 5.1.14. (Conditions over the wiggles to get $\#\mathcal{U}_1(\varphi, I) = 1$).
Let $F \in \mathcal{K}$ have N wiggles. Let $I = [A, D]$ be an interval with $A \geq 6$ and $|I| \geq \nu_0$, where ν_0 is the constant from Proposition 4.1.8. In addition, suppose that

$$I \not\subset [r_k(F) - \nu_0, R_k(F) + \nu_0]$$

for any $k \geq 0$. Then

$$\#\mathcal{U}_1(\varphi, I) = 1.$$

Proof. Let T be the tract of F. First, observe that the hypothesis implies that there is $t, t + \nu_0 \in [A, D]$ such that

$$R_k(F) + \nu_0 < t < t + \nu_0 < r_{k+1}(F)$$

for some $-1 \leq k \leq N$. So, in particular, the intersection of T with each of the vertical lines at the real parts t and $t + \nu_0$ is connected. Let ρ be maximal so that the vertical geodesic

$$\gamma = F^{-1}(\{z \in \mathbb{H} \colon\ |z| = \rho\})$$

contains a point at real part t, then all of points on γ have real part at least t. Therefore, γ separates all points of T which have real part less than t from all points at real part greater that $t + \nu_0$ by Proposition 4.1.8.

Now, if $J \in \mathcal{U}_1(\varphi, I)$, this implies that $\varphi(J) = I$. Then $F^{-1}(J)$ connects two points at real parts t and $t + \nu_0$, and thus $F^{-1}(J) \cap \gamma \neq \emptyset$, which implies that $\rho \in J$. By Lemma 5.1.9, the elements of $\mathcal{U}_1(\varphi, I)$ have pairwise disjoint interiors. However, by Observation 5.1.8 we know that the elements of $\mathcal{U}_1(\varphi, I)$ can only intersect at their endpoints. Hence, the claim is proved. ☐

Definition 5.1.15. (Larger quadruples).
Let $Q = [A, B, C, D]$ and $\tilde{Q} = [\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}]$ be two quadruples. We say that $Q \prec \tilde{Q}$

if the following holds:

$$\tilde{A} < A, \quad B < \tilde{B}, \quad \tilde{C} < C \quad \text{and} \quad D < \tilde{D}.$$

In a similar way, for two intervals $I = [A, D]$ and $\tilde{I} = [\tilde{A}, \tilde{D}]$, we write $I \prec \tilde{I}$ if

$$\tilde{A} < A \quad \text{and} \quad D < \tilde{D}.$$

Additionally, we denote $|\tilde{Q} - Q|_{\text{conf}} := \max(A - \tilde{A}, \tilde{B} - B, C - \tilde{C}, \tilde{D} - D)$.

The following results are regarding functions in the class $\mathcal{K}$ that are (N, Γ)-close. Let us recall that for different elements F, F_j, $\tilde{F}$ in $\mathcal{K}$, their one-dimensional projections are denoted by φ, φ_j, $\tilde{\varphi}$ respectively (see Notation 4.1.6).

We start by seeing that for fixed n, the elements of $\mathcal{U}_n(\tilde{\varphi},, I)$ are uniformly bounded for a function $\tilde{F}$ (N, Γ)-close to F.

Proposition 5.1.16. (Elements of $\mathcal{U}_n(\tilde{\varphi}, I)$ are to the left of Γ).
Let $F \in \mathcal{K}$ have N wiggles. Let $I = [A, D] \subset [6, \infty)$ be an interval and let $n_0 \geq 0$. Then there is $\Gamma > R_{N-1}(F)$ with the following property.

Suppose $\tilde{F} \in \mathcal{K}$ is (N, Γ)-close to F. Then for $n \leq n_0$, all of the elements of $\mathcal{U}_n(\tilde{\varphi}, I)$ are to the left of Γ.

Proof. Let Γ be the constant from Proposition 4.2.3 with $K = D$. If $x \in J$, where $J \in \mathcal{U}_n(\tilde{\varphi}, I)$ for some $n \leq n_0$, then $\tilde{\varphi}^n(x) \leq K$. Hence $x \leq \Gamma$ as required. □

The following result shows us how to approximate elements between $\mathcal{U}_n(\varphi, I)$ and $\mathcal{U}_n(\tilde{\varphi}, \tilde{I})$ that are (N, Γ)-close in $\mathcal{K}$.

Proposition 5.1.17. (Approximation on elements of $\mathcal{U}_n(\varphi, I)$).
Let $F \in \mathcal{K}$ have N wiggles. Let $I \prec \tilde{I}$ be two intervals and let $n_0 \geq 0$. Then there is $\Gamma > R_{N-1}(F)$ so that the following holds.

If $\tilde{F} \in \mathcal{K}$ is (N, Γ)-close to F, then for every $n \geq 0$ and every $\tilde{J} \in \mathcal{U}_n(\tilde{\varphi}, \tilde{I})$ that is to the left of $2R_N(\tilde{F})$, there is $J \in \mathcal{U}_n(\varphi, I)$ with $J \subset \tilde{J}$.

In particular, for $n \leq n_0$,

$$\#\mathcal{U}_n(\tilde{\varphi}, \tilde{I}) \leq \#\mathcal{U}_n(\varphi, I). \tag{5.2}$$

Proof. Denote $I = [A, D]$ and $\tilde{I} = [\tilde{A}, \tilde{D}]$. Set $\varepsilon := \max(A - \tilde{A}, \tilde{D} - D) > 0$. Let us take $\tau > \tilde{D}$ and Γ according to Proposition 4.2.10. Let $\tilde{F} \in \mathcal{K}$ be as in the statement of the proposition and let $n \geq 0$. Take $\tilde{J} \in \mathcal{U}_n(\tilde{\varphi}, \tilde{I})$ such that it is to the left of $2R_N(\tilde{F})$. Then, by Proposition 4.2.10, for all $t \in \tilde{J} = [\tilde{a}, \tilde{d}]$,

$$|\varphi^n(t) - \tilde{\varphi}^n(t)| \leq \varepsilon. \tag{5.3}$$

So, in particular, from (5.3), we have

$$|\varphi^n(\tilde{a}) - \tilde{\varphi}^n(\tilde{a})| \leq \varepsilon \quad \text{and} \quad |\varphi^n(\tilde{d}) - \tilde{\varphi}^n(\tilde{d})| \leq \varepsilon. \tag{5.4}$$

By Observation 5.1.8, we have that $\tilde{\varphi}^n(\tilde{a}) = \tilde{A}$ and $\tilde{\varphi}^n(\tilde{d}) = \tilde{D}$. Then, it follows from (5.4) that

$$\varphi^n(\tilde{a}) \leq \tilde{A} + \varepsilon \leq A \quad \text{and} \quad D \leq \tilde{D} - \varepsilon \leq \varphi^n(\tilde{d}).$$

This means that $\varphi^n(\tilde{J}) \supset I$. Therefore, by Lemma 5.1.2, there is a interval $J \subseteq \tilde{J}$ such that $J \in \mathcal{U}_n(\varphi, I)$, and the claim holds.

Note that, in particular, every $\tilde{J}$ is to the left of Γ by Proposition 5.1.16. On the other hand, all elements of $\mathcal{U}_n(\varphi, I)$ are to the left of Γ by definition. Therefore,(5.2) holds as claimed. This complete the proof. □

We conclude this section with the version of the preceding result for intervals that are mapped crookedly.

Proposition 5.1.18. (Approximations on elements of $\widehat{\mathcal{U}}_n(\varphi, Q)$)
Let $F \in \mathcal{K}$ have N wiggles, and let $Q \prec \tilde{Q}$ be quadruples. Let $n_0 \geq 0$. Then there is $\Gamma > R_{N-1}(F)$ with the following property.

Suppose that $\tilde{F} \in \mathcal{K}$ is (N, Γ)-close to F, that $n \leq n_0$, that $\tilde{J} \in \mathcal{U}_n(\tilde{\varphi}, \tilde{Q})$, and that $J \in \mathcal{U}_n(\varphi, Q)$ with $\tilde{J} \supseteq J$. If J is mapped crookedly over Q by φ^n, then $\tilde{J}$ is mapped crookedly over $\tilde{Q}$ by $\tilde{\varphi}^n$.

In particular,

$$\#\mathcal{U}_n(\tilde{\varphi}, \tilde{Q}) - \#\widehat{\mathcal{U}}_n(\tilde{\varphi}, \tilde{Q}) \leq \#\mathcal{U}_n(\varphi, Q) - \#\widehat{\mathcal{U}}_n(\varphi, Q). \tag{5.5}$$

Proof. Recall that $Q = [A, B, C, D]$ and $\tilde{Q} = [\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}]$. We set $\varepsilon := |\tilde{Q} - Q|_{\text{conf}}$ as in Definition 5.1.15. We take $\tau > \tilde{D}$ and Γ according to Proposition 4.2.10. Then assume $\tilde{F} \in \mathcal{K}$ is as in the statement of the proposition. Let $\tilde{J} \in \mathcal{U}_n(\tilde{\varphi}, \tilde{Q})$ and let J be as in the hypothesis. Then, by Proposition 4.2.10, for all $t \in \tilde{J}$,

$$|\varphi^n(t) - \tilde{\varphi}^n(t)| \leq \varepsilon. \tag{5.6}$$

Since J is mapped crookedly over Q by φ^n, this means that there are preimages t_B and $t_{\tilde{B}}$ of B under φ^n surrounding a preimage t_C of C under φ^n, that is $t_C \in [t_B, t_{\tilde{B}}]$. So by (5.6), in particular we have

$$\begin{aligned} &|\varphi^n(t_B) - \tilde{\varphi}^n(t_B)| \leq \varepsilon, \\ &|\varphi^n(t_{\tilde{B}}) - \tilde{\varphi}^n(t_{\tilde{B}})| \leq \varepsilon \quad \text{and} \\ &|\varphi^n(t_C) - \tilde{\varphi}^n(t_C)| \leq \varepsilon. \end{aligned} \tag{5.7}$$

Then, (5.7) implies that

$$\begin{aligned} &\tilde{\varphi}^n(t_B) \leq B + \varepsilon \leq \tilde{B}, \\ &\tilde{\varphi}^n(t_{\tilde{B}}) \leq B + \varepsilon \leq \tilde{B} \quad \text{and} \\ &\tilde{\varphi}^n(t_C) \leq \tilde{C} - \varepsilon \leq C. \end{aligned} \tag{5.8}$$

Since $F^{-n}(J)$ is an arc that connects a point at the real part A to a point at the real part D. So by (5.8), we have that there are preimages $s_B < s_{\tilde{B}}$ of $\tilde{B}$ under $\tilde{\varphi}^n$ such that $[t_B, t_{\tilde{B}}] \cap [s_B, s_{\tilde{B}}] \neq \emptyset$. Further, there is also a preimage s_C of C under $\tilde{\varphi}^n$. By the last equation of (5.8), we obtain that $s_C \in [s_B, s_{\tilde{B}}]$. Hence, $\tilde{J}$ is mapped crookedly over $\tilde{Q}$.

Lastly, observe that (5.5) follows immediately from the first part together with Proposition 5.1.17, and thus the proof is complete. □

5.2 The main construction

Recall that our goal is to show that there is a disjoint-type entire function f so that every Julia continuum of f is a pseudo-arc, Theorem 1.0.4. So we start by constructing the function $F \in \mathcal{H}_{\nu_0}$ that satisfies the hypotheses of Theorem 3.2.7.

As already mentioned, the idea of the construction is to start with the strip and inductively add a new wiggle. In particular, in the N-th step of the construction, we will have obtained a function in $\mathcal{K}$ with N wiggles. These functions converge to a limit function $F \in \mathcal{K}$ with infinitely many wiggles, which is the desired function.

The outline of the construction is as follows. In Proposition 5.2.1, we set up the induction, that is; we define $\tilde{F} \in \mathcal{K}$ with N wiggles in such a way that the number of intervals mapped crookedly increases. Next, applying Proposition 5.2.1 repeatedly, we can achieve crookedness over a prescribed quadruple for all sufficiently large times, Proposition 5.2.2. We then establish Theorem 5.2.3 which provides us the existence of the function F in $\mathcal{K}$ so that $\widehat{\mathcal{J}}(F)$ is a pseudo-arc.

Proposition 5.2.1. (Increasing the number of intervals mapped crookedly). *Let $F \in \mathcal{K}$ have N wiggles. Let Q and $\tilde{Q}$ be quadruples such that $Q \prec \tilde{Q}$ and let $\Gamma > 0$. There exits a non-negative integer k such that*

$$\#\mathcal{U}_m(\varphi, Q) - \#\widehat{\mathcal{U}}_m(\varphi, Q) = k$$

for sufficiently large $m \in \mathbb{N}$.

Suppose that $k \geq 1$. Then there exists a function $\tilde{F} \in \mathcal{K}$ such that $\tilde{F}$ is (N, Γ)-close to F and there exists $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$,

$$\#\mathcal{U}_n(\tilde{\varphi}, \tilde{Q}) - \#\widehat{\mathcal{U}}_n(\tilde{\varphi}, \tilde{Q}) \leq k - 1.$$

Proof. Let $Q = [A, B, C, D]$ and $\tilde{Q} = [\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}]$. Set $\varepsilon = |\tilde{Q} - Q|_{\text{conf}}$ as in Definition 5.1.15 and set $\delta := \min(B - A, D - C)$. By Lemma 3.2.2, let $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$, all of the elements of $\mathcal{U}_m(\varphi, Q)$ are to the right of the last wiggle of F and, additionally, those elements have length at least ν_0. Thus, for all $m \geq m_0$ we have

$$\#\mathcal{U}_m(\varphi, Q) = \#\mathcal{U}_{m_0}(\varphi, Q)$$

by Proposition 5.1.14 and Lemma 5.1.11. Further, it follows from Corollary

5.1.13 that $\#\widehat{\mathfrak{U}}_m(\varphi, Q)$ is non-decreasing. So, in particular a number k as in the statement of the proposition does exist.

By assumption, if m_0 is picked sufficiently large, we have

$$\#\widehat{\mathfrak{U}}_m(\varphi, Q) = \#\widehat{\mathfrak{U}}_{m_0}(\varphi, Q) = \#\mathfrak{U}_{m_0}(\varphi, Q) - k$$

for all $m \geq m_0$.

Assume that $k > 0$ and, choose $\Gamma_1 > \Gamma$ sufficiently large so that Propositions 5.1.16, 5.1.17 and 5.1.18 apply for m_0. Further, let $\Gamma_2 > \Gamma_1$ be so large that Proposition 4.2.10 holds for $\tau = \Gamma_1$. We now choose $n_1 \in \mathbb{N}$ sufficiently large such that the following hold.

(i) $n_1 > m_0$.

(ii) All elements of $\mathfrak{U}_{n_1}(\varphi, Q)$ are to the right of $\Gamma_2 + \nu_0 + 1$.

(iii) $n_1 > \frac{\log(2\nu_0+2) - \log\delta}{\log 2}$.

Now, let $z, w \in T$ with $z_1 = \operatorname{Re} F^{n_1}(z), w_1 = \operatorname{Re} F^{n_1}(w) \in \overline{T}$ such that $\operatorname{Re} z = A$, $\operatorname{Re} w = B$. Then $|z - w| \geq B - A \geq \delta$. By Lemma 3.2.2, we obtain

$$\begin{aligned} |z_1 - w_1| &\geq 2^{n_1}|\varphi^{n_1}(z_1) - \varphi^{n_1}(w_1)| \\ &\geq 2^{n_1}|A - B| \geq 2^{n_1}\delta > 2\nu_0 + 2. \end{aligned} \tag{5.9}$$

The last inequality holds by (iii). Analogously, we apply the same argument to the points C and D.

Let $I_1 = [A_1, D_1]$ be the right-most element of $\mathfrak{U}_{n_1}(\varphi, Q)$ that is not in $\widehat{\mathfrak{U}}_{n_1}(\varphi, Q)$. Now we can choose two points $B_1 < C_1$ such that

$$\{\varphi^{n_1}(B_1), \varphi^{n_1}(C_1)\} = \{B, C\}$$

and

$$\min(B_1 - A_1, D_1 - C_1) > 2\nu_0 + 2, \tag{5.10}$$

by equation (5.9). Let $Q_1 = [A_1, B_1, C_1, D_1]$ be the resulting quadruple (see Figure 5.1).

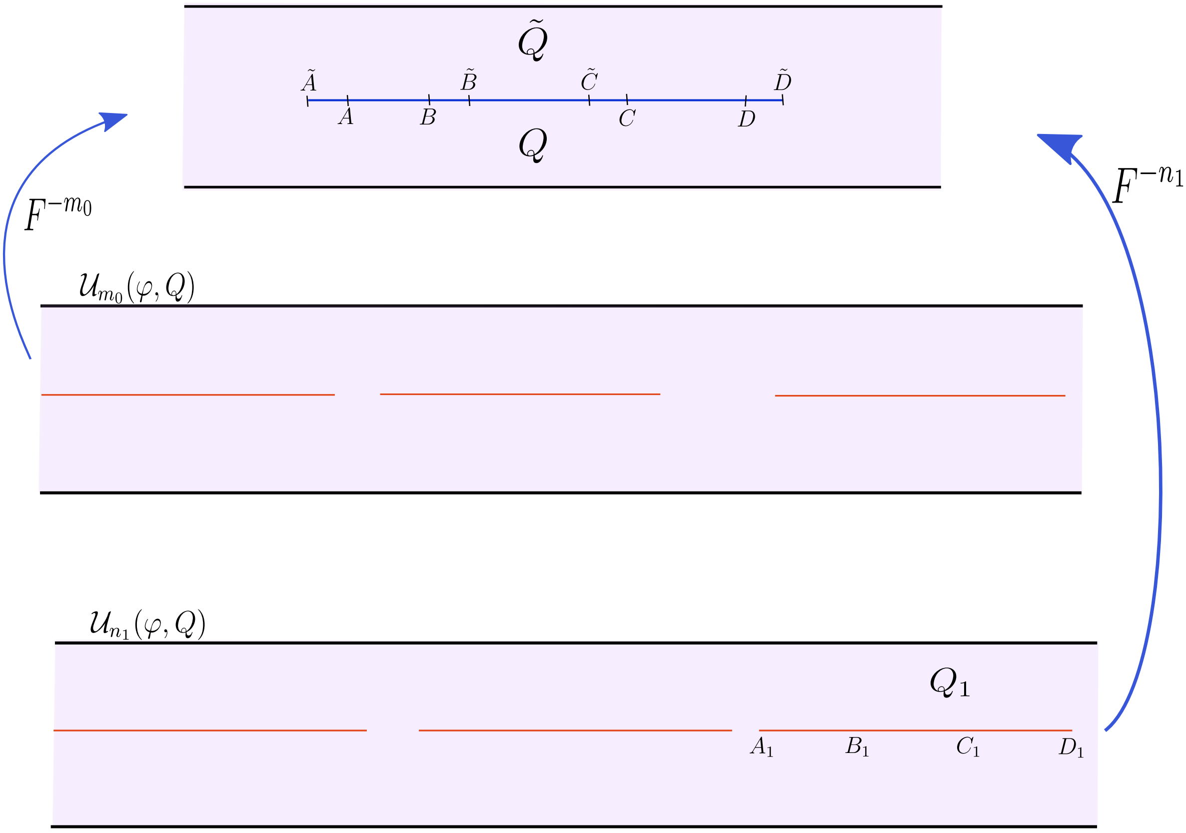


Figure 5.1: Illustration of the setting of $\mathfrak{U}_{m_0}(\varphi, Q)$ and $\mathfrak{U}_{n_1}(\varphi, Q)$. The orange lines represent intervals in their respective sets. Observe that up to this stage, $\tilde{F}$ has not yet been defined. However, we have set Q_1, where a wiggle will be added, and hence $\tilde{T}$ and $\tilde{F}$ will be determined.

We now define $\tilde{F} \in \mathcal{K}$ that is (N, Γ_2)-close to F by setting $r_N(\tilde{F}) := B_1 - \nu_0$ and $R_N(\tilde{F}) := C_1 + \nu_0$. Next, we want to prove the following properties for the one-dimensional projection $\tilde{\varphi}$ of the function $\tilde{F}$.

Claim 1. The following properties hold for φ and $\tilde{\varphi}$:

(1) $\#\mathfrak{U}_n(\tilde{\varphi}, \tilde{Q}) = \#\mathfrak{U}_{m_0}(\tilde{\varphi}, \tilde{Q}) \leq \#\mathfrak{U}_{m_0}(\varphi, Q)$ for $n \geq m_0$.

(2) Every element of $\mathfrak{U}_{m_0}(\tilde{\varphi}, \tilde{Q})$ that contains an element of $\widehat{\mathfrak{U}}_{m_0}(\varphi, Q)$ is in $\widehat{\mathfrak{U}}_{m_0}(\tilde{\varphi}, \tilde{Q})$.

(3) $\mathfrak{U}_1(\tilde{\varphi}, Q_1) = \widehat{\mathfrak{U}}_1(\tilde{\varphi}, Q_1)$.

Proof of Claim. By Proposition 5.1.17, we have that $\#\mathfrak{U}_{m_0}(\tilde{\varphi}, \tilde{Q}) \leq \#\mathfrak{U}_{m_0}(\varphi, Q)$. On the other hand, for $n \geq m_0$ there is no interval of $\mathfrak{U}_n(\varphi, Q)$ which is contained in

$$[r_N(\tilde{F}) - \nu_0, R_N(\tilde{F}) + \nu_0].$$

This will be shown by contradiction. Suppose that there is an interval $J_1 \in \mathfrak{U}_n(\varphi, Q)$ such that $J_1 \subseteq [r_N(\tilde{F}) - \nu_0, R_N(\tilde{F}) + \nu_0]$. Then, (5.10) implies that $J_1 \subseteq [A_1, D_1]$. However $[A_1, D_1] \in \mathfrak{U}_{n_1}(\varphi, Q)$ which is a contradiction to Observation 5.1.7. Further, by Proposition 5.1.17, we also obtain that there is no interval of $\mathfrak{U}_n(\tilde{\varphi}, \tilde{Q})$ contained in $[r_N(\tilde{F}) - \nu_0, R_N(\tilde{F}) + \nu_0]$. Therefore, by Proposition 5.1.14 we have

$$\#\mathfrak{U}_n(\tilde{\varphi}, \tilde{Q}) \leq \#\mathfrak{U}_{m_0}(\tilde{\varphi}, \tilde{Q}) \leq \#\mathfrak{U}_{m_0}(\varphi, Q)$$

for $n \geq m_0$. Hence (1) is proved.

Observe that (2) follows immediately from Proposition 5.1.18.

Finally, let $J_2 \in \mathfrak{U}_1(\tilde{\varphi}, Q_1)$, this implies that $\operatorname{Re} \tilde{F}^{-1}(J_2) = \tilde{\varphi}(J_2) = Q_1$. Then $\tilde{F}^{-1}(J_2)$ is an arc which connects a point at the real part $A_1 < r_N(\tilde{F})$ to a point at the real part $D_1 > R_N(\tilde{F})$ (see Figure 5.2). By the shape of the tract, we can find $x_1, x_2, y \in J_2$ with $x_1 < y < x_2$ such that $\tilde{\varphi}(x_1) = B_1 = \tilde{\varphi}(x_2)$ and $\tilde{\varphi}(y) = C_1$, which means that $\tilde{\varphi}$ does indeed map J_2 crookedly to Q_1. Hence, $J_2 \in \widehat{\mathfrak{U}}_1(\tilde{\varphi}, Q_1)$ and (3) is proved. △

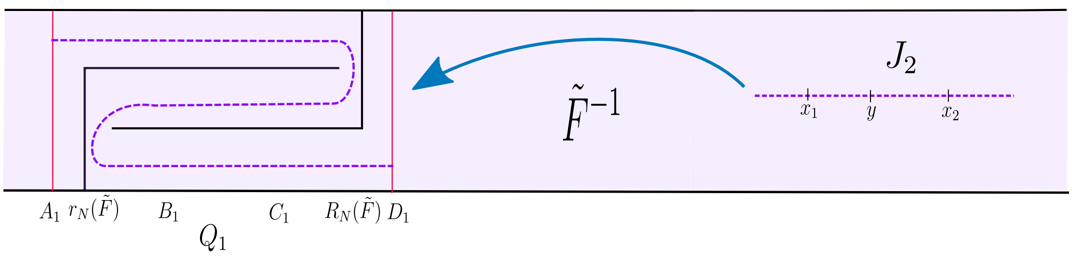


Figure 5.2: Schematic of the proof of (3), where $\tilde{F}$ is defined by adding a wiggle on Q_1

We will now conclude the proof of Proposition 5.2.1 as follows. Let $n > n_1$ and set $n_2 := n - m_0$. Let $J \in \mathfrak{U}_n(\tilde{\varphi}, \tilde{Q})$, note that $\tilde{\varphi}^{n_2}(J) \in \mathfrak{U}_{m_0}(\tilde{\varphi}, \tilde{Q})$, by Lemma 5.1.11(i). On the other hand, there is $I \in \mathfrak{U}_{m_0}(\varphi, Q)$ such that

$$\tilde{\varphi}^{n_2}(J) \supset I,$$

by Proposition 5.1.17. Moreover, by Claim 1(1), there is at most one such J for every I. Now, if $I \in \widehat{\mathfrak{U}}_{m_0}(\varphi, Q)$, then by the above and Claim 1(2), $\tilde{\varphi}^{n_2}(J) \in \widehat{\mathfrak{U}}_{m_0}(\tilde{\varphi}, \tilde{Q})$, and thus $J \in \widehat{\mathfrak{U}}_n(\tilde{\varphi}, \tilde{Q})$ by Lemma 5.1.12.

Claim 2. Let I and J be as above. If $I = \varphi^{n_1 - m_0}([A_1, D_1])$, then $J \in \widehat{\mathfrak{U}}_n(\tilde{\varphi}, \tilde{Q})$.

Proof of Claim. Set $\tilde{I}_l := \tilde{\varphi}^{n_1 - l}([A_1, D_1])$ and $J_l := \tilde{\varphi}^{n-l}(J)$ for $l = m_0, \ldots, n_1$. First, we want to show that

$$\tilde{I}_l \subset \operatorname{int}(J_l) \quad \text{for} \quad l = m_0, \ldots, n_1. \tag{5.11}$$

So we prove (5.11) by induction over l. Consider the case $l = m_0$. By Proposition 4.2.10, we have that for all $t \in [A_1, D_1]$,

$$|\tilde{\varphi}^{n_1 - m_0}(t) - \varphi^{n_1 - m_0}(t)| \leq \varepsilon,$$

and in particular $\tilde{I}_{m_0} \cap I \subset \tilde{I}_{m_0} \cap J_{m_0} \neq \emptyset$.

It follows from the previous argument that $\varphi^{m_0}(\tilde{I}_{m_0}) = \varphi^{n_1}([A_1, D_1])$ which is contained in $[\tilde{A}, \tilde{D}] = \tilde{\varphi}^n(J)$. Therefore, $\tilde{I}_{m_0} \cap J_{m_0} \neq \emptyset$, but it does not contain either of its endpoints. Hence, $\tilde{I}_{m_0} \subset \operatorname{int}(J_{m_0})$.

Now suppose the claim holds for $l < n_1$. Note that the interval

$$\tilde{I}_l \not\subset [r_N(\tilde{F}) - \nu_0, R_N(\tilde{F}) + \nu_0].$$

By Proposition 5.1.14, $\#\mathfrak{U}_1(\tilde{\varphi}, \tilde{I}_l) = 1$. Since both intervals $\tilde{I}_{l+1}$ and J_{l+1} map over $\tilde{I}_l$, then these both intervals contain the unique element of $\mathfrak{U}_1(\tilde{\varphi}, \tilde{I}_l)$, and therefore they have non-empty intersection. On the other hand, the endpoints of J_{l+1} map to the endpoints of J_l under $\tilde{\varphi}$. By the inductive hypothesis, $\tilde{I}_l \subset \operatorname{int}(J_l)$, and therefore $\tilde{I}_{l+1} \subset \operatorname{int}(J_{l+1})$. Hence (5.11) is proved.

In particular $J_{n_1} = \tilde{\varphi}^{n - n_1}(J) \supset [A_1, D_1]$. By Claim 1, it follows that J is mapped crookedly over Q_1 by $\tilde{\varphi}^{n-n_1}$. Now, by choice of Γ_2 and Proposition 4.2.10, one of $\tilde{\varphi}^{n_1}(B_1)$ and $\tilde{\varphi}^{n_1}(C_1)$ is between $\tilde{A}$ and $\tilde{B}$, and the other between $\tilde{C}$ and D. Thus J is mapped crookedly over $\tilde{Q}$ by $\tilde{\varphi}^n$. △

Hence, if J as above that does not belong to $\widehat{\mathcal{U}}_n(\tilde{\varphi}, \tilde{Q})$, then $I \notin \widehat{\mathcal{U}}_{m_0}(\varphi, Q)$ and also $I \neq \varphi^{n_1-m_0}([A_1, D_1])$. There are $k-1$ such intervals I, and therefore there are at most $k-1$ intervals J in $\mathcal{U}_n(\tilde{\varphi}, \tilde{Q}) \setminus \widehat{\mathcal{U}}_n(\tilde{\varphi}, \tilde{Q})$, and this completes the proof. □

Proposition 5.2.2. (Creating crookedness over prescribed quadruples).
Let $F \in \mathcal{K}$ have N wiggles, let Q be a quadruple and $\Gamma > 0$.

Then there is $\tilde{F} \in \mathcal{K}$ with $\hat{N}$ wiggles and $\hat{N} > N$, that is (N, Γ)-close to F, and such that

$$\#\mathcal{U}_n(\tilde{\varphi}, Q) = \#\widehat{\mathcal{U}}_n(\tilde{\varphi}, Q)$$

for some $n \geq 0$.

Proof. Let Q_0 be a quadruple such that $Q_0 \prec Q$, and let $k \geq 0$ such that $\mathcal{U}_n(\varphi, Q_0)$ contains k elements not mapped crookedly, that is,

$$\#\mathcal{U}_n(\varphi, Q_0) - \#\widehat{\mathcal{U}}_n(\varphi, Q_0) = k$$

for all n sufficiently large, as in Proposition 5.2.1. Let

$$Q_0 \prec Q_1 \prec \ldots \prec Q_k = Q$$

be a sequence of quadruples. We now apply Proposition 5.2.1 inductively to obtain a sequence of functions $F_0 = F, F_1, \ldots, F_\ell =: \tilde{F}$ with $\ell \leq k$, and F_j with $(N+j)$ wiggles for $j \in \{0, \ldots, k\}$, such that

$$k_j := \lim_{n\to\infty} \Big(\#\mathcal{U}_n(\varphi_j, Q_j) - \#\widehat{\mathcal{U}}_n(\varphi_j, Q_j)\Big)$$

is strictly decreasing, and $k_\ell = 0$. Therefore, $\tilde{F}$ is the desired function and the proof is complete. □

We are now ready to show one of the main results of this work.

Theorem 5.2.3. (Model function with pseudo-arcs).
There is $F \in \mathcal{K}$ such that the Julia continuum $\widehat{\mathfrak{J}}(F)$ of F is a pseudo-arc. Moreover, F can be chosen such that

$$\liminf_{r\to\infty} \max_{\operatorname{Re}\xi=r} \frac{\log \operatorname{Re} F(\xi)}{r} = \frac{1}{2}. \tag{5.12}$$

Proof. Let $(Q_k)_{k=1}^{\infty}$ be an enumeration of the countably many quadruples stated in the hypotheses of Theorem 3.2.7. We construct inductively a sequence of functions $(F_k)_{k=0}^{\infty}$ that have N_k wiggles, for an increasing sequence $(N_k)_{k\geq 0}$.

Here $N_0 = 0$ and $F_0 \colon S \to \mathbb{H}$, where $S = \{x + iy \colon x > 4,\ |y| < \pi\}$, is the unique conformal isomorphism in $\mathcal{K}$ with $F_0(5) = 5$ and $F_0(\infty) = \infty$. Further, we recursively construct a sequence $(\Gamma_k)_{k=0}^{\infty}$ related to the functions F_k in such a way that

(i) F_{k+1} is (N_k, Γ_k)-close to F_k;

(ii) For every $k \geq 1$, there is some $n_k \in \mathbb{N}$ having the following property: If $F \in \mathcal{K}$ such that F is (N_k, Γ_k)-close to F_k, then the one dimensional projection φ of F has the property $\mathcal{U}_n(\varphi, Q_k) = \widehat{\mathcal{U}}_n(\varphi, Q_k)$ for all $n \geq n_k$.

First, the number Γ_0 can be chosen in an arbitrary manner.

Suppose that F_k and Γ_k have been constructed. We now apply Proposition 5.2.2 to the function F_k, the number Γ_k and a slightly smaller quadruple $\tilde{Q}_{k+1} \prec Q_{k+1}$, to obtain function F_{k+1} that has N_{k+1} wiggles. Furthermore, this function has the property that

$$\mathcal{U}_{n_{k+1}}(\varphi_{k+1}, \tilde{Q}_{k+1}) = \widehat{\mathcal{U}}_{n_{k+1}}(\varphi_{k+1}, \tilde{Q}_{k+1})$$

for some $n_{k+1} \geq 0$. Further, by Proposition 5.1.18 there is $\Gamma_{k+1} > 0$ such that $\mathcal{U}_{n_{k+1}}(\varphi, Q_{k+1}) = \widehat{\mathcal{U}}_{n_{k+1}}(\varphi, Q_{k+1})$ for every F that is (N_{k+1}, Γ_{k+1})-close to F_{k+1}. This concludes the inductive construction.

Now, let $F \in \mathcal{K}$ be the limit of the functions F_k, that is, the tract T of F has $N(F) = \infty$ wiggles, and F is defined by the sequence $(r_{N_k}(F_{k+1}), R_{N_k}(F_{k+1}))_{k=0}^{\infty}$. By Corollary 5.1.13 and property (ii) of the inductive construction, we have obtained

$$\mathfrak{U}_n(\varphi, Q_k) = \widehat{\mathfrak{U}}_n(\varphi, Q_k)$$

for all $n \geq n_k$ and $k \geq 0$. Therefore, F satisfies the hypotheses of Theorem 3.2.7 (see also Remark 5.1.4), and hence $\widehat{\mathfrak{J}}(F)$ is a pseudo-arc. We now show the following.

Claim 1. If Γ_k is chosen sufficiently large, depending on $R_{N_k-1} := R_{N_k-1}(F) = R_{N_k-1}(F_k)$, then F additionally satisfies (5.12).

Proof of Claim. Set $\Gamma_k \geq k \cdot \big(\nu_0/2 + \Lambda(R_{N_k-1}+1) + \nu_0\big)$, where ν_0 is the constant from Proposition 4.1.8 and Λ is the constant from Proposition 4.1.9. Let $a \in T_k$ such that $\operatorname{Re} a = R_{N_k-1} + 1$, and furthermore $|F(a)|$ is maximal. Next, let $\zeta \in T_k$ with $\operatorname{Re}\zeta = \Gamma_k - \nu_0$. We may also assume that Γ_k is chosen so large that $\operatorname{Re}\zeta > \operatorname{Re} a + 2\nu_0$.

On the other hand, we know that $T \cap \{\operatorname{Re} z < \Gamma_k\} = T_k \cap \{\operatorname{Re} z < \Gamma_k\}$ by Remark 4.2.2, this implies that $a, \zeta \in T$. In particular, the vertical geodesic γ_ζ in T through ζ separates the vertical γ_a from infinity, and thus $|F(\zeta)| > |F(a)|$.

Recall that $\Gamma_k \leq r_k$ by definition, this implies that $\Gamma_k - \nu_0 \leq r_k - \nu_0$, and thus a, ζ belong to the straight part of T.

Let $Q \subset T$ be the unique component whose boundary contains both γ_a and γ_ζ. Consider Q as a quadrilateral whose sides are γ_a, γ_ζ and two parts of the boundary of T. Next, we can apply the principal branch of the logarithm, and get a conformal map

$$\log F \colon T \to \big\{x + iy \colon |y| < \pi/2\big\}.$$

Since the geodesics γ_a and γ_ζ under F are semi-circles, and so Q is mapped by F to a quadrilateral that is a half-annulus, then log maps this half-annulus conformally to the rectangle

$$R := \big\{x + iy \colon \log|F(a)| < x < \log|F(\zeta)| \text{ and } |y| < \pi/2\big\}.$$

The conformal modulus of the rectangle R is

$$\operatorname{mod}(R) = \frac{1}{\pi} \log \frac{|F(\zeta)|}{|F(a)|} \tag{5.13}$$

On the other hand, since γ_ζ has diameter at most ν_0 together with the structure of the tract, we have that

$$Q \subset \left\{x + iy \colon 0 < x < \operatorname{Re}\zeta + \nu_0 \ \text{ and } \ |y| < \pi\right\}.$$

Additionally, let us note that every curve connecting the vertical sides of the set $\left\{x + iy \colon 0 < x < \operatorname{Re}\zeta + \nu_0 \ \text{ and } \ |y| < \pi\right\}$ contains a subcurve that connects the vertical sides of Q. Hence, by this observation and the monotonicity of the modulus [Ahl66, Theorem 2, p. 11], we get that

$$\operatorname{mod}(Q) \leq \frac{\operatorname{Re}\zeta + \nu_0}{2\pi}. \tag{5.14}$$

Since the modulus is conformally invariant, that is, $\operatorname{mod}(R) = \operatorname{mod}(Q)$, from (5.13) and (5.14), we obtain

$$\log|F(\zeta)| \leq \frac{\operatorname{Re}\zeta}{2} + \frac{\nu_0}{2} + \log|F(a)| \leq \frac{\operatorname{Re}\zeta}{2} + \frac{\nu_0}{2} + \Lambda \cdot (R_{N_k - 1} + 1). \tag{5.15}$$

So, by the choice of Γ_k, we have obtained that

$$\frac{\nu_0}{2} + \Lambda \cdot (R_{N_{k-1}} + 1) \leq \frac{\Gamma_k - \nu_0}{k} = \frac{\operatorname{Re}\zeta}{k}, \tag{5.16}$$

and thus by (5.15) and (5.16), we get

$$\log|F(\zeta)| \leq \operatorname{Re}\zeta \cdot \left(\frac{1}{2} + \frac{1}{k}\right). \tag{5.17}$$

Set $\rho_k := \Gamma_k - \nu_0$. Therefore, by (5.17), we get

$$\begin{aligned}\liminf_{r\to\infty} \max_{\operatorname{Re}\zeta=r} \frac{\log \operatorname{Re} F(\zeta)}{r} &\leq \liminf_{k\to\infty} \max_{\operatorname{Re}\zeta=\rho_k} \frac{\log|F(\zeta)|}{\rho_k} \\ &\leq \liminf_{k\to\infty} \max_{\operatorname{Re}\zeta=\rho_k} \frac{\operatorname{Re}\zeta}{\rho_k} \cdot \left(\frac{1}{2}+\frac{1}{k}\right) = \frac{1}{2}.\end{aligned}$$

Thus, (5.12) holds as claimed. △

Hence, the proof of Theorem 5.2.3 is complete. □

By applying Theorem 3.1.7, we obtain a disjoint-type entire function g. This function will have an invariant Julia continuum that is a pseudo-arc, because the Julia continuum $\widehat{\mathfrak{J}}(F)$ of the function F from Theorem 5.2.3 is an invariant Julia continuum of the periodic extension $\hat{F}$ of F (see Observation 3.1.6). To show that all Julia continua of g are pseudo-arcs, as claimed in the main theorem, we must show that all Julia continua of $\hat{F}$ are pseudo-arcs. This is attained by the following result.

Theorem 5.2.4. ([Rem16, Corollary 8.7]).
Let $F \in \mathcal{B}^p_{\log}$ be a disjoint type function with a unique tract up to translation of 2π, and such that T has Euclidean bounded decorations.

If one bounded-address Julia continuum of F is a pseudo-arc, then every Julia continuum of F is a pseudo-arc.

Remark. In [Rem16, Section 8], this result is stated for bounded decorations, that is, in Definition 4.1.1 instead to take the Euclidean diameter, it is the hyperbolic diameter in the right half plane. Note that Euclidean bounded decorations together with disjoint type implies that the hyperbolic diameter is bounded. So, having Euclidean bounded decorations implies bounded decorations in the sense of [Rem16], and Theorem 5.2.4 does indeed apply in our setting.

Using these previous results, we are now ready to prove Theorem 1.0.4.

Proof of Theorem 1.0.4. Let $F \in \mathcal{K}$ be as in Theorem 5.2.3, and let $\hat{F}$ be its $2\pi i$-periodic extension. Since the Julia continuum $\widehat{\mathfrak{J}}(F)$ of F is a pseudo-arc, thus by Theorem 5.2.4, every Julia continuum of $\hat{F}$ is a pseudo-arc.

On the other hand, by Theorem 3.1.7, there is a disjoint-type function g such that every Julia continuum of g is homeomorphic to a Julia continuum of $\hat{F}$. Hence, every Julia continuum of $\mathcal{J}(g)$ is also a pseudo-arc.

To complete the proof, we are left to show that $\hat{F}$ has Euclidean bounded decorations. Recall that $\tilde{F}\colon \tilde{T} \to \mathbb{H}_1$ where $\tilde{T} = F^{-1}(\mathbb{H}_1)$, and the periodic extension $\hat{F}\colon \hat{\mathfrak{T}} \to \mathbb{H}_1$ is given by $\hat{F}(z + 2\pi im) = \tilde{F}(z)$ for all $z \in \tilde{T}$, $m \in \mathbb{Z}$, where $\hat{\mathfrak{T}}$ is the $2\pi im$ translations of $\tilde{T}$ (see Observation 3.1.6 for all details). The vertical geodesics in $\tilde{T}$ are the preimages of the semi-circles

$$\Upsilon := \left\{z \in \mathbb{H}_1 \colon |z - 1| = R\right\}$$

for all $R \geq 0$ by $\tilde{F}$. Observe that the semi-circles Υ are not geodesics in $\mathbb{H}$. However, the two endpoints of any geodesic in Υ lie on some geodesic on $\mathbb{H}$, which is also semi-circle centered at 0 of radius $\sqrt{1 + R^2}$. Now, note that hyperbolic distance in $\mathbb{H}$ between any point in Υ and the second semi-circle is at most 1, this implies that the Euclidean distance in T between a point of the vertical geodesic of $\tilde{T}$ and the vertical geodesics in T connecting its endpoints is bounded by 2. Hence $\tilde{F}$ has Euclidean bounded decorations, and thus $\hat{F}$ also has Euclidean bounded decorations and the proof is complete. $\square$

We now conclude with the proof of Theorem 1.0.6

Proof of Theorem 1.0.6. We want to show that the disjoint-type entire function g constructed in the proof of Theorem 1.0.4 has finite lower order. From Theorem 3.1.7, recall that $\Theta = f \circ \psi$, where $f \in \mathcal{B}$, ψ is a quasiconformal map, Θ is a universal covering on the tract of Θ and

$$\Theta(\exp z) = \exp F(z), \tag{5.18}$$

where F is the function from Theorem 5.2.3. Furthermore, we obtain a disjoint-type entire function by setting $g = \lambda f$ for sufficiently small λ.

Since ψ is Hölder continuous at ∞, there is some $p > 1$ such that

$$|z|^{1/p} \leq |\psi(z)| \leq |z|^p \tag{5.19}$$

for sufficiently large z. This also holds for the inverse ψ^{-1}. Then for $z \in \exp(T)$ we define

$$M(r, \Theta) := \max\{|\Theta(z)| \colon |z| \leq r\}.$$

Set $\rho = \log r$, and note that $M(\rho, F) = \max\{\operatorname{Re} F(\xi) \colon \xi \in T \text{ and } \operatorname{Re}\xi = \rho\}$ and thus,

$$\log M(r, \Theta) = M(\rho, F). \tag{5.20}$$

This equality holds by the maximum principle and the fact that the points on ∂T get mapped to imaginary axis by F, so the real part of those points are 0.

Recall that $V(2) = \{z \in V \colon \operatorname{Re}\Theta(z) > e^2\}$, where $V = \exp(\tilde{T})$ (see Theorem 3.1.7 for more details). Further, f is bounded on $\mathbb{C}\setminus\psi(V(2))$ by the aforementioned theorem. So, this allows us to bound $\log\log|f(z)|$ in terms of the maximum modulus $M(r, \Theta)$ for points $z \in \psi(V(2))$ and sufficiently large $|z| = r$, that is, by the above and (5.19) we get

$$\begin{aligned}\log\log|f(z)| &= \log\log|\Theta(\psi^{-1}(z))| \\ &\leq \log\log M(|\psi^{-1}(z)|, \Theta) \\ &\leq \log\log M(|z|^p, \Theta) \\ &\leq \log\log M(r^p, \Theta).\end{aligned} \tag{5.21}$$

Therefore, since f is bounded on $\mathbb{C}\setminus\psi(V(2))$ and by (5.21), it follows that

$$\begin{aligned}\liminf_{r\to\infty}\frac{\log\log M(r, f)}{\log r} &\leq \liminf_{r\to\infty}\frac{\log\log M(r^p, \Theta)}{\log r} \\ &\leq \liminf_{\tilde{r}\to\infty}\frac{\log\log M(\tilde{r}, \Theta)}{\log\tilde{r}}\end{aligned} \tag{5.22}$$

where $\tilde{r} = r^p$. By Theorem 5.2.3 and (5.20), we know that

$$\liminf_{r^p\to\infty}\frac{\log\log M(r^p, \Theta)}{p\log r} = \frac{1}{2}.$$

Since $|g(z)| < |f(z)|$ for all z, then the estimate from (5.22), also holds for g, that is, $M(r,g) \leq M(r^p,\Theta)$ for r sufficiently large. So,

$$\liminf_{r\to\infty} \frac{\log\log M(r,g)}{\log r} = \frac{p}{2} < \infty$$

Hence, g has finite lower order of growth, as required. $\square$

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