Birationally Rigid Complete Intersections of Codimension Three

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Contents

Abstract

In this work, we study the birational geometry of Fano complete intersections of codimension three. In particular, we establish that they are birational superrigid, given certain regularity conditions. We also provide an estimate for the codimension of the set of such complete intersections with non-regular points.

Furthermore, we show, using the $4n^2$-inequality for complete intersection singularities, and the technique of hypertangent divisors, that in the parameter space of $(M+3)$-dimensional Fano complete intersections of codimension three, the codimension of the complement to the set of birationally superrigid complete intersections is at least

$$\frac{(M-10)(M-11)}{2} - 2$$

for $M \geq 30$. We also determine the minimal dimension such that a regular complete intersection V is birationally superrigid given the removal of the last $a \in \{1, 2, 3, 4, 5\}$ hypertangent divisors.

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Introduction

In this thesis, we investigate the birational superrigidity of Fano complete intersections of codimension three in projective space.

A rigorous definition of the term "birational rigidity" first appeared in [22]. Varying definitions occur in the literature, such as those found in [2], [14], and [1]. Birational rigidity was first proven for smooth three-dimensional quartics in $\mathbb{P}^4$ in the paper of V.A. Iskovskikh and Yu.I. Manin on three-dimensional quartics [16]. In the decades since, birational rigidity has been shown in families of non-singular Fano hypersurfaces (see [3], [4], and [23]), non-singular Fano complete intersections (see [25]), and other non-singular varieties (see [1]). Regarding singular Fano varieties, birational rigidity was demonstrated for quartic threefolds with simple singularities in [20] and [31]. Fano hypersurfaces $V^d \subset \mathbb{P}^d$ of index one and degree $d = 4, 5, 6, 7, 8$, were proven to be birationally (super)rigid by [1]. Evans & Pukhlikov proved the birational superrigidity of singular Fano complete intersections of codimension 2 in [6], and they proved the same for codimensions 20 and above in [7]. Furthermore, in [7], the authors discussed complete intersections with multi-quadratic singularities, and introduced the notion of correct multi-quadratic singularities. We adapt these notions to the context of complete intersections of codimension 3 in Section 3.0.1, and in Chapter 3.

In the first chapter, we embark on a review of definitions and terminology from basic algebraic geometry. The chapter begins with a perfunctory introduction to varieties and rational maps, and proceeds to a brief discussion of divisors and intersection theory. All of these concepts are fundamental to the rest of the text.

The second chapter begins with a brief introduction to birational (super)rigidity, and beyond that, it ventures into some of the methods and techniques which we use to demonstrate birational (super)rigidity more generally. The first of these methods, which will be covered in the chapter, one which is fundamental to all of the methods and techniques discussed, is the threshold of canonical adjunction. This leads into a discussion of the method of maximal singularities, the primary method which we use to prove birational (super)rigidity in this text. This includes coverage of the $4n^2$-inequality, and the $4n^2$-inequality for complete intersection singularities. We also introduce the technique of hypertangent divisors, and the connectedness principle of Shokurov & Kollár (see [32], [18]).

The third and final chapter contains Theorem 24, the main result of the thesis. This result establishes that if $V \subset \mathbb{P}^{M+3}$ is a complete intersection of codimension 3 given by three polynomials f_1, f_2 and f_3, then V is birationally superrigid so long as it satisfies certain regularity conditions. The $4n^2$-inequality and the technique of hypertangent divisors feature prominently in the proof. We also provide an estimate for the codimension of the set of such complete intersections with non-regular points. Using the projection method (see [26], Chapter 3), we show that for M large enough, the codimension of this set is given by a quadratic polynomial in M. In particular, we are interested in determining the minimal dimension M_i, for some $i \in \{0, 1, 2\}$ (see Section 3.3), such that for $M \geq M_i$, a complete intersection V satisfying said regularity conditions is birationally superrigid (given the removal of a certain number of hypertangent divisors).

Chapter 1

Background

The main purpose of this chapter is to perform a brief review of some fundamental topics of algebraic geometry which will feature prominently in the rest of the text. For brevity, we assume that the reader is familiar with some introductory concepts in algebraic geometry which are covered in [30], [11], [33], [15] and many other texts. Everywhere in this text, we work over the field of complex numbers $\mathbb{C}$, and unless otherwise specified, the term "variety" refers to an irreducible projective variety.

1.1 Rational Varieties & Maps

Definition 1 (Regular Map of Affine Varieties)**.** Let $X \subset$ and Y be closed algebraic subsets of $\mathbb{A}^n$ and $\mathbb{A}^m$. A function from X to Y is a *regular map* if there exist $f_1, \dots, f_m \in \mathbb{C}[X]$, such that

$$\varphi(\underline{x}) = \Big(f_1(\underline{x}), \dots, f_m(\underline{x})\Big),$$

for any $\underline{x} \in X \subset \mathbb{A}^n$.

Definition 2 (Regular Map of Projective Varieties)**.** Let $X \subset$ and Y be closed algebraic subsets of $\mathbb{P}^n$ and $\mathbb{P}^m$. A function from X to Y is a *regular map* if there exist homogeneous polynomials $f_1, \dots, f_m \in \mathbb{C}[X]$, such that

$$\varphi(\underline{x}) = \Big(f_1(\underline{x}) : \dots : f_m(\underline{x})\Big),$$

for any $\underline{x} \in X \subset \mathbb{P}^n$.

Definition 3 (Rational Map)**.** Let X and Y be varieties. A rational map f from X to Y, written $f : X \dashrightarrow Y$, is a morphism $f : U \to Y$ from a non-empty open subset $U \subset X$ to Y.

Remark 1. A rational map $f : X \dashrightarrow Y$ is an equivalence class of morphisms from non-empty open subsets of X to Y. Two rational maps $f_1 : U_1 \to Y$ and $f_2 : U_2 \to Y$ with $U_1,\ U_2 \subset X$ are considered equivalent if

$$f_1|_{U_1 \cap U_2} = f_2|_{U_1 \cap U_2}.$$

Definition 4 (Rational Variety)**.** A variety $X \subset \mathbb{P}^N$, where $N = \dim X$, is said to be *rational* if either of the following equivalent conditions holds:

(i) there exists a non-empty Zariski open set $U \subset X$ which is isomorphic to an open subset $V \subset \mathbb{P}^N$;

(ii) the field $\mathbb{C}(X)$ of rational functions on the variety X is isomorphic to the field of rational functions in N independent variables over $\mathbb{C}$. That is, $\mathbb{C}(X) \cong \mathbb{C}(t_1, \ldots, t_N)$.

Definition 5 (Birational Varieties)**.** We say that a variety X and is *birational/birationally equilvalent* to Y (and vice versa) if one of the following conditions holds:

(i) there are rational maps $X \dashrightarrow Y$ and $Y \dashrightarrow X$, which are inverse to each other;

(ii) there are subsets $U \subset X$ and $V \subset Y,$ such that $U \cong V$;

(iii) the function fields $\mathbb{C}(X)$ and $\mathbb{C}(Y)$ are isomorphic.

Remark 2. Note that by combining Definition 4 (i) and Definition 5 (ii), it is clear that a variety is rational iff it is birational to $\mathbb{P}^n$ for some n.

Definition 6 (Rational Connectedness)**.** We say that a variety $X \subset \mathbb{P}^N$ is *rationally connected* if any two points $x_1, x_2 \in X$ in general position, can be connected by some irreducible rational curve. That is, there is a map $\mathbb{P}^1 \to X$ such that

$$f(t_1) = x_1 \text{ and } f(t_2) = x_2,$$

for some $t_1, t_2 \in \mathbb{P}^1$.

Definition 7 (Non-singular Variety)**.** Let X be a projective variety, and $x \in X$ be a (closed) point. Also let $\mathcal{O}_{X,x}$ be the local ring of functions at x with residue field κ and maximal ideal $\mathfrak{m}_x$. We say that X is non-singular at x if

$$\dim_\kappa \mathcal{O}_{X,x} = \dim \mathfrak{m}_x/\mathfrak{m}_x^2.$$

The variety X is said to *non-singular* if it is non-singular at every point $x \in X$.

Definition 8 (Exceptional Subvariety)**.** Let $f : X \dashrightarrow Y$ be a regular birational map. A subvariety $Z \subset X$ is *exceptional* with respect to f if $\operatorname{codim} Z = 1$, but $\operatorname{codim} f(Z) \geq 2$.

Definition 9 (Centre of an Exceptional Divisor)**.** Let X be a normal variety and $f : Y \dashrightarrow X$ be a birational morphism. A prime divisor $E \subset Y$ is called a *divisor over* X. The closure $\overline{f(E)}$ of its image is called the *centre* of E in X.

Definition 10 (Birational Contraction)**.** A birational map $f : X \dashrightarrow S$ *contracts* a (prime) divisor $D \subset X$ if f is defined at the generic point of D and $f(D) \subset S$ is of codimension ≥ 2. The map f is called a *birational contraction* if f^{-1} does not contract any divisor.

Theorem 1. *Let $f : X \to Y$ be a regular birational map. For $x \in X$, assume that $y = f(x)$ is a non-singular point of Y and that the inverse map $g = f^{-1}$ is not regular at y. Then there is an exceptional subvariety $Z \subset X$ with $Z \ni x$.*

Proof. See Theorem 2.16 of Section 4.4 in Chapter 1 of [30]. □

Corollary 1. If $f : X \to Y$ is a regular birational map between non-singular varieties which is not an isomorphism, then f has an exceptional subvariety.

Blowing Up

Blowing up is a ubiquitous process in birational geometry. It plays many different roles in algebraic geometry, and it is fundamental to the resolution of singularities, and the resolution of the indeterminacy of rational maps.

Example 1 (Blowing Up a Point in $\mathbb{A}^n$). After setting a coordinate system to establish an origin $o = \{x_1 = \cdots = x_n = 0\}$, consider for $\lambda \in \mathbb{C}$, the set

$$\left\{ \Big((x_1, \ldots, x_n), (y_1 : \cdots : y_n) \Big) \;\Big|\; (x_1, \ldots, x_n) = \lambda(y_1, \ldots, y_n) \right\} \subset \mathbb{A}^n \times \mathbb{P}^{n-1}.$$

Assuming that $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \neq (0, \ldots, 0)$, this occurs when there exists some $\lambda \in \mathbb{C}$ such that $x_i = \lambda y_i$ for all $i = 1, \ldots, n$. Furthermore, if $y_j \neq 0$, then $x_i y_j = x_j y_i$ for all i and j such that $y_j \neq 0$. The blow-up $\widetilde{\mathbb{A}^n} \subset \mathbb{P}^{n-1} \times \mathbb{A}^n$ is given by

$$\{x_i y_j = x_j y_i \mid i, j = 1, \ldots, n\}.$$

We have a so-called exceptional divisor $E \cong \mathbb{P}^{n-1} \times \{o\}$, and there is an isomorphism between $\mathbb{A}^n \backslash \{o\}$ and $\widetilde{\mathbb{A}^n} \backslash E$.

Other Examples of Birational Maps.

Example 2 (Cremona Transformation in $\mathbb{P}^2$). Let $P_1 = (1 : 0 : 0), P_2 = (0 : 1 : 0)$, and $P_3 = (0 : 0 : 1)$ be points in $\mathbb{P}^2$ over $\mathbb{C}$. Let U be the open set $\mathbb{P}^2 \backslash \{P_1, P_2, P_3\}$. The map

$$f : U \longrightarrow \mathbb{P}^2$$

$$(x : y : z) \longmapsto (xy : xz : yz)$$

is called the Cremona transformation of the plane $\mathbb{P}^2$.

Example 3 (Rational Functions). A rational map $f : X \to \mathbb{A}^1 = k$ is a *rational function* on X. Hence a rational function is given by a function $\varphi \in \mathcal{O}_X(U)$ on some non-empty open subset $U \subset X$, where two such regular functions define the same rational function if and only if they agree on a non-empty open subset(see Remark 1).

1.2 Complete Intersections

Definition 11 (Complete Intersection). A subvariety $X = X_{d_1 \cdots d_r} \subset \mathbb{P}^n$ is said to be a *complete intersection of varieties* if and only if the homogeneous ideal I of X is generated by $r = \operatorname{codim} X$ homogeneous polynomials $f_1, f_2, \ldots, f_r$. In some neighbourhood of a point $x \in X \subset \mathbb{P}^n$, the complete intersection X can be written as $V(f_1, \ldots, f_r)$. That is, it is the intersection of r hypersurfaces $\{f_i = 0\}$ in $\mathbb{P}^n$.

Suppose that x is a singular point on a complete intersection $X \subset \mathbb{P}^n$ of codimension r. We say that x is a *singularity of type* $\underline{\mu} = (\mu_1, \ldots, \mu_r)$ if in some neighbourhood of x, the polynomials $f_1, \ldots, f_r$ can be written as

$$\begin{aligned} f_1 &= f_{1,\mu_1} + f_{1,\mu_1+1} + \cdots + f_{1,d_1} \\ f_2 &= f_{2,\mu_2} + f_{2,\mu_2+1} + \cdots + f_{2,d_2} \\ &\vdots \\ f_r &= f_{r,\mu_r} + f_{r,\mu_r+1} + \cdots + f_{r,d_r}, \end{aligned}$$

where f_{i,μ_i+j} is a homogeneous polynomial, and $d_1, \ldots, d_r$ are the degrees of $f_1, \ldots, f_r$ respectively. Note that this definition depends on the polynomials $f_1, \ldots, f_r$.

Remark 3. By imposing certain conditions on the leading terms f_{i,μ_i}, this definition can be made invariant. For example, if $\mu_1 = \cdots = \mu_r = 1$, and the linear forms $f_{i,1}$ are linearly independent, then the point is non-singular. We postpone these additional considerations until Chapter 3.

Definition 12 (Isolated Singularity)**.** A singular point x in a variety X is said to be *isolated* if there is a neighbourhood of x which contains no other singular point.

1.3 Divisors

Definition 13 (Prime Divisor)**.** A prime divisor on X is an irreducible subvariety of codimension 1.

Definition 14 (Weil Divisor)**.** Let X be an irreducible variety, and let $\mathrm{Div}(X)$ denote the free group generated by all prime divisors of X. A collection of irreducible closed subvarieties $D_1, \ldots, D_r$ of codimension 1 in X with respective multiplicities $a_1, \ldots, a_r$ is called a *Weil divisor*, or simply, a *divisor* on X.

A divisor D is written

$$D = \sum_{i=1}^{r} a_i D_i. \tag{1.1}$$

If $a_i \geq 0$ for all i, then D is said to be *effective*. In addition, if all the $a_i \neq 0$ for all the a_i, then the variety

$$\bigcup_{a_i \neq 0} D_i$$

is called the *support* of D and is denoted by $\mathrm{Supp}(D)$.

Definition 15 (Local Equations of a Subvariety)**.** Let Y be a subvariety of X, and let x be a point in Y. Functions $f_1, \ldots, f_n \in \mathcal{O}_x$ are called local equations of Y if there exists an affine neighbourhood U of x such that $f_1, \ldots, f_m \in \mathbb{C}[U]$, and the ideal $\mathfrak{a}_V = (f_1, \ldots, f_m)$ in $\mathbb{C}[U]$, where $V = Y \cap U$.

Consider the ideal $\mathfrak{a}_{Y,x} \subset \mathcal{O}_x$ which consists of functions $f \in \mathcal{O}_x$ that vanish on Y in some neighbourhood of x. For an affine variety X,

$$\mathfrak{a}_{Y,x} = \left\{ f = \frac{u}{v} \,\middle|\, u, v \in \mathbb{C}[X] \text{ with } u \in \mathfrak{a}_Y \text{ and } v(x) \neq 0 \right\},$$

and if all components of Y pass through x, then $\mathfrak{a}_Y = \mathfrak{a}_{Y,x} \cap \mathbb{C}[X]$.

Lemma 2. *Functions $f_1, \dots, f_m \in \mathcal{O}_x$ are local equations of Y in a neighbourhood of x if and only if $\mathfrak{a}_{Y,x} = (f_1, \dots, f_m)$.*

Proof. See page 107 of [30]. □

Theorem 3. *An irreducible subvariety $Y \subset X$ of codimension 1 is defined by a local equation in a neighbourhood of any non-singular point $x \in X$.*

Proof. See page 107 of [30]. □

Lemma 4. *Let A be a Noetherian ring, and $\mathfrak{a} \subset A$ be an ideal such that every element of $1 + \mathfrak{a}$ is invertible in A. Then*

$$\bigcap_{n>0} (\mathfrak{b} + \mathfrak{a}^n) = \mathfrak{b}$$

for an arbitrary ideal $\mathfrak{b} \subset A$.

Proof. See [30], Appendix 6. □

Definition 16 (Simple Normal Crossing Divisor)**.** Let X be a smooth variety of dimension n. A Weil divisor $D = \sum_i D_i$ on X is a *simple normal crossing divisor* if each D_i is smooth, and for every $p \in X$, D is cut out by $x_1 \cdots x_r$ where $x_1, \dots, x_r$ are independent local parameters (see Definition 34) in $\mathcal{O}_{X,p}$ for some $r \leq n$.

Definition 17 (Normal Crossing Divisor)**.** A Weil divisor $D = \sum_i D_i$ on X is a *normal crossing divisor* if for every $x \in D$, the local ring $\mathcal{O}_{X,x}$ is regular, and there exist $x_1, \dots, x_n \in I(x)$, and $1 \leq r \leq n$ such that D is cut out by $x_1 \cdots x_n \in \mathcal{O}_{X,x}$.

1.3.1 The Order of Vanishing of a Function along a Prime Divisor

Definition 18 (Discrete Valuations & Valuation Rings)**.** Let X be a variety. A *discrete valuation* on $\mathbb{C}(X)$ is a surjective function

$$v : \mathbb{C}(X)^* \to \mathbb{Z}$$

such that

(i) v is a ring homomorphism.

(ii) $v(x+y) \geq \min(v(x), v(y))$, where $x + y \neq 0$.

Furthermore, we can extend the function v to $0 \in \mathbb{C}(X)$ by setting $v(0) = \infty$. This extension satisfies the two conditions above, and

$$R = \{x \in \mathbb{C}(X) \mid v(x) \geq 0\}$$

is called the *valuation ring* of v.

If D is a prime divisor, then we associate it with some integer $v_D(f)$. Supposing that X is non-singular in codimension 1, then $\operatorname{Sing} X$ is of codimension ≥ 2.

Furthermore, let D be an irreducible codimension 1 subvariety of X, and let U be some affine open set intersecting D which consists only of non-singular points. Then by Theorem 3, D is defined in U by a local equation which we will denote by π. Thus by Lemma 2, $\mathfrak{a}_D = (\pi)$ in $\mathbb{C}[U]$. And for any regular function $f \neq 0$ on U, there exists a unique integer $r > 0$ such that $f \in (\pi^r)$ and $f \neq (\pi^{r+1})$. If this were not the case, then $f \in \bigcap_r (\pi)^r$, which would imply by Lemma 4, that $f = 0$. The function

$$\nu_D : \mathbb{C}^* \longrightarrow \mathbb{Z}$$

$$f \longmapsto r$$

has the properties of a discrete valuation. Consequently, if X is irreducible, then any function $f \in \mathbb{C}(X)$ can be written in the form $\frac{g}{h}$ with g, h regular on U. In particular, if

$f \neq 0$, then we set

$$\nu_D(f) = \nu_D(g) - \nu_D(h).$$

Definition 19 (Order of Vanishing)**.** If $\varphi \in \mathbb{C}(X)$, then the order of vanishing of φ along a divisor D on X, denoted by $v_D(\varphi)$, is the unique integer $r \geq 0$ such that $\varphi \in (\pi)^r \backslash (\pi^{r+1})$.

Remark 4. $v_D(\varphi) = 0 \Longrightarrow \varphi \in (\pi^0)\backslash(\pi^1) = \mathbb{C}[X]\backslash(\pi)$.

Definition 20 (Principal Weil Divisor)**.** A *principal Weil divisor* is a divisor D of the form

$$D = \sum \nu_C(f)C,$$

for some $f \in \mathbb{C}(X)$, where the sum is over all prime divisors C of a variety X. It may also be called the *divisor of* f, and is denoted by $\operatorname{div}(f)$. We denote the set of all principal divisors of X by $\operatorname{Prin} X$.

Lemma 5. $\operatorname{Prin} X$ *is a subgroup of* $\operatorname{Div} X$.

Proof. This follows from the fact that $\operatorname{div} fg = \operatorname{div} f + \operatorname{div} g$ and $\operatorname{div} f^{-1} = -\operatorname{div} f$. □

Definition 21 (Linear Equivalence of Divisors)**.** Two divisors D_1 and D_2 are *linearly equivalent* if their difference is principal. We will denote a linear equivalence between D_1 and D_2 by $D_1 \equiv D_2$.

Definition 22 (Divisor Class Group)**.** The quotient group consisting of all divisors of X modulo linear equivalence is called the *divisor class group*, and is denoted by $\operatorname{Cl} X$. An element of $\operatorname{Cl} X$ is called a divisor class.

Definition 23 (Refinement of an Open Cover)**.** Let T be a topological space, and let $\{U_i \subset T\}_{i\in I}$ be an open cover of T. Then a *refinement* of this open cover is a set of open subsets $\{V_j \subset T\}_{j\in J}$ which is an open cover in itself, and such that for each $j \in J$ there exists an $i \in I$ with $V_j \subset U_i$.

Definition 24 (Cartier Divisor)**.** Let $\{U_i\}$ be an open cover of an irreducible variety X. A *Cartier divisor* or *locally principal divisor* on X denoted by $\{(U_i, f_i)\}_i$ is a collection $\{f_i\}$ of rational functions, each defined on some U_i, such that:

i) f_i is not identically zero for all i.

ii) f_i/f_j and f_j/f_i are both regular on $U_i \cap U_j$

Each function f_i is called a *local equation* for D at any point $x \in U_i$. Local equations $\{f_i\}$ and $\{f_j\}$ on open sets U_i, and U_j define the same Cartier divisor on $U_i \cap U_j$ if f_i/f_j and f_j/f_i are regular on $U_i \cap U_j$.

Remark 5. The Cartier divisors on a variety X form an abelian group: for two Cartier divisors we can take a common refinement of the open covers and take the product of the functions which define the divisors.

Definition 25 (Picard Group)**.** Let X be a quasi-projective variety. The *Picard group* $(\operatorname{Pic} X, +)$ of X is the quotient group of the group of *Cartier divisors modulo linear equivalence.*

Definition 26 (Ample & Very Ample Divisors)**.** We say that a divisor D on a projective variety X is *very ample* if there is a closed embedding $X \subset \mathbb{P}^N$ into projective space such that D is linearly equivalent to a hyperlane section of X. Furthermore, a divisor D on X is *ample* if mD is *very ample* for some $m > 0$.

Definition 27 ($\mathbb{Q}$-divisors)**.** A $\mathbb{Q}$-linear combination of prime divisors is called a $\mathbb{Q}$-*divisor.*

Definition 28 ($\mathbb{Q}$-Cartier Divisor)**.** A $\mathbb{Q}$-divisor D is said to be $\mathbb{Q}$-*Cartier* if mD is a Cartier divisor, for some $m \in \mathbb{N}$.

Definition 29 ($\mathbb{Q}$-factorial singularities)**.** A variety X has $\mathbb{Q}$-factorial singularities if all Weil divisors on X are $\mathbb{Q}$-Cartier.

Definition 30 (Multiplicity at a Point)**.** Let $F \subset \mathbb{A}^n$ be a hypersurface, such that in some neighbourhood of a point $x \in F$, it is given by the zero set of a regular function f. Then in that neighbourhood, we can choose local coordinates $z = (z_1, \ldots, z_n)$ on X in a neighbourhood of x, and write the decomposition

$$f(z) = f_0 + f_1(z) + f_2(z) + f_3(z) + \ldots$$

where $f_i(z)$ is a homogeneous polynomial of degree i. The hypersurface F has *multiplicity* m at x, denoted by $\operatorname{mult}_x X$, if

$$f_0 = f_1(z) = f_2(z) = \cdots = f_{m-1}(z) = 0,$$

and $f_m \neq 0$.

Definition 31 (Multiplicity Along a Subvariety)**.** If Z and Y are subvarieties of F, then *the multiplicity of Z along Y* is

$$\operatorname{mult}_Z Y = \min\Big\{ \operatorname{mult}_x Y \mid x \in Z \Big\},$$

where $\operatorname{mult}_x Y$ is the multiplicity of Y at x.

1.3.2 Differential Forms in Affine Space

Let x be a point in $\mathbb{A}^n$ for some positive integer n. Consider the action of vectors $\underline{v} = (v_1, \ldots, v_n)$ at x on functions $f \in \mathcal{O}_{\mathbb{A}^n,x}$:

$$\underline{v}(f) = v_1 \frac{\partial f}{\partial z_1}(x) + v_2 \frac{\partial f}{\partial z_2}(x) + \cdots + v_n \frac{\partial f}{\partial z_n}(x).$$

Then there is a linear differential operator

$$\left(\underline{v}_1 \frac{\partial f}{\partial z_1} + v_2 \frac{\partial f}{\partial z_2} + \cdots + v_n \frac{\partial f}{\partial z_n}\right) f \bigg|_{z=p}.$$

This operator obeys the following properties:

(1) Linearity: $\underline{v}(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 \underline{v}(f_1) + \lambda_2 \underline{v}(f_2)$ for all $\lambda_1, \lambda_2 \in \mathbb{C}$.

(2) The Leibnitz rule: $\underline{v}(f_1 f_2) = f_1(x)\underline{v}(f_2) + f_2(x)\underline{v}(f_1)$.

Definition 32 (Tangent Space)**.** The tangent space to a variety X is the locus of points on lines tangent to X at x. We will denote it by $T_x X$.

Theorem 6. *The tangent space $T_x\mathbb{A}^n$ of $\mathbb{A}^n$ at a point x is the set of maps $\{\underline{v} : \mathcal{O}_{\mathbb{A}^n,x} \to \mathbb{C}\}$ satisfying the above properties.*

Proof. It is apparent that $T_x\mathbb{A}^n$ is contained in the set of all such functions, so we will focus on showing that all such functions exist in the tangent space.

Let $\underline{v} : \mathcal{O}_{\mathbb{A}^n,x} \longrightarrow \mathbb{C}$ be a map satisfying the two properties above. If $f = 1$, then by Leibnitz rule,

$$\underline{v}(1) = \underline{v}(1^2) = 1 \cdot \underline{v}(1) + 1 \cdot \underline{v}(1) = 2\underline{v}(1)$$

$$\Rightarrow \underline{v}(1) = 0 \Rightarrow \underline{v}(f) = 0 \text{ for any } f \in \mathbb{C}.$$

Again, by Leibnitz rule, note that

$$f, g \in \mathfrak{m}_x \text{ (that is to say that } f(x) = g(x) = 0) \Longrightarrow \underline{v}(fg) = 0, \text{ so } \underline{v}(\mathfrak{m}_x^2) = 0.$$

Assuming for simplicity that $x = (0, \ldots, 0)$, write for $g, h \in \mathbb{C}[z_*]$

$$\begin{aligned} g(z_*) &= g_0 + g_1(z_1, \ldots, z_n) + g_{\geq 2}(z_1, \ldots, z_n) \\ h(z_*) &= h_0 + h_1(z_1, \ldots, z_n) + h_{\geq 2}(z_1, \ldots, z_n), \end{aligned}$$

where

- g_0, h_0 are constants,
- g_1 and h_1 are linear terms, and
- g_2 and h_2 are linear combinations of some monomials of degree at least 2.

Furthermore, for $h_0 = h(x) \neq 0$, we have

$$f = \frac{g}{h} = \frac{g_0}{h_0} + \frac{1}{h_0^2}(g_1 h_0 - g_0 h_1) + (\text{a rational function in } \mathfrak{m}_x^2).$$

Then it follows that

$$f(x) = \frac{g_0}{h_0} \text{ and } \frac{1}{h^2}(g_1 h_0 - g_0 h_1) = \sum_{i=1}^{n} \frac{\partial f}{\partial z_i}(x) z_i,$$

so

$$f(z) = f(x) + \sum_{i=1}^{n} \frac{\partial f}{\partial z_i}(x) z_i + f_{\geq 2}(z), \text{ with } f_{\geq 2} \in \mathfrak{m}_x^2.$$

Consequently,

$$\underline{v}(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial z_i}(x) \underline{v}(z_i).$$

Set $\underline{v}(z_i) = v_i \in \mathbb{C}$ to complete the proof. □

Remark 6. If $x = (a_1, \ldots, a_n)$, then $v_i = \underline{v}(z_i - a_i)$.

Remark 7. For any subvariety $X \subset \mathbb{A}^n$, $T_x X = \{\underline{v} : \mathcal{O}_{\mathbb{A}^n,x} \to \mathbb{C}\}$ satisfies linearity and the Leibnitz rule for $x \in X$.

Definition 33 (The Cotangent Space)**.** The dual vector space of $T_x\mathbb{A}^n$ is called the *cotangent space*, and it is denoted by $T_x^*\mathbb{A}^n = (T_x\mathbb{A}^n)^*$. For $f \in \mathcal{O}_{\mathbb{A}^n,x}$, set $df \in T_x^*\mathbb{A}^n$ or $df(x)$ as

$$df(\underline{v}) = \underline{v}(f)$$

for all $\underline{v} \in T_x\mathbb{A}^n$.

Remark 8. The elements of $T_x^*\mathbb{A}^n$ are called *covectors*. Furthermore, there is a natural isomorphism $T_x^*\mathbb{A}^n \cong \mathfrak{m}_x/\mathfrak{m}_x^2$, and in turn there is a natural isomorphism $T_x\mathbb{A}^n \cong (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$.

Theorem 7.

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial z_i}(x)\; dz_i.$$

Proof. It would be necessary to show that for every vector $\underline{v}$, the LHS and RHS give the same number when applied to $\underline{v}$. It is enough to check this for $\underline{v} = \frac{\partial}{\partial z_i}$ (a basis) when it is clear. □

Note that df satisfies linearity and the Leibnitz rule:

(1) Linearity: $d(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 df_1 + \lambda_2 df_2$.

(2) The Leibnitz rule: $d(fg)(x) = f(x)dg(x) + g(x)df(x)$.

Definition 34 (Local Parameters)**.** Regular functions $u_1, \ldots, u_{n-1} \in \mathcal{O}_{X,x}$ form a *system of local parameters* at the point x if

$$u_1(x) = \ldots u_{n-1}(x) = 0$$

and

$$du_1(x), \ldots, du_{n-1}(x)$$

form a basis in T_x^*X.

Definition 35 (Regular 1-form)**.** If $U \subset \mathbb{A}^n$ be a Zariski open set, then a regular 1-form on U is a map

$$\omega : U \to \bigsqcup_{x \in U} T_x^*\mathbb{A}^n$$
$$x \longmapsto \omega(x) \in T_x^*\mathbb{A}^n,$$

where

$$\omega = \alpha_1(z_1, \ldots, z_n)dz_1 + \cdots + \alpha_n(z_1, \ldots, z_n)dz_n,$$

and $\alpha_1, \ldots, \alpha_n$ are regular functions on U.

More generally: let $F \subset \mathbb{A}^n$ be a non-singular hypersurface. That is, that $F = \{f = 0\}$, and $f \in \mathbb{C}[z_1, \ldots, z_n]$. Also set $U_i = \left\{x \in \mathbb{A}^n \;\middle|\; \frac{\partial f}{\partial z_i} \neq 0\right\}$, for $i = 1, \ldots, n$. If x is a point $x = (a_1, \ldots, a_n) \in U_i \cap F$, the functions $(z_j - a_j)$, for $j \neq i$, form a system of local parameters on F at x. Consequently, the wedge products

$$dz_{j_1} \wedge \cdots \wedge dz_{j_k},$$

where $k \leq n-1$ and $1 \leq j_1 < \cdots < j_k \leq n$ (none of the j_α is i), form a basis of $\bigwedge^k T_x^*F$.

Definition 36 (Regular k-form)**.** A regular k-form ω on X is a map

$$\omega : U \longrightarrow \bigsqcup_{x \in U} \bigwedge^k T_x^*X$$

$$x \longmapsto \omega(x) \in \bigwedge^k T_x^*X,$$

such that $\omega(x)$ can be written as

$$\sum_{1 \leq j_1 \leq \cdots \leq j_k \leq n} c_{i,j_1,\ldots,j_k}(z_1, \ldots, z_n)\, dz_{j_1} \wedge \ldots dz_{j_k},$$

where none of the j_α is i, and $c_{i,j_1,\ldots,j_k}$ is a regular function on $U_i \cap X$. We denote the vector space of k-forms on U by $\Omega^k[U]$.

1.3.3 Differential Forms in Projective Space

In this section, let $X \subset \mathbb{P}^n$ be a non-singular projective variety, and recall that

$$\mathbb{P}^n = \bigcup_{i=0}^{n} U_i, \text{ where } U_i = \{x_i \neq 0\} = \mathbb{A}^n.$$

Definition 37 (Regular Differential Forms)**.** A regular k-form ω on X is a map

$$\omega : X \longrightarrow \bigsqcup_{x \in X} \bigwedge^k T_x^* X$$

$$\omega : x \longmapsto \omega(x) \in \bigwedge^k T_x^* X,$$

such that on each open set $U_i \cap X$ it can be written as

$$\sum_{1 \leq j_1 \leq \cdots \leq j_k \leq n} c_{i,j_1,\ldots,j_k}\, dz_{j_1} \wedge \ldots dz_{j_k},$$

where none of the j_α is i and $c_{i,j_1,\ldots,j_k}$ is a regular function on $U_i \cap X$.

Remark 9. Regular differential forms on X form a module over $\mathbb{C}[X]$ which we denote by $\Omega[X]$.

Lemma 8. *Every non-singular point $x \in X$ of an n-dimensional variety has a neighbourhood U such that $\Omega^k[U]$ is a free $\mathbb{C}[U]$-module of rank $\binom{n}{k}$.*

Proof. See Page 196 of [30]. □

By Lemma 8, the module $\Omega^n[U]$ is of rank 1 over $\mathbb{C}[U]$. Consequently, if $\omega \in \Omega^n[U]$, then

$$\omega = g du_1 \wedge \cdots \wedge du_n, \text{ where } g \in \mathbb{C}[U]. \tag{1.2}$$

If $v_1, \ldots, v_n$ are regular functions on X such that $v_1 - v_1(x), \ldots, v_n - v_n(x)$ are local parameters at any $x \in U$, then

$$\Omega^1[U] = \mathbb{C}[U]dv_1 \oplus \cdots \oplus \mathbb{C}[U]dv_n.$$

In particular,

$$du_i = \sum_{j=1}^{n} h_{ij} dv_j \text{ for } i = 1, \ldots, n. \tag{1.3}$$

As $d_x u_1, \ldots, d_x u_n$ constitute a basis of the cotangent space $(T_x \mathbb{A}^n)^*$ for each $x \in U$, we see from Equation 1.3 that $\det |h_{ij}(x)| \neq 0$. This determinant is the *Jacobian determinant* of the functions $u_1, \ldots, u_n$ with respect to $v_1, \ldots, v_n$, and we denote it by $J\left(\frac{u_1,\ldots,u_n}{v_1,\ldots,v_n}\right)$. Furthermore, note that for all $x \in U$,

$$J\left(\frac{u_1, \ldots, u_n}{v_1, \ldots, v_n}\right) \in \mathbb{C}[U], \quad \text{and} \quad J\left(\frac{u_1, \ldots, u_n}{v_1, \ldots, v_n}\right)(x) \neq 0.$$

Substituting Equation 1.3 into Equation 1.2,

$$\omega = gJ\left(\frac{u_1, \ldots, u_n}{v_1, \ldots, v_n}\right) dv_1 \wedge \cdots \wedge dv_n. \tag{1.4}$$

Definition 38 (Rational k-form)**.** Suppose we have a pair (ω, U) consisting of an k-form $\omega \in \Omega^k[X]$ and an open set $U \subset X$. We introduce an equivalence relation such that $(\omega, U) \sim (\omega', U')$ if $\omega = \omega'$ on $U \cap U'$. An equivalence class under this relation is said to be a *rational k-form* on X, and we denote the set of all rational k-forms on X by $\Omega^k(X)$.

Consider a rational differential n-form on an n-dimensional non-singular variety X. It is of the form $\omega = g du_1 \wedge \cdots \wedge du_n$ in some neighbourhood of a point x. So if we cover X by affine sets U_i such that on each U_i, then we have such an expression as $\omega = g^{(i)} du_1^{(i)} \wedge \cdots \wedge du_n^{(i)}$. Furthermore, by Equation 1.4,

$$g^{(j)} = g^{(i)} J\left(\frac{u_1^{(i)}, \ldots, u_n^{(i)}}{v_1^{(j)}, \ldots, v_n^{(j)}}\right).$$

As the Jacobian determinant is regular and non-zero in $U_i \cap U_j$, the system of functions $g^{(i)}$ on U_i is compatible with Definition 24, and defines a Cartier divisor on X. We call this the divisor of ω, and denote it by $\operatorname{div}\omega$. This divisor satisfies the following properties:

(a) $\operatorname{div}(f\omega) = \operatorname{div} f + \operatorname{div}\omega$ for $f \in \mathbb{C}(X)$.

(b) $\operatorname{div}\omega \geq 0$ if and only if $\omega \in \Omega^n[X]$.

Definition 39 (Canonical Class)**.** Following Lemma 8, $\Omega^n(X)$ is a 1-dimensional vector space over $\mathbb{C}(X)$. So any form $\omega \in \Omega^n(X)$, it follows that $\omega = f\omega_1$, where ω_1 is some non-zero form in $\Omega^n(X)$. By property (a) above, all forms in $\Omega^n(X)$ are linearly equivalent, and therefore exist in a single divisor class, K_X, called the *canonical class* of X. A divisor in this class (that is, one associated to a non-zero rational differential n-form on X) is called a *canonical divisor* of X.

Theorem 9 (The Adjunction Formula)**.** *If X is a non-singular variety, and $Y \subset X$ is an irreducible closed subvariety of codimension 1, then*

$$K_Y = (K_X + Y)|_Y.$$

Proof. See [15], II, 8.20. □

Definition 40 (Fano Variety)**.** A smooth projective variety X is called a *Fano variety*, if its *anti-canonical class* $-K_X$ is ample. Furthermore, if a normal projective variety X has singular points, and some positive multiple $-nK_X$ (for $n \in \mathbb{N}$) of its anti-canonical class $-K_X$ is an ample Cartier divisor, then X is called a *singular Fano variety.* Assuming for a (possibly singular) Fano variety X, that $\operatorname{Pic} X = \mathbb{Z}H$, then $K_X = -rH$, and we call the integer r the *index* of X. Furthermore, when $r = 1$, we say that X is a *primitive* Fano variety.

The adjunction formula allows us to compute canonical classes for certain kinds of varieties.

Example 4. Suppose that $X \subset \mathbb{P}^N$ is a non-singular complete intersection of codimension $k \in \mathbb{N}$ given by equations of degree $(d_1, \ldots, d_k)$. We use the adjunction formula to compute its canonical class as follows:

$$\begin{aligned} K_X &= (K_{\mathbb{P}^n} + X)|_X \\ &= -(N+1)H_X + d_1 H_X + d_2 H_X + \cdots + d_k H_X \\ &= \left(\sum_{i=1}^{k} d_k - N - 1 \right) H_X, \end{aligned}$$

where H_X is a hyperplane section of X.

In particular, if

$$N = \sum_{i=1}^{k} d_k, \text{ then } K_X = -H_X.$$

Theorem 10. *Let D be a Cartier divisor on a projective variety X. Then D is ample on X if and only if*

$$(D^{\dim Y} \cdot Y) > 0,$$

where $Y \subset X$ is any subvariety of X, and $r = \dim Y$.

Proof. See [21]. □

Remark 10. If we again have a smooth complete intersection X of codimension k given by equations of degree $(d_1, \ldots, d_k)$, and $N = \sum_{i=1}^{k} d_i$, then

$$\left((-K_X)^{\dim Y} \cdot Y \right) = (H^{\dim Y} \cdot Y) = \deg Y > 0.$$

By Theorem 10, $-K_X$ is ample, and X is Fano.

Remark 11. A complete intersection in $\mathbb{P}^N$ given by equations of degree $(d_1, \ldots, d_k)$ is Fano if $\sum_{i=1}^{k} d_i < N + 1$.

1.3.4 The Linear System Associated to a Divisor

For an arbitrary divisor D on a non-singular variety X, consider the set which consists of all functions $f \in \mathbb{C}(X)$ such that

$$\operatorname{div} f + D \geq 0. \tag{1.5}$$

This set is a vector space over $\mathbb{C}$ equipped with the same operations on functions. Indeed, if $D = \sum_i n_i C_i$ then Equation 1.5 is the same as saying that $\nu_{C_i}(f) \geq -n_i$, and $\nu_C(f) \geq 0$, for $C \neq C_i$.

Let $\mathcal{L}$ be an invertible sheaf on X, and $s \in \Gamma(X, \mathcal{L})$ be a nonzero section of $\mathcal{L}$.

Definition 41 (Riemann-Roch Space)**.** The space of functions which satisfy 1.5 is called the *associated vector space of* D or *Riemann-Roch space of* D. It is denoted by $\mathcal{L}(D)$ or $\mathcal{L}(X, D)$. The former notation will be used when the variety in question is clear. So if X is non-singular, and $D \in \operatorname{Div}(X)$,

$$\mathcal{L}(D) = \{f \in \mathbb{C}(X)^* \mid \operatorname{div}(f) + D \geq 0\} \cup \{0\}.$$

The dimension of $\mathcal{L}(D)$, also called the dimension of D, is denoted by $l(D)$.

Definition 42 (Linear Systems)**.** Let $W \subset \mathcal{L}(D)$ be a vector subspace. Then the set

$$|D| := \Big\{D + \operatorname{div}(f) \mid f \in W \backslash \{0\}\Big\}$$

is called a *linear system*. It is the set of all effective divisors linearly equivalent to D.

Consider the map

$$H^0(S, \mathcal{O}_S(D)) \to |D|$$

where

$$f \mapsto \operatorname{div}(f) + D$$

(where $H^0(S, \mathcal{O}_S(D))$ is the vector space of all rational functions f such that $\operatorname{div}(f) + D \geq 0$). This map is surjective, since if $E \in |D|$, then $E = \operatorname{div}(f) + D$ for some f (E is linearly equivalent to D). If $\operatorname{div}(f) + D = \operatorname{div}(g) + D$, then $\operatorname{div}(f/g) = 0$. It follows that $f = \lambda g$

for some $\lambda \in \mathbb{C}$. Therefore the fibers are precisely the one dimensional linear subspaces of $H^0(S, \mathcal{O}_S(D))$, and so

$$|D| = \mathbb{P}\Big(H^0(S, \mathcal{O}_S(D))\Big).$$

Remark 12. Linear systems of dimension $1, 2$, or 3 are called *pencils*, *nets*, or *webs* respectively.

Theorem 11. *Linearly equivalent divisors are of equal dimension.*

Proof. Suppose that $D_1 \sim D_2$. Then by definition, $D_1 - D_2 = \operatorname{div} g$, for some $g \in \mathbb{C}(X)$. If $f \in \mathcal{L}(D_1)$ then $\operatorname{div} f + D_1 \geq 0$. We can see that

$$\operatorname{div}(fg) + D_2 = \operatorname{div} f + D_1 \geq 0,$$

which implies that $fg \in \mathcal{L}(D+2)$ and $g \cdot \mathcal{L}(D_1) = \mathcal{L}(D_2)$. This defines an isomorphism of vector spaces $\mathcal{L}(D_1)$ and $\mathcal{L}(D_2)$. □

Definition 43 (Base Locus of a Linear System)**.** The *base locus* or *base set* of a linear system Σ, denoted by $\operatorname{Bs}(\Sigma)$, on a variety is the subvariety of points, called *basepoints*, which lie on all the divisors in Σ. It is the set of points where all the elements of the linear system vanish.

Definition 44 (Fixed & Moving Parts of a Linear System)**.** Let X be a normal projective variety, and D be a Cartier divisor on X. A point $x \in X$ is called a *fixed point* of $|D|$ if $x \in \operatorname{Supp} D'$ for any $D' \in |D|$. The *fixed part* F of $|D|$ is the "biggest" effective divisor linearly equivalent to D. It is denoted by $\operatorname{Fix} D$, and is defined as follows:

$$\operatorname{Fix} D := \inf\{L \mid L \sim D,\ L \geq 0\},$$

where we say that

$$D' \leq D \text{ if } \operatorname{Supp} D' \subset \operatorname{Supp} D.$$

The complement of $\operatorname{Fix} D$ in $|D|$ is called the *moving part* $\operatorname{Mov} D$ of D.

Definition 45 (Fixed Components of a Linear System)**.** A *fixed component* of a linear system $|L|$ is an effective divisor D'' on X such that, for every $D \in |L|$,

$$D = D' + D''$$

where D' is an effective divisor.

1.3.5 Logarithmic Pairs

Definition 46 (Boundary $\mathbb{Q}$-divisor)**.** A Weil $\mathbb{Q}$-divisor $\triangle = \sum d_i D_i$ is called a *boundary* $\mathbb{Q}$*-divisor* if the D_i are the distinct irreducible components of $\triangle$, and $0 \leq d_i \leq 1$.

Definition 47 (Log Pair)**.** A *logarithmic pair* (*log pair* for short) is an ordered pair $(X, \triangle)$ where X is a normal variety, $K_X + \triangle$ is $\mathbb{Q}$-Cartier, and $\triangle$ is a boundary $\mathbb{Q}$-divisor.

Definition 48 (Log Smooth Pairs)**.** We say that a pair $(X, \triangle)$ is *log smooth* if it has global simple normal crossings.

Definition 49 (Log Resolution)**.** Let $(X, \triangle)$ be a log pair. A *log resolution* of the pair is a birational map $\pi : Y \dashrightarrow X$ of projective varieties such that the pair $(Y,\ \Gamma = \widetilde{\triangle} + E)$ is log smooth, and $E = \sum E_i$, where $\widetilde{\triangle}$ is the strict transform of $\triangle$ and E is the exceptional divisor.

Definition 50 (Discrepancy of an Exceptional Divisor)**.** Let π be a log resolution (which is projective, and therefore proper). Also let $(X, \triangle)$ be a pair, where $\triangle$ is a $\mathbb{Q}$-divisor (which need not be effective) such that $(K_X + \triangle)$ is $\mathbb{Q}$-Cartier.

Furthermore, we write

$$K_Y = \pi^*(K_X + \triangle) + \sum a(E_i, X, \triangle)E_i,$$

where the sum runs over the prime divisors of Y, $\pi_*(K_Y) = K_X$, and the numbers $a(E_i, X, \triangle) \in \mathbb{Q}$ are called the *discrepancies* of the divisor $E_i \subset Y$.

Remark 13. If the centre of E_i is a component, say D_i of $\triangle = \sum d_i D_i$, then $a(E_i, X, \triangle) := -d_i$. If E_i is not an exceptional divisor of π and its center is not a component of $\triangle$, then we define $a(E_i, X, \triangle) := 0$. Thus the above sum is a finite sum.

The divisor E induces a discrete valuation on $\mathbb{C}(X)$, that is a function

$$\mathrm{ord}_E : \mathbb{C}(X) \to \mathbb{Z} \cup \{\infty\}.$$

Let $D \subset X$ be an effective divisor. We obtain the *multiplicity* $\nu_E(D)$ by applying the discrete valuation ord_E to D, and if we denote the set of exceptional divisors of φ by ε, then

$$\varphi^* D = D^+ + \sum_{E \in \varepsilon} \nu_E(D) E. \tag{1.6}$$

For the corresponding canonical class K_{X^+} we get,

$$K_{X^+} = \varphi^* K_X + \sum_{E \in \varepsilon} a(E) E, \tag{1.7}$$

where $a(E)$ is the discrepancy of E.

Theorem 12 (Lefschetz Hyperplane Theorem)**.** *Let X be a non-singular projective variety of dimension n, and let D be any ample, effective divisor on X. Then the map*

$$r_i : H^i(X, \mathbb{Z}) \to H^i(D, \mathbb{Z})$$

is injective when $i = n - 1$, and an isomorphism for $i \leq n - 2$.

Proof. See page 156 of [13]. □

1.4 Intersection Theory

In Chapter 3, we apply intersection theory. For more on the subject, see [15] and [10].

Definition 51 (k-cycles)**.** Let X be an n-dimensional projective variety, and k be a positive integer less than or equal to n. A k-cycle on X is a $\mathbb{Z}$-linear combination of irreducible subvarieties of dimension k. The group of k-cycles is the free Abelian group generated by subvarieties of dimension k, and it is denoted by $Z_k(X)$.

Furthermore, a k-cycle $Z = \sum_i n_i Y_i$ is said to be *effective* if all non-zero coefficients n_i are non-negative.

Definition 52 (Degree Map)**.** Let D be a zero-cycle on a projective variety X, that is $D = \sum_i a_i x_i$, where the x_i's are points on X. We define the *degree map*

$$\deg : Z_0(X) \longrightarrow \mathbb{Z},$$

$$D \longmapsto \sum_i a_i.$$

The *degree* of D is its image under this map.

Definition 53 (Intersection of Cycles)**.** Let k_1 and k_2 be integers less than or equal to n, such that $k_1 + k_2 \geq n$. And let A and B be k_1 and k_2 cycles of a non-singular variety X respectively. Suppose that the intersection of these cycles is *proper*. That is to say, suppose that

$$\dim(A \cap B) = \dim A + \dim B - \dim X.$$

If $\mathcal{C}$ is the set $\{C_1, \ldots, C_m\}$ of irreducible components of $A \cap B$, then we define the intersection $(A \cdot B)$ of A and B to be

$$(A \cdot B) = \sum_{C_i \in \mathcal{C}} \mathrm{mult}_i(A, B) C_i,$$

where $\mathrm{mult}_i(A, B)$ is called the *intersection multiplicity* of A and B along C_i.

Remark 14. Intersection multiplicity is defined in different ways in the literature. These include Serre's formula [[8], Theorem 2.7], and the definition provided in [10], Chapter 7]. However, as complete intersections are the primary object of study in this text, we can take advantage of Proposition 18.13 from [9] and assume that the varieties we consider are Cohen-Macauley. This in turn allows us to take advantage of Proposition 8.2 of [10], and conclude that $\mathrm{mult}_{C_i}(A, B)$ equals the length of $\mathcal{O}_{C_i, A\cap B}$.

Definition 54 (Numerical Equivalence)**.** We say that k-cycles Y_1 and Y_2 are *numerically equivalent* if, for any cycle C of codimension k, we have

$$\deg(Y_1 \cdot C) = \deg(Y_2 \cdot C).$$

Definition 55 (Numerical Chow Groups)**.** Let X be a non-singular integral quasi-projective variety. The group $A_k(X) := Z_k(X)/\sim$, where $\sim$ is the relation of numerical equivalence, is termed *numerical Chow groups.* The intersection product induces a graded ring structure on

$$\bigoplus_{k=0}^{n} A_k(X).$$

This ring is called the Chow ring of X, and it is denoted by $A(X)$.

Remark 15. We can also analogously define the group of k-cocycles $A^k(X) := A_{n-k}(X)$.

Definition 56 (Pseudo-effective Cone)**.** Set $A^1_{\mathbb{R}}(X) = \operatorname{Pic} X \otimes \mathbb{R}$. We define the cones $A^1_+X \subset A^1_{\mathbb{R}}X$ of *pseudo-effective classes*, and $A^1_{\text{mob}}X \subset A^1_{\mathbb{R}}X$, as the closed cones (with respect to the standard real topology of $A^1_{\mathbb{R}}(X) \cong \mathbb{R}^k$), generated by the classes of effective divisors and mobile divisors respectively.

Definition 57 (Pseudo-effective Divisor)**.** A divisor is *pseudo-effective* if it is in the pseudo-effective cone.

Chapter 2

Birational Rigidity & Superrigidity

In this chapter, we give a brief introduction to, and discuss major theorems on birational rigidity and superrigidity. In the beginning of this chapter, we invoke adjunctions of the canonical class of a variety, and define the so-called *threshold of canonical adjunction*, where such adjunctions, roughly speaking, stop. This is the notion of *termination of canonical adjunction*. We will then look at one of the implications of the termination of canonical adjunction, the so-called Noether-Fano inequality. Then we will immediately use all this information to define the key notions of birational rigidity and superrigidity. We also delve into some methods for proving that a variety is birationally superrigid, most of which can be found in the book [26] by Pukhlikov. Then we pivot towards the main method for proving that a variety is birationally superrigid: the so-called method of maximal singularities (covered in Subsection 2.2.1). The method of maximal singularities is the main method for establishing a sufficient condition for birational rigidity and superrigidity. Other techniques which we discuss include inversion of adjunction, the technique of hypertangent divisors (discussed in Subsection 2.2.5), the $4n^2$-inequality (see Subsection 2.2.3), as well as its adaptation for complete intersection singularities (Subsection 2.2.4). Then we proceed in the final section, to discuss these methods as they apply to complete intersections for complete intersections of codimension three, which are investigated in Chapter 3.

2.1 Definitions

This section introduces several concepts which are fundamental to the methods and techniques which we use to approach the question of birational rigidity. In particular, we will introduce so-called geometric discrete valuations, and the threshold of canonical adjunction.

2.1.1 The Threshold of Canonical Adjunction

A smooth, projective, rationally connected variety V satisfies the classical condition of termination of adjunction of the canonical class. That is, for any effective divisor D on V, the linear system $|D + mK_V|$ is empty for $m >> 0$, since K_V is negative on every family of rational curves sweeping out V, whereas an effective divisor is non-negative on such a family.

Definition 58 (Threshold of Canonical Adjunction)**.** For a rationally connected projective variety V, let $A^1V = \operatorname{Pic} V$ be its Picard group. Furthermore, let $A^1_+V \subset A^1V \otimes \mathbb{R}$ denote the cone of pseudo-effective classes. The *threshold of canonical adjunction* of a divisor D on V is

$$c(D, V) := \sup\{\varepsilon \in \mathbb{Q}_{\geq 0} \mid D + \varepsilon K_V \in A^1_+V\}.$$

The threshold is independent of linear equivalence of divisors, and so for a non-empty linear system Σ on V, we can set

$$c(D, V) = c(\Sigma, V).$$

Definition 59 (Virtual Threshold of Canonical Adjunction)**.** For a mobile linear system Σ on a variety V, define the *virtual threshold of canonical adjunction* by

$$c_{\text{virt}} = \inf_{V^\sharp \dashrightarrow V}\{c(\Sigma^\sharp, V^\sharp)\},$$

where $V^\sharp$ is a smooth projective model of V, $\Sigma^\sharp$ is the strict transform of the system Σ on $V^\sharp$, and this infimum is taken over all birational morphisms $V^\sharp \dashrightarrow V$.

Remark 16. The virtual threshold is a birational invariant of the pair (V, Σ). That is, if $\chi : V \dashrightarrow V^+$ is a birational map, then $\Sigma^+ = \chi_* \Sigma$ is the strict transform of (the system) Σ with respect to χ^{-1}, then we get $c_{\text{virt}}(\Sigma) = c_{\text{virt}}(\Sigma^+)$.

Definition 60 (Birational Rigidity & Superrigidity)**.** We say that a variety V is *birationally superrigid* if for every mobile linear system Σ on V,

$$c(\Sigma, V) = c_{\text{virt}}(\Sigma).$$

Furthermore, we say that V is *birationally rigid* if for any mobile linear system Σ on V, there is a birational self-map $\delta \in \operatorname{Bir} V$ such that

$$c_{\text{virt}}(\Sigma) = c(\delta_* \Sigma, V).$$

If V is a rationally connected variety, then we can prove birational superrigidty by contradiction as follows: for some mobile linear system Σ on V, suppose that

$$c_{\text{virt}}(\Sigma) < c(\Sigma). \tag{2.1}$$

(This assumption is necessary because without it, the variety is birationally superrigid).

Then there is a birational map $\varphi : V^+ \to V$ such that

$$c_{\text{virt}}(\Sigma^+, V^+) < c(\Sigma),$$

where Σ^+ is the strict transform of Σ. This implies that there is a divisor $E \subset V^+$ which is contracted by φ. The alternative would have been for φ to be an isomorphism in codimension 1, and this would resulted in the equality

$$c(D, V) = c(D^+, V^+).$$

By Definition 59, this result would contradict our earlier assumption that $c_{\text{virt}}(\Sigma) < c(\Sigma)$. So there are exceptional divisors $E \subset V^+$ contracted by the morphism φ, and each of these exceptional divisors determines a discrete valuation ord_E on the field $\mathbb{C}(V)$ of rational functions.

Remark 17. The valuation corresponding to an exceptional divisor E is independent of the model V^+ of V. Given a second birational map $\varphi^\# : V^\# \to V$, the birational map $(\varphi^\#)^{-1} \circ \varphi : V^+ \to V$ is an isomorphism at a general point of E, $(\varphi^\#)^{-1} \circ \varphi(E) = E^\# \subset V^\#$ is an exceptional divisor of $\varphi^\#$. So $\mathrm{ord}_E = \mathrm{ord}_E^\#$.

2.2 Methods & Techniques

We now look at the main methods used to prove birational superrigidity, beginning with the main method: the method of maximal singularities.

2.2.1 The Method of Maximal Singularities

In this section, we introduce the method of maximal singularities, and we go on to discuss it in the context of Fano varieties. We will ask whether for a geometric discrete valuation ν_E of $\mathbb{C}(V)$, there is a mobile linear system Σ such that $c(\Sigma) > 0$, and for which ν_E is a maximal singularity. We will also see how termination of canonical adjunction leads to the so-called Noether-Fano inequality. In particular, we show that if a variety is not birationally superrigid, then it must have a maximal singularity.

Definition 61 (Geometric Discrete Valuation)**.** A discrete valuation ν_E of $\mathbb{C}(V)$ is said to be *geometric* if there is a birational map $\widetilde{V} \to V$, and some exceptional divisor $E \subset \widetilde{V}$ such that $\nu_E = \mathrm{ord}_E$ is the order of vanishing of E.

Definition 62. For a variety V and an effective $\mathbb{Q}$-divisor D, a log pair (V, D) is said to be:

- *canonical* if for any geometric discrete valuation ν_E, the inequality $\nu_E(D) \leq a(E, V)$ holds.

- *log canonical* if for any geometric discrete valuation ν_E, the inequality

$$\nu_E(D) \leq a(E, V) + 1$$

holds.

- *terminal* if for any geometric discrete valuation ν_E, the inequality $\nu_E(D) < a(E, V)$ holds.

- *log terminal* if for any geometric discrete valuation ν_E, the inequality

$$\nu_E(D) < a(E, V) + 1$$

holds.

Remark 18. These concepts can be defined only in terms of the generalised discrepancies defined in Definition 50, but we leave the definition in the more transparent form above.

Remark 19. Henceforth, unless otherwise specified, let $n = c(\Sigma, V)$.

Definition 63 (Maximal Singularities, Maximal Subvarieties, & The Noether-Fano Inequality)**.** A *maximal singularity* is a geometric discrete valuation ord_E on $\mathbb{C}(V)$, and of the linear system Σ, such that the *Noether-Fano inequality*

$$\nu_E(\Sigma) > n \cdot a(E),$$

holds. Furthermore, an irreducible subvariety $Y \subset V$ of codimension ≥ 2 is said to be a *maximal subvariety* of Σ if

$$\mathrm{mult}_Y \Sigma > n(\mathrm{codim}\, Y - 1),$$

where $\mathrm{mult}_Y \Sigma = \mathrm{mult}_Y D$, for all $D \in \Sigma$.

Remark 20. The Noether-Fano inequality can be re-formulated in the language of $\mathbb{Q}$-divisors as follows. If $D \in \Sigma$ be a general divisor, then the log pair $\left(V, \frac{1}{n}D\right)$ is not canonical, that is, it has a non-canonical singularity $E \subset V^+$, which satisfies the inequality $\nu_E\left(\frac{1}{n}D\right) > a(E)$.

2.2.2 Resolutions of Discrete Valuations

A geometric discrete valuation induced by an exceptional divisor can be resolved using a birational morphism which generates a non-singular prime divisor.

Suppose V is a projective variety with a birational map $\phi : V^+ \dashrightarrow V$, which contracts an exceptional divisor E to a centre $B = \phi(E) \subset V$, where $B \not\subset \operatorname{Sing} V$, and $\operatorname{Sing} V$ is the singular locus of V. Furthermore, if $\operatorname{codim} B \geq 2$, then one can construct a unique sequence of blow-ups associated with an exceptional divisor $E \subset V^+$.

Proposition 1. If $\sigma_B : V(B) \to V$ is the blow-up of the subvariety $B \subset V$, and $E(B) = \sigma_B^{-1}(B)$ the corresponding exceptional divisor, then one of the following two alternatives applies:

- we have a birational isomorphism $(\sigma_B^{-1} \circ \phi) : V^+ \dashrightarrow V(B)$ in a neighbourhood of the generic point of the divisor E and $(\sigma_B^{-1} \circ \phi)(E) = E(B)$.
- $B^+ = (\sigma_B^{-1} \circ \phi)(E)$ is an irreducible subvariety of codimension ≥ 2 and $B^+ \subset E(B)$ and $\sigma_B(B^+) = B$.

In the event of the latter scenario, we have a blow-up $\sigma_{B^+} : V(B) \to B^+$ of $V(B)$ along B^+. We begin an iterative process where we take the composition $(\sigma_B \circ \sigma_{B^+})$ and check which of the two scenarios apply to it, and continue to repeat this. This creates a sequence of blowups

$$\begin{array}{ccc} V_i & \longrightarrow & V_{i-1} \\ \varphi_{i,i-1} \Big\uparrow & & \Big\uparrow \\ E_i & \longrightarrow & B_{i-1} \end{array},$$

for $i = 1, 2 \ldots,$ where $V_0 = V$, $B_0 = \text{centre}(E, V)$ is the centre of E on V_j, $E_i = \varphi_{i,i-1}^{-1}(B_{i-1})$ is the exceptional divisor of V_i over V_{i-1}, and B_i is the centre of the blow-up $\varphi_{i+1,i}$. So in general, for $i > j$, set

$$\varphi_{i,j} = \varphi_{j+1,j} \circ \cdots \circ \varphi_{i,i+1} : V_i \longrightarrow V_j,$$

and we naturally have an identity $\varphi_{i,i} = \text{id}_{V_i}$.

Proposition 2. This sequence of blow-ups terminates. So for some $K \geq 1$, the first scenario of Proposition 1 holds. That is, for a blow-up $\sigma_{K,0}$, and an exceptional divisor E_K, $(\sigma_{K,0}^{-1} \circ \phi) = E_K$.

The sequence of blow-ups above is called the *resolution of the discrete valuation* ν_E with respect to the model V.

The Oriented Graph Structure on the Exceptional Divisors

Let $\{E_1, \ldots, E_K\}$ be the set of exceptional divisors which was obtained in Section 2.2.2 from the resolution of the discrete valuation ν_E. Given the set of blow-ups $\varphi_{i,i-1}$ above, with corresponding exceptional divisors $\{E_1, \ldots, E_K\}$, the following structure of an oriented graph exists: between vertices E_i and E_j is an oriented edge (an arrow) which we will denote by $i \to j$, if $i > j$ and $B_{i-1} \subset E_j^{i-1}$.

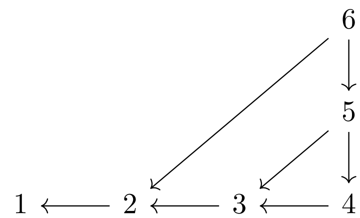
This graph structure is used to formalize the operation of computing the strict transforms of exceptional divisors. We have:

$$E_j^i = \varphi_{i,j}^* E_j - \sum_{j \leftarrow k \leq i} \varphi_{i,k}^* E_k, \tag{2.2}$$

where $\varphi_{i,j}$ is the composition of blow-ups which featured on the previous page.

It is also possible to compute the pullback in terms of strict transforms. Denote by $p_{i,j}$, the number of paths from E_i to E_j, for $i \geq j$. Also set, $p_{i,i} = 1$.

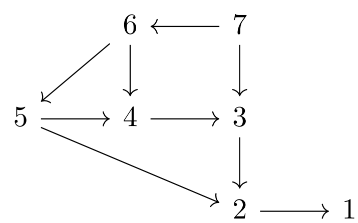
Example 5.



We have

$$p_{6,1} = p_{6,2} = p_{5,2} + 1 = p_{4,2} + p_{3,2} + 1 = 1 + 1 + 1 = 3.$$

Example 6.



$$p_{7,1} = p_{7,2} = p_{6,2} + 1 = p_{5,2} + p_{4,2} + 1 = 2 + 1 + 1 = 4$$

Theorem 13. *The following decomposition holds:*

$$\varphi_{i,j}^* E_j = \sum_{k=j}^{i} p_{kj} E_k^i.$$

Proof. This is proven by induction on $i \geq j$. If $i = j$, then there is nothing to prove. If $i = j+1$, then $\varphi_{j+1,j}^* E_j = E_j^{j+1} + E_{j+1}$, since $B_j \subset E_j$ and the divisor E_j is non-singular at the generic point of B_j. For $i \geq j+2$ we get:

$$E_j^i = \varphi_{i,i-1}^*(\varphi_{i-1,j}^* E_j) = \varphi_{i,i-1}^*\left(\sum_{k=j}^{i-1} p_{kj} E_k^{i-1}\right) = \sum_{k=j}^{i-1} p_{kj} E_k^i + \left(\sum_{k=j,\ B_{i-1}\subset E_k^{i-1}} p_{kj}\right) E_i.$$

We see that the following equality holds:

$$p_{ij} = \sum_{i \to k} p_{kj}$$

(in each path from i to j mark the first vertex of the graph after $i : i \to k \to \cdots \to j$). □

The p_{ij} are combinatorial invariants which give explicit presentations for multiplicities and discrepancies. Let Σ^j be the strict transform of the linear system Σ on V_j. Set $\nu_j = \operatorname{mult}_{B_{j-1}} \Sigma^{j-1}$ and $\beta_j = \operatorname{codim} B_{j-1} - 1$. We get

$$\nu_{E_K}(\Sigma) = \nu_E(\Sigma) = \sum_{i=1}^{K} p_{K_i} \nu_i, \quad \text{and} \quad a(E) = \sum_{i=1}^{K} p_{K_i} \beta_i.$$

Remark 21. Recall that $B_{j-1} = \varphi_{i,i-1}(E_i)$. Also remember that in our attempt to define $\nu_E(D)$ and $a(E)$, for $D \in \Sigma$, we let $\nu_E(\Sigma) = \nu_E(D) = \operatorname{mult}_Y \Sigma$, where $E = \varphi^{-1}(Y)$ was the exceptional divisor of the blowup $\varphi : V^+ \to V$.

Setting for convenience of notations $p_i = p_{K_i}$, we get the traditional form of the Noether-Fano inequality:

$$\sum_{i=1}^{K} p_i \nu_i > n \sum_{i=1}^{K} p_i \beta_i. \tag{2.3}$$

2.2.3 The $4n^2$-inequality

Let $i = 0, \ldots, N-1$, where $N \in \mathbb{N}$. Given a resolution of a maximal singularity $E \subset V^+$, there are two alternative possibilities regarding the non-decreasing dimensions $\dim B_i$, of the centres of the individual blow-ups:

$$\dim B_0 = \cdots = \dim B_{N-1},$$

or

$$\dim B_0 \; < \; \dim B_{N-1}.$$

In the second of these scenarios, we say that E is an *infinitely near singularity.* Focusing on the latter, suppose we have a resolution of a maximal singularity $E \subset V^+$.

Theorem 14. *Suppose E is an infinitely near maximal singularity on a mobile linear system Σ, and $\dim B_0 = \dim B_{N-1}$. Then the subvariety $B = B_0$ (which is the centre of the singularity E on V), is a maximal subvariety of Σ.*

Proof. See Proposition 2.1 in Section 2.1 of Chapter 2 in [26]. □

Remark 22. An important corollary of this fact is that if the centre $B \subset V$ of the maximal singularity E is not a maximal subvariety of Σ, then E is an infinitely near maximal singularity.

Again setting $B = B_0$, we consider the algebraic cycle of the scheme-theoretic self-intersection $Z = (D_1 \circ D_2)$ of a linear system Σ on V, where $D_1, D_2 \in \Sigma$ are general divisors.

Theorem 15 (The $4n^2$-inequality)**.** *Suppose E is an infinitely near maximal singularity of the system Σ on V, and say it has a centre B with codimension ≥ 3. Also let $Z = (D_1 \circ D_2)$ be a self-intersection of Σ, with $n = c(\Sigma) > 0$ as the threshold of canonical adjunction, and recall that the Noether-Fano inequality holds. Then the estimate*

$$\operatorname{mult}_B Z > 4n^2$$

holds.

Proof. Let D and Q be prime divisors on V. Furthermore, let D^B and Q^B be their strict transforms on $V(B)$ (under σ_B). For a general subvariety $Y \subset V$, we denote its strict transform on $V(B)$ by Y^B.

Returning to the discrete valuation ν, we divide the resolution $\varphi_{i,i-1} : V_i \to V_{i-1}$ into:

1. the *lower part*, with $i \in \{1, \ldots, L\}$ for $L \leq K$, corresponding to the blowups with $\operatorname{codim} B_{i-1} \geq 3$ and;

2. the *upper part*, for the indices $i \in \{L+1, \dots, K\}$ corresponding to the blowups with $\operatorname{codim} B_{i-1} = 2$. In the case $L = K$, the upper part is empty.

Let $D_1, D_2 \in \Sigma$ be two different general divisors. Define a sequence of codimension 2 cycles on V_i by setting:

$$\begin{aligned} D_1 \circ D_2 &= Z_0, \\ D_1^1 \circ D_2^2 &= Z_0^1 + Z_1, \\ &\vdots \\ D_1^i \circ D_2^i &= (D_i^{i-1} \circ D_2^{i-1})^i + Z_i, \\ &\vdots \end{aligned}$$

where $Z_i \subset E_i$. So more generally, we get

$$D_1^i \circ D_2^i = Z_0^i + Z_1^i + \cdots + Z_{i-1}^i + Z_i, \text{ for any } i \leq L.$$

Furthermore, for any $j > i, j \leq L$ set,

$$\begin{aligned} m_{i,j} &= \operatorname{mult}_{B_{j-1}}(Z_i^{j-1}) \\ d_i &= \deg Z_i. \end{aligned}$$

We have a system of equalities

$$\nu_i^2 + d_i = m_{0,i} + \cdots + m_{i-1,i}, \quad \text{for } i = 1, \dots, L.$$

Furthermore,

$$d_L \geq \sum_{i=L+1}^{K} \nu_i^2 \deg[(\varphi_{i-1,L})_* B_{i-1}] \geq \sum_{i=L+1}^{K} \nu_i^2.$$

Lemma 16. *If $m_{ij} > 0$, then $i \to j$.*

Proof. If $m_{ij} > 0$, then some component of Z_i^{j-1} contains B_{j-1}. However, $Z_i^{j-1} \subset E_i^{j-1}$. $\square$

Lemma 17. *For any $i \geq 1$, and $j \leq L$ we have $m_{ij} \leq d_i$.*

Proof. The cycles B_a are non-singular at their generic points. However, since the maps $\varphi_{a,b} : B_a \to B_b$ are surjective, we can count multiplicities at generic points. Taking into account that the multiplicities are non-increasing with respect to blowing up of a non-singular subvariety, we reduce the claim to the "obvious" case of a hypersurface in a projective space. $\square$

Theorem 18. *The inequality*

$$\sum_{i=1}^{L} p_i m_{0,i} \geq \sum_{i=1}^{L} p_i \nu_i^2 + p_L \sum_{i=L+1}^{K} \nu_i^2$$

holds.

Proof. For $1 \leq i \leq L$, we have

$$p_i(\nu_i^2 + d_i) = p_i(m_{0,i} + \cdots + m_{i-1,i}),$$

by Equation 2.2.3. We take the sum

$$\sum_{i=1}^{L} p_i \nu_i^2 + \sum_{i=1}^{L} p_i d_i = \sum_{i=1}^{L} p_i m_{0,i} + \sum_{i=2,j+1}^{L} p_i m_{j,i}.$$

It follows from the above that

$$\sum_{i=2,\, j+1 \leq i}^{L} p_i m_{j,i} = \sum_{i=2,\, m_{j,i} \neq 0}^{L} p_i m_{j,i} \leq \sum_{i=2,\, i \to j}^{L} p_i d_j \leq \sum_{j=1}^{L-1} p_i d_j.$$

$\square$

Corollary 2. Set $m = m_{0,1} = \mathrm{mult}_B(D_1 \circ D_2)$, where $D_i \in \Sigma$. Then we have the following inequality:

$$m\left(\sum_{i=1}^{L} a(i)\right) \;\geq\; \sum_{i=1}^{L} a(i)\nu_i^2 \;+\; a(L)\sum_{i=L+1}^{K} \nu_i^2. \tag{2.4}$$

Corollary 3. We have

$$m\left(\sum_{i=1}^{L} p_i\right) \geq \sum_{i=1}^{K} p_i\nu_i^2.$$

Proof. This holds $p_i \leq p_L$, where $i \geq L+1$. □

Recall that in this setting, the Noether-Fano inequality takes the form

$$\sum_{i=1}^{K} p_i\nu_i > n\sum_{i=1}^{K} p_i\beta_i,$$

and note that the right-hand side of Equation 2.4 is strictly greater than the value of the quadratic form $\sum_{i=1}^{K} p_i\nu_i^2$ at the point

$$\nu_1 = \cdots = \nu_K = \frac{\sum_{i=1}^{K} p_i\delta_i n}{\sum_{i=1}^{K} p_i}.$$

Set

$$\Sigma_l = \sum_{\delta_j \geq 2} p_j, \qquad \Sigma_u = \sum_{\delta_j = 1} p_j.$$

In these notations we get:

$$\mathrm{mult}_B\, Z > \frac{(2\Sigma_l + \Sigma_u)^2}{\Sigma_l(\Sigma_l + \Sigma_u)} n^2.$$

□

2.2.4 The $4n^2$-inequality for Complete Intersection Singularities

In this subsection, we discuss an extension of the $4n^2$-inequality (see Theorem 15) to singularities of complete intersections. This was first discovered by *Pukhlikov* [28], with the result being that on a complete intersection, the multiplicity of the self-intersection of a mobile linear system with a maximal singularity exceeds $4n^2\mu$, where μ is the multiplicity of the singular point. This result plays a crucial role in the proof of Theorem 24, the main result of this work.

Generic Complete Intersection Singularities

Let (V, o) be a germ of a generic complete intersection singularity of codimension l and type $\underline{\mu} = (\mu_1, \dots, \mu_l)$, where

$$\dim V = M \ \geq \ l + \mu_1 + \dots + \mu_l + 3.$$

Also let $o = (0, \dots, 0)$ be the origin of the affine space $\mathbb{A}^{M+l}$, and $q_{j,i}$ be homogeneous polynomials of degree i in the coordinates $z_1, \dots, z_{M+l}$.

Consider the germ (V, o) given by a system of l analytic equations

$$\begin{aligned} 0 &= q_{1,\mu_1} + q_{1,\mu_1+1} + \dots \\ &\dots \\ 0 &= q_{1,\mu_l} + q_{1,\mu_l+1} + \dots, \end{aligned}$$

in $\mathbb{C}^{M+l}$, where

$$2 \leq \mu_1 \leq \dots \leq \mu_l \ \text{ for } \ l \geq 1.$$

Now set

$$|\underline{\mu}| = \mu_1 + \dots + \mu_l,$$

and recall that we assume that $M \geq l + |\mu| + 3$. Furthermore, let P be a linear subspace in $\mathbb{C}^{M+l}$ of dimension $2l + |\mu| + 3$, and let us denote $V \cap P$ as V_P.

Remark 23. For more on the basis for the assumption that $M \geq l + |\mu| + 3$, see page 3 of [28].

Definition 64 (Generic Complete Intersection Singularity)**.** A complete intersection singularity (V, o) is *generic* if for any linear subspace P of dimension $2l + |\mu| + 3$:

- the singularity $o \in V_P$ is isolated.
- $\dim V_P = l + |\underline{\mu}| + 3$.
- the blow-up of the point $\varphi_P : V_P^+ \to V_P$, with exceptional divisor $Q_P = \varphi_P^{-1}(o)$, is non-singular in some neighbourhood of Q_P.
- Q_P is a non-singular complete intersection

  $$Q_P = \{q_{1,\mu_1} = \cdots = q_{l,\mu_l} = 0\} \subset \mathbb{P}^{2l+|\mu|+2}$$

  of codimension l and type $\mu = (\mu_1, \ldots, \mu_l)$.

Statement & Proof of the Result.

Proposition 3. Let Σ be a mobile linear system on V. Assume that for some positive $n \in \mathbb{Q}$ the pair $\left(V, \frac{1}{n}\Sigma\right)$ is not canonical at o but canonical outside this point. Then the self-intersection $Z = (D_1 \circ D_2)$ of the system Σ satisfies the inequality

$$\operatorname{mult}_o Z > 4n^2 \operatorname{mult}_o V. \tag{2.5}$$

The proof of this result, as shown in [28], depends on a number of minor results (in the form of Lemmas 19 and 20). For the remainder of the section, (V, o) is assumed to be generic.

Proof. Let P be a general linear subspace of dimension $2l+|\mu|+3$ and Σ_P be the restriction of Σ onto P. By Theorem 23, that is inversion of adjunction, the pair

$$\left(V_P, \frac{1}{n}\Sigma_P\right)$$

is not canonical. Suppose

$$Z_P = Z|_P = (Z \circ V_P)$$

is the self-intersection of Σ_P, and $\mathrm{mult}_o Z = \mathrm{mult}_o Z_P$. We assume that $M = l + |\mu| + 3$, so that $P = \mathbb{C}^{M+l}$. Let $\Pi \ni o$ be a general linear subspace of dimension $|\mu| + 3$ and $V_\Pi = V \cap \Pi$. Since $\dim V = M$, it follows that $o \in V_\Pi \subset \Pi = \mathbb{C}^{|\mu|+3}$ is an isolated singularity of codimension l. Let $\varphi_\Pi : V_\Pi^+ \to V_\Pi$ be the blow-up of the point o with $Q_\Pi = \varphi^{-1}(o)$ being the exceptional divisor, and $Q_\Pi \subset \mathbb{P}^{|\mu|+2}$ is a non-singular complete intersection of type $|\mu|$ and codimension l. Notice that by the adjunction formula(see Theorem 9), $a(Q_\Pi, V_\Pi) = 2$.

Let $D \in \Sigma$ be a general divisor and $D^+ \in \Sigma^+$ its strict transform on V^+, then

$$D^+ \sim -\nu Q, \text{ for } \nu > 0.$$

The inequality holds immediately for $\nu > 2n$, as

$$\mathrm{mult}_o Z \ \geq \ \nu^2\mu \ \geq \ 4n^2\mu.$$

Henceforth, we suppose that $\nu \leq 2n$. If we restrict the divisors D and Q to the linear subspace Π, so that $D|_{V_\Pi} = D_\Pi$ and $D_\Pi^+ \sim -\nu Q_\Pi$.

Since by assumption, $\left(V, \frac{1}{n}\Sigma\right)$ is non-canonical at the point o, and is canonical elsewhere, we can use the theorem on inversion of adjunction (Theorem 23). By the theorem, the pair

$$\left(V_\Pi, \frac{1}{n}D_\Pi\right)$$

is not log canonical at o, and is therefore also not canonical by Definition 62. It follows for some exceptional divisor E_Π over V_Π,

$$\nu_{E_\Pi}\left(\frac{1}{n}D_\Pi\right) > a(E_\pi, V_\Pi),$$

and consequently, the Noether-Fano inequality, that is

$$\mathrm{ord}_{E_\Pi}(D_\Pi) > n \cdot a(E_\pi, V_\Pi),$$

holds. Given that $\nu \leq 2n$, and $a(Q_\Pi, V_\Pi) = 2$, it follows that $E_\Pi \neq Q_\Pi$, since the alternative would create a contradiction with the Noether-Fano inequality. By Lemma 4.1 from Section 4 in Chapter 2 of [26], E_Π is a non-log canonical (and so not canonical) singularity of the pair

$$\left(V_\Pi^+,\ \frac{1}{n}D_\Pi^+ + \frac{\nu - 2n}{n}Q_\Pi\right).$$

Denote by $\triangle_\Pi \subset Q_\Pi$ the centre of E_Π on V_Π, an irreducible subvariety in Q_Π.

Lemma 19. *If* $\mathrm{codim}(\triangle_\Pi \subset Q_\Pi) = 1$, *then the estimate*

$$\mathrm{mult}_o Z \geq 8n^2\mu$$

holds.

Proof. Note $\mathrm{mult}_o Z = \mathrm{mult}_o Z_\Pi$. Then by Proposition 4.1 from Section 4 in Chapter 2 of [26], we have

$$\mathrm{mult}_o Z_\Pi \ \geq\ \nu^2\mu + 4\Big(3 - \frac{\nu}{n}\Big)n^2\mu = 8n^2\mu + \mu(2n-\nu)^2.$$

The proof is complete. □

Clearly, if $\mathrm{codim}(\triangle_\Pi \subset Q_\Pi) = 1$, then Proposition 3 is satisfied. So we can assume that $\mathrm{codim}(\triangle_\Pi \subset Q_\Pi) \geq 2$.

We conclude that for some exceptional divisor E over V with the centre at o the Noether-Fano type inequality

$$\mathrm{ord}_E\,\Sigma > n\Big(2\,\mathrm{ord}_E\,Q + a(E, V^+)\Big) \tag{2.6}$$

is satisfied. Furthermore, the centre $\triangle \subset Q$ of E on V has codimension at least 2 and dimension at least $2l$.

We now construct a resolution

$$V_0 = V \leftarrow V_1 = V^+ \leftarrow V_2 \leftarrow \cdots \leftarrow V_K,$$

of a discrete valuation given by E where $V_1 = V^+,\ E_1 = Q,\ B_o = o$, and $B_1 = \triangle$ so that E_K defines the discrete valuation ord_E, and all the constructions discussed in Section 2.2.2 hold for a blow-up $\varphi_{i,i-1} : V_i \to V_{i-1}$, which is the blow-up of the centre B_{i-1} of E on V_{i-1}, where $i \geq 2$ (as V_i are non-singular at the generic point of B_i). Set

$$E_i = \varphi^{-1}_{i,i-1}(B_{i-1}) \subset V_i$$

to be an exceptional divisor such that $E_1 = Q$.

As in the proof of the $4n^2-$inequality, we take the graph associated with this resolution and split the sequence $\varphi_{i,i-1},\ i = 1, \ldots, K$, of blow-ups into:

- a lower part, consisting of $\{1, 2, \ldots, L\}$, corresponding to centres B_{i-1} such that $\operatorname{codim} B_{i-1} \geq 3$, and;
- an upper part consisting $\{L + 1, \ldots, K\}$, corresponding to centres B_{i-1} such that $\operatorname{codim} B_{i-1} \geq 2$.

As we did during the proof of the $4n^2-$inequality, we denote the strict transform of a variety V_i by setting

$$\nu_i = \operatorname{mult}_{B_i-1} \Sigma^i \text{ for any } i = 2, \ldots, K.$$

Let Γ be the oriented graph of the resolution of the singularity E and p_{ij} be the number of paths from the vertex i to the vertex j. Set $p_{ii} = 1$, and $p_i = p_{Ki},\ i = 1, \ldots, K$. With this in mind, the Noether-Fano type inequality 2.6 becomes

$$\sum_{i=1}^{K} p_i \nu_i \ > \ \left(2p_1 + \sum_{i=2}^{K} p_i \delta_i\right), \tag{2.7}$$

where $\nu_1 = \nu$ and $\delta_i = \mathrm{codim}(B_{i-1} \subset V_{i-1})$.

Lemma 20. *Let $Y \subset \mathbb{P}^N$ be a non-singular complete intersection of codimension $l \geq 1$, $S \subset Y$ an irreducible subvariety of codimension $a \geq 1$, and $B \subset Y$ an irreducible subvariety of dimension $a \cdot l$, where the estimate $N \geq (l+1)(a+1)$ is satisfied. Then the inequality*

$$\mathrm{mult}_B\, S \leq m$$

holds, where $m > 1$ is defined by the condition $S \sim mH_Y^a$ and $H_Y \in A^1Y$ is the class of a hyperplane sections of Y.

Proof. The case $l = 1$ was given in [25]. The argument for the arbitrary l was discussed in [31]. □

We immediate apply Lemma 20 to Σ_Q^1, and conclude that $\nu_1 \geq \nu_2$, since $\dim B_1 = \dim \triangle \geq 2l$. Based on the resolution, we get

$$\nu_2 \geq \nu_3 \geq \cdots \geq \nu_K.$$

Let $D_1, D_2 \in \Sigma$ be general divisors, and

$$Z = (D_1 \circ D_2)$$

be their scheme-theoretic intersection, the self-intersection of the linear system Σ. We now use the technique of counting multiplicities as follows: write

$$(D_1^i \circ D_2^i) = \left(D_1^{i-1} \circ D_2^{i-1}\right)^i + Z_i,$$

for $i \geq 1$, where Z_i is supported on E_i so can be seen as an effective divisor on E_i. Thus for any $i \leq L$, we get

$$(D_1^i \circ D_2^i) = Z_0^i + Z_1^i + \cdots + Z_{i-1}^i + Z_i.$$

For any $j > i$, where $j \leq L$ set

$$m_{i,j} = \mathrm{mult}_{B_{j-1}}\, Z_i^{j-1},$$

and $d_i = \deg Z_i$ for $i = 2, \ldots, L$. Furthermore, for Z_1 on $E_1 = Q$ we have the relation

$$Z_1 \sim d_1 H_Q,$$

where $d_1 \in \mathbb{Z}_{\geq 0}$, and H_Q is the divisor class of the hyperplane section of $Q \subset \mathbb{P}^{4l+2}$. Following [[26], Chapter 2], we have the following list of equalities

$$\begin{array}{rcl} \nu_1^2 + d_1 & = & m_{0,1} \\ \nu_2^2 + d_2 & = & m_{0,2} + m_{1,2} \\ & \ldots & \\ \nu_i^2 + d_i & = & m_{0,i} + \cdots + m_{i-1,i} \\ & \ldots & \end{array}$$

$i = 2, 3, \ldots, L$ where

$$d_L \geq \sum_{i=L+1}^{K} \nu_i^2$$

holds.

Lemma 21. *The inequalities*

$$d_1 \geq m_{1,2},$$

and

$$m_{0,1} \geq \mu m_{0,2}$$

hold.

Proof. The first inequality follows from Lemma 20, the fact that $Z_i \sim d_1 H_Q$, and $\dim B_1 \geq 2l$. In the case of the second inequality, note the numerical equivalence

$$\begin{array}{rcl} (Z^1 \circ E_1) & \sim & \dfrac{1}{\mu} \deg(Z^1 \circ E_1) H_Q^2 \\ & \sim & \dfrac{1}{\mu} m_{0,1} H_Q^2. \end{array}$$

Now we apply Lemma 20 to $Z^1 \circ E_1$, and we get

$$m_{0,2} = \text{mult}_{B_1} Z_0^1 \leq \frac{1}{\mu} m_{0,1},$$

which is the intended result. □

Note that $m_{0,1} \geq \mu m_{0,i}$ for $i \geq 3$ as

$$m_{0,2} \geq m_{0,3} \geq \cdots \geq m_{0,L}.$$

We set some additional notation by setting,

$$m^*_{i,j} = \mu m_{i,j}$$

for $(i,j) \neq (0,1)$ and $m^*_{0,1} = m_{0,1}$, and

$$d^*_i = \mu d_i$$

for $i = 1, \dots, L$.

So by substitution into the earlier list of equalities, we obtain

$$\begin{aligned}
\mu\nu_1^2 + d_1^* &= m^*_{0,1} \\
\mu\nu_2^2 + d_2^* &= m^*_{0,2} + m^*_{1,2} \\
&\cdots \\
\mu\nu_i^2 + d_i^* &= m^*_{0,i} + \cdots + m^*_{i-1,i} \\
&\cdots
\end{aligned}$$

where $i = 2, 3, \dots, L$ and

$$d^*_L \ \geq \ \mu \sum_{i=L+1}^{K} \nu_i^2$$

holds.

□

2.2.5 The Technique of Hypertangent Divisors

This is a technique that will feature very heavily in the proof of birational superrigidity in Chapter 3. The technique of hypertangent divisors is generally based on certain regularity conditions. We will now state these conditions in a general form, and introduce their use when we discuss hypertangent divisors. In Chapter 3, we will use this technique to exclude infinitely near maximal singularities for both non-singular, as well as for all three forms of singular complete intersections of codimension 3 discussed in Section 3.0.1.

Regularity Conditions

We follow the notations from Section 3.0.1. Let us arrange the forms

$$q_{i,j},\ i \in \{1,2,3\},\ \text{with } j > 2,$$

in the standard order, corresponding to the lexicographic order of pairs (i,j). That is, that a pair (i_1, j_1) precedes (i_2, j_2), if $j_1 < j_2$ or $j_1 = j_2$ but $i_1 < i_2$.

This results in some sequence

$$h_1, h_2, \ldots, h_{M+3-l} \tag{2.8}$$

of $M+3-l$ forms in $z_* = (z_1, \ldots, z_{M+3})$ of non-decreasing degrees.

Definition 65. The point $o \in V$ is *regular* if, for some integer function N_l in l, the sequence of polynomials obtained from the first $N_l + 3$ terms of sequence 2.8 is a regular sequence in $\mathcal{O}_{\mathbb{P},o}$.

Hypertangent Divisors

To exclude a maximal singularity E, we can use the construct of *hypertangent divisors*, and *hypertangent linear systems*. We will now explain how the technique of using hypertangent divsors for this purpose works, and apply it in a manner similar to its use in the paper [7]. In Chapter 3, we use the technique of hypertangent divisors as part of the exclusion of maximal singularities and therefore the proof of superrigidty. This technique, and the associated hypertangent linear systems were developed for excluding infinitely near maximal singularities.

Fix a point o in an affine chart $\mathbb{C}^{M+3}$ of the space $\mathbb{P} = \mathbb{P}^{M+3}$ with coordinates $(z_1, \ldots, z_{M+3})$ and origin o. Let $j > 2$ be an integer. Now recall from Section 3.0.1 that for some $l \in \{0, 1, 2, 3\}$ and a subset $I \subset \{1, 2, 3\}$, such that $|I| = 3 - l$, the linear forms $q_{i,1}, i \in I$, are linearly independent, whereas the other forms $q_{i,1}, i \notin I$, are their linear combinations.

Denote by

$$f_{i,\alpha} = q_{i,1} + \cdots + q_{i,\alpha}, \tag{2.9}$$

the truncated i-th equation of the polynomials in the triple $\underline{f}$, and the $q_{i,j}$ are forms of degree for i in z_*, and $j \in \{1, \alpha\}$.

Definition 66 (j-th hypertangent system)**.** Define the *j-th hypertangent system* at the point o as follows:

$$\Lambda(j) = \left\{ \left(\sum_{i \in I} q_{i,1} s_{i,j-1} + \sum_{i=1}^{3} \sum_{\alpha=2}^{d_i - 1} f_{i,\alpha} s_{i,j-\alpha} \right) \Bigg|_V = 0 \right\},$$

where $s_{i,j-\alpha}$ run independently through the set of homogeneous polynomials of degree $j - \alpha$ in the variables z_* (if $j - \alpha < 0$, then $s_{j-\alpha} = 0$).

Alternatively, a non-empty linear system Σ on V is *hypertangent* (with respect to a point $o \in V$) if $\Sigma^+ \subset |kH - lE|$, where $l \geq k + 1$, and Σ^+ is the strict transform of the system Σ on V^+.

Now set for $j \geq 2$:

$$\begin{aligned} c(j) &= 3 - l + \#\Big\{(i,\alpha) \mid i = 1,2,3 \;\; 1 \leq \alpha \leq \min\{j, d_i - 1\}\Big\}, \\ c(1) &= 3 - l, \\ c(0) &= 0. \end{aligned}$$

Also set

$$m(j) = c(j) - c(j-1), \text{ for } j \in \{1, \dots, d_3 - 1\},$$

and take general divisors

$$D_{j,1}, \dots, D_{j,m(j)}$$

in the linear system $\Lambda(j)$. We put these divisors into the lexicographic order of the pairs (j, α), where

$$(j_1, \alpha_1) < (j_2, \alpha_2), \text{ if } \alpha_1 < \alpha_2, \text{ or } \alpha_1 = \alpha_2 \text{ and } j_1 < j_2.$$

So we obtain a sequence

$$R_1, \dots, R_{M-l} \tag{2.10}$$

of effective divisors on V.

Definition 67 (Hypertangent Divisors)**.** Denote the affine space $\mathbb{A}^{M+3}_{(z_1,\dots,z_{M+3})} \subset \mathbb{P}$. Consider the divisors

$$D_i = \overline{\{f_{i,\alpha}|_{\mathbb{A}\cap V} = 0\}}, \text{ for } i = 1,2,3 \text{ and } 1 \leq \alpha \leq d_i - 1.$$

These are said to be *hypertangent divisors*: if $H \in \operatorname{Pic} V$ is the class of a hyperplane section, then clearly

$$D_i \in |iH| \text{ and } \operatorname{mult}_o D_i \geq i + 1,$$

since in the affine part of V

$$f_{i,\alpha}\Big|_{\mathbb{A}\cap V} = -(q_{i,\alpha+1} + \cdots + q_{i,d_i})\Big|_V.$$

Alternatively, an effective divisor D on V is said to be *hypertangent* with respect to the point o, if $D^+ \in |mH - lE|$ where $l \geq m+1$, and D^+ is the strict transform of the divisor D on V^+. The number $\beta(D) = l/m > 1$ is called the *slope* of the divisor D.

Key to the technique is the construction of a sequence of irreducible subvarieties

$$Y_0 = Y, Y_1, \ldots, Y_M,$$

indexed by a sequence of distinct indices $i(1), i(2), \ldots, i(M)$ such that:

- $Y_{k+1} \subset Y_k$, and $\dim Y_k = M + 1 - k$, and $\operatorname{codim}_V Y_k = k + 2$;
- $Y_k \subset D_{i(k)}$, so that $(Y_k \circ D_{i(k)})$ is an effective cycle on V and Y_{k+1} is one of its irreducible components;
- the following estimate holds:

$$\frac{\mathrm{mult}_o}{\deg} Y_{k+1} \geq \frac{\mathrm{mult}_o}{\deg} Y_k \cdot \frac{i(k)+1}{i(k)}$$

  for each $k = 1, \ldots, M+2$.

Having introduced the regularity conditions, we retrieve N_l, which represents the number of divisors in the sequence derived from the regularity conditions.

Proposition 4. We have the equality

$$\operatorname{codim}_o \left(\left(\bigcap_{j=1}^{N_l} |R_j| \right) \subset V \right) = N_l, \tag{2.11}$$

where codim_o denotes the codimension in an arbitrarily small neighbourhood of the point o.

Proof. As stated above,

$$f_{i,\alpha}|_V = -(q_{i,\alpha+1} + \cdots + q_{i,d_i})\Big|_V \tag{2.12}$$

for $1 \leq \alpha \leq d_i$, where the codimension of the base locus of the tangent linear system $\Lambda(1)$ near the point o is equal to $(3-l)$ and the hypertangent linear system $\Lambda(j),\ j > 2$ is equal to

$$(3-l) + \operatorname{codim}\left(\Big\{q_{i,\alpha} = 0 \mid i \in \{1,2,3\},\ 1 \leq \alpha \leq 1 + \min\{j, d_i - 1\}\Big\}\right).$$

Therefore, for general hypertangent divisors R_*, the equality

$$\operatorname{codim}\left(\Big(\bigcap_{j=1}^{i} |R_j|\Big) \subset V\right) = i$$

follows from the regularity of the subsequence

$$h_1, \ldots, h_i$$

of the sequence 2.10. □

Remark 24. Following from Proposition 4, let $\mathcal{D}$ be a finite set of hypertangent divisors, such that $\#\mathcal{D} \leq \dim V - 1$. Applying the technique of hypertangent divisors depends on the equality of the codimension of the base locus of the hypertangent divisors in $\mathcal{D}$ to the expected codimension at every point. That is,

$$\operatorname{codim}_o\left(\bigcap_{D \in \mathcal{D}} D\right) = \#\mathcal{D},$$

holds. In particular, the technique works when a general variety satisfies this condition at every point.

2.2.6 The Connectedness Principle

The connectedness principle of Shokurov & Kollár has a number of useful applications in birational geometry. In particular, we will be using it to prove the theorem on inversion of adjunction introduced below.

Theorem 22. *Let V and W be normal varieties and $h : V \to W$ be a proper morphism with connected fibres. Also let $D = \sum d_i D_i$ be a $\mathbb{Q}$-divisor on V. Furthermore, suppose that $-(K_V + D)$ is h-nef and h-big. Now assume that the map*

$$f : Y \xrightarrow{g} V \xrightarrow{h} W$$

is a resolution of singularities of the pair (V, D). If we set

$$K_Y = g^*(K_V + D) + \sum e_i E_i,$$

then the support

$$\bigcup_{e_i \leq -1} e_i E_i$$

of $\sum_{e_i \leq -1} e_i E_i$ is connected in a neighbourhood of any fibre of the morphism f.

Proof. See Theorem 17.4 in [18] □

2.2.7 Inversion of Adjunction

This result was proved by Shokurov in dimension three in [32], and by Kollár [18] in arbitrary dimension. We make use of the version of the theorem discussed here in a number of proofs in this text, and it features more generally in the presentation of the theory of birational rigidity in [26].

Theorem 23 (Inversion of Adjunction)**.** *Let $x \in V$ be a germ of a $\mathbb{Q}$-factorial terminal variety V, and let D be an effective $\mathbb{Q}$-divisor, the support of which contains the point x. Let $R \subset V$ be an irreducible subvariety of codimension one, $R \subset \operatorname{Supp} D$, and, moreover, R is a Cartier divisor. Assume that the pair (V, D) is not canonical at the point x, but that it is canonical outside that point. That is, suppose that the point x is an isolated centre of non-canonical singularities of that pair. Then the pair $(R, D_R = D|_R)$ is not log canonical at the point x.*

Proof. Say $D = \sum_{i \in I} d_i D_i$, and let D_i be irreducible components of D such that $d_i \in \mathbb{Q}_{>0}$ for all $i \in I$. By assumption, the pair (V, D) is canonical outside of the point o. So for all geometric valuations E, the inequality $\nu_E(D) > n \cdot a(E)$ holds. Taking a general irreducible subvariety $B \subset \operatorname{Supp} D_i$ for some i of codimension 1, then $a(E) = 1$. But $\nu_E(D_i) > 1$, which implies $d_i \leq 1$ for all $\in I$. The condition to be non-canonical at o is a strict inequality so that D can be replaced by $\frac{1}{1+\epsilon} D$ for some small $\epsilon \in \mathbb{Q}_+$, so we can assume $d_i < 1$ for all $i \in I$.

Let $\varphi : \widetilde{V} \to V$ be a resolution of singularities of the pair $(V, D + R)$. Now

$$K_{\widetilde{V}} = \varphi^*(K_V + D + R) + \sum_{j \in J} e_j E_j - \sum_{i \in I} d_i \widetilde{D}_i - \widetilde{R}, \tag{2.13}$$

where $E_j, j \in J$, are exceptional divisors of φ, and $\widetilde{D}, \widetilde{R}$ are strict transforms of D and R respectively. Set

$$b_j = \operatorname{ord}_{E_j} \varphi^* D, \ a_j = a(E_j, V), \ \text{for } j \in J.$$

Note that by construction,

$$e_j = a(E_j) - \operatorname{ord}_{E_j} \varphi^* D - \operatorname{ord}_{E_j} \varphi^* R.$$

For some $J' \subset J$, we have

$$\varphi^{-1}(o) = \bigcup_{j \in J'} E_j,$$

and since $o \in R$, we have $\operatorname{ord}_{E_j} \varphi^* R \geq 1$ for all $j \in J'$.

Recall that by assumption (V, D) is not canonical at the point o

$$\operatorname{ord}_{E_j} \varphi^* D > a(E_j) \text{ for some } j \in J',$$

and consequently

$$\operatorname{ord}_{E_j} \varphi^* D > e_j + \operatorname{ord}_{E_j} \varphi^* D + \operatorname{ord}_{E_j} \varphi^* R.$$

Therefore, $e_j < -1$ for some $j \in J'$. Following the connectedness principle, we note that since the coefficient of $\widetilde{R}$ is -1 in Equation 2.13, and we have established that there is an index r such that $e_r < -1$, it follows that $\widetilde{R} \cap E_r \neq \emptyset$.

Combining Equation 2.13 and the adjunction formula, we have

$$\begin{aligned} K_{\widetilde{R}} &= (K_V + \widetilde{R})|_{\widetilde{R}} \\ &= \varphi_R^*\Big(K_V + D + R\Big)\Big|_{\widetilde{R}} + \sum_{j\in J} e_j E_j|_{\widetilde{R}} - \sum_{i\in I} d_i \widetilde{D}_i|_{\widetilde{R}}. \end{aligned}$$

We apply the adjunction formula again, by noting that $K_R = (K_V + R)|_R$. And we set $\varphi_R = \varphi|_{\widetilde{R}} : \widetilde{R} \to R$ to be the restriction of φ onto R. So we have

$$\begin{aligned} K_{\widetilde{R}} &= (K_V + \widetilde{R})|_{\widetilde{R}} \\ &= \varphi_R^*(K_R + D|_R) + \Big(\sum_{j\in J} e_j E_j|_{\widetilde{R}} - \sum_{i\in I} d_i \widetilde{D}_i|_{\widetilde{R}}\Big). \end{aligned}$$

Now, for $l \in J^+$, there is at least one prime divisor of the form $E_l|_{\widetilde{R}}$ with coefficient $e_l < -1$. Therefore, the pair $(R, D_R = D|_R)$ is not log canonical at the point x. □

Chapter 3

Birational Rigidity of Complete Intersections of Codimension Three

One of the aims of this chapter is to demonstrate birational superrigidity for all singular complete intersections of codimension three. We begin by with Section 3.0.1, where we introduce and discuss complete intersections of codimension three and their singularities.

3.0.1 Complete Intersections of Codimension Three with Singularities

Denote by the symbol $\mathbb{P}$ the complex projective space $\mathbb{P}^{M+3}$, for positive M. For any integral triple $\underline{d} = (d_1, d_2, d_3)$ such that $d_1 \leq d_2 \leq d_3$, set $M = |d| - 3$, where

$$|d| = d_1 + d_2 + d_3.$$

Now let

$$\mathcal{P}(\underline{d}) = \mathcal{P}_{d_1,M+4} \times \mathcal{P}_{d_2,M+4} \times \mathcal{P}_{d_3,M+4}$$

be the space of triples of homogeneous polynomials (f_1, f_2, f_3) of degree d_1, d_2, and d_3 respectively, in the coordinates $(x_0 : \cdots : x_{M+3})$ on $\mathbb{P} = \mathbb{P}^{M+3}$. Now set

$$\mathcal{P}_{\text{fact}}(\underline{d}) \subset \mathcal{P}(\underline{d})$$

to be the set of triples $\underline{f} = (f_1, f_2, f_3)$ such that the zero set

$$V(\underline{f}) = \{f_1 = f_2 = f_3 = 0\} \subset \mathbb{P}$$

is an irreducible, reduced and factorial complete intersection of codimension 3.

Fixing a point $o \in \mathbb{P}$, we take a triple $(f_1, f_2, f_3) \in \mathcal{P}(\underline{d})$ with $o \in V = V(f_1, f_2, f_3)$. Then we fix a system of affine coordinates $(z_1, \ldots, z_{M+3})$ on an affine chart $\mathbb{C}^{M+3}$ of the space $\mathbb{P}$ with origin at the point o. It follows that f_1, f_2 and f_3 are of the form

$$\begin{aligned} f_1 &= q_{1,1} + q_{1,2} + \cdots + q_{1,d_1}, \\ f_2 &= q_{2,1} + q_{2,2} + \cdots + q_{2,d_2}, \\ f_3 &= q_{3,1} + q_{3,2} + \cdots + q_{3,d_3} \end{aligned}$$

where $q_{i,j}$ are homogeneous of degree j in $z_* = (z_1, \ldots, z_{M+3})$.

Complete intersections and the singularities which can occur on them, are fundamental to the main results of our work. These singularities are termed correct multi-quadratic singularities. A more general definition of these singularities can be found in Section 0.2 of [7], and the definition we provide here is a special case (where $k = 3$ - see [7], where the codimension of the complete intersection is denoted by k).

Definition 68 (Correct multi-quadratic singularity)**.** A point $o \in V$ is a *correct multi-quadratic singularity of type* 2^l, *where* $l \in \{0, 1, 2, 3\}$, if the following conditions are satisfied:

- $\dim\langle q_{1,1}, q_{2,1}, q_{3,1}\rangle = 3 - l$.
- for a general linear subspace $P \subset \mathbb{P}$ of dimension $\max\{8, 3l+6\}$, containing the point o, the intersection $V_P = V \cap P$ has an isolated singularity at the point o.
- for the blow-up $\varphi_P : V_P^+ \to V_P$ of the point o the exceptional divisor $Q_P = \varphi^{-1}(o)$ is a non-singular complete intersection of type 2^l in the $\max\{4+l, 4l+2\}$-dimensional projective space.

Remark 25. Correct multi-quadratic singularities of type 2^0 are non-singular points and the definition below is adjusted for this purpose. Furthermore, note that a multi-quadratic singularity of type 2^1 is a quadratic hypersurface singularity. For more details on correct multi-quadratic singularities, see [7].

We proceed to give an explicit description of what it means for the polynomials f_1, f_2, f_3 to define a variety with at most correct multi-quadratic singularities.

Definition 69 (Complete Intersection with At Most Correct Multi-quadratic Singularities)**.** An irreducible, reduced complete intersection $V = V(\underline{f})$ has at most correct multi-quadratic singularities if every point $o \in V$ is either non-singular, or a correct multi-quadratic singularity of type 2^l, where $l \in \{0, 1, 2, 3\}$. The set of triples $\underline{f} \in \mathcal{P}(\underline{d})$ such that $V(\underline{f})$ has at most multi-quadratic singularities is denoted $\mathcal{P}_{\mathrm{mq}}(\underline{d})$.

Remark 26. It was shown in [7] that a variety with singularities of this type is factorial and its singularities are terminal.

Assume that for $l \geq 0$,

$$\dim\langle q_{1,1}, q_{2,1}, q_{3,1}\rangle = 3 - l,$$

and let $I \subset \{1, 2, 3\}$ be a subset with $|I| = 3 - l$ such that the linear forms $\{q_{i,1} \mid i \in I\}$ are linearly independent. Set $\Pi \subset \mathbb{C}^{M+3}$ to be the subspace

$$\Pi = \{q_{i,1} = 0 \mid i \in I\} \sim \mathbb{C}^{M+l}.$$

By assumption, for every $j \in J = \{1, 2, 3\}\backslash I$, there are (uniquely determined) constants $\lambda_{j,i},\ i \in I$, such that

$$q_{j,1} = \sum_{i \in I} \lambda_{j,i} q_{i,1}.$$

Now set, for every $j \in J$,

$$q^*_{j,2} = \left(q_{j,2} - \sum_{i \in I} \lambda_{j,i} q_{i,2} \right)\Bigg|_{\Pi}.$$

Quadratic Singularities

Suppose that exactly two of the linear forms $q_{1,1}, q_{2,1}$ and $q_{3,1}$ are linearly independent, that the forms in question are $q_{1,1}$ and $q_{2,1}$, and that

$$q_{3,1} = \lambda_{3,1} q_{1,1} + \lambda_{3,2} q_{2,1}$$

for some $\lambda_{3,1},\ \lambda_{3,2} \in \mathbb{C}$.

Then we have the equalities

$$\begin{aligned} q_{1,1}\Big|_{\{f_1\,=\,f_2\,=\,0\}} &= -(q_{1,2} + \cdots + q_{1,d_1})\Big|_{\{f_1\,=\,f_2\,=\,0\}} \\ q_{2,1}\Big|_{\{f_1\,=\,f_2\,=\,0\}} &= -(q_{2,2} + \cdots + q_{2,d_2})\Big|_{\{f_1\,=\,f_2\,=\,0\}}, \end{aligned}$$

and as a result

$$q_{3,1}\Big|_{\{f_1\,=\,f_2\,=\,0\}} = -\lambda_{3,1}(q_{1,2} + \cdots + q_{1,d_1}) - \lambda_{3,2}(-q_{2,2} - \cdots - q_{2,d_2})\Big|_{\{f_1\,=\,f_2\,=\,0\}}.$$

It follows that

$$f_3 - \lambda_{3,1} f_1 - \lambda_{3,2} f_2 \;=\; \Big(q_{3,2} - \lambda_{3,1} q_{1,2} - \lambda_{3,2} q_{2,2}\Big) + \ldots,$$

where the ellipsis represents higher order terms. Isolating the quadratic terms, the quadratic singularity is given by the quadratic form

$$\Big(q_{3,2} - \lambda_{3,1} q_{1,2} - \lambda_{3,2} q_{2,2}\Big)\Big|_{\{q_{1,1}\,=\,q_{2,1}\,=\,0\}}.$$

Following Definition 68, we intersect $V = V(f_1, f_2, f_3)$ with a general linear subspace $P \subset \mathbb{P}$ of dimension

$$\max\Big\{8,\ 3(1)+6\Big\} = 9.$$

The intersection $V_P = V \cap P$ is of dimension 6, and it has an isolated singularity at o. When we blow-up V_P at the point o, the resulting exceptional divisor $Q_P = \varphi^{-1}(o)$ is a non-singular quadric of dimension 5 in $\mathbb{P}^6$.

Bi-quadratic Singularities

Let us assume that $q_{1,1} \not\equiv 0$, and

$$q_{2,1} = \lambda_{2,1} q_{1,1}, \text{ and } q_{3,1} = \lambda_{3,1} q_{1,1},$$

for some $\lambda_{2,1}, \lambda_{3,1} \in \mathbb{C}$.

Note that

$$q_{2,1}|_{\{f_1=0\}} = \lambda_{2,1} q_{1,1}|_{\{f_1=0\}} = -\lambda_{2,1}(q_{1,2} + \cdots + q_{1,d_k}).$$

Restricting f_2 to the set $\{f_1 = 0\}$,

$$f_2\Big|_{\{f_1=0\}} = (f_2 - \lambda_{2,1} f_1)\Big|_{\{f_1=0\}} = \Big((q_{2,2} - \lambda_{2,1} q_{1,1}) + \dots\Big)\Big|_{\{f_1\,=\,0\}}.$$

Isolating the quadratic terms, the restriction $f_2\Big|_{\{f_1=0\}}$ starts with the quadratic form

$$(q_{2,2} - \lambda_{2,1} q_{1,2})\Big|_{\{q_{1,1}\,=\,0\}}. \tag{3.1}$$

Repeating this for f_3, we have

$$\begin{aligned} f_3\Big|_{\{f_1=0\}} &= (f_3 - \lambda_{3,1} f_1)\Big|_{\{f_1=0\}} \\ &= \Big((q_{3,2} - \lambda_{3,1} q_{1,2}) + \dots\Big)\Big|_{\{f_1\,=\,0\}}, \end{aligned}$$

which starts with the quadratic form

$$(q_{3,2} - \lambda_{2,1} q_{1,2})\Big|_{\{q_{1,1}\,=\,0\}}. \tag{3.2}$$

So we have two quadratic polynomials, restricted to the hyperplane $\{q_{1,1} = 0\}$. Following Definition 68, we intersect V with a general linear subspace $P \subset \mathbb{P}$ of dimension

$$\max\Big\{8, 3(2) + 6\Big\} = 12.$$

The intersection $V_P = V \cap P$ is of dimension 9, and it has an isolated singularity at o. When we blow-up V_P at the point o, the resulting exceptional divisor $Q_P = \varphi^{-1}(o)$ is a non-singular complete intersection of two quadrics given by the quadratic forms

$$(q_{2,2} - \lambda_{2,1} q_{1,2})\Big|_{P\,\cap\,\{q_{1,1}=0\}}$$

and

$$(q_{3,2} - \lambda_{3,1} q_{1,2})\Big|_{P\,\cap\,\{q_{1,1}=0\}}$$

in $\mathbb{P}^{10}$, and this is non-singular complete intersection of dimension 8.

Multi-quadratic Singularities

Suppose that we have,

$$q_{1,1} = q_{2,1} = q_{3,1} = 0,$$

so that the quadratic forms $q_{1,2}, q_{2,2}, q_{3,2}$ define a complete intersection of the type $2 \cdot 2 \cdot 2$.

Following Definition 68, we intersect

$$V = V(f_1, f_2, f_3)$$

with a general linear subspace $P \subset \mathbb{P}$ of dimension

$$\max\Big\{8, 3(3) + 6\Big\} = 15.$$

The intersection $V_P = V \cap P$ is of dimension 12, and it has an isolated singularity at o. When we blow-up V_P at the point o, the resulting exceptional divisor $Q_P = \varphi^{-1}(o)$ is a non-singular complete intersection of three quadrics of dimension 8 in $\mathbb{P}^{14}$.

Henceforth, we denote the quadratic, bi-quadratic, and multi-quadratic singular loci on V by $\mathrm{QSing}\, V$, $\mathrm{BSing}\, V$ and $\mathrm{MSing}\, V$.

3.1 Statement of the Result.

Let us recall some notation from Section 3.0.1. Recall first that we denote the complex projective space $\mathbb{P}^{M+3}$, for positive M by $\mathbb{P}$. Furthermore, we have set integral triples $\underline{d} = (d_1, d_2, d_3)$ such that

$$d_1 \leq d_2 \leq d_3,$$

and

$$|d| = d_1 + d_2 + d_3 = M + 3.$$

We also considered

$$\mathcal{P}(\underline{d}) = \mathcal{P}_{d_1,M+4} \times \mathcal{P}_{d_2,M+4} \times \mathcal{P}_{d_3,M+4},$$

which is the space of triples of homogeneous polynomials (f_1, f_2, f_3) in the coordinates $(x_0 : \cdots : x_{M+3})$ on $\mathbb{P} = \mathbb{P}^{M+3}$ of degrees d_1, d_2, and d_3 respectively.

Remark 27. The condition that $|d| = M+3$ has been imposed to ensure that the complete intersection is of index 1. This is due to the adjunction formula for complete intersections in the projective space. By this we mean Example 4, where $k = 3$, and $M + 3$.

The following claim is the main result of the chapter:

Theorem 24. *There exists a Zariski open subset $\mathcal{P}_{reg}(\underline{d}) \subset \mathcal{P}_{\text{fact}}(\underline{d})$ such that:*

(i) for every triple $\underline{f} \in \mathcal{P}_{reg}(\underline{d})$, the Fano variety $V = V(\underline{f})$ has terminal singularities, and is birationally superrigid, and;

(ii) the estimate

$$\operatorname{codim}\left(\left(\mathcal{P}(\underline{d})\Big\backslash\mathcal{P}_{\mathrm{reg}}(\underline{d})\right)\subset\mathcal{P}(\underline{d})\right)\geq\frac{(M-10)(M-11)}{2}-2$$

holds.

Let $\mathcal{P}_{\mathrm{mq}}$ denote the set of triples $(f_1, f_2, f_3) \in \mathcal{P}(\underline{d})$ such that $V(f_1, f_2, f_3)$ is an irreducible reduced complete intersection in $\mathbb{P}$ with every point $o \in V(f_1, f_2, f_3)$ being non-singular or a correct multi-quadratic singularity of type 2^l, $l \in \{1, 2, 3\}$ (in the sense of Definition 68). The following theorem follows from Theorem 0.2 in [7]:

Theorem 25. *The following inequality holds:*

$$\operatorname{codim}\Big((\mathcal{P}(\underline{d})\backslash\mathcal{P}_{\mathrm{mq}}(\underline{d})\subset\mathcal{P}(\underline{d})\Big)\geq\frac{(M-11)(M-10)}{2}-2.$$

3.2 Proof of Birational Rigidity I: Exclusion of Infinitely Near Maximal Singularities

We now begin the process of proving birational superrigidity. In Section 3.2.1, we impose certain so-called regularity conditions on our complete intersections, resulting in a Zariski open set $\mathcal{P}_{\mathrm{reg}}(\underline{d})$. Then we will prove birational superrigidity by the method of maximal singularities, and the technique of hypertangent divisors.

Fix a mobile linear system $\Sigma \subset |nH|$ on V, where H is the class of a hyperplane section. We will show that the linear system Σ has no maximal subvarieties. Then in Subsection 2.2.5, we introduce the construct of hypertangent divisors, which will play a key role in the process of excluding infinitely near maximal singularities on Σ. This exclusion process will consider in turn, each of the following four mutually exclusive scenarios:

- Σ has an infinitely near maximal singularity, whose centre B on V is not within the singular locus, henceforth denoted by $\operatorname{Sing} V$ (Subsection 3.2.2).

- Σ has an infinitely near maximal singularity, and $B \subset \operatorname{Sing} V$, but B is not in the locus of bi-quadratic or multi-quadratic points (Subsection 3.2.3).
- Σ has an infinitely near maximal singularity, and B is inside the locus of bi-quadratic points, but not inside the locus of multi-quadratic points (Subsection 3.2.4).
- Σ has an infinitely near maximal singularity, but its centre is singular and in the locus of multi-quadratic points (Subsection 3.2.5).

By excluding all of these cases (in the order above), we will conclude that a mobile linear system on $V = V(\underline{f})$ cannot have a maximal singularity, thereby implying the birational superrigidity of V.

3.2.1 Regular Complete Intersections

As mentioned in Section 2.2.5, to prove birational superrigidity, we impose some conditions of general position, called the regularity conditions, on the variety concerned. Recall that we discussed the three separate forms of singular complete intersections in Section 3.0.1. Note that we treat the non-singular case as a correct multi-quadratic singularity (see Definition 68) of type 2^l for $l = 0$.

For a triple $(\underline{f})$, we will now state the regularity conditions upon which our use of the technique of hypertangent divisors will be based.

(R.1) Let $\mathcal{D}$ be a finite set of hypertangent divisors, such that $\#\mathcal{D} \leq \dim V - 1$. Applying the technique of hypertangent divisors depends on the codimension of the base locus of the hypertangent divisors in $\mathcal{D}$ being equal to the expected codimension at every point. In particular, the technique works when a general variety in the family under consideration satisfies this condition at every point.

Now we will state the regularity condition $(R(a))$, where $a \in \{1, 2, 3, 4, 5\}$. The condition is dependent on the value of the parameter a, whose significance, and chosen values will be explained in later sections.

$(R(a))$ Let us arrange the forms

$$q_{i,j},\ i \in \{1,2,3\},\ \text{with } j > 2,$$

in the standard order, corresponding to the lexicographic order of pairs (i,j). That is, a pair (i_1, j_1) precedes (i_2, j_2), if $j_1 < j_2$, or $j_1 = j_2$ but $i_1 < i_2$. Concerning the forms above, this results is the sequence

$$q_{1,2}|_\Pi,\ q_{2,2}|_\Pi,\ \ldots,\ q_{3,d_3}|_\Pi,$$

and we proceed to remove the last $a \in \{1,2,3,4,5\}$ hypertangent divisors

$$q_{1,2}|_\Pi,\ q_{2,2}|_\Pi,\ \underbrace{\ldots,\ q_{3,d_3}|_\Pi}_{a \text{ divisors removed}},$$

so that the sub-sequence that remains is regular in $\mathcal{O}_{\mathbb{P},o}$.

Henceforth, the set of triples $(f_1, f_2, f_3) \in \mathcal{P}_{\text{mq}}(\underline{d})$, such that the regularity condition $(R(a))$ is satisfied at every point $o \in V(f_1, f_2, f_3)$ will be denoted by $\mathcal{P}_{\text{reg}}(\underline{d})$. So we can obtain $\mathcal{P}_{\text{reg}}(\underline{d})$ from $\mathcal{P}_{\text{mq}}(\underline{d})$ by removing some additional subsets.

We are now in a position to pivot to the beginning of the proof of Theorem 24 (i) by excluding maximal singularities on Σ.

Proposition 5. The linear system Σ on V has no maximal subvarieties.

Proof. The proof is virtually identical to the one found in Section 1.1 of [6]. □

3.2.2 Exclusion of Infinitely Near Maximal Singularities I: Centres Not Contained in the Singular Locus

In this section, we show that there are no infinitely near maximal singularities of a mobile linear system Σ on V whose centres are not contained in the singular locus of V.

Theorem 26. *The centre B of a maximal singularity E is contained in the singular locus* $\operatorname{Sing} V$.

Proof. Assume otherwise: $B \not\subset \operatorname{Sing} V$. That is, that $o \notin \operatorname{Sing} V$ for a general point $o \in B$. Since $\operatorname{codim}(B \subset V) \geq 3$, the $4n^2$-inequality holds, so for a general point $o \in B$,

$$\frac{\operatorname{mult}_o Z}{\deg Z} > \frac{4n^2}{n^2 \deg Z} = \frac{4}{d_1 d_2 d_3}.$$

Having established this, let us show that the reverse inequality holds to establish the needed contradiction:

Lemma 27. *For any non-singular point $o \in V$ and any irreducible subvariety Y of codimension 2, the inequality*

$$\frac{\operatorname{mult}_o}{\deg} Y \leq \frac{4}{d_1 d_2 d_3} \tag{3.3}$$

holds.

Proof. Suppose that this inequality does not hold, and let us denote three general divisors in a hypertangent system Λ_i by $D_{i,1}, D_{i,2}$, and $D_{i,3}$. Set $Y_2 = Y$, and let us construct a sequence of irreducible subvarieties $Y_i \ni o$ of codimension i in $V = V(f_1, f_2, f_3)$.

We consider the first hypertangent system (ie. the tangent system), which consists of tangent hyperplanes at the point o. Given the intersection of three tangent hyperplanes at the point, we see by the regularity condition, that the base locus $\operatorname{Bs}\Lambda_1$ is of codimension 3, and Y is of codimension 2. Since a general tangent divisor cannot contain Y_2, we can intersect Y_2 with one tangent divisor, and get Y_3, an irreducible component of $(Y_2 \circ D_{1,1})$, with the maximal value of $\operatorname{mult} / \deg$.

We have

$$\frac{\operatorname{mult}_o}{\deg} Y_3 \; > \; \frac{2}{1} \cdot \frac{\operatorname{mult}}{\deg} Y_2.$$

Now, Y_3 and $\operatorname{Bs}\Lambda_1$ are of codimension 3. However, due to our regularity conditions, and Lefschetz theorem, we can say that the intersection of the three tangent divisors is an irreducible subvariety, and has multiplicity 8. Since the ratio of multiplicity to degree of Y_3 exceeds that of $\mathrm{Bs}\,\Lambda_1$, it follows that $Y_3 \neq \mathrm{Bs}\,\Lambda_1$. So $\mathrm{Bs}\,\Lambda_1$ does not contain Y_3, and as a result, a general divisor in the tangent system cannot contain Y_3. We can take the component of $(Y_3 \circ D_{1,2})$ with maximal $\mathrm{mult}_o \,/\, \deg$, and denote it by Y_4.

We have the estimate

$$\frac{\mathrm{mult}_o}{\deg} Y_4 \;>\; \left(\frac{2}{1}\right)^2 \cdot \frac{\mathrm{mult}}{\deg} Y_2.$$

At this juncture, we cannot intersect Y_4 with one more divisor $D_{1,3} \in \Lambda_1$ because now $\mathrm{codim}\, Y_4 \;>\; \mathrm{codim}\,\mathrm{Bs}\,\Lambda_1$.

It is now necessary to shift to the second hypertangent system Λ_2, which is composed of hypertangent divisors cut out by quadrics. Now, $\mathrm{codim}(\mathrm{Bs}\,\Lambda_2) = 6$, since the base locus is obtained by taking

$$q_{1,1} = q_{2,1} = q_{3,1} = q_{1,2} = q_{2,2} = q_{3,2} = 0.$$

So we can intersect Y_4 with a general hypertangent divisor in Λ_2, to obtain a codimension 5 subvariety $Y_5 \subset (Y_4 \circ D_{2,1})$ of maximal ratio of multiplicity over degree. We intersect again to get $Y_6 \subset (Y_5 \circ D_{2,2})$, which is of codimension 6, satisfying the inequality

$$\frac{\mathrm{mult}_o}{\deg} Y_6 \;>\; \frac{\mathrm{mult}}{\deg} Y_4 \cdot \left(\frac{3}{2}\right)^2.$$

Now

$$\mathrm{codim}\, Y_6 = \mathrm{codim}(\mathrm{Bs}\,\Lambda_2) = 6,$$

so we cannot intersect Y_6 with a third hypertangent divisor in Λ_2. Consequently, we migrate to the hypertangent system Λ_3, and successively intersect with three hypertangent divisors in that system. This results in the estimate

$$\frac{\mathrm{mult}_o}{\deg} Y_9 \;>\; \left(\frac{3}{2}\right)^2 \cdot \left(\frac{4}{3}\right)^3 \cdot \frac{\mathrm{mult}}{\deg} Y_4.$$

This process continues until we reach $\Lambda_{d_1 - 1}$, which is the last hypertangent system where we take three general divisors to intersect with subvarieties of the sequence. After this, in all hypertangent systems until Λ_{d_2-1}, we intersect with two divisors because the codimension of the base loci of these systems come only from f_2 and f_3. Then until the final stage of the construction (that is, prior to Λ_{d_3-1}), we intersect with only one hypertangent divisor in each system.

If we let ρ be the product of slopes of the last a hypertangent divisors(these divisors are omitted), then we get the following inequality:

$$\frac{\text{mult}}{\text{deg}} Y_{M-a} >$$

$$\frac{1}{\rho} \cdot \frac{4}{d_1 d_2 d_3} \cdot \left(\frac{2}{1}\right)^2 \cdot \left(\frac{3}{2}\right)^2 \cdot \left(\frac{4}{3} \cdots \frac{d_1}{d_1-1}\right)^3 \cdot \left(\frac{d_1+1}{d_1} \cdots \frac{d_2}{d_2-1}\right)^2 \cdot \left(\frac{d_2+1}{d_2} \cdots \frac{d_3}{d_3-1}\right)$$

$$= \frac{4}{3\rho}.$$

Here also, we note that in Section 3.3, we explicitly compute for each $a \in \{1, 2, 3, 4, 5\}$, the value

$$M_i(a) = i \bmod 3, \quad i = 0, 1, 2,$$

such that for

$$M \geq M_i(a) \text{ and } M \equiv i \bmod 3,$$

the inequality $4 \geq 3\rho$ holds, in turn implying that $\text{mult}_o Y_{M-a} > \deg Y_{M-a}$, which cannot be. For instance, it is shown in Section 3.3, that when we remove the last 4 hypertangent divisors, we have

$$\frac{\text{mult}_o}{\text{deg}} Y_{N-4} > \frac{4}{3 \cdot \frac{d_3}{d_3-1} \cdot \frac{d_3-1}{d_3-2} \cdot \frac{d_2-1}{d_2-2} \cdot \frac{d_1-1}{d_1-2}} = \frac{4(d_3-2)(d_2-2)(d_1-2)}{3d_3(d_2-1)(d_1-1)} > 1$$

for:

$$\begin{cases} M \geq 45, & \text{when } M \cong 0 \bmod 3 \\ M \geq 43, & \text{when } M \cong 1 \bmod 3 \cdot \\ M \geq 41 & \text{when } M \cong 2 \bmod 3 \end{cases}$$

It would follow from this that $\mathrm{mult}_o Y_{N-4} > \deg Y_{N-4}$, which is impossible.

As a consequence,

$$\frac{\mathrm{mult}_o}{\deg} Y \ > \ \frac{4}{d_1 d_2 d_3}.$$

□

There is no infinitely near maximal singularity with centre $B \not\subset \mathrm{Sing}\, V$. □

3.2.3 Exclusion of Infinitely Near Maximal Singularities II: Singular Centres Not In the Loci of Bi-quadratic and Multi-quadratic points.

We have just established that the centre of an infinitely near maximal singularity on Σ must be contained in the singular locus. Now we turn to the task of showing that there is no infinitely near maximal singularity E with centre B, a general point of which is contained in $\mathrm{QSing}\, V$.

Theorem 28.

$$B \subset \mathrm{BSing}\, V \ \cup \ \mathrm{MSing}\, V.$$

Proof. Assume the converse: a general point $o \in B$ is in QSing, since $B \subset \mathrm{Sing}\, V$. By assumption, the $4n^2$-inequality for complete intersection singularities (see [28]) is satisfied:

$$\mathrm{mult}_o Z > 4 \cdot 2 \cdot n^2 = 8n^2.$$

We re-write this as

$$\frac{\mathrm{mult}_o Z}{\deg Z} > \frac{8n^2}{n^2 \deg V} = \frac{8}{d_1 d_2 d_3}.$$

We now move to show the reverse inequality:

Lemma 29. *For any non-singular point $o \in V$ and any irreducible subvariety Y of codimension 2, the inequality*

$$\frac{\mathrm{mult}_o}{\deg} Y \leq \frac{8}{d_1 d_2 d_3} \tag{3.4}$$

holds.

Proof. Suppose this inequality does not hold. Recall from Subsection 3.0.1 that quadratic singularities have the following characterisation: there are two linearly independent linear forms

$$q_{i,1}, \quad \text{with } i \in I \subset \{1,2,3\}, \text{ and } |I| = 2,$$

so say $q_{1,1}$ and $q_{2,1}$, and furthermore

$$q_{3,1} = \lambda_{3,1} q_{1,1} + \lambda_{3,2} q_{2,1}.$$

In general, the tangent system Λ_1 is defined by

$$\Lambda_1 = \left\{ \left(\sum_{i \in I} q_{i,1} s_{i,0} \right) \Big|_V = 0 \right\},$$

where $I \subset \{1,2,3\}$, $|I| = 3 - l$, and the $s_{i,o}$ are constants (see Subsection 68).

By the Lefschetz theorem (Theorem 12), $\operatorname{codim} \operatorname{Bs} \Lambda_1 = \operatorname{codim} Y_2 = 2$. Furthermore, due to the regularity conditions and Lefschetz theorem, the intersection of the two tangent divisors is an irreducible subvariety, and has multiplicity 4. Since the ratio of multiplicity to degree of Y_2 exceeds that of $\operatorname{Bs} \Lambda_1$, it follows that $Y_2 \neq \operatorname{Bs} \Lambda_1$, and in particular, $Y_2 \not\subset \operatorname{Bs} \Lambda_1$. For this reason, a general divisor $D'_{1,1}$ in Λ_1 will not contain Y_2, and so we can intersect it with Y_2 to obtain $Y_3 \subset (Y_2 \circ D'_{1,1})$, which is an irreducible component of Y_2 with maximal mult / deg at the point o.

As a result,

$$\frac{\operatorname{mult}_o}{\deg} Y_3 > \frac{2}{1} \cdot \frac{\operatorname{mult}_o}{\deg} Y_2.$$

However, we now have a situation where $\operatorname{codim} Y_3 > \operatorname{codim} \operatorname{Bs} \Lambda_1$, and so we cannot intersect a general divisor in Λ_1 with Y_3. This requires the introduction of the second hypertangent system Λ_2, where $\operatorname{codim} Y_3 < \operatorname{codim} \operatorname{Bs} \Lambda_2 = 5$.

We can therefore intersect Y_3 with a general divisor $D'_{2,1} \in \Lambda_2$ to obtain a codimension 4 subvariety $Y_4 \subset (Y_3 \circ D'_{2,1})$. We can do it again, by intersecting Y_4 with a second general divisor $D'_{2,2}$ to result in a codimension 5 subvariety $Y_5 \subset (Y_4 \circ D'_{2,2})$. At this stage, we have the estimate

$$\frac{\mathrm{mult}_o}{\deg} Y_5 > \frac{2}{1} \cdot \left(\frac{3}{2}\right)^2 \cdot \frac{\mathrm{mult}_o}{\deg} Y_2.$$

Since $\mathrm{Bs}\,\Lambda_2$ and Y_5 are of equal codimension, we have to move to the next hypertangent system, which is Λ_3 and is of codimension 8. A general divisor in Λ_3 does not contain Y_4. We can take a scheme theoretic intersection $(Y_5 \circ D'_{3,1})$ to get Y_6, which is of codimension 6. We obtain Y_7 after intersecting Y_6 with a general divisor $D'_{3,2} \in \Lambda_3$. We reach the limit with this hypertangent system when we get the subvariety $Y_8 \subset (Y_7 \circ D'_{3,3})$, and

$$\frac{\mathrm{mult}_o}{\deg} Y_8 > \frac{2}{1} \cdot \left(\frac{3}{2}\right)^2 \cdot \left(\frac{4}{3}\right)^2 \cdot \frac{\mathrm{mult}_o}{\deg} Y_2.$$

We move to Λ_4, and continue the process until Λ_{d_3}, and the resulting ratio of multiplicity over degree for the final subvariety Y_{M-1-a} is :

$$\frac{\mathrm{mult}_o}{\deg} Y_{M-1-a} >$$

$$\frac{1}{\rho} \cdot \frac{8}{d_1 d_2 d_3} \cdot \left(\frac{2}{1}\right)\left(\frac{3}{2}\right)^2\left(\frac{4}{3}\right)^3\left(\frac{5}{4} \cdots \frac{d_1}{d_1-1}\right)^3\left(\frac{d_1+1}{d_1} \cdots \frac{d_2}{d_2-1}\right)^2\left(\frac{d_2+1}{d_2} \cdots \frac{d_3}{d_3-1}\right)$$

$$= \frac{1}{\rho} \cdot \frac{8 \cdot d_1 \cdot d_2}{3 \cdot 2 \cdot d_1 \cdot d_2} = \frac{4}{3\rho},$$

where ρ has the same meaning as in the non-singular case.

Again we will compute in Section 3.3, the value

$$M_i(a) \equiv i \bmod 3, \ i = 0, 1, 2,$$

such that for

$$M \geq M_i(a), \ \text{and } M \equiv i \bmod 3,$$

the inequality $4 \geq 3\rho$ holds for each $a \in \{1,2,3,4,5\}$. As in the non-singular case, this results in a contradiction, from which it follows that

$$\frac{\mathrm{mult}_o}{\deg} Y \leq \frac{8}{d_1 d_2 d_3}.$$

□

We conclude that there is no infinitely near maximal singularity with centre $B \subset \mathrm{QSing}\, V$. □

3.2.4 Exclusion of Infinitely Near Maximal Singularities III: Singular Centres in the Locus of Bi-quadratic Points, But Not in the Locus of Multi-quadratic Points.

We now want to show that there is no infinitely near maximal singularity E with singular centre B which is also in the locus of bi-quadratic singularities.

Theorem 30. *The centre B of a maximal singularity E is not contained in the loci of bi-quadratic points. That is, $B \not\subset \mathrm{BSing}$.*

Proof. Assume the converse: say $B \subset \mathrm{BSing}$. Then a general point $o \in B$ is a correct bi-quadratic singularity, and by (see [28]), we have the inequality

$$\mathrm{mult}_o Z > 4 \cdot (2 \cdot 2) \cdot n^2 = 16n^2.$$

We re-write this as

$$\frac{\mathrm{mult}_o Z}{\deg Z} > \frac{16n^2}{n^2 \deg V} = \frac{16}{d_1 d_2 d_3}.$$

To prove that this is not the case, we once again look to show the reverse inequality.

Lemma 31. *For any non-singular point $o \in V$ and any irreducible subvariety Y of codimension 2, the inequality*

$$\frac{\mathrm{mult}_o}{\deg} Y \leq \frac{16}{d_1 d_2 d_3} \tag{3.5}$$

holds.

Proof. Suppose that the inequality does not hold. Recall from Subsection 3.0.1 that in the bi-quadratic case the linear forms $q_{i,1}$ are proportional and at least one of them is non-zero, say

$$q_{1,1} \not\equiv 0,$$

and

$$\begin{aligned} q_{2,1} &= \lambda_2 q_{1,1} \\ q_{3,1} &= \lambda_3 q_{1,1} \end{aligned}$$

So in this case the tangent system consists of the divisor $\{q_{1,1}|_V = 0\}$, and we cannot intersect $Y = Y_2$ with this divisor as Y_2 may be contained in it.

We move promptly to the next hypertangent system Λ_2, so that

$$\operatorname{codim} Y_2 < \operatorname{codim} \mathrm{Bs}\, \Lambda_2 = 4,$$

and we can therefore intersect Y_2 with a general divisor $D''_{2,1}$ to obtain a codimension 3 subvariety $Y_3 \subset (Y_2 \circ D''_{2,1})$. Then we go on to intersect Y_3 with a second general divisor $D''_{2,2}$ to generate $Y_4 \subset (Y_3 \circ D''_{2,2})$.

We have

$$\frac{\mathrm{mult}_o}{\deg} Y_4 > \left(\frac{3}{2}\right)^2 \cdot \frac{\mathrm{mult}_o}{\deg} Y_2.$$

At this juncture, both the base locus of the system Λ_2, and the subvariety Y_4 are of codimension 4. We cannot intersect Y_4 by a general divisor in Λ_2, and so we shift our attention to the hypertangent system Λ_3, which is cut out by cubics.

The codimension of $\mathrm{Bs}\,\Lambda_3$ is 7, so a general divisor in it does not contain Y_4. Consequently, we can take a scheme theoretic intersection $(Y_4 \circ D''_{3,1})$ to get Y_5, which is of codimension 5. We do this again when we obtain Y_6 after intersecting Y_5 with a general divisor $D''_{3,2} \in \Lambda_3$. We intersect Y_6 with a third general divisor $D''_{3,3} \in \Lambda_3$ to generate the codimension 7 subvariety Y_7.

Now,

$$\frac{\mathrm{mult}_o}{\deg} Y_7 > \left(\frac{3}{2}\right)^2 \cdot \left(\frac{4}{3}\right)^3 \cdot \frac{\mathrm{mult}_o}{\deg} Y_2,$$

so we move to Λ_4 and repeat the process until we reach the end of this process, where we obtain the inequality

$$\frac{\mathrm{mult}_o}{\deg} Y_{M-a-2} >$$

$$\frac{1}{\rho} \cdot \frac{16}{d_1 d_2 d_3} \cdot \left(\frac{3}{2}\right)^2 \cdot \left(\frac{4}{3}\right)^3 \cdot \left(\frac{5}{4} \cdots \frac{d_1}{d_1 - 1}\right)^3 \cdot \left(\frac{d_1+1}{d_1} \cdots \frac{d_2}{d_2-1}\right)^2 \cdot \left(\frac{d_2+1}{d_2} \cdots \frac{d_3}{d_3-1}\right)$$

$$= \frac{1}{\rho} \cdot \frac{16 \cdot d_1 \cdot d_2 \cdot d_3}{3 \cdot d_1 \cdot d_2 \cdot d_3 \cdot 2 \cdot 2} = \frac{4}{3\rho}.$$

As mentioned in Section 3.2.2, and 3.2.3, we show in Section 3.4 that $4 \geq 3\rho$ for

$$M \geq M_i(a), \text{ where } M \equiv i \bmod 3, \text{ and } a \in \{1,2,3,4,5\},$$

for explicitly computed values of $M_i(a)$. This is impossible, and so

$$\frac{\mathrm{mult}_o}{\deg} Y \leq \frac{16}{d_1 d_2 d_3}.$$

□

Since the $4n^2$-inequality (for complete intersection singularities) does not hold, there is no infinitely near maximal singularity with centre $B \subset \mathrm{BSing}\, V$.

□

3.2.5 Exclusion of Infinitely Near Maximal Singularities IV: Singular Centres In the Locus of Multi-quadratic points.

We want to show that there is no infinitely near maximal singularity E with singular centre B which is in the locus of multi-quadratic singularities.

Theorem 32. *The centre B of a maximal singularity E is not contained in the locus of multi-quadratic points. That is, $B \not\subset \operatorname{MSing} V$.*

Proof. Assume otherwise: say $B \subset \operatorname{MSing}$. As usual, let $o \in B$ be a general point. In the multi-quadratic case, under our assumptions, we can use the $4n^2$-inequality for complete intersection singularities as follows:

$$\operatorname{mult}_o Z > 4 \cdot (2 \cdot 2 \cdot 2) \cdot n^2 = 32n^2.$$

Alternatively,

$$\frac{\operatorname{mult}_o Z}{\deg Z} > \frac{32n^2}{n^2 \deg V} = \frac{32}{d_1 d_2 d_3}.$$

As in all the previous instances, we will be showing that the reverse inequality is in fact the case. In the present case, we aim to show that:

Lemma 33. *For any non-singular point $o \in V$ and any irreducible subvariety Y of codimension 2, the inequality*

$$\frac{\operatorname{mult}_o}{\deg} Y \leq \frac{32}{d_1 d_2 d_3} \tag{3.6}$$

holds.

Proof. Recall that $Y = Y_2$. Again, we assume that

$$\frac{\operatorname{mult}_o}{\deg} Y_2 > \frac{32}{d_1 d_2 d_3}.$$

From Subsection 3.0.1, with multi-quadratic singularities we have

$$q_{1,1} = q_{2,1} = q_{3,1} = 0,$$

so we start to apply the technique of hypertangent divisors from the second hypertangent system Λ_2. By the regularity condition, the base locus $\mathrm{Bs}\,\Lambda_2$ is of codimension 3. Since

$$\operatorname{codim} \mathrm{Bs}\,\Lambda_2 > \operatorname{codim} Y_2,$$

a general divisor $D'''_{2,1} \in \Lambda_2$ does not contain Y_2, and so we can intersect Y_2 with any general divisor in Λ_2 to yield $Y_3 \subset (Y_2 \circ \Lambda_2)$. We have generated the slope $\frac{3}{2}$ once. We have

$$\frac{\mathrm{mult}_o}{\deg} Y_3 > \frac{3}{2} \cdot \frac{\mathrm{mult}_o}{\deg} Y_2.$$

Since

$$\operatorname{codim} \mathrm{Bs}\,\Lambda_2 = \operatorname{codim} Y_3 = 3,$$

we have to abandon Λ_2 in favour of the next hypertangent system Λ_3. The base locus of this system is of codimension 6, exceeding that of Y_3, and therefore a general divisor in it Λ_3 cannot contain Y_3. So we can intersect a general divisor $D'''_{3,1}$ with Y_3 to yield $Y_4 \subset (Y_3 \circ \Lambda_3)$. Comparing the codimensions of Y_4 and Λ_3, we can intersect Y_4 with another general divisor $D'''_{3,2}$ to give us the codimension 5 subvariety $Y_5 \subset (Y_4 \circ \Lambda_3)$. We can perform one more intersection, to obtain $Y_6 \subset (Y_5 \circ D'''_{3,3})$.

As a result,

$$\frac{\mathrm{mult}_o}{\deg} Y_6 > \frac{3}{2} \cdot \left(\frac{4}{3}\right)^3 \cdot \frac{\mathrm{mult}_o}{\deg} Y_2.$$

Noting that $\operatorname{codim} \mathrm{Bs}\,\Lambda_3 = \operatorname{codim} Y_6$, we shift to Λ_4, whose base locus is of codimension 9. We intersect a general divisor $D'''_{4,1}$ with Y_6, and get $Y_7 \subset (Y_6 \circ D'''_{4,1})$, which is of codimension 7. After another application of the procedure, we have $Y_8 \subset (Y_7 \circ D'''_{4,2})$, for a second general divisor $D'''_{4,2}$. There is room for one more intersection, and that is $(Y_8 \circ D'''_{4,3}) \supset Y_9$, where $D'''_{4,3}$ is again a general divisor.

At this stage, we see that

$$\frac{\mathrm{mult}_o}{\deg} Y_9 > \frac{3}{2} \cdot \left(\frac{4}{3}\right)^3 \cdot \left(\frac{5}{4}\right)^3 \cdot \frac{\mathrm{mult}_o}{\deg} Y_2.$$

Having once again hit upon a subvariety of equal codimension to the base locus of our hypetangent system Λ_4, we move to the next system, which is Λ_5, and then we are free to continue the procedure.

We get the inequality

$$\frac{\text{mult}_o}{\deg} Y_{M-3-a} >$$

$$\frac{1}{\rho} \cdot \frac{32}{d_1 d_2 d_3} \cdot \left(\frac{3}{2}\right) \cdot \left(\frac{4}{3}\right)^3 \cdot \left(\frac{5}{4} \cdots \frac{d_1}{d_1-1}\right)^3 \cdot \left(\frac{d_1+1}{d_1} \cdots \frac{d_2}{d_2-1}\right)^2 \cdot \left(\frac{d_2+1}{d_2} \cdots \frac{d_3}{d_3-1}\right)$$

$$= \frac{1}{\rho} \cdot \frac{32 \cdot d_1 \cdot d_2 \cdot d_3}{3 \cdot 3 \cdot d_1 \cdot d_2 \cdot d_3 \cdot 2} = \frac{16}{9\rho}.$$

□

Refer to Section 3.3 for the proof that $16 \geq 9\rho$ for $M \geq M_i(a)$. Now we have a contradiction, and so there is no infinitely near maximal singularity with centre $B \subset \operatorname{MSing} V$. □

We have excluded all possibilities for the centre B of a maximal singularity of the linear system Σ, since in all instances, the assumption that Σ has a maximal singularity results in a contradiciton. This proves Theorem 24(i), the birational superrigidity of $V = V(\underline{f})$ for $\underline{f} \in \mathcal{P}_{\text{reg}}(\underline{d})$.

3.3 Proof of Birational Rigidity II: Minimal Dimension for Birational Rigidity of Regular Complete Intersections

In this section, we determine the minimal dimensions

$$M_i(a) \equiv i \bmod 3,\ i = 0, 1, 2,$$

such that given the removal of the last a hypertangent divisors, a complete intersection V, satisfying the regularity condition $(R(a))$, is birationally superrigid. It was shown in [7], that it is sufficient to consider this question in the following scenarios:

(1) where the degrees of f_1, f_2, f_3 are equal. That is

$$d_1 = d_2 = d_3 = d, \text{ so that } M \equiv 0 \bmod 3.$$

(2) where the degrees are "almost equal". That is,

(2.1) $d_1 = d_2$, and $d_3 = d_2 + 1$, so that $M \equiv 1 \bmod 3$,

(2.2) $d_2 = d_3 = d_1 + 1$, so that $M \equiv 2 \bmod 3$.

The reason why, for a given dimension M, we only need to consider cases (1), (2.1) or (2.2), (depending on the congruence class of $M(\bmod 3)$), is that when the degrees d_1, d_2, d_3 are the closest to each other, the product ρ of the slopes of the last a missing hypertangent divisors (in the notation of Section 3.2) is the highest. So if the technique of hypertangent divisors proves birational superrigidity in one of the three cases above, then it proves it for any triple of degrees d_1, d_2, d_3 such that

$$d_1 \leq d_2 \leq d_3, \text{ and, } d_1 + d_2 + d_3 = M + 3.$$

Consequently, in Section 3.4, and in the rest of Section 3.3, we consider only the cases (1), (2.1), and (2.2) in our computations.

Remark 28. These sections are subdivided in such a way that for each of these cases, we proceed to consider the results as we cycle through values of $a \in \{1, 2, 3, 4, 5\}$, where a is the number of removed hypertangent divisors. Furthermore, when the value of a is clear, the number $M_i(a)$ will be written simply as M_i.

3.3.1 Equal Degrees

By assumption,

$$d = d_1 = d_2 = d_3 = \frac{M+3}{3}.$$

Removing the last hypertangent divisor

When $a = 1$, we remove the hypertangent divisor which corresponds to $\frac{d_3}{d_3-1}$, and we have:

$$\begin{aligned}\frac{\text{mult}_o}{\deg}Y > \frac{4}{3 \cdot \frac{d_3}{d_3-1}} &= \frac{4(d_3-1)}{3 \cdot d_3} \\ &= \frac{4\left(\frac{M}{3}\right)}{3\left(\frac{M}{3}+1\right)} = \frac{4M}{9\left(\frac{M}{3}+1\right)} \geq 1\end{aligned}$$

So we have $M \geq 9$.

Removing the Last Two Hypertangent Divisors

When $a = 2$, then we remove the hypertangent divisors corresponding to the slopes

$$\frac{d_3}{d_3-1} \text{ and } \frac{d_2}{d_2-1},$$

and since the degrees are equal, we have

$$\begin{aligned}\frac{\text{mult}_o}{\deg}Y > \frac{4}{3 \cdot \frac{d_3}{d_3-1} \cdot \frac{d_2}{d_2-1}} &= \frac{4(d_3-1)(d_2-1)}{3 \cdot d_2 \cdot d_3} \\ &= \frac{4\left(\frac{M}{3}\right)^2}{3\left(\frac{M}{3}+1\right)^2} = \frac{4 \cdot \frac{M^2}{9}}{3\left(\frac{M}{3}+1\right)^2} \ \geq \ 1.\end{aligned}$$

We have the inequalities

$$M^2 - 18M - 27 \geq 0, \text{ and } M \geq 19.392.$$

The minimum value of $M_0 \geq M$ required for supperigidity is 21, for $M_0 \equiv 0(\text{mod})3$.

Removing the Last Three Hypertangent Divisors

If $a = 3$, then we remove the hypertangent divisors corresponding to

$$\frac{d_3}{d_3-1}, \frac{d_2}{d_2-1}, \text{ and } \frac{d_1}{d_1-1}.$$

Following the removal of the corresponding slopes, we have:

$$\begin{aligned}\frac{\text{mult}_o}{\deg}Y > \frac{4}{3\cdot\frac{d_3}{d_3-1}\cdot\frac{d_2}{d_2-1}\cdot\frac{d_1}{d_1-1}} &= \frac{4(d_1-1)(d_2-1)(d_3-1)}{3\cdot d_1\cdot d_2\cdot d_3} \\ &= \frac{4\left(\frac{M}{3}\right)^3}{3\left(\frac{M}{3}+1\right)^3} = \frac{4\cdot\frac{M^3}{27}}{3\left(\frac{M^3}{27}+\frac{M^2}{3}+M+1\right)} \geq 1.\end{aligned}$$

In the end, we have

$$M^3-27M^2-81M-81\geq 0, \quad \text{and} \quad M\geq 29.80850.$$

The minimum value of $M_0 \geq M$ for V supperigid is 30, since we have imposed the condition that $M_0 \equiv 0(\text{mod})3$.

Removing the Last Four Hypertangent Divisors

If $a=4$, we remove the hypertangent divisors corresponding to the slopes

$$\frac{d_3}{d_3-1}, \frac{d_2}{d_2-1}, \frac{d_1}{d_1-1} \text{ and } \frac{d_3-1}{d_3-2},$$

and we have:

$$\begin{aligned}\frac{\text{mult}_o}{\deg}Y > \frac{4}{3\cdot\frac{d_3}{d_3-1}\cdot\frac{d_2}{d_2-1}\cdot\frac{d_1}{d_1-1}\cdot\frac{d_3-1}{d_3-2}} &= \frac{4(d_3-2)(d_2-1)(d_1-1)}{3\cdot d_3\cdot d_2\cdot d_1} \\ &= \frac{4\left(\frac{M-3}{3}\right)\left(\frac{M}{3}\right)^2}{3\left(\frac{M+3}{3}\right)^3} \geq 1.\end{aligned}$$

We have the inequality

$$M^3-39M^2-81M-81\geq 0, \text{ and } M\geq 41.023.$$

The minimum value of $M_0 \geq M$ for which $M_0 \equiv 0(\text{mod})3$, is 42.

Removing the Last Five Hypertangent Divisors

If $a = 5$, then we remove the divisors corresponding to

$$\frac{d_3}{d_3-1},\ \frac{d_2}{d_2-1},\ \frac{d_1}{d_1-1},\ \frac{d_3-1}{d_3-2}\ \text{ and }\ \frac{d_2-1}{d_2-2}.$$

So we have:

$$\begin{aligned}\frac{\text{mult}_o}{\deg}Y > \frac{4}{3\cdot\frac{d_3}{d_3-1}\cdot\frac{d_2}{d_2-1}\cdot\frac{d_1}{d_1-1}\cdot\frac{d_3-1}{d_3-2}\cdot\frac{d_2-1}{d_2-2}} &= \frac{4}{3\cdot\frac{d_3}{d_3-2}\cdot\frac{d_2}{d_2-2}\cdot\frac{d_1}{d_1-1}} \\ &= \frac{4(d_3-2)(d_2-2)(d-1)}{3\cdot d_3\cdot d_2\cdot d_1} \\ &= \frac{4\left(\frac{M}{3}\right)\left(\frac{M}{3}-1\right)^2}{3\left(\frac{M}{3}+1\right)^3} \geq 1.\end{aligned}$$

For V superrigid, we have

$$M^3 - 51M^2 - 45M - 81 \geq 0,$$

and $M \geq 51.8945$. The least value $M_0 \geq M$ where $M_0 \equiv 0 (\mathrm{mod}\, 3)$ is 54.

Now we move towards the two cases where the degrees are "almost equal". We begin by consider the scenario where

$$\begin{aligned} d_1 &= d_2 \\ d_3 &= d_2 + 1. \end{aligned}$$

3.3.2 "Almost equal" Degrees : $M \equiv 1$ (mod 3)

As mentioned before, for $M \equiv 1$ (mod 3), we have

$$\begin{aligned} d_1 &= d_2 \\ d_3 &= d_2 + 1. \end{aligned}$$

Furthermore,

$$\begin{aligned} M+3 &= d_1+d_2+d_3 \\ M+3 &= d_2+d_2+(d_2+1)=3d_2+1, \end{aligned}$$

$$\Longrightarrow d_1=d_2=\frac{M+2}{3}, \quad \text{and} \quad d_3=\frac{M+5}{3}.$$

Removing the Last Hypertangent Divisor

When $a=1$, we remove the last hypertangent divisor, which corresponds to $\frac{d_3}{d_3-1}$, and at the conclusion of the technique of hypertangent divisor, we have:

$$\begin{aligned} \frac{\text{mult}_o}{\deg}Y > \frac{4}{3\cdot\frac{d_3}{d_3-1}} &= \frac{4(d_3-1)}{3\cdot d_3} \\ &= \frac{4\left(\frac{M+2}{3}\right)}{3\left(\frac{M+5}{3}\right)} = \frac{4(M+2)}{3(M+5)} \geq 1, \end{aligned}$$

and we have $M \geq 7$, and naturally, we set $M_1 = 7$.

Removing the Last Two Hypertangent Divisors

For $a=2$, we remove the divsors which correspond to

$$\frac{d_3}{d_3-1} \text{ and } \frac{d_2}{d_2-1}.$$

So the technique of hypertangent divisors ends with the inequality:

$$\begin{aligned} \frac{\text{mult}_0}{\deg}Y > \frac{4}{3\cdot\frac{d_3}{d_3-1}\cdot\frac{d_2}{d_2-1}} = \frac{4(d_3-1)(d_2-1)}{3\cdot d_3\cdot d_2} &= \frac{4\left(\frac{M+2}{3}\right)\left(\frac{M-1}{3}\right)}{3\left(\frac{M+5}{3}\right)\left(\frac{M+2}{3}\right)} \\ &= \frac{4(M-1)}{3(M+5)} \geq 1 \end{aligned}$$

So $M \geq 19$, and we set $M_1 = 19$.

Removing the Last Three Hypertangent Divisors

For $a = 3$, we remove the divisors corresponding to

$$\frac{d_3}{d_3 - 1},\ \frac{d_2}{d_2 - 1},\ \text{and}\ \frac{d_1}{d_1 - 1}.$$

We have as a result the inequality:

$$\begin{aligned}\frac{\mathrm{mult}_o}{\deg} Y > \frac{4}{3 \cdot \frac{d_3}{d_3-1} \cdot \frac{d_2}{d_2-1} \cdot \frac{d_1}{d_1-1}} &= \frac{4}{3 \cdot \frac{d_3}{d_2-1} \cdot \frac{d_1}{d_1-1}} \\ &= \frac{4(d_2 - 1)(d_1 - 1)}{3 \cdot d_3 \cdot d_1} \\ &= \frac{4\left(\frac{M-1}{3}\right)^2}{3\left(\frac{M+2}{3}\right)\left(\frac{M+5}{3}\right)} \geq 1.\end{aligned}$$

So for V superrigid, we have

$$M^2 - 29M - 26 \geq 0.$$

Therefore, $M \geq 29.87$ and we set $M_1 = 31$, since here $M \equiv 1 (\mathrm{mod}\, 3)$. This is the minimum value of M required for superrigidity.

Removing the Last Four Hypertangent Divisors

We remove the four divisors corresponding to

$$\frac{d_3}{d_3 - 1},\ \frac{d_2}{d_2 - 1},\ \frac{d_1}{d_1 - 1}\ \text{and}\ \frac{d_3 - 1}{d_{3'} - 2}.$$

At the end of the technique of hypertangent divisors, we have:

$$\begin{aligned}\frac{\mathrm{mult}_o}{\deg}Y > \frac{4}{3\cdot\frac{d_3}{d_3-1}\cdot\frac{d_2}{d_2-1}\cdot\frac{d_1}{d_1-1}\cdot\frac{d_3-1}{d_3-2}} &= \frac{4(d_3-1)(d_2-1)(d_1-1)(d_3-2)}{3\cdot d_3\cdot d_2\cdot d_1\cdot(d_3-1)}\\ &= \frac{4(d_2-1)(d_1-1)(d_3-2)}{3\cdot d_3\cdot d_1\cdot(d_3-1)}\\ &= \frac{4\left(\frac{M-1}{3}\right)^3}{3\left(\frac{M+5}{3}\right)\left(\frac{M+2}{3}\right)^2}\geq 1\end{aligned}$$

That is,

$$4(M-1)^3 \geq 3(M+5)(M+2)^2,$$

and

$$M^3-39M^2-60M-64\geq 0.$$

The lowest value of M for which this inequality holds, is 40.52. So we set $M_1 = 43$.

Removing the Last Five Hypertangent Divisors

In the case of $a = 5$, we remove divisors corresponding to the slopes

$$\frac{d_3}{d_3-1},\ \frac{d_2}{d_2-1},\ \frac{d_1}{d_1-1},\ \frac{d_3-1}{d_3-2},\ \text{and}\ \frac{d_2-1}{d_2-2}.$$

After removing the hypertangent divisors corresponding to these slopes, we have the inequality:

$$\begin{aligned}\frac{\mathrm{mult}_o}{\deg}Y > \frac{4}{3\cdot\frac{d_3}{d_3-1}\cdot\frac{d_2}{d_2-1}\cdot\frac{d_1}{d_1-1}\cdot\frac{d_3-1}{d_3-2}\cdot\frac{d_2-1}{d_2-2}} &= \frac{4}{3\cdot\frac{d_3}{d_2-1}\cdot\frac{d_1}{d_1-1}\cdot\frac{d_3-1}{d_2-2}}\geq 1\\ &= \frac{4(d_2-1)^2(d_2-2)}{3\cdot d_3\cdot d_1\cdot(d_3-1)}\\ &= \frac{4\left(\frac{M-1}{3}\right)^2\left(\frac{M-4}{3}\right)}{3\cdot\left(\frac{M+5}{3}\right)\left(\frac{M+2}{3}\right)^2}\\ &= \frac{4(M-1)^2(M-4)}{3(M+5)(M+2)^2}\geq 1.\end{aligned}$$

So we have:

$$M^3 - 51M^2 - 36M - 76 \geq 0.$$

The highest value of M for which this inequality is satisfied is 51.724. So we set $M_1 = 52$.

3.3.3 "Almost equal" Degrees : $M \equiv 2(\mathbf{mod}3)$

Once more, when $M \equiv 2(\mathrm{mod}3)$,

$$d_2 = d_3 = d_1 + 1.$$

Since,

$$\begin{aligned} M + 3 &= d_1 + d_2 + d_3 \\ &= d_1 + (d_1 + 1) + (d_1 + 1) \\ &= 3d_1 + 2. \end{aligned}$$

Then we have

$$\implies d_1 = \frac{M+1}{3}, \quad \text{and} \quad d_2 = d_3 = \frac{M+4}{3}.$$

Removing the Last Hypertangent Divisor

When $a = 1$, we remove the hypertangent divisor corresponding to $\frac{d_3}{d_3-1}$, with the technique of hypertangent divisors concluding as follows:

$$\begin{aligned} \frac{\mathrm{mult}_o}{\deg} Y > \frac{4}{3 \cdot \frac{d_3}{d_3-1}} &= \frac{4(d_3-1)}{3 \cdot d_3} \\ &= \frac{4\left(\frac{M+1}{3}\right)}{3\left(\frac{M+4}{3}\right)} \\ &= \frac{4M+4}{3M+12} \geq 1. \end{aligned}$$

Since $M \geq 8$, we set $M_2 = 8$.

Removing the Last Two Hypertangent Divisors

When $a = 2$, we remove the hypertangent divisors corresponding to the slopes

$$\frac{d_3}{d_3 - 1} \text{ and } \frac{d_2}{d_2 - 1},$$

and at the end of the technique of hypertangent divisors, we have :

$$\begin{aligned}\frac{\text{mult}_o}{\deg} Y > \frac{4}{3 \cdot \frac{d_2}{d_2 - 1} \cdot \frac{d_3}{d_3 - 1}} &= \frac{4(d_2 - 1)(d_3 - 1)}{3 \cdot d_2 \cdot d_3} \\ &= \frac{4\left(\frac{M+1}{3}\right)^2}{3\left(\frac{M+4}{3}\right)^2} \geq 1\end{aligned}$$

So we have the inequality

$$M^2 - 16M - 44 \geq 0,$$

and

$$M \geq 18.392.$$

The least value of $M_2 \geq M$ such that $M_2 \equiv 2 (\text{mod } 3)$ is 20, and so we set $M_2 = 20$.

Removing the Last Three Hypertangent Divisors

We remove those hypertangent divisors corresponding to

$$\frac{d_3}{d_3 - 1}, \frac{d_2}{d_2 - 1}, \text{ and } \frac{d_1}{d_1 - 1} = \frac{d_3 - 1}{d_3 - 2},$$

and at the end of the technique of hypertangent divisors, we have the inequality:

$$\begin{aligned}\frac{\text{mult}_o}{\deg} Y > \frac{4}{3 \cdot \frac{d_3}{d_3 - 1} \cdot \frac{d_2}{d_2 - 1} \cdot \frac{d_3 - 1}{d_3 - 2}} &= \frac{4(d_2 - 1)(d_3 - 2)}{3 d_2 d_3} \\ &= \frac{4\left(\frac{M+1}{3}\right)\left(\frac{M-2}{3}\right)}{3\left(\frac{M+4}{3}\right)^2} \\ &= \frac{4(M^2 - M - 2)}{3(M^2 + 8M + 16)} \geq 1.\end{aligned}$$

The result is that,

$$M^2 - 28M - 56 \geq 0,$$

and

$$M \geq 29.875.$$

The minimum value of $M_2 \geq M$ which is congruent to $2 \pmod 3$ is 32, and we set $M_2 = 32$.

Removing the Last Four Hypertangent Divisors

When $a = 4$, we remove the divisors which correspond to the slopes

$$\frac{d_3}{d_3 - 1},\ \frac{d_2}{d_2 - 1},\ \frac{d_1}{d_1 - 1},\ \text{and}\ \frac{d_3 - 1}{d_3 - 2}.$$

This removal of hypertangent divisors and slopes leads to the inequality:

$$\begin{aligned}
\frac{\text{mult}_o}{\deg} Y > \frac{4}{3 \cdot \frac{d_3}{d_3-1} \cdot \frac{d_2}{d_2-1} \cdot \frac{d_1}{d_1-1} \cdot \frac{d_3-1}{d_3-2}} &= \frac{4(d_3-1)(d_2-1)(d_1-1)(d_3-2)}{3 \cdot d_1 \cdot d_3 \cdot d_2 \cdot (d_3-1)} \\
&= \frac{4(d_2-1)(d_1-1)(d_3-2)}{3 \cdot d_3 \cdot d_2 \cdot (d_3-1)} \\
&= \frac{4\left(\frac{M+1}{3}\right)\left(\frac{M-2}{3}\right)^2}{3\left(\frac{M+4}{3}\right)^2\left(\frac{M+1}{3}\right)} \\
&= \frac{4(M+1)(M-2)^2}{3(M+4)^2(M+1)} \geq 1.
\end{aligned}$$

We have,

$$M^3 - 39M^2 - 72M - 32 \geq 0,$$

and

$$M \geq 40.785.$$

The lowest $M_2 \geq M$ which is congruent to 2 (mod) 3, is 41. So we set $M_2 = 41$.

Removing the Last Five Hypertangent Divisors

For $a = 5$, we remove the hypertangent divisors corresponding to the slopes

$$\frac{d_3}{d_3-1},\ \frac{d_2}{d_2-1},\ \frac{d_1}{d_1-1},\ \frac{d_3-1}{d_3-2},\ \text{ and }\ \frac{d_2-1}{d_2-2}.$$

So we have:

$$\begin{aligned}\frac{\mathrm{mult}_o}{\deg}Y > \frac{4}{3\cdot\frac{d_3}{d_3-1}\cdot\frac{d_2}{d_2-1}\cdot\frac{d_1}{d_1-1}\cdot\frac{d_3-1}{d_3-2}\cdot\frac{d_2-1}{d_2-2}} &= \frac{4(d_3-1)(d_2-1)(d_1-1)(d_3-2)(d_2-2)}{3\cdot d_3\cdot d_2\cdot d_1\cdot(d_3-1)\cdot(d_2-1)} \\ &= \frac{4(d_1-1)(d_3-2)(d_2-2)}{3\cdot d_3\cdot d_2\cdot d_1} \\ &= \frac{4\left(\frac{M-2}{3}\right)^3}{3\left(\frac{M+4}{3}\right)^2\left(\frac{M+1}{3}\right)} \\ &= \frac{4(M-2)^3}{3(M+4)^2(M+1)} \geq 1\end{aligned}$$

Ultimately, this means that

$$\implies M^2 - 51M - 80 \geq 0$$

$$M \geq 51.496.$$

The minimum value of $M_2 \geq M$ where M is congruent to 2(mod)3, is 53, and so we set $M_2 = 53$.

Let us summarise the results of this section.

3.3.4 Results

For M congruent to $0, 1$, and 2 modulo 3, superrigidity holds respectively for:

(i) M at least $9, 7$, and 8, when we remove the final hypertangent divisor.

(ii) M at least $21, 19$, and 20, when we remove the last two Hypertangent Divisors.

(iii) M at least $30, 31$, and 32, when the last 3 hypertangent divisors are removed.

(iv) M at least $42, 43$, and 41, with the removal of the last 4 hypertangent divisors.

(v) M not less than $54, 52$, and 53, when the last 5 hypertangent divisors are removed.

3.4 Estimation of Codimension of Complete Intersections With Non-Regular Points

In this section, we prove the claim of Theorem 24(ii). We will show that for M large enough (with an explicit bound for how large it should be) the codimension of the complement $\mathcal{P}(\underline{d})\backslash\mathcal{P}_{\rm reg}(\underline{d})$ is at least

$$\frac{(M-10)(M-11)}{2} - 2.$$

In order to estimate the codimension of triples (f_1, f_2, f_3) such that $V(f_1, f_2, f_3)$ does not satisfy the regularity condition $(R(a))$, we use the projection method (explained for instance, in Chapter 3 of [26]). The details of this will be given below. In this section, the integral parameter a takes the three values $a \in \{3, 4, 5\}$ as the estimates for $a = 1, 2$ are too weak. We make a few preliminary remarks:

Remark 29. We make use of the observation that was made in [7]: that the worst estimates of the codimension of the set of non-regular triples (f_1, f_2, f_3) are obtained in the cases when the degrees d_1, d_2, d_3 are equal or "almost equal" in the sense of Section 3.3. So for a fixed dimension M, we will consider only these cases.

Remark 30. We localize the problem: fix a point $o \in \mathbb{P}$ and obtain estimates of the set of triples such that the complete intersection $V(f_1, f_2, f_3)$ contains the point o, and is non-regular there. Now we use an observation from [7] that the worst estimates correspond to the case when $o \in V(f_1, f_2, f_3)$ is a non-singular point. Given that the point $o \in \mathbb{P}$ is arbitrary and varies in $(M+3)$-projective space, but the condition that

$$f_1(o) = f_2(o) = f_3(o) = 0$$

gives codimension 3 for the triple (f_1, f_2, f_3), we conclude that if the codimension in the local problem is c, then in the global problem, it is $c - M$. For this reason, we will consider the local problem (for the case of equal or "almost equal" degrees) for a fixed point o, and find out the lower bound for dimensions M such that

$$c \geq \frac{(M-10)(M-11)}{2} - 2 + M = \frac{1}{2}(M^2 - 19M + 106).$$

We now begin the proof of Theorem 24 (ii).

Proof. **3.4.1 Equal Degrees**

Fix a non-singular point $o \in V$ and a system of affine coordinates $z_1, \ldots, z_{M+3}$ on an affine chart of $\mathbb{P}^{M+3}$ with the origin at o. Suppose that

$$d_1 = d_2 = d_3 = d,$$

and as a result

$$M + 3 = 3d.$$

Now,

$$\begin{aligned} f_1 &= q_{1,1} + q_{1,2} + \cdots + q_{1,d}, \\ f_2 &= q_{2,1} + q_{2,2} + \cdots + q_{2,d}, \\ f_3 &= q_{3,1} + q_{3,2} + \cdots + q_{3,d}. \end{aligned}$$

Looking at the equations above, we have $3d$ homogeneous polynomials of the form $q_{i,j}$. We order them lexicographically as follows:

$$q_{1,1}, q_{2,1}, q_{3,1}, q_{1,2}, q_{2,2}, q_{3,2}, \ldots, q_{1,d}, q_{2,d}, q_{3,d},$$

and we want to know whether this sequence is regular or not when we remove a few final terms. Since the point $o \in V$ is non-singular, we have a linear subspace

$$\Pi = T_o V = \{q_{1,1} = q_{2,1} = q_{3,1} = 0\} \sim \mathbb{C}^M.$$

We fix Π, and form the following sequence of homogenous polynomials in Π

$$q_{1,2}|_\Pi,\ q_{2,2}|_\Pi,\ q_{3,2}|_\Pi,\ \dots\ q_{3,d}|_\Pi.$$

We want to estimate the codimension of the set of such sequences with the last $a \in \{3,4,5\}$ polynomials removed, which are not regular (in the space of all sequences).

Projectivizing Π to $\mathbb{P}^{M-1}$, we have

$$3(d-1) - a = M - a$$

homogeneous polynomials on $\mathbb{P}^{M-1}$. We use the projection method (explained and used in [26], Chapter 3): we fix the first polynomial in the sequence where the regularity condition is violated. Beginning with the first polynomial in the sequence, violation of the regularity conditions means that $q_{1,2} \equiv 0$ on $\mathbb{P}^{M-1}$ which yields the codimension

$$\binom{M-1+2}{2} = \binom{M+1}{2}.$$

Then we project to a hyperplane, which is $\mathbb{P}^{M-2}$, and the codimension of non-regular polynomials is $\binom{M}{2}$ if regularity is violated at $q_{2,2}$. At the next step, we project again to $\mathbb{P}^{M-3}$ to get codimension $\binom{M-1}{2}$ if regularity is violated at $q_{3,2}$. This brings us to the end of the quadrics, and their minimum estimate is $\binom{M-1}{2}$.

Furthermore, the minimum estimate for the cubics is

$$\binom{M-1-2}{3} = \binom{M-3}{3},$$

and more generally, after say, k iterations of this, we have a minimum codimension of

$$\binom{M-2k-1}{k+2}.$$

This process generates the following sequence of mimimum estimates for each degree up to $d-2$ (if regularity is violated for the first time at that degree):

$$\binom{M-1}{2},\ \binom{M-3}{3},\ \binom{M-5}{4},\ldots,\ \binom{d+4}{d-2}=\binom{d+4}{6}. \tag{3.7}$$

It is necessary to consider the fact that the intended sequence of polynomials is obtained by removing the last $a \in \{3,4,5\}$ members. Without that, the last 6 codimensions would have been

$$\binom{d+4}{d-1}=\binom{d+4}{5},\quad \binom{d+3}{d-1}=\binom{d+3}{4},\quad \binom{d+2}{d-1}=\binom{d+2}{3},$$

and

$$\binom{d+2}{d}=\binom{d+2}{2},\quad \binom{d+1}{d}=\binom{d+1}{1}=d+1\,,\quad \binom{d}{d}=1.$$

If we remove the last $a=3$ polynomials, we have to add the number $\binom{d+2}{3}$, to the sequence (3.7) of minimum estimates. If $a=4$, we add the number $\binom{d+3}{4}$, and if $a=5$, we add $\binom{d+4}{5}$.

Let us denote what remains of the sequence (3.7) after the removal of the last a polynomials by (3.7a). Our aim is to find the minimal element of this sequence for each of the three values of a under consideration. We use the following well known properties of binomial coefficients $\binom{A}{B}$: that it is an increasing function of A when B is fixed, and an increasing function of B when A is fixed, provided that $B \leq \left[\frac{A}{2}\right]$.

So examining these binomial coefficients and using the equality

$$\binom{A}{B}=\binom{A}{A-B}$$

(so that we ensure that the bottom number does not exceed half of the top one), we see that the top row of numbers reduces steadily, but the bottom row increases steadily until a certain point, and then subsides until we reach the value of 3. As explained above, we need to find the lower bound for the dimension M such that the minimal element of the sequence (3.7) is

$$\geq \frac{(M-11)(M-10)}{2} + (M-2) \ = \ \frac{1}{2}(M^2 - 19M + 106).$$

Using the properties of binomial coefficients mentioned above, we see that, when we remove the last $a = 3$ hypertangent divisors, the minimal element of the sequence of binomial coefficients is

$$\min\left\{\binom{M-1}{2}, \binom{M-3}{3}, \binom{d+2}{3}\right\}.$$

When we remove $a = 4$ hypertangent divisors, the minimal element of the sequence is

$$\min\left\{\binom{M-1}{2}, \binom{M-3}{3}, \binom{M-5}{4}, \binom{d+3}{4}\right\}.$$

And finally, when we remove $a = 5$, we look for

$$\min\left\{\binom{M-1}{2}, \binom{M-3}{3}, \binom{M-5}{4}, \binom{M-7}{5}, \binom{d+4}{5}\right\}.$$

So set:

$$P_0 = P_0(M) = \frac{(M-11)(M-10)}{2} + (M-2) = \frac{1}{2}(M^2 - 19M + 106).$$

The main method of comparing these polynomials will be to simply compare the values of these polynomials before and after the points where their curves intersect, with a view to determining in particular, the range of values of M for which $P_0(M)$ is smaller or equal to the remaining binomial coefficients. In both cases where the degrees of f_1, f_2 and f_3 are “almost equal”, we take the same approach, except that we acknowledge the numerical differences in degree during the computations. Before we begin these comparisons, let us fix some notation.

First let:

$$
\begin{aligned}
P_1 &= \binom{M-1}{2} = \frac{(M-1)(M-2)}{2}, \\
P_2 &= \binom{M-3}{3} = \frac{(M-3)(M-4)(M-5)}{6}, \\
P_3 &= \binom{M-5}{4} = \frac{(M-5)(M-6)(M-7)(M-8)}{24}, \\
\text{and } \ P_4 &= \binom{M-7}{5} = \frac{(M-7)(M-8)(M-9)(M-11)}{120},
\end{aligned}
$$

For $a \in \{3,4,5\}$, we will use the symbols $P_E^{(a)}, P_{A1}^{(a)}$, and $P_{A2}^{(a)}$ to denote the binomial coefficient that we compare with P_0, P_1, P_2, P_3 and/or P_4 when M is $0, 1$, and 2 modulo 3. In the case of equal degrees, $P_E^{(a)}$ denotes the polynomial which is added to sequence 3.7, when the last a polynomials are removed. For example, we set

$$P_E^{(3)} = \binom{d+2}{3}, \text{ and } \ P_E^{(4)} = \binom{d+3}{4}.$$

Similarly, for $M \equiv 1 \pmod 3$ and $M \equiv 2 \pmod 3$, we denote the polynomial which is added to sequence 3.7, when the last a polynomials are removed by $P_{A1}^{(a)}$ and $P_{A2}^{(a)}$ respectively.

Removing the Last Three Hypertangent Divisors

Since

$$M + 3 = 3d,$$

$$
\begin{aligned}
P_E^{(3)} = \binom{\frac{M}{3}+3}{3} &= \frac{\left(\frac{M}{3}+3\right)\left(\frac{M}{3}+2\right)\left(\frac{M}{3}+1\right)}{6} \\
&= \frac{M^3}{162} + \frac{M^2}{9} + \frac{11}{18}M + 1
\end{aligned}
$$

We will now find out for what values of M, we have $P_0 = \min\left\{P_1, P_2, P_E^{(3)}\right\}$.

$$\begin{cases} P_2 < P_1 \leq P_E^{(3)} \leq P_0, & \text{for } 0 < M \leq 6 \\ P_2 < P_E^{(3)} < P_1 \leq P_0, & \text{for } 6 < M \leq 6.5, \\ P_2 < P_E^{(3)} \leq P_0 < P_1, & \text{for } 6.5 < M \leq 6.675, \\ P_2 \leq P_0 < P_E^{(3)} < P_1, & \text{for } 6.675 < M \leq 7.891, \\ P_0 < P_2 \leq P_E^{(3)} < P_1, & \text{for } 7.891 < M \leq 9, \\ P_0 < P_E^{(3)} < P_2 \leq P_1, & \text{for } 9 < M \leq 10.104, \\ P_0 < P_E^{(3)} < P_1 < P_2, & \text{for } M > 10.104 \end{cases}$$

If we remove the last three hypertangent divisors, then codim $\left(\mathcal{P}(\underline{d})\backslash\mathcal{P}_{\text{mq}}(\underline{d})\right) \geq P_0$, for $M \geq 7.891$. However, since we require that $M \equiv 0 (\text{mod } 3)$, we set $M \geq 9$.

Let us now consider what happens when we remove four hypertangent divisors.

Removing the Last Four Hypertangent Divisors

Since we are removing 4 polynomials, we compare $P_E^{(4)}, P_0$, and P_2. Let us write $P_E^{(4)}$ as a polynomial:

$$\begin{aligned} P_E^{(4)} &= \binom{\frac{M}{3}+4}{4} \\ &= \frac{\left(\frac{M}{3}+4\right)\left(\frac{M}{3}+3\right)\left(\frac{M}{3}+2\right)\left(\frac{M}{3}+1\right)}{24} \\ &= \frac{(M+12)(M+9)(M+6)(M+3)}{1944}. \end{aligned}$$

Let us see for what values of M, we have $P_0 = \min\left\{P_1, P_2, P_E^{(4)}\right\}$.

We have

$$\begin{cases} P_2 < P_1 < P_E^{(4)} \leq P_3 < P_0 \ \text{ for } 0 < M \leq 3 \\ P_2 < P_1 \leq P_3 < P_E^{(4)} \leq P_0, \ \text{ for } 3 < M \leq 3.603 \\ P_2 \leq P_3 < P_1 < P_E^{(4)} < P_0, \ \text{ for } 3.603 < M \leq 5 \\ P_3 < P_2 < P_1 < P_E^{(4)} \leq P_0, \ \text{ for } 5 < M \leq 5.879 \\ P_3 < P_2 < P_1 \leq P_0 < P_E^{(4)}, \ \text{ for } 5.879 < M \leq 6.5 \\ P_3 < P_2 \leq P_0 < P_1 < P_E^{(4)} \ \text{ for } 6.5 < M \leq 7.891 \\ P_0 < P_3 < P_2 < P_1 \leq P_E^{(4)}, \ \text{ for } 7.891 < M \leq 10.104 \\ P_0 < P_3 < P_1 < P_2 \leq P_E^{(4)}, \ \text{ for } M > 11.027 \end{cases}$$

When we remove the last four hypertangent divisors, then $\operatorname{codim}\left(\mathcal{P}(\underline{d})\backslash\mathcal{P}_{\mathrm{mq}}(\underline{d})\right) \geq P_0$, for $M \geq 7.891$, and again we set $M \geq 9$.

Removing the Last Five Hypertangent Divisors

Set

$$P_E^{(5)} = \binom{5+d-1}{d-1} = \binom{d+4}{5}.$$

Then

$$\begin{aligned} P_E^{(5)} &= \binom{\frac{M}{3}+5}{5} \\ &= \frac{\left(\frac{M}{3}+5\right)\left(\frac{M}{3}+4\right)\left(\frac{M}{3}+3\right)\left(\frac{M}{3}+2\right)\left(\frac{M}{3}+1\right)}{120} \\ &= \frac{(M+15)(M+12)(M+9)(M+6)(M+3)}{29160} \end{aligned}$$

The relevant comparisons break down as follows:

$$\begin{cases} P_2 < P_1 < P_E^{(5)} < P_3 \leq P_4 < P_0, & \text{for } \ 0 < M \leq 1.847 \\ P_2 < P_1 < P_E^{(5)} \leq P_3 < P_4 < P_0, & \text{for } \ 1.847 < M \leq 2.896 \\ P_2 < P_1 \leq P_3 < P_4 < P_0 \leq P_E^{(5)}, & \text{for } \ 2.896 < M \leq 3.233 \\ P_2 < P_1 \leq P_3 < P_4 < P_E^{(5)} \leq P_0, & \text{for } \ 3.233 < M \leq 3.603 \\ P_2 < P_3 < P_1 \leq P_4 < P_E^{(5)} < P_0, & \text{for } \ 3.603 < M \leq 4.088 \\ P_2 \leq P_3 < P_4 < P_1 < P_E^{(5)} < P_0, & \text{for } \ 4.088 < M \leq 5 \\ P_3 < P_2 \leq P_4 < P_1 < P_0 < P_E^{(5)}, & \text{for } \ 5 < M \leq 5.65 \\ P_3 < P_4 \leq P_2 < P_1 \leq P_0 < P_E^{(5)}, & \text{for } \ 5.65 < M \leq 6.5 \\ P_3 \leq P_4 < P_2 < P_0 < P_1 < P_E^{(5)}, & \text{for } \ 6.5 < M \leq 7 \\ P_3 < P_4 < P_2 \leq P_0 < P_1 < P_E^{(5)}, & \text{for } \ 7 < M \leq 7.891 \\ P_3 < P_4 < P_0 < P_2 \leq P_1 < P_E^{(5)}, & \text{for } \ 7.891 < M \leq 10.104 \\ P_4 < P_3 \leq P_0 < P_2 < P_1 < P_E^{(5)}, & \text{for } \ 10.104 < M \leq 10.424 \\ P_4 < P_0 < P_3 \leq P_1 < P_2 < P_E^{(5)}, & \text{for } \ 10.424 < M \leq 12.874 \\ P_4 < P_0 < P_1 < P_3 \leq P_2 < P_E^{(5)}, & \text{for } \ 12.874 < M \leq 15.231 \\ P_4 \leq P_0 < P_1 < P_2 < P_3 < P_E^{(5)}, & \text{for } \ 15.231 < M \leq 17 \\ P_0 < P_4 \leq P_1 < P_2 < P_3 < P_E^{(5)}, & \text{for } \ 17 < M \leq 21.19 \\ P_0 < P_1 \leq P_4 < P_2 < P_3 < P_E^{(5)}, & \text{for } \ M > 21.19 \end{cases},$$

Since $\operatorname{codim}\left(\mathcal{P}(\underline{d})\backslash\mathcal{P}_{\mathrm{mq}}(\underline{d})\right) \geq P_0$, for $M > 17$, and we require M to be divisible by 3, we set $M \geq 18$.

3.4.2 "Almost equal" Degrees : $M \equiv 1 \pmod 3$

We now move to consider the case, where

$$\begin{aligned} d_1 &= d_2, \\ d_3 &= d_2 + 1. \end{aligned}$$

Removing the Last Three Hypertangent Divisors

Set

$$P_{A1}^{(3)} = \binom{d_3+2}{d_3-1} = \binom{d_3+2}{3} = \binom{d_1+3}{3}.$$

Then

$$\begin{aligned} P_{A1}^{(3)} &= \binom{\frac{M}{3}+\frac{5}{3}+2}{3} \\ &= \frac{\left(\frac{M}{3}+\frac{2}{3}+3\right)\left(\frac{M}{3}+\frac{2}{3}+2\right)\left(\frac{M}{3}+\frac{2}{3}+1\right)}{6} \\ &= \frac{(M+11)(M+8)(M+5)}{162} \\ &= \frac{1}{162}(M^3+24M^2+183M+440). \end{aligned}$$

Let us see for what values of M is $P_0 = \min\left\{P_1, P_2, P_{A1}^{(3)}\right\}$.

$$\begin{cases} P_2 < P_1 \leq P_{A1}^{(3)} \leq P_0, & \text{for } 0 < M \leq 5.698 \\ P_2 < P_1 \leq P_0 < P_{A1}^{(3)}, & \text{for } 5.698 < M \leq 6.5, \\ P_2 \leq P_0 < P_1 < P_{A1}^{(3)}, & \text{for } 6.5 < M \leq 7.891, \\ P_0 < P_2 < P_1 \leq P_{A1}^{(3)}, & \text{for } 7.891 < M \leq 9.598, \\ P_0 < P_2 \leq P_{A1}^{(3)} < P_1, & \text{for } 9.598 < M \leq 10, \\ P_0 < P_{A1}^{(3)} < P_2 \leq P_1, & \text{for } 10 < M \leq 10.104, \\ P_0 < P_{A1}^{(3)} < P_1 < P_2, & \text{for } M > 10.104 \end{cases}$$

So $\operatorname{codim}\left(\mathcal{P}(\underline{d})\backslash\mathcal{P}_{\rm mq}(\underline{d})\right) \geq P_0$ for $M > 7.891$. As we require that $M \equiv 1 \pmod 3$, we set $M_0 \geq 10$.

Removing the Last Four Hypertangent Divisors

Set

$$P_{A1}^{(4)} = \binom{d_3+3}{d_3-1} = \binom{d_3+3}{4}.$$

We have

$$
\begin{aligned}
P_{A1}^{(4)} &= \binom{\frac{M}{3}+4}{4} \\
&= \binom{\frac{M}{3}+\frac{5}{3}+3}{4} \\
&= \frac{\left(\frac{M}{3}+\frac{5}{3}+3\right)\left(\frac{M}{3}+\frac{5}{3}+2\right)\left(\frac{M}{3}+\frac{5}{3}+1\right)\left(\frac{M}{3}+\frac{5}{3}\right)}{24} \\
&= \frac{(M+14)(M+11)(M+8)(M+5)}{1944}.
\end{aligned}
$$

Since we are removing the last 4 terms, we have to compare the values of $P_{A1}^{(4)}, P_0$, and P_3. The breakdown of the values of the polynomials under consideration, and their corresponding values of M is:

$$
\begin{cases}
P_2 < P_1 < P_3 \leq P_{A1}^{(4)} < P_0, & \text{for } 0 < M \leq 2.5 \\
P_2 < P_1 \leq P_3 < P_{A1}^{(4)} \leq P_0, & \text{for } 2.5 < M \leq 3.603 \\
P_2 < P_1 < P_3 < P_{E}^{(4)} \leq P_0, & \text{for } 3.603 < M \leq 4.776 \\
P_2 < P_3 < P_1 < P_0 \leq P_{A1}^{(4)}, & \text{for } 4.776 < M \leq 5 \\
P_3 < P_2 < P_1 \leq P_0 < P_{A1}^{(4)}, & \text{for } 5 < M \leq 6.5 \\
P_3 < P_2 \leq P_0 < P_1 < P_{A1}^{(4)}, & \text{for } 6.5 < M \leq 7.891 \\
P_3 < P_0 < P_2 \leq P_1 < P_{A1}^{(4)}, & \text{for } 7.891 < M \leq 10.104 \\
P_3 < P_0 < P_1 \leq P_2 < P_{A1}^{(4)}, & \text{for } 7.891 < M \leq 10.104 \\
P_3 \leq P_0 < P_2 < P_1 < P_{A1}^{(4)}, & \text{for } 10.104 < M \leq 10.424 \\
P_0 < P_3 \leq P_1 < P_2 < P_{A1}^{(4)}, & \text{for } 10.424 < M \leq 12.874 \\
P_0 < P_1 < P_3 < P_2 \leq P_{A1}^{(4)}, & \text{for } 12.874 < M \leq 13.326 \\
P_0 < P_1 < P_3 \leq P_{A1}^{(4)} < P_2, & \text{for } 13.326 < M \leq 14.5 \\
P_0 < P_1 < P_{A1}^{(4)} < P_3 \leq P_2, & \text{for } 14.5 < M \leq 15.231 \\
P_0 < P_1 < P_{A1}^{(4)} < P_2 < P_3, & \text{for } M > 15.231
\end{cases}
$$

The value $M_1 \equiv 1 (\bmod 3)$ such that for $M \geq M_1$, the quadric $P_0(M)$ gives the minimum codimension is 13.

Removing the Last Five Hypertangent Divisors

Set

$$P_{A1}^{(5)} = \binom{5+(d_3-1)}{d_3-1} = \binom{d_3+4}{d_3-1}.$$

Then

$$\begin{aligned} P_{A1}^{(5)} &= \binom{\frac{M}{3}+\frac{5}{3}+4}{5} \\ &= \frac{\left(\frac{M}{3}+\frac{5}{3}+4\right)\left(\frac{M}{3}+\frac{5}{3}+3\right)\left(\frac{M}{3}+\frac{5}{3}+2\right)\left(\frac{M}{3}+\frac{5}{3}+1\right)\left(\frac{M}{3}+\frac{5}{3}\right)}{120} \\ &= \frac{(M+17)(M+14)(M+11)(M+8)(M+5)}{29160} \end{aligned}$$

We have:

$$
\begin{cases}
P_2 < P_1 < P_4 < P_3 \le P_0 < P_{A1}^{(5)}, & \text{for } 0 < M \le 0.576 \\
P_2 < P_1 < P_{A1}^{(5)} < P_3 \le P_4 < P_0, & \text{for } 0.576 < M \le 1.847 \\
P_2 < P_1 < P_{A1}^{(5)} \le P_3 < P_4 < P_0, & \text{for } 1.847 < M \le 2.323 \\
P_2 < P_1 < P_3 < P_{A1}^{(5)} \le P_4 < P_0, & \text{for } 2.323 < M \le 2.473 \\
P_2 < P_1 \le P_3 < P_4 < P_{A1}^{(5)} < P_0, & \text{for } 2.473 < M \le 3.603 \\
P_2 < P_3 \le P_1 < P_4 < P_{A1}^{(5)} < P_0, & \text{for } 3.603 < M \le 4.088 \\
P_2 < P_3 < P_4 < P_1 < P_{A1}^{(5)} \le P_0, & \text{for } 4.088 < M \le 4.151 \\
P_2 \le P_3 < P_4 < P_1 < P_0 \le P_{A1}^{(5)}, & \text{for } 4.151 < M \le 5 \\
P_3 < P_2 \le P_4 < P_1 < P_0 \le P_{A1}^{(5)}, & \text{for } 5 < M \le 5.65 \\
P_3 < P_4 < P_2 < P_1 \le P_0 < P_{A1}^{(5)}, & \text{for } 5.65 < M \le 6.5 \\
P_3 \le P_4 < P_2 < P_0 < P_1 < P_{A1}^{(5)}, & \text{for } 6.5 < M \le 7 \\
P_3 < P_4 < P_2 \le P_0 < P_1 < P_{A1}^{(5)}, & \text{for } 7 < M \le 7.891 \\
P_3 \le P_4 < P_2 \le P_0 < P_1 < P_{A1}^{(5)}, & \text{for } 7.891 < M \le 8 \\
P_4 < P_3 < P_0 < P_2 \le P_1 < P_{A1}^{(5)}, & \text{for } 8 < M \le 10.104 \\
P_4 < P_3 \le P_0 < P_1 \le P_2 < P_{A1}^{(5)}, & \text{for } 10.104 < M \le 10.424 \\
P_4 < P_0 < P_3 \le P_1 < P_2 < P_{A1}^{(5)}, & \text{for } 10.104 < M \le 12.874 \\
P_4 < P_0 < P_1 < P_3 \le P_2 < P_{A1}^{(5)}, & \text{for } 12.874 < M \le 15.231 \\
P_4 < P_0 < P_1 < P_2 < P_3 \le P_{A1}^{(5)}, & \text{for } 15.231 < M \le 17 \\
P_0 < P_4 \le P_1 < P_2 < P_3 \le P_{A1}^{(5)}, & \text{for } 17 < M \le 21.19 \\
P_0 < P_1 < P_4 < P_2 < P_3 \le P_{A1}^{(5)}, & \text{for } M > 21.19
\end{cases},
$$

The value $M_1 \equiv 1 \pmod 3$ such that for $M \ge M_1$, the quadric $P_0(M)$ is the minimum codimension is 19.

3.4.3 "Almost equal" Degrees : $M \equiv 2 \ (\textbf{mod}\ 3)$

Finally, let us examine the case where

$$d_2 = d_3 = d_1 + 1.$$

Removing the Last Three Hypertangent Divisors

Set

$$P_{A2}^{(3)} = \binom{d_3 + 2}{3}.$$

Then since $d_3 = d_1 + 1$, and we have

$$\begin{aligned} P_{A2}^{(3)} = \binom{d_1+3}{3} &= \binom{\frac{M}{3} + \frac{1}{3} + 2}{3} \\ &= \frac{\left(\frac{M+10}{3}\right)\left(\frac{M+7}{3}\right)\left(\frac{M+4}{3}\right)}{6} \\ &= \frac{(M+10)(M+7)(M+4)}{162}. \end{aligned}$$

We have:

$$\begin{cases} P_2 < P_1 < P_{A2}^{(3)} \leq P_0, & \text{for } 0 < M \leq 6.178 \\ P_2 < P_1 \leq P_0 < P_{A2}^{(3)}, & \text{for } 6.178 < M \leq 6.5, \\ P_2 < P_0 < P_1 \leq P_{A2}^{(3)}, & \text{for } 6.5 < M \leq 7.563, \\ P_2 < P_0 < P_{A2}^{(3)} \leq P_1 <, & \text{for } 7.563 < M \leq 7.891, \\ P_0 < P_2 \leq P_{A2}^{(3)} < P_1, & \text{for } 7.891 < M \leq 9.5, \\ P_0 < P_{A2}^{(3)} < P_2 \leq P_1, & \text{for } 9.5 < M \leq 10.104, \\ P_0 < P_{A2}^{(3)} < P_1 < P_2, & \text{for } M > 10.104 \end{cases}$$

If we remove the last three hypertangent divisors, then codim $\left(\mathcal{P}(\underline{d})\backslash\mathcal{P}_{\text{mq}}(\underline{d})\right) \geq P_0$, for $M \geq 7.891$. However, since we require that $M \equiv 0(\text{mod } 3)$, we set $M \geq 8$.

Removing the Last Four Hypertangent Divisors

Set
$$P^{(4)}_{A2} = \binom{4+(d_3-1)}{d_3-1} = \binom{d_3+3}{4}.$$

$$\begin{aligned} P^{(4)}_{A2} &= \binom{\frac{M}{3}+\frac{4}{3}+3}{4} \\ &= \frac{\left(\frac{M}{3}+\frac{4}{3}+3\right)\left(\frac{M}{3}+\frac{4}{3}+2\right)\left(\frac{M}{3}+\frac{4}{3}+1\right)\left(\frac{M}{3}+\frac{4}{3}\right)}{24} \\ &= \frac{(M+13)(M+10)(M+7)(M+4)}{1944} \end{aligned}$$

We will compare the values of $P^{(4)}_{A2}, P_0, P_1, P_2$ and P_3:

$$\begin{cases} P_2 < P_1 < P^{(4)}_{A2} < P_3 \leq P_0, & \text{for } 0 < M \leq 0.576 \\ P_2 < P_1 < P_3 \leq P^{(4)}_{E} < P_0, & \text{for } 0.576 < M \leq 2.75 \\ P_2 < P_1 \leq P_3 < P^{(4)}_{E} \leq P_0, & \text{for } 2.75 < M \leq 3.603 \\ P_2 \leq P_3 < P_1 < P^{(4)}_{E} < P_0, & \text{for } 3.603 < M \leq 5 \\ P_3 < P_2 < P_1 < P^{(4)}_{E} \leq P_0, & \text{for } 5 < M \leq 5.324 \\ P_3 < P_2 < P_1 \leq P_0 < P^{(4)}_{E}, & \text{for } 5.324 < M \leq 6.5 \\ P_3 < P_2 \leq P_0 < P_1 < P^{(4)}_{E}, & \text{for } 6.5 < M \leq 7.891 \\ P_3 < P_0 < P_2 \leq P_1 < P^{(4)}_{E}, & \text{for } 7.891 < M \leq 10.104 \\ P_3 \leq P_0 < P_2 < P_1 < P^{(4)}_{E}, & \text{for } 10.104 < M \leq 10.424 \\ P_0 < P_3 < P_1 < P_2 \leq P^{(4)}_{E}, & \text{for } 10.424 < M \leq 12.134 \\ P_0 < P_1 \leq P_3 < P^{(4)}_{E} < P_2, & \text{for } 12.134 < M \leq 12.874 \\ P_0 < P_1 < P_3 \leq P^{(4)}_{E} < P_2, & \text{for } 12.874 < M \leq 14 \\ P_0 < P_1 < P^{(4)}_{E} < P_3 \leq P_2, & \text{for } 14 < M \leq 15.231 \\ P_0 < P_1 < P^{(4)}_{E} < P_3 < P_2, & \text{for } M > 15.231 \end{cases}$$

The value $M_1 \equiv 1 (\bmod 3)$ such that for $M \geq M_1$, the quadric $P_0(M)$ gives the minimum codimension is 11.

Removing the Last Five Hypertangent Divisors

Let us set:

$$
\begin{aligned}
P_{A2}^{(5)} = \binom{5+d_3-1}{d_3-1} &= \binom{d_3+4}{d_3-1} \\
&= \binom{\frac{M}{3}+\frac{4}{3}+4}{5} \\
&= \frac{\left(\frac{M}{3}+\frac{4}{3}+4\right)\left(\frac{M}{3}+\frac{4}{3}+3\right)\left(\frac{M}{3}+\frac{4}{3}+2\right)\left(\frac{M}{3}+\frac{4}{3}+1\right)\left(\frac{M}{3}+\frac{4}{3}\right)}{120} \\
&= \frac{(M+16)(M+13)(M+10)(M+7)(M+4)}{29160}.
\end{aligned}
$$

We again compare the various values of $P_2, P_{A2}^{(5)}, P_3$ and P_0 against one another as follows:

$$
\begin{cases}
P_2 < P_1 < P_{A2}^{(5)} < P_4 < P_3 \leq P_0, & \text{for } 0 < M \leq 0.576 \\
P_2 < P_1 < P_{A2}^{(5)} < P_4 \leq P_3 < P_0, & \text{for } 0.576 < M \leq 1.847 \\
P_2 < P_1 < P_{A2}^{(5)} \leq P_3 < P_4 < P_0, & \text{for } 1.847 < M \leq 2.611 \\
P_2 < P_1 < P_3 < P_{A2}^{(5)} \leq P_4 < P_0, & \text{for } 2.611 < M \leq 2.854 \\
P_2 < P_1 \leq P_3 < P_{A2}^{(5)} < P_0, & \text{for } 2.854 < M \leq 3.603 \\
P_2 < P_3 < P_1 \leq P_{A2}^{(5)} < P_0, & \text{for } 3.603 < M \leq 4.088 \\
P_2 < P_3 < P_4 < P_1 < P_{A2}^{(5)} \leq P_0, & \text{for } 4.088 < M \leq 4.75 \\
P_2 \leq P_3 < P_4 < P_1 < P_0 < P_{A2}^{(5)}, & \text{for } 4.75 < M \leq 5 \\
P_3 < P_2 \leq P_4 < P_1 < P_0 < P_{A2}^{(5)}, & \text{for } 5 < M \leq 5.65 \\
P_3 < P_4 < P_2 < P_1 \leq P_0 < P_{A2}^{(5)}, & \text{for } 5.65 < M \leq 6.5 \\
P_3 \leq P_4 < P_2 < P_0 < P_1 \leq P_{A2}^{(5)}, & \text{for } 6.5 < M \leq 7 \\
P_3 \leq P_4 < P_2 \leq P_0 < P_1 < P_{A2}^{(5)}, & \text{for } 7 < M \leq 7.891 \\
P_3 \leq P_4 < P_0 < P_2 < P_1 < P_{A2}^{(5)}, & \text{for } 7.891 < M \leq 8 \\
P_4 < P_3 < P_0 < P_2 \leq P_1 < P_{A2}^{(5)}, & \text{for } 8 < M \leq 10.104 \\
P_4 < P_3 \leq P_0 < P_1 < P_2 < P_{A2}^{(5)}, & \text{for } 10.104 < M \leq 10.424 \\
P_4 < P_0 < P_1 \leq P_3 < P_2 < P_{A2}^{(5)}, & \text{for } 10.424 < M \leq 12.874 \\
P_4 < P_0 < P_1 < P_3 \leq P_2 < P_{A2}^{(5)}, & \text{for } 12.874 < M \leq 15.231 \\
P_4 < P_0 < P_1 < P_2 < P_3 \leq P_{A2}^{(5)}, & \text{for } 15.231 < M \leq 16.656 \\
P_4 \leq P_0 < P_1 < P_2 < P_3 \leq P_{A2}^{(5)}, & \text{for } 16.656 < M \leq 17 \\
P_0 \leq P_4 < P_1 < P_2 < P_3 \leq P_{A2}^{(5)}, & \text{for } 17 < M \leq 21.19 \\
P_0 \leq P_1 < P_4 < P_2 < P_3 \leq P_{A2}^{(5)}, & \text{for } 17 < M \leq 21.19
\end{cases}
$$

The quadric $P_0(M)$ is the minimum estimate of the codimension for values of M exceeding 17.

3.4.4 Results

Let

$$C = \operatorname{codim}\left(\left(\mathcal{P}(\underline{d})\Big\backslash\mathcal{P}_{\mathrm{reg}}(\underline{d})\right) \subset \mathcal{P}(\underline{d})\right).$$

We have shown that

$$C \geq \frac{(M-11)(M-10)}{2} + (M-2)$$

for each $a \in \{3,4,5\}$, for certain values of M. In particular, when M is congruent to $0, 1$, and 2:

(i) if we remove the last 3 hypertangent divisors,

$$C \geq \frac{(M-11)(M-10)}{2} + (M-2)$$

for M not less than $9, 10$, and 8 respectively.

(ii) when we remove 4 divisors, then

$$C \geq \frac{(M-11)(M-10)}{2} + (M-2)$$

for M not less than $9, 13$ and 11 respectively.

(iii) when $a = 5$,

$$C \geq \frac{(M-11)(M-10)}{2} + (M-2)$$

for M at least $18, 19$ and 17 respectively.

This completes the proof of Theorem 24(ii). □

Remark 31. There are scenarios that could cause the estimates of codimension to be stronger:

1) If the point $o \in V$ is singular, then the dimension of Π increases. This would result in stronger estimates when we approximate the codimension of the set of non-regular sequences of polynomials with the last a polynomials removed.

2) When the triple (d_1, d_2, d_3) is such that the degrees are neither equal nor "almost equal", then again, the estimates are stronger. This is because the degree j of the polynomial $q_{i,j}|_\Pi$ in the lexicographically ordered sequence takes the minimal value for a given M if the degrees are equal or most equal.

3.4.5 Concluding Remarks

Let us aggregate the information contained in Subsections 3.3.4 & 3.4.4:

We have obtained the minimum dimensions M necessary for birational superrigidity of regular complete intersections for each $a \in \{1, 2, 3, 4, 5\}$ in the three cases of equal and "almost equal" degrees. We also have estimates of the minimum dimensions for which P_0 is the lowest estimate of the codimension C.

Reviewing Subsection 3.4.4, we see that the most optimal result is obtained by removing the last three hypertangent divisors. This is because, for "almost equal degrees", we see that, in comparison with $a = 4, 5$, removing the last three hypertangent divisors provides us with the lowest values of M for which

$$\operatorname{codim}\left(\left(\mathcal{P}(\underline{d})\Big\backslash\mathcal{P}_{\text{reg}}(\underline{d})\right) \subset \mathcal{P}(\underline{d})\right) \geq \frac{(M-11)(M-10)}{2} + (M-2).$$

Furthermore, for equal degrees, we need M to be at least 9 when $a = 3$, and $a = 4$. In other words, when the degrees are equal, it is still preferable to remove 3 hypertangent divisors because removing an additional hypertangent divisor results in no change to the minimum value of M needed for C to be the lowest codimension.

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