Homogenization of the Lake Equations and the Related Topics

湖方程均质化及相关课题

Thesis Submitted in Accordance with

the Requirements of

the University of Liverpool

For the Degree of

Doctor of Philosophy

by

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NOVEMBER 2022

ABSTRACT

The purpose of this thesis is to study the homogenization of the lake equation. It generates reaction effect and the reaction term is induced by homogenization due to the weak convergence. It can be characterized by L. Tartar's method of oscillating test function. We begin by the homogenization of a Stokes equation perturbed by a drift. By means of constructing a similar homogenized equation we figure out the weak limit of the important term, this leads to a limit equation with an extra zero-order term. We then turn to the homogenization of an anelastic Stokes system arising from the lake equations and prove the existence of the solution of the equation using Lax-Milgram Theorem. The homogenization of the anelastic Stokes equation lays a fundation for the homogenization of the lake equation. We finally give the proof of the existence and uniquess of the solution of lake equation using Faedo-Galerkin method and study the homogenization of the lake equation by constructing a homogenized equation of the test function according to Tartar's method.

Keywords: Homogenization; Viscous Lake Equations; Navier Boundary Conditions; Anelastic Stokes System.

摘要

本文研究湖方程的均质化。在均质化过程中，产生了一些反应效应，其中的反应项是由因弱收敛而引出的均质化产生的。其过程可用L.Tartar的振荡测试函数方法描述。 我们开始给出一个带有漂移扰动项的Stokes方程的均质化。通过构造一个类似的均质方程可以推导出其中重要项的弱极限，进而得出带有一个零阶项的极限方程。其后，我们推导了由湖方程引出的一个滞弹性Stokes系统的均质化，并且用Lax-Milgram定理证明该方程解的存在性。这为最终研究湖方程的均质化奠定了一定基础。我们最终运用Faedo-Galerkin方法证明了湖方程解的存在唯一性，并且通过Tartar的方法构造了一个有关测试函数的均质方程来研究湖方程的均质化。

关键词：均质化；粘性湖方程；纳维边界条件；滞弹性斯托克斯系统。

ACKNOWLEDGEMENT

First of all, I wish to express heartfelt gratitude to my principal supervisor, Professor Chi-kun Lin, for his proffesional guidance and valuable suggestions. The precious present he gave to me was not only knowledge, but also meticulous academic spirit and strong mind during my doctoral study. The invaluable experiences and brilliant methods he imparted to me have motivated and enlightened current project in many ways. His instruction and encouragement will become an important influence in my life. I would also like to show sincere appreciation to my co-supervisor Dr. Xinyao Yang who was my supervisor during the final year progress in my undergraduate period. She wisely instructed me and shared many usefull skills to me through the early research life. Without her direction, I could not enter the field of mathematics. I am also grateful to Dr. Gayane Piliposyan for her support and encouragement. Her offer of help and suggestions benefit me a lot. I am also thankful to University of Liverpool (UoL) and Xi'an Jiaotong-Liverpool University (XJTLU) for their support on this research. Finally, I want to express my appreciation to my parents for their unconditional love and support. They devoted most of their time and energy to cultivating me and created an excellent atmosphere for learning.

Contents

Chapter 1

Introduction

The theory of homogenization is a distinct mathematical subject set forth during the last century. This theory has many significant applications in filtration, disperse media, mechanics of perforated and composite materials, it can be also found in some other branches of physics, modern technology and mechanics. Originally, the theory of homogenization was developed in order to give an account of the macroscopic behaviour of composite materials. The most striking characteristic of composite materials is the fact that they consists of at least two finely mixed constituents. Due to their properties, the composite materials have a wide range of industrial uses nowadays. In general, composite materials have a better behaviour than the average behaviour of their individual constituents. Typical examples are the superconducting multifilamentary composites which are used in the synthesis of optical fibres.

In the ordinary way, in comparison to their global dimension, the heterogeneity in composite materials are relatively small. Hence, there are two scales used to characterize the material, one is the microscopic, which describes the heterogeneity, and the another one is the macroscopic, which is used to describe the global behaviour of the composite materials. In addition, the composite can be viewed as a homogeneous material in macroscopical sight.

This brings up an important topic, homogenization, which is aimed at researching accurately the macroscopic properties of the composite materials by considering the properties of the microscopic structure. That is to say, the heterogeneous medium can be replaced by the homogenized material. Moreover, the global characteristics of the homogenized material are a good approximation of the heterogeneous medium. From the perspective of mathematics, the heterogeneous medium can be regarded as a sequence of the solutions of a boundary value problem depending on a parameter lying in a probability space, and the homogenized material is the limit of the sequence for an appropriate weak topology. This signifies mainly that the solutions of a boundary value problem converge weakly but may not converge strongly (due to the oscillations) to the solution of a limit boundary value problem which is clearly defined. The aim of homogenization is to research how oscillations of coefficients of PDEs generate oscillations in their solutions.

Currently, a vast literature is available on the mathematical aspects of the homogenization theory. In Tartar's [59, 60], the author talked about the background of homogenization theory and the early work done by Évariste Sanchez-Palencia [51, 52]. In consideration of the periodic geometry that mixtures of materials show, Évariste Sanchez-Palencia applied variational methods to identify the first term of an asymptotic expansion when the period length ε tends to 0. These generated equations have the same form as the original equations, but have an essential relation with anisotropic materials, even when the materials used are all isotropic. In Cioranescu and Donato's [16], they introduced the framework of homogenization theory started with the variational approach of partial differential equations and then focused on the periodic homogenization of linear partial differential equations. They also revisited the oscillating test function method due to Tartar [57, 58], which is the main tool we will use in this research. In addition, some other monographs can be referred to study the theory of homogenization, in particular: Allaire [3, 4], Bachvalov [5], Bachvalov and Panasenko [6], Oleinik [63], Zhikov, Kozlov and Oleinik [64].

When solving the problems of partial differential equations, the theory of homogenization can be understood as the 'averaging' of partial differential equations, which has been usually associated with the method of weak convergence. Weak convergence method is a significant technique proposed for researching nonlinear partial differential equations. In Evans' [23], the author gave an introduction to this method. When solving some nonlinear partial differential equations, namely

$$T(u) = f, \tag{1.0.1}$$

where T represents a given nonlinear operator, u is the unknown and f is a given function. To research the existence of a solution u of (1.0.1), one can construct a suitable group of solvable approximating problems. We write these problems as

$$T_i(u_i) = f_i \quad \text{for} \quad i = 1, 2, 3, \cdots, \tag{1.0.2}$$

where T_i denotes a nonlinear operator which is close to T by some means for $i \to \infty$, the function f_i is close to f, and u_i is a solution of (1.0.2). Now, the expectation is that the functions $\{u_i\}_{i=1}^{\infty}$ will converge to a solution u of the equation (1.0.1).

In point of fact, the operators T_i may represent discretizations, singular regularizations, finite dimensional projections, gradients of approximate energy functionals, systems collapsing in the limit to a single equation, and so on. In effect, it is usually not very hard to invent a class of reasonable approximation which can be solved in fact for a given nonlinear partial differential equation such as (1.0.1). The approach is to prove that the solutions of (1.0.2) converge to a solution of (1.0.1) indeed. But the nonlinearity is an impediment for solving this problem. Nevertheless, the functions $\{u_i\}_{i=1}^{\infty}$ usually satisfy some uniform estimates, and likewise these best available boundary conditions are usually not too strong. In consideration of these relatively poor estimates, it can be usually shown only that the functions $\{u_i\}_{i=1}^{\infty}$ (or a subsequence) converge weakly to a limit u in some function space

$$u_i \overset{\mathrm{w}}{\rightharpoonup} u \quad \text{as} \quad i \to \infty. \tag{1.0.3}$$

This kind of weak convergence is almost always a certain problem, given the strong nonlinearities. Even if one has constructed approximate operators T_i in some way which tend to T, it does not necessarily mean in application that the weak convergence (1.0.3) can imply

$$T_i(u_i) \to T(u) \quad \text{as} \quad i \to \infty. \tag{1.0.4}$$

The obstacle in the procedure is that weak convergence does not have a good behaviour in the case of nonlinearities, and yet such weak convergence is obviously the best result we can obtain.

Early in the late 1940s, a Brinkman-type law [11] has been introduced when researching the Stokes or Navier-Stokes equations. The law was obtained from the Stokes equations by adding a term proportional to the velocity to the momentum equation. Therefore, when deriving the homogenization process, Brinkman's law is widely applied, this means that there is usually a linear zero-order term for the velocity appears in the limit problem.

While studying homogenization, the main obstruction is to identify the limit of the terms containing products of two sequences convergence weekly in some function space. The main tool we used to overcome this obstacle is the oscillating test functions method introduced by L. Tartar [57, 58] in 1970s. The idea of Tartar's method is to construct an appropriate collection of oscillating test functions, then by subtraction some terms cancel and passing to the limit one can derive the limit equation of the target problem using the limit expressions obtained [58] (see also [16] for the elementary introduction). The method of the oscillating test functions plays a dominant role in the present research.

Along with the oscillating test functions method, there are many other methods usually been used in homogenization problems, such as the multiple-scale method, the two-scale convergence method introduced by Nguetseng [50] and developed by Allaire [3, 4], and the approaches of G-convergence and H-convergence to the non-periodic case introduced by Spagnolo [54] and Tartar [56] repectively, the more details can be also found in Murat's [49].

The present research was inspired from the work done by M. Briane and P. Gérard [13] as well. In [13], the authors not only revised Tartar's problem but also proposed a new method based on a parametrix of the Laplace operator to overcome the lack of integrability of the given oscillating drift term.

The objective in this thesis is to show the homogenization of a kind of viscous lake equations based on the method of weak convergence. The two-dimensional viscous lake equations can be written as

$$\partial_t(b\boldsymbol{u}) + \operatorname{div}(b\boldsymbol{u}\otimes\boldsymbol{u}) + b\eta\boldsymbol{u} + b\nabla h = \operatorname{div}(\nu b\Sigma(\boldsymbol{u})) + b\boldsymbol{f}, \tag{1.0.5}$$

$$\operatorname{div}(b\boldsymbol{u}) = 0, \tag{1.0.6}$$

for $(\boldsymbol{x}, t) = (x_1, x_2, t) \in \Omega \times (0, \infty)$ with $\Omega \subset \mathbb{R}^2$, a bounded and smooth domain. Here, $\Sigma(\boldsymbol{u}) = 2D(\boldsymbol{u}) - \operatorname{div}(\boldsymbol{u})\, I$ and $D(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathrm{T}})$ is the deformation tensor.

The viscous lake equations (anelastic limit) have been derived by D. Levermore and M. Sammartino in [37] as the shallow water limit of the three-dimensional Navier-Stokes equations, in order to model the evolution of the vertically averaged horizontal components of the velocity to the incompressible three-dimensional viscous fluid confined to a shallow basin with varying bottom topography. The viscous lake equation was started from three-dimensional incompressible Euler flow, the so-called lake equations

$$\begin{cases} \partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = 0, & (x,t) \in Q_T, \\ \operatorname{div}(b\boldsymbol{u}) = 0, & (x,t) \in Q_T, \\ b\boldsymbol{u} \cdot \nabla = 0, & (\sigma,t) \in S_T, \\ \boldsymbol{u}(x,0) = \boldsymbol{u}_0(x), & x \in \Omega, \end{cases} \tag{1.0.7}$$

where $Q_T = \Omega \times [0,T)$, $S_T = \partial\Omega \times [0,T)$ for all $0 < T < \infty$. The lake equations have been derived in [14, 15, 28] to model the evolution of the vertically averaged horizontal components of the three-dimensional velocity to the incompressible Euler equations confined to a shallow basin with a varying bottom topography. Some results for Euler equations can be employed from Evans and Müller [24], Semmes [53]. More information about the lake equation can be found in publications such as Bresch and Métivier [9], Lacave, Nguyen and Paudaser [33], Jiu and Niu [30], Lions [43] and so on. There are also some literature regarding the compressible and incompressible flows, for example, Bresch, Gisclon and Lin [10], DiPerna and Majda [20, 21, 22].

During the last decades, extensive research has been done into the homogenization of different types of fluid flow models, and they are still topics of research today. In Marchenko and Hruslov's [46], they considered the homogenization of the stationary incompressible Navier-Stokes equation under distribution of identical obstacles, and they were the first to prove that Brinkman's law describe the limiting behavior of Stokes flow in a periodically perforated domain for a particular scaling of the holes. Afterwards, in his seminal PhD thesis [1], Allaire analyzed the homogenization of the Navier-Stokes equations in open sets perforated with tiny holes. Nowadays, numerous publications can be found on fluid flow homogenization. Homogenization of stationary Navier-Stokes equations was investigated in, for example, Hillairet [29] and Lu [45]. For compressible fluids, both stationary and time-dependent cases were considered in Feireisl and Lu [25], Lu and Schwarzacher [44] and so on. There are also articles investigating the homogenization of the evolutionary Navier-Stokes system such as Mikelič [48] and Feireisl, Namlyeyeva and Nečasová [26].

The viscous lake equations considered in this thesis have the following equations

$$\partial_t(b\boldsymbol{u}^\varepsilon) - \frac{1}{2}(\boldsymbol{u}^\varepsilon \cdot \boldsymbol{v}^\varepsilon)\nabla b + b\,\mathrm{curl}(\boldsymbol{v}^\varepsilon)J\boldsymbol{u}^\varepsilon + b\nabla p^\varepsilon = \mathrm{div}(b\Sigma(\boldsymbol{u}^\varepsilon)) + b\boldsymbol{f}, \tag{1.0.8}$$

$$\mathrm{div}(b\boldsymbol{u}^\varepsilon) = 0, \tag{1.0.9}$$

for $(\boldsymbol{x}, t) \in Q_T = \Omega \times [0, T)$, with the Navier boundary conditions

$$\boldsymbol{u}^\varepsilon \cdot \boldsymbol{n} = 0, \qquad \boldsymbol{\tau} \cdot \left(\nabla\boldsymbol{u}^\varepsilon + (\nabla\boldsymbol{u}^\varepsilon)^{\mathrm{T}}\right) \cdot \nabla = -\beta\boldsymbol{\tau} \cdot \boldsymbol{u}^\varepsilon, \tag{1.0.10}$$

for $(\boldsymbol{\sigma}, t) \in S_T = \partial\Omega \times [0, T)$, and the initial condition

$$\boldsymbol{u}^\varepsilon(\boldsymbol{x}, 0) = \boldsymbol{u}_0^\varepsilon(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Omega. \tag{1.0.11}$$

Here we assume the eddy viscosity coefficient $\nu = 1$, turbulent drag coefficient $\eta = 0$ and the nonlinear term $\mathrm{div}(b\boldsymbol{u}^\varepsilon \otimes \boldsymbol{u}^\varepsilon)$ is replaced by

$$-\frac{1}{2}(\boldsymbol{u}^\varepsilon \cdot \boldsymbol{v}^\varepsilon)\nabla b + b\,\mathrm{curl}(\boldsymbol{v}^\varepsilon)J\boldsymbol{u}^\varepsilon \tag{1.0.12}$$

based on some identities. It is remarked that (1.0.10) are usually called the (general) Navier boundary conditions, which were first used by Navier in 1827 (see [7, 42]) and mean that there is a stagnant layer of fluid close to the wall allowing a fluid to slip, and the slip velocity is proportional to the shear stress.

Before deriving the homogenization of viscous lake equations, we considered the homogenization of some important equations in the early chapters on the basis of the homogenization results of a two dimensional equivalent of the perturbed Stokes problem given in section 3 of [13].

In Chapter 2, we proposed the homogenization of a Stokes equation with a drift term in a bounded smooth domain $\Omega \subset \mathbb{R}^2$,

$$\begin{cases} \partial_t \boldsymbol{u}^\varepsilon - \Delta \boldsymbol{u}^\varepsilon + \mathrm{curl}(\boldsymbol{v}^\varepsilon)J\boldsymbol{u}^\varepsilon + \nabla p^\varepsilon = \boldsymbol{f}, & (\boldsymbol{x}, t) \in Q_T, \\ \mathrm{div}(\boldsymbol{u}^\varepsilon) = 0, & (\boldsymbol{x}, t) \in Q_T, \\ \boldsymbol{u}^\varepsilon(\boldsymbol{\sigma}, t) = 0, & (\boldsymbol{\sigma}, t) \in S_T, \\ \boldsymbol{u}^\varepsilon(\boldsymbol{x}, 0) = \boldsymbol{u}_0^\varepsilon(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \end{cases} \tag{1.0.13}$$

where $Q_T = \Omega \times [0, T)$, $S_T = \partial\Omega \times [0, T)$ for all $0 < T < \infty$ and J is the rotation matrix of angle $90°$. The equation arises from the Navier-Stokes equation where the nonlinear term $(\boldsymbol{u}^\varepsilon \cdot \nabla)\boldsymbol{u}^\varepsilon$ is replaced by $(\boldsymbol{v}^\varepsilon \cdot \nabla)\boldsymbol{u}^\varepsilon$. We prove the limit $\boldsymbol{u}$ satisfies the Brinkman [11] Stokes equation

$$\begin{cases} \partial_t \boldsymbol{u} - \Delta \boldsymbol{u} + \operatorname{curl}(\boldsymbol{v}) J \boldsymbol{u} + \nabla p + M \boldsymbol{u} = \boldsymbol{f}, & (\boldsymbol{x}, t) \in Q_T, \\ \operatorname{div}(\boldsymbol{u}) = 0, & (\boldsymbol{x}, t) \in Q_T, \\ \boldsymbol{u}(\boldsymbol{\sigma}, t) = 0, & (\boldsymbol{\sigma}, t) \in S_T, \\ \boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \end{cases} \tag{1.0.14}$$

where M is the positive definite symmetric matrix-valued function.

In Chapter 3, we consider an anelastic Stokes system with drift term arising from the lake equations

$$-\operatorname{div}(b\Sigma(\boldsymbol{u}^\varepsilon)) - \frac{1}{2}(\boldsymbol{u}^\varepsilon \cdot \boldsymbol{v}^\varepsilon)\nabla b + b \operatorname{curl}(\boldsymbol{v}^\varepsilon) J \boldsymbol{u}^\varepsilon + b \nabla p^\varepsilon = b \boldsymbol{f}, \tag{1.0.15}$$

$$\operatorname{div}(b \boldsymbol{u}^\varepsilon) = 0, \qquad \boldsymbol{x} \in \Omega \tag{1.0.16}$$

with the Navier boundary conditions

$$\boldsymbol{u}^\varepsilon \cdot \boldsymbol{n} = 0, \qquad \boldsymbol{\tau} \cdot \left(\nabla \boldsymbol{u}^\varepsilon + (\nabla \boldsymbol{u}^\varepsilon)^{\mathrm{T}}\right) \cdot \nabla = -\beta \boldsymbol{\tau} \cdot \boldsymbol{u}^\varepsilon, \qquad \boldsymbol{x} \in \partial\Omega\,. \tag{1.0.17}$$

This is the analogue of the Stokes equation for the lake equation where the eddy viscosity coefficient $\nu = 1$, turbulent drag coefficient $\eta = 0$ and the nonlinear term $\operatorname{div}(b \boldsymbol{u}^\varepsilon \otimes \boldsymbol{u}^\varepsilon)$ is replaced by

$$-\frac{1}{2}(\boldsymbol{u}^\varepsilon \cdot \boldsymbol{v}^\varepsilon)\nabla b + b \operatorname{curl}(\boldsymbol{v}^\varepsilon) J \boldsymbol{u}^\varepsilon. \tag{1.0.18}$$

We show that the sequence $\boldsymbol{u}^\varepsilon$ converges weakly in $H^1_b(\Omega)$ to the solution $\boldsymbol{u}$ of the Brinkman equation

$$\begin{cases} -\operatorname{div}(b\Sigma(\boldsymbol{u})) - \frac{1}{2}(\boldsymbol{u} \cdot \boldsymbol{v})\nabla b + b \operatorname{curl}(\boldsymbol{v}) J \boldsymbol{u} + b \nabla p + 2M \boldsymbol{u} = b \boldsymbol{f} \\ \operatorname{div}(b \boldsymbol{u}) = 0, \qquad \boldsymbol{x} \in \Omega \end{cases}$$

with the Navier boundary conditions

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0, \qquad \boldsymbol{\tau} \cdot \left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathrm{T}}\right) \cdot \nabla = -\beta \boldsymbol{\tau} \cdot \boldsymbol{u}, \qquad \boldsymbol{x} \in \partial\Omega\,.$$

In Chapter 4, based on the result in the previous chapters, we propose the homogenization limit of the lake equaitons;

$$\begin{cases} \partial_t(b\boldsymbol{u}) - \dfrac{1}{2}(\boldsymbol{u}\cdot\boldsymbol{v})\nabla b + b\,\mathrm{curl}(\boldsymbol{v})J\boldsymbol{u} + b\nabla p + 2M\boldsymbol{u} = \mathrm{div}(b\Sigma(\boldsymbol{u})) + b\boldsymbol{f} \\ \mathrm{div}(b\boldsymbol{u}) = 0, \qquad (\boldsymbol{x},t)\in Q_T \\ \boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0(\boldsymbol{x}), \qquad \boldsymbol{x}\in\Omega \end{cases}$$

with the Navier boundary conditions

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0, \qquad \boldsymbol{\tau}\cdot\big(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathrm{T}}\big)\cdot\nabla = -\beta\boldsymbol{\tau}\cdot\boldsymbol{u}, \qquad (\boldsymbol{\sigma},t)\in S_T\,.$$

The proof follows Tartar's method using the oscillating test function

$$\begin{cases} \partial_t(b\boldsymbol{w}_\lambda^\varepsilon) + b\ \mathrm{div}\big((\boldsymbol{v}^\varepsilon - \boldsymbol{v})\otimes\boldsymbol{\lambda}\big) + b\nabla q_\lambda^\varepsilon = \mathrm{div}(b\Sigma(\boldsymbol{w}_\lambda^\varepsilon)), \quad \boldsymbol{x}\in\Omega \\ \mathrm{div}(b\boldsymbol{w}_\lambda^\varepsilon) = 0, \quad \boldsymbol{x}\in\Omega \\ \boldsymbol{w}_\lambda^\varepsilon(x,0) = \boldsymbol{w}_{\lambda 0}^\varepsilon(x), \quad \boldsymbol{x}\in\Omega \\ \boldsymbol{w}_\lambda^\varepsilon\cdot\boldsymbol{n} = 0, \quad \boldsymbol{x}\in\partial\Omega. \end{cases} \tag{1.0.19}$$

Some classical results for the Navier-Stokes equations and the knowledge of fluid mechanics we making use of refer to Bensoussan, Lions and Papanicolaou [8], Dafni [18], DiPerna and Majda [21, 22], Majda and Bertozzi [47] and Tsutsui [62].

Chapter 2

Homogenization of a Stokes Equation with a Drift Term

In this chapter we study the homogenization of a two-dimensional nonstationary Stokes equation with an oscillating drift term varied from the Navier-Stokes equation. The purpose of the model is to understand how the oscillations of $\boldsymbol{v}^\varepsilon$ will create oscillations in $\nabla \boldsymbol{u}^\varepsilon$ and to discover the equation satisfied by the weak limit $\boldsymbol{u}$ of $\boldsymbol{u}^\varepsilon$. In this chapter, Tartar's approach based on oscillating test functions method leads to a limit equation with an extra zero-order term. In Section 2.1, we give an introduction to the work done by L. Tartar and show the main results. In Section 2.2, we prove the existence and uniqueness theorem for the solution of the drift nonstationary Stokes equation using Faedo-Galerkin method. Section 2.3 is devoted to the proof of the homogenization result by imitating Tartar approach and constructing a nonstationary linearized Stokes equation.

2.1 Introduction and Main Results

The aim of this chapter is to deal with the homogenization of the non-stationary Stokes equation with an oscillating drift term in a bounded smooth domain $\Omega \subset \mathbb{R}^2$,

$$\begin{cases} \partial_t \boldsymbol{u}^\varepsilon - \Delta \boldsymbol{u}^\varepsilon + \operatorname{curl}(\boldsymbol{v}^\varepsilon) J \boldsymbol{u}^\varepsilon + \nabla p^\varepsilon = \boldsymbol{f}, & (\boldsymbol{x}, t) \in Q_T, \\ \operatorname{div}(\boldsymbol{u}^\varepsilon) = 0, & (\boldsymbol{x}, t) \in Q_T, \\ \boldsymbol{u}^\varepsilon(\boldsymbol{\sigma}, t) = 0, & (\boldsymbol{\sigma}, t) \in S_T, \\ \boldsymbol{u}^\varepsilon(\boldsymbol{x}, 0) = \boldsymbol{u}_0^\varepsilon(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \end{cases} \tag{2.1.1}$$

where $Q_T = \Omega \times [0, T)$ and $S_T = \partial\Omega \times [0, T)$ for all $0 < T < \infty$.

The homogenization of partial differential equations was first studied by L. Tartar at the end of the Seventies. He developed a method based on oscillating test functions to study homogenization. In his paper [58], the author talked about oscillations of a function to describe a sequence of functions which converges weakly but may not converge strongly. Homogenization is then concerned with understanding how oscillations of coefficients of a partial differential equation create oscillations in its solution.

In [13], the authors revisited a homogenization problem studied by Tartar. The paper considered a scalar equation and a two-dimensional perturbed Stokes equation with a L^2-bounded oscillating drift.

The equation (2.1.1) arises from the Navier-Stokes equation (we assume that the viscosity and the density of the fluid are both equal to 1)

$$\partial_t \boldsymbol{u}^\varepsilon - \Delta \boldsymbol{u}^\varepsilon + (\boldsymbol{u}^\varepsilon \cdot \nabla) \boldsymbol{u}^\varepsilon + \nabla p^\varepsilon = \boldsymbol{f}. \tag{2.1.2}$$

Thanks to Tartar's work [57, 58], for the nonlinear term $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}$ with the divergence free velocity $\boldsymbol{u}$ in the Navier-Stokes equation, we have the identity

$$(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u}) = \operatorname{curl}(\boldsymbol{u})\times\boldsymbol{u} + \nabla\Big(\frac{1}{2}\,|\boldsymbol{u}|^2\Big), \tag{2.1.3}$$

Furthermore, the equivalent of transformation (2.1.3) in two-dimension is

$$(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \operatorname{div}(\boldsymbol{u}\otimes\boldsymbol{u}) = \operatorname{curl}(\boldsymbol{u})J\boldsymbol{u} + \nabla\Big(\frac{1}{2}\,|\boldsymbol{u}|^2\Big), \tag{2.1.4}$$

where $\operatorname{curl}(\boldsymbol{u}) := \partial_1 u_2 - \partial_2 u_1$ and $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

To overcome the trouble caused by the nonlinearity, we replace the first $\boldsymbol{u}$ in the nonlinear term $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}$ with a given vector-valued divergence free function v, hence the nonlinear term changes to $(\boldsymbol{v}\cdot\nabla)\boldsymbol{u}$. Finally we can replace (2.1.2) to the drift nonstationary Stokes equation (2.1.1).

In this chapter, we introduce the spaces $\mathbb{H}(\Omega)$, $\mathbb{V}(\Omega)$ and $\mathbb{V}^{-1}(\Omega)$ defined by

$$\mathbb{H}(\Omega) = \left\{\boldsymbol{u}\in L^2(\Omega) \mid \operatorname{div}(\boldsymbol{u}) = 0\right\},$$
$$\mathbb{V}(\Omega) = \left\{\boldsymbol{u}\in H_0^1(\Omega) \mid \operatorname{div}(\boldsymbol{u}) = 0\right\},$$

and

$$\mathbb{V}^{-1}(\Omega) = \left\{\boldsymbol{u}\in H^{-1}(\Omega) \mid \operatorname{div}(\boldsymbol{u}) = 0\right\}.$$

We define the solution space associated with (2.1.1);

$$V^2(Q_T) = \left\{\boldsymbol{u}\in L^2([0,T];\mathbb{V}(\Omega)), \quad \boldsymbol{u}' = \frac{\partial\boldsymbol{u}}{\partial t}\in L^2([0,T];\mathbb{V}^{-1}(\Omega))\right\}$$

which is a Banach space with respect to the norm

$$\|\boldsymbol{u}\|_{V^2(Q_T)} = \|\boldsymbol{u}\|_{L^2([0,T];\mathbb{V}(\Omega))} + \|\boldsymbol{u}'\|_{L^2([0,T];\mathbb{V}^{-1}(\Omega))}.$$

We begin with the existence theorem of the solution for the drift nonstationary Stokes equation.

Theorem 2.1.1. *Let $\boldsymbol{f} \in L^2(Q_T)$. Assume that $\boldsymbol{v}^\varepsilon$ is bounded in $W^{1,0,r}_{x,t}(Q_T) \cap C^\infty_{x,t}(Q_T)$, with $r > 2$ and $div(\boldsymbol{v}^\varepsilon) = 0$. Then, there exists*

$$\boldsymbol{u}^\varepsilon \in V^2(Q_T) \tag{2.1.5}$$

and

$$p^\varepsilon \in L^{\frac{2r}{2+r}}\big(Q_T\big). \tag{2.1.6}$$

Moreover, $\boldsymbol{u}^\varepsilon$ satisfies the weak formulation of (2.1.1) *given by*

$$\langle \boldsymbol{u}^{\varepsilon\prime}(t), \boldsymbol{\varphi}(\boldsymbol{x})\rangle_{\mathbb{V}^{-1}(\Omega),\mathbb{V}(\Omega)} + \int_\Omega \nabla \boldsymbol{u}^\varepsilon(\boldsymbol{x},t) : \nabla \boldsymbol{\varphi}(\boldsymbol{x})\, d\boldsymbol{x} \\ + B(\boldsymbol{u}^\varepsilon(\boldsymbol{x},t), \boldsymbol{v}^\varepsilon(\boldsymbol{x},t), \boldsymbol{\varphi}(\boldsymbol{x})) = \langle \boldsymbol{f}(\boldsymbol{x},t), \boldsymbol{\varphi}(\boldsymbol{x})\rangle\,, \tag{2.1.7}$$

for every $\boldsymbol{\varphi} \in \mathbb{V}(\Omega)$. The trilinear form $B(\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon, \boldsymbol{\varphi})$ in (2.1.7) *is defined by*

$$B(\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon, \boldsymbol{\varphi}) \stackrel{\text{def}}{\equiv} \langle (\nabla \boldsymbol{u}^\varepsilon)^{\mathrm{T}} \boldsymbol{v}^\varepsilon, \boldsymbol{\varphi}\rangle - \int_\Omega (\boldsymbol{v}^\varepsilon \otimes \boldsymbol{u}^\varepsilon) : \nabla \boldsymbol{\varphi}\, d\boldsymbol{x} \tag{2.1.8}$$

Here the term $\mathrm{curl}(\boldsymbol{v}^\varepsilon) J \boldsymbol{u}^\varepsilon$ *in the nonstationary Stokes equation* (2.1.1) *is replaced by the extension of the identity* (2.1.4) *for any divergence free functions $\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon$*

$$\mathrm{curl}(\boldsymbol{v}^\varepsilon) J \boldsymbol{u}^\varepsilon = div(\boldsymbol{v}^\varepsilon \otimes \boldsymbol{u}^\varepsilon) + (\nabla \boldsymbol{u}^\varepsilon)^{\mathrm{T}} \boldsymbol{v}^\varepsilon - \nabla(\boldsymbol{v}^\varepsilon \cdot \boldsymbol{u}^\varepsilon). \tag{2.1.9}$$

The initial condition has to be understood in $L^2(\Omega)$,

$$\lim_{t\to 0} \|\boldsymbol{u}^\varepsilon(t)\|_{L^2(\Omega)} = \|\boldsymbol{u}^\varepsilon_0\|_{L^2(\Omega)}\,.$$

Moreover, since $B(\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon, \boldsymbol{u}^\varepsilon) = (\nabla \boldsymbol{u}^\varepsilon)^{\mathrm{T}} \boldsymbol{v}^\varepsilon \cdot \boldsymbol{u}^\varepsilon - (\boldsymbol{v}^\varepsilon \otimes \boldsymbol{u}^\varepsilon) : \nabla \boldsymbol{u}^\varepsilon = 0$, for all $t \in [0, T)$, the energy inequality holds

$$\|\boldsymbol{u}^\varepsilon(t)\|^2_{\mathbb{H}(\Omega)} + \int_0^{\mathrm{T}} \|\boldsymbol{u}^\varepsilon(\tau)\|^2_{\mathbb{V}(\Omega)}\, d\tau \leqslant \|\boldsymbol{u}^\varepsilon_0\|^2_{L^2(\Omega)} + C^2_\Omega \int_0^T \|\boldsymbol{f}(\tau)\|^2_{L^2(\Omega)} d\tau \tag{2.1.10}$$

where C_Ω is the Poincaré constant.

It is worth noticing that the force terms in $\operatorname{curl}(\boldsymbol{v}^\varepsilon)J\boldsymbol{u}^\varepsilon$ occur because of a magnetic force $\boldsymbol{u}^\varepsilon \times \boldsymbol{b}$ as well. Although this force does not work directly it creates oscillations which dissipate energy at a microscopic level, an effect that will appear in the homogenization limit equation. To derive the homogenization limit of (2.1.1) as suggested by Tartar [58] (See also [13] for the revised method.), we need to construct the homogenized nonstationary Stokes equation of the test function $\boldsymbol{w}^\varepsilon_\lambda$, for $\boldsymbol{\lambda} \in \mathbb{R}^2$,

$$\begin{cases} \partial_t \boldsymbol{w}^\varepsilon_\lambda - \Delta \boldsymbol{w}^\varepsilon_\lambda + \operatorname{div}\big((\boldsymbol{v}^\varepsilon - \boldsymbol{v}) \otimes \boldsymbol{\lambda}\big) + \nabla q^\varepsilon_\lambda = 0, & (\boldsymbol{x}, t) \in Q_T, \\ \operatorname{div}(\boldsymbol{w}^\varepsilon_\lambda) = 0, & (\boldsymbol{x}, t) \in Q_T, \\ \boldsymbol{w}^\varepsilon_\lambda(\sigma, t) = 0, & (\boldsymbol{\sigma}, t) \in S_T, \\ \boldsymbol{w}^\varepsilon_\lambda(x, 0) = \boldsymbol{w}^\varepsilon_{0\lambda}(\boldsymbol{x}), & \boldsymbol{x} \in \Omega. \end{cases} \tag{2.1.11}$$

The main result of this chapter is the following homogenization limit of (2.1.1).

Theorem 2.1.2. *Assume the same hypothesis of Theorem 2.1.1. In addition, suppose*

$$\boldsymbol{v}^\varepsilon \overset{\mathrm{w}}{\rightharpoonup} \boldsymbol{v} \quad \text{weakly in} \quad W^{1,0,r}_{x,t}(Q_T) \cap C^\infty_{x,t}(Q_T), \quad r > 2 \tag{2.1.12}$$

and

$$\boldsymbol{u}^\varepsilon_0 \overset{\mathrm{w}}{\rightharpoonup} \boldsymbol{u}_0 \quad \text{weakly in} \quad L^2(\Omega) \tag{2.1.13}$$

then we can extract a subsequence (still denoted by $\{\boldsymbol{u}^\varepsilon\}_\varepsilon$), such that

$$\begin{cases} \boldsymbol{u}^\varepsilon \to \boldsymbol{u} \quad \text{in} \quad L^2(Q_T) \\ p^\varepsilon \overset{\mathrm{w}}{\rightharpoonup} p \quad \text{weakly in} \quad L^{\frac{2r}{2+r}}(Q_T)\,. \end{cases} \tag{2.1.14}$$

Moreover, the limit u satisfies the Brinkman Stokes equation

$$\begin{cases} \partial_t \boldsymbol{u} - \Delta \boldsymbol{u} + \operatorname{curl}(\boldsymbol{v}) J \boldsymbol{u} + \nabla p + M \boldsymbol{u} = \boldsymbol{f}, & (\boldsymbol{x}, t) \in Q_T, \\ div\, (\boldsymbol{u}) = 0, & (\boldsymbol{x}, t) \in Q_T, \\ \boldsymbol{u}(\boldsymbol{\sigma}, t) = 0, & (\boldsymbol{\sigma}, t) \in S_T, \\ \boldsymbol{u}(x, 0) = \boldsymbol{u}_0(x), & \boldsymbol{x} \in \Omega, \end{cases} \tag{2.1.15}$$

where M is the positive definite symmetric matrix-valued function defined by

$$\begin{cases} (\nabla \boldsymbol{w}_\lambda^\varepsilon)^{\mathrm{T}} \boldsymbol{v}^\varepsilon \overset{\mathrm{w}}{\rightharpoonup} M \boldsymbol{\lambda} \quad \textit{weakly in} \quad L^{\frac{r}{2-2r}}(Q_T) \\ \nabla \boldsymbol{w}_\lambda^\varepsilon : \nabla \boldsymbol{w}_\mu^\varepsilon \overset{\mathrm{w}}{\rightharpoonup} M \boldsymbol{\lambda} \cdot \boldsymbol{\mu} \quad \textit{weakly} * \textit{in } \mathcal{M}(Q_T) \textit{ and in } L^{\frac{r}{2-2r}}(Q_T), \end{cases} \tag{2.1.16}$$

for $\boldsymbol{\lambda}$, $b\mu \in \mathbb{R}^2$. Moreover, the zero-order term of (2.1.15) *is given by the convergence*

$$\begin{cases} (\nabla \boldsymbol{u}^\varepsilon)^{\mathrm{T}} (\boldsymbol{v}^\varepsilon - \boldsymbol{v}) \overset{\mathrm{w}}{\rightharpoonup} M \boldsymbol{u} \quad \textit{weakly in} \quad L^{2-2r}(Q_T) \\ \nabla \boldsymbol{u}^\varepsilon : \nabla \boldsymbol{w}_\lambda^\varepsilon \overset{\mathrm{w}}{\rightharpoonup} M \boldsymbol{u} \cdot \boldsymbol{\lambda} \quad \textit{weakly} * \textit{in} \quad \mathcal{M}(Q_T) \textit{ and in } L^{2-2r}(Q_T). \end{cases} \tag{2.1.17}$$

The energy relation for the limit equation (2.1.15) *is*

$$\begin{aligned} \|\boldsymbol{u}(t)\|^2_{\mathbb{H}(\Omega)} + \int_0^T \|\boldsymbol{u}(\tau)\|^2_{\mathbb{V}(\Omega)}\, d\tau + M \int_0^T \|\boldsymbol{u}(\tau)\|^2_{\mathbb{V}(\Omega)} d\tau \\ \leqslant \|\boldsymbol{u}_0\|^2_{L^2(\Omega)} + C_\Omega^2 \int_0^t \|\boldsymbol{f}(\tau)\|^2_{L^2(\Omega)} d\tau\,. \end{aligned} \tag{2.1.18}$$

The rest of the chapter is organized as follows. In Section 2.2, we prove the existence and uniqueness theorem for the solution for the drift nonstationary Stokes equation (2.1.1). Section 2.3 is devoted to the proof of the main theorem (Theorem 2.1.2).

Notation. In this chapter, $\nabla \boldsymbol{u} := \left(\frac{\partial \boldsymbol{u}_i}{\partial \boldsymbol{x}_j}\right)_{1\leqslant i,j\leqslant N}$ for $\boldsymbol{u} : \mathbb{R}^N \to \mathbb{R}^N$. $L^q(\Omega), (q \geqslant 1)$ denotes the classical Lebesgue space with norm $\|\boldsymbol{f}\|_q = (\int_\Omega |\boldsymbol{f}|^q dx)^{1/q}$, the Sobolev space of functions with all its k-th partial derivatives in $L^2(\Omega)$ will be denoted by $H^k(\Omega)$, and its dual space is $H^{-k}(\Omega)$. We use $\langle \boldsymbol{f}, \boldsymbol{g}\rangle = \int_\Omega \boldsymbol{f}\boldsymbol{g}dx$ to denote the standard inner product on the Hilbert space $L^2(\Omega)$. Given any Banach space $\mathbb{X}$ with norm $\|\cdot\|_{\mathbb{X}}$ and $p \geq 1$, the space of measurable functions $\boldsymbol{u} = \boldsymbol{u}(t)$ from $[0, T)$ into $\mathbb{X}$ such that $\|\boldsymbol{u}\|_{\mathbb{X}} \in L^q([0, T))$ will be denoted $L^q([0, T); \mathbb{X})$. And $C([0, T); \text{w-}H^k(\Omega))$ will denote the space of continuous function from $[0, T)$ into w-$H^k(\Omega)$. This means that for every $\boldsymbol{\varphi} \in H^{-k}(\Omega)$, the function $\langle \boldsymbol{\varphi}, \boldsymbol{u}(t)\rangle$ is in $C([0, T))$.

2.2 Proof of the Existence and Uniqueness

We will prove the existence and uniqueness theorem for solution for the drift nonstationary Stokes equation (2.1.1) in this section. The strategy of the proof is motivated by Leray's seminal work on the incompressible Navier-Stokes equations [35], see also [61, 17, 19]. It proceeds in six steps. We can construct a sequence of approximate solutions by any method that yields a consistent weak formulation and an energy relation. We will employ the Faedo-Galerkin method to approximate our drift Stokes equation by a sequence of Cauchy problems for suitable systems of ODEs in finite dimensional spaces. The structure of these steps was enlightened by my supervisor Prof. Lin [38, 39, 40, 41].

Step 1: *Construction of approximate solutions $\{\boldsymbol{u}_m^\varepsilon\}_m$ by the Faedo-Galerkin method.*

First, we select a countable orthonormal basis $\{\boldsymbol{e}_i\}_{i=1}^{\infty}$ of the space $H_0^1(\Omega) \cap C_c^{\infty}(\Omega)$. For any $m \in \mathbb{N}$, we define the approximate solution $\boldsymbol{u}_m^{\varepsilon}(\boldsymbol{x}, t)$ of (2.1.1) by

$$\boldsymbol{u}_m^{\varepsilon}(\boldsymbol{x}, t) = \sum_{i=1}^{m} c_{im}^{\varepsilon}(t)\boldsymbol{e}_i(\boldsymbol{x}), \tag{2.2.1}$$

with $c_{im}^{\varepsilon}(t) \in H^1[0, T), 0 < T < \infty$.

Let now introduce, for any $m \in \mathbb{N}$, the finite dimensional approximate problem for (2.1.1)

$$\begin{cases} \langle \boldsymbol{u}_m^{\varepsilon}{}'(\boldsymbol{x}, t), \boldsymbol{e}_k \rangle + \int_\Omega \nabla \boldsymbol{u}_m^{\varepsilon} : \nabla \boldsymbol{e}_k \, d\boldsymbol{x} + B(\boldsymbol{u}_m^{\varepsilon}, \boldsymbol{v}^{\varepsilon}, \boldsymbol{e}_k) = \langle \boldsymbol{f}(\boldsymbol{x}, t), \boldsymbol{e}_k \rangle \,, \\ \qquad\qquad\qquad\qquad \text{in } \mathcal{D}'[0, T), \ \forall k = 1, \cdots, m \\ \boldsymbol{u}_m^{\varepsilon}(\boldsymbol{x}, 0) = \boldsymbol{u}_{0m}^{\varepsilon}(\boldsymbol{x}), \end{cases} \tag{2.2.2}$$

where $\boldsymbol{u}_{0m}^{\varepsilon}$ is the orthogonal projection of $\boldsymbol{u}_0^{\varepsilon}$ onto the space spanned by $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_m\}$ in $H_0^1(\Omega)$.

From the initial condition in this problem, we have

$$\sum_{i=1}^{m} c_{im}^{\varepsilon}(0)\boldsymbol{e}_i = \boldsymbol{u}_m^{\varepsilon}(0) = \boldsymbol{u}_{0m}^{\varepsilon} = \sum_{i=1}^{m} \langle \boldsymbol{u}_0^{\varepsilon}, \boldsymbol{e}_i \rangle \, \boldsymbol{e}_i,$$

which implies $c_{im}^{\varepsilon}(0) = \langle \boldsymbol{u}_0^{\varepsilon}, \boldsymbol{e}_i \rangle$, since $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_m\}$ are linearly independent.

From classical results concerning Hilbert spaces, we have that

$$\boldsymbol{u}_{0m}^{\varepsilon}(\boldsymbol{x}) \to \boldsymbol{u}_0^{\varepsilon}(\boldsymbol{x}) \text{ strongly in } L^2(\Omega)$$

as $m \to \infty$ so that

$$\|\boldsymbol{u}_{0m}^{\varepsilon}\|_{L^2(\Omega)} \leqslant \|\boldsymbol{u}_0^{\varepsilon}\|_{L^2(\Omega)}.$$

Moreover, applying the Cauchy-Schwarz inequality and the Poincaré inequality, these solutions will satisfy the regularized version of the energy relation (2.1.10) as the equality

$$\|\boldsymbol{u}_m^\varepsilon(t)\|_{\mathbb{H}(\Omega)}^2 + \int_0^t \|\boldsymbol{u}_m^\varepsilon(\tau)\|_{\mathbb{V}(\Omega)}^2 d\tau = \|\boldsymbol{u}_{0m}^\varepsilon\|_{L^2(\Omega)}^2 + C_\Omega^2 \int_0^T \|\boldsymbol{f}(\tau)\|_{L^2(\Omega)}^2 d\tau \tag{2.2.3}$$

for all $t \in [0, T)$.

Consequently, problem (2.2.2) is a system of m linear ordinary differential equations of the first order with unknowns $c_{1m}, \cdots, c_{mm}$. By Picard's local existence theory, the system of linear ODEs (2.2.2) has a unique solution on some interval $(0, t_m), 0 < t_m < T$. Furthermore, the energy relation (2.2.3) provides a global $L^2(\Omega)$ bound on the solutions, ensuring that they are global. The detail is referred to [61].

Step 2: *Show that $\{\boldsymbol{u}_m^\varepsilon\}_m$ converges strongly to $\{\boldsymbol{u}^\varepsilon\}$ in $L^2(Q_T)$ as $m \to \infty$.*

Thanks to the energy inequality (2.1.10) and the energy space (2.1.5), we can obtain

(a) $\{\boldsymbol{u}_m^\varepsilon\}_m$ is bounded in $L^2\big([0, T); \mathbb{V}(\Omega)\big)$;

(b) $\{\partial_t \boldsymbol{u}_m^\varepsilon\}_m$ is bounded in $L^2\big([0, T); \mathbb{V}^{-1}(\Omega)\big)$;

(c) $\mathbb{V}(\Omega)$ is compactly embedded in $\mathbb{H}(\Omega)$ and $L^2(\Omega)$ is continuously embedded in $\mathbb{V}^{-1}(\Omega)$.

From the Aubin-Lions lemma (see [61]), we can know that the injection of the space $L^2\big([0, T); \mathbb{V}(\Omega)\big)$ into $L^2(Q_T)$ is compact, which implies that $\{\boldsymbol{u}_m^\varepsilon\}_m$ admits a strongly converging subsequence in $L^2(Q_T)$.

Step 3: *Passage to the limit as $m \to \infty$.*

We want to pass to the limit as $m \to \infty$ in (2.2.2) using the energy estimate (2.2.3). We recall that at the present time $\varepsilon > 0$ is fixed, and we are only concerned with a passage to the limit as $m \to \infty$. First, Step 2 and the energy estimate (2.2.3) ensures that we can extract a subsequence (still denoted by m), such that

$$\begin{cases} \boldsymbol{u}_m^\varepsilon \to \boldsymbol{u}^\varepsilon & \text{strongly in} \quad L^2(Q_T) \\ \boldsymbol{u}_m^{\varepsilon\,\prime} \xrightarrow{\text{w}} \boldsymbol{u}^{\varepsilon\prime} & \text{weakly in} \quad L^2\big([0,T);\mathbb{V}^{-1}(\Omega)\big) \\ \boldsymbol{u}_{0m}^\varepsilon \to \boldsymbol{u}_0^\varepsilon & \text{strongly in} \quad L^2(\Omega). \end{cases} \tag{2.2.4}$$

Now let $\boldsymbol{\varphi}(\boldsymbol{x}) \in \mathbb{V}(\Omega)$ and choose a $\psi(t) \in \mathcal{D}[0,T)$ such that $\psi(0) = 1$ and $\psi(T) = 0$. Multiply the equation in (2.2.2) by $\langle \boldsymbol{\varphi}, \boldsymbol{e}_i \rangle_{L^2(\Omega)} \psi$ and sum over k from 1 to m. After integration in t over $[0,T)$, we get

$$\begin{aligned} - \int_{Q_T} \boldsymbol{u}_m^\varepsilon(\boldsymbol{x},t)\psi'(t)\boldsymbol{\varphi}(\boldsymbol{x})\, dxdt + \int_{Q_T} \nabla \boldsymbol{u}_m^\varepsilon(\boldsymbol{x},t) : \nabla\boldsymbol{\varphi}(\boldsymbol{x})\psi(t)\, d\boldsymbol{x}dt \\ + \int_0^T B(\boldsymbol{u}_m^\varepsilon(\boldsymbol{x},t), \boldsymbol{v}^\varepsilon(\boldsymbol{x},t), \boldsymbol{\varphi}(\boldsymbol{x}))\psi(t)\, dt \\ = \int_{Q_T} \boldsymbol{f}(\boldsymbol{x},t)\psi(t)\boldsymbol{\varphi}(\boldsymbol{x})\, d\boldsymbol{x}dt + \langle \boldsymbol{u}_{0m}^\varepsilon(\boldsymbol{x}), \boldsymbol{\varphi}(\boldsymbol{x})\rangle. \end{aligned} \tag{2.2.5}$$

Note that for the first term in this equation, due to the following identity

$$\begin{aligned} &\int_{Q_T} \partial_t \boldsymbol{u}_m^\varepsilon \psi(t)\boldsymbol{\varphi}(\boldsymbol{x})\, d\boldsymbol{x}dt \\ =& \int_{Q_T} \Big(\frac{d}{dt}\big(\boldsymbol{u}_m^\varepsilon(\boldsymbol{x},t)\psi(t)\big) - \boldsymbol{u}_m^\varepsilon(\boldsymbol{x},t)\psi'(t)\Big)\boldsymbol{\varphi}(\boldsymbol{x})\, dxdt \\ =& \int_\Omega \big(\boldsymbol{u}_m^\varepsilon(\boldsymbol{x},T)\psi(T) - \boldsymbol{u}_m^\varepsilon(\boldsymbol{x},0)\psi(0)\big)\boldsymbol{\varphi}(\boldsymbol{x})\, d\boldsymbol{x} - \int_{Q_T} \boldsymbol{u}_m^\varepsilon \psi'(t)\boldsymbol{\varphi}(\boldsymbol{x})\, d\boldsymbol{x}dt \\ =& - \langle \boldsymbol{u}_{0m}^\varepsilon(\boldsymbol{x}), \boldsymbol{\varphi}(\boldsymbol{x})\rangle - \int_{Q_T} \boldsymbol{u}_m^\varepsilon(\boldsymbol{x},t)\psi'(t)\boldsymbol{\varphi}(\boldsymbol{x})\, d\boldsymbol{x}dt. \end{aligned}$$

We now let $m \to \infty$ here. All the terms pass to the limit, thanks to convergences (2.2.4), we have

$$\int_{Q_T} \boldsymbol{u}_m^\varepsilon(\boldsymbol{x},t)\psi'(t)\boldsymbol{\varphi}(\boldsymbol{x})\,d\boldsymbol{x}dt \to \int_{Q_T} \boldsymbol{u}^\varepsilon(\boldsymbol{x},t)\psi'(t)\boldsymbol{\varphi}(\boldsymbol{x})\,d\boldsymbol{x}dt \quad \text{as} \quad m\to\infty\,,$$

$$\int_{Q_T} \nabla\boldsymbol{u}_m^\varepsilon : \nabla\boldsymbol{\varphi}(\boldsymbol{x})\psi(t)\,d\boldsymbol{x}dt \to \int_{Q_T} \nabla\boldsymbol{u}^\varepsilon : \nabla\boldsymbol{\varphi}(\boldsymbol{x})\psi(t)\,d\boldsymbol{x}dt \quad \text{as} \quad m\to\infty\,,$$

$$\int_0^T B(\boldsymbol{u}_m^\varepsilon, \boldsymbol{v}^\varepsilon, \boldsymbol{\varphi})\psi(t)\,dt \to \int_0^T B(\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon, \boldsymbol{\varphi})\psi(t)\,dt \quad \text{as} \quad m\to\infty\,,$$

and

$$\langle \boldsymbol{u}_{0m}^\varepsilon(\boldsymbol{x}), \boldsymbol{\varphi}(\boldsymbol{x})\rangle \to \langle \boldsymbol{u}_0^\varepsilon(\boldsymbol{x}), \boldsymbol{\varphi}(\boldsymbol{x})\rangle \quad \text{as} \quad m\to\infty\,.$$

Summing up the above convergence results, we finally get that $\boldsymbol{u}^\varepsilon$ satisfies the integral identity

$$\begin{aligned} -\int_{Q_T} \boldsymbol{u}^\varepsilon(\boldsymbol{x},t)\psi'(t)\boldsymbol{\varphi}(\boldsymbol{x})\,d\boldsymbol{x}dt + \int_{Q_T} \nabla\boldsymbol{u}^\varepsilon(\boldsymbol{x},t) : \nabla\boldsymbol{\varphi}(\boldsymbol{x})\psi(t)\,d\boldsymbol{x}dt \\ + \int_0^T B(\boldsymbol{u}^\varepsilon(\boldsymbol{x},t), \boldsymbol{v}^\varepsilon(\boldsymbol{x},t), \boldsymbol{\varphi}(\boldsymbol{x}))\psi(t)\,dt \\ = \int_{Q_T} \boldsymbol{f}(\boldsymbol{x},t)\psi(t)\boldsymbol{\varphi}(\boldsymbol{x})\,d\boldsymbol{x}dt + \langle \boldsymbol{u}_0^\varepsilon(\boldsymbol{x}), \boldsymbol{\varphi}(\boldsymbol{x})\rangle, \end{aligned} \tag{2.2.6}$$

which is exactly the variational equation in (2.1.7) since ψ and $\boldsymbol{\varphi}$ are arbitrary respectively in $\mathcal{D}[0,T)$ and $H_0^1(\Omega)$. This shows that $\boldsymbol{u}^\varepsilon$ is the weak solution of (2.1.1).

Step 4: *The energy inequality.*

To recover the energy inequality (2.1.10) from the energy relation (2.2.3), first we note that the regularized initial data $\boldsymbol{u}_{0m}^\varepsilon(x)$ converges to $\boldsymbol{u}_0^\varepsilon(x)$ strongly in $L^2(\Omega)$ as m tends to infinite so that

$$\|\boldsymbol{u}_{0m}^\varepsilon\|_{L^2(\Omega)} \to \|\boldsymbol{u}_0^\varepsilon\|_{L^2(\Omega)}\,.$$

The convergence of $\boldsymbol{u}_m^\varepsilon$ in $L^2(Q_T)$ together with the fact that the norm of the weak limit of a sequence is a lower bound for the inferior limit of the norms, yields

$$\|\boldsymbol{u}^\varepsilon(t)\|_{L^2}^2 \leqslant \liminf_{m\to\infty} \|\boldsymbol{u}_m^\varepsilon(t)\|_{L^2}^2 , \qquad 0 < t < T < \infty .$$

Similarly, the convergence of $\boldsymbol{u}_m^\varepsilon$ in $L^2(Q_T)$ implies

$$\int_0^T \|\nabla \boldsymbol{u}^\varepsilon(\tau)\|_{L^2}^2 d\tau \leqslant \liminf_{m\to\infty} \int_0^T \|\nabla \boldsymbol{u}_m^\varepsilon(\tau)\|_{L^2}^2 d\tau .$$

By combining the above inequalities, we obtain from (2.2.3) the energy inequality

$$\|\boldsymbol{u}^\varepsilon(t)\|_{\mathbb{H}}^2 + \int_0^t \|\boldsymbol{u}^\varepsilon(\tau)\|_{\mathbb{V}}^2 \, d\tau \leqslant \|\boldsymbol{u}_0^\varepsilon\|_{L^2(\Omega)}^2 + C_\Omega^2 \int_0^T \|\boldsymbol{f}(\tau)\|_{L^2(\Omega)}^2 d\tau \quad (2.2.7)$$

for all $t \in [0, T)$, where C_Ω is the Poincaré constant which is independent of ε.

Step 5: *Proof of the uniqueness.*

Since the nonstationary Stokes equation (2.1.1) is linear, the uniqueness follows immediately from the energy inequality. Let $\boldsymbol{u}_1^\varepsilon, \boldsymbol{u}_2^\varepsilon$ be two solutions corresponding to the same data of (2.1.1) then their difference $\boldsymbol{u}^\varepsilon \equiv \boldsymbol{u}_1^\varepsilon - \boldsymbol{u}_2^\varepsilon$ satisfies the equation

$$\begin{cases} \partial_t \boldsymbol{u}^\varepsilon + \operatorname{curl}(\boldsymbol{v}^\varepsilon) J \boldsymbol{u}^\varepsilon = \Delta \boldsymbol{u}^\varepsilon, & (\boldsymbol{x}, t) \in Q_T, \\ \boldsymbol{u}^\varepsilon(\boldsymbol{\sigma}, t) = 0, & (\boldsymbol{\sigma}, t) \in S_T, \\ \boldsymbol{u}^\varepsilon(\boldsymbol{x}, 0) = 0, & \boldsymbol{x} \in \Omega , \end{cases} \quad (2.2.8)$$

and the energy inequality

$$\|\boldsymbol{u}^\varepsilon(t)\|_{L^2}^2 + \int_0^T \|\nabla \boldsymbol{u}^\varepsilon(\tau)\|_{L^2}^2 d\tau \leqslant 0 . \quad (2.2.9)$$

The energy inequality (2.2.9) implies that

$$\|\boldsymbol{u}^\varepsilon(t)\|_{L^2}^2 = 0, \qquad 0 \leqslant t < \infty\,,$$

and the uniqueness is proved.

Step 6: *The existence of the pressure p^ε.*

Since $\boldsymbol{u}^\varepsilon \in W$ exists and is unique in the nonstationary drift Stokes equation, we have the following equation for p^ε

$$\nabla p^\varepsilon = \Delta \boldsymbol{u}^\varepsilon - \partial_t \boldsymbol{u}^\varepsilon - \mathrm{curl}(\boldsymbol{v}^\varepsilon) J \boldsymbol{u}^\varepsilon + f, \quad (\boldsymbol{x}, t) \in Q_T. \tag{2.2.10}$$

Taking divergence to the both sides of the equation, we have

$$\mathrm{div}\,(\nabla p^\varepsilon) = \Delta(\mathrm{div}\,(\boldsymbol{u}^\varepsilon)) - \partial_t(\mathrm{div}\,(\boldsymbol{u}^\varepsilon)) - \mathrm{div}\,(\mathrm{curl}(\boldsymbol{v}^\varepsilon) J \boldsymbol{u}^\varepsilon) + \mathrm{div}\,\boldsymbol{f}.$$

Since $\boldsymbol{u}^\varepsilon$ is divergence free, the above equation turns to the Poisson equation

$$\Delta p^\varepsilon = -\mathrm{div}\,(\mathrm{curl}(\boldsymbol{v}^\varepsilon) J \boldsymbol{u}^\varepsilon). \tag{2.2.11}$$

Then the existence and uniqueness of the Poisson equation ensures that there exists a unique p^ε such that the equation (2.1.1) holds in $\mathcal{D}'(\Omega)$. From the boundedness of $\boldsymbol{v}^\varepsilon$ in $W^{1,0,r}_{x,t}(Q_T)$ and $\boldsymbol{u}^\varepsilon$ in $L^2\big([0,T); \mathbb{V}(\Omega)\big)$, we have

$$\mathrm{curl}(\boldsymbol{v}^\varepsilon) J \boldsymbol{u}^\varepsilon \in L^{\frac{2r}{2+r}}\big([0,T); L^{\frac{2r}{2+r-2r}}(\Omega)\big), \tag{2.2.12}$$

then it follows immediately that,

$$\Delta p^\varepsilon = -\mathrm{div}\,(\mathrm{curl}(\boldsymbol{v}^\varepsilon) J \boldsymbol{u}^\varepsilon) \in L^{\frac{2r}{2+r}}\big([0,T); W^{-1,\frac{2r}{2+r-2r}}(\Omega)\big). \tag{2.2.13}$$

Finally from the regularity property we deduce the boundedness of p^ε

$$p^\varepsilon \in L^{\frac{2r}{2+r}}\big([0,T); W^{1,\frac{2r}{2+r-2r}}(\Omega)\big). \tag{2.2.14}$$

Since the space $W^{1,\frac{2r}{2+r-2r}}(\Omega) \subset L^{\frac{2r}{2+r-2r}}(\Omega)$ is a compact embedding and $L^{\frac{2r}{2+r-2r}}(\Omega) \subset L^{\frac{2r}{2+r}}(\Omega)$ for $r > 2$, we can assert that

$$p^\varepsilon \in L^{\frac{2r}{2+r}}(Q_T). \tag{2.2.15}$$

That is, p^ε is bounded in $L^{\frac{2r}{2+r}}(Q_T)$. Due to the reflexivity of the space $L^{\frac{2r}{2+r}}(Q_T)$ for $r > 2$, up to a subsequence the following convergence holds

$$p^\varepsilon \overset{\text{w}}{\rightharpoonup} p \quad \text{weakly in} \quad L^{\frac{2r}{2+r}}(Q_T). \tag{2.2.16}$$

This completes the proof of Theorem 2.1.1.

2.3 Proof of the Homogenization Result

In this section, we will give the proof of homogenization limit theorem 2.1.2. We first rewrite the weak formulation as

$$\begin{aligned} -\int_{Q_T} \boldsymbol{u}^\varepsilon(\boldsymbol{x},t)\psi'(t)\boldsymbol{\varphi}(\boldsymbol{x})\,d\boldsymbol{x}dt + \int_{Q_T} \nabla \boldsymbol{u}^\varepsilon(\boldsymbol{x},t) : \nabla\boldsymbol{\varphi}(\boldsymbol{x})\psi(t)\,d\boldsymbol{x}dt \\ + \int_0^T B(\boldsymbol{u}^\varepsilon(\boldsymbol{x},t), \boldsymbol{v}^\varepsilon(\boldsymbol{x},t), \boldsymbol{\varphi}(\boldsymbol{x}))\psi(t)\,dt \\ = \int_{Q_T} \boldsymbol{f}(\boldsymbol{x},t)\psi(t)\boldsymbol{\varphi}(\boldsymbol{x})\,d\boldsymbol{x}dt + \langle \boldsymbol{u}_0^\varepsilon(\boldsymbol{x}), \boldsymbol{\varphi}(x)\rangle, \end{aligned} \tag{2.3.1}$$

for every $\boldsymbol{\varphi}(\boldsymbol{x}) \in H_0^1(\Omega)$ and some $\psi(t) \in \mathcal{D}[0,T)$ such that $\psi(0) = 1$ and $\psi(T) = 0$. Meanwhile, we have the energy inequality

$$\|\boldsymbol{u}^\varepsilon(t)\|^2_{\mathbb{H}(\Omega)} + \int_0^t \|\boldsymbol{u}^\varepsilon(\tau)\|^2_{\mathbb{V}(\Omega)}\,d\tau \leqslant \|\boldsymbol{u}_0^\varepsilon\|^2_{L^2(\Omega)} + C_\Omega^2 \int_0^T \|\boldsymbol{f}(\tau)\|^2_{L^2(\Omega)}d\tau. \tag{2.3.2}$$

Similar to the Step 2 above, we obtain the convergence of $\boldsymbol{u}^\varepsilon$ as $\varepsilon \to 0$,

$$\boldsymbol{u}^\varepsilon \to \boldsymbol{u} \quad \text{strongly in} \quad L^2(Q_T), \tag{2.3.3}$$

due to Aubin-Lions lemma, which shows that the injection of the space $L^2\big([0,T);\mathbb{V}(\Omega)\big)$ into $L^2(Q_T)$ is compact.

Convergence results of the test function.

To derive the homogenization limit of (2.1.1), we imitate Tartar approach and construct a nonstationary linearized Stokes equation using the test function $\boldsymbol{w}^\varepsilon_\lambda$ for $\boldsymbol{\lambda} \in \mathbb{R}^2$

$$\begin{cases} \partial_t \boldsymbol{w}^\varepsilon_\lambda - \Delta \boldsymbol{w}^\varepsilon_\lambda + \operatorname{div}\big((\boldsymbol{v}^\varepsilon - \boldsymbol{v}) \otimes \boldsymbol{\lambda}\big) + \nabla q^\varepsilon_\lambda = 0, & (\boldsymbol{x}, t) \in Q_T, \\ \operatorname{div}(\boldsymbol{w}^\varepsilon_\lambda) = 0, & (\boldsymbol{x}, t) \in Q_T, \\ \boldsymbol{w}^\varepsilon_\lambda(\boldsymbol{\sigma}, t) = 0, & (\boldsymbol{\sigma}, t) \in S_T, \\ \boldsymbol{w}^\varepsilon_\lambda(\boldsymbol{x}, 0) = \boldsymbol{w}^\varepsilon_{0\lambda}(\boldsymbol{x}), & \boldsymbol{x} \in \Omega. \end{cases} \tag{2.3.4}$$

According to the results in [34] (see Theorem 6, p. 100), suppose that $\partial\Omega \in C^1$. Let $\boldsymbol{v}^\varepsilon \in W^{1,0,r}_{x,t}(Q_T) \cap C^\infty_{x,t}(Q_T)$ with $r > 2$. The spaces $W^{k,l,p}_{x,t}(Q_T)$ occuring here consists of all the elements of $L^p(Q_T)$, which possess generalized derivatives with respect to x up to order k, and with respect to t up to order l (inclusive) in $L^p(Q_T)$. The norm in this space is defined as

$$\|\boldsymbol{u}\|_{W^{k,l,p}_{x,t}(Q_T)} = \Big(\int_{Q_T} \big(\sum_{\alpha=0}^{k} |D_x{}^\alpha \boldsymbol{u}|^p + \sum_{\alpha=0}^{l} |D_t{}^\alpha \boldsymbol{u}|^p \big) d\boldsymbol{x} dt \Big)^{\frac{1}{p}}.$$

Then we have

$$\boldsymbol{F}(\boldsymbol{x}, t) = -\operatorname{div}\big((\boldsymbol{v}^\varepsilon - \boldsymbol{v}) \otimes \boldsymbol{\lambda}\big) \in L^r(Q_T) \tag{2.3.5}$$

and

$$\alpha(\boldsymbol{x}) = \boldsymbol{w}^\varepsilon_{0\lambda}(\boldsymbol{x}) \in W^{2-\frac{2}{r},r}(\Omega). \tag{2.3.6}$$

We also assume that

$$\boldsymbol{w}^\varepsilon_{0\lambda} \overset{\mathrm{w}}{\rightharpoonup} \boldsymbol{w}_{0\lambda} = 0 \quad \text{weakly in} \quad W^{2-\frac{2}{r},r}(\Omega) \qquad \text{as} \quad \varepsilon \to 0\,. \tag{2.3.7}$$

By the differentiability properties of the nonstationary linearized Navier-Stokes equation, the problem (2.3.4) possesses a unique solution $\boldsymbol{w}_\lambda^\varepsilon$, q_λ^ε, such that

$$\boldsymbol{w}_\lambda^\varepsilon \in W_{x,t}^{2,1,r}(Q_T), \qquad \nabla q_\lambda^\varepsilon \in L^r(Q_T), \tag{2.3.8}$$

and the following estimates holds

$$\|\boldsymbol{w}_\lambda^\varepsilon\|_{W_{x,t}^{2,1,r}(Q_T)} + \|\nabla q_\lambda^\varepsilon\|_{L^r(Q_T)} \leqslant C\big(\,\|\boldsymbol{F}\|_{L^r(Q_T)} + \|\boldsymbol{w}_{0\lambda}^\varepsilon\|_{W^{2-\frac{2}{r},r}(\Omega)}\big). \tag{2.3.9}$$

Since

$$\boldsymbol{v}^\varepsilon \overset{\mathrm{w}}{\rightharpoonup} \boldsymbol{v} \quad \text{weakly in} \quad W_{x,t}^{1,0,r}(Q_T) \cap C_{x,t}^\infty(Q_T), \tag{2.3.10}$$

this shows that in bounded smooth domain Ω,

$$\boldsymbol{F} = -\mathrm{div}\big((\boldsymbol{v}^\varepsilon - \boldsymbol{v}) \otimes \boldsymbol{\lambda}\big) \overset{\mathrm{w}}{\rightharpoonup} 0 \qquad \text{as} \quad \varepsilon \to 0\,. \tag{2.3.11}$$

Combining with (2.3.7), we obtain

$$\|\boldsymbol{w}_\lambda^\varepsilon\|_{W_{x,t}^{2,1,r}(Q_T)} + \|\nabla q_\lambda^\varepsilon\|_{L^r(Q_T)} \leqslant C\big(\,\|\boldsymbol{F}\|_{L^r(Q_T)} + \|\boldsymbol{w}_{0\lambda}^\varepsilon\|_{W^{2-\frac{2}{r},r}(\Omega)}\big) \to 0. \tag{2.3.12}$$

By maximum principle, we are led to the conclusion that

$$\begin{cases} \boldsymbol{w}_\lambda^\varepsilon \overset{\mathrm{w}}{\rightharpoonup} 0 \quad \text{weakly in} \quad W_{x,t}^{2,1,r}(Q_T) \\ q_\lambda^\varepsilon \overset{\mathrm{w}}{\rightharpoonup} 0 \quad \text{weakly in} \quad W_{x,t}^{1,0,r}(Q_T)/\mathbb{R}\,. \end{cases} \tag{2.3.13}$$

Then in bounded smooth domain Ω,

$$(\nabla \boldsymbol{w}_\lambda^\varepsilon)^{\mathrm{T}} \overset{\mathrm{w}}{\rightharpoonup} 0 \quad \text{weakly in} \quad W^{1,r}(Q_T). \tag{2.3.14}$$

However, the weak convergence (2.3.10) of $\boldsymbol{v}^\varepsilon$ and (2.3.14) does not guarantee the convergence of the product $(\nabla \boldsymbol{w}_\lambda^\varepsilon)^{\mathrm{T}}\boldsymbol{v}^\varepsilon$. Indeed, we have the following convergence instead,

$$(\nabla \boldsymbol{w}_\lambda^\varepsilon)^{\mathrm{T}}\boldsymbol{v}^\varepsilon \overset{\mathrm{w}}{\rightharpoonup} 0 \cdot v + M\lambda = M\lambda \quad \text{weakly in} \quad L^q(Q_T), \tag{2.3.15}$$

where $\frac{1}{q} = \frac{1}{r} - 1 + \frac{1}{r} - 1 = \frac{2-2r}{r}$ and the term M is a positive definite symmetric matrix-valued function, it is a kind of measure. The more details of measures can be found in [12]. Similarly,

$$\nabla \boldsymbol{w}_\lambda^\varepsilon : \nabla \boldsymbol{w}_\mu^\varepsilon \overset{\text{w}}{\rightharpoonup} M\boldsymbol{\lambda}\cdot\boldsymbol{\mu} \quad \text{weakly } * \text{ in } \mathcal{M}(Q_T) \text{ and in } L^{\frac{r}{2-2r}}(Q_T). \tag{2.3.16}$$

(2.3.15) and (2.3.16) are exactly the convergence results (2.1.16).

Let $\varphi(\boldsymbol{x}) \in C_c^\infty(\Omega)$ be a scalar function, choose $\psi(t) \in \mathcal{D}[0,T)$ with $\psi(0) = 1$ and $\psi(T) = 0$. Following Tartar's oscillating test function method we put $\varphi\psi\boldsymbol{w}_\lambda^\varepsilon$ and $\varphi\psi\boldsymbol{u}^\varepsilon$ as test function in equation (2.1.1) and (2.3.4) respectively. The weak formulation of (2.1.1) is

$$\begin{aligned}
&\int_{Q_T} \partial_t \boldsymbol{u}^\varepsilon \cdot \varphi\psi\boldsymbol{w}_\lambda^\varepsilon \, d\boldsymbol{x}dt \\
&+ \int_{Q_T} \nabla\boldsymbol{u}^\varepsilon : \nabla\boldsymbol{w}_\lambda^\varepsilon \varphi\psi \, d\boldsymbol{x}dt + \int_{Q_T} \nabla\boldsymbol{u}^\varepsilon : (\boldsymbol{w}_\lambda^\varepsilon(\nabla\varphi)^{\mathrm{T}})\psi \, d\boldsymbol{x}dt \\
&+ \int_{Q_T} (\nabla\boldsymbol{u}^\varepsilon)^{\mathrm{T}}\boldsymbol{v}^\varepsilon \cdot \varphi\psi\boldsymbol{w}_\lambda^\varepsilon \, d\boldsymbol{x}dt - \int_{Q_T} (\boldsymbol{v}^\varepsilon \otimes \boldsymbol{u}^\varepsilon) : \nabla\boldsymbol{w}_\lambda^\varepsilon \varphi\psi \, d\boldsymbol{x}dt \\
&- \int_{Q_T} (\boldsymbol{v}^\varepsilon \otimes \boldsymbol{u}^\varepsilon) : (\boldsymbol{w}_\lambda^\varepsilon(\nabla\varphi)^{\mathrm{T}})\psi \, dxdt + \int_{Q_T} (\boldsymbol{v}^\varepsilon \cdot \boldsymbol{u}^\varepsilon)\nabla\varphi \cdot \boldsymbol{w}_\lambda^\varepsilon \psi \, d\boldsymbol{x}dt \\
&- \int_{Q_T} p^\varepsilon(\nabla\varphi \cdot \boldsymbol{w}_\lambda^\varepsilon)\psi \, d\boldsymbol{x}dt = \int_{Q_T} \boldsymbol{f}\varphi\psi\boldsymbol{w}_\lambda^\varepsilon \, d\boldsymbol{x}dt.
\end{aligned} \tag{2.3.17}$$

As $\boldsymbol{w}_\lambda^\varepsilon \overset{\text{w}}{\rightharpoonup} 0$ weakly in $W_{x,t}^{2,1,r}(Q_T)$, some of the terms in (2.3.17) vanish except the second and the fifth terms, consequently the formulation (2.3.17) turns to

$$\int_{Q_T} \nabla\boldsymbol{u}^\varepsilon : \nabla\boldsymbol{w}_\lambda^\varepsilon \varphi\psi \, d\boldsymbol{x}dt - \int_{Q_T} (\boldsymbol{v}^\varepsilon \otimes \boldsymbol{u}^\varepsilon) : \nabla\boldsymbol{w}_\lambda^\varepsilon \varphi\psi \, d\boldsymbol{x}dt = o(1). \tag{2.3.18}$$

On the other hand, the weak formulation of (2.3.4) is

$$
\begin{aligned}
&-\int_{Q_T} \boldsymbol{w}_\lambda^\varepsilon \cdot \partial_t\big(\psi(t)\boldsymbol{u}^\varepsilon\big)\varphi\, d\boldsymbol{x}dt - \langle \boldsymbol{w}_{0\lambda}^\varepsilon \boldsymbol{u}_0^\varepsilon, \varphi\rangle \\
&+\int_{Q_T} \nabla\boldsymbol{w}_\lambda^\varepsilon : \nabla\boldsymbol{u}^\varepsilon\varphi\psi\, d\boldsymbol{x}dt - \int_{Q_T} \boldsymbol{w}_\lambda^\varepsilon \cdot \operatorname{div}\big(\boldsymbol{u}^\varepsilon(\nabla\varphi)^{\mathrm{T}}\big)\psi\, d\boldsymbol{x}dt \\
&+\int_{Q_T} \big((\boldsymbol{v}^\varepsilon-\boldsymbol{v})\otimes\lambda\big) : \nabla\boldsymbol{u}^\varepsilon\varphi\psi\, d\boldsymbol{x}dt - \int_{Q_T} q_\lambda^\varepsilon(\nabla\varphi\cdot\boldsymbol{u}^\varepsilon)\, d\boldsymbol{x}dt \\
&-\int_{Q_T} \big((\boldsymbol{v}^\varepsilon-\boldsymbol{v})\otimes\lambda\big) : \big(\boldsymbol{u}^\varepsilon(\nabla\varphi)^{\mathrm{T}}\big)\psi\, d\boldsymbol{x}dt = 0.
\end{aligned}
\tag{2.3.19}
$$

The convergence (2.3.10) and the strong convergence of $\boldsymbol{u}^\varepsilon$ in $L^2(Q_T)$ implies

$$
\int_\Omega \big((\boldsymbol{v}^\varepsilon-\boldsymbol{v})\otimes\boldsymbol{\lambda}\big) : \big(\boldsymbol{u}^\varepsilon(\nabla\varphi)^{\mathrm{T}}\big)\psi\, d\boldsymbol{x} \to 0 \quad \text{for all} \quad 0\leqslant t<T. \tag{2.3.20}
$$

By energy estimate and the Cauchy-Schwarz inequality

$$
\begin{aligned}
&\left|\int_\Omega \big((\boldsymbol{v}^\varepsilon-\boldsymbol{v})\otimes\boldsymbol{\lambda}\big) : \big(\boldsymbol{u}^\varepsilon(\nabla\varphi)^{\mathrm{T}}\big)\psi\, d\boldsymbol{x}\right| \\
\leqslant& |\psi|\, \|(\boldsymbol{v}^\varepsilon-\boldsymbol{v})\otimes\boldsymbol{\lambda}\|_{L^\infty} \|\nabla\varphi\|_{L^2} \|\boldsymbol{u}^\varepsilon(t)\|_{\mathbb{H}} \\
\leqslant& C\, \|\boldsymbol{u}^\varepsilon(t)\|_{\mathbb{H}}
\end{aligned}
\tag{2.3.21}
$$

for all $0\leqslant t<T$. Therefore, by Lebesgue dominated theorem

$$
\int_{Q_T} \big((\boldsymbol{v}^\varepsilon-\boldsymbol{v})\otimes\boldsymbol{\lambda}\big) : \big(\boldsymbol{u}^\varepsilon(\nabla\varphi)^{\mathrm{T}}\big)\psi\, d\boldsymbol{x}dt \to 0 \quad \text{as} \quad \varepsilon\to 0. \tag{2.3.22}
$$

Then applyingthe convergences (2.3.7) and (2.3.13), for the same reason as before we can rewrite the weak formulation (2.3.19) as

$$
\int_{Q_T} \nabla\boldsymbol{w}_\lambda^\varepsilon : \nabla\boldsymbol{u}^\varepsilon\varphi\psi\, d\boldsymbol{x}dt - \int_{Q_T} \big((\boldsymbol{v}^\varepsilon-\boldsymbol{v})\otimes\boldsymbol{\lambda}\big) : \nabla\boldsymbol{u}^\varepsilon\varphi\psi\, d\boldsymbol{x}dt = o(1). \tag{2.3.23}
$$

Note that for a matrix A and vectors $\boldsymbol{v}, \boldsymbol{u}$, the following identity holds

$$
(\boldsymbol{v}\otimes\boldsymbol{u}) : A = A^{\mathrm{T}}\boldsymbol{v}\cdot\boldsymbol{u}.
$$

In this way the weak formulations (2.3.18) and (2.3.23) can be written as

$$\int_{Q_T} \nabla \boldsymbol{u}^\varepsilon : \nabla \boldsymbol{w}^\varepsilon_\lambda \varphi\psi \, d\boldsymbol{x}dt - \int_{Q_T} (\nabla \boldsymbol{w}^\varepsilon_\lambda)^{\mathrm{T}} \boldsymbol{v}^\varepsilon \cdot \boldsymbol{u}^\varepsilon \varphi\psi \, d\boldsymbol{x}dt = o(1) \quad (2.3.24)$$

and

$$\int_{Q_T} \nabla \boldsymbol{w}^\varepsilon_\lambda : \nabla \boldsymbol{u}^\varepsilon \varphi\psi \, d\boldsymbol{x}dt - \int_{Q_T} (\nabla \boldsymbol{u}^\varepsilon)^{\mathrm{T}} (\boldsymbol{v}^\varepsilon - \boldsymbol{v}) \cdot \boldsymbol{\lambda} \varphi\psi \, d\boldsymbol{x}dt = o(1). \quad (2.3.25)$$

Then the equation (2.3.24) gives the weak convergence

$$\nabla \boldsymbol{u}^\varepsilon : \nabla \boldsymbol{w}^\varepsilon_\lambda \xrightarrow{\mathrm{w}} (\nabla \boldsymbol{w}^\varepsilon_\lambda)^{\mathrm{T}} \boldsymbol{v}^\varepsilon \cdot \boldsymbol{u}^\varepsilon \qquad \text{in} \quad \mathcal{D}'(Q_T). \quad (2.3.26)$$

Taking difference of (2.3.24) and (2.3.25) yields the weak convergence

$$(\nabla \boldsymbol{u}^\varepsilon)^{\mathrm{T}} (\boldsymbol{v}^\varepsilon - \boldsymbol{v}) \cdot \boldsymbol{\lambda} \xrightarrow{\mathrm{w}} (\nabla \boldsymbol{w}^\varepsilon_\lambda)^{\mathrm{T}} \boldsymbol{v}^\varepsilon \cdot \boldsymbol{u}^\varepsilon \qquad \text{in} \quad \mathcal{D}'(Q_T). \quad (2.3.27)$$

We may now consider the convergence of the term $(\nabla \boldsymbol{w}^\varepsilon_\lambda)^{\mathrm{T}} \boldsymbol{v}^\varepsilon \cdot \boldsymbol{u}^\varepsilon$. Recall the result we obtained before

$$(\nabla \boldsymbol{w}^\varepsilon_\lambda)^{\mathrm{T}} \boldsymbol{v}^\varepsilon \xrightarrow{\mathrm{w}} 0 \cdot \boldsymbol{v} + M\boldsymbol{\lambda} = M\boldsymbol{\lambda} \quad \text{weakly in} \quad L^{\frac{r}{2-2r}}(Q_T), \quad (2.3.28)$$

and the strong convergence of $\boldsymbol{u}^\varepsilon$

$$\boldsymbol{u}^\varepsilon \to \boldsymbol{u} \quad \text{strongly in} \quad L^2(Q_T), \quad (2.3.29)$$

we conclude that

$$(\nabla \boldsymbol{w}^\varepsilon_\lambda)^{\mathrm{T}} \boldsymbol{v}^\varepsilon \cdot \boldsymbol{u}^\varepsilon \xrightarrow{\mathrm{w}} M\boldsymbol{u} \cdot \boldsymbol{\lambda} \quad \text{weakly in} \quad L^q(Q_T), \quad (2.3.30)$$

where $\frac{1}{q} = \frac{2-2r}{r} + \frac{1}{2} = \frac{1}{2-2r}$.

According to (2.3.26), (2.3.27) and (2.3.30), it follows that

$$\begin{cases} (\nabla \boldsymbol{u}^\varepsilon)^{\mathrm{T}} (\boldsymbol{v}^\varepsilon - \boldsymbol{v}) \xrightarrow{\mathrm{w}} M\boldsymbol{u} & \text{weakly in} \quad L^{2-2r}(Q_T) \\ \nabla \boldsymbol{u}^\varepsilon : \nabla \boldsymbol{w}^\varepsilon_\lambda \xrightarrow{\mathrm{w}} M\boldsymbol{u} \cdot \boldsymbol{\lambda} & \text{weakly} * \text{in} \quad \mathcal{M}(Q_T) \text{ and in } L^{2-2r}(Q_T). \end{cases} \quad (2.3.31)$$

Passage to the limit as $\epsilon \to 0$.

The task is now to discuss the convergence of the weak formulation (2.3.1) as $\varepsilon \to 0$. First, it follows directly from the hypothesis (2.1.13) that

$$\langle \boldsymbol{u}_0^\varepsilon(\boldsymbol{x}), \boldsymbol{\varphi}(\boldsymbol{x})\rangle \to \langle \boldsymbol{u}_0(\boldsymbol{x}), \boldsymbol{\varphi}(\boldsymbol{x})\rangle \quad \text{as} \quad \varepsilon \to 0\,. \tag{2.3.32}$$

For the first term in the weak formulation (2.3.1), by energy inequality and Cauchy-Schwarz inequality, we have

$$|\langle \boldsymbol{u}^\varepsilon(t), \psi'(t)\boldsymbol{\varphi}(\boldsymbol{x})\rangle| \leqslant \sup_{0\leqslant t<T} |\psi'(t)|\, \|\boldsymbol{u}^\varepsilon(t)\|_{\mathbb{H}(\Omega)}\, \|\boldsymbol{\varphi}\|_{L^2(\Omega)} \leqslant C\, \|\boldsymbol{u}^\varepsilon(t)\|_{\mathbb{H}(\Omega)}\,. \tag{2.3.33}$$

Then the convergence (2.3.3) and Lebesgue dominated convergence theorem ensure the convergence

$$\int_{Q_T} \boldsymbol{u}^\varepsilon(\boldsymbol{x},t)\psi'(t)\boldsymbol{\varphi}(\boldsymbol{x})\, d\boldsymbol{x}dt \to \int_{Q_T} \boldsymbol{u}(\boldsymbol{x},t)\psi'(t)\boldsymbol{\varphi}(\boldsymbol{x})\, d\boldsymbol{x}dt \quad \text{as} \quad \varepsilon \to 0\,. \tag{2.3.34}$$

Since $\boldsymbol{u}^\varepsilon$ is bounded in $L^2\big([0,T);\mathbb{V}(\Omega)\big)$, on account of the fact that $\mathbb{V}(\Omega)$ is compactly embedded into $\mathbb{H}(\Omega)$, we have

$$\nabla \boldsymbol{u}^\varepsilon \overset{\text{w}}{\rightharpoonup} \nabla \boldsymbol{u} \quad \text{weakly in} \quad L^2(Q_T). \tag{2.3.35}$$

By energy estimate and the Cauchy-Schwarz inequality

$$\begin{aligned}|\,\langle \nabla \boldsymbol{u}^\varepsilon(t), \nabla\boldsymbol{\varphi}\psi\rangle\,| \leqslant& \sup_{0\leqslant t<T} |\psi|\, \|\nabla \boldsymbol{u}^\varepsilon(t)\|_{L^2(\Omega)}\, \|\nabla\boldsymbol{\varphi}\|_{L^2(\Omega)}\\ \leqslant& C\, \|\nabla \boldsymbol{u}^\varepsilon(t)\|_{L^2(\Omega)}\end{aligned}$$

for all $0 \leqslant t < T$. It follows by Lebesgue's dominated convergence theorem that

$$\int_{Q_T} \nabla \boldsymbol{u}^\varepsilon(\boldsymbol{x},t) : \nabla\boldsymbol{\varphi}(\boldsymbol{x})\psi(t)\, dxdt \to \int_{Q_T} \nabla \boldsymbol{u}(x,t) : \nabla\boldsymbol{\varphi}(\boldsymbol{x})\psi(t)\, d\boldsymbol{x}dt. \tag{2.3.36}$$

It remains to show the convergence of the trilinear form

$$B(\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon, \boldsymbol{\varphi}) = \langle (\nabla \boldsymbol{u}^\varepsilon)^{\mathrm{T}} \boldsymbol{v}^\varepsilon, \boldsymbol{\varphi} \rangle - \int_\Omega (\boldsymbol{v}^\varepsilon \otimes \boldsymbol{u}^\varepsilon) : \nabla \boldsymbol{\varphi} \, dx. \tag{2.3.37}$$

For the first term in the trilinear form, we rewrite it as

$$\langle (\nabla \boldsymbol{u}^\varepsilon)^{\mathrm{T}} \boldsymbol{v}^\varepsilon, \boldsymbol{\varphi} \rangle = \langle (\nabla \boldsymbol{u}^\varepsilon)^{\mathrm{T}} (\boldsymbol{v}^\varepsilon - \boldsymbol{v}), \boldsymbol{\varphi} \rangle + \langle (\nabla \boldsymbol{u}^\varepsilon)^{\mathrm{T}} \boldsymbol{v}, \boldsymbol{\varphi} \rangle. \tag{2.3.38}$$

We first consider the first term in the RHS of (2.3.38), it follows from (2.3.31) that

$$\langle (\nabla \boldsymbol{u}^\varepsilon)^{\mathrm{T}} (\boldsymbol{v}^\varepsilon - \boldsymbol{v}), \boldsymbol{\varphi} \rangle \to \langle M \boldsymbol{u}, \boldsymbol{\varphi} \rangle. \tag{2.3.39}$$

In addition, the convergence $\nabla \boldsymbol{u}^\varepsilon \overset{\mathrm{w}}{\rightharpoonup} \nabla \boldsymbol{u}$ weakly in $L^2(Q_T)$ implies

$$\langle (\nabla \boldsymbol{u}^\varepsilon)^{\mathrm{T}} \boldsymbol{v}, \boldsymbol{\varphi} \rangle \to \langle (\nabla \boldsymbol{u})^{\mathrm{T}} \boldsymbol{v}, \boldsymbol{\varphi} \rangle. \tag{2.3.40}$$

Together with (2.3.39) and (2.3.40), we obtain the limit of $\langle (\nabla \boldsymbol{u}^\varepsilon)^{\mathrm{T}} \boldsymbol{v}^\varepsilon, \boldsymbol{\varphi} \rangle$,

$$\langle (\nabla \boldsymbol{u}^\varepsilon)^{\mathrm{T}} \boldsymbol{v}^\varepsilon, \boldsymbol{\varphi} \rangle \to \langle (\nabla \boldsymbol{u})^{\mathrm{T}} \boldsymbol{v}, \boldsymbol{\varphi} \rangle + \langle M \boldsymbol{u}, \boldsymbol{\varphi} \rangle. \tag{2.3.41}$$

On the other hand, from the strong convergence (2.3.3) of $\boldsymbol{u}^\varepsilon$ and the weak convergence (2.1.12) of $\boldsymbol{v}^\varepsilon$ impliy the convergence

$$\boldsymbol{v}^\varepsilon \otimes \boldsymbol{u}^\varepsilon \overset{\mathrm{w}}{\rightharpoonup} \boldsymbol{v} \otimes \boldsymbol{u} \quad \text{weakly in} \quad L^{\frac{2r}{2+r}}\big([0,T); L^{\frac{2r}{2+r-2r}}(\Omega)\big). \tag{2.3.42}$$

Thus we can derive the limit of the second term in the trilinear form

$$\int_\Omega (\boldsymbol{v}^\varepsilon \otimes \boldsymbol{u}^\varepsilon) : \nabla \boldsymbol{\varphi} \, d\boldsymbol{x} \to \int_\Omega (\boldsymbol{v} \otimes \boldsymbol{u}) : \nabla \boldsymbol{\varphi} \, d\boldsymbol{x} \quad \text{as} \quad \varepsilon \to 0\,. \tag{2.3.43}$$

Putting (2.3.41) and (2.3.43) together, we have the convergence of the trilinear form

$$B(\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon, \boldsymbol{\varphi}) \to B(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{\varphi}) + \langle M \boldsymbol{u}, \boldsymbol{\varphi} \rangle \quad \text{as} \quad \varepsilon \to 0\,. \tag{2.3.44}$$

We can deduce from the energy estimate and Cauchy-Schwarz inequality that

$$\begin{aligned} \left|\langle(\nabla \boldsymbol{u}^{\varepsilon})^{\mathrm{T}}\boldsymbol{v}^{\varepsilon},\boldsymbol{\varphi}\rangle\psi\right| \leqslant & \sup_{0\leqslant t<T}|\psi|\,\|\boldsymbol{v}^{\varepsilon}\|_{L^{\infty}}\,\|\nabla \boldsymbol{u}^{\varepsilon}(t)\|_{L^2}\,\|\boldsymbol{\varphi}\|_{L^2} \\ \leqslant & C\,\|\nabla \boldsymbol{u}^{\varepsilon}(t)\|_{L^2} \end{aligned} \tag{2.3.45}$$

and

$$\begin{aligned} \left|\int_{\Omega}(\boldsymbol{v}^{\varepsilon}\otimes\boldsymbol{u}^{\varepsilon}):\nabla\boldsymbol{\varphi}\,d\boldsymbol{x}\psi\right| \leqslant & \sup_{0\leqslant t<T}|\psi|\,\|\boldsymbol{v}^{\varepsilon}\|_{L^{\infty}}\,\|\boldsymbol{u}^{\varepsilon}(t)\|_{\mathbb{H}}\,\|\nabla\boldsymbol{\varphi}\|_{L^2} \\ \leqslant & C\,\|\boldsymbol{u}^{\varepsilon}(t)\|_{\mathbb{H}}\,. \end{aligned} \tag{2.3.46}$$

Combining (2.3.32), (2.3.34), (2.3.36) and (2.3.44), by Lebesgue dominated theorem we conclude that the limit $\boldsymbol{u}$ satisfies the weak formulation

$$\begin{aligned} -\int_{Q_T}&\boldsymbol{u}(\boldsymbol{x},t)\psi'(t)\boldsymbol{\varphi}(\boldsymbol{x})\,d\boldsymbol{x}dt + \int_{Q_T}\nabla\boldsymbol{u}(\boldsymbol{x},t):\nabla\boldsymbol{\varphi}(\boldsymbol{x})\psi(t)\,d\boldsymbol{x}dt \\ &+ \int_0^T B(\boldsymbol{u}(\boldsymbol{x},t),\boldsymbol{v}(\boldsymbol{x},t),\boldsymbol{\varphi}(\boldsymbol{x}))\psi(t)\,dt + \int_{Q_T} M\boldsymbol{u}(\boldsymbol{x},t)\boldsymbol{\varphi}(\boldsymbol{x})\psi(t)\,d\boldsymbol{x}dt \\ &= \int_{Q_T}\boldsymbol{f}(\boldsymbol{x},t)\psi(t)\boldsymbol{\varphi}(\boldsymbol{x})\,d\boldsymbol{x}dt + \langle\boldsymbol{u}_0(\boldsymbol{x}),\boldsymbol{\varphi}(\boldsymbol{x})\rangle. \end{aligned} \tag{2.3.47}$$

Moreover, it follows from (2.2.16) that

$$\nabla p^{\varepsilon} \overset{\mathrm{w}}{\rightharpoonup} \nabla p \quad \text{weakly in} \quad L^{\frac{2r}{2+r}}([0,T);W^{-1,\frac{2r}{2+r}}(\Omega)). \tag{2.3.48}$$

Therefore we can write the limit equation of the nonstationary perturbed Stokes equation as

$$\partial_t\boldsymbol{u} - \Delta\boldsymbol{u} + \operatorname{curl}(\boldsymbol{v})J\boldsymbol{u} + \nabla p + M\boldsymbol{u} = \boldsymbol{f}. \tag{2.3.49}$$

In addition, for the divergence of $\boldsymbol{u}^{\varepsilon}$, the weak convergence of $\boldsymbol{u}^{\varepsilon}$ to $\boldsymbol{u}$ in $L^2\big([0,T);\mathbb{V}(\Omega)\big)$, and the fact that $\mathbb{V}(\Omega)$ is compactly embedded into $\mathbb{H}(\Omega)$ give that

$$\operatorname{div}\boldsymbol{u}^{\varepsilon} \overset{\mathrm{w}}{\rightharpoonup} \operatorname{div}\boldsymbol{u} \quad \text{weakly in} \quad L^2(Q_T). \tag{2.3.50}$$

Finally, the desired homogenization equation (2.1.15) is obtained.

Convergence of the Energy:

The energy associated with (2.1.1) is given by

$$E^\varepsilon(\boldsymbol{u}^\varepsilon)(t) \overset{\text{def}}{\equiv} \|\boldsymbol{u}^\varepsilon(t)\|^2_{\mathbb{H}(\Omega)} + \int_0^t \|\boldsymbol{u}^\varepsilon(\tau)\|^2_{\mathbb{V}(\Omega)}\, d\tau. \tag{2.3.51}$$

First we have the compactness of $E^\varepsilon(\boldsymbol{u}^\varepsilon)$.

Lemma 2.3.1. *$E^\varepsilon(\boldsymbol{u}^\varepsilon)$ is a relatively compact set in $C[0,T)$ for all $0 < T < \infty$.*

Proof. As discussed in the previous sections, according to the Arzelà-Ascoli theorem, it is equivalent to show

(a) $|E^\varepsilon(\boldsymbol{u}^\varepsilon)(t)| < C_1$, for all $t \in [0,T)$.

(b) $|E^\varepsilon(\boldsymbol{u}^\varepsilon)(t+h) - E^\varepsilon(\boldsymbol{u}^\varepsilon)(t)| \leqslant \theta(h)$, uniformly with respect to ε, for all $t \in [0,T)$ and for all $h > 0$, where θ tends to zero as h goes to zero.

(a) follows immediately by Cauchy-Schwarz and the energy inequalities. For the second statement (b), observe that the weak formulation yields

$$\begin{aligned}|E^\varepsilon(\boldsymbol{u}^\varepsilon)(t+h) - E^\varepsilon(\boldsymbol{u}^\varepsilon)(t)| &\leqslant \left| \int_t^{t+h} |\langle \boldsymbol{f}(t), \boldsymbol{u}^\varepsilon(t)\rangle dt \right| \\ &\leqslant h^{\frac{1}{2}} \|\boldsymbol{f}\|_{L^2(Q_T)} \|\boldsymbol{u}^\varepsilon\|_{L^2((0,T);\mathbb{V}(\Omega))}\end{aligned}$$

where we have use the Cauchy-Schwarz inequality, energy relation and the assumption of the initial condition and nonhomogeneous term f. ■

Hence there exists a subsequence still denoted by $\{E^\varepsilon(\boldsymbol{u}^\varepsilon)\}_\varepsilon$ and a function $E(\boldsymbol{u}) \in C[0,T)$ such that

$$E^\varepsilon(\boldsymbol{u}^\varepsilon) \to E(\boldsymbol{u}) \quad \text{in} \quad C[0,T)\,.$$

The same discussion as Step 4 in Section 2.2, we have

$$\|\boldsymbol{u}(t)\|_{\mathbb{H}(\Omega)}^2 + \int_0^t \|\boldsymbol{u}(\tau)\|_{\mathbb{V}(\Omega)}^2 \, d\tau \leqslant \liminf_{\varepsilon \to \infty} E^\varepsilon(\boldsymbol{u}^\varepsilon)(t)$$

$$\leqslant \|\boldsymbol{u}_0\|_{L^2}^2 + C_\Omega^2 \int_0^T \|\boldsymbol{f}(\tau)\|_{L^2(\Omega)}^2 d\tau \,.$$

As we have seen, the product of two weakly convergent sequences does not converge in general, to the product of the limits, and this is the principal difficulty when trying to characterize the limit $E(\boldsymbol{u})$;

$$\lim_{\varepsilon \to 0} E^\varepsilon(\boldsymbol{u}^\varepsilon)(t) \stackrel{\text{def}}{\equiv} E(\boldsymbol{u})(t) \neq \|\boldsymbol{u}(t)\|_{\mathbb{H}(\Omega)}^2 + \int_0^t \|\boldsymbol{u}(\tau)\|_{\mathbb{V}(\Omega)}^2 \, d\tau \,.$$

The weak convergence would only give a compactness of $E^\varepsilon(\boldsymbol{u}^\varepsilon)$ in $C[0,T)$, so it is difficult to find the limit energy from (2.3.51) directly. However, using the homogenization limit or (2.3.31), we have

$$E(\boldsymbol{u})(t) = \|\boldsymbol{u}(t)\|_{\mathbb{H}(\Omega)}^2 + \int_0^t \|\boldsymbol{u}(\tau)\|_{\mathbb{V}(\Omega)}^2 \, d\tau + M \int_0^t \|\boldsymbol{u}(\tau)\|_{\mathbb{V}(\Omega)}^2 d\tau \,. \tag{2.3.52}$$

The extra M-term is induced by homogenization.

Chapter 3

Homogenization of an Anelastic Stokes System arising from the Lake Equations

In this chapter, we will study the Homogenization of an Anelastic Stokes System arising from the Lake Equations. This is the analogue of the Stokes equation for the lake equation where the eddy viscosity coefficient $\nu = 1$, turbulent drag coefficient $\eta = 0$ and the nonlinear term $\operatorname{div}(b\boldsymbol{u}^\varepsilon \otimes \boldsymbol{u}^\varepsilon)$ is replaced by $-\frac{1}{2}(\boldsymbol{u}^\varepsilon \cdot \boldsymbol{v}^\varepsilon)\nabla b + b\,\operatorname{curl}(\boldsymbol{v}^\varepsilon)J\boldsymbol{u}^\varepsilon$. In Section 3.1, we introduce a two-dimensional viscous lake equation and deduce some important identities. In Section 3.2, we discuss the anelastic Stokes equation with drift term and prove the existence of the weak solution to the boundary value problem of the anelastic Stokes system using Lax-Milgram Theorem. In Section 3.3, we study the homogenization of the anelastic Stokes equation with drift term and give the homogenization result.

3.1 Introduction

We consider the two-dimensional viscous lake equations

$$\partial_t(b\boldsymbol{u}) + \operatorname{div}(b\boldsymbol{u}\otimes\boldsymbol{u}) + b\eta\boldsymbol{u} + b\nabla h = \operatorname{div}(\nu b\Sigma(\boldsymbol{u})) + b\boldsymbol{f}, \qquad (3.1.1)$$

$$\operatorname{div}(b\boldsymbol{u}) = 0, \qquad (3.1.2)$$

for $(\boldsymbol{x},t) = (x_1,x_2,t) \in \Omega\times(0,\infty)$ with $\Omega\subset\mathbb{R}^2$, a bounded and smooth domain. The vector field $\boldsymbol{u} = (u_1,u_2)^{\mathrm{T}}$ is a function of $(\boldsymbol{x},t)\in\mathbb{R}^2\times\mathbb{R}^+$ denoting the fluid velocity, the scalar function $h = h(\boldsymbol{x},t)$ stands for the surface height, $\boldsymbol{f}(\boldsymbol{x},t)$ is the wind forcing defined over $\Omega\times[0,\infty)$, $\boldsymbol{u}\otimes\boldsymbol{u} = (u_iu_j)$, and

$$\begin{aligned}\Sigma(\boldsymbol{u}) &= 2D(\boldsymbol{u}) - \operatorname{div}(\boldsymbol{u})\, I \\ &= \begin{bmatrix}\partial_1u_1-\partial_2u_2 & \partial_1u_2+\partial_2u_1\\ \partial_1u_2+\partial_2u_1 & -\partial_1u_1+\partial_2u_2\end{bmatrix}\end{aligned} \qquad (3.1.3)$$

where I is the 2×2 identity matrix and $D(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathrm{T}})$ is the deformation tensor. Here $\nu(\boldsymbol{x})$ and $\eta(\boldsymbol{x})$ are positive eddy viscosity coefficient and a non-negative turbulent drag coefficient defined over Ω, the bottom function $b = b(\boldsymbol{x})$ is a given smooth function to denote the depth of the basin, which is assumed to be non-degenerate, i.e., that there exist two positive constants b_1 and b_2 such that

$$0 < b_1 \leqslant b(\boldsymbol{x}) \leqslant b_2, \qquad \boldsymbol{x}\in\overline{\Omega}\,. \qquad (3.1.4)$$

This means that lakes and oceans have vertical lateral boundaries, like swimming pools. The initial condition is assumed to be weighted incompressible

$$\boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0(\boldsymbol{x}), \quad \operatorname{div}(b\boldsymbol{u}_0) = 0, \qquad \boldsymbol{x}\in\Omega. \qquad (3.1.5)$$

We also impose the Navier boundary conditions as

$$\boldsymbol{u}\cdot\boldsymbol{n}=0,\quad \nu\boldsymbol{\tau}\cdot\left(\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^{\mathrm{T}}\right)\cdot\nabla=-\beta\boldsymbol{\tau}\cdot\boldsymbol{u},\qquad \boldsymbol{x}\in\partial\Omega\,,\quad(3.1.6)$$

where $\boldsymbol{n}(\boldsymbol{x})$ and $\boldsymbol{\tau}(\boldsymbol{x})$ are the outward unit normal and a unit tangent to $\partial\Omega$ at $\boldsymbol{x}$ and $\beta(\boldsymbol{x})$ is a non-negative turbulent boundary drag coefficient defined on $\partial\Omega$. (3.1.6) are usually called the (general) Navier boundary conditions, which were first used by Navier in 1827 (see [7, 42]) and mean that there is a stagnant layer of fluid close to the wall allowing a fluid to slip, and the slip velocity is proportional to the shear stress. Since b depends on $\boldsymbol{x}$ only, $b=b(\boldsymbol{x})$ and satisfies (3.1.2), assuming $\boldsymbol{f}=0$, we can rewrite (3.1.1)–(3.1.3) as

$$\partial_t\boldsymbol{u}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}+\eta\boldsymbol{u}+\nabla h=b^{-1}\mathrm{div}(\nu b\Sigma(\boldsymbol{u})),\qquad \mathrm{div}(b\boldsymbol{u})=0,\quad(3.1.7)$$

for $(\boldsymbol{x},t)\in Q_T=\Omega\times[0,T)$. This system has been derived in [28, 36, 37] to model the evolution of the vertical averaged horizontal component of the three-dimensional velocity to the incompressible Euler equations confined to a shallow basin with varying bottom topography.

Notice that we have the identity

$$2\mathrm{div}(b\boldsymbol{u}\otimes\boldsymbol{u})-\nabla(b|\boldsymbol{u}|^2)=\begin{bmatrix}2\partial_2(bu_1u_2)+\partial_1(bu_1^2)-\partial_1(bu_2^2)\\ 2\partial_1(bu_1u_2)+\partial_2(bu_2^2)-\partial_2(bu_1^2)\end{bmatrix}.\quad(3.1.8)$$

Using the b-weighted divergence free condition $\mathrm{div}(b\boldsymbol{u})=0$, the first and second components of (3.1.8) can be rewritten respectively as

$$\begin{cases}2\partial_2(bu_1u_2)+\partial_1(bu_1^2)-\partial_1(bu_2^2)=-|\boldsymbol{u}|^2\partial_1 b-2b^2u_2\omega_b\\ 2\partial_1(bu_1u_2)+\partial_2(bu_2^2)-\partial_2(bu_1^2)=-|\boldsymbol{u}|^2\partial_2 b+2b^2u_1\omega_b\end{cases}\quad(3.1.9)$$

where

$$\omega_b=\frac{1}{b}\mathrm{curl}(\boldsymbol{u})=\frac{1}{b}\nabla\times\boldsymbol{u}=\frac{1}{b}(\partial_1u_2-\partial_2u_1)\quad(3.1.10)$$

is the potential vorticity associated with the lake equation (3.1.1)–(3.1.2). Therefore (3.1.8) can be rewritten as

$$\begin{aligned} 2\mathrm{div}(b\boldsymbol{u}\otimes\boldsymbol{u}) - \nabla(b|\boldsymbol{u}|^2) &= -|\boldsymbol{u}|^2\nabla b + 2b^2\omega_b\boldsymbol{u}^\perp \\ &= -|\boldsymbol{u}|^2\nabla b + 2b\,\mathrm{curl}(\boldsymbol{u})J\boldsymbol{u} \end{aligned} \tag{3.1.11}$$

where

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{u}^\perp = J\boldsymbol{u} = \begin{bmatrix} -u_2 \\ u_1 \end{bmatrix}. \tag{3.1.12}$$

We can extend (3.1.11) to any two b-weighted divergence free vector functions $\boldsymbol{u}$ and $\boldsymbol{v}$, i.e., $\mathrm{div}(b\boldsymbol{u}) = \mathrm{div}(b\boldsymbol{v}) = 0$;

$$\begin{aligned} 2\mathrm{div}(b\boldsymbol{u}\otimes\boldsymbol{v}) - \nabla(b\boldsymbol{u}\cdot\boldsymbol{v}) &= \begin{bmatrix} 2\partial_2(bu_1v_2) + \partial_1(bu_1v_1) - \partial_1(bu_2v_2) \\ 2\partial_1(bu_2v_1) + \partial_2(bu_2v_2) - \partial_2(bu_1v_1) \end{bmatrix} \\ &= -(\boldsymbol{u}\cdot\boldsymbol{v})\nabla b + b\,\mathrm{curl}(\boldsymbol{u})J\boldsymbol{v} - b(\nabla\boldsymbol{v})\boldsymbol{u} + b(\nabla\boldsymbol{u})^{\mathrm{T}}\boldsymbol{v}\,. \end{aligned} \tag{3.1.13}$$

Note that the extra term $-|\boldsymbol{u}|^2\nabla b$ in (3.1.11) (or $-(\boldsymbol{u}\cdot\boldsymbol{v})\nabla b$ in (3.1.13)) shows the nonlinear effect induced by the varying topography of the lake equation. When $\boldsymbol{v} = \boldsymbol{u}$, we have

$$-(\nabla\boldsymbol{u})\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathrm{T}}\boldsymbol{u} = (\nabla\times\boldsymbol{u})\boldsymbol{u}^\perp = \mathrm{curl}(\boldsymbol{u})J\boldsymbol{u} \tag{3.1.14}$$

and (3.1.13) will reduce to (3.1.11). In particular, when $b = 1$, $\omega_b = \omega = \mathrm{curl}(\boldsymbol{u})$, (3.1.11) and (3.1.13) will become

$$\begin{aligned} 2\mathrm{div}(\boldsymbol{u}\otimes\boldsymbol{u}) - \nabla(|\boldsymbol{u}|^2) &= \begin{bmatrix} \partial_1 & 2\partial_2 \\ -\partial_2 & 2\partial_1 \end{bmatrix} \begin{bmatrix} u_1^2 - u_2^2 \\ u_1u_2 \end{bmatrix} \\ &= 2\omega\boldsymbol{u}^\perp = 2\mathrm{curl}(\boldsymbol{u})J\boldsymbol{u} \end{aligned} \tag{3.1.15}$$

and

$$2\mathrm{div}(\boldsymbol{u}\otimes\boldsymbol{v}) - \nabla(\boldsymbol{u}\cdot\boldsymbol{v}) = \mathrm{curl}(\boldsymbol{u})J\boldsymbol{v} - (\nabla\boldsymbol{v})\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathrm{T}}\boldsymbol{v} \tag{3.1.16}$$

respectively.

In this chapter, we employ the following notations. The weighted square integrable space $L_b^2(\Omega)$ consists of all measurable functions $\boldsymbol{f}$ that satisfy

$$\int_{\Omega} |\boldsymbol{f}(\boldsymbol{x})|^2 \, b(\boldsymbol{x}) d\boldsymbol{x} < \infty.$$

The resulting $L_b^2(\Omega)$-norm of $\boldsymbol{f}$ is defined by

$$\|\boldsymbol{f}\|_{L_b^2(\Omega)} = \left(\int_{\Omega} |\boldsymbol{f}(\boldsymbol{x})|^2 \, b(\boldsymbol{x}) d\boldsymbol{x} \right)^{\frac{1}{2}}.$$

The space $L_b^2(\Omega)$ is a Hilbert space equipped with the following inner product

$$\langle \boldsymbol{f}, \gamma \rangle_b = \int_{\Omega} \boldsymbol{f} \cdot \gamma b(\boldsymbol{x}) d\boldsymbol{x}.$$

The weighted Sobolev space $H_b^1(\Omega)$ consists of all functions $\boldsymbol{f}$ with weak derivatives $\nabla \boldsymbol{f}$ satisfying

$$\|\boldsymbol{f}\|_{H_b^1(\Omega)} = \left(\int_{\Omega} (|\boldsymbol{f}(\boldsymbol{x})|^2 + |\nabla \boldsymbol{f}(\boldsymbol{x})|^2) b(\boldsymbol{x}) d\boldsymbol{x} \right)^{\frac{1}{2}} < \infty.$$

We also introduce the spaces $\mathbb{H}_b(\Omega)$, $\mathbb{V}_b(\Omega)$ and $\mathbb{V}_b^{-1}(\Omega)$ defined by

$$\mathbb{H}_b(\Omega) = \left\{ \boldsymbol{u} \in L_b^2(\Omega) \mid \operatorname{div}(b\boldsymbol{u}) = 0, \quad \boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad \boldsymbol{x} \in \partial\Omega \right\},$$

$$\mathbb{V}_b(\Omega) = \left\{ \boldsymbol{u} \in H_b^1(\Omega) \mid \operatorname{div}(b\boldsymbol{u}) = 0, \quad \boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad \boldsymbol{x} \in \partial\Omega \right\},$$

and

$$\mathbb{V}_b^{-1}(\Omega) = \left\{ \boldsymbol{u} \in H_b^{-1}(\Omega) \mid \operatorname{div}(b\boldsymbol{u}) = 0, \quad \boldsymbol{u} \cdot \boldsymbol{n} = 0, \quad \boldsymbol{x} \in \partial\Omega \right\}.$$

3.2 Anelastic Stokes equation with drift term

One can extend Tartar's pioneer work on the homogenization of Stokes equation perturbed by a drift [57, 58] to the lake equation;

$$-\mathrm{div}(b\Sigma(\boldsymbol{u}^\varepsilon)) - \frac{1}{2}(\boldsymbol{u}^\varepsilon \cdot \boldsymbol{v}^\varepsilon)\nabla b + b\,\mathrm{curl}(\boldsymbol{v}^\varepsilon)J\boldsymbol{u}^\varepsilon + b\nabla p^\varepsilon = b\boldsymbol{f}, \quad (3.2.1)$$

$$\mathrm{div}(b\boldsymbol{u}^\varepsilon) = 0, \qquad \boldsymbol{x} \in \Omega \quad (3.2.2)$$

with the Navier boundary conditions

$$\boldsymbol{u}^\varepsilon \cdot \boldsymbol{n} = 0, \qquad \boldsymbol{\tau} \cdot \left(\nabla \boldsymbol{u}^\varepsilon + (\nabla \boldsymbol{u}^\varepsilon)^{\mathrm{T}}\right) \cdot \nabla = -\beta \boldsymbol{\tau} \cdot \boldsymbol{u}^\varepsilon, \qquad \boldsymbol{x} \in \partial\Omega\,. \quad (3.2.3)$$

This is the analogue of the Stokes equation for the lake equation (3.1.1)–(3.1.3) where the eddy viscosity coefficient $\nu = 1$, turbulent drag coefficient $\eta = 0$ and the nonlinear term $\mathrm{div}(b\boldsymbol{u}^\varepsilon \otimes \boldsymbol{u}^\varepsilon)$ is replaced by

$$-\frac{1}{2}(\boldsymbol{u}^\varepsilon \cdot \boldsymbol{v}^\varepsilon)\nabla b + b\,\mathrm{curl}(\boldsymbol{v}^\varepsilon)J\boldsymbol{u}^\varepsilon \quad (3.2.4)$$

based on the identities (3.1.8)–(3.1.16). Here we assume $\boldsymbol{v}^\varepsilon \in L^\infty_b(\Omega)$. Note that from (3.1.3) we have

$$\begin{aligned}
\mathrm{div}(b\Sigma(\boldsymbol{u}^\varepsilon)) &= \mathrm{div}\begin{bmatrix} b(\partial_1 u_1^\varepsilon - \partial_2 u_2^\varepsilon) & b(\partial_1 u_2^\varepsilon + \partial_2 u_1^\varepsilon) \\ b(\partial_1 u_2^\varepsilon + \partial_2 u_1^\varepsilon) & b(-\partial_1 u_1^\varepsilon + \partial_2 u_2^\varepsilon) \end{bmatrix} \\
&= \begin{bmatrix} \partial_1[b(\partial_1 u_1^\varepsilon - \partial_2 u_2^\varepsilon)] + \partial_2[b(\partial_1 u_2^\varepsilon + \partial_2 u_1^\varepsilon)] \\ \partial_1[b(\partial_1 u_2^\varepsilon + \partial_2 u_1^\varepsilon)] + \partial_2[b(-\partial_1 u_1^\varepsilon + \partial_2 u_2^\varepsilon)] \end{bmatrix} \qquad (3.2.5) \\
&= \begin{bmatrix} \nabla b \cdot \nabla u_1^\varepsilon + \nabla b \cdot \nabla^\perp u_2^\varepsilon + b\Delta u_1^\varepsilon \\ \nabla b \cdot \nabla u_2^\varepsilon - \nabla b \cdot \nabla^\perp u_1^\varepsilon + b\Delta u_2^\varepsilon \end{bmatrix} \\
&= \nabla b \cdot \left(\nabla \boldsymbol{u}^\varepsilon - \nabla^\perp (\boldsymbol{u}^\varepsilon)^\perp\right) + b\Delta \boldsymbol{u}^\varepsilon
\end{aligned}$$

where $\nabla^\perp = J\nabla = \begin{bmatrix} -\partial_2 \\ \partial_1 \end{bmatrix}$ is the orthogonal gradient then

$$\begin{aligned} \operatorname{div}\big(b\Sigma(\boldsymbol{u}^\varepsilon)\big) \cdot \boldsymbol{v}^\varepsilon &= \nabla b \cdot \big(\nabla \boldsymbol{u}^\varepsilon - \nabla^\perp (\boldsymbol{u}^\varepsilon)^\perp\big) \cdot \boldsymbol{v}^\varepsilon + b\Delta \boldsymbol{u}^\varepsilon \cdot \boldsymbol{v}^\varepsilon \\ &= \nabla b \cdot (v_1^\varepsilon \nabla u_1^\varepsilon + v_2^\varepsilon \nabla u_2^\varepsilon) + \nabla b \cdot (v_1^\varepsilon \nabla^\perp u_2^\varepsilon - v_2^\varepsilon \nabla^\perp u_1^\varepsilon) \\ &\qquad + b(v_1^\varepsilon \Delta u_1^\varepsilon + v_2^\varepsilon \Delta u_2^\varepsilon). \end{aligned} \tag{3.2.6}$$

We will integrate (3.2.6) separately. First

$$\int_\Omega \nabla b \cdot \nabla \boldsymbol{u}^\varepsilon \cdot \boldsymbol{v}^\varepsilon d\boldsymbol{x} = \int_\Omega \nabla b \cdot \Big(\sum_{j=1}^2 v_j^\varepsilon \nabla u_j^\varepsilon\Big) d\boldsymbol{x} = \sum_{j=1}^2 \int_\Omega \nabla u_j^\varepsilon \cdot v_j^\varepsilon \nabla b d\boldsymbol{x} \,. \tag{3.2.7}$$

Next, since $\operatorname{div}(b\boldsymbol{v}^\varepsilon) = 0$, we have

$$\begin{aligned} &- \big(\nabla b \cdot \nabla^\perp (\boldsymbol{u}^\varepsilon)^\perp\big) \cdot \boldsymbol{v}^\varepsilon = -\nabla b \cdot \big(v_1^\varepsilon \nabla^\perp(-u_2^\varepsilon) + v_2^\varepsilon \nabla^\perp u_1^\varepsilon\big) \\ &= -v_1^\varepsilon \partial_1 b \partial_2 u_2^\varepsilon + v_1^\varepsilon \partial_2 b \partial_1 u_2^\varepsilon + v_2^\varepsilon \partial_1 b \partial_2 u_1^\varepsilon - v_2^\varepsilon \partial_2 b \partial_1 u_1^\varepsilon \\ &= -(v_1^\varepsilon \partial_1 b \partial_2 u_2^\varepsilon + v_2^\varepsilon \partial_2 b \partial_2 u_2^\varepsilon) + v_2^\varepsilon \partial_2 b \partial_2 u_2^\varepsilon + v_1^\varepsilon \partial_2 b \partial_1 u_2^\varepsilon \\ &\quad + v_2^\varepsilon \partial_1 b \partial_2 u_1^\varepsilon - (v_2^\varepsilon \partial_2 b \partial_1 u_1^\varepsilon + v_1^\varepsilon \partial_1 b \partial_1 u_1^\varepsilon) + v_1^\varepsilon \partial_1 b \partial_1 u_1^\varepsilon \\ &= -(\boldsymbol{v}^\varepsilon \cdot \nabla b)(\partial_2 u_2^\varepsilon + \partial_1 u_1^\varepsilon) + \partial_1 b (v_1^\varepsilon \partial_1 u_1^\varepsilon + v_2^\varepsilon \partial_2 u_1^\varepsilon) \\ &\quad + \partial_2 b (v_1^\varepsilon \partial_1 u_2^\varepsilon + v_2^\varepsilon \partial_2 u_2^\varepsilon) \\ &= b(\operatorname{div} \boldsymbol{v}^\varepsilon)(\operatorname{div} \boldsymbol{u}^\varepsilon) + \partial_1 b (v_1^\varepsilon \partial_1 u_1^\varepsilon + v_2^\varepsilon \partial_2 u_1^\varepsilon) + \partial_2 b (v_1^\varepsilon \partial_1 u_2^\varepsilon + v_2^\varepsilon \partial_2 u_2^\varepsilon) \\ &= b(\operatorname{div} \boldsymbol{v}^\varepsilon)(\operatorname{div} \boldsymbol{u}^\varepsilon) + \operatorname{div}(b\boldsymbol{v}^\varepsilon \cdot \nabla \boldsymbol{u}^\varepsilon) - b(\nabla \boldsymbol{u}^\varepsilon)^{\mathrm{T}} : (\nabla \boldsymbol{v}^\varepsilon)^{\mathrm{T}} \,. \end{aligned} \tag{3.2.8}$$

Here $A : B$ denotes the matrix product

$$A : B \stackrel{\text{def}}{\equiv} \operatorname{tr}(AB^{\mathrm{T}}) = \sum_{i,j=1}^2 a_{ij} b_{ij} \,. \tag{3.2.9}$$

Integrating (3.2.8) over Ω and using the divergence theorem we have

$$
\begin{aligned}
-\int_\Omega \nabla b\cdot\nabla^\perp(\boldsymbol{u}^\varepsilon)^\perp\cdot\boldsymbol{v}^\varepsilon d\boldsymbol{x} = \int_\Omega (\operatorname{div}\boldsymbol{v}^\varepsilon)(\operatorname{div}\boldsymbol{u}^\varepsilon) b d\boldsymbol{x} \\
-\int_\Omega (\nabla\boldsymbol{u}^\varepsilon)^{\mathrm{T}} : (\nabla\boldsymbol{v}^\varepsilon)^{\mathrm{T}} b d\boldsymbol{x} + \int_{\partial\Omega} (b\boldsymbol{v}^\varepsilon\cdot\nabla\boldsymbol{u}^\varepsilon)\cdot\nabla ds\,.
\end{aligned}
\tag{3.2.10}
$$

Since $\Omega\subset\mathbb{R}^2$ and $\boldsymbol{u}^\varepsilon\cdot\nabla = \boldsymbol{v}^\varepsilon\cdot\nabla = 0$ on $\partial\Omega$, we have (see [32] Lemma 4.1)

$$
(\boldsymbol{v}^\varepsilon\cdot\nabla\boldsymbol{u}^\varepsilon)\cdot\nabla = -\kappa\boldsymbol{u}^\varepsilon\cdot\boldsymbol{v}^\varepsilon = -\kappa(\boldsymbol{u}^\varepsilon\cdot\boldsymbol{\tau})(\boldsymbol{v}^\varepsilon\cdot\boldsymbol{\tau}), \qquad \boldsymbol{x}\in\partial\Omega \tag{3.2.11}
$$

where $\kappa(\boldsymbol{x})$ is the curvature of $\partial\Omega$ at $\boldsymbol{x}$, and thus (3.2.10) becomes

$$
\begin{aligned}
-\int_\Omega \nabla b\cdot\nabla^\perp(\boldsymbol{u}^\varepsilon)^\perp\cdot\boldsymbol{v}^\varepsilon d\boldsymbol{x} = \int_\Omega b(\operatorname{div}\boldsymbol{v}^\varepsilon)(\operatorname{div}\boldsymbol{u}^\varepsilon) b d\boldsymbol{x} \\
-\int_\Omega (\nabla\boldsymbol{u}^\varepsilon)^{\mathrm{T}} : (\nabla\boldsymbol{v}^\varepsilon)^{\mathrm{T}} b d\boldsymbol{x} - \kappa\int_{\partial\Omega} (\boldsymbol{u}^\varepsilon\cdot\boldsymbol{\tau})(\boldsymbol{v}^\varepsilon\cdot\boldsymbol{\tau}) b ds.
\end{aligned}
\tag{3.2.12}
$$

Similarly, by Green's identity we have

$$
\begin{aligned}
&\int_\Omega b\Delta\boldsymbol{u}^\varepsilon\cdot\boldsymbol{v}^\varepsilon d\boldsymbol{x} = \sum_{j=1}^2\int_\Omega b v_j^\varepsilon\Delta u_j^\varepsilon d\boldsymbol{x} = \sum_{j=1}^2\int_\Omega b v_j^\varepsilon(\mathrm{div}\nabla u_j^\varepsilon) d\boldsymbol{x} \\
&= \sum_{j=1}^2\int_\Omega b v_j^\varepsilon(\mathrm{div}\nabla u_j^\varepsilon) + \nabla(b v_j^\varepsilon)\cdot\nabla u_j^\varepsilon d\boldsymbol{x} - \sum_{j=1}^2\int_\Omega \nabla(b v_j^\varepsilon)\cdot\nabla u_j^\varepsilon d\boldsymbol{x} \\
&= \sum_{j=1}^2\int_\Omega \mathrm{div}(b v_j^\varepsilon\nabla u_j^\varepsilon) d\boldsymbol{x} - \sum_{j=1}^2\int_\Omega \nabla u_j^\varepsilon\cdot(b\nabla v_j^\varepsilon + v_j^\varepsilon\nabla b) d\boldsymbol{x} \\
&= \int_\Omega \mathrm{div}(b\boldsymbol{v}^\varepsilon\cdot\nabla\boldsymbol{u}^\varepsilon) d\boldsymbol{x} - \sum_{j=1}^2\int_\Omega (b\nabla u_j^\varepsilon\cdot\nabla v_j^\varepsilon + \nabla u_j^\varepsilon\cdot v_j^\varepsilon\nabla b) d\boldsymbol{x} \\
&= \int_{\partial\Omega} (b\boldsymbol{v}^\varepsilon\cdot\nabla\boldsymbol{u}^\varepsilon)\cdot\nabla ds - \sum_{i,j=1}^2\int_\Omega \partial_i u_j^\varepsilon\partial_i v_j^\varepsilon b d\boldsymbol{x} - \sum_{j=1}^2\int_\Omega \nabla u_j^\varepsilon\cdot v_j^\varepsilon\nabla b d\boldsymbol{x} \\
&= \int_{\partial\Omega} (\boldsymbol{v}^\varepsilon\cdot\nabla\boldsymbol{u}^\varepsilon)\cdot\nabla b ds - \int_\Omega \nabla\boldsymbol{u}^\varepsilon : \nabla\boldsymbol{v}^\varepsilon b d\boldsymbol{x} - \sum_{j=1}^2\int_\Omega \nabla u_j^\varepsilon\cdot v_j^\varepsilon\nabla b d\boldsymbol{x}
\end{aligned}
$$

which becomes

$$\int_\Omega b\Delta \boldsymbol{u}^\varepsilon \cdot \boldsymbol{v}^\varepsilon d\boldsymbol{x} = -\kappa \int_{\partial\Omega} (\boldsymbol{u}^\varepsilon \cdot \boldsymbol{\tau})(\boldsymbol{v}^\varepsilon \cdot \boldsymbol{\tau}) bds - \int_\Omega \nabla \boldsymbol{u}^\varepsilon : \nabla \boldsymbol{v}^\varepsilon bd\boldsymbol{x} - \sum_{j=1}^{2} \int_\Omega \nabla u_j^\varepsilon \cdot v_j^\varepsilon \nabla bd\boldsymbol{x} \tag{3.2.13}$$

after using (3.2.11) again. Now adding (3.2.7), (3.2.12) and (3.2.13) together and using the identity

$$\begin{aligned}\Sigma(\boldsymbol{u}^\varepsilon) : \Sigma(\boldsymbol{v}^\varepsilon) &= \mathrm{tr}\Big[\big(2D(\boldsymbol{u}^\varepsilon) - \operatorname{div} \boldsymbol{u}^\varepsilon I\big)\big(2D(\boldsymbol{v}^\varepsilon) - \operatorname{div} \boldsymbol{v}^\varepsilon I\big)^{\mathrm{T}}\Big] \\ &= 4D(\boldsymbol{u}^\varepsilon) : D(\boldsymbol{v}^\varepsilon) - 2\operatorname{div} \boldsymbol{u}^\varepsilon \operatorname{div} \boldsymbol{v}^\varepsilon \\ &= 2\Big(\nabla \boldsymbol{u}^\varepsilon : \nabla \boldsymbol{v}^\varepsilon + (\nabla \boldsymbol{u}^\varepsilon)^{\mathrm{T}} : (\nabla \boldsymbol{v}^\varepsilon)^{\mathrm{T}} - \operatorname{div} \boldsymbol{u}^\varepsilon \operatorname{div} \boldsymbol{v}^\varepsilon\Big)\end{aligned} \tag{3.2.14}$$

we have

$$\begin{aligned}&-\int_\Omega \operatorname{div}(b\Sigma(\boldsymbol{u}^\varepsilon)) \cdot \boldsymbol{v}^\varepsilon d\boldsymbol{x} \\ &= -\int_\Omega (\operatorname{div} \boldsymbol{u}^\varepsilon)(\operatorname{div} \boldsymbol{v}^\varepsilon) bd\boldsymbol{x} + 2\int_{\partial\Omega} \kappa(\boldsymbol{u}^\varepsilon \cdot \boldsymbol{\tau})(\boldsymbol{v}^\varepsilon \cdot \boldsymbol{\tau}) bds \\ &\quad + \int_\Omega \nabla \boldsymbol{u}^\varepsilon : \nabla \boldsymbol{v}^\varepsilon bd\boldsymbol{x} + \int_\Omega (\nabla \boldsymbol{u}^\varepsilon)^{\mathrm{T}} : (\nabla \boldsymbol{v}^\varepsilon)^{\mathrm{T}} bd\boldsymbol{x} \\ &= \frac{1}{2}\int_\Omega \Sigma(\boldsymbol{u}^\varepsilon) : \Sigma(\boldsymbol{v}^\varepsilon) bd\boldsymbol{x} + 2\int_{\partial\Omega} \kappa(\boldsymbol{u}^\varepsilon \cdot \boldsymbol{\tau})(\boldsymbol{v}^\varepsilon \cdot \boldsymbol{\tau}) bds\,.\end{aligned} \tag{3.2.15}$$

Therefore we can define the bilinear form $a(\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon) : \mathbb{V}_b \times \mathbb{V}_b \mapsto \mathbb{R}$ as

$$a(\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon) = \frac{1}{2}\int_\Omega \Sigma(\boldsymbol{u}^\varepsilon) : \Sigma(\boldsymbol{v}^\varepsilon) bd\boldsymbol{x} + 2\int_{\partial\Omega} \kappa(\boldsymbol{u}^\varepsilon \cdot \boldsymbol{\tau})(\boldsymbol{v}^\varepsilon \cdot \boldsymbol{\tau}) bds\,. \tag{3.2.16}$$

Since

$$\begin{aligned}\frac{1}{2}\Sigma(\boldsymbol{u}^\varepsilon) : \Sigma(\boldsymbol{u}^\varepsilon) &= 2D(\boldsymbol{u}^\varepsilon) : D(\boldsymbol{u}^\varepsilon) - (\operatorname{div} \boldsymbol{u}^\varepsilon)^2 \\ &= (\partial_1 u_2^\varepsilon + \partial_2 u_1^\varepsilon)^2 + (\partial_1 u_1^\varepsilon - \partial_2 u_2^\varepsilon)^2 \geqslant 0,\end{aligned}$$

it is easy to see that $a(\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon)$ is coercive ([31], [36], [55]);

$$\begin{aligned}
a(\boldsymbol{u}^\varepsilon, \boldsymbol{u}^\varepsilon) &= \frac{1}{2}\int_\Omega \Sigma(\boldsymbol{u}^\varepsilon) : \Sigma(\boldsymbol{u}^\varepsilon) b d\boldsymbol{x} + 2\int_{\partial\Omega} \kappa(\boldsymbol{u}^\varepsilon \cdot \boldsymbol{\tau})(\boldsymbol{u}^\varepsilon \cdot \boldsymbol{\tau}) b ds \\
&\geqslant b_1\left(\frac{1}{2}\int_\Omega \Sigma(\boldsymbol{u}^\varepsilon) : \Sigma(\boldsymbol{u}^\varepsilon) d\boldsymbol{x} + 2\int_{\partial\Omega} \kappa|\boldsymbol{u}^\varepsilon \cdot \boldsymbol{\tau}|^2 ds\right) \\
&= b_1\int_\Omega (\partial_1 u_2^\varepsilon + \partial_2 u_1^\varepsilon)^2 + (\partial_1 u_1^\varepsilon - \partial_2 u_2^\varepsilon)^2 d\boldsymbol{x} \\
&\quad + 2b_1\int_{\partial\Omega} \kappa|\boldsymbol{u}^\varepsilon \cdot \boldsymbol{\tau}|^2 ds \\
&= b_1\|\nabla \boldsymbol{u}^\varepsilon\|^2_{L^2_b(\Omega)} - b_1\int_{\partial\Omega} \kappa|\boldsymbol{u}^\varepsilon|^2 ds + 2b_1\int_{\partial\Omega} \kappa|\boldsymbol{u}^\varepsilon|^2 ds \\
&= b_1\|\nabla \boldsymbol{u}^\varepsilon\|^2_{L^2_b(\Omega)} + c_1\|\boldsymbol{u}^\varepsilon\|^2_{L^2_b(\Omega)} \\
&\geqslant C\|\boldsymbol{u}^\varepsilon\|^2_{H^1_b(\Omega)}\,,
\end{aligned} \tag{3.2.17}$$

where $C = \min\{b_1, c_1\}$.

From the representation formula (3.1.13)

$$\begin{aligned}
b\,\mathrm{curl}(\boldsymbol{v}^\varepsilon) J\boldsymbol{u}^\varepsilon =& 2\mathrm{div}(b\boldsymbol{u}^\varepsilon \otimes \boldsymbol{v}^\varepsilon) - \nabla(b\boldsymbol{u}^\varepsilon \cdot \boldsymbol{v}^\varepsilon) \\
& + (\boldsymbol{u}^\varepsilon \cdot \boldsymbol{v}^\varepsilon)\nabla b + b(\nabla \boldsymbol{u}^\varepsilon)^{\mathrm{T}}\boldsymbol{v}^\varepsilon - b(\nabla \boldsymbol{v}^\varepsilon)\boldsymbol{u}^\varepsilon
\end{aligned} \tag{3.2.18}$$

we define the integrals $I_i, i = 1, \cdots, 4$ by

$$\begin{aligned}
I_1 &= -\frac{1}{2}\int_\Omega (\boldsymbol{u}^\varepsilon \cdot \boldsymbol{v}^\varepsilon)\nabla \log b \cdot \boldsymbol{\varphi} b d\boldsymbol{x}, \\
I_2 &= -2\int_\Omega \boldsymbol{u}^\varepsilon \otimes \boldsymbol{v}^\varepsilon : \nabla\boldsymbol{\varphi} b d\boldsymbol{x}, \\
I_3 &= \int_\Omega (\nabla \boldsymbol{u}^\varepsilon)\boldsymbol{v}^\varepsilon \cdot \boldsymbol{\varphi} b d\boldsymbol{x}, \\
I_4 &= \int_\Omega (\nabla \boldsymbol{v}^\varepsilon)^{\mathrm{T}}\boldsymbol{u}^\varepsilon \cdot \boldsymbol{\varphi} b d\boldsymbol{x}\,.
\end{aligned} \tag{3.2.19}$$

Therefore the variational formulation of (3.2.1)–(3.2.3) is given by

$$a(\boldsymbol{u}^\varepsilon, \boldsymbol{\varphi}) + (\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon, \boldsymbol{\varphi}) = L(\boldsymbol{\varphi}) \tag{3.2.20}$$

where $(\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon, \boldsymbol{\varphi}) = I_1 + I_2 + I_3 + I_4$ is the trilinear form and the linear form $L : \mathbb{V}_b \mapsto \mathbb{R}$ defined by

$$L(\boldsymbol{\varphi}) \overset{\text{def}}{\equiv} \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle_b = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} b d\boldsymbol{x}, \tag{3.2.21}$$

for all $\boldsymbol{\varphi} \in H_b^1(\Omega)$ satisfying $\operatorname{div}(b\boldsymbol{\varphi}) = 0$ in Ω and $\boldsymbol{\varphi} \cdot \nabla = 0$ on $\partial\Omega$. For $\boldsymbol{u}^\varepsilon, \boldsymbol{\varphi} \in \mathbb{V}_b$, by Cauchy-Schwarz inequality and trace theorem we have

$$\begin{aligned} |a(\boldsymbol{u}^\varepsilon, \boldsymbol{\varphi})| \leqslant & 2\|D(\boldsymbol{u}^\varepsilon)\|_{L_b^2(\Omega)} \|D(\boldsymbol{\varphi})\|_{L_b^2(\Omega)} + 2\|\operatorname{div} \boldsymbol{u}^\varepsilon\|_{L_b^2(\Omega)} \|\operatorname{div} \boldsymbol{\varphi}\|_{L_b^2(\Omega)} \\ & + 2\|\kappa\|_{L^\infty(\partial\Omega)} \|\boldsymbol{u}^\varepsilon\|_{L_b^2(\partial\Omega)} \|\boldsymbol{\varphi}\|_{L_b^2(\partial\Omega)} \\ \leqslant & C' \|\boldsymbol{u}^\varepsilon\|_{H_b^1(\Omega)} \|\boldsymbol{\varphi}\|_{H_b^1(\Omega)} \end{aligned} \tag{3.2.22}$$

Now we need to estimate the trilinear term $(\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon, \boldsymbol{\varphi})$. Using Cauchy-Schwarz inequality and Poincaré inequality, we have

$$\begin{aligned} |I_1| = & \frac{1}{2} \left| \int_\Omega (\boldsymbol{u}^\varepsilon \cdot \boldsymbol{v}^\varepsilon) \nabla \log b \cdot \boldsymbol{\varphi} b d\boldsymbol{x} \right| \\ & \leqslant \frac{1}{2} \|\boldsymbol{v}^\varepsilon\|_{L^\infty(\Omega)} \|\nabla \log b\|_{L^\infty(\Omega)} \|\boldsymbol{u}^\varepsilon\|_{L_b^2(\Omega)} \|\boldsymbol{\varphi}\|_{L_b^1(\Omega)} \\ & \leqslant C_1 \|\boldsymbol{u}^\varepsilon\|_{H_b^1(\Omega)} \|\boldsymbol{\varphi}\|_{H_b^1(\Omega)} , \end{aligned}$$

$$\begin{aligned} |I_2| = \left| 2 \int_\Omega \boldsymbol{u}^\varepsilon \otimes \boldsymbol{v}^\varepsilon : \nabla \boldsymbol{\varphi} b d\boldsymbol{x} \right| & \leqslant 2\|\boldsymbol{u}^\varepsilon \otimes \boldsymbol{v}^\varepsilon\|_{L_b^2(\Omega)} \|\nabla \boldsymbol{\varphi}\|_{L_b^2(\Omega)} \\ & \leqslant \|\boldsymbol{v}^\varepsilon\|_{L^\infty(\Omega)} \|\boldsymbol{u}^\varepsilon\|_{L_b^2(\Omega)} \|\nabla \boldsymbol{\varphi}\|_{L_b^2(\Omega)} \\ & \leqslant C_2 \|\boldsymbol{u}^\varepsilon\|_{H_b^1(\Omega)} \|\boldsymbol{\varphi}\|_{H_b^1(\Omega)} , \end{aligned}$$

$$|I_3| = \left|\int_\Omega (\nabla \boldsymbol{u}^\varepsilon)\boldsymbol{v}^\varepsilon \cdot \boldsymbol{\varphi} b d\boldsymbol{x}\right| \leqslant \|\boldsymbol{v}^\varepsilon\|_{L^\infty(\Omega)} \|\nabla \boldsymbol{u}^\varepsilon\|_{L_b^2(\Omega)} \|\boldsymbol{\varphi}\|_{L_b^2(\Omega)}$$
$$\leqslant C_3 \|\boldsymbol{u}^\varepsilon\|_{H_b^1(\Omega)} \|\boldsymbol{\varphi}\|_{H_b^1(\Omega)}$$

and

$$|I_4| = \left|\int_\Omega (\nabla \boldsymbol{v}^\varepsilon)^{\mathrm{T}}\boldsymbol{u}^\varepsilon \cdot \boldsymbol{\varphi} b d\boldsymbol{x}\right| \leqslant \|\nabla \boldsymbol{v}^\varepsilon\|_{L^\infty(\Omega)} \|\boldsymbol{u}^\varepsilon\|_{L_b^2(\Omega)} \|\boldsymbol{\varphi}\|_{L_b^2(\Omega)}$$
$$\leqslant C_4 \|\boldsymbol{u}^\varepsilon\|_{H_b^1(\Omega)} \|\boldsymbol{\varphi}\|_{H_b^1(\Omega)}.$$

Hence

$$|(\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon, \boldsymbol{\varphi})| \leq C_0 \|\boldsymbol{u}^\varepsilon\|_{H_b^1(\Omega)} \|\boldsymbol{\varphi}\|_{H_b^1(\Omega)}. \tag{3.2.23}$$

Therefore $B(\boldsymbol{u}^\varepsilon, \boldsymbol{\varphi}) = a(\boldsymbol{u}^\varepsilon, \boldsymbol{\varphi}) + (\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon, \boldsymbol{\varphi})$ is bounded. Let $\boldsymbol{\varphi} = \boldsymbol{u}^\varepsilon$ in (3.2.23) we get

$$|(\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon, \boldsymbol{u}^\varepsilon)| \leq C_0 \|\boldsymbol{u}^\varepsilon\|^2_{H_b^1(\Omega)}.$$

Choosing $\widetilde{C} = C - C_0 > 0$, combine with the coercivity of $a(\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon)$, we have

$$B(\boldsymbol{u}^\varepsilon, \boldsymbol{u}^\varepsilon) \geqslant \widetilde{C} \|\boldsymbol{u}^\varepsilon\|_{H_b^1(\Omega)}. \tag{3.2.24}$$

Therefore $B(\boldsymbol{u}^\varepsilon, \boldsymbol{\varphi})$ is coercive.

Then, according to the Lax-Milgram theorem there exists a unique solution $\boldsymbol{u}^\varepsilon$ of equation (3.2.1) in $H_b^1(\Omega)$. To determine the existence of the pressure p^ε, we take the divergence of (3.2.1), then the pressure p^ε satisfies the nonhomogeneous strictly elliptic equation of the divergence form

$$\operatorname{div}\big(b\nabla p^\varepsilon\big) = \operatorname{div}\Big(\operatorname{div}(b\Sigma(\boldsymbol{u}^\varepsilon)) + \frac{1}{2}(\boldsymbol{u}^\varepsilon \cdot \boldsymbol{v}^\varepsilon)\nabla b - b\operatorname{curl}(\boldsymbol{v}^\varepsilon) J\boldsymbol{u}^\varepsilon + b\boldsymbol{f}\Big) \equiv F. \tag{3.2.25}$$

Since $u^\varepsilon \in H_b^1(\Omega)$, it follows that $\operatorname{div}(b\Sigma(\boldsymbol{u}^\varepsilon)) \in H_b^{-1}(\Omega)$ and $F \in H_b^{-2}(\Omega)$. By regularity theorem of elliptic equations (see Theorem 9.15 in[27]), the space of p^ε has two more regularities than the space of F, then we can assert that there exists a unique solution $p^\varepsilon \in L^2_{loc}(\Omega)/\mathbb{R}$.

Thus we have the following result:

Theorem 3.2.1. *Let $\Omega \subset \mathbb{R}^2$ be a smooth simply connected bounded domain with nonnegative curvature $\kappa(\boldsymbol{x}) \geqslant 0$ for all $\boldsymbol{x} \in \partial\Omega$. For $\varepsilon > 0$ fixed there exists at least one weak solution $(\boldsymbol{u}^\varepsilon, p^\varepsilon)$ to the boundary value problem of the anelastic Stokes system* (3.2.1)–(3.2.3).

3.3 Homogenization of the Anelastic Stokes Equation

In this section we will find the limit equation of weak formulation (3.2.20) for equation (3.2.1)-(3.2.3) when $\varepsilon \to 0$.

First, using the coercivity of the bilinear form and Cauchy-Schwarz inequality, we get the energy estimate

$$\begin{aligned} C\left\|\boldsymbol{u}^\varepsilon\right\|^2_{H^1_b(\Omega)} &\le a(\boldsymbol{u}^\varepsilon, \boldsymbol{u}^\varepsilon) + (\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon, \boldsymbol{u}^\varepsilon) = \left|\langle \boldsymbol{f}, \boldsymbol{u}^\varepsilon\rangle\right|_b \\ &\le \left\|\boldsymbol{f}\right\|_{L^2_b(\Omega)} \left\|\boldsymbol{u}^\varepsilon\right\|_{L^2_b(\Omega)} \le \left\|\boldsymbol{f}\right\|_{L^2_b(\Omega)} \left\|\boldsymbol{u}^\varepsilon\right\|_{H^1_b(\Omega)}, \end{aligned}$$

that is,

$$\left\|\boldsymbol{u}^\varepsilon\right\|^2_{H^1_b(\Omega)} \le C_1 \left\|\boldsymbol{f}\right\|_{L^2_b(\Omega)}. \tag{3.3.1}$$

So we can deduce that $\boldsymbol{u}^\varepsilon$ is bounded in $H^1_b(\Omega)$ and weakly converges, up to a subsequence $\{\boldsymbol{u}^\varepsilon\}_\varepsilon$, to a function $\boldsymbol{u}$ in $H^1_b(\Omega)$, due to the reflexivity of the Sobolev space $H^1_b(\Omega)$

$$\boldsymbol{u}^\varepsilon \xrightarrow{\text{w}} \boldsymbol{u} \quad \text{weakly} \quad \text{in } H^1_b(\Omega). \tag{3.3.2}$$

Furthermore, it follows that

$$\begin{cases} \boldsymbol{u}^\varepsilon \to \boldsymbol{u} \quad \text{strongly} \quad \text{in } L_b^2(\Omega) \\ \nabla \boldsymbol{u}^\varepsilon \xrightarrow{\text{w}} \nabla \boldsymbol{u} \quad \text{weakly} \quad \text{in } L_b^2(\Omega) \\ \Delta \boldsymbol{u}^\varepsilon \xrightarrow{\text{w}} \Delta \boldsymbol{u} \quad \text{weakly} \quad \text{in } H_b^{-1}(\Omega). \end{cases} \tag{3.3.3}$$

For $p^\varepsilon \in L^2_{loc}(\Omega)/\mathbb{R}$, since the space $L^2_{loc}(\Omega)/\mathbb{R}$ is reflexive, we find

$$p^\varepsilon \xrightarrow{\text{w}} p \quad \text{weakly} \quad \text{in } L^2_{loc}(\Omega)/\mathbb{R}. \tag{3.3.4}$$

For $\boldsymbol{v}^\varepsilon \in L_b^\infty(\Omega)$, since $L_b^\infty(\Omega) \subset L_b^r(\Omega)$ for $1 \leqslant r \leqslant \infty$

$$\begin{cases} \boldsymbol{v}^\varepsilon \xrightarrow{\text{w}} \boldsymbol{v} \quad \text{weakly} \quad \text{in } L_b^r(\Omega) \\ \nabla \boldsymbol{v}^\varepsilon \xrightarrow{\text{w}} \nabla \boldsymbol{v} \quad \text{weakly} \quad \text{in } W_b^{-1,r}(\Omega). \end{cases} \tag{3.3.5}$$

By convergences (3.3.3)–(3.3.5), we can derive the following convergence

$$\int_\Omega \nabla b \cdot (\nabla \boldsymbol{u}^\varepsilon \boldsymbol{\varphi}) d\boldsymbol{x} \to \int_\Omega \nabla b \cdot (\nabla \boldsymbol{u} \boldsymbol{\varphi}) d\boldsymbol{x}, \tag{3.3.6}$$

$$\int_\Omega \nabla b \cdot [\nabla^\perp (\boldsymbol{u}^\varepsilon)^\perp \boldsymbol{\varphi}] d\boldsymbol{x} \to \int_\Omega \nabla b \cdot [\nabla^\perp (\boldsymbol{u})^\perp \boldsymbol{\varphi}] d\boldsymbol{x} \tag{3.3.7}$$

and

$$\int_\Omega b \Delta \boldsymbol{u}^\varepsilon \cdot \boldsymbol{\varphi} d\boldsymbol{x} \to \int_\Omega b \Delta \boldsymbol{u} \cdot \boldsymbol{\varphi} d\boldsymbol{x}. \tag{3.3.8}$$

Combining (3.3.6)–(3.3.8) together we have

$$\int_\Omega \operatorname{div}(b\Sigma(\boldsymbol{u}^\varepsilon)) \cdot \boldsymbol{\varphi} d\boldsymbol{x} \to \int_\Omega \operatorname{div}(b\Sigma(\boldsymbol{u})) \cdot \boldsymbol{\varphi} d\boldsymbol{x}. \tag{3.3.9}$$

In the same way, we obtain

$$\int_\Omega \frac{1}{2} (\boldsymbol{u}^\varepsilon \cdot \boldsymbol{v}^\varepsilon) \nabla b \cdot \boldsymbol{\varphi} d\boldsymbol{x} \to \int_\Omega \frac{1}{2} (\boldsymbol{u} \cdot \boldsymbol{v}) \nabla b \cdot \boldsymbol{\varphi} d\boldsymbol{x}. \tag{3.3.10}$$

We now turn to the third term $b\,\mathrm{curl}(\boldsymbol{v}^\varepsilon)J\boldsymbol{u}^\varepsilon$ in (3.2.1). By the representation formula (3.2.18)

$$\begin{aligned} b\,\mathrm{curl}(\boldsymbol{v}^\varepsilon)J\boldsymbol{u}^\varepsilon =& 2\mathrm{div}(b\boldsymbol{u}^\varepsilon \otimes \boldsymbol{v}^\varepsilon) - \nabla(b\boldsymbol{u}^\varepsilon \cdot \boldsymbol{v}^\varepsilon) \\ & + (\boldsymbol{u}^\varepsilon \cdot \boldsymbol{v}^\varepsilon)\nabla b + b(\nabla \boldsymbol{u}^\varepsilon)^{\mathrm{T}}\boldsymbol{v}^\varepsilon - b(\nabla \boldsymbol{v}^\varepsilon)\boldsymbol{u}^\varepsilon. \end{aligned} \tag{3.3.11}$$

It is easy to deduce the following convergences

$$\int_\Omega 2\mathrm{div}(b\boldsymbol{u}^\varepsilon \otimes \boldsymbol{v}^\varepsilon)d\boldsymbol{x} \to \int_\Omega 2\mathrm{div}(b\boldsymbol{u} \otimes \boldsymbol{v}) \cdot \boldsymbol{\varphi} d\boldsymbol{x}, \tag{3.3.12}$$

$$\int_\Omega \nabla(b\boldsymbol{u}^\varepsilon \cdot \boldsymbol{v}^\varepsilon) \cdot \boldsymbol{\varphi} d\boldsymbol{x} \to \int_\Omega \nabla(b\boldsymbol{u} \cdot \boldsymbol{v}) \cdot \boldsymbol{\varphi} d\boldsymbol{x}, \tag{3.3.13}$$

$$\int_\Omega (\boldsymbol{u}^\varepsilon \cdot \boldsymbol{v}^\varepsilon)\nabla b \cdot \boldsymbol{\varphi} d\boldsymbol{x} \to \int_\Omega (\boldsymbol{u} \cdot \boldsymbol{v})\nabla b \cdot \boldsymbol{\varphi} d\boldsymbol{x}, \tag{3.3.14}$$

and

$$\int_\Omega (\nabla \boldsymbol{v}^\varepsilon)\boldsymbol{u}^\varepsilon \cdot \boldsymbol{\varphi} b\, d\boldsymbol{x} \to \int_\Omega (\nabla \boldsymbol{v})\boldsymbol{u} \cdot \boldsymbol{\varphi} b\, d\boldsymbol{x}. \tag{3.3.15}$$

Since the weak convergence of $\nabla \boldsymbol{u}^\varepsilon \overset{\mathrm{w}}{\rightharpoonup} \nabla \boldsymbol{u}$ in $L^2_b(\Omega)$ and $\boldsymbol{v}^\varepsilon \overset{\mathrm{w}}{\rightharpoonup} \boldsymbol{v}$ in $L^r_b(\Omega)$ do not guarantee the convergence of the product $b(\nabla \boldsymbol{u}^\varepsilon)^{\mathrm{T}}\boldsymbol{v}^\varepsilon$. So we have to proceed to determine the limit of $b(\nabla \boldsymbol{u}^\varepsilon)^{\mathrm{T}}\boldsymbol{v}^\varepsilon$.

According to Tartar's oscillating test function method, we construct a similar homogenized equation using the test function $\boldsymbol{w}^\varepsilon_\lambda$ with $\boldsymbol{\lambda} \in \mathbb{R}^2$

$$\begin{cases} -\mathrm{div}\big(b\Sigma(\boldsymbol{w}^\varepsilon_\lambda)\big) + b\ \mathrm{div}\big((\boldsymbol{v}^\varepsilon - \boldsymbol{v}) \otimes \boldsymbol{\lambda}\big) + b\nabla q^\varepsilon_\lambda = 0, \quad \boldsymbol{x} \in \Omega \\ \mathrm{div}(b\boldsymbol{w}^\varepsilon_\lambda) = 0, \quad \boldsymbol{x} \in \Omega \\ \boldsymbol{w}^\varepsilon_\lambda \cdot \nabla = 0, \quad \boldsymbol{x} \in \partial\Omega. \end{cases} \tag{3.3.16}$$

First, we write down the weak formulation of (3.3.16)

$$a(\boldsymbol{w}^\varepsilon_\lambda, \boldsymbol{\varphi}) = \langle \boldsymbol{f}^\varepsilon, \boldsymbol{\varphi} \rangle_b\,, \tag{3.3.17}$$

where the bilinear form

$$a(\boldsymbol{w}_\lambda^\varepsilon, \boldsymbol{\varphi}) = \frac{1}{2}\int_\Omega \Sigma(\boldsymbol{w}_\lambda^\varepsilon) : \Sigma(\boldsymbol{\varphi}) b\, d\boldsymbol{x} + 2\int_{\partial\Omega} \kappa(\boldsymbol{w}_\lambda^\varepsilon \cdot \boldsymbol{\tau})(\boldsymbol{\varphi} \cdot \boldsymbol{\tau}) b\, ds,$$

and

$$\boldsymbol{f}^\varepsilon = \operatorname{div}\big((\boldsymbol{v}^\varepsilon - \boldsymbol{v}) \otimes \boldsymbol{\lambda}\big).$$

Using the result (3.2.17) in Section 3.2 we can obtain the coercivity of the bilinear form

$$\begin{aligned} a(\boldsymbol{w}_\lambda^\varepsilon, \boldsymbol{w}_\lambda^\varepsilon) &= \frac{1}{2}\int_\Omega \Sigma(\boldsymbol{w}_\lambda^\varepsilon) : \Sigma(\boldsymbol{w}_\lambda^\varepsilon) b\, d\boldsymbol{x} + 2\int_{\partial\Omega} \kappa(\boldsymbol{w}_\lambda^\varepsilon \cdot \boldsymbol{\tau})(\boldsymbol{w}_\lambda^\varepsilon \cdot \boldsymbol{\tau}) b\, d\boldsymbol{x} \\ &\geq C \left\|\boldsymbol{w}_\lambda^\varepsilon\right\|_{H_b^1(\Omega)}^2 . \end{aligned} \tag{3.3.18}$$

Meanwhile, we have

$$|a(\boldsymbol{w}_\lambda^\varepsilon, \boldsymbol{\varphi})| \leq C_1 \left\|\boldsymbol{w}_\lambda^\varepsilon\right\|_{H_b^1(\Omega)} \left\|\boldsymbol{\varphi}\right\|_{H_b^1(\Omega)} . \tag{3.3.19}$$

Therefore by the Lax-Milgram Theorem there exists a unique solution $\boldsymbol{w}_\lambda^\varepsilon \in H_b^1(\Omega)$ for (3.3.16). Moreover, we have the energy estimate

$$\begin{aligned} C \left\|\boldsymbol{w}_\lambda^\varepsilon\right\|_{H_b^1(\Omega)}^2 &\leq a(\boldsymbol{w}_\lambda^\varepsilon, \boldsymbol{w}_\lambda^\varepsilon) = |\langle \boldsymbol{f}^\varepsilon, \boldsymbol{w}_\lambda^\varepsilon \rangle_b| \\ &\leq \left\|\boldsymbol{f}^\varepsilon\right\|_{L_b^2(\Omega)} \left\|\boldsymbol{w}_\lambda^\varepsilon\right\|_{L_b^2(\Omega)} \leq \left\|\boldsymbol{f}^\varepsilon\right\|_{L_b^2(\Omega)} \left\|\boldsymbol{w}_\lambda^\varepsilon\right\|_{H_b^1(\Omega)} , \end{aligned} \tag{3.3.20}$$

that is,

$$\left\|\boldsymbol{w}_\lambda^\varepsilon\right\|_{H_b^1(\Omega)} \leq C' \left\|\boldsymbol{f}^\varepsilon\right\|_{L_b^2(\Omega)} . \tag{3.3.21}$$

In addition, since $\boldsymbol{v}^\varepsilon \overset{\text{w}}{\rightharpoonup} \boldsymbol{v}$ in $L_b^r(\Omega)$, it follows that

$$\left\|\boldsymbol{f}^\varepsilon\right\|_{L_b^2(\Omega)} = \left\|\operatorname{div}\big((\boldsymbol{v}^\varepsilon - \boldsymbol{v}) \otimes \boldsymbol{\lambda}\big)\right\|_{L_b^2(\Omega)} \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{3.3.22}$$

Therefore

$$\boldsymbol{w}_\lambda^\varepsilon \overset{\text{w}}{\rightharpoonup} 0 \quad \text{weakly in } H_b^1(\Omega). \tag{3.3.23}$$

In this way

$$\int_\Omega \operatorname{div}(b\Sigma(\boldsymbol{w}_\lambda^\varepsilon)) \cdot \boldsymbol{\varphi} d\boldsymbol{x} = -\int_\Omega \Sigma(\boldsymbol{w}_\lambda^\varepsilon) : (\nabla\boldsymbol{\varphi}) b \, d\boldsymbol{x} \to 0,$$

and so

$$\operatorname{div}(b\Sigma(\boldsymbol{w}_\lambda^\varepsilon)) \xrightharpoonup{\text{w}} 0 \quad \text{weakly in } H_b^{-1}(\Omega). \tag{3.3.24}$$

Then for q_λ^ε, it follows that

$$b\nabla q_\lambda^\varepsilon = \operatorname{div}\big(b\Sigma(\boldsymbol{w}_\lambda^\varepsilon)\big) - b \operatorname{div}\big((\boldsymbol{v}^\varepsilon - \boldsymbol{v}) \otimes \boldsymbol{\lambda}\big) \xrightharpoonup{\text{w}} 0 \quad \text{weakly in } H_b^{-1}(\Omega),$$

and

$$\nabla q_\lambda^\varepsilon \xrightharpoonup{\text{w}} 0 \quad \text{weakly in } H_b^{-1}(\Omega).$$

According to the regularity theorem of elliptic equations, we have the convergence of q^ε

$$q_\lambda^\varepsilon \xrightharpoonup{\text{w}} 0 \quad \text{weakly in } L^2_{loc}(\Omega)/\mathbb{R}. \tag{3.3.25}$$

Let $\varphi \in C_c^\infty(\Omega)$ be a scarlar function with $\nabla\varphi = 0$. Following the Tartar's method we put $\varphi\boldsymbol{w}_\lambda^\varepsilon$ as test function in equation (3.2.1) and $\varphi\boldsymbol{u}^\varepsilon$ in equation (3.3.16), using the fact that $\boldsymbol{w}_\lambda^\varepsilon \xrightharpoonup{\text{w}} 0$ we have

$$\int_\Omega \Sigma(\boldsymbol{u}^\varepsilon) : \Sigma(\boldsymbol{w}_\lambda^\varepsilon)\varphi b \, d\boldsymbol{x} - 2\int_\Omega (\boldsymbol{u}^\varepsilon \otimes \boldsymbol{v}^\varepsilon) : \nabla(\boldsymbol{w}_\lambda^\varepsilon)\varphi b \, d\boldsymbol{x} = o(1) \tag{3.3.26}$$

and

$$\int_\Omega \Sigma(\boldsymbol{w}_\lambda^\varepsilon) : \Sigma(\boldsymbol{u}^\varepsilon)\varphi b \, d\boldsymbol{x} - \int_\Omega \big((\boldsymbol{v}^\varepsilon - \boldsymbol{v}) \otimes \boldsymbol{\lambda}\big) : \boldsymbol{u}^\varepsilon \varphi b \, d\boldsymbol{x} = o(1). \tag{3.3.27}$$

In addition, it is fact that

$$\boldsymbol{u}^\varepsilon \otimes \boldsymbol{v}^\varepsilon : (\nabla\boldsymbol{w}_\lambda^\varepsilon) = (\nabla\boldsymbol{w}_\lambda^\varepsilon)^{\mathrm{T}} \boldsymbol{v}^\varepsilon \cdot \boldsymbol{u}^\varepsilon$$

$$\big((\boldsymbol{v}^\varepsilon - \boldsymbol{v}) \otimes \boldsymbol{\lambda}\big) : \nabla\boldsymbol{u}^\varepsilon = (\nabla\boldsymbol{u}^\varepsilon)^{\mathrm{T}}(\boldsymbol{v}^\varepsilon - \boldsymbol{v}) \cdot \boldsymbol{\lambda}$$

In consequence, we rewrite (3.3.26) and (3.3.27) as

$$\int_\Omega \Sigma(\boldsymbol{u}^\varepsilon):\Sigma(\boldsymbol{w}^\varepsilon_\lambda)\varphi b\,d\boldsymbol{x} - 2\int_\Omega (\nabla \boldsymbol{w}^\varepsilon_\lambda)^{\mathrm{T}}\boldsymbol{v}^\varepsilon\cdot\boldsymbol{u}^\varepsilon\varphi b\,d\boldsymbol{x} = o(1) \tag{3.3.28}$$

$$\int_\Omega \Sigma(\boldsymbol{w}^\varepsilon_\lambda):\Sigma(\boldsymbol{u}^\varepsilon)\varphi b d\boldsymbol{x} - \int_\Omega (\nabla \boldsymbol{u}^\varepsilon)^{\mathrm{T}}(\boldsymbol{v}^\varepsilon-\boldsymbol{v})\cdot\boldsymbol{\lambda}\varphi b d\boldsymbol{x} = o(1). \tag{3.3.29}$$

The equation (3.3.28) shows that

$$\Sigma(\boldsymbol{u}^\varepsilon):\Sigma(\boldsymbol{w}^\varepsilon_\lambda) \overset{\mathrm{w}}{\rightharpoonup} 2(\nabla \boldsymbol{w}^\varepsilon_\lambda)^{\mathrm{T}}\boldsymbol{v}^\varepsilon\cdot\boldsymbol{u}^\varepsilon \quad \text{weakly in} \quad \mathcal{D}'(\Omega). \tag{3.3.30}$$

The task is now to determine the convergence of $(\nabla \boldsymbol{w}^\varepsilon_\lambda)^{\mathrm{T}}\boldsymbol{v}^\varepsilon$. The convergence $\nabla \boldsymbol{w}^\varepsilon_\lambda \overset{\mathrm{w}}{\rightharpoonup} 0$ in $L^2_b(\Omega)$ and $\boldsymbol{v}^\varepsilon \overset{\mathrm{w}}{\rightharpoonup} \boldsymbol{v}$ in $L^r_b(\Omega)$ do not guarantee the convergence of $(\nabla \boldsymbol{w}^\varepsilon_\lambda)^{\mathrm{T}}\boldsymbol{v}^\varepsilon$. Indeed, we have the following convergence instead.

Lemma 3.3.1. *There exists a positive definite symmetric matrix-valued function M such that (up to a subsequence)*

$$\begin{cases} (\nabla \boldsymbol{w}^\varepsilon_\lambda)^{\mathrm{T}}\boldsymbol{v}^\varepsilon \overset{\mathrm{w}}{\rightharpoonup} 0\cdot\boldsymbol{v} + M\boldsymbol{\lambda} = M\boldsymbol{\lambda} \quad \textit{weakly in } L_b^{\frac{2r}{2+r}}(\Omega) \\ \nabla \boldsymbol{w}^\varepsilon_\lambda : \nabla \boldsymbol{w}^\varepsilon_\mu \overset{\mathrm{w}}{\rightharpoonup} M\boldsymbol{\lambda}\cdot\boldsymbol{\mu} \quad \textit{weakly} * \textit{in} \quad \mathcal{M}(\Omega) \textit{ and } L_b^{\frac{2r}{2+r}}(\Omega) \end{cases} \tag{3.3.31}$$

for $\boldsymbol{\lambda},\boldsymbol{\mu}\in\mathbb{R}^2$. Moreover, since $\boldsymbol{u}^\varepsilon \to \boldsymbol{u}$ in $L^2_b(\Omega)$, we have

$$(\nabla \boldsymbol{w}^\varepsilon_\lambda)^{\mathrm{T}}\boldsymbol{v}^\varepsilon\cdot\boldsymbol{u}^\varepsilon \overset{\mathrm{w}}{\rightharpoonup} M\boldsymbol{u}\cdot\boldsymbol{\lambda} \quad \textit{weakly} * \textit{in} \quad \mathcal{M}(\Omega) \textit{ and } L_b^{\frac{r}{1+r}}(\Omega). \tag{3.3.32}$$

Then we conclude from (3.3.30) that

$$\Sigma(\boldsymbol{u}^\varepsilon):\Sigma(\boldsymbol{w}^\varepsilon_\lambda) \overset{\mathrm{w}}{\rightharpoonup} 2M\boldsymbol{u}\cdot\boldsymbol{\lambda} \quad \text{weakly} * \text{in} \quad \mathcal{M}(\Omega) \text{ and } L_b^{\frac{r}{1+r}}(\Omega). \tag{3.3.33}$$

On the other hand, the equation (3.3.29) shows that the term

$$(\nabla \boldsymbol{u}^\varepsilon)^{\mathrm{T}}(\boldsymbol{v}^\varepsilon-\boldsymbol{v})\cdot\boldsymbol{\lambda}$$

has the same weak limit with $\Sigma(\boldsymbol{w}_\lambda^\varepsilon) : \Sigma(\boldsymbol{u}^\varepsilon)$, hence

$$(\nabla \boldsymbol{u}^\varepsilon)^{\mathrm{T}}(\boldsymbol{v}^\varepsilon - \boldsymbol{v}) \cdot \lambda \overset{\mathrm{w}}{\rightharpoonup} 2M\boldsymbol{u} \cdot \boldsymbol{\lambda} \quad \text{weakly} * \text{in} \quad \mathcal{M}(\Omega) \text{ and } L_b^{\frac{r}{1+r}}(\Omega), \tag{3.3.34}$$

thus

$$(\nabla \boldsymbol{u}^\varepsilon)^{\mathrm{T}}(\boldsymbol{v}^\varepsilon - \boldsymbol{v}) \overset{\mathrm{w}}{\rightharpoonup} 2M\boldsymbol{u} \quad \text{weakly in} \quad L_b^{\frac{r}{1+r}}(\Omega). \tag{3.3.35}$$

Consequently,

$$(\nabla \boldsymbol{u}^\varepsilon)^{\mathrm{T}}\boldsymbol{v}^\varepsilon = (\nabla \boldsymbol{u}^\varepsilon)^{\mathrm{T}}(\boldsymbol{v}^\varepsilon - \boldsymbol{v}) + (\nabla \boldsymbol{u}^\varepsilon)^{\mathrm{T}}\boldsymbol{v} \overset{\mathrm{w}}{\rightharpoonup} 2M\boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathrm{T}}\boldsymbol{v}. \tag{3.3.36}$$

Moreover, since the Navier boundary conditions

$$\boldsymbol{u}^\varepsilon \cdot \boldsymbol{n} = 0, \qquad \boldsymbol{\tau} \cdot \left(\nabla \boldsymbol{u}^\varepsilon + (\nabla \boldsymbol{u}^\varepsilon)^{\mathrm{T}}\right) \cdot \nabla = -\beta \boldsymbol{\tau} \cdot \boldsymbol{u}^\varepsilon, \qquad \boldsymbol{x} \in \partial\Omega,$$

are linear, the weakly compactness of u^ε in $H_b^1(\Omega)$ guarantees the passage of limit of Navier boundary conditions, in this way

$$\boldsymbol{u}^\varepsilon \cdot \boldsymbol{n} \overset{\mathrm{w}}{\rightharpoonup} \boldsymbol{u} \cdot \boldsymbol{n} = 0 \tag{3.3.37}$$

and

$$\boldsymbol{\tau} \cdot \left(\nabla \boldsymbol{u}^\varepsilon + (\nabla \boldsymbol{u}^\varepsilon)^{\mathrm{T}}\right) \cdot \nabla + \beta \boldsymbol{\tau} \cdot \boldsymbol{u}^\varepsilon \overset{\mathrm{w}}{\rightharpoonup} \boldsymbol{\tau} \cdot \left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathrm{T}}\right) \cdot \nabla + \beta \boldsymbol{\tau} \cdot \boldsymbol{u} = 0\,. \tag{3.3.38}$$

Finally we get:

Theorem 3.3.2. *The solution* $\boldsymbol{u}^\varepsilon$ *of* (3.2.1) *converges weakly in* $H_b^1(\Omega)$ *to the solution* $\boldsymbol{u}$ *of the Brinkman equation*

$$\begin{cases} -\operatorname{div}(b\Sigma(\boldsymbol{u})) - \dfrac{1}{2}(\boldsymbol{u} \cdot \boldsymbol{v})\nabla b + b \operatorname{curl}(\boldsymbol{v})J\boldsymbol{u} + b\nabla p + 2M\boldsymbol{u} = b\boldsymbol{f} \\ \operatorname{div}(b\boldsymbol{u}) = 0, \qquad \boldsymbol{x} \in \Omega \end{cases}$$

with the Navier boundary conditions

$$\boldsymbol{u}\cdot\boldsymbol{n}=0, \qquad \boldsymbol{\tau}\cdot\left(\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^{\mathrm{T}}\right)\cdot\nabla=-\beta\boldsymbol{\tau}\cdot\boldsymbol{u}, \qquad \boldsymbol{x}\in\partial\Omega\,,$$

where M is the positive definite symmetric matrix-valued function defined by

$$\begin{cases}(\nabla\boldsymbol{w}_\lambda^\varepsilon)^{\mathrm{T}}\boldsymbol{v}^\varepsilon \overset{\mathrm{w}}{\rightharpoonup} 0\cdot\boldsymbol{v}+M\boldsymbol{\lambda}=M\boldsymbol{\lambda} \quad \textit{weakly in } L_b^{\frac{2r}{2+r}}(\Omega)\\ \nabla\boldsymbol{w}_\lambda^\varepsilon:\nabla\boldsymbol{w}_\mu^\varepsilon \overset{\mathrm{w}}{\rightharpoonup} M\boldsymbol{\lambda}\cdot\boldsymbol{\mu} \quad \textit{weakly} * \textit{ in } \quad \mathcal{M}(\Omega) \textit{ and } L_b^{\frac{2r}{2+r}}(\Omega)\end{cases}$$

for $\boldsymbol{\lambda},\boldsymbol{\mu}\in\mathbb{R}^2$.

Chapter 4

Homogenization of the Viscous Lake Equations

In this chapter, we discuss the viscous lake equation with the Navier boundary conditions and the initial condition. In Section 4.1, we give the proof of the existence and uniqueness of the solution of lake equation using Faedo-Galerkin method. In Section 4.2, we study the homogenization of the viscous lake equation by constructing a homogenized equation of the test function $\boldsymbol{w}_\lambda^\varepsilon$ according to Tartar's method.

4.1 Existence and Uniqueness of the Solution for the Viscous Lake equation

In this section we discuss the viscous lake equation

$$\partial_t(b\boldsymbol{u}^\varepsilon) - \frac{1}{2}(\boldsymbol{u}^\varepsilon \cdot \boldsymbol{v}^\varepsilon)\nabla b + b\,\mathrm{curl}(\boldsymbol{v}^\varepsilon)J\boldsymbol{u}^\varepsilon + b\nabla p^\varepsilon = \mathrm{div}(b\Sigma(\boldsymbol{u}^\varepsilon)) + b\boldsymbol{f}, \tag{4.1.1}$$

$$\mathrm{div}(b\boldsymbol{u}^\varepsilon) = 0, \qquad (\boldsymbol{x}, t) \in Q_T \tag{4.1.2}$$

with the Navier boundary conditions

$$\boldsymbol{u}^\varepsilon \cdot \boldsymbol{n} = 0, \qquad \boldsymbol{\tau} \cdot \left(\nabla \boldsymbol{u}^\varepsilon + (\nabla \boldsymbol{u}^\varepsilon)^{\mathrm{T}}\right) \cdot \nabla = -\beta \boldsymbol{\tau} \cdot \boldsymbol{u}^\varepsilon, \qquad (\boldsymbol{\sigma}, t) \in S_T \tag{4.1.3}$$

and the initial condition

$$\boldsymbol{u}^\varepsilon(\boldsymbol{x}, 0) = \boldsymbol{u}_0^\varepsilon(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Omega \tag{4.1.4}$$

where $Q_T = \Omega \times [0, T)$ with $\Omega \subset \mathbb{R}^2$, a bounded and smooth domain, and $S_T = \partial\Omega \times [0, T)$ for all $0 < T < \infty$.

We first state the existence and uniqueness result for (4.1.1).

Theorem 4.1.1. *Assume* $\boldsymbol{f} \in L^2(Q_T)$, $\boldsymbol{v}^\varepsilon \in L^\infty(Q_T)$, *and* $\boldsymbol{u}_0^\varepsilon \in L^2(\Omega)$. *We define the solution space associated with* (4.1.1)*;*

$$V^2(Q_T) = \left\{ \boldsymbol{u} \in L^2([0, T); \mathbb{V}_b(\Omega)), \quad \boldsymbol{u}' = \frac{\partial \boldsymbol{u}}{\partial t} \in L^2([0, T); \mathbb{V}_b^{-1}(\Omega)) \right\}$$

which is a Banach space with respect to the norm

$$\|\boldsymbol{u}\|_{V^2(Q_T)} = \|\boldsymbol{u}\|_{L^2([0,T);\mathbb{V}_b(\Omega))} + \|\boldsymbol{u}'\|_{L^2([0,T);\mathbb{V}_b^{-1}(\Omega))} . \tag{4.1.5}$$

Then there exists a pair of function $(\boldsymbol{u}^\varepsilon, p^\varepsilon)$ *such that*

$$\boldsymbol{u}^\varepsilon \in V^2(Q_T) \quad \textit{and} \quad \nabla p^\varepsilon \in W^{-1,\infty}([0, T); \mathbb{V}_b^{-1}(\Omega)) \tag{4.1.6}$$

and satisfies the weak formulation of (4.1.1) *in the sense of distribution*

$$\begin{aligned} -\int_{Q_T} \boldsymbol{u}^\varepsilon(\boldsymbol{x}, t)\psi'(t)\boldsymbol{\varphi}(\boldsymbol{x})b(\boldsymbol{x})d\boldsymbol{x}dt + \int_0^T a(\boldsymbol{u}^\varepsilon(\boldsymbol{x}, t), \boldsymbol{\varphi})\psi(t)dt \\ + \int_0^T (\boldsymbol{u}^\varepsilon(\boldsymbol{x}, t), \boldsymbol{v}^\varepsilon(\boldsymbol{x}, t), \boldsymbol{\varphi}(\boldsymbol{x}))\psi(t)dt \\ = \int_{Q_T} \boldsymbol{f}(\boldsymbol{x}, t)\psi(t)\boldsymbol{\varphi}(\boldsymbol{x})b(\boldsymbol{x})d\boldsymbol{x}dt + \langle \boldsymbol{u}_0^\varepsilon(\boldsymbol{x}), \boldsymbol{\varphi}(\boldsymbol{x})\rangle_b, \end{aligned} \tag{4.1.7}$$

for all $T \in (0,\infty)$ *and for all* $\boldsymbol{\varphi} \in C_c^\infty(\Omega)$ *with* $div(b\boldsymbol{\varphi}) = 0$ *in* Ω *and* $\boldsymbol{\varphi} \cdot \nabla = 0$ *on* $\partial\Omega$*, and* $\psi \in \mathcal{D}[0,T)$*, where* $\psi(0) = 1$ *and* $\psi(T) = 0$*. Here the bilinear form* $a(\boldsymbol{u}^\varepsilon, \boldsymbol{\varphi})$ *is defined by*

$$a(\boldsymbol{u}^\varepsilon, \boldsymbol{\varphi}) = \frac{1}{2}\int_\Omega \Sigma(\boldsymbol{u}^\varepsilon) : \Sigma(\boldsymbol{\varphi}) b(\boldsymbol{x}) d\boldsymbol{x} + 2\int_{\partial\Omega} \kappa(\boldsymbol{u}^\varepsilon \cdot \boldsymbol{\tau})(\boldsymbol{\varphi} \cdot \boldsymbol{\tau}) b(\boldsymbol{x}) ds\,, \tag{4.1.8}$$

and the trilinear form

$$\begin{aligned}&(\boldsymbol{u}^\varepsilon(\boldsymbol{x},t), \boldsymbol{v}^\varepsilon(\boldsymbol{x},t), \boldsymbol{\varphi}(\boldsymbol{x}))\\ &\qquad = -\frac{1}{2}\int_\Omega (\boldsymbol{u}^\varepsilon \cdot \boldsymbol{v}^\varepsilon)\nabla b \cdot \boldsymbol{\varphi} d\boldsymbol{x} + \int_\Omega b\,\mathrm{curl}(\boldsymbol{v}^\varepsilon) J\boldsymbol{u}^\varepsilon \cdot \boldsymbol{\varphi} d\boldsymbol{x},\end{aligned}$$

Moreover, there exists a constant c *depending on* Ω *and* T *such that*

$$\|\boldsymbol{u}^\varepsilon\|_{V^2(Q_T)} + \|\boldsymbol{u}^\varepsilon\|_{L^\infty([0,T);L^2(\Omega))} \leqslant c\big(\|\boldsymbol{u}_0^\varepsilon\|_{L^2(\Omega)} + \|\boldsymbol{f}\|_{L^2(Q_T)}\big)\,. \tag{4.1.9}$$

We will prove the existence and uniqueness theorem for the solution for the viscous lake equation. The strategy of the proof is motivated by Leray's seminal work on the incompressible Navier-Stokes equations [35], see also [61]. It proceeds in seven steps. We can construct a sequence of approximate solutions by any method that yields a consistent weak formulation and an energy relation. We will employ the Faedo-Galerkin method to approximate our drift Stokes equation by a sequence of Cauchy problems for suitable systems of ODEs in finite dimensional spaces.

Step 1: *Construction of approximate solution* $\boldsymbol{u}_m^\varepsilon$ *by the Faedo-Galerkin method.*

Take a countable orthonormal basis $\{\boldsymbol{e}_i\}_{i=1}^\infty$ of the space $H_b^1(\Omega) \cap C_c^\infty(\Omega)$. For any $m \in \mathbb{N}$, we define the approximate solution by eigenfunction expansion

$$\boldsymbol{u}_m^\varepsilon(\boldsymbol{x},t) = \sum_{i=1}^{m} c_{im}^\varepsilon(t)\boldsymbol{e}_i(\boldsymbol{x}). \tag{4.1.10}$$

The coefficients $c_{im}^\varepsilon(t)$ will satisfy the m nonlinear ordinary differential equations by inserting $\boldsymbol{u}_m^\varepsilon$, $\boldsymbol{e}_i$ for $i,k = 1,2,\cdots,m$, in (4.1.1)-(4.1.8).

Let now introduce, for any $m \in \mathbb{N}$, the finite dimensional approximate problem for (4.1.1)

$$\begin{aligned}&\langle \boldsymbol{u}_m^\varepsilon(t_2), \boldsymbol{e}_i\rangle_b - \langle \boldsymbol{u}_m^\varepsilon(t_1), \boldsymbol{e}_i\rangle_b = I_1 + I_2 + I_3\\ &\equiv -\int_0^T a(\boldsymbol{u}_m^\varepsilon, \boldsymbol{e}_i)dt - \int_0^T (\boldsymbol{u}_m^\varepsilon, \boldsymbol{v}^\varepsilon, \boldsymbol{e}_i(\boldsymbol{x}))dt + \int_{Q_T} \boldsymbol{f}(\boldsymbol{x},t)\boldsymbol{e}_i d\boldsymbol{x}dt\end{aligned} \tag{4.1.11}$$

in $\mathcal{D}'[0,T)$, for all $0 < T < \infty$ and for all $i = 1,\cdots,m$ with initial condition

$$\boldsymbol{u}_m^\varepsilon(\boldsymbol{x},0) = \boldsymbol{u}_{0m}^\varepsilon(\boldsymbol{x}), \tag{4.1.12}$$

where $\boldsymbol{u}_{0m}^\varepsilon$ is the orthogonal projection of $\boldsymbol{u}_0^\varepsilon$ onto the space spanned by $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_m\}$ in $H_0^1(\Omega)$.

From the initial condition in this problem, we have

$$\sum_{i=1}^{m} c_{im}^\varepsilon(0)\boldsymbol{e}_i = \boldsymbol{u}_m^\varepsilon(0) = \boldsymbol{u}_{0m}^\varepsilon = \sum_{i=1}^{m} \langle \boldsymbol{u}_0^\varepsilon, \boldsymbol{e}_i\rangle \boldsymbol{e}_i,$$

which implies $c_{im}^\varepsilon(0) = \langle \boldsymbol{u}_0^\varepsilon, \boldsymbol{e}_i\rangle$, since $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_m\}$ are linearly independent.

From classical results concerning Hilbert spaces, we have that

$$\boldsymbol{u}_{0m}^\varepsilon(\boldsymbol{x}) \to \boldsymbol{u}_0^\varepsilon(\boldsymbol{x}) \text{ strongly in } L^2(\Omega)$$

as $m \to \infty$ so that

$$\|\boldsymbol{u}_{0m}^\varepsilon\|_{L^2(\Omega)} \leqslant \|\boldsymbol{u}_0^\varepsilon\|_{L^2(\Omega)}.$$

Moreover, these solutions will satisfy the regularized version of the energy relation (4.1.9) as the equality

$$\|\boldsymbol{u}_m^\varepsilon(T)\|_{H(\Omega)}^2 + C\int_0^T \|\boldsymbol{u}_m^\varepsilon(\tau)\|_{V(\Omega)}^2 d\tau = \|\boldsymbol{u}_{0m}^\varepsilon\|_{L^2(\Omega)}^2 + \frac{1}{C}\int_0^T \|\boldsymbol{f}(\tau)\|_{L^2(\Omega)}^2 d\tau. \tag{4.1.13}$$

Consequently, problem (4.1.11) is a system of m linear ordinary differential equations of the first order with unknowns $c_{1m}, \cdots, c_{mm}$. By Picard's local existence theory, the system of linear ODEs (4.1.11) has a unique solution on some interval $(0, t_m), 0 < t_m < T$. Furthermore, the energy relation (4.1.13) provides a global $L^2(\Omega)$ bound on the solutions, ensuring that they are global. The detail is referred to [61].

Step 2: *Show that the sequence* $\{\boldsymbol{u}_m^\varepsilon\}_m$ *is a relatively compact set in*

$$C([0,T]; \text{w-}\mathbb{H}_b(\Omega)) \cap \text{w-}L^2_{loc}([0,T]; \mathbb{V}_b(\Omega)).$$

We deduce from the energy bound (4.1.13) that

$$\{\boldsymbol{u}_m^\varepsilon\}_m \quad \text{is bounded in} \quad L^\infty([0,T]; \mathbb{H}_b(\Omega)) \tag{4.1.14}$$

$$\{\boldsymbol{u}_m^\varepsilon\}_m \quad \text{is bounded in} \quad L^2_{loc}([0,T]; \mathbb{V}_b(\Omega)). \tag{4.1.15}$$

Because norm bounded sets are relatively compact in weak-$*$ topologies, which is the same as the weak topologies on these reflexive spaces, then we obtain from (4.1.15) that

$$\{\boldsymbol{u}_m^\varepsilon\}_m \quad \text{is relatively compact in} \quad L^2\big([0,T]; \mathbb{V}_b(\Omega)\big). \tag{4.1.16}$$

We conclude from the classical compactness argument that there exists a subsequence of $\{\boldsymbol{u}_m^\varepsilon\}_m$, which is still denoted by $\{\boldsymbol{u}_m^\varepsilon\}_m$ and

$$\boldsymbol{u}^\varepsilon \in L^\infty([0,T), \mathbb{H}_b(\Omega)) \cap L^2\big([0,T); \mathbb{V}_b(\Omega)\big)$$

such that

$$\begin{aligned} \boldsymbol{u}_m^{\varepsilon} &\overset{\text{w}}{\rightharpoonup} \boldsymbol{u}^{\varepsilon} \quad \text{weakly} * \text{ in} \quad L^{\infty}\left([0,T);\mathbb{H}_b(\Omega)\right) \\ \boldsymbol{u}_m^{\varepsilon} &\overset{\text{w}}{\rightharpoonup} \boldsymbol{u}^{\varepsilon} \quad \text{weakly in} \quad L^2\big([0,T);\mathbb{V}_b(\Omega)\big). \end{aligned} \tag{4.1.17}$$

The uniform bound (4.1.14) also shows that $\{\boldsymbol{u}_m^{\varepsilon}(t)\}$ is a relatively compact set in w-$\mathbb{H}_{\text{b}}(\Omega)$ for all $t \geqslant 0$. However, compactness requires more than just boundedness because of the strong topology over the time variable t. We appeal to the Arzelà-Ascoli theorem, which asserts that $\{\boldsymbol{u}_m^{\varepsilon}\}_m$ is a relatively compact set in v if and only if

(a) $\{\boldsymbol{u}_m^{\varepsilon}(t)\}_m$ is a relatively compact set in w-$\mathbb{H}_b(\Omega)$ for all $t \geqslant 0$;

(b) $\{\boldsymbol{u}_m^{\varepsilon}\}_m$ is equicontinuous in $C\big([0,\infty);\text{w-}\mathbb{H}_b(\Omega)\big)$.

Condition (a) is satisfied. In order to establish (b), we will estimate the three integrals I_1, I_2 and I_3 of (4.1.11) separately. Let $B \subset C_c^{\infty}(\Omega)$ be an enumerable set which is dense in $\mathbb{H}_b(\Omega)$, then from the energy relation (4.1.13) for any $\boldsymbol{e}_i \in B$, due to the boundedness result (3.2.22) in Section 3.2

$$|a(\boldsymbol{u}_m^{\varepsilon}, \boldsymbol{e}_i)| \leq C \, \|\boldsymbol{u}_m^{\varepsilon}\|_{H_b^1(\Omega)} \, \|\boldsymbol{e}_i\|_{H_b^1(\Omega)} \, ,$$

using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} |I_1| &\leq \alpha \, \|\boldsymbol{e}_i\|_{H_b^1(\Omega)} \int_{t_1}^{t_2} \|\boldsymbol{u}_m^{\varepsilon}\|_{H_b^1(\Omega)} \, d\tau \\ &\leq \alpha \, \|\boldsymbol{e}_i\|_{H_b^1(\Omega)} \, |t_2 - t_1|^{\frac{1}{2}} \, \Big(\int_0^{\infty} \|\boldsymbol{u}_m^{\varepsilon}\|^2_{H_b^1(\Omega)} \, d\tau \Big)^{\frac{1}{2}} \leq \; C_1 \, |t_2 - t_1|^{\frac{1}{2}} \, . \end{aligned}$$

Similarly, from the estimate (3.2.23) in Section 3.2

$$|(\boldsymbol{u}_m^{\varepsilon}, \boldsymbol{v}^{\varepsilon}, \boldsymbol{e}_i)| \leq C_0 \|\boldsymbol{u}_m^{\varepsilon}\|_{H_b^1(\Omega)} \|\boldsymbol{e}_i\|_{H_b^1(\Omega)} \tag{4.1.18}$$

it follows that

$$
\begin{aligned}
|I_2| &\leq \beta \, \|\boldsymbol{e}_i\|_{H^1_b(\Omega)} \int_{t_1}^{t_2} \|\boldsymbol{u}^\varepsilon_m\|_{H^1_b(\Omega)} \, d\tau \\
&\leq \beta \, \|\boldsymbol{e}_i\|_{H^1_b(\Omega)} \, |t_2 - t_1|^{\frac{1}{2}} \, \big(\int_0^\infty \|\boldsymbol{u}^\varepsilon_m\|^2_{H^1_b(\Omega)} \, d\tau \big)^{\frac{1}{2}} \leq \; C_2 \, |t_2 - t_1|^{\frac{1}{2}} \, ,
\end{aligned}
$$

For I_3, we also see that

$$
|I_3| \leq \|\boldsymbol{e}_i\|_{L^2(\Omega)} \, |t_2 - t_1|^{\frac{1}{2}} \left(\int_{t_1}^{t_2} \|\boldsymbol{f}\|^2_{L^2_b(\Omega)} \, dt \right)^{\frac{1}{2}} \leq \; C_3 \, |t_2 - t_1|^{\frac{1}{2}} \ .
$$

Collecting the above estimates,

$$
| \left\langle \boldsymbol{u}^\varepsilon_m(t_2), \boldsymbol{e}_i \right\rangle_b - \left\langle \boldsymbol{u}^\varepsilon_m(t_1), \boldsymbol{e}_i \right\rangle_b | \leqslant C|t_2 - t_1|^{\frac{1}{2}} \, , \qquad i = 1, 2, 3, \cdots
$$

Therefore

$$
| \left\langle \boldsymbol{u}^\varepsilon_m(t_2), \boldsymbol{e}_i \right\rangle_b - \left\langle \boldsymbol{u}^\varepsilon_m(t_1), \boldsymbol{e}_i \right\rangle_b | \to 0 \quad \text{as} \quad |t_2 - t_1| \to 0.
$$

Since $\{\boldsymbol{e}_i\}_{i=1}^\infty$ forms an enumerable dense subset of $\mathbb{H}_b(\Omega)$, it follows from the density argument that

$$
|\langle u^\varepsilon_m(t_2), \boldsymbol{\varphi} \rangle_b - \langle u^\varepsilon_m(t_1), \boldsymbol{\varphi} \rangle_b| \to 0 \quad \text{ as } |t_2 - t_1| \to 0 \tag{4.1.19}
$$

for all $\boldsymbol{\varphi} \in H^1_b(\Omega) \cap C^\infty_c(\Omega)$. This proves the equi-continuity of $\boldsymbol{u}^\varepsilon_m$ in the space $C([0, T), \text{w-}\mathbb{H}_\text{b}(\Omega))$.

Step 3: *Show that the sequence* $\{\boldsymbol{u}^\varepsilon_m\}_m$ *is a relatively compact set in*

$$
L^2_{loc}\left([0, T); \mathbb{H}_b(\Omega)\right)
$$

endowed with the strong topology.

The proof is based on the following embedding

$$
C\left([0, T); \text{w-}\mathbb{H}_\text{b}(\Omega)\right) \cap \text{w-}L^2_{loc}\left([0, T); \mathbb{V}_b(\Omega)\right) \hookrightarrow L^2_{loc}\left([0, T); \mathbb{H}_b(\Omega)\right) \tag{4.1.20}
$$

is continuous. The key step is the Rellich lamma for weighted Sobolev space, which states that $\mathbb{V}_b(\Omega) \hookrightarrow \mathbb{H}_b(\Omega)$ is a compact embedding. Step 2 states that $\{\boldsymbol{u}_m^\varepsilon\}_m$ is a relatively compact set in $C([0,T); \text{w-}L_b^2(\Omega))$ and w-$L_{loc}^2([0,T); \mathbb{V}_b(\Omega))$, and because the compact operator maps weakly convergent sequences into strong convergent sequences, it follows that $\{\boldsymbol{u}_m^\varepsilon\}_m$ is strongly convergent in $L_{loc}^2([0,T); \mathbb{H}_b(\Omega))$.

Step 4: *Passage to the limit as $m \to \infty$.*

We want to pass to the limit as $m \to \infty$ in (4.1.11) using the energy estimate (4.1.13). We recall that at the present time $\varepsilon > 0$ is fixed, and we are only concerned with a passage to the limit as $m \to \infty$. First, Step 2 and the energy estimate (4.1.13) ensures that there exists a subsequence of $\{\boldsymbol{u}_m^\varepsilon\}_m$, which we still denote by $\{\boldsymbol{u}_m^\varepsilon\}_m$, such that

$$\begin{cases} \boldsymbol{u}_m^\varepsilon \overset{\text{w}}{\rightharpoonup} \boldsymbol{u}^\varepsilon \quad \text{weakly} * \text{ in} \quad L^\infty([0,T); \mathbb{H}_b(\Omega)) \\ \boldsymbol{u}_m^\varepsilon \overset{\text{w}}{\rightharpoonup} \boldsymbol{u}^\varepsilon \quad \text{weakly in} \quad L^2\big([0,T); \mathbb{V}_b(\Omega)\big) \\ \boldsymbol{u}_{0m}^\varepsilon \to \boldsymbol{u}_0^\varepsilon \quad \text{strongly in} \quad L^2(\Omega). \end{cases} \tag{4.1.21}$$

Now let $\boldsymbol{\varphi}(\boldsymbol{x}) \in \mathbb{V}_b(\Omega)$ and choose a $\psi(t) \in \mathcal{D}[0,T)$ such that $\psi(0) = 1$ and $\psi(T) = 0$. Multiply the equation in (4.1.11) by $\langle \boldsymbol{\varphi}(\boldsymbol{x}), \boldsymbol{e}_i \rangle_b \psi(t)$ and sum over i from 1 to m. After integration in t over $[0,T)$, we get

$$\begin{aligned} -\int_{Q_T} \boldsymbol{u}_m^\varepsilon(\boldsymbol{x},t)\psi'(t)\boldsymbol{\varphi}(\boldsymbol{x})b(\boldsymbol{x})d\boldsymbol{x}dt + \int_0^T a(\boldsymbol{u}_m^\varepsilon, \boldsymbol{\varphi})\psi(t)dt \\ + \int_0^T (\boldsymbol{u}_m^\varepsilon, \boldsymbol{v}^\varepsilon, \boldsymbol{\varphi}(\boldsymbol{x}))\psi(t)dt \\ = \int_{Q_T} \boldsymbol{f}(\boldsymbol{x},t)\psi(t)\boldsymbol{\varphi}(\boldsymbol{x})b(\boldsymbol{x})d\boldsymbol{x}dt + \langle \boldsymbol{u}_0^\varepsilon(\boldsymbol{x}), \boldsymbol{\varphi}(\boldsymbol{x}) \rangle_b. \end{aligned} \tag{4.1.22}$$

Note that for the first term in this equation, due to the following identity

$$\begin{aligned}
&\int_{Q_T} \partial_t \boldsymbol{u}_m^\varepsilon \psi(t) \boldsymbol{\varphi}(\boldsymbol{x}) b(\boldsymbol{x}) d\boldsymbol{x} dt \\
&\quad = \int_{Q_T} \Big(\frac{d}{dt} \big(\boldsymbol{u}_m^\varepsilon(\boldsymbol{x}, t) \psi(t) \big) - \boldsymbol{u}_m^\varepsilon(\boldsymbol{x}, t) \psi'(t) \Big) \boldsymbol{\varphi}(\boldsymbol{x}) b(\boldsymbol{x}) d\boldsymbol{x} dt \\
&\quad = \int_{\Omega} \big(\boldsymbol{u}_m^\varepsilon(\boldsymbol{x}, T) \psi(T) - \boldsymbol{u}_m^\varepsilon(\boldsymbol{x}, 0) \psi(0) \big) \boldsymbol{\varphi}(\boldsymbol{x}) b(\boldsymbol{x}) d\boldsymbol{x} \\
&\qquad - \int_{Q_T} \boldsymbol{u}_m^\varepsilon \psi'(t) \boldsymbol{\varphi}(\boldsymbol{x}) b(\boldsymbol{x}) d\boldsymbol{x} dt \\
&\quad = -\langle \boldsymbol{u}_{0m}^\varepsilon(\boldsymbol{x}), \boldsymbol{\varphi}(\boldsymbol{x}) \rangle_b - \int_{Q_T} \boldsymbol{u}_m^\varepsilon(\boldsymbol{x}, t) \psi'(t) \boldsymbol{\varphi}(\boldsymbol{x}) b(\boldsymbol{x}) d\boldsymbol{x} dt.
\end{aligned}$$

We now let $m \to \infty$ here. All the terms pass to the limit, thanks to the convergence (4.1.21), we have

$$\int_{Q_T} \boldsymbol{u}_m^\varepsilon(\boldsymbol{x}, t) \psi'(t) \boldsymbol{\varphi}(\boldsymbol{x}) b(\boldsymbol{x}) d\boldsymbol{x} dt \to \int_{Q_T} \boldsymbol{u}^\varepsilon(\boldsymbol{x}, t) \psi'(t) \boldsymbol{\varphi}(\boldsymbol{x}) b(\boldsymbol{x}) d\boldsymbol{x} dt$$

$$\int_0^T a(\boldsymbol{u}_m^\varepsilon, \boldsymbol{\varphi}) \psi(t) dt \to \int_0^T a(\boldsymbol{u}^\varepsilon, \boldsymbol{\varphi}) \psi(t) dt$$

$$\int_0^T (\boldsymbol{u}_m^\varepsilon, \boldsymbol{v}^\varepsilon, \boldsymbol{\varphi}(\boldsymbol{x})) \psi(t) dt \to \int_0^T (\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon, \boldsymbol{\varphi}(\boldsymbol{x})) \psi(t) dt$$

and

$$\langle \boldsymbol{u}_{0m}^\varepsilon(\boldsymbol{x}), \boldsymbol{\varphi}(\boldsymbol{x}) \rangle_b \to \langle \boldsymbol{u}_0^\varepsilon(\boldsymbol{x}), \boldsymbol{\varphi}(\boldsymbol{x}) \rangle_b$$

Summing up the above convergence results, we finally get that $\boldsymbol{u}^\varepsilon$ satisfies the integral identity

$$
\begin{aligned}
-\int_{Q_T} \boldsymbol{u}^\varepsilon(\boldsymbol{x},t)\psi'(t)\boldsymbol{\varphi}(\boldsymbol{x})b(\boldsymbol{x})d\boldsymbol{x}dt + \int_0^T a(\boldsymbol{u}^\varepsilon, \boldsymbol{\varphi})\psi(t)dt \\
+ \int_0^T (\boldsymbol{u}^\varepsilon, \boldsymbol{v}^\varepsilon, \boldsymbol{\varphi}(\boldsymbol{x}))\psi(t)dt \\
= \int_{Q_T} \boldsymbol{f}(\boldsymbol{x},t)\psi(t)\boldsymbol{\varphi}(\boldsymbol{x})b(\boldsymbol{x})d\boldsymbol{x}dt + \langle \boldsymbol{u}_0^\varepsilon(\boldsymbol{x}), \boldsymbol{\varphi}(\boldsymbol{x})\rangle_b,
\end{aligned}
\tag{4.1.23}
$$

which is exactly the variational equation in (4.1.7) since ψ and $\boldsymbol{\varphi}$ are arbitrary respectively in $\mathcal{D}[0,T)$ and $H_0^1(\Omega)$. This shows that $\boldsymbol{u}^\varepsilon$ is the weak solution of (4.1.1).

Step 5: *Existence of the pressure p^ε.*

Now we are in a position to show the existence of the pressure p^ε. First, we set, for $t \in [0,T)$

$$
\begin{cases}
\widetilde{\boldsymbol{u}}^\varepsilon(t) = \displaystyle\int_0^t \boldsymbol{u}^\varepsilon(\tau)d\tau, \\
\boldsymbol{\beta}^\varepsilon(t) = \displaystyle\int_0^t -\frac{1}{2}(\boldsymbol{u}^\varepsilon(\tau)\cdot\boldsymbol{v}^\varepsilon)\nabla b + b\,\mathrm{curl}(\boldsymbol{v}^\varepsilon)J\boldsymbol{u}^\varepsilon(\tau)d\tau, \\
\widetilde{\boldsymbol{f}}(t) = \displaystyle\int_0^t \boldsymbol{f}(\tau)d\tau.
\end{cases}
\tag{4.1.24}
$$

Then for $\boldsymbol{u}^\varepsilon \in$ w-$L^2_{loc}([0,T);\mathbb{V}_b(\Omega))$ is a solution of (4.1.1),

$$
\widetilde{\boldsymbol{u}}^\varepsilon, \boldsymbol{\beta}^\varepsilon, \widetilde{\boldsymbol{f}} \in C([0,T);\mathbb{V}_b^{-1}(\Omega)). \tag{4.1.25}
$$

Integrating (4.1.1) for $t \in [0,T)$; setting

$$
\widetilde{p}^\varepsilon(t) = \int_0^t p^\varepsilon(\tau)d\tau, \tag{4.1.26}
$$

we obtain

$$b\boldsymbol{u}^\varepsilon(t) - b\boldsymbol{u}_0^\varepsilon - \operatorname{div}(b\Sigma(\widetilde{\boldsymbol{u}}^\varepsilon(t))) + \boldsymbol{\beta}^\varepsilon(t) + b\nabla\widetilde{p}^\varepsilon(t) = \widetilde{\boldsymbol{f}}(t). \tag{4.1.27}$$

Then from the regularity property of elliptic equations (see Proposition I.1.1 and Proposition I.1.2 in [61]), we get for every $t \in [0,T)$, the existence of some function $\widetilde{p}^\varepsilon(t)$,

$$\widetilde{p}^\varepsilon(t) \in L_b^2(\Omega) \tag{4.1.28}$$

such that the formulation (4.1.27) satisfied. Since the gradient operator is an isomorphism from $L^2(\Omega)/\mathbb{R}$ into $H^{-1}(\Omega)$, observing that

$$b\nabla\widetilde{p}^\varepsilon(t) = \widetilde{\boldsymbol{f}}(t) - \boldsymbol{\beta}^\varepsilon(t) + \operatorname{div}(b\Sigma(\widetilde{\boldsymbol{u}}^\varepsilon(t))) - b\boldsymbol{u}^\varepsilon(t) + b\boldsymbol{u}_0^\varepsilon, \tag{4.1.29}$$

we conclude that $\nabla\widetilde{p}^\varepsilon(t)$ belongs to $C([0,T); H_b^{-1}(\Omega))$ and therefore

$$\widetilde{p}^\varepsilon \in C([0,T); L_b^2(\Omega)). \tag{4.1.30}$$

This enables us to differentiate (4.1.27) in sence of distribution in Q_T, thus we obtain (4.1.1). The pressure appears in general as a distribution on Q_T defined by (4.1.26) and (4.1.30). By application of Proposition I.1.2 in [61] it follows that

$$p^\varepsilon \in L^\infty([0,T); H_b^1(\Omega)). \tag{4.1.31}$$

Step 6: *The sequence $\{p^\varepsilon\}_\varepsilon$ converges weakly $*$ in $L^\infty([0,T); H_b^1(\Omega))$.*

From (4.1.1), we can represent p^ε as

$$b\nabla p^\varepsilon = -\partial_t(b\boldsymbol{u}^\varepsilon) + \frac{1}{2}(\boldsymbol{u}^\varepsilon \cdot \boldsymbol{v}^\varepsilon)\nabla b - b\operatorname{curl}(\boldsymbol{v}^\varepsilon)J\boldsymbol{u}^\varepsilon + \operatorname{div}(b\Sigma(\boldsymbol{u}^\varepsilon)) + b\boldsymbol{f}. \tag{4.1.32}$$

From the results (3.2.22) and (3.2.23) in Section 3.2, for $\boldsymbol{u}^{\varepsilon}, \boldsymbol{\varphi} \in \mathbb{V}_b$ and every $t \in [0, T)$,

$$|a(\boldsymbol{u}^{\varepsilon}(t), \boldsymbol{\varphi})| = \left| -\int_{\Omega} \operatorname{div}(b\Sigma(\boldsymbol{u}^{\varepsilon}(t))) \cdot \boldsymbol{\varphi} d\boldsymbol{x} \right| \leqslant C' \|\boldsymbol{u}^{\varepsilon}(t)\|_{H_b^1(\Omega)} \|\boldsymbol{\varphi}\|_{H_b^1(\Omega)} \tag{4.1.33}$$

and

$$|(\boldsymbol{u}^{\varepsilon}(t), \boldsymbol{v}^{\varepsilon}(t), \boldsymbol{\varphi})| \leq C_0 \|\boldsymbol{u}^{\varepsilon}(t)\|_{H_b^1(\Omega)} \|\boldsymbol{\varphi}\|_{H_b^1(\Omega)} . \tag{4.1.34}$$

On the other hand, applying the Cauchy-Schwarz inequality and the Poincaré inequality,

$$|\langle \boldsymbol{f}(t), \boldsymbol{\varphi} \rangle_b| \leqslant \|\boldsymbol{f}(t)\|_{L_b^2(\Omega)} \|\boldsymbol{\varphi}\|_{L_b^2(\Omega)} \leqslant c \|\boldsymbol{f}(t)\|_{H_b^1(\Omega)} \|\boldsymbol{\varphi}\|_{H_b^1(\Omega)} , \tag{4.1.35}$$

using the weak formulation (4.1.7),

$$\begin{aligned} &\left| -\int_{\Omega} \partial_t \boldsymbol{u}^{\varepsilon}(t) \cdot \boldsymbol{\varphi} b d\boldsymbol{x} \right| \\ = &\left| -a(\boldsymbol{u}^{\varepsilon}(t), \boldsymbol{\varphi}) - (\boldsymbol{u}^{\varepsilon}(t), \boldsymbol{v}^{\varepsilon}(t), \boldsymbol{\varphi}) + \langle \boldsymbol{f}(t), \boldsymbol{\varphi} \rangle_b \right| \leqslant C \|\boldsymbol{\varphi}\|_{H_b^1(\Omega)} \end{aligned} \tag{4.1.36}$$

for every $t \in [0, T)$. Therefore, the terms in RHS of the identity (4.1.32) are all bounded in $H_b^1(\Omega)$. Therefore for all $t \in [0, T)$, $b\nabla p^{\varepsilon}(t)$ is also bounded in space $H_b^1(\Omega)$. Consequently, there exists a subsequence of $\{p^{\varepsilon}\}_{\varepsilon}$ still denoted by $\{p^{\varepsilon}\}_{\varepsilon}$ and $p \in L^{\infty}([0, T); H_b^1(\Omega))$ such that

$$p^{\varepsilon} \xrightarrow{\text{w}} p \quad \text{weakly} * \text{ in} \quad L^{\infty}([0, T); H_b^1(\Omega)). \tag{4.1.37}$$

Step 7: *The energy inequality.*

To recover the energy inequality (4.1.10) from the energy relation (4.1.13), first we note that the regularized initial data $\boldsymbol{u}_{0m}^{\varepsilon}(\boldsymbol{x})$ converges to $\boldsymbol{u}_0^{\varepsilon}(\boldsymbol{x})$ strongly in $L^2(\Omega)$ as m tends to infinite so that

$$\|\boldsymbol{u}_{0m}^{\varepsilon}\|_{L^2(\Omega)} \to \|\boldsymbol{u}_0^{\varepsilon}\|_{L^2(\Omega)} .$$

The convergence of $\boldsymbol{u}_m^\varepsilon$ in $C([0,T), \text{w-}\mathbb{H}_{\text{b}}(\Omega))$ together with the fact that the norm of the weak limit of a sequence is a lower bound for the inferior limit of the norms, yields

$$\|\boldsymbol{u}^\varepsilon(t)\|^2_{L^2_b(\Omega)} \leqslant \liminf_{m\to\infty} \|\boldsymbol{u}^\varepsilon_m(t)\|^2_{L^2_b(\Omega)}\,, \qquad 0 < t < T < \infty\,.$$

Then we obtain from (4.1.13) the energy inequality

$$\|\boldsymbol{u}^\varepsilon(T)\|^2_{H(\Omega)} + C\int_0^T \|\boldsymbol{u}^\varepsilon(\tau)\|^2_{V(\Omega)} d\tau \leqslant \|\boldsymbol{u}^\varepsilon_0\|^2_{L^2(\Omega)} + \frac{1}{C}\int_0^T \|f(\tau)\|^2_{L^2(\Omega)} d\tau. \tag{4.1.38}$$

Step 8 : *Proof of the uniqueness.*

Since the lake equation (4.1.1) is linear, the uniqueness follows immediately from the energy inequality. Let $\boldsymbol{u}_1^\varepsilon, \boldsymbol{u}_2^\varepsilon$ be two solutions corresponding to the same data of (4.1.1) then their difference $\boldsymbol{U}^\varepsilon \equiv \boldsymbol{u}_1^\varepsilon - \boldsymbol{u}_2^\varepsilon$ satisfies the equation

$$\begin{cases} \partial_t(b\boldsymbol{U}^\varepsilon) - \frac{1}{2}(\boldsymbol{U}^\varepsilon\cdot\boldsymbol{v}^\varepsilon)\nabla b + b\,\text{curl}(\boldsymbol{v}^\varepsilon)J\boldsymbol{U}^\varepsilon = \text{div}\big(b\Sigma(\boldsymbol{U}^\varepsilon)\big), \\ \text{div}(b\boldsymbol{U}^\varepsilon) = 0, \qquad (\boldsymbol{x},t)\in Q_T, \\ \boldsymbol{U}^\varepsilon(\boldsymbol{\sigma},t) = 0, \qquad (\boldsymbol{\sigma},t)\in S_T, \\ \boldsymbol{U}^\varepsilon(\boldsymbol{x},0) = 0, \qquad \boldsymbol{x}\in\Omega\,, \end{cases} \tag{4.1.39}$$

and the energy inequality

$$\|\boldsymbol{U}^\varepsilon(t)\|^2_{L^2_b} + \int_0^t \|\boldsymbol{U}^\varepsilon(\tau)\|^2_{H^1_b} d\tau \leqslant 0\,. \tag{4.1.40}$$

The energy inequality (4.1.40) implies that

$$\|\boldsymbol{U}^\varepsilon(t)\|^2_{L^2_b} = 0, \qquad 0 \leqslant t < \infty\,,$$

and the uniqueness is proved. This completes the proof of Theorem 4.1.1.

4.2 Homogenization of the Viscous Lake equation

Now we will discuss the convergence of the weak formulation (4.1.7) as $\varepsilon \to 0$. First, we give the following lemmas.

Lemma 4.2.1. $\{\boldsymbol{u}^\varepsilon\}_\varepsilon$ *is a relatively compact set in*

$$C([0,T); \text{w-}\mathbb{H}_b(\Omega)) \cap \text{w-}L^2_{loc}([0,T); \mathbb{V}_b(\Omega)).$$

Lemma 4.2.2. *The sequence* $\{\boldsymbol{u}^\varepsilon\}_\varepsilon$ *is a relatively compact set in*

$$L^2_{loc}([0,T); \mathbb{H}_b(\Omega))$$

endowed with the strong topology.

The proof of Lemma 4.2.1 and Lemma 4.2.2 are the same as Step 2 and Step 3 in Section 4.1. The detail is omitted.

Lemma 4.2.1 ensures that there exists a subsequence of $\{\boldsymbol{u}^\varepsilon\}_\varepsilon$, which we still denote by $\{\boldsymbol{u}^\varepsilon\}_\varepsilon$, and $\boldsymbol{u} \in C\big([0,T); \mathbb{H}_b(\Omega)\big)$ such that

$$\boldsymbol{u}^\varepsilon \to \boldsymbol{u} \quad \text{in} \quad C\big([0,T); \text{w-}\mathbb{H}_b(\Omega)\big) \quad \text{as} \quad \varepsilon \to 0. \tag{4.2.1}$$

We deduce from Lemma 4.2.2 that $\{\boldsymbol{u}^\varepsilon\}_\varepsilon$ is bounded in $L^2_{loc}([0,T); \mathbb{V}_b(\Omega))$. This implies that there exists a subsequence of $\{\boldsymbol{u}^\varepsilon\}_\varepsilon$ (still denoted by $\{\boldsymbol{u}^\varepsilon\}_\varepsilon$) and a function

$$\boldsymbol{u} \in L^2_{loc}\big([0,T); \mathbb{V}_b(\Omega)\big)$$

such that

$$\begin{cases} \boldsymbol{u}^\varepsilon \overset{\text{w}}{\rightharpoonup} \boldsymbol{u} & \text{in} \quad \text{w-}L^2_{loc}\big([0,T); \text{w-}\mathbb{V}_b(\Omega)\big) \\ \nabla\boldsymbol{u}^\varepsilon \overset{\text{w}}{\rightharpoonup} \nabla\boldsymbol{u} & \text{in} \quad \text{w-}L^2_{loc}\big([0,T); \text{w-}\mathbb{H}_b(\Omega)\big) \\ \Delta\boldsymbol{u}^\varepsilon \overset{\text{w}}{\rightharpoonup} \Delta\boldsymbol{u} & \text{in} \quad \text{w-}L^2_{loc}\big([0,T); \text{w-}\mathbb{V}_b^{-1}(\Omega)\big). \end{cases} \tag{4.2.2}$$

In addition, we suppose the weak convergence of the initial condition

$$\boldsymbol{u}_0^\varepsilon \overset{\text{w}}{\rightharpoonup} \boldsymbol{u}_0 \quad \text{weakly in} \quad L^2(\Omega). \tag{4.2.3}$$

Then the convergence (4.2.1)–(4.2.3) imply the following convergences

$$\begin{gathered}\int_\Omega \boldsymbol{u}^\varepsilon(\boldsymbol{x},t)\psi'(t)\boldsymbol{\varphi}(\boldsymbol{x})b(\boldsymbol{x})d\boldsymbol{x} \to \int_\Omega \boldsymbol{u}(\boldsymbol{x},t)\psi'(t)\boldsymbol{\varphi}(\boldsymbol{x})b(\boldsymbol{x})d\boldsymbol{x}\\ \int_\Omega \operatorname{div}(b\Sigma(\boldsymbol{u}^\varepsilon))\cdot\boldsymbol{\varphi}d\boldsymbol{x} \to \int_\Omega \operatorname{div}(b\Sigma(\boldsymbol{u}))\cdot\boldsymbol{\varphi}d\boldsymbol{x}\\ \langle \boldsymbol{u}_0^\varepsilon(\boldsymbol{x}), \boldsymbol{\varphi}(\boldsymbol{x})\rangle_b \to \langle \boldsymbol{u}_0(\boldsymbol{x}), \boldsymbol{\varphi}(\boldsymbol{x})\rangle_b\end{gathered} \tag{4.2.4}$$

for all $\boldsymbol{\varphi} \in L^2(\Omega) \cap C_c^\infty(\Omega)$ and $\psi \in \mathcal{D}[0,T)$. By energy estimate and the Cauchy-Schwarz inequality

$$\begin{aligned}\left|\int_\Omega \boldsymbol{u}^\varepsilon(\boldsymbol{x},t)\psi'(t)\boldsymbol{\varphi}(\boldsymbol{x})bd\boldsymbol{x}\right| &\leqslant \sup_{0\leqslant t<T} |\psi'(t)| \, \|\boldsymbol{u}^\varepsilon\|_{L_b^2(\Omega)} \|\boldsymbol{\varphi}\|_{L^2(\Omega)}\\ &\leqslant C \, \|\boldsymbol{u}^\varepsilon\|_{L_b^2(\Omega)}\end{aligned} \tag{4.2.5}$$

for all $0 < t < \infty$. Then the convergence (4.2.4) and Lebesgue dominated convergence theorem ensure the convergence

$$\int_{Q_T} \boldsymbol{u}^\varepsilon(\boldsymbol{x},t)\psi'(t)\boldsymbol{\varphi}(\boldsymbol{x})b(\boldsymbol{x})d\boldsymbol{x}dt \to \int_{Q_T} \boldsymbol{u}(\boldsymbol{x},t)\psi'(t)\boldsymbol{\varphi}(\boldsymbol{x})b(\boldsymbol{x})d\boldsymbol{x}dt. \tag{4.2.6}$$

Similarly, from the convergence (4.2.4), the boundedness result (3.2.22) and the Lebesgue dominated convergence theorem imply the convergence

$$\int_0^T a(\boldsymbol{u}^\varepsilon(\boldsymbol{x},t),\boldsymbol{\varphi})\psi(t)dt \to \int_0^T a(\boldsymbol{u}(\boldsymbol{x},t),\boldsymbol{\varphi})\psi(t)dt. \tag{4.2.7}$$

Now we need to consider the convergence of the trilinear term

$$\int_0^T (\boldsymbol{u}^\varepsilon(\boldsymbol{x},t), \boldsymbol{v}^\varepsilon(\boldsymbol{x},t), \boldsymbol{\varphi}(\boldsymbol{x}))\psi(t)dt \tag{4.2.8}$$

in the weak formulation (4.1.7). As we shown in Section 3.3, the convergences (3.3.12)-(3.3.15) also hold for $\boldsymbol{u}^\varepsilon(\boldsymbol{x},t)$ and $(\boldsymbol{x},t) \in Q_T$. The boundedness result (3.2.23) of the trilinear form and the Lebesgue dominated convergence theorem ensure the convergences of the terms in the trilinear form expect the term $b(\nabla \boldsymbol{u}^\varepsilon(\boldsymbol{x},t))^{\mathrm{T}}\boldsymbol{v}^\varepsilon$. For the same reason as in Section 3.3, we have to determine the limit of $b(\nabla \boldsymbol{u}^\varepsilon(\boldsymbol{x},t))^{\mathrm{T}}\boldsymbol{v}^\varepsilon$. In this way, we imitate Tartar approach and construct a homogenized equation of the test function $\boldsymbol{w}^\varepsilon_\lambda$ for $\boldsymbol{\lambda} \in \mathbb{R}^2$

$$\begin{cases} \partial_t(b\boldsymbol{w}^\varepsilon_\lambda) + b\ \mathrm{div}\big((\boldsymbol{v}^\varepsilon - \boldsymbol{v}) \otimes \boldsymbol{\lambda}\big) + b\nabla q^\varepsilon_\lambda = \mathrm{div}(b\Sigma(\boldsymbol{w}^\varepsilon_\lambda)), \\ \mathrm{div}(b\boldsymbol{w}^\varepsilon_\lambda) = 0, \quad \boldsymbol{x} \in \Omega \\ \boldsymbol{w}^\varepsilon_\lambda(x,0) = \boldsymbol{w}^\varepsilon_{\lambda 0}(x), \quad \boldsymbol{x} \in \Omega \\ \boldsymbol{w}^\varepsilon_\lambda \cdot \boldsymbol{n} = 0, \quad \boldsymbol{x} \in \partial\Omega. \end{cases} \tag{4.2.9}$$

Moreover, we assume that

$$\boldsymbol{w}^\varepsilon_{\lambda 0} \overset{\mathrm{w}}{\rightharpoonup} \boldsymbol{w}_{\lambda 0} = 0 \quad \text{weakly in} \quad L^2(\Omega) \qquad \text{as} \quad \varepsilon \to 0\,. \tag{4.2.10}$$

First, we give the energy estimate of (4.2.9) as

$$\begin{aligned} &\|\boldsymbol{w}^\varepsilon_\lambda(T)\|^2_{L^2_b(\Omega)} + C\int_0^T \|\boldsymbol{w}^\varepsilon_\lambda(\tau)\|^2_{H^1_b(\Omega)} dt \\ &\qquad\qquad \leqslant \|\boldsymbol{w}^\varepsilon_{\lambda 0}\|^2_{L^2(\Omega)} + \int_{Q_T} \boldsymbol{F}^\varepsilon \cdot \boldsymbol{w}^\varepsilon_\lambda b(\boldsymbol{x}) d\boldsymbol{x} dt \end{aligned} \tag{4.2.11}$$

where $\boldsymbol{F}^\varepsilon = -\mathrm{div}\big((\boldsymbol{v}^\varepsilon - \boldsymbol{v}) \otimes \boldsymbol{\lambda}\big)$. Since $\boldsymbol{v}^\varepsilon \overset{\mathrm{w}}{\rightharpoonup} \boldsymbol{v}$ weakly in $L^\infty(Q_T)$ and $L^\infty(Q_T) \subset L^r(Q_T)$ for $1 \leqslant r < \infty$, then

$$\boldsymbol{v}^\varepsilon \overset{\mathrm{w}}{\rightharpoonup} \boldsymbol{v} \quad \text{weakly in} \quad L^r(Q_T)\,.$$

This shows that for bounded smooth domain Ω,

$$\boldsymbol{F}^\varepsilon = -\mathrm{div}\big((\boldsymbol{v}^\varepsilon - \boldsymbol{v}) \otimes \boldsymbol{\lambda}\big) \overset{\mathrm{w}}{\rightharpoonup} 0 \quad \text{as} \quad \varepsilon \to 0\,,$$

or

$$\int_{Q_T} \boldsymbol{F}^\varepsilon \cdot \boldsymbol{w}_\lambda^\varepsilon b(\boldsymbol{x}) d\boldsymbol{x} dt \to 0 \quad \text{as} \quad \varepsilon \to 0\,. \tag{4.2.12}$$

Then we obtain

$$\begin{aligned} &\|\boldsymbol{w}_\lambda^\varepsilon\|_{L^\infty([0,T);\mathbb{H}_b(\Omega))} + \|\boldsymbol{w}_\lambda^\varepsilon\|_{L^2([0,T);\mathbb{V}_b(\Omega))} \\ &\qquad\qquad \leqslant \|\boldsymbol{w}_{\lambda 0}^\varepsilon\|_{L^2(\Omega)}^2 + \int_{Q_T} \boldsymbol{F}^\varepsilon \cdot \boldsymbol{w}_\lambda^\varepsilon b(\boldsymbol{x}) d\boldsymbol{x} dt \end{aligned} \tag{4.2.13}$$

Therefore,

$$\{\boldsymbol{w}_\lambda^\varepsilon\} \quad \text{is bounded in} \quad L^\infty([0,T);\mathbb{H}_b(\Omega)) \cap L^2([0,T);\mathbb{V}_b(\Omega)). \tag{4.2.14}$$

Using the same method as Step 2 and 3 in Section 4.1, we can assert that the sequence $\{\boldsymbol{w}_\lambda^\varepsilon\}_\varepsilon$ is a relatively compact set in

$$C([0,T);\text{w-}\mathbb{H}_b(\Omega)) \cap \text{w-}L^2_{loc}([0,T);\mathbb{V}_b(\Omega))$$

and in strong topology of $L^2_{loc}([0,T);\mathbb{H}_b(\Omega))$. Hence there exists a subsequence of $\{\boldsymbol{w}_\lambda^\varepsilon\}_\varepsilon$, which we still denote by $\{\boldsymbol{w}_\lambda^\varepsilon\}_\varepsilon$, and $\boldsymbol{w}_\lambda \in C\big([0,T);\mathbb{H}_b(\Omega)\big)$ such that

$$\begin{cases} \boldsymbol{w}_\lambda^\varepsilon \to 0 \quad \text{in} \quad C\big([0,T);\text{w-}\mathbb{H}_b(\Omega)\big) \\ \boldsymbol{w}_\lambda^\varepsilon \overset{\text{w}}{\rightharpoonup} 0 \quad \text{in} \quad \text{w-}L^2_{loc}\big([0,T);\text{w-}\mathbb{V}_b(\Omega)\big) \end{cases} \tag{4.2.15}$$

as $\varepsilon \to 0$. Now back to the equation (4.2.9), using the convergence of $\boldsymbol{w}_\lambda^\varepsilon$, same as convergences (4.2.4) we obtain for $\varepsilon \to 0$

$$\begin{gathered} \int_{Q_T} \boldsymbol{w}_\lambda^\varepsilon(\boldsymbol{x},t)\psi'(t)\boldsymbol{\varphi}(\boldsymbol{x})b(\boldsymbol{x})d\boldsymbol{x}dt \to \int_{Q_T} \boldsymbol{w}_\lambda(\boldsymbol{x},t)\psi'(t)\boldsymbol{\varphi}(\boldsymbol{x})b(\boldsymbol{x})d\boldsymbol{x}dt \\ \int_{Q_T} \operatorname{div}(b\Sigma(\boldsymbol{w}_\lambda^\varepsilon)) \cdot \boldsymbol{\varphi} d\boldsymbol{x} \to \int_{Q_T} \operatorname{div}(b\Sigma(\boldsymbol{w}\lambda)) \cdot \boldsymbol{\varphi} d\boldsymbol{x} \\ \langle \boldsymbol{w}_{\lambda 0}^\varepsilon(\boldsymbol{x}), \boldsymbol{\varphi}(\boldsymbol{x})\rangle_b \to \langle \boldsymbol{w}_{\lambda 0}(\boldsymbol{x}), \boldsymbol{\varphi}(\boldsymbol{x})\rangle_b. \end{gathered} \tag{4.2.16}$$

From the equation (4.2.9), q^ε_λ satisfies the following equation

$$b\nabla q^\varepsilon_\lambda = -\partial_t(b\boldsymbol{w}^\varepsilon_\lambda) + \mathrm{div}(b\Sigma(\boldsymbol{w}^\varepsilon_\lambda)) - b\ \mathrm{div}\big((\boldsymbol{v}^\varepsilon - \boldsymbol{v}) \otimes \boldsymbol{\lambda}\big), \tag{4.2.17}$$

then by the convergences (4.2.12) and (4.2.16), we have

$$b\nabla q^\varepsilon_\lambda = -\partial_t(b\boldsymbol{w}^\varepsilon_\lambda) + \mathrm{div}(b\Sigma(\boldsymbol{w}^\varepsilon_\lambda)) - b\ \mathrm{div}\big((\boldsymbol{v}^\varepsilon - \boldsymbol{v}) \otimes \boldsymbol{\lambda}\big) \xrightharpoonup{\mathrm{w}} 0. \tag{4.2.18}$$

Due to the regularity theorem of the elliptic equations,

$$q^\varepsilon_\lambda \xrightharpoonup{\mathrm{w}} 0 \quad \text{weakly} * \text{ in} \quad L^\infty([0,T]; H^1_b(\Omega)). \tag{4.2.19}$$

The weak formulation of (4.1.1) is

$$\begin{aligned}
\int_{Q_T} \Sigma(\boldsymbol{u}^\varepsilon) :&\Sigma(\boldsymbol{w}^\varepsilon_\lambda)\varphi(\boldsymbol{x})\psi(t)b(\boldsymbol{x})d\boldsymbol{x}dt \\
&- 2\int_{Q_T} (\boldsymbol{u}^\varepsilon \otimes \boldsymbol{v}^\varepsilon) : \nabla \boldsymbol{w}^\varepsilon_\lambda \varphi(\boldsymbol{x})\psi(t)b(\boldsymbol{x})d\boldsymbol{x}dt \\
= &- \int_{Q_T} \boldsymbol{u}^\varepsilon(\boldsymbol{x},t)\psi'(t)\varphi\boldsymbol{w}^\varepsilon_\lambda b(\boldsymbol{x})d\boldsymbol{x}dt \\
&- \frac{1}{2}\int_{Q_T} \Sigma(\boldsymbol{u}^\varepsilon) : \Sigma(\boldsymbol{w}^\varepsilon_\lambda \nabla\varphi\, I)\psi(t)b(\boldsymbol{x})d\boldsymbol{x}dt \\
&- 2\int_{S_T} \kappa(\boldsymbol{u}^\varepsilon \cdot \boldsymbol{\tau})(\boldsymbol{w}^\varepsilon_\lambda \cdot \boldsymbol{\tau})\varphi(\boldsymbol{x})\psi(t)b(\boldsymbol{x})dsdt \\
&+ \frac{1}{2}\int_{Q_T} (\boldsymbol{u}^\varepsilon \cdot \boldsymbol{v}^\varepsilon)\nabla \log b \cdot \boldsymbol{w}^\varepsilon_\lambda \varphi\psi(t)b(\boldsymbol{x})d\boldsymbol{x}dt \\
&- \int_{Q_T} (\nabla \boldsymbol{u}^\varepsilon)\boldsymbol{v}^\varepsilon \cdot \boldsymbol{w}^\varepsilon_\lambda \varphi\psi b(\boldsymbol{x})d\boldsymbol{x}dt \\
&- \int_{Q_T} (\nabla \boldsymbol{v}^\varepsilon)^{\mathrm{T}}\boldsymbol{u}^\varepsilon \cdot \boldsymbol{w}^\varepsilon_\lambda \varphi\psi b(\boldsymbol{x})d\boldsymbol{x}dt + \int_{Q_T} \boldsymbol{f}\boldsymbol{w}^\varepsilon_\lambda \varphi\psi b(\boldsymbol{x})d\boldsymbol{x}dt \\
&+ \langle \boldsymbol{u}^\varepsilon_0(\boldsymbol{x}), \varphi\boldsymbol{w}^\varepsilon_\lambda \rangle_b\,.
\end{aligned} \tag{4.2.20}$$

Here we use the identity

$$\operatorname{div}(b\varphi \boldsymbol{w}_\lambda^\varepsilon) = \varphi \operatorname{div}(b\boldsymbol{w}_\lambda^\varepsilon) + b\boldsymbol{w}_\lambda^\varepsilon \nabla\varphi = 0$$

so we let $\nabla\varphi = 0$. Due to the convergence $\boldsymbol{w}_\lambda^\varepsilon \xrightarrow{\text{w}} 0$ the RHS of the equation (4.2.20) tends to 0, consequently the formulation (4.2.20) can be written as

$$\begin{aligned} &\int_{Q_T} \Sigma(\boldsymbol{u}^\varepsilon) : \Sigma(\boldsymbol{w}_\lambda^\varepsilon)\varphi(\boldsymbol{x})\psi(t)b(\boldsymbol{x})d\boldsymbol{x}dt \\ &- 2\int_{Q_T} (\boldsymbol{u}^\varepsilon \otimes \boldsymbol{v}^\varepsilon) : \nabla \boldsymbol{w}_\lambda^\varepsilon \varphi(\boldsymbol{x})\psi(t)b(\boldsymbol{x})d\boldsymbol{x}dt = o(1). \end{aligned} \tag{4.2.21}$$

In the same way, we write the weak formulation of (4.2.9) as

$$\begin{aligned} &\int_{Q_T} \Sigma(\boldsymbol{w}_\lambda^\varepsilon) : \Sigma(\boldsymbol{u}^\varepsilon)\varphi(\boldsymbol{x})b(\boldsymbol{x})d\boldsymbol{x}dt \\ &- \int_{Q_T} \big((\boldsymbol{v}^\varepsilon - \boldsymbol{v}) \otimes \boldsymbol{\lambda}\big) : \nabla \boldsymbol{u}^\varepsilon \varphi(\boldsymbol{x})b(\boldsymbol{x})d\boldsymbol{x}dt = o(1). \end{aligned} \tag{4.2.22}$$

Recall that

$$(\boldsymbol{u}^\varepsilon \otimes \boldsymbol{v}^\varepsilon) : \nabla \boldsymbol{w}_\lambda^\varepsilon = (\nabla \boldsymbol{w}_\lambda^\varepsilon)^{\mathrm{T}} \boldsymbol{v}^\varepsilon \cdot \boldsymbol{u}^\varepsilon$$

and

$$\big((\boldsymbol{v}^\varepsilon - \boldsymbol{v}) \otimes \boldsymbol{\lambda}\big) : \nabla \boldsymbol{u}^\varepsilon = (\nabla \boldsymbol{u}_\lambda^\varepsilon)^{\mathrm{T}} (\boldsymbol{v}^\varepsilon - \boldsymbol{v}) \cdot \lambda.$$

Therefore, the equations (4.2.21) an (4.2.22) can be rewritten as

$$\begin{aligned} &\int_{Q_T} \Sigma(\boldsymbol{u}^\varepsilon) : \Sigma(\boldsymbol{w}_\lambda^\varepsilon)\varphi(\boldsymbol{x})\psi(t)b(\boldsymbol{x})d\boldsymbol{x}dt \\ &- 2\int_{Q_T} (\nabla \boldsymbol{w}_\lambda^\varepsilon)^{\mathrm{T}} \boldsymbol{v}^\varepsilon \cdot \boldsymbol{u}^\varepsilon \varphi(\boldsymbol{x})\psi(t)b(\boldsymbol{x})d\boldsymbol{x}dt = o(1) \end{aligned} \tag{4.2.23}$$

and

$$\begin{aligned} &\int_{Q_T} \Sigma(\boldsymbol{w}_\lambda^\varepsilon) : \Sigma(\boldsymbol{u}^\varepsilon)\varphi(\boldsymbol{x})b(\boldsymbol{x})d\boldsymbol{x}dt \\ &- \int_{Q_T} (\nabla \boldsymbol{u}_\lambda^\varepsilon)^{\mathrm{T}} (\boldsymbol{v}^\varepsilon - \boldsymbol{v}) \cdot \lambda \varphi(\boldsymbol{x})b(\boldsymbol{x})d\boldsymbol{x}dt = o(1). \end{aligned} \tag{4.2.24}$$

The remaining steps proceeds along the same lines as given in Section 3.3 with modification. We have the following lemma

Lemma 4.2.3. *For $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathbb{R}^2$ and $1 \leqslant r < \infty$, there exists a positive definite symmetric matrix-valued function M such that (up to a subsequence)*

$$\begin{cases} (\nabla \boldsymbol{w}_\lambda^\varepsilon)^{\mathrm{T}} \boldsymbol{v}^\varepsilon \overset{\mathrm{w}}{\rightharpoonup} 0 + M\boldsymbol{\lambda} = M\boldsymbol{\lambda} \quad \textit{weakly in} \quad W_1 \\ \nabla \boldsymbol{w}_\lambda^\varepsilon : \nabla \boldsymbol{w}_\mu^\varepsilon \overset{\mathrm{w}}{\rightharpoonup} M\boldsymbol{\lambda} \cdot \boldsymbol{\mu} \quad \textit{weakly} * \textit{in} \quad \mathcal{M}(Q_T) \quad \textit{and} \quad W_2 \end{cases} \tag{4.2.25}$$

where $W_1 = L_{loc}^{\frac{2r}{2+r}}\big([0,T); L_b^{\frac{2r}{2+r}}(\Omega)\big)$ and $W_2 = L_{loc}^1\big([0,T); L_b^1(\Omega)\big)$. Moreover, since $\boldsymbol{u}^\varepsilon \to \boldsymbol{u}$ in $C\left([0,T); \mathrm{w}\text{-}\mathbb{H}_{\mathrm{b}}(\Omega)\right)$, we have

$$(\nabla \boldsymbol{w}_\lambda^\varepsilon)^{\mathrm{T}} \boldsymbol{v}^\varepsilon \cdot \boldsymbol{u}^\varepsilon \overset{\mathrm{w}}{\rightharpoonup} M\boldsymbol{u} \cdot \boldsymbol{\lambda} \quad \textit{weakly} * \textit{in} \quad \mathcal{M}(Q_T) \tag{4.2.26}$$

and weakly in $L_{loc}^{\frac{2r}{2+r}}\big([0,T); L_b^{\frac{r}{1+r}}(\Omega)\big)$.

Taking the difference of (4.2.23) and (4.2.24), we derive that

$$(\nabla \boldsymbol{u}^\varepsilon)^{\mathrm{T}}(\boldsymbol{v}^\varepsilon - \boldsymbol{v}) \cdot \lambda \overset{\mathrm{w}}{\rightharpoonup} 2(\nabla \boldsymbol{w}_\lambda^\varepsilon)^{\mathrm{T}} \boldsymbol{v}^\varepsilon \cdot \boldsymbol{u}^\varepsilon.$$

From (4.2.26),

$$(\nabla \boldsymbol{u}^\varepsilon)^{\mathrm{T}}(\boldsymbol{v}^\varepsilon - \boldsymbol{v}) \cdot \lambda \overset{\mathrm{w}}{\rightharpoonup} 2M\boldsymbol{u} \cdot \lambda,$$

thus

$$(\nabla \boldsymbol{u}^\varepsilon)^{\mathrm{T}}(\boldsymbol{v}^\varepsilon - \boldsymbol{v}) \overset{\mathrm{w}}{\rightharpoonup} 2M\boldsymbol{u} \quad \text{in} \quad \mathrm{w}\text{-}L_{loc}^{\frac{2r}{2+r}}\big([0,T); \mathrm{w}\text{-}L_b^1(\Omega)\big). \tag{4.2.27}$$

Consequently,

$$(\nabla \boldsymbol{u}^\varepsilon)^{\mathrm{T}} \boldsymbol{v}^\varepsilon = (\nabla \boldsymbol{u}^\varepsilon)^{\mathrm{T}}(\boldsymbol{v}^\varepsilon - \boldsymbol{v}) + (\nabla \boldsymbol{u}^\varepsilon)^{\mathrm{T}} \boldsymbol{v} \overset{\mathrm{w}}{\rightharpoonup} 2M\boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathrm{T}} \boldsymbol{v}. \tag{4.2.28}$$

Moreover, for the Navier boundary conditions, due to the linearity and the weak compactness of u^ε we can write the limit of the Navier boundary conditions

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0, \qquad \boldsymbol{\tau} \cdot \left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathrm{T}}\right) \cdot \nabla = -\beta \boldsymbol{\tau} \cdot \boldsymbol{u}, \qquad \boldsymbol{x} \in \partial\Omega\,.$$

In consequence, the homogenization of the viscous lake equation can be represented as:

Theorem 4.2.4. *Let* $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathbb{R}^2$. *The solution* $\boldsymbol{u}^\varepsilon$ *of* (4.1.1) *converges weakly to the solution* $\boldsymbol{u}$ *of the Brinkman equation*

$$\begin{cases} \partial_t(b\boldsymbol{u}) - \dfrac{1}{2}(\boldsymbol{u}\cdot\boldsymbol{v})\nabla b + b\,\mathrm{curl}(\boldsymbol{v})J\boldsymbol{u} + b\nabla p + 2M\boldsymbol{u} = \mathrm{div}(b\Sigma(\boldsymbol{u})) + b\boldsymbol{f} \\ \mathrm{div}(b\boldsymbol{u}) = 0, \qquad\qquad (\boldsymbol{x}, t) \in Q_T \\ \boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Omega \end{cases}$$

with the Navier boundary conditions

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0, \qquad \boldsymbol{\tau}\cdot\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^{\mathrm{T}}\right)\cdot\nabla = -\beta\boldsymbol{\tau}\cdot\boldsymbol{u}, \qquad (\boldsymbol{\sigma}, t) \in S_T\,,$$

where M *is the positive definite symmetric matrix-valued function defined by*

$$\begin{cases} (\nabla\boldsymbol{w}_\lambda^\varepsilon)^{\mathrm{T}}\boldsymbol{v}^\varepsilon \overset{\mathrm{w}}{\rightharpoonup} 0\cdot\boldsymbol{v} + M\boldsymbol{\lambda} = M\boldsymbol{\lambda} \quad \textit{weakly in } W_1 \\ \nabla\boldsymbol{w}_\lambda^\varepsilon : \nabla\boldsymbol{w}_\mu^\varepsilon \overset{\mathrm{w}}{\rightharpoonup} M\boldsymbol{\lambda}\cdot\boldsymbol{\mu} \quad \textit{weakly} * \textit{in} \quad \mathcal{M}(Q_T) \textit{ and } W_2 \end{cases}$$

where W_1, W_2 *are defined in Lemma 4.2.3.*

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