Non-stationary response of nonlinear systems with singular parameter matrices subject to combined deterministic and stochastic excitation

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Abstract

A new technique is proposed for determining the response of multi-degree-of-freedom nonlinear systems with singular parameter matrices subject to combined deterministic and non-stationary stochastic excitation. Singular matrices in the governing equations of motion potentially account for the presence of constraints equations in the system. Further, they also appear when a redundant coordinates modeling is adopted to derive the equations of motion of complex multi-body systems. In this regard, the system response is decomposed into a deterministic and a stochastic component corresponding to the two components of the excitation. Then, two sets of differential equations are formulated and solved simultaneously to compute the system response. The first set pertains to the deterministic response component, whereas the second one pertains to the stochastic component of the response. The latter is derived by utilizing the generalized statistical linearization method for systems with singular matrices, while a formula for determining the time-dependent equivalent elements of the generalized statistical linearization methodology is also derived. The efficiency of the proposed technique is demonstrated by pertinent numerical examples. Specifically, a vibration energy

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harvesting device subject to combined deterministic and modulated white noise excitation and a structural nonlinear system with singular parameter matrices subject to combined deterministic and modulated white and colored noise excitations are considered.

1 Introduction

Assessing the reliability of nonlinear multi-degree-of-freedom (MDOF) systems subject to combined deterministic and stochastic loading constitutes a persistent challenge in random vibration, which finds a plethora of applications in several engineering fields. Indicatively, these span from vibration energy harvesting (e.g., [1, 2, 3]) to the problem of turbine blades vibration under turbulent flow (e.g., [4, 5]), or nonlinear vibration of beams and plates (e.g., [6]), and vibration of gear systems (e.g., [7]).

In this context, considerable research effort has been put over the last decades into developing methodologies and techniques aiming at determining the response of nonlinear MDOF systems subject to combined deterministic and stochastic excitation. This has been done by utilizing and combining standard deterministic and stochastic analysis tools such as, indicatively, the harmonic balance and statistical linearization or Gaussian closure methods (e.g., [8, 9, 10, 7, 11, 12]), the harmonic balance and stochastic averaging methods (e.g., [13]), and the equivalent linearization and deterministic or stochastic averaging methods (e.g., [14, 15]). Further, the need for more accurate media behavior modeling dictated by recent advances in theoretical and applied mechanics (e.g., [16]) has propelled the use of fractional calculus which, in turn, resulted to the development of pertinent frameworks (e.g., [6, 17]). Yet, most of the approaches available in the literature to-date treat systems whose stochastic excitation component is modeled as a stationary stochastic process. However, a more accurate modeling of the applied stochastic excitation component necessitates considering the non-stationary characteristics corresponding to excitations often met in nature, such as wave, wind and earthquake loads. This has recently led to the extension of relevant tools, and approaches accounting for non-stationary stochastic excitations (e.g., [18]) and non-stationary excitations described by evolutionary power spectrum forms (e.g., [19]) have been proposed.

An additional aspect of the response determination problem for MDOF systems subject to combined deterministic and stochastic excitation relates to the complexity of the system under consideration. In this regard, a technique accounting for singular parameter matrices and constraints in the equations governing the dynamics of the MDOF system has been recently developed in [20]. Examples of such systems are often met in engineering applications including, indicatively, systems with massless joints (e.g., [21, 22]), oscillators modeled via additional auxiliary state equations (e.g., [23]), energy harvesting devices (e.g., [24]) and specific classes of non-viscously damped systems (e.g., [25]). Hence, utilizing tools from the theory of generalized matrix inverses (e.g., [26]) has led to the extension of known input-output (excitation-response) expressions in random vibration theory, and subsequently, to the development of various frameworks for determining the response of MDOF linear and nonlinear systems (e.g., [27, 28, 29, 30, 31]), conducting joint time-frequency analysis of the system response

(e.g., [32, 33]), or solving random eigenvalue problems for systems with singular random parameter matrices [34].

In this paper, the technique developed in [20] is extended to MDOF nonlinear systems with singular parameter matrices subject to combined deterministic and non-stationary stochastic excitation. This is done by formulating and solving simultaneously two sets of differential equations, corresponding to the deterministic and the stochastic components of the response, respectively. An additional contribution relates to the generalization of the expression derived in [28] to determine the time-dependent equivalent elements of the generalized statistical linearization methodology for systems with singular parameter matrices. Three numerical examples are considered to assess the reliability of the proposed technique. These include a vibration energy harvesting device subject to combined deterministic and modulated white noise excitation and a structural nonlinear system with singular parameter matrices subject to combined deterministic and modulated white and colored noise excitations. The obtained results are compared with pertinent Monte Carlo simulation (MCS) data as well as with corresponding results obtained by the approach proposed in [18].

2 Mathematical formulation

2.1 Nonlinear MDOF systems with singular parameter matrices

The governing equations of motion of an l-DOF nonlinear system subject to combined deterministic and non-stationary stochastic excitation are given by

$$\mathbf{M_x}\ddot{\mathbf{x}} + \mathbf{C_x}\dot{\mathbf{x}} + \mathbf{K_x}\mathbf{x} + \mathbf{\Phi_x}(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}) = \mathbf{f}_{d,\mathbf{x}}(t) + \mathbf{Q_x}(t), \tag{1}$$

where $\mathbf{x}$ denotes the (possibly dependent) l-dimensional response displacement vector and $\dot{\mathbf{x}}$, $\ddot{\mathbf{x}}$ are the response velocity and acceleration l-dimensional vectors, respectively. Further, $\mathbf{M_x}$, $\mathbf{C_x}$ and $\mathbf{K_x}$ correspond to the $l \times l$ mass, damping and stiffness matrices of the system, whereas $\mathbf{\Phi_x}(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}})$ denotes the l-dimensional nonlinear vector of the system. Lastly, $\mathbf{f}_{d,\mathbf{x}}(t)$ and $\mathbf{Q_x}(t)$ are the l-dimensional vectors of the deterministic and the zero-mean non-stationary stochastic excitation, respectively. It is noted that considering a zero-mean excitation is rather for simplicity and not restrictive for the ensuing analysis, which can be generalized also to the case of a nonzero-mean process (e.g., [35]).

Considering, next, that the l-DOF system of Eq. (1) is subject to additional constraints [36, 27]

$$\mathbf{A}(\mathbf{x}, \dot{\mathbf{x}}, t)\ddot{\mathbf{x}} = \mathbf{b}(\mathbf{x}, \dot{\mathbf{x}}, t), \tag{2}$$

Eq. (1) is recast into

$$\bar{\mathbf{M}}_\mathbf{x}\ddot{\mathbf{x}} + \bar{\mathbf{C}}_\mathbf{x}\dot{\mathbf{x}} + \bar{\mathbf{K}}_\mathbf{x}\mathbf{x} + \bar{\mathbf{\Phi}}_\mathbf{x}(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}) = \bar{\mathbf{f}}_{d,\mathbf{x}}(t) + \bar{\mathbf{Q}}_\mathbf{x}(t). \tag{3}$$

In Eq. (3), $\bar{\mathbf{M}}_\mathbf{x}, \bar{\mathbf{C}}_\mathbf{x}$ and $\bar{\mathbf{K}}_\mathbf{x}$ denote the augmented $(l + m) \times l$ mass, damping and stiffness matrices of the system given by [31, 33]

$$\bar{\mathbf{M}}_\mathbf{x} = \begin{bmatrix} \mathbf{J}\mathbf{M_x} \\ \mathbf{A} \end{bmatrix}, \ \bar{\mathbf{C}}_\mathbf{x} = \begin{bmatrix} \mathbf{J}\mathbf{C_x} \\ \mathbf{E} \end{bmatrix}, \ \bar{\mathbf{K}}_\mathbf{x} = \begin{bmatrix} \mathbf{J}\mathbf{K_x} \\ \mathbf{L} \end{bmatrix}, \tag{4}$$

the augmented $(l+m)$-dimensional nonlinearity vector has the form

$$\bar{\mathbf{\Phi}}_{\mathbf{x}} = \begin{bmatrix} \mathbf{J}\mathbf{\Phi}_{\mathbf{x}} \\ \mathbf{0}_{m\times 1} \end{bmatrix}, \tag{5}$$

whereas the augmented $(l+m)$-dimensional vectors of the applied deterministic and stochastic excitations are given by

$$\bar{\mathbf{f}}_{d,\mathbf{x}}(t) = \begin{bmatrix} \mathbf{J}\mathbf{f}_{d,\mathbf{x}}(t) \\ \mathbf{0}_{m\times 1} \end{bmatrix} \tag{6}$$

and

$$\bar{\mathbf{Q}}_{\mathbf{x}}(t) = \begin{bmatrix} \mathbf{J}\mathbf{Q}_{\mathbf{x}}(t) \\ \mathbf{F} \end{bmatrix}, \tag{7}$$

respectively. For the derivation of the system parameter matrices in Eq. (4), as well as the excitation vectors in Eq. (7), the system constraints Eq. (2) is written, for simplicity, in the form $\mathbf{A}\ddot{\mathbf{x}}+\mathbf{E}\dot{\mathbf{x}}+\mathbf{L}\mathbf{x}=\mathbf{F}$, with $\mathbf{A}$, $\mathbf{E}$, $\mathbf{L}$ denoting $m\times l$ matrices and $\mathbf{F}$ denoting an m-dimensional vector (e.g., [27]). Further, $\mathbf{J}$ represents an $l\times l$ matrix connecting the constraints Eq. (2) with the system governing equations of motion Eq. (1) (e.g., [30, 33]). A detailed derivation of Eqs. (3-7) can be found in [27, 29, 20].

2.2 System response determination

In this section, a semi-analytical technique is proposed for determining the response of MDOF systems with singular parameter matrices subject to combined deterministic and non-stationary stochastic excitation. This is attained by decomposing the nonlinear system into two subsystems, i.e., one subject to the non-stationary stochastic excitation and one subject to the deterministic excitation. The former is simplified by resorting to the generalized statistical linearization method for systems with singular parameter matrices [28, 29], followed by a state variable treatment. This involves the formulation of a time-dependent matrix differential equation, whose solution yields the standard deviation of the stochastic component of the response. Further, a set of deterministic differential equations corresponding to the subsystem subject to the deterministic excitation, and thus, governing the deterministic response component, is derived and solved simultaneously with the matrix differential equation above. This can be done by resorting to any standard numerical scheme, such as the Runge–Kutta method.

2.2.1 Generalized statistical linearization based framework

Consider the augmented system in Eq. (3) which is subject to combined deterministic and non-stationary stochastic excitation. The system response is decomposed into two components, namely the stochastic and the deterministic one, accounting, respectively, for the corresponding components of the excitation. That is

$$\mathbf{x}(t) = \mathbf{x}_s(t) + \mathbf{x}_d(t), \tag{8}$$

where the l-dimensional vector $\mathbf{x}_s(t)$ denotes the zero-mean stochastic component of the response and the l-dimensional vector $\mathbf{x}_d(t)$ represents the deterministic response

component. Then, taking into account that the stochastic displacement component is modeled as a zero-mean process, substituting Eq. (8) into Eq. (3) and ensemble averaging yields

$$\bar{\mathbf{M}}_{\mathbf{x}}\ddot{\mathbf{x}}_d + \bar{\mathbf{C}}_{\mathbf{x}}\dot{\mathbf{x}}_d + \bar{\mathbf{K}}_{\mathbf{x}}\mathbf{x}_d + \mathrm{E}\left[\bar{\mathbf{\Phi}}_{\mathbf{x}}(\mathbf{x}_s + \mathbf{x}_d, \dot{\mathbf{x}}_s + \dot{\mathbf{x}}_d, \ddot{\mathbf{x}}_s + \ddot{\mathbf{x}}_d)\right] = \bar{\mathbf{f}}_{d,\mathbf{x}}(t), \quad (9)$$

where $\mathrm{E}[\cdot]$ denotes the expectation operator. Eq. (9) constitutes a subsystem of deterministic differential equations to be solved for computing the deterministic response of the system. Then, subtracting Eq. (9) from Eq. (3) yields a subsystem of equations subject to non-stationary stochastic excitation, namely

$$\bar{\mathbf{M}}_{\mathbf{x}}\ddot{\mathbf{x}}_s + \bar{\mathbf{C}}_{\mathbf{x}}\dot{\mathbf{x}}_s + \bar{\mathbf{K}}_{\mathbf{x}}\mathbf{x}_s + \tilde{\mathbf{\Phi}}_{\mathbf{x}} = \bar{\mathbf{Q}}_{\mathbf{x}}(t), \quad (10)$$

where

$$\tilde{\mathbf{\Phi}}_{\mathbf{x}} = \bar{\mathbf{\Phi}}_{\mathbf{x}}(\mathbf{x}_s + \mathbf{x}_d, \dot{\mathbf{x}}_s + \dot{\mathbf{x}}_d, \ddot{\mathbf{x}}_s + \ddot{\mathbf{x}}_d) - \mathrm{E}\left[\bar{\mathbf{\Phi}}_{\mathbf{x}}(\mathbf{x}_s + \mathbf{x}_d, \dot{\mathbf{x}}_s + \dot{\mathbf{x}}_d, \ddot{\mathbf{x}}_s + \ddot{\mathbf{x}}_d)\right]. \quad (11)$$

Clearly, the nonlinear terms in Eqs. (9) and (10) consist of a deterministic and a stochastic response components, which are intertwined. Therefore, Eqs. (9) and (10) constitute a coupled set of differential equations to be solved for determining the system response.

In this regard, the generalized statistical linearization method is applied and a linear system equivalent to the subsystem of Eq. (10) is defined as

$$\left(\bar{\mathbf{M}}_{\mathbf{x}} + \bar{\mathbf{M}}_e(t)\right)\ddot{\mathbf{x}}_s + \left(\bar{\mathbf{C}}_{\mathbf{x}} + \bar{\mathbf{C}}_e(t)\right)\dot{\mathbf{x}}_s + \left(\bar{\mathbf{K}}_{\mathbf{x}} + \bar{\mathbf{K}}_e(t)\right)\mathbf{x}_s = \bar{\mathbf{Q}}_{\mathbf{x}}(t), \quad (12)$$

where $\bar{\mathbf{M}}_e(t)$, $\bar{\mathbf{C}}_e(t)$ and $\bar{\mathbf{K}}_e(t)$ are the time-varying $(l+m)\times l$ mass, damping and stiffness matrices of the equivalent linear system. Then, the error function is defined as the difference between the nonlinear system in Eq. (10) and the equivalent linear system in Eq. (12), and it is minimized by adopting a mean square minimization criterion in conjunction with the Gaussian response assumption [35].

Clearly, one of the advantages of the standard statistical linearization method relates to its capacity to provide closed-form expressions for determining the equivalent linear elements of Eq. (12). In this context, consider that $\mathbf{m}_{i*}^{e\mathrm{T}}(t)$, $\mathbf{c}_{i*}^{e\mathrm{T}}(t)$ and $\mathbf{k}_{i*}^{e\mathrm{T}}(t)$ for $i = 1, 2, \ldots, l+m$ denote the i-th row of the $(l+m)\times l$ time-varying matrices $\bar{\mathbf{M}}_e(t)$, $\bar{\mathbf{C}}_e(t)$ and $\bar{\mathbf{K}}_e(t)$, respectively, and that $\hat{\mathbf{x}} = \begin{bmatrix}\mathbf{x}_s & \dot{\mathbf{x}}_s & \ddot{\mathbf{x}}_s\end{bmatrix}^{\mathrm{T}}$ is a $3l$-dimensional vector with "T" denoting the matrix transpose operation. A key aspect in determining $\mathbf{m}_{i*}^{e\mathrm{T}}(t)$, $\mathbf{c}_{i*}^{e\mathrm{T}}(t)$ and $\mathbf{k}_{i*}^{e\mathrm{T}}(t)$ is that the covariance matrix $\mathrm{E}[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}]$ is invertible [35]. However, due to the possibly dependent coordinates utilized to model the system governing equations of motion in Eq. (1), $\mathrm{E}[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}]$ is singular. Nevertheless, generalized expressions for the equivalent elements have been proposed in [28, 29] for the case where the system is subject to stationary stochastic excitation, as well as in [20] for MDOF systems subject to combined deterministic and stationary stochastic excitation.

In this regard, the equivalent linear elements for systems with singular parameter matrices and subject to deterministic and non-stationary stochastic excitations are given by

$$\begin{bmatrix}\mathbf{k}_{i*}^{e\mathrm{T}}(t)\\ \mathbf{c}_{i*}^{e\mathrm{T}}(t)\\ \mathbf{m}_{i*}^{e\mathrm{T}}(t)\end{bmatrix} = \mathrm{E}[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}]^{+}\mathrm{E}[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}]\mathrm{E}\begin{bmatrix}\frac{\partial\tilde{\mathbf{\Phi}}_{\mathbf{x}}(i)}{\partial\mathbf{x}}\\ \frac{\partial\tilde{\mathbf{\Phi}}_{\mathbf{x}}(i)}{\partial\dot{\mathbf{x}}}\\ \frac{\partial\tilde{\mathbf{\Phi}}_{\mathbf{x}}(i)}{\partial\ddot{\mathbf{x}}}\end{bmatrix}, \quad (13)$$

for $i = 1, 2, \ldots, l + m$, where "+" denotes the Moore-Penrose generalized inverse matrix operation (e.g., [26]). In passing, note that an arbitrary term should also be included in Eq. (13) due to utilizing the generalized inverse matrix theory for its derivation. Thus, Eq. (13) corresponds, in essence, to a family of solutions for the equivalent linear elements rather than a unique expression. Nevertheless, it has been proved in [28] that the solution derived by setting the arbitrary term equal to zero is at least as good as any other solution corresponding to a nonzero value for the arbitrary term. Therefore, Eq. (13) constitutes the counterpart of the expression in [28] used for determining the time-dependent equivalent elements of the generalized statistical linearization methodology for systems with singular parameter matrices. The interested reader is directed to [28, 29] for details on the derivation of Eq. (13); corresponding expressions accounting for joint time-frequency response analysis of nonlinear systems with singular matrices are found in [32].

Finally, it is noted that the response covariance matrix $\mathrm{E}[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}]$ as well as its Moore-Penrose generalized matrix inverse $\mathrm{E}[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}]^{+}$ are required for computing the equivalent linear elements in Eq. (13), and subsequently, for determining the system response. In addition, it is readily seen that the equivalent linear elements are time-dependent, and thus, in contrast to the stationary case [28, 29], a set of differential equations is derived and solved in the ensuing analysis. This is attained by utilizing a state variable formulation, which leads to a matrix differential equation governing the time-variant covariance matrix of the system response.

2.2.2 State variable analysis for MDOF systems with singular parameter matrices

In this section, the state variable formulation developed in [27] for MDOF systems with singular parameter matrices is further extended to treat the linear system with time-dependent equivalent elements in Eq. (12). Ultimately, this leads to a time-varying matrix differential equation to be solved for determining the standard deviation of the non-stationary response component.

In this regard, suppose for simplicity that $\bar{\mathbf{M}}_{\mathbf{x},t} = \bar{\mathbf{M}}_{\mathbf{x}} + \bar{\mathbf{M}}_e(t)$, $\bar{\mathbf{C}}_{\mathbf{x},t} = \bar{\mathbf{C}}_{\mathbf{x}} + \bar{\mathbf{C}}_e(t)$ and $\bar{\mathbf{K}}_{\mathbf{x},t} = \bar{\mathbf{K}}_{\mathbf{x}} + \bar{\mathbf{K}}_e(t)$. Then, taking into account the properties of the generalized matrix inverse theory (e.g., [26]), Eq. (12) yields

$$\ddot{\mathbf{x}} = \bar{\mathbf{M}}_{\mathbf{x},t}^{+}\left(-\bar{\mathbf{C}}_{\mathbf{x},t}\dot{\mathbf{x}} - \bar{\mathbf{K}}_{\mathbf{x},t}\mathbf{x} + \bar{\mathbf{Q}}_{\mathbf{x}}(t)\right) + \left(\mathbf{I} - \bar{\mathbf{M}}_{\mathbf{x},t}^{+}\bar{\mathbf{M}}_{\mathbf{x},t}\right)\mathbf{y}, \tag{14}$$

where $\mathbf{y}$ is an arbitrary l-dimensional vector. Clearly, the presence of $\mathbf{y}$ in Eq. (14) defines a family of equations for the response acceleration. Nevertheless, it is noted that for the special case when the $(l+m)\times l$ matrix $\bar{\mathbf{M}}_{\mathbf{x},t}$ has full rank, i.e., $\mathrm{rank}(\bar{\mathbf{M}}_{\mathbf{x},t}) = l$, its Moore-Penrose generalized matrix inverse simplifies to $\bar{\mathbf{M}}_{\mathbf{x},t}^{+} = \left(\bar{\mathbf{M}}_{\mathbf{x},t}^{*}\bar{\mathbf{M}}_{\mathbf{x},t}\right)^{-1}\bar{\mathbf{M}}_{\mathbf{x},t}^{*}$, where "$*$" denotes the conjugate transpose matrix operation. Substituting the latter into Eq. (14), the arbitrary part becomes zero and Eq. (14) is recast into the state space form

$$\dot{\mathbf{p}} = \bar{\mathbf{G}}_{\mathbf{x}}(t)\mathbf{p} + \mathbf{q}_{\mathbf{x}}, \tag{15}$$

where

$$\bar{\mathbf{G}}_{\mathbf{x}}(t) = \begin{bmatrix} \mathbf{0}_{l\times l} & \mathbf{I}_{l\times l} \\ -\bar{\mathbf{M}}_{\mathbf{x},t}^{+}\bar{\mathbf{K}}_{\mathbf{x},t} & -\bar{\mathbf{M}}_{\mathbf{x},t}^{+}\bar{\mathbf{C}}_{\mathbf{x},t} \end{bmatrix} \tag{16}$$

is a $2l \times 2l$ matrix with time-dependent elements and

$$\mathbf{p} = \begin{bmatrix} \mathbf{x}_s \\ \dot{\mathbf{x}}_s \end{bmatrix}, \ \mathbf{q}_{\mathbf{x}} = \begin{bmatrix} \mathbf{0}_{l\times 1} \\ \bar{\mathbf{M}}^{+}_{\mathbf{x},t}\bar{\mathbf{Q}}_{\mathbf{x}}(t) \end{bmatrix} \tag{17}$$

are $2l$-dimensional vectors. The interested reader is also referred to [22], where a generalized state variable formulation for MDOF systems with fractional derivative terms and singular parameter matrices is introduced.

Next, for an initially at rest system it is assumed that the time-dependent system response vector $\mathbf{p}$ in Eq. (17) is a zero-mean stochastic process. Then, defining the $2l \times 2l$ matrix of the system response variance $\mathbf{V} = \mathrm{E}[\mathbf{p}\mathbf{p}^{\mathrm{T}}]$ and resorting to the standard theory of linear systems (e.g., [37]), the general solution of the state space equation Eq. (15) is derived. It takes the form

$$\dot{\mathbf{V}}(t) = \mathbf{V}\bar{\mathbf{G}}^{\mathrm{T}}_{\mathbf{x}}(t) + \bar{\mathbf{G}}_{\mathbf{x}}(t)\mathbf{V} + \int_0^t \exp\left(\bar{\mathbf{G}}_{\mathbf{x}}(t-\tau)\right)\left(\mathbf{w}(t,\tau) + \mathbf{w}^{\mathrm{T}}(t,\tau)\right)\mathrm{d}\tau, \tag{18}$$

where

$$\mathbf{w}(t,\tau) = \begin{bmatrix} \mathbf{0}_{l\times l} & \mathbf{0}_{l\times l} \\ \mathbf{0}_{l\times l} & \bar{\mathbf{M}}^{+}_{\mathbf{x},t}\mathbf{w}_{\bar{\mathbf{Q}}_{\mathbf{x}}}(t,\tau)(\bar{\mathbf{M}}^{+}_{\mathbf{x},t})^{\mathrm{T}} \end{bmatrix} \tag{19}$$

is the $2l \times 2l$ covariance matrix of the system excitation with $\mathbf{w}_{\bar{\mathbf{Q}}_{\mathbf{x}}}(t,\tau) = \mathrm{E}[\bar{\mathbf{Q}}_{\mathbf{x}}\bar{\mathbf{Q}}^{\mathrm{T}}_{\mathbf{x}}]$ denoting the $(l+m) \times (l+m)$ covariance matrix of $\bar{\mathbf{Q}}_{\mathbf{x}}$.

2.2.3 Solution of the proposed matrix differential equation subject to modulated white noise

In this section, the zero-mean non-stationary excitation in Eq. (1) is modeled as the product of a stationary excitation with a modulated time-function. That is

$$\mathbf{Q}_{\mathbf{x}}(t) = \mathbf{a}(t)\mathbf{Q}_{\mathbf{x},s}(t), \tag{20}$$

where $\mathbf{a}(t)$ is a deterministic $l \times n$ matrix of modulating functions and $\mathbf{Q}_{\mathbf{x},s}(t)$ is an n-dimensional stationary stochastic process. Therefore, the $(l+m) \times (l+m)$ covariance matrix of the excitation in Eq. (19) takes the form

$$\mathbf{w}_{\bar{\mathbf{Q}}_{\mathbf{x}}}(t,\tau) = \begin{bmatrix} \mathbf{J}\mathbf{a}(t)\mathbf{w}_{\mathbf{Q}_{\mathbf{x},s}}(t-\tau)\mathbf{a}^{\mathrm{T}}(t)\mathbf{J}^{\mathrm{T}} & \mathbf{J}\mathbf{a}(t)\mathbf{Q}_{\mathbf{x},s}\mathbf{F}^{\mathrm{T}} \\ \mathbf{F}\mathbf{Q}^{\mathrm{T}}_{\mathbf{x},s}\mathbf{a}^{\mathrm{T}}(t)\mathbf{J}^{\mathrm{T}} & \mathbf{F}\mathbf{F}^{\mathrm{T}} \end{bmatrix}, \tag{21}$$

where $\mathbf{w}_{\mathbf{Q}_{\mathbf{x},s}}(t-\tau) = \mathrm{E}[\mathbf{Q}_{\mathbf{x},s}\mathbf{Q}^{\mathrm{T}}_{\mathbf{x},s}]$. Eq. (21) is further simplified if the stationary excitation $\mathbf{Q}_{\mathbf{x},s}(t)$ in Eq. (20) is modeled as a Gaussian white noise process with $\mathbf{w}_{\mathbf{Q}_{\mathbf{x},s}}(t-\tau) = \delta(t-\tau)\mathbf{S}$, where $\mathbf{S}$ is a real, symmetric and non-negative $n \times n$ matrix of constants, and $\delta(\cdot)$ denotes the Dirac delta function. Thus, taking into account Eq. (21), the matrix differential equation in Eq. (18) becomes

$$\dot{\mathbf{V}}(t) = \mathbf{V}\bar{\mathbf{G}}^{\mathrm{T}}_{\mathbf{x}}(t) + \bar{\mathbf{G}}_{\mathbf{x}}(t)\mathbf{V} + \mathbf{\Theta}(t), \tag{22}$$

where

$$\mathbf{\Theta}(t) = \begin{bmatrix} \mathbf{0}_{l\times l} & \mathbf{0}_{l\times l} \\ \mathbf{0}_{l\times l} & \bar{\mathbf{M}}^{+}_{\mathbf{x},t}\mathbf{w}_{\bar{\mathbf{Q}}_{\mathbf{x}}}(t,\tau)\left(\bar{\mathbf{M}}^{+}_{\mathbf{x},t}\right)^{\mathrm{T}} \end{bmatrix} \tag{23}$$

is a $2l \times 2l$ matrix with

$$\mathbf{w}_{\bar{\mathbf{Q}}_{\mathbf{x}}}(t,\tau) = \begin{bmatrix} \mathbf{J}\mathbf{a}(t)\mathbf{S}\mathbf{a}^{\mathrm{T}}(t)\mathbf{J}^{\mathrm{T}} & \mathbf{J}\mathbf{a}(t)\mathbf{Q}_{\mathbf{x},s}\mathbf{F}^{\mathrm{T}} \\ \mathbf{F}\mathbf{Q}_{\mathbf{x},s}^{\mathrm{T}}\mathbf{a}^{\mathrm{T}}(t)\mathbf{J}^{\mathrm{T}} & \mathbf{F}\mathbf{F}^{\mathrm{T}} \end{bmatrix}. \tag{24}$$

Clearly, in the special case where the system excitation is a stationary process, Eq. (22) degenerates to the standard Lyapunov matrix differential equation governing the covariance matrix of the system response (e.g., [35, 27]).

The matrix differential equation Eq. (22) in conjunction with the generalized equivalent linear elements derived by Eq. (13) constitute a coupled set of equations to be solved for determining the response of the subsystem subject to the non-stationary excitation. The deterministic component of the response is derived by considering Eq. (9), i.e., the subsystem subject to the deterministic excitation. Overall, the differential equations corresponding to the deterministic and stochastic response components are solved simultaneously by resorting to any standard numerical algorithm, such as the Runge-Kutta method.

2.2.4 Solution of the proposed matrix differential equation subject to modulated colored noise

In this section, the non-stationary non-white system excitation is modeled by considering additional auxiliary linear filter equations. In general, linear and nonlinear filters are widely used to model non-white excitation processes in engineering dynamics in various cases, such as the Kanai-Tajimi excitation, or even to provide sufficiently accurate approximations in cases where the excitation power spectrum cannot be represented in the time domain as the response of a filter (e.g., [38, 39, 40]).

In this regard, each one of the nonzero elements of the stationary excitation vector $\mathbf{Q}_{\mathbf{x},s}(t)$ in Eq. (20) are considered as the output of a linear r-order filter equation whose input is a Gaussian white noise process. Specifically, the filter equations are

$$v_{r-1}u^{(r-1)} + v_{r-2}u^{(r-2)} + \cdots + v_0 u^{(0)} = Q_s(t) \tag{25}$$

and

$$u^{(r)} + \lambda_{r-1}u^{(r-1)} + \cdots + \lambda_0 u^{(0)} = w(t), \tag{26}$$

where λ_i and v_i ($i = 0, 1, \ldots, r-1$) denote the filter coefficients, $w(t)$ is a white noise process with constant power spectrum density S_0, and the superscript "(j)" denotes the j-th order derivative ($j = 0, 1, \ldots, r$).

Next, assuming that $\mathbf{v} = \begin{bmatrix} v_0 & v_1 & \cdots & v_{r-1} \end{bmatrix}^{\mathrm{T}}$ is the vector of the filter constants and that $\mathbf{u} = \begin{bmatrix} u^{(0)} & u^{(1)} & \cdots & u^{(r-1)} \end{bmatrix}^{\mathrm{T}}$ represents the pre-filter output, combining Eqs. (7), (20) and (25) yields

$$\bar{\mathbf{D}}_{\mathbf{x}}\mathbf{u} = \bar{\mathbf{Q}}_{\mathbf{x}}(t), \tag{27}$$

where

$$\bar{\mathbf{D}}_{\mathbf{x}} = \begin{bmatrix} \mathbf{0}_{l\times r} \\ \bar{\mathbf{M}}_{\mathbf{x},t}^{+}\bar{\mathbf{P}}a(t)\mathbf{v}^{\mathrm{T}} \end{bmatrix} \tag{28}$$

and

$$\bar{\mathbf{Q}}_{\mathbf{x}} = \begin{bmatrix} \mathbf{0}_{l\times 1} \\ \bar{\mathbf{M}}_{\mathbf{x},t}^{+}\bar{\mathbf{P}}a(t)Q_s(t) \end{bmatrix}. \tag{29}$$

The $(l+m)$-dimensional vector $\bar{\mathbf{P}}$ in Eqs. (28-29) corresponds to the nonzero elements of the excitation $\bar{\mathbf{Q}}_{\mathbf{x}}$ in Eq. (7). Therefore, Eq. (7) is equivalently written as

$$\bar{\mathbf{Q}}_{\mathbf{x}}(t) = a(t)Q_s(t)\bar{\mathbf{P}} \tag{30}$$

with $a(t)$ denoting a time-modulating function and

$$\bar{\mathbf{P}} = \begin{bmatrix} \mathbf{J}\mathbf{I}_{\bar{\mathbf{P}}} \\ (a(t)Q_s(t))^{-1}\,\mathbf{F} \end{bmatrix}. \tag{31}$$

For instance, assuming for simplicity that $\mathbf{Q}_{\mathbf{x}}(t)$ in Eq. (20) contains only a single zero-mean process in its first entry yields $\mathbf{I}_{\bar{\mathbf{P}}} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^{\mathrm{T}}$, and $\mathbf{J}$, $\mathbf{F}$ correspond to the $l \times l$ matrix and m-dimensional vector of Eq. (7), respectively. Further, Eq. (26) is written in the standard state variable form

$$\dot{\mathbf{u}} = \mathbf{\Lambda}\mathbf{u} + \mathbf{w}_s, \tag{32}$$

where

$$\mathbf{\Lambda} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \\ -\lambda_0 & -\lambda_1 & \cdots & -\lambda_{r-1} \end{bmatrix} \tag{33}$$

denotes an $r \times r$ matrix and $\mathbf{w}_s = \begin{bmatrix} 0 & 0 & \cdots & w(t) \end{bmatrix}^{\mathrm{T}}$ is an r-dimensional vector.

Overall, the governing equations of the system under consideration are derived by combining the equations of the original system defined in Eq. (15) and the filter equations Eqs. (25-26). Specifically, considering the new variable $\mathbf{z} = \begin{bmatrix} \mathbf{p}^{\mathrm{T}} & \mathbf{u}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$, the augmented state space system is written as

$$\dot{\mathbf{z}} = \bar{\mathbf{N}}\mathbf{z} + \mathbf{W}, \tag{34}$$

where

$$\bar{\mathbf{N}} = \begin{bmatrix} \bar{\mathbf{G}}_{\mathbf{x}} & \bar{\mathbf{D}}_{\mathbf{x}} \\ \mathbf{0}_{r\times 2l} & \mathbf{\Lambda} \end{bmatrix}, \ \mathbf{W} = \begin{bmatrix} \mathbf{0}_{2l\times 1} \\ \mathbf{w}_s \end{bmatrix}. \tag{35}$$

Finally, denoting by $\mathbf{V} = \mathrm{E}[\mathbf{z}\mathbf{z}^{\mathrm{T}}]$ the response covariance matrix, the matrix differential equation corresponding to Eq. (22) for the case where the system excitation is modeled as modulated colored noise takes the form

$$\dot{\mathbf{V}} = \mathbf{V}\bar{\mathbf{N}}^{\mathrm{T}}(t) + \bar{\mathbf{N}}(t)\mathbf{V} + \mathbf{W}_s, \tag{36}$$

where $\mathbf{W}_s = \mathrm{diag}(0, 0, \ldots, 2\pi S_0)$ is a $(2l + r) \times (2l + r)$ diagonal matrix.

The matrix differential equation Eq. (36) is considered in conjunction with Eq. (13) to determine the stochastic response component. Moreover, similar to the formulation in section 2.2.3, the deterministic component of the response is computed by considering the subsystem subject to the deterministic excitation in Eq. (9). Finally, the Runge-Kutta method is used to solve simultaneously the set of differential equations governing the stochastic and the deterministic response.

2.2.5 Mechanization of the proposed technique

The mechanization of the proposed technique is concisely described by the following steps:

1. Consider Eq. (8) to decompose the system response into deterministic and stochastic parts. Then, form the subsystems of deterministic and stochastic differential equations defined by Eqs. (9) and (10), respectively.

2. Apply the generalized statistical linearization methodology in section 2.2.1 to derive the equivalent linear system in Eq. (12) corresponding to the nonlinear stochastic differential equation Eq. (10). This is done by utilizing Eq. (13) for determining the time-varying equivalent linear elements.

3. Apply the state variable analysis for systems with singular parameter matrices in section 2.2.2. First, construct matrix $\bar{\mathbf{G}}_{\mathbf{x}}$ in Eq. (16). Then,

Case 1: Nonlinear system subject to modulated white noise.
determine matrix $\mathbf{\Theta}$ in Eq. (23), and thus, formulate the matrix differential equation Eq. (22).

Case 2: Nonlinear system subject to modulated colored noise.
determine matrices $\bar{\mathbf{D}}_{\mathbf{x}}$ and $\mathbf{\Lambda}$ in Eqs. (28) and (33), respectively, and thus, construct matrix $\bar{\mathbf{N}}$ in Eq. (35). Next, form the matrix differential equation Eq. (36).

4. Finally, solve simultaneously the matrix differential equation derived in step 3, i.e., Eq. (22) for the white noise excitation, or Eq. (36) for the colored noise excitation, in conjunction with the deterministic differential equation Eq. (9) derived in step 1. This can be done by resorting to any standard numerical algorithm, such as the Runge-Kutta method.

3 Numerical examples

In this section, three numerical examples are used to demonstrate the validity of the proposed technique and assess its reliability. The first one pertains to a nonlinear piezoelectric energy harvesting device subject to combined deterministic and modulated white noise excitation. The technique is applied to determine the response displacement and induced voltage of the device, while a comparison with pertinent MCS data (500 realizations) is used to demonstrate the accuracy of the obtained results. The second example refers to a 2-DOF nonlinear structural system with singular parameter matrices subject to combined deterministic and modulated white noise excitation, whereas in the third example the same system is considered subject to combined deterministic and modulated colored noise excitation. In both cases the results obtained by the proposed technique are compared with corresponding results obtained by the standard approach in [18].

3.1 Nonlinear energy harvesting device subject to combined deterministic and modulated white noise excitation

In this example, the proposed technique is used for determining the response of a typical nonlinear piezoelectric energy harvesting device. Such devices consist of a mechanical part, which is usually a cantilever beam moving as a result of applied excitation and a corresponding piezoelectric part, which is used to transform the mechanical energy into electric current or voltage. They often operate in tandem with large scale infrastructure such as bridges and high-rise buildings (e.g., [41]), which, in turn, are potentially subject to combined deterministic and non-stationary stochastic excitation (e.g., [42]).

The coupled electro-mechanical equations governing the dynamics of the system subject to combined deterministic and non-stationary excitation are given by

$$\ddot{q} + 2\zeta\dot{q} + \frac{dU(q)}{dq} + \kappa^2 y = f_d(t) + Q(t) \tag{37}$$

$$\dot{y} + \alpha y - \dot{q} = 0 \tag{38}$$

where q, $\dot{q}$ and $\ddot{q}$ denote the response displacement, velocity and acceleration of the mechanical part, and y is the induced voltage of a capacitive harvester (e.g., [43, 44, 31]). ζ denotes the damping coefficient of the mechanical part, κ is a coupling coefficient, α is a constant and $U(q)$ represents the potential function. The nonlinear function of the system is given by

$$\frac{dU(q)}{dq} = q + \lambda q^2 + \delta q^3, \tag{39}$$

where λ and δ are coefficients classifying a typical harvesting device into distinctive classes; the interested reader is directed to [45, 44] for a detailed discussion. Further, assume that the deterministic component of the excitation is given by $f_d = f_{d,1}\sin(\omega_d t)$. The non-stationary stochastic excitation component is modeled as a modulated white noise stochastic process $Q(t) = a(t)Q_s$, where $a(t) = A\exp(-\mu t)$ with $A, \mu > 0$ is a time-modulating function and $Q_s(t)$ is a Gaussian white noise process with $\mathrm{E}\left[Q_s(t)Q_s(t+\tau)\right] = 2\pi S_0\delta(\tau)$.

Next, the proposed technique is used to treat the system of Eqs. (37-39). In this regard, considering the coordinates vector $\mathbf{x}(t) = \begin{bmatrix} q(t) & y(t) \end{bmatrix}^{\mathrm{T}}$, Eqs. (37-39) are written in the form of Eq. (1), where

$$\mathbf{M_x} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ \mathbf{C_x} = \begin{bmatrix} 2\zeta & 0 \\ -1 & 1 \end{bmatrix}, \ \mathbf{K_x} = \begin{bmatrix} 1 & \kappa^2 \\ 0 & \alpha \end{bmatrix}, \tag{40}$$

$$\mathbf{\Phi_x} = \begin{bmatrix} \lambda q^2 + \delta q^3 \\ 0 \end{bmatrix} \tag{41}$$

and

$$\mathbf{f}_{d,\mathbf{x}} = \begin{bmatrix} f_d(t) \\ 0 \end{bmatrix}, \ \mathbf{Q_x} = \begin{bmatrix} Q(t) \\ 0 \end{bmatrix}. \tag{42}$$

Clearly, the matrix $\mathbf{M_x}$ in Eq. (40) is singular, and thus, a direct treatment of the system of Eqs. (37-39) is not possible. However, in the ensuing analysis a solution is derived

in a direct manner by resorting to the generalized matrix inverse theory. Specifically, considering that Eq. (38) constitutes the constraints equation of the harvesting device (e.g., [44, 31, 33]) and differentiating it once with respect to time, Eq. (2) is formulated, where

$$\mathbf{A} = \begin{bmatrix} -1 & 1 \end{bmatrix}, \ \mathbf{E} = \begin{bmatrix} 0 & \alpha \end{bmatrix}, \ \mathbf{L} = \begin{bmatrix} 0 & 0 \end{bmatrix} \tag{43}$$

and

$$F = 0. \tag{44}$$

Further, the $l \times l$ matrix $\mathbf{J}$ in Eqs. (4) and (5) interconnecting the constraints to the system governing equations takes the form

$$\mathbf{J} = \mathbf{I}_l - \mathbf{A}^{+}\mathbf{A}, \tag{45}$$

where $\mathbf{I}_l$ denotes the $l \times l$ identity matrix. The interested reader is directed to indicative Refs. [27, 22, 31] for more details. Therefore, the system of Eqs. (37-39) is equivalently written in the form of Eq. (3), where

$$\bar{\mathbf{M}}_{\mathbf{x}} = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0 \\ -1 & 1 \end{bmatrix}, \ \bar{\mathbf{C}}_{\mathbf{x}} = \begin{bmatrix} -0.5\alpha & 0.5 \\ -0.5\alpha & 0.5 \\ 0 & \alpha \end{bmatrix}, \ \bar{\mathbf{K}}_{\mathbf{x}} = \begin{bmatrix} 0.5 & 0.5\kappa^2 + \alpha \\ 0.5 & 0.5\kappa^2 + \alpha \\ 0 & 0 \end{bmatrix}, \tag{46}$$

$$\bar{\mathbf{\Phi}}_{\mathbf{x}}(\mathbf{x}) = \left(\lambda q^2 + \delta q^3\right) \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix} \tag{47}$$

and

$$\bar{\mathbf{f}}_{d,\mathbf{x}} = f_{d,1} \sin\left(\omega_d t\right) \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix}, \ \bar{\mathbf{Q}}_{\mathbf{x}} = Q(t) \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix}. \tag{48}$$

Next, considering that the system response consists of a stochastic and a deterministic component, namely $\mathbf{x}_s = \begin{bmatrix} q_s & y_s \end{bmatrix}^{\mathrm{T}}$ and $\mathbf{x}_d = \begin{bmatrix} q_d & y_d \end{bmatrix}^{\mathrm{T}}$, ensemble averaging the nonlinear vector in Eq. (47) yields

$$\mathrm{E}[\bar{\mathbf{\Phi}}_{\mathbf{x}}] = \left(\lambda \sigma_{q_s}^2 + \lambda q_d^2 + 3\delta \sigma_{q_s}^2 q_d + \delta q_d^3\right) \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix}. \tag{49}$$

Then, applying the generalized statistical linearization method with $\hat{\mathbf{x}} = \mathbf{x}_s$ the equivalent linear matrix $\bar{\mathbf{K}}_e$ is determined by Eq. (13) in the form

$$\bar{\mathbf{K}}_e = \left(\lambda q_d + 1.5\delta \left(\sigma_{q_s}^2 + q_{d_2}^2\right)\right) \begin{bmatrix} R(1,1) & R(2,1) \\ R(1,1) & R(2,1) \\ 0 & 0 \end{bmatrix}, \tag{50}$$

where $R(i,j)$, $i,j = 1,2$, denotes the (i,j) element of the matrix $\mathrm{E}[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}]^{+}\mathrm{E}[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}]$. For the numerical evaluation, the following set of parameter values are used for the system $\zeta = 0.1$, $\kappa = 3.25$, $\alpha = 0.8$, $\delta = 0.2$, $\lambda = 2\sqrt{\delta} \approx 0.89$, and $f_{d_1} = 0.1$,

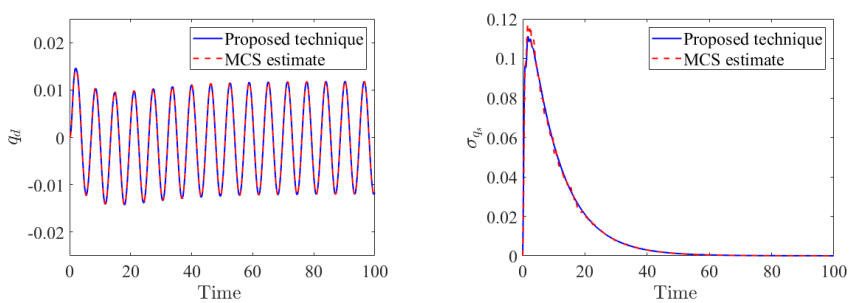
$\omega_d = 1$, $A = 1$, $\mu = 0.1$ and $S_0 = 0.2/\pi$ for the excitation. In this regard, the matrix differential equation Eq. (22) is formed, where

$$\mathbf{w}_{\bar{\mathbf{Q}}_{\mathbf{x}}} = \exp\left(-0.2t\right) \begin{bmatrix} 0.0159 & 0.0159 & 0 \\ 0.0159 & 0.0159 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{51}$$

and

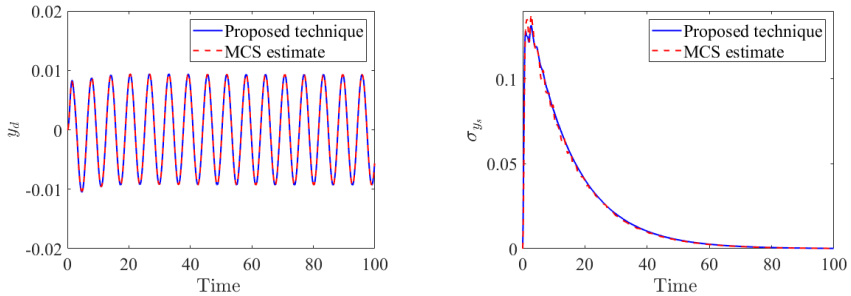
$$\mathbf{\Theta}(t) = \exp\left(-0.2t\right) \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.0637 & 0.0637 \\ 0 & 0 & 0.0637 & 0.0637 \end{bmatrix}. \tag{52}$$

Finally, the deterministic response component and the standard deviation of the stochastic response component for both the mechanical and the piezoelectric parts of the device are determined by considering the coupled set of Eqs. (9) and (22). Specifically, 10 differential equations governing the stochastic response of the system are derived by Eq. (22), whereas 4 additional differential equations governing the deterministic response are derived by Eq. (9). These are solved simultaneously by the Runge-Kutta method. The solid lines in Figs. 1(a) and 1(b) show the obtained results corresponding to the mechanical part of the device, namely the deterministic response displacement and the standard deviation of the stochastic response displacement, respectively. Further, the solid lines in Figs. 2(a) and 2(b) correspond to the piezoelectric part of the device. Fig. 2(a) shows the deterministic component of the induced voltage y, whereas Fig. 2(b) shows the standard deviation of the stochastic component of y. The obtained results are compared and found in good agreement with MCS data (500 realizations) generated by the spectral representation method [46], with a signal duration $T_0 = 100$ s and a cut-off frequency equal to 2π rad/s.



(a) (b)

Fig. 1: Response of the mechanical part of the nonlinear energy harvesting device described by Eqs. (37-39) subject to combined deterministic and modulated white noise excitation: (a) deterministic response displacement; (b) standard deviation of the stochastic response displacement. MCS data (500 realizations) are included for comparison.



(a) (b)

Fig. 2: Response of the piezoelectric part of the nonlinear energy harvesting device described by Eqs. (37-39) subject to combined deterministic and modulated white noise excitation: (a) deterministic component of the induced voltage; (b) standard deviation of the stochastic component of the induced voltage. MCS data (500 realizations) are included for comparison.

3.2 2-DOF nonlinear structural system with singular parameter matrices subject to combined deterministic and modulated white noise excitation

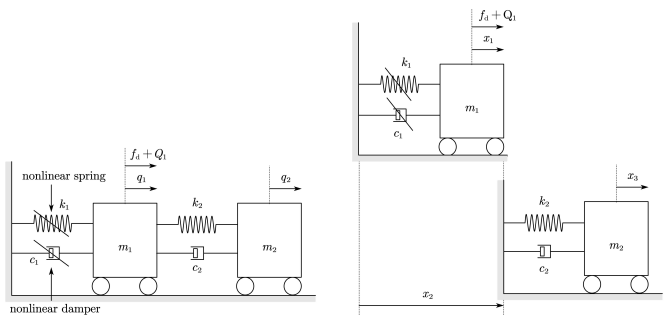
The 2-DOF nonlinear structural system in Fig. 3(a) is considered, where mass m_1 is connected to the foundation with a nonlinear spring with stiffness coefficient k_1 and a nonlinear damper with damping coefficient c_1. The corresponding forces are $k_1 q_1(1+\varepsilon_1 q_1^2)$ and $c_1\dot{q}_1(1+\varepsilon_2\dot{q}_1^2)$, respectively, where ε_1 and ε_2 are positive constants, and q_1 denotes the response displacement of mass m_1. Further, mass m_2 is connected to mass m_1 via a linear spring and a linear damper with stiffness and damping coefficients k_2 and c_2, respectively. q_2 denotes the response displacement of mass m_2. The system is subject to a combined deterministic and non-stationary stochastic excitation, which is applied on mass m_1. The deterministic excitation component is $f_d = f_{d,1}\sin(\omega_d t)$. The stochastic excitation is modeled as a modulated white noise $Q_1(t) = a(t)Q_s(t)$, where $a(t) = A\exp(-\mu t)$ is a time-modulating function with $t \geq 0$ and $A, \mu > 0$, and $Q_s(t)$ is a Gaussian white noise process with $\mathrm{E}\left[Q_s(t)Q_s(t+\tau)\right] = 2\pi S_0\delta(\tau)$.

The system governing equations of motion are derived by considering the (generalized) coordinates vector $\mathbf{q} = \begin{bmatrix} q_1 & q_2 \end{bmatrix}^{\mathrm{T}}$. The mass, damping and stiffness matrices of the system are given by

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 \\ m_2 & m_2 \end{bmatrix}, \ \mathbf{C} = \begin{bmatrix} c_1 & -c_2 \\ 0 & c_2 \end{bmatrix}, \ \mathbf{K} = \begin{bmatrix} k_1 & -k_2 \\ 0 & k_2 \end{bmatrix}. \tag{53}$$

Further, the system nonlinearity is written as

$$\mathbf{\Phi}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}) = \begin{bmatrix} \varepsilon_1 k_1 q_1^3 + \varepsilon_2 c_1 \dot{q}_1^3 \\ 0 \end{bmatrix}, \tag{54}$$



(a) (b)

Fig. 3: (a) A 2-DOF nonlinear structural system subject to combined deterministic and non-stationary stochastic excitation. (b) The nonlinear system of Fig. 3(a) modeled by employing an additional redundant coordinate.

whereas the deterministic and non-stationary excitation vectors are

$$\mathbf{f}_d = \begin{bmatrix} f_{d,1}\sin(\omega_d t) \\ 0 \end{bmatrix}, \ \mathbf{Q} = \begin{bmatrix} Q_1 \\ 0 \end{bmatrix}. \tag{55}$$

For the numerical evaluation, the system parameters take the values $m_1 = m_2 = 1$, $c_1 = c_2 = 0.2$, $k_1 = k_2 = 1$, $\varepsilon_1 = \varepsilon_2 = 0.1$, $S_0 = \frac{0.2}{\pi}$, $A = 1$, $\mu = 0.1$ and the excitation parameter values are $f_{d,1} = 1$, $\omega_d = 1$. The deterministic response component and the standard deviation of the stochastic response component of the nonlinear system are derived by applying the standard technique proposed in [18]. The obtained results for the response displacement and the response velocity for each of the system DOFs are shown by dashed line in Figs. 4 and 5, respectively.

Next, the system governing equations of motion are derived by adopting a redundant coordinates modeling. The nonlinear system in Fig. 3(a) is decomposed into its constituent parts as seen in Fig. 3(b), and considering the coordinates vector $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^{\mathrm{T}}$, the equation of motion Eq. (1) is formed. Further, differentiating twice with respect to time the constraints equation connecting the two subsystems in Figs. 3(a) and 3(b), i.e., $x_2 = x_1 + d$, where d denotes the length of mass m_1, Eq. (2) is formed, where

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}, \tag{56}$$

$\mathbf{E} = \mathbf{L} = \mathbf{0}_{1\times 3}$ and $F = 0$. In addition, matrix $\mathbf{J}$ in Eq. (45) becomes

$$\mathbf{J} = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{57}$$

In this regard, the parameter matrices in Eq. (4) are given by

$$\bar{\mathbf{M}}_{\mathbf{x}} = \begin{bmatrix} 0.5 & 0.5 & 0.5 \\ 0.5 & 0.5 & 0.5 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}, \ \bar{\mathbf{C}}_{\mathbf{x}} = \begin{bmatrix} 0.1 & 0 & 0 \\ 0.1 & 0 & 0 \\ 0 & 0 & 0.2 \\ 0 & 0 & 0 \end{bmatrix}, \ \bar{\mathbf{K}}_{\mathbf{x}} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0.5 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \tag{58}$$

whereas the nonlinearity of Eq. (5) becomes

$$\bar{\mathbf{\Phi}}_{\mathbf{x}}(\mathbf{x}, \dot{\mathbf{x}}, \ddot{\mathbf{x}}) = \left(k_1\varepsilon_1 x_1^3 + c_1\varepsilon_2 \dot{x}_1^3\right) \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \end{bmatrix}^{\mathrm{T}}. \tag{59}$$

Lastly, the deterministic and non-stationary stochastic excitation components are given by

$$\bar{\mathbf{f}}_{d,\mathbf{x}} = f_{d,1} \sin(\omega_d t) \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \end{bmatrix}^{\mathrm{T}} \tag{60}$$

and

$$\bar{\mathbf{Q}}_{\mathbf{x}} = Q_1(t) \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \end{bmatrix}^{\mathrm{T}}, \tag{61}$$

respectively.

Then, for the application of the proposed technique the system response is decomposed into a deterministic component $\mathbf{x}_d = \begin{bmatrix} x_{d,1} & x_{d,2} & x_{d,3} \end{bmatrix}^{\mathrm{T}}$ and a corresponding stochastic component $\mathbf{x}_s = \begin{bmatrix} x_{s,1} & x_{s,2} & x_{s,3} \end{bmatrix}^{\mathrm{T}}$. Next, ensemble averaging the nonlinear function in Eq. (59), i.e.,

$$\mathrm{E}\left[\bar{\mathbf{\Phi}}_{\mathbf{x}}\right] = \begin{bmatrix} 0.5k_1\varepsilon_1 \left(x_{d,1}^3 + 3x_{d,1}\sigma_{x_{s,1}}^2\right) + 0.5c_1\varepsilon_2 \left(\dot{x}_{d,1}^3 + 3\dot{x}_{d,1}\sigma_{\dot{x}_{s,1}}^2\right) \\ 0.5k_1\varepsilon_1 \left(x_{d,1}^3 + 3x_{d,1}\sigma_{x_{s,1}}^2\right) + 0.5c_1\varepsilon_2 \left(\dot{x}_{d,1}^3 + 3\dot{x}_{d,1}\sigma_{\dot{x}_{s,1}}^2\right) \\ 0 \\ 0 \end{bmatrix}, \tag{62}$$

Eq. (9) is formed for the subsystem subject to deterministic excitation, while the generalized statistical linearization method is applied to treat the subsystem subject to non-stationary excitation. Thus, considering the 6-dimensional vector $\hat{\mathbf{x}} = \begin{bmatrix} \mathbf{x}_s & \dot{\mathbf{x}}_s \end{bmatrix}^{\mathrm{T}}$, the equivalent linear elements in Eq. (13) are given by

$$\bar{\mathbf{K}}_e = 1.5k_1\varepsilon_1 \left(x_{d,1}^2 + \sigma_{x_{s,1}}^2\right) \begin{bmatrix} R(1,1) & R(2,1) & R(3,1) \\ R(1,1) & R(2,1) & R(3,1) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{63}$$

and

$$\bar{\mathbf{C}}_e = 1.5c_1\varepsilon_2 \left(\dot{x}_{d,1}^2 + \sigma_{\dot{x}_{s,1}}^2\right) \begin{bmatrix} R(4,4) & R(5,4) & R(6,4) \\ R(4,4) & R(5,4) & R(6,4) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{64}$$

where $R(i,j)$, $i,j = 1,2,\ldots,6$ denotes the (i,j) element of matrix $\mathrm{E}[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}]^{+}\mathrm{E}[\hat{\mathbf{x}}\hat{\mathbf{x}}^{\mathrm{T}}]$ of Eq. (13).

Then, following the presentation in section 2.2.3, the matrix differential equation Eq. (22) is formed, where

$$\mathbf{w}_{\bar{\mathbf{Q}}_{\mathbf{x}}} = \frac{\exp\left(-0.2t\right)}{20\pi} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{65}$$

and

$$\mathbf{\Theta}(t) = \frac{\exp(-0.2t)}{5\pi} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{bmatrix}. \tag{66}$$

This leads to 21 differential equations pertaining to the determination of the stochastic response component, which are solved simultaneously with 6 additional differential equations derived by Eq. (9). The set of all differential equations is solved by the Runge-Kutta method. The obtained results for the deterministic and stochastic components of the response displacement and velocity for both DOFs of the system are shown by solid line in Figs. 4 and 5, respectively. Clearly, these are in total agreement with the corresponding results (dashed line) obtained by the standard method proposed in [18].

3.3 2-DOF nonlinear structural system with singular parameter matrices subject to combined deterministic and modulated colored noise excitation

In this section, the system shown in Figs. 3(a) and 3(b) is subject to combined deterministic and non-stationary stochastic excitation, with the latter modeled as modulated colored noise. Similar to the case in section 3.2, the deterministic excitation component is given by $f_{d,1}\sin(\omega_d t)$. The stochastic component is modeled as $Q_1(t) = a(t)Q_s$, where $a(t) = A\exp(-\mu t)$ is a time-modulating function with $t \geq 0$ and $A, \mu > 0$, and Q_s is a non-white stochastic excitation process with a Kanai-Tajimi power spectrum

$$S(\omega) = \frac{1 + 4\xi_g \frac{\omega^2}{\omega_g^2}}{\left(1 - \frac{\omega^2}{\omega_g^2}\right)^2 + 4\xi_g \frac{\omega^2}{\omega_g^2}} S_0. \tag{67}$$

The values of the parameters in the power spectrum of Eq. (67) are $\xi_g = 0.5$, $\omega_g = 1$ and $S_0 = 0.2\pi$.

Next, the technique proposed in section 2.2.4 is applied for determining the response of the system. In this regard, Eqs. (25) and (26) reduce to a second order linear filter with coefficients λ_0, λ_1, v_0 and v_1, given by

$$v_1 u^{(1)} + v_0 u^{(0)} = Q_s(t) \tag{68}$$

and

$$u^{(2)} + \lambda_1 u^{(1)} + \lambda_0 u^{(0)} = w(t), \tag{69}$$

where $v_0 = \omega_g^2, v_1 = 2\zeta_g\omega_g, \lambda_0 = \omega_g^2, \lambda_1 = 2\zeta_g\omega_g$, and $w(t)$ is a white noise process. Further, since the excitation is applied only on the first DOF of the system (see Fig. 3(b)), $\mathbf{I}_{\bar{\mathbf{P}}}$ in Eq. (31) is equal to $\mathbf{I}_{\bar{\mathbf{P}}} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$ and using Eq. (45), Eq. (31) yields $\bar{\mathbf{P}} = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$. This leads to the computation of the 6×2 matrix $\bar{\mathbf{D}}_{\mathbf{x}}$

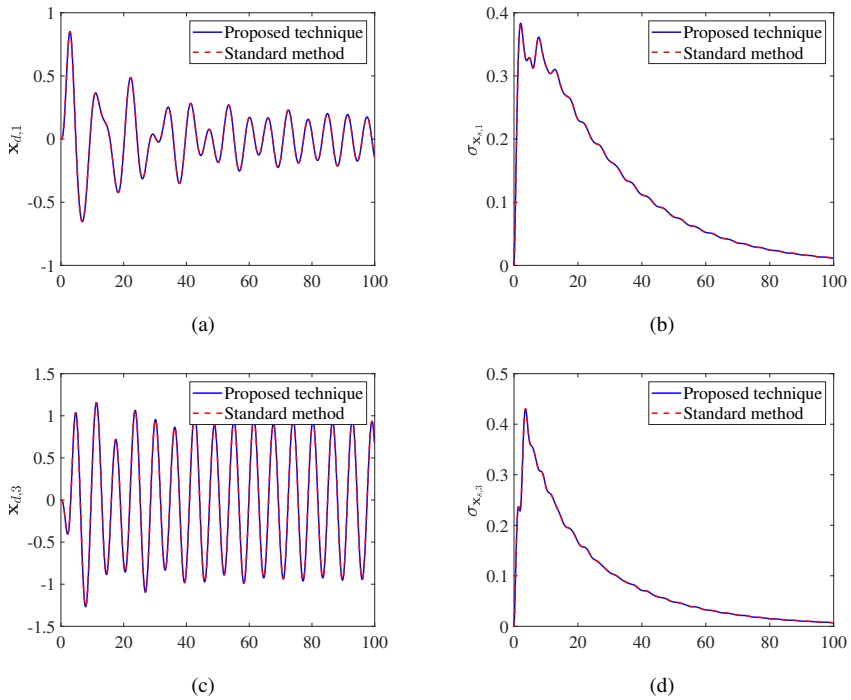


Fig. 4: Response of the 2-DOF nonlinear structural system in Figs. 3(a) and 3(b) subject to combined deterministic and modulated white noise excitation: (a) deterministic response displacement of the 1st DOF; (b) standard deviation of the stochastic response displacement of the 1st DOF; (c) deterministic response displacement of the 3rd DOF; and (d) standard deviation of the stochastic response displacement of the 3rd DOF. Results obtained by the proposed technique (solid line) vs corresponding results obtained by the method in [18] (dashed line).

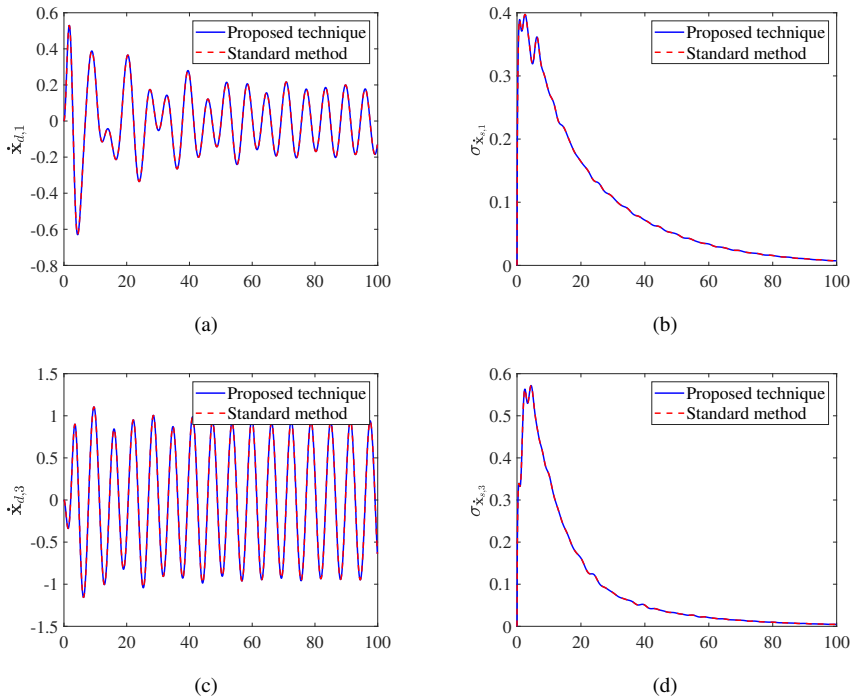


Fig. 5: Response of the 2-DOF nonlinear structural system in Figs. 3(a) and 3(b) subject to combined deterministic and modulated white noise excitation: (a) deterministic response velocity of the 1st DOF; (b) standard deviation of the stochastic response velocity of the 1st DOF; (c) deterministic response velocity of the 3rd DOF; and (d) standard deviation of the stochastic response velocity of the 3rd DOF. Results obtained by the proposed technique (solid line) vs corresponding results obtained by the method in [18] (dashed line).

in Eq. (28). In addition, Eqs. (63-64) are used for computing the 6×6 matrix $\bar{\mathbf{G}}_{\mathbf{x}}(t)$ in Eq. (16), whereas $\mathbf{\Lambda}$ is readily found by Eq. (33).

Finally, taking into account Eq. (35), the matrix differential equation Eq. (36) is formed and solved simultaneously with the deterministic sub-equations derived by Eq. (9). Overall, Eq. (36) yields 36 differential equations governing the stochastic response component, whereas Eq. (9) yields 6 additional differential equations governing the deterministic component of the response. The set of differential equations is solved by the Runge–Kutta method. The results obtained for the deterministic and the stochastic components of the response displacement and velocity for both DOFs of the system are shown by solid line in Figs. 6 and 7, respectively. To demonstrate the validity of the proposed technique, corresponding results obtained by the standard method in [18] are also included in Figs. 6 and 7 for comparison. The latter are depicted by dashed line and practically coincide with the results obtained by the proposed technique.

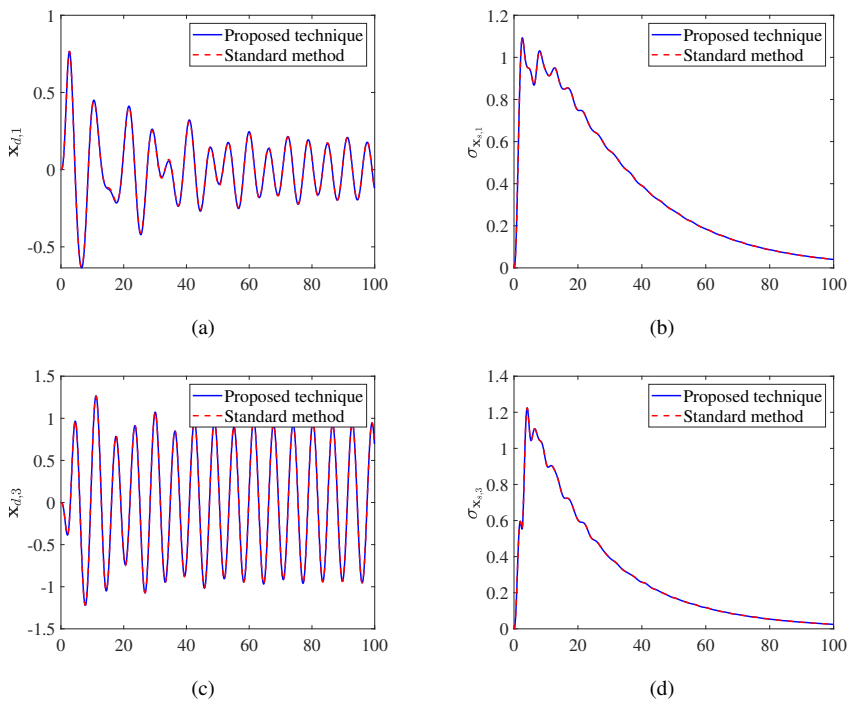


Fig. 6: Response of the 2-DOF nonlinear structural system in Figs. 3(a) and 3(b) subject to combined deterministic and modulated colored noise excitation: (a) deterministic response displacement of the 1st DOF; (b) standard deviation of stochastic response displacement of the 1st DOF; (c) deterministic response displacement of the 3rd DOF; and (d) standard deviation of the stochastic response displacement of the 3rd DOF. Results obtained by the proposed technique (solid line) vs corresponding results obtained by the method in [18] (dashed line).

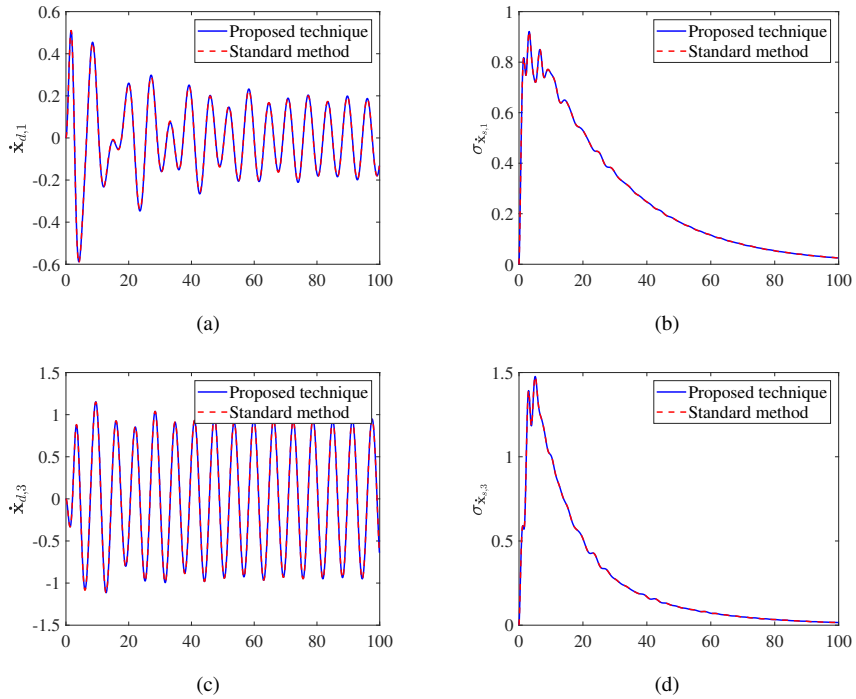


Fig. 7: Response of the 2-DOF nonlinear structural system in Figs. 3(a) and 3(b) subject to combined deterministic and modulated colored noise excitation: (a) deterministic response velocity of the 1st DOF; (b) standard deviation of stochastic response velocity of the 1st DOF; (c) deterministic response velocity of the 3rd DOF; and (d) standard deviation of stochastic response velocity of the 3rd DOF. Results obtained by the proposed technique (solid line) vs corresponding results obtained by the method in [18] (dashed line).

4 Concluding remarks

In this paper, a new technique has been proposed for determining the response of MDOF systems with singular parameter matrices subject to combined deterministic and non-stationary stochastic excitations. The appearance of singular matrices in the equations of motion pertain to additional constraints equations in the system, or to a redundant coordinates modeling of its governing dynamics. Further, the stochastic excitation component is modeled as a non-stationary process driven by the need to develop response analysis frameworks accounting for the non-stationary characteristics of excitations such as wave, wind and earthquake loads.

In this regard, the MDOF nonlinear system has been decomposed into two subsystems based on the applied excitation, and a coupled set of equations has been derived and solved to determine the system response. First, a subsystem of deterministic equa-

tions governing the response of the system subject to deterministic excitation has been derived. Next, the generalized statistical linearization method has been utilized to treat the nonlinear subsystem subject to non-stationary stochastic excitation. This has been done in conjunction with a state space formulation, which resulted a matrix differential equation governing the stochastic response. The latter has been solved simultaneously with the deterministic equation above by applying a standard Runge-Kutta numerical scheme. In addition, a closed form expression for determining the time-dependent equivalent elements of the generalized statistical linearization methodology ([28]) has been derived. Overall, the proposed technique can be construed as an extension of the approach in [20] to systems subject to combined deterministic and non-stationary stochastic excitation. It has been assessed by considering three numerical examples including a vibration energy harvesting device subject to combined deterministic and modulated white noise excitation, and a structural nonlinear system with singular parameter matrices subject to combined deterministic and modulated white and colored noise excitations. The reliability of the obtained results has been demonstrated by comparisons to MCS data and corresponding results obtained by the approach proposed in [18].

CRediT authorship contribution statement

Peihua Ni: Methodology, Software, Writing – original draft, Visualization. **Vasileios C. Fragkoulis:** Conceptualization, Methodology, Writing – review & editing, Supervision, Project administration, Funding acquisition. **Fan Kong:** Conceptualization, Methodology, Writing – review & editing, Funding acquisition. **Ioannis P. Mitseas:** Conceptualization, Methodology, Writing – review & editing, Supervision, Funding acquisition. **Michael Beer:** Supervision, Project administration, Funding acquisition.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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