Portfolio Selection Under Systemic Risk*

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Abstract

This paper proposes a novel methodology to construct optimal portfolios that explicitly incorporates the occurrence of systemic events. Investors maximize a modified Sharpe ratio that is conditional on a systemic event, with the latter interpreted as a low market return environment. We solve the portfolio allocation problem analytically under the absence of short-selling restrictions and numerically when short-selling restrictions are imposed. This approach for obtaining an optimal portfolio allocation is made operational by embedding it in a multivariate dynamic setting using dynamic conditional correlation and copula models. We evaluate the out-of-sample performance of our portfolio empirically on the US stock market over the period 2007 to 2020 using ex-post final wealth paths and systemic risk metric against mean-variance, equally-weighted, and global minimum variance portfolios. Our portfolio maximizing a modified Sharpe ratio outperforms all competitors under market distress and remains competitive in non-crisis periods.

Keywords: conditional volatility models; portfolio allocation; Sharpe ratio; systemic risk; conditional tail risk.

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1 Introduction

Systemic risk is defined as the risk of collapse of an entire financial system, as opposed to risk associated with any single individual entity or component of the system. It also refers to the risk imposed by poorly understood interlinkages and interdependencies between assets and institutions in the financial market, where the failure of a single entity or cluster of entities can trigger the failure of more institutions, see Allen and Carletti (2013).

The global financial crisis of 2007-2008 and subsequent crises (e.g. COVID-19 crisis) provide ample evidence of the importance of containing this risk. More formally, Ben Bernanke, as previous Chairman of the US Federal Reserve, defined systemic risk as "*developments that threaten the stability of the financial system as a whole and consequently the broader economy, not just that of one or two institutions.*" For a brief discussion on the elements of a systemic risk monitor that help identify risks to financial stability, readers can consult Liang (2013). In this paper, we formally incorporate the occurrence of systemic events to the construction of optimal portfolios. This new approach is better suited to accommodate market turbulences and, as a result of this, it is able to outperform popular alternatives such as the classical mean-variance, global minimum variance and equally-weighted portfolios out-of-sample.

Prevalent financial regulations such as Basel capital requirements seek to control firms' individual risks without accounting for systemic events (Acharya et al. 2017). Empirical evidence shows, however, that the interconnection among financial institutions has increased significantly in recent years, generating the risk of potential system-wide distress with major knock-on effects on the real economy. Financial institutions in the same sector have linkages and connections which can become channels for spreading poor performance from one to the others. A message that stems from this literature, see Board (2016), is that it is necessary for regulators to monitor systemically important financial institutions (SIFIs) whose failures may impose negative spillover effects on the wider financial system. Benoit et al. (2013, 2017) differentiate between two distinct approaches that measure the systemic risk contribution of financial institutions. The first method looks at different sources of systemic risk such as financial contagion, bank panics, liquidity problems, etc. It relies on the use of confidential data directly provided by financial institutions to regulators. Following this idea, various regulatory models are proposed to identify the transmission channels of systemic risk and supervise inter-bank behaviors with the aim of enhancing the stability of the financial system. Gourieroux et al. (2012), for example, propose a new regulation mechanism which requires periodic reporting by financial institutions of their structural information, which is used to quantify the bilateral exposures concerning equities, lendings, or derivatives. The second method depends on market trading data such as the prices of stocks, bonds, and CDSs.

Many financial economists have developed their own measures to quantify firms' contribution to the overall risk of the financial system (see, for example, Acharya et al. 2017, Acharya et al. 2012, Brownlees and Engle 2016 and Adrian and Brunnermeier 2016). While distinguished from traditional risk measures, the systemic risk measures proposed by these authors focus on the *interconnection* among financial firms. Prominent systemic risk measures are the CATFIN of Allen et al. (2012), the CoVaR of Adrian and Brunnermeier (2016) and its extension to a multivariate setting by Girardi and Ergün (2013), the SRISK of Brownlees and Engle (2012, 2016) and its extension to a multifactor model by Engle et al. (2014), the systemic expected shortfall (SES) of Acharya et al. (2017), and econometric measures of connectedness and systemic risk in finance and insurance sectors, such as Hong et al. (2009), Battiston et al. (2012), Billio et al. (2012), Helbing (2013), Ang and Longstaff (2013), Diebold and Yılmaz (2014), and Hautsch et al. (2014). Bisias et al. (2012) present a survey that covers over thirty systemic risk indices.

Although the existing systemic risk measures are helpful for financial regulators, portfolio managers are still looking for practical guidance under which they can account for systemic events during their decision-making process. A general approach for constructing optimal portfolios is to maximize a reward-to-risk ratio. Modern portfolio theory pioneered by Markowitz (1952) stresses the idea that portfolio diversification leads to a risk reduction. Following this idea, Tobin (1958) developed further the concept of optimal portfolio allocation by arguing that agents would diversify their asset allocation. An alternative strategy to solve the optimal portfolio allocation exercise is to maximize the investors' expected utility, which was first proposed by Von Neuman and Morgenstern (1944). In this framework, the optimal portfolio decision is obtained as a result of the maximization of the expected utility derived from the portfolio return.

Unfortunately, none of these two paradigms is devised to properly take into account the occurrence of systemic events. Both approaches incorporate the possibility of joint dependence between the assets within the portfolio through the presence of cross-correlation between the returns on the portfolio constituents or through more sophisticated measures considering joint dependence in the tails. A seminal example is the literature on optimal portfolio allocation under tail quantile restrictions using value-at-risk (VaR) and expected shortfall (ES), see Duffie and Pan (1997) and Jorion (2007) for a comprehensive review of VaR models. More specifically, in an optimal asset allocation context, tail quantiles act as constraints in the asset allocation optimization exercise. These mean-risk models discussed in Fishburn (1977) can be considered as an extension of standard mean-variance formulations that interpret portfolio risk as the probability of tail events and that implicitly incorporate the occurrence of such events through VaR measures. The relevant literature includes Basak and Shapiro (2001), Campbell et al. (2001), Bassett et al. (2004), Engle and Manganelli (2004) and Ibragimov and Walden (2007), as seminal examples.

Whereas the macroprudential literature has made substantial progress in developing monitoring tools for assessing the underlying systemic risk in a financial system (see Tente et al. 2019, among others), the portfolio management literature has not evolved in parallel. This branch of the empirical finance literature has not explored systematically yet the implications of systemic events on the construction of investment portfolios. Our main contribution in this paper is to bring the attention of academics and financial practitioners to this important problem that has been overlooked until recently. To do this, we apply methods from the emerging macroprudential literature on systemic risk to the optimal portfolio allocation problem.

The marginal expected shortfall (MES) proposed by Brownlees and Engle (2012) has received much attention recently. This measure accounts for the comovements between individual firms and the market under stressed market conditions. It is defined as the expected percentage loss of a firm's equity value in times of a market decline. Motivating by this measure of systemic risk, we propose a modified mean-variance objective function to reflect investor's risk-return tradeoff. In particular, we propose a modified Sharpe ratio that is conditional on a systemic event, with the latter interpreted as a low market return environment. We solve the portfolio allocation problem analytically under the absence of short-selling restrictions and numerically when short-selling restrictions are imposed. This approach for obtaining an optimal portfolio allocation is made operational by embedding it in a multivariate dynamic setting. To do this, we consider two different processes for modelling multivariate financial returns and set up the portfolio allocation problem in an out-of-sample setting. The first model fits the return data to a GARCH type process and models the joint dependence between the return vector of portfolio constituents and the market portfolio using a dynamic conditional correlation (DCC) model introduced in Engle (2002). The second approach models the joint dependence between returns on portfolio assets and the market index using a Student's t-copula model. In contrast to standard approaches for portfolio selection, our proposed methodology is conditional on the occurrence of systemic events. To do this, we simulate the multivariate returns using a Monte Carlo scenario generation method.

We evaluate the portfolio performance on the US stock market. We choose a group of large financial institutions as portfolio assets, and the S&P 500 Index as benchmark rate. Our out-of-sample evaluation period spans from the beginning of 2007 to the end of 2020, hence covering two major financial crises with important systemic events (i.e. the bankruptcy of Lehman Brothers and the outbreak of COVID-19). We compare the ex-post wealth paths and portfolio-level systemic risk metric against three competitors. The first competitor is the unconditional Sharpe ratio that represents the classic mean-variance approach, the second portfolio is the naive equally-weighted portfolio that reflects full diversification and is shown to work well in financial applications (DeMiguel et al. 2007), and the third competitor is the global minimum variance portfolio (GMVP) which is often shown to outperform the mean-variance portfolio in many empirical studies (see, for example, Jagannathan and Ma 2003 and DeMiguel et al. 2009). The results of our empirical study show the outperformance of our portfolio against these three competitors in terms of profitability and systemic risk, especially during crisis periods.

The rationale for the excellent performance of our model is its positive exposure to assets that are more resilient in periods of market distress. Our portfolio clearly outperforms competitors under market distress and remains competitive in non-crisis periods. Interestingly, the proposed portfolio is less diversified than benchmark portfolios during crisis times since we only invest on a few stocks with low long-run MES level. In these periods, our strategy invests on those stocks that are expected to experience small losses under stressed market conditions. Underdiversification is the result of optimal strategies aiming to minimize exposure to systemic events. This is done by reducing the set of eligible assets to a small group of stocks with small systemic risk. This empirical finding provides an alternative interpretation to the presence of underdiversification observed in financial markets, see Mitton and Vorkink (2007) and references therein. Interestingly, our results can also be related to a recent literature on time series, see Farmer et al. (2019), which finds pockets of predictability. These pockets are short periods of time over which there is predictability of returns within longer periods with little or no evidence of predictability. In our setting, we interpret these *pockets* as periods of systemic risk that drive the overall performance of the proposed portfolio based on the maximization of a conditional Sharpe ratio objective function.

Our paper also contributes to a relatively scarce literature on systemic risk-based portfolio selection. There are a few studies on the implications of systemic risk in the investment decisions of financial institutions. Biglova et al. (2014) study portfolio selection under systemic risk using the Co-Rachev ratio as objective function. In their setting, systemic risk takes place when all assets in the investment portfolio are distressed, i.e., below their individual VaR thresholds. However, this definition can be ambiguous since the poor performance of individual assets in a portfolio does not necessarily imply a poor state of the whole financial system. Another exception is Capponi and Rubtsov (2022). These authors consider the problem of maximizing portfolio returns conditional on a systemic event given by the realization of an extremely adverse market outcome. These authors seek the portfolio that performs best in a low return environment and when the market is in distress. To solve the portfolio allocation problem, Capponi and Rubtsov (2022) impose the restrictive assumption that the distribution of the portfolio and market returns follows a bivariate Student's t distribution. More importantly, none of these papers explicitly focus on finding the best trade-off between return and risk under stressed market conditions. Our paper bridges this gap.

The rest of the paper is organized as follows. Section 2 introduces our novel objective function defined as a modified Sharpe ratio conditional on the occurrence of systemic events. Section 3 presents the investors' optimal portfolio allocation problem under systemic risk. This section derives analytically the solution without short-selling restrictions and proposes numerical methods to obtain the solution under the presence of short-selling restrictions. Section 4 introduces the simulation of return scenarios under a DCC model and a Student's t-copula for modelling the joint conditional distribution of asset and market portfolio returns. Section 5 discusses an application of our optimal asset allocation strategy to a portfolio of 23 assets and presents several robustness checks. Conclusions are in Section 6. Appendix A reviews several prominent systemic risk measures. A description of the simulation of return scenarios is provided in Appendix B. The last appendix collects the figures.

2 Our objective function under market distress

The mean-variance framework developed by Markowitz (1952) is one of the cornerstones for portfolio theory. Optimal portfolios are obtained by maximizing the expected return on an investment portfolio conditional on a given level of risk that is proxied by the variance of the portfolio return. Alternative formulations consider risk measures given by tail events such as VaR and ES, see Duffie and Pan (1997) and Jorion (2007) for a comprehensive review of VaR models. In these models the objective function is the expected portfolio return that is constrained by a tail quantile restriction on the asset allocation optimization exercise.

Based on these objective functions, the literature in financial economics has developed performance measures to evaluate investment strategies. A natural performance measure based on the seminal mean-variance framework is the Sharpe ratio (Sharpe 1966), which is originally proposed for measuring the performance of mutual funds. This measure is defined as the ratio between the expected portfolio excess return (i.e. the expected portfolio return minus risk-free rate) and its standard deviation. Sharpe (1994) later revised this measure by referring the portfolio performance with respect to a certain benchmark rate R_b, which can change over time, such that the revised Sharpe ratio is defined as

$$SR(R_p) = \frac{E(R_p - R_b)}{std(R_p - R_b)}. \tag{1}$$

In the remainder of this paper, when referring to the Sharpe ratio, we will consider expression (1). It is typical to use the Sharpe Ratio to evaluate and compare the ex-post portfolio performance among different investment strategies.

Interestingly, Biglova et al. (2009) argue that the maximization of the Sharpe ratio allows one to obtain a market portfolio that is optimal in the sense that it is not dominated in stochastic dominance of second order by non-satiable risk-averse investors. This result suggests that using the Sharpe ratio and related performance measures as the investor's objective function in a portfolio allocation setting is a fruitful strategy (see Rachev et al. 2008 for a review of performance measures). The choice of a performance measure allows one to explicitly introduce the risk measure along with the corresponding reward measure in the portfolio choice optimization problem without having to specify a risk aversion coefficient.

Although the Sharpe Ratio works well in Gaussian settings, it is not a suitable performance measure in settings characterised by skewness and heavy tails of the return distributions. In order to capture higher moments of the return distributions on the performance of investment portfolios, many authors have developed their own ratios such as Gini ratio (Shalit and Yitzhaki 1984), Mean Absolute Deviation ratio (Konno and Yamazaki 1991), Mini-max ratio (Young 1998), Sortino-Satchell ratio (Sortino and Satchell 2001), Rachev ratio (Biglova et al. 2004) and others (see Farinelli et al. 2008 for a detailed survey). In this paper, we focus on tail risk measures capturing systemic risk. In particular, we propose a conditional performance measure that incorporates the occurrence of systemic risk without imposing any distributional assumptions.

Our objective function for optimal portfolio allocation is inspired by the conditional performance measure proposed by Biglova et al. (2014). These authors study the portfolio selection problem in the presence of systemic risk and propose a conditional version of Rachev ratio (CoRR), which is defined as:

$$CoRR(R_p;\alpha,\beta) = \frac{E(R_p - R_b|R_1 \geq -VaR_{1-\beta}(R_1), ..., R_n \geq -VaR_{1-\beta}(R_n))}{-E(R_p - R_b|R_1 \leq -VaR_{\alpha}(R_1), ..., R_n \leq -VaR_{\alpha}(R_n))}, \quad (2)$$

where $VaR_q(X) = -inf\{x|P(X \leq x) > q\}$ is the VaR of the random variable X that is interpreted as a financial return on an investment portfolio. The interpretation of this measure is different from standard systemic risk formulations. CoRR does not link systemic risk to the occurrence of distress in the financial system, instead, it evaluates portfolio performance conditional on the occurrence of idiosyncratic events in all assets in the portfolio (i.e. all asset returns are above (or below) their individual VaR levels). Moreover, CoRR takes the expected portfolio return as a reward measure conditional on all asset prices comoving in the tail. This assumption may be difficult to be satisfied in practice and might lead to an empty set if the set of assets in the portfolio is sufficiently large.

Unlike Biglova et al. (2014), we define a systemic event when the return on the market index is below a certain threshold C over a time horizon h. Following the related literature, we assume that there exists a benchmark systemic risk index, which is the S&P 500 Index in our case, that reflects broad market conditions. The goal of our investors is to maximize the Sharpe ratio conditional on the systemic risk index being below a threshold level C between t and $t+h$, and we set the horizon h to one month (i.e. 22 trading days). Our investment strategy aims to find portfolios that perform best under stressed market conditions.

We start by introducing several assumptions and notations used throughout the paper. In our economy there is no risk-free asset and there are $N \geq 2$ risky assets (firms) with stochastic simple returns denoted by $R_t = (R_{1,t}, ..., R_{N,t})^T$. The return on the financial system is proxied by a market portfolio return $R_{m,t}$. The logarithmic returns of the firm i and the market are denoted, respectively, as $r_{i,t} = log(1+R_{i,t})$ and $r_{m,t} = log(1+R_{m,t})$. The mean vector of returns is denoted by $\mu_t = E(R_t)$, while $\Sigma_t = E[(R_t - \mu_t)(R_t - \mu_t)^T]$ represents the covariance matrix of returns. The vector of portfolio weights is denoted by $W_t = (\omega_{1,t}, ..., \omega_{N,t})^T$ such that $\sum_{i=1}^{N} \omega_{i,t} = 1$. Let $R_{p,t} = W_t^T R_t$ be an investment portfolio with expected return given by $\mu_{p,t} = W_t^T \mu_t$. Similarly, $\mu_{m,t}$ and $\sigma_{m,t}$ denote the expected return and standard deviation of the market portfolio return reflecting the performance of the financial system. The column vector $\sigma_t = (\sigma_{1m,t}, ..., \sigma_{Nm,t})^T$ contains covariances of each asset with the market portfolio. Hereafter, we use $I\{x\}$ to denote the indicator function that equals 1 if condition x is met and 0 otherwise. $\mathbf{1}$ and $\mathbf{0}$ are column vectors of ones and zeros, respectively, whose dimension are understood from the context.

In the next section we will be concerned with building portfolios under stressed market scenarios. Different definitions of SE can be adopted. For instance, Acharya et al. (2017) consider SE as extreme tail events that happen rarely on a daily basis. In particular, they focus on those "moderately bad days" defined as the worst 5% of daily market outcomes, $SE_t = \{R_{m,t} \leq -VaR_{5\%}(R_{m,t})\}$, while Biglova et al. (2014) define SE as all assets in the portfolio being below their individual VaR levels, $SE_t = \{R_{1,t} \leq -VaR_\alpha(R_{1,t}), ..., R_{N,t} \leq -VaR_\alpha(R_{N,t})\}$. We follow Brownlees and Engle (2012, 2016) and define a systemic event as a severe drop of the market index below a threshold C over a time horizon h, that is:

$$SE_{t:t+h} = \{R_{m,t:t+h} < C\}, \tag{3}$$

where $R_{m,t:t+h}$ is the multiperiod simple market return between t and $t+h$. We also follow related literature and define the magnitude of the market decline (C) as a function of the length of the time horizon (h). Acharya et al. (2012) set C equal to -2% and h equal to one trading day to estimate the daily MES; Brownlees and Engle (2016) set C equal to -10% and h equal to one month for computing the monthly MES (i.e. LRMES); Engle et al. (2014) focus on long-run market stress and fix C equal to -40% and h equal to six months. In the empirical section, we use $C = 0$ and -40% as threshold values, which on a monthly basis correspond to $C = 0$ and -6.7% respectively.

We construct a new performance measure that will be used to build optimal portfolios under stressed market conditions. To do this, we incorporate systemic risk directly into the reward and risk measures. In order to account for the interconnection between individual assets and the financial market we propose to use the first and second moments of the excess portfolio return conditional on the occurrence of a systemic event. Our new performance measure is defined as:

$$CoSR_t(R_{p,t}) := \frac{CoER_t(R_{p,t})}{CoSD_t(R_{p,t})} = \frac{W_t^T \mu_{t|SE} - \mu_{m,t|SE}}{\sqrt{W_t^T \Sigma_{t|SE} W_t + \sigma^2_{m,t|SE} - 2W_t^T \sigma_{t|SE}}}. \tag{4}$$

Following the spirit of the Sharpe ratio and similar performance measures, the CoSR is defined as a ratio of a conditional reward measure over a conditional risk measure. The conditional reward measure CoER is defined as

$$\begin{aligned} CoER_t(R_{p,t}) : &= E_t(R_{p,t:t+h} - R_{m,t:t+h} | SE_{t:t+h}), \\ &= E_t(W_t^T R_{t:t+h} - R_{m,t:t+h} | SE_{t:t+h}), \\ &= W_t^T \mu_{t|SE} - \mu_{m,t|SE}, \end{aligned} \tag{5}$$

where $\mu_{t|SE} = E_t(R_{t:t+h}|SE_{t:t+h})$ denotes the column vector of conditional expected returns on individual assets, while $\mu_{m,t|SE} = E_t(R_{m,t:t+h}|SE_{t:t+h})$ represents the conditional expected market return. Inspired by the formulation of LRMES, we add the market index as a benchmark to enable us measure portfolio performance under stressed market scenarios. Analogously, we define the risk measure CoSD as the conditional second moment of the portfolio excess return, that is:

$$\begin{aligned} CoSD_t(R_{p,t}) :&= [Var_t(R_{p,t:t+h} - R_{m,t:t+h}|SE_{t:t+h})]^{1/2} \\ &= \left[Var_t(W_t^T R_{t:t+h} - R_{m,t:t+h}|SE_{t:t+h})\right]^{1/2} \\ &= \left(W_t^T \Sigma_{t|SE} W_t + \sigma^2_{m,t|SE} - 2W_t^T \sigma_{t|SE}\right)^{1/2}, \end{aligned} \tag{6}$$

where $\Sigma_{t|SE} = Var_t(R_{t:t+h}|SE_{t:t+h})$ denotes the conditional covariance matrix of asset returns, $\sigma^2_{m,t|SE} = Var_t(R_{m,t:t+h}|SE_{t:t+h})$ denotes the conditional variance of market return, and $\sigma_{t|SE} = cov_t(R_{t:t+h}, R_{m,t:t+h}|SE_{t:t+h})$ is the column vector of conditional covariances between individual assets and the market portfolio.

3 Portfolio allocation under systemic risk

In this section, we present the portfolio allocation problem of an investor that is concerned with maximizing the modified Sharpe ratio conditional on the market being under distress. We describe first the generic portfolio optimization problem when the investor's objective function is given by a performance measure $\rho(\cdot)$. In this setting, the investor's optimal portfolio is obtained as

$$W^* = \arg\underset{W}{max}\, \rho(R_p), \quad \text{s.t. } \mathbf{1}^T W = 1. \tag{7}$$

Different performance measures $\rho(\cdot)$ will lead to different optimal portfolios. In the empirical application, we will consider the Sharpe ratio as the relevant objective function of interest under short-selling restrictions ($W \geq \mathbf{0}$).

In what follows, we present the optimization problem of an investor with objective function given by the CoSR measure defined above. To simplify the problem, we note that this measure can be expressed as a function of the portfolio weights as $CoSR = \frac{W^T\mu}{\sqrt{W^T\Sigma W}}$, with $\mu = E(R - R_m \cdot \mathbf{1}|SE)$ and $\Sigma = Var(R - R_m \cdot \mathbf{1}|SE)$ be the conditional mean vector and conditional covariance matrix of excess returns on individual assets respectively. The solution to the optimization problem is

$$W^{CoSR} = \arg\underset{W}{max}\, \{CoSR\}, \quad \text{s.t. } \mathbf{1}^T W = 1. \tag{8}$$

This portfolio optimization problem can be solved analytically under the absence of short-selling constraints. To do this, we first solve for the conditional efficient frontier among all assets. That is, given a desired conditional expected excess return level e, we find the portfolio weights W^* that minimize the risk measure.[1] The optimization problem becomes

$$W^* = \arg \underset{W}{min} \ \frac{1}{2} CoSD, \quad \text{s.t. } \mu^T W = e, \text{ and } \mathbf{1}^T W = 1. \tag{9}$$

Expression (9) is a convex optimization problem since the objective function is convex and is subject to affine constraints. Furthermore, the Slater's condition is satisfied, hence the first order conditions are necessary and sufficient for an optimum. The Lagrangian of this problem is $\mathcal{L} = \frac{1}{2} W^T \Sigma W - \lambda_1(\mu^T W - e) - \lambda_2(\mathbf{1}^T W - 1)$, that yields the following first order condition with respect to W: $\frac{\partial \mathcal{L}}{\partial W} = \Sigma W - \lambda_1 \mu - \lambda_2 \mathbf{1} = 0$. Assuming that Σ is full rank, we obtain $W = \lambda_1 \Sigma^{-1} \mu + \lambda_2 \Sigma^{-1} \mathbf{1}$. Now we need to solve for multipliers λ_1 and λ_2. Using the portfolio constraints $\mu^T W = e$ and $\mathbf{1}^T W = 1$, we have

$$\begin{cases} \lambda_1 \mu^T \Sigma^{-1} \mu + \lambda_2 \mu^T \Sigma^{-1} \mathbf{1} = e, \\ \lambda_1 \mathbf{1}^T \Sigma^{-1} \mu + \lambda_2 \mathbf{1}^T \Sigma^{-1} \mathbf{1} = 1. \end{cases} \tag{10}$$

Let $s_{\mu\mu} = \mu^T \Sigma^{-1} \mu$, $s_{1\mu} = \mu^T \Sigma^{-1} \mathbf{1}$ and $s_{11} = \mathbf{1}^T \Sigma^{-1} \mathbf{1}$, and $A = \begin{pmatrix} s_{\mu\mu} & s_{1\mu} \\ s_{1\mu} & s_{11} \end{pmatrix}$, with $A = \widetilde{\mu}^T \Sigma^{-1} \widetilde{\mu}$, and $\widetilde{\mu} = (\mu \ 1)^T$. The system of equations (10) can be rewritten in matrix form as $A\lambda = \widetilde{e}$, with $\lambda = (\lambda_1 \ \lambda_2)^T$ and $\widetilde{e} = (e \ 1)^T$. The matrix A is positive definite and, hence, invertible such that $\lambda = A^{-1} \widetilde{e}$. Replacing the value of W obtained above, we obtain the optimal portfolio weights $W^* = \Sigma^{-1} \widetilde{\mu} A^{-1} \widetilde{e}$. The portfolio W^* is the minimum conditional variance portfolio for a given conditional mean e and such that $\mathbf{1}^T W = 1$ is satisfied. The conditional variance frontier can be expressed as

$$CoSD^* = W^{*T} \Sigma W^* = \widetilde{e}^T A^{-1} \widetilde{e} = \frac{s_{11} e^2 - 2 s_{1\mu} e + s_{\mu\mu}}{s_{11} s_{\mu\mu} - s_{1\mu}^2}. \tag{11}$$

Now we can find the portfolio with maximum CoSR among all portfolios W^* located on

[1] The conditional variance of the portfolio's excess return (i.e., CoSD) is divided by two in the optimization problem. This is merely for algebraic convenience and does not change the solution to the optimization problem.

the efficient frontier. Hence, the optimization problem (8) can be written as

$$W^{CoSR} = \arg\max_{W^*} \frac{CoER}{CoSD^*} = \arg\max_{W^*} \frac{e}{\sqrt{\frac{s_{11}e^2 - 2s_{1\mu}e + s_{\mu\mu}}{s_{11}s_{\mu\mu} - s_{1\mu}^2}}}. \tag{12}$$

The first order condition of this problem with respect to the objective expected reward e is $\partial \left(\frac{e}{\sqrt{s_{11}e^2 - 2s_{1\mu}e + s_{\mu\mu}}} \right) / \partial e = 0$, which yields $e = \frac{s_{\mu\mu}}{s_{1\mu}}$. Therefore, the optimal portfolio weights defining the CoSR portfolio satisfy

$$W^{CoSR} = \Sigma^{-1}\widetilde{\mu}^T A^{-1} \begin{pmatrix} \frac{s_{\mu\mu}}{s_{1\mu}} \\ 1 \end{pmatrix} = \begin{pmatrix} \Sigma^{-1}\mu & \Sigma^{-1}\mathbf{1} \end{pmatrix} \begin{pmatrix} \frac{1}{s_{1\mu}} \\ 0 \end{pmatrix} = \frac{\Sigma^{-1}\mu}{\mu^T \Sigma^{-1}\mathbf{1}}. \tag{13}$$

It is often the case that we want to place additional constraints on the optimization - for instance we might want to restrict the portfolio weights so that none of the weights are greater than 25% of the overall wealth invested in the portfolio, or we might want to prohibit short selling allowing only long positions. This is a realistic scenario in settings characterised by systemic risk in which financial regulators ban short-selling to reduce short-term investment with speculative motives. Unfortunately, under short-selling restrictions ($W \geq \mathbf{0}$) the optimization problem (8) cannot be solved analytically and thus a numerical procedure must be employed. In our empirical application, we use the Solver function *fmincon* built in Matlab.

4 Simulation of return scenarios

Although CoSR has no closed-form expression in dynamic models when short-selling restrictions are imposed, we can still use a Monte Carlo simulation-based procedure to implement our systemic risk-based portfolio. The dynamic CoSR measure can be calculated using its empirical analog calculated from simulated returns over the subset of simulated crisis scenarios.

This section discusses two alternative multivariate settings to model dynamics of the returns of constituents of the investment portfolio and the market portfolio. First, we consider a semiparametric model in which the conditional mean and covariance matrix of the vector of returns is modelled parametrically. The return distribution is left unmodelled beyond these two moments and will be simulated using naive nonparametric bootstrap methods. As a robustness check, we also use a fully parametric model that allows for heavy tails and joint tail dependence in return distributions. To do this, we consider a Student's t-copula model for modelling the multivariate conditional distribution of returns.

The following subsections describe both approaches to generate the vector of assets and market portfolio returns. A detailed algorithm describing the simulation scheme is presented in Appendix B.

4.1 GARCH-DCC Modelling

The DCC model proposed by Engle (2002) can be seen as an extension to the constant conditional correlation (CCC) model developed by Bollerslev (1990), which captures the time-varying correlation of multivariate data. In this subsection, we use the GARCH-DCC model to describe the volatility dynamics and conditional correlations between returns on portfolio assets and the market index.

Let r_t be an $(N+1) \times 1$ vector of logarithmic returns. The last return, $r_{N+1,t}$ is the return on the market index, i.e. $r_{N+1,t} = r_{m,t}$. We propose an AR(1)-GJR-GARCH(1,1) model for the dynamics of returns such that

$$\begin{aligned} r_{i,t} &= \alpha_{i,0} + \alpha_{i,\mu}\, r_{i,t-1} + \xi_{i,t}, \\ \xi_{i,t} &= \sigma_{i,t}\ \varepsilon_{i,t}, \end{aligned} \tag{14}$$

where $\xi_{i,t}$ is the error term and $\varepsilon_{i,t}$ is an innovation process with $E_{t-1}(\varepsilon_{i,t}) = 0$ and $E_{t-1}(\varepsilon^2_{i,t}) = 1$; $\alpha_{i,0}$ and $\alpha_{i,\mu}$ are the parameters of the autoregressive process with $|\alpha_{i,\mu}| < 1$ to ensure stationarity of the process $r_{i,t}$ for $i = 1, \dots, N+1$. The DCC model of Engle (2002) is estimated in two steps. In the first step, the univariate GARCH models for each time series of returns are fitted and estimates of their conditional variances are thus obtained. In the second step, the standardized residuals $\varepsilon_{i,t} = \xi_{i,t}/\sigma_{i,t}$ are used to estimate the time-varying correlation matrix. More formally, the conditional variance process is defined as $H_t = D_t P_t D_t$, with $P_t = [\rho_{ij,t}]$ the conditional correlation matrix and D_t a diagonal matrix with time-varying standard deviations on the diagonal. Thus,

$$\begin{aligned} D_t &= diag(\sigma_{1,t}, ..., \sigma_{N+1,t}), \\ P_t &= diag(q^{-1/2}_{11,t}, ..., q^{-1/2}_{N+1\,N+1,t})\, Q_t\, diag(q^{-1/2}_{11,t}, ..., q^{-1/2}_{N+1\,N+1,t}). \end{aligned} \tag{15}$$

To capture potential leverage effects that may be empirically relevant in periods of financial distress, the idiosyncratic conditional variance terms $\sigma^2_{i,t}$ are modelled as univariate GJR-GARCH models. For the GJR-GARCH(1,1) model the elements of H_t can be expressed as:

$$\sigma^2_{i,t} = \omega_i + (\alpha_i + \gamma_i I\{\xi_{i,t-1} < 0\})\xi^2_{i,t-1} + \beta_i\, \sigma^2_{i,t-1},\ \ i = 1, ..., N+1. \tag{16}$$

The quantity $Q_t = [q_{ij,t}]$ in (15) is a symmetric positive definite matrix which is specified as

$$Q_t = (1 - \theta_1 - \theta_2)\bar{Q} + \theta_1 \varepsilon_{t-1}\varepsilon^T_{t-1} + \theta_2 Q_{t-1}, \tag{17}$$

where $\bar{Q} = E(\varepsilon_t\, \varepsilon_t^T)$ is the unconditional covariance matrix of the standardized residuals ε_t obtained from the first step estimation; θ_1 and θ_2 are non-negative scalars satisfying $0 < \theta_1 + \theta_2 < 1$. The correlation estimator is given by $\rho_{ij,t} = \frac{q_{ij,t}}{\sqrt{q_{ii,t} q_{jj,t}}}$. Hereafter, we will refer to the above specified model as GARCH-DCC.

4.2 GARCH-Copula Modelling

An $(N+1)$-dimensional copula C is a multivariate distribution function on $[0, 1]^{N+1}$ with standard uniform marginal distributions. Following Sklar's theorem (Sklar 1959), any multivariate distribution, in our case the multivariate distribution function of the innovations of the above GARCH processes, can be decomposed into univariate margins and a certain copula, that is

$$F_{\varepsilon_1,...,\varepsilon_{N+1}}(u_1, ..., u_{N+1}) = C\left(F_{\varepsilon_1}(u_1), ..., F_{\varepsilon_{N+1}}(u_{N+1})\right), \tag{18}$$

where u_i is uniformly distributed on $(0, 1)$, $F_{\varepsilon_1,...,\varepsilon_{N+1}}$ denotes the joint cumulative distribution function and F_{ε_i} the corresponding marginal distribution functions of the innovations ε_i, for $i = 1, \ldots, N+1$.

In this subsection, we use a t-copula function to model the mutual dependence among standardized residuals. This copula function is given by

$$C^t_{\nu,\boldsymbol{\rho}}(u_1, ..., u_{N+1}) = \int_{-\infty}^{t_\nu^{-1}(u_1)} \cdots \int_{-\infty}^{t_\nu^{-1}(u_{N+1})} \frac{\Gamma(\frac{\nu+N+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{(\nu\pi)^{N+1}|\boldsymbol{\rho}|}} \left(1 + \frac{\boldsymbol{x}'\boldsymbol{\rho}^{-1}\boldsymbol{x}}{\nu}\right)^{-\frac{\nu+N+1}{2}} d\boldsymbol{x}, \tag{19}$$

where Γ is the gamma function, $\boldsymbol{\rho}$ is a correlation matrix, ν represents the degree of freedom both in margins and copula function. Note that if the t-copula and univariate t margins share the same degree of freedom ν, then we obtain a multivariate t distribution with ν degree of freedom as in (19). In our case, we assume that $F_{\varepsilon_1}, ..., F_{\varepsilon_{N+1}}$ are univariate t distributions with different degree of freedom parameters $\nu_1, ..., \nu_{N+1}$, thus we obtain a multivariate distribution function F_ν which has been termed as meta-t_ν distribution function (see Fang et al. 2002 for more details). In the following, we will refer to this model as GARCH-Copula.

4.3 CoSR estimation

To obtain the estimator of CoSR, we first estimate individual elements contained in μ_t based on the Monte Carlo average of the simulated arithmetic h-period firm returns, that is

$$\widehat{\mu}_{i,t} = \frac{\sum_{s=1}^{S} R_{i,t:t+h}^{s} \, I\{R_{m,t:t+h}^{s} < C\}}{\#SE}, \tag{20}$$

where S is the number of Monte Carlo simulations and $\#SE = \sum_{s=1}^{S} I\{R_{m,t:t+h}^{s} < C\}$ is the number of scenarios out of S affected by market distress. For each asset in the portfolio the filtered mean vector (average h-period ahead return conditional on a market distress episode) is given by $\widehat{\mu}_t = (\widehat{\mu}_{1,t}, ..., \widehat{\mu}_{N,t})^T$. Similarly, $\mu_{m,t}$ can be estimated as

$$\widehat{\mu}_{m,t} = \frac{\sum_{s=1}^{S} R_{m,t:t+h}^{s} \, I\{R_{m,t:t+h}^{s} < C\}}{\#SE}. \tag{21}$$

Thus the estimator of CoER can be written as

$$\widehat{CoER}_t = W_t^T \widehat{\mu}_t - \widehat{\mu}_{m,t}, \tag{22}$$

where W_t denotes the vector of portfolio weights that is known at time t. As for the CoSD, we first estimate the covariance matrix $\Sigma_{t|SE}$ using the Monte Carlo sample counterpart, with element (i, j) defined as

$$\widehat{\Sigma}_{t(i,j)|SE} = \frac{\sum_{s=1}^{S} \left(R_{i,t:t+h}^{s} - \widehat{\mu}_{i,t}\right) \left(R_{j,t:t+h}^{s} - \widehat{\mu}_{j,t}\right) I\{R_{m,t:t+h}^{s} < C\}}{\#SE - 1} \tag{23}$$

for $i, j = 1, \ldots, N$. Then, we estimate $\sigma^2_{m,t|SE}$ as

$$\widehat{\sigma}^2_{m,t|SE} = \frac{\sum_{s=1}^{S} \left(R^s_{m,t:t+h} - \widehat{\mu}_{m,t}\right)^2 I\{R^s_{m,t:t+h} < C\}}{\#SE - 1}. \tag{24}$$

Analogously, we obtain the estimator of $\sigma_{im,t|SE}$ as

$$\widehat{\sigma}_{im,t|SE} = \frac{\sum_{s=1}^{S} \left(R^s_{i,t:t+h} - \widehat{\mu}_{i,t}\right) \left(R^s_{m,t:t+h} - \widehat{\mu}_{m,t}\right) I\{R^s_{m,t:t+h} < C\}}{\#SE - 1}, \tag{25}$$

and hence $\widehat{\sigma}_{t|SE} = (\widehat{\sigma}_{1,t}, ..., \widehat{\sigma}_{N,t})^T$. Combining the above estimators together, we obtain the estimator of CoSD, that is

$$\widehat{CoSD}_t = \left(W_t^T \widehat{\Sigma}_{t|SE} W_t + \widehat{\sigma}^2_{m,t|SE} - 2W_t^T \widehat{\sigma}_{t|SE}\right)^{1/2}. \tag{26}$$

The estimator of $CoSR_t$ is expressed as $\widehat{CoSR}_t = \frac{\widehat{CoER}_t}{\widehat{CoSD}_t}$.

5 Empirical analysis

This section illustrates the performance of our systemic risk-based optimal portfolios. We compare the ex-post final wealth and cumulative logarithmic returns of portfolios obtained by maximizing two performance measures: the traditional Sharpe ratio (SR) corresponding to the mean-variance strategy and our CoSR measure that incorporates systemic events. We also add the naive equally-weighted portfolio $\omega_i = 1/N$, for $i = 1, \ldots, N$, and the GMVP as benchmarks. Finally, we compute portfolio's LRMES as the relevant portfolio-level systemic risk measure, which is defined below as the weighted sum of LRMES across the portfolio constituents.

5.1 Dataset

We use stock price data from the US market. Our sample contains 23 big financial firms that are either SIFIs or non-SIFIs. The Financial Stability Board (FSB), in consultation with Basel Committee on Banking Supervision (BCBS) and national authorities, has just identified the latest list of global systemically important financial institutions (G-SIFIs) in November of 2020.[2] The overall number of G-SIFIs contained in the list is

[2] https://www.fsb.org/wp-content/uploads/P111120.pdf

30, specifically 20 of them are traded on the US market. Besides, the Board of Governors of the Federal Reserve System also maintains a list of domestic systemically important financial institutions (D-SIFIs). This list includes those financial institutions not being big enough for G-SIFIs status, but still possess high enough domestic systemically importance, making them subject to the most stringent annual Stress Test (USA-ST) from the Federal Reserve. Despite the lack of any official D-SIFIs designation, the institutions being subject to the USA-ST can be considered to be D-SIFIs in the US.[3] According to the list released by Federal Reserve as of March 2014, 17 banks traded on US stock market were identified as D-SIFIs.[4]

The intensity of the computational simulation methods that we propose makes difficult to work with large sets of assets. In addition, the definition of the systemic risk measures also involves knowledge of financial information on firms beyond the stock price, which is not readily available for some firms. These two factors reduce the number of firms that we can consider in our empirical application. Thus we consider 16 firms within the group of SIFIs contained in the above two lists. All firms within the top three buckets (3.5%, 2.5% and 2.0%) of G-SIFIs list are included in our dataset.[5] A few remarks on computational complexity are given in the last section of Appendix B. In addition to the SIFIs, we also add 7 non-SIFIs into our dataset since we aim to find the best tradeoff between risk and return rather than only minimizing the underlying systemic risk of our portfolios. Our choice of non-SIFIs is motivated by Brownlees and Engle (2016), these authors also use these firms in their empirical study on systemic risk.

Historical return data on the stocks included in our dataset are retrieved from the Wharton Database website[6] over the period from January 3, 2000 to December 31, 2020 (5284 daily observations for each stock), and the panel is balanced since all firms have been trading continuously during the sample period. The price sequences are adjusted for splits based on split adjustment factors reported by both CRSP and Compustat. We proxy the market index with the S&P Composite Index, which will be later used as our benchmark when solving the portfolio optimization problem.

The full list of tickers and company names grouped by subindustry are Deposito-

[3] https://www.govinfo.gov/content/pkg/CHRG-113hhrg80873/pdf/CHRG-113hhrg80873.pdf
[4] https://www.federalreserve.gov/newsevents/press/bcreg/ccar_20140326.pdf
[5] The bucket approach is defined in Table 2 of the Basel Committee document (see https://www.bis.org/publ/bcbs255.pdf).
[6] https://wrds-www.wharton.upenn.edu/

ries: Bank of America (BAC), Citigroup (C), Synovus Financial (SNV), Truist Financial Corporation (TFC), HSBC Holdings (HSBC), JP Morgan Chase & Co (JPM), Barclays (BCS), Morgan Stanley (MS), State Street (STT), ING Groep (ING), Keycorp (KEY), Northern Trust (NTRS), PNC Financial Services (PNC) and Wells Fargo & Co (WFC); Insurance companies: Lincoln National (LNC), Progressive (PGR) and Global Life (GL); Broker-Dealers companies: Goldman Sachs (GS) and Schwab Charles (SCHW); and other financial companies: American Express (AXP), Franklin Resources (BEN), Blackrock (BLK) and Capital One Financial (COF). The reason for only including large financial institutions in our dataset is that they are more exposed to systemic risk than non-financial firms, especially during crisis times.

For illustrative purposes, Figure 1 presents a descriptive analysis of two big financial institutions (Citigroup and Goldman Sachs) as well as two non-financial counterparts (Squibb and Boeing). The main aim of this exercise is to highlight the systemic risk of large financial institutions as opposed to non-financial firms of similar size. By doing so, we aim to motivate the importance of our portfolio strategy for portfolios of assets that exhibit large individual systemic risk.

The left panel of Figure 1 reports the relative price movements for these firms. The initial level of each price series has been normalized to unity to facilitate the comparison of relative performance, and no dividend adjustments are explicitly taken into account. The evolution of S&P 500 Index in the out-of-sample period (2007-2020) is reported in the top right panel of Figure 1. The S&P 500 Index has experienced four dramatic declines over the analyzed period. The first one happened during 2007-2009 due to the subprime crisis, the second one took place over 2010-2012 due to the European sovereign debt crisis, the third one occurred at the beginning of 2016 due to a decline in oil prices, and the latest one broke out at the beginning of 2020 due to the Covid-19 pandemic. The bottom panels of Figure 1 illustrate the dynamics of SRISK and LRMES (see Appendix A for definitions of both systemic risk measures) for these four firms over the evaluation period. During the subprime crisis, both financial firms suffered great losses with a drawdown of around 80%, while the non-financial firms performed much better, with relatively small drops in asset prices.

The comparison of the SRISK and LRMES measures between financial and non-financial firms during the different crisis episodes reveals that financial firms contribute more to the overall market disruption than non-financial firms. We also observe the buildup of the systemic risk measure at the start of the different crises for the two financial firms but not for the non-financial firms. In particular, the SRISK of non-financial firms delivers lower volatilities and is always below zero throughout the out-of-sample period. It is interesting to note, for example, that despite the increase in the SRISK of Boeing during the Covid-19 pandemic its value remains negative. Brownlees and Engle (2016) argue that a negative SRISK indicates that the firm faces expected capital surpluses conditional on a market decline, i.e. the firm functions well and does not contribute to the overall systemic risk during times of crisis. Similar insights are obtained from the analysis of the dynamics of LRMES. This measure displays quite different patterns across firm groups over time. The LRMES of financial firms increases significantly before each crisis, which reflects the fact that the interconnections between financial institutions and the market become stronger during difficult times. However, the LRMES of non-financial firms does not exhibit violent fluctuations before or during crisis times. The lack of sensitivity of both systemic risk measures for both non-financial firms confirms the weak linkage between non-financial firms and the market.

These results show that our objective function is more relevant when the universe of assets includes large firms that are potentially systemic, although not necessarily classified as SIFIs. Therefore, in the remaining, we only focus on large financial firms when studying optimal portfolio allocation under market distress periods since these firms are more likely to affect and be affected by market declines during systemic risk episodes.

5.2 Empirical methodology

We demonstrate the superiority of the proposed portfolio selection procedure under stressed market conditions by comparing the results of the portfolios obtained from maximizing our CoSR measure against competitors used in the literature. We backtest our model over the period January 2007 to December 2020. The backtesting period has been chosen to include most of the recent financial crises. In particular, we use a rolling window of 1,500 daily historical returns to estimate the model parameters and then simulate 30,000 return scenarios from the above processes for each asset contained in the portfolio at the beginning of each month.

The portfolio optimization problem (8) with short-selling constraints is solved on a monthly basis by maximizing the proposed performance ratio CoSR based on generated return scenarios. To generate the return scenarios, we follow the two strategies discussed above. First, we apply a GARCH-DCC model for the dynamics of returns. After fitting the model, we use nonparametric bootstrap to resample the standardized residuals. These pseudo-samples are used as inputs of the GARCH and DCC filters respectively, to get the simulated monthly returns. The second approach is to use a GARCH-Copula model. After fitting the model, we simulate 30,000 independent random trials of mutually dependent standardized residuals over a one-month horizon based on the fitted t-copula. Using the simulated standardized residuals as inputs to the GARCH filter, we obtain 30,000 simulated monthly cumulative returns. We can estimate the reward and risk measures using the generated return distributions, i.e. compute the first and second conditional moments by filtering realizations that satisfy the SE condition. In particular, following Capponi and Rubtsov (2022), we choose the following two specifications for the systemic event threshold C: i) $C = 0$, i.e., rebalancing occurs when the market index experiences negative returns, and ii) $C = -6.7\%$ for monthly rebalancing, which corresponds to a 40% decrease in the market index over a six-month period. Although the second specification better captures a SE (i.e. a significant drop in the market index), we still want to see the differences in portfolio allocation between milder and stronger definitions of systemic risk. Thus we also test our portfolios on less severe market declines, which are represented by the first specification.

For comparison purposes, we also evaluate the performance of our CoSR portfolio against three other performance criteria, namely the mean-variance (SR) portfolio obtained from maximizing the Sharpe ratio, the equally-weighted portfolio (1/N), and the GMVP. The first refers to the portfolio on the mean-variance efficient frontier that has the highest expected return per unit of risk, the second strategy represents a well-diversified portfolio of assets, and the last is the portfolio on the mean-variance efficient frontier with minimum variance. Moreover, the portfolio strategy maximizing CoSR is related to the SR portfolio since it is obtained by adjusting the latter to account for systemic risk events (see equations (5) and (6)). To avoid the construction of portfolios with large negative allocations to all assets under stressed market conditions, we assume that short-selling is not allowed in our model. Furthermore, we assume that our investors have an initial wealth of $FW_0 = 1$ and an initial cumulative logarithmic return $CR_0 = 0$ at the beginning of the backtesting period.

Three main steps are performed to calculate the ex-post final wealth and cumulative return at the k-th recalibration ($k = 0, 1, 2, ..., 168$). Firstly, we choose a performance ratio. Second, we generate return scenarios based on the algorithms described above and obtain the solution W^*_{k+1} to the optimization problem (7). This step is performed in Matlab using the *fmincon* function. Following Kresta et al. (2015), we randomly choose 20 starting points in order to find the global instead of local minimum when solving (7). Secondly, the ex-post final wealth is given by

$$FW_{k+1} = FW_k(1 + W_k^{*T} R_{k+1}), \tag{27}$$

where R_{k+1} is the ex-post vector of simple returns between k and $k + 1$. Thirdly, the ex-post cumulative logarithmic return is given by

$$CR_{k+1} = CR_k + ln(1 + W_k^{*T} R_{k+1}). \tag{28}$$

Note that the latter measure reports the cumulative performance of the portfolio net of wealth. That is, expression (27) implies that $FW_{K+1} = FW_0 \prod_{k=0}^{K} (1 + W_k^{*T} R_{k+1})$. Then, taking logs, we obtain $\ln FW_{K+1} - \ln FW_0 = \sum_{k=0}^{K} \ln(1 + W_k^{*T} R_{k+1})$. Therefore, the growth in wealth due to the cumulative return on the portfolio is given by expression (28), with $CR_0 = 0$.

By repeatedly computing FW_{k+1} and CR_{k+1} for different performance ratios we obtain the wealth and cumulative return path evolutions over the evaluation period and the final wealth and total return accumulated at the end of the period. For simplicity, we neglect transaction costs for now. The influence of transaction costs will be further studied later.

5.3 Empirical results

In this section, we present the backtesting results. First, we show the results of the portfolio optimization exercise using the GARCH-DCC and GARCH-Copula models, respectively. Second, we study the influence of adding transaction costs to the results. We also compute confidence intervals to our estimates of final wealth paths to account for the uncertainty arising from model estimation.

The empirical results of the portfolio optimization backtesting using the GARCH-DCC model are depicted in Figure 2. There are several noticeable features from these figures. Firstly, all portfolios perform badly during the 2007-2008 financial crisis, no matter which model is chosen. In general, the CoSR portfolio with $C = -6.7\%$ outperforms the other competitors throughout the evaluation period. Final wealth is maximized when investors use the CoSR as objective function, the second strategy is the SR portfolio and the worst performance with regards to final wealth is the GMVP. In contrast, when the systemic event is defined by a milder threshold (i.e. $C = 0$) the results vary. In this scenario the CoSR portfolio does not outperform the competing portfolios consistently but it is still more resilient to crises than the other three portfolios. Losses are significantly smaller during these periods. This observation also reflects the importance of choosing a proper systemic event threshold for portfolio selection. Conditioning on a mild threshold may jeopardize return at the expense of a more conservative portfolio allocation.

Table 1 confirms that the CoSR portfolio with $C = -6.7\%$ provides the best performance. An investor would multiply their wealth by 2.280 using the SR strategy, by 1.343 using the equally-weighted strategy, by 1.323 using the GMVP, while following the proposed CoSR strategy the final wealth would be around triple (2.794 for $C = 0$ and 3.021 for $C = -6.7\%$ respectively). Similarly, the annual return of the CoSR portfolio with $C = -6.7\%$ is 8.22%, which is about two percentage points above the SR portfolio given by 6.06%. The annual return for the equally-weighted portfolio and the GMVP are 2.13% and 2.02% respectively.

Another factor the investor would care about is the risk of the strategy. The CoSR strategy not only outperforms the other competing strategies in terms of profitability but also the maximum drawdown decreases, which is an important indicator of portfolio performance for portfolio managers. While SR, 1/N and GMVP strategies lost near 70% (74.22%, 71.74%, and 67.21%, respectively) of their values during the 2007-2008 financial crisis, the maximum drawdown of CoSR was around 60% for both thresholds. Similar findings are obtained for the other three major crisis episodes. In these cases there is also a drop in profitability of the strategy but this drop is smaller compared to the 2007-2009 period. To add robustness to the results, we repeat the analysis for the copula model. The results are very similar to those obtained for the GARCH-DCC model. The empirical results of the portfolio optimization backtesting are depicted in

Table 1: Final wealth and maximum drawdown of particular wealth paths based on GARCH-DCC model.

Strategy	SR	CoSR(C=0)	CoSR(C=-6.7%)	1/N	GMVP
Final Wealth	2.280	2.794	3.021	1.343	1.323
Annual Return	6.06%	7.62%	8.22%	2.13%	2.02%
Maximum Drawdown	74.22%	61.55%	58.75%	71.74%	67.21%

Figure 3. There are several noticeable features from this figure. All portfolios perform badly during 2007-2008 financial crisis, no matter which model is chosen. The SR, 1/N and GMVP strategies lose almost all of their value during that period, while the CoSR portfolio performs much better but still lose more than 50% of its value. The SR portfolio is a serious competitor and reports similar profitability figures to the CoSR during the first half of the evaluation period, however, from the second semester of 2016, the CoSR portfolio consistently beats the SR portfolio. Overall, the CoSR portfolio has a strong upward trend in profitability that results in superior performance over time. This strong performance is due to its relatively stable performance in times of market downturns. Table 2 summarizes earnings and maximum drawdown of different strategies. The SR portfolio provides the worst performance in terms of final wealth whereas the maximum drawdown is comparable to the maximum drawdown of the equally-weighted portfolio (73.86% for SR and 71.74% for 1/N). Both systemic event thresholds provide similar performance, where the CoSR portfolio with $C = 0$ provides the highest value of final wealth (annual return) and the lowest maximum drawdown.

Table 2: Final wealth and maximum drawdown of particular wealth paths based on GARCH-Copula model.

Strategy	SR	CoSR(C=0)	CoSR(C=-6.7%)	1/N	GMVP
Final Wealth	1.299	2.423	2.134	1.343	1.423
Annual Return	1.88%	6.53%	5.56%	2.13%	2.55%
Maximum Drawdown	73.86%	59.98%	61.07%	71.74%	67.43%

Portfolio diversification for portfolios of SIFI firms:

As an additional robustness exercise, we repeat the portfolio allocation exercise for the subset of the firms in our study that are classified by the Financial Stability Board (FSB) and the Basel Committee on Banking Supervision (BCBS) as G-SIFIs and by the Board of

Governors of the Federal Reserve System as D-SIFIs. In particular, we consider 16 firms. This exercise may be interesting to highlight the importance of portfolio diversification in a setting where all the assets in the portfolio are affected by systemic risk. Note that in the above exercises some firms were within the pool of SIFIs but others were not.

The results of this exercise are reported in Figure 4. The top panel of this figure shows the outperformance of the CoSR portfolio compared to the competitors for the GARCH-DCC model, however, as expected, there is a sizeable drop in profitability for all portfolios compared to the portfolios also considering non-SIFIs, see Figure 2. The analysis of the GARCH-Copula model shows similar results, however, in this case, the equally-weighted portfolio is the top contender, followed by the CoSR with threshold C equal to zero.

Portfolio turnover and transaction cost:

We use the definition of portfolio turnover in Kirby and Ostdiek (2012), which is consistent with the concept used in the mutual fund industry. This measure provides an indication of the variability of the portfolio weights over time. Table 3 reports the turnover rates for all the portfolios under investigation. This table shows that portfolio optimization strategies based on the maximization of CoSR are characterized by relatively high turnover rates. Unsurprisingly, the turnover rates are much smaller for the equally-weighted portfolio than for the remaining competitors. In contrast, both CoSR portfolios take larger values, which suggests that these portfolios are more flexible than the competitors to adapt to changes in market conditions.

Table 3: Comparison of turnover rates.

Strategy	SR	CoSR(C=0)	CoSR(C=-6.7%)	1/N	GMVP
GARCH-DCC	0.250	0.356	0.298	0.025	0.236
GARCH-Copula	0.241	0.431	0.361	0.025	0.253

On the other hand, an increase in portfolio turnover entails an increase in transaction costs due to higher fees and other costs derived from modifying the portfolio allocation. We proceed to analyze the impact on portfolio performance of including these costs. To do this, we recompute the ex-post final wealth and the total return for all portfolios considering proportional transaction costs. In order to stress test the impact of transaction costs, we adopt 5 basis points as proportional transaction costs. Tables 4 and 5 report the results in this case. Figures 5 and 6 also illustrate the difference in portfolio performance for the DCC and copula models, respectively. The presence of transaction costs does not alter the results.[7]

Table 4: Recomputation results based on GARCH-DCC model with transaction costs.

Strategy	SR	CoSR(C=0)	CoSR(C=-6.7%)	1/N	GMVP
Final Wealth	2.186	2.632	2.873	1.338	1.272
Annual Return	5.74%	7.16%	7.83%	2.10%	1.73%
Maximum Drawdown	74.33%	61.76%	58.96%	71.77%	67.37%

Table 5: Recomputation results based on GARCH-Copula model with transaction costs.

Strategy	SR	CoSR(C=0)	CoSR(C=-6.7%)	1/N	GMVP
Final Wealth	1.247	2.254	2.008	1.338	1.364
Annual Return	1.59%	5.98%	5.11%	2.10%	2.24%
Maximum Drawdown	73.99%	60.23%	61.31%	71.77%	67.60%

Portfolio systemic risk measure:

Our portfolios are constructed to maximize the Sharpe ratio conditional on the market being under distress. This ratio can be viewed as a measure of risk-adjusted profitability under market distress, with the latter interpreted as a systemic event. In order to assess the underlying systemic risk of such portfolios we define portfolio's LRMES as

$$LRMES_{p,t} = \sum_{i=1}^{N} \omega_{i,t} LRMES_{i,t}. \quad (29)$$

This measure is a weighted combination of the LRMES of the individual firms at each point in time. Interestingly, the portfolio's LRMES can be interpreted as the expected percentage drop in portfolio value under stressed market conditions. Thus a lower value of $LRMES_p$ reflects a lower level of potential loss during crisis times. This quantity can be estimated based on the generated return scenarios obtained from the GARCH-DCC and GARCH-Copula models.

[7] Unreported results show the effect of transaction costs of different magnitude on portfolio performance. More specifically, we obtain the results of the CoSR portfolio with $C = -6.7\%$ for the best performing strategy - GARCH-DCC approach - assuming transaction costs that range from 0 to 10 basis points. The results confirm the profitability of the CoSR strategy across different levels of the transaction costs. The CoSR strategy always outperforms the 1/N portfolio and GMVP.

Figure 7 displays portfolios' LRMES paths obtained from different investment strategies over the out-of-sample evaluation period for the GARCH-DCC and GARCH-Copula models, respectively. The LRMES of the CoSR portfolio is relatively stable across the evaluation period and is always lower than for the other benchmark portfolios. This forward-looking measure can serve as an early warning indicator or monitoring tool for both portfolio managers and financial regulators who aim to control the losses of their portfolios, especially during crisis times.

An important feature of the portfolio allocation exercise is to study the variation of the portfolio across assets and over time. The optimal weights are shown in Figure 8. Here, we set $C = -6.7\%$ for both GARCH-DCC and GARCH-Copula models when computing optimal weights and LRMES. Firms that receive greater allocations of wealth under the optimal CoSR portfolio strategy are more attractive from a systemic risk-return perspective. Interestingly, the empirical results in Figure 8 show that the optimal CoSR portfolio is less diversified than the SR portfolio during crisis times after accounting for systemic risk. For instance, the CoSR portfolio implies a relatively high investment proportion in PGR while the SR portfolio invests more in BEN across the evaluation period. An interpretation of this result is that investors anticipate a systemic risk event in advance. As a result, investors prefer to sacrifice diversification benefits and gain from the reduced exposure of their portfolios to stressed market conditions (see also Capponi and Rubtsov 2022). These insights of the model provide an alternative interpretation to the presence of underdiversification compared to standard mean-variance efficient allocations, see Mitton and Vorkink (2007) and references therein. In our model, underdiversification takes place because the CoSR portfolio is less likely to suffer great losses during a market slide. Figure 9 shows that the LRMES of PGR is always lower than the LRMES of BEN. This difference becomes even larger during distress episodes.

5.4 Estimation effects on optimal portfolio allocation

Throughout the study, we have considered two different specifications (GARCH-DCC and GARCH-Copula) to model the joint dynamics of financial returns. This exercise has provided robustness to our results against the presence of model uncertainty. Another related exercise is to study the impact of parameter uncertainty. In this case the objective is to assess the impact of parameter estimation on the outcome of the model. In our setting, the outcomes of the model are estimates of the final wealth and portfolio return. This exercise is particularly important in our setting as our model is heavily parametrized as it is custom in multivariate time series models. Alternative nonparametric solutions suffer instead from the curse of dimensionality as the number of variables in the model grows beyond a few dimensions.

In this section we assess the impact of estimation error. The uncertainty arises because of the parameter estimation error but also because of the randomness in choosing starting values in the portfolio optimization. As mentioned before, we randomly choose 20 starting points when solving portfolio optimization problems in order to find global instead of local optima. In the backtesting exercise we follow a rolling window approach starting initially from the beginning of 2007 with a window size of 1,500 observations. After fitting the different models within each window, the estimated parameters for predicting one-month ahead returns are obtained before re-estimating the same model with additional observations. The prediction of returns obtained from each model and the corresponding portfolio optimization are done on a monthly basis by updating the in-sample dataset. Motivated by the need of gauging the underlying estimation uncertainty, the whole procedure is repeated multiple times with the same methodology. By doing so, we obtain multiple portfolio path realizations throughout the out-of-sample period.

Figures 10 and 11 show the ex-post final wealth paths for different strategies after accounting for estimation uncertainty. For instance, the curve "CoSR_Average" reflects the average of the 200 portfolio paths, which is embedded into the corresponding 90% confidence bounds centered around the average. The grey shadow area reflects the uncertainty arising from the model estimation, return prediction and portfolio optimization underlying the 200 simulation exercises. The corresponding results for other competitors are also displayed therein. The results of both approaches displayed in Figures 10 and 11 confirm the statistical significance of the previous evidence on the superiority of the CoSR portfolios over the competing benchmark portfolios in all cases.

5.5 An alternative objective function for portfolio allocation

An alternative strategy to incorporate systemic risk in the portfolio allocation problem is to replace the denominator in (1) by the LRMES of portfolio's excess return. By doing this, we develop a new performance measure that we call mean-MES ratio (MMR):

$$\begin{aligned} MMR_t(R_{p,t}) &:= \frac{E_t(R_{p,t:t+h} - R_{m,t:t+h})}{-E_t(R_{p,t:t+h} - R_{m,t:t+h}|SE_{t:t+h})}, \\ &= \frac{W_t^T \mu_t - \mu_{m,t}}{\mu_{m,t|SE} - W_t^T \mu_{t|SE}}. \end{aligned} \tag{30}$$

If we set $C = VaR_\alpha(R_{m,t:t+h})$, then the above expression can be rewritten as

$$MMR_t(R_{p,t}) = \frac{W_t^T \mu_t - \mu_{m,t}}{LRMES_{p,t} - ES_{m\alpha,t}}, \tag{31}$$

where $ES_{m\alpha,t} = ES_\alpha(R_{m,t:t+h})$, and ES_α is defined as $ES_\alpha(X) = -E(X|X \leq VaR_\alpha(X))$ if we assume a continuous distribution for the probability law of X. The risk measure in the denominator can be decomposed into the difference between portfolio's LRMES and the ES of market return. MMR is able, by construction, to measure the tradeoff between portfolio's mean return and systemic risk, which formulates a new mean-ES model that accounts for systemic risk.

In what follows, we present the backtesting results for the portfolios obtained under the MMR objective function. We first show the results of the portfolio optimization exercise using GARCH-DCC and GARCH-Copula models, respectively. Then we study the systemic risk of MMR portfolio and compare against the CoSR portfolio proposed as our main objective function above. We also compute the confidence intervals of the ex-post final wealth paths to account for the uncertainty arising from the model estimation procedure.

Backtesting results

The backtesting results of GARCH-DCC and GARCH-Copula model including the MMR optimal portfolios are illustrated in Figure 12 and 13, respectively. These portfolios provide the best out-of-sample performance in terms of cumulative return over the evaluation period. The second competitors are the CoSR portfolios studied earlier whereas the remaining competitors perform clearly below these two investment portfolios that are focused on minimizing the effect of systemic events. Table 6 extends Table 1 by replacing the CoSR statistics by the MMR values. An investor will multiply his/her wealth by 2.280 using SR strategy, by 1.343 using 1/N strategy, by 1.323 using GMVP, while following the proposed MMR portfolio the final wealth would be more than sextuple (6.627) for $C = 0$ and triple (3.734) for $C = -6.7\%$. Similarly, the MMR portfolio with $C = 0$ gives an annual return of 14.46%, which is more than double the annual return of the SR portfolio (6.06%). The MMR portfolio with $C = -6.7\%$ performs slightly worse but still beats the other competitors with an annual return of 9.87%. The annual return for the naive and GMVP are 2.13% and 2.02%, respectively.

To add robustness to the results, we repeat the analysis using the GARCH-Copula model. The backtesting results are illustrated in Figure 13. The SR, 1/N and GMVP lost almost all of their value during that period, while the MMR portfolios perform much better but still lost more than half of their value. The MMR portfolio with $C = 0$ provides the best performance, which is the same as we conclude from the GARCH-DCC model. However, the level of profitability is much lower compared to the previous counterparts. The CoSR presents strong performance in the second part of the evaluation period clearly beating the other portfolios but not the MMR in terms of profitability.

Table 7 summarizes the earnings and maximum drawdown of the different portfolios. Results for the CoSR portfolios are found in Table 2 and not reported here again. The MMR portfolio with mild systemic event threshold provides the best performance in terms of final wealth (3.222), while the MMR portfolio with $C = -6.7\%$ gives the lowest maximum drawdown (63.30%) among the competitors.

Table 6: Backtesting results based on GARCH-DCC model.

Strategy	SR	MMR(C=0)	MMR(C=-6.7%)	1/N	GMVP
Final Wealth	2.280	6.627	3.734	1.343	1.323
Annual Return	6.06%	14.46%	9.87%	2.13%	2.02%
Maximum Drawdown	74.22%	34.96%	41.45%	71.74%	67.21%

Table 7: Backtesting results based on GARCH-Copula model.

Strategy	SR	MMR(C=0)	MMR(C=-6.7%)	1/N	GMVP
Final Wealth	1.299	3.222	2.040	1.343	1.423
Annual Return	1.88%	8.72%	5.22%	2.13%	2.55%
Maximum Drawdown	73.86%	63.57%	63.30%	71.74%	67.43%

The MMR portfolio is clearly a strong portfolio candidate under market distress in terms of cumulative return, however, its exposure to systemic risk is significantly larger than for the CoSR portfolio. Figure 14 presents the dynamics of the LRMES of the different portfolios over the evaluation period. For both GARCH-DCC and GARCH-Copula methodologies and different values of C, the CoSR portfolio exhibits values of the LRMES statistic well below the other portfolios. This observation provides strong support to the CoSR against the MMR portfolio once we jointly consider the profitability measures given by the ex-post final wealth and cumulative return and the systemic risk measure given by portfolio's LRMES.

Another advantage of CoSR strategies compared to MMR portfolios is the excess variability in final wealth and cumulative return of the latter class of investment strategies. The results of the robustness exercise for both GARCH-DCC and GARCH-Copula accounting for estimation uncertainty obtained from 200 trials are illustrated in Figure 15 and 16, respectively. The solid lines reflect the average of 200 portfolio paths, while the shaded areas represent the corresponding 90% confidence bounds centered around the average. The simulations suggest that MMR portfolios tend to suffer bigger losses than CoSR portfolios under market distress after accounting for estimation uncertainty. It is also worth noting that MMR portfolios are more sensitive to estimation error than the CoSR strategies, which makes their performance more volatile (the variance of the final wealth paths is much bigger than other competitors).

6 Conclusions

Although the existing systemic risk measures are helpful for financial regulators, portfolio managers are still looking for practical guidance under which they can account for systemic events during their decision-making process. A general approach for constructing optimal portfolios is to maximize a reward-to-risk ratio. In this paper we propose a systemic Sharpe ratio as the investor's objective function that conditions on the market return being under the threshold of a systemic event. By doing so, we propose a methodology for portfolio construction that explicitly incorporates the sensitivity of portfolio performance to systemic risk events. Using this objective function, we solve the portfolio allocation problem analytically under the absence of short-selling restrictions and numerically when short-selling restrictions are imposed. This approach for obtaining an optimal portfolio allocation is made operational by embedding it in a dynamic setting and simulating the returns on the portfolio assets using Monte Carlo return scenario analysis.

We have applied the above model to a basket of 23 assets of big financial firms trading in the US stock market over an out-of-sample evaluation period spanning 2007 to 2020. The results of the empirical study confirm the outperformance of our systemic risk portfolio against the standard mean-variance formulation, the naive equally-weighted portfolio, and the global minimum variance portfolio. The systemic risk portfolio is, by construction, more resilient in periods of market distress and remains competitive in non-crisis periods. This portfolio is less diversified than benchmark portfolios during crisis times. In these periods, the systemic risk strategy invests on those stocks that are expected to experience a small loss under stressed market conditions. In contrast to an emerging literature that suggests that the presence of underdiversification in financial markets is a rational response to a preference for positive skewness, we find that investors take conservative positions on a few stocks that are resilient against systemic risk to shield against potential large drawdowns in portfolio value.

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Appendix A - A brief review of systemic risk measures

This section reviews the different definitions of systemic risk measures proposed in the macro-finance literature. In particular, we review four prominent market-based measures that constitute the building blocks of an emerging literature on systemic risk. First, we review the MES and SES measures introduced in Acharya et al. (2017). The first measure is defined as the expected decrease of an institution's net equity return conditional on a market decline:

$$MES_{i,t|t-1} = -E_{t-1}(R_{i,t}|R_{m,t} < C), \tag{32}$$

where C denotes a given threshold defining the magnitude of a systemic event and $E_{t-1}(\cdot)$ denotes the expectation operator conditional on the information available to the individual up to time $t-1$. This measure gauges how a specific institution's risk exposure contributes to the system-wide risk. Financial institutions with higher MES contribute more to the overall risk of the financial market, thus these institutions can be seen as systemically dangerous. The SES extends the MES and measures the amount an institution's equity would drop below its target level (defined as the prudential capital fraction k of assets) in case of a future crisis when aggregate capital is less than k times aggregate assets:

$$\frac{SES_{i,t|t-1}}{W_{i,t}} = kL_{i,t} - 1 - E_{t-1}\left(R_{i,t}\middle| \sum_{i=1}^{N} W_{i,t} < k\sum_{i=1}^{N} A_{i,t}\right), \tag{33}$$

where $L_{i,t} = (A_{i,t}/W_{i,t})$ is the quasi leverage ratio, $A_{i,t} = (D_{i,t} + W_{i,t})$ is the total assets, $D_{i,t}$ is the book value of debt, $W_{i,t}$ is the market value of equity, and N denotes the number of financial firms within the system. Acharya et al. (2017) also show that the conditional expectation term can be expressed as a linear function of MES:

$$SES_{i,t|t-1} = W_{i,t}(kL_{i,t} - 1 + \theta MES_{i,t|t-1} + \Delta_i), \tag{34}$$

where θ and Δ_i are constant terms (see also Benoit et al. 2017). The higher the SES is, the higher the contribution of the financial institution to the system's overall risk. Acharya et al. (2017) provide detailed theoretical justification for the positive correlation between SES and a firm's MES and leverage.

The second measure that we review is SRISK. It was initially proposed by Acharya et al. (2010), and later extended to a conditional version by Brownlees and Engle (2012, 2016). SRISK measures the expected capital shortfall conditional on a systemic event and can be expressed as a function of the institution's size, leverage, and long-run MES (LRMES):

$$SRISK_{i,t+h|t} = max\left[0; W_{i,t}\big(kL_{i,t} - 1 + (1-k)LRMES_{i,t+h|t}\big)\right], \tag{35}$$

where $LRMES_{i,t+h|t}$ denotes the expected equity loss of firm i conditional on a systemic event over a time horizon h (usually one month or six months):

$$LRMES_{i,t+h|t}(C) = -E_t(R_{i,t:t+h}|R_{m,t:t+h} < C). \tag{36}$$

Here we denote the multiperiod simple firm (market) return between time t and $t+h$ as $R_{i,t:t+h}$ ($R_{m,t:t+h}$) and the systemic event (SE) as $\{R_{m,t:t+h} < C\}$. SRISK extends the MES by taking account of both size and debt of financial institutions. The institution with highest SRISK is seen as the main contributor to a crisis and is taken as the most systemically important. It is worth noting that the mathematical expressions for SRISK and SES are almost the same, since they are both comprised of three components: firm size, leverage, and marginal risk. At the aggregate level, SRISK can be thought of as a stress test on the whole financial system, where the adverse case scenario is defined as a 10% (40%) decrease of the market index over a one-month (six-month) time horizon.

Another important systemic risk measure is ΔCoVaR. The marginal risk measure given by VaR is used for measuring the tail risk of a portfolio or an individual firm, however, it fails to take into account spillover effects from other institutions and is highly pro-cyclical. To overcome these shortcomings, Adrian and Brunnermeier (2016) modify the VaR measure and present ΔCoVaR. This measure is defined as the difference between the VaR of the financial system conditional on a particular institution being in distress and the VaR of the financial system conditional on that institution being in its median state:

$$\Delta CoVaR_{i,t}(\alpha) = CoVaR_t^{m|C(r_{i,t})} - CoVaR_t^{m|C_{Median}(r_{i,t})}, \tag{37}$$

where the CoVaR corresponds to the VaR of the market return conditional on a certain event $C(r_{i,t})$ observed for firm i:

$$P\big(R_{m,t} \leq CoVaR_t^{m|C(r_{i,t})} \big| C(r_{i,t})\big) = \alpha. \tag{38}$$

Various definitions of $C(r_{i,t})$ can be adopted to define a systemic event for firm i, for example, Adrian and Brunnermeier (2016) define it as the institution's loss equals to its VaR, while Girardi and Ergün (2013) consider the case where the institution's loss exceeds its VaR level. A higher ΔCoVaR indicates higher systemic risk of the financial institution for a given level of idiosyncratic risk.

Appendix B - Simulation of stock returns

Algorithm for GARCH-DCC Model

This section describes the simulation algorithm for constructing return predictions based on nonparametric bootstrap approach (see Brownlees and Engle 2016). Specifically, we are interested in computing the portfolio's expected excess return over next month conditional on a systemic event during that period. In the following, we assume that all parameters are known, while in practice we estimate the model parameters using all available information up to the current time. Let h denote the length of the forecasting horizon on which the returns will be simulated, which in our case is 22 trading days. The details of the bootstrap procedure are discussed as follows:

- Construct the GJR-DCC standardized innovations: $\epsilon_t = (L_t)^{-1}\varepsilon_t, \quad t = 1, ..., T$, where L_t denotes the lower-triangular matrix in the Cholesky decomposition of the correlation matrix R_t.

- Sample with replacement $S \times h$ vectors of standardized innovations ϵ_t. This provides us with S pseudo-samples, $\epsilon^s_{T+t}, \quad t = 1, ..., h,\ s = 1, ..., S$, of length h (i.e. 22 days) of the GJR-DCC innovations.

- Use the pseudo-samples obtained from the previous step as inputs of the GJR-DCC filters, and set initial values as the last values of returns $r_{i,T}$, error terms $\xi_{i,T}$, pseudo correlation matrix R_T and variances $\sigma^2_{i,T}$. This yields S simulated paths of logarithmic returns between period $T+1$ and $T+h$ for all firms and the market index conditional on the realized process up to T, that is, we obtain $r^s_{i,T+t}|\mathcal{F}_T, \quad t = 1, ..., h\,, \quad s = 1, ..., S$, with $\mathcal{F}_T$ denoting the information set available to the individual at time T.

- Calculate the arithmetic multi-period return for each simulated path as

$$R^s_{i,T:T+h} = exp\left(\sum_{t=1}^{h} r^s_{i,T+t}\right) - 1. \tag{39}$$

The simulated monthly returns are utilized to solve the portfolio optimization problems specified in (7) and (9).

Algorithm for GARCH-Copula Model

In the GARCH-Copula setting, we generate future return scenarios according to the following procedure:

- Given the standardized residuals $\varepsilon_t = (\varepsilon_{1,t}, ..., \varepsilon_{N+1,t})$ $(t = 1, ..., T)$ from the fitted GARCH models, we estimate the cumulative distribution function (CDF) F_{ν_i} of each series with a univariate t location-scale distribution, $\varepsilon_{i,t} \sim F_{\nu_i} \quad t = 1, ..., T$.

- Transform the marginal distribution functions to uniforms with the empirical CDFs: $u_{i,t} = F_{\nu_i}(\varepsilon_{i,t}) \quad t = 1, ..., T$, where $u_{i,t} \sim U(0, 1)$.

- Given the transformed uniform margins, now we estimate the scalar degrees of freedom parameter ν and the linear correlation matrix $\boldsymbol{\rho}$ of the t-copula using the Matlab function *copulafit* (see The MathWorks 2019).

- Simulate mutually dependent returns by first simulating the corresponding dependent standardized residuals ϵ^s_{T+t} $(t = 1, ..., h,\ s = 1, ..., S)$. To do so, we first simulate dependent uniform variates u^s_{T+t} $(t = 1, ..., h,\ s = 1, ..., S)$ using the Matlab function *copularnd* (see The MathWorks 2019).

- Transform those simulated uniform variates u^s_{T+t} into standardized residuals via the inverse marginal CDF of each series: $\epsilon^s_{i,T+t} = F^{-1}_{\nu_i}(u^s_{i,T+t}) \quad t = 1, ..., h,\ s = 1, ..., S$, where $F^{-1}_{\nu_i}$ is the inverse CDF of the fitted ith marginal distribution. This step delivers simulated standardized residuals (pseudo samples) consistent with those obtained from the GARCH filter above. Note that these residuals are independent in time but dependent at any point in time.

- Using simulated standardized residuals $\epsilon^s_{i,T+t}$ as the i.i.d. input noise process, we reintroduce the autocorrelation as well as heteroskedasticity observed in historical return data via the Matlab function *filter* (see The MathWorks 2019). We set initial values as the last values of returns $r_{i,T}$, standardized residuals $\varepsilon_{i,T}$ and variances $\sigma^2_{i,T}$. This yields S simulated paths of logarithmic returns between period $T+1$ and $T+h$ for all firms and the market index conditional on the realized process up to T, that is

$$r^s_{i,T+t}|\ \mathcal{F}_T, \quad t = 1, ..., h, \quad s = 1, ..., S. \tag{40}$$

- Calculate the arithmetic multi-period return for each simulated path as

$$R^s_{i,T:T+h} = exp\left(\sum_{t=1}^{h} r^s_{i,T+t}\right) - 1. \tag{41}$$

Given the simulated return distributions, we can easily compute the conditional (unconditional) moments of portfolio excess returns and solve the portfolio optimization problems specified in (7) and (9).

Computational Remarks

Despite the computational benefits of scenario analysis, return simulation and solving the nonlinear programming problem can still be a computationally exhaustive task - ultimately depending on the sample size of Monte Carlo simulation and the sample size of robustness check. Getting access to rich computing resources is crucial. This application is feasible only through access to high performance clusters (HPC) or cloud computing resources.[8] Specifically, RAM could be a limiting factor in our case since we have to cache all simulated return scenarios in order to dynamically solve the subsequent portfolio optimization problems. Due to the computational intensity, we set the sample size of return simulations as 30,000 and the number of iterations as 200. The single run of model estimation and return simulation procedure took around 5.7 hours on a MacBook

[8] The robustness exercise made use of the facilities of the Hamilton HPC Service of Durham University (https://www.dur.ac.uk/arc/platforms/).

Pro 2018 with 2.6 GHz 6-Core i7 processor and 16 GB RAM, while the multiple runs for purpose of robustness exercise took around 2.5 days on HPC. Solving the portfolio optimization problems took around 7 minutes and 25.3 hours for single run and multiple runs, respectively. In order to accelerate computation speed, we also employ the Parallel Computing Toolbox build in Matlab.[9]

Appendix C - Figures

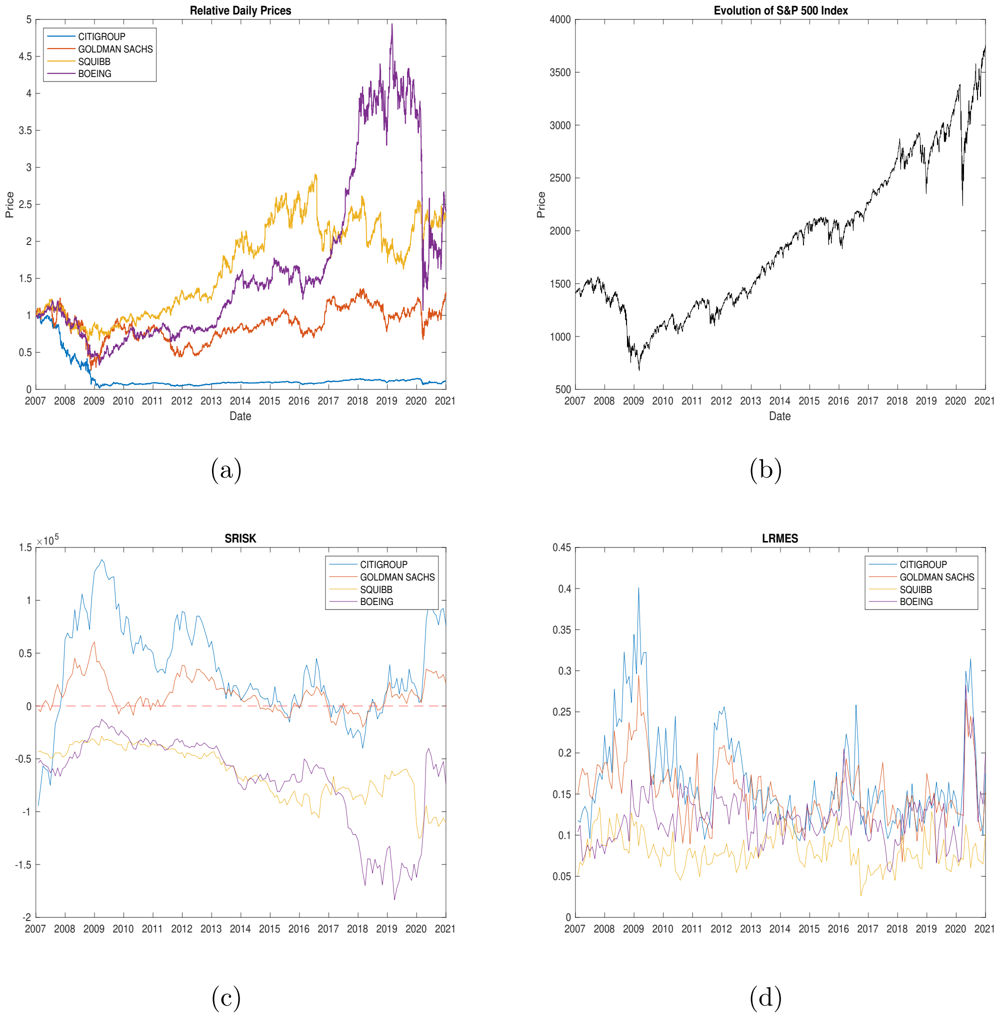


Figure 1: Left top panel reports the relative price developments between Jan 3, 2007 and Dec 31, 2020. Right top panel reports the S&P 500 Index. Dynamics of SRISK index on the left bottom panel and LRMES on the right bottom panel. In term of the simulation approach, here we follow Brownlees and Engle (2016).

[9] https://www.mathworks.com/products/parallel-computing.html

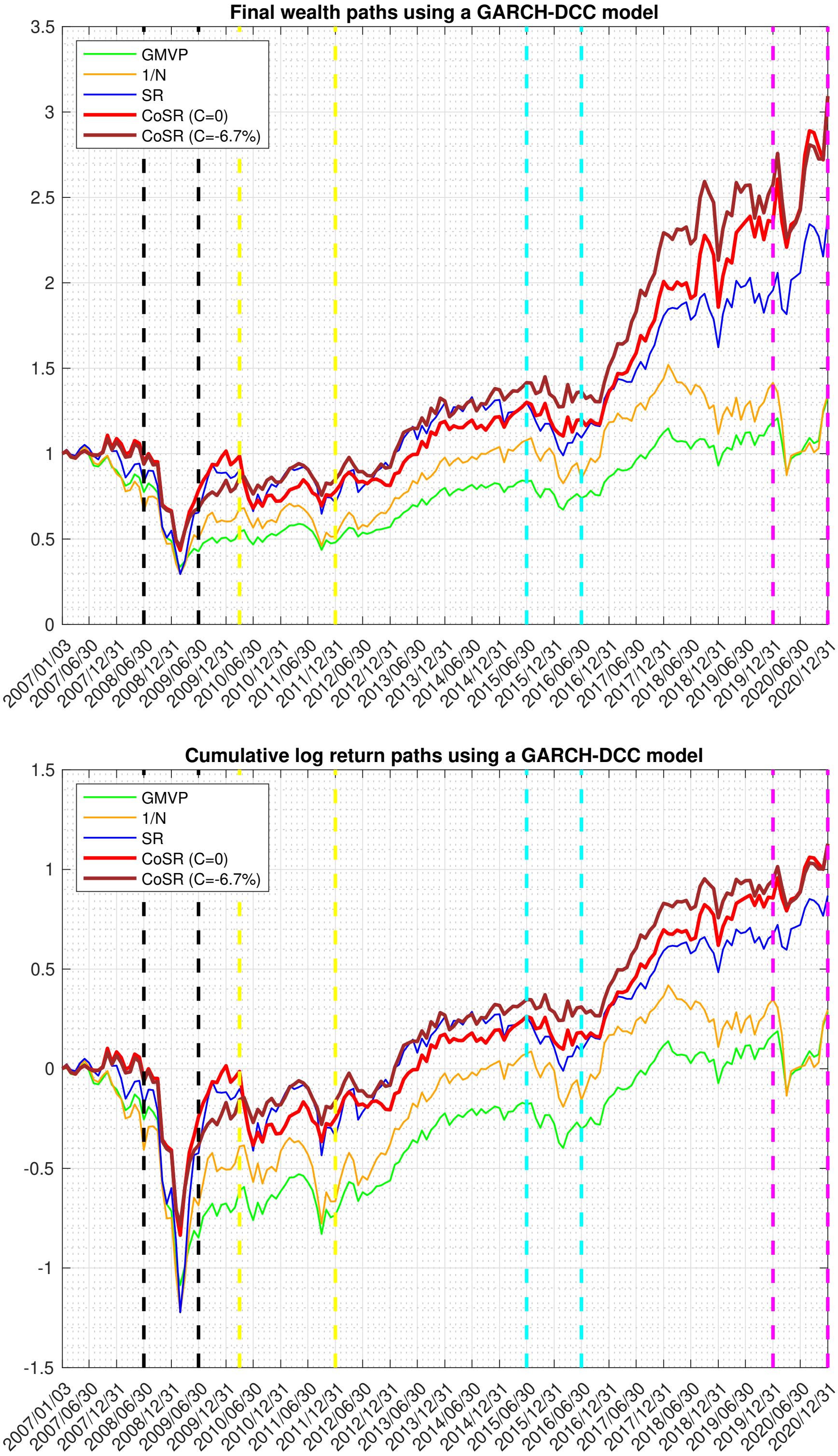


Figure 2: Top panel compares *ex post* final wealth paths and bottom panel compares the *ex post* cumulative return obtained using different strategies based on GARCH-DCC model.

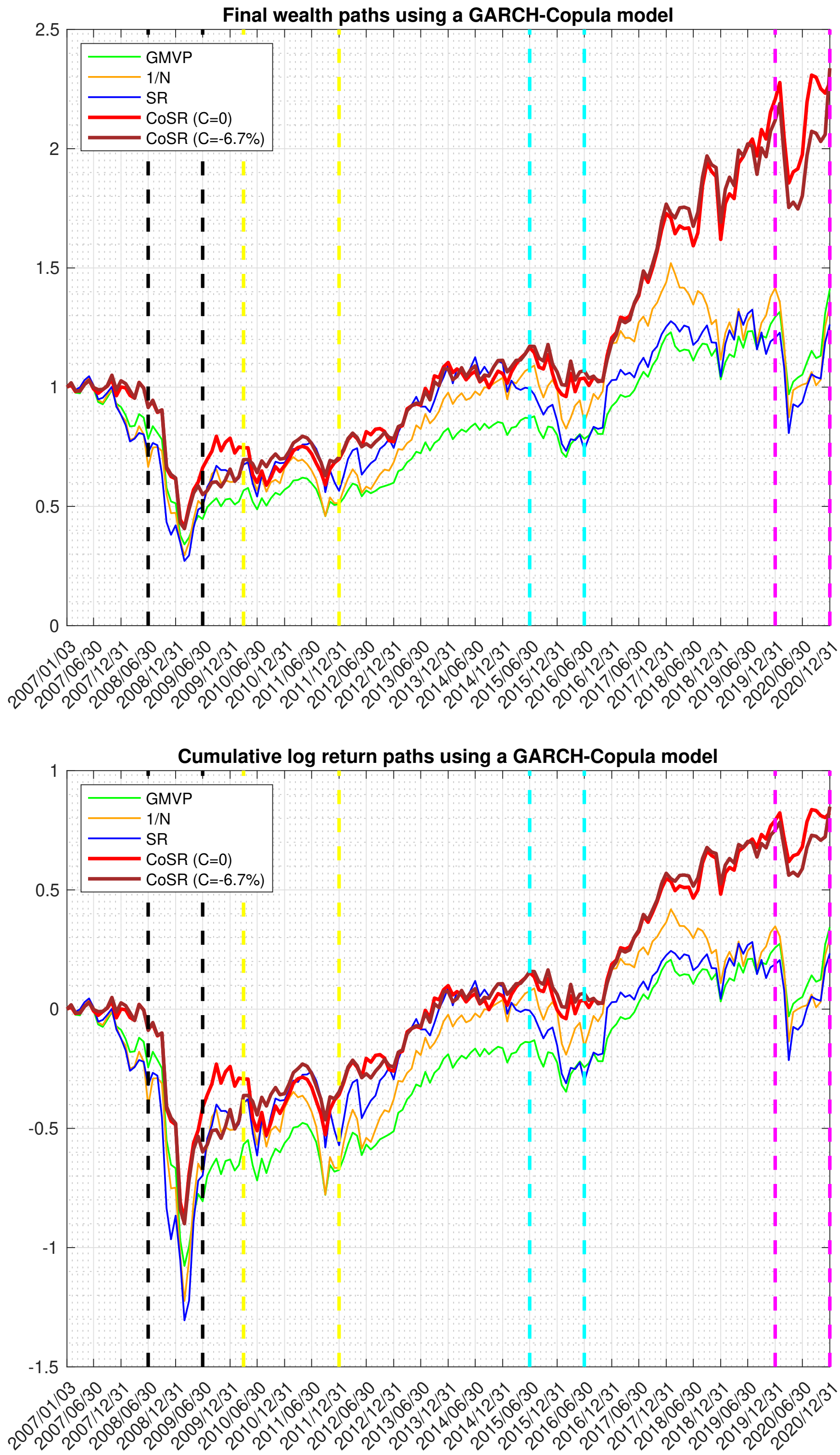


Figure 3: Ex-post final wealth (left panel) and ex-post cumulative return (right panel) paths obtained using different strategies based on GARCH-Copula model.

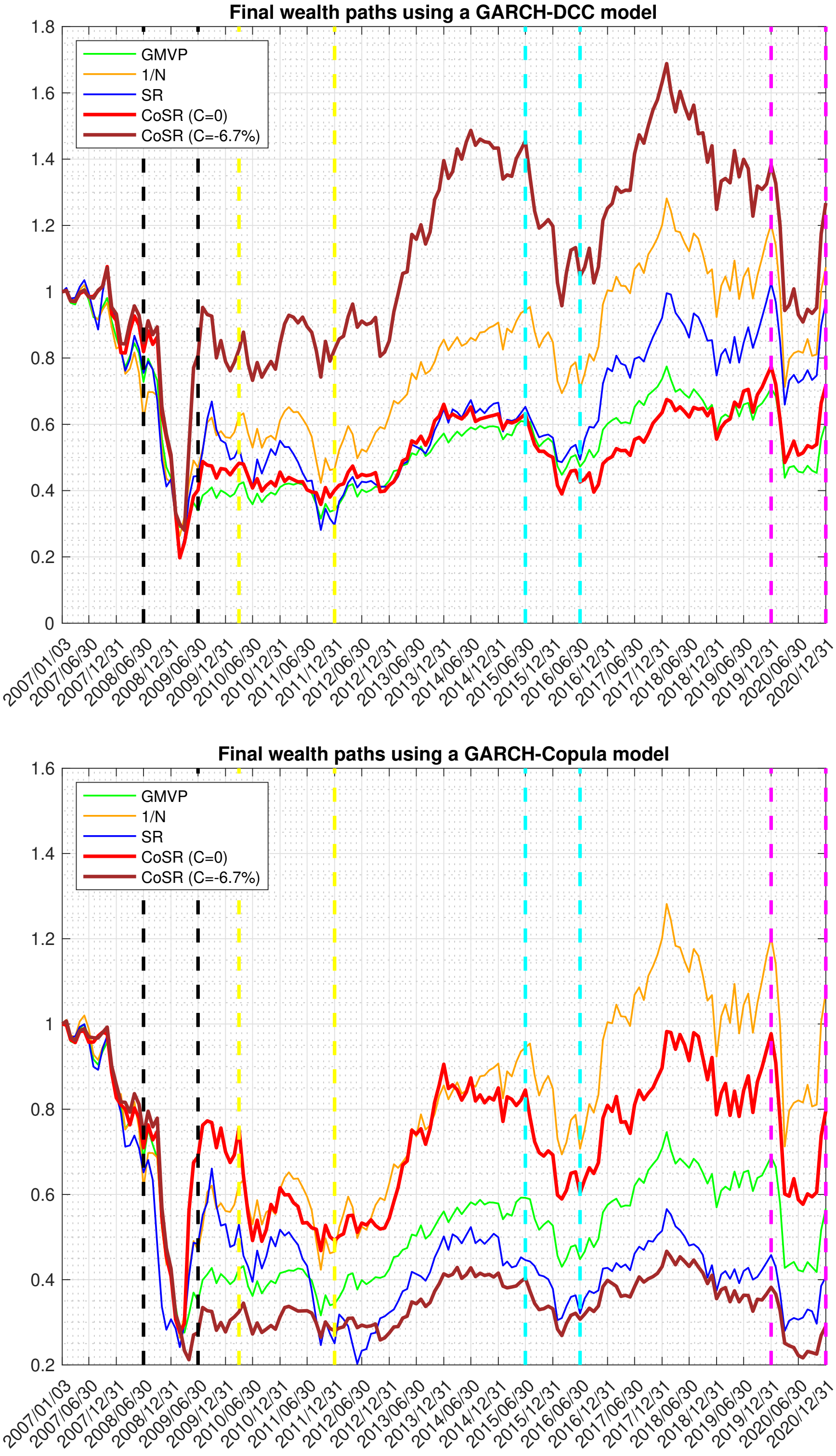


Figure 4: Ex-post final wealth paths obtained using **only** SIFIs as portfolio assets based on GARCH-DCC model (top panel) and GARCH-Copula model (bottom panel) respectively.

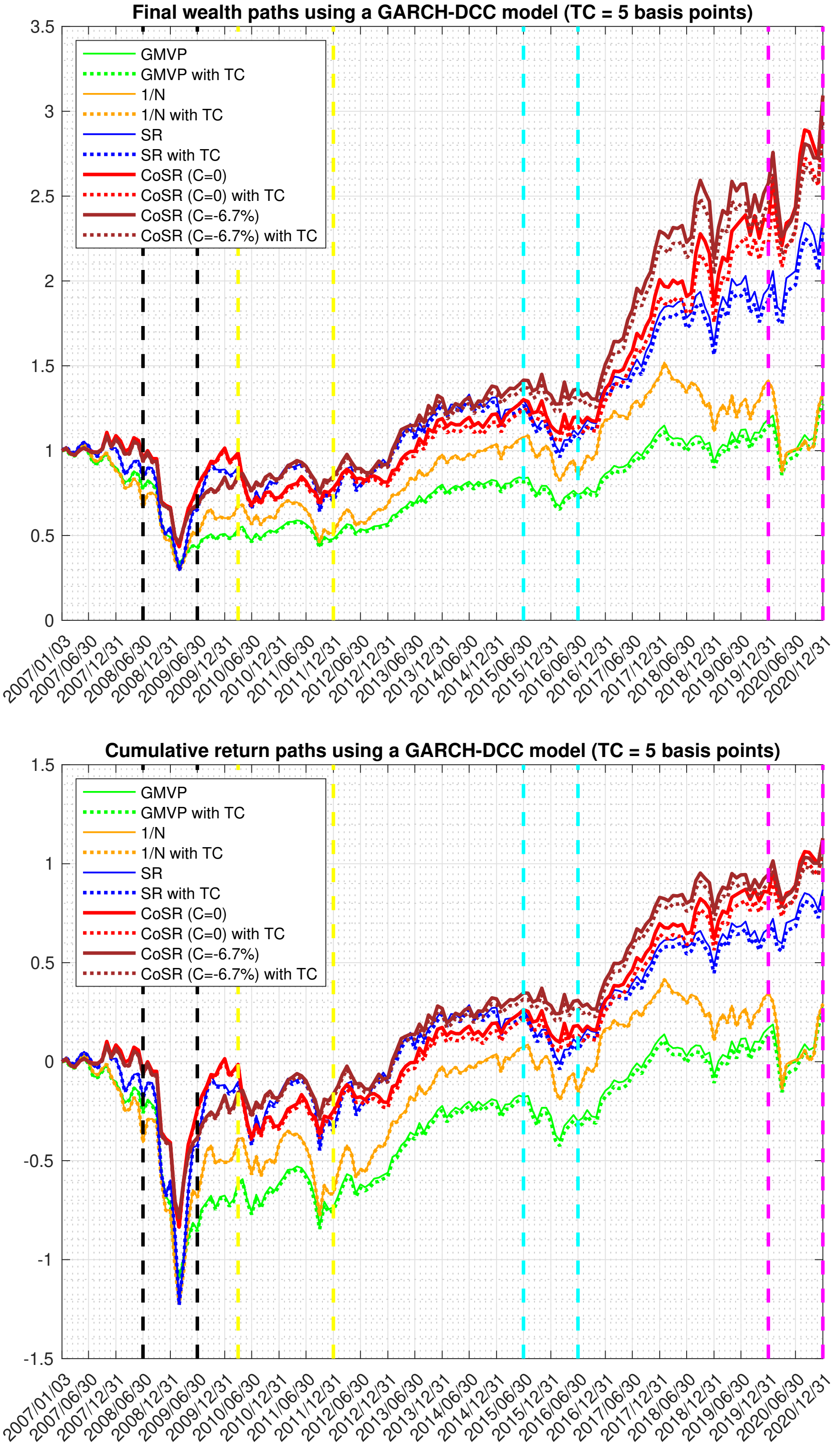


Figure 5: Ex-post final wealth (top panel) and ex-post cumulative return (bottom panel) paths obtained using different strategies based on GARCH-DCC model with proportional transaction costs.

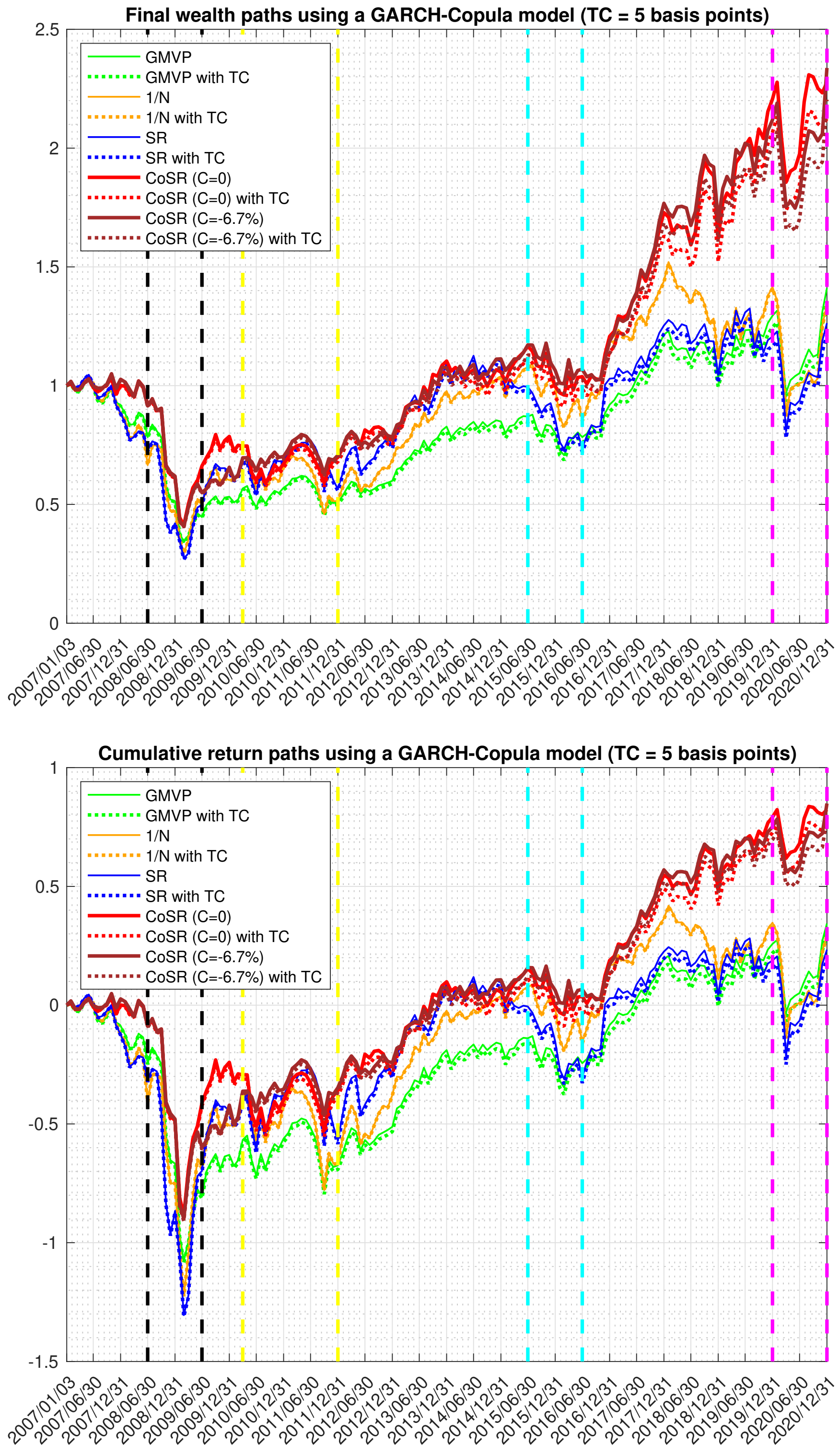


Figure 6: Ex-post final wealth (top panel) and ex-post cumulative return (bottom panel) paths obtained using different strategies based on GARCH-Copula model with proportional transaction costs.

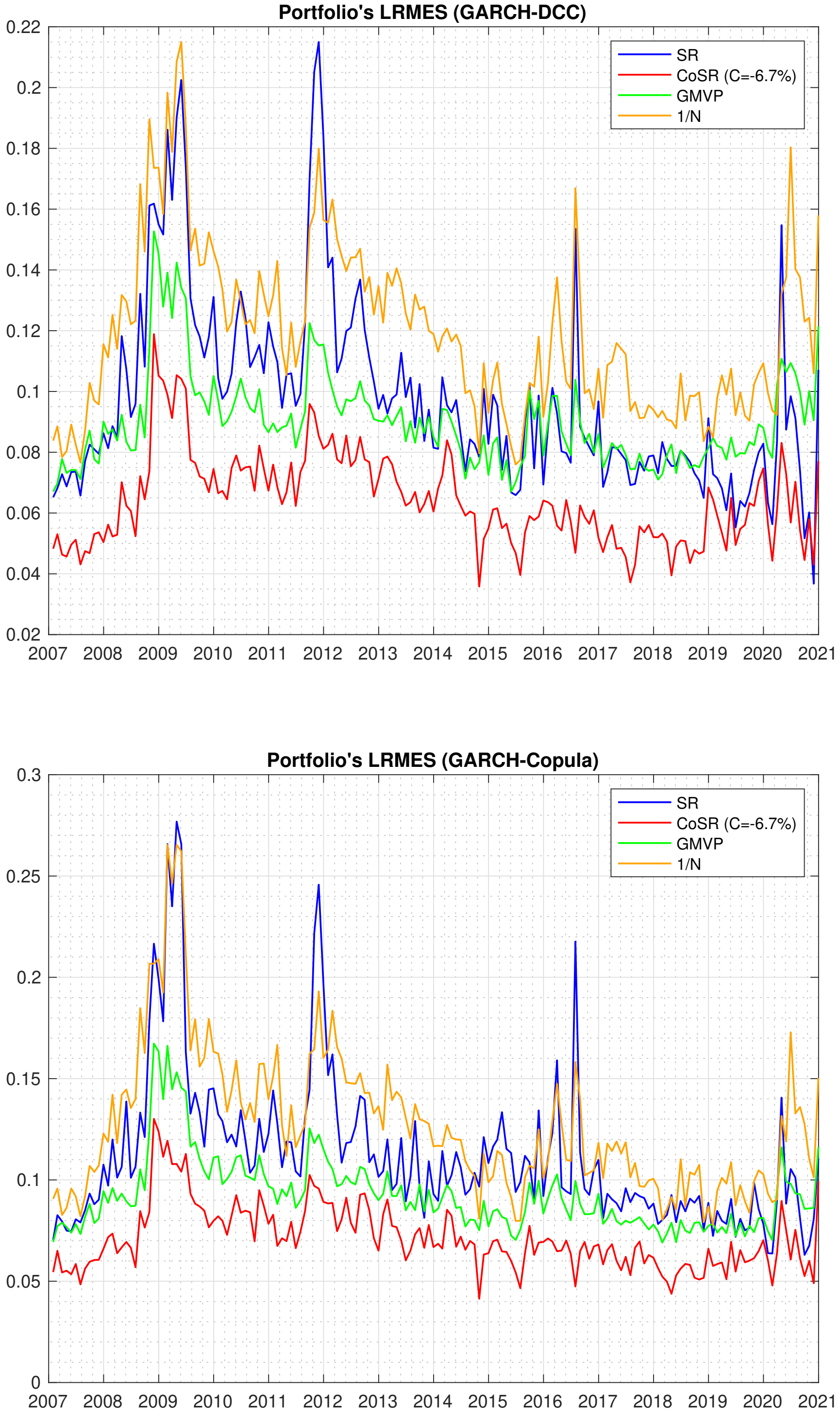


Figure 7: Portfolio's LRMES paths based on GARCH-DCC (top panel) and GARCH-Copula model (bottom panel).

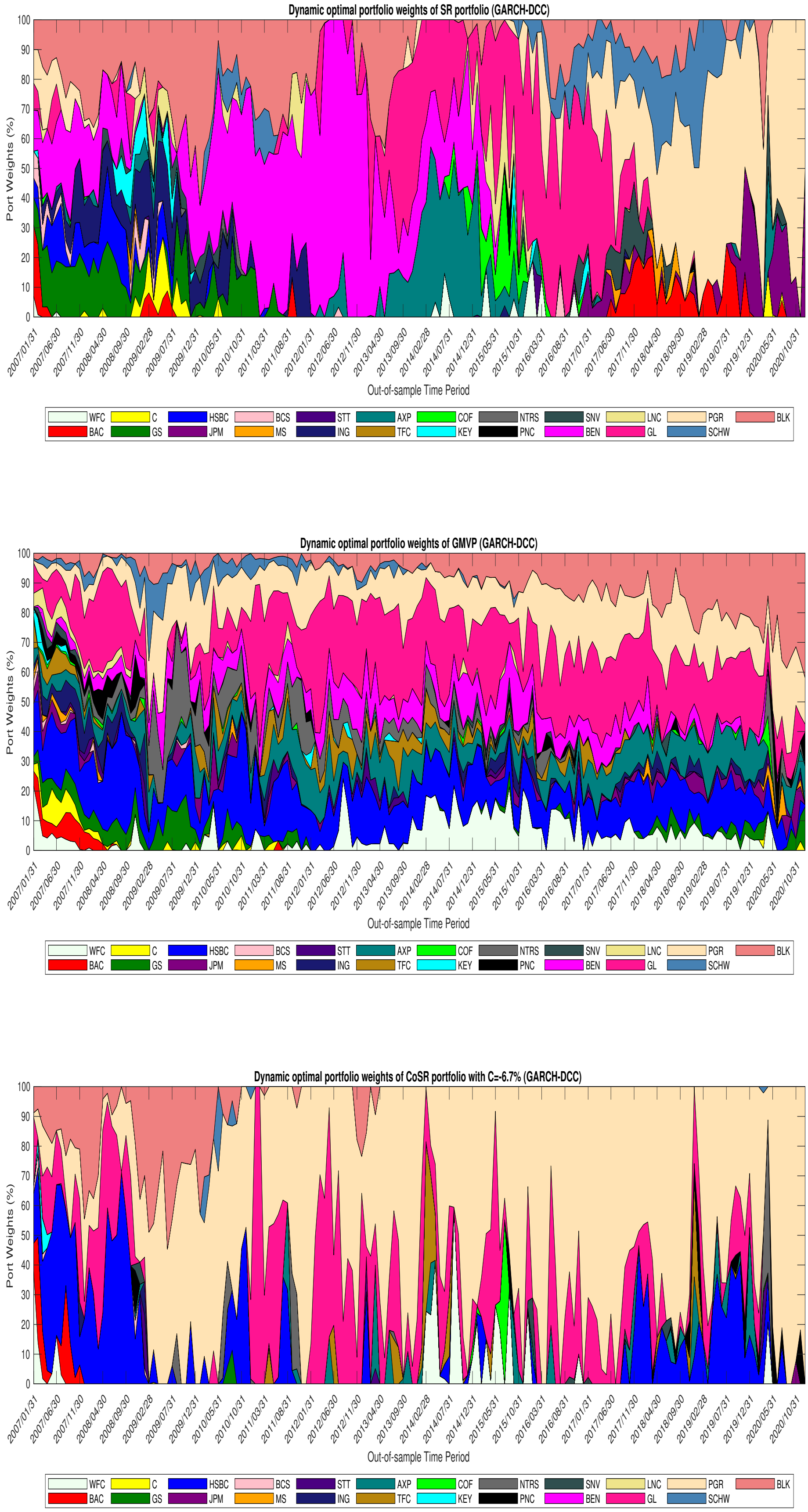


Figure 8: Time-varying portfolios' composition based on GARCH-DCC model. Top panel reports the portfolio weights under the SR strategy, middle panel under the GMVP strategy, and bottom panel under the CoSR strategy, respectively.

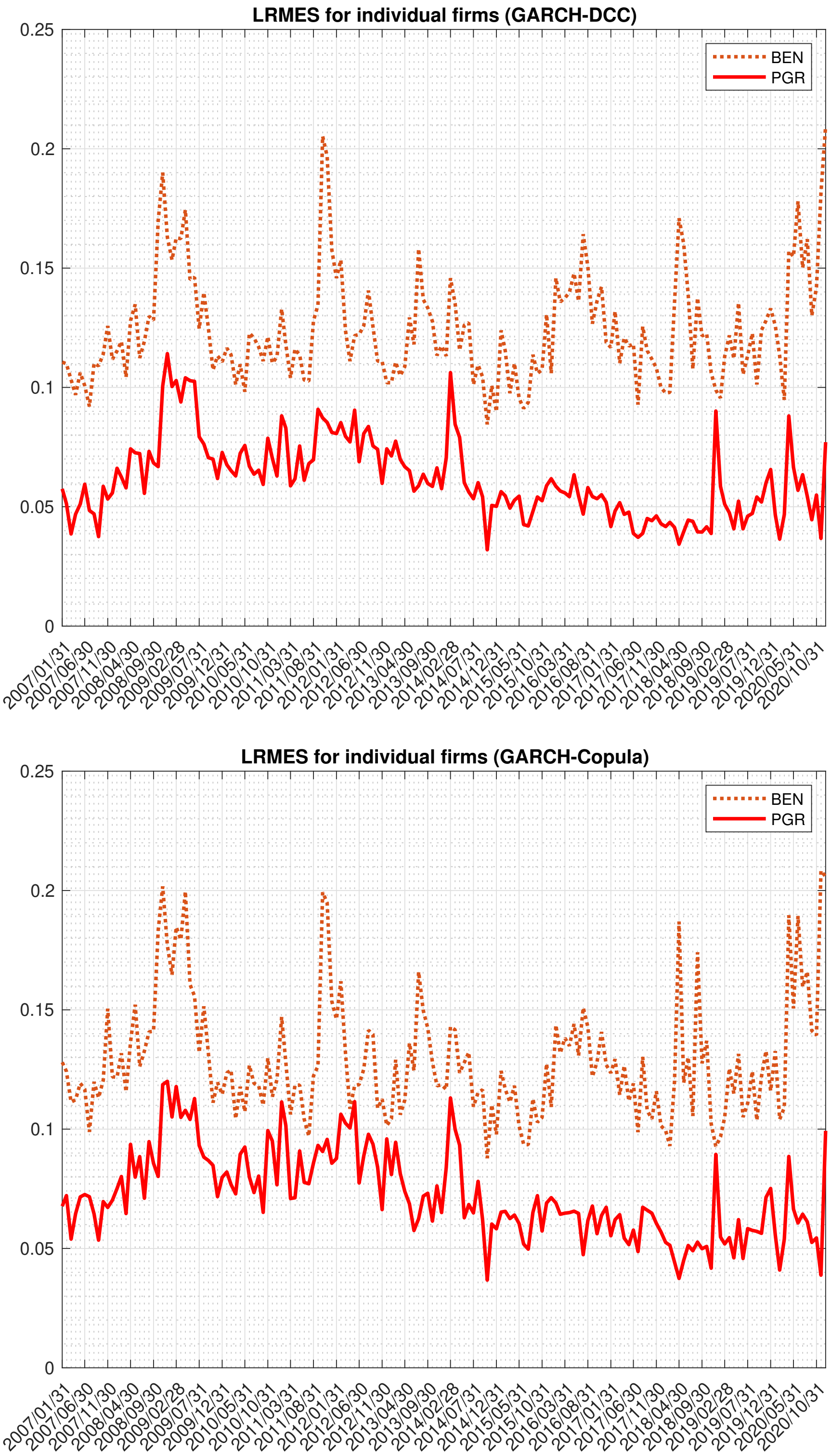


Figure 9: Comparison of individual firm's LRMES based on GARCH-DCC model (top panel) and GARCH-Copula model (bottom panel).

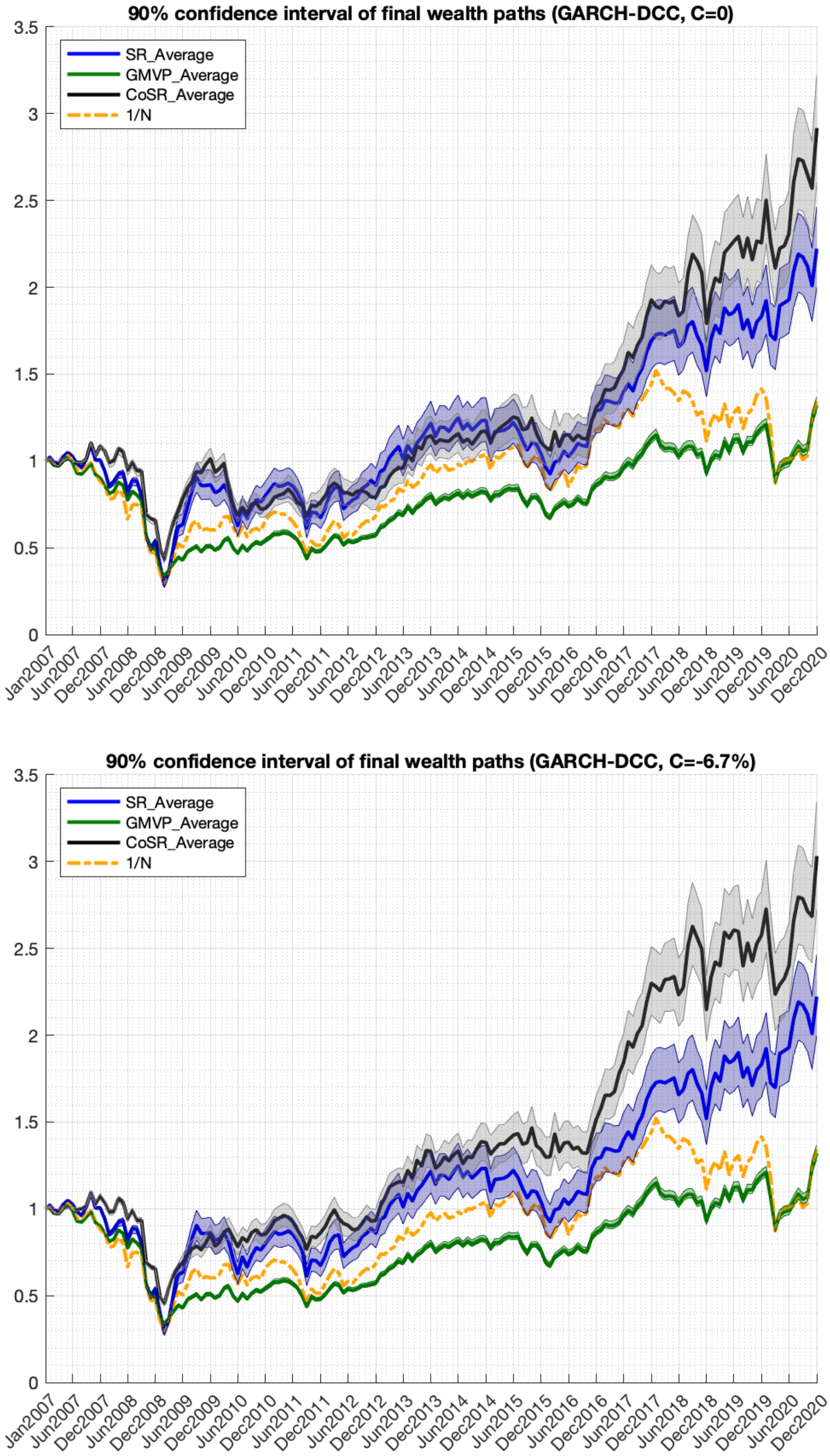
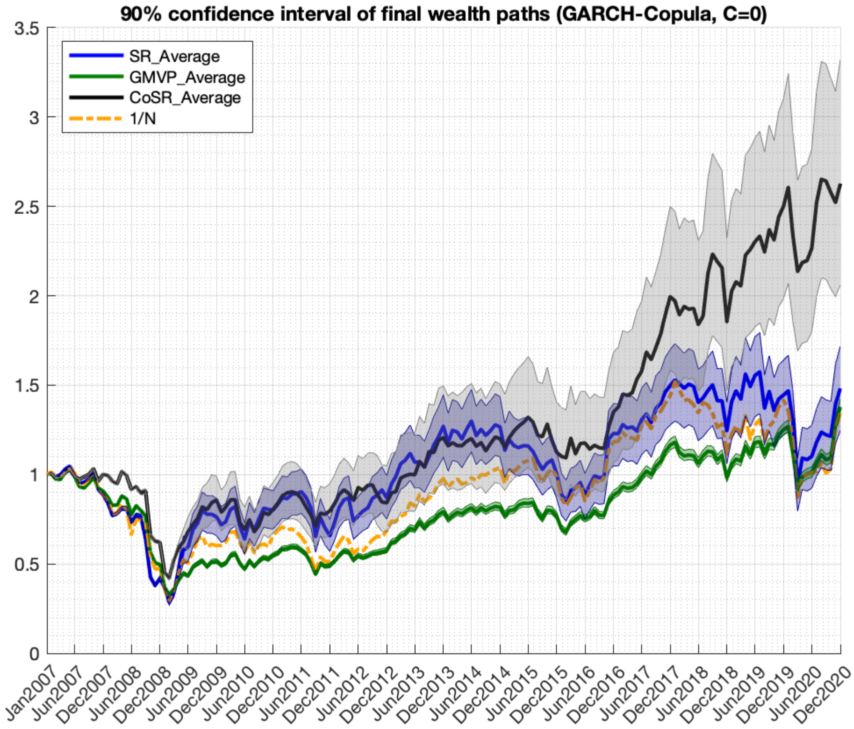


Figure 10: Comparison between different strategies accounting for estimation uncertainty using the GARCH-DCC model. Top panel considers a systemic event given by $C = 0$ and bottom panel considers a systemic event given by $C = -6.7\%$.



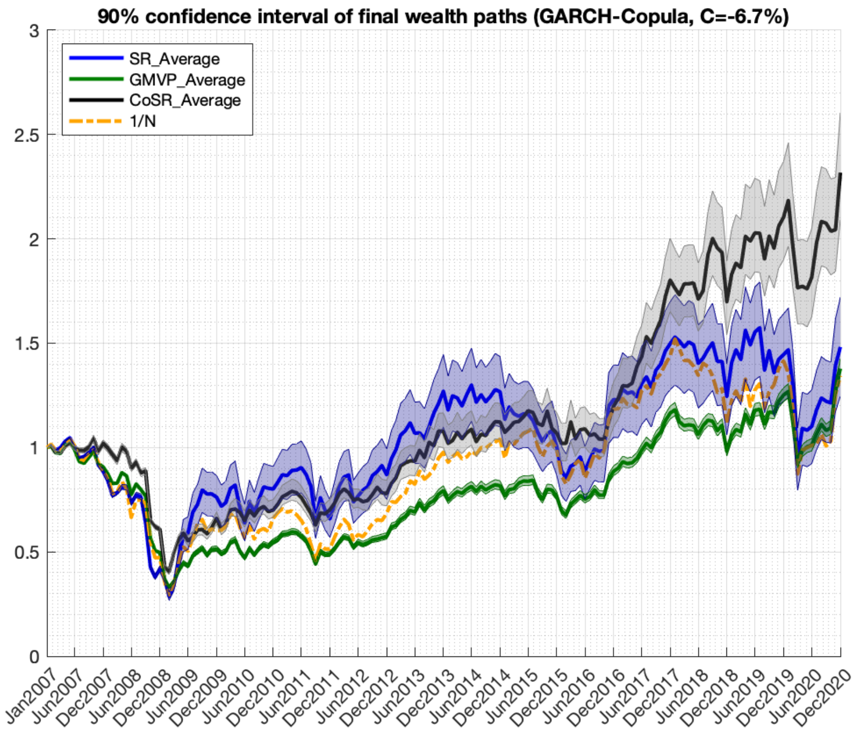


Figure 11: Comparison between different strategies accounting for estimation uncertainty using the GARCH-Copula model. Top panel considers a systemic event given by $C = 0$ and bottom panel considers a systemic event given by $C = -6.7\%$.

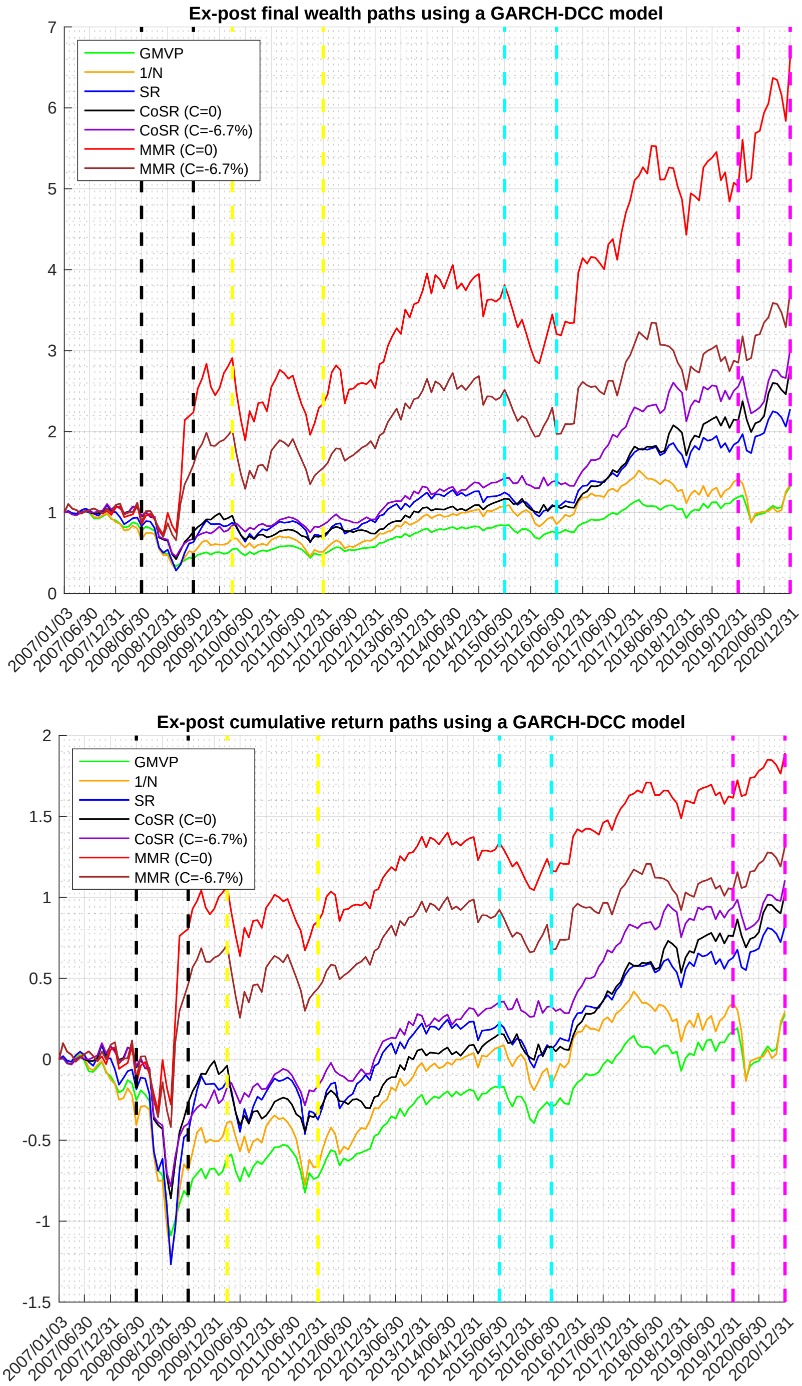


Figure 12: Ex-post final wealth (top panel) and ex-post cumulative return (bottom panel) paths obtained using different strategies based on GARCH-DCC model ($S = 30,000$).

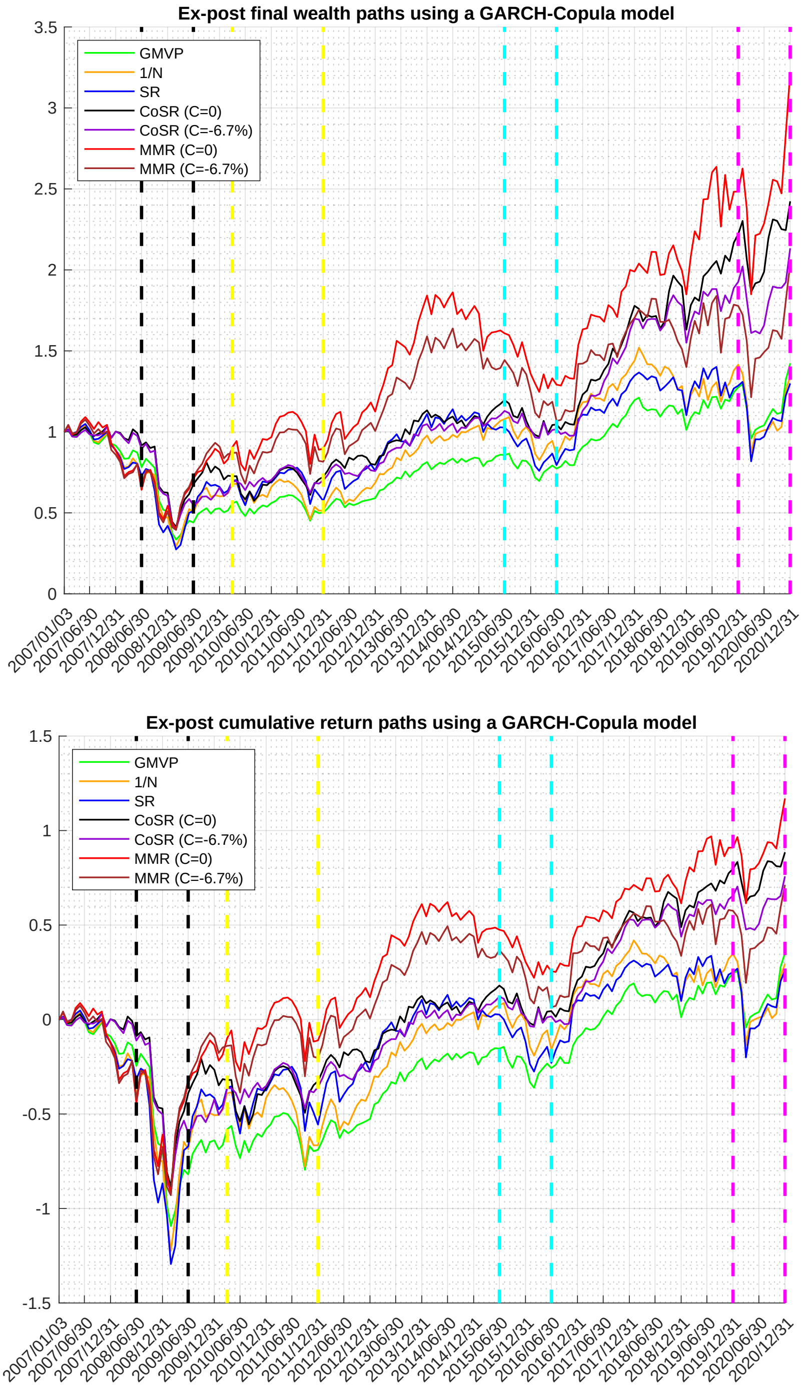


Figure 13: Ex-post final wealth (top panel) and ex-post cumulative return (bottom panel) paths obtained using different strategies based on GARCH-Copula model ($S = 30,000$).

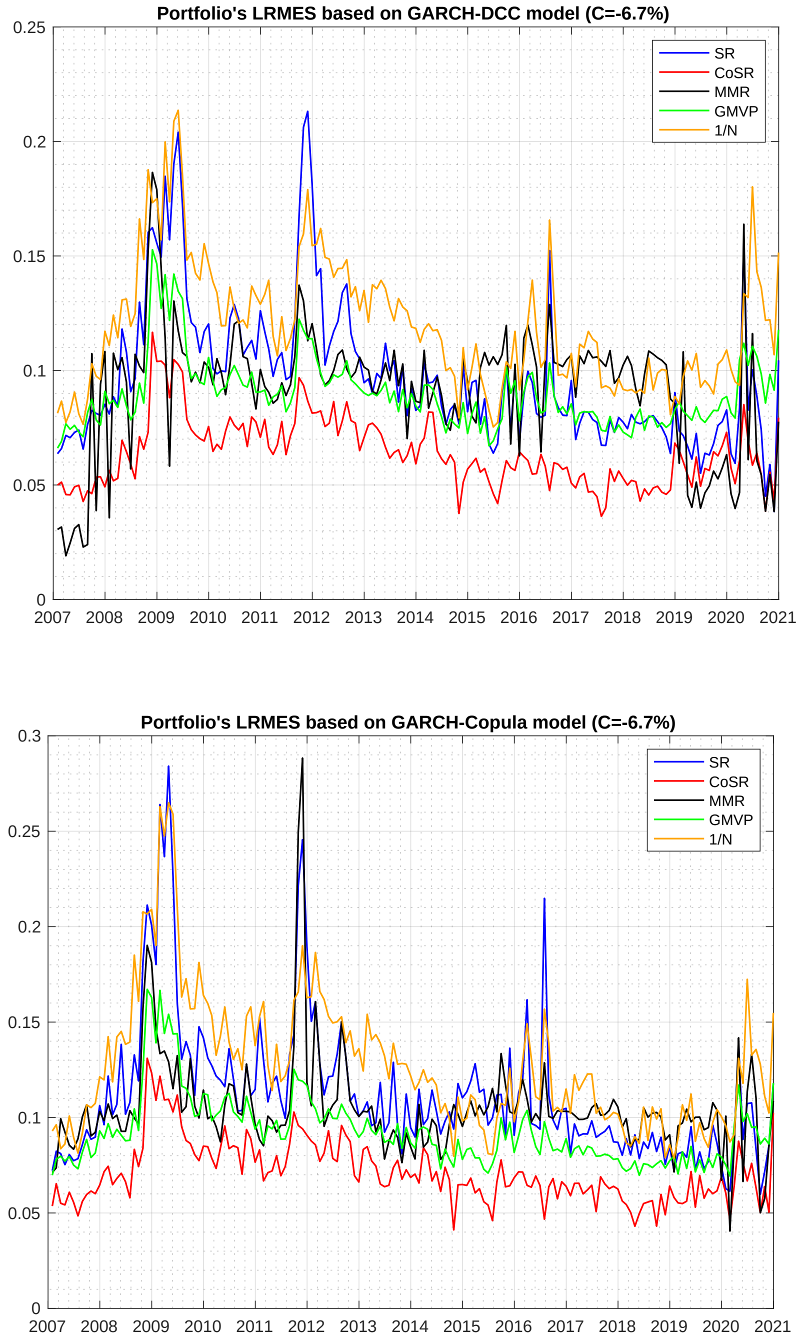


Figure 14: Portfolio's LRMES paths based on GARCH-DCC (top panel) and GARCH-Copula model (bottom panel).

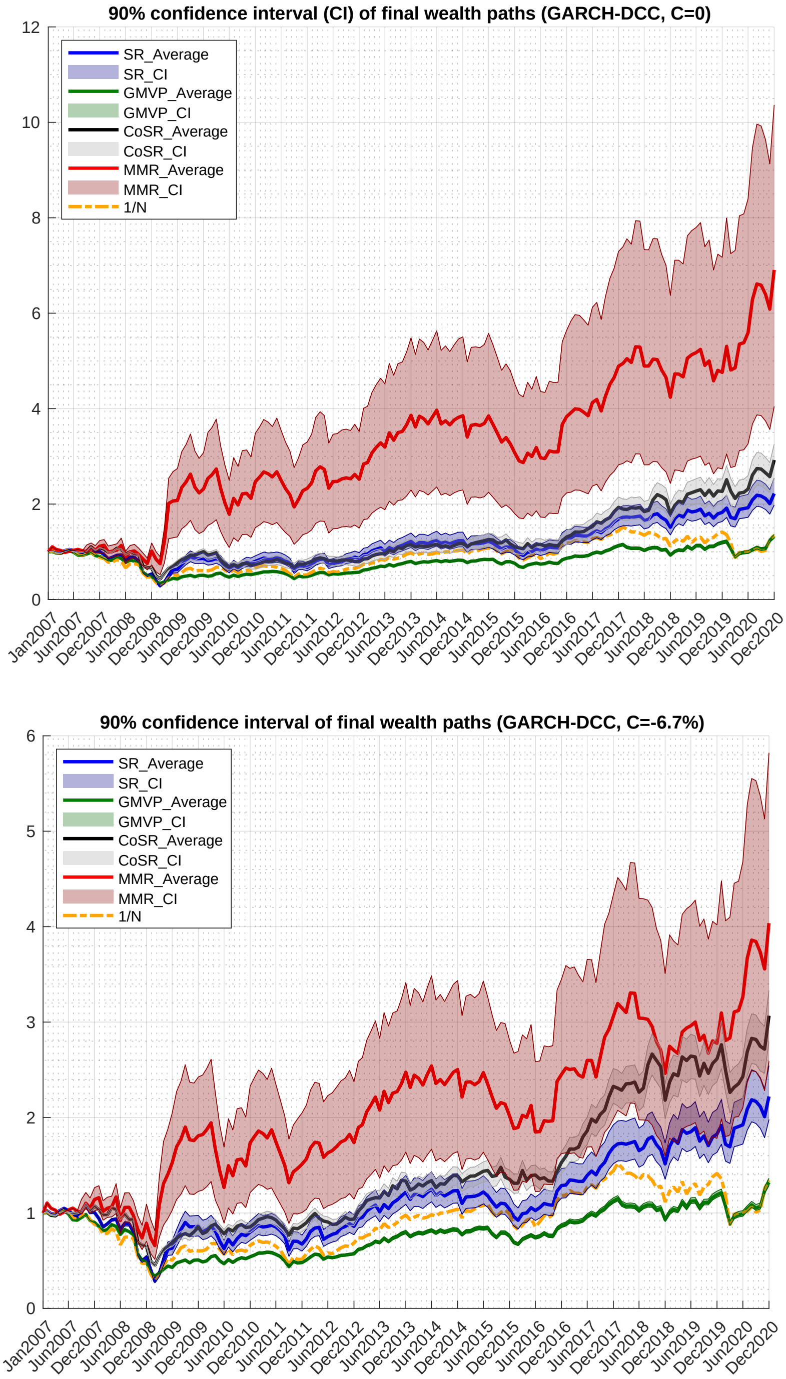


Figure 15: Comparison between different strategies accounting for estimation uncertainty using the GARCH-DCC model. Top panel considers a systemic event given by $C = 0$ and bottom panel considers a systemic event given by $C = -6.7\%$.

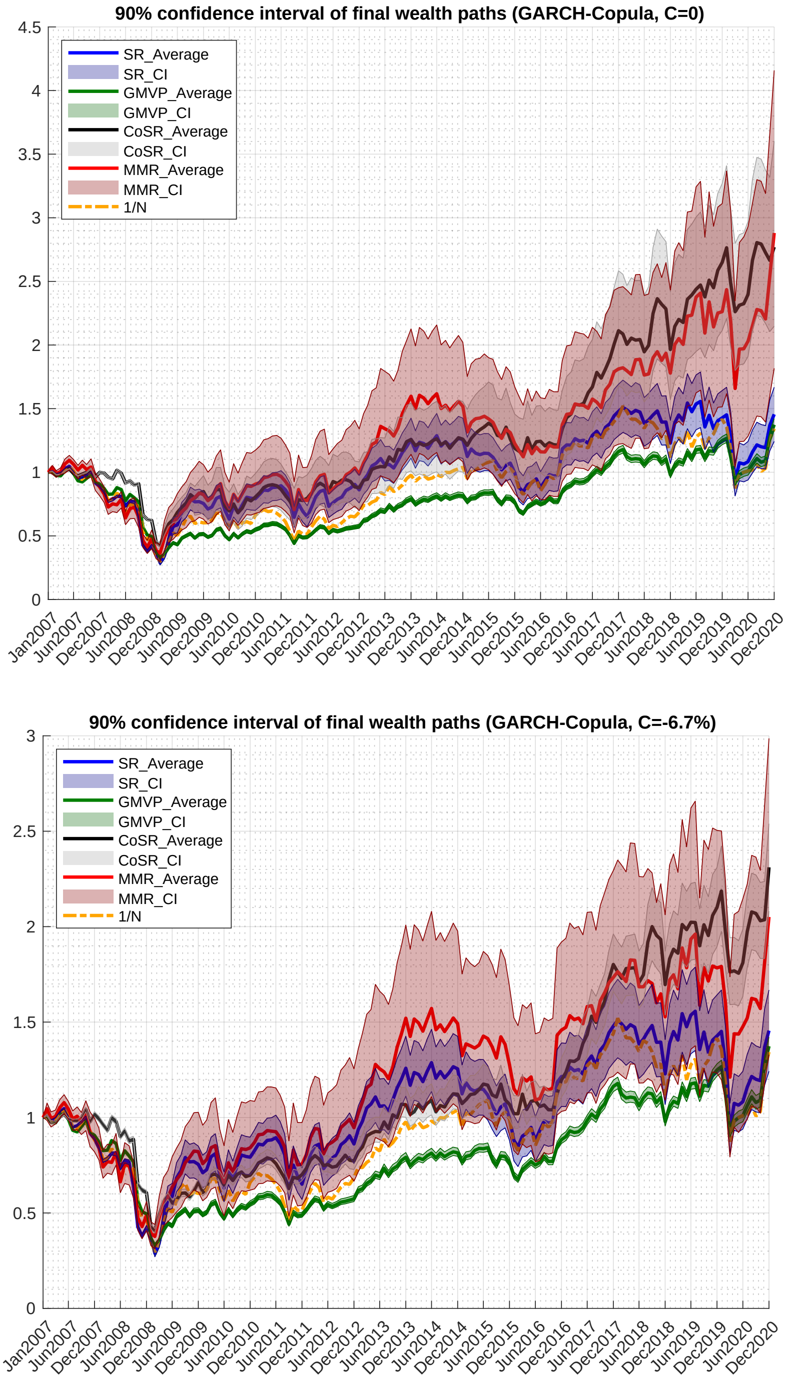


Figure 16: Comparison between different strategies accounting for estimation uncertainty using the GARCH-Copula model. Top panel considers a systemic event given by $C = 0$ and bottom panel considers a systemic event given by $C = -6.7\%$.