Robust risk-sensitive control

Haochen Hua[1], Bujar Gashi[2], and Moyu Zhang[3]

Abstract

We introduce a *risk-sensitive* generalisation of the mixed H_2/H_∞ control problem for linear stochastic systems with additive noise. Two criteria of *exponential-quadratic* form are used to generalise the usual quadratic criteria. The solutions are found in a linear state-feedback form for both the finite and the infinite horizon formulations in terms of coupled Riccati differential and algebraic equations. A change of measures for both criteria and completion of squares method is used to derive the solutions, and explicit sufficient conditions for the admissibility of controls are derived. An application to the problem of *robust* portfolio control in a market with random interest rate subject to a disturbance is also given.

Keywords: Stochastic mixed H_2/H_∞ control; Risk-sensitive control; Robust portfolio control.

1. Introduction

Let $(\Omega, \mathcal{F}, (\mathcal{F}(t), t \geq 0), \mathbb{P})$ be a given complete probability space on which a d-dimensional standard Brownian motion $(W(t), t \geq 0)$ is defined. We assume that $\mathcal{F}(t)$ is the augmentation of $\sigma\{W(s)|0 \leq s \leq t\}$ by all the $\mathbb{P}$-null sets of $\mathcal{F}$. Consider the following linear stochastic control system:

[1]The College of Energy and Electrical Engineering, Hohai University, Nanjing, China; Email: Huahc16@tsinghua.org.cn.

[2]Department of Mathematical Sciences, The University of Liverpool, Liverpool, L69 7ZL, UK; Email: Bujar.Gashi@liverpool.ac.uk.

[3]Postdoctoral Research Center, Guangdong Yuecai Investment Holdings, Guangzhou, China; Email: moyu_zhang0728@163.com.

$$\begin{cases} dx(t) = [A(t)x(t) + B_2(t)u(t) + B_1(t)v(t)]dt + A_1(t)dW(t), \quad t \geq 0, \\ z(t) = \begin{bmatrix} C(t)x(t) \\ \\ D(t)u(t) \end{bmatrix}, \quad \forall t \geq 0, \\ \\ x(0) = x_0 \in \mathbb{R}^n \quad \text{is given.} \end{cases} \tag{1.1}$$

Here $x(\cdot)$ is the *state* of the system, $u(\cdot)$ is the *control* process, $v(\cdot)$ is the *disturbance*, and $z(\cdot)$ is the *output* of the system. We assume that:

$$A(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times n}), \quad B_2(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times n_u}), \quad B_1(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times n_v}),$$

$$A_1(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times d}), \quad C(\cdot) \in L^\infty(0,T;\mathbb{R}^{m_c\times n}), \quad D(\cdot) \in L^\infty(0,T;\mathbb{R}^{m_d\times n_u}),$$

where $L^\infty(0,T;E)$ denotes the set of all E-valued uniformly bounded functions. The control process $u(\cdot)$ and the disturbance $v(\cdot)$ are assumed to be adapted square-integrable processes, and that ensures (1.1) has a unique strong solution. Further consider the following two *quadratic* criteria:

$$I_1(u(\cdot),v(\cdot)) \;:=\; \mathbb{E}\left[\int_0^T z'(t)z(t)dt\right], \tag{1.2}$$

$$I_2(u(\cdot),v(\cdot)) \;:=\; \mathbb{E}\left[\int_0^T (\theta^2 v'(t)v(t) - z'(t)z(t))dt\right], \tag{1.3}$$

for some positive θ. The linear state-feedback stochastic *mixed* H_2/H_∞ *control problem* is to find the optimal pair $(u^*(\cdot), v^*(\cdot))$ of the *optimal control* $u^*(\cdot)$ and of the *worst case disturbance* $v^*(\cdot)$ that are a *Nash equilibrium*, i.e. that satisfy the following two inequalities:

$$I_1(u^*(\cdot),v^*(\cdot)) \;\leq\; I_1(u(\cdot),v^*(\cdot)), \tag{1.4}$$

$$I_2(u^*(\cdot),v^*(\cdot)) \;\leq\; I_2(u^*(\cdot),v(\cdot)). \tag{1.5}$$

Inequality (1.4) indicates that $u^*(\cdot)$ minimizes the quadratic cost of the output (i.e. the *output energy*) under the worst-case disturbance, and this corresponds to the "H_2" part of the problem. Inequality (1.5) ensures that under the worst-case disturbance the effect of the disturbance on the output, as measured by the quadratic cost I_2, is bounded, and this corresponds to the "H_∞" part of the problem. This problem was considered in [31] where initially the case of a *deterministic* system was solved, i.e. the case when $A_1(t) = 0$ for all $t \geq 0$. Such a case admits an explicit linear state-feedback solution in terms of a pair of coupled Riccati differential equations. In [31] it was further shown that such a solution is also the solution to the stochastic case, which in particular means that the Nash equilibrium $(u^*(\cdot), v^*(\cdot))$ does *not* depend on the stochastic noise *intensity* $A_1(\cdot)$.

There have been many further developments on the stochastic mixed H_2/H_∞ control problem. The emphasis has been on the nonlinear systems and systems with multiplicative noise (see, e.g., [8, 32, 50, 51, 48, 49, 25]) as well as applications (see, e.g., [24, 23, 22]). However, the criteria in these further developments have remained quadratic. On the other hand, another well-known branch of control theory is the *risk-sensitive* control. Here the optimality criterion is of *exponential-quadratic* form. The risk-sensitive control problem for linear stochastic systems with additive noise was introduced by Jacobson [26] who found an explicit closed-form solution in a linear state-feedback in the case of full observations. For risk-sensitive control with partial observations see, e.g., [3, 7, 40], for discrete-time systems see, e.g., [46, 47], for connections with robust control see, e.g., [20, 44, 14, 40], for the risk-sensitive maximum principle see, e.g., [30, 27, 35], for the risk-sensitive control of mean-filed systems see, e.g. [35, 36, 33], for the Hamilton-Jacobi-Bellman equation of risk-sensitive control see [37], for the risk-sensitive differential games see, e.g., [2, 13, 21, 34, 38, 42, 43], and for more general exponential criteria that admit explicit closed-form solutions see [9, 10, 16, 19]. The risk-sensitive control is particularly suitable for optimal investment problems, see, e.g., [15, 4, 11, 9, 16, 12, 19].

In this paper, we *generalise* the linear state-feedback stochastic mixed H_2/H_∞ control problem by replacing the quadratic criteria (1.2) and (1.3) with the following two exponential-quadratic criteria:

$$
\begin{aligned}
J_1(u(\cdot), v(\cdot)) &:= \frac{1}{\gamma_1}\mathbb{E}\left\{\exp\left[\frac{\gamma_1}{2}\int_0^T z'(t)M_1(t)z(t)dt\right]\right\}, \qquad (1.6)\\
J_2(u(\cdot), v(\cdot)) &:= \frac{1}{\gamma_2}\mathbb{E}\left\{\exp\left[\frac{\gamma_2}{2}\int_0^T (\theta^2 v(t)'N(t)v(t) - z'(t)M_2(t)z(t))dt\right]\right\},
\end{aligned}
$$

where $\gamma_1, \gamma_2 \in \mathbb{R}$ and

$$
M_1(\cdot), M_2(\cdot) \in L^\infty(0, T; \mathbb{R}^{(m_c+m_d)\times(m_c+m_d)}), \quad N(\cdot) \in L^\infty(0, T; \mathbb{R}^{n_v\times n_v}),
$$

are given symmetric matrices. The aim now is to find a Nash equilibrium $(u^*(\cdot), v^*(\cdot))$ that satisfies the following inequalities:

$$J_1(u^*(\cdot), v^*(\cdot)) \le J_1(u(\cdot), v^*(\cdot)), \quad \forall u(\cdot) \in \mathcal{A}_u \tag{1.7}$$
$$J_2(u^*(\cdot), v^*(\cdot)) \le J_2(u^*(\cdot), v(\cdot)), \quad \forall v(\cdot) \in \mathcal{A}_v. \tag{1.8}$$

for some suitably defined admissible sets of linear state-feedback controls and disturbances $\mathcal{A}_u$ and $\mathcal{A}_v$, respectively. This is clearly a risk-sensitive generalisation of the stochastic mixed H_2/H_∞ control problem of [31] which now appears as the special case corresponding to $\gamma_1 \to 0$, $\gamma_2 \to 0$, and the cost matrices M_1, M_2, N, being identity matrices. The motivation for introducing this generalisation is that we obtain an explicit solution to the problem, and different from [31], the corresponding coupled Riccati equations depend on intensity $A_1(\cdot)$ of random noise. Moreover, there are applications, such as vehicle active suspensions [6], and optimal investment [15, 5, 4, 11, 9, 16, 12, 19], where the exponential-quadratic cost functional appears. If systems in such applications are subject to disturbances, then our results can be used for their control.

The problem of finding the linear state-feedback Nash equilibrium (u^*, v^*) that satisfies (1.7) and (1.8) subject to (1.1, and its solution method, appear to be new and most closely related to existing results on risk-sensitive *differential games* as follows. In [2], a risk-sensitive differential game was considered where two exponential criteria are used for a general nonlinear system with a state-multiplicative noise. Certain *limiting cases* of such a problem are considered which are equivalent to the risk-sensitive control problem or several independent risk-sensitive control problems. In [21], an *existence* result for a class of risk-sensitive differential game is given in terms of backward stochastic differential equations, whereas in [13] and [38] a *single* criterion risk-sensitive differential game is considered.In [42], [43], a risk-sensitive differential game of several players where each uses the same criterion is considered. In [34] a maximum-principle for a risk-sensitive differential games of a general nonlinear system with a control and state multiplicative noise was considered which gives necessary and sufficient conditions for an *open-loop* controls to be optimal. The risk-sensitive differential games in *infinite* horizon have not been considered in the above mentioned papers.

The main contributions of the present paper are as follows.

• We solve the finite-horizon risk-sensitive H_2/H_∞ control problem, which is the problem of finding a Nash equilibrium (u^*, v^*) that satisfies (1.7) and (1.8) subject to (1.1). The solution is of a linear-state feedback form and depends on solutions to two coupled Riccati differential equations that are more general than those in [31]. The change of measure for both criteria and the completion of squares method is used to derive the solution under two Novikov integrability conditions. Thus, this represents an expansion of the approach used in risk-sensitive control problems that require a single criterion to the current two criteria setting. We also give a general result on the *uniqueness* of solution to the resulting coupled Riccati differential equations. The solution is found on the assumption of admissibility of the Nash equilibrium, and sufficient conditions for this to hold are also derived. These contributions are the subject of section 2.

• We solve the *infinite-horizon* risk-sensitive H_2/H_∞ control which instead of the criteria J_1 and J_2 uses certain infinite horizon versions of average type with a general weigh. Our solution method is similar to the finite horizon, but with the additional difficulty of dealing with the stability requirements, which are of *exponential* type under two different probability measures. The solution method is an expansion to a two risk-sensitive criteria setting of the approach used in [16] and [19] for the infinite-horizon risk-sensitive control problems. The explicit state-feedback solution is obtained in terms of certain coupled Riccati algebraic equations, and under the assumption of admissibility of the Nash equilibrium. We further give sufficient conditions for the admissibility of the Nash equilibrium which also ensure the stability of the system. These contributions are the subject of section 3.

• We formulate an investment problem in a market with a *random interest rate* that is subject to a *disturbance*, and an investor that uses the power utility from terminal wealth as a criterion of optimal investment and requires a robustness of expected growth. This is an expansion of an optimal investment problem with stochastic interest rate as considered in, e.g. [16, 18, 17, 19], by taking into the consideration the effect of disturbances on the interest rate model. Although the stochastic control system in this problem is with multiplicative noise and non-exponential-quadratic criteria, we show that this problem can be reformulated as an example of control problem considered in section 2, and is solved as an application of our results. We further give two numerical examples for both the finite and infinite horizon formulations of this application, which in particular illustrate that our assumptions are reasonable and can be satisfied. These contributions are the subject of section 4.

Finally, note that throughout the paper we have suppressed argument t where convenient for notational simplicity.

2. Finite horizon risk-sensitive H_2/H_∞ control problem

We confine ourselves to *linear* state feedback controls and disturbances (as in [31]). The sets of all such controls and disturbances are defined as:

$$\mathcal{U} \ := \ \{u(\cdot): \quad u(t) = K_u(t)x(t) \quad \text{where } K_u(\cdot) \in L^\infty(0,T;\mathbb{R}^{n_u\times n})\},$$

$$\mathcal{V} \ := \ \{v(\cdot): \quad v(t) = K_v(t)x(t) \quad \text{where } K_v(\cdot) \in L^\infty(0,T;\mathbb{R}^{n_v\times n})\}.$$

Let the matrices M_1 and M_2 have the following partition:

$$M_1(t) = \begin{bmatrix} M_{11}(t) & M_{12}(t) \\ M_{12}'(t) & M_{13}(t) \end{bmatrix}, \quad M_2(t) = \begin{bmatrix} M_{21}(t) & M_{22}(t) \\ M_{22}'(t) & M_{23}(t) \end{bmatrix}, \quad t \in [0,T],$$

where $M_{11}(\cdot), M_{21}(\cdot) \in L^\infty(0,T;\mathbb{R}^{m_c\times m_c})$, $M_{12}(\cdot), M_{22}(\cdot) \in L^\infty(0,T;\mathbb{R}^{m_c\times m_d})$, $M_{13}(\cdot), M_{23}(\cdot) \in L^\infty(0,T;\mathbb{R}^{m_d\times m_d})$. We make the following *positivity* assumption.

Asssumption 1. *$D'(t)M_{13}(t)D(t) > 0$ and $N(t) > 0$ for a.e. $t \in [0,T]$.*

In the special case of $M_{13}(t) = I$ and $N(t) = I$, this assumption is the same as in [31]. We also make the following assumption on a pair of coupled Riccati differential equations that appear in our solution.

Asssumption 2. *There exists a solution pair $(P_1(\cdot), P_2(\cdot))$ to the following coupled Riccati differential equations:*

$$\begin{cases} \dot{P}_1 + P_1[A - B_2(D'M_{13}D)^{-1}D'M_{12}'C] \\ +[A - B_2(D'M_{13}D)^{-1}D'M_{12}'C]'P_1 \\ -\theta^{-2}(P_1B_1N^{-1}B_1'P_2 + P_2B_1N^{-1}B_1'P_1) \\ +P_1[\gamma_1 A_1A_1' - B_2(D'M_{13}D)^{-1}B_2']P_1 \\ +C'[M_{11} - M_{12}D(D'M_{13}D)^{-1}D'M_{12}]C = 0, \quad t \in [0,T], \\ P_1(T) = 0, \end{cases} \tag{2.1}$$

$$
\begin{cases}
\dot{P}_2 - P_2(\theta^{-2}B_1N^{-1}B_1' - \gamma_2 A_1A_1')P_2 \\
+P_2[A - B_2(D'M_{13}D)^{-1}D'M_{12}'C] \\
+(A - B_2(D'M_{13}D)^{-1}D'M_{12}'C)'P_2 - P_2B_2(D'M_{13}D)^{-1}B_2'P_1 \\
-P_1B_2(D'M_{13}D)^{-1}B_2'P_2 \\
+[C'M_{22}D(D'M_{13}D)^{-1}B_2' \\
-C'M_{12}D(D'M_{13}D)^{-1}D'M_{23}D(D'M_{13}D)^{-1}B_2']P_1 \\
+P_1[C'M_{22}D(D'M_{13}D)^{-1}B_2' \\
-C'M_{12}D(D'M_{13}D)^{-1}D'M_{23}D(D'M_{13}D)^{-1}B_2']' \\
-P_1B_2(D'M_{13}D)^{-1}D'M_{23}D(D'M_{13}D)^{-1}B_2'P_1 - C'M_{21}C \\
+C'M_{22}D(D'M_{13}D)^{-1}D'M_{12}'C + C'M_{12}D(D'M_{13}D)^{-1}D'M_{22}'C \\
-C'M_{12}D(D'M_{13}D)^{-1}D'M_{23}D(D'M_{13}D)^{-1}D'M_{12}'C = 0, \quad t \in [0,T], \\
P_2(T) = 0.
\end{cases} \tag{2.2}
$$

Theorem 1. *If a solution pair $(P_1(\cdot), P_2(\cdot))$ to the coupled Riccati backward differential equations (2.1) and (2.2) exists, then it must be unique.*

Proof. For notational simplicity, by defining the matrices

$$
\begin{aligned}
\hat{A} &:= A - B_2(D'M_{13}D)^{-1}D'M_{12}'C, \\
\hat{A}_1 &:= \gamma_1 A_1A_1' - B_2(D'M_{13}D)^{-1}B_2', \\
\hat{N} &:= \theta^{-2}B_1N^{-1}B_1', \\
\hat{C} &:= C'[M_{11} - M_{12}D(D'M_{13}D)^{-1}D'M_{12}]C, \\
\underline{A} &:= C'M_{22}D(D'M_{13}D)^{-1}B_2' \\
&\quad - C'M_{12}D(D'M_{13}D)^{-1}D'M_{23}D(D'M_{13}D)^{-1}B_2', \\
\underline{C} &:= -C'M_{21}C + C'M_{22}D(D'M_{13}D)^{-1}D'M_{12}'C \\
&\quad + C'M_{12}D(D'M_{13}D)^{-1}D'M_{22}'C, \\
&\quad - C'M_{12}D(D'M_{13}D)^{-1}D'M_{23}D(D'M_{13}D)^{-1}D'M_{12}'C, \\
\underline{B}_2 &:= B_2(D'M_{13}D)^{-1}D'M_{23}D(D'M_{13}D)^{-1}B_2', \\
\underline{N} &:= B_2(D'M_{13}D)^{-1}B_2', \\
\underline{A}_1 &:= \theta^{-2}B_1N^{-1}B_1' - \gamma_2 A_1A_1'.
\end{aligned}
$$

we can write equations (2.1) and (2.2) as:

$$
\begin{cases}
\dot{P}_1 + P_1\hat{A} + \hat{A}'P_1 - P_1\hat{N}P_2 - P_2\hat{N}P_1 + P_1\hat{A}_1P_1 + \hat{C} = 0, \quad t \in [0,T], \\
P_1(T) = 0,
\end{cases}
$$

$$\begin{cases} \dot{P}_2 - P_2\underline{A}_1P_2 + P_2\hat{A} + \hat{A}'P_2 - P_2\underline{N}P_1 \\ -P_1\underline{N}P_2 - P_1\underline{B}_2P_1 + P_1\underline{A}' + \underline{A}P_1 + \underline{C} = 0, \quad t \in [0,T], \\ P_2(T) = 0. \end{cases}$$

These two equations can be further be written as:

$$\begin{cases} \dot{P}_1 + P_1\hat{A} + \hat{A}'P_1 + \begin{bmatrix} P_1 & P_2 \end{bmatrix} \hat{M} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} + \hat{C} = 0, \quad t \in [0,T], \\ P_1(T) = 0, \end{cases} \tag{2.3}$$

$$\begin{cases} \dot{P}_2 + P_2\hat{A} + \hat{A}'P_2 + P_1\underline{A}' + \underline{A}P_1 \\ + \begin{bmatrix} P_1 & P_2 \end{bmatrix} \underline{M} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} + \underline{C} = 0, \quad t \in [0,T], \\ P_2(T) = 0, \end{cases} \tag{2.4}$$

where

$$\hat{M} := \begin{bmatrix} \hat{A}_1 & -\hat{N} \\ -\hat{N} & 0 \end{bmatrix}, \quad \underline{M} := \begin{bmatrix} -B_2 & -\underline{N} \\ -\underline{N} & -\underline{A}_1 \end{bmatrix}.$$

We first assume that there are two solution pairs to the equations (2.3) and (2.4), and then we prove that those are actually the same. Thus, let there exist two solution pairs (X_1, X_2) and (Y_1, Y_2) to the equations (2.3) and (2.4). This means that the following hold:

$$\begin{cases} \dot{X}_1 + X_1\hat{A} + \hat{A}'X_1 + \begin{bmatrix} X_1 & X_2 \end{bmatrix} \hat{M} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \hat{C} = 0, \quad t \in [0,T], \\ X_1(T) = 0, \end{cases} \tag{2.5}$$

$$\begin{cases} \dot{Y}_1 + Y_1\hat{A} + \hat{A}'Y_1 + \begin{bmatrix} Y_1 & Y_2 \end{bmatrix} \hat{M} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} + \hat{C} = 0, \quad t \in [0,T], \\ Y_1(T) = 0, \end{cases} \tag{2.6}$$

$$\begin{cases} \dot{X}_2 + X_2\hat{A} + \hat{A}'X_2 + X_1\underline{A}' + \underline{A}X_1 \\ + \begin{bmatrix} X_1 & X_2 \end{bmatrix} \underline{M} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \underline{C} = 0, \quad t \in [0,T], \\ X_2(T) = 0, \end{cases} \tag{2.7}$$

$$
\begin{cases}
\dot{Y}_2 + Y_2\hat{A} + \hat{A}'Y_2 + Y_1\underline{A}' + \underline{A}Y_1 \\
+ \begin{bmatrix} Y_1 & Y_2 \end{bmatrix} \underline{M} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} + \underline{C} = 0, \quad t \in [0,T], \\
Y_2(T) = 0,
\end{cases} \tag{2.8}
$$

By defining $\Delta_1 := X_1 - X_2$ and $\Delta_2 := Y_1 - Y_2$, and taking the difference of (2.5) with (2.6) and the difference of (2.7) with (2.8) we obtain, respectively:

$$
\begin{cases}
\dot{\Delta}_1 + \Delta_1\hat{A} + \hat{A}'\Delta_1 + \begin{bmatrix} \Delta_1 & \Delta_2 \end{bmatrix} \hat{M} \begin{bmatrix} X_1 + Y_1 \\ X_2 + Y_2 \end{bmatrix} = 0, \quad t \in [0,T], \\
\Delta_1(T) = 0,
\end{cases} \tag{2.9}
$$

$$
\begin{cases}
\dot{\Delta}_2 + \Delta_2\hat{A} + \hat{A}'\Delta_2 + \Delta_1\underline{A}' + \underline{A}\Delta_1 \\
+ \begin{bmatrix} \Delta_1 & \Delta_2 \end{bmatrix} \underline{M} \begin{bmatrix} X_1 + Y_1 \\ X_2 + Y_2 \end{bmatrix} = 0, \quad t \in [0,T], \\
\Delta_2(T) = 0.
\end{cases} \tag{2.10}
$$

As X_1, Y_1, X_2, Y_2 are assumed to be known, equations (2.9) and (2.10) are a linear system of differential equations in Δ_1 and Δ_2 with the unique solution $\Delta_1(t) = \Delta_2(t) = 0$ for all $t \in [0,T]$. This implies that $X_1(t) = X_2(t)$ and $Y_1(t) = Y_2(t)$ for all $t \in [0,T]$.

□

The main aim of this section is to show that there exists a unique Nash equilibrium $(u^*(\cdot), v^*(\cdot))$ such that inequalities (1.7) and (1.8) hold, and it is given by:

$$
u^*(t) := -(D'M_{13}D)^{-1}(C'M_{12}D + P_1B_2)'x(t), \quad t \in [0,T], \tag{2.11}
$$

$$
v^*(t) := -\theta^{-2}N^{-1}B_1'P_2x(t), \quad t \in [0,T]. \tag{2.12}
$$

If the noise $v^*(\cdot)$ is applied to system (1.1), then for any $u(\cdot) \in \mathcal{U}$ we define:

$$\begin{aligned}
\theta_u'(t) &:= -\gamma_1 x'(t) P_1(t) A_1(t), \quad t \in [0,T], \\
Z_u(t) &:= \exp\left[-\int_0^t \theta_u'(\tau) dW(\tau) - \frac{1}{2}\int_0^t \theta_u'(\tau)\theta_u(\tau) d\tau \right], \quad t \in [0,T], \\
Z_u &:= Z_u(T), \\
\widetilde{\mathbb{P}}_u(\alpha) &:= \int_\alpha Z_u(\omega) d\mathbb{P}(\omega), \quad \forall \alpha \in \mathcal{F}.
\end{aligned}$$

A sufficient condition for $\widetilde{\mathbb{P}}_u$ to be a probability measure is the following Novikov condition:

$$\mathbb{E}\left\{ \exp\left[\frac{\beta_u}{2} \int_0^T \theta_u'(t)\theta_u(t) dt \right] \right\} < \infty, \tag{2.13}$$

for some $\beta_u > 0$. We now define the admissible set of controls as:

$$\mathcal{A}_u := \{ u(\cdot) \in \mathcal{U} \quad \text{that satisfy (2.13)} \}.$$

Similarly, if control $u^*(\cdot)$ is applied to system (1.1), then for any $v(\cdot) \in \mathcal{V}$ we define:

$$\begin{aligned}
\theta_v'(t) &:= -\gamma_2 x'(t) P_2(t) A_1, \quad t \in [0,T], \\
Z_v(t) &:= \exp\left[-\int_0^t \theta_v'(\tau) dW(\tau) - \frac{1}{2}\int_0^t \theta_v'(\tau)\theta_v(\tau) d\tau \right], \quad t \in [0,T], \\
Z_v &:= Z_v(T), \\
\widetilde{\mathbb{P}}_v(\alpha) &:= \int_\alpha Z_v(\omega) d\mathbb{P}(\omega), \qquad \forall \alpha \in \mathcal{F}.
\end{aligned}$$

The sufficient Novikov condition for $\widetilde{\mathbb{P}}_v$ to be a probability measure is:

$$\mathbb{E}\left\{ \exp\left[\frac{\beta_v}{2} \int_0^T \theta_v'(t)\theta_v(t) dt \right] \right\} < \infty, \tag{2.14}$$

for some $\beta_v > 0$. The set of admissible disturbances is defined as:

$$\mathcal{A}_v := \{ v(\cdot) \in \mathcal{V} \quad \text{that satisfy (2.14)} \}.$$

Asssumption 3. $(u^*(\cdot), v^*(\cdot)) \in \mathcal{A}_u \times \mathcal{A}_v$.

Theorem 2. *There exists a unique pair* $(u(\cdot), v(\cdot)) \in \mathcal{A}_u \times \mathcal{A}_v$ *that satisfies inequalities (1.7) and (1.8), and that pair is* $(u^*(\cdot), v^*(\cdot))$*. In this case we have:*

$$\begin{aligned} J_1(u^*(\cdot), v^*(\cdot)) &= \frac{1}{\gamma_1} \exp\left[\frac{\gamma_1}{2} x'(0)P_1(0)x(0) + \frac{\gamma_1}{2}\int_0^T tr(A_1' P_1 A_1) dt\right], \\ J_2(u^*(\cdot), v^*(\cdot)) &= \frac{1}{\gamma_2} \exp\left[\frac{\gamma_2}{2} x'(0)P_2(0)x(0) + \frac{\gamma_2}{2}\int_0^T tr(A_1' P_2 A_1) dt\right]. \end{aligned}$$

Proof. We first consider $J_1(u(\cdot), v^*(\cdot))$ with $u(\cdot) \in \mathcal{A}_u$. Since

$$\begin{aligned} 0 &= x'(0)P_1(0)x(0) + \int_0^T [x'\dot{P}_1 x + 2x'P_1(Ax + B_2 u + B_1 v^*) \\ &+ tr(A_1' P_1 A_1)]dt + \int_0^T 2x' P_1 A_1 dW, \end{aligned}$$

we can write $J_1(u(\cdot), v^*(\cdot))$ as:

$$J_1(u(\cdot), v^*(\cdot)) = \frac{1}{\gamma_1}\mathbb{E} \exp\left[\frac{\gamma_1}{2} x'(0)P_1(0)x(0) + \frac{\gamma_1}{2}\int_0^T tr(A_1' P_1 A_1) dt\right.$$

$$+\frac{\gamma_1}{2}\int_0^T x'(\dot{P}_1 + C'M_{11}C + P_1 A + A'P_1 - \theta^{-2} P_1 B_1 N^{-1} B_1' P_2$$

$$-\theta^{-2} P_2 B_1 N^{-1} B_1' P_1) x dt$$

$$+\frac{\gamma_1}{2}\int_0^T (u'D'M_{13}Du + 2x'(C'M_{12}D + P_1 B_2)u) dt$$

$$\left.+\frac{\gamma_1}{2}\int_0^T 2x' P_1 A_1 dW(t)\right].$$

Let $\widetilde{\mathbb{E}}_u$ denote the expectation under the probability measure $\widetilde{\mathbb{P}}_u$. Due to

Assumption 2, for any $u(\cdot) \in \mathcal{A}_u$ we can write $J_1(u(\cdot), v^*(\cdot))$ as:

$$J_1(u(\cdot), v^*(\cdot)) = \frac{1}{\gamma_1}\widetilde{\mathbb{E}}_u \exp\left[\frac{\gamma_1}{2}x'(0)P_1(0)x(0) + \frac{\gamma_1}{2}\int_0^T tr(A_1'P_1A_1)dt\right.$$
$$+\frac{\gamma_1}{2}\int_0^T x'(\dot{P}_1+C'M_{11}C+P_1A+A'P_1-\theta^{-2}P_1B_1N^{-1}B_1'P_2$$
$$-\theta^{-2}P_2B_1N^{-1}B_1'P_1)xdt$$
$$\left.+\frac{\gamma_1}{2}\int_0^T (x'\gamma_1P_1A_1A_1'P_1x + u'D'M_{13}Du + 2x'(C'M_{12}D + P_1B_2)u)dt\right].$$

The completion of squares gives:

$$\begin{aligned}&u'D'M_{13}Du + 2x'(C'M_{12}D + P_1B_2)u\\&= [u + (D'M_{13}D)^{-1}(C'M_{12}D + P_1B_2)'x]'\\&\times D'M_{13}D[u + (D'M_{13}D)^{-1}(C'M_{12}D + P_1B_2)'x]\\&-x'(C'M_{12}D + P_1B_2)(D'M_{13}D)^{-1}(C'M_{12}D + P_1B_2)'x.\end{aligned}$$

Due to Assumption 1, for all $u(\cdot) \in \mathcal{A}_u$ we have:

$$J_1(u(\cdot), v^*(\cdot)) = \frac{1}{\gamma_1}\widetilde{\mathbb{E}}_u \exp\left[\frac{\gamma_1}{2}x'(0)P_1(0)x(0) + \frac{\gamma_1}{2}\int_0^T tr(A_1'P_1A_1)dt\right.$$
$$+\frac{\gamma_1}{2}\int_0^T[u+(D'M_{13}D)^{-1}(C'M_{12}D+P_1B_2)'x]$$
$$\left.\times' D'M_{13}D[u+(D'M_{13}D)^{-1}(C'M_{12}D+P_1B_2)'x]dt\right],$$
$$\geq \frac{1}{\gamma_1}\exp\left[\frac{\gamma_1}{2}x'(0)P_1(0)x(0) + \frac{\gamma_1}{2}\int_0^T tr(A_1'P_1A_1)dt\right],$$

with equality if and only if $u(t) = u^*(t)$ for $a.e.$ $t \in [0, T]$ $a.s.$.

We now consider $J_2(u^*(\cdot), v(\cdot))$ for all $v(\cdot) \in \mathcal{A}_v$. For notational convenience we define the matrix K^* as:

$$K^*(t) := -(D'(t)M_{13}(t)D(t))^{-1}(C'(t)M_{12}(t)D(t) + P_1(t)B_2(t))', \quad t \in [0, T].$$

Since

$$
\begin{aligned}
0 &= x'(0)P_2(0)x(0) + \int_0^T [x'\dot{P}_2 x + 2x'P_2(Ax + B_2K^*x + B_1 v) \\
&+ tr(A_1'P_2A_1)]dt + \int_0^T 2x'P_2A_1 dW(t),
\end{aligned}
$$

we can write $J_2(u^*(\cdot), v(\cdot))$ as:

$$
J_2(u^*(\cdot), v(\cdot)) = \frac{1}{\gamma_2}\mathbb{E}\exp\Bigg[\frac{\gamma_2}{2}x'(0)P_2(0)x(0) + \frac{\gamma_2}{2}\int_0^T tr(A_1'P_2A_1)dt
$$

$$
+\frac{\gamma_2}{2}\int_0^T x'(\dot{P}_2 + P_2A + A'P_2 + P_2B_2K^* + (K^*)'B_2'P_2
$$

$$
-C'M_{21}C - C'M_{22}DK^* - (K^*)'D'M_{22}'C - (K^*)'D'M_{23}DK^*)xdt
$$

$$
+\frac{\gamma_2}{2}\int_0^T (\theta^2 v'Nv + 2x'P_2B_1v)dt + \frac{\gamma_2}{2}\int_0^T 2x'P_2A_1dW(t)\Bigg].
$$

If we denote by $\widetilde{\mathbb{E}}_v$ the expectation under the probability measure $\widetilde{\mathbb{P}}_v$, then for any $v(\cdot) \in \mathcal{A}_v$ we have:

$$
J_2(u^*(\cdot), v(\cdot)) = \frac{1}{\gamma_2}\widetilde{\mathbb{E}}_v\exp\Bigg[\frac{\gamma_2}{2}x'(0)P_2(0)x(0) + \frac{\gamma_2}{2}\int_0^T tr(A_1'P_2A_1)dt
$$

$$
+\frac{\gamma_2}{2}\int_0^T x'(\dot{P}_2 + P_2A + A'P_2 + P_2B_2K^* + (K^*)'B_2'P_2
$$

$$
-C'M_{21}C - C'M_{22}DK^* - (K^*)'D'M_{22}'C
$$

$$
-(K^*)'D'M_{23}DK^* + \gamma_2 P_2A_1A_1'P_2)xdt
$$

$$
+\frac{\gamma_2}{2}\int_0^T (\theta^2 v'Nv + 2x'P_2B_1v)dt + \frac{\gamma_2}{2}\int_0^T 2x'P_2A_1dW(t)\Bigg].
$$

The completion of squares gives:

$$
\begin{aligned}
\theta^2 v'Nv + 2x'P_2B_1v &= (v + \theta^{-2}N^{-1}B_1'P_2x)'\theta^2 N(v + \theta^{-2}N^{-1}B_1'P_2x) \\
&- \theta^{-2}x'P_2B_1N^{-1}B_1'P_2x(t).
\end{aligned}
$$

Due to our assumption on $P_2(\cdot)$, for any $v(\cdot) \in \mathcal{A}_v$ we have:

$$\begin{aligned} J_2(u^*(\cdot), v(\cdot)) &= \frac{1}{\gamma_2}\widetilde{\mathbb{E}}_v \exp\left[\frac{\gamma_2}{2}x'(0)P_2(0)x(0) + \frac{\gamma_2}{2}\int_0^T tr(A_1'P_2A_1)dt\right. \\ &+ \left.\frac{\gamma_2}{2}\int_0^T (v+\theta^{-2}N^{-1}B_1'P_2x)'\theta^2 N(v+\theta^{-2}N^{-1}B_1'P_2x)dt\right]. \\ &\geq \frac{1}{\gamma_2}\exp\left[\frac{\gamma_2}{2}x'(0)P_2(0)x(0) + \frac{\gamma_2}{2}\int_0^T tr(A_1'P_2A_1)dt\right], \end{aligned}$$

with equality if and only if $v(t) = v^*(t)$ for *a.e.* $t \in [0,T]$ *a.s.*. □

We now give some sufficient conditions for Assumption 3 to hold, which means sufficient conditions for (2.13) and (2.14) to hold under the pair $(u^*(\cdot), v^*(\cdot))$. We define the matrix A^* as:

$$A^* := A - B_2(D'M_{13}D)^{-1}(C'M_{12}D + P_1B_2)' - \theta^{-2}B_1N^{-1}B_1'P_2, \quad t \in [0,T]. \tag{2.15}$$

Consider the following Lyapunov and Riccati differential equations, respectively:

$$\begin{cases} d\Sigma = [A^*\Sigma + \Sigma(A^*)' + A_1A_1']\,dt, \quad t \in [0,T], \\ \\ \Sigma(0) = 0, \end{cases} \tag{2.16}$$

$$\begin{aligned} dQ_1 &= -\left[Q_1A^* + (A^*)'Q_1 + 2\alpha_1Q_1A_1A_1'Q_1\right. \\ &+ \left.\beta_u\gamma_1^2P_1A_1A_1'P_1/2\right]dt, \end{aligned} \tag{2.17}$$

$$\begin{aligned} dQ_2 &= -\left[Q_2A^* + (A^*)'Q_2 + 2\beta_1Q_2A_1A_1'Q_2\right. \\ &+ \left.\beta_v\gamma_2^2P_2A_1A_1'P_2/2\right]dt, \end{aligned} \tag{2.18}$$

for $t \in [0,T]$, where α_1 and β_1 are positive constants.

Theorem 3. *Let $\Sigma(t) > 0$ for all $t \in (0,T]$, and let α_1, α_2, β_1, β_2, be positive constants such that: $\alpha_1^{-1} + \alpha_2^{-1} = 1$ and $\beta_1^{-1} + \beta_2^{-1} = 1$. If there exist $Q_1(0)$, $Q_2(0) \in \mathbb{R}^{n\times n}$ for which equations (2.17) and (2.18) have solutions Q_1 and Q_2 satisfying:*

$$Q_1(T) + \Sigma^{-1}(T)/2\alpha_2 > 0, \quad Q_2(T) + \Sigma^{-1}(T)/2\beta_2 > 0,$$

then $(u^*(\cdot), v^*(\cdot)) \in \mathcal{A}_u \times \mathcal{A}_v$.

Proof. Here we show that condition (2.13) holds, and similarly it can be shown that condition (2.14) also holds. Under the pair $(u^*(\cdot), v^*(\cdot))$ the state equation of (1.1) becomes:

$$\begin{aligned} dx &= [A - B_2(D'M_{13}D)^{-1}(C'M_{12}D + P_1B_2)' - \theta^{-2}B_1N^{-1}B_1'P_2]xdt \\ &+ A_1dW = A^*xdt + A_1dW, \quad t \in [0, T]. \end{aligned}$$

Thus, the system state $x(T)$ under the pair $(u^*(\cdot), v^*(\cdot))$ has the distribution $x(T) \sim N(\mu(T), \Sigma(T))$, where the mean $\mu(t) := \mathbb{E}[x(t)]$, $t \in [0, T]$, is the solution to the linear differential equation $d\mu = A^*\mu dt$, $\mu(0) = x_0$. From the differential of quadratic form $x'(t)Q_1(t)x(t)$ we obtain the following (by integrating from 0 to T):

$$\begin{aligned} 0 &= -x'(T)Q_1(T)x(T) + x'(0)Q_1(0)x(0) + \int_0^T tr(A_1'Q_1A_1)dt \\ &+ \frac{1}{\alpha_1}\Big[-\frac{1}{2}\int_0^T (-2\alpha_1 x'Q_1A_1)(-2\alpha_1 x'Q_1A_1)'dt - \int_0^T (-2\alpha_1 x'Q_1A_1)dW \Big] \\ &+ \int_0^T x'\big[\dot{Q}_1 + Q_1A^* + (A^*)'Q_1 + 2\alpha_1 Q_1A_1A_1'Q_1\big]xdt. \end{aligned}$$

We now have:

$$\begin{aligned} &\frac{\beta_u}{2}\int_0^T \theta_u'\theta_u dt = \int_0^T \frac{\beta_u\gamma_1^2}{2}x'P_1A_1A_1'P_1xdt \\ &= -x'(T)Q_1(T)x(T) + x'(0)Q_1(0)x(0) + \int_0^T tr(A_1'Q_1A_1)dt \\ &+ \frac{1}{\alpha_1}\Big[-\frac{1}{2}\int_0^T (-2\alpha_1 x'Q_1A_1)(-2\alpha_1 x'Q_1A_1)'dt - \int_0^T (-2\alpha_1 x'Q_1A_1)dW \Big]. \end{aligned}$$

Using the Hölder's inequality, we have the following for condition (2.13):

$$
\begin{aligned}
&\mathbb{E}\left[\exp\left\{\frac{\beta_u}{2}\int_0^T \theta_u'\theta_u dt\right\}\right] \le \exp\left\{x'(0)Q_1(0)x(0) + \int_0^T tr(A_1'Q_1A_1)dt\right\} \\
&\times\left(\mathbb{E}\left[\exp\left\{-\frac{1}{2}\int_0^T (-2\alpha_1 x'Q_1A_1)(-2\alpha_1 x'Q_1A_1)'dt\right.\right.\right. \\
&\left.\left.\left.-\int_0^T (-2\alpha_1 x'Q_1A_1)dW\right\}\right]\right)^{\frac{1}{\alpha_1}} \\
&\times\left(\mathbb{E}\left[\exp\left\{-\alpha_2 x'(T)Q_1(T)x(T)\right\}\right]\right)^{\frac{1}{\alpha_2}} \\
&\le \exp\left\{x'(0)Q_1(0)x(0) + \int_0^T tr(A_1'Q_1A_1)dt\right\} \\
&\times\left(\mathbb{E}\left[\exp\left\{-\alpha_2 x'(T)Q_1(T)x(T)\right\}\right]\right)^{\frac{1}{\alpha_2}} \\
&= \exp\left\{x'(0)Q_1(0)x(0) + \int_0^T tr(A_1'Q_1A_1)dt\right\} \\
&\times\left(\left|2\alpha_1 Q_1(T) + \Sigma^{-1}(T)\right|^{\frac{1}{2}}\left|\Sigma(T)\right|^{-\frac{1}{2}}\right)^{\frac{1}{\alpha_2}} \\
&\times\left(\exp\left\{\frac{1}{2}\mu'(T)\Sigma^{-1}(T)\left[2\alpha_2 Q_1(T) + \Sigma^{-1}(T)\right]^{-1}\Sigma^{-1}(T)\mu(T)\right.\right. \\
&\left.\left.-\frac{1}{2}\mu'(T)\Sigma^{-1}(T)\mu(T)\right\}\right)^{\frac{1}{\alpha_2}} \\
&< \infty.
\end{aligned}
$$

□

3. Infinite horizon risk-sensitive H_2/H_∞ control problem

Here we consider an *infinite* horizon version of our problem. The derivation is similar to the finite horizon, but it is more involved due to certain *stability* requirements, which are absent in the finite horizon case.

Asssumption 4. *The matrices A, A_1, B_1, B_2, C, D, M_1, M_2, N, are constant, $D'M_{13}D > 0$ and $N > 0$.*

We consider the following two functionals, which are the infinite horizon versions of $J_1(u(\cdot), v(\cdot))$ and $J_2(u(\cdot), v(\cdot))$:

$$
\begin{aligned}
J_1^\infty(u(\cdot), v(\cdot)) &= \lim_{T\to\infty} \frac{1}{f_1(T)\gamma_1} \ln \mathbb{E}\left\{\exp\left[\frac{\gamma_1}{2}\int_0^T z'(t)M_1 z(t)dt\right]\right\}, \\
J_2^\infty(u(\cdot), v(\cdot)) &= \lim_{T\to\infty} \frac{1}{f_2(T)\gamma_2} \ln \mathbb{E}\left\{\exp\left[\frac{\gamma_2}{2}\int_0^T (\theta^2 v'(t)Nv(t)\right.\right. \\
&- \left.\left. z'(t)M_2 z(t))dt\right]\right\}.
\end{aligned}
$$

where $f_1(T)$ and $f_2(T)$ are given positive functions. Our aim is to find a Nash equilibrium $(u^*(\cdot), v^*(\cdot))$ such that the following inequalities hold:

$$
J_1^\infty(u^*(\cdot), v^*(\cdot)) \leq J_1^\infty(u(\cdot), v^*(\cdot)), \quad \forall u(\cdot) \in \mathcal{A}_u^\infty \tag{3.1}
$$

$$
J_2^\infty(u^*(\cdot), v^*(\cdot)) \leq J_2^\infty(u^*(\cdot), v(\cdot)), \quad \forall v(\cdot) \in \mathcal{A}_v^\infty. \tag{3.2}
$$

for some suitably defined sets $\mathcal{A}_u^\infty$ and $\mathcal{A}_v^\infty$. We confine ourselves only to linear *constant* state feedback controls and disturbances. The sets of all such controls and disturbances are defined as:

$$
\mathcal{U}^\infty := \{u(\cdot): \quad u(t) = K_u x(t) \quad \text{where } K_u \in \mathbb{R}^{n_u\times n}\},
$$

$$
\mathcal{V}^\infty := \{v(\cdot): \quad v(t) = K_v x(t) \quad \text{where } K_v \in \mathbb{R}^{n_v\times n}\}.
$$

Asssumption 5. *There exists a real solution pair (P_1, P_2) to the following coupled Riccati algebraic equations:*

$$
\left\{
\begin{array}{l}
P_1[A - B_2(D'M_{13}D)^{-1}D'M_{12}'C] + [A - B_2(D'M_{13}D)^{-1}D'M_{12}'C]'P_1 \\
-\theta^{-2}(P_1B_1N^{-1}B_1'P_2 + P_2B_1N^{-1}B_1'P_1) \\
+P_1[\gamma_1 A_1A_1' - B_2(D'M_{13}D)^{-1}B_2']P_1 \\
+C'[M_{11} - M_{12}D(D'M_{13}D)^{-1}D'M_{12}]C = 0,
\end{array}
\right. \tag{3.3}
$$

$$
\begin{cases}
-P_2(\theta^{-2}B_1N^{-1}B_1' - \gamma_2 A_1A_1')P_2 + P_2[A - B_2(D'M_{13}D)^{-1}D'M_{12}'C] \\
+(A - B_2(D'M_{13}D)^{-1}D'M_{12}'C)'P_2 - P_2B_2(D'M_{13}D)^{-1}B_2'P_1 \\
-P_1B_2(D'M_{13}D)^{-1}B_2'P_2 \\
+[C'M_{22}D(D'M_{13}D)^{-1}B_2' \\
-C'M_{12}D(D'M_{13}D)^{-1}D'M_{23}D(D'M_{13}D)^{-1}B_2']P_1 \\
+P_1[C'M_{22}D(D'M_{13}D)^{-1}B_2' \\
-C'M_{12}D(D'M_{13}D)^{-1}D'M_{23}D(D'M_{13}D)^{-1}B_2']' \\
-P_1B_2(D'M_{13}D)^{-1}D'M_{23}D(D'M_{13}D)^{-1}B_2'P_1 - C'M_{21}C \\
+C'M_{22}D(D'M_{13}D)^{-1}D'M_{12}'C + C'M_{12}D(D'M_{13}D)^{-1}D'M_{22}'C \\
-C'M_{12}D(D'M_{13}D)^{-1}D'M_{23}D(D'M_{13}D)^{-1}D'M_{12}'C = 0.
\end{cases} \tag{3.4}
$$

Asssumption 6. *The functions $f_1(T)$ and $f_2(T)$ are such that:*

$$
\lim_{T\to\infty} \frac{tr(A_1'P_1A_1)T + x'(0)P_1x(0)}{2f_1(T)} = g_1 \in \mathbb{R},
$$

$$
\lim_{T\to\infty} \frac{tr(A_1'P_2A_1)T + x'(0)P_2x(0)}{2f_2(T)} = g_2 \in \mathbb{R}.
$$

We show later in this section that there exists a unique pair $(u_\infty^*(\cdot), v_\infty^*(\cdot))$ that satisfies the inequalities (3.1) and (3.2), and is given by:

$$
u_\infty^*(t) \ := \ -(D'M_{13}D)^{-1}(C'M_{12}D + P_1B_2)'x(t), \quad t \in [0,\infty), \tag{3.5}
$$

$$
v_\infty^*(t) \ := \ -\theta^{-2}N^{-1}B_1'P_2x(t), \quad t \in [0,\infty). \tag{3.6}
$$

If the noise $v_\infty^*(\cdot)$ is applied to the system (1.1), then for any $u(\cdot) \in \mathcal{U}^\infty$ we define:

$$
\begin{aligned}
\theta_u'(t) \ &:= \ -\gamma_1 x'(t)P_1A_1, \quad t \in [0,T] \\
Z_u(t) \ &:= \ \exp\left[-\int_0^t \theta_u'(\tau)dW(\tau) - \frac{1}{2}\int_0^t \theta_u'(\tau)\theta_u(\tau)d\tau \right], \quad t \in [0,T], \\
Z_u \ &:= \ Z_u(T), \\
\widetilde{\mathbb{P}}_u(\alpha) \ &:= \ \int_\alpha Z_u(\omega)d\mathbb{P}(\omega), \quad \forall \alpha \in \mathcal{F}.
\end{aligned}
$$

A sufficient condition for $\widetilde{\mathbb{P}}_u$ to be a probability measure is the following Novikov condition:

$$\mathbb{E}\left\{\exp\left[\frac{\beta_u}{2}\int_0^T \theta_u'(t)\theta_u(t)dt\right]\right\} < \infty, \tag{3.7}$$

for some $\beta_u > 0$. The controls $u(\cdot)$ are restricted further, so that the following *stability* condition holds:

$$\lim_{T\to\infty}\frac{1}{f_1(T)\gamma_1}\ln\widetilde{\mathbb{E}}_u\left\{\exp\left[-\frac{\gamma_1}{2}x(T)'P_1x(T)\right]\right\} = h_1 \in \mathbb{R}. \tag{3.8}$$

We can now formulate the admissible set of controls as:

$\mathcal{A}_u^\infty := \{u(\cdot) \in \mathcal{U} \quad$ such that (3.7) holds for all $T \in (0,\infty)$, and (3.8) holds$\}$.

Similarly, if the control $u_\infty^*(\cdot)$ is applied to the system (1.1), then for any $v(\cdot) \in \mathcal{V}^\infty$ we define:

$$\begin{aligned}
\theta_v'(t) &:= -\gamma_2 x'(t)P_2A_1, \quad t\in[0,T],\\
Z_v(t) &:= \exp\left[-\int_0^t \theta_v'(\tau)dW(\tau) - \frac{1}{2}\int_0^t \theta_v'(\tau)\theta_v(\tau)d\tau\right], \quad t\in[0,T],\\
Z_v &:= Z_v(T),\\
\widetilde{\mathbb{P}}_v(\alpha) &:= \int_\alpha Z_v(\omega)d\mathbb{P}(\omega), \qquad \forall \alpha\in\mathcal{F}.
\end{aligned}$$

The sufficient Novikov condition for $\widetilde{\mathbb{P}}_v$ to be a probability measure is:

$$\mathbb{E}\left\{\exp\left[\frac{\beta_v}{2}\int_0^T \theta_v'(t)\theta_v(t)dt\right]\right\} < \infty, \tag{3.9}$$

for some $\beta_v > 0$. We further restrict the set of permitted disturbances, so that the following holds:

$$\lim_{T\to\infty}\frac{1}{f_2(T)\gamma_2}\ln\widetilde{\mathbb{E}}_v\left\{\exp\left[-\frac{\gamma_2}{2}x(T)'P_2x(T)\right]\right\} = h_2 \in \mathbb{R}. \tag{3.10}$$

The set of disturbances that we consider can now be defined as:

$\mathcal{A}_v := \{v(\cdot) \in \mathcal{V} \quad$ such that (3.9) holds for all $T \in (0,\infty)$, and (3.10) holds$\}$.

Asssumption 7. $(u^*_\infty(\cdot), v^*_\infty(\cdot)) \in \mathcal{A}_u^\infty \times \mathcal{A}_v^\infty$.

Theorem 4. *There exists a unique pair* $(u^*_\infty(\cdot), v^*_\infty(\cdot))$ *that satisfies the inequalities (3.1) and (3.2), and is given by (3.5) and (3.6). In this case we have:*

$$\begin{aligned} J_1^\infty(u^*(\cdot), v^*(\cdot)) &= g_1 + h_1, \\ J_2^\infty(u^*(\cdot), v^*(\cdot)) &= g_2 + h_2. \end{aligned}$$

Proof. Let us first consider $J_1^\infty(u(\cdot), v^*_\infty(\cdot))$ with $u(\cdot) \in \mathcal{A}_u^\infty$. Since

$$\begin{aligned} 0 &= -x'(T)P_1x(T) + x'(0)P_1x(0) + \int_0^T [2x'P_1(Ax + B_2u + B_1v^*_\infty) \\ &+ tr(A_1'P_1A_1)]dt + \int_0^T 2x'P_1A_1dW, \end{aligned}$$

we can write $J_1^\infty(u(\cdot), v^*_\infty(\cdot))$ as:

$$\begin{aligned} &J_1^\infty(u(\cdot), v^*_\infty(\cdot)) = \lim_{T\to\infty} \frac{1}{f_1(T)\gamma_1} \ln \mathbb{E} \exp \Bigg[\frac{\gamma_1}{2} x'(0)P_1x(0) + \frac{\gamma_1}{2} tr(A_1'P_1A_1)T \\ &+ \frac{\gamma_1}{2} \int_0^T x'(C'M_{11}C + P_1A + A'P_1 \\ &- \theta^{-2}P_1B_1N^{-1}B_1'P_2 - \theta^{-2}P_2B_1N^{-1}B_1'P_1)xdt \\ &- \frac{\gamma_1}{2} x'(T)P_1x(T) + \frac{\gamma_1}{2} \int_0^T (u'D'M_{13}Du + 2x'(C'M_{12}D + P_1B_2)u)dt \\ &+ \frac{\gamma_1}{2} \int_0^T 2x'P_1A_1dW(t) \Bigg]. \end{aligned}$$

For any $u(\cdot) \in \mathcal{A}_u^\infty$ we can write $J_1^\infty(u(\cdot), v_\infty^*(\cdot))$ as:

$$J_1^\infty(u(\cdot), v_\infty^*(\cdot)) = \lim_{T\to\infty} \frac{1}{f_1(T)\gamma_1} \ln \widetilde{\mathbb{E}}_u \exp\left[\frac{\gamma_1}{2}x'(0)P_1x(0) + \frac{\gamma_1}{2}tr(A_1'P_1A_1)T\right.$$

$$+\frac{\gamma_1}{2}\int_0^T x'(C'M_{11}C + P_1A + A'P_1 - \theta^{-2}P_1B_1N^{-1}B_1'P_2$$

$$-\theta^{-2}P_2B_1N^{-1}B_1'P_1)xdt$$

$$+\frac{\gamma_1}{2}\int_0^T (\gamma_1 x'P_1A_1A_1'P_1x + u'D'M_{13}Du + 2x'(C'M_{12}D + P_1B_2)u)dt$$

$$\left.-\frac{\gamma_1}{2}x'(T)P_1x(T)\right].$$

The completion of squares gives:

$$\begin{aligned}
&u'D'M_{13}Du + 2x'(C'M_{12}D + P_1B_2)u \\
&= [u + (D'M_{13}D)^{-1}(C'M_{12}D + P_1B_2)'x]' \\
&\times D'M_{13}D[u + (D'M_{13}D)^{-1}(C'M_{12}D + P_1B_2)'x] \\
&-x'(C'M_{12}D + P_1B_2)(D'M_{13}D)^{-1}(C'M_{12}D + P_1B_2)'x.
\end{aligned}$$

For all $u(\cdot) \in \mathcal{A}_u^\infty$ we have:

$$\begin{aligned}
J_1^\infty(u(\cdot), v_\infty^*(\cdot)) &= \lim_{T\to\infty} \frac{1}{f_1(T)\gamma_1} \ln \widetilde{\mathbb{E}}_u \exp\left[\frac{\gamma_1}{2}x'(0)P_1x(0) + \frac{\gamma_1}{2}tr(A_1'P_1A_1)T\right. \\
&- \left.\frac{\gamma_1}{2}x'(T)P_1x(T) + \frac{\gamma_1}{2}\int_0^T (u - u_\infty^*)'D'M_{13}D(u - u_\infty^*)dt\right], \\
&\geq g_1 + h_1,
\end{aligned}$$

with equality if and only if $u(t) = u_\infty^*(t)$ for $a.e.$ $t \in [0, \infty)$.

We now consider $J_2^\infty(u_\infty^*(\cdot), v(\cdot))$ for all $v(\cdot) \in \mathcal{A}_v^\infty$. For notational convenience we define the matrix K^* as:

$$K_\infty^*(t) := -(D'M_{13}D)^{-1}(C'M_{12}D + P_1B_2)', \quad t \in [0, \infty).$$

Since

$$
\begin{aligned}
0 &= -x'(T)P_2x(T) + x'(0)P_2x(0) + \int_0^T [2x'P_2(Ax + B_2K^*x + B_1v) \\
&+ tr(A_1'P_2A_1)]dt + \int_0^T 2x'P_2A_1dW(t),
\end{aligned}
$$

we can write $J_2^\infty(u_\infty^*(\cdot), v(\cdot))$ as:

$$
J_2^\infty(u_\infty^*(\cdot), v(\cdot)) = \lim_{T\to\infty} \frac{1}{f_2(T)\gamma_2} \ln \mathbb{E} \exp \left[\frac{\gamma_2}{2} x'(0)P_2x(0) \right.
$$

$$
+\frac{\gamma_2}{2} \int_0^T tr(A_1'P_2A_1)dt
$$

$$
+\frac{\gamma_2}{2} \int_0^T x'(P_2A + A'P_2 + P_2B_2K_\infty^* + (K_\infty^*)'B_2'P_2
$$

$$
-C'M_{21}C - C'M_{22}DK_\infty^* - (K_\infty^*)'D'M_{22}'C - (K_\infty^*)'D'M_{23}DK_\infty^*)xdt
$$

$$
\left. -\frac{\gamma_2}{2} x'(T)P_2x(T) + \frac{\gamma_2}{2} \int_0^T (\theta^2 v'Nv + 2x'P_2B_1v)dt + \frac{\gamma_2}{2} \int_0^T 2x'P_2A_1dW(t) \right].
$$

If we denote by $\widetilde{\mathbb{E}}_v$ the expectation under the probability measure $\widetilde{\mathbb{P}}_v$, then for any $v(\cdot) \in \mathcal{A}_v$ we have:

$$
J_2^\infty(u_\infty^*(\cdot), v(\cdot)) = \lim_{T\to\infty} \frac{1}{f_2(T)\gamma_2} \ln \widetilde{\mathbb{E}}_v \exp \left[\frac{\gamma_2}{2} x'(0)P_2x(0) \right.
$$

$$
+\frac{\gamma_2}{2} \int_0^T tr(A_1'P_2A_1)dt + \frac{\gamma_2}{2} \int_0^T x'(P_2A + A'P_2 + P_2B_2K^* + (K_\infty^*)'B_2'P_2
$$

$$
\begin{aligned}
&-C'M_{21}C - C'M_{22}DK_\infty^* - (K_\infty^*)'D'M_{22}'C \\
&-(K_\infty^*)'D'M_{23}DK_\infty^* + \gamma_2P_2A_1A_1'P_2)xdt
\end{aligned}
$$

$$
\left. -\frac{\gamma_2}{2} x'(T)P_2x(T) + \frac{\gamma_2}{2} \int_0^T (\theta^2 v'Nv + 2x'P_2B_1v)dt + \frac{\gamma_2}{2} \int_0^T 2x'P_2A_1dW(t) \right].
$$

The completion of squares gives:

$$\begin{aligned}\theta^2 v'Nv + 2x'P_2B_1v &= (v+\theta^{-2}N^{-1}B_1'P_2x)'\theta^2N(v+\theta^{-2}N^{-1}B_1'P_2x)\\ &- \theta^{-2}x'P_2B_1N^{-1}B_1'P_2x(t).\end{aligned}$$

Due to our assumption on P_2, for any $v(\cdot)\in\mathcal{A}_v^\infty$ we have:

$$\begin{aligned}&J_2^\infty(u^*(\cdot),v(\cdot)) = \lim_{T\to\infty}\frac{1}{f_2(T)\gamma_2}\ln\widetilde{\mathbb{E}}_v\exp\Bigg[\frac{\gamma_2}{2}x'(0)P_2x(0)\\ &+\frac{\gamma_2}{2}\int_0^T tr(A_1'P_2A_1)dt - \frac{\gamma_2}{2}x'(T)P_2x(T)\\ &+\frac{\gamma_2}{2}\int_0^T (v+\theta^{-2}N^{-1}B_1'P_2x)'\theta^2N(v+\theta^{-2}N^{-1}B_1'P_2x)dt\Bigg].\\ &\geq g_2+h_2,\end{aligned}$$

with equality if and only if $v(t)=v_\infty^*(t)$ for $a.e.\ t\in[0,\infty)$ $a.s.$. □

We now give sufficient conditions for Assumption 5 to hold, i.e. to have $(u_\infty^*(\cdot),v_\infty^*(\cdot))\in\mathcal{A}_u^\infty\times\mathcal{A}_v^\infty$. If conditions of Theorem 3 hold for all $T\in(0,\infty)$ in the current case of constant coefficients and Nash equilibrium (u_∞^*,v_∞^*), then the requirements (3.7) and (3.9) are satisfied for all $T\in(0,\infty)$. In remains to find (sufficient) conditions under which the stability requirements (3.8) and (3.10) are satisfied under the pair (u_∞^*,v_∞^*). Note first that by the Girsanov theorem processes $\widetilde{W}_u$ and $\widetilde{W}_v$ defined as:

$$\begin{aligned}\widetilde{W}_u(t) &:= W(t)-\int_0^t\gamma_1A_1'P_1x(s)ds,\quad t\geq 0,\\ \widetilde{W}_v(t) &:= W(t)-\int_0^t\gamma_1A_1'P_2x(s)ds,\quad t\geq 0,\end{aligned}$$

are Brownian motions under the probability measures $\widetilde{\mathbb{P}}_u$ and $\widetilde{\mathbb{P}}_v$, respectively, where x is the solution of the state equation in (1.1) under the pair (u_∞^*,v_∞^*). By substituting (u_∞^*,v_∞^*) in the state equation of (1.1) we obtain the following two forms of that equation:

$$\begin{aligned}dx(t) &= A_ux(t)dt+A_1d\widetilde{W}_u(t),\quad t\geq 0,\\ dx(t) &= A_vx(t)dt+A_1d\widetilde{W}_v(t),\quad t\geq 0,\end{aligned}$$

where

$$\begin{aligned} A_u &:= A - B_2(D'M_{13}D)^{-1}(C'M_{12}D + P_1B_2)' \\ &- \theta^{-2}B_1N^{-1}B_1'P_2 + \gamma_1 A_1A_1'P_1, \\ A_v &:= A - B_2(D'M_{13}D)^{-1}(C'M_{12}D + P_1B_2)' \\ &- \theta^{-2}B_1N^{-1}B_1'P_2 + \gamma_2 A_1A_1'P_2. \end{aligned}$$

Let the mean and covariance of $x(t)$ under the probability measures $\widetilde{\mathbb{P}}_u$ and $\widetilde{\mathbb{P}}_v$ be defined as, respectively:

$$\begin{aligned} \mu_u(t) &:= \widetilde{\mathbb{E}}_u[x(t)], \quad \Sigma_u(t) := \widetilde{\mathbb{E}}_u[(x(t)-\mu_u(t))(x(t)-\mu_u(t))'], \\ \mu_v(t) &:= \widetilde{\mathbb{E}}_v[x(t)], \quad \Sigma_v(t) := \widetilde{\mathbb{E}}_v[(x(t)-\mu_u(t))(x(t)-\mu_v(t))'], \end{aligned}$$

These are solutions to the following linear differential equations (for $t \geq 0$):

$$d\mu_u = A_u\mu_u dt, \quad d\Sigma_u = [A_u\Sigma_u + \Sigma_u A_u' + A_1A_1']\, dt, \tag{3.11}$$

$$d\mu_v = A_v\mu_v dt, \quad d\Sigma_v = [A_v\Sigma_v + \Sigma_v A_v' + A_1A_1']\, dt, \tag{3.12}$$

with $\mu_u(0) = \mu_v(0) = x_0$, $\Sigma_u(0) = \Sigma_v(0) = 0$. For any $T > 0$ we define:

$$H_u(T) := [\gamma_1\Sigma_u^{1/2}(T)P_1\Sigma_u^{1/2}(T)+I]^{-1}, \quad H_v(T) := [\gamma_2\Sigma_v^{1/2}(T)P_2\Sigma_v^{1/2}(T)+I]^{-1}.$$

Theorem 5. *Let $\Sigma_u(t) > 0$, $\Sigma_v(t) > 0$, $H_u(t) > 0$, and $H_v(t) > 0$, for all $t > 0$. The stability conditions (3.8) and (3.10) hold under (u_∞^*, v_∞^*) if:*

$$\lim_{T\to\infty} \frac{\ln|H_u(T)| + \mu_u'(T)\Sigma_u^{-1}(T)[H_u(T)-I]\Sigma_u^{-1}(T)\mu_u(T)}{2f_1(T)\gamma_1} = h_1 \tag{3.13}$$

$$\lim_{T\to\infty} \frac{\ln|H_v(T)| + \mu_v'(T)\Sigma_v^{-1}(T)[H_v(T)-I]\Sigma_v^{-1}(T)\mu_v(T)}{2f_2(T)\gamma_2} = h_2. \tag{3.14}$$

Proof. We only derive condition (3.13) as the derivation of condition (3.14) proceeds similarly. Under the probability measure $\mathbb{P}_u$ it holds that $x(t) \sim N(\mu_u(t), \Sigma_u(t))$. Therefore:

$$\widetilde{\mathbb{E}}_u\left[e^{-\gamma_1 x'(T)P_1x(T)/2}\right] = \int_{\mathbb{R}^n} e^{-\gamma_1 x'P_1x/2}\frac{e^{(x-\mu_u(T))'\Sigma_u^{-1}(T)(x-\mu_u(T))/2}}{(2\pi)^{-n/2}|\Sigma(T)|^{-1/2}}dx$$

$$= \frac{|[\gamma_1P_1 + \Sigma_u^{-1}(T)]^{-1}|^{1/2}}{|\Sigma_u(T)|^{1/2}}$$

$$\times \exp\left\{\frac{1}{2}\mu_u'(T)\left[\Sigma_u^{-1}(T)(\gamma_1P_1+\Sigma_u^{-1}(T))^{-1}\Sigma_u^{-1}(T) - \Sigma_u^{-1}(T)\right]\mu_u(T)\right\},$$

Taking the limit as $T \to \infty$ of logarithm of this expression divided by $\gamma_1 f_1(T)$, gives (3.13). However, this is just the stability condition (3.8). $\square$

4. Robust portfolio control

Here we illustrate the application of our results to the problem of *robust investment* or *robust portfolio control* in a market with a *stochastic* interest rate that is subject to a disturbance. For background on the optimal investment problem see, for example, [29], [28]. Thus, consider a market of a *bank account* with price $S_0(t)$ and l stocks with prices $S_i(t)$, $i = 1, ..., l$. We assume that:

$$\begin{cases} dS_0(t) = S_0(t)r(t)dt, \quad t \in [0,T], \\ dS_i(t) = S_i(t)[\mu_i(t)dt + \sigma_i'(t)dW(t)], \quad i = 1, ...l, \quad t \in [0,T] \\ S_i(0) > 0, \quad i = 0, 1, ..., l, \quad \text{are given.} \end{cases} \tag{4.1}$$

Here r is the interest rate, μ_i is the appreciation rate, whereas the d-dimensional vector process σ_i is the volatility of stock, and must be such that equations (4.1) have unique strong solutions. We make further assumptions on these coefficients below.

In this market consider an *investor* with an initial wealth of $y_0 > 0$. Let $n_i(t)$, $i = 0, ..., l$, denote the number of shares the investor holds in asset i at time t. The value of investor's portfolio of assets is thus:

$$y(t) := \sum_{i=0}^{l} n_i(t)S_i(t), \quad t \in [0,T].$$

This portfolio is called *self-financing* if (see, e.g. [29], [28]):

$$\begin{cases} dy(t) = \sum_{i=0}^{l} n_i(t)dS_i(t), \quad t \in [0,T], \\ y(0) = y_0. \end{cases} \tag{4.2}$$

After substituting the differentials of $S_i(t)$ into this equation, and defining $\tilde{u}_i(t) = n_i(t)S_i(t)$, $t \in [0,T]$, $i = 1, ..., l$, we obtain:

$$dy(t) = [r(t)y(t) + H(t)\tilde{u}(t)]dt + \tilde{u}'(t)\sigma(t)dW(t), \quad t \in [0,T], \tag{4.3}$$

where $\tilde{u}(t) := [\tilde{u}_1(t), ..., \tilde{u}_l(t)]'$, $\sigma'(t) := [\sigma_1(t), ..., \sigma_l(t)]$, and $H(t) := [\mu_1(t) - r(t), ..., \mu_l(t) - r(t)]$. It is clear that portfolio (4.3) is an example of a stochastic control system with state y being investor's *wealth*.

A typical *optimal portfolio control* problem is the following maximization of expected *power utility* from terminal wealth problem:

$$\begin{cases} \max\limits_{\tilde{u}(\cdot)\in\mathcal{A}_p} \mathbb{E}\left[y^{\lambda}(T)\right], \\ \\ \text{s.t. } (4.3) \text{ and } y(t) > 0 \text{ for all } t \in [0,T] \ a.s., \end{cases} \tag{4.4}$$

for some $\lambda \in (0,1)$ and a suitable admissible set of controls $\mathcal{A}_p$. This problem is well studied when the coefficients are *bounded* processes (see, for example, [29], [28]), and it admits a linear state-feedback solution. For the case of unbounded coefficients see, e.g., [18], and for other optimality criteria see, e.g, [9], [16].

We consider a market with $\sigma(\cdot) \in L^\infty(0,T;\mathbb{R}^{l\times d})$. The n-dimensional *factor* process $\tilde{x}$ is the solution to the following linear stochastic differential equation with the added disturbance $\tilde{v}$:

$$\begin{cases} d\tilde{x}(t) = \left[\widetilde{A}(t)\tilde{x}(t) + \widetilde{B}(t)\tilde{v}(t)\right]dt + \widetilde{A}_1(t)dW(t), \quad t \in [0,T], \\ \\ \tilde{x}(0) \in \mathbb{R}^n, \end{cases} \tag{4.5}$$

Here $\widetilde{A}(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times n})$, $\widetilde{B}(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times n_v})$, $\widetilde{A}_1(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times d})$, whereas the disturbance $\tilde{v}(\cdot)$ is assumed to be an adapted and square-integrable process. The interest rate is defined as:

$$r(t) := \tilde{x}'(t)\widetilde{Q}(t)\tilde{x}(t), \quad t \in [0,T],$$

for some non-negative $\widetilde{Q}(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times n})$. Moreover, similarly to [4], [5], we assume that:

$$H'(t) := \widetilde{L}'(t)\tilde{x}(t), \quad t \in [0,T],$$

where $\widetilde{L}(\cdot) \in L^\infty(0,T;\mathbb{R}^{l\times n})$. Our interest rate model without the disturbance is the so-called *quadratic-affine term-structure model (QATSM)* (see, e.g., [9]). The inclusion of disturbance $\tilde{v}$ in our model is a generalization of QATSM that enables taking into consideration uncertainties in the model.

We are thus faced with the problem of controlling investor's wealth in a market with random interest rate subject to a disturbance. This problem has not been considered before. Note that the wealth equation (4.3) is a stochastic control system with *multiplicative* noise and *unbounded* coefficients (due to r and H), there is a positivity constraint on the state (the wealth is required to be positive), and the typical optimality criterion in (4.4) is *not* quadratic. This means that existing methods, such as those in [8] and [51], that consider systems with multiplicative noise and disturbances, cannot be applied. On the other hand, although this appears to be a different problem from our finite-horizon risk-sensitive H_2/H_∞ control problem of section 2, it can be reformulated as an *example* of such a problem, as shown below.

As the control process $\tilde{u}$ is confined to ensure $y(t) > 0$ for all $t \geq 0$ *a.s.*, the logarithm of $y(t)$ is well-defined, and its differential is:

$$\begin{aligned} d\log y(t) &= \frac{1}{y(t)}\left[r(t)y(t) + H(t)\tilde{u}(t)\right]dt - \frac{1}{2}\frac{1}{y^2(t)}\tilde{u}'(t)\sigma(t)\sigma'(t)\tilde{u}(t)dt \\ &+ \frac{1}{y(t)}\tilde{u}'(t)\sigma(t)dW(t). \end{aligned}$$

By defining $c(t) := \tilde{u}(t)/y(t)$ for $t \in [0,T]$ as a new control process, we have:

$$d\log y(t) = [r(t) + H(t)c(t) - c'(t)\sigma(t)\sigma'(t)c(t)/2]dt + c'(t)\sigma(t)dW(t).$$

Integration of both sides from 0 to T gives:

$$\begin{aligned} y(T) &= \exp\left[\int_0^T [r(t) + H(t)c(t) - c'(t)\sigma(t)\sigma'(t)c(t)/2]dt \right. \\ &+ \left. \int_0^T c'(t)\sigma(t)dW(t)\right], \end{aligned} \tag{4.6}$$

where we have taken $y_0 = 1$ for simplicity. The expected power utility from terminal wealth can thus be written as:

$$\begin{aligned}\mathbb{E}\left[y^{\lambda}(T)\right] &= \mathbb{E}\left\{\exp\left[\int_0^T \lambda[\tilde{x}'(t)\tilde{Q}(t)\tilde{x}(t) + \tilde{x}'(t)\widetilde{L}(t)c(t)\right.\right. \\ &- \left.\left. c'(t)\sigma(t)\sigma'(t)c(t)/2]dt + \int_0^T \lambda c'(t)\sigma(t)dW(t)\right]\right\}. \quad (4.7)\end{aligned}$$

In order to write (4.7) in the form of criterion (1.6), we introduce the following change of measure:

$$\begin{aligned}\theta_c &:= -\lambda\sigma'(t)c(t), \quad t \in [0,T], \\ Z_c(t) &:= \exp\left[-\int_0^t \theta_c'(\tau)dW(\tau) - \frac{1}{2}\int_0^t \theta_c'(\tau)\theta_c(\tau)d\tau\right], \quad t \in [0,T], \\ Z_c &:= Z_c(T), \\ \widetilde{\mathbb{P}}_c(\alpha) &:= \int_\alpha Z_c(\omega)d\mathbb{P}(\omega), \quad \forall \alpha \in \mathcal{F}.\end{aligned}$$

The expected power utility from terminal wealth (4.7) can now be written as:

$$\begin{aligned}\mathbb{E}\left[y^{\lambda}(T)\right] &= \widetilde{\mathbb{E}}_c\left\{\exp\left[\int_0^T [\tilde{x}'(t)\lambda\tilde{Q}(t)\tilde{x}'(t) + \lambda\tilde{x}'(t)\widetilde{L}(t)c(t)\right.\right. \\ &- \left.\left.\lambda(1-\lambda)c'(t)\sigma(t)\sigma'(t)c(t)/2]dt\right]\right\}. \\ &= \widetilde{\mathbb{E}}_c\left\{\exp\left[-\frac{1}{2}\int_0^T \tilde{z}'(t)\tilde{M}(t)\tilde{z}(t)dt\right]\right\},\end{aligned}$$

where $\widetilde{\mathbb{E}}_c$ is the expectation under probability measure $\widetilde{\mathbb{P}}_c$ and

$$\tilde{z}(t) := \begin{bmatrix} \tilde{x}(t) \\ \\ c(t) \end{bmatrix}, \quad \tilde{M}(t) := \begin{bmatrix} -\lambda\tilde{Q}(t)/2 & -\lambda\tilde{L}(t) \\ -\lambda\tilde{L}'(t) & \lambda(1-\lambda)\sigma(t)\sigma'(t) \end{bmatrix}, \quad t \in [0,T].$$

The process W_c defined as:

$$W_c(t) := \int_0^t \theta_c(s)ds + W(t), \quad t \in [0,T],$$

is a standard Brownian motion under $\widetilde{\mathbb{P}}_c$ by Grisanov theorem. The equation of factor process (4.5) can be written as:

$$\begin{cases} d\tilde{x}(t) = \left[\widetilde{A}(t)\tilde{x}(t) + \lambda\sigma'(t)c(t) + \widetilde{B}(t)\tilde{v}(t)\right] dt \\ \qquad\qquad +\widetilde{A}_1(t)dW_c(t), \quad t \in [0,T], \\ \tilde{x}(0) \in \mathbb{R}^n, \end{cases} \tag{4.8}$$

For the purpose of robust portfolio control, we introduce the following two criteria:

$$\begin{aligned} \tilde{J}_1(c(\cdot),\tilde{v}(\cdot)) &:= -\widetilde{\mathbb{E}}_c\left\{\exp\left[-\frac{1}{2}\int_0^T \tilde{z}'(t)\tilde{M}_1(t)\tilde{z}(t)dt\right]\right\}, \\ \tilde{J}_2(c(\cdot),\tilde{v}(\cdot)) &:= \frac{1}{\tilde{\gamma}}\widetilde{\mathbb{E}}_c\left\{\exp\left[\frac{\tilde{\gamma}}{2}\int_0^T \left[\tilde{\theta}^2\tilde{v}'(t)\tilde{N}\tilde{v}(t) - \tilde{z}'(t)\tilde{M}_2(t)\tilde{z}(t)\right] dt\right]\right\}, \end{aligned}$$

for given symmetric matrices

$$\tilde{M}_1(\cdot), \tilde{M}_2(\cdot) \in L^\infty(0,T;\mathbb{R}^{(n+l)\times(n+l)}), \quad \tilde{N}(\cdot) \in L^\infty(0,T;\mathbb{R}^{n_v\times n_v}).$$

If $\tilde{M}_1(t) = \tilde{M}(t)$ for $t \in [0,T]$, then the minimization of $\tilde{J}_1(c(\cdot),\tilde{v}(\cdot))$ is *equivalent* to the maximization of the power utility from terminal wealth. The interpretation of $\tilde{J}_2(c(\cdot),\tilde{v}(\cdot))$ depends on the particular choice of its coefficients. Once such a choice is the following. The solution of the wealth equation (4.6) under the probability measure $\widetilde{\mathbb{P}}_c$ is:

$$\begin{aligned} y(T) &= \exp\left[\int_0^T [\tilde{x}'(t)\tilde{Q}(t)\tilde{x}(t) + \tilde{x}'(t)\tilde{L}(t)c(t)\right. \\ &+ \left.(2\lambda-1)c'(t)\sigma(t)\sigma'(t)c(t)/2]dt + \int_0^T c'(t)\sigma(t)dW_c(t)\right]. \end{aligned}$$

The expected *growth rate* of investor's wealth is thus:

$$\begin{aligned} G(c(\cdot),\tilde{v}(\cdot)) &:= \widetilde{\mathbb{E}}_c\left\{\int_0^T [\tilde{x}'(t)\tilde{Q}(t)\tilde{x}(t) + \tilde{x}'(t)\tilde{L}(t)c(t)\right. \\ &+ \left.(2\lambda-1)c'(t)\sigma(t)\sigma'(t)c(t)/2]dt\right\} \\ &= \widetilde{\mathbb{E}}_c\left[\int_0^T \tilde{z}'(t)\bar{M}(t)\tilde{z}(t)dt\right], \end{aligned}$$

where

$$\bar{M}(t) := \begin{bmatrix} \tilde{Q}(t) & \tilde{L}(t)/2 \\ \tilde{L}'(t)/2 & (2\lambda - 1)\sigma(t)\sigma'(t)/2 \end{bmatrix}, \quad t \in [0, T].$$

As investor's wealth y is subjected to disturbance $\tilde{v}$ (through the factor process $\tilde{x}$), we can require that the ratio of expected growth rate and expected energy of disturbance be *greater* than some pre-specified value η^2 for all possible disturbances. This suggests the consideration of criterion:

$$\begin{aligned} &\widetilde{\mathbb{E}}_c \left[\int_0^T \eta^2 \tilde{v}'(t)\tilde{v}(t)dt \right] + G(c(\cdot), \tilde{v}(\cdot)) \\ &= \widetilde{\mathbb{E}}_c \left\{ \int_0^T \left[\eta^2 \tilde{v}'(t)\tilde{v}(t) + \tilde{z}'(t)\bar{M}(t)\tilde{z}(t) \right] dt \right\}, \end{aligned}$$

which is to be minimized. This corresponds to the criterion $\tilde{J}_2(c(\cdot), \tilde{v}(\cdot))$ with the following coefficients:

$$\tilde{\gamma} \to 0^+, \quad \tilde{\theta}^2 = \eta^2, \quad \tilde{N}(t) = I_{n_v \times n_v}, \quad \tilde{M}_2(t) = -\bar{M}(t), \quad t \in [0, T].$$

The aim now to find the linear state-feedback pair $(c^*(\cdot), \tilde{v}^*(\cdot))$ such that the following two inequalities hold:

$$\begin{aligned} \tilde{J}_1(c^*(\cdot), \tilde{v}^*(\cdot)) &\leq \tilde{J}_1(c(\cdot), \tilde{v}^*(\cdot)), \\ \tilde{J}_2(c^*(\cdot), \tilde{v}^*(\cdot)) &\leq \tilde{J}_2(c^*(\cdot), \tilde{v}(\cdot)). \end{aligned}$$

However, this is an example of the control problem considered in section 2 with the following coefficients (for all $t \in [0, T]$):

$$\begin{cases} A(t) = \tilde{A}(t), \quad B_2(t) = \lambda\sigma'(t), \quad B_1(t) = \tilde{B}(t), \\ A_1(t) = \tilde{A}_1(t), \quad C(t) = I_{n \times n}, \quad D(t) = I_{l \times l} \\ M_1(t) = \tilde{M}_1(t), \quad M_2(t) = \tilde{M}_2(t), \quad N(t) = \tilde{N}(t) = I_{n_v \times n_v}, \\ \gamma_1 = -1, \quad \gamma_2 = \tilde{\gamma} \to 0^+, \quad \theta^2 = \eta^2. \end{cases} \tag{4.9}$$

If assumptions of section 2 hold for the above coefficients with $\tilde{M}_1(t) = \tilde{M}(t)$ and $\tilde{M}_2(t) = -\bar{M}(t)$ for all $t \in [0, T]$, which implies that we must have

$\sigma(t)\sigma'(t) > 0$ for all $t \in [0,T]$, then the required pair $(c^*(\cdot), \tilde{v}^*(\cdot))$ is given by Theorem 2 as:

$$\begin{aligned} c^*(t) &:= -\frac{1}{1-\lambda}(\sigma\sigma')^{-1}(\lambda\sigma P_1 - \lambda\tilde{L}')\tilde{x}(t), \quad t \in [0,T], \\ \tilde{v}^*(t) &:= -\tilde{\theta}^{-2}\tilde{N}\tilde{B}'P_2\tilde{x}(t), \quad t \in [0,T], \end{aligned}$$

where $P_1(t)$ and $P_2(t)$ are solutions to Riccati equations in Assumption 2 corresponding to the coefficients (4.9). This implies that investor's *trading strategy*, i.e. the number of shares held on each asset at time t should be:

$$\begin{aligned} n_i^*(t) &= \tilde{u}_i^*(t)/S_i(t) = c_i^*(t)y(t)/S_i(t), \quad t \in [0,T], \quad i = 1, ..., l, \\ n_0^*(t) &= y(t) - \sum_{i=1}^{l} n_i^*(t)S_i(t), \quad t \in [0,T]. \end{aligned}$$

As the trading strategy is of a linear-feedback form with respect to investors wealth y, the solution to equation (4.3) under such a trading strategy is strictly positive, as required.

4.1. Numerical example in finite horizon

Here we give a numerical example illustrating that all assumption of section 2 are satisfied in the the case of the robust portfolio control problem. We assume the following numerical values:

$$\ell = 1, \quad d = 1, \quad n = 1, \quad n_v = 1, \quad \lambda = 0.5,$$

$$\sigma(t) = 1, \quad \widetilde{A}(t) = 0, \quad \widetilde{B}(t) = 1, \quad \widetilde{A}_1(t) = \sqrt{2},$$

$$\widetilde{L}(t) = 0, \quad \widetilde{Q}(t) = 0.02, \quad \widetilde{N}(t) = 1,$$

$$\tilde{\gamma} \in (0, 0.1), \quad \eta^{-2} = 0.25(1 - 8\tilde{\gamma}), \quad T = 1.$$

This gives:

$$\tilde{M}_1(t) = \tilde{M}(t) = \begin{bmatrix} -\frac{0.02}{4} & 0 \\ 0 & 0.25 \end{bmatrix}, \quad \tilde{M}_2(t) = -\bar{M}(t) = \begin{bmatrix} -0.02 & 0 \\ 0 & 0 \end{bmatrix}.$$

In the notation of section 2, we thus have: $A(t)=0$, $B_2(t)=0.5$, $B_1(t)=1$, $A_1(t)=\sqrt{2}$, $C(t)=1$, $D(t)=1$, $\gamma_1=-1$, $\gamma_2\in(0,0.1), \theta^{-2}=0.25(1-8\gamma_2)$, $T=1$,

$$M_1(t)=\begin{bmatrix}-\frac{0.02}{4} & 0\\ 0 & 0.25\end{bmatrix}, \quad M_2(t)=\begin{bmatrix}-0.02 & 0\\ 0 & 0\end{bmatrix}.$$

It is clear that Assumption 1 holds. We now show that Assumption 2 also holds by finding the explicit solutions to the coupled Riccati differential equations (2.1) and (2.2), which in this case are:

$$\dot{P}_1(t)-2\eta^{-2}P_2(t)P_1(t)-3P_1^2(t)-0.005=0, \quad P_1(1)=0, \tag{4.10}$$

$$\dot{P}_2(t)-(\eta^{-2}-2\tilde{\gamma})P_2^2(t)-2P_2(t)P_1(t)+0.02=0, \quad P_2(1)=0. \tag{4.11}$$

If we define $Z(t):=P_2(t)/4$, $t\in[0,1]$, then these two equations can be written as:

$$\begin{gathered}\dot{P}_1(t)-8\eta^{-2}Z(t)P_1(t)-3P_1^2(t)-0.005=0, \quad P_1(1)=0,\\ \dot{Z}(t)-4(\eta^{-2}-2\tilde{\gamma})Z^2(t)-2Z(t)P_1(t)+0.005=0, \quad P_2(1)=0.\end{gathered}$$

Let $V(t):=P_1(t)+Z(t)$, $t\in[0,1]$. By adding the above two equations, we obtain the equation of V as:

$$\dot{V}(t)-V(t)[3P_1(t)+4(\eta^{-2}-2\tilde{\gamma})Z(t)]=0 \quad V(1)=0.$$

As this is a linear equation in V, its unique solution is $V(t)=0$, $t\in[0,1]$. This implies that it is necessary to have:

$$P_1(t)=-Z(t)=-\frac{1}{4}P_2(t), \quad t\in[0,1].$$

By substituting $P_2(t)=-4P_1(t)$ in (4.10), we obtain the following *uncoupled* Riccati differential equation:

$$\dot{P}_1-(1+16\tilde{\gamma})P_1^2-0.005=0, \quad P_1(1)=0.$$

The explicit solution to this equation is:

$$P_1(t)=\sqrt{\frac{0.005}{1+16\tilde{\gamma}}}\tan\left[-(1-t)\sqrt{0.005(1+16\tilde{\gamma})}\right], \quad t\in[0,1],$$

from which we obtain the solution to equation (4.11) as

$$P_2(t) = -4\sqrt{\frac{0.005}{1+16\tilde{\gamma}}}\tan\left[-(1-t)\sqrt{0.005(1+16\tilde{\gamma})}\right], \quad t \in [0,1].$$

The graphs of these two solutions for $\tilde{\gamma} = 1/15$ are given in Figure 1.

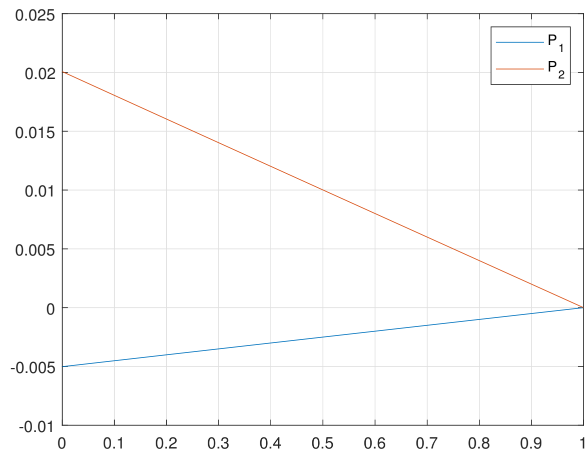

Figure 1. Solutions P_1 and P_2.

We now focus on the corresponding Lyapunov differential equation (2.16) and the Riccati differential equations (2.17) and (2.18) and show that the sufficient conditions of Theorem 3 hold. For the matrix A^* in (2.15), which in this example is a scalar, we have:

$$A^*(t) = -8\tilde{\gamma}P_1(t).$$

The Lyapunov differential equation (2.16) in this case becomes:

$$d\Sigma(t) = [-16\tilde{\gamma}P_1(t)\Sigma(t) + 2]\,dt, \quad \Sigma(0) = 0.$$

From the explicit solution to this equation

$$\Sigma(t) = 2\int_0^t e^{\int_\tau^t -16\tilde{\gamma}P_1(s)ds}d\tau, \quad t \in [0,1],$$

it is clear that $\Sigma(1) > 0$. The Riccati differential equations (2.17) and (2.18) in this case are:

$$dQ_1 = -[2Q_1A^*(t) + 4\alpha_1 Q_1^2 + \beta_u(P_1(t))^2]dt, \tag{4.12}$$
$$dQ_2 = -[2Q_2A^*(t) + 4\beta_1 Q_2^2 + \beta_v\tilde{\gamma}16(P_1(t))^2]dt. \tag{4.13}$$

By choosing $\beta_1 = \alpha_1$ and $\beta_v = \beta_u/16\tilde{\gamma}$, which is possible, we see that equation (4.13) becomes the same as equation (4.12. Thus, we only focus in showing that there exists a $Q_1(0)$ such that $Q_1(1) = 0$, which will imply that there exists a $Q_2(0)$ such that $Q_2(1) = 0$, and these, together with the fact that $\Sigma(t) > 0$, are sufficient for the requirements of Theorem 3 to hold. We seek the solution to equation (4.12) in the form:

$$Q_1(t) = e^{-Xt} - Y(t), \tag{4.14}$$

for some positive constant X and a differentiable function Y with $Y(1) = e^{-X}$, yet to be determined. By substituting (4.14) in (4.12), we obtain:

$$\dot{Y} + \left[2A^*(t) + 8\alpha_1 e^{-Xt}\right] Y - 4\alpha_1 Y^2 + e^{-Xt} Q_Y(t) = 0 \tag{4.15}$$

where

$$Q_Y(t) := X - 2A^*(t) - 4\alpha_1 e^{-2Xt} - \beta_u e^{2Xt}(P_1(t))^2, \quad t \in [0, 1].$$

As we are considering only positive X, the following holds (for any $\tilde{\gamma} \in (0, 0.1)$):

$$\begin{aligned} X - 2A^*(t) - 4\alpha_1 e^{-2Xt} &\geq X - 2A^*(t) - 4\alpha_1 \\ &= X + 16\tilde{\gamma} P_1(t) - 4\alpha_1 \\ &\geq X + 16P_1^* - 4\alpha_1, \end{aligned}$$

where

$$P_1^* = \sqrt{\frac{0.005}{1 + 16\tilde{\gamma}}} \tan\left[-\sqrt{0.005(1 + 16\tilde{\gamma})}\right].$$

As $P_1^* < 0$, we choose X such that (for some $\alpha_1 > 1$):

$$X > -16P_1^* + 4\alpha_1.$$

By also choosing β_u such that:

$$0 < \beta_u < \frac{X + 16P_1^* - 4\alpha_1}{e^X (P_1^*)^2} \leq \frac{X - 2A^*(t) - 4\alpha_1 e^{-2Xt}}{e^{Xt}(P_1(t))^2} \tag{4.16}$$

we ensure that $Q_Y(t) \geq 0$ for all $t \in [0, 1]$. The Riccati backward differential equation:

$$\dot{Y} + \left[2A^*(t) + 8\alpha_1 e^{-Xt}\right] Y - 4\alpha_1 Y^2 + e^{-Xt} Q_Y(t) = 0, \quad Y(1) = e^{-X},$$

has a unique solution $Y(t) \geq 0$ for all $t \in [0,1]$ (see, for example, [1]). Thus, by choosing $Q_1(0) = 1 - Y(0)$, we conclude that there exists a solution to equation (4.12) such that $Q_1(1) = 0$. As this implies the existence of $Q_2(0)$ such that $Q_2(1) = 0$, we conclude that the conditions

$$Q_1(1) + \Sigma^{-1}(1)/2\alpha_2 > 0, \quad Q_2(1) + \Sigma^{-1}(1)/2\beta_2 > 0,$$

of Theorem 3 hold. By Theorem 2 and Theorem 3 it now follows that c^* and $\tilde{v}^*$ are:

$$\begin{aligned} c^*(t) &= -\sqrt{\frac{0.005}{1+16\tilde{\gamma}}}\tan\left[-(1-t)\sqrt{0.005(1+16\tilde{\gamma})}\right]\tilde{x}^*(t), \quad t \in [0,1] \\ \tilde{v}^*(t) &= (1-8\tilde{\gamma})\sqrt{\frac{0.005}{1+16\tilde{\gamma}}}\tan\left[-(1-t)\sqrt{0.005(1+16\tilde{\gamma})}\right]\tilde{x}^*(t), \quad t \in [0,1]. \end{aligned}$$

Here $\tilde{x}^*$ is the solution to the stochastic differential equation

$$d\tilde{x}^*(t) = \tilde{v}^*(t)dt + \sqrt{2}dW(t), \quad \tilde{x}^*(0) \in \mathbb{R},$$

which corresponds to the factor process (4.5) under the worst-case-distrbance (WCD) $\tilde{v}(t) = \tilde{v}^*(t)$. The robust trading strategy n_0^* and n_1^* is thus:

$$n_1^*(t) = c^*(t)y^*(t)/S_1^*(t), \quad n_0^*(t) = y^*(t) - n_1^*(t)S_1(t), \quad t \in [0,1],$$

where the equations for the stock price S_1^* and investor's wealth y^* under the WCD are:

$$\begin{aligned} dS_1^*(t) &= S_1^*(t)[0.02(\tilde{x}^*(t))^2dt + dW(t)], \quad S_1^*(0) > 0, \quad t \in [0,1], \\ dy^*(t) &= 0.02(\tilde{x}^*(t))^2y^*(t)dt + c^*(t)y^*(t)dW(t), \quad y^*(0) = y_0, \quad t \in [0,1]. \end{aligned}$$

We choose $\tilde{\gamma} = 1/15$, $y_0 = 1$, $S(0) = 1$, and $\tilde{x}^*(0) = 1$ for illustration. We also consider the investor's wealth under an *example* of the not the worst case disturbance (NWCD) $\tilde{v}(t) = 2\tilde{x}^\star(t)$ with $\tilde{x}^\star(t)$ being the solution to the equation

$$d\tilde{x}^\star(t) = \tilde{v}^\star(t)dt + \sqrt{2}dW(t), \quad \tilde{x}^\star(0) = 1.$$

The stock price under NWCD and the wealth of the investor who applies the control designed for the WCD in the case of a NWCD, i.e.

$$c^\star(t) = -\sqrt{\frac{0.005}{1+16\tilde{\gamma}}}\tan\left[-(1-t)\sqrt{0.005(1+16\tilde{\gamma})}\right]\tilde{x}^\star(t), \quad t \in [0,1]$$

are:

$$\begin{aligned} dS_1^\star(t) &= S_1^\star(t)[0.02(\tilde{x}^\star(t))^2 dt + dW(t)], \quad S_1^\star(0) = 1, \quad t \in [0,1], \\ dy^\star(t) &= 0.02(\tilde{x}^\star(t))^2 y^\star(t)dt + c^\star(t)y^\star(t)dW(t), \quad y^\star(0) = 1, \quad t \in [0,1]. \end{aligned}$$

In the graphs below we have given one realisation of the above processes under the WCD and NWCD as follows: in Figure 2 the factor process, in Figure 3 the logarithm of the stock price, and in Figure 4 the logarithm of the investor's wealth. As can be expected, the investor's wealth under the NWCD is larger then under WCD.

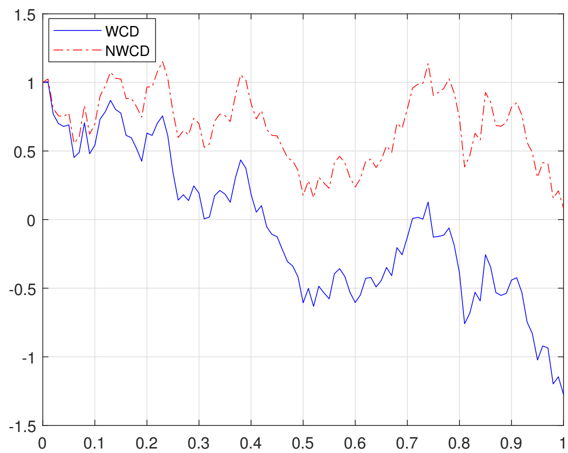

Figure 2. The factor process under WCD (x^*) and under NWCD ($x^\star$).

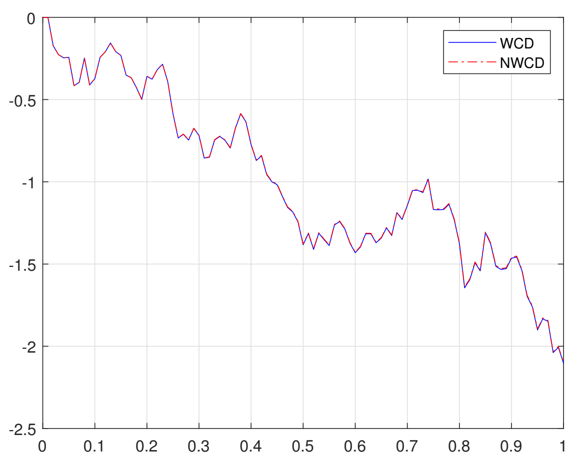

Figure 3. The logarithm of the stock price under WCD (S_1^*) and under NWCD ($S_1^\star$).

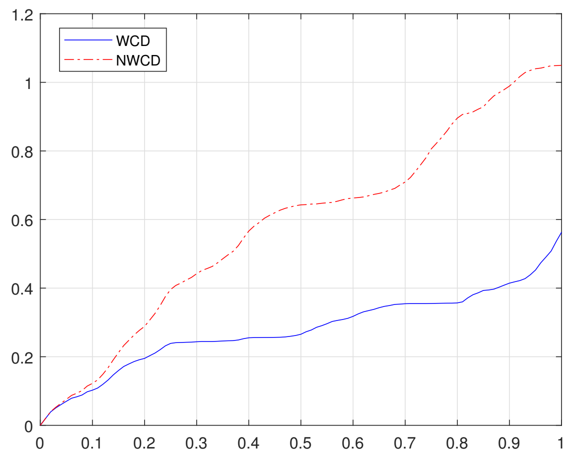

Figure 4. The logarithm of the investor's wealth under WCD (y^*) and under NWCD ($y^\star$).

4.2. Numerical example in infinite horizon

Here we consider an infinite horizon version of the robust portfolio control problem, where we assume that all coefficients are constant and instead of $\tilde{J}_1$ and $\tilde{J}_2$ we use their infinite horizon versions:

$$\begin{aligned}
\tilde{J}_1^\infty(c(\cdot),\tilde{v}(\cdot)) &:= \lim_{T\to\infty} -\frac{1}{f_1(T)} \log \widetilde{\mathbb{E}}_c \left\{ \exp\left[-\frac{1}{2}\int_0^T \tilde{z}'(t)\tilde{M}_1\tilde{z}(t)dt \right] \right\}, \\
\tilde{J}_2^\infty(c(\cdot),\tilde{v}(\cdot)) &:= \lim_{T\to\infty} \frac{1}{f_2(T)\log\tilde{\gamma}} \widetilde{\mathbb{E}}_c \left\{ \exp\left[\frac{\tilde{\gamma}}{2}\int_0^T \left[\tilde{\theta}^2\tilde{v}'(t)\tilde{N}\tilde{v}(t) \right.\right.\right. \\
&- \left.\left.\left. \tilde{z}'(t)\tilde{M}_2\tilde{z}(t)\right]dt \right] \right\},
\end{aligned}$$

We assume $f_1(T) = f_2(T) = T$ and the following numerical values: $\ell = 1$, $d = 1$, $n = 1$, $\lambda = 11/12$, $\sigma = 3/11$, $\widetilde{A} = 0$, $\widetilde{B} = 1$, $\widetilde{A}_1 = \sqrt{16.5}$, $\widetilde{L} = 0$, $\widetilde{Q} = 1$, $\widetilde{N} = 1$, $\eta = \sqrt{22}$, $\tilde{\gamma} = 1$. This gives:

$$\tilde{M}_1 = \tilde{M} = \begin{bmatrix} -\frac{11}{24} & 0 \\ 0 & \frac{1}{176} \end{bmatrix}, \quad \tilde{M}_2 = -\bar{M} = \begin{bmatrix} -1 & 0 \\ 0 & -\frac{15}{484} \end{bmatrix}.$$

In the notation of section 3, we thus have: $A = 0$, $B_2 = 1/4$, $B_1 = 1$, $A_1 = \sqrt{16.5}$, $C = 1$, $D = 1$, $\theta = \sqrt{22}$, $\gamma_1 = -1$, $\gamma_2 = 1$, $N = 1$,

$$M_1 = \begin{bmatrix} -\frac{11}{24} & 0 \\ 0 & \frac{1}{176} \end{bmatrix}, \quad M_2 = \begin{bmatrix} -1 & 0 \\ 0 & -\frac{15}{484} \end{bmatrix}.$$

The Assumption 4 clearly holds. We now show that Assumption 5 also holds by finding real solutions to the Riccati algebraic equations (3.3) and (3.4), which in the current case are, respectively:

$$-44P_1P_2 - 27.5P_1^2 - \frac{11}{24} = 0, \tag{4.17}$$

$$-5.5P_2^2 - 22P_1P_2 + 60P_1^2 + 1 = 0. \tag{4.18}$$

By multiplying both sides of equation (4.18) with $-11/24$ and defining $Z := -11P_2/24$, we can rewrite the above two equations as:

$$96P_1Z - 27.5P_1^2 - \frac{11}{24} = 0, \tag{4.19}$$

$$12Z^2 - \frac{121}{12}P_1Z - 27.5P_1^2 - \frac{11}{24} = 0.$$

The difference of these two equations is:

$$12Z^2 - \left(\frac{121}{12} + 96\right)P_1Z = 0. \tag{4.20}$$

The two possible solutions in Z of this equation are:

$$Z_1 = 0, \quad Z_2 = \frac{697}{72}P_1.$$

Substituting Z_1 in (4.19) gives:

$$-27.5P_1^2 - \frac{11}{24} = 0,$$

which has no real solutions. Substituting Z_2 in (4.19) gives:

$$\left(96\frac{697}{72} - 27.5\right)P_1^2 - \frac{11}{24} = 0.$$

Its two possible solutions, denoted P_1^+ and P_1^-, are:

$$P_1^{\pm} = \pm 0.006797.$$

Through Z_2, this gives the following two values for P_2:

$$P_2^{\mp} = \mp 0.14411.$$

In summary, the real solutions to the coupled Riccati algebraic equations (4.17) and (4.18) are:

$$(0.006797, -0.14411), \qquad (-0.006797, 0.14411),$$

and thus Assumption 5 holds. Due to our choice of functions f_1 and f_2, the Assumption 6 clearly holds.

Next we show that the conditions of Theorem 3 can be satisfied for $(P_1, P_2) = (-0.006797, 0.14411)$ and all $T \in (0, \infty)$. The corresponding matrix A^*, which in this case is a scalar, is $A^* = -3.16419$. The Lyapunov differential equation (2.16) becomes:

$$\begin{cases} d\Sigma(t) = [-6.3284\Sigma(t) + 16.5]\, dt, \quad t \in [0, \infty), \\ \\ \Sigma(0) = 0, \end{cases}$$

Its explicit solution is:

$$\Sigma(t) = 16.5 \int_0^t e^{-6.3284(t-\tau)} d\tau, \quad t \in [0, \infty),$$

form which it follows that $\Sigma(T) > 0$ for all $T \in (0, \infty)$. The corresponding Riccati differential equations (2.17) and (2.18), respectively, are:

$$dQ_1(t) = -\left[-6.3284 Q_1(t) + 33\alpha_1 Q_1^2(t) + 8.25\beta_u P_1^2\right] dt, \tag{4.21}$$
$$dQ_2(t) = -\left[-6.3284 Q_2(t) + 33\beta_1 Q_2^2(t) + 8.25\beta_v P_2^2\right] dt, \tag{4.22}$$

for $t \in [0, \infty)$, where α_1 and β_1 are positive constants. It is sufficient to seek constant solutions to these two equations. Thus, by assuming that Q_1 and Q_2 are constant, equations (4.21) and (4.22) become the following algebraic equations, respectively:

$$0 = -6.3284 Q_1 + 33\alpha_1 Q_1^2 + 8.25\beta_u P_1^2, \tag{4.23}$$
$$0 = -6.3284 Q_2 + 33\beta_1 Q_2^2 + 8.25\beta_v P_2^2. \tag{4.24}$$

If we choose $\alpha_1 = 2$, $\beta_1 = 2$, $\beta_u = 398.0113$, $\beta_v = 0.8854$, then the solutions to equations (4.23) and (4.24) are:

$$Q_1 = Q_2 = 0.0479.$$

As these are both positive, by choosing $\alpha_2 = 2$ and $\beta_2 = 2$, it follows that conditions of Theorem 3

$$Q_1(T) + \Sigma^{-1}(T)/2\alpha_2 > 0, \quad Q_2(T) + \Sigma^{-1}(T)/2\beta_2 > 0,$$

hold for all $T \in (0, \infty)$.

It remains to show that the stability conditions of Theorem 5 hold. In the current example we have $A_u = -3.052$ and $A_v = -0.786375$. The solutions to the corresponding equations (3.11) are:

$$\mu_u(t) \;\; = \;\; e^{A_u t}\tilde{x}(0), \quad \Sigma_u(t) = \frac{A_1^2}{-2A_u}\left(1 - e^{2A_u t}\right), \quad t > 0.$$

Since A_u is negative, it follows that:

$$\begin{aligned} &\lim_{T\to\infty} \frac{\ln|H_u(T)| + \mu_u^2(T)\Sigma_u^{-2}(T)[H_u(T) - I]}{-2T} \\ &= \lim_{T\to\infty} \frac{\ln|[-P_1\Sigma_u(T) + 1]^{-1}| + \mu_u^2(T)\Sigma_u^{-2}(T)[[-P_1\Sigma_u(T) + 1]^{-1} - I]}{-2T} = 0, \end{aligned}$$

and thus the asymptotic relation (3.13) holds with $h_1 = 0$. As A_v is also negative, the proof that the asymptotic relation (3.14) also holds with $h_2 = 0$ proceeds very similarly to the above.

It now follows from Theorem 4 that the solution to our infinite horizon robust portfolio control problem is given by:

$$c_\infty^*(t) = 0.299068\tilde{x}(t), \quad \tilde{v}_\infty^*(t) = -3.17042\tilde{x}(t), \quad t \geq 0,$$

where $\tilde{x}$ is the solution to the following stochastic differential equation

$$\begin{cases} d\tilde{x}(t) = \tilde{v}_\infty^*(t)dt + \sqrt{16.5}dW(t), \quad t \geq 0, \\ \\ \tilde{x}(0) \in \mathbb{R}. \end{cases}$$

The optimal trading strategy n_0^* and n_1^* is thus:

$$n_1^*(t) = c_\infty^*(t)y(t)/S_1(t), \quad n_0^*(t) = y(t) - n_1^*(t)S_1(t), \quad t \geq 0.$$

5. Conclusions

We introduced a risk-sensitive generalisation to the mixed H_2/H_∞ control problem for linear stochastic systems with additive noise. Both the finite and infinite horizon cases are considered, and explicit solutions in terms of coupled differential and algebraic equations of Riccati type are obtained. An application to a robust portfolio control problem is given. There are many possibilities for further research in this direction, and these include the consideration of nonlinear systems (as in [9], [17]), more general risk-sensitive criteria (as in [9], [10], [19]), or systems with multiplicative noise and random coefficients. This case appears to be particularly challenging as the current solution method breaks down (note that after the change of measure in this case one ends up with x^4 terms rather than x^2 terms, and thus the completion of squares method does not apply). Another approach would be the use of other criteria as in [16]), where non-quadratic terms appear is the cost functional.

Acknowledgement

The authors are grateful to the two reviewers for their very careful reading of the article, for the suggestions, and for the remarks that lead to an improved version of the paper.

References

[1] Anderson, B. D. O., & Moore, J. B., (1990). *Optimal control: linear quadratic methods*, Prentice Hall.

[2] Başar, T., (1999). Nash equilibria of risk-sensitive nonlinear stochastic differential games, *J Optim Theory Appl.*, 100, 479-498.

[3] Bensoussan, A., & Van Schuppen, J. H., (1985). Optimal control of partially observable stochastic systems with an exponential-of-integral performance index, *SIAM J. Control Optim.*, 23, 599-613.

[4] Bielecki, T. R., Pliska, S. R., & Sheu, S. J., (2005). Risk sensitive portfolio management with Cox–Ingersoll–Ross interest rates: the HJB equation, *SIAM J. Control and Optimization*, 44, 1811-1843.

[5] Bielecki, T. R., Pliska, S., & Yong, J., (2004). Optimal investment decisions for a portfolio with a rolling horizon bond and a discount bond, *Inter. J. Theor. App. Fin.*, 8, 871-913.

[6] Brezas P., & Smith, M. C., (2014). Linear quadratic and risk-sensitive control for vehicle active suspensions, *IEEE Tran. Automatic Control*, 22, 543-556.

[7] Charalambous, C. D., & Hibey, J. L., (1996). On the application of minimum principle for solving partially observable risk sensitive control problems, *Systems & Control Letters*, 27, 169-179.

[8] Chen, B. S., & Zhang, W., (2004). Stochastic H_2/H_∞ control with state-dependent noise, *IEEE Trans. Automatic Control*, 49, 45-57.

[9] Date, P., & Gashi, B., (2013). Risk-sensitive control for a class of nonlinear systems with multiplicative noise, *Systems & Control Letters*, 62, 989-999.

[10] Date, P., & Gashi, B., (2014). Generalised risk-sensitive control with full and partial state observation, *Journal of Mathematical Modelling and Algorithms in Operations Research*, 13, 87-101.

[11] Davis, M. H. A., & Lleo, S., (2011). Jump-diffusion risk-sensitive asset management I: diffusion factor model, *SIAM J. Fin. Math.*, 2, 22-54.

[12] Davis, M. H. A., & Lleo, S., (2021). Risk-sensitive benchmarked asset management with expert forecasts, *Mathematical Finance*, 31, 1162-1189.

[13] Duncan, T. E., (2016). Linear exponential quadratic stochastic differential games, *IEEE Trans. Automatic Control*, 61, 2550-2552.

[14] Dupuis, P., James, M. R., & Petersen I. R., (2000). Robust properties of risk-sensitive control, *Mathematics of Control, Signals, and Systems*, 13, 318-332.

[15] Flemings, W. H., & Sheu, S. J., (2002). Risk-sensitive control and an optimal investment model II, *Annals of Applied Probability*, 12, 730-767.

[16] Gashi, B., (2019). Optimal stochastic regulators with state-dependent weights, *Systems & Control Letters*, 134C, 104522.

[17] Gashi, B., & Hua, H., (2021). Optimal regulators for a class of nonlinear stochastic systems, *I. J. of Control*, doi:10.1080/00207179.2021.1982014.

[18] Gashi, B. & Li, J., (2019). Integrability of exponential process and its application to backward stochastic differential equations, *IMA J. Man. Math.*, 30, 335-365.

[19] Gashi, B. & Zhang, M., (2022). Indefinite risk-sensitive control, *E. J. of Control*, doi.org/10.1016/j.ejcon.2022.100741.

[20] Glover, K., & Doyle, J., (1988). State space formulae for all stabilizing controllers that satisfy an H_∞ norm bound and relation to risk-sensitivity, *Systems & Control Letters*, 11, 167-172.

[21] Hamadèe, S., & Mu, R., (2020), Risk-sensitive nonzero-sum stochastic differential game with unbounded coefficients, *Dyn Games Appl.*, 1-25.

[22] Hua, H., Cao, J., Yang, G., & Ren, G., (2018). Voltage control for uncertain stochastic nonlinear system with application to energy internet: non-fragile robust H_∞ approach, *Journal of Mathematical Analysis and Applications*, 463, 93-110.

[23] Hua, H., Qin, Y., Geng, J., Hao, C. , & Cao, J., (2019). Robust mixed H_2/H_∞ controller design for energy routers in energy internet, *Energies*, 12, Art. no. 340.

[24] Hua, H., Qin, Y., He, Z., Li, L., & Cao, J., Energy sharing and frequency regulation in energy Internet via mixed H_2/H_∞ control with Markovian jump, *CSEE Journal of Power and Energy Systems*, preprint, DOI: 10.17775/CSEEJPES.2019.01900.

[25] Huang, Y., Zhang, W., & Feng, G., (2008). Infinite horizon H_2/H_∞ control for stochastic systems with Markovian jumps, *Automatica*, 44, 857-863.

[26] Jacobson, D. H., (1973). Optimal stochastic linear systems with exponential criteria and their relation to deterministic differential games, *IEEE Trans. Automatic Control*, 18, 124-131.

[27] Shi, J., & Wu, Z., (2011). A risk sensitive stochastic maximum principle for optimal control of jump diffusions and its applications, *Acta Math. Scie.*, 31B, 419-433.

[28] Karatzas, I., & Shreve, S. E. , (1998). *Methods of mathematical finance*, Springer.

[29] Korn, R., *Optimal portfolios*, (1997). World Scientific.

[30] Lim, A. E. B., & Zhou, X. Y., (2005). A new risk-sensitive maximum principle, *IEEE Trans. Automatic Control*, 50, 958-966.

[31] Limebeer, D. J. N., Anderson, B. D. O., & Hendel, B., (1994). A Nash game approach to mixed H_2/H_∞ control, *IEEE Trans. Automatic Control*, 39, 69-82.

[32] Ma, L., Wang, Z., Bo, Y., & Guo, Z., (2011). A game theory approach to mixed H_2/H_∞ control for a class of stochastic time-varying systems with randomly occurring nonlinearities, *System & Control Letters*, 60, 1009-1015.

[33] Ma, H., & Wang, W., (2022). Stochastic maximum principle for partially observed risk-sensitive optimal control problems of mean-field forward-backward stochastic differential equations, *Optimal Control Appl. Meth.*, 43, 532-552.

[34] Moon, J., (2019). Necessary and sufficient conditions of risk-sensitive optimal control and differential games for stochastic differential delayed equations, *Int. J. Robust Nonlinear Control*, 29, 4812-4827.

[35] Moon, J., (2020). The risk-sensitive maximum principle for controlled forward-backward stochastic differential equations, *Automatica*, 120, 109069.

[36] Moon, J., (2021). Risk-sensitive maximum principle for stochastic optimal control of mean-field type Markov regime-switching jump-diffusion systems, *Int. J. Robust Nonlinear Control*, 31, 2141-2167.

[37] Moon, J., (2021). Generalized risk-sensitive optimal control and Hamilton-Jacobi-Bellman equation, *IEEE Trans. Autom. Control*, 66, 2319-2325.

[38] Moon J., Duncan T. E., Başar T., (2019). Risk-sensitive zero-sum differential games, *IEEE Trans. Autom. Control*, 64, 1503-1518.

[39] Moon, J., & Kim, Y., (2019). Linear-exponential-quadratic control for mean field stochastic systems, *IEEE Trans. on Automatic Control*, 12, 5094-5100.

[40] Pan, Z., & Başar, T, (1996). Model simplification and optimal control of stochastic singularly perturbed systems under exponentiated quadratic cost, *SIAM J. Control and Optimization*, 34 , 1734-1766.

[41] Sheng, L., Zhang, W., & Gao, M., (2014). Relationship between Nash equilibrium strategies and H_2/H_∞ control of stochastic Markov jump systems with multiplicative noise, *IEEE Trans. Automatic Control*, 59, 2592-2597.

[42] Tembine H., Zhu Q. Y., & Başar, T., (2013). Risk-sensitive mean-field games, *IEEE Trans Automat Contr.*, 59, 835-850.

[43] Tembine H, Zhu Q. Y, & Başar, T., (2011). Risk-sensitive mean-field stochastic differential games, *IFAC Proc.*, 44, 3222-3227.

[44] Ugrinovskii, V., & Petersen, I. R., (2002). Robust output feedback stabilization via risk-sensitive control, *Automatica*, 38, 945-955.

[45] Wang, J. W., & Wang, J. M., (2019). Mixed H_2/H_∞ sampled-data output feedback control design for a semi-linear parabolic PDE in the sense of spatial L_∞ norm, *Automatica*, 103, 282-293.

[46] Whittle, P., (1981). Risk-sensitive linear/quadratic/Gaussian control, *Adv. App. Prob.*, 13, 764-777.

[47] Whittle, P., (1990). *Risk-sensitive optimal control*, Wiley,

[48] Wu, C., Chen, B., & Zhang, W., (2019). Multiobjective H_2/H_∞ control design of the nonlinear mean-field stochastic jump-diffusion systems via fuzzy approach, *IEEE Trans. on Fuzzy Systems*, 27, 686-700.

[49] Yan, Z., Song, Y., & Park, J. H., (2021). Finite-time H_2/H_∞ control for linear Itô stochastic Markovian jump systems: mode-dependent approach, *IET Control Theory & Applications*, 14, 3557-3567.

[50] Zhang, W., & Feng, G., (2008). Nonlinear stochastic H_2/H_∞ control with (x, u, v)-dependent noise: infinite horizon case, *IEEE Trans. on Automatic Control*, 53, 1323-1328.

[51] Zhang, W., Zhang, H., & Chen, B. S., (2006). Stochastic H_2/H_∞ control with (x, u, v)-dependent noise: finite horizon case, *Automatica*, 42, 1891-1898.