On the existence of highly organized communities in networks of locally interacting agents

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Abstract. In this paper we investigate phenomena of spontaneous emergence or purposeful formation of highly organized structures in networks of related agents. We show that the formation of large organized structures requires exponentially large, in the size of the structures, networks. Our approach is based on Kolmogorov, or descriptional, complexity of networks viewed as finite size strings. We apply this approach to the study of the emergence or formation of simple organized, hierarchical, structures based on Sierpinski Graphs and we prove a Ramsey type theorem that bounds the number of vertices in Kolmogorov random graphs that contain Sierpinski Graphs as subgraps. Moreover, we show that Sierpinski Graphs encompass close-knit relationships among their vertices that facilitate fast spread and learning of information when agents in their vertices are engaged in pairwise interactions modelled as two person games. Finally, we generalize our findings for any organized structure with succinct representations. Our work can be deployed, in particular, to study problems related to the security of networks by identifying conditions which enable or forbid the formation of sufficiently large insider-subnetworks with malicious common goal to overtake the network or cause disruption of its operation.

Keywords: Organized Threat Structures, Network Threats, Sierpinksi Triangle, Sierpinski Graphs, Kolmogorov Complexity, Ramsey Theory.

1 Introduction

In this paper, we address the problem of the formation or emergence of certain configurations or patterns, termed *highly organized structures* in our context, in evolving networks of agents (e.g. social networks) based on combining the concepts of *Ramsey Numbers in graphs* and *Kolmogorov Complexity*. Our work does not rely on random graph or asymptotics techniques and focuses on finite graphs and can be used to study when and how organized structures appear in finite size networks of interacting agents, answering questions such as the following: (i) given a network of interacting agents of a specific size, how large can an organized subnetwork be in which a common goal diffuses to overtake the whole network? (ii) can a subgroup of a specific organization structure appear in networks of a given size? and (iii) how large a network of interacting agent should be before it becomes vulnerable to the possibility of formation of large subnetworks of closely-interacting malicious agents?

Broadly speaking, *Ramsey theory* refers to mathematical statements that a specific structure (e.g. string, graph, number sequence etc.) is certain to contain a large, highly organised, substructure. Examples of such statements stems from several disciplines, including mathematical logic, number theory, analysis, and graph theory. One of the most well-known such statements in number theory, proved by van der Waerden in 1927 (see [15]) is the following: for *any* fixed positive integers r and k, there exists some positive integer n such that if the integers $\{1, 2, \ldots, n\}$ are colored using r different colors, then there exist *at least* k monochromatic integers in arithmetic progression. The *least* such value of n is called the *Van der Waerden number* $W(r, k)$. In this paper, our focus is on the Ramsey Theory of *graphs*. This research area was inaugurated in 1930 by Ramsey in [12] in which he stated and proved the, so called, *Ramsey's Theorem*: for *any* graph H, there exists a natural number n such that for *any* colouring of the edges of K_n (i.e. the clique on n vertices) with two colours, K_n contains a *monochromatic* copy of H as a subgraph, not necessarily induced. The *least* such value n is called the *Ramsey number* of H and is denoted by $r(H)$ (if $H = K_t$, we simply write $r(t)$). For our purposes, we consider the Ramsey numbers for the class of *Sierpinski Graphs*. This class of graphs has *bounded* maximum degree for its vertices. This fact leads to *linear*, in the number of vertices of H, upper bounds for the Ramsey numbers of these graphs. This follows from a more general result of Chvátal, Rödl, Szemerédi and Trotter in [4] which states that if H is a graph on n vertices and maximum degree Δ, then the Ramsey number

$r(H)$ is bounded by $c(\Delta)n$ for some constant $c(\Delta)$ depending only on Δ. That is, the Ramsey number of bounded-degree graphs grows linearly in the number of vertices. Moreover, a linear bound also holds for the *induced* Ramsey numbers which are defined as the Ramsey numbers only, now, the monochromatic copy of H needs to form an *induced* subgraph, which is a stricter form of subgraph that allows the existence of edges in the subgraph H if and only if they are edges of H. Other edges, which may exist among vertices of H as edges of the larger graph should not exist, as it may be the case in the general subgraph notion. The *Induced Ramsey Theorem* states that $r_{\mathrm{ind}}(H)$ exists for every graph H (see, e.g., Chapter 9.3 in [6]).

On the other hand, the *Kolmogorov complexity* of a *finite object* (most often a bit sequence modelling a finite structure such as a graph) is defined as the *minimum* number of bits into which the object can be compressed without losing information, i.e. so as the compressed string is recoverable through some algorithm, running on a reference machine, which is most often the universal Turing machine (see [9]). Note the the word *complex* in the context of Kolmogorov complexity reflects, rather, *lack* of organization, pattern, and "sophistication" as opposed to the meaning the word complex has in everyday language as describing an object or entity of a high degree of organization and fine structural details. Thus, an object of *high* Kolmogorov complexity is, actually, of low complexity in the usual sense since they have all *randomness* properties while objects of *low* Kolmogorov complexity lack randomness, having internal structural organization and many regularities. Thus, they can be *succinctly* described or *compressed*, in the terminology of Kolmogorov complexity theory. With regard to this concept of complexity, it was proposed and developed, independently, by Andrei Kolmogorov, Ray Solomonoff, and Gregory Chaitin ([8, 10, 3]). This concept addresses the complexity of *finite objects*, in contrast with classical complexity theory, like NP-completeness, that addresses the complexity of *infinite* sets of finite objects, i.e. languages in the complexity-theoretical terminology.

Based on Ramsey theory and Kolmogorov Complexity arguments, we investigate, in this paper, phenomena of existence or formation of highly organized structures, such as organizations, people's networks, and societal patterns, in evolving networks of agents. As a specific case, we focus on the class of *Sierpinski Graphs* which have, also, the important property of being *close-knit*, i.e. agents interconnected with connections based on these graphs are very "cohesive". Close-knit graph families play an important role in diffusion processes in graphs (see [13]) in contexts such

as the emergence of shared and agreed upon innovations or ideas over large population of agents..

2 Closeness properties of graphs

Agent interactions and the study of their long term evolution as well as properties using such models, can reveal crucial information about the dynamics of the social fabric and the way it expands as well as how *organized* substructures emerge, *inescapably*, due to these dynamics. For instance, in this paper we show that, under certain conditions, one may locate in large structures containing interacting agents, large substructures with certain "member closeness" properties. We show, for instance, in this paper that large *close-knit* regular substructures emerge, as the network expands, as a form of local "island" of regularity and social organization, even if the network of agents evolves in *unpredictable* (i.e. random) ways on a global scale. Close-knit agent structures, as Young showed in [13], play an important role in *diffusing* throughout society ideas and beliefs that are, initially, held by only a relatively small group of interacting agents.

Young's mathematical result examines how an idea or opinion spreads over a population of interacting agents whose social structure has a certain "coherence" property and whose members interact through a two person "innovation acceptance" game.

In [13], Young defined a parameter of networks of agents, modelled as *graphs*, that describes their "coherence" as well as their vulnerability to outsiders' views against the innovation to be diffused. A group of agents S (i.e. nodes in a given graph G with no isolated vertices) is *close-knit* if the following condition is true for every $S' \subseteq S, S' \neq \emptyset$, given appropriate values for r (see discussion below):

$$\min_{S' \subseteq S} \frac{d(S', S)}{\sum_{i \in S'} d_i} \geq r \tag{1}$$

where $d(S', S)$ is the internal connections of S' i.e. the number of links $\{i, j\}$ where $i \in S'$ and $j \in S$ while d_i is the total number of links that agent i possesses (i.e. its degree in the graph). It is obvious from Inequality (1) that for every subset of a *close-Knit* group S, the ratio of internal degree to the total degree is at least r. Intuitively, to have a large such ratio in a group of agents we need, relatively, many internal connections and few external connections.

Given a positive integer k and a real number $0 \leq r \leq 1/2$, we call a graph G (r, k)-close-knit if every agent belongs to a group S of cardinality

at most k which is r-close-knit as defined by Inequality (1). Finally, a *class* of graphs is *close-knit* if for every $0 \leq r \leq 1/2$ there exists an integer k, possibly depending on r, such that every graph in the class is (r, k)-close-knit.

Moreover, the agents of the population are involved in playing a two person game on a regular basis in which pairs of interacting agents compete for a payoff against the other (non zero-sum game). According to the game, each time two agents interact they, independently, choose to either adopt or not adopt the innovation, based on the payoffs they can receive from their choices. The game is assumed to have a *risk dominant* Nash equilibrium in which *both* players adopt the innovation (see [13] for the details).

Based on these two elements, Young proved that for classes of close-knit agent graphs *all* community members will eventually adopt the innovation, i.e. the risk dominant Nash equilibrium of the game, in a number of interactions ("time") which is bounded and independent from the size of the community. In other words, if all the subsets of the community members have strong pairwise links and, at the same time, are weakly connected to outsiders (who may even be negative towards adopting the innovation), then a group of initiators will manage, in the end, to convince all population members to adopt the innovation.

3   Sierpinski triangle based group formations

In this section we describe a class of graphs based on the *Sierpinski Triangle* fractal and prove that it forms a *close-knit graph family*.

Definition 1 (Sierpinski Triangle gasket of level l). *Given an integer l, $l \geq 1$, we define the Sierpinski Triangle gasket of level l or, simply, Sierpinski Triangle of level l as follows: for $l = 1$ the Sierpinski Triangle is an equilateral triangle while for $l > 1$, the Sierpinski Triangle of level l is composed of three copies of a Sierpinski Triangle of level $l-1$ connected at their corners.*

In Figure 1 we see a few of the first Sierpinski Triangles, for $l = 1, 2, 3, 4, 5, 6$ and 7. Based on the Sierpinski Triangles, we can define a corresponding family of graphs, called *Sierpinski Graphs*.

Definition 2 (Sierpinski graphs). *The Sierpinski Graph of level l, $l \geq 1$, denoted by S_l is formed as follows: if $l = 1$ then the Sierpinski Graph of level 1 is formed if we replace the three vertices and the edges of the*

Fig. 1. Sierpinski Triangles, for $l = 1, 2, 3, 4, 5, 6$ and 7 respectively.

Sierpinski Triangle of level 1 with graph vertices and edges otherwise, for $l > 1$, the Sierpinski Graph of level 1 is formed by three copies of the Sierpinkski Graph of level $l - 1$ by identifying their vertices corresponding to the corners of the corresponding Sierpinski Triangle of level $l - 1$.

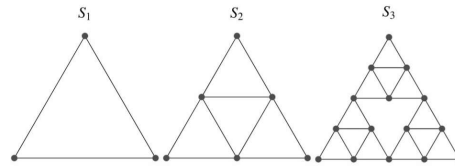In Figure 2, we see the Sierpinski Graphs S_1, S_2 and S_3. In particular,

Fig. 2. The Sierpinski Graphs S_1, S_2 and S_3.

in Figure 3 we see how the Sierpinski graph of level 4, S_4, is composed of three copies of the Sierpinski Graph of level 3, S_3, which is shown in a dashed enclosure. It is not hard to see that the following properties hold
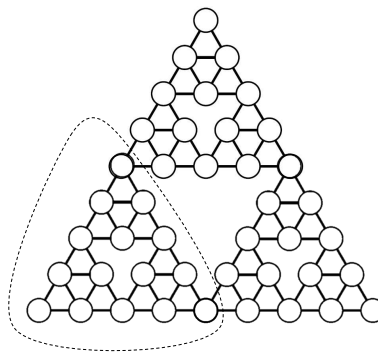

Fig. 3. The Sierpinski Graph of level 4.

for S_l:

- For every value of $l > 1$, all vertices of S_l have degree 4 except three vertices (the "corner" ones) which have degree 2.
- The number of vertices of S_l is $n_l = \frac{3}{2}(3^{l-1} + 1)$.
- The number of edges of S_l is $m_l = 3^l$.

The degree and connectivity properties helps satisfy Inequality (1). Intuitively, having too many nodes with large degrees, exposes a group of agents to much external interference. This affects negatively the close-knit property. We prove the following:

Theorem 1. *The family of Sierpinski Graphs is close-knit.*

Proof. Let $S_l = (V, E)$ be a Sierepinski Graph of level l, $l \geq 1$. Let S be a subset of its set of vertices V, such that $|S| = k$ and S' a nonempty subset of S. Each of the vertices in S and S' have degree either 2 or 4. Let, also, i, $0 \leq i \leq 3$ be the number of vertices of degree 2 in S'. We compute the numerator and denominator of the fraction in Inequality (1).

If by α, $0 \leq \alpha \leq 1$, we denote the fraction of edges between vertices in S' and vertices in $S - S'$, then with respect to the numerator of the fraction in Inequality (1), we have

$$d(S', S) = \alpha[4(k - i) + 2i] + (1 - \alpha)\,\frac{4(k - i) + 2i}{2} \tag{2}$$

$$= \alpha[4(k - i) + 2i] + (1 - \alpha)\,[2(k - i) + i]. \tag{3}$$

since edges between vertices in S' should be counted only once for the calculation of $d(S', S)$ in (3). As for the denominator in Inequality (1), we have

$$\sum_{i \in S'} d_i = 4(k - i) + 2i. \tag{4}$$

From (3) and (4) we have

$$\frac{d(S', S)}{\sum_{i \in S'} d_i} = \frac{\alpha[4(k - i) + 2i] + (1 - \alpha)\,[2(k - i) + i]}{4(k - i) + 2i}. \tag{5}$$

After some straightforward manipulations, (5) reduces to $\frac{1+\alpha}{2}$.

Therefore, Inequality (1) holds for any r and k. We conclude, that S_l, for $l \geq 1$, i.e. the Sierpinski Graphs, form a close-knit family of graphs. $\square$

4 Existence of highly organized structures in networks of interacting agents

In this section, we investigate the conditions upon which highly organized structures can appear in networks of interacting agents. These structures

may represent any organized community of individuals with a common aim or shared beliefs. These conditions, loosely speaking, rely on the structures possessing certain recognizable *regularity properties* that can be used to *succintly* describe them. These *regularity* and *succinctness* conditions model the "*organized structure*" notion. For concreteness, we will fix our organized, regular, structures to belong in the class of *Sierpinski Graphs* which, in addition, have the property of being *close-knit* (see Section 3).

More specifically, we will study the problem of the existence of Sierpinski Graphs, as subgraphs, in sufficiently large graphs, which can model evolving interacting agents, social networks and societies. In this section we deploy techniques from Kolmogorov Complexity and Ramsey Theory. In what follows, we briefly state the main definitions and some useful results from *Kolmogorov Complexity* and *Ramsey Theory*. We, then, apply both theories in order to investigate conditions that enable or hinder the emergence of organized substructures in evolving structures which are modelled as graphs.

4.1 Kolmogorov Complexity

In informal terms (see [8, 10, 3]) the *Kolmogorov Complexity* of a (binary) string x is the *length* of the *shortest algorithmic description* of x. In other words, the *Kolmogorov Complexity*, denoted by $C(x)$, of a finite string x is the length of the *shortest program* (or Turing machine in general) encoding as a binary sequence of bits, which produces x as output, without taking any input. Similarly, the *conditional* Kolmogorov Complexity of x given y, denoted by $C(x|y)$, is the length of the shortest program which produces x as output given y as input. It can be shown that $C(x)$ is, in some sense, *universal* in that it does not depend on the choice of the programming language or Turing machine model, up to fixed additive constant, which depends on this choice but not on x.

In this paper, our focus is on graphs. We can deploy the notion and properties of Kolmogorov Complexity by encoding graphs with strings as follows (see, e.g., [9]):

Definition 3. *Each labelled graph $G = (V, E)$ on n nodes $V = \{1, 2, \ldots, n\}$ can be represented (up to automorphism) by a binary string $E(G)$ of length $\binom{n}{2}$. We simply assume a fixed ordering of the $\binom{n}{2}$ possible edges in an n-node graph, e.g. lexicographically, and let the ith bit in the string indicate presence (1) or absence (0) of the ith edge. Conversely, each binary string of length $\binom{n}{2}$ encodes an n-node graph. Hence we can identify each such graph with its binary string representation.*

Definition 4. *A labelled graph G on n nodes has* randomness deficiency *at most $\delta(n)$, and is called $\delta(n)$-random, if it satisfies*

$$C(E(G)|n) \geq \binom{n}{2} - \delta(n). \tag{6}$$

Also, the following holds (see, e.g., [9]):

Lemma 1. *A fraction of at least $1 - 1/2^{\delta(n)}$ of all labelled graphs G on n nodes is $\delta(n)$-random. In particular, for $\delta(n) = \log n$, a fraction of $(1 - \frac{1}{n})$ of all graphs on n vertices is $\log n$-random.*

Definition 5. *Let $G = (V, E)$ be a labelled graph on n nodes. Consider a labelled graph H on k nodes $\{1, 2, \ldots, k\}$. Each subset of k nodes of G induces a subgraph G_k of G. The subgraph G_k is an ordered labelled occurrence of H when we obtain H by relabelling the nodes $i_1 < i_2 < \cdots < i_k$ of G_k as $1, 2, \ldots, k$.*

4.2 Ramsey Theory

We first provide the definition of the r-colouring of a set in Ramsey Theory.

Definition 6. *(r-colouring) Let S be a set and $r \in Z_+$. An r-colouring of S is a function $f : S \to \{1, 2, \ldots, r\}$.*

As discussed in the introduction, Ramsey Theory is concerned with questions involving the appearance of certain patterns in sufficiently large graphs. For instance, Ramsey Theory started with the following question: given a graph H, determine the Ramsey number $r(H)$, which is defined as the *smallest* natural number n such that *any* two-colouring of $E(K_n)$ contains a monochromatic copy of H. In [5] the following is proved:

Theorem 2. *There exists a constant c such that any graph H on k vertices with maximum degree Δ satisfies*

$$r(H) \leq k \, 2^{c\Delta \log \Delta}.$$

In this paper, we are interested in the *Induced Ramsey Number* of a given graph H, which is defined as follows:

Definition 7. *(Induced Ramsey numbers) A graph H is an induced subgraph of a graph H if $V(H) \subset V(G)$ and two vertices of H are adjacent if and only if they are adjacent in G. The induced Ramsey number $r_{ind}(H)$ is defined as the* minimum *integer for which there is a graph G on $r_{ind}(H)$ vertices such that every two-colouring of the edges of G contains an induced monochromatic copy of H.*

Note that an induced monochromatic copy of H is, also, an *induced* copy of H as an induced subgraph, in the ordinary graph-theoretical sense, regardless of the colour of the edges of G.

There is a number of results that provide upper bounds on induced Ramsey numbers for sparse graphs. For instance, Beck in [1] focused on the case when H is a tree. Also, Haxell, Kohayakawa, and Łuczak [7] showed that the cycle of length k has induced Ramsey number linear in k. Moreover, Łuczak and Rödl [11] proved that the induced Ramsey number of a graph with bounded maximum degree is at most polynomial in the number of its vertices, settling a conjecture of Trotter. More precisely, they proved the following:

Theorem 3. *For every integer d, there is a constant c_d such that every graph H on k vertices and maximum degree at most d satisfies $r_{ind}(H) \leq k^{c_d}$.*

The proof provides an upper bound on c_d that is a tower of 2's of height proportional to d^2. Since a Sierpinski Graph has maximum degree equal to 4, then an immediate corollary from Theorem 3, applied for $d = 4$, is the following:

Corollary 1. *Let $H = S_l$ be a labelled Sierpinski Graph of level l with n_l vertices. Then $r_{ind}(S_l) \leq n_l^{c_d}$, where c_d is a positive constant, independent from n_l and, thus, from l.*

4.3 The emergence of organized subgraphs in evolving graphs

Although random graphs (see e.g. [2]) are a powerful tool for proving limit properties, i.e. properties that hold in the limit as the graph size grows asymptotically, its main limitation is that it does not say anything about *specific*, finite, graph instances, for a fixed size n. Kolmogorov complexity theory, on the other hand, focuses on the study of specific, finite, objects (graphs in our case). Based on this theory, we can prove the following:

Theorem 4. *Let S_l be a labelled Sierpinski Graph on n_l vertices. Let G be a labelled incompressible graph on n vertices that contains S_l as an induced subgraph. Then $n \geq 2^{\frac{n_l-1}{2}}$.*

Proof. Let G to be a labelled *incompressible* graph with n vertices whose encoding $E(G)$ as a binary string (see Definition 3) has length $l(E(G))$. Since G is incompressible, it holds

$$C(E(G)|n, P) \geq \frac{n(n-1)}{2} \tag{7}$$

where P is a program that can reconstruct $E(G)$ from the value n and an alternative encoding $E'(G)$ as described below.

Since G contains, from our assumption, S_l as an *induced* subgraph, we can describe G by forming an alternative encoding $E'(G)$ constructed from $E(G)$ as follows:

1. We add to the encoding $E(G)$ the description of the Sierpinski Graph subgraph of G. In order to describe such a graph of n_l vertices in the graph G of n vertices, we need $\log \binom{n}{n_l}$ bits to denote the subset of the n_l vertices and $\log n_l!$ bits to denote the specific ordering $i_1 i_2 \ldots i_{n_l-1} i_{n_l}$ that provides the structure of a Sierpinski Graph, as we see it in the example graph of Figure 2, i.e. we form the Sierpinski Graph on n_l vertices and label them from top to bottom and from left to right (e.g. see Figure 2 for an indication of the "top" and the "bottom" of a Sierpinski Graph). In total, to describe this Sierpinski Graph subgraph, we need at most $\log \binom{n}{n_l} + \log n_l!$ bits and, since $\binom{n}{n_l} \leq \frac{n_l^n}{n_l!}$, we need, at most

$$\log \frac{n_l^n}{n_l!} + \log n_l! = n_l \log n \tag{8}$$

    bits.
2. We delete, from $E(G)$, *all* the bits that encode the edges of the Sierpinski Graph subgraph, saving $\frac{n_l(n_l-1)}{2}$ bits.

Now, it is easy to provide an algorithm P that, given $E'(G)$, constructs the Sierpinski Graph subgraph on n_l vertices given as input the number n of vertices of the graph, the value of n_l and the specific ordering $i_1 i_2 \ldots i_{n_l-1} i_{n_l}$, of the vertices of the Sierpinski Graph.

The length of the new encoding is, at most

$$l(E'(G)) = l(E(G)) + n_l \log n - \frac{n_l(n_l-1)}{2}. \tag{9}$$

Given the value of n, the program P can reconstruct $E(G)$ from $E'(G)$. Thus

$$C(E(T)|n, P) \leq l(E'(T)). \tag{10}$$

Since G is incompressible, it must hold $l(E'(G)) \geq l(E(G))$. From (9), this can, only, hold if

$$n_l \log n \geq \frac{n_l(n_l-1)}{2} \Leftrightarrow n \geq 2^{\frac{n_l-1}{2}} \tag{11}$$

which is the required. $\qquad\square$

Theorem 4 states that no incompressible graph with *fewer* than $2^{\frac{n_l-1}{2}}$ vertices can contain S_l as an *induced* subgraph. Consequently, an incompressible graph cannot contain S_l as a *monochromatic* induced subgraph either, in *any* two colouring of its edges. Thus, from Corollary 1 and Theorem 4, we have the following:

Theorem 5. *No incompressible graph on $r_{ind}(S_l)$ vertices can contain S_l as an induced subgraph except, possibly, for a finite set of values for l.*

Proof. Let G be an incompressible graph on n vertices that contains S_l as an induced subgraph. Since $r_{\text{ind}}(S_l) \leq n_l^{c_d}$ from Corollary 1, it follows that $n \leq n_l^{c_d}$. Also, the bound $n \geq 2^{\frac{n_l-1}{2}}$ holds from Theorem 4. Thus, it follows that $n \leq n_l^{c_d}$ can only hold for a finite set of values for n_l, whose cardinality depends on the constant c_d, since the growth rate of n is exponential in n_l while the growth rate of the bound $n_l^{c_d}$ for $r_{\text{ind}}(S_l)$ is polynomial in n_l. $\qquad\square$

Moreover, based on Lemma 1, we obtain the following *stronger*, than Theorem 5, result:

Theorem 6. *Almost all graphs on $r_{ind}(S_l)$ vertices (a fraction of $(1 - \frac{1}{r_{ind}(S_l)})$ of them) are such that no two-colouring of their edges contains an induced monochromatic copy of S_l, as l grows.*

Proof. Following the same line of proof as in Theorem 4, we now start with a labelled graph G such that

$$C(E(G)|n, P) \geq \frac{n(n-1)}{2} - \log n. \tag{12}$$

These graphs form a fraction of at least $(1 - \frac{1}{n})$ of all labelled graphs on n vertices, according to Lemma 1. For $n = r_{\text{ind}}(S_l)$ these graphs form a ratio of at least $(1 - \frac{1}{r_{\text{ind}}(S_l)})$ of all graphs with $r_{\text{ind}}(S_l)$ vertices. What is stated below, applies to *all* of these graphs which, as l tends to infinity, including *almost all* possible graphs on $r_{\text{ind}}(S_l)$ vertices.

   The rest of the proof follows closely the proof of Theorem 4, setting $n = r_{\text{ind}}(S_l)$, but now the following inequality must be satisfied instead of Inequality (11):

$$n_l \log n + \log n \geq \frac{n_l(n_l-1)}{2} \Leftrightarrow n \geq 2^{\frac{n_l(n_l-1)}{2(n_l+1)}} \Leftrightarrow r_{\text{ind}}(S_l) \geq 2^{\frac{n_l(n_l-1)}{2(n_l+1)}}. \tag{13}$$

However, since $r_{\text{ind}}(S_l) \leq n_l^{c_d}$ for some constant c_d depending only on the maximum degree d of the graph vertices, according to Theorem 3,

Inequality (13) would require $n_l^{cd} \geq 2^{\frac{n_l(n_l-1)}{2(n_l+1)}}$, which cannot hold from some value of l onwards. $\square$

Definition 8 (Size Constructible Graphs). *Let $\mathcal{F}$ be a family of graphs. We call $\mathcal{F}$ size constructible if each of the graphs in $\mathcal{F}$ can be uniquely constructed by an algorithm $P_{\mathcal{F}}$ which takes as inputs the graph's size, i.e. the number of vertices of the graph, and, possibly, a permutation that gives some ordering information about the vertices.*

A direct consequence of Definition 8 is that the family of Sierpinski Graphs is size constructible. This is due to the fact that a Sierpinski Graph S_l with n_l vertices can be described with only information n_l and the permutation that denotes the ordering of its vertices which the reconstruction algorithm P that we described in the proof of Theorem 4 uses in order to reconstruct the graph.

Then, Theorem 4 can be generalized as follows:

Theorem 7. *Let $\mathcal{F}$ be a family of size constructible graph. Let Let G be a labelled incompressible graph on n vertices that contains as an induced subgraph a graph in $\mathcal{F}$ with k vertices. Then $n \geq 2^{\frac{k-1}{2}}$ if $P_{\mathcal{F}}$ requires information about the ordering of the vertices and $n \geq k2^{\frac{k}{2}}\left(\frac{1}{e\sqrt{2}} - o(1)\right)$ if $P_{\mathcal{F}}$ does not require this information.*

Proof. If the algorithm $P_{\mathcal{F}}$ needs, except from k, also the ordering of the vertices, then a graph in $\mathcal{F}$ can be described using $k \log n + \log k!$ bits, as in the case of Sierpinski Graphs in Theorem 4, thus the proof of this theorem also applies to the case of $\mathcal{F}$.

If $P_{\mathcal{F}}$ does not need the ordering information, then instead of needing the number of bits given in Equation (8) in the proof of Theorem 4, we need, at most

$$\log \frac{n}{k!} = k \log n - \log k!. \tag{14}$$

bits since the ordering information, i.e. $\log k!$ bits, is not required.

Thus, the following inequality must hold for an incompressible graph G, as in Inequality (11) for the Sierpinski Graphs in the proof of Theorem 4):

$$k \log n - \log k! \geq \frac{k(k-1)}{2}. \tag{15}$$

Using Stirling's approximation $k! \approx (\frac{k}{e})^k \sqrt{2\pi k}$ and solving for n, Inequality (15) leads to

$$n \geq k2^{\frac{k}{2}}\left(\frac{1}{e\sqrt{2}} - o(1)\right)$$

which is the required. □

In our context, i.e. the emergence of an organized, close-knit, community in the form of a Sierpinski Graph, the results above have certain ineresting consequences. First of all, Theorem 4 and Theorem 7 (the generalization of Theorem 4) show that the existence or purposeful formation of orgaqnized structures, such as the Sierpinski Graphs, in *incompressible*, i.e. random-like, networks of interacting agents requires the networks to be *exponentially large* with respect to the size of the organized structure. Smaller interconnection networks, i.e. *polynomially large* in the size of Sierpinski Graph, almost certainly do not contain such organized structures.

5 Induced Ramsey numbers for incompressible graphs

In this section we investigate Ramsey Numbers in the context of Kolmogorov random graphs. More specifically, we prove bounds on the size of Kolmogorov Random graphs so as to contain a Sierpinski Graph (or other size constructible graph) as a subgraph.

Definition 9. *(Induced Ramsey numbers for incompressible graphs) The induced Ramsey number for incompressible graphs, $r_{ind}^{INC,\delta(n)}(H)$, is defined as the* minimum *integer for which there is a Kolmogorov Random graph G with randomness deficiency at least $\delta(n)$ on $r_{ind}^{INC,\delta(n)}(H)$ vertices such that every 2-colouring of $E(G)$ contains an induced monochromatic copy of H.*

Theorem 8. *Let H be a size constructible graph on k vertices and of maximum degree at most d, whose description needs vertex ordering information (a similar result holds for size constructible graphs that do not require such information). Let $n_1 = 2^{\frac{k-1}{2}} - 1$ and $n_2 = r_{ind}(H)$. Then for $\delta(n) \geq n_1 n_2 + \binom{n_2}{2}$, it holds that (i) $r_{ind}^{INC,\delta(n)}(H)$ exists, and (ii) $r_{ind}^{INC,\delta(n)}(H) < 2^{\frac{k-1}{2}} + r_{ind}(H)$.*

Proof. Let G_1 be any *incompressible* graph on $n_1 = 2^{\frac{k-1}{2}} - 1$ vertices, so that (according to Theorem 7) it does not contain H as an induced subgraph. Since G_1 is incompressible, it holds $C(E(G_1)) \geq \binom{n_1}{2}$.

Let, also, G_2 be any graph on $n_2 = r_{\text{ind}}(H) \leq k^{c_d}$ vertices such that for every two-colouring of its edges it contains an induced monochromatic copy of H (the existence of such a graph is guaranteed by Theorem 3).

Let $n = n_1 + n_2$. We focus on the graphs G with n vertices which, simply, consist of one copy of G_1 and one copy of G_2, that is, two subgraphs isomorphic to G_1 and G_2. There are no additional edges except those in these two subgraphs.

Obviously, every two-colouring of $E(G)$ contains an induced monochromatic copy of H, since G_2 does. This proves that $r_{\text{ind}}^{\text{INC},\delta(n)}(H)$ exists for Kolmogorov random, i.e. incompressible, graphs (at least for the deficiency function $\delta(n)$ which will be defined below).

Also, G has $n = n_1 + n_2 = 2^{\frac{k-1}{2}} - 1 + r_{\text{ind}}(H)$ vertices. We will show that for appropriate randomness deficiency function $\delta(n) = \delta(n_1, n_2)$ the following holds, i.e. G is $\delta(n_1, n_2)$-incompressible:

$$C(E(G)) \geq \binom{n_1 + n_2}{2} - \delta(n_1, n_2)$$

$$\text{with } \delta(n_1, n_2) \geq \binom{n_2}{2} + n_1 n_2. \tag{16}$$

Assume, towards a contradiction, that $C(E(G)) < \binom{n_1+n_2}{2} - \delta(n_1, n_2)$ for a randomness deficiency function $\delta(n_1, n_2)$ that obeys the inequality in (16). Then the following holds:

$$C(E(G)) < \binom{n_1 + n_2}{2} - \delta(n_1, n_2) \leq$$

$$\binom{n_1 + n_2}{2} - \left[\binom{n_2}{2} + n_1 n_2\right] = \binom{n_1}{2}. \tag{17}$$

We will describe an algorithm that can reconstruct G_1 from a description of G of $C(E(G)) < \binom{n_1}{2}$ bits contradicting our assumption that $C(E(G_1)) \geq \binom{n_1}{2}$, i.e. that G_1 is incompressible.

Let us assume that we have a description of G of $C(E(G)) < \binom{n_1}{2}$ bits. Then we can reconstruct G_1 as follows. We first reconstruct G using its description of $C(E(G))$ bits. We, then, compute its *connected components* using, e.g., a *Depth First Search (DFS)* algorithm (see [14]). Let these components be $G_{i_1}, G_{i_2}, \ldots, G_{i_s}$. One of these components must be the graph G_2. The union of the rest of the components should be the graph G_1. However, it is possible that only *one* component exists, beyond G_2, if G_1 is a *connected* graph. Our next step is to identify G_2. If we succeed in this task, then we have identified G_1: it is the graph that is composed of vertices and edges of G which do not belong in G_2.

In order to identify G_2 we work on the components $G_{i_1}, G_{i_2}, \ldots, G_{i_s}$, one at a time. We should be cautious here, since G_2 may not be a *connected* graph and, thus, we need to locate *all* its components. The crucial

observation is that G_2 cannot contain a component such that no two-colouring of its edges contains an induced monochromatic copy of H. Otherwise, we could dispense with this component and have a smaller graph satisfy the definition of $r_{\text{ind}}(H)$, contradicting its *minimality* requirement.

Let G_{i_l} be the currently examined component. We produce all possible edge 2-colourings of this component and for *each* of them we check whether G_2 contains a monochromatic copy of H. Observe that this can be true only for G_2 components since all other components belong to G_1, which is an incompressible graph with less than $2^{\frac{k-1}{2}}$ vertices and thus, according to Theorem 4, its edge 2-colourings (and, consequently, its components' 2-colourings) cannot contain a monochromatic copy of H. Thus, having identified G_2 we can reconstruct G_1 as the subgraph of G containing the rest of the components. In this way, we have managed to reconstruct G_1 using less than $\binom{n_1}{2}$ bits, which is a contradiction. Consequently, $C(E(G)) \geq \binom{n_1+n_2}{2} - \delta(n_1, n_2)$, that is G is $\delta(n_1, n_2)$-incompressible.

In conclusion, G is a graph of $n = n_1 + n_2$ which, for $\delta(n) \geq \delta(n_1, n_2)$, is $\delta(n)$-incompressible and all the 2-colourings of its edges contain an induced monochromatic copy of H. Thus, it follows that $r_{\text{ind}}^{\text{INC},\delta(n)}(H) \leq n = 2^{\frac{k-1}{2}} - 1 + r_{\text{ind}}(H)$ or, in simpler form, $r_{\text{ind}}^{\text{INC},\delta(n)}(H) < 2^{\frac{k-1}{2}} + r_{\text{ind}}(H)$. $\square$

From Theorems 4 and 8, the following is derived:

Corollary 2. *For every graph H on k vertices and maximum degree at most d, it holds that $2^{\frac{k-1}{2}} \leq r_{ind}^{INC,\delta(n)}(H) < 2^{\frac{k-1}{2}} + r_{ind}(H)$.*

Note that, since $n = n_1 + n_2$, the canonical representation of G has $|E(G)| = \binom{n_1+n_2}{2}$ bits while we set $\delta(n) = \delta(n_1, n_2) = n_1 n_2 + \binom{n_2}{2}$. Thus, considering k as a varying parameter, $\delta(n)$ is in the order of $\sqrt{|E(G)|}$. We could not prove Theorem 8 for deficiency functions $\delta(n)$ smaller than this, e.g. $\delta(n) = \log |E(G)|$. However, we believe that this is not possible since when a graph contains some regularity, in our case the Sierpinski Graph S_l on n_l vertices, its complexity drops since the regularity can be deployed in reducing the size of its description.

6 Conclusions and directions for further research

Society is a complex human creation composed of multifaceted autonomous, interacting, agents and their interrelationships. It is a common theme

in numerous research works that out of evolving (even randomly) interactions and relationships important phenomena and patterns emerge in complex networks of agents. Such results rely on well established mathematical and physical theories about models of interacting agents, such as Random Graph theory and Complex Network theory.

In this paper we considered two well established mathematical theoretical frameworks targeting the concept of complexity of finite objects, the *society and its interacting agents* in our case: *Kolmogorov Complexity* and *Ramsey Theory.* Both of these theories, each from another perspective, study the conditions upon which certain substructures appear (or do not appear) in large, evolving, structures such as societal networks of interacting agents, also deriving estimates on how large the structures should become in order to contain such regularities. We applied elements of these theories in order to study the appearance of regular structures or patterns, in evolving societies, that have certain desirable properties. One of these properties, which was among our targets, is *close-knittedness* or, in other words, the property that describes structures of communicating agents whose members interact closely with each other and defend, strongly, their group's coherence and views against the containing, larger, structure.

More specifically, with Theorem 5 we proved that the Sierpinski Graph, as a highly organized structure, cannot emerge in evolving graphs (societies) unless they reach a sufficient size, exponential in the size of the organized structure. With Theorem 8 and Corollary 2 we gave bounds on the size the graphs (societies) that contains with certainty any organized structure of bounded degree (i.e. relationships) among its vertices (i.e. society members).

We hope that our work will contribute to the further exploitation of the rich mathematical theories of complex structures and their long-term evolutionary properties. In this way, we feel we can strengthen the efforts towards the study of Social Sciences with methodologies stemming from exact sciences and formal systems.

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