T-duality and the signature of 4-dimensional spacetime

Maxime Médevielle

Supervised by
Dr Thomas Mohaupt

University of Liverpool
Department of Mathematical Sciences
March 2023

Thesis submitted in accordance with the requirements of the University of Liverpool for the degree of Doctor in Philosophy

Examiners:

Prof. Chris Hull FRS (Imperial College London, United Kingdom)

Prof. Alon Faraggi (University of Liverpool, United Kingdom)

Abstract

This thesis is centered around the study of geometric aspects of string theory and supergravity. We will focus on $\mathcal{N} = 2, D = 4$ supergravity theories in arbitrary spacetime signature. We will obtain these supergravity theories as Calabi-Yau compactifications of Hull's exotic Type II theories. In ten dimensions these theories are related to each other via T-duality and S-duality. We will obtain the four dimensional duality web of theories as a projection of the ten-dimensional one. Moreover the T-duality relations between theories define maps, from the vectormultiplet geometries to the hypermultiplet geometries, called c-maps, which will be characterized and classified.

We will then turn to the study of solutions of such theories. We will be interested in non-extremal black hole and cosmological solutions exhibiting planar symmetry. Such solutions will be T-dualized and we will interpret their behavior once embedded in string theory.

Authorship Declaration

I hereby declare that the material presented in this doctoral thesis is the result of my own research activity together with my collaborators. All references to other people's work are cited explicitly. All my work has been carried out in the String Phenomenology Group in the Department of Mathematical Sciences at the University of Liverpool during my doctoral studies.

The results presented in this thesis are based on two papers [1, 2]. Their details are as follows:

1. M. Médevielle, T. Mohaupt and G. Pope,
   Type II Calabi-Yau compactifications, T-duality and special geometry in general spacetime signature,
   JHEP 02 (2022) 048 [arXiv:2111.09017 [hep-th]].
   Location in thesis: Chapter 4.

2. M. Médevielle and T. Mohaupt,
   T-duality across non-extremal horizons,
   to appear.
   Location in thesis: Chapter 5.

The following paper was published by the author during the PhD but will not be presented in this thesis [3]:

3. H. Erbin and M. Médevielle,
   Closed string theory without level-matching at the free level,
   JHEP 03 (2023) 091 [arXiv:2209.05585 [hep-th]].

À ma Mamusiu

Acknowledgements

I would first like to express my gratitude towards my supervisor Thomas Mohaupt. Thank you for giving me the opportunity to explore such a fascinating subject, giving me the freedom to explore anything I found interesting to quench my curiosity. I feel privileged for having collaborated and discussed physics and maths with someone with such a deep knowledge of string theory. I also thank Harold Erbin for collaborating with me on a project that allowed me to get a glimpse of what working on string field theory looks like.

I thank the members of the String and Beyond the Standard Model Phenomenology group in Liverpool – Alon Faraggi, Radu Tatar, Susha Parameswaran, Ed Hardy and Juri Smirnov – for our lively Tuesday group meetings and paper discussions. I am particularly indebted to Alon, who always made sure that I never lose my interest in the classical double copy and that I always keep in mind that the Higgs could be composite.

Life in Liverpool would have been very dull if it weren't for the friends I made in the department. I'd first like to thank Jack, my academic big brother, who always had his office door open during my first year, whenever I needed help regarding dimensional reductions, c-maps, dualizations and many other topics. Flavio can always offer me good company despite questionable taste in movies. Thank you for the many laughs, movie nights, occasional walks at the Albert dock, and overall for your friendship. Thank you also to Viktor, who welcomed me warmly when I first arrived in Liverpool and who made me feel at ease in the department when I didn't know anyone. I feel blessed for the wonderful dishes you cooked on occasion. I wish you and Saghar much joy in your soon-to-be newlywed life, and if any of you ever need japanese lessons you know who to call. I'd like to thank Martin for being a very quirky and entertaining office mate in my first year. I will remember forever the day you decided to have a whole block of Brie and a Redbull for dinner.

Ben is very very cool. Thank you for mentoring my chess career and giving me enough advice to allow me to beat you on many many intense blitz games (let the record show I have just beaten Ben's notorious English opening 5 minutes ago in a 7+5). Thank you for the interesting discussions, whether they were about politics, philosophy of science or string theory (the shrewd reader will understand that Ben is an intellectual (nerd?)).

I feel sorry that Covid delayed by one year my meeting Francesco. Thank you for the delicious amatriciana, and for always popping by my office to discuss One Piece theories (which were arguably more complex than string theory and QCD combined). I wish you all the best in your research on semi-leptonic decay renormalization schemes at seventeen loops. I can now admit that I never understood a single word of what you were saying, but seeing you smile when explaining your research to me was enjoyable enough. I'd like to thank Joaquim, the most spanish of portuguese, for always filling awkward silences with awkward sounds. Thank you for your unbounded interest in french culture. I hope one day you pursue this PhD in linguistics and etymology. I'd like to thank Marco, the prettiest and most scouse of italians, for the unwavering kindness and empathy you have always shown me even during my most challenging times. Your impersonations are unparalleled and I always love when you come up with a new one. I wait eagerly for the day when you will play Smash at 200% of your abilities so that I can witness the full potential of your N-Mewtwo.

I would like to express to Bruno my deepest and sincerest gratitude. Being my roommate for three years is no easy fit but, as everything else, Bruno did it with great success. You are in many regards an inspiration and it is no coincidence you were always the one I turned to whenever I needed advice. I still have a long way to go but I like to think that most of the personal growth I achieved during this PhD was thanks to you. I am sorry for complaining so much – I am fRench after all. Thank you for the time spent watching Ozark, SnK, movies... Of course thank you for the legendary shrimp rice (with a kick). Thank you for the late night discussions about string theory or life. Our friendship is very dear to me and I wish you all the successes that you deserve in your future endeavors (and for once I did not use the word "deserve" lightly).

I'd like to thank Vine Court for the salt and pepper chicken.

Now in french. Merci à Yoann, Juliette et Qamar pour nos soirées à chaque fois que je revenais en France. Merci d'avoir écouté sans relâches mes nombreuses complaintes sur mon avenir professionnel. Merci particulièrement à Juliette et Yoann pour nous avoir accueillis à chaque occasion et pour ne pas avoir perdu patience quand Qamar et moi rentrions dans des discussions bouillonantes sur One Piece. Je souhaite tout le bonheur à Qamar dans sa nouvelle vie de jeune marié.

Je tiens à remercier Gabin, mon compagnon d'infortune de l'Imperial. Le quatrième étage de Gabor Hall aurait été bien plus morne si tu n'avais pas été là. On a probablement tenu jusqu'au bout grâce au salut offert par le chicken katsu curry de Kimiko (no picklz pour toi et sumo size pour moi...). Merci pour les après-midi passées au parc du Luxembourg (malgré tes retards incessants) à regarder toujours le même vieux mec jouer au tennis sans faillir. Merci de m'avoir initié au monde de Twitch, tu es le seul responsable de plusieurs milliers d'heures que j'ai passées à procrastiner. Merci pour les sessions Rocket League hardstuck Plat III-Diam I. Je nous mets au défi d'un jour passer Champion en duo. Merci de m'avoir laissé spectate tes games d'échecs afin que je puisse me délecter des tes moves les plus clownesques. Enfin, un très grand merci pour ton aide numérique, particulièrement indispensable sur Cadabra.

Je tiens a remercier mon vieux pote Maxime. Tu m'as toujours soutenu même dans les moments difficiles et m'a toujours accepté tel que je suis. Je prends toujours plaisir à écouter quand tu me racontes tes dernières aventures. L'histoire retiendra qu'au moment d'écrire cette thèse, tu n'es plus qu'à une seule victoire pour atteindre l'objectif des 100 games gagnées au foot.

Je tiens à remercier toute ma famille pour tout son soutien depuis toujours. Je remercie Rémi, Anne, Marc et Marie, la branche marseillaise de mon arbre généalogique, de compenser mon introversion par leur extraversion et ainsi apporter un équilibre certain dans l'Univers. Je remercie Manon et Jocelyn qui sont venus me rendre visite en Angleterre par deux fois, pour m'avoir accueilli de nombreux après-midi quand j'étais de passage a Paris. Merci aussi de m'avoir hebergé de nombreuses fois pour me permettre d'attraper mes vols de retour tôt le matin. Merci pour les jeux de société, notamment les games dantesques sur Love Letter. Je remercie ma grande soeur Manon tout particulièrement pour toute sa gentillesse et sa joie de vivre rayonnante. Je remercie mon autre grande soeur adorée Sophie, pour tout le soutien, les nombreux encouragements et les barres de rire. Merci pour les nombreuses recettes de cuisine qui m'ont permis de manger autre chose que des pâtes et du riz. Je te souhaite tout le succès et tout le bonheur du monde.

Je tiens enfin a remercier tout particulièrement ma Maman. Sans son soutien et sa gentillesse indéfectibles, il m'aurait été impossible d'accomplir quoi que ce soit. Te dédier cette thèse est une evidence et le témoignage de ma profonde gratitude pour tout l'amour que tu m'as apporté.

Contents

Chapter 1

Introduction

So are we quarks, strings,
branes or what?

New York Times, 22/09/1998

Unification of the observable fundamental phenomena of nature is one of the primary goals of physics, and has historically been a fruitful way to make paradigm shifting discoveries. The first unification dates back to Isaac Newton in the 17th century with the formulation of the universal law of attraction, which unified under a single theoretical framework the motion of celestial bodies in space and the motion of massive bodies on the surface of the Earth. At the beginning of the 20th century Albert Einstein, with his theory of special relativity, unified the concept of space and time to form a single entity called spacetime. He then incorporated gravity in this framework in 1915 with his theory of general relativity which led to a revolution in astrophysics and cosmology.
Thermodynamics first unified the concepts of heat and work as two manifestations of energy. Subsequently a microscopic description of these phenomena was given by Ludwig Boltzmann with the advent of statistical physics and the atomic hypothesis. James Clerk Maxwell, with the equations that bear his name, unified electric and magnetic phenomena. The quantum revolution allowed for a unification of all atomic phenomena. Subsequently, electromagnetism and quantum phenomena were also unified in a field theoretic framework dubbed quantum electrodynamics (QED) initiated by Paul Dirac and then developed by people like Tomonaga, Schwinger, Feynman and Dyson. In the late 60's, Glashow, Salam and Weinberg described, by incorporating the Higgs mechanism (whose associated boson was discovered at the LHC in 2012), the unification of QED and the weak nuclear force into the electroweak interaction. This laid the foundation for the standard model of particle physics which, in its current form, is the most precise and experimentally verified theory of physics.

Even with all these spectacular successes, we know that the standard model is not the end of the story as there are several loose ends. First and foremost, the standard model does not account for gravity. Indeed, a quantum field theory for gravitation only works at low energies because of its non-renormalizability. In order to probe phenomena like black hole singularities or the Big Bang, a genuinely new quantum gravity theory is required. The standard model also fails to explain other phenomena such as matter/antimatter asymmetry, dark matter, dark energy, neutrino masses, hierarchy problems.

One of the main goals of current research in theoretical high energy physics is to find a single unifying framework that can describe all the matter content in the universe and all fundamental interactions. Such framework is sometimes referred to as a "theory of everything". String theory is currently the leading candidate for such a theory.
String theory was initially proposed in the form of the so called "dual resonance model" by Veneziano in 1968 as a way to describe the strong nuclear interaction. The Veneziano formula for the scattering of mesons was generalized for N-particles by Nambu, Nielsen and Susskind who provided a physical interpretation in terms of an inifinite number of simple harmonic oscillators describing the motion of an extended one-dimensional string, hence the name "string theory". String theory was a popular research topic in hadron physics until it fell out of fashion in 1973 with the rise of quantum chromodynamics (QCD) as the theory of the strong nuclear force, which fitted better with experimental evidence such as the electron/proton deep inelastic scattering.
In 1974, John H. Schwarz and Joël Scherk, and independently Tamiaki Yoneya, found that the spectrum of the quantum string contained a particle that exactly matched the graviton. This lead to string theory being reinterpreted not as a theory of nuclear forces but as a theory of quantum gravity. The early string models were still facing challenges, one being that the bosonic string theory has a critical spacetime dimension $D = 26$ and a tachyonic vacuum.
In 1984, the first superstring revolution marked a period of important discoveries that reignited the interest in string theory. It was understood that string theory was capable of describing all elementary particles and interactions between them thus making it a promising candidate for a theory of everything. The revolution started with the discovery of anomaly cancellation in type I string theory via the Green-Schwarz mechanism (named after Michal Green and John H. Schwarz). Shortly after came the construction of the heterotic string theories, by David Gross, Jeffrey Harvey, Emil Martinec and Ryan Rohm. It was understood by Philip Candelas, Gary Horowitz, Andrew Strominger and Edward Witten that phenomenologically realistic models could be obtained by compactifying these theories on Calabi-Yau manifolds. By 1985, the five supersymmetric string theories (type I, Type IIA and IIB, heterotic $SO(32)$ and $E_8 \times E_8$) were constructed.
In the early 1990's, Edward Witten found evidence that the different superstrings were different limits of an 11-dimensional theory whose low energy description is the unique

11-dimensional supergravity theory. This theory became known as M-theory. He relied heavily on earlier works on non-perturbative dualities, primarily the work of Sen on S-duality [4] and the work of Hull and Townsend on U-duality [5]. These discoveries began the second superstring revolution. The different versions of superstring theory were unified by various dualities and equivalences such as T-, S- and U-duality. The non-perturbative nature of D-branes culminated at the same period and plays a central role in string dualities. Proposals for a fundamental non-perturbative definition of string theory emerged, amongst which we can mention the matrix model of Banks, Fischler, Shenker and Susskind (BFSS), or the AdS/CFT correspondence by Juan Maldacena, which gives a direct stringy realization of the holographic principle (for a detailed account of the history of string theory see [6]). All these discoveries point to a rethinking of the role of spacetime in string theory. As Andrew Strominger pointed out "The notion of space ... and time... and dimension are not absolute" [6]. He compares the situation with the phases of $H_2 0$ and their temperature dependance: in various regimes, water switches between solid, liquid and steam. In the case of the dimensionality of spacetime, there is a dependence on the energy of the system: it becomes another dynamical parameter similarly to how the spacetime metric is made dynamical in classical general relativity. John Schwarz went further and claimed "The remarkable role of duality symmetries and their geometrically non-intuitive implications suggest to me that the theory might look very algebraic in structure without evident geometric properties so that no space-time manifold is evident in its formulation. In this case, the existence of space-time would have to emerge as a property of a class of solutions. Other solutions might not have any such interpretation". The emergence of spacetime from more fundamental degrees of freedom is an idea that is not specific to string theory but common with other approaches to quantum gravity such as loop quantum gravity and causal sets. One particularly interesting question, and which plays a central role in this thesis, is the role of spacetime signature. Indeed, if spacetime were to be an emergent phenomenon, something would need to dictate not only its dimensionality, but also its signature. Moreover, just like dimensionality can be dynamical in string theory, can dynamical signature change occur?

In his "Orthogonal" trilogy [7–9], famous hard science fiction writer Greg Egan imagined a story taking place in a universe where there are four fundamentally identical dimensions, in other words, a universe with Euclidean signature. In his book "Dichronauts" [10], he explored a universe with two distinct dimensions of time (split signature in four dimensions is often referred to as Klein space). Such considerations have often times been fuel to science fiction books and philosophical treaties alike. Although the topic of signature change has been explored in the theoretical physics literature, it remains somewhat controversial. Indeed, considering spacetimes of arbitrary signatures leads rapidly to fundamental issues clashing with causality and unitarity. Even if signature change is not permitted in the real world, it can have interesting theoretical and even practical use

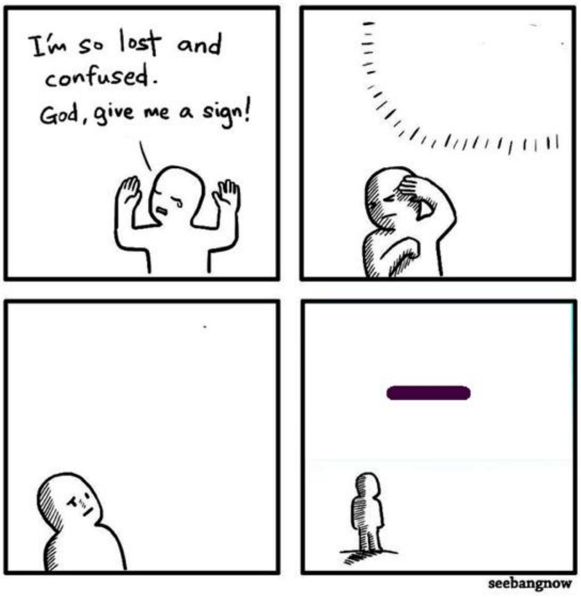
(arbitrary signatures have for example found applications in the study of exotic forms of matter and metamaterials, see [11, 12] for example).
We do not aim to give a comprehensive account of the literature on signature change as it is too extensive. Instead, we will only present a few interesting discussions that have been raised in the topic so as to contextualise the work done in this thesis.
First and foremost, it is important to state that even fundamental considerations of causality can not trivially rule out the possibility of other signatures. Indeed, it has been shown by Craig and Weinstein [13, 14] that the ultrahyperbolic wave equation (the wave equation in other signatures), possesses a well defined initial value problem. Initial data on a mixed hypersurface obeying a particular nonlocal constraint evolves deterministically in the remaining time dimension.
The most familiar case of signature change which is familiar for anyone dealing with quantum field theories is the transition from Lorentzian to Euclidean signature. Indeed, to deal with path integrals, it is a standard procedure to work with Euclidean signature by performing a Wick rotation. However, this is usually interpreted as a mathematical trick and not a proper hint of a fundamental Euclidean signature of the metric. This interpretation tends to change when considering gravity into the mix. Indeed, in "Euclidean quantum gravity", the Euclidean path integral is taken seriously with the hope of dealing with topology change (in the path integral the spacetime signature is *a priori* unconstrained). The interplay between Euclidean and Lorentzian signature is explored from several avenues in the context of cosmology. The famous "no-boundary proposal" of Hartle-Hawking [15] deals with the cosmological singularity by describing a transition in the early universe from a state with Euclidean signature to one with Lorentzian signature. Such a signature transition has also been proposed in Vilenkin's quantum tunneling proposal [16] for the wave function of the early Universe. Interestingly, recent numerical studies of string matrix models (see [17–19]) have also observed a similar behavior. Arbitrary signature change and compact extra timelike dimensions have also been discussed in the cosmology context by Sakharov [20]. Other applications of extra timelike dimensions in the Kaluza-Klein context have also been studied (see for example [21–23] or more recently [24] for the case of an oscillating universe)
The use of split signature in four dimensions to study Lorentzian physics has been discussed in [25]. Klein space has also found applications in different contexts like twistor theory [26], the double copy [27], and the study of black holes [28].
If one were to accept the physical possibility of signature fluctuations, then this raises the question of what determines the currently observed spacetime signature in our patch of the universe. Several explanations have been proposed amongst which we can mention [29] which treats one metric component as a genuine quantum field. The signature of spacetime is then determined dynamically by its expectation value and it was found that (under some assumptions) Lorentzian four-dimensional spacetime is selected. In [30]

it is argued that the signature of spacetime is constrained by the consistency of electromagnetism. Tegmark brought forth anthropic arguments in order to explain the observed signature [31](although as mentioned earlier the question of determinism in the ultrahyperbolic case is non-trivial). In the string context, the brane scan for branes of arbitrary world-volume signature does not single out Lorentzian signature but suggests that it plays a preferred role [32]. Lastly, group theoretic arguments based on representation theory can be invoked to explain the observed spacetime signature [33, 34]

If signature changing events are possible, the metric at the junction would degenerate and strictly speaking the Einstein equations would cease to apply. Several proposals for "junction conditions" have been proposed and lead to intense debates in the litterature sometimes refered to as the "signature change controversy" (see for example [35–46]). Ultimately, there are strong hints that the issue of signature change can only be settled in a full fledged quantum gravity framework. Indeed, as explained in [47], any model of QFT on curved backgrounds suffers from infinite particle production at the junction. Without a proper understanding of the backreaction any attempt to describe dynamical signature change is moot. Just like the question of topology change and chronology protection, the question of signature change seems to ultimately require quantum gravity in order to be settled.

Exotic spacetime signatures have been explored in the string theory context in various places. We can mention F-theory, a non-perturbative completion of Type IIB string theory, which is formally formulated in (10+2) dimensions (although interpreting the extra timelike coordinate as genuinely geometric is debatable). The $\mathcal{N} = 2$ superstring also gives rise to a 4D theory in split signature [48–50]. Itzhak Bars is a known proponent of "two-time physics" which originates from string theory and is motivated by unifying symmetry arguments (see [51] for a review). Braneworld scenarios also support the possibility of dynamic signature change [52]. Of primary interest to us in this thesis is the possibility of signature change induced by string dualities. Indeed Hull found that considering exotic dualities leads one to string theories of arbitrary spacetime signature [53]. Moreover it was found that exotic negative branes (branes with negative tension), induce a bubble of signature change around them [54], and these spacetimes are exactly described by the exotic theories found by Hull. It has also been argued from algebraic considerations [55–57] that different spacetime signatures sit on the same footing in the context of the E_{11} formulation of M-theory [58].

We conclude by mentioning that there are considerations of going even beyond arbitrary signatures of the metric. Indeed, several hints suggest that there are ways of making sense of complex spacetime metrics. This was initially explored in the QFT context [59] but is also discussed in the quantum gravity context through the lense of the Euclidean path integral (see for example [60]).



Thesis outline This thesis is organised as follows:

In Chapter 2 we will review the classical and quantum bosonic string. We will rederive the important results relevant for this work, in particular the construction of the massless spectrum. We will study the bosonic string compactified on a circle and introduce the central concept of T-duality. We will then turn to the supersymmetric string with a focus on Type II theories and the T-duality relation between them. We then conclude with an overview of other superstring theories and dualities relating them.

Chapter 3 will be devoted to presenting the necessary background of supergravity and geometry. We will first present supersymmetry algebras and their representation theory to construct supersymmetric field theories. Then, we will show how to perform a dimensional reduction in the field theory setting, a process known as Kaluza-Klein theory. We will describe Calabi-Yau manifolds, which are a prime choice for the compactification space in superstring theory. We will then introduce the "special geometry" of the Calabi-Yau moduli spaces.

In Chapter 4 we will present the new results that constitute the main work of this thesis. We will first review the relevant material about vector and hypermultiplets in arbitrary spacetime signatures. We will then perform the Calabi-Yau compactifications of the exotic Type II supergravity theories that were constructed by Hull by performing timelike T-duality. We will be focusing on the sign flips happening in the Lagrangian in order to interpret them in terms of special geometry of the scalar sector of the supergravity theories in 4 dimensions. We will classify the different T-duality relations happening in 4 dimensions in order to construct the 4-dimensional duality web.

In Chapter 5 we will present some preliminary results of work to appear, which studies the global action of T-duality on spacetimes with non-extremal Killing horizons. These solutions are black hole and cosmological solutions of Einstein (anti)-Maxwell theory with planar symmetry. In order to perform T-duality, a 4-dimensional version of the Type II Buscher rules will be derived. An embedding of the dualized solutions in Type II* supergravity will also be discussed.

Finally we will conclude with potential future projects furthering the work done in this thesis. In particular we will discuss embedding this work in Double Field Theory, an effective field theoretic framework which makes T-duality manifest.

Chapter 2

Bosonic and Type II string theory

There is geometry in the
humming of the strings.

Pythagoras

In this chapter we will introduce the classical and quantum string theory. We focus on the bosonic string because it allows to study many aspects of string theory while avoiding complications due to to the presence of fermions in the case of the superstring. We will be focusing primarily on the closed string as its spectrum contains the gravitational sector of the theory. We will then introduce T-duality, a fundamental symmetry of string theory that plays a major role in this thesis. Finally we will describe the Type IIA and Type IIB string theories, which are T-dual to each other. Our main references are [61–63].

2.1 Classical bosonic string theory

We start with the study of the classical relativistic bosonic string. Our aim is to write an action for such a string and study its dynamics.

2.1.1 Nambu-Goto action

We start with the classical relativistic string, which means a one-dimensional object which traces a *worldsheet* in spacetime. This worldsheet Σ is a surface embedded in Minkowski spacetime $\mathbb{M}$

$$X : \Sigma \to X(\sigma, \tau) \in \mathbb{M}, \tag{2.1}$$

where we chose linear coordinates associated to a Lorentz frame $X = X^\mu$ with $\mu = 0, 1, ..., D-1$ and D being the dimension of spacetime. We choose on the worldsheet local coordinates $\sigma = (\sigma^0, \sigma^1) = (\sigma^\alpha)$. σ^0 is time-like and σ^1 is space-like. We introduce the following notation

$$\dot{X} = \partial_0 X^\mu = \left(\frac{\partial X^\mu}{\partial \sigma^0}\right), \qquad X' = \partial_1 X^\mu = \left(\frac{\partial X^\mu}{\partial \sigma^1}\right). \tag{2.2}$$

For a free string that does not split or join, the topology of the worldsheet is either a strip (open string) or a cylinder (closed string). The coordinate σ^1 takes values in $[0, \pi]$ and σ^0 takes values in $[\sigma^0_{(1)}, \sigma^0_{(2)}] \subset \mathbb{R}$. We want the action describing this physical system to not depend on the choice of coordinates on the worldsheet, in other words we want the action to exhibit reparametrization invariance. One obvious physical quantity of the worldsheet that does not depend on coordinates is the area of the worldsheet, so we start with an action proportional to the worldsheet area

$$S[X] = -TA(\Sigma) = -T \int_\Sigma d^2 A, \tag{2.3}$$

where T is the *string tension*. The Minkowski metric $\eta_{\mu\nu}$ induces on the worldsheet Σ a metric $g_{\alpha\beta}$ by *pullback*

$$g_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}. \tag{2.4}$$

We can therefore define the invariant area element on Σ

$$d^2 A = d^2\sigma \sqrt{|\det g_{\alpha\beta}|}. \tag{2.5}$$

We can now write the action for the string as

$$S_{NG}[X] = -T \int d^2\sigma \sqrt{|\det \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}|}. \tag{2.6}$$

This is the *Nambu-Goto action*. We notice that it is also manifestly invariant under the Poincaré transformations of $\mathbb{M}$ (for a review on spacetime symmetries see chapter 3).

We can write the action more explicitly for computational purposes

$$S_{NG} = \int d^2\sigma \mathcal{L} = -T \int d^2\sigma \sqrt{(\dot{X} \cdot X')^2 - (\dot{X})^2 (X')^2}. \tag{2.7}$$

We define *worldsheet momentum densities* as

$$P^\alpha_\mu := \frac{\partial \mathcal{L}}{\partial \partial_\alpha X^\mu}. \tag{2.8}$$

and we find

$$P^0_\mu := \frac{\partial \mathcal{L}}{\partial \dot{X}^\mu} = T \frac{(X')^2 \dot{X}_\mu - (\dot{X} \cdot X') X'_\mu}{\sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2 (X')^2}}, \tag{2.9}$$

$$P^1_\mu := \frac{\partial \mathcal{L}}{\partial X'^\mu} = T \frac{\dot{X}^2 X'_\mu - (\dot{X} \cdot X') \dot{X}_\mu}{\sqrt{(\dot{X} \cdot X')^2 - \dot{X}^2 (X')^2}}. \tag{2.10}$$

The equations of motion are obtained by applying the variational principle on the Nambu-Goto action, which means that the action has to be invariant under $X \to X + \delta X$, keeping the endpoints of the string fixed: $\delta X(\sigma^0 = \sigma^0_{(1)}) = \delta X(\sigma^0 = \sigma^0_{(2)}) = 0$. Carrying out the variation gives

$$\delta S = \int d^2\sigma (P^0_\mu \delta \dot{X}^\mu + P^1_\mu \delta X'^\mu). \tag{2.11}$$

Performing an integration by parts gives two boundary terms

$$\delta S = \int_0^\pi d\sigma^1 [P^0_\mu \delta X^\mu]^{\sigma^0_{(2)}}_{\sigma^0_{(1)}} + \int_{\sigma^0_{(1)}}^{\sigma^0_{(2)}} d\sigma^0 [P^1_\mu \delta X^\mu]^{\sigma^1=\pi}_{\sigma^1=0} - \int d^2\sigma \partial_\alpha P^\alpha_\mu \delta X^\mu. \tag{2.12}$$

The first term vanishes because of the variational principle, which requires that the initial and final positions are kept fixed. However, we need to impose other boundary conditions for the second term to vanish as well, namely

$$\int_{\sigma^0_{(1)}}^{\sigma^0_{(2)}} d\sigma^0 [P^1_\mu \delta X^\mu]^{\sigma^1=\pi}_{\sigma^1=0} = 0. \tag{2.13}$$

There are 3 different possibilities to satisfy this constraint:

1. *Periodic* boundary conditions

$$X(\sigma^1) = X(\sigma^1 + \pi). \tag{2.14}$$

This means that the string is *closed*.

2. *Neumann* boundary conditions

$$P^1_\mu|_{\sigma^1=0,\pi} = 0. \tag{2.15}$$

These constraints imply that δX^μ don't have restrictions at the boundaries so these describe strings whose momentum is conserved at the endpoints, and can therefore move freely.

3. *Dirichlet* boundary conditions

$$P^0_i|_{\sigma^1=0,\pi} = 0. \tag{2.16}$$

This implies that the tangential component of the world-sheet momentum vanishes at the boundary, meaning that the endpoints of the string are kept fixed in the i-th direction

$$X^i(\sigma^1 = 0) = x^i_0, \quad X^i(\sigma^1 = \pi) = x^i_1. \tag{2.17}$$

The translation invariance of Minkowski is therefore broken and momentum is not conserved at the ends of the string. The way to restore this conservation is to couple these open strings with new dynamical objects called *D-branes*.

A detailed introduction of D-branes is beyond the scope of this work and much remains to be understood since these objects are inherently non-perturbative, we will only comment on a few general properties that these objects possess. When we want to specify their dimensionality, we call them Dp-branes, whence a D0-brane is a D-particle, a D1-brane is a D-string (which is not the same object as the fundamental string), a D2-brane is a membrane,... and a (D-1)-brane is space-filling. Dirichlet conditions were rarely considered, and the existence of D-branes was shown using T-duality in [64, 65]. Indeed, since T-duality exchanges Neumann and Dirichlet, such boundary conditions necessarily appear in regions of the moduli space of the open string. It was then shown by Polchinski that D-branes are the objects charged under the Ramond-Ramond fields, and that they correspond to the black p-branes of the supergravity effective theory, thus triggering the second superstring revolution and rapid advances in the non-perturbative understanding of string theory. When considering stacks of D-branes that coincide, the field theories living on the world-volume of these branes are described by non-abelian gauge theories, making them a prime choice for string phenomenology in order to recover the gauge group of the standard model. (For a review on D-branes see [66, 67]).

One might wonder if Dirichlet boundary conditions in the time direction make sense. As it turns out, after a Wick rotation, these objects play the role of instantons. In field theory, instantons are solutions of the Euclidean equations of motion with a finite action thus corresponding to saddle points of the Euclidean path integral. When expanding around these saddle points, instantons contribute as non-perturbative corrections to observables. Indeed, for gauge theories (and string theory), the coupling constant g enters with a weight of $e^{-\frac{1}{g^2}}$ which is not analytic at $g = 0$ and is therefore not captured by perturbation theory. Euclidean Dp-branes can contribute in Type II theories when their world-volume are along compact directions (a direct computation of instanton effects in Type IIA/B on Calabi-Yau threefolds was achieved recently [68, 69]). Timelike Dirichlet boundary conditions can also play a role in Lorentzian signature if one considers time-like T-duality, in which case D-branes are time-like T-dual to E-branes (Euclidean branes), which are not necessarily compact.

Coming back to our variational principle, after imposing the various boundary conditions, we finally obtain the equations of motion which take the form

$$\partial_\alpha P^\alpha_\mu = 0. \tag{2.18}$$

which might look complicated if written explicitly, but if we were to choose a coordinate system where $\dot{X} \cdot X' = 0, \dot{X}^2 = -1, X'^2 = 1$ then these would reduce to a simple wave equation. We are interested in getting the quantum theory of the relativistic string, however the presence of the square root in the action makes the process cumbersome, we therefore turn to the Polyakov action.

2.1.2 Polyakov action

One can write an action called the Polyakov action, classicaly equivalent to the Nambu-Goto action, that is more suitable for quantisation. The square root is eliminated by adding an auxiliary field $h_{\alpha\beta}$ into the action

$$S_P[X,h] = -\frac{T}{2}\int d^2\sigma\sqrt{h}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu}, \tag{2.19}$$

where $h_{\alpha\beta}$ has signature $(-+)$. This action, just like the Nambu-Goto action, exhibits spacetime Poincaré invariance and reparametrisation invariance on the worldsheet. However, this action exhibits a new local symmetry: Weyl invariance

$$h_{\alpha\beta}(\sigma) \to e^{2\Lambda(\sigma)}h_{\alpha\beta}(\sigma). \tag{2.20}$$

This symmetry is a special property of strings, and requiring that it holds in the quantum case will impose stringent constraints on our theory.

Once again, we obtain the equations of motion from the variational principle on the Polyakov action

$$\frac{1}{\sqrt{h}}\partial_\alpha\left(\sqrt{h}h^{\alpha\beta}\partial_\beta X^\mu\right) = 0, \tag{2.21}$$

$$\partial_\alpha X^\mu\partial_\beta X_\mu - \frac{1}{2}h_{\alpha\beta}h^{\gamma\delta}\partial_\gamma X^\mu\partial_\delta X_\mu = 0. \tag{2.22}$$

The same boundary conditions (periodic, Neumann, Dirichlet) as the Nambu-Goto action need to be imposed for boundary terms to vanish. The first equation is a two-dimensional wave equation on the Riemannian manifold $(\Sigma, h_{\alpha\beta})$ which can be reformulated as

$$\Box X^\mu = 0 \iff \nabla_\alpha\nabla^\alpha X^\mu = 0, \tag{2.23}$$

where ∇_α is the covariant derivative with respect to the worldsheet metric $h_{\alpha\beta}$.

The Polyakov action allows us to take an alternative viewpoint because it takes the form of a two-dimensional field theory of free massless scalar fields. Interpreted this way, Σ is a two-dimensional spacetime on which D scalar fields X live and take value in the *target space* $\mathbb{M}$. This is called the worldsheet point of view. Definining the energy-momentum tensor

$$T_{\alpha\beta} := -\frac{4}{T}\frac{1}{\sqrt{h}}\frac{\delta S_P}{h^{\alpha\beta}} = 2\partial_\alpha X^\mu\partial_\beta X_\mu - h_{\alpha\beta}h^{\gamma\delta}\partial_\gamma X^\mu\partial_\delta X_\mu, \tag{2.24}$$

we can reinterpret one of the equations of motion as

$$T_{\alpha\beta} = 0. \tag{2.25}$$

This is a constraint that has to be imposed on the solutions of the other equation of motion.

We can bring expressions to a standard form by performing a gauge fixing. Indeed, the metric $h_{\alpha\beta}$ has 3 independent components and three local symmetries (reparametrisation of two coordinates and Weyl transformations), suggesting that we can completely fix the form of the metric. This is true locally because any two-dimensional Riemannian metric can be written as the product of a flat metric and a conformal factor, which can be removed using a Weyl transformation

$$h_{\alpha\beta} \to e^{2\Omega(\sigma)}\eta_{\alpha\beta} \to \eta_{\alpha\beta}. \tag{2.26}$$

In this *conformal gauge* the equations of motion for the field X^μ are

$$\Box X^\mu = -(\partial_0^2 - \partial_1^2)X^\mu = 0. \tag{2.27}$$

We recognize the two-dimensional wave equation in flat space-time whose general solution is

$$X^\mu(\sigma) = X_L^\mu(\sigma^0 + \sigma^1) + X_R^\mu(\sigma^0 - \sigma^1), \tag{2.28}$$

describing decoupled left- and right-moving waves. Any solution of this equation needs to be supplemented by constraints in order to be a solution of string theory. The first constraint is the boundary conditions that were introduced earlier (periodic, Neumann, Dirichlet). The other constraint is the h equation $T_{\alpha\beta} = 0$ which now needs to be enforced by hand

$$T_{01} = T_{10} = 2\dot{X}X' = 0, \qquad T_{00} = T_{11} = \dot{X}^2 + X'^2 = 0. \tag{2.29}$$

For the Polyakov action the canonical momenta are

$$\Pi^\mu = \frac{\partial \mathcal{L}_P}{\partial \dot{X}_\mu} = T\dot{X}^\mu, \tag{2.30}$$

and the canonical Hamiltonian is

$$H_{can} = \int_0^\pi d\sigma^1(\dot{X}\Pi - \mathcal{L}_P) = \frac{T}{2}\int_0^\pi d\sigma^1(\dot{X}^2 + X'^2), \tag{2.31}$$

which clearly vanishes on-shell because of the constraints.

The general solution of the wave equations suggests to introduce *lightcone coordinates*

$$\sigma^\pm := \sigma^0 \pm \sigma^1. \tag{2.32}$$

In this coordinate system the wave equation takes the form

$$-4\partial_+\partial_- X^\mu = 0, \tag{2.33}$$

which makes it manifest that the general solution decomposes as independent left- and right-moving waves. The constraints become

$$T_{++} = 2\partial_+ X^\mu \partial_+ X_\mu = 0 \iff \dot{X}_L^2 = 0, \tag{2.34}$$

$$T_{--} = 2\partial_- X^\mu \partial_- X_\mu = 0 \iff \dot{X}_R^2 = 0. \tag{2.35}$$

2.1.3 Explicit solution

We are now ready to study the solutions in detail. In this section, we will focus on the case of periodic boundary conditions, the reason being as we will show later, that closed string excitations describe gravitational degrees of freedom and therefore play a central role in this work. The most general solution of the two-dimensional wave equation periodic in σ^1 can be parametrised as

$$X^\mu(\sigma) = x^\mu + L_S^2 p^\mu \sigma^0 + \frac{i}{2} L_S \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{-2in\sigma^-} + \frac{i}{2} L_S \sum_{n\neq 0} \frac{1}{n} \tilde{\alpha}_n^\mu e^{-2in\sigma^+}, \tag{2.36}$$

where $x^\mu, p^\mu \in \mathbb{R}$, $(\alpha_n^\mu)^* = \alpha_{-n}^\mu$ and $(\tilde{\alpha}_n^\mu)^* = \tilde{\alpha}_{-n}^\mu$. The *string length* L_S is defined as

$$L_S = \frac{1}{\sqrt{\pi T}} = \sqrt{2\alpha'}, \tag{2.37}$$

but we will from now on work in string units $L_S = c = \hbar = 1$ unless specified otherwise. α' is called the *Regge slope parameter* and it is standard to show dimensionful formulas using this quantity. We can compute the total momentum

$$P^\mu = T \int_0^\pi d\sigma^1 \dot{X}^\mu = p^\mu, \tag{2.38}$$

and the motion of the centre of mass

$$x_{CM}^\mu = \frac{1}{\pi} \int_0^\pi d\sigma^1 X^\mu(\sigma) = x^\mu + p^\mu \sigma^0, \tag{2.39}$$

which matches the world-line of a massive relativistic particle

$$x^\mu(\tau) = x^\mu(0) + \frac{dx^\mu}{d\tau}(0)\tau. \tag{2.40}$$

The physical interpretation of the parameters is now clear, p^μ is the total momentum of the string whose centre of mass behaves like a relativistic particle, which is a straight line in Minkowski spacetime in the free case. The motion of the relativistic string decomposes into two parts: a *zero mode* part corresponding to the motion of the centre of mass and the remaining terms that describe left- and right-moving waves.

In order to impose the constraint, we can evaluate the conserved charges. We recall that, on-shell, the energy momentum tensor is conserved which in light-cone coordinates gives

$$\partial_- T_{++} = 0, \quad \partial_+ T_{--} = 0 \tag{2.41}$$

Since the left and right-moving sectors are independent we focus on just T_{++}. Because of the periodic boundary condition we have that $T_{++}(\sigma^+ + \pi) = T_{++}(\sigma^+)$, therefore we can create infinitely many conserved currents by multiplying T_{++} with an arbitrary (smooth) periodic function since

$$\partial_-(f(\sigma^+)T_{++}) = 0, \tag{2.42}$$

whose corresponding conserved charge is

$$L_f = T \int_0^\pi d\sigma^1 f(\sigma^+) T_{++}. \tag{2.43}$$

The function f being periodic, we can expand it in a Fourier series with basis $\{e^{2im\sigma^+}|m \in \mathbb{Z}\}$ providing a basis $\{\tilde{L}_m|m \in \mathbb{Z}\}$ for the conserved charges. Being conserved, we can evaluate them at $\sigma^0 = 0$ and using the constraint $T_{\pm\pm} = 2(\partial_\pm X)^2$ we find

$$\tilde{L}_m := \frac{1}{4\pi} \int_0^\pi d\sigma^1 e^{2im\sigma^1} T_{++} = \frac{1}{2} \sum_{n=-\infty}^{\infty} \tilde{\alpha}_{m-n} \cdot \tilde{\alpha}_n, \tag{2.44}$$

$$L_m := \frac{1}{4\pi} \int_0^\pi d\sigma^1 e^{-2im\sigma^1} T_{--} = \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{m-n} \cdot \alpha_n, \tag{2.45}$$

where we have defined $\alpha_0^\mu = \tilde{\alpha}_0^\mu = \frac{1}{2} p^\mu$. The constraints $T_{\pm\pm} = 0$ imply that

$$L_m = \tilde{L}_m = 0. \tag{2.46}$$

The canonical Hamiltonian is

$$H = \frac{1}{2\pi} \int_0^\pi d\sigma^1 (\dot{X}^2 + X'^2) = L_0 + \tilde{L}_0 = \frac{1}{2} \sum_{n=-\infty}^{\infty} (\alpha_{-n} \cdot \alpha_n + \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n) = \frac{p^2}{4} + N + \tilde{N} = 0, \tag{2.47}$$

where we defined *total occupation numbers*

$$N = \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n, \quad \tilde{N} = \sum_{n=1}^{\infty} \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n. \tag{2.48}$$

This provides us with the *mass shell condition*

$$M^2 = -p^2 = 4(N + \tilde{N}). \tag{2.49}$$

The same type of analysis can be performed for Neumann and Dirichlet boundary conditions, as well as the case of a non-oriented string. Having investigated the classical relativistic string, it is now time to turn to the quantum theory.

2.2 Quantization of the bosonic string

We now want to turn to the quantum theory of the relativistic string. The most rigorous procedure is to apply the BRST formalism, but for our purposes old covariant quantization will be enough to access the relevant properties of the theory that we want to study.

2.2.1 The Fock space

We start by imposing the equal time commutation relations on the string coordinates

$$[X^\mu(\sigma^0,\sigma^1),\Pi^\nu(\sigma^0,\sigma'^1)] = i\eta^{\mu\nu}\delta_\pi(\sigma^1-\sigma'^1), \tag{2.50}$$

$$[X^\mu(\sigma^0,\sigma^1),X^\mu(\sigma^0,\sigma^1)] = [\Pi^\nu(\sigma^0,\sigma'^1),\Pi^\nu(\sigma^0,\sigma'^1)] = 0, \tag{2.51}$$

where

$$\delta_\pi(\sigma^1) = \frac{1}{\pi}\sum_{k=-\infty}^{\infty} e^{-2ik\sigma^1} = \delta_\pi(\sigma^1+\pi) \tag{2.52}$$

is the δ-function of period π. These can be shown to be equivalent to

$$[x^\mu,p^\nu] = i\eta^{\mu\nu}, \quad [\alpha^\mu_m,\alpha^\nu_n] = m\eta^{\mu\nu}\delta_{m+n,0}, \quad [\tilde{\alpha}^\mu_m,\tilde{\alpha}^\nu_n] = m\eta^{\mu\nu}\delta_{m+n,0} \tag{2.53}$$

with all other commutation relations vanishing. Since X^μ is hermitian we have

$$(x^\mu)^\dagger = x^\mu, \quad (p^\mu)^\dagger = p^\mu, \quad (\alpha^\mu_m)^\dagger = \alpha^\mu_{-m}, \quad (\tilde{\alpha}^\mu_m)^\dagger = \tilde{\alpha}^\mu_{-m}. \tag{2.54}$$

We see that we indeed get an infinite set of harmonic oscillators, so let us now turn to the construction of the Fock space. We start by defining the ground state $|0\rangle$ defined as

$$\alpha^\mu_n|0\rangle = 0, \quad \tilde{\alpha}^\mu_n|0\rangle = 0, \quad p^\mu|0\rangle = 0, \quad n>0. \tag{2.55}$$

The oscillator eigenstates are generated by acting on the ground state with creation operators. Assuming the vacuum is normalized as $\langle 0|0\rangle = 1$, the scalar products of two oscillator states is

$$(\alpha^\mu_{-m}|0\rangle, \alpha^\nu_{-n}|0\rangle) = \langle 0|\alpha^\mu_m\alpha^\nu_{-n}|0\rangle = m\eta^{\mu\nu}\delta_{m,n}. \tag{2.56}$$

We notice that this scalar product is not positive definite, but we still haven't imposed the constraints. We will explore how to impose the constraints in the quantum theory in the next section which will allow us to restrict the Fock space to the space of physical states $\mathcal{F}_{\text{phys}}$. By combining momentum and oscillator eigenstates we can write down a basis for the Fock space of a closed string

$$\mathcal{B} = \{\alpha^{\mu_1}_{-m_1}\alpha^{\mu_2}_{-m_2}\ldots\tilde{\alpha}^{\nu_1}_{-n_1}\tilde{\alpha}^{\nu_2}_{-n_2}\ldots|k\rangle \in \mathcal{F}|k\in\mathbb{R}^D, \mu_i,\nu_i = 0,...,D-1, m_j,n_j = 1,2,...\} \tag{2.57}$$

2.2.2 Imposing the constraints

The constraints in the classical theory are $L_m = \tilde{L}_m = 0$ and are imposed on solutions as initial conditions. In the quantum theory, the constraints select the *physical space* $\mathcal{F}_{phys} \subset \mathcal{F}$. Since $L^\dagger_m = L_{-m}$, it is sufficient to impose

$$L_m|\phi\rangle = 0, m>0 \implies |\phi\rangle \in \mathcal{F}_{phys} \subset \mathcal{F}. \tag{2.58}$$

The reason why we do not impose these constraints directly as an operator equation is the same reason as for the Gupta-Bleuler quantisation of QED. Indeed, imposing the constraint at the level of operators will spoil commutation relations, so instead we impose that the matrix elements between physical states vanish.

The case $m = 0$ needs to be treated separately because it has an ordering ambiguity. Indeed the normal ordered operator is

$$L_0^{\mathrm{NO}} = \frac{1}{2} : \alpha_{-n} \cdot \alpha_n := \frac{1}{8} p^2 + N, \tag{2.59}$$

while the classical ordering gives

$$L_0^{\mathrm{CO}} = \frac{1}{2} \sum_{-\infty}^{\infty} \alpha_{-n} \cdot \alpha_n = \frac{1}{8} p^2 + \frac{1}{2} \sum_{n=1}^{\infty} (\alpha_n \cdot \alpha_{-n} + \alpha_{-n} \cdot \alpha_n) \tag{2.60}$$

which entails

$$L_0^{\mathrm{CO}} - L_0^{\mathrm{NO}} = \frac{D}{2} \sum_{n=1}^{\infty} n. \tag{2.61}$$

We know that $L_0 + \tilde{L}_0$ is the worldsheet Hamiltonian so this signifies an ambiguity in the energy of the ground state. In QFT, normal ordering is imposed, because it renders a vanishing energy for the ground state which is the only consistent choice with the Poincaré invariance of Minkowski spacetime. However, the worlsheet has finite spatial extension and the energy of the ground state now depends on the volume, this is the *Casimir effect*. To take into account this effect, we introduce shift parameters a and $\tilde{a}$ such that we arrive at the following definition for physical states

$$|\phi\rangle \in \mathcal{F}_{\mathrm{phys}} \iff (L_0 - a)|\phi\rangle = (\tilde{L}_0 - \tilde{a})|\phi\rangle = 0, \tag{2.62}$$

$$L_m|\phi\rangle = \tilde{L}_m|\phi\rangle = 0, m > 0. \tag{2.63}$$

Actually this space is not positive definite but only positive semi-definite

$$\langle\phi|\phi\rangle \geq 0. \tag{2.64}$$

The reason is because there are non-trivial null states in $\mathcal{F}_{\mathrm{phys}}$, whose existence is related to the residual symmetry under conformal transformations which is not fixed in the conformal gauge. Therefore we can define an equivalence relation $\sim$, identifying states which differ only by a null state. Therefore the true physical space is

$$\mathcal{H} = \mathcal{F}_{\mathrm{phys}} / \sim \tag{2.65}$$

Finally, in order to ensure positive definiteness and unitarity in a Minkowski background, it can be shown that one must impose the following conditions: $D = 26$ and $a = \tilde{a} = 1$. This is the *no-ghost theorem* (see [70] for details on how to derive it in the old covariant formulation). Let us interpret physically the L_0 and $\tilde{L}_0$ constraints, imposing them on a physical state gives

$$(L_0 - 1)|\phi\rangle = 0 \implies \left(\frac{1}{8}p^2 + N\right)|\phi\rangle = |\phi\rangle \implies \frac{1}{8}k^2 + N = 1, \tag{2.66}$$

and identical for $\tilde{L}_0$. This gives

$$\frac{1}{8}M^2 = N - 1 = \tilde{N} - 1, \tag{2.67}$$

which can be rearranged as *the mass shell condition*

$$\frac{1}{4}M^2 = N + \tilde{N} - 2, \tag{2.68}$$

and the *level-matching condition*

$$N = \tilde{N}. \tag{2.69}$$

This constraint physically means that for a closed string, the left- and right-moving excitations contribute equally to the mass of the string. Actually, this constraint holds off-shell and is a consequence of translation invariance in σ^1 (see [3,71] for discussions on removing the constraint in a string field theoretic context).

2.2.3 Massless spectrum of the closed bosonic string

We are now ready to start studying the spectrum of states of the bosonic string. Before looking at the excited states, comments are in order with respect to the ground state. Indeed, according to the mass shell condition, the ground state is characterized by $N = \tilde{N} = 0$ and so has a negative mass squared $M^2 = -8$. This means that the vacuum of the closed bosonic string is tachyonic, signalling an instability, and that the perturbative expansion was not performed on the true vacuum of the theory. The fate of the open string tachyon was studied in detail using string field theoretic techniques (see [72] for a review). The case of the closed string tachyon is much less understood.
Nonetheless, we aim to have a theory with space-time fermions, which is achieved by considering *superstrings*. In these theories the tachyon is projected out of the spectrum so they do not suffer from tachyon instabilities. We will therefore pursue our study of the bosonic string, ignoring tachyon-related problems, as the lessons we can learn in this context can be transposed to the supersymmetric case.

The general form of a massless closed string state is

$$|\zeta, k\rangle = \zeta_{\mu\nu}\alpha^{\mu}_{-1}\tilde{\alpha}^{\nu}_{-1}|k\rangle. \tag{2.70}$$

The only constraints that impose additional conditions are

$$L_1|\zeta, k\rangle = \tilde{L}_1|\zeta, k\rangle = 0, \tag{2.71}$$

which lead to

$$k^\mu \zeta_{\mu\nu} = 0. \tag{2.72}$$

We can decompose this state into symmetric and anti-symmmetric parts $\zeta_{\mu\nu} = s_{\mu\nu} + b_{\mu\nu}$, where

$$s_{\mu\nu} = \zeta_{(\mu\nu)} = \frac{1}{2}(\zeta_{\mu\nu} + \zeta_{\nu\mu}), \quad b_{\mu\nu} = \zeta_{[\mu\nu]} = \frac{1}{2}(\zeta_{\mu\nu} - \zeta_{\nu\mu}). \tag{2.73}$$

We focus on the symmetric part for now, so the states

$$s_{\mu\nu}\alpha^\mu_{-1}\tilde{\alpha}^\nu_{-1}|k\rangle, \tag{2.74}$$

are physical if they satisfy

$$k^2 = 0, \quad k^\mu s_{\mu\nu} = 0. \tag{2.75}$$

A state with the following polarization

$$\sigma_{\mu\nu} = k_\mu \zeta_\nu + k_\nu \zeta_\mu, \quad k^\mu \zeta_\mu = 0, \tag{2.76}$$

is a null state. This induces the following residual gauge symmetry

$$s_{\mu\nu} \to s_{\mu\nu} + k_\mu \zeta_\nu + k_\nu \zeta_\mu, \tag{2.77}$$

where $k^\mu \zeta_\mu = 0$. Since the trace s^μ_μ is a Lorentz scalar, we decompose the polarization tensor into a traceless symmetric part and a scalar part. We choose an auxiliary null vector $\bar{k}$, linearly independent from k and we impose $k\bar{k} = 1$ in order to parametrize physical states. We get the following decomposition

1. The traceless part is

$$\psi_{\mu\nu} = s_{\mu\nu} - \frac{1}{D-2} s^\rho_\rho(\eta_{\mu\nu} - k_\mu \bar{k}_\nu - k_\nu \bar{k}_\mu). \tag{2.78}$$

2. The trace part is

$$\phi_{\mu\nu} = \frac{1}{D-2} s^\rho_\rho(\eta_{\mu\nu} - k_\mu \bar{k}_\nu - k_\nu \bar{k}_\mu). \tag{2.79}$$

We can check that this is a correct decomposition because

$$s_{\mu\nu} = \psi_{\mu\nu} + \phi_{\mu\nu}, \quad \eta^{\mu\nu}\psi_{\mu\nu} = 0, \quad \eta_{\mu\nu}\phi_{\mu\nu} = s^\rho_\rho, \tag{2.80}$$

with the trace part of $s_{\mu\nu}$ being physical as $k^\mu \phi_{\mu\nu} = 0$ and is not null. This field corresponds to a scalar field called *the dilaton*, we will come back to the role of this field in string theory in a moment. The physical state condition means that the tracless part is transversal. For a standard representative of the momentum for a massless field $k = (k^0, 0, ..., k^0)$ this gives

$$k^0 \psi_{00} + k^0 \psi_{0,D-1} = 0 \Rightarrow \psi_{0,D-1} = -\psi_{00}, \tag{2.81}$$

and so on. Taking into account the symmetry we can give the following matrix form

$$(\psi_{\mu\nu})=\begin{pmatrix} \psi_{00} & \psi_{01} & \psi_{02} & \dots & \psi_{0,D-2} & -\psi_{00} \\ \psi_{01} & \psi_{11} & \psi_{12} & \dots & \psi_{1,D-2} & -\psi_{01} \\ \vdots & & & & & \vdots \\ \psi_{0,D-2} & \psi_{1,D-2} & \psi_{2,D-2} & \dots & \psi_{D-2,D-2} & -\psi_{0,D-2} \\ -\psi_{00} & -\psi_{01} & -\psi_{02} & \dots & -\psi_{0,D-2} & \psi_{00} \end{pmatrix}.$$

We can remove the null part according to (2.76) in which case we are left with the transverse part representing the physical states

$$(\psi_{\mu\nu}^{\text{transv}})=\begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & \psi_{11} & \psi_{12} & \dots & \psi_{1,D-2} & 0 \\ \vdots & & & & & \vdots \\ 0 & \psi_{1,D-2} & \psi_{2,D-2} & \dots & \psi_{D-2,D-2} & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix},$$

with $\psi_{11} + \psi_{22} + \dots + \psi_{D-2,D-2} = 0$.

The total number of states is decomposed as follows

$$\frac{1}{2}D(D+1) - 1 = \left(\frac{1}{2}(D-2)(D-1) - 1\right) + (D-1) + D \tag{2.82}$$

We can interpret it the following way:

- D unphysical components are removed by the physical condition $k^\mu s_{\mu\nu} = 0$.
- $D-1$ degrees of freedom are removed because of the residual gauge invariance, parametrized by the vector ζ_μ subject to $k^\mu \zeta_\mu = 0$.
- We are left with $\frac{1}{2}(D-2)(D-1) - 1$ physical states represented by $(\psi_{\mu\nu}^{\text{transv}})$

The field $\psi_{\mu\nu}$ describes a massless symmetric tensor which corresponds in $D = 4$ to a massless spin-2 particle with helicity eigenstates $h = \pm 2$. To show that this state has the same kinematic properties as a graviton, one needs to study a linearisation of the vacuum Einstein equations, we therefore refer the reader to [73] for details.

Let us now turn to the antisymmetric part of the massless closed string state which will be of the form

$$b_{\mu\nu}\alpha_{-1}^{\mu}\tilde{\alpha}_{-1}^{\nu}|k\rangle, \tag{2.83}$$

which is physical if

$$k^2 = 0, \quad k^\mu b_{\mu\nu} = 0. \tag{2.84}$$

The null states are given by

$$\beta_{\mu\nu} = k_\mu \zeta_\nu - k_\nu \zeta_\mu, \quad \text{where } k^\mu \zeta_\mu = 0. \tag{2.85}$$

The decomposition of states can be done in a similar fashion as previously

$$\frac{1}{2}D(D-1) = \frac{1}{2}(D-2)(D-3) + (D-2) + (D-1). \tag{2.86}$$

This decomposition is understood the following way:

- $(D-1)$ unphysical components are removed by the physical condition $k^\mu b_{\mu\nu} = 0$ (one is trivially satisfied when $\mu = \nu$ because of the anti-symmetry of $b_{\mu\nu}$).
- $(D-2)$ degrees of freedom removed because of residual gauge invariance. As before $k^\mu \zeta_\mu = 0$ fixes one component, and another one can be fixed using the fact the the gauge invariance is itself gauge invariant under $\zeta_\mu \to \zeta_\mu + Ck_\mu$ which can be used to fix another component.
- We are left with $\frac{1}{2}(D-2)(D-3)$ physical states.

This antisymmetric tensor field is known as the Kalb-Ramond field. Its importance comes from the fact that the fundamental string is charged under this field. When we will introduce T-duality, we will see that this field gets mixed with the metric. In 4 dimensions the Kalb-Ramond field can be dualized, in the sense of Hodge duality, into a scalar where it is then named the *universal string axion*. This fact will be of importance when we will derive a 4D version of Buscher rules in Chapter 5.

Finally we turn to the trace part whose corresponding physical field is *the dilaton* ϕ. This field plays a crucial role in string theory as its vacuum expectation value (vev) determines the coupling constant of the theory. To see this we study the following action

$$S[g_{\mu\nu}, \phi] = \frac{1}{2\kappa^2} \int d^D x \sqrt{g} e^{-2\phi} \left(R + 4\partial_\mu \phi \partial^\mu \phi\right), \tag{2.87}$$

which is a string effective action describing the dilaton coupled to gravitons. The action is given in the *string frame* so the Einstein-Hilbert term has an extra factor $e^{-2\phi}$ compared to the usual *Einstein frame* metric.

The equations of motion are solved for $g_{\mu\nu} = \eta_{\mu\nu}$ and arbitrary constant dilaton $\phi = \phi_0$. This means that Minkowski space is not a unique solution, but actually a 1 parameter family of solutions parametrized by the vev of the dilaton. Massless scalar fields with arbitrary vevs are called *moduli*, and the space of inequivalent ground states is *the moduli space of vacua*.

We observe that shifting the dilaton $\phi \to \phi + \phi_0$ is equivalent to rescaling the gravitational coupling $\kappa \to \kappa e^{\phi_0}$. Therefore, as long as we do not specify the vev of the dilaton, κ has no physical meaning. We can therefore define κ_0 to be the value at which the gravitational coupling equals the string scale parameter α'. By dimensional analysis we have

$$\kappa_0 = (\alpha')^{(D-2)/4}. \tag{2.88}$$

We can now parametrise different values of the coupling constant using the vev ϕ_0 of the dilaton

$$\kappa = (\alpha')^{(D-2)/4} e^{\phi_0}. \tag{2.89}$$

We will not give a detailed account of how interactions and amplitudes are described in string theory. For this we refer the reader to any standard textbook on string theory. What is relevant for us to know is that the scattering of closed oriented strings are represented by connected oriented surfaces, and there is a unique basic interaction describing the splitting of a closed string or the fusion of two.

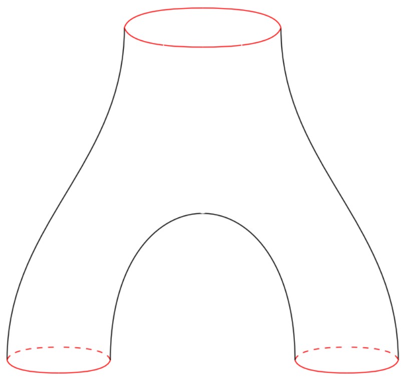


Figure 2.1: The closed string vertex.

When performing the path integral, the sum over inequivalent surfaces includes a sum over topologies. In the closed string case, the topology of the surface is encoded by the Euler characteristic $\chi = 2 - 2g$ where g is the genus of the surface. Increasing the genus of the worldsheet by one means that the surface has two new vertices so a surface of genus g and M external states has $M - \chi(g)$ vertices, and therefore carries a factor $\kappa^{M-\chi(g)}$, where κ is the string coupling constant (we give it the same name as the gravitational one because considering graviton scattering allows us to identify them).

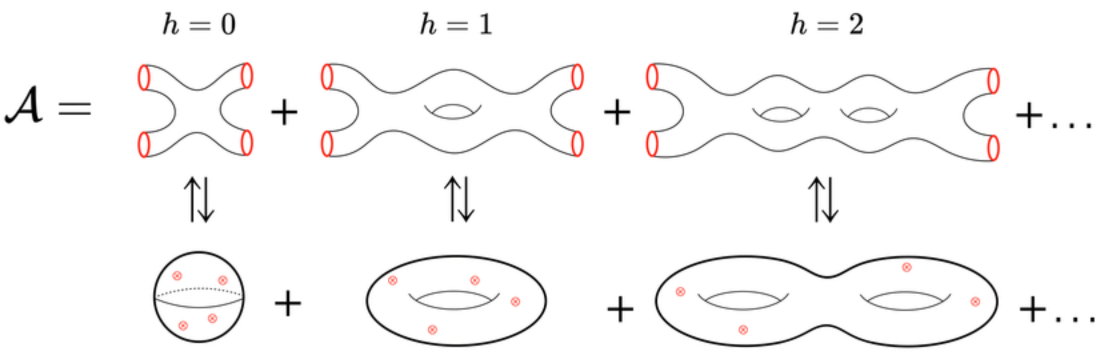


Figure 2.2: Genus h expansion of a four-point amplitude. The boundary circles are mapped to points thanks to the conformal structure of the surfaces.

Now exploiting equation (2.89), an M-particle string amplitude can be written as $e^{(M-\chi(g))\phi_0}$ and we indeed see that we can define the dimensionless string coupling constant as $g_S = e^{\phi_0}$ thus justifying the claim that the dilaton vev encodes the perturbative expansion of string scattering amplitudes.

2.2.4 Curved backgrounds

So far we have only discussed string theory in a flat Minkoswki background. We now turn to string theory in a curved background. The Polyakov action is now

$$S[X,h|G] = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{h}h^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu G_{\mu\nu}(X). \tag{2.90}$$

This action describes a map between the worldsheet with metric $h_{\alpha\beta}$ to the curved target spacetime with background metric $G_{\mu\nu}$. Ultimately, since the closed string has a graviton, we expect this background to actually be dynamical. It can also be shown that the background can be constructed as a coherent state of gravitons.
In what follows, we will consider the fully generalized Polyakov action, where the string is also coupled to a background B-field and a background dilaton

$$S[X,h|G,B,\phi] = S[X,h|G] + S[X,h|B] + S[X,h|\phi]. \tag{2.91}$$

The coupling with the B-field is described by a two-dimensional Wess-Zumino term

$$S[X,h|B] = -\frac{1}{4\pi\alpha'}\int d^2\sigma\epsilon^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu B_{\mu\nu}(X). \tag{2.92}$$

The coupling with the dilaton is

$$S[X,h|\phi] = \frac{1}{4\pi}\int d^2\sigma\sqrt{h}R_h\phi(X), \tag{2.93}$$

where R_h is the world-sheet Ricci scalar. For a constant dilaton background $\phi = \phi_0$, we can perform the integral explicitly. Indeed, due to the Gauss-Bonnet theorem, the Einstein-Hilbert action in 2 dimensions is purely topological. We therefore find that

$$\frac{1}{4\pi}\int d^2\sigma\sqrt{h}R_h\phi_0 = \chi(\Sigma)\phi_0 = (2-2g)\phi_0. \tag{2.94}$$

In the path integral, this term contributes to $e^{-S[X,h|\phi]} = e^{-(2-2g)\phi_0}$ which shows again that the vev of the dilaton accounts for the genus dependance in scattering amplitudes.

When imposing the conformal gauge, the theory is not free as it was in the Minkoswki case, it is therefore non-trivial to check that the theory is renormalisable and conformal. The renormalization procedure typically introduces an energy scale which breaks conformal invariance. However in string theory conformal symmetry is gauged so it would be anomalous and thus inconsistent. We need to understand under which circumstances conformal invariance is preserved in the quantum theory. The energy scale μ dependence of the coupling constants g is encoded in the β-functions as

$$\beta^{(g)} = \frac{\partial g}{\partial \log \mu} = \mu \frac{\partial g}{\partial \mu}. \tag{2.95}$$

In order to be scale invariant, all the β-functions need to vanish and it was shown by Polchinski that in 2D, combined with Poincaré invariance and unitarity, this implies conformal invariance [74] (To be accurate we only require *local* Weyl invariance, we also actually have functionals since the couplings are field dependent so we now write them $\bar{\beta}$-functionals). The conditions to preserve conformal invariance at the quantum level are therefore

$$\bar{\beta}[G_{\mu\nu}(X)] = \bar{\beta}[B_{\mu\nu}(X)] = \bar{\beta}[\phi(X)] = 0. \tag{2.96}$$

These equations can be evaluated in α' perturbation theory and we obtain

$$\alpha' R^G_{\mu\nu} + 2\alpha' \nabla_\mu \nabla_\nu \phi - \frac{\alpha'}{4} H_{\mu\lambda\omega} H_\nu{}^{\lambda\omega} + \mathcal{O}(\alpha'^2) = 0, \tag{2.97}$$

$$-\frac{\alpha'}{2} \nabla^\omega H_{\omega\mu\nu} + \alpha' \nabla^\omega \phi H_{\omega\mu\nu} + \mathcal{O}(\alpha'^2) = 0, \tag{2.98}$$

$$-\frac{\alpha'}{2} \nabla^2 \phi + \alpha' \nabla_\omega \phi \nabla^\omega \phi - \frac{\alpha'}{24} H_{\mu\nu\lambda} H^{\mu\nu\lambda} + \mathcal{O}(\alpha'^2) = 0. \tag{2.99}$$

We see for example, that at the lowest level the first equations give us Einstein equations with the B-field and dilaton as sources.

More generally, all the $\bar{\beta}$-functional equations enforce that the background fields satisfy their equation of motions. In other words, the quantum consistency of the theory enforces the background fields to be on-shell. All these equations of motion can be derived from the following effective action

$$S_{\text{eff}} = \frac{1}{2\kappa^2} \int d^D x \sqrt{G} e^{-2\phi} \left(R^G + 4\partial_\mu \phi \partial_\nu \phi - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \right). \tag{2.100}$$

Field configurations that solve the $\bar{\beta}$-equations are said to be *consistent string backgrounds*. The simplest solution is

$$G_{\mu\nu} = \eta_{\mu\nu}, \quad B_{\mu\nu} = 0 \quad \phi = \text{const}. \tag{2.101}$$

This is an exact solution at all orders in α' since this is exactly the Polyakov action which defines a conformal field theory. If we only impose $B_{\mu\nu} = 0$ and $\phi = \text{const}$ then we have

$$\bar{\beta}[G_{\mu\nu(X)}] = 0 \Rightarrow \alpha' R_{\mu\nu} + \frac{1}{2} \alpha'^2 R_{\mu\alpha\beta\gamma} R_\nu{}^{\alpha\beta\gamma} + \mathcal{O}((\alpha')^3) = 0. \tag{2.102}$$

It is clear that at first order in α', string theory recovers the vacuum Einstein equations. But we now see that string theory also provides higher order curvature corrections to

Einstein equations. (Interestingly, in the Type II theories that we will introduce later on, the α'^2 corrections are absent and the first corrections appear at order α'^3).
We have seen that consistency requires the spacetime dimension to be $D = 26$ for the bosonic string. However the universe has 4 large dimensions. To make contact with the real world, it is standard to *compactify* the extra dimensions, which means that the spacetime manifold takes the form $\mathbb{M} \times X$ where X is a compact manifold whose size is smaller than the length scales we can currently probe today. The first $\bar{\beta}$-equation tells us that the spacetime must be Ricci-flat (when there are no fluxes). But besides Minkoswki, there is a large number of other Ricci-flat solutions (alluding to the vastness of the string theory landscape).
The simplest choice is to just consider a torus where the extra dimensions are periodically identified. This simple example already exhibits many interesting properties and in the next section we will explore one of them, T-duality, in detail. In the context of the superstrings (where $D = 10$), a prime choice for the compact space are *Calabi-Yau* manifolds. The reason is that the resulting lower dimensional theory preserves some of the supersymmetry and therefore allows one to have better analytic control of the resulting theory. A description of Calabi-Yau manifolds will be given in Chapter 3.

The requirement that $D = 26$ is actually not true in full generality. To see this, we relax this condition and study again the $\bar{\beta}$-equation of the dilaton which now takes the form

$$\bar{\beta}(\phi) = \frac{D-26}{6} - \frac{\alpha'}{2}\nabla_\mu\nabla^\mu\phi + \alpha'\nabla_\mu\phi\nabla^\mu\phi - \frac{\alpha'}{24}H_{\mu\nu\rho}H^{\mu\nu\rho} + \mathcal{O}(\alpha'^2) = 0. \tag{2.103}$$

We can see that the Weyl anomaly arises as the leading order term in the dilaton's $\bar{\beta}$-equation and the effective action now becomes

$$S_{\text{eff}} = \frac{1}{2\kappa^2}\int d^Dx\sqrt{G}e^{-2\phi}\left(R^G - \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} + 4\partial_\mu\phi\partial_\nu\phi - \frac{2(D-26)}{3\alpha'}\right). \tag{2.104}$$

This method for getting $D \neq 26$ world is however not straightforward because Minkoswki spacetime is no longer a solution of the theory. There are also reasons to expect the perturbative regime to be less under control. The best way to understand how we can get non-critical string theories is to look at the BRST quantisation. Indeed, each string coordinate carries a central charge $c = 1$ but the Faddeev-Popov ghosts that arise when gauge fixing carry a charge $c_{\text{ghost}} = -26$ and therefore the consistency requirement is really $c_{\text{total}} = c + c_{\text{ghost}} = 0$. Therefore we have two choices:

- critical string theory where the string coordinates CFT has $c = 26$ and thus cancels the conformal anomaly.
- non-critical string theory where $c \neq 26$ where the conformal mode now becomes a dynamical field. This is what is known as Liouville string theory.

2.3 String compactifications

In this thesis, we work in the framework of the critical string, and we adopt a *geometric approach* to extra dimensions (in contrast to a CFT/algebraic approach). By this we mean that we will consider compactifications of the higher dimensional theory on a compact manifold. For simplicity, we are going to study the case of a circle compactification of a single spacetime coordinate. This simple example is enough to explore many properties of string compactifications, including T-duality. In a nutshell, T-duality can be understood as a target-space duality, we will show that by considering theories defined on two different backgrounds, they are actually physically indistinguishable.

2.3.1 Spectrum of the closed strings on $\mathcal{S}^1$

In this subsection we study the worldsheet theory of a closed string on a background of the form

$$\mathcal{M}_{D-1} \times S^1_R, \tag{2.105}$$

where R is the radius of the circle. In later chapters, we will explore this from the spacetime and effective actions point of view. We denote the internal coordinate by X subject to

$$X \simeq X + 2\pi R. \tag{2.106}$$

Thus the string is closed if

$$X(\sigma^0, \sigma^1 + \pi) = X(\sigma^0, \sigma^1) + 2\pi R m, \quad m \in \mathbb{Z}. \tag{2.107}$$

The parameter m is known as the *winding number* and says how many times does the string wind around the circle, the sign accounting for direction. As is standard from quantum mechanics, a periodic direction implies that momentum is quantized in this direction, indeed

$$e^{ikX} \simeq e^{ik(X+2\pi Rm)} \to k \cdot 2\pi R m \in 2\pi\mathbb{Z} \to k = \frac{n}{R}, n \in \mathbb{Z}. \tag{2.108}$$

The parameter n is the *momentum number*.

The expansion of the coordinates X^μ for $\mu = 0, ..., D-2$ does not change from the previous section, however the internal coordinate gets modified

$$X(\sigma^0, \sigma^1) = x + p\sigma^0 + 2p\sigma^0 + 2w\sigma^1 + \dots, \tag{2.109}$$

Defining *left-* and *right-moving* momenta as

$$p_L = \frac{1}{2}p + w, \quad p_R = \frac{1}{2}p - w, \tag{2.110}$$

the mode expansion becomes

$$X(\sigma^+,\sigma^-) = x + p_L\sigma^+ + p_R\sigma^- + \dots . \tag{2.111}$$

so we can express the compact coordinate entirely in terms of left and right movers

$$X(\sigma^+,\sigma^-) = X_L(\sigma^+) + X_R(\sigma^-). \tag{2.112}$$

Splitting the operator x into left- and right-moving parts we have

$$[x_L, p_L] = i = [x_R, p_R]. \tag{2.113}$$

We write k_L and k_R the eigenvalues of p_L and p_R and take the values

$$k_L = \frac{1}{2}k + w, \quad k_R = \frac{1}{2}k - w, \quad \text{where } k = \frac{n}{R}, \quad w = mR, \tag{2.114}$$

so that momentum eigenstates are written as

$$e^{ik_\mu x^\mu} e^{ik_L x_L} e^{ik_R x_R} |0\rangle = |k^\mu, (k_L, k_R)\rangle = |k^\mu, (k, w)\rangle = |k^\mu, (n, m), R\rangle. \tag{2.115}$$

We can therefore write a generic state of the Fock space as

$$\alpha^{\mu_1}_{-m_1} \dots \alpha_{-k_1} \dots \tilde{\alpha}^{\nu_1}_{-n_1} \dots \tilde{\alpha}_{-l_1} \dots |k^\mu, (k_L, k_R)\rangle, \quad m_i, n_i, k_j, l_j > 0. \tag{2.116}$$

Rederiving the mass formula, we now have contributions from internal momenta and the level-matching formula also gets modified and we find

$$M^2 = 4\left(N + \tilde{N} + \frac{1}{2}k_L^2 + \frac{1}{2}k_R^2 - 2\right), \tag{2.117}$$

$$N + \frac{1}{2}k_L^2 = \tilde{N} + \frac{1}{2}k_R^2, \tag{2.118}$$

which can be expressed as

$$M^2 = \frac{2}{\alpha'}(N + \tilde{N} - 2) + \frac{n^2}{R^2} + \frac{m^2R^2}{\alpha'^2}, \tag{2.119}$$

$$N - \tilde{N} = mn, \tag{2.120}$$

where we have reintroduced the α' parameter. We see that the level-matching condition does not depend on R so the spectrum does not change when we vary continuously the value of R. We now see a new *modulus* appear, the radius R of the circle. We could naively think that the moduli space is

$$\mathcal{M}_{S^1} = R \in]0, \infty[\tag{2.121}$$

We will see in a moment that T-duality manifests itself by making these circles not all distinct. For now, we want to look at the spectrum of massless states. As usual there is the issue of the tachyon to deal with in the bosonic string, but we are interested in the properties shared with the superstring. The states that are massless for all values of R are necessarily states with $N = \tilde{N} = 1$ and they have no winding or internal momentum $m = n = 0$. We therefore have the following states and their corresponding spacetime fields:

- $\alpha_{-1}^{\mu}\tilde{\alpha}_{-1}^{\nu}|(0,0),R\rangle \rightarrow$ these are the states we've studied before and correspond to the metric, B-field and dilaton.

- $(\alpha_{-1}^{\mu}\tilde{\alpha}_{-1}+\alpha_{-1}\tilde{\alpha}_{-1}^{\mu})|(0,0),R\rangle \rightarrow$ the corresponding spacetime field V_{μ} is a $U(1)$ gauge field called the Kaluza-Klein vector and comes from the higher dimensional metric.

- $(\alpha_{-1}\tilde{\alpha}_{-1}^{\nu}-\alpha_{-1}^{\nu}\tilde{\alpha}_{-1})|(0,0),R\rangle \rightarrow$ the corresponding spacetime field A_{ν} is also a $U(1)$ gauge field called the winding vector and comes from the higher dimensional B-field.

- $\alpha_{-1}\tilde{\alpha}_{-1}|(0,0),R\rangle \rightarrow$ the corresponding spacetime field σ is a scalar field called the Kaluza-Klein scalar and comes from the higher dimensional metric. This scalar field is actually a modulus that parametrizes the size of the compactification circle.

To see why the KK scalar parametrizes the size of the circle, we consider the physical radius that can actually be measured which we call ρ which we obtain when computing the length $l = 2\pi\rho$ of the circle using the background metric (see Chapter 3 for more details on the Kaluza-Klein ansatz):

$$l = \int_0^{2\pi R} dx e^{\langle\sigma\rangle} = 2\pi R e^{\langle\sigma\rangle} = 2\pi\rho. \tag{2.122}$$

We can always compensate a change of the parametric radius R by a change of the vev $\langle\sigma\rangle$ and vice versa. To parametrize distinct circles we can either fix R and vary $\langle\sigma\rangle$, or fix $\langle\sigma\rangle$ and vary R. For the time being, we will fix $\langle\sigma\rangle$ and use R, though it is useful to keep in mind that changing R amounts to changing the vev of the Kaluza-Klein scalar σ.

Besides these generic massless states, there are others that become massless at specific points in the moduli space, which means for specific values of R. By inspecting the mass formula (we reintroduce α') we can first look at winding modes with $m \neq 0$ (which enforces $n = 0$ according to level-matching) and the mass of these states is

$$M^2 = \left(\frac{mR}{\alpha'}\right)^2 - \frac{4}{\alpha'} \tag{2.123}$$

which are massless when $R^2 = 4\alpha'/m^2$. Similarly, we can consider KK modes of the tachyon with no winding ($m = 0$) which have mass

$$M^2 = \frac{n^2}{R^2} - \frac{4}{\alpha'} \tag{2.124}$$

which are massless when $R^2 = n^2\alpha'/4$.

There is a very specific value of the radius where we get the richest spectrum of massless states. This is known as the *self-dual radius* $R = \sqrt{\alpha'}$ for reasons that will become clear once we introduced T-duality. At this specific radius the solutions of the level matching condition with $M^2 = 0$ are now given by

- $N = \tilde{N} = 1$ with $m = n = 0$ that we have described above
- $N = \tilde{N} = 0$ with $n = \pm 2$ and $m = 0$. These are KK modes of the tachyon which correspond to spacetime scalars with charges $(\pm 2, 0)$ under the $U(1) \times U(1)$ gauge symmetry.
- $N = \tilde{N} = 0$ with $n = 0$ and $m = \pm 2$. These are winding modes of the tachyon which correspond to spacetime scalars with charges $(0, \pm 2)$ under $U(1) \times U(1)$.
- $N = 1$ and $\tilde{N} = 0$ with $n = m = \pm 1$. These are two new vector fields $\alpha^{\mu}_{-1}|0, p\rangle$. They carry charge $(\pm 1, \pm 1)$ under $U(1) \times U(1)$.
- $N = 0$ and $\tilde{N} = 1$ with $n = -m = \pm 1$. These are two further vector fields $\tilde{\alpha}^{\mu}_{-1}|0, p\rangle$. They carry charge $(\pm 1, \mp 1)$ under $U(1) \times U(1)$.

Looking at the charges, we see that at the self dual radius the theory develops an enhanced gauge symmetry

$$U(1) \times U(1) \to SU(2) \times SU(2). \tag{2.125}$$

We therefore have four extra massless charged vector bosons under $U(1) \times U(1)$. These fit into the adjoint representation $(\mathbf{3}, \mathbf{0}) \oplus (\mathbf{0}, \mathbf{3})$ of the group $SU(2) \times SU(2)$ which indicates an enhanced, non-abelian gauge symmetry at this point. This can point towards interpreting T-duality for arbitrary radii as the remnant after symmetry breaking of a gauge symmetry.

2.3.2 T-duality

Looking more closely at the mass formula

$$M^2 = \frac{2}{\alpha'}(N + \tilde{N} - 2) + \frac{n^2}{R^2} + \frac{m^2 R^2}{\alpha'^2}, \tag{2.126}$$

we can see that it is invariant under the exchange of momenta and windings

$$m \leftrightarrow n \tag{2.127}$$

as long as we also invert the radius of the circle in string units

$$R \to \frac{\sqrt{\alpha'}}{R}. \tag{2.128}$$

A string moving along a circle of radius R has the same spectrum as a string moving along a circle of radius $R' = \frac{\sqrt{\alpha'}}{R}$. This means that if we want to specify the radius of the circle, we first have to identify which modes are windings and which are momenta. For large radii $R \gg \sqrt{\alpha'}$, classical geometry is valid and the initial identification is correct. However, for radii $R \ll \sqrt{\alpha'}$, it becomes more natural to reinterpret the winding modes

as momentum modes and vice versa, and now the limit $R \to 0$ is reintrepreted as an alternative decompactification limit where $R' = \frac{\sqrt{\alpha'}}{R} \to \infty$. In this limit, the winding modes are reinterpreted as momentum modes and the original momentum modes become infinitely heavy and decouple.
In the stringy regime where $R \approx \sqrt{\alpha'}$ the tower of momentum states and winding states become comparable in mass, and the geometric interpretation becomes ambiguous. Since small and large radii are related by symmetry, the string length $\sqrt{\alpha'}$ can be interpreted as a minimal length scale of the theory. It is crucial to realise that this duality relies on the existence of the winding modes, whence making it a purely stringy effects because we do not see such a phenomenon in field theory, where particles can not wind around the compact direction.
It can be shown that T-duality is still valid for the full conformal field theory on the worldsheet, thus making it an exact perturbative duality for the full interacting theory. In position space, T-duality acts as a "chiral reflection"

$$X_L \to X_L \quad \text{and} \quad X_R \to -X_R. \tag{2.129}$$

This transformation does not have an interpretation in classical Riemannian geometry, therefore in order to describe strings on toroidal backgrounds, a generalization is needed. Two related ideas have been put forward in order to do this

- *Generalized geometry* is a framework where the tangent bundle T of a manifold is replaced by $T \oplus T^*$ (T^* being the cotangent bundle) and the Lie bracket is replaced by a Courant bracket (see [75] for a review).
- *Doubled Geometry* is a framework where the whole spacetime manifold is doubled. The effective field theory defined on such geometries, dubbed *Double Field Theory* is then able to capture the winding modes as degrees of freedom which are treated on an equal footing as the momentum modes (see [76, 77] for reviews).

Let's take a moment to look at the behavior of an open string under T-duality. We saw that T-duality as a chiral reflection acts as

$$X(\sigma^0,\sigma^1) = X_L(\sigma^0+\sigma^1)+X_R(\sigma^0-\sigma^1) \to X'(\sigma^0,\sigma^1) = X_L(\sigma^0+\sigma^1)-X_R(\sigma^0-\sigma^1), \tag{2.130}$$

which entails

$$\partial_0 X = \partial_1 X'. \tag{2.131}$$

This means that under T-duality, Neumann and Dirichlet boundary conditions are exchanged. Dualizing a circle transverse to a Dp-brane turns it into a D(p+1) brane in the dual theory. Conversely, a Dp-brane wrapped around a circle gives a D(p-1)-brane in the dual theory. This is actually how D-branes were first discovered, by studying T-duality in the open string sector [64, 65].

Here, we could derive the T-duality transformation rules from the worldsheet perspective of the massless background fields known as the *Buscher rules*. However we will be interested in deriving these rules in the Type II superstring case including the Ramond sector as well as timelike isometries. We therefore refer to appendix B for a derivation of generalized Type II Buscher rules from a spacetime perspective.

2.4 Superstring theory

As we have mentioned already, the bosonic string can not be the end of the story. First of all, we want the spectrum to include fermionic degrees of freedom in order to describe the matter content of the universe. Second of all, we saw that the bosonic string has a tachyonic vacuum rendering the theory unstable. To answer these problems, we will now introduce the superstring. Many aspects of the superstring are similar to what was done for the bosonic string. We will therefore not give a detailed account and only highlight the main properties and novelties, and focus particularly on the Type II theories, which are the most relevant ones for the present work.

2.4.1 The Ramond-Neveu-Schwarz string

We will start by generalizing the Polyakov action in conformal gauge to make it supersymmetric. We will not introduce supersymmetry here, and refer the reader to Chapter 3 for a more detailed account. The resulting world-sheet theory is therefore a two-dimensional supersymmetric field theory. We recall that from the worldsheet point of view, the string coordinates X^μ are scalar fields which transform under a global internal $SO(1, D-1)$ symmetry. Therefore the natural choice if we want fermions is to add two dimensional-spinor fields ψ^μ which also transform as $SO(1, D-1)$ vectors. In 2D Minkowski space, the gamma matrices can be chosen to be real as

$$\gamma^0 = \left((\gamma^0)_a{}^b\right) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^1 = \left((\gamma^1)_a{}^b\right) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \{\gamma^\alpha, \gamma^\beta\} = 2\eta^{\alpha\beta}. \tag{2.132}$$

In 2D, we can impose a reality and chirality condition at the same time on the spinors thus making them Majorana-Weyl spinors. In order to generalize the gauge-fixed Polyakov action, it is natural to add the action for massless spinors, whence giving us the *Ramond-Neveu-Schwarz action*

$$S_{RNS} = -\frac{1}{2\pi}\int d^2\sigma \left(\partial_\alpha X^\mu \partial^\alpha X_\mu + i\bar{\psi}^\mu \gamma^\alpha \partial_\alpha \psi_\mu\right). \tag{2.133}$$

This action is invariant under the following two-dimensional supersymmetry transformations

$$\delta X^\mu = i\bar{\epsilon}\psi^\mu, \quad \delta\psi^\mu = \gamma^\alpha \partial_\alpha X^\mu \epsilon, \tag{2.134}$$

where $\epsilon = (\epsilon_a)$ is the supersymmetry parameter. As usual we apply the variational principle on this action and we get the following equations of motion

$$\Box X^\mu = 0, \quad i\gamma^\alpha \partial_\alpha \psi^\mu = 0. \tag{2.135}$$

We still have our usual wave equation for the string coordinates, and we see that the spinors satisfy a massless version of the Dirac equation which, written explicitly, is

$$i \begin{pmatrix} 0 & -\partial_0 + \partial_1 \\ \partial_0 + \partial_1 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.136}$$

By writing $\psi_- := \psi_1$ and $\psi_+ := \psi_2$ we can rewrite this in the following illuminating way

$$\partial_+ \psi_- = 0, \quad \partial_- \psi_+ = 0. \tag{2.137}$$

The purely positive chirality spinors are therefore purely left moving $\psi_+ = \psi_+(\sigma^+)$ while negative chirality spinors are purely right-moving $\psi_- = \psi_-(\sigma^-)$. We can therefore think of $\psi_\pm$ as the supersymmetric partners of the left- and right-moving parts of the string coordinates X^μ. We can now derive the conserved currents, we again have an energy momentum tensor associated with reparametrisation invariance

$$T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{i}{4}\bar{\psi}^\mu \gamma_\alpha \partial_\beta \psi_\mu - \frac{i}{4}\bar{\psi}^\mu \gamma_\beta \partial_\alpha \psi_\mu - \text{Trace}, \tag{2.138}$$

where the Trace term are further terms rendering the current traceless. We now also have a current associated with the supersymmetry which takes the form

$$J_\alpha = -\frac{1}{2}\gamma^\beta \gamma_\alpha \psi^\mu \partial_\beta X_\mu. \tag{2.139}$$

The resulting worldsheet theory is conformally invariant.

The systematic way to study the superstring is to supersymmetrise the Polyakov action *before* gauge fixing. The resulting action is invariant under *local* supersymmetric transformations. Theories of this kind are called *supergravity* theories, and will be the subject of the next chapter. For our purposes, it is enough to know that upon imposing the *superconformal gauge* $h_{\alpha\beta} = \eta_{\alpha\beta}, \psi_\alpha = 0$, the locally supersymmetric action reduces to the RNS action with the additional constraints

$$T_{\alpha\beta} = 0, \quad J_\alpha = 0. \tag{2.140}$$

These currents satisfy conservation equations

$$\partial^\alpha T_{\alpha\beta} = 0, \quad \partial^\alpha J_\alpha = 0 \tag{2.141}$$

and

$$\eta^{\alpha\beta} T_{\alpha\beta} = 0, \quad \gamma^\alpha J_\alpha = 0. \tag{2.142}$$

which means that the gauge-fixed theory is superconformal.

Just like in the bosonic case, we determine the boundary conditions the spinor fields ψ^μ have to satisfy by requiring vanishing boundary terms in the variation of the action. Again focussing on the closed string case, where X^μ satisfy periodic boundary conditions, the worldsheet spinors can satisfy either periodic or anti-periodic boundary conditions. Since left- and right-moving components of the worldsheet spinors decouple, we can impose the boundary conditions independently. There are therefore 4 combinations of boundary conditions

$$\psi^\mu_+(\sigma^0, \sigma^1 + \pi) = \pm\psi^\mu_+(\sigma^0, \sigma^1), \quad \psi^\mu_-(\sigma^0, \sigma^1 + \pi) = \pm\psi^\mu_-(\sigma^0, \sigma^1). \tag{2.143}$$

The periodic boundary conditions are called *Ramond boundary conditions* and the anti-periodic ones are the *Neveu-Schwarz boundary conditions*. We know the equations of motion as well as the boundary conditions, so we can now write down the corresponding mode expansions. The mode expansions of X^μ remain unchanged and for the closed string, the fermions give

$$\psi^\mu_- = \sum_{m\in\mathbb{Z}} d^\mu_m e^{-2im\sigma^-} \quad (\mathrm{R}), \qquad \psi^\mu_+ = \sum_{m\in\mathbb{Z}} \tilde{d}^\mu_m e^{-2im\sigma^+} \quad (\mathrm{R}), \tag{2.144}$$

$$\psi^\mu_- = \sum_{r\in\mathbb{Z}+\frac{1}{2}} b^\mu_r e^{-2ir\sigma^-} \quad (\mathrm{NS}), \qquad \psi^\mu_+ = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \tilde{b}^\mu_r e^{-2ir\sigma^+} \quad (\mathrm{NS}), \tag{2.145}$$

which combine in 4 different ways. States in the NS-NS and R-R sector are spacetime bosons while states in the NS-R and R-NS sector are spacetime fermions.
The covariant quantisation leads to the following anti-commutation relations for the fermionic modes

$$\{d^\mu_m, d^\nu_n\} = \eta^{\mu\nu}\delta_{m+n,0} \quad (\mathrm{R}), \qquad \{b^\mu_r, b^\nu_s\} = \eta^{\mu\nu}\delta_{r+s,0} \quad (\mathrm{NS}), \tag{2.146}$$

where $m, n \in \mathbb{Z}, r, s \in \mathbb{Z} + \frac{1}{2}$ and analogous relations for $\tilde{d}^\mu_m, \tilde{b}^\mu_r$.

We can now construct the corresponding Fock space for each boundary conditions. Physical states as usual are defined by imposing the constraints (2.140), which we express in terms of the Fourier modes

$$\begin{aligned}
L_m &= \frac{1}{2}\sum_{n=-\infty}^{\infty} :\alpha_{-n}\cdot\alpha_{m+n}: + \frac{1}{2}\begin{cases} \sum_{n\in\mathbb{Z}}(n+\frac{m}{2}) :d_{-n}\cdot d_{m+n}: & (\mathrm{R}), \\ \sum_{r\in\mathbb{Z}+\frac{1}{2}}(r+\frac{m}{2}) :b_{-r}\cdot b_{m+r}: & (\mathrm{NS}), \end{cases} \\
F_m &= \sum_{n\in\mathbb{Z}} \alpha_{-n}\cdot d_{m+n} \quad (\mathrm{R}), \\
G_r &= \sum_{n\in\mathbb{Z}} \alpha_{-n}\cdot b_{r+n} \quad (\mathrm{NS}).
\end{aligned}$$

In the case of the closed string this set of relations is supplemented with a second set for $\tilde{L}_m, \tilde{F}_m, \tilde{G}_r$. F_m and G_r correspond to the Fourier modes of the supercurrent J_α. As in the bosonic case, the theory has a critical dimension which in the supersymmetric case is $D = 10$. The physical state conditions are now

$$L_m|\phi\rangle = 0, \quad m > 0, \quad (L_0 - a)|\phi\rangle = 0, \quad a = \begin{cases} 0 & \text{(R)}, \\ \frac{1}{2} & \text{(NS)}, \end{cases} \tag{2.147}$$

$$F_m|\phi\rangle = 0, m \geq 0 \quad \text{(R)} \tag{2.148}$$

$$G_r|\phi\rangle = 0, r > 0 \quad \text{(NS)} \tag{2.149}$$

We are now ready to construct the Fock space of the RNS string. We will describe how this is done for the open string, as the closed string case just amounts to two chiral sectors each equivalent to the open string. For the NS boundary conditions the momentum states $|k\rangle$ satisfy

$$\alpha^\mu_m|k\rangle = 0, \quad b^\mu_r|k\rangle = 0, \quad m, r > 0, \tag{2.150}$$

and the mass of the state is computed using the L_0-constraint

$$\left(L_0 - \frac{1}{2}\right)|\phi\rangle = 0 \Rightarrow \alpha' M^2 = N - \frac{1}{2}, \tag{2.151}$$

where

$$N = \sum_{m=1}^{\infty} \alpha_{-m} \cdot \alpha_m + \sum_{r=\frac{1}{2}}^{\infty} r b_{-r} \cdot b_r. \tag{2.152}$$

As in the bosonic case, the true physical states are obtained by imposing the other constraints and by dividing out the residual gauge equivalences.
The lowest mass state $|k\rangle$, for which $N = 0$ and $\alpha' M^2_{\text{open}} = \frac{1}{4}\alpha' M^2_{\text{closed}} = -\frac{1}{2}$, corresponds to a scalar tachyon. We will see shortly how this state is projected out of the spectrum. The next excited states is $b^\mu_{-\frac{1}{2}}|k\rangle$, for which $N = \frac{1}{2}$ and $\alpha' M^2_{\text{open}} = \frac{1}{4}\alpha' M^2_{\text{closed}} = 0$, corresponds to a massless vector.

For R-boundary conditions, the momentum eigenstates satisfy

$$\alpha^\mu_m|k\rangle = 0, \quad d^\mu_m|k\rangle = 0, \quad m > 0, \tag{2.153}$$

and the L_0 constraint gives

$$L_0|\phi\rangle = 0 \Rightarrow \alpha' M^2 = N, \tag{2.154}$$

where

$$N = \sum_{m=1}^{\infty} \alpha_{-m} \cdot \alpha_m + \sum_{m=1}^{\infty} m d_{-m} \cdot d_m. \tag{2.155}$$

We note that

$$[N, d^\mu_0] = 0, \quad \{d^\mu_0, d^\nu_0\} = \eta^{\mu\nu}. \tag{2.156}$$

Using the operators $d_0^\mu, \mu = 0, \dots, 9$, we can form $2^{10} = 1024$ independent, energetically degenerate states

$$(d_0^0)^{n_0}(d_0^1)^{n_1} \dots (d_0^9)^{n_9}|k\rangle, \quad n_0, \dots, n_9 = \{0, 1\}. \tag{2.157}$$

These states form a reducible representation of the Clifford algebra $Cl(1, 9)$. The dimension of an irreducible representation of $Cl(p, q)$ is

$$\dim Cl(p, q) = 2^{[(p+q)/2]}. \tag{2.158}$$

For $Cl(1, 9)$ the unique (up to equivalence) irreducible representation has dimension $2^5 = 32$. Following [78], an explicit re-arrangement of the $SO(1, 9)$ operators into fermionic creation and annihilation operators is given by

$$d_0^\pm = \frac{i}{\sqrt{2}}(d_0^0 \pm d_0^1), \quad d_i^\pm = \frac{i}{\sqrt{2}}(d_0^{2i} \pm i d_0^{2i+1}), \quad i = 1, \dots, 4. \tag{2.159}$$

We can now define a highest weight state $|0\rangle_R$ by

$$d_i^+|0\rangle_R = 0 \tag{2.160}$$

and create a Clifford representation by acting with the fermionic operators d_i^-, with basis

$$(d_0^-)^{m_0} \dots (d_4^-)^{m_4}|0\rangle_R, \quad m_0, \dots, m_4 = \{0, 1\}. \tag{2.161}$$

Since the degenerate ground state transforms in a spinor representation of the spacetime rotation group, it is a spacetime fermion. The oscillators being spacetime vectors their application cannot change bosonic states into fermionic ones and vice versa.

2.4.2 Type II superstrings

When studying a specific superstring model, we want to include the NS-NS sector of the closed string since this is where there is a graviton. However, we just saw that the vacuum is still tachyonic. The way to fix this problem is by performing a projection of states on the spectrum, a process called *GSO-projection*. This projection can be derived from several consistency requirements (for example modular invariance or spacetime supersymmetry, see [79] for details). Relevant for us, is that there are only 2 supersymmetric modular invariant theories of closed strings with spacetime fermions that allows for an open string sector: Type IIA and Type IIB superstring theories. Both of these theories have the 4 sectors NS-NS, R-R, NS-R and R-NS in their Hilbert space.

In the NS-sector, the GSO-projection operator is

$$P_{\text{GSO}}^{\text{NS}} = -(-1)^{\sum_{r=-\frac{1}{2}}^{\infty} b_{-r} \cdot b_r}. \tag{2.162}$$

States with $P_{\text{GSO}} = 1$ remain in the spectrum, and so all states with an even number of b-modes are projected out. Moreover we see than the tachyon is projected out since

$$P_{\text{GSO}}^{\text{NS}}|k\rangle = -|k\rangle, \tag{2.163}$$

so $b^{\mu}_{-\frac{1}{2}}|k\rangle$ becomes the new ground state of the NS-sector, which is not tachyonic but massless. For the R-sector, the projection operator is

$$P_{\text{GSO}}^{\text{R}} = \bar{\Gamma}(-1)^{\sum_{m=1}^{\infty} d_{-m}\cdot d_m}, \tag{2.164}$$

where Γ are the spacetime γ-matrices and

$$\bar{\Gamma} = \Gamma_0\Gamma_1 \dots \Gamma_9 \tag{2.165}$$

is the 10D chirality operator satisfying

$$\bar{\Gamma}^2 = 1 \text{ and } \{\bar{\Gamma}, \Gamma^{\mu}\} = 0. \tag{2.166}$$

Spinors that satisfy

$$\bar{\Gamma}\psi = \pm\psi \tag{2.167}$$

are said to have positive or negative chirality and spinors with a definite chirality are called Weyl-spinors. The GSO-projection therefore projects out states with a negative chirality combined with an odd-number of d-operators, or positive chirality with an even number of d-operators.

In the case of the closed string, there is an additional choice, which is the relative chirality of the vacuum in the R-sectors. If they have opposite chirality, the resulting theory is the Type IIA superstring which is non-chiral with $(\mathcal{N}_+, \mathcal{N}_-) = (1, 1)$ Poincaré supersymmetry algebra. If they have the same chirality, then the resulting theory is the Type IIB superstring which is chiral with $(\mathcal{N}_+, \mathcal{N}_-) = (2, 0)$ Poincaré supersymmetry algebra.

The massless supergravity theories with such susy algebras are unique, called Type IIA and IIB supergravity, and correspond to the massless effective field descriptions of the corresponding string theories (for details on supersymmetry algebras and Type II supergravities, see chapter 3). We won't describe these theories here in detail but we can describe their massless spectrum. We will give the decomposition into irreducible representations of the Lorentz group $SO(1, 9)$ and its little group $SO(8)$. The ground state $b^{\mu}_{-\frac{1}{2}}|k\rangle$ of the NS-sector is a Lorentz vector which is the [10] representation. The NS-NS sector is common to both Type II theories:

$$SO(1, 9) : [10] \times [10] = [54] + [45] + [1], \tag{2.168}$$

$$SO(8) : [8] \times [8] = [35] + [28] + [1]. \tag{2.169}$$

We already know these universal states from the bosonic string, as they correspond to the graviton, Kalb-Ramond field and the dilaton.
For spinor indices, restricting to physical states means that we restrict the representations to Majorana-Weyl representations, which have real dimension 16 such that in the R-sector, the vacuum is a Majorana-Weyl spinor $[16]_{\pm}$. Moreover the Dirac equation further eliminates half of the degrees of freedom such that spinors have 8 on-shell degrees of freedom.
States in the NS-R and R-NS sectors are decompositions of the product between a vector and a spinor representation, which give a vector-spinor and a spinor and so we get

$$SO(1,9) : [10] \times [16]_{\pm} = [144]_{\mp} + [16]_{\mp}, \tag{2.170}$$

$$SO(8) : [8] \times [8]_{\pm} = [56]_{\mp} + [8]_{\mp}. \tag{2.171}$$

In Type IIA, these sectors contain two *gravitini* $\psi^{\mu}_{\pm}$ and two *dilatini* $\lambda_{\pm}$ which are of opposite chirality since we've already mentioned that this theory is non-chiral. Type IIB has the same type of fields however, they have the same chirality.

Finally, for the R-R sector, we need to decompose the product of two spinors. For Dirac or Majorana spinors, this reduces to the sum of all possible antisymmetric tensor representations. We also need to take into account that some antisymmetric tensors can be further decomposed into (anti)-self-dual parts (in the sense of Hodge duality). In the case of Type IIA we get

$$SO(1,9) : [16]_{+} \times [16]_{-} = [1] + [45] + [210]. \tag{2.172}$$

This corresponds to antisymmetric tensors G_0, G_2, G_4 of rank $0, 2, 4$ respectively. The on-shell condition for these fields are Maxwell-like equations

$$\partial^{\mu_1} G_{\mu_1 \dots \mu_{p+2}} = 0, \quad \partial_{[\mu} G_{\mu_1 \dots \mu_{p+2}]} = 0. \tag{2.173}$$

The interpretation is that these fields are actually gauge invariant field strength and not gauge potentials. The decomposition in terms of transverse degrees of freedom is then

$$SO(8) : [8]_{+} \times [8]_{-} = [8] + [56]. \tag{2.174}$$

which represent the gauge potentials C_1, C_3 of G_2, G_4. The zero-form field strength G_0 does not carry any local degrees of freedom and therefore does not have an associated gauge potential. Its existence is related to a massive deformation of Type IIA [80], but this goes beyond the scope of this work and in the rest we will ignore this term.
For Type IIB we get the following decomposition

$$SO(1,9) : [16]_{+} \times [16]_{+} = [10] + [120] + [126], \tag{2.175}$$

$$SO(8) : [8]_{+} \times [8]_{+} = [1] + [28] + [35]. \tag{2.176}$$

Field strength	G_1	G_2	G_3	G_4	$G_5 = \star G_5$
IIA/IIB	IIB	IIA	IIB	IIA	IIB
electric	D(-1)	D0	D1	D2	D3
magnetic	D7	D6	D5	D4	D3

Table 2.1: Half BPS-branes in Type IIA/B string theories.

This therefore corresponds to the field strengths G_1, G_3, G_5 of rank $1, 3, 5$ with associated gauge potentials C_0, C_2, C_4. It is important to note that the rank 5 field strength is subject to a self-duality condition $G_5 = \star G_5$, whence dividing by 2 its number of degrees of freedom.

If we add up the sectors, we get 128 bosonic degrees of freedom and 128 fermionic ones. These are the supergravity multiplet which describe the whole field spectrum of the Type IIA and IIB supergravity theories. As mentioned before, the fundamental string is charged under the Kalb-Ramond B-field, but it is neutral under the R-R fields. Instead, as we alluded to earlier, it is the D-branes of string theory which are charged under these fields, and therefore act as sources for these R-R fields. This means that we can infer a D-brane spectrum for these Type II theories. To be more precise, Dp-branes exist in both theories for arbitrary p, but are stable only for even p in IIA and odd p in IIB since they have an associated conserved charge.
The stable D-branes preserve half of the supersymmetry and are therefore called *half BPS-branes* (see chapter 3 for details), this allows one to make exact statements which hold non-perturbatively.

2.4.3 Type II T-duality

We saw that for the bosonic string, performing a T-duality on a circle of radius R mapped the theory to itself on a background with a circle of radius $\frac{\alpha'}{R}$. In this sense the theory is self dual. However, in the case of the Type II theories, they are not mapped to themselves but *to each other*. If several coordinates are compactified, then we can perform several T-dualities, one on each coordinate, so on an even torus, a Type II theory is then dual to itself (on the dual torus). Going back to the case of a single circle compactification, we perform a T-duality on X^9, the compact coordinate of radius R. The transformation on the bosonic coordinates is the same as in the bosonic string case

$$X_L^9 \to X_L^9 \quad \text{and} \quad X_R^9 \to -X_R^9, \tag{2.177}$$

which exchanges momentum and winding modes. In the RNS formalism, supersymmetry requires the worldsheet spinors to transform in the same way as their bosonic partners, that is

$$\psi_L^9 \to \psi_L^9 \quad \text{and} \quad \psi_R^9 \to -\psi_R^9. \tag{2.178}$$

This has the effect of changing the chirality of the R-sector ground state in the right-moving sector. As we have seen, the relative chirality between left- and right-moving sectors is what distinguishes the Type IIA and Type IIB theories. Since only one sector is flipped, we understand that Type IIA on a circle of radius R is T-dual to Type IIB on a circle of radius $\tilde{R}$. We can also infer what happens to the half-BPS branes of the respective theories. Type IIA has even Dp-branes and are mapped to the odd Dp-branes of Type IIB theory. If the dualized coordinate X^9 is longitudinal to the Dp-brane, the duality gives a $D(p-1)$-brane, if it is transverse to the Dp-brane, the duality gives a $D(p+1)$-brane.

T-duality is a perturbative duality of Type II theories, which means that it holds order by order in the perturbative expansion of the respective theories. To see this we can derive the relation between the coupling constants of the two theories. It is sufficient to look at the low energy NS-NS sector, but the relation holds generally. We have

$$S_{IIA} = \frac{1}{g_S^2}\int d^{10}x\mathcal{L}_{\text{NS}}, \quad S_{IIB} = \frac{1}{\tilde{g}_S^2}\int d^{10}x\mathcal{L}_{\text{NS}}. \tag{2.179}$$

which, compactified on circles of radii R and $\tilde{R}$ gives

$$S_{IIA} = \frac{2\pi R}{g_S^2}\int d^{9}x\mathcal{L}_{\text{NS}}, \quad S_{IIB} = \frac{2\pi \tilde{R}}{\tilde{g}_S^2}\int d^{9}x\mathcal{L}_{\text{NS}}, \tag{2.180}$$

Because of T-duality, these expression are the same, and since we know that $R\tilde{R} = \alpha'$ we conclude that the couplings satisfy

$$\tilde{g}_S = \frac{\sqrt{\alpha'}}{R} g_S. \tag{2.181}$$

2.4.4 The web of dualities

Let us conclude this chapter with general remarks and an overview of the current state of superstring theory. First of all, besides the two Type II theories, there are other consistent theories in 10D:

- The Type I superstring is a theory of open and closed *non-oriented* strings. It can be constructed by performing an *orientifold projection* (which implements the modding out by world-sheet parity) of Type IIB theory with 32 D9-branes. The orientifold projection breaks the spacetime supersymmetry down to $\mathcal{N} = 1$.

- The Heterotic string theories are theories obtained when we consider the RNS model but with chiral supersymmetry $(1, 0)$. The critical dimensions of each chiral sector then differ by $26 - 10 = 16$, so the 16 unpaired worldsheet bosons need to be compactified and satisfy certain boundary conditions. Two heterotic theories are constructed this way, one with gauge group $E_8 \times E_8$ and one with gauge group $Sp(32)/\mathbb{Z}_2$ (usually refered to $SO(32)$ as they are locally isomorphic).

- The non-supersymmetric string theories, with the heterotic $SO(16) \times SO(16)$ being the only modular invariant and tachyon-free one. Since there is no evidence for supersymmetry in experiments so far, and considering how still poorly understood supersymmetry breaking is in string theory, these less-studied non supersymmetric theories are becoming increasingly more interesting for phenomenology.

All these theories are initially defined perturbatively. However, as we have seen in the case of Type II theories, they are not independent and some equivalences can be found. It can be shown that the two supersymmetric heterotic theories are T-dual to each other. Other types of dualities have been discovered which further interconnects the different string theories. S-duality is a non-perturbative duality which can be thought of as a generalization of electric-magnetic duality, therefore relating the weak coupling regime of one theory to the strong coupling of another. Type I and heterotic $SO(32)$ are S-dual, and TypeIIB is self-dual. F-theory [81] (see [82] for a review) has been proposed as a non-perturbative definition of Type IIB theory, which geometrizes the $SL(2, \mathbb{Z})$ S-duality symmetry as being the group of large diffeomorphisms of an internal torus of a 12-dimensional theory. F-theory compactified on an elliptically fibered manifold allows one to study non-perturbative vacua of string theory. It has been shown by Witten [83], building on work on S-duality by Sen [4] and on U-duality by Hull and Townsend [5], that the strong coupling regime of Type IIA is eleven dimensional. This theory, which is not stringy, has been dubbed *M-theory*. Only its low energy limit is known, 11D supergravity, which is the only supersymmetric field theory in 11 dimensions. When compactified on a torus, 11D supergravity exhibits a manifest *U-duality* which includes T- and S-duality, and can map the coupling (dilaton) of one theory to geometric properties (moduli) of another theory. M-theory compactified on an interval, known as Hořava–Witten theory [84, 85], gives the Heterotic $E_8 \times E_8$ string theory. This intricate web of theories is nowadays interpreted as a vast moduli space of the underlying fundamental theory, the elusive M-theory. In some sense, different theories are just manifestions of particular limits of M-theory.

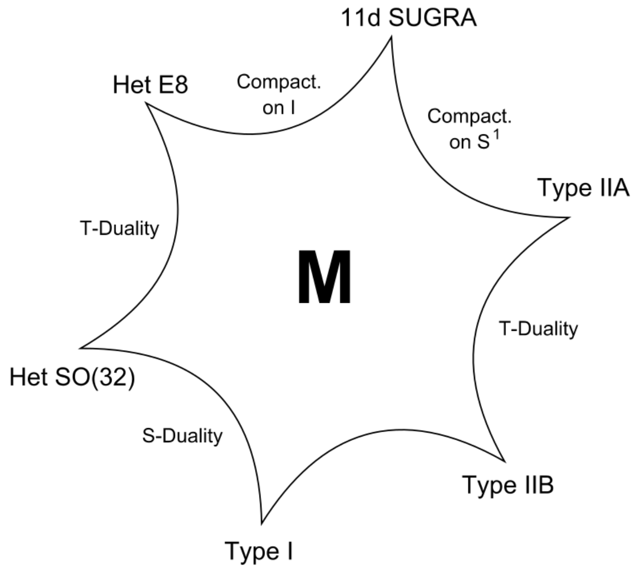


Figure 2.3: The Superstring duality web

Several proposals have been put forward to provide a non-perturbative definition of M-theory

- Matrix models like the BFSS model [86] (or IKKT model for Type IIB [87]). In this theory, the fundamental degrees of freedom are the D0-branes of Type IIA and any calculation of M-theory can be rephrased as a quantum mechanical matrix model calculation.
- Supermembranes [88, 89] which are similar in spirit to string theory except that the fundamental degrees of freedom are now two-dimensional objects that sweep a world-volume. Their critical dimension is $D = 11$ but the quantum theory is still poorly understood. It was shown that their double dimensional reduction to 10 dimensions recovers the Green-Schwarz superstring [90].
- AdS/CFT [91] states that string/M-theory on certain anti de-Sitter backgrounds is dual to certain conformal field theories in one less dimension (whence making it a direct realisation of the holographic principle). Since this is a weak-strong duality, it allows to define M-theory non-perturbatively, albeit on certain backgrounds. The AdS/CFT correspondence has found numerous applications like quantum information and the black hole information paradox, QCD confinement, condensed matter,...

Each of these approaches provide insights into the fundamental nature of M-theory, but a precise definition is still missing.

One of the goals of string theory is to derive the standard model of particle physics, and compute as many of its parameters as possible as well as fitting with current cosmological data. The vast possible choices of different compactifications makes the *landscape* of different vacua extremely large. However, there is no known dynamical principle in string theory that selects a particular vacuum. In more recent years, an alternative viewpoint was to consider the *swampland* of string theory. The swampland is in a sense the complement of the landscape, the set of low energy theories which cannot be UV-completed into a consistent theory of quantum gravity. Studying common features that appear in the swampland gives us insight into consistency requirement that any quantum gravity theory should have, for example the absence of global symmetries (see [92] for review).

Considering the very high energy scale of the massive excitations of strings, only the lowest energy excitations can be probed in current experiments. Moreover, in most compactifications, only the massless spectrum can be derived exactly. As mentioned before, the low energy effective theory of strings are supergravity theories. Thanks to supersymmetry and dualities, they are perfect arenas to explore various properties of string theory, including its non-perturbative regime. With that goal in mind, we introduce the supergravity formalism in the next chapter.

Chapter 3

Supergravity

> There are endless possibilities when you see the geometry in nature and your environment.
>
> *Monir Farmanfarmaian*

This chapter is dedicated to the study of field theories with gauged supersymmetry, and some of their geometric aspects. Supersymmetry is a property of certain theories that relate bosonic and fermionic degrees of freedom. When the supersymmetry is local, which means when the supersymmetry parameter is spacetime dependent, then it can be shown that the theory automatically includes gravity whence the name of *supergravity theories*. We will introduce the main features of these theories, such as their algebras and representation theory. We will present *dimensional reduction* in this field theoretic context as it is a ubiquitous tool to study supergravity. We will then give a description of a special type of manifold, dubbed *Calabi-Yau manifolds*, for string theory compactifications that preserves some supersymmetry. Finally we will present the *special geometry* of Calabi-Yau threefolds' moduli spaces.

3.1 Supersymmetry

Supersymmetry is an extension of the usual spacetime symmetries that one imposes on field theories. The spacetime symmetries of a relativistic theory possess the Poincaré group as symmetry group. Lie groups are often studied through their associated Lie algebras that contain most of the necessary information about the group itself. We will start with studying the Lorentz algebra and associated representation theory, to build up towards the full Poincaré superalgebra. Here we will only present supersymmetry with

Lorentzian signature but supersymmetry in other signatures will play a role in the next chapter (for a discussion of supersymmetry algebras in arbitrary spacetime dimensions and signature, see [93]). This section is mainly based on [94].

3.1.1 The Poincaré superalgebra

We start with 4-dimensional Minkowski space equipped with the flat metric $\eta_{\mu\nu}$ with the following Lorentz transformations: $x^\mu \to \Lambda^\mu{}_\nu x^\nu$ where

$$\Lambda^T \eta \Lambda = \eta. \tag{3.1}$$

These matrices form the Lorentz group $SO(1,3)$. We are interested in the representations of that group, in particular the spinor ones. Technically speaking, the Lorentz group doesn't have any, but its double cover $Spin(1,3)$ does. These two groups share the same algebra $so(1,3)$. A Lorentz transformation can be written as

$$\Lambda = \exp(-\frac{i}{2}\omega_{\mu\nu} M^{\mu\nu}), \tag{3.2}$$

where $\omega_{\mu\nu}$ are six paramaters and $M^{\mu\nu} = -M^{\nu\mu}$ are the antisymmetric generators. These matrices generate as wanted the $so(1,3)$ algebra

$$[M^{\mu\nu}, M^{\rho\sigma}] = i(\eta^{\nu\rho} M^{\mu\sigma} - \eta^{\nu\sigma} M^{\mu\rho} + \eta^{\mu\sigma} M^{\nu\rho} - \eta^{\mu\rho} M^{\nu\sigma}). \tag{3.3}$$

The Lorentz generators can be decomposed in terms of rotations J_i and boosts K_i defined as

$$J_i = \frac{1}{2}\epsilon_{ijk} M_{jk} \quad \text{and} \quad K_i = M_{0i}, \tag{3.4}$$

where $i, j = 1, 2, 3$ are contracted using δ_{ij} and $\epsilon_{ijk} = +1$. We find from the Lorentz algebra that these generators obey

$$[J_i, J_j] = i\epsilon_{ijk} J_k, \quad [J_i, K_j] = i\epsilon_{ijk} K_k, \quad [K_i, K_j] = -i\epsilon_{ijk} J_k. \tag{3.5}$$

We can see that the rotations form an $su(2)$ sub-algebra, which is expected since $SO(3) \cong SU(2)/\mathbb{Z}_2$. If we perform the following linear combination

$$A_i = \frac{1}{2}(J_i + iK_i) \quad \text{and} \quad B_i = \frac{1}{2}(J_i - iK_i), \tag{3.6}$$

both operators are hermitian and we find two mutually commuting $su(2)$ algebras inside $so(1,3)$

$$[A_i, A_j] = i\epsilon_{ijk} A_k, \quad [B_i, B_j] = i\epsilon_{ijk} B_k, \quad [A_i, B_j] = 0. \tag{3.7}$$

The representations of $SU(2)$ are well known from quantum mechanics: they are labelled by an integer or half-integer $j \in \frac{1}{2}\mathbb{Z}$ which is called *spin* in the context of rotations. The dimension of the representation is $2j + 1$. Since we have two copies of $su(2)$, the representations of the Lorentz algebra have therefore two labels (j_1, j_2) and has dimension $(2j_1 + 1)(2j_2 + 1)$. We can enumerate the simplest of these representations

$$(0,0) : \text{scalar} \tag{3.8}$$
$$(\frac{1}{2},0) : \text{left-handed Weyl spinor} \tag{3.9}$$
$$(0,\frac{1}{2}) : \text{right-handed Weyl spinor} \tag{3.10}$$
$$(\frac{1}{2},\frac{1}{2}) : \text{vector} \tag{3.11}$$
$$(1,0) : \text{self-dual 2-form} \tag{3.12}$$
$$(0,1) : \text{anti-self-dual 2-form} \tag{3.13}$$

The physical spin of a particle is the quantum number under rotations $\vec{J}$, that is $j = j_1 + j_2$. It turns out that the generators A_i and B_i are complex conjugates

$$(A_i)^* = -B_i. \tag{3.14}$$

So really it is the *complexified* Lorentz algebra that is isomorphic to two copies of $su(2)$. We sometimes keep this in mind by writing the real section as

$$so(1,3) \cong su(2) \times su(2)^*. \tag{3.15}$$

For us it means that a complex conjugate of a representation exchanges the two quantum numbers

$$(j_1, j_2)^* = (j_2, j_1). \tag{3.16}$$

The scalar and vector representations $(0,0)$ and $(1,1)$ are real, but the left- and right-handed Weyl spinors $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ are exchanged under complex conjugation.

The continuous symmetry of Minkowski spacetime also include the spacetime translations generated by P^μ. They commute between themselves and the commutation relations with the Lorentz generators are given by

$$[P^\mu, P^\nu] = 0 \quad \text{and} \quad [M^{\mu\nu}, P^\sigma] = i(P^\mu \eta^{\nu\sigma} - P^\nu \eta^{\mu\sigma}), \tag{3.17}$$

which shows that P^μ transforms as a vector of the Lorentz group. Together with the Lorentz commutation relations these form the Poincaré algebra. Field theories can also exhibit additional continuous symmetries like the $SU(N)$ gauge groups of the standard model. If we call the generators of such symmetries T, then we will find that they are always Lorentz scalars, which means that they commute with the Poincaré generators

$$[P^\mu, T] = [M^{\mu\nu}, T] = 0. \tag{3.18}$$

This is the essence of the Coleman-Mandula theorem [95]. The theorem states that in any spacetime dimension greater than $d = 1 + 1$, the symmetry group of the S-matrix of any interacting quantum field theory must factorise as

$$\text{Poincaré} \times \text{Internal}.$$

The theorem comes with a set of underlying assumptions like causality and locality. However some other assumptions can be relaxed to find loopholes of the theorem. One interesting loophole is the case where all particles are massless, in which case the Poincaré group is enhanced into the conformal group. Another loophole, which is the one relevant for us, is supersymmetry. Indeed, one of the assumptions of the theorem is that the symmetries of the theory are Lie groups generated by Lie algebras. However, supersymmetry is described by a similar yet different mathematical structure called a Lie superalgebra, or $\mathbb{Z}_2$-graded Lie algebra in the mathematical litterature. In a nutshell, it means that we allow for both commutation and anticommutation relations inside our algebra. This is the essence of the Haag-Lopuszanski-Sohnius theorem [96] which generalizes the Coleman-Mandula theorem.

Supersymmetric theories have a new conserved charge called the *supercharge* which is a left-handed Weyl spinor Q_α and its right-handed counterpart $\bar{Q}_{\dot{\alpha}}$. The case where there are multiple supercharges is known as *extended supersymmetry* but for now we focus on theories with a single supercharge ie. $\mathcal{N} = 1$ supersymmetry. The anti-commutation relation, the one that allows us to escape the Coleman-Mandula theorem, is

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu, \tag{3.19}$$

where σ^μ are the Pauli matrices defined as

$$\sigma^\mu = (1, \sigma^i) \quad \text{with} \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.20}$$

We also have the following commutation relations to complete the supersymmetry algebra

$$[M^{\mu\nu}, Q_\alpha] = (\sigma^{\mu\nu})_\alpha{}^\beta Q_\beta, \quad [M^{\mu\nu}, \bar{Q}^{\dot{\alpha}}] = (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \bar{Q}^{\dot{\beta}}, \quad [Q_\alpha, P^\mu] = \{Q_\alpha, Q_\beta\} = 0, \tag{3.21}$$

where the generators of the Lorentz group in the left-handed and right-handed spinor representation are, respectively,

$$(\sigma^{\mu\nu})_\alpha{}^\beta = \frac{i}{4}(\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu)_\alpha{}^\beta \quad \text{and} \quad (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} = (\bar{\sigma}^\mu \sigma^\nu - \bar{\sigma}^\nu \sigma^\mu)^{\dot{\alpha}}{}_{\dot{\beta}} \tag{3.22}$$

We saw that the Coleman-Mandula theorem states that internal symmetries must commute with spacetime symmetries of the Poincaré group. However, they don't necessarily commute with the supercharge Q_α. Indeed all internal symmetries must commute with the supercharge with the exception of internal $U(1)$ symmetries that act as

$$Q_\alpha \to e^{-i\lambda} Q_\alpha \quad \text{and} \quad \bar{Q}_{\dot{\alpha}} \to e^{i\lambda} \bar{Q}_{\dot{\alpha}}. \tag{3.23}$$

This $U(1)$ symmetry is known as *R-symmetry* and is by definition the automorphism group of the supersymmetry algebra which commutes with the Lorentz group. If we call the generator R then the commutation relations are written

$$[R, Q_\alpha] = -Q_\alpha \quad \text{and} \quad [R, \bar{Q}_{\dot{\alpha}}] = +\bar{Q}_{\dot{\alpha}}. \tag{3.24}$$

For extended supersymmetry, the R-symmetry group can be larger as we will see in a moment.

3.1.2 Representations

Now that we have derived the supersymmetry algebra it is time to turn to its representations. We first start by constructing the irreducible representations of the Poincaré group, which we commonly call "particles". Irreducible representations are labelled by their eigenvalue of the Casimir. Casimir operators are operators that commute with all the generators of the group. The Poincaré group has two such Casimirs

$$C_1 = P_\mu P^\mu \quad \text{and} \quad C_2 = W_\mu W^\mu, \tag{3.25}$$

where $W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma}$ is the *Pauli-Lubanski vector*. The representations are therefore labelled by the eigenvalues of C_1 and C_2. The first one is simply the mass m of the particle $C_1 = m^2$. We now need to seperate between the massless and massive cases:

- Massive particles: We can always boost to the rest frame of the particle so that $P^\mu = (m, 0, 0, 0)$. In this frame the Pauli-Lubanski vector is given by

  $$W^0 = 0 \quad \text{and} \quad W^i = -mJ^i, \tag{3.26}$$

  where J^i are the generators of rotations. This means that $C_2 = -m^2 J^2$ and so is specified by the eigenvalues of J^2. Massive particles are therefore characterized by their mass m and spin j.

- Massless particles: In that case $C_2 = 0$, so both Casimirs vanish. We can however still characterize representations. We choose a frame where $P^\mu = (E, 0, 0, E)$. There, $W^\mu = M_{12} P^\mu$ so the porportionality constant between W and P is determined by the eigenvalue of the $U(1)$ rotation in the (x^1, x^2)-plane. This eigenvalue is called the *helicity*, $h = 0, \frac{1}{2}, 1, \ldots$. Massless particles are therefore characterized by their mass $m = 0$ and helicity h.

The symmetry group that survives after boosting to a preferred frame is known as the *little group*. In the case of massless particles, a state $|p_\mu, h\rangle$ needs to come paired up with the state $|p_\mu, -h\rangle$ in order to preserve CPT invariance.

We turn to the representations of the $\mathcal{N} = 1$ supersymmetry algebra. We start with the massless case. C_1 is still a Casimir so all particles in a multiplet will have the same mass. However C_2 is no longer a Casimir, the representations of the algebra can therefore contain particles of different spins. Another Casimir operator can be constructed, but in what follows, we will construct the representations by starting from a particle and acting on it with successive susy generators until a representation of the full superalgebra is built up.

We consider a state of a massless particle of helicity h in a preferred frame. Restricted on such a state, the susy algebra becomes

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu = 2E(1+\sigma^3)_{\alpha\dot{\alpha}} = 4E \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} . \tag{3.27}$$

From this, and the fact that any state in a supersymmetric theory is necessarily positive, we infer that Q_2 and $\bar{Q}_2$ annihilate this state. Therefore to build up the representation we only need to act with Q_1 and $\bar{Q}_1$. These act like fermionic creation and annihilation operators. Therefore the representation is straightforward and consists of two states: The starting state, which satisfies $Q_1|p_\mu, h\rangle = 0$, and $\bar{Q}_1|p_\mu, h\rangle$. The supersymmetry multiplet then consists of these two states, and it can be shown thanks to the commutation relation that the second state has a helicity $\frac{1}{2}$ lower. All in all the multiplet has two states: $|p_\mu, h\rangle$ and $|p_\mu, h - \frac{1}{2}\rangle$. As before, CPT invariance ensures that for each state, there is a corresponding state with the opposite helicity.

The different representations differ by the helicity of the starting state, we can list:

- Starting with $h = \frac{1}{2}$ we have

h	$-\frac{1}{2}$	0	$+\frac{1}{2}$
multiplicity	1	2	1

  This *chiral* multiplet is comprised of a single Weyl spinor and a complex scalar.

- Starting with $h = 1$ we have

h	-1	$-\frac{1}{2}$	$+\frac{1}{2}$	+1
multiplicity	1	1	1	1

  This *vector* multiplet is comprised of a photon as well as a single Weyl spinor.

- Starting with $h = 2$ we have

h	-2	$-\frac{3}{2}$	$+\frac{3}{2}$	+2
multiplicity	1	1	1	1

This is known as the *supergravity* multiplet as it is comprised of a graviton and its supersymmetric partner, the *gravitino* of spin $\frac{3}{2}$.

We could continue these constructions, but for massless fields with helicity greater than 2, there are strong restrictions which prohibit these fields from interacting in Minkowski space (although one can study higher-spin field theories in (anti)-de Sitter backgrounds).

We will not describe the construction for massive representations as they are not relevant for this work, the main difference in the construction is that now both Q_1 and Q_2 can act as fermionic creation operators.

The scalar sector of the basic supergravity theories in 4 dimensions will all contain a non-linear sigma model for the scalar sector, which means that the Lagrangian will have a term of the form

$$e^{-1}\mathcal{L}_{bos} = \cdots - \frac{1}{2}g_{IJ}(\varphi)\partial_\mu\phi^I\partial^\mu\varphi^J. \tag{3.28}$$

This metric defines a target space geometry "data" that defines a particular theory. In the case of $\mathcal{N} = 1, D = 4$ supergravity, the target space geometry of the scalars is a Kähler-Hodge manifold (see [97] for details).

3.1.3 Extended supersymmetry

We now turn to the case of extended supersymmetry. This means that we now have a collection of $\mathcal{N}$ supercharges

$$Q^I_\alpha \quad \text{and} \quad \bar{Q}^I_{\dot{\alpha}}, \quad I = 1, \ldots, \mathcal{N}. \tag{3.29}$$

Each supercharge keeps the same commutation relations with the Poincaré generators and among supercharges we have

$$\{Q^I_\alpha, \bar{Q}^J_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}}P_\mu\delta^{IJ}. \tag{3.30}$$

The anti-commutator of the supercharges with themselves becomes

$$\{Q^I_\alpha, Q^J_\beta\} = \epsilon_{\alpha\beta}Z^{IJ} \quad \text{and} \quad \{\bar{Q}^I_{\dot{\alpha}}, \bar{Q}^J_{\dot{\beta}}\} = \epsilon_{\dot{\alpha}\dot{\beta}}(Z^\dagger)^{IJ}. \tag{3.31}$$

$Z^{IJ} = -Z^{JI}$ are called the *central charges* as they commute with all other elements of the algebra.

We also saw that for $\mathcal{N} = 1$ we had a $U(1)$ R-symmetry group that changes the phase of the supercharges. In the extended case, the R-symmetry group can rotate supercharges among themselves. The R-symmetry group is $U(2) \cong U(1)\times SU(2)$ in the $\mathcal{N} = 2$ case and $SU(4)$ in the $\mathcal{N} = 4$ case (we will also see in the next chapter that the R-symmetry group depends on the signature of spacetime). The construction of massless representations is similar to what was performed in the $\mathcal{N}=1$ case still with Q_2^I and $\bar{Q}_2^I$ annihilating the starting state $|\Omega\rangle$ and (3.29) enforces $Z^{IJ}|\Omega\rangle = 0$ which means that the central charge doesn't play a role for the massless states. The Q_1^I and $\bar{Q}_1^I$ now form a collection of $\mathcal{N}$ fermionic creation and annihilation operators, and we build up the representation by acting on the starting state with successive creation operators

$$\begin{aligned}
&|\Omega\rangle \\
&\bar{Q}_1^{I\dagger}|\Omega\rangle \\
&\bar{Q}_1^{I\dagger}\bar{Q}_1^{J\dagger}|\Omega\rangle \\
&\dots \\
&\bar{Q}_1^{1\dagger}\dots\bar{Q}_1^{\mathcal{N}\dagger}|\Omega\rangle.
\end{aligned}$$

The starting state $|\Omega\rangle$ has helicity h. Acting with p excitation operators there are different states, each with helicity $h-p/2$. Therefore the full multiplet has $2^{\mathcal{N}}$ different states. Finally, adding the CPT conjugates leaves us with $2^{\mathcal{N}+1}$ states inside a supermultiplet. Let us now look at specific examples with an emphasis on the $\mathcal{N}=2$ case as it is the one of prime interest in this work.

$\mathcal{N}=2$ **supersymmetry** As before, we start with a state of given helicity h.

- Starting with $h=1/2$, there are two states in the first level, namely $\bar{Q}_1^1|\Omega\rangle$ and $\bar{Q}_1^2|\Omega\rangle$ each with $h=0$ and a single state in the final level $\bar{Q}_1^1\bar{Q}_1^2|\Omega\rangle$ with $h=-\frac{1}{2}$. After CPT conjugation we have

h	-$\frac{1}{2}$	0	+$\frac{1}{2}$
multiplicity	2	4	2

  This is known as a *hypermultiplet*. We can see that it is comprised of two chiral multiplets.

- Starting with $h=0$ we have

h	-1	-$\frac{1}{2}$	0	+$\frac{1}{2}$	+1
multiplicity	1	2	2	2	1

  This is the $\mathcal{N}=2$ vector multiplet which is comprised of an $\mathcal{N}=1$ vectormultiplet and an $\mathcal{N}=1$ chiral multiplet.

- If we start with $h=2$ we have

h	-2	-$\frac{3}{2}$	-1	+1	+$\frac{3}{2}$	+2
multiplicity	1	2	1	1	2	1

This is the $\mathcal{N}=2$ supergravity multiplet which is comprised of an $\mathcal{N}=1$ supergravity multiplet and an $\mathcal{N}=1$ vector multiplet.

The target space geometry of the scalar sector of $\mathcal{N}=2, D=4$ supergravity is special Kähler for the vector multiplets and quaternionic Kähler for the hypermultiplets. We will describe these geometries in later sections and next chapter as they will play a central role.

$\mathcal{N}=4$ **supersymmetry**

- Starting with $h=1$ we get

h	-1	$-\frac{1}{2}$	0	$+\frac{1}{2}$	+1
multiplicity	1	4	6	4	1

  This is comprised of an $\mathcal{N}=2$ vectormultiplet with an $\mathcal{N}=2$ hypermultiplet and is the only $\mathcal{N}=4$ multiplet that does not include gravity.

- Starting with $h=2$ we get

h	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$+\frac{1}{2}$	+1	$+\frac{3}{2}$	+2
multiplicity	1	2	2	2	2	2	2	2	1

  This is the $\mathcal{N}=4$ supergravity multiplet comprised of an $\mathcal{N}=2$ supergravity multiplet and an $\mathcal{N}=2$ vectormultiplet.

In $\mathcal{N}=4, D=4$ supergravity, the target space geometry of the scalar sector is

$$\mathcal{M}=\frac{SU(1,1)}{U(1)}\times\frac{SO(6,n)}{SU(4)\times SO(n)}, \tag{3.32}$$

where n is the number of vectormultiplets.

$\mathcal{N}=8$ **supersymmetry** Beyond $\mathcal{N}=4$ there are no multiplets with helicites $h\leq 1$. This means that we are necessarily working with a supergravity theory with local supersymmetry. Beyond $\mathcal{N}=8$, the multiplets have helicities $h>2$ and so they are usually not considered in Minkowski backgrounds as already mentioned. In that sense, $\mathcal{N}=8$ is the maximal amount of supersymmetries possible. The theory has a unique multiplet

h	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$+\frac{1}{2}$	+1	$+\frac{3}{2}$	+2
multiplicity	1	8	28	56	70	56	28	8	1

and the target space of the scalar sector is completely fixed by supersymmetry

$$\mathcal{M} = \frac{E_{7,7}}{SU(8)}. \tag{3.33}$$

This theory is of particular interest in string theory as it can be obtained as a dimensional reduction from 11D supergravity or 10D Type II supergravity.

In the case of massive representations the supercharges never act trivially on the states so one gets twice the amount of fermionic creation and annihilation operators compared to massless representations. If the supersymmetry algebra contains central charges $Z_i, i = 1, \dots, l$ we can order them by size $|Z_1| \geq |Z_2| \dots$ It can be shown that all states in a supersymmetry representation satisfy the mass bound $M \geq |Z_1| \geq |Z_2| \dots$, called the *BPS bound*. If one or several of the bounds is saturated, then the number of supercharges that act non-trivially get reduced, hence reducing the dimension of the representation. Such representations that saturate the BPS bound are called *short* or *BPS representations*. The case in which all bounds are saturated gives the shortest BPS representation possible, which has as many states as a massless representation.

The existence of short multiplets is an extremely powerful tool to study the strong coupling regime of extended supersymmetric quantum field theories because the mass of the states is protected by supersymmetry. It is the use of these techniques that lead to the study of the strong coupling regime of Type IIA string theory and the emergence of M-theory [83].

We will not give a detailed account of supersymmetry representations in higher dimensions, we refer the reader to [97] and [98] for details. We simply mention that the 11D supergravity theory, which is maximal in this dimension, has a field content comprised of a metric G_{MN}, a 3-form gauge field A_{MNP} and a Majorana vector-spinor Ψ_M and has the following bosonic action

$$S^{11D} = \frac{1}{2\kappa_{11}^2} \int d^{11}x \sqrt{|G|} \left(R - \frac{1}{2}|F_4|^2 \right) - \frac{1}{6} \int A_3 \wedge F_4 \wedge F_4. \tag{3.34}$$

Doing a dimensional reduction (technique introduced in the next section), the 11D supersymmetry splits into a left- and right-chiral one, and thus becomes the supergravity multiplet of Type IIA supergravity introduced in the previous chapter whose bosonic action is

$$\begin{aligned} S_{IIA} =& \frac{1}{2\kappa_{10}^2} \int \mathrm{d}^{10}x \sqrt{|g|} \left(e^{-2\Phi} \left[\mathcal{R} - \frac{1}{2} |H_3|^2 + 4(\nabla\Phi)^2 \right] - \frac{1}{2} |F_2|^2 - \frac{1}{2} \left| \tilde{F}_4 \right|^2 \right) \\ & - \frac{1}{4\kappa_{10}^2} \int B_2 \wedge F_4 \wedge F_4. \end{aligned} \tag{3.35}$$

The other theory with 32 supercharges in 10 dimensions is the one with two supercharges of the same chirality, which is the Type IIB supergravity also introduced in the previous chapter and whose bosonic action is

$$\begin{aligned}S_{IIB} =& \frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}x\sqrt{|g|}e^{-2\Phi}\left[\mathcal{R} - \frac{1}{2}\left|H_3\right|^2 + 4(\nabla\Phi)^2\right] \\ &- \frac{1}{4\kappa_{10}^2}\int \mathrm{d}^{10}x\sqrt{|g|}\left[\left|F_1\right|^2 + \left|\tilde{F}_3\right|^2 + \frac{1}{2}\left|\tilde{F}_5\right|^2\right] - \frac{1}{4\kappa_{10}^2}\int B_2\wedge F_3\wedge F_5. \end{aligned} \tag{3.36}$$

Both theories reduce to the same 9D maximal supergravity theory, which is at the heart of the T-duality relation between them (see appendix B for an explicit derivation).

We have so far mentioned several times of going from a higher dimensional supergravity theory to a lower dimensional one. We now turn to this process called *dimensional reduction*, which is a powerful technique and was revived with the advent of supergravity, in the 80s [99].

3.2 Kaluza-Klein dimensional reduction

The original idea of Kaluza-Klein theory dates back to the 1920s when Kaluza and Klein considered the dimensional reduction of 5D gravity as a way to unify, in 4 dimensions, gravity with Maxwell's electromagnetism. This section follows [100].

Let us start by considering the simple dimensional reduction of a scalar field satisfying a massless wave equation in higher dimension

$$\hat{\Box}\hat{\phi} = 0, \tag{3.37}$$

where $\hat{\Box} = \partial^{\hat{\mu}}\partial_{\hat{\mu}}$. The hat symbol is here to make it explicit that the objects are defined in the higher dimensional theory. We want to compactify one of the coordinates, which we call z on a circle S^1 of radius L (we assume for now that this coordinate is spacelike and we will show how the final results are modified in the timelike case). The field is expanded in Fourier modes

$$\hat{\phi}(x,z) = \sum_n \phi_n(x)e^{inz/L}, \tag{3.38}$$

where x denotes collectively all the lower dimensional coordinates. We see that the equation of motion becomes

$$\Box\phi_n - \frac{n^2}{L^2}\phi_n = 0, \tag{3.39}$$

which is the wave equation of a scalar field of mass $|n|/L$.

Therefore a higher dimensional massless field reduces to a massless field and an infinite tower of massive states. In dimensional reduction, we assume that we probe the theory at an energy scale lower than the mass of the first massive state. In the effective field theory all massive states are truncated out, and the fields do not depend on the extra dimension.

This truncation is said to be a *consistent* truncation. In a nutshell, it means that the fields that we kept cannot be sources for the fields that were truncated, therefore the lower dimensional equations of motion are consistent with the higher dimensional ones.

Note that this argument does not work in the timelike reduction case. To see this, let's consider again our wave equation this time with parameter ϵ

$$\Box\phi_n - \epsilon\frac{n^2}{L^2}\phi_n = 0, \tag{3.40}$$

where $\epsilon = +1$ in the spacelike case and $\epsilon = -1$ in the timelike case. In the latter case, the modes with $n > 0$ are seen from the 4D perspective as tachyons, and therefore not only does the effective theory argument breaks down but we might run into causality and unitarity problem. To avoid these problems, we follow the argument of [101]. In essence, if we consider plane waves $\phi_n = \phi_n^0 \exp[-i(E_n t - \mathbf{p}_n \mathbf{x})]$ then the enery momentum relationship reads

$$E_n^2 - \mathbf{p}_n^2 - \epsilon\frac{n^2}{L^2} = 0. \tag{3.41}$$

If we consider the timelike case $\epsilon = -1$, we can have a second interpretation that avoids the aforementioned problems. Instead of interpreting the n^2/L^2 as a mass, we instead group it with the energy term

$$\bar{E}_n^2 = E_n^2 + \frac{n^2}{L^2}. \tag{3.42}$$

With this interpretation, we no longer have a tachyonic tower of states, instead the excitations differ from the $n = 0$ mode just in their energies. In the spacelike case, KK modes correspond to different particles, but in the timelike case, they can be viewed as excited energy states of the same particle.

3.2.1 Einstein-Hilbert term reduction

We will be interested in the dimensional reduction of supergravity theories, so let us start with the reduction of the Einstein-Hilbert term itself. The full fledged computation is very tedious so we will only go through the main steps without delving too much in cumbersome computational details.

We start with the Einstein-Hilbert action in (d+1) dimensions. We will represent higher dimensional quantities with a hat. So our starting point is

$$\mathcal{L}_{EH} = \sqrt{-\hat{g}}\hat{R}. \tag{3.43}$$

We can therefore decompose the metric in a Fourier series as such

$$\hat{g}_{\hat{\mu}\hat{\nu}}(x,z) = \sum_n g_{\mu\nu}^{(n)}(x)e^{inz/L}. \tag{3.44}$$

Performing the dimensional reduction means we lose the dependence on z and keep only the zero mode. We therefore decompose our higher metric using the Kaluza-Klein ansatz

$$ds^2_{(d+1)} = e^{2\alpha\phi} ds^2_d + e^{2\beta\phi}(dz + V)^2, \tag{3.45}$$

where α and $\beta \neq 0$ are parameters that we will choose appropriately, V is the Kaluza-Klein vector and ϕ is the Kaluza-Klein scalar. We can write the metric in matrix form. To do this we express the metric the following way

$$\begin{aligned} ds^2_{d+1} &= e^{2\alpha\phi} ds^2_d + e^{2\beta\phi}(dz + V_\mu dx^\mu)^2 \\ &= g_{\mu\nu} dx^\mu dx^\nu + e^{2\beta\phi} dz^2 + e^{2\beta\phi} V_\mu V_\nu dx^\mu dx^\nu + 2e^{2\beta\phi} V_\mu dz dx^\mu \\ \Rightarrow \hat{g}_{\hat{\mu}\hat{\nu}} dx^{\hat{\mu}} dx^{\hat{\nu}} &= e^{2\beta\phi} dz^2 + 2e^{2\beta\phi} V_\mu dz dx^\mu + \left(e^{2\alpha\phi} g_{\mu\nu} + e^{2\beta\phi} V_\mu V_\nu\right) dx^\mu dx^\nu . \end{aligned}$$

From this we can read off the matrix components

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} e^{2\alpha\phi} g_{\mu\nu} + e^{2\beta\phi} V_\mu V_\nu & e^{2\beta\phi} V_\mu \\ e^{2\beta\phi} V_\nu & e^{2\beta\phi} \end{pmatrix} . \tag{3.46}$$

The reduction of the Ricci scalar is simpler in the vielbein formalism so we now aim to get a reduction ansatz for the $d+1$ vielbein. By definition

$$\hat{g}_{\hat{\mu}\hat{\nu}} = \hat{e}_{\hat{\mu}}{}^{\hat{a}} \hat{e}_{\hat{\nu}}{}^{\hat{b}} \hat{\eta}_{\hat{a}\hat{b}}. \tag{3.47}$$

We consider first the components $\hat{\mu} = \mu$ and $\hat{\nu} = \nu$

$$\begin{aligned} \hat{g}_{\mu\nu} &= \hat{e}_\mu{}^{\hat{a}} \hat{e}_\nu{}^{\hat{b}} \hat{\eta}_{\hat{a}\hat{b}} \\ \Rightarrow e^{2\alpha\phi} g_{\mu\nu} + e^{2\beta\phi} V_\mu V_\nu &= \hat{e}_\mu{}^a \hat{e}_\nu{}^b \eta_{ab} + \hat{e}_\mu{}^z \hat{e}_\nu{}^z . \end{aligned}$$

These equations are satisfied if we choose

$$\hat{e}_\mu{}^z = e^{\beta\phi} V_\mu \qquad \text{and} \qquad \hat{e}_\mu{}^a = e^{\alpha\phi} e_\mu{}^a . \tag{3.48}$$

We now consider the components $\hat{\mu} = \mu$ and $\hat{\nu} = z$ and we get

$$\begin{aligned} \hat{g}_{\mu z} &= \hat{e}_\mu{}^{\hat{a}} \hat{e}_z{}^{\hat{b}} \hat{\eta}_{\hat{a}\hat{b}} \\ \Rightarrow e^{2\beta\phi} V_\mu &= e^{\alpha\phi} e_\mu{}^a \hat{e}_z{}^b \eta_{ab} + e^{\beta\phi} V_\mu \hat{e}_z{}^z . \end{aligned}$$

which is satisfied for $\hat{e}_z{}^b = 0$ and $\hat{e}_z{}^z = e^{\beta\phi}$. Therefore the ansatz for our vielbein decomposition is

$$\hat{e}_{\hat{\mu}}{}^{\hat{a}} = \begin{pmatrix} e^{\alpha\phi} e_\mu{}^a & 0 \\ e^{\beta\phi} V_\mu & e^{\beta\phi} \end{pmatrix} . \tag{3.49}$$

This ansatz is unique up to Lorentz transformations. We can now relate directly the d and $d+1$ dimensional vielbeins

$$\hat{e}^a = \hat{e}_{\hat{\mu}}{}^a dx^{\hat{\mu}} = \hat{e}_\mu{}^a dx^\mu + \hat{e}^a_z dz = e^{\alpha\phi} e_\mu{}^a dx^\mu + 0 = e^{\alpha\phi} e^a , \tag{3.50}$$

$$\hat{e}^z = \hat{e}_{\hat{\mu}}{}^z dx^{\hat{\mu}} = \hat{e}_z{}^z dz + \hat{e}_\mu{}^z dx^\mu = e^{\beta\phi} dz + e^{\beta\phi} V_\mu dx^\mu = e^{\beta\phi}(V + dz). \tag{3.51}$$

Setting the torsion to zero we compute the following 1-forms

$$\begin{aligned} d\hat{e}^a &= \alpha d\phi e^{\alpha\phi} \wedge e^a + e^{\alpha\phi} de^a \\ &= \alpha \partial_b \phi e^{\alpha\phi} e^b \wedge e^a + e^{\alpha\phi}(-\omega^a{}_b \wedge e^b) \\ &= \alpha \partial_b \phi e^{-\alpha\phi} \hat{e}^b \wedge \hat{e}^a - \omega^a{}_b \wedge \hat{e}^b, \end{aligned}$$

where we have exploited Cartan's first structure equation. We compute the other one-form

$$\begin{aligned} d\hat{e}^z &= d(e^{\beta\phi}(dz + V)) \\ &= \beta d\phi e^{\beta\phi} \wedge (dz + V) + e^{\beta\phi} dV \\ &= \beta e^{-\alpha\phi} \partial_a \phi \hat{e}^a \wedge \hat{e}^z + e^{(\beta - 2\alpha)\phi} \frac{1}{2} F_{ab} \hat{e}^a \wedge \hat{e}^b. \end{aligned}$$

Where we defined $F = dV$ such that

$$F = \frac{1}{2} F_{ab} e^a \wedge e^b = \frac{1}{2}(\partial_a V_b - \partial_b V_a) e^a \wedge e^b \tag{3.52}$$

We can now derive the spin connection components using Cartan's first structure equation

$$-\hat{\omega}^z{}_a \wedge \hat{e}^a = d\hat{e}^z \implies \hat{\omega}^{az} = -\beta e^{-\alpha\phi} \partial^a \phi \hat{e}^z - \frac{1}{2} e^{(\beta - 2\alpha)\phi} F^a{}_b \hat{e}^b. \tag{3.53}$$

Moreover we write

$$d\hat{e}^a = -\hat{\omega}^a{}_b \wedge \hat{e}^b - \hat{\omega}^a{}_z \wedge \hat{e}^z = \alpha e^{-\alpha\phi} \partial_b \phi \hat{e}^b \wedge \hat{e}^a - \omega^a{}_b \wedge e^b. \tag{3.54}$$

We can therefore write

$$-\hat{\omega}^a{}_b \wedge \hat{e}^b = -\omega^a{}_b \wedge \hat{e}^b - \alpha e^{-\alpha\phi} \partial_b \phi \hat{e}^a \wedge \hat{e}^b + \frac{1}{2} e^{(\beta - 2\alpha)\phi} F^a{}_b \hat{e}^z \wedge \hat{e}^b + \alpha e^{-\alpha\phi} \partial_a \phi \underbrace{\hat{e}^b \wedge \hat{e}^b}_{=0}. \tag{3.55}$$

So all in all we have

$$\hat{\omega}^{ab} = \omega^{ab} + \alpha e^{-\alpha\phi} \left(\partial^b \phi \hat{e}^a - \partial^a \phi \hat{e}^b \right) - \frac{1}{2} e^{(\beta - 2\alpha)\phi} F^{ab} \hat{e}^z. \tag{3.56}$$

Having derived the reduction formulas we are now ready to procede with the reduction of the curvature two-form. Before this, we want to choose the values of the parameters α and β. We want to obtain the dimensionally reduced Lagrangian in Einstein frame and we want the scalar kinetic term to have the canonical normalisation. This imposes the following choices for our parameters

$$\alpha^2 = \frac{1}{2(D-1)(D-2)}, \quad \beta = -(D-2)\alpha. \tag{3.57}$$

To get the reduction of the curvature we use Cartan's second structure equation

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b. \tag{3.58}$$

Carrying out the reduction is straightforward but cumbersome so we give directly the reduced formulas

$$\hat{R}_{ab} = e^{-2\alpha\phi}\left(R_{ab} - \frac{1}{2}\partial_a\phi\partial_b\phi - \alpha\eta_{ab}\Box\phi\right) - \frac{1}{2}e^{-2D\alpha\phi}F_a{}^cF_{bc}, \tag{3.59}$$

$$\hat{R}_{az} = \frac{1}{2}e^{(D-3)\alpha\phi}\nabla^b\left(e^{-2(D-1)\alpha\phi}F_{ab}\right), \tag{3.60}$$

$$\hat{R}_{zz} = (D-2)\alpha e^{-2\alpha\phi}\Box\phi + \frac{1}{4}e^{-2D\alpha\phi}F^2. \tag{3.61}$$

Finally we can get the curvature scalar by contracting the indices $\hat{R} = \eta^{\hat{a}\hat{b}}\hat{R}_{\hat{a}\hat{b}}$ which gives

$$\begin{aligned}\hat{R} &= \eta^{ab}\left(e^{-2\alpha\phi}(R_{ab} - \frac{1}{2}\partial_a\phi\partial_b\phi - \alpha\eta_{ab}\Box\phi) - \frac{1}{2}e^{-2D\alpha\phi}F_a{}^cF_{bc}\right)\\ &\quad + (D-2)\alpha e^{-2\alpha\phi}\Box\phi + \frac{1}{4}e^{-2D\alpha\phi}F^2\\ &= e^{-2\alpha\phi}\left(R - \frac{1}{2}(\partial\phi)^2 + (D-3)\alpha\Box\phi - \frac{1}{4}e^{-2D\alpha\phi}F^2\right).\end{aligned} \tag{3.62}$$

The determinant of the metric reduces as follows

$$\sqrt{-\hat{g}} = e^{(\beta+D\alpha)\phi}\sqrt{-g} = e^{2\alpha\phi}\sqrt{-g}. \tag{3.63}$$

Therefore all in all the dimensional reduction of the Einstein-Hilbert action is

$$\begin{aligned}S_{EH} &= \frac{1}{16\pi G_N^{D+1}}\int d^{D+1}\sqrt{-\hat{g}}\hat{R}\\ &= \frac{2\pi L}{16\pi G_N^{D+1}}\int d^Dx\sqrt{-g}\left(R - \frac{1}{2}(\nabla\phi)^2 - \frac{\epsilon}{4}e^{-2(D-1)\alpha\phi}F^2\right),\end{aligned} \tag{3.64}$$

where we have reintroduced the ϵ parameter to take into account the timelike reduction case.

3.2.2 p-form gauge field kinetic term reduction

Now that we have seen how to reduce the Einstein-Hilbert term in the action, we turn to the reduction of the kinetic term of antisymmetric p-form fields, as they are ubiquitous in supergravity. The general form of this term is

$$S = \int -\frac{1}{2}\hat{F}_p \wedge \star\hat{F}_p. \tag{3.65}$$

We can infer just from the index structure that a p-form field will give rise, after an S^1 reduction, to a p-form and a $(p-1)$-form. We choose the following reduction ansatz for the associated gauge potential

$$\hat{A}_{p-1}(x,z) = A_{p-1}(x) + A_{p-2}(x) \wedge dz. \tag{3.66}$$

We could identify the lower dimensional field strength directly from this but it turns out to not be a convenient choice. Instead we add and substract a term so that

$$\begin{aligned}\hat{F}_p &= dA_{p-1} - dA_{p-2} \wedge V + dA_{p-2} \wedge (dz + V) \\ &\equiv \tilde{F}_p + F_{p-1} \wedge (dz + V),\end{aligned} \tag{3.67}$$

where V is the Kaluza-Klein vector and

$$\tilde{F}_p = dA_{p-1} - dA_{p-2} \wedge V, \quad F_{p-1} = dA_{p-2}. \tag{3.68}$$

We can now reduce the field strength in a vielbein basis such that

$$\begin{aligned}\hat{F} &= \frac{1}{p!}\hat{F}_{\hat{\mu}_1\dots\hat{\mu}_p}\hat{e}^{\hat{\mu}_1} \wedge \cdots \wedge \hat{e}^{\hat{\mu}_p} \\ &= \frac{e^{n\alpha\phi}}{n!}\hat{F}_{\mu_1\dots\mu_p}e^{\mu_1} \wedge \cdots \wedge e^{\mu_p} + \frac{e^{((n-1)\alpha+\beta)\phi}}{(n-1)!}\hat{F}_{\mu_1\dots\mu_{p-1}z}e^{\mu_1} \wedge \cdots \wedge e^{\mu_{p-1}} \wedge (dz+V) \\ &\equiv \frac{1}{n!}\tilde{F}_{\mu_1\dots\mu_n}e^{\mu_1} \wedge \cdots \wedge e^{\mu_p} + \frac{1}{(n-1)!}F_{\mu_1\dots\mu_{p-1}}e^{\mu_1} \wedge \cdots \wedge e^{\mu_{p-1}} \wedge (dz+V),\end{aligned} \tag{3.69}$$

from which we can read off

$$\hat{F}_{\mu_1\dots\mu_p} = e^{-n\alpha\phi}\tilde{F}_{\mu_1\dots\mu_p}, \quad \hat{F}_{\mu_1\dots\mu_{p-1}z} = e^{(D-p-1)\alpha\phi}F_{\mu_1\dots\mu_{p-1}} \tag{3.70}$$

where we have expressed β in terms of α. Finally, bearing in mind the factor coming from the reduction of the determinant of the metric we have after reduction (and reintroducing the ϵ parameter)

$$S = \int -\frac{1}{2}\hat{F}_p \wedge \star\hat{F}_p = \int -\frac{1}{2}e^{-2(p-1)\alpha\phi}\tilde{F}_p \wedge \star\tilde{F}_p - \frac{\epsilon}{2}e^{2(D-p)\alpha\phi}F_{p-1} \wedge \star F_{p-1}. \tag{3.71}$$

We will not show explicitly how to reduce topological terms as they are more case dependent. However we show the reduction of Type II Chern-Simons term in detail in appendix B where we derive the generalized Buscher rules.

3.3 Calabi-Yau manifolds

The only manifold that has been discussed so far for the extra dimensions of spacetime was the circle S^1. One can consider toroidal backgrounds, where the extra dimensions are just product of circles, but when string theories are compactified on tori, none of the supersymmetry is broken leading to theories in 4 dimensions with either $\mathcal{N} = 4$ or $\mathcal{N} = 8$. These theories are very far from phenomenological since supersymmetry is broken in the real world. This objective of reducing supersymmetry after compactification leads us, as we will see, to consider a very special type of manifold called *Calabi-Yau* manifolds, which have the very interesting property of breaking supersymmetry partially. This allows one to have theories that are more viable phenomenologically, while still being computationally tractable thanks to the remaining amount of supersymmetry. This section is mainly based on [63] and we refer the reader to appendix A for relevant definitions and concepts of geometry and topology.

3.3.1 Definition

A Calabi-Yau n-fold is a Kähler manifold with n complex dimensions and vanishing first Chern class

$$c_1 = \frac{1}{2\pi}[\mathcal{R}] = 0. \tag{3.72}$$

It was conjectured by Calabi, and subsequently proven by Yau [102], that any compact Kähler manifold with $c_1 = 0$ admits a Kähler metric with $SU(N)$ holonomy. We will see soon that manifolds with $SU(N)$ holonomy are necessarily Ricci flat, therefore metrics with such holonomy correspond precisely to Kähler manifolds of vanishing first Chern class.

Betti numbers are fundamental topological invariants associated to manifolds (see appendix A). The Betti number b_p is the dimension of the p-th de Rham cohomology $H^p(M)$ of the manifold M. If the manifold is equipped with a metric, these numbers count the number of linearly independant harmonic p-forms on M. For Kähler manifolds, they can be decomposed into *Hodge numbers* $h^{p,q}$, which count the number of harmonic (p,q)-forms on M

$$b_k = \sum_{p=0}^{k} h^{p,k-p}. \tag{3.73}$$

Calabi-Yau n-folds are characterized by the value of their Hodge numbers[1]. Symmetries and dualities relate different Hodge numbers and therefore only a subset are independant. The Hodge numbers satisfy

$$h^{p,0} = h^{n-p,0}. \tag{3.74}$$

This comes from the fact that the spaces $H^p(M)$ and $H^{n-p}(M)$ are isomorphic. Complex conjugation gives

$$h^{p,q} = h^{q,p}, \tag{3.75}$$

and Poincaré duality gives

$$h^{p,q} = h^{n-q,n-p}. \tag{3.76}$$

Any compact connected manifold has $h^{0,0} = 1$, which corresponds to constant functions. A *simply connected* manifold has vanishing *fundamental group* (the first homotopy group),

[1]This characterization is not complete since inequivalent Calabi-Yau manifolds sometimes have the same Hodge numbers. Wall proved that Calabi-Yau threefolds can be characterized up to homotopy by the Euler characteristic, second Chern class and triple intersection numbers [103].

and therefore vanishing first homology group, whence[2]

$$h^{1,0} = h^{0,1} = 0. \tag{3.77}$$

Since we will compactify theories from 10 to 4 dimensions, the manifolds of interest for us are Calabi-Yau threefolds. In this case, the complete cohomological description of the manifold only requires specifying $h^{1,1}$ and $h^{2,1}$. We can display the full set of Hodge numbers in the following *Hodge diamond*

$$\begin{array}{ccccccc} & & & h^{3,3} & & & \\ & & h^{3,2} & & h^{2,3} & & \\ & h^{3,1} & & h^{2,2} & & h^{1,3} & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ & & h^{1,0} & & h^{0,1} & & \\ & & & h^{0,0} & & & \end{array} = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{2,2} & & 0 & \\ 1 & & h^{2,1} & & h^{1,2} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

Using relations discussed above one can express the Euler characteristic of the Calabi-Yau threefold as

$$\chi = \sum_{p=0}^{6} (-1)^p b_p = 2(h^{1,1} - h^{2,1}). \tag{3.78}$$

3.3.2 Mirror symmetry

We have already seen in the context of T-duality that the geometry probed by strings is very different from the geometry seen by point particles. A similar phenomenon appears in the context of Calabi-Yau manifolds that goes by the name of *mirror symmetry*. The mirror map associates with almost[3] any Calabi-Yau threefold M another Calabi-Yau threefold W such that

$$H^{p,q}(M) = H^{3-p,q}(W). \tag{3.79}$$

This conjecture implies in particular that $h^{1,1}(M) = h^{2,1}(W)$ and vice-versa. The precise statement of the conjecture is too technical and beyond the scope of this work, what is important for us is that this conjecture implies that Type IIA compactified on M is exactly equivalent to Type IIB on W. This can be at first value a very surprising statement since the two Calabi-Yau threefolds are in general very different from the classical geometry

[2] Calabi-Yau manifolds that are not simply connected can be constructed by modding out by discrete freely acting isometry groups. In cases of interest, these groups are finite and thus the resulting Calabi-Yau still satisfies $h^{1,0} = h^{0,1} = 0$, see [104] for details.

[3] The few cases where it fails, the mirror still exist but it does not define a Calabi-Yau threefold.

point of view. Even the most basic topology of the manifolds are different since their Euler characteristics are related by

$$\chi(M) = -\chi(W). \tag{3.80}$$

We can give a heuristic of how mirror symmetry works in simple cases through the lense of T-duality. We have already seen that Type IIA and Type IIB compactified on a circle are T-dual to each-other, which can be interpeted as a form of mirror symmetry. Let's now consider the case of the torus $T^2 = S^1 \times S^1$ where the first circle has radius R_1 and the second circle has radius R_2. We can intepret the torus as an S^1 fibration over an S^1 base. We can characterize this torus by its complex-structure and Kähler-structure parameters (see next section for details)

$$\tau = i\frac{R_2}{R_1} \quad \text{and} \quad \rho = iR_1R_2. \tag{3.81}$$

The shape, or complex structure of the torus, is described by τ while the size, or Kähler structure, is described by ρ. If we perform a T-duality on the fiber circle we know that $R_1 \to \frac{1}{R_1}$ such that the resulting torus now has parameters

$$\tilde{\tau} = iR_1R_2 \quad \text{and} \quad \tilde{\rho} = i\frac{R_2}{R_1}. \tag{3.82}$$

This shows that under the mirror map, complex-structure and Kähler-structure parameters are interchanged, which also happens in the Calabi-Yau threefold case. Actually interpreting mirror symmetry as a manifestation of T-duality is possibly more than just heuristic. Calabi-Yau threefolds that have a mirror are conjectured to be T^3 fibrations over a base B, which is the essence of the SYZ conjecture [105]. Mirror symmetry would then be a fiber-wise T-duality on all three directions of the T^3. Since the number of dualities is odd, even and odd forms are interchanged and so the $(1,1)$ and $(2,1)$ cohomologies are interchanged, as expected from mirror symmetry. In 1991, mirror symmetry was used to solve important problems in enumerative geometry [106] that until then resisted the mathematical community and is just one of many examples of the fruitfulness of string theory in pure mathematics.

3.3.3 Conditions for unbroken supersymmetry

We will now show how the condition of unbroken supersymmetry in the lower dimensional theory imposes on us the use of Calabi-Yau manifolds as the manifold on which dimensions are compactified. We assume that spacetime is decomposed into a product of a non-compact four-dimensional spacetime M_4 and a six-dimensional internal manifold M

$$M_{10} = M_4 \times M. \tag{3.83}$$

Our convention for indices is such that x^M correspond to coordinates of M^{10}, x^μ corresponds to M^4 and y^m corresponds to M. The fact that the resulting theory preserves some supersymmetry imposes constraints on the vacua that arise. Each of the supersymmetry charges Q_α generates an infinitesimal transformation of all the fields with an assiocated infinitesimal parameter ϵ_α. A particular background will be left invariant under unbroken supersymmetries. The invariance of the bosonic fields is trivial because the supersymmetry transformation will contain at least one fermionic field, which are known to vanish on classical backgrounds. The nontrivial conditions are therefore coming from the fermionic sector

$$\delta_\epsilon(\text{fermionic fields}) = 0. \tag{3.84}$$

If the expectation value of the fermions still vanish after performing a supersymmetry variation, then one obtains a solution of the bosonic equations of motion that preserves supersymmetry for the backgrounds considered. In fact, a solution to the supersymmetry constraints is always a solution of the equations of motion (but the converse is not true).

The supersymmetry transformations of Type IIA supergravity are

$$\delta\lambda = \left(-\frac{1}{3}\Gamma^M\partial_M\Phi\Gamma_{11} + \frac{1}{6}\mathbf{H} - \frac{1}{4}e^\Phi\mathbf{F}^{(2)} + \frac{1}{12}e^\Phi\tilde{\mathbf{F}}^{(4)}\Gamma_{11}\right)\varepsilon \tag{3.85}$$

$$\delta\Psi_M = \left(\nabla_\mu - \frac{1}{4}\mathbf{H}_M\Gamma_{11} - \frac{1}{8}e^\Phi F_{NP}\Gamma_M{}^{NP}\Gamma_{11} + \frac{1}{8}e^\Phi\mathbf{F}^{(4)}\Gamma_M\right)\varepsilon, \tag{3.86}$$

where $\mathbf{A} = \frac{1}{p!}A_{M_1\ldots M_p}\Gamma^{M_1\ldots M_p}$ and all the spinors are of Majorana type. For Type IIB the transformations are

$$\delta\lambda = \frac{1}{2}(\partial_M\Phi - ie^\Phi\partial_M C_0)\Gamma^M\varepsilon + \frac{1}{4}\left(ie^\Phi\tilde{\mathbf{F}}^{(3)} - \mathbf{H}\right)\varepsilon^\star \tag{3.87}$$

$$\delta\Psi_M = \left(\nabla_\mu + \frac{i}{8}e^\Phi\mathbf{F}^{(1)}\Gamma_M + \frac{i}{16}e^\Phi\tilde{\mathbf{F}}^{(5)}\Gamma_M\right)\varepsilon - \frac{1}{8}\left(2\mathbf{H}_M + ie^\Phi\tilde{\mathbf{F}}^{(3)}\Gamma_M\right)\varepsilon^\star, \tag{3.88}$$

where the spinors are of Weyl type. If we focus on the case of a compactification without fluxes for the Kalb-Ramond field or for the Ramond-Ramond fields, we see that the supersymmetry transformations for the gravitini reduce simply to

$$\delta\Psi_\mu = \nabla_\mu\epsilon = 0. \tag{3.89}$$

This is known as the *Killing spinor equation*, it means that ϵ is a covariantly conserved spinor. We can decompose the supersymmetry parameter into external and internal parts as

$$\epsilon(x,y) = \zeta(x)\otimes\eta(y). \tag{3.90}$$

If we restrict ourselves to the internal manifold we have

$$\nabla_m\eta = 0. \tag{3.91}$$

This leads to the following *integrability condition*

$$[\nabla_m, \nabla_n]\eta = \frac{1}{4} R_{mnpq}\Gamma^{pq}\eta = 0. \tag{3.92}$$

This in turn implies that the internal manifold M is Ricci-flat

$$R_{mn} = 0. \tag{3.93}$$

For an orientable six-dimensional spin manifold[4], if one parallel transports a spinor η around a closed curve it will generically be rotated by a matrix $Spin(6) = SU(4)$ which is the generic *holonomy group*. A real spinor on such manifolds has eight components, which can be decomposed into two irreducible $SU(4)$ representations

$$\mathbf{8} = \mathbf{4} \oplus \bar{\mathbf{4}}, \tag{3.94}$$

where the $\mathbf{4}$ and $\bar{\mathbf{4}}$ represent spinors of opposite chirality which are complex conjugates of each other. Thus, a spinor of definite chirality has four complex components. A spinor that is covariantly conserved remains unchanged after being parallely transported along a closed curve. The largest subgroup of $SU(4)$ for which a spinor of definite chirality can be invariant is $SU(3)$. This is because the $\mathbf{4}$ has an $SU(3)$ decomposition

$$\mathbf{4} = \mathbf{3} \oplus \mathbf{1}, \tag{3.95}$$

and the singlet is invariant under $SU(3)$ transformations. Since the condition for unbroken supersymmetry in four dimensions is equivalent to the existence of a covariantly constant spinor on the internal manifold, it follows that the manifold should have $SU(3)$ holonomy.

Let us now show that the manifold is of Kähler type. We start by decomposing the covariantly constant spinor

$$\epsilon(x, y) = \zeta_+ \otimes \eta_+(y) + \zeta_- \otimes \eta_-(y), \tag{3.96}$$

where the fields $\eta_\pm(y)$ correspond to the covariantly constant spinors coming from the singlet pieces of the $\mathbf{4}$ and $\bar{\mathbf{4}}$ and $\zeta_\pm$ are two-component constant Weyl spinors on M_4. These spinors being covariantly constant they can be normalized $\eta_+^\dagger \eta_+ = \eta_-^\dagger \eta_- = 1$ and so we define the tensor

$$J_m{}^n = i\eta_+^\dagger \gamma_m \eta_+ = -i\eta_-^\dagger \gamma_m \eta_-, \tag{3.97}$$

which satisfies

$$J_m{}^n J_n{}^p = -\delta_m{}^p. \tag{3.98}$$

This implies that the manifold is almost complex, and J is the almost complex structure. Since the spinors $\eta_\pm$ and the metric are covariantly constant, the almost complex structure is covariantly constant as well

$$\nabla_m J_n{}^p = 0. \tag{3.99}$$

[4]A spin manifold is a manifold on which spinors can be defined

This implies that the associated Nijenhuis tensor vanishes identically $N^p{}_{mn} = 0$ making J a complex structure. This means that we can write local complex coordinates z^a and $\bar{z}^a$ such that

$$J_a{}^b = i\delta_a{}^b, \quad J_{\bar{a}}{}^{\bar{b}} = -i\delta_{\bar{a}}{}^{\bar{b}}, \quad \text{and} \quad J_a{}^{\bar{b}} = J_{\bar{a}}{}^b = 0. \tag{3.100}$$

From this we see easily that

$$g_{mn} = J_m{}^k J_n{}^l g_{kn}, \tag{3.101}$$

which implies that the metric is hermitian with respect to the complex structure J. We can therefore write

$$J_{mn} = J_m{}^k g_{kn}, \tag{3.102}$$

which defines the following antisymmetric two-form

$$J = \frac{1}{2} J_{mn} dx^m \wedge dx^n, \tag{3.103}$$

whose components are related to the metric according to

$$J_{a\bar{b}} = i g_{a\bar{b}}. \tag{3.104}$$

Moreover it is closed as

$$dJ = \partial J + \bar{\partial} J = i\partial_a g_{b\bar{c}} dz^a \wedge dz^b \wedge dz^{\bar{c}} + i\partial_{\bar{a}} g_{b\bar{c}} dz^{\bar{a}} \wedge dz^b \wedge dz^{\bar{c}} = 0. \tag{3.105}$$

As a result, the internal manifold is of Kähler type and J is the Kähler form. We have shown that the internal manifold is a Kähler, Ricci-flat compact manifold with $SU(3)$ holonomy and therefore, according to Yau's theorem, has vanishing first Chern-class. Thus concludes our proof that requiring partial supersymmetry breaking leads the compact manifold to be Calabi-Yau.

We can now consider possible fermion bilinears. The only non-zero possibility which is consistent with both chirality and symmetry is

$$\Omega_{abc} = \eta^T \gamma_{abc} \eta_-. \tag{3.106}$$

This can be used to define a nowhere-vanishing $(3,0)$-form

$$\Omega = \frac{1}{6} \Omega_{abc} dz^a \wedge dz^b \wedge dz^c \tag{3.107}$$

which is closed. Since η and the metric are covariantly conserved, it satisfies $\nabla_{\bar{d}} \Omega_{abc} = 0$. The connection terms vanish for a Kähler manifold and therefore one has $\bar{\partial}\Omega = 0$. It is obvious that $\partial\Omega = 0$ since there are only three holomorphic dimensions. Therefore $d\Omega = (\partial + \bar{\partial})\Omega = 0$ which means that it is indeed closed. It can also be shown that Ω is not exact. A Calabi-Yau manifold has $h^{3,0} = 1$ and Ω is a unique representative of the $(3,0)$ cohomology class. The existence of a holomorphic $(3,0)$-form implies that the manifold has a vanishing first Chern class, again showing that the manifold is Calabi-Yau.

3.3.4 Deformations of Calabi-Yau manifolds

As we mentioned previously, specifying Hodge numbers is not sufficient to completely determine a Calabi-Yau manifold. Some Calabi-Yau manifolds are related to each other by smooth transformations of the parameters that characterize their size and shape. These parameters are moduli. In this section, we will explain how these moduli parametrize the space of possible choices of the undetermined vacuum expectation values of massless scalar fields in 4D. This space is known as the *moduli space* of the Calabi-Yau compactification.

Antisymmetric tensor-field deformations We consider the spectrum of fluctuations about a given Calabi-Yau of fixed Hodge numbers. We start with the simple case of antisymmetric tensor fields. These are of importance in this work because the Type II supergravity theories contain various $p-$form fields. We can for example focus on the case of the Kalb-Ramond B-field. Since it is part of the Neveu-Schwarz sector, this field is common to both Type II theories, whose equation of motion is[5]

$$\Delta B_{p-1} = d \star dB_{p-1} = 0. \tag{3.108}$$

After compactification on the Calabi-Yau threefold the Laplacian is written as

$$\Delta = \Delta_4 + \Delta_6, \tag{3.109}$$

and the number of massless four-dimensional fields is given by the number of zero modes of the internal Laplacian Δ_6. These are counted by the Betti numbers b_p. Therefore after compactification we have in 4D two-forms, one-forms and zero-forms coming from the B-field, which we can summarize in the following table

B_{MN}	$\mathrm{B}_{\mu\nu}$	$\mathrm{B}_{\mu n}$	B_{mn}
p-form fields in 4D	p=2	p=1	p=0
# of fields in 4D	$\mathrm{b}_0 = 1$	$\mathrm{b}_1 = 0$	$\mathrm{b}_2 = h^{1,1}$

Here the b_2 scalar fields are moduli that come from the B-field. More generally, a p-form field will give rise to b_p moduli fields.

Metric deformations The ten-dimensional metric gives rise to a four-dimensional metric and a set of moduli. In Calabi-Yau compactifications there are no massless vector fields coming from the metric since $b_1 = 0$. This is related to the fact that the massless gauge fields are in one to one correspondence with the continuous isometries of the manifold, but Calabi-Yau manifolds don't have any. We analyse the deformations of the metric by performing a small variation

$$g_{mn} \to g_{mn} + \delta g_{mn}, \tag{3.110}$$

[5]assuming vanishing background field

and imposing that the new background still satisfies the Calabi-Yau conditions. In particular the new background has to be Ricci-flat such that

$$R_{mn}(g + \delta g) = 0. \tag{3.111}$$

This leads to differential equations for δg and the number of solutions count the number of ways to deform the background metric while preserving supersymmetry. The coefficients of these solutions are the moduli and they parameterize the changes of shape and size of the Calabi-Yau manifold. Some metric deformations only describe diffeomorphisms and we are not interested in these. To eliminate them we make the following gauge fixing

$$\nabla^m \delta g_{mn} = \frac{1}{2} \nabla_n \delta g^m_m. \tag{3.112}$$

We then expand (3.111) to first order in δg which leads to the *Lichnerowicz equation*

$$\nabla^k \nabla_k \delta g_{mn} + 2R_{m\ n}^{\ p\ q} \delta g_{pq} = 0. \tag{3.113}$$

The equations for the mixed components $\delta g_{a\bar{b}}$ and pure components δg_{ab} actually decouple.

- $\delta g_{a\bar{b}}$: (3.111) reduces to $(\Delta \delta g)_{a\bar{b}} = 0$. We can view $\delta g_{a\bar{b}}$ as the components of a (1,1)-form

  $$\delta g_{a\bar{b}} dz^a \wedge d\bar{z}^{\bar{b}}. \tag{3.114}$$

  Therefore the allowed metric variations of the form $\delta g_{a\bar{b}}$ correspond to harmonic (1,1)-forms. We can therefore expand in a basis

  $$\delta g_{a\bar{b}} = \sum_{\alpha=1}^{h^{1,1}} \tilde{t}^\alpha b^\alpha_{a\bar{b}}, \quad \tilde{t}^\alpha \in \mathbb{R}. \tag{3.115}$$

  These moduli (called Kähler moduli since they transform the Kähler form) have to be chosen so that the deformed metric is still positive definite. Positivity of the Kähler metric is equivalent to

  $$\int_{M_r} \underbrace{J \wedge \cdots \wedge J}_{\text{r-times}} > 0, \quad r = 1, 2, 3, \tag{3.116}$$

  where M_r is any complex r-dimensional submanifold of the Calabi-Yau. The subset of moduli that satisfy this condition is called the *Kähler cone* since if J satisfies the condition, so does λJ for any $\lambda \in \mathbb{R}_+$.

- δg_{ab}: In this case (3.111) reduces to $\Delta_{\bar{\partial}} \delta g^a = 0$ where $\delta g^a = \delta g^a_{\bar{b}} d\bar{z}^{\bar{b}}$ and $\delta g^a_{\bar{b}} = g^{a\bar{c}} \delta g_{\bar{c}\bar{b}}$. This (0,1)-form has values in $T^{1,0}(M)$, the holomorphic tangent bundle which we write T_M. The corresponding cohomology group is $H^{0,1}_{\bar{\partial}}(M, T_M)$. For a new metric to be Kähler, there must be a coordinate system where it has only mixed components. Holomorphic coordinate transformations do not change the type of index and so δg_{ab} can only be removed by a non-holomorphic transformation. The new metric is therefore Kähler with respect to a new complex structure. Therefore elements of $H^{0,1}_{\bar{\partial}}(M, T_M)$ correspond to deformations of the complex structure. Using the unique holomorphic (3,0) form, we can define an isomorphism between $H^{0,1}_{\bar{\partial}}(M, T_M)$ and $H^{2,1}_{\bar{\partial}}(M)$ by defining the following complex (2,1) forms

$$\Omega_{abc}\delta g^c_{\bar{d}} dz^a \wedge dz^b \wedge d\bar{z}^{\bar{d}}, \tag{3.117}$$

which is harmonic if (3.111) is satisfied. We can again expand the complex structure deformations in a basis

$$\Omega_{abc}\delta g^c_{\bar{d}} = \sum_{\alpha=1}^{h^{2,1}} t^\alpha b^\alpha_{ab\bar{d}}, \tag{3.118}$$

where the t^α are called the complex structure moduli.

The Kähler deformations moduli can be combined with the B-field moduli

$$(\delta B_{i\bar{j}} + i\delta g_{i\bar{j}}) dz^i \wedge d\bar{z}^{\bar{j}} = \sum_{\alpha=1}^{h^{1,1}} \tilde{t}^\alpha b^\alpha, \tag{3.119}$$

where $\tilde{t}^\alpha$ are now complex with their imaginary parts still restricted by the positive definiteness of the metric. This is what is known as the *complexification of the Kähler form* and we can write a complexified Kähler form

$$\mathcal{J} = B + iJ. \tag{3.120}$$

3.4 Special Geometry

The mathematics needed to describe the Calabi-Yau moduli spaces is known as *special geometry* which is described in this section. It is of particular interest for us because in the next chapter, we will study the theories obtained in 4D after compactifying Type II supergravity theories on a Calabi-Yau threefold, with a particular interest on the scalar geometry of the moduli. We follow [63] as well as the original paper [107].

The moduli space has a natural metric defined on it which is given by the sum of two pieces

$$ds^2 = \frac{1}{2V}\int g^{a\bar{b}} g^{c\bar{d}} \left[\delta g_{ac}\delta g_{\bar{b}\bar{d}} + (\delta g_{a\bar{d}}\delta g_{c\bar{b}} - \delta B_{a\bar{d}}\delta B_{c\bar{b}})\right] \sqrt{g} d^6 x. \tag{3.121}$$

For Calabi-Yau threefolds, the moduli space is locally isometric to the product of the moduli space of complex structures and the moduli space of Kähler structures

$$\mathcal{M}(M) = \mathcal{M}^{2,1}(M) \times \mathcal{M}^{1,1}(M). \tag{3.122}$$

3.4.1 Complex structure moduli space

Under deformations of the complex structure, the Calabi-Yau metric components change according to

$$\delta g_{\bar{a}\bar{b}} = -\frac{1}{||\Omega||^2}\bar{\Omega}_{\bar{a}}{}^{cd}(\chi_\alpha)_{cd\bar{b}}\delta t^\alpha, \quad \text{where} \quad ||\Omega||^2 = \frac{1}{6}\Omega_{abc}\bar{\Omega}^{abc}. \tag{3.123}$$

where t^α are local coordinates for the complex structure moduli space, with $\alpha = 1, \ldots, h^{2,1}$ and the χ_α are a set of (2,1)-forms. We therefore write the metric on the complex structure moduli space as

$$ds^2 = 2G_{\alpha\bar{\beta}}\delta t^\alpha \delta \bar{t}^{\bar{\beta}}, \tag{3.124}$$

and using these equations as well as (3.123) we can express this metric, called the Weil-Petersson metric, through scalar products of harmonic forms

$$G_{\alpha\bar{\beta}} = \frac{1}{V}(\chi_\alpha, \bar{\chi}_{\bar{\beta}}) = \frac{-i\int_M \chi_\alpha \wedge \bar{\chi}_{\bar{\beta}}}{i\int_M \Omega \wedge \bar{\Omega}}, \tag{3.125}$$

Under a change in complex structure the holomorphic $(3,0)$-form Ω becomes a linear combination of a $(3,0)$-form and $(2,1)$-forms since dz becomes a linear combination of dz and $d\bar{z}$. More precisely we have

$$\partial_\alpha \Omega = K_\alpha \Omega + \chi_\alpha, \tag{3.126}$$

where $\partial_\alpha = \frac{\partial}{\partial t^\alpha}$ and K_α depends on the coordinates t^α but not on the coordinates of the Calabi-Yau manifold M. Using this relation, we can see that we can derive the complex structure moduli space metric from the following Kähler potential

$$\mathcal{K}^{2,1} = -\log\left(i\int \Omega \wedge \bar{\Omega}\right). \tag{3.127}$$

We now exploit a relation between complex structures on M and the periods of the holomorphic top-form Ω. Choosing a complex structure on M amounts to specifying a decomposition of the third de-Rham cohomology group into Dolbeaut cohomology groups

$$H^3(M) = H^{3,0}_{\bar{\partial}}(M) \oplus H^{2,1}_{\bar{\partial}}(M) \oplus H^{1,2}_{\bar{\partial}}(M) \oplus H^{0,3}_{\bar{\partial}}(M). \tag{3.128}$$

Such a decomposition is obtained by picking one of the $b_3 = 1 + h^{2,1} + h^{2,1} + 1$ harmonic forms and declaring it to be the holomorphic top-form.

We now choose a basis (A^I, B_I), $I = 0, \ldots, h^{2,1}$ of the third homology group $H_3(M, \mathbb{Z})$ of M, with normalization

$$A^I \cdot B_J = \delta^I_J = -B_J \cdot A^I, \quad \text{and} \quad A^I \cdot A^J = B_I \cdot B_J = 0. \tag{3.129}$$

where $\cdot$ is the intersection product (which is defined by counting intersection points between submanifolds, weigthed with orientation). The Poincaré dual cohomology basis is denoted by (α_I, β^I) and we have

$$\int_{A^J} \alpha_I = \int \alpha_I \wedge \beta_J = \delta_I^J \quad \text{and} \quad \int_{B_J} \beta^I = \int \beta^I \wedge \alpha_J = -\delta_J^I. \tag{3.130}$$

The group of transformations that preserves these properties is the symplectic modular group $Sp(2h^{2,1}+2;\mathbb{Z})$. We can define coordinates X^I on the moduli space by using the A periods of the holomorphic three-form Ω

$$X^I = \int_{A^I} \Omega \quad \text{with} \quad I = 0, \ldots, h^{2,1}. \tag{3.131}$$

The number of coordinates defined this way is one more than the number of moduli fields. However, the coordinates X^I are only defined up to a complex rescaling , since the holomorphic three-form can be rescaled as $\Omega \to \lambda\Omega, \lambda \in \mathbb{C}^*$. We therefore define

$$t^\alpha = \frac{X^\alpha}{X^0} \quad \text{with} \quad \alpha = 1, \ldots, h^{2,1}. \tag{3.132}$$

This gives the right number of coordinates to describe the complex-structure moduli. Since we have the right number of coordinates, the B periods

$$F_I = \int_{B_I} \Omega \tag{3.133}$$

must be functions of X, that is $F_I = F_I(X)$. It follows that

$$\Omega = X^I \alpha_I - F_I(X)\beta^I. \tag{3.134}$$

A consequence of (3.126) is

$$\int \Omega \wedge \partial_I \Omega = 0, \tag{3.135}$$

which implies

$$F_I = X^J \frac{\partial F_J}{\partial X^I} = \frac{1}{2}\frac{\partial}{\partial X^I}(X^J F_J), \tag{3.136}$$

or equivalently

$$F_I = \frac{\partial F}{\partial X^I} \quad \text{where} \quad F = \frac{1}{2} X^I F_I. \tag{3.137}$$

As a result, all the B periods are expressed in terms of a single function F called the *prepotential*. Moreover, since

$$2F = X^I \frac{\partial F}{\partial X^I}, \tag{3.138}$$

it implies that F is homogeneous of degree two. Using the general rule for closed three-forms

$$\int_M \alpha \wedge \beta = -\sum_I \left(\int_{A^I} \alpha \int_{B^I} \beta - \int_{A^I} \beta \int_{B^I} \alpha \right), \tag{3.139}$$

we can write the Kähler potential as a function of the prepotential

$$\mathcal{K}^{2,1} = -\log\left[i\int_M \Omega\wedge\bar{\Omega}\right] = -\log\left[-i(X^I(z)\bar{F}_I(z) - F_I(z)\bar{X}^I(z))\right]. \qquad (3.140)$$

The cases where we can express the Kähler potential in such a fashion is what characterizes *special geometry*[6].

3.4.2 Kähler structure moduli space

We now turn to the Kähler structure moduli and we will show that this moduli space is again described by special geometry. We first consider an inner product on the space of (1,1) cohomology classes defined by

$$G(\rho,\sigma) = \frac{1}{2V}\int_M \rho_{a\bar{d}}\sigma_{\bar{b}c}g^{a\bar{b}}g^{c\bar{d}}\sqrt{g}d^6x = \frac{1}{2V}\int_M \rho\wedge\star\sigma, \qquad (3.141)$$

where $\star$ is the Hodge-star operator on the Calabi-Yau manifold. We define the cubic form

$$\kappa(\rho,\sigma,\tau) = \int_M \rho\wedge\sigma\wedge\tau. \qquad (3.142)$$

Using the following identity

$$\star\,\sigma = -J\wedge\sigma + \frac{1}{4V}\kappa(\sigma,J,J)J\wedge J, \qquad (3.143)$$

we can rewrite

$$G(\rho,\sigma) = -\frac{1}{2V}\kappa(\rho,\sigma,J) + \frac{1}{8V^2}\kappa(\rho,J,J)\kappa(\sigma,J,J). \qquad (3.144)$$

We recall that the complexified Kähler form can be expanded in terms of harmonic $(1,1)$-forms as follows

$$\mathcal{J} = B + iJ = \tilde{t}^\alpha b_\alpha \quad \text{with} \quad \alpha = 1,\ldots,h^{1,1}. \qquad (3.145)$$

Therefore the metric on the moduli space is then

$$G_{\alpha\bar{\beta}} = \frac{1}{2}G(b_\alpha,b_\beta) = \frac{\partial}{\partial\tilde{t}^\alpha}\frac{\partial}{\partial\bar{\tilde{t}}^\beta}\mathcal{K}^{1,1}, \qquad (3.146)$$

where

$$e^{-\mathcal{K}^{1,1}} = \frac{4}{3}\int J\wedge J\wedge J = 8V. \qquad (3.147)$$

These equations show that the space spanned by $\tilde{t}^\alpha$ is a Kähler manifold and the Kähler potential is given by the logarithm of the volume of the Calabi-Yau.

[6]To be more precise, in the cases in which the prepotential does not exist, in special Kähler manifolds, one can always perform a duality transformation to a symplectic frame where a prepotential exists

We also define the *intersection numbers*

$$\kappa_{\alpha\beta\gamma} = \kappa(b_\alpha, b_\beta, b_\gamma) = \int b_\alpha \wedge b_\beta \wedge b_\gamma, \tag{3.148}$$

in order to form

$$G(\tilde{t}) = \frac{1}{6}\frac{\kappa_{\alpha\beta\gamma}\tilde{t}^\alpha \tilde{t}^\beta \tilde{t}^\gamma}{\tilde{t}^0} = \frac{1}{6\tilde{t}^0}\int \mathcal{J}\wedge\mathcal{J}\wedge\mathcal{J}, \tag{3.149}$$

where we have introduced $\tilde{t}^0$ in order to make it a homogeneous function of degree two. We can finally write

$$\mathcal{K}^{1,1} = -\log(8V) = -\log\left[i\left(\tilde{t}^A\frac{\partial\bar{G}}{\partial\bar{\tilde{t}}^A} - \bar{\tilde{t}}^A\frac{\partial G}{\partial\tilde{t}^A}\right)\right]. \tag{3.150}$$

It is understood that in this formula the right hand-side is evaluated at $\tilde{t}^0 = 1$. We again have the Kähler potential expressed in terms of a prepotential $G(\tilde{t})$ so we indeed see that the Kähler structure moduli space is described by special geometry.

3.4.3 Special geometry of $\mathcal{N} = 2, D = 4$ supergravity

When compactifying Type II supergravity theories on Calabi-Yau threefolds, one obtains $\mathcal{N} = 2$ supergravity theories in four dimensions coupled to vectormultiplets and hypermultiplets. In the next chapter we will delve into the resulting scalar geometries of these theories. We will therefore not give a detailed account of these geometries here but only mention how the different moduli are distributed in the different supermultiplets. We will also mention how the results are affected when considering higher order corrections.

Moduli are either allocated to vector multiplets, where the geometry of the target space is (projective) special Kähler, or to hypermultiplets, where the geometry is quaternionic Kähler[7]. The dimension of a quaternionic Kähler manifold is divisible by four. Hypermultiplets contain a mixture of moduli of the metric, moduli resulting from reducing p-form gauge fields, and, for Type II string theory, the dilaton and the axion obtained from dualizing the Kalb-Ramond two-form. In IIA compactifications, complex structure moduli sit in hypermultiplets and Kähler moduli sit in vector multiplets while in IIB compactifications it is the other way around. The special Kähler spaces $\mathcal{M}^{1,1}$ and $\mathcal{M}^{2,1}$ are Kähler submanifolds of the scalar target manifolds, at least to lowest order in α'.

Type IIA compactified on a Calabi-Yau threefold M results in a four-dimensional theory containing $h^{1,1}$ abelian vectormultiplets and $h^{2,1}+1$ hypermultiplets. So the moduli space again takes locally the product form

$$\mathcal{M}^{1,1}(M) \times \mathcal{M}^{2,1}(M). \tag{3.151}$$

[7]Note that in general a quaternionic Kähler manifold is not Kähler. However, in our context, the manifolds of hypermultiplets will always be special quaternionic Kähler.

Each vectormultiplet contains two real scalar fields so the dimension of $\mathcal{M}^{1,1}(M)$ is $2h^{1,1}$. This space is a special-Kähler manifold with a holomorphic prepotential. Each hypermultiplet contains four real scalars so the dimension of $\mathcal{M}^{2,1}(M)$ is $4(h^{2,1}+1)$.

Type IIB compactified on a Calabi-Yau threefold W yields $h^{2,1}$ abelian vectormultiplets and $h^{1,1}+1$ hypermultiplets. The corresponding moduli space takes the form

$$\mathcal{M}^{1,1}(W) \times \mathcal{M}^{2,1}(W). \tag{3.152}$$

Here the case is opposite to Type IIA, because now we have $\mathcal{M}^{2,1}(W)$ which is special Kähler and $\mathcal{M}^{1,1}(W)$ which is quaternionic Kähler.

An important consequence of the product structure of the moduli space is that the complex structure prepotential F is exact in α'. Indeed, the α' expansion is an expansion in terms of the Calabi-Yau volume V which belongs to $\mathcal{M}^{1,1}(M)$ and it is independent of position in $\mathcal{M}^{2,1}(M)$, i.e. the complex structure. We can now exploit the fact that Type IIA and Type IIB compactified on mirror Calabi-Yau manifolds are conjectured to be equivalent and deduce the consequences for the moduli spaces. First of all we have

$$\mathcal{M}^{1,1}(M) = \mathcal{M}^{2,1}(W) \quad \text{and} \quad \mathcal{M}^{1,1}(W) = \mathcal{M}^{2,1}(M). \tag{3.153}$$

We note that the prepotential of the Type IIB vectormultiplets is independent of the Kähler moduli and the dilaton. As a result, it is exact both in α' and g_S. Mirror symmetry maps the complex structure moduli space of Type IIB compactified on W to the Kähler-structure moduli space of Type IIA compactified on the mirror M. The Type IIA side receives α' corrections. This means that a purely classical result is mapped to an infinite series of quantum corrections. Said another way, a classical computation of the periods of Ω in W is mapped to a problem of counting holomorphic curves in M. Both sides should be exact in g_S since the IIA dilaton is not part of $\mathcal{M}^{1,1}(M)$ and the IIB dilaton is not part of $\mathcal{M}^{2,1}(W)$.

Chapter 4

Type II Calabi-Yau compactifications in general spacetime signature

> What is time? If nobody asks me, I know; but if I were desirous to explain it to one that should ask me, plainly I know not.
>
> *Augustine of Hippo*

In this chapter we will present the results of the work performed in [1]. We obtain the bosonic Lagrangians of vector and hypermultiplets coupled to four-dimensional $\mathcal{N} = 2$ supergravity in signatures $(0, 4)$, $(1, 3)$ and (2,2) by compactification of Type II string theories in signatures (0,10), (1,9) and (2,8) on a Calabi-Yau threefold. Depending on the signature and the distinctions between Type IIA/IIA*/IIB/IIB*/IIB' the resulting scalar geometries are special Kähler or special para-Kähler for vector multiplets and quaternion-Kähler or para-quaternion Kähler for hypermultiplets. By spacelike and timelike reductions we obtain three-dimensional $\mathcal{N} = 4$ supergravity theories coupled to two sets of hypermultiplets. We determine the c-maps relating vector to hypermultiplets, and show how the four-dimensional theories are related by spacelike, timelike and mixed, signature-changing T-dualities.

4.1 Introduction

String theory is a web of perturbatively defined theories which are related to each other by various dualities. In particular, ten-dimensional Type II string theories, which have the maximal amount of supersymmetry, are related to each other by T-duality and S-duality. If one includes timelike T-duality, then besides the familiar Type IIA and Type IIB theories there exist two further theories in Lorentz signature, Type IIA* and Type IIB*, and there also exist further Type II theories in all possible ten-dimensional spacetime signatures [53, 108, 109]. The formal properties of these theories as well as their potential applications in model building and cosmology have been investigated further in [54, 110]. Exotic Type II theories have unusual features and their ultimate role in string theory remains to be understood. From the point of view of symmetries and string geometry, it is natural to include them. Timelike dimensional reduction is a valid solution-generating technique, and timelike T-dualities exist whenever one can find an alternative dimensional up-lift. Symmetries which become manifest in dimensional reduction give information about the hidden symmetries of the full theory [111]. Including time in the reduction as a strategy for uncovering the full symmetry structure underlying string theory has been advocated in [112]. In the frameworks of doubled and exceptional geometry and field theory, Type II* theories seem to be on the same footing as the conventional ones [113, 114]. Since Type II and Type II* theories have the same Euclideanized version [108], it is natural to think of them as resulting from the same underlying Euclidean partition function.

When combining timelike T-duality with S-duality, string symmetries also lead to backgrounds with non-Lorentzian signatures. While their interpretation is challenging, they cannot be discarded ad hoc, since string theory is believed to be a single theory with all of its consistent backgrounds connected by physical processes. The relevant question is therefore whether vacua with exotic signature can be generated, and evidence for this has been presented in [54]. It has also been argued that string theories in exotic signature can be defined holographically as duals of gauge theories based on Lie supergroups [54]. We also note that the network of Type II string theories and of the related eleven-dimensional M-theories realizes all possible ten- and eleven-dimensional supersymmetry algebras with 32 real supercharges [115], so that when allowing exotic signatures in string theory, all maximally symmetric supergravities in all signatures can be realized as limits. Moreover, the maximally supersymmetric supergravity theories in ten and eleven dimensions can all be related to real forms of a single complex ortho-symplectic Lie superalgebra [116, 117]. Finally, the inclusion of non-Lorentzian signatures is natural from the point of view of the Euclidean approach to quantum gravity, since complex saddle points contribute to the functional integral. Recently, the role of complex spacetime metrics in quantum field theory and quantum gravity has been emphasised in [60, 118, 119]. As a natural extension, one can complexify all fields, which would imply to consider all Type II theories as part of a single complex configuration space. We remark that complex saddle points can contribute to Euclidean path integrals for scalar fields, and that there are examples where actions with inverted kinetic terms can be viewed as arising from manipulating integration contours in complexified field space, see [120] for an elementary example.

Calabi-Yau compactifications of Lorentz signature Type IIA/B string theories give rise to four-dimensional $\mathcal{N} = 2$ supergravity theories with vector and hypermultiplets [121–123]. This is a much studied class of theories which while not phenomenologically realistic, has rich and complex dynamics, since the scalar geometry is not rigid but depends on functions of the scalar fields. $\mathcal{N} = 2$ supersymmetry still severely restricts the quantum and stringy corrections that these functions can receive, so that one often can find exact non-perturbative results. Applications range from the study of field theories, to black holes and their entropy, and to the AdS/CFT correspondence. It is therefore interesting to extend these studies to the Calabi-Yau compactifications of exotic Type II theories.

The Calabi-Yau compactification of the Type IIA theory in Euclidean signature $(0, 10)$ has been worked out in [124]. Moreover, the vector multiplet sectors of five- and four-dimensional supergravity in arbitrary signature have been found in [125] through an analysis of Killing spinor equations combined with the reductions of eleven-dimensional supergravity theories in signatures (1,10), (2,9) and (5,6) on Calabi-Yau threefolds, followed by the reduction to four dimensions on spacelike and timelike circles.

In this chapter we will obtain the bosonic actions for four-dimensional $\mathcal{N} = 2$ supergravity coupled to vector and hypermultiplets in signatures $(0, 4), (1, 3)$ and $(2, 2)$ by compactification of Type II string theories in signatures $(0, 10), (1, 9), (2, 8)$ on Calabi-Yau threefolds. Carrying out these reductions is straightforward since for Type IIA one can adapt the computations of [122, 124], while the corresponding results for Type IIB are fixed by mirror symmetry. Therefore, our main focus is the interpretation of the relative sign flips between terms in the resulting four-dimensional Lagrangians in terms of the special geometry of their vector and hypermultiplet manifolds. These geometries vary between signatures and between Type II and Type II*. There is an intimate relation between these signs and the variation of the R-symmetry groups of the underlying supersymmetry algebras between signatures, and between standard and twisted (type-*) supersymmetry algebras.

As is well known, in signature $(1, 3)$ the scalar geometry of vector multiplets coupled to supergravity is special Kähler, while the geometry of hypermultiplets is quaternionic-Kähler, see [97, 126–128] for review. In Euclidean signature the scalar geometry of vector multiplets becomes special para-Kähler, which is reflected by a change of the abelian factor of the R-symmetry group from $U(1)$ to $SO(1, 1)$ [129, 130]. In three Euclidean dimensions the geometry of hypermultiplets is para-quaternionic Kähler [131]. In [132] it was observed that there exists a twisted version of the $\mathcal{N} = 2$ supersymmetry algebra, which has a non-compact R-symmetry group. The vector multiplet Lagrangian differs from the standard one by a sign flip of some kinetic terms, which is analogous to the difference between Type II and Type II* theories. The same type of sign flip was observed in [125], when reducing five-dimensional vector multiplets coupled to supergravity from signature $(2, 3)$ to signature $(1, 3)$. One should therefore expect that $\mathcal{N} = 2$ theories which realize the twisted $\mathcal{N} = 2$ algebra can be obtained as Calabi-Yau compactifications of Type II* theories. In this chapter we will verify this explicitly, as part of obtaining a complete list of scalar geometries for four-dimensional $\mathcal{N} = 2$ supergravity with vector and hypermultiplets for all signatures through the dimensional reduction of Type II theories on Calabi-Yau threefolds.

In addition, we will perform all possible spacelike and timelike dimensional reductions from four to three dimensions. After reduction, vector multiplets can be dualized into hypermultiplets, so that one obtains a scalar manifold which is the product of two hypermultiplet manifolds. The map relating a vector multiplet manifold to a hypermultiplet by reduction is know as the c-map [133,134]. By starting in arbitrary signature and including timelike reductions, one obtains variants of the c-map, which we describe for all possible cases. Whenever a dimensional reduction can be combined with a different dimensional lifting (equivalently, whenever the same three-dimensional theory can be obtained from two different four-dimensional theories by reduction), this realizes a T-duality between the underlying Type II string theories. We map out the complete network of spacelike, timelike and mixed T-dualities, where mixed T-dualities combine spacelike reduction/lifting with timelike lifting/reduction and thus change the four-dimensional signature.

We briefly mention further motivations and future application of the work contained in this chapter. One is the study of solutions to four-dimensional $\mathcal{N} = 2$ theories with twisted supersymmetry and with non-Lorentzian signature, as well as their dimensional uplifts to ten and eleven dimensions. In particular, according to [135, 136], there is a correspondence between the planar cosmological solutions in $\mathcal{N} = 2$ vector multiplet theories that can be embedded into Type II string theory, and planar black hole solutions in vector multiplet theories realizing the twisted $\mathcal{N} = 2$ supersymmetry algebra, which, as we show in this chapter, can be embedded into Type II*. Both solutions can be related to the same four-dimensional Euclidean partition function, which explains that their Killing horizons satisfy the same thermodynamic relations [136]. This is consistent with Type II and Type II* having the same Euclideanized form [108]. For other work on solutions of exotic $\mathcal{N} = 2$ theories see [125, 137–141].

Another potential application is topological string theory. Standard Type II Calabi-Yau compactifications allow two topological twists, which define two topological worldsheet theories, the A-model and the B-model, which are sensitive to the Kähler and complex structure moduli respectively. Since Calabi-Yau compactifications of exotic Type

II theories work analogously to standard Type II theories, and given that we will show that the geometry of the resulting moduli spaces can be determined in the supergravity approximation, we expect that a world-sheet perspective for these compactifications can be developed too. Topological string theories encode a subsector of the full string theories, and may also be related to a 'topological phase' of string theory, where more of its symmetries become manifest [142]. We remark that in such a topological phase, the expectation value of the spacetime metric is zero, which makes it natural that phases with non-Lorentzian signature coexist in the theory with conventional Lorentzian phases.

The outline of this chapter is as follows. We start from the classification of four-dimensional $\mathcal{N} = 2$ and three-dimensional $\mathcal{N} = 4$ supersymmetry algebras and explain how most of the qualitative features of the scalar geometries of vector and hypermultiplets as well as their mutual relations by T-dualities can already be predicted by inspection of the R-symmetry groups. We present the bosonic vector and hypermultiplet Lagrangians, and explain the effects of changing the supersymmetry algebra on the scalar geometries. This includes a brief review of special para-Kähler and para-quaternion-Kähler geometries, which replace the familar special Kähler and quaternion-Kähler geometries for certain signatures. We perform all possible spacelike and timelike reductions from signatures (0,4), (1,3) and (2,2) to signatures (0,3) and (1,2), and show that the six resulting c-maps which map vector multiplet manifolds to hypermultiplet manifolds fall into three distinct classes, depending on whether the resulting hypermultiplet manifold is quaternionic-Kähler, para-quaternionic-Kähler with a special Kähler base or para-quaternionic-Kähler with a special para-Kähler base.

Then we review Type II string theories in ten dimensions and catalogue the relative sign flips of their kinetic terms. Next, we explain how these sign flips affect compactifications on Calabi-Yau threefolds, and obtain the corresponding sign flips of the resulting four-dimensional vector and hypermultiplet Lagrangians. While in the main part of the chapter we just trace the kinetic terms, we provide a full derivation in appendix C. Here we use that the reduction of all individual terms is available from the work of [122] on the reduction of Type IIA with signature (1,9) and of [124] on the reduction of Type IIA with signature (0,10). Combining the results from dimensional reduction with the previous results on c-maps we obtain six types of T-dualities by identifying all possible combinations of reductions from four to three with 'oxidations' from three to four dimensions. These T-dualities organise into two orbits, one which relates Type IIA/IIB/IIA*/IIB* through 'pure' – that is spatial or timelike T-dualities, the other which relates Type IIA$_{(0,10)}$/IIB'$_{(1,9)}$/IIA$_{(2,8)}$ through 'mixed', signature changing T-dualities. This separation coincides with the one between worldsheet theories with Lorentzian and with Euclidean signature [53]. Both orbits could only be related through the S-duality between Type IIB* and Type IIB', which is not expected to be valid for generic $\mathcal{N} = 2$ compactifications, though it may be realized for non-generic '$\mathcal{N} = 4$-like' compactifications.

4.2 Vector and hypermultiplets in four and three dimensions

4.2.1 Supersymmetry algebras in four and three dimensions

Four-dimensional $\mathcal{N} = 2$ supersymmetry algebras, that is four-dimensional supersymmetry algebras with eight real supercharges,[1] have been classified for arbitrary signature in [132]. They are completely characterized by their R-symmetry groups, which we list in Table 4.1. While the $\mathcal{N} = 2$ algebra is unique in Euclidean signature $(0, 4)$ and in neutral signature (2,2), there are two non-isomorphic algebras in Lorentz signature (1,3).[2] Besides the standard $\mathcal{N} = 2$ algebra with compact R-symmetry group $U(2)$ there exists a second algebra with non-compact R-symmetry $U(1, 1)$, which we will refer to as the twisted $\mathcal{N} = 2$ algebra. The change of the R-symmetry group reflects itself in certain sign flips in the bosonic Lagrangian [132], which are similar to those which distinguish Type II and Type II* string theories [108]. We will see later that theories realizing the twisted $\mathcal{N} = 2$ algebra are obtained by the compactification of Type II* string theories on Calabi-Yau threefolds. The uniqueness of the supersymmetry algebras in Euclidean and neutral signature reflects the uniqueness of Type II string theories in signatures (0,10) and (2,8), from which such theories can again be obtained as Calabi-Yau compactifications.

Three-dimensional $\mathcal{N} = 4$ supersymmetry algebras have been classified, for arbitrary signature in [115], and are again characterized uniquely by their R-symmetry groups, see Table 4.2. The embeddings $U^*(2) \subset SO^*(4)$, $U(2) \subset O(4)$, $U(1, 1) \subset O(2, 2)$ and $GL(2, \mathbb{R}) \subset O(2, 2)$ indicate how these algebras are related to four-dimensional $\mathcal{N} = 2$ algebras by spacelike or timelike dimensional reduction, see Table 4.3. There is no candidate for a dimensional lift of the algebra with R-symmetry $O(1, 3)$. In the following sections we will review vector and hypermultiplets, in particular, the geometry of their scalar manifolds, and how this geometry is tied to the R-symmetry group.

[1] In Euclidean signature this is the smallest supersymmetry algebra. Our convention is to count supersymmetries in multiples of Majorana spinors, irrespective of whether Majorana spinors exist is the particular signature. This convention is natural if one considers supersymmetry algebras in different signatures at the same time.

[2] We use the mostly plus convention, $so(1, 3)$ means that the metric has 3 positive eigenvalues and 1 negative eigenvalue.

Signature	R-symmetry	VM geometry	HM geometry
$(0,4)$	$U(2)^* \cong SO(1,1) \times SU(2)$	SPK	QK
$(1,3)$	$U(2) \cong U(1) \times SU(2)$	SK_+	QK
	$U(1,1) \cong U(1) \times SU(1,1)$	SK_-	PQK
$(2,2)$	$GL(2,\mathbb{R}) \cong SO(1,1) \times SL^{\pm}(2,\mathbb{R})$	SPK	PQK

Table 4.1: Four-dimensional $\mathcal{N}=2$ supersymmetry algebras, their R-symmetry groups and their scalar geometries. We use the acronyms SK = special Kähler, SPK = special para-Kähler, QK = quaternionic Kähler and PQK = para-quaternionic Kähler. See Section 4.2.2 for further explanations.

Signature	R-symmetry	HM_1 geometry	HM_2 geometry
$(0,3)$	$SO^*(4) \cong SL(2,\mathbb{R}) \times SU(2)$	PQK	QK
$(1,2)$	$O(4) \cong SU(2) \times SU(2)$	QK	QK
	$O(1,3)$	–	–
	$O(2,2) \cong SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$	PQK	PQK

Table 4.2: Three-dimensional $\mathcal{N}=4$ supersymmetry algebras, their R-symmetry groups and their scalar geometries. See Section 4.2.3 for further explanations.

4.2.2 Vector multiplets

We start in signature (1,3) with the standard $\mathcal{N}=2$ supersymmetry algebra with R-symmetry group $U(2) \cong U(1) \times SU(2)$. A vector multiplet contains a complex scalar z, an $SU(2)$ doublet of spinors, and a gauge field $\mathcal{A}_\mu$. The scalar and gauge field are neutral under $SU(2)$. Under the $U(1)$, the scalars, spinors and vectors carry charges $\mp 1, \mp\frac{1}{2}, 0$ respectively. The scalar manifold is an affine special Kähler manifold for rigid supersymmetry and a projective special Kähler manifold for local supersymmetry. We refer to [127] for a review of special geometry which uses the same conventions and terminology as used in this chapter. Both types of special Kähler geometries (SK geometries) have in common that the Kähler metric $g_{\alpha\bar{\beta}}(z,\bar{z})$, $\alpha,\beta = 1,\ldots,n_V$ of the scalar manifold can be expressed in terms of a holomorphic function $\mathcal{F}(z^\alpha)$, called the prepotential. Special Kähler geometry is intimately related to the invariance of the field equations under symplectic transformations, which generalize and contain electric-magnetic duality transformations [143, 144]. We are interested in the case where the n_V vector multiplets are coupled to $\mathcal{N}=2$ supergravity. The supergravity multiplet contains one further vector field $\mathcal{A}^0_\mu$. A simple, linear action of the symplectic group $Sp(2n_V+2,\mathbb{R})$ is obtained by taking certain field-dependent linear combinations A^I_μ, $I=0,1,\ldots,n_V$ of the vector fields $\mathcal{A}^0_\mu$, A^α_μ. The associated field strengths $F^I_{\mu\nu}$, when combined with their duals $G_{I|\mu\nu}$

Signature	R-symmetry	Geometry	Reduction	R-symmetry	Geometry	c-map
(0,4)	$SO(1,1) \times SU(2)$	SPK × QK	(0,4) → (0,3)	$SL(2,\mathbb{R}) \times SU(2)$	PQK × QK	Euclidean c-map
(1,3)	$U(1) \times SU(2)$	SK × QK	(1,3) → (0,3)	$SL(2,\mathbb{R}) \times SU(2)$	PQK × QK	Temporal c-map
	$U(1) \times SU(1,1)$	SK × PQK	(1,3) → (0,3)	$SU(2) \times SU(1,1)$	QK × PQK	Twisted temporal c-map
	$U(1) \times SU(2)$	SK × QK	(1,3) → (1,2)	$SU(2) \times SU(2)$	QK × QK	(spatial) c-map
	$U(1) \times SU(1,1)$	SK × PQK	(1,3) → (1,2)	$SU(1,1) \times SU(1,1)$	PQK × PQK	Twisted (spatial) c-map
(2,2)	$SO(1,1) \times SL^{\pm}(2,\mathbb{R})$	SPK × PQK	(2,2) → (1,2)	$SU(1,1) \times SU(1,1)$	PQK × PQK	Neutral c-map

Table 4.3: Dimensional reduction from four to three dimensions for all inequivalent signatures: R-symmetry groups, scalar geometries, and type of c-map.

form a vector $(F^I_{\mu\nu}, G_{I|\mu\nu})$ which transforms linearly under $Sp(2n_V+2,\mathbb{R})$. The dual field strengths are dependent quantities, which are defined as $G^{\pm}_{I|\mu\nu} = \partial\mathcal{L}/\partial F^{\pm|I}_{\mu\nu}$, where $\mathcal{L}$ is the Lagrangian, and where $F^{\pm|I}_{\mu\nu}$ and $G^{\pm}_{I|\mu\nu}$ are the (anti-)selfdual parts of $F^I_{\mu\nu}$ and $G_{I|\mu\nu}$. We remark that the linear action of the symplectic group is obvious if one uses the gauge equivalence between $\mathcal{N}=2$ Poincaré supergravity with n_V vector and n_H hypermultiplets to $\mathcal{N}=2$ conformal supergravity with n_V+1 vector and n_H+1 hypermultiplets. In the superconformal setting A^I_μ are the vector fields of the n_V+1 superconformal vector multiplets. The corresponding scalars X^I allow a symplectically covariant description of the scalar sector. In terms of the X^I the prepotential is a holomorphic function $F(X)$ which is homogeneous of degree 2, $F(\lambda X) = \lambda^2 F(X)$. Combining the scalars X^I with $F_I = \partial F/\partial X^I$ one obtains another symplectic vector (X^I, F_I). The scalars z^α can be recovered as ratios $z^\alpha = X^\alpha/X^0$. The couplings between scalar and vector fields are encoded in a complex matrix $\mathcal{N}_{IJ} = \mathcal{R}_{IJ} + i\mathcal{I}_{IJ}$, which can be expressed in terms of the prepotential. The kinetic terms for the scalar and vector fields are positive definite if $g_{\alpha\bar{\beta}}$ is positive definite and if $\mathcal{I}_{IJ}$ is negative definite (in our convention).

The Lagrangian for the bosonic degrees of the supergravity multiplet and of n_V vector multiplets takes the form

$$L_{G+VM} = \frac{1}{2}\star R_4 - g_{\alpha\bar{\beta}}(z,\bar{z})dz^\alpha \wedge \star d\bar{z}^{\bar{\beta}} - \frac{\lambda}{4}\mathcal{I}_{IJ}F^I \wedge \star F^J + \frac{1}{4}\mathcal{R}_{IJ}F^I \wedge F^J \,, \tag{4.1}$$

where $\lambda = -1$.

We now turn to the modifications which occur if we change the supersymmetry algebra. In signature (1,3) we have the twisted algebra with R-symmetry group $U(1,1)$. For this algebra the Lagrangian takes exactly the same form, but with $\lambda = 1$, that is, the signs of the kinetic terms for all vector fields are flipped [132].[3] While the scalar manifold remains the same (for a given prepotential), we will use the notation $\mathrm{SK}_\pm = \mathrm{SK}_{\mp\lambda}$ to keep track of the relative sign between scalar and vector fields. Note that SK_+ corresponds to the case with standard kinetic terms, $\lambda = -1$.

Something more drastic happens in signatures (0,4) and (2,2), where special Kähler geometry is replaced by special para-Kähler geometry. We will provide a concise summary and refer to the review [127] as well as the original papers [129, 130, 132] for details. Para-complex geometries are modelled on the para-complex numbers (also called split complex numbers) in the same way as complex geometries are modelled on the complex numbers. The para-complex numbers are obtained by replacing the complex unit i, which satisfies $i^2 = -1$ and $\bar{i} = -i$ by the para-complex unit e, which satisfies $e^2 = 1$ and $\bar{e} = -e$. This allows one to define 'para-analogues' of almost complex, complex, Hermitian, Kähler

[3] This sign flip had already been observed in [145] by comparing the reductions of vector multiplets coupled to supergravity from signatures (1,4) and (2,3) to signature (1,3). See [132] for a detailed explanation how this sign flip is related to the underlying R-symmetry groups.

and of affine and projective special Kähler geometry. For example, an almost para-complex structure J on an even-dimensional real manifold $\mathcal{M}$ is an endomorphism field $J \in \mathrm{End}(T\mathcal{M})$ which satisfies $J^2 = \mathrm{Id}_{T\mathcal{M}}$ and has an equal number of eigenvalues ± 1. If J is integrable, $\mathcal{M}$ admits local para-complex coordinates $z^i = x^i + ey^i$, and is a para-complex manifold. Special para-Kähler (SPK) geometry is the para-analogue of special Kähler geometry. All usual formulae take the same form (assuming some care in placing factors e), with the prepotential now a para-holomorphic function of para-complex scalar fields z^α.

The change from complex to para-complex target geometry is reflected by the change in the abelian factor of the R-symmetry group. For special Kähler targets, the infinitesimal action of $U(1) \subset U(2)$ is given by multiplication by the complex structure I. Similarly the infinitesimal action of $SO(1,1) \subset U(1,1)$ is given by multiplication by the para-complex structure J [129]. Thus Table 4.1 tells us immediately that the vector multiplet geometry is SK for signature (1,3) but SPK for signature (0,4) and (2,2). This can also be verified by explicit construction of the vector multiplet representations, which in addition fixes the relative sign between the scalar and vector field terms [129, 132]. As we will review later, SPK geometry arises when reducing Euclidean IIA supergravity on a Calabi-Yau threefold [124]. Note that if the scalar manifold is SPK, this relative sign does not really matter, that is we can take $\lambda = -1$ or $\lambda = 1$, because this sign can be flipped by a local field redefinition [132]. This reflects that in signatures $(0,4)$ and $(2,2)$ the supersymmetry algebra is unique, whereas in signature $(1,3)$ there are two inequivalent supersymmetry algebras, whose vector multiplet representations are distinguished by the relative sign between scalar and vector field terms. In Minkowski signature sign flips of the gauge kinetic term map solutions of one theory to solutions of the other. For planar Reissner-Nordstrom-like solutions, this defines a map which exchanges the regions inside and outside horizons, and maps cosmological to black hole solutions [136]. In contrast, in Euclidean and neutral signature solutions with flipped vector kinetic terms are related to one another by a field redefinition [145].

4.2.3 Hypermultiplets

Hypermultiplets exist in all dimensions $D \leq 6$. Their field content is four real scalars and a doublet of spinors. The scalar geometry does not change under dimensional reduction. In Lorentz signature the scalar geometry is hyper-Kähler (HK) in the rigid case and quaternion-Kähler (QK) in the local case. A detailed review in conventions close to ours can be found in [97]. In both cases the scalar manifold $\mathcal{N}$ carries the action of a quaternionic structure, which is spanned (at least locally) by three complex structures I_i, $i = 1, 2, 3$, which satisfy the quaternionic algebra, that is they mutually anticommute and satisfy $I_i I_j = I_k$ for i, j, k cyclic. Hypermultiplet scalars are charged under a non-abelian subgroup $SU(2)$ of the R-symmetry group, and the infinitesimal action of $SU(2)$ is given by multiplication with the complex structures I_i. The corresponding finite action is given by the unit quaternions, $a1 + bI_1 + cI_2 + dI_3$, where $a^2 + b^2 + c^2 + d^2 = 1$, which form a group isomorphic to $SU(2)$. Three-dimensonal hypermultiplets can be obtained from four-dimensional vector multiplets by dimensional reduction. This induces a map between (generic) SK manifolds and (non-generic) QK manifolds. This map is known as the c-map [133,134]. The resulting QK manifolds contain the SK manifold they are constructed from as a totally geodesic submanifold, and the QK manifold is a group bundle over an SK base.

Table 4.3 shows that in various four- and three-dimensional signatures a factor $SU(2)$ of the R-symmetry group is replaced by $SU(1,1)$ relative to the standard Lorentz signature algebra. This indicates that the quaternionic structure of the HM scalar manifold is replaced by a para-quaternionic structure. The para-quaternions (also called split quaternions) are obtained by replacing two of the three complex units by para-complex units. The para-quaternionic algebra is isomorphic to the algebra $\mathbb{R}(2)$ of real 2×2 matrices, and the group of unit para-quaternions is isomorphic to $SU(1,1)$. The para-analogues of hyper-Kähler (HK) and quaternion-Kähler (QK) geometry are called para-hyper-Kähler (PHK) and para-quaternion-Kähler (PQK) geometry. We refer to [131, 146] and the review [127] for details. As we will discuss below, there are versions of the c-map which map SK and SPK manifolds to PQK manifolds.

One case where we expect that the hypermultiplet geometry is PQK is signature (0,3). This has been verified explicitly by dimensional reduction from signature (1,3) to signature (0,3), which defines the temporal c-map, and from signature (0,4) to signature (0,3), which defines the Euclidean c-map [131]. More generally the results of [131] imply the following: suppose that $\mathcal{M}_{2n_V}$ is a (projective) SK or SPK manifold with coordinates $z^\alpha = x^\alpha + i_{\epsilon_1} y^\alpha$, metric $g_{\alpha\bar{\beta}}$, $\alpha, \beta = 1, \ldots, n_V$ and vector coupling matrix $\mathcal{N}_{IJ} = \mathcal{R}_{IJ} + i_{\epsilon_1}\mathcal{I}_{IJ}$, where $\epsilon_1 = -1$, $i_{-1} = i$ for SK and $\epsilon_1 = 1$, $i_1 = e$ for SPK. In the SK case we assume that $g_{\alpha\bar{\beta}}$ is positive definite and that $\mathcal{I}_{IJ}$ is negative definite.[4] Consider the two-parameter family of bosonic Lagrangians for $n_H = n_V + 1$ hypermultiplets,

$$\begin{aligned} L_{HM}^{(\epsilon_1,\epsilon_2)} =& - g_{\alpha\bar{\beta}} dz^\alpha \wedge \star d\bar{z}^{\bar{\beta}} - \frac{1}{4} d\varphi \wedge \star d\varphi \\ &+ \epsilon_1 e^{-2\varphi} \left[d\tilde{\phi} + \frac{1}{2}\left(\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I\right)\right] \wedge \star \left[d\tilde{\phi} + \frac{1}{2}\left(\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I\right)\right] \\ &- \frac{\epsilon_2}{2} e^{-\varphi} \left[\mathcal{I}_{IJ} d\zeta^I \wedge \star d\zeta^J - \epsilon_1 \mathcal{I}^{IJ} \left(d\tilde{\zeta}_I + \mathcal{R}_{IK} d\zeta^K\right) \wedge \star \left(d\tilde{\zeta}_I + \mathcal{R}_{IK} d\zeta^K \right)\right] , \end{aligned} \tag{4.2}$$

where $\epsilon_2 = \pm 1$ is a second parameter. It was shown in [131], that the resulting HM manifold $\mathcal{N}_{4n_H} = \mathcal{N}_{4n_V+4}$ is QK for $(\epsilon_1 = -1, \epsilon_2 = -1)$ and PQK for the other three

[4] If this condition is relaxed one obtains QK manifolds of indefinite signature. See [147] for special geometry with indefinite signature SK and QK target spaces.

Domain	Image	Parameters	c-map
SK	QK	$\epsilon_1 = -1, \epsilon_2 = -1$	spatial
SK	PQK_{SK}	$\epsilon_1 = -1, \epsilon_2 = 1$	temporal
SPK	PQK_{SPK}	$\epsilon_1 = 1, \epsilon_2 = \pm 1$	Euclidean

Table 4.4: As far as the scalar geometries are concerned, there are three distinct c-maps. The parameters refer to the hypermultiplet 'master Lagrangian' (4.2).

cases. Moreover for ($\epsilon_1 = -1, \epsilon_2 = 1$) the PQK manifold is a group bundle over an SK base (the space parametrized by the complex scalars z^α), while for ($\epsilon_1 = 1, \epsilon_2 = \pm 1$) it is a group bundle over an SPK base (the space parametrized by the para-complex scalars z^α). Finally, the manifolds with ($\epsilon_1 = 1, \epsilon_2 = \pm 1$) are isometric (keeping the base manifold fixed). Thus there are three inequivalent cases: QK, PQK with an SK base and PQK with an SPK base, see Table 4.4 for a summary. If we need to empasize the base we will write PQK_{SK} or PQK_{SPK}. We remark that the c-maps $\mathcal{M}_{2n_V} \to \mathcal{N}_{4n_V+4}$ are maps between S(P)K manifolds and Q(P)K manifolds, which are well defined on their own, that is without reference to supermultiplets, Lagrangians and dimensional reduction. In particular the resulting QK/PQK manifolds are admissible (though non-generic) HM target manifolds in all signatures for dimensions up to six, provided that they are compatible with the R-symmetry group. We will see that QK/PQK manifolds of all of these types appear in Type II compactifications on Calabi-Yau threefolds.

4.2.4 Reduction to three dimensions

Let us consider the dimensional reduction of $\mathcal{N} = 2$ supergravity with n_V vector multiplets and n_H hypermultiplets to three dimensions. The field content and scalar geometry of the HM sector does not change, while vector multiplets can be dualized into hypermultiplets after reduction. Moreover, the bosonic degrees of freedom of the supergravity multiplet, that is the metric and the graviphoton, give rise to an additional hypermultiplet, so that we end up with three-dimensional $\mathcal{N} = 4$ supergravity with $(n_V + 1) + n_H$ hypermultiplets. Since the four-dimensional HMs play a passive role, we only need to consider the bosonic Lagrangian (4.1) for gravity and n_V vector multiplets. There are four different starting points: signature (1,3) with either the standard or twisted $\mathcal{N} = 2$ algebra, signature (0,4) and signature (2,2). The scalar geometry and relative signs are encoded in two parameters: $\epsilon_1 = \mp 1$ distinguishes between SK (signature (1,3)) and SPK (signatures (0,4), (2,2)), while $\lambda = \pm 1$ encodes the relative sign between scalar and vector terms. As mentioned earlier, the choice of this sign is only relevant in signature (1,3), since in the other signatures it can be changed by a field redefinition. We introduce another parameter $\epsilon = \mp 1$, which distinguishes between spacelike reduction and timelike reduction. After the reduction, the Einstein-Hilbert term is non-dynamical, and the local degrees of freedom of the four-dimensional metric reside in the KK-scalar φ and the scalar $\tilde{\phi}$ which is dual to the KK-vector. The four-dimensional vector fields A^I_μ decompose into scalars ζ^I and three-dimensional vector fields which we dualize into scalars $\tilde{\zeta}_I$. Together with the (para-)complex scalars z^α, this is the field content of $n_H + 1$ hypermultiplets. The computation is the same as in [134] and [131], except that we now include the case $\lambda = +1$. The Lagrangian takes the form

$$\begin{aligned}\mathbf{e}^{-1}L_3 = &\frac{1}{2}R_3 - \frac{1}{4}\partial_\mu\phi\partial^\mu\phi - g_{\alpha\bar{\beta}}\partial_\mu z^\alpha\partial^\mu\bar{z}^{\bar{\beta}} \\ &+ \epsilon_1 e^{-2\phi}\left[\partial^\rho\tilde{\phi} + \frac{1}{2}\left(\zeta^I\partial^\rho\tilde{\zeta}_I + \tilde{\zeta}_I\partial^\rho\zeta^I\right)\right]\left[\partial_\rho\tilde{\phi} + \frac{1}{2}\left(\zeta^I\partial_\rho\tilde{\zeta}_I - \tilde{\zeta}_{\mathcal{I}}\partial_\rho\zeta^I\right)\right] \\ &+ \frac{\lambda\epsilon}{2}e^{-\phi}\left[\mathcal{I}_{IJ}\partial_\mu\zeta^I\partial^\mu\zeta^J - \epsilon_1\mathcal{I}^{IJ}\left(\partial^\rho\tilde{\zeta}_I - \mathcal{R}_{IK}\partial^\rho\zeta^K\right)\left(\partial_\rho\tilde{\zeta}_J - \mathcal{R}_{JL}\partial_\rho\zeta^L\right)\right] .\end{aligned} \tag{4.3}$$

By comparison to (4.2) we read off that $\lambda\epsilon = -\epsilon_2$. All of these spaces are either QK, PQK$_{SK}$ or PQK$_{PSK}$. Starting with four theories in four dimensions, we have six different cases.

1. Start with VMs in signature (1,3), with the standard $\mathcal{N} = 2$ algebra, and reduce over space, (1,3) $\to$ (1,2). Then $\epsilon_1 = -1$ and $\epsilon = -1$, $\lambda = -1$ which implies $\epsilon_2 = -1$. This is the standard ('spatial') c-map of [134] which maps $SK_+ \to QK$.

2. Start with VMs in signature (1,3), with the standard $\mathcal{N} = 2$ algebra, and reduce over time, (1,3) $\to$ (0,3). Then $\epsilon_1 = -1$ and $\epsilon = 1$, $\lambda = -1$ which implies $\epsilon_2 = 1$. This is the temporal c-map [131], which maps $SK_+ \to PQK_{SK}$.

3. Start with VMs in signature (0,4) and reduce over space, (0,4) $\to$ (0,3). Then $\epsilon_1 = 1$ and $\epsilon = -1, \lambda = \pm 1$ which implies $\epsilon_2 = \pm 1$. This is the Euclidean c-map [131], which maps $SPK \to PQK_{SPK}$.

4. Start with VMs in signature (2,2) and reduce over time, (2,2) $\to$ (1,2). Then $\epsilon_1 = 1$ and $\epsilon = 1, \lambda = \pm 1$ which implies $\epsilon_2 = \pm 1$. This works like the Euclidean c-map, $SPK \to PQK_{SPK}$, but if we want to emphasize the context of dimensional reduction, that is, that we reduce over time rather than space, we will call it the neutral c-map.

5. Start with VMs in signature (1,3), with the twisted $\mathcal{N} = 2$ algebra, and reduce over space, (1,3) $\to$ (1,2). Then $\epsilon_1 = -1$ and $\epsilon = -1$, $\lambda = 1$ which implies $\epsilon_2 = -1$. This maps SK to PQK with a SK base: $SK_- \to PQK_{SK}$. Thus the sign flip between scalar and vector fields exchanges the roles of the spatial and temporal c-map. In the case at hand we obtain a PQK manifold from an SK manifold through spatial

4d signature	Source	3d signature	Image	c-map
(0,4)	SPK	(0,3)	PQK_{SPK}	Euclidean
(1,3)	SK_+	(0,3)	PQK_{SK}	temporal
(1,3)	SK_-	(0,3)	QK	twisted temporal ≅ spatial
(1,3)	SK_+	(1,2)	QK	spatial
(1,3)	SK_-	(1,2)	PQK_{SK}	twisted spatial ≅ temporal
(2,2)	SPK	(1,2)	PQK_{SPK}	neutral ≅ Euclidean

Table 4.5: When reducing four-dimensional vector multiplets to three dimensions, there are six distinct cases, although there are only three distinct types of hypermultiplet manifolds that arise from the construction.

reduction. While this works like the temporal c-map as far as the scalar geometries are concerned, we will call this the twisted spatial c-map if we want to emphasize the context of dimensional reduction, that is, that we reduce over space, but start with flipped four-dimensional gauge kinetic terms.

6. Start with VMs in signature (1,3), with the twisted $\mathcal{N}=2$ algebra, and reduce over time, (1,3) $\to$ (0,3). Then $\epsilon_1=-1$ and $\epsilon=1$, $\lambda=1$ which implies $\epsilon_2=1$. This maps SK to QK despite that we are reducing over time: $SK_-\to QK$. While this works like the spatial c-map as far as the manifolds are concerned, we will call this the twisted temporal c-map if we need to emphasize the context of dimensional reduction.

See Table 4.5 for a summary.

If we start with a theory of n_V vector and n_H hypermultiplets in four dimensions, with scalar manifold $\mathcal{M}_{2n_V}\times\tilde{\mathcal{N}}_{4n_H}$, reduction to three dimensions leads us to a theory with $(n_V+1)+n_H$ hypermultiplets, where the two hypermultiplet manifolds form a direct product:

$$\mathcal{M}_{2n_V}\times\tilde{\mathcal{N}}_{4n_H}\to\mathcal{N}_{4n_V+4}\times\tilde{\mathcal{N}}_{4n_H}\ .$$

If both factors are 'in the image of the c-map', the three-dimensional theory can be lifted to a different four-dimensional theory with $n'_V=n_H-1$ vector multiplets and $n'_H=n_V+1$ hypermultiplets.

$$\mathcal{M}_{2n_V}\times\tilde{\mathcal{N}}_{4n_H}\to\tilde{\mathcal{N}}_{4n_V+4}\times\tilde{\mathcal{N}}_{4n_H}\leftarrow\tilde{\mathcal{N}}_{4n_V+4}\times\tilde{\mathcal{M}}_{2n_H-2}=\tilde{\mathcal{N}}_{4n'_H}\times\tilde{\mathcal{M}}_{2n'_V}\ . \tag{4.4}$$

In the context of string theory, the relations between the four-dimensional theories are T-dualities, which we call spacelike, timelike and mixed depending on how they combine spacelike/timelike reduction with spacelike/timelike oxidation. Which T-dualities exist depends on the details of the HM sectors of the four-dimensional theories. Therefore we will now consider the Calabi-Yau compactifications of Type II theories in signature (0,10), (1,9) and (2,8), which give rise to four-dimensional $\mathcal{N} = 2$ theories in signatures (0,4), (1,3) and (2,2).

4.3 Ten-dimensonal Type II string theories

As explained before, Type IIA and Type IIB string theory are related by T-duality. When performing a timelike T-duality (meaning a T-duality where the compact dimension is a time-like coordinate), one finds that Type IIA and Type IIB are no longer T-dual to each other and instead they are respectively dual to two new theories dubbed IIB* and IIA*, as summarized in Figure 4.1 [108].

$$
\begin{tikzcd}
\mathrm{IIB}^* \arrow[r, leftrightarrow, "T_S"] \arrow[d, leftrightarrow, "T_T"'] & \mathrm{IIA}^* \arrow[d, leftrightarrow, "T_T"] \\
\mathrm{IIA} \arrow[r, leftrightarrow, "T_S"] & \mathrm{IIB}
\end{tikzcd}
$$

Figure 4.1: Diagram showing the relationship between II and II* theories, where $T_S(T_T)$ denotes a spacelike (timelike) T-duality.

At first glance, this procedure raises questions since we are considering theories on backgrounds with closed time-like curves. Moreover, the dual theories have ghosts in their low energy limits. However there is no known mechanism in string theory that explicitly prevents the formation of closed timelike curves, in the spirit of Hawking's chronological protection conjecture [148]. The presence of ghosts in the low energy effective theories is not necessarily inconsistent either.

To see why, let us consider Yang-Mills on a timelike circle which gives a Euclidean theory with a scalar A_0 from the reduction that is ghost-like. If we were to consider the whole theory without truncation, then in the sector without Wilson lines, the ghost can be gauged away, and only appears as a ghost from the low energy point of view. The situation for the II* theories could be similar. Indeed, truncated to their supergravity limits these theories appear to have ghosts, but if the whole tower of string states is kept it could happen that these ghosts are an artefact of the low energy description, and the whole theory is itself consistent. Indeed if the procedure of timelike T-duality is valid, then the II* theories are equivalent to the regular Type II theories which are ghost free.

The difference between Type IIA and Type IIA* and between Type IIB and Type IIB* lies in certain phase factors, which at the level of the effective supergravity Lagrangian manifest themselves in sign flips of the kinetic terms of the R-R fields, as well as factors of powers of i in the fermionic terms. For Type IIB/IIB* the scalar manifolds are different, namely $SL(2,\mathbb{R})/SO(2)$ for Type IIB and $SL(2,\mathbb{R})/SO(1,1)$ for Type IIB*. It was observed that supersymmetry is realized in Type II* theories in a modified, twisted form, which can be interpreted as a generalized $O(p,q)$ Majorana condition [53]. In [115] the R-symmetry groups for supersymmetry algebras in arbitrary dimension and signature were classified, which allows to put this observation into a wider context. It was found that in certain signatures there exist several non-isomorphic supersymmetry algebras with the same number of supercharges (and, where applicable the same chirality properties), whose R-symmetry groups are different real forms of the same complex Lie group. For example, in signature (1,9), chiral supersymmetry algebras with $\mathcal{N}$ left-moving (or right-moving) supercharges are real forms of a complex supersymmetry algebra with R-symmetry group $O(\mathcal{N},\mathbb{C})$. Real supersymmetry algebras are obtained by imposing $O(p,q)$ Majorana conditions, with $p+q=\mathcal{N}$, which leads to real supersymmetry algebras with R-symmetry group $O(p,q)$. For ten-dimensional chiral supersymmetry algebras with 32 real supercharges the two possible cases are $O(2)$ and $O(1,1)$ which correspond to Type IIB/IIB*. There also are two inequivalent non-chiral algebras, which have the same discrete R-symmetry group but differ by a relative sign in the reality condition imposed on left- and right-moving supercharges, corresponding to Type IIA/IIA*. Similarly, $\mathcal{N}$-extended supersymmetry algebras in four-dimensions are real forms of a complex supersymmetry algebra with R-symmetry $GL(\mathcal{N},\mathbb{C})$ and the reality conditions defining real supersymmetry algebras in signature (1,3) lead to R-symmetry groups of the form $U(p,q)$, $p+q=\mathcal{N}$. For $\mathcal{N}=2$ the two possibilities are $U(2)$ and $U(1,1)$. For completeness we note that for $\mathcal{N}=2$ the reality conditions defining real supersymmetry algebras in signatures (0,4) and (2,2) lead to unique algebras with R-symmetry $U^*(2)$ and $GL(2,\mathbb{R})$, respectively, see [115] for details.

All Type II theories have the same NS-NS sector which consists of the graviton G_{MN}, Kalb-Ramond field B_{MN} and dilaton Φ. The R-R sector of Type IIA/IIA* contains a one form C_1 and a three-form C_3 while the R-R sector of Type IIB/IIB* contains a zero-form C_0, a two-form C_2 and a four-form C_4 whose field strength is self-dual or anti-self-dual, $*G_5=\pm G_5$.[5] The difference between the bosonic actions of Type II and Type II* is a sign flip of the kinetic terms for all fields in R-R sector, see Tables 4.6 and 4.7.[6]

[5] We will specify our choice of sign below.

[6] The information for Type IIA is taken from Table 1 in [53]. Note that their notation for signature is (s,t), where s corresponds to positive eigenvalues of the metric, and t to negative eigenvalues of the

Type	G_{MN}	B_{MN}	Φ	C_1	C_3
$\mathrm{IIA}_{(1,9)}$	$+$	$+$	$+$	$+$	$+$
$\mathrm{IIA}^*_{(1,9)}$	$+$	$+$	$+$	$-$	$-$
$\mathrm{IIA}_{(0,10)}$	$+$	$-$	$+$	$-$	$+$
$\mathrm{IIA}_{(2,8)}$	$+$	$-$	$+$	$+$	$-$

Table 4.6: Relative signs for kinetic terms in ten-dimensional Type IIA theories. A $+$ sign corresponds to a standard kinetic term in Lorentz signature, thus discarding the overall $-$ sign with which these terms appear in the action when using the mostly plus convention for the metric.

Type	G_{MN}	B_{MN}	Φ	C_0	C_2	C_4
$\mathrm{IIB}_{(1,9)}$	$+$	$+$	$+$	$+$	$+$	$+$
$\mathrm{IIB}^*_{(1,9)}$	$+$	$+$	$+$	$-$	$-$	$-$
$\mathrm{IIB'}_{(1,9)}$	$+$	$-$	$+$	$-$	$+$	$-$

Table 4.7: Relative signs for kinetic terms in ten-dimensional Type IIB theories. A $+$ indicates the standard sign for a theory in Lorentz signature using the mostly plus convention.

The bosonic actions for Type IIA/IIA* take the form [53]

$$S^{IIA/IIA^*}_{(1,9)} = \int d^{10}x\sqrt{|G|}e^{-2\Phi}\left(R + 4(\partial\Phi)^2 - H^2 + \lambda G_2^2 + \lambda G_4^2\right) + \cdots \tag{4.5}$$

where we omitted the Chern-Simons terms, and where H, G_2, G_4 are the field strength of B, C_1, C_3, respectively. Type IIA corresponds to $\lambda = -1$ while Type IIA* corresponds to $\lambda = 1$. Taking into account that we use the mostly plus convention for the metric, this means that all bosonic fields have positive kinetic energy for $\lambda = -1$. In Table 4.6 this corresponds to a row where all entries are +, that is we discard the overall minus sign that these terms have in the action. Generally, in this and other tables, we record sign flips relative to standard kinetic terms in Lorentz signature, which correspond to a row with only + signs. Note that the kinetic term of the dilaton in the action (4.5) has a + sign, since we are in the string frame. When going to the Einstein frame, this sign flips, showing that the dilaton has positive kinetic energy.

For Type IIB/IIB* there is no simple covariant action, since the five-form field strength is self-dual. However one can use a pseudo-action, whose variation gives the field equation except the self-duality condition $G_5 = \pm * G_5$, which is then imposed by hand [53]:[7]

$$S^{IIB/IIB^*}_{(1,9)} = \int d^{10}x\sqrt{|G|}e^{-2\Phi}\left(R + 4(\partial\Phi)^2 - H^2 + \lambda G_1^2 + \lambda G_3^2 + \lambda G_5^2\right) + \cdots \tag{4.6}$$

where $G_{p+1} = dC_p + \cdots$ are the field strength and where again we only display the Maxwell-like terms. Type IIB corresponds to $\lambda = -1$, where all kinetic terms have their standard sign, while Type IIB* exhibits a sign flip for all R-R fields.

Type II string theories exist for all ten-dimensional signatures. We will use the notation Type $\mathrm{II}_{(t,s)}$ for a theory where the metric has t negative and s positive eigenvalues. Since we prefer the mostly plus convention for Lorentz signature, we will usually refer to the t directions as timelike and the s directions as spacelike. Theories where t and s are exchanged have been shown to be equivalent [53]. The unique theories in signatures (0,10) and (2,8) are IIA theories, denoted $\mathrm{IIA}_{(0,10)}$ and $\mathrm{IIA}_{(2,8)}$. These theories are non-chiral and have the same R-R sector as Type $\mathrm{IIA}_{(1,9)}$. Their actions have the same structure as the Type $\mathrm{IIA}_{(1,9)}$ action, but with some sign flips for the Maxwell-like terms [53], which are listed in Table 4.6. In both signatures the B-field has a flipped kinetic term, while in the R-R sector either C_1 or C_3 has a sign flip:

$$S^{IIA}_{(0,10)} = \int d^{10}x\sqrt{|G|}e^{-2\Phi}\left(R + 4(\partial\Phi)^2 + H^2 + G_2^2 - G_4^2\right) + \cdots\ , \tag{4.7}$$

metric, while we use (t, s).

[7]The sign in the self-duality relation is correlated with the sign of terms which we have not displayed (Chern-Simons and fermionic terms), see for example [149] for a general discussion. Full bosonic Type II Lagrangians, which however use a different notation and normalization for the bosonic fields, can be found in [54]. In their conventions the sign is correlated with whether the worldvolume theories of fundamental strings and D-strings are Lorentzian or Euclidean, resulting in a (+)-sign for Type IIB and a (−)-sign for Type IIB*/IIB'.

$$S^{IIA}_{(2,8)} = \int d^{10}x\sqrt{|G|}e^{-2\Phi}\left(R + 4(\partial\Phi)^2 + H^2 - G_2^2 + G_4^2\right) + \cdots \,. \tag{4.8}$$

Type II string theories in different signatures are related by what we call mixed T-dualities, that is T-dualities which combine a spacelike/timelike reduction with a timelike/spacelike oxidation (lifting). For this to work one needs to make use of the S-dual of the Type IIB* theory, which is called Type IIB'. As shown in [53] Buscher T-duality along an isometric direction $X^\sharp$ in the target space of the worldsheet sigma model preserves the sign of the term $G_{\sharp\sharp}\partial_\alpha X^\sharp\partial^\alpha X^\sharp$ if the worldsheet theory has Lorentzian signature, but reverses it if the worldsheet theory has Euclidean signature. In the Type IIB* theory, D-branes are replaced by E-branes, which have a Euclidean worldvolume. If one applies S-duality, fundamental IIB*-strings and E-strings are exchanged, so that in the resulting IIB'-theory fundamental strings have a Euclidean worldvolume. The Type IIA$_{(0,10)}$ and Type IIA$_{(2,8)}$ theories are then obtained from the Type IIB'$_{(1,9)}$ theory by T-dualities which involve a timelike/spacelike reduction combined by a spacelike/timelike oxidation [53]. At the supergravity level, S-duality exchanges the B-field and the R-R two-form C_2, resulting in the sign flips recorded in Table 4.7,

$$S^{IIB'}_{(1,9)} = \int d^{10}x\sqrt{|G|}e^{-2\Phi}\left(R + 4(\partial\Phi)^2 + H^2 + G_1^2 - G_3^2 + G_5^2\right) + \cdots \tag{4.9}$$

Note that while G_5 is an S-duality singlet, Φ and C_0 parametrize the indefinite signature coset space $SL(2,\mathbb{R})/SO(1,1)$ on which S-duality acts non-linearly. In summary, the complete duality web of Type II theories in arbitrary signatures is depicted below

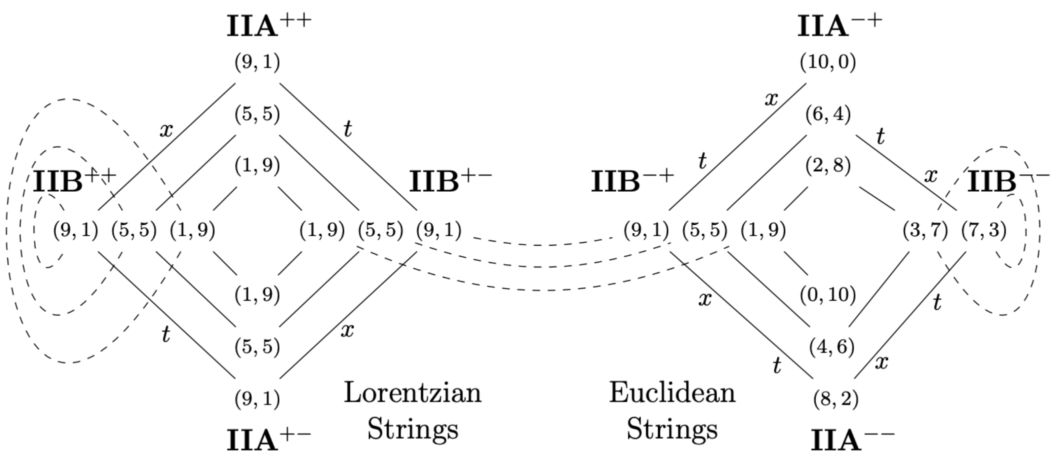


Figure 4.2: 10 dimensional duality web of Type II theories in arbitrary signatures. Full lines represent T-dualities and dashed lines S-duality. The first sign in exponents is + if the fundamental string is Lorentzian and - if it is Euclidian. The second sign tells the same information about the D1/D2-brane. Diagram adopted from [110]

4.4 Type II Calabi-Yau compactifications

By compactification of Type II string theories one obtains $\mathcal{N} = 2$ supergravity coupled to vector and hypermultiplets. Since we always compactify six spatial dimensions to go from signatures (0,10), (1,9), (2,8) to signatures (0,4), (1,3), (2,2), the only essential difference in these reductions is between Type IIA and Type IIB, which are distinguished, as far as bosonic degrees of freedom are concerned, by the field content of their R-R sectors. Otherwise the bosonic Type IIA/IIA* actions only differ from one another by relative sign flips that one has to follow through, and the same applies to the Type IIB/IIB*/IIB' theories. Since the mechanics of Calabi-Yau compactifications is well known from the standard cases of IIA$_{(1,9)}$ [122] and IIB$_{(1,9)}$ [123], we will not go through the computational details but highlight how the ten-dimensional sign flips modify the resulting four-dimensional actions. More details are given in appendix C. The reduction of the Euclidean IIA$_{(0,10)}$ theory was worked out in detail in [124].

4.4.1 Type IIA Calabi-Yau compactifications

We start with Type IIA theories, where we have the cases Type IIA$_{(1,9)}$, IIA$^*_{(1,9)}$, IIA$_{(0,10)}$ and IIA$_{(2,8)}$. We first consider aspects which work the same in all cases.

The metric. In a general real six-fold compactification, the massless four-dimensional fields resulting from the reduction of the metric G_{MN} are the four-dimensional metric $g_{\mu\nu}$, vector fields, and scalar fields. Massless vector fields are on one-to-one correspondence with Killing vector fields, and since CY3-folds (Calabi-Yau threefolds) do not have isometries, there are no massless vectors in our case. The massless scalar fields are in one-to-one with deformations of the six-fold metric which preserve Ricci-flatness. For CY3-folds these deformations are, due to the existence of a holomorphic (3,0)-form, in one-to-one correspondence with the deformations of the complex structure and of the (real) Kähler form. This gives rise to $h^{2,1}$ complex scalars z^α, $\alpha = 1, \ldots, h^{2,1}$ and $h^{1,1}$ real scalars y^A, $A = 1, \ldots, h^{1,1}$, where $h^{i,j}$ are the Hodge numbers of the CY3-fold.

***p*-form fields.** A ten-dimensional p-form decomposes into products of four-dimensional p'-forms and six-dimensional p''-forms, where $p' + p'' = p$. Massless p'-forms are in one-to-one correspondence with harmonic p''-forms, which are counted by the Betti-numbers $b_{p''}$ of the compact space. For a CY3-fold the Betti numbers are related to the Hodge numbers by $b_{p''} = \sum_{i+j=p''} h^{i,j}$. Moreover, for a CY3-fold

$$h^{0,0} = h^{3,0} = h^{0,3} = h^{3,3} = 1 \,, \quad h^{1,0} = h^{0,1} = h^{3,2} = h^{2,3} = 0 \,,$$

so that the only numbers that vary between CY3s are $h^{1,1} = h^{2,2} \geq 1$ and $h^{1,2} = h^{2,1} \geq 0$.

The B-field. The B-field B_{MN} gives rise to $h^{1,1}$ real scalars x^A, as well as a four-dimensional B-field, which we dualize into a scalar $\tilde{\phi}$.

10d	4d	
G_{MN}	$g_{\mu\nu}$	Metric
	z^α	Complex structure moduli
	y^A	(Real) Kähler moduli
B_{MN}	$b_{\mu\nu} \sim \tilde{\phi}$	Universal axion
	x^A	$h^{1,1}$ real scalars
Φ	φ	Dilaton

Table 4.8: Massless fields in Type II Calabi-Yau compactifications, NS-NS sector.

10d	4d	
C_M	$\mathcal{A}^0_\mu$	vector
C_{MNP}	$C_{\mu np} \sim \mathcal{A}^A_\mu$	$h^{1,1}$ vectors
	$C_{mnp} \sim \zeta^I, \tilde{\zeta}_I$	$2h^{2,1}+2$ scalars

Table 4.9: Massless fields in Type IIA Calabi-Yau compactifications, R-R sector.

The dilaton. The ten-dimensional dilaton Φ gives rise to a four-dimensional scalar φ, which differs from Φ by a field-redefinition. Essentially, one absorbs a factor proportional to the volume of the internal space, in order that the four-dimensional action acquires standard form.

The R-R sector. The R-R one-form C_M gives rise to a vector $\mathcal{A}^0_\mu$. The R-R three-form C_{MNP} gives rise to $h^{1,1}$ vectors $\mathcal{A}^A_\mu$ and $2h^{2,1}+2$ real scalars $\zeta^I, \tilde{\zeta}_I$, $I = 0, 1, \ldots, h^{2,1}$. The massless fields originating from the NS-NS sector are summarized in Table 4.8, those from the R-R sector in Table 4.9.

Collecting all these fields, this is the bosonic field content of the $\mathcal{N} = 2$ Poincaré supergravity multiplet, $(g_{\mu\nu}, \mathcal{A}^0_\mu)$, of $n_V = h^{1,1}$ vector multiplets $(y^A, x^A, \mathcal{A}^A_\mu)$, and of $n_H = h^{2,1}+1$ hypermultiplets $(z^\alpha, \varphi, \tilde{\phi}, \zeta^I, \tilde{\zeta}_I)$. The signs of the kinetic terms of the scalar fields can be inferred from those of the higher-dimensional ones, and are listed in Table 4.10.

Signs are taken relative to the standard $\mathrm{IIA}_{(1,9)}$ theory, where all kinetic terms have the standard sign, denoted $+$. In $\mathrm{IIA}^*_{(1,9)}$ half of the signs in the HM sector are flipped, so that the HM scalar manifold has neutral signature. In the Euclidean $\mathrm{IIA}_{(0,10)}$ theory only the signs of the scalars x^A which descend from the B-field are flipped, which gives the VM manifold neutral signature. Finally in the $\mathrm{IIA}_{(2,8)}$ case, we have signs flips for $x^A, \zeta^I, \tilde{\zeta}_I$, so that both VM and HM scalar manifold have neutral signature.[8]

[8]Note that in both cases the sign flip of the four-dimensional B-field $b_{\mu\nu}$ is compensated by a second sign flip when we dualize this two-form into the scalar $\tilde{\phi}$. Dualization flips the sign of the kinetic term

	y^A	x^A	z^α	φ	$\tilde{\phi}$	ζ^I	$\tilde{\zeta}_I$	$\mathcal{A}^0_\mu$	$\mathcal{A}^A_\mu$
$\mathrm{IIA}_{(1,9)}$	+	+	+	+	+	+	+	+	+
$\mathrm{IIA}^*_{(1,9)}$	+	+	+	+	+	−	−	−	−
$\mathrm{IIA}_{(0,10)}$	+	−	+	+	+	+	+	−	+
$\mathrm{IIA}_{(2,8)}$	+	−	+	+	+	−	−	+	−

Table 4.10: Signs of the kinetic terms for scalar and vector fields resulting from Type IIA CY3 compactifications. A + indicates a standard kinetic term. The fields y^A, x^A are the vector multiplet scalars.

The ten-dimensional sign flips also affect the four-dimensional vector kinetic terms. For Type $\mathrm{IIA}^*_{(1,9)}$ the signs of all vector kinetic terms are flipped, whereas for Type $\mathrm{IIA}_{(0,10)}$ only the sign of $\mathcal{A}^0_\mu$ is flipped, while for Type $\mathrm{IIA}_{(2,8)}$ only the signs of $\mathcal{A}^A_\mu$ are flipped. Thus the vector kinetic terms have signatures $(+)^{h^{1,1}+1}$, $(-)^{h^{1,1}+1}$, $(+)^{h_{1,1}}(-)$, and $(+)(-)^{h^{1,1}}$, respectively.

The interactions of these fields are encoded in certain coupling matrices that one obtains when performing the dimensional reduction. For a four-dimensional $\mathcal{N}=2$ theory these coupling matrices can be interpreted as geometrical data on the scalar manifolds $\mathcal{M}_{2h^{1,1}}$ of the vector and $\mathcal{N}_{4h^{2,1}+4}$ of hypermultiplets, which have real dimensions $2h^{1,1}$ and $4h^{2,1}+4$, respectively. These are the geometries that we have reviewed in the previous section. At this point the sign flips become relevant since they determine the signatures of the metrics of $\mathcal{M}_{2h^{1,1}}$ and $\mathcal{N}_{4h^{2,1}+4}$.

The vector multiplet sector

Let us first consider the vector multiplet scalars y^A and x^A. In signature $(1,9)$ their kinetic terms come with same sign, and the manifold $\mathcal{M}_{2h^{1,1}}$ can be shown to be a complex manifold. The real scalars y^A and x^A can be combined into complex scalars $z^A = y^A + ix^A$, which provide holomorphic coordinates for $\mathcal{M}_{2h^{1,1}}$. The scalar fields y^A parametrize the moduli space of real Kähler forms J on the CY3, while x^A parametrize the deformations of the internal components of the B-field, which corresponds to a harmonic (1,1)-form on the CY3. The combined moduli space parametrized by z^A can be viewed as a complexification of the real moduli space of Kähler forms, and is usually just called the Kähler moduli space. This space carries itself a Kähler metric $g_{A\bar{B}}(z,\bar{z})$, which appears in the four-dimensional action as the generalized kinetic term (sigma model) of the scalars z^A, that is $\mathcal{L} \sim g_{A\bar{B}}(z,\bar{z})\partial_\mu z^A \partial^\mu \bar{z}^{\bar{B}}$. Moreover, this Kähler metric is not generic, but special, because its Kähler potential $K(z,\bar{z})$ can be obtained from a holomorphic prepotential $\mathcal{F}(z)$. Thus

for a p-form field if and only if the metric has an even number of negative eigenvalues.

$g_{A\bar{B}}(z,\bar{z})$ is a (projective) special Kähler metric or SK metric for short.

Let us next look at the four-dimensional vector fields, still restricting ourselves to signature (1,9). We have obtained $h^{1,1}+1$ vector fields, of which one, $\mathcal{A}^0_\mu$, belongs to the supergravity multiplet and is called the graviphoton, while the others, $\mathcal{A}^A_\mu$, belong to the $h^{1,1}$ vector multiplets. We denote the corresponding field strength by $\mathcal{F}^0_{\mu\nu}$ and $\mathcal{F}^A_{\mu\nu}$. As explained before, the vector fields can be rearranged into linear combinations A^Σ_μ, $\Sigma = 0,\dots,n_V = h^{1,1}$ so that the field strength $F^\Sigma_{\mu\nu}$ together with their duals $G_{\Sigma|\mu\nu}$ form a symplectic vector. By carrying out the reduction explicitly, one finds that the couplings between scalars and vectors are encoded by the complex coupling matrix $\mathcal{N}_{\Sigma\Lambda} = \mathcal{R}_{\Sigma\Lambda} + i\mathcal{I}_{\Sigma\Lambda}$, which depends on the scalars z^A through the prepotential $\mathcal{F}(z^A)$.

The resulting bosonic Lagrangian for the supergravity multiplet and n_V vector multiplets has the form (4.1), and the only difference between Type IIA and Type IIA* is the overall sign flip for the vector fields A^Σ_μ. Since we use a convention where $\mathcal{I}_{\Sigma\Lambda}$ is negative definite, Type IIA corresponds to $\lambda=-1$, while Type IIA* corresponds to $\lambda=1$:

$$L^{(1,3)IIA/IIA^*}_{G+VM} = \frac{1}{2}\star R_4 - \bar{g}_{A\bar{B}}(z,\bar{z})dz^A\wedge\star d\bar{z}^B - \frac{\lambda}{4}\mathcal{I}_{\Sigma\Lambda}F^\Sigma\wedge\star F^\Lambda + \frac{1}{4}\mathcal{R}_{\Sigma\Lambda}F^\Sigma\wedge F^\Lambda\,, \tag{4.10}$$

where $A,B=1,\dots,n_V=h^{1,1}$ and $\Lambda,\Sigma=0,\dots,n_V=h^{1,1}$. For $\lambda=-1$ this is the standard result of [122]. For the Type IIA* the sign flips in the ten-dimensional Lagrangian induce a sign flip in the four-dimensional Maxwell term.

In signatures $(0,10)$ and $(2,8)$ Table 4.10 shows that the metric of $\mathcal{M}_{2h^{1,1}}$ has neutral signature. The four-dimensional $\mathcal{N}=2$ supersymmetry algebra requires SPK geometry for the vector multiplets in these cases. The case (0,10) $\to$ (0,4) has been worked out in full detail in [124]. As far as the vector multiplet sector is concerned, the only difference between this and the case (2,8) $\to$ (2,2) is an overall sign flip of the Maxwell term. Note that in both cases the vector kinetic terms have Lorentz signature and therefore are indefinite. As mentioned before, the overall sign of the Maxwell term is conventional in the sense that it can be flipped by a field redefinition. Therefore we can take either value of $\lambda=\pm1$ in the following Lagrangian:

$$L^{(0,4),(2,2)}_{G+VM} = \frac{1}{2}\star R_4 - \bar{g}_{A\bar{B}}(z,\bar{z})dz^A\wedge\star d\bar{z}^B + \frac{\lambda}{4}\mathcal{I}_{\Sigma\Lambda}F^\Sigma\wedge\star F^\Lambda + \frac{1}{4}\mathcal{R}_{\Sigma\Lambda}F^\Sigma\wedge F^\Lambda\,. \tag{4.11}$$

Compared to (4.10) the scalar geometry is now SPK and the scalar fields z^A are para-complex fields. The couplings $\bar{g}_{A\bar{B}}$, $\mathcal{I}_{\Sigma\Lambda}$ and $\mathcal{R}_{\Sigma\Lambda}$ are determined by the standard formulae of special geometry, but using a para-holomorphic instead of a holomorphic prepotential, see [124, 129, 130] for details.

The hypermultiplet sector

The scalars z^α, $\alpha = 1, \ldots, h^{2,1}$ parametrize the deformations of the complex structure of the CY3 metric. They provide coordinates on a special Kähler submanifold of the hypermultiplet manifold, with metric $g_{\alpha\bar{\beta}}(z, \bar{z})$ and prepotential $\mathcal{F}(z^\alpha)$.

The additional scalars φ (dilaton), $\tilde{\phi}$ (axion) and $\zeta^I, \tilde{\zeta}_I$, $I = 0, \ldots h^{2,1}$ (R-R scalars) extend this SK manifold either to a quaternion-Kähler manifold (QK manifold) or to a para-quaternion-Kähler manifold (PQK manifold). Which case is realized depends on the signs of the kinetic terms of the R-R scalars. The HM Lagrangian takes the form

$$\begin{aligned} L^{IIA}_{HM} = &- \tilde{G}_{\alpha\bar{\beta}} dz^\alpha \wedge \star d\bar{z}^{\bar{\beta}} - \frac{1}{4} d\varphi \wedge \star d\varphi \\ &- e^{-2\varphi} \left[d\tilde{\phi} + \frac{1}{2} \left(\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I \right) \right] \wedge \star \left[d\tilde{\phi} + \frac{1}{2} \left(\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I \right) \right] \\ &- \frac{\lambda}{2} e^{-\varphi} \left[\mathcal{I}_{IJ} d\zeta^I \wedge \star d\zeta^J + \mathcal{I}^{IJ} \left(d\tilde{\zeta}_I + \mathcal{R}_{IK} d\zeta^K \right) \wedge \star \left(d\tilde{\zeta}_I + \mathcal{R}_{IK} d\zeta^K \right) \right] , \end{aligned} \tag{4.12}$$

where $\lambda = -1$ for Type IIA$_{(1,9)}$ [122] and Type IIA$_{(0,10)}$ [124] and $\lambda = 1$ for Type IIA$^*_{(1,9)}$ and Type IIA$_{(2,8)}$. The coupling matrices $\mathcal{I}_{IJ}$ and $\mathcal{R}_{IJ}$ depend on the complex scalars z^α and are determined by the prepotential $\mathcal{F}(z^\alpha)$ by the same formulae as vector field couplings for vector multiplets. This reflects that the HM manifolds resulting from CY3 compactifications are not generic, but of a special type, which can be obtained from SK manifolds by the c-map. In the case at hand the SK manifold is the complex structure moduli space, and the c-map is either the standard c-map or the temporal c-map. These c-maps can be defined using the reduction of a Lorentz signature VM Lagrangian to three dimensions either over space or over time. That the same types of HM manifolds occur in CY3 compactifications is no coincidence, but related to the fact that T-duality of Type II CY3 compactifications exchanges VMs and HMs, as we will see later. The HM geometry is QK for Type IIA$_{(1,9)}$ and Type IIA$_{(0,10)}$ (and positive definite, since in our convention $\mathcal{I}_{IJ}$ is negative definite), and PQK for Type IIA$^*_{(1,9)}$ and Type IIA$_{(2,8)}$. Note that in all cases the submanifold parametrized by the scalars z^α is an SK manifold.

4.4.2 Type IIB Calabi-Yau compactifications

We now turn to Type IIB compactifications. The NS-NS sector is the same as for Type IIA. In the R-R sector the zero form C_0 gives rises to a scalar c. The two-form C_2 gives rise to a two-form $C_{\mu\nu}$ which we dualize into a scalar a, and to $h^{1,1}$ scalars u^A, $A = 1, \ldots, h^{1,1}$. Taking into account the self-duality of the five-form G_5, the four-form C_4 gives rise to $h^{1,1}$ two-forms $C^A_{\mu\nu}$ which we dualize to scalars v^A, and $1 + h^{2,1}$ vectors $\mathcal{A}^0_\mu$ and $\mathcal{A}^\alpha_\mu$. The first vector is associated to the harmonic (3,0)-form of the CY3, while the other vectors correspond to the harmonic (2,1)-forms. See Table 4.11 for a summary.

10 d	4 d	
C	c	scalar
C_{MN}	$C_{\mu\nu} \sim a$ $C_{mn} \sim u^A$	scalar $h^{1,1}$ scalars
C_{MNPQ}	$C_{\mu\nu mn} \sim v^A$ $C_{\mu mnp} \sim \mathcal{A}^0_\mu, \mathcal{A}^A_\mu$	$h^{1,1}$ scalars $1 + h^{2,1}$ vector fields

Table 4.11: Massless fields in Type IIB Calabi-Yau compactifications, R-R sector.

Together with the NS-NS fields, these fields are the bosonic content of the supergravity multiplet, $(g_{\mu\nu}, \mathcal{A}^0_\mu)$, of $h^{2,1}$ vector multiplets (z^α, A^α_μ), and of $h^{1,1}+1$ hypermultiplets $(y^A, x^A, \varphi, \tilde{\phi}, u^A, v^A, c, a)$. The vector fields can be rearranged into linear combinations A^I_μ with field strength $F^I_{\mu\nu}$ which together with the dual field strength $G_{I|\mu\nu}$ form a symplectic vector. The relative signs between the kinetic terms are determined by those between the ten-dimensional fields and are listed in Table 4.12.

While one can perform the reduction of Type IIB theories explicitly, see for example [123] for $\mathrm{IIB}_{(1,9)}$, we can infer the result by using mirror symmetry and tracing sign flips. As is well known, Type $\mathrm{IIA}_{(1,9)}$ compactified on a CY3 with Hodge numbers $(h^{1,1}, h^{2,1})$ is equivalent to $\mathrm{IIB}_{(1,9)}$ compactified on the mirror CY3 with Hodge numbers $(h'^{1,1} = h^{2,1}, h'^{2,1} = h^{1,1})$. Both theories have $n_V = h^{1,1} = h'^{2,1}$ vector multiplets and $n_H = h^{2,1}+1 = h'^{1,1}+1$ hypermultiplets. In IIA compactifications complex structure moduli sit in hypermultiplets and Kähler moduli in vector multiplets while in IIB compactifications it is the other way round. We would like to compactify the IIB theory on the same CY3 as the IIA theory, but this is the same as compactifying the IIA theory on the mirror. The resulting theory has $n_V = h^{2,1}$ vector multiplets and $n_H = h^{1,1}+1$ hypermultiplets, and the action can be brought to our preferred standard form of a vector and hypermultiplet action. To adapt results from Type IIB to Type IIB* and Type IIB', we then only have to trace the effect of the ten-dimensional sign flips.

As a result, the bosonic Lagrangian for the supergravity multiplet and the $n_V = h_{2,1}$ vector multiplets takes the form

$$L^{IIB/IIB^*/IIB'}_{G+VM} = \frac{1}{2} \star R_4 - \bar{g}_{\alpha\bar{\beta}}(z,\bar{z}) dz^\alpha \wedge \star d\bar{z}^{\bar{\beta}} - \frac{\lambda}{4} \mathcal{I}_{IJ} F^I \wedge \star F^J + \frac{1}{4} \mathcal{R}_{IJ} F^I \wedge F^J \, , \tag{4.13}$$

where $\lambda = -1$ for IIB, and $\lambda = 1$ for IIB* and IIB'. The geometry is SK, with the two cases distinguished by an overall sign flip of the gauge fields.

In the hypermultiplet sector we can rearrange the scalars into linear combinations

	z^α	y^A	x^A	φ	$\tilde{\phi}$	c	a	u^A	v_A	A^I_μ
IIB	+	+	+	+	+	+	+	+	+	+
IIB*	+	+	+	+	+	−	−	−	−	−
IIB'	+	+	−	+	−	−	+	+	−	−

Table 4.12: Signs of the kinetic terms for scalar and vector fields resulting from Type IIB CY3 compactifications. A + indicates a standard kinetic term. The fields z^α are the vector multiplet scalars.

$\zeta^I \sim c, v^A$ and $\tilde{\zeta}_I \sim a, u^A$. The IIB HM Lagrangians take the form

$$\begin{aligned} L^{IIB/IIB^*/IIB'}_{HM} =& -\tilde{G}_{A\bar{B}} dz^A \wedge \star d\bar{z}^{\bar{B}} - \frac{1}{4} d\varphi \wedge \star d\varphi \\ &+ \epsilon_1 e^{-2\varphi} \left[d\tilde{\phi} + \frac{1}{2}\left(\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I\right)\right] \wedge \star \left[d\tilde{\phi} + \frac{1}{2}\left(\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I\right)\right] \\ &- \frac{\epsilon_2}{2} e^{-\varphi} \left[\mathcal{I}_{IJ} d\zeta^I \wedge \star d\zeta^J - \epsilon_1 \mathcal{I}^{IJ} \left(d\tilde{\zeta}_I + \mathcal{R}_{IK} d\zeta^K \right) \wedge \star \left(d\tilde{\zeta}_I + \mathcal{R}_{IK} d\zeta^K \right) \right] . \end{aligned} \tag{4.14}$$

For Type IIB the scalars $z^A = y^A + ix^A$ are complex and parametrize an SK submanifold. The parameters ϵ_1, ϵ_2 take the values $(\epsilon_1, \epsilon_2) = (-1, -1)$, the HM manifold is positive definite and QK. For Type IIB* the signs of all R-R scalars are flipped, and $(\epsilon_1, \epsilon_2) = (-1, 1)$. The scalars z^A are again complex and span an SK submanifold, and the HM manifold is PQK. When going from IIB* to IIB', the signs of $x^A, \tilde{\phi}$ and of $\tilde{\zeta}_I \sim a, u^A$ are flipped. The scalars y^A and x^A now combine into para-complex scalars $z^A = y^A + ex^A$ which parametrize an SPK submanifold. We now have $(\epsilon_1, \epsilon_2) = (1, 1)$. This is again a PQK manifold, but with a PSK submanifold instead of an SK manifold. Thus the S-duality relating Type IIB* to Type IIB' changes the HM manifolds in a significant way, while keeping it consistent with the same supersymmetry algebra.

Comparing Type IIA with Type IIB compactifications on the same CY3, we see that, loosely speaking, vector and hypermultiplet get exchanged. A Type IIA compactification has $n_V = h^{1,1}$ vector and $n_H = h^{2,1} + 1$ hypermultiplets, while a Type IIB compactification has $n'_V = h^{2,1}$ vector and $n'_H = h^{1,1} + 1$ hypermultiplets. As indicated by the Hodge numbers, complex structure moduli of the CY3 metric end up in HMs for Type IIA and in VMs for Type IIB, while (para)-complexified Kähler moduli end up in VMs for Type IIB and in HMs for Type IIA. In both cases there is a universal HM which contains the dilaton, axion and two R-R fields. Moreover, in both cases all model dependence, that is the dependence on the choice of the CY3, is encoded in two functions, the holomorphic prepotential of the complex structure moduli space and the (para)-holomorphic prepotential of the (para)complexified Kähler moduli space.[9] The complex structure moduli space is of course always complex, but the Kähler moduli space becomes para-complex

[9]To see that the two prepotentials are on the same footing one must go beyond a simple dimensional

for Type IIA$_{(0,10)}$ and Type IIA$_{(2,8)}$ as well as Type IIB'$_{(1,9)}$ due to the sign flip of the Kalb-Ramond field. The HM manifolds are completely determined by their distinguished S(P)K submanifold through a c-map. This structure is consistent with certain pairs of compactifications being 'on the same moduli' after compactification to three dimensions. As a result, Type II compactifications are mutually related by T-dualities transverse to the CY3. This will be studied in detail in the next section.

4.5 T-duality

We can now combine the results about c-maps with those about CY3 compactifications to determine how the four-dimensional theories resulting from Type II CY3 compactifications are related by T-duality. Type IIA$_{(1,9)}$ string theory compactified on a circle of radius R, measured in string units $\sqrt{\alpha'}$, is equivalent to Type IIB$_{(1,9)}$ compactified on a circle of radius $1/R$ [64, 65]. Moreover, Type IIB$_{(1,9)}$ string theory on ten-dimensional Minkowski space can be obtained as an alternative decompactification limit $R \to 0$ of the circle compactified Type IIA$_{(1,9)}$ theory, with winding modes playing the roles of momentum modes, and vice versa. This is what is meant when saying that the uncompactified theories 'are T-dual to each other.' T-duality extends to backgrounds which include a compact factor transverse to the circle. In particular Type IIA$_{(1,9)}$ compactified on $X \times S^1_R$, is equivalent to Type IIB$_{(1,9)}$ compactified on $X \times S^1_{1/R}$, where X is the same CY3. By taking the alternative decompactification limit $R \to \infty$, one can map the four-dimensional effective field theories for Type IIA$_{(1,9)}$ and Type IIB$_{(1,9)}$, compactified on the same CY3 X, to one another, and the relation between the respective vector and hypermultiplet sectors is given by the c-map and its inverse [133]. Timelike T-dualities and mixed T-dualities which combine spacelike/timelike reduction with timelike/spacelike oxidation, together with S-duality, relate all ten-dimensional Type II theories to one another [53, 108]. In this section we extend these T-dualities to CY3 compactifications. We remark that it is straightforward though somewhat tedious to work out the explicit relations between the fields of two T-dual four-dimensional effective field theories. T-duality operates naturally in the string frame, and therefore we would need to convert our actions from the Einstein frame to the string frame, perform the reductions of T-dual theories over circles of radii R and $1/R$, and then read off the relations between the fields. While the explicit map between fields is needed for some applications, in particular for mapping solutions from one theory to solutions of a T-dual theory, we will only be interested in how the various Type II CY3 compactifications are related to each other by T-duality and S-duality. For this it is sufficient to match the hypermultiplet manifolds that we get after reduction to

reduction and include the α'-corrections to the Kähler moduli space. We refer to [150] for a review of string theory on Calabi-Yau manifolds.

three dimensions, as this shows that both four-dimensional theories reduce to the same three-dimensional theory. All that we need for this comparison was worked out in section 2. Explicit maps between the fields will be given in the next chapter where we will study the action of T-duality on solutions of the four-dimensional effective field theories.

4.5.1 Signature (1,3) and spacelike/timelike T-duality

To start exploring the web of relations between four-dimensional theories we begin with the CY3 compactification of the Type $\mathrm{IIA}_{(1,9)}$ theory.

- Type $\mathrm{IIA}_{(1,9)}$ string theory on a Calabi-Yau threefold has $n_V = h^{1,1}$ vector and $n_H = h^{2,1} + 1$ hypermultiplets. It realizes the standard $\mathcal{N} = 2$ algebra with R-symmetry $U(2) \cong U(1) \times SU(2)$ and the scalar manifold has the form
$$\mathcal{M}^{IIA} = \mathcal{M}^{SK_+}_{2h^{1,1}} \times \tilde{\mathcal{N}}^{QK}_{4h^{2,1}+4} \ .$$
Upon spacelike reduction the scalar manifold becomes the product of two QK manifolds
$$\mathcal{M}^{(1,2)} = \mathcal{N}^{QK}_{4h^{1,1}+4} \times \tilde{\mathcal{N}}^{QK}_{4h^{2,1}+4} \ .$$
If one swaps the roles of the two factors and lifts back over space, one obtains
$$\mathcal{M}^{IIB} = \mathcal{N}^{QK}_{4h^{1,1}+4} \times \tilde{\mathcal{M}}^{SK_+}_{2h^{2,1}}$$
as required for a Calabi-Yau compactification of Type IIB string theory. This is the standard, spatial T-duality between Type IIA and Type IIB, extended to their Calabi-Yau compactifications. It employs the standard, spatial c-map in both directions.

- If we start again with Type IIA, but perform a timelike reduction, we obtain a theory in signature (0,3) with scalar target
$$\mathcal{M}^{(0,3)} = \mathcal{N}^{PQK}_{4h^{1,1}+4} \times \tilde{\mathcal{N}}^{QK}_{4h^{2,1}+4} \ ,$$
where the first factor is now PQK rather than QK. Swapping the two factors and lifting back over time we obtain a scalar manifold of the form
$$\mathcal{M}^{IIB^*} = \mathcal{N}^{PQK}_{4h^{1,1}+4} \times \tilde{\mathcal{M}}^{SK_-}_{2h^{2,1}} \ .$$
Note that after the oxidation we have flipped gauge field terms (recorded as SK_-) since we need such a sign in order to obtain a QK manifold by timelike reduction. The resulting four-dimensional theory realizes the twisted Lorentz signature algebra with R-symmetry $U(1,1) \cong U(1) \times SU(1,1)$. Thus we obtain the timelike T-duality between Type IIA and Type IIB*, extended to their Calabi-Yau compactifications. It employs the temporal c-map for reduction and the twisted temporal c-map for oxidation.

- If we start with Type IIA* the initial scalar manifold is

$$\mathcal{M}^{IIA^*} = \mathcal{M}^{SK_-}_{2h^{1,1}} \times \tilde{\mathcal{N}}^{PQK_{SK}}_{4h^{2,1}+4} .$$

Upon space-like reduction this becomes

$$\mathcal{M}^{(1,2)} = \mathcal{N}^{PQK_{SK}}_{4h^{1,1}+4} \times \tilde{\mathcal{N}}^{PQK_{SK}}_{4h^{2,1}+4} ,$$

which lifts back to

$$\mathcal{M}^{IIB^*} = \mathcal{N}^{PQK_{SK}}_{4h^{1,1}+4} \times \tilde{\mathcal{M}}^{SK_-}_{2h^{2,1}} .$$

This realizes the spacelike T-duality between Type IIA* and Type IIB*, extended to their Calabi-Yau compactifications. Here we employ the twisted spatial c-map in both directions.

- If we start with Type IIA* and reduce over time we obtain instead

$$\mathcal{M}^{(0,3)} = \mathcal{N}^{QK}_{4h^{1,1}+4} \times \tilde{\mathcal{N}}^{PQK_{SK}}_{4h^{2,1}+4} ,$$

which lifts back to

$$\mathcal{M}^{IIB} = \mathcal{N}^{QK}_{4h^{1,1}+4} \times \tilde{\mathcal{M}}^{SK_+}_{2h^{2,1}} ,$$

and we realize the timelike T-duality between Calabi-Yau compactifications of Type IIA* and Type IIB. Here we use the twisted temporal c-map for reduction and the temporal c-map for oxidation.

The relations between the four-dimensional theories are summarized by the lower face of the cubic diagram in Figure 4.3.

4.5.2 Mixed T-dualities and signature change

Let us now mix spacelike/timelike reduction with timelike/spacelike oxidation in order to relate four-dimensional theories across signatures. If we start with the CY3 compactification of Type IIA$_{(0,10)}$ we have a scalar manifold which is the product of an SPK and a QK manifold:

$$\mathcal{M}^{IIA,(0,4)} = \mathcal{M}^{SPK}_{2h^{1,1}} \times \tilde{\mathcal{N}}^{QK}_{4h^{2,1}+4} .$$

Upon spacelike reduction to signature $(0,3)$, the scalar manifold becomes

$$\mathcal{M}^{(0,3)} = \mathcal{N}^{PQK_{SPK}}_{4h^{1,1}+4} \times \tilde{\mathcal{N}}^{QK}_{4h^{2,1}+4} ,$$

where the first PQK manifold has an SPK base. This step involves the Euclidean c-map. We now need to identify a IIB theory that gives rise to the same scalar manifold upon timelike reduction. To obtain the QK manifold $\tilde{\mathcal{N}}^{QK}_{4h^{2,1}+4}$ by timelike reduction we need

$$\begin{tikzcd}
& \mathrm{IIB}^* \arrow[rr, leftrightarrow, "T_S"] \arrow[dl, leftrightarrow, "T_T"'] \arrow[dd, leftrightarrow, "CY3"] & & \mathrm{IIA}^* \arrow[dl, leftrightarrow, "T_T"] \arrow[dd, leftrightarrow, "CY3"] \\
\mathrm{IIA} \arrow[rr, leftrightarrow, "T_S"] \arrow[dd, leftrightarrow, "CY3"'] & & \mathrm{IIB} \arrow[dd, leftrightarrow, "CY3"] & \\
& \mathrm{B}^* \arrow[rr, leftrightarrow, "C_S"'] \arrow[dl, leftrightarrow, "C_T"] & & \mathrm{A}^* \arrow[dl, leftrightarrow, "C_T"] \\
\mathrm{A} \arrow[rr, leftrightarrow, "C_S"'] & & \mathrm{B} &
\end{tikzcd}$$

Figure 4.3: The spacelike and timelike T-dualities T_S, T_T between the four Type II string theories in ten-dimensional Minkwoski space induce relations between the four-dimensional supergravity theories, denoted A, B, A*, B* obtained by compatification on the same Calabi-Yau threefold. The number n_V of vector multiplets and n_H of hypermultiplets is related to the Hodge number of the Calabi-Yau threefold by $(n_V, n_H) = (m, n) = (h_{1,1}, h_{2,1} + 1)$ for type-A and $(n_V, n_H) = (m', n') = (h_{2,1}, h_{1,1} + 1)$ for type-B. The theories denoted A^*, B^* have the same structure, but a modified supersymmetry algebra with a non-compact R-symmetry group which results in sign flips in the Lagrangian and modifications of the scalar geometry. The maps relating the four-dimensional theories are denoted C_S, C_T, depending on whether they use a spacelike or timelike reduction and oxidation.

to start with vector multiplets which have SK geometry and a sign flip between scalar and vector term, denoted SK$_-$. This could be either the CY3 compactification of IIB* or IIB'. The map $\tilde{\mathcal{M}}^{SK_-}_{2h^{2,1}} \to \tilde{\mathcal{N}}^{QK}_{4h^{2,1}+4}$ is the twisted version of the temporal c-map. The HM manifold of the partner theory must match $\mathcal{N}^{PQK_{SPK}}_{4h^{1,1}+4}$, that is, it must be a PQK manifold with an SPK base. Therefore we need to choose IIB', which has a scalar manifold of the type

$$\mathcal{M}^{IIB'} = \mathcal{N}^{PQK_{SPK}}_{4h^{2,1}+4} \times \tilde{\mathcal{M}}^{SK_-}_{2h^{1,1}} \,.$$

This shows the existence of a mixed T-duality relating the CY3 compactifications of Type IIA$_{(0,10)}$ and Type IIB'$_{(1,9)}$, which uses the Euclidean c-map for reduction and the twisted temporal c-map for oxidation.

If we reduce the IIB' theory over space, the resulting scalar manifold is

$$\mathcal{M}^{(1,2)} = \mathcal{N}^{PQK_{SPK}}_{4h^{2,1}} \times \tilde{\mathcal{N}}^{PQK_{SK}}_{4h^{1,1}+4} \,,$$

where $\tilde{\mathcal{M}}^{SK_-}_{2h^{1,1}} \to \tilde{\mathcal{N}}^{PQK_{SK}}_{4h^{1,1}+4}$ is the twisted version of the spatial c-map.

By lifting this back over time we obtain the scalar manifold of the CY3 compactification of IIA$_{(2,8)}$,

$$\mathcal{M}^{IIA,(2,2)} = \mathcal{M}^{SPK}_{2h^{1,1}} \times \tilde{\mathcal{N}}^{PQK_{SK}}_{4h^{2,1}+4} \ .$$

This step involves the inverse of the neutral c-map which maps SPK to PQK through a timelike reduction. In summary we have shown the existence of a mixed T-duality relating the CY3 compactifications of Type IIB'$_{(1,9)}$ and Type IIA$_{(2,8)}$ which uses the twisted spatial c-map for reduction and the neutral c-map for oxidation. We summarize the six T-dualities which relate Type IIA and Type IIB theories in four dimensions in Table 4.13. Note that under T-duality the compactifications organise into two orbits: the orbit of 'pure' T-dualities relating IIA/IIA*/IIB/IIB* in signature (1,3), and the orbit of mixed T-dualities relating Type IIA theories in signatures (0,4) and (2,2) to the Type IIB' theory in signatures (1,3). In order to connect these two orbits to one another we would need to use the duality between IIB* and IIB', which is an S-duality. It is important to notice that we are now working at the level of the 4-dimensional effective field theory and we are therefore strictly speaking about the S-duality that arises generically in 4 dimensions for theories with $\mathcal{N} = 4$ supersymmetry, which is different from the 10D S-duality of the fundamental theory. For example the 4D S-duality acts on the 4-dimensional dilaton, which differs from the 10-dimensional one by a rescaling from the Calabi-Yau volume. Moreover the 10D S-duality acts non-linearly on the coset space defined by Φ and C_0 whereas in 4D S-duality acts on the coset space defined by ϕ and $\tilde{\phi}$. Since CY3 backgrounds only preserve four-dimensional $\mathcal{N} = 2$ supersymmetry, there is not good reason to expect that S-duality is valid, and therefore we should expect that there are two distinct classes of compactifications. The relation of the backgrounds within each orbit relies on T-duality for backgrounds of the form CY3 $\times$ S^1 which is an established perturbative symmetry of string theory. Note however, that there are special, non-generic $\mathcal{N} = 2$ compactifications which are '$\mathcal{N} = 4$-like' and exhibit S-duality. For this class all Type II CY3 compactifications should form a single orbit at the non-perturbative level by combining pure T-dualities, mixed T-dualities and S-duality. Note that for S-duality to work, there has to exist, for the HM manifolds of these special models, an isomorphism of PQK manifolds, which replaces an SK base with an SPK base. Such an isomorphism can only exist in special case when the structure of the prepotential for the S(P)K base is very simple.

Let us finally point out that considering the other ten-dimensional signatures will add nothing new. The Type II$_{(5,5)}$ theories cannot be compactified on CY3 folds, and the theories IIA$_{(10,0)}$, IIA$^*{}_{(9,1)}$, IIA$_{(8,2)}$ are related to those we have considered by an overall sign change of the metric, which maps $(t, s) \to (s, t)$. Theories related in this way have been shown to be equivalent [53]. The Type IIA$_{(4,6)}$ and Type IIA$_{(6,4)}$ reduce to theories in signature (4,0) and (0,4) which are equivalent to those we have considered. From the higher-dimensional point of view the chain of mixed T-dualities is projected onto the

IIA in 4d	Scalar mfd.	3d sig.	Scalar mfd.	IIB in 4d	Scalar mfd.
$\mathrm{IIA}_{(0,4)}$	SPK $\times$ QK	(0,3)	$\mathrm{PQK}_{SPK} \times$ QK	$\mathrm{IIB'}_{(1,3)}$	$\mathrm{PQK}_{SPK} \times \mathrm{SK}_-$
$\mathrm{IIA}_{(1,3)}$	$\mathrm{SK}_+ \times$ QK	(0,3)	$\mathrm{PQK}_{SK} \times$ QK	$\mathrm{IIB}^*_{(1,3)}$	$\mathrm{PQK}_{SK} \times \mathrm{SK}_-$
$\mathrm{IIA}^*_{(1,3)}$	$\mathrm{SK}_- \times \mathrm{PQK}_{SK}$	(0,3)	QK $\times \mathrm{PQK}_{SK}$	$\mathrm{IIB}_{(1,3)}$	QK $\times \mathrm{SK}_+$
$\mathrm{IIA}_{(1,3)}$	$\mathrm{SK}_+ \times$ QK	(1,2)	QK $\times$ QK	$\mathrm{IIB}_{(1,3)}$	QK $\times \mathrm{SK}_+$
$\mathrm{IIA}^*_{(1,3)}$	$\mathrm{SK}_- \times \mathrm{PQK}_{SK}$	(1,2)	$\mathrm{PQK}_{SK} \times \mathrm{PQK}_{SK}$	$\mathrm{IIB}^*_{(1,3)}$	$\mathrm{PQK}_{SK} \times \mathrm{SK}_-$
$\mathrm{IIA}_{(2,2)}$	SPK $\times \mathrm{PQK}_{SK}$	(1,2)	$\mathrm{PQK}_{SPK} \times \mathrm{PQK}_{SK}$	$\mathrm{IIB'}_{(1,3)}$	$\mathrm{PQK}_{SPK} \times \mathrm{SK}_-$

Table 4.13: Summary of T-dualities between Type IIA and Type IIB Calabi-Yau compactifications for all (inequivalent) four-dimensional signatures. In the middle we specify the three-dimensional theory to which the T-dual four-dimensional theories reduce.

chain we have described. Since the four-dimensional theory in signature (0,4) is unique, the difference between Type $\mathrm{IIA}_{(0,10)}$ and Type $\mathrm{IIA}_{(4,6)}$ is lost from the four-dimensional perspective. Similarly the Type $\mathrm{IIB}_{(3,7)}$ and Type $\mathrm{IIB}_{(7,3)}$ reduce to theories in signatures $(3,1)$ of (1,3), which are of the Type IIB' type, and the distinction between Type $\mathrm{IIB'}_{(3,7)}$ and Type $\mathrm{IIB'}_{(1,9)}$ is lost from the four-dimensional perspective.[10] We have already pointed out that in order to relate such a theory to Type IIB*, and thus to the other T-duality orbit, we need to use S-duality, which can only be expected to be a symmetry for non-generic $\mathcal{N} = 2$ compactifications. Signatures (3,7) and (7,3) have a unique theory of Type IIB', but they can be related through mixed T-dualities to signatures (1,9) and (9,1) where by using S-duality they can be related to Type IIB* and from there by T-dualities to Type IIA/IIA* and Type IIB. Thus for $\mathcal{N} = 2$ compactifications which preserve S-duality, we can connect the CY3 compactifications of all Type II theories in signatures (0,10) , . . . (4,6) to one another (and the same applies to signature obtained by an overall sign flip). All in all, for generic $\mathcal{N} = 2$ we have to expect two disjoint T-duality orbits, as shown in the following figure.

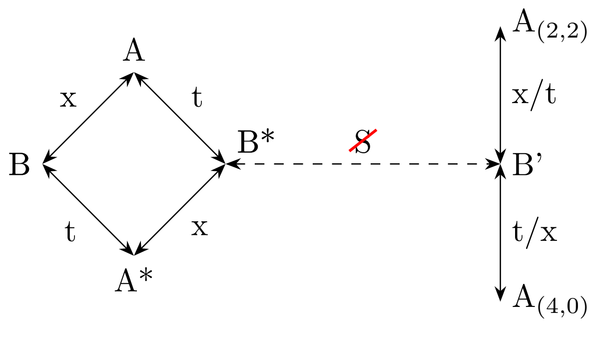


Figure 4.4: 4-dimensional duality web of Type II theories in arbitrary signatures compactified on Calabi-Yau threefolds.

[10]The actions of $\mathrm{IIB'}_{(1,9)/(9,1)}$ and $\mathrm{IIB'}_{(3,7)/(7,3)}$ differ by an overall sign flip of the R-R fields [53, 54]. Upon compactification on a CY3, this results in a flip of the parameter ϵ_2 in the hypermultiplet sector, but the resulting hypermultiplet manifolds are isometric.

4.6 Outlook

In this chapter we have obtained the CY3 compactifications of all ten-dimensional Type II string theories and analyzed how they are related to one another by T-duality and S-duality. At the level of symmetries and effective supergravity, we get a full and satisfactory picture which is consistent with the idea that exotic string theories and their compactifications fit into an extended string theory landscape and allow one to realize all maximally supergravity theories in all signatures as limits. Of course, admitting backgrounds with multiple time directions, as well as inverted kinetic terms in the effective action raises conceptual questions. Instead of repeating the arguments of [53, 54, 108, 110], let us ask what new insights and future directions result from our work.

Apart from symmetry considerations, a reason to consider the inclusion of exotic string theories into the string theory landscape is string universality, and the observation that bubbles of exotic spacetime signature can be generated [54]. One future direction is to explore in more detail whether and how dynamical signature change can be realized as a physical process in string theory. It would be interesting to relate this to the recent work [60, 118, 119] on complex spacetime metrics, which was in part motivated by earlier work [59] on topology change. Gravity with a dynamical signature has recently been discussed in [151] within the framework of Einstein-Cartan gravity.

To make a preliminary remark on this topic, we note that the T-duality orbit along which spacetime signature can change in Type II CY3 compactifications is connected to the orbit of the standard IIA/IIB compactifications and their IIA*/IIB* partners by S-duality, which we cannot expect to be valid in generic $\mathcal{N} = 2$ compactifications. This suggests that in backgrounds with less than $\mathcal{N} = 4$ supersymmetry, there is, generically (with the exception of special '$\mathcal{N} = 4$ like' backgrounds), a separation between a phase with Lorentzian string worldsheets and fixed Lorentzian spacetime signature, and a phase with Euclidean string worldsheets and arbitrary spacetime signature.[11] Thus signature change may only be relevant cosmologically if the universe goes through a phase of high unbroken supersymmetry.[12]

Based on the results of this chapter, solutions of exotic string theories can now be explored systematically from a four-dimensional $\mathcal{N} = 2$ perspective in addition to the ten-dimensional perspective. Future work will include how solutions transform under dualities, whether solutions of different theories can be connected to one another, including the question of dynamical signature change. Staying within the class of theories with

[11] Lorentizan string worldsheets are also possible in neutral signature (5,5), but this signature does not give rise to CY3 compactifications. Also, to connect it to the standard IIA/IIB theories, one needs to use S-duality.

[12] Our universe could still be a brane world embedded into a higher-dimensional universe with multiple time directions, an option that has been explored in [110].

Lorentzian string worldsheets, there are interesting questions regarding the relation between solutions in Type II and Type II*. The results of this chapter imply that the dual pair of planar cosmological and black hole solutions described in [135, 136] lifts to a 'dual pair' of solutions in Type IIA and Type IIA*. This raises the question whether these 'dual solutions' can be related by T-duality (which is not obvious). Both solutions have horizon thermodynamics that can be related to the same Euclidean thermal partition functions, and one can now ask whether the Type II embedding provides insights into the underlying microscopic physics. One could also study whether these solutions correspond to admissible saddle points, in the sense of [60, 118, 119], of the Euclidean path integral.

Chapter 5

T-duality across non-extremal horizons

> I think mathematics has been overused.
>
> *Radu Tatar*

In this chapter we will present some results of a work to appear [2]. Our aim is to study the global action of T-duality on spacetimes with non-extremal Killing horizons which means that the Killing vector changes between spacelike and timelike. In particular we want to investigate what happens to the singularities and the horizons after performing T-duality. Indeed, it has been observed that singularities and horizons are exchanged after performing T-duality [152], and we want to check this for our solutions. The solutions that we study are black hole and cosmological solutions of Einstein (anti-)Maxwell theory with planar symmetry, they can essentially be thought of as a planar version of Reissner-Nordström. This chosen class of metric is suitable for our investigation because of the relative simple structure of the metrics. In order to apply T-duality to these solutions, we derive a 4-dimensional version of Buscher rules[1]. including spacelike/timelike reductions as well as including fields that come from the Ramond-Ramond sector once embedded into string theory.

[1]In this chapter we will use the terms T-duality and Buscher rules interchangeably. However one has to keep in mind that the true T-duality between Type II theories only applies when the background has a compact circle, the theories being physically inequivalent in the decompactification limit. The derivation of Buscher rules only requires a background with a (not necessarily compact) isometry. In that sense, Buscher rules are more general than the $O(d, d, \mathbb{Z})$ T-duality, and should therefore be thought of as a solution generating technique rather than a strict physical equivalence between theories.

5.1 Introduction

In [135, 136], some planar solutions of the STU and anti-STU model were studied. The *anti* theory is identical to the original theory up to sign flips for some kinetic terms. Such sign flips have been discussed and parametrized in the previous chapter. The *STU* model is an $\mathcal{N} = 2, D = 4$ supergravity theory coupled to three vectormultiplets. The prepotential has the simple form $F = STU$ where S, T, U are respectively the scalar fields belonging in each vectormultiplet. The corresponding scalar target space is $\left(\frac{SU(1,1)}{U(1)}\right)^3$.

These solutions describe respectively cosmological and black hole solutions. By setting the scalar fields of these solutions to be constant one obtains solutions of Einstein (anti-) Maxwell theory which can famously be obtained as a consistent truncation of $\mathcal{N} = 2, D = 4$ supergravity. In order to dualize solutions, we need to have a 4-dimensional version of the T-duality Buscher rules that includes the Ramond sector. It is known how to derive these rules in 10 dimensions from a spacetime perspective [116] which has the merit of incorporating the Ramond sector (see appendix B for an explicit derivation). The procedure is to perform a dimensional reduction on the first action either on space or time, perform a field redefinition in 3 dimensions, and then uplift to the other action in 4 dimensions. We will derive these rules explicitly in the case of Einstein-Maxwell coupled to the universal hypermultiplet (UHM). We will explain in a moment the necessity to include this multiplet but for now let us pause to state explicitly which framework we have in mind to discuss these solutions.

In this chapter, all results and dualization rules are performed in a purely 4-dimensional perspective (provided we know which field corresponds to the 4D dilaton) and can therefore be studied independently of the details of the embedding into string theory since we are looking at a universal sector. The $\mathcal{N} = 2$ language is used to interpret the different fields coming from the UHM. The UHM is a hypermultiplet that is always present when compactifying Type II theories on Calabi-Yau manifolds and is comprised of the dilaton, the universal axion, and two scalar fields coming from the Ramond-Ramond sector. Einstein-Maxwell is a common subsector of $\mathcal{N} = 2$ and $\mathcal{N} = 8$ supergravity but the STU model is not obtainable as a generic Calabi-Yau compactification (even though it can be obtained from a specific self-mirror Calabi-Yau orbifold see [153]). However, this theory can be seen as a consistent truncation of $\mathcal{N} = 8, D = 4$ supergravity whose scalar coset is $\frac{E_{7(7)}}{SU(8)}$ [154], whence making it embeddable into Type II supergravities which allows for an explicit dimensional uplift. Therefore in the rest of the chapter we will use the $\mathcal{N} = 2$ language to talk about the fields from the universal hypermultiplet but whenever an explicit uplift will be needed, we will adopt the $\mathcal{N} = 8$ point of view.

Even though the solutions of interest do not support scalar fields, we need to include the UHM scalar sector in the action in order to have a matching of degrees of freedom.

This can be understood from two different perspectives. From the supergravity point of view, reducing the supergravity multiplet to 3 dimensions gives the UHM. Therefore if we want to uplift to 4 dimensions to a different theory, the dual theory will need to be a gravity theory coupled to the UHM. From the point of view of Calabi-Yau manifolds, the supergravity multiplet+UHM is the field content that is universal irrespective of the specific choice of manifold. Formally, it can be seen as the field content of a Calabi-Yau with all geometric moduli frozen $h^{1,1} = h^{2,1} = 0$ (even though such Calabi-Yau manifolds don't exist). The c-map works as long as the number of vectormultiplets is greater or equal to zero and the number of hypermultiplets is greater or equal to one. The case at hand is therefore the minimal case where there are no vectormultiplets and a single hypermultiplet. This is a special case because performing the spacelike c-map on Einstein-Maxwell-UHM will dualize the theory to itself making the c-map a self-duality in this case where the scalar manifold is in the image of the c-map. Generically, the c-map relates a theory with n_V vectormultiplets and n_H hypermultiplets to a theory with $n'_V = n_H - 1$ vectormultiplets and $n'_H = n_V + 1$ hypermultiplets.

5.2 4-dimensional Buscher rules

Our starting point is therefore the Einstein-Maxwell-UHM action in the Einstein frame which has the form

$$\begin{aligned}\mathcal{S}_4 = \int d^4x\sqrt{\hat{g}_E}\Big(&\hat{R}_E - \frac{1}{2}\hat{g}_E^{\mu\rho}\hat{g}_E^{\nu\lambda}\hat{F}_{\mu\nu}\hat{F}_{\rho\lambda} - 2\hat{g}_E^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi \\ &- 2e^{4\varphi}\,[\,\partial^\mu\tilde{\varphi} + \frac{1}{2}(\zeta\partial^\mu\tilde{\zeta} - \tilde{\zeta}\partial^\mu\zeta)]\,[\,\partial_\mu\tilde{\varphi} + \frac{1}{2}(\zeta\partial_\mu\tilde{\zeta} - \tilde{\zeta}\partial_\mu\zeta)] \\ &- e^{2\varphi}\left[\partial_\mu\zeta\partial^\mu\zeta + \partial^\rho\tilde{\zeta}\partial_\rho\tilde{\zeta}\right]\Big)\,.\end{aligned} \tag{5.1}$$

Notice that compared to the previous chapter, the dilaton has been rescaled as $\varphi \to -2\varphi$ and there is an overall factor of 2 in order to fit with string theory interpretation. This action describes the metric and graviphoton coming from the supergravity multiplet, and the universal hypermultiplet comprised of 4 scalars which are the dilaton φ, the universal axion $\tilde{\varphi}$ and two Ramond-Ramond scalars ζ and $\tilde{\zeta}$. As explained before, we are here assuming a Calabi-Yau embedding but even if the embedding is a toroidal one, there should be a U-duality frame adapted to an $\mathcal{N} = 2$ truncation. The target space geometry of the universal hypermultiplet is $U(2,1)/U(2) \times U(1)$ in the spacelike reduction case and $U(2,1)/(U(1,1) \times U(1))$ in the timelike reduction case [131].

T-duality is naturally expressed in the string frame so we need to perform the following conformal transformation

$$\hat{g}^E_{\mu\nu} = e^{-2\varphi}\hat{g}^S_{\mu\nu}. \tag{5.2}$$

The transformation rule of the Ricci scalar under a conformal rescaling is a known formula which in our case gives

$$\hat{R}^E = e^{2\varphi}\hat{R}^S + 6e^{2\varphi}\Delta_S\varphi - 6\hat{g}_S^{\mu\nu}e^{2\varphi}\partial_\mu\varphi\partial_\nu\varphi. \tag{5.3}$$

After a suitable integration by parts we obtain the following action in 4 dimensions in string frame

$$\begin{aligned}\mathcal{S}_4 = \int d^4x\sqrt{\hat{g}_S}\Big[&e^{-2\varphi}\Big(\hat{R}_S + 4\hat{g}_S^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi\Big) - \frac{1}{2}\hat{g}_S^{\mu\rho}\hat{g}_S^{\nu\lambda}\hat{F}_{\mu\nu}\hat{F}_{\rho\lambda}\\ &- 2e^{2\varphi}\,[\,\partial^\mu\tilde{\varphi} + \frac{1}{2}(\zeta\partial^\mu\tilde{\zeta} - \tilde{\zeta}\partial^\mu\zeta)]\,[\,\partial_\mu\tilde{\varphi} + \frac{1}{2}(\zeta\partial_\mu\tilde{\zeta} - \tilde{\zeta}\partial_\mu\zeta)]\\ &- \Big(\partial_\mu\zeta\partial^\mu\zeta + \partial^\rho\tilde{\zeta}\partial_\rho\tilde{\zeta}\Big)\Big].\end{aligned} \tag{5.4}$$

We need to perform one more transformation. This is because the universal axion $\tilde{\varphi}$ is actually the Kalb-Ramond B-field, that was dualized (in the sense of Hodge duality) into a scalar because we are in 4 dimensions. From the point of view of T-duality it is more natural to have the B-field as a 2-form so we need to undualize $\tilde{\varphi}$. We therefore perform the usual procedure of adding a Lagrange multiplier that enforces the Bianchi identity as an equation of motion (see appendix D for details).

All in all our final action in 4 dimensions in string frame is

$$\begin{aligned}\mathcal{S}_4 = \int d^4x\sqrt{\hat{g}_S}\Big[&e^{-2\varphi}\Big(\hat{R}_S + 4\hat{g}_S^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi\Big) - \frac{1}{2}\hat{g}_S^{\mu\rho}\hat{g}_S^{\nu\lambda}\hat{F}_{\mu\nu}\hat{F}_{\rho\lambda}\\ &- \frac{1}{12}e^{-2\varphi}\hat{H}_{\mu\nu\rho}\hat{H}^{\mu\nu\rho} - \frac{1}{6}(\zeta\partial^\lambda\tilde{\zeta} - \tilde{\zeta}\partial^\lambda\zeta)\epsilon^{\mu\nu\rho\lambda}\hat{H}_{\mu\nu\rho}\\ &- \Big(\partial_\mu\zeta\partial^\mu\zeta + \partial^\rho\tilde{\zeta}\partial_\rho\tilde{\zeta}\Big)\Big]\end{aligned} \tag{5.5}$$

We are now ready to perform the dimensional reduction. Our reduction ansatzes are the following

- For the metric
$$\hat{g}^S_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} g^S_{\mu\nu} + e^{2\sigma}V_\mu V_\nu & e^{2\sigma}V_\nu \\ e^{2\sigma}V_\mu & e^{2\sigma}\end{pmatrix}$$
- For the Kalb-Ramond field
$$\hat{B}^S_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} B_{\mu\nu} - V_{[\mu}A'_{\nu]} & A'_\nu \\ -A'_\mu & 0\end{pmatrix}$$
For convenience we write $\hat{H}_3 = \tilde{H}_3 + F' \wedge (dy + V)$ where $F' = dA'$ and $\tilde{H}_3 = H_3 - dV \wedge A'$.
- And for the gauge field $\hat{A}_\mu = \xi dy + (A_\mu - \xi V_\mu)dx^\mu$. The scalars reduce trivially.

The resulting action after reduction is therefore

$$\begin{aligned}\mathcal{S}_3 = \int d^3x\sqrt{g_S}\Bigg(& e^{-2\bar{\varphi}}R_S + 4e^{-2\bar{\varphi}}g_S^{\mu\nu}\partial_\mu\bar{\varphi}\partial_\nu\bar{\varphi} - e^{-2\bar{\varphi}}g_S^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma - \frac{1}{4}e^{-2(\bar{\varphi}-\sigma)}V_{\mu\nu}V^{\mu\nu} \\ & - \frac{1}{2}e^{\sigma}(F_{\mu\nu} - 2\partial_{[\mu}\xi V_{\nu]})(F^{\mu\nu} - 2\partial^{[\mu}\xi V^{\nu]}) - e^{-\sigma}\partial_\mu\xi\partial^\mu\xi \\ & - \frac{1}{12}e^{-2\bar{\varphi}}\tilde{H}_{\mu\nu\rho}\tilde{H}^{\mu\nu\rho} - \frac{1}{4}e^{-2(\bar{\varphi}+\sigma)}F'_{\mu\nu}F'^{\mu\nu} \\ & - \frac{1}{2}\epsilon^{\mu\nu y\lambda}F'_{\mu\nu}(\zeta\partial_\lambda\tilde{\zeta} - \tilde{\zeta}\partial_\lambda\zeta) \\ & - e^{\sigma}\left[\partial_\mu\zeta\partial^\mu\zeta + \partial^\rho\tilde{\zeta}\partial_\rho\tilde{\zeta}\right]\Bigg)\end{aligned} \tag{5.6}$$

where $\bar{\varphi} = \varphi - \frac{1}{2}\sigma$. This action has a symmetry that is not yet manifest in this parametrization. To make it manifest we first perform the following field redefinition $A^*_\mu = A_\mu - \xi V_\mu$ therefore $F_{\mu\nu} - 2\partial_{[\mu}\xi V_{\nu]} \to F^*_{\mu\nu} + \xi V_{\mu\nu}$. Moreover we need to dualize this gauge field into a scalar, and the procedure is the same as before, we add the following Lagrange multiplier

$$\mathcal{L}_m = -\epsilon^{\mu\nu\rho}F^{*\mu\nu}\partial_\rho\tilde{\xi}. \tag{5.7}$$

After more manipulations we finally obtain the reduced action

$$\begin{aligned}\mathcal{S}_3 = \int d^3x\sqrt{g_S}\Bigg(& e^{-2\bar{\varphi}}R_S + 4e^{-2\bar{\varphi}}g_S^{\mu\nu}\partial_\mu\bar{\varphi}\partial_\nu\bar{\varphi} - e^{-2\bar{\varphi}}g_S^{\mu\nu}\partial_\mu\sigma\partial_\nu\sigma - \frac{1}{4}e^{-2(\bar{\varphi}-\sigma)}V_{\mu\nu}V^{\mu\nu} \\ & - e^{-\sigma}\left[\partial_\mu\xi\partial^\mu\xi + \partial^\rho\tilde{\xi}\partial_\rho\tilde{\xi}\right] \\ & + \epsilon^{\mu\nu\rho}\xi V_{\mu\nu}\partial_\rho\tilde{\xi} \\ & - \frac{1}{12}e^{-2\bar{\varphi}}\tilde{H}_{\mu\nu\rho}\tilde{H}^{\mu\nu\rho} - \frac{1}{4}e^{-2(\bar{\varphi}+\sigma)}F'_{\mu\nu}F'^{\mu\nu} \\ & - \epsilon^{\mu\nu y\lambda}F'_{\mu\nu}\zeta\partial_\lambda\tilde{\zeta} \\ & - e^{\sigma}\left[\partial_\mu\zeta\partial^\mu\zeta + \partial^\rho\tilde{\zeta}\partial_\rho\tilde{\zeta}\right]\Bigg),\end{aligned} \tag{5.8}$$

which is manifestly invariant under the following transformations

$$\bar{\varphi} \to \bar{\varphi}, \sigma \to -\sigma, \tilde{\sigma} \leftrightarrow \tilde{\varphi}, \tilde{\xi} \leftrightarrow \tilde{\zeta}, \xi \leftrightarrow \zeta, A'_\mu \leftrightarrow -V_\mu, B_{\mu\nu} \to B_{\mu\nu} + A'_\mu V_\nu - A'_\nu V_\mu. \tag{5.9}$$

We can then see that for all the fields in the NS-NS sector, all the rules in 4 dimensions are identical to the ones we find in 10 dimensions (see appendix B). For the R-R fields, we see easily that the rules give

$$\zeta^{4D} = \zeta^{3D} \to \xi = \hat{A}_y. \tag{5.10}$$

However, for the transformation of $\tilde{\zeta}$ we can not give the rules directly for the field but only its derivative because of the dualization condition in 3 dimensions. This is not an issue since the action possesses a shift isometry, even though it is not manifest, see [134, 155] for details.

After some computations we find that the 4 dimensional Buscher rules for the R-R sector fields are

$$\zeta' = \hat{A}_y \tag{5.11}$$

$$\partial_\lambda \tilde{\zeta}' = \frac{1}{2}\hat{\epsilon}_{y\mu\nu\lambda}\sqrt{\hat{g}_{yy}}\left[\hat{F}^{\mu\nu} + \hat{A}_y\left(\partial^\mu\left(\frac{\hat{g}^{\nu y}}{\hat{g}_{yy}}\right) - \partial^\nu\left(\frac{\hat{g}^{\mu y}}{\hat{g}_{yy}}\right)\right)\right] \tag{5.12}$$

$$\hat{A'}_y = \zeta \tag{5.13}$$

$$\hat{F'}^{\mu\nu} = -\sqrt{\hat{g}_{yy}}\hat{\epsilon}^{y\mu\nu\rho}\partial_\rho\tilde{\zeta} + \zeta\left(\partial^\mu(\hat{B}_{\nu y}) - \partial^\nu(\hat{B}_{\mu y})\right) \tag{5.14}$$

One can check explicitly that the same symmetries found before are still present in the cases where the duality relates theories with flipped kinetic signs, other signatures and with timelike reduction. The resulting Buscher rules will then have some sign flips which however are not relevant in our analysis.

5.3 Dualization of the cosmological solution

We recall the cosmological solution of Einstein-Maxwell theory [135, 136] :

$$\begin{array}{c} ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2\left(dx^2 + dy^2\right), \quad f(r) = \left(-\frac{2M}{r} + \frac{e^2}{r^2}\right) \\ e = \sqrt{Q^2 + P^2}, \quad F = -\frac{Q}{r^2}dt \wedge dr + P dx \wedge dy \end{array} \tag{5.15}$$

whose Penrose diagram is (after maximal extension)

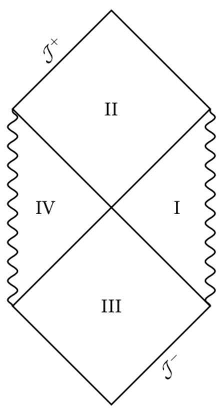


Figure 1. Cosmological solution

We will restrict ourselves to a purely electric solution which means that in our case we have $A = -\frac{e}{r}dt$.

This solution was obtained in the Einstein frame, but it takes the same form in string frame as the solution does not support a dilaton. We can therefore apply the Buscher rules derived previously. The metric describes the two regions I and II, one exterior with a spacelike isometry and one interior with a timelike isometry. We want to dualize the interior patch I so we perform a timelike duality and the resulting metric is

$$ds'^2 = -\frac{dt^2}{f(r)} + \frac{dr^2}{f(r)} + r^2(dx^2 + dy^2) \tag{5.16}$$

Since the starting metric is diagonal, we still have a trivial B-field in the dualized solution. The new dilaton is

$$\varphi' = -\frac{1}{2}\ln|f(r)|. \tag{5.17}$$

Moreover we have

$$\zeta' = -\frac{e}{r}, \tag{5.18}$$

and

$$\partial_\lambda \tilde{\zeta}' = 0. \tag{5.19}$$

We also have trivially

$$A'_y = 0 \quad \text{and} \quad F'_{\mu\nu} = 0. \tag{5.20}$$

By construction this is a solution but we want to perform a sanity check that this is indeed a solution of the equations of motion of the target theory, namely Einstein anti-UHM. To do this we go back to Einstein frame where the metric takes the form

$$ds'^2 = -dt^2 + dr^2 + (e^2 - 2Mr)(dx^2 + dy^2) \tag{5.21}$$

The corresponding Einstein tensor is

$$G_{\mu\nu} = \begin{pmatrix} M^2(2Mr-e^2)^{-2} & 0 & 0 & 0 \\ 0 & M^2(2Mr-e^2)^{-2} & 0 & 0 \\ 0 & 0 & M^2(2Mr-e^2)^{-1} & 0 \\ 0 & 0 & 0 & M^2(2Mr-e^2)^{-1} \end{pmatrix} \tag{5.22}$$

We also have

$$\zeta' = -\sqrt{2}\frac{e}{r} \tag{5.23}$$

The reason why a factor of $\sqrt{2}$ appeared is because the original solutions were derived with the usual normalisation, but our action has a factor 1/2 instead of 1/4 in front of the Maxwell term.

The stress energy tensor is

$$T_{\mu\nu} = 4(\partial_\mu\varphi'\partial_\nu\varphi' - \frac{1}{2}g_{\mu\nu}(\partial^\rho\varphi'\partial_\rho\varphi')) - 2e^{2\varphi'}\left(\partial_\mu\zeta'\partial_\nu\zeta' - \frac{1}{2}g_{\mu\nu}(\partial^\rho\zeta'\partial_\rho\zeta')\right) \tag{5.24}$$

where the sign flip comes from the fact that we are in the anti-UHM theory. Applied to our solution we find

$$T_{\mu\nu} = 4\left(\left[\frac{e^2 - Mr}{e^2 r - 2Mr^2}\right]^2 \delta_{rr} - \frac{1}{2} g_{\mu\nu}\left[\frac{e^2 - Mr}{e^2 r - 2Mr^2}\right]^2\right) \tag{5.25}$$

$$- 2\left(-\frac{2M}{r} + \frac{e^2}{r^2}\right)^{-1}\left(\frac{2e^2}{r^4}\delta_{rr} - \frac{1}{2} g_{\mu\nu}\frac{2e^2}{r^4}\right) \tag{5.26}$$

We evaluate for the diagonal components and find

$$T_{tt} = T_{rr} = \frac{2M^2}{(e^2 - 2Mr)^2} \tag{5.27}$$

and

$$T_{xx} = T_{yy} = \frac{2M^2}{2Mr - e^2} \tag{5.28}$$

We therefore indeed find that our solution satisfies as expected the equations of motions

$$G_{\mu\nu} = 2T_{\mu\nu} \tag{5.29}$$

where again the factor of 2 comes from our normalisation convention.

5.4 Analysis of the dualized solution

Let us first display the profile of the dilaton

$$\varphi' = -\frac{1}{2}\ln|f(r)| = -\frac{1}{2}\ln|\frac{-2M}{r} + \frac{e^2}{r^2}| \tag{5.30}$$

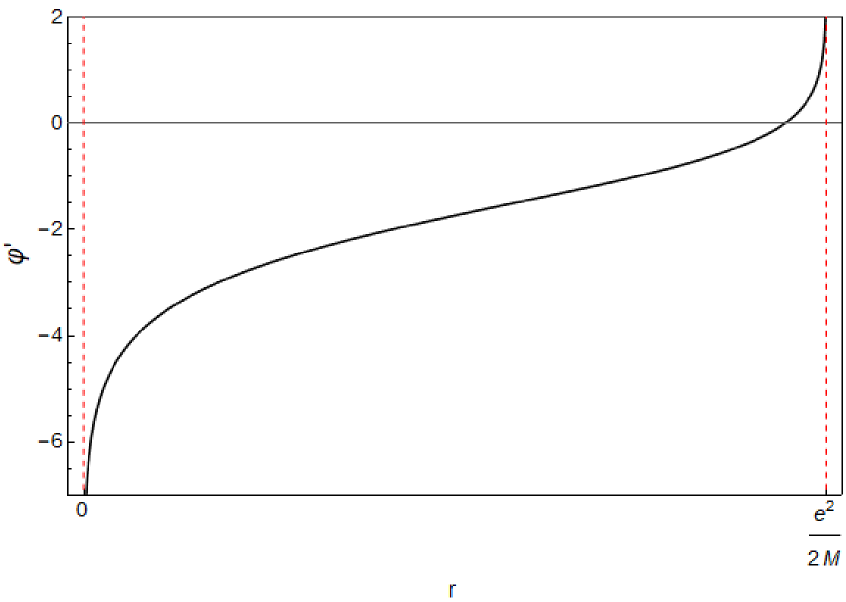


Figure 2. Dilaton profile of the dualized interior patch of the cosmological solution.

The dilaton becomes divergent at the loci $r = 0$ and $r = e^2/2M$.

Now let us look at curvature. In Einstein frame the Ricci tensor is

$$R_{\mu\nu} = \frac{2M^2}{(e^2 - 2Mr)^2}\delta_{rr} \tag{5.31}$$

so Ricci scalar is

$$R = \frac{2M^2}{(e^2 - 2Mr)^2} \tag{5.32}$$

In the string frame we have

$$R_{\mu\nu} = \mathrm{diag}(\frac{e^4 - 2e^2Mr + 2M^2r^2}{r^2(e^2 - 2Mr)^2}, \frac{3e^4 - 10e^2Mr + 6M^2r^2}{r^2(e^2 - 2Mr)^2}, -\frac{e^2 - 2Mr}{r^2}, -\frac{e^2 - 2Mr}{r^2}) \tag{5.33}$$

$$R = \frac{4M^2}{r^2(-e^2 + 2Mr)} \tag{5.34}$$

We can see that both scalar invariants blow up at the position where the horizon was in the cosmological solution, and is now a curvature singularity. This is an expected feature of T-duality which exchanges horizons and singularities [152].

At $r = 0$, the Ricci scalar is regular in Einstein frame but singular in string frame which may look less problematic than the image of the horizon. However, the relation between Einstein and string frame becomes singular

$r \to 0$	$\varphi' \to -\infty$	$g_S'^{(4)} = e^{\varphi'} \to 0$
$r \to r_*$	$\varphi' \to +\infty$	$g_S'^{(4)} = e^{\varphi'} \to \infty$

It does not matter whether we consider Einstein or string frame as the "right frame" since the question is whether or not a solution is regular as a whole. In the swampland program, the distance conjecture states that for any effective field theory coupled to gravity that has a moduli space, if a modulus moves towards a point at an infinite geodesic distance in moduli space, then one encounters an infinite tower of states which becomes exponentially light (for more details see [156]). For the string coupling, it implies a singular ratio between the string scale set by the string tension $T = \frac{1}{2\pi\alpha'}$ and by the Planck scale set by gravitational interactions with coupling κ. We know from equation (2.89) that in D spacetime dimensions we have the following relation

$$\kappa = (\alpha')^{(D-2)/4} e^{\varphi'}. \tag{5.35}$$

Reexpressed in terms of mass and length scales this gives

$$e^{\frac{\varphi'}{4}} = \frac{\kappa^{1/4}_{(10D)}}{\sqrt{\alpha'}} \propto \frac{L_P^{(10D)}}{L_S} = \frac{M_S}{M_P^{(10D)}}. \tag{5.36}$$

Irrespective of the details of the uplift of the solution to 10D IIB* theory, we know that the 4D and 10D dilaton are related by

$$e^{-2\varphi'_{(4D)}} \propto \mathcal{V} e^{-2\varphi'_{(10D)}}, \tag{5.37}$$

where $\mathcal{V}$ is the volume of the internal space. However the volume of the internal space is proportional to a scalar field that does not sit in the universal hypermultiplet, this means that for our solutions the internal volume is kept constant. Approaching $r = 0$ therefore means that we have

$$\frac{M_S}{M_P^{(10D)}} \propto e^{\varphi'_{(10)}/4} \to 0. \tag{5.38}$$

If we keep M_P fixed and take $M_S \to 0, \alpha' \to \infty$ then from the string mass formula

$$\frac{M^2}{M_S^2} \propto \alpha' M^2 = 4\left(N + \tilde{N} + \frac{1}{2}k_L^2 + \frac{1}{2}k_R^2 + const\right), \tag{5.39}$$

We see that for a state with fixed N and $\tilde{N}$ this limit implies that $M \to 0$. In this limit the whole tower of massive string excitations "collapses" and become all massless. This is known as a tensionless string limit as $T = \frac{1}{2\pi\alpha'} \to 0$ (this limit of string theory has connections with higher spin theories and Vasiliev gravity see [157]). This signals that our effective description of the theory breaks down as expected from the swampland distance conjecture.

For $r \to e^2/2M$ (which means close to the image of the horizon) we find instead that $\varphi' \to +\infty$ so that $g'_S \to +\infty$ which is a strong coupling limit. If we are permitted to perform S-duality, for example by embedding into a toroidal compactification or using an "exact" STU-model (one in which the prepotential does not receive higher order corrections, these models are known to exist see [153] and [158]) then this is equivalent to $g''_S = \frac{1}{g'_S} \to 0$ in the S-dual IIB'-theory, and we again have a tensionless string limit.

Lastly we want to see if these singular regions can be reached using geodesics. We want first to investigate if $r = 0$ is located at a finite distance interval for a spacelike geodesic, meaning that we have $dt = dx = dy = 0$ and whence

$$ds'^2 = \frac{1}{-2M/r + e^2/r^2} dr^2 \tag{5.40}$$

which gives

$$\lambda = -\left(\frac{-e^2\sqrt{e^2}}{3M^2} + \frac{e^2\sqrt{e^2 - 2MR}}{3M^2} + \frac{R\sqrt{e^2 - 2MR}}{3M}\right) \tag{5.41}$$

where λ is the affine parameter and which here is finite. We now want to probe the same region with a radial null geodesic. We write the metric in the following way

$$-hdt^2 + hdr^2 + r^2(dx^2 + dy^2) \tag{5.42}$$

We look at a null geodesic parametrized as

$$X^\mu = (-r + r_0, r, 0, 0) \quad \text{and} \quad \dot{X}^\mu = (-1, 1, 0, 0) \tag{5.43}$$

for which the non vanishing Christoffel symbols are

$$\Gamma^t_{rt} = \frac{1}{2}h^{-1}\dot{h}, \quad \Gamma^x_{rx} = r^{-1} \quad \Gamma^r_{tt} = \frac{1}{2}h^{-1}\dot{h} \quad \Gamma^r_{rr} = \frac{1}{2}h^{-1}\dot{h} \quad \Gamma^r_{xx} = \frac{1}{2}h^{-1}(-2r) \tag{5.44}$$

Geodesic equation for t:

$$\Gamma^t_{rt}\dot{X}^r\dot{X}^t + \Gamma^t_{tr}\dot{X}^t\dot{X}^r \tag{5.45}$$

$$= 2.\frac{1}{2}h^{-1}\dot{h}(-1)(1) = -h^{-1}\dot{h} = h^{-1}\dot{h}\dot{X}^t. \tag{5.46}$$

Geodesic equation for r:

$$\Gamma^r_{tt}\dot{X}^t\dot{X}^t + \Gamma^r_{rr}\dot{X}^r\dot{X}^r + \Gamma^r_{xx}\dot{X}^x\dot{X}^x + \Gamma^x_{yy}\dot{X}^y\dot{X}^y \tag{5.47}$$

$$= \Gamma^r_{tt} + \Gamma^r_{rr} = h^{-1}\dot{h}\dot{X}^r \tag{5.48}$$

so all in all we have

$$\ddot{X}^\mu + \Gamma^\mu_{\nu\rho}\dot{X}^\nu\dot{X}^\rho = F(r)\dot{X}^\mu \tag{5.49}$$

with $F(r) = h^{-1}\dot{h}$. This is the geodesic equation for a non-affine parameter. To find the affine parameter we use

$$\frac{d\lambda}{dr} = e^{\int^r F(s)ds} = h(r) \tag{5.50}$$

so

$$\lambda = \int_R^0 h(r) = \int_R^0 \frac{1}{-2M/r + e^2/r^2} = \left[-\frac{2Mr(Mr+e^2) + e^4\ln(|e^2 - 2Mr|)}{8M^3}\right]_R^0 \tag{5.51}$$

which is finite.

If we now want to probe the image of the horizon we do

$$\lambda = \int_R^{e^2/2M} h(r) = \int_R^{e^2/2M} \frac{1}{-2M/r + e^2/r^2} = \left[-\frac{2Mr(Mr+e^2) + e^4\ln(|e^2 - 2Mr|)}{8M^3}\right]_R^{e^2/2M} \tag{5.52}$$

which diverges.

5.5 Dualization of the black hole solution and other patches

We repeat the process of the previous section without details. The black hole solution is

$$\begin{gathered} ds^2 = +f(r)dt^2 - \tfrac{dr^2}{f(r)} + r^2\left(dx^2 + dy^2\right), \quad f(r) = \left(-\tfrac{2M}{r} + \tfrac{e^2}{r^2}\right) \\ e = \sqrt{Q^2 + P^2}, \quad F = -\tfrac{Q}{r^2}dt\wedge dr + Pdx\wedge dy \end{gathered} \tag{5.53}$$

whose Penrose diagram is (when maximally extended)

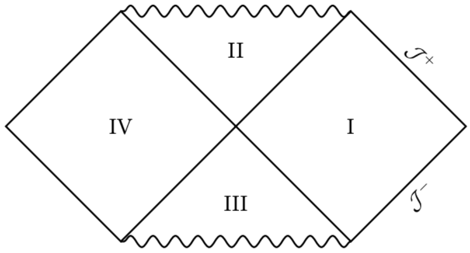


Figure 2

now we dualize on space (internal patch) so the dual solution is

$$ds'^2 = \left(\frac{1}{f(r)}dt^2 - \frac{1}{f(r)}dr^2 + r^2\left(dx^2 + dy^2\right)\right) \tag{5.54}$$

and the Ramond-Ramond sector is the same as the case of the dualized cosmological solution. In Einstein frame the metric takes the form:

$$ds'^2 = dt^2 - dr^2 + (e^2 - 2Mr)(dx^2 + dy^2) \tag{5.55}$$

As previously we can check that this is indeed a solution to the equations of motion. This solution has the same Einstein tensor except for a sign flip in the xx and yy components

$$G_{\mu\nu} = \begin{pmatrix} M^2(2Mr - e^2)^{-2} & 0 & 0 & 0 \\ 0 & M^2(2Mr - e^2)^{-2} & 0 & 0 \\ 0 & 0 & -M^2(2Mr - e^2)^{-1} & 0 \\ 0 & 0 & 0 & -M^2(2Mr - e^2)^{-1} \end{pmatrix} \tag{5.56}$$

the stress energy tensor for this solution is

$$T_{\mu\nu} = 4\left(\left[\frac{e^2 - Mr}{e^2 r - 2Mr^2}\right]^2 \delta_{rr} + \frac{1}{2}g_{\mu\nu}\left[\frac{e^2 - Mr}{e^2 r - 2Mr^2}\right]^2\right) \tag{5.57}$$

$$- 2\left(-\frac{2M}{r} + \frac{e^2}{r^2}\right)^{-1}\left(\frac{2e^2}{r^4}\delta_{rr} + \frac{1}{2}g_{\mu\nu}\frac{2e^2}{r^4}\right) \tag{5.58}$$

and when we evaluate its components we find indeed that $G_{\mu\nu} = 2T_{\mu\nu}$.

The dilaton has the same profile as before. The probing of the singularity and horizon is unaffected by the sign flips so the interpretations are identical as in the previous case.

Fo the exterior patches with $r > e^2/2M$, the spacelike dual of the cosmological solution is the same as in the timelike case except the patch lives in Einstein-UHM and not Einstein anti-UHM. The same goes for the timelike dual of the black hole solution.

We want to look at the radial null geodesics in order to probe $r = e^2/2M$ again. The integrals are the same except the bounds are exchanged, so the behavior is the same up to a sign however it is interesting to note that one distinction with the comsological case is that the singularity in the original black hole solution was spacelike and not timelike.

We make a final remark on possibly relating the dualized solutions. Let us focus on the dualized interior patches that are solutions of Einstein anti-UHM. In [136] it was found that the Euclidian action and grand potential as well as other thermodynamic relations are the same for the original cosmological and black hole solutions, except for the range of some of the variables. This could indicate that the two solutions could be interpreted as two phases of the same underlying solution. If we compare the metrics of the interior patches in Einstein frame after dualization we have

- dual of cosmological solution

$$ds'^2 = \left(-dt^2 + dr^2 + \left(e^2 - 2Mr\right)\left(dx^2 + dy^2\right)\right) \tag{5.59}$$

- dual of black hole solution

$$ds'^2 = \left(dt^2 - dr^2 + \left(e^2 - 2Mr\right)\right)\left(dx^2 + dy^2\right)\right) \tag{5.60}$$

The solutions are identical up to sign flips in the first two components, therefore they cannot be identified right away. However they could still be two different localizations of the same E-brane system. Indeed, in [135] the uplift of the cosmological solution was performed where it was found that it corresponds (in the extremal case) to a D1 and D5 brane system with momentum along common direction and a Taub-NUT space when uplifted to Type IIB supergravity. Uplifted to 11D supergravity, it is described by a system of 3 M5 branes intersecting over a string with a PP-wave superimposed along the intersection direction. Performing a dimensional reduction of this sytem would then give the IIA embedding which would correspond to a D0-D4-D4-D4 system as described in [159]. It is therefore natural to expect that the solution of Einstein- anti Maxwell will uplift to a Euclidean version of this brane system in Type IIB* theory. A 10 dimensional analysis of these solutions will therefore shed light on the microscopic relation between the cosmological and black hole solution where the different solutions could be interpreted as different localizations of the same underlying brane system. To illustrate this, let's take the example of a D1- and D2-brane, which are T-dual to eachother. The D2-brane metric is

$$ds^2 = H^{-1/2}(y_1, ..., y_7)(-dt^2 + dx^2 + dy^2) + H^{1/2}(y_1, ..., y_7)(dy_1^2 + ... + dy_7^2), \tag{5.61}$$

where H is a function harmonic on the transverse space which means $\Delta^{\perp} H = 0$. Performing a T-duality along the y direction we get, according to the Buscher rules

$$ds^2 = H^{-1/2}(y_1, ..., y_7)(-dt^2 + dx^2) + H^{1/2}(y_1, ..., y_7)(dy_1^2 + ... + dy_7^2 + dy^2). \tag{5.62}$$

In order to obtain the metric for the D1 brane we therefore need to "localize" by assuming that the harmonic function H now depends on the coordinate y. Assuming this we indeed get the metric of a D1-brane

$$ds^2 = H^{-1/2}(y_1, ..., y_7, y)(-dt^2 + dx^2) + H^{1/2}(y_1, ..., y_7, y)(dy_1^2 + ... + dy_7^2 + dy^2). \quad (5.63)$$

Therefore what we mean by our solutions being two different localizations means that choosing appropriate coordinate dependencies of the E-brane system will recover both metrics (5.59) and (5.60).

Chapter 6

Outlook

人の夢は終わらねェ!

マーシャル・D・ティーチ

We started this thesis by introducing the classical and quantum bosonic string, rederiving its massless spectrum and studying its compactification on a circle, which lead to the important concept of T-duality. After describing the supersymmetric string focusing on Type II theories we showed how they are related by T-duality. We presented the necessary tools from supergravity and geometry, reviewing supersymmetry algebras and their representation theory, and introducing Kaluza-Klein dimensional reduction in this supergravity setting. We finally described Calabi-Yau manifolds as well as the "special geometry" of their moduli spaces.

With these tools, we were able to perform Calabi-Yau compactifications of Hull's exotic Type II supergravity theories and interpret the sign flips of the resulting 4-dimensional Lagrangians of $\mathcal{N} = 2, D = 4$ supergravity in terms of special geometry of the scalar sector. The different c-maps between the vector multiplet and the hypermultiplet geometries were classified and the 4-dimensional duality web was obtained as a projection of the one found in 10D. We then presented the action of T-duality on some spacetimes with non-extremal Killing horizons, which are solutions of Einstein (anti)-Maxwell theory with planar symmetry, which required the derivation of a 4-dimensional version of Buscher rules. We concluded with a discussion of the embedding of the dualized solutions into Type II* supergravity.

One could go further and embed these solutions in a T-duality covariant framework. Double Field Theory (DFT) has initially been constructed as the massless effective theory of the closed string field theory on a double toroidal background. This background is truly doubled in the sense that it is comprised of the torus coordinates as well as coordinates dual to the winding modes [160]. In this way DFT incorporates T-duality as a manifest symmetry in an effective field theory framework. At first glance, this might seem absurd – indeed T-duality is strictly stringy in nature because the string is able to wind around compact dimensions, which a particle can not do. However this is precisely taken into account in DFT thanks to the doubled coordinates. When the compactification scale is much higher than the string scale, the winding modes are not present at low energies since it is hard for strings to wrap cycles. In DFT, this corresponds to the case where the fields do not depend on the dual coordinates. Oppositely, in the T-dual description where the compactification scale is small compared to the string scale, the momentum modes get heavy and DFT only depends on the dual coordinates.

The fundamental fields of DFT are the metric g_{ij}, the Kalb-Ramond B-field b_{ij} and the dilaton ϕ which in the string theory context correspond to the universal massless Neveu-Schwarz sector. However, in DFT we need quantities covariant under the T-duality transformations which correspond to the group $O(d,d)$. In DFT the metric and B-field are therefore unified in a single object called the *generalized metric* defined as

$$\mathcal{H}_{M,N} = \begin{pmatrix} g^{ij} & -g^{ik}b_{kj} \\ b_{ik}g^{kj} & g_{ij} - b_{ik}g^{kl}b_{lj} \end{pmatrix}, \tag{6.1}$$

where $M, N = 1, \ldots, 2D$. Written in terms of the generalized metric, the DFT action takes the form

$$\begin{aligned} S = \int dX e^{-2d} \big(& 4\mathcal{H}^{MN}\partial_M\partial_N d - \partial_M\partial_N\mathcal{H}^{MN} \\ & - 4\mathcal{H}^{MN}\partial_M d\partial_N d + 4\partial_M\mathcal{H}^{MN}\partial_N d \\ & + \frac{1}{8}\mathcal{H}^{MN}\partial_M\mathcal{H}^{KL}\partial_N\mathcal{H}_{KL} \\ & - \frac{1}{2}\mathcal{H}^{MN}\partial M\mathcal{H}^{KL}\partial_K\mathcal{H}_{NL} + \Delta(\mathrm{SC})\mathcal{R} \end{aligned} \tag{6.2}$$

where the field d is a T-scalar combining the dilaton and the determinant of the metric, with $e^{-2d} = \sqrt{g}e^{-2\phi}$. The last term vanishes upon imposing the independence of the fields with respect to the dual coordinates (known as the strong constraint).

When all the dimensions are non-compact and the dual coordinates are projected out, one finds that the DFT action reduces to the familiar 10D supergravity action

$$S = \int dx\sqrt{g}e^{-2\phi}\left(R + 4(\partial\phi)^2 - \frac{1}{12}H_{ijk}H^{ijk}\right). \tag{6.3}$$

Since this action is invariant under diffeomorphisms and gauge transformations of the two-form, the DFT action must be invariant under *generalized diffeomorphisms* incorporating the two. Interestingly, an extension of DFT known as exceptional field theory, that incorporates the full U-duality group as a manifest symmetry, has also been proposed. For reviews on these extended field theories (ExFT), we refer the reader to [76, 161, 162].

Embedding our solutions into these extended geometric frameworks would allow us to probe interesting features of spacetime from a genuinely stringy point of view. Embedding branes in ExFT has been explored in the past (see for example [163]). Moreover our planar solutions support Ramond-Ramond fields, and DFT can be extended to support this sector [113]. One particularly interesting aspect to investigate would be to see if, in this framework, our non-extremal solutions with Ramond fluxes exhibit a resolution of spacetime singularities for example in the spirit of [164].

Appendix A

Basics of geometry and topology

We give here some basic geometry and topology needed for this thesis as well as fixing conventions. We follow [63] but refer the reader to [165] for more details and a more rigorous approach.

Real manifolds A real d-dimensional Riemannian manifold is a space that locally looks like Euclidean space $\mathbb{R}^d$. A manifold is defined by considering a covering with open sets, on which local coordinates systems are introduced. The manifold is constructed by pasting the open sets together and in regions when two overlap, the two sets of local coordinates are related by smooth transition functions. Many topological aspects of real manifolds can be studied using homology and cohomology. In the following we will assume that our manifold M is compact without boundary.

A p-form A_p is an antisymmetric tensor of rank p. The components of A_p are

$$A_p = \frac{1}{p!} A_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}, \tag{A.1}$$

where $\wedge$ is the wedge product (an antisymmetrized tensor product). The possible values of p are $p = 0, 1, \dots, d$ where d is the dimension of M.

The *exterior derivative* d gives a linear map from the space of p-forms to the space of $(p+1)$ forms

$$dA_p = \frac{1}{p!} \partial_{\mu_1} A_{\mu_2 \dots \mu_{p+1}} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{p+1}}. \tag{A.2}$$

An important property is that $d^2 = 0$ making this operator nilpotent. A p-form is called *closed* if

$$dA_p = 0, \tag{A.3}$$

and *exact* if there exists a globally defined $(p-1)$-form A_{p-1} such that

$$A_p = dA_{p-1}. \tag{A.4}$$

A closed p-form can always be written locally in the form dA_{p-1} but not necessarily globally. Therefore a closed form is not necessarily exact but an exact form is always closed.

We denote the space of closed p-forms on M by $C^p(M)$ and the space of exact p-forms on M by $Z^p(M)$. Then the p-th *de Rham cohomology group* $H^p(M)$ is defined to be the following quotient space

$$H^p(M) = C^p(M)/Z^p(M). \tag{A.5}$$

$H^p(M)$ is therefore the space of closed forms in which two forms that only differ by an exact form are considered to be equivalent. The dimension of H^p is called the *Betti number*. These numbers are topological invariants characterizing the manifold. Another important topological invariant is the *Euler characteristic* which can be expressed as

$$\chi(M) = \sum_{i=0}^{d} (-1)^i b_i(M). \tag{A.6}$$

The Betti numbers also give the dimensions of the *homology groups*, which are defined in a similar way to the cohomology groups. The analog of the exterior derivative is the boundary operator δ which acts on submanifolds of M. If N is a submanifold of M, then δN is its boundary. This operator associate with each submanifold its boundary (with signs to take into account the orientation). This operator is also nilpotent since the boundary of a boundary is always 0. It can therefore be used to define homology groups of M in the same way as the cohomology groups were defined using the exterior derivative. Arbitrary linear combinations of submanifolds of dimension p are called p-chains. A chain that has no boundary is called *closed* and a chain that is a boundary is called *exact*. A closed chain z_p, also called a *cycle*, satisfies

$$\delta z_p = 0. \tag{A.7}$$

The simplicial homology group $H_p(M)$ is defined to consist of equivalence classes of p-cycles. Two p-cycles are equivalent if and only if their difference is a boundary.

Poincaré duality Given a real manifold M, let A be an arbitrary p-form and let N be an arbitrary $(p+1)$-chain. Stokes theorem states

$$\int_N dA = \int_{\partial N} A. \tag{A.8}$$

This formula provides an isomorphism between $H^p(M)$ and $H_{d-p}(M)$ called *Poincaré duality*. To every closed p-form A there corresponds a $(d-p)$-cycle N with the property

$$\int_M A \wedge B = \int_N B, \tag{A.9}$$

for all closed $(d-p)$-forms B. Poincaré duality allows us to determine the Betti numbers of a manifold by counting the non-trivial cycles of the manifold. For example, S^N has Betti numbers $b_0 = 1, b_1 = 0, \ldots, b_N = 1$.

Riemannian geometry We now consider manifolds endowed with a metric of indefinite signature, making the manifold a *pseudo-Riemannian manifold*. The metric is a symmetric tensor characterized by an infinitesimal line element

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu. \tag{A.10}$$

The metric tensor can be expressed in terms of the *frame*. Frames are d linearly independant one-forms e^α that are defined locally on M. In a basis of one-forms we write

$$e^\alpha = e^\alpha_\mu dx^\mu. \tag{A.11}$$

The components e^α_μ form a matrix called the *vielbein*. Let $\eta_{\alpha\beta}$ be the flat metric then we have the following relation

$$g_{\mu\nu} = \eta_{\alpha\beta} e^\alpha_\mu \epsilon^\beta_\nu. \tag{A.12}$$

Next we define the Laplace operator that acts on p-forms which is given by

$$\Delta_p = d^\dagger d + d d^\dagger = (d + d^\dagger)^2, \tag{A.13}$$

where

$$d^\dagger = (-1)^{dp+d+1} \star d \star \tag{A.14}$$

for Euclidian signature and with an extra sign for Lorentzian signature. The Hodge $\star$-operator acting on p-forms is defined as

$$\star\, (dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}) = \frac{\varepsilon^{\mu_1 \cdots \mu_p \mu_{p+1} \cdots \mu_d}}{(d-p)!|g|^{1/2}} g_{\mu_{p+1}\nu_{p+1}} \cdots g_{\mu_d \nu_d} dx^{\nu_{p+1}} \wedge \cdots \wedge dx^{\nu_d}. \tag{A.15}$$

The Levi-Civita symbol ε transforms as a tensor density while $\varepsilon/|g|^{1/2}$ is a tensor. A p-form A is said to be *harmonic* if and only if

$$\Delta_p A = 0. \tag{A.16}$$

Harmonic p-forms are in one to one correspondence with the elements of $H^p(M)$. Indeed, if we consider a harmonic form A_p and a positive-definite scalar product then

$$(A_p, (dd^\dagger + d^\dagger d) A_p) = 0 \implies (d^\dagger A_p, d^\dagger A_p) + (dA_p, dA_p) = 0. \tag{A.17}$$

which means that A_p is both closed and co-closed. The Hodge theorem states that on a compact manifold with a positive-definite metric, a p-form has a unique decomposition into harmonic, exact and co-exact pieces

$$A_p = A_p^h + dA^e_{p-1} + d^\dagger A^{ce}_{p+1} \tag{A.18}$$

If we consider a closed form A_p then by definition $dA_p = 0$. We can write

$$0 = (dA_p, A^{ce}_{p+1}) = (dd^\dagger A^{ce}_{p+1}, A^{ce}_{p+1}) = (d^\dagger A^{ce}_{p+1}, d^\dagger A^{ce}_{p+1}) \implies d^\dagger A^{ce}_{p+1} = 0. \tag{A.19}$$

Therefore a closed form can always be written in the form

$$A_p = A_p^h + dA_{p-1}^e. \tag{A.20}$$

Since the Hodge dual turns a closed p-form into a co-closed $(d-p)$-form and *vice versa*, then the Hodge dual defines an isomorphism between the space of harmonic p-forms and the space of harmonic $(d-p)$-forms, therefore

$$b_p = b_{d-p}. \tag{A.21}$$

Another fundamental geometric concept is the *connection*. There are actually two of them, the *affine connection* and the *spin connection*. They are used in forming covariant derivatives so as to map tensors to tensors. Their expressions can be deduced from the requirement that the vielbein is covariantly constant

$$\nabla_\mu e_\nu^\alpha = \partial_\mu e_\nu^\alpha - \Gamma^\rho_{\mu\nu} e_\rho^\alpha + \omega_\mu{}^\alpha{}_\beta e_\nu^\beta = 0. \tag{A.22}$$

This equation determines the affine connection Γ and the spin connection ω up to a contribution characterized by a *torsion tensor*. In this work we will assume that the torsion vanishes, therefore the affine connection is the Levi-Civita connection given by

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\lambda}(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}). \tag{A.23}$$

The formula for the spin connection is

$$\omega_\mu{}^\alpha{}_\beta = -e_\beta^\nu(\partial_\mu e_\nu^\alpha - \Gamma^\lambda_{\mu\nu} e_\lambda^\alpha). \tag{A.24}$$

The curvature tensor can be constructed from either of the two connections. We construct it from the spin connection ω, which is a Lie-algebra valued one-form $\omega^\alpha{}_\beta = \omega_\mu{}^\alpha{}_\beta dx^\mu$. The algebra is $SO(N)$ in the Riemannian case, and a non-compact form of it in the indefinite signature case. We can regard this connection as a Yang-Mills gauge field and thus, the curvature two-form associated to this connection is just the corresponding field-strength

$$R^\alpha{}_\beta = d\omega^\alpha{}_\beta + \omega^\alpha{}_\gamma \wedge \omega^\gamma{}_\beta. \tag{A.25}$$

Written with the base-space components included we have $R_{\mu\nu}{}^\alpha{}_\beta$. The indices can be moved up and down by contracting with metrics and vielbeins and one can form $R^\mu{}_{\nu\rho\lambda}$ which coincides with the *Riemann curvature tensor* that one constructs from the affine connection. Contracting a pair of indices gives the *Ricci tensor*

$$R_{\nu\lambda} = R^\mu{}_{\nu\mu\lambda}, \tag{A.26}$$

and a further contraction gives the *scalar curvature*

$$R = g^{\mu\nu} R_{\mu\nu}. \tag{A.27}$$

Holonomy groups The holonomy group of a Riemannian manifold M of dimension d describes the way objects transform under parallel transport around closed curves. These objects can be tensors or spinors, in our case we will focus on spinors since the manifolds we consider are endowed with a spin structure. The spinors are more informative since the most general transformation of a vector is a rotation, which is an element of $SO(d)$ [1]. The corresponding transformation for a spinor is an element of the covering group $Spin(d)$. If we parallel transport a spinor along a closed curve, it is rotated from its original orientation

$$\varepsilon \to U\varepsilon, \tag{A.28}$$

where U is an element of $Spin(d)$. If we consider doing two consecutive paths each time leaving and returning to the same point then

$$\varepsilon \to U_1 U_2 \varepsilon = U_3 \varepsilon. \tag{A.29}$$

As a result, the U matrices build a group called the *holonomy group* $\mathcal{H}(M)$ of the manifold M.

The generic holonomy group of a Riemannian spin manifold M of real dimension d is $Spin(d)$. One can consider special classes of manifolds where the holonomy group $\mathcal{H}(M)$ is only a subgroup of $Spin(d)$. Such manifolds are called manifolds of *special holonomy*. This lead to a complete classification by Berger [166]

- $\mathcal{H} \subseteq U(d/2)$ if and only if M is Kähler
- $\mathcal{H} \subseteq SU(d/2)$ if and only if M is Calabi-Yau
- $\mathcal{H} \subseteq Sp(d/4)$ if and only if M is hyper-Kähler
- $\mathcal{H} \subseteq Sp(d/4) \cdot Sp(1) \nsubseteq Sp(d/4)$ if and only if M is quaternionic Kähler
- $\mathcal{H} \subseteq G_2$ is possible in 7 dimensions
- $\mathcal{H} \subseteq Spin(7)$ is possible in 8 dimensions

For the first two cases d needs to be a multiple of two and the next two cases it needs to be a multiple of four. We will describe Kähler manifolds in the following sections. Calabi-Yau manifolds are introduced in detail in chapter 3. The special Kähler geometries are also introduced there. The manifolds of G_2 holonomy have been explored as possible compact spaces for compactification of M-theory (see [167] for a recent review) and the manifolds of $Spin(7)$ holonomy for compactification of F-theory (see for example [168])

[1] We assume that the manifold is orientable

Complex manifolds A complex manifold of complex dimension n is a special case of a real manifold of dimension $d = 2n$. It is defined in an analogous manner using complex local coordinates systems. In this case, the transition functions are biholomorphic which means that they and their inverse are holomorphic functions. We denote complex local coordinates as $z^a(a = 1, \ldots, n)$ and their complex conjugates $\bar{z}^{\bar{a}}$.

A complex manifold admits a tensor J which in complex coordinates has components

$$J_a{}^b = i\delta_a{}^b, \qquad J_{\bar{a}}{}^{\bar{b}} = -i\delta_{\bar{a}}{}^{\bar{b}}, \qquad J_a{}^{\bar{b}} = J_{\bar{a}}{}^b = 0. \tag{A.30}$$

These equations are preserved by holomorphic changes of variables, so they describe a globally well-defined tensor.

If one has a real manifold M of dimensions $2n$, there are requirements for it to define a complex manifold. The first requirement is the existence of a tensor $J^m{}_n$ called an *almost complex structure* that satisfies

$$J_m{}^n J_n{}^p = -\delta_m{}^p. \tag{A.31}$$

The second condition is that the almost complex structure is a *complex structure*. The obstruction to this is given by the *Nijenhuis tensor*

$$N^p{}_{mn} = J_m{}^q \partial_{[q} J_{n]}{}^p - J_n{}^q \partial_{[q} J_{m]}{}^p. \tag{A.32}$$

When this tensor is identically zero, J is a complex structure. Then it is possible to choose complex coordinates in every open set such that J satisfies (A.30) and the transition functions are biholomorphic.

On a complex manifold one can define (p, q)-forms as

$$A_{p,q} = \frac{1}{p!q!} A_{a_1 \ldots a_p \bar{b}_1 \ldots \bar{b}_q} dz^{a_1} \wedge \cdots \wedge dz^{a_p} \wedge d\bar{z}^{\bar{b}_1} \wedge \cdots \wedge d\bar{z}^{\bar{z}\bar{b}_q}. \tag{A.33}$$

The real exterior derivative can be decomposed into holomorphic and anti-holomorphic pieces

$$d = \partial + \bar{\partial}, \tag{A.34}$$

with

$$\partial = dz^a \frac{\partial}{\partial z^a} \quad \text{and} \quad \bar{\partial} = d\bar{z}^{\bar{a}} \frac{\partial}{\partial \bar{z}^{\bar{a}}}. \tag{A.35}$$

∂ and $\bar{\partial}$ are called *Dolbeaut operators* which are maps from (p, q)-form to $(p+1, q)$-forms and $(p, q+1)$-forms, respectively. Each of them is nilpotent

$$\partial^2 = \bar{\partial}^2 = 0, \tag{A.36}$$

and anticommute

$$\partial\bar{\partial} + \bar{\partial}\partial = 0. \tag{A.37}$$

Complex geometry We now consider a complex Riemannian manifold. The metric tensor in terms of complex coordinates is given by

$$ds^2 = g_{ab}dz^a dz^b + g_{a\bar{b}}dz^a d\bar{z}^{\bar{b}} + g_{\bar{a}b}d\bar{z}^{\bar{a}}dz^b + g_{\bar{a}\bar{b}}d\bar{z}^a d\bar{z}^{\bar{b}}. \tag{A.38}$$

The reality of the metric implies that g_{ab} and $g_{\bar{a}\bar{b}}$ are complex conjugates and so are $g_{a\bar{b}}$ and $g_{\bar{a}b}$. A *hermitian manifold* is a special case where

$$g_{ab} = g_{\bar{a}\bar{b}} = 0. \tag{A.39}$$

The Dolbeaut cohomology group $H^{p,q}_{\bar{\partial}}(M)$ of a hermitian manifold consists of equivalent classes of $\bar{\partial}$-closed (p,q)-forms. The dimension of $H^{p,q}_{\bar{\partial}}(M)$ is called the *Hodge number* $h^{p,q}$. We can define the Laplacians

$$\Delta_\partial = \partial\partial^\dagger + \partial^\dagger\partial \quad \text{and} \quad \Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger\bar{\partial}. \tag{A.40}$$

A *Kähler manifold* is defined to be a hermitian manifold on which the *Kähler form*

$$J = ig_{a\bar{b}}dz^a \wedge d\bar{z}^{\bar{b}} \tag{A.41}$$

is closed

$$dJ = 0. \tag{A.42}$$

It follows that the metric on these manifold satisfies $\partial_a g_{b\bar{c}} = \partial_b g_{a\bar{c}}$, as well as the complex conjugate relation. The metric can therefore be written locally as

$$g_{a\bar{b}} = \frac{\partial}{\partial z^a}\frac{\partial}{\partial \bar{z}^{\bar{b}}}\mathcal{K}(z,\bar{z}), \tag{A.43}$$

where $\mathcal{K}(z,\bar{z})$ is the *Kähler potential* and thus

$$J = i\partial\bar{\partial}\mathcal{K}. \tag{A.44}$$

The Kähler potential is defined up to the addition of an arbitrary holomorphic and anti-holomorphic functions $f(z)$ and $\bar{f}(\bar{z})$ since

$$\tilde{\mathcal{K}}(z,\bar{z}) = \mathcal{K}(z,\bar{z}) + f(z) + \bar{f}(\bar{z}) \tag{A.45}$$

leads to the same metric. On Kähler manifolds, the various Laplacians all become identical

$$\Delta_d = 2\Delta_{\bar{\partial}} = 2\Delta_\partial. \tag{A.46}$$

The various possible choices of cohomology groups (based on d, ∂ and $\bar{\partial}$) each have a unique harmonic representative of the corresponding type just like in the real case. In Kähler manifolds they all become identical

$$H^{p,q}_{\bar{\partial}}(M) = H^{p,q}_{\partial}(M) = H^{p,q}(M). \tag{A.47}$$

As a consequence, the Hodge and Betti numbers are related by

$$b_k = \sum_{p=0}^{k} h^{p,k-p}. \tag{A.48}$$

If ω is a (p,q)-form on a Kähler manifold with n complex dimensions, then the complex conjugate form $\omega^\star$ is a (q,p)-form. It follows that

$$h^{p,q} = h^{q,p}. \tag{A.49}$$

Similarly, if ω is a (p,q)-form then $\star\omega$ is a $(n-p, n-q)$-form, which implies

$$h^{n-p,n-q} = h^{p,q}. \tag{A.50}$$

In terms of complex local coordinates, only the mixed components of the Ricci tensor are nonvanishing for a hermitian manifold. Therefore, one can define a (1,1)-form, called the *Ricci form*, by

$$\mathcal{R} = iR_{a\bar{b}}dz^a \wedge d\bar{z}^{\bar{b}}. \tag{A.51}$$

For a hermitian manifold, the exterior derivative of the Ricci form is proportional to the torsion. For Kähler manifolds, the torsion always vanishes and therefore the Ricci form is always closed

$$d\mathcal{R} = 0. \tag{A.52}$$

It is therefore a representative of a cohomology class belonging to $H^{1,1}(M)$. This class is called the *first Chern class*

$$c_1 = \frac{1}{2\pi}[\mathcal{R}]. \tag{A.53}$$

Appendix B

Derivation of generalized Type II Buscher rules

This section is dedicated to the derivation of the generalized Type II Buscher rules. This means that we derive the rules for arbitrary signature, spacelike and timelike dualities, and we include the Ramond-Ramond sector rules. The procedure is to take the Type IIA and Type IIB supergravity actions, dimensionally reduce them, and then compare the actions in the lower dimension in order to have a dictionary between the 10D fields. This section follows [54] giving more details for pedagogical purposes.

We start with the 10D Type IIA theories actions in string frame as follows

$$\begin{aligned} S =& \frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}x\sqrt{|\hat{g}|}\left(e^{-2\hat{\Phi}}\left[\hat{\mathcal{R}}-\frac{\alpha}{2}|\hat{H}_3|^2+4(\nabla\hat{\Phi})^2\right]-\frac{\alpha\beta}{2}|\hat{F}_2|^2-\frac{\beta}{2}|\hat{\tilde{F}}_4|^2\right) \\ &-\frac{1}{4\kappa_{10}^2}\int \hat{B}_2\wedge\hat{F}_4\wedge\hat{F}_4. \end{aligned} \tag{B.1}$$

The parameters α, β and γ are sign parameters which indicate respectively if the fundamental string has Lorentzian or Euclidean worldsheet, the D2-brane has Lorentzian or Euclidean worldvolume and if the compact dimension is spacelike or timelike.
We use the following reduction ansatzes

$$\begin{aligned} &\mathrm{d}s_{10}^2 = \mathrm{d}s_9^2+\gamma e^{2\sigma}\left(\mathrm{d}y+A_1\right)^2, \\ &\hat{\tilde{F}}_4 = \tilde{F}_4+\tilde{F}_3\wedge(\mathrm{d}y+A_1), \quad \hat{C}_3 = C_3+C_2\wedge(\mathrm{d}y+A_1) \\ &\hat{\tilde{F}}_2 = \tilde{F}_2+F_1\wedge(\mathrm{d}y+A_1), \quad \hat{C}_1 = C_1+C_0\wedge(\mathrm{d}y+A_1) \\ &\hat{H}_3 = \tilde{H}_3+H_2\wedge(\mathrm{d}y+A_1), \quad \hat{B}_2 = B_2+B_1\wedge(\mathrm{d}y+A_1) \end{aligned} \tag{B.2}$$

where $\gamma=\pm1$ specifies the signature of the compact dimension, and $G_2=dA_1$, $H_p = dB_{p-1}$, $F_p=dC_{p-1}$ and

$$\begin{aligned} &\tilde{F}_4 \equiv F_4+G_2\wedge C_2-\tilde{H}_3\wedge C_1, \quad \tilde{F}_2\equiv F_2+G_2\wedge C_0 \\ &\tilde{F}_3 \equiv F_3+H_2\wedge C_1-\tilde{H}_3\wedge C_0, \quad \tilde{H}_3\equiv H_3-G_2\wedge B_1 \end{aligned} \tag{B.3}$$

These satisfy the modified Bianchi identities

$$\begin{aligned} \mathrm{d}\tilde{F}_4 &= G_2 \wedge \tilde{F}_3 + \tilde{H}_3 \wedge \tilde{F}_2,\ \mathrm{d}\tilde{F}_2 = G_2 \wedge F_1, \\ \mathrm{d}\tilde{F}_3 &= H_2 \wedge \tilde{F}_2 + \tilde{H}_3 \wedge F_1,\ \mathrm{d}\tilde{H}_3 = -G_2 \wedge H_2. \end{aligned} \tag{B.4}$$

The reduction of the Ricci scalar and the gauge kinetic term have been performed explicitly in the general case in chapter 3 and we refer the reader to that section for details. In this case we obtain

$$\begin{aligned} &\hat{R} = R - \frac{1}{2}\gamma e^{2\sigma}\left|G_2\right|^2 - 2e^{-\sigma}\nabla^2 e^{\sigma}, \quad \sqrt{|\hat{g}|} = e^{\sigma}\sqrt{|g|} \\ &\left|\hat{\tilde{F}}_4\right|^2 = \left|\tilde{F}_4\right|^2 + \gamma e^{-2\sigma}\left|\tilde{F}_3\right|^2, \qquad \left|\hat{\tilde{F}}_2\right|^2 = \left|\tilde{F}_2\right|^2 + \gamma e^{-2\sigma}\left|\tilde{F}_1\right|^2 \\ &\left|\hat{H}_3\right|^2 = \left|\tilde{H}_3\right|^2 + \gamma e^{-2\sigma}\left|H_2\right|^2. \end{aligned} \tag{B.5}$$

Plugging in all these in the action we find

$$\begin{aligned} S = \frac{1}{2\kappa_9^2}\int \mathrm{d}^9x\sqrt{|\hat{g}|}\Bigg(e^{\sigma-2\hat{\Phi}}\Bigg[R - \frac{1}{2}\gamma e^{2\sigma}|G_2|^2 - 2e^{-\sigma}\nabla^2 e^{\sigma} - \frac{\alpha}{2}(|\tilde{H}_3|^2 + \gamma e^{-2\sigma}|H_2|^2) + 4(\nabla\hat{\Phi})^2\Bigg] \\ - \frac{\alpha\beta}{2}e^{\sigma}(|\tilde{F}_2|^2 + \gamma e^{-2\sigma}|\tilde{F}_1|^2) - \frac{\beta}{2}e^{\sigma}(|\tilde{F}_4|^2 + \gamma e^{-2\sigma}|\tilde{F}_3|^2) - \frac{1}{4\kappa_{10}^2}\int \hat{B}_2 \wedge \hat{F}_4 \wedge \hat{F}_4, \end{aligned} \tag{B.6}$$

where $\kappa_9^2 = \kappa_{10}^2/(2\pi R)$. We manipulate the Ricci part doing an integration by parts

$$\begin{aligned} \int d^9x e^{\sigma}\sqrt{g}e^{-2\hat{\Phi}}(-2)e^{-\sigma}\nabla^2 e^{\sigma} &= \int d^9x\sqrt{g}e^{-2\hat{\Phi}}\frac{1}{\sqrt{g}}(-2)\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu e^{\sigma}) \\ &= \int d^9x 2\partial_\mu(e^{-2\hat{\Phi}})\sqrt{g}g^{\mu\nu}e^{\sigma}\partial_\nu\sigma \\ &= \int d^9x(-4)e^{-2\hat{\Phi}+\sigma}\partial_\mu\hat{\Phi}\partial_\nu\sigma\sqrt{g}g^{\mu\nu}. \end{aligned}$$

Moreover the Chern-Simons term is reduced as follows

$$\begin{aligned} CS &= -\frac{1}{4k_{10}^2}\int \hat{B}_2 \wedge \hat{F}_4 \wedge \hat{F}_4 \\ &= -\frac{1}{4k_{10}^2}\int \hat{B}_2 \wedge d\hat{C}_3 \wedge d\hat{C}_3 \\ &= -\frac{1}{4k_{10}^2}\int B_1 \wedge dy \wedge d(C_3 + C_2 \wedge A_1) \wedge d(C_3 + C_2 \wedge A_1) \\ &+ 2(B_2 + B_1 \wedge A_1) \wedge d(C_2 \wedge dy) \wedge d(C_3 + C_2 \wedge A_1) \\ &= -\frac{1}{4k_{10}^2}\int B_1 \wedge d(C_3 + C_2 \wedge A_1) \wedge d(C_3 + C_2 \wedge A_1) \wedge dy \\ &+ 2(B_2 + B_1 \wedge A_1) \wedge dC_2 \wedge d(C_3 + C_2 \wedge A_1) \wedge dy \\ &= -\frac{2\pi R}{4k_{10}^2}\int B_1 \wedge (dC_3 + dC_2 \wedge A_1 + C_2 \wedge dA_1) \wedge (dC_3 + dC_2 \wedge A_1 + C_2 \wedge dA_1) \end{aligned}$$

$$
\begin{aligned}
&+ 2B_2 \wedge dC_2 \wedge (dC_3 + dC_2 \wedge A_1 + C_2 \wedge dA_1)\\
&+ 2B_1 \wedge A_1 \wedge dC_2 \wedge (dC_3 + dC_2 \wedge A_1 + C_2 \wedge dA_1)\\
&= -\frac{1}{4k_9^2}\int B_1 \wedge \dot{F}_4 \wedge \dot{F}_4\\
&+ B_1 \wedge dC_2 \wedge A_1 \wedge (\dot{F}_4 + dC_2 \wedge A_1)\\
&+ B_1 \wedge (\dot{F}_4 + dC_2 \wedge A_1) \wedge dC_2 \wedge A_1\\
&+ 2B_2 \wedge F_3 \wedge \dot{F}_4\\
&+ 2B_1 \wedge A_1 \wedge dC_2 \wedge (\dot{F}_4 + dC_2 \wedge A_1)\\
&= -\frac{1}{4k_9^2}\int B_1 \wedge \dot{F}_4 \wedge \dot{F}_4 + 2B_2 \wedge F_3 \wedge \dot{F}_4,
\end{aligned}
$$

where we have defined $\dot{F}_4 = F_4 + G_2 \wedge C_2$. So all in all the 9D Type IIA actions are

$$
\begin{aligned}
S = \frac{1}{2\kappa_9^2}\int \mathrm{d}^9x\sqrt{|g|}e^{\sigma-2\Phi}\left[\mathcal{R} + 4\nabla\Phi\cdot\nabla(\Phi-\sigma) - \frac{\alpha}{2}|\tilde{H}_3|^2 - \frac{\alpha\gamma}{2}e^{-2\sigma}|H_2|^2 - \frac{\gamma}{2}e^{2\sigma}|G_2|^2\right]\\
-\frac{1}{4\kappa_9^2}\int \mathrm{d}^9x\sqrt{|g|}\left[\alpha\beta\gamma e^{-\sigma}|F_1|^2 + \alpha\beta e^{\sigma}|\tilde{F}_2|^2 + \beta\gamma e^{-\sigma}|\tilde{F}_3|^2 + \beta e^{\sigma}|\tilde{F}_4|^2\right]\\
-\frac{1}{4\kappa_9^2}\int\left[B_1 \wedge \dot{F}_4 \wedge \dot{F}_4 + 2B_2 \wedge F_3 \wedge \dot{F}_4\right].
\end{aligned}
\tag{B.7}
$$

We now turn to the reduction of the Type IIB theories. The 10D actions are

$$
\begin{aligned}
S =&\frac{1}{2\kappa_{10}^2}\int \mathrm{d}^{10}x\sqrt{|\hat{g}|}e^{-2\hat{\Phi}}\left[\hat{\mathcal{R}} - \frac{\alpha}{2}|\hat{H}_3|^2 + 4(\nabla\hat{\Phi})^2\right]\\
&-\frac{1}{4\kappa_{10}^2}\int \mathrm{d}^{10}x\sqrt{|\hat{g}|}\left[\alpha\beta|\hat{F}_1|^2 + \beta|\hat{\tilde{F}}_3|^2 + \frac{\alpha\beta}{2}|\hat{\tilde{F}}_5|^2\right] - \frac{1}{4\kappa_{10}^2}\int \hat{B}_2 \wedge \hat{F}_3 \wedge \hat{F}_5,
\end{aligned}
\tag{B.8}
$$

where $\hat{\tilde{F}}_3 = F_3 - H_3 \wedge C_0$ and $\hat{\tilde{F}}_5 = F_5 - H_3 \wedge C_2$. As usual with the Type IIB actions, the equations of motion derived from these actions need to be supplemented with the self-duality constraint $\hat{\tilde{F}}_5 = \alpha\beta \star \hat{\tilde{F}}_5$.

The reduction of the Neveu-Schwarz sector is identical as previously, and the reduction of the kinetic terms of the Ramond-Ramond sector is very similar so we do not detail them. We however give the details for the reduction of the Chern-Simons term. If it were to be performed directly, the computation becomes quite cumbersome. Instead we use a trick to write the Chern Simons term as a boundary term

$$
-\frac{1}{4\kappa_{10}^2}\int_{X_{11}} \hat{H}_3 \wedge \hat{\tilde{F}}_3 \wedge \hat{\tilde{F}}_5,
\tag{B.9}
$$

which as we can see is equal to our Chern-Simons term

$$
-\frac{1}{4\kappa_{10}^2}\int_{X_{11}} \hat{H}_3 \wedge \hat{\tilde{F}}_3 \wedge \hat{\tilde{F}}_5
$$

$$
\begin{aligned}
&= -\frac{1}{4\kappa_{10}^2}\int_{X_{11}} \hat{H}_3 \wedge (F_3 - H_3 \wedge C_0) \wedge (F_5 - H_3 \wedge C_2)\\
&= -\frac{1}{4\kappa_{10}^2}\int_{X_{11}} \hat{H}_3 \wedge F_3 \wedge F_5\\
&= -\frac{1}{4\kappa_{10}^2}\int_{X_{11}} d\left(\hat{B}_2 \wedge F_3 \wedge F_5\right)\\
&= -\frac{1}{4\kappa_{10}^2}\int_{X_{10}} \hat{B}_2 \wedge F_3 \wedge F_5 = S_{CS}^{IIB}.
\end{aligned}
$$

We now proceed with the dimensional reduction

$$
\begin{aligned}
S_{CS} = &-\frac{1}{4\kappa_{10}^2}\int_{X_{11}} H_2 \wedge dy \wedge (\tilde{F}_3 + \tilde{F}_2 \wedge A_1) \wedge (\tilde{F}_5 + \tilde{F}_4 \wedge A_1)\\
&+ (\tilde{H}_3 + H_2 \wedge A_1) \wedge \tilde{F}_2 \wedge dy \wedge (\tilde{F}_5 + \tilde{F}_4 \wedge A_1)\\
&+ (\tilde{H}_3 + H_2 \wedge A_1) \wedge (\tilde{F}_3 + \tilde{F}_2 \wedge A_1) \wedge \tilde{F}_4 \wedge dy\\
&= -\frac{1}{4\kappa_{10}^2}\int_{X_{11}} H_2 \wedge \tilde{F}_3 \wedge \tilde{F}_5 \wedge dy + H_2 \wedge \tilde{F}_3 \wedge \tilde{F}_4 \wedge A_1 \wedge dy\\
&+ H_2 \wedge \tilde{F}_2 \wedge A_1 \wedge \tilde{F}_5 \wedge dy + H_2 \wedge \tilde{F}_2 \wedge A_1 \wedge \tilde{F}_4 \wedge A_1 \wedge dy\\
&- \tilde{H}_3 \wedge \tilde{F}_2 \wedge \tilde{F}_5 \wedge dy - \tilde{H}_3 \wedge \tilde{F}_2 \wedge \tilde{F}_4 \wedge A_1 \wedge dy\\
&- H_2 \wedge A_1 \wedge \tilde{F}_2 \wedge \tilde{F}_5 \wedge dy - H_2 \wedge A_1 \wedge \tilde{F}_2 \wedge \tilde{F}_4 \wedge A_1 \wedge dy\\
&+ \tilde{H}_3 \wedge \tilde{F}_3 \wedge \tilde{F}_4 \wedge dy + \tilde{H}_3 \wedge \tilde{F}_2 \wedge A_1 \wedge \tilde{F}_4 \wedge dy\\
&+ H_2 \wedge A_1 \wedge \tilde{F}_3 \wedge \tilde{F}_4 \wedge dy + H_2 \wedge A_1 \wedge \tilde{F}_2 \wedge A_1 \wedge \tilde{F}_4 \wedge dy\\
&= -\frac{1}{4\kappa_9^2}\int_{X_{10}} \left[H_2 \wedge \tilde{F}_3 \wedge \tilde{F}_5 - \tilde{H}_3 \wedge \tilde{F}_2 \wedge \tilde{F}_5 + \tilde{H}_3 \wedge \tilde{F}_3 \wedge \tilde{F}_4\right].
\end{aligned}
$$

We now need to exploit the following relations

$$d\left[\tilde{F}_4 \wedge \tilde{F}_5\right] = -H_2 \wedge \tilde{F}_3 \wedge \tilde{F}_5 + \tilde{H}_3 \wedge \tilde{F}_2 \wedge \tilde{F}_5 - G_2 \wedge \tilde{F}_4 \wedge \tilde{F}_4 + \tilde{H}_3 \wedge \tilde{F}_3 \wedge \tilde{F}_4, \tag{B.10}$$

$$d\left[A_1 \wedge \dot{F}_4 \wedge \dot{F}_4 - 2\dot{B}_2 \wedge F_3 \wedge \dot{F}_4\right] = G_2 \wedge \tilde{F}_4 \wedge \tilde{F}_4 - 2\tilde{H}_3 \wedge \tilde{F}_3 \wedge \tilde{F}_4, \tag{B.11}$$

where $\dot{F}_4 \equiv F_4 - H_2 \wedge C_2$ and $\dot{B}_2 \equiv B_2 + B_1 \wedge A_1$. The proof of the first one goes as follows

$$
\begin{aligned}
&d\left[\tilde{F}_4 \wedge \tilde{F}_5\right] = d\tilde{F}_4 \wedge \tilde{F}_5 + \tilde{F}_4 \wedge d\tilde{F}_5\\
&= d(F_4 - H_2 \wedge C_2 - \tilde{H}_3 \wedge C_1) \wedge \tilde{F}_5 + \tilde{F}_4 \wedge d(F_5 - G_2 \wedge C_3 - \tilde{H}_3 \wedge C_2)\\
&= -H_2 \wedge F_3 \wedge \tilde{F}_5 - d\tilde{H}_3 \wedge C_1 \wedge \tilde{F}_5 + \tilde{H}_3 \wedge F_2 \wedge \tilde{F}_5\\
&- \tilde{F}_4 \wedge G_2 \wedge F_4 - \tilde{F}_4 \wedge d\tilde{H}_3 \wedge C_2 + \tilde{F}_4 \wedge \tilde{H}_3 \wedge F_3\\
&= -H_2 \wedge F_3 \wedge \tilde{F}_5 + H_2 \wedge G_2 \wedge C_1 \wedge \tilde{F}_5 + H_2 \wedge \tilde{H}_3 \wedge C_0 \wedge \tilde{F}_5\\
&- H_2 \wedge G_2 \wedge C_1 \wedge \tilde{F}_5 - H_2 \wedge \tilde{H}_3 \wedge C_0 \wedge \tilde{F}_5\\
&+ \tilde{H}_3 \wedge F_2 \wedge \tilde{F}_5 - \tilde{H}_3 \wedge H_2 \wedge C_0 \wedge \tilde{F}_5
\end{aligned}
$$

$$
\begin{aligned}
&+ \tilde{H}_3 \wedge H_2 \wedge C_0 \wedge \tilde{F}_5 \\
&- G_2 \wedge \tilde{F}_4 \wedge F_4 + G_2 \wedge \tilde{F}_4 \wedge H_2 \wedge C_2 + G_2 \wedge \tilde{F}_4 \wedge \tilde{H}_3 \wedge C_1 \\
&- G_2 \wedge \tilde{F}_4 \wedge H_2 \wedge C_2 - G_2 \wedge \tilde{F}_4 \wedge \tilde{H}_3 \wedge C_1 \\
&+ \tilde{H}_3 \wedge F_3 \wedge \tilde{F}_4 - \tilde{H}_3 \wedge G_2 \wedge C_1 \wedge \tilde{F}_4 - \tilde{H}_3 \wedge \tilde{H}_3 \wedge C_0 \wedge \tilde{F}_4 \\
&+ \tilde{H}_3 \wedge G_2 \wedge C_1 \wedge \tilde{F}_4 + \tilde{H}_3 \wedge \tilde{H}_3 \wedge C_0 \wedge \tilde{F}_4 \\
&- d\tilde{H}_3 \wedge C_1 \wedge \tilde{F}_5 - \tilde{F}_4 \wedge d\tilde{H}_3 \wedge C_2 \\
&= -H_2 \wedge \tilde{F}_3 \wedge \tilde{F}_5 + \tilde{H}_3 \wedge \tilde{F}_2 \wedge \tilde{F}_5 - G_2 \wedge \tilde{F}_4 \wedge \tilde{F}_4 + \tilde{H}_3 \wedge \tilde{F}_3 \wedge \tilde{F}_4 \\
&- H_2 \wedge G_2 \wedge C_1 \wedge \tilde{F}_5 - H_2 \wedge \tilde{H}_3 \wedge C_0 \wedge \tilde{F}_5 + \tilde{H}_3 \wedge H_2 \wedge C_0 \wedge \tilde{F}_5 - G_2 \wedge \tilde{F}_4 \wedge H_2 \wedge C_2 \\
&- G_2 \wedge \tilde{F}_4 \wedge \tilde{H}_3 \wedge C_1 + \tilde{H}_3 \wedge G_2 \wedge C_1 \wedge \tilde{F}_4 - d\tilde{H}_3 \wedge C_1 \wedge \tilde{F}_5 - \tilde{F}_4 \wedge d\tilde{H}_3 \wedge C_2 .
\end{aligned}
$$

In the last equality, the first line is what we want and we can see that the last two lines vanish

$$
\begin{aligned}
=& - H_2 \wedge G_2 \wedge C_1 \wedge \tilde{F}_5 - G_2 \wedge \tilde{F}_4 \wedge \tilde{H}_3 \wedge C_1 - d\tilde{H}_3 \wedge C_1 \wedge \tilde{F}_5 - \tilde{F}_4 \wedge d\tilde{H}_3 \wedge C_2 \\
=& - H_2 \wedge G_2 \wedge C_1 \wedge \tilde{F}_5 - G_2 \wedge \tilde{F}_4 \wedge H_2 \wedge C_2 + G_2 \wedge H_2 \wedge C_1 \wedge \tilde{F}_5 + \tilde{F}_4 \wedge G_2 \wedge H_2 \wedge C_2 = 0.
\end{aligned}
$$

The proof of the second relation goes as

$$
\begin{aligned}
&d\left[A_1 \wedge \dot{F}_4 \wedge \dot{F}_4 - 2\dot{B}_2 \wedge F_3 \wedge \dot{F}_4\right] = dA_1 \wedge \dot{F}_4 \wedge \dot{F}_4 - 2A_1 \wedge d\dot{F}_4 \wedge \dot{F}_4 \\
&- 2d\dot{B}_2 \wedge F_3 \wedge \dot{F}_4 - 2\dot{B}_2 \wedge F_3 \wedge d\dot{F}_4 \\
&= G_2 \wedge \dot{F}_4 \wedge \dot{F}_4 - 2A_1 \wedge d\dot{F}_4 \wedge \dot{F}_4 - 2d(B_2 + B_1 \wedge A_1) \wedge F_3 \wedge \dot{F}_4 + 2\dot{B}_2 \wedge F_3 \wedge d\dot{F}_4 \\
&= G_2 \wedge (\tilde{F}_4 + \tilde{H}_3 \wedge C_1) \wedge (\tilde{F}_4 + \tilde{H}_3 \wedge C_1) - 2A_1 \wedge (-H_2 \wedge dC_2) \wedge (F_4 - H_2 \wedge C_2) \\
&- 2H_3 \wedge F_3 \wedge \dot{F}_4 - 2H_2 \wedge A_1 \wedge F_3 \wedge \dot{F}_4 + 2B_1 \wedge G_2 \wedge F_3 \wedge \dot{F}_4 + 2\dot{B}_2 \wedge F_3 \wedge (-H_2 \wedge dC_2) \\
&= G_2 \wedge \tilde{F}_4 \wedge \tilde{F}_4 + 2G_2 \wedge \tilde{F}_4 \wedge \tilde{H}_3 \wedge C_1 + 2A_1 \wedge H_2 \wedge F_3 \wedge F_4 - 2A_1 \wedge H_2 \wedge F_3 \wedge H_2 \wedge C_2 \\
&- 2H_3 \wedge F_3 \wedge \dot{F}_4 - 2H_2 \wedge A_1 \wedge F_3 \wedge \dot{F}_4 + 2B_1 \wedge G_2 \wedge F_3 \wedge \dot{F}_4 - 2\dot{B}_2 \wedge F_3 \wedge H_2 \wedge F_3 \\
&= G_2 \wedge \tilde{F}_4 \wedge \tilde{F}_4 - 2\tilde{H}_3 \wedge F_3 \wedge \dot{F}_4 + 2G_2 \wedge \tilde{F}_4 \wedge \tilde{H}_3 \wedge C_1 \\
&= G_2 \wedge \tilde{F}_4 \wedge \tilde{F}_4 - 2\tilde{H}_3 \wedge \tilde{F}_3 \wedge \hat{F}_4 \\
&= G_2 \wedge \tilde{F}_4 \wedge \tilde{F}_4 - 2\tilde{H}_3 \wedge \tilde{F}_3 \wedge \tilde{F}_4 .
\end{aligned}
$$

We know that the action of Type IIB supergravity needs to be supplemented with self-duality condition for the 5-form field strength in order to account for the correct number of degrees of freedom. We need to get the corresponding 9-dimensional version of that constraint. We start with the 10D constraint which is written as

$$\tilde{F}_5^{(10)} = \alpha\beta \star \tilde{F}_5^{(10)} . \tag{B.12}$$

The decomposition of the higher dimensional Hodge star is as follows (to obtain this formula one can write the expression in components and use the reduction ansatzes)

$$\hat{\star}\tilde{F}_5^{(10)} = -\gamma e^{-\sigma} \star \tilde{F}_4 + e^{\sigma}(A_1 + dy) \wedge \star\tilde{F}_5 \tag{B.13}$$

which implies

$$(\tilde{F}_5 + \tilde{F}_4 \wedge A_1) + \tilde{F}_4 \wedge dy = -\alpha\beta\gamma(e^{-\sigma} \star \tilde{F}_4 - \gamma e^{\sigma} A_1 \wedge \star\tilde{F}_5) + \alpha\beta e^{\sigma} \star \tilde{F}_5 \wedge dy. \quad \text{(B.14)}$$

Identifying we get

$$\alpha\beta e^{\sigma} \star \tilde{F}_5 = \tilde{F}_4 \Rightarrow \star^2 \tilde{F}_5 = \alpha\beta e^{-\sigma} \star \tilde{F}_4. \quad \text{(B.15)}$$

However, we have (see for example [165]): $\star^2 \tilde{F}_5 = -\gamma \tilde{F}_5$. Therefore we get

$$\tilde{F}_5 = -\alpha\beta\gamma e^{-\sigma} \star \tilde{F}_4. \quad \text{(B.16)}$$

This means that the potentials are not independant. The terms that depend on $\tilde{F}_5$ are

$$\hat{S} = -\frac{1}{4\kappa_9^2} \int \left[\frac{\alpha\beta}{2} e^{\sigma} \tilde{F}_5 \wedge \star\tilde{F}_5 - \tilde{F}_4 \wedge \tilde{F}_5 \right], \quad \text{(B.17)}$$

and the equation of motion for the field F_5 yields

$$\delta\hat{S} = -\frac{1}{4\kappa_9^2} \int \delta F_5 \wedge \left(e^{\sigma}\alpha\beta \star \tilde{F}_5 - \tilde{F}_4 \right), \quad \text{(B.18)}$$

which is proportional to the self-duality constraint. Therefore we can add a Lagrange multiplier that enforces it and treat F_5 as an auxiliary field Λ_5, which, when integrated out, finally gives

$$\hat{S} = -\frac{1}{4\kappa_9^2} \int \frac{\alpha\beta\gamma}{2} e^{-\sigma} \tilde{F}_4 \wedge \star\tilde{F}_4. \quad \text{(B.19)}$$

So overall the reduced IIB actions are

$$\begin{aligned} S =& \frac{1}{2\kappa_9^2} \int \mathrm{d}^9 x \sqrt{|g|} e^{\sigma - 2\Phi} \left[\mathcal{R} + 4\nabla\Phi \cdot \nabla(\Phi - \sigma) - \frac{\alpha}{2} |\tilde{H}_3|^2 - \frac{\alpha\gamma}{2} e^{-2\sigma} |H_2|^2 - \frac{\gamma}{2} e^{2\sigma} |G_2|^2 \right] \\ & - \frac{1}{4\kappa_9^2} \int \mathrm{d}^9 x \sqrt{|g|} \left[\alpha\beta e^{\sigma} |F_1|^2 + \beta\gamma e^{-\sigma} |\tilde{F}_2|^2 + \beta e^{\sigma} |\tilde{F}_3|^2 + \alpha\beta\gamma e^{-\sigma} |\tilde{F}_4|^2 \right] \\ & - \frac{1}{4\kappa_9^2} \int \left[-A_1 \wedge \hat{F}_4 \wedge \hat{F}_4 + 2\hat{B}_2 \wedge F_3 \wedge \hat{F}_4 \right]. \end{aligned} \quad \text{(B.20)}$$

We have obtained the reduced actions of Type IIA and Type IIB theories so we are now ready to derive the generalized Buscher rules. In order to go from one action to the other, we need to perform the following field redefinitions

$$\begin{aligned} &\sigma \to -\sigma, \\ &\Phi \to \Phi - \sigma, \\ &A_1 \to -B_1, \\ &B_1 \to -A_1, \\ &B_2 \to B_2 + B_1 \wedge A_1. \end{aligned} \quad \text{(B.21)}$$

Moreover the sign parameters are related as follows

$$IIA_{\gamma}^{\alpha,\beta} \to IIB_{\alpha\gamma}^{\alpha,\alpha\beta\gamma}. \quad \text{(B.22)}$$

We can show explicitly that the $|\tilde{H}_3|^2$ term is invariant

$$
\begin{aligned}
|\tilde{H}_3|^2 &= (dB_2 - dA_1 \wedge B_1) \wedge \star(dB_2 - dA_1 \wedge B_1) \\
&\to (d(B_2 + B_1 \wedge A_1) - dB_1 \wedge A_1) \wedge \star(d(B_2 + B_1 \wedge A_1) - dB_1 \wedge A_1) \\
&= (dB_2 + dB_1 \wedge A_1 - B_1 \wedge dA_1 - dB_1 \wedge A_1) \wedge \star(dB_2 + dB_1 \wedge A_1 - B_1 \wedge dA_1 - dB_1 \wedge A_1) \\
&= (dB_2 - G_2 \wedge B_1) \wedge \star(dB_2 - G_2 \wedge B_1) \\
&= |\tilde{H}_3|^2.
\end{aligned}
$$

We now prove all the rules starting with the Neveu-Schwarz sector.
For the dilaton

$$
\hat{\Phi}' = \Phi' \to \Phi - \sigma = \Phi - \frac{1}{2}\ln|\alpha\gamma g_{yy}| = \Phi - \frac{1}{2}\ln|g_{yy}|. \tag{B.23}
$$

As a side note we can show that this formula is consistent with the relationship between coupling constants. Indeed

$$
\begin{aligned}
\tilde{\Phi} &= \Phi - \frac{1}{2}\ln|g_{yy}| \\
\implies e^{\langle\tilde{\Phi}\rangle} &= e^{\langle\Phi\rangle - \frac{1}{2}\ln|\langle g_{yy}\rangle|} \\
\implies \tilde{g_S} &= g_S\frac{1}{e^{\langle\sigma\rangle}} = \frac{\sqrt{\alpha'}}{R}g_S,
\end{aligned}
$$

after restoring string units which indeed matches (2.181).[1]

For the metric

$$
\begin{aligned}
\hat{g}'_{mn} &= g'_{mn} + \gamma e^{2\sigma'}A'_{1m}A'_{1n} \\
&\to g_{mn} + \alpha\gamma e^{-2\sigma}(-B_{1m})(-B_{1n}) \\
&= \hat{g}_{mn} - \gamma e^{2\sigma}(A_{1m}A_{1n}) + \alpha\gamma e^{-2\sigma}B_{1m}B_{1n} \\
&= \hat{g}_{mn} + \frac{\alpha\hat{B}_{ym}\hat{B}_{yn} - \hat{g}_{ym}\hat{g}_{yn}}{\hat{g}_{yy}}.
\end{aligned} \tag{B.24}
$$

For the Kaluza-Klein vector

$$
\hat{g}'_{my} = \gamma e^{2\sigma'}A'_{1m} \to -\alpha\gamma e^{-2\sigma}B_{1m} = -\alpha\frac{\hat{B}_{my}}{g_{yy}} \implies g'_{ym} = \alpha\frac{\hat{B}_{ym}}{g_{yy}}. \tag{B.25}
$$

For the Kaluza Klein scalar

$$
\hat{g}'_{yy} = \gamma e^{2\sigma'} \to \alpha\gamma e^{-2\sigma} = \alpha\frac{1}{\hat{g}_{yy}}. \tag{B.26}
$$

[1] To be accurate, the true parameter that should appear is the physical radius ρ that we introduced before but most references do not distinguish between the physical radius ρ and the parametric radius R.

For the Kalb-Ramond B-field

$$
\begin{aligned}
\hat{B'}_{mn} &= B'_{2mn} + (B'_1 \wedge A'_1)_{mn} \\
&= B'_{2mn} + (B'_{1m}A'_{1n} - B'_{1n}A'_{1m}) \\
&\to B_{2mn} + (B_{1m}A_{1n} - A_{1m}B_{1n}) + (A_{1m}B_{1n} - A_{1n}B_{1m}) \\
&= B_{2mn} \\
&= \hat{B}_{mn} - (B_{1m}A_{1n} - A_{1m}B_{1n}) \\
&= \hat{B}_{mn} - \left(\hat{B}_{my}\gamma e^{2\sigma}\hat{g}_{ny} - \hat{B}_{ny}\gamma e^{2\sigma}\hat{g}_{my}\right) \\
&= \hat{B}_{mn} - \frac{\left(\hat{B}_{my}\hat{g}_{ny} - \hat{B}_{ny}\hat{g}_{my}\right)}{\hat{g}_{yy}}.
\end{aligned}
\tag{B.27}
$$

For the winding vector

$$
\hat{B}'_{my} = B'_{1m} \to -A_{1m} = -\gamma e^{-2\sigma} g_{my} = -\frac{g_{my}}{g_{yy}} \implies B'_{ym} = \frac{g_{ym}}{g_{yy}}. \tag{B.28}
$$

Now we give the Ramond-Ramond sector rules. We give the IIA fields expressed in terms of IIB fields but the reverse works with the same logic

$$
\hat{C}^A_1|_y = C^A_0 \to C^B_0 = \hat{C}^B_0, \tag{B.29}
$$

$$
\hat{C}^A_1|_{\not{y}} = C^A_1 + C^A_0 \wedge A^A_1 \to C^B_1 + C^B_0 \wedge (-B^B_1) = \hat{C}^B_2|_y - \hat{C}^B_0 \wedge \hat{B}^B_2|_y, \tag{B.30}
$$

$$
\hat{C}^A_3|_y = C^A_2 \to C^B_2 + C^B_1 \wedge A^B_1 - C^B_1 \wedge A^B_1 = \hat{C}^B_2|_{\not{y}} - \hat{C}^B_2|_y \wedge \frac{\hat{g}^B_{ym}}{\hat{g}^B_{yy}}, \tag{B.31}
$$

$$
\begin{aligned}
\hat{C}^A_3|_{\not{y}} &= C^A_3 + C^A_2 \wedge A^A_1 \to C^B_3 + C^B_2 \wedge (-B^B_1) \\
&= \hat{C}_4|_y + (C^B_2 + C^B_1 \wedge A^B_1 - C^B_1 \wedge A^B_1) \wedge (-B^B_1) \\
&= \hat{C}^B_4|_y - \hat{C}^B_2|_{\not{y}} \wedge \hat{B}^B_2|_y + \hat{C}^B_2|_y \wedge \frac{\hat{g}^B_{ym}}{\hat{g}^B_{yy}} \wedge \hat{B}^B_2|_y,
\end{aligned}
\tag{B.32}
$$

where $|_y$ and $|_{\not{y}}$ indicates if one of the coordinates is the one on which we reduce or not. Thus concludes the derivation of the generalized Type II Buscher rules.

Appendix C

Details of Type II on Calabi-Yau

In this appendix we provide some more details on Calabi-Yau compactifications of Type IIA theories. Since this has been worked out in detail for $\mathrm{IIA}_{(1,9)}$ in [122] and for $\mathrm{IIA}_{(0,10)}$ in [124] all we need in order to include the additional cases of $\mathrm{IIA}^*_{(1,9)}$ and $\mathrm{IIA}_{(2,8)}$, is to trace the ten-dimensional sign flips through the computation. The results from the reduction of the individual terms is taken from [124] whose conventions and notation we follow. We refer to [104, 106, 127] for further background on CY3 compactifications and special geometry.

C.1 Ten-dimensional Lagrangians

In Section 4.3 we used the string frame parametrization of [108] to display the various Type IIA Lagrangians.[1] In order to use the results of [122] and [124] on CY3 compactifications, we use the following Einstein frame parametrization:

$$\begin{aligned} S_{IIA} = \int_{M_{10}} & \frac{1}{2} \star R_{10} - \frac{9}{16} d\log\phi \wedge \star d\log\phi - \frac{\alpha_1}{4}\phi^{\frac{9}{4}} dV \wedge \star dV \\ & - \frac{\alpha_2}{2}\phi^{-\frac{3}{2}} H_3 \wedge \star H_3 - \frac{\alpha_3}{2}\phi^{\frac{3}{4}} (F_4 + dV \wedge B_2) \wedge \star (F_4 + dV \wedge B_2) \\ & - \frac{\sqrt{2}}{2} (F_4 + dV \wedge B_2) \wedge F_4 \wedge B_2 - \frac{\sqrt{2}}{6} dV \wedge B_2 \wedge dV \wedge B_2 \wedge B_2 \, . \end{aligned} \tag{C.1}$$

Note that the dilaton has been redefined according to $\Phi \propto \log\phi$. Moreover we now use the same notations for form-fields as in [122] and [124]: V is the RR one-form, while F_4 is the four-form field strength. The three parameters α_i, $i = 1, 2, 3$ encode the various sign flips, see Table C.1.

[1] Complete bosonic string frame (pseudo-)Lagrangians for all Type II theories can be found in the appendix of [54].

Type	α_1	α_2	α_3
$\mathrm{IIA}_{(1,9)}$	1	1	1
$\mathrm{IIA}^*_{(1,9)}$	-1	1	-1
$\mathrm{IIA}_{(0,10)}$	-1	-1	1
$\mathrm{IIA}_{(2,8)}$	1	-1	-1

Table C.1: Relative signs for kinetic terms in ten-dimensional Type IIA theories. A plus sign corresponds to a standard kinetic term in Lorentz signature, as realized in $\mathrm{IIA}_{(1,9)}$.

As a quick check, note that this action is consistent with Table 4.6. We also note that the three signs are not independent, since $\alpha_3 = \alpha_1\alpha_2$. This reflects that the three signs encode four independent theories, rather than six. For Type $\mathrm{IIA}_{(0,10)}$ and Type $\mathrm{IIA}_{(1,9)}$ it was shown in [124] that these Lagrangians arise from dimensional reduction of eleven-dimensional supergravity with signature (1,10). For signature (1,9) one recovers the Lagrangian of [122].[2] The Lagrangians for Type $\mathrm{IIA}^*_{(1,9)}$ and Type $\mathrm{IIA}_{(2,8)}$ are obtained by a similar computation starting from the Lagrangian for eleven-dimensional supergravity in signature (2,9) given in [108]. Since these Lagrangians only differ by relative signs, and since in all cases we reduce on the same manifold, we can use the results of [122] and [124] for the individual terms, and afterwards assemble them into four distinct four-dimensional Lagrangians.

C.2 Reduction of the graviton-dilaton sector

Since the Einstein-Hilbert and dilaton term are the same for all cases we can discuss them at once.

Following [122] and [124] the reduction of

$$S_{EH+\phi} = \int_{M_{10}} \frac{1}{2} \star R_{10} - \frac{9}{16} d\log\phi \wedge \star \log\phi$$

results in

$$S_{EH+\phi} = \int_{M_4} \frac{1}{2} \star R_4 - \frac{1}{2} G_{AB}(v) dv^A \wedge \star dv^B - \frac{1}{4} d\varphi \wedge \star d\varphi - g_{\alpha\bar{\beta}}(z,\bar{z}) dz^\alpha \wedge \star d\bar{z}^{\bar{\beta}} \ . \quad \text{(C.2)}$$

Here z^α are the complex structure moduli with their special Kähler metric $g_{\alpha\beta}$ and v^A are the real Kähler moduli with their special real metric $G_{AB}(v)$. Following the notation

[2] As remarked in [124] the second term in the third line is absent in [122], but present in [121]. It is straightforward to check that this term is generated by the field redefinition described explicitly in [124].

of [124] we define the following integrals

$$\mathcal{K} = \int_{CY_3} J \wedge J \wedge J \,, \qquad \mathcal{K}_A = \int_{CY_3} V^A \wedge J \wedge J \,,$$
$$\mathcal{K}_{AB} = \int_{CY_3} V^A \wedge V^B \wedge J \,, \qquad \mathcal{K}_{ABC} = \int_{CY_3} V^A \wedge V^B \wedge V^C \,.$$

Using the expansion $J = M^A V^A$ of the Kähler form, we obtain

$$\mathcal{K} = \mathcal{K}_{ABC} M^A M^B M^C =: (\mathcal{K}MMM) \,, \quad \mathcal{K}_A = \mathcal{K}_{ABC} M^B M^C =: (\mathcal{K}MM)_A \,,$$
$$\mathcal{K}_{AB} = \mathcal{K}_{ABC} M^C =: (\mathcal{K}M)_{AB} \,,$$

so that

$$G_{AB}(M) = -3 \left(\frac{(\mathcal{K}M)_{AB}}{(\mathcal{K}MMM)} - \frac{3}{2} \frac{(\mathcal{K}MM)_A (\mathcal{K}MM)_B}{(\mathcal{K}MMM)^2} \right) \,. \tag{C.3}$$

The real Kähler moduli z^α are related to the M^A by the field redefinition

$$M^A = \sqrt{2} \phi^{-3/4} v^A \,. \tag{C.4}$$

The four-dimensional dilaton φ is related to the ten-dimensional dilaton ϕ by

$$\varphi = \log \left(2 \mathcal{V} \phi^{-3} \right) \,.$$

Further details are given in [124].

C.3 Contribution of the B-field to the vector multiplet sector

Next we turn to terms descending from the B-field kinetic term, which arise from taking the internal part of the B-field to be a harmonic two-form. Following [122] and [124] we say that terms which arise from harmonic two-forms belong to the H^2-cohomology sector. These are precisely the terms which contribute to the gravity plus vector multiplet sector of the four-dimensional theory. The ten-dimensional term takes the form

$$S_{H^2(B_2)} = \alpha_2 \int_{M_{10}} -\frac{1}{2} \phi^{-3/2} H_3 \wedge \star H_3 \Big|_{H^2}$$

where the sign depends on which theory we start with. We denote the projection of terms onto the H^2-cohomology sector by $|_{H^2}$. Decomposing $B_2|_{H^2} = a^A V^A$ and integrating over the CY3, we obtain

$$S_{H^2(B_2)} = \alpha_2 \int_{M_4} -\frac{1}{2} G_{AB}(v) da^A \wedge \star da^B \,,$$

where a^A are four-dimensional scalar fields. We can combine this term with one of the terms obtained from the reduction of the Einstein-Hilbert term to obtain

$$\int_{M_4} -\frac{1}{2} G_{AB}(v) \left(dv^A \wedge \star dv^B + \alpha_2 da^A \wedge \star da^B \right) .$$

Making the field redefinition

$$v^A = \frac{1}{2^{1/6}} y^A, \qquad a^A = -\frac{1}{2^{1/6}} x^A, \qquad \mathcal{K}_{ABC} = c_{ABC} , \tag{C.5}$$

we can rewrite this contribution as

$$\int_{M_4} -\bar{g}_{AB}(y) \left(dx^A \wedge \star dx^B + \alpha_2 dy^A \wedge \star dy^B \right)$$

where we have defined the new coupling matrix by

$$\bar{g}_{AB} := \frac{1}{2} \alpha_2 G_{AB} = -\frac{3}{2} \alpha_2 \left(\frac{(cy)_{AB}}{(cyyy)} - \frac{3}{2} \frac{(cyy)_A (cyy)_B}{(cyyy)^2} \right) . \tag{C.6}$$

C.4 Contribution of R-R-kinetic terms to the vector multiplet sector

Next we consider the contribution of the kinetic term of the R-R three-form, including its Chern-Simons like improvement term, to the H^2-sector, that is to the four-dimensional vector multiplets. Depending on which theory we start with, the ten-dimensional term is

$$S_{H^2}(A_3) = \alpha_3 \int_{M_{10}} \frac{1}{2} \phi^{3/4} \left(F_4 + dV \wedge B_2 \right) \wedge \star \left(F_4 + dV \wedge B_2 \right) \Big|_{H^2} .$$

Following [124] we decompose F_4 and B_2 as

$$F_4 \big|_{H^2} = \mathcal{F}^A \wedge V^A, \qquad B_2 \big|_{H^2} = a^A V^A, \tag{C.7}$$

where $\mathcal{F}^A$ are four-dimensional field strengths, $\mathcal{F}^A = d\mathcal{A}^A$. Inserting this into the above action, we obtain an action ready to integrate over:

$$S_{H^2}(A_3) = \alpha_3 \int_{M_4} \frac{1}{2} \phi^{3/4} \left(\mathcal{F}^A + a^A dV \right) \wedge \star \left(\mathcal{F}^B + a^B dV \right) \int_{CY_3} V^A \wedge \star V^B .$$

Performing the integral we obtain

$$S_{H^2}(A_3) = \alpha_3 \int_{M_4} \frac{\sqrt{2}}{3!} \mathcal{K}(v) G_{AB}(v) \left(\mathcal{F}^A + a^A dV \right) \wedge \star \left(\mathcal{F}^B + a^B dV \right) \Big|_{H^2} .$$

Similarly, the reduction of the kinetic term of the R-R one-form

$$S_V = \alpha_1 \int_{M_{10}} \frac{1}{4} \phi^{\frac{9}{4}} dV \wedge \star dV$$

becomes

$$S_V = \alpha_1 \int_{M_4} \frac{\sqrt{2}}{2 \cdot 3!} \mathcal{K}(v) \mathcal{F}^0 \wedge \star \mathcal{F}^0$$

after integration over the CY3, where we have set $\mathcal{F}^0 = dV$ and used the field redefinition (C.4).

Using that $\alpha_3 = \alpha_1 \alpha_2$, we can combine terms as

$$\begin{aligned} S_{H^2}(A_3) + S_V =& \alpha_1 \int_{M_4} \sqrt{2} \left(\frac{1}{12} (\mathcal{K}vvv) + \frac{\alpha_2}{6} (\mathcal{K}vvv) G_{AB} a^A a^B \right) \mathcal{F}^0 \wedge \star \mathcal{F}^0 \\ &+ \frac{\sqrt{2}\alpha_2}{3} (\mathcal{K}vvv) G_{AB} a^B \mathcal{F}^A \wedge \star \mathcal{F}^0 + \frac{\sqrt{2}\alpha_2}{6} (\mathcal{K}vvv) G_{AB} \mathcal{F}^A \wedge \star \mathcal{F}^B , \end{aligned}$$

where $(\mathcal{K}vvv) := \mathcal{K}_{ABC} v^A v^B v^C$. Rescaling the gauge fields

$$\mathcal{F}^A = \frac{1}{2^{1/6}} F^A$$

as well as the scalars, and using (C.6), we can express the above as

$$\begin{aligned} S_{H^2}(A_3) + S_V =& \alpha_1 \int_{M_4} \frac{1}{2} (cyyy) \left(\frac{1}{6} + \frac{2}{3} (gxx) \right) F^0 \wedge \star F^0 \\ &- \frac{2}{3} (cyyy) (gx)_A F^A \wedge \star F^0 + \frac{1}{3} (cyyy) g_{AB} F^A \wedge \star F^B . \end{aligned}$$

C.5 Contribution of the topological terms to the vector multiplet sector

The final contribution to the gravity plus vector multiplet sector comes from the topological terms

$$S_{H^2(top)} = \int_{M_{10}} - \frac{\sqrt{2}}{2} (F_4 + dV \wedge B_2) \wedge F_4 \wedge B_2 - \frac{\sqrt{2}}{6} dV \wedge B_2 \wedge dV \wedge B_2 \wedge B_2 \Big|_{H^2} .$$

According to [124], after reduction and field redefinitions this takes the form

$$S_{H^2(top)} = \int_{M_4} \frac{1}{6} \left[3(cx)_{AB} \mathcal{F}^A \wedge \mathcal{F}^B - 3(cxx)_A \mathcal{F}^A \wedge \mathcal{F}^0 + (cxxx) \mathcal{F}^0 \wedge \mathcal{F}^0 \right] .$$

C.6 Final result for the gravity and vector multiplet sector

Combining everything obtained so far gives us the bosonic Lagrangian for the gravity multiplet and the vector multiplets:

$$\begin{aligned} S_{G+VM} = \int_{M_4} & \frac{1}{2} \star R_4 - \bar{g}_{AB}(y) \left(dx^A \wedge \star dx^B + \alpha_2 dy^A \wedge \star dy^B \right) \\ & - \alpha_1 \left[\frac{1}{2}(cyyy) \left(\frac{1}{6} + \frac{2}{3}(gxx) \right) \mathcal{F}^0 \wedge \star \mathcal{F}^0 \right. \\ & \left. - \frac{2}{3}(cyyy)(gx)_A \mathcal{F}^A \wedge \star \mathcal{F}^0 + \frac{1}{3}(cyyy) g_{AB} \mathcal{F}^A \wedge \star \mathcal{F}^B \right] \\ & + \frac{1}{6} \left[3(cx)_{AB} \mathcal{F}^A \wedge \mathcal{F}^B - 3(cxx)_A \mathcal{F}^A \wedge \mathcal{F}^0 + (cxxx) \mathcal{F}^0 \wedge \mathcal{F}^0 \right] . \end{aligned}$$

As shown in [130], one can introduce complex fields $z^A = x^A + iy^A$ if $\alpha_2 = 1$ and para-complex fields $z^A = x^A + ey^A$ if $\alpha_2 = -1$. Then the second term in the first line becomes a sigma model

$$\int_{M_4} -\bar{g}_{A\bar{B}}(z, \bar{z}) dz^A \wedge \star d\bar{z}^{\bar{B}}$$

with a target space which is projective special Kähler for $\alpha_2 = 1$ and projective special para-Kähler for $\alpha_2 = -1$. More generally, the results of [130] imply that the full Lagrangian can be rewritten into the form

$$S_{G+VM} = \int_{M_4} \frac{1}{2} \star R_4 - \bar{g}_{A\bar{B}}(z, \bar{z}) dz^A \wedge \star d\bar{z}^{\bar{B}} + \frac{\alpha_1}{4} \mathcal{I}_{\Sigma\Lambda} F^\Sigma \wedge \star F^\Lambda + \frac{1}{4} \mathcal{R}_{\Sigma\Lambda} F^\Sigma \wedge F^\Lambda \ ,$$

where $\Sigma, \Lambda = 0, 1, \ldots, n_V = h_{1,1}$, and where the vector coupling matrices $\mathcal{I}_{\Lambda\Sigma}$ and $\mathcal{R}_{\Lambda\Sigma}$ can be expressed by a holomorphic or para-holomorphic prepotential through the standard formulae of special geometry. This completes the derivation of the bosonic gravity plus vector multiplet Lagrangians for the four Type IIA theories. Our result indeed matches (4.10) and (4.11). For signature (1,3), where $\alpha_2 = 1$ and $\alpha_1 = \alpha_3 = -\lambda = \pm 1$ we obtain a complex, (projective) SK scalar manifold, and the sign of the Maxwell term distinguishes between Type IIA and Type IIA*. For signatures (0,4) and (2,2), where $\alpha_2 = -1$, we obtain a para-complex, (projective) SPK manifold, and both signatures differ by the sign of the Maxwell term, which is controlled by $\alpha_1 = -\alpha_3 = -\lambda$. However, in these signatures the supersymmetry algebra is unique and the sign can be changed by a field redefinition [132].

C.7 Contribution of the kinetic R-R-terms to the hypermultiplet sector

We now turn to contributions from terms where the internal part is a harmonic 3-form. So far we have discussed one such term, which arises from the reduction of the Einstein-Hilbert term. This results in the sigma model $-\tilde{G}_{\alpha\bar{\beta}}(z,\bar{z})dz^{\alpha}\wedge\star d\bar{z}^{\bar{\beta}}$, where z^{α} parametrize the deformations of the complex structure of the CY3, with (projective) SK metric $g_{\alpha\bar{\beta}}(z,\bar{z})$. The R-R one-form does not contribute, but there are contributions for the R-R three-form and from the Kalb-Ramond field.

We start with the contribution of the kinetic term of the four-form field strength $F_4 = dA_3$,

$$S_{H^3}(A_3) = \int_{M_{10}} \frac{-\alpha_3}{2}\phi^{3/4}F_4\wedge\star F_4\Big|_{H^3} .$$

It has been shown in [122], [124] that

$$F_4|_{H^3} = d\check{A} = 2^{1/4}d\zeta^I\wedge\alpha_I + 2^{1/4}d\tilde{\zeta}_I\wedge\beta^I = P^I\wedge\Phi_I + \bar{Q}\wedge\bar{\Omega} + \text{h.c.} \tag{C.8}$$

where the complex one-forms $P^I, \bar{Q}\in\Omega^1(M_4)\otimes\mathbb{C}$ can be expressed in terms of special geometry data associated with complex structure moduli as

$$P^I = i2^{1/4}\left(d\tilde{\zeta}_J + \mathcal{N}_{JK}d\zeta^K\right)N^{IJ}, \qquad \bar{Q} = -i2^{1/4}\frac{X^I}{(XN\bar{X})}\left(d\tilde{\zeta}_I + \mathcal{N}_{IJ}d\zeta^J\right) . \tag{C.9}$$

While the four-dimensional hypermultiplet scalars $\zeta^I, \tilde{\zeta}_I$ are defined through the expansion of $d\check{A}$ in terms of α_I, β^I, the expansion of $d\check{A}$ in terms of Φ_I, Ω is used to carry out the integration over the CY3:

$$S_{H^3}(A_3) = \int_{M_4} -\alpha_3\phi^{3/4}P^I\wedge\star\bar{P}^J\int_{CY_3}\Phi_I\wedge\star\bar{\Phi}_J + \int_{M_4}\alpha_3\phi^{3/4}\bar{Q}\wedge\star Q\int_{CY_3}\bar{\Omega}\wedge\star\Omega\,.$$

Following [124] this can be evaluated and ultimately brought to the form

$$S_{H^3}(A_3) = \int_{M_4}\frac{\alpha_3}{2}e^{-\varphi}\left[\mathcal{I}_{IJ}d\zeta^I\wedge\star d\zeta^J + \mathcal{I}^{IJ}\left(d\tilde{\zeta}_I + \mathcal{R}_{IK}d\zeta^K\right)\wedge\star\left(d\tilde{\zeta}_J + \mathcal{R}_{JK}d\zeta^K\right)\right] ,$$

where φ is the four-dimensional dilaton.

C.8 Contribution of topological terms and of the B-field to the hypermultiplet sector

The topological contribution to the H^3-sector comes from

$$S_{H^3(\text{top})} = \int_{M_{10}} -\frac{\sqrt{2}}{2}F_4\wedge F_4\wedge B_2\Big|_{H^3}$$

Inserting in the expansions of these fields, we obtain

$$S_{H^3(\text{top})} = \int_{M_4} -\sqrt{2}\mathcal{B}_2 \wedge P^I \wedge \bar{P}^J \int_{CY_3} \Phi_I \wedge \bar{\Phi}_J + \int_{M_4} \sqrt{2}\mathcal{B}_2 \wedge \bar{Q} \wedge Q \int_{CY_3} \bar{\Omega} \wedge \Omega \, .$$

Following [124] this can be evaluated and ultimately be brought to the form

$$S_{H^3(\text{top})} = -\int_{M_4} 2\mathcal{B}_2 \wedge d\zeta^I \wedge d\tilde{\zeta}_I = \int_{M_4} 2\mathcal{H}_3 \wedge \zeta^I d\tilde{\zeta}_I \, , \tag{C.10}$$

where $\mathcal{H}_3 = d\mathcal{B}_2$ is the field strength of the four-dimensional Kalb-Ramond field $\mathcal{B}_2$.

To this we add the contribution from the reduction of the kinetic term of the ten-dimensional Kalb-Ramond field. (Here the internal part is a zero-form, so this belongs to the 'H^0 sector.')

$$S_{H^3(\text{top})} + S_{H^0(B_2)} = \int_{M_4} 2\mathcal{H}_3 \wedge \zeta^I d\tilde{\zeta}_I - \alpha_2 e^{2\varphi} \mathcal{H}_3 \wedge \star \mathcal{H}_3 \, .$$

Following [124] we dualize the four-dimensional Kalb-Ramond field $\mathcal{B}_2$ into $\tilde{\phi}$.

$$S_{H^3(\text{top})} + S_{H^0(\tilde{\phi})} = -\int_{M_4} e^{-2\varphi} \left[d\tilde{\phi} + \frac{1}{2}\left(\zeta^I d\tilde{\zeta}_I - \tilde{\zeta} d\zeta^I\right)\right] \wedge \star \left[d\tilde{\phi} + \frac{1}{2}\left(\zeta^I d\tilde{\zeta}_I - \tilde{\zeta} d\zeta^I\right)\right] \, .$$

Note that α_2 has cancelled, because in signatures (0,10) and (2,8), where $\alpha_2 = -1$, the Hodge dualization generates an additional sign compared to signature (1,9), where $\alpha_2 = 1$.

C.9 Final result for the hypermultiplet sector

By collecting all terms contributing to the hypermultiplet sector we obtain

$$\begin{aligned} S_H = \int_{M_4} & - \tilde{G}_{\alpha\bar{\beta}} dz^\alpha \wedge \star d\bar{z}^{\bar{\beta}} - \frac{1}{4} d\varphi \wedge \star d\varphi \\ & - e^{-2\varphi} \left[d\tilde{\phi} + \frac{1}{2}\left(\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I\right)\right] \wedge \star \left[d\tilde{\phi} + \frac{1}{2}\left(\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I\right)\right] \\ & + \frac{\alpha_3}{2} e^{-\varphi} \left[\mathcal{I}_{IJ} d\zeta^I \wedge \star d\zeta^J + \mathcal{I}^{IJ} \left(d\tilde{\zeta}_I + \mathcal{R}_{IK} d\zeta^K \right) \wedge \star \left(d\tilde{\zeta}_I + \mathcal{R}_{IK} d\zeta^K \right)\right] \end{aligned}$$

where $\alpha, \beta = 1, \ldots, h_{2,1} = n_V - 1$ and $I, J = 1, \ldots, n_V = h_{2,1} + 1$. This indeed agrees with (4.12) upon identifying $\alpha_3 = -\lambda$. This completes the derivation of the four Type IIA hypermultiplet Lagrangians from dimensional reduction. For Type $\text{IIA}_{(1,9)}$ and Type $\text{IIA}_{(0,10)}$, where $\alpha_3 = -\lambda = 1$, the geometry is QK ($\alpha_3 = 1$), while for Type $\text{IIA}^*(1,9)$ and Type $\text{IIA}(2,8)$ where $\alpha_3 = -\lambda = -1$, the geometry is PQK. In both cases the distinguished submanifold is the SK manifold provided by the complex structure moduli space.

Appendix D

Undualizing the Kalb-Ramond field in 4 dimensions

If we were to directly substitute the universal axion by its Hodge dual at the level of the action the resulting kinetic term might get sign errors. The correct way to dualize fields at the level of the action is to promote the associated field strength as the fundamental field and add a Lagrange multiplier to the action that enforces the Bianchi identity. In our case we therefore write

$$S=\int d^4x\sqrt{\hat{g}_S}(-2)e^{2\varphi}\,[\,\partial^\mu\tilde{\varphi}+\frac{1}{2}(\zeta\partial^\mu\tilde{\zeta}-\tilde{\zeta}\partial^\mu\zeta)]\,[\,\partial_\mu\tilde{\varphi}+\frac{1}{2}(\zeta\partial_\mu\tilde{\zeta}-\tilde{\zeta}\partial_\mu\zeta)]+K\epsilon^{\mu\nu\rho\lambda}\hat{H}_{\mu\nu\rho}(\partial_\lambda\tilde{\varphi}), \tag{D.1}$$

where K is an undetermined constant that we will fix by requiring canonical normalization. We have

$$\begin{aligned}\delta_{\partial_\sigma\tilde{\varphi}}S&=\int d^4x\sqrt{\hat{g}_S}(-2)e^{2\varphi}2\,[\,\partial^\sigma\tilde{\varphi}+\frac{1}{2}(\zeta\partial^\sigma\tilde{\zeta}-\tilde{\zeta}\partial^\sigma\zeta)]+K\epsilon^{\mu\nu\rho\sigma}\hat{H}_{\mu\nu\rho}=0\\ &\Rightarrow\partial^\sigma\tilde{\varphi}=-\frac{1}{2}(\zeta\partial^\sigma\tilde{\zeta}-\tilde{\zeta}\partial^\sigma\zeta)+\frac{1}{4}Ke^{-2\varphi}\epsilon^{\mu\nu\rho\sigma}\hat{H}_{\mu\nu\rho}.\end{aligned}$$

Plugging back in the action we get:

$$\begin{aligned}S=\int d^4x\sqrt{\hat{g}_S}(-2)e^{2\varphi}&\left[\frac{1}{4}Ke^{-2\varphi}\epsilon^{\mu\nu\rho\sigma}\hat{H}_{\mu\nu\rho}\right]\left[\frac{1}{4}Ke^{-2\varphi}\epsilon_{\mu'\nu'\rho'\sigma}\hat{H}^{\mu'\nu'\rho'}\right]\\ &+K\epsilon^{\mu\nu\rho\lambda}\hat{H}_{\mu\nu\rho}\frac{1}{4}Ke^{-2\varphi}\epsilon_{\mu'\nu'\rho'\lambda}\hat{H}^{\mu'\nu'\rho'}-\frac{1}{2}K(\zeta\partial^\lambda\tilde{\zeta}-\tilde{\zeta}\partial^\lambda\zeta)\epsilon^{\mu\nu\rho\lambda}\hat{H}_{\mu\nu\rho}.\end{aligned}$$

We use the following formula

$$\epsilon^{\mu\nu\rho\lambda}\epsilon_{\mu'\nu'\rho'\lambda}=-1!\delta^{\mu\nu\rho}_{\mu'\nu'\rho'}=-6\delta^\mu_{\mu'}\delta^\nu_{\nu'}\delta^\rho_{\rho'}, \tag{D.2}$$

and we end up with

$$S=\int d^4x\sqrt{\hat{g}_S}\frac{-6K^2}{8}e^{-2\varphi}\hat{H}_{\mu\nu\rho}\hat{H}^{\mu\nu\rho}-\frac{K}{2}(\zeta\partial^\lambda\tilde{\zeta}-\tilde{\zeta}\partial^\lambda\zeta)\epsilon^{\mu\nu\rho\lambda}\hat{H}_{\mu\nu\rho}. \tag{D.3}$$

We pick $K = \frac{1}{3}$ and get

$$S = \int d^4x \sqrt{\hat{g}_S} \Big(-\frac{1}{12} e^{-2\varphi} \hat{H}_{\mu\nu\rho} \hat{H}^{\mu\nu\rho} - \frac{1}{6} (\zeta \partial^\lambda \tilde{\zeta} - \tilde{\zeta} \partial^\lambda \zeta) \epsilon^{\mu\nu\rho\lambda} \hat{H}_{\mu\nu\rho} \Big). \tag{D.4}$$

So the final 4D Einstein-Maxwell-UHM action in string frame is

$$\begin{aligned} \mathcal{S}_4 = \int d^4x \sqrt{\hat{g}_S} \Big[& e^{-2\varphi} \Big(\hat{R}_S + 4 \hat{g}_S^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \Big) - \frac{1}{2} \hat{g}_S^{\mu\rho} \hat{g}_S^{\nu\lambda} \hat{F}_{\mu\nu} \hat{F}_{\rho\lambda} \\ & - \frac{1}{12} e^{-2\varphi} \hat{H}_{\mu\nu\rho} \hat{H}^{\mu\nu\rho} - \frac{1}{6} (\zeta \partial^\lambda \tilde{\zeta} - \tilde{\zeta} \partial^\lambda \zeta) \epsilon^{\mu\nu\rho\lambda} \hat{H}_{\mu\nu\rho} \\ & - \Big(\partial_\mu \zeta \partial^\mu \zeta + \partial^\rho \tilde{\zeta} \partial_\rho \tilde{\zeta} \Big) \Big]. \end{aligned} \tag{D.5}$$

Bibliography

[1] M. Médevielle, T. Mohaupt, and G. Pope, "Type-II Calabi-Yau compactifications, T-duality and special geometry in general spacetime signature," *JHEP* **02** (2022) 048, arXiv:2111.09017 [hep-th].

[2] M. Médevielle and T. Mohaupt, "T-duality across non-extremal horizons," *to appear* .

[3] H. Erbin and M. Médevielle, "Closed string theory without level-matching at the free level," arXiv:2209.05585 [hep-th].

[4] A. Sen, "Strong - weak coupling duality in four-dimensional string theory," *Int. J. Mod. Phys. A* **9** (1994) 3707–3750, arXiv:hep-th/9402002.

[5] C. M. Hull and P. K. Townsend, "Unity of superstring dualities," *Nucl. Phys. B* **438** (1995) 109–137, arXiv:hep-th/9410167.

[6] D. Rickles, *A brief history of string theory: From dual models to M-theory*. The frontiers collection. Springer, Heidelberg, Germany, 2014.

[7] G. Egan, *The Clockwork Rocket*. Night Shade, 2011.

[8] G. Egan, *The Eternal Flame*. Night Shade, 2012.

[9] G. Egan, *The Arrows of Time*. Night Shade, 2014.

[10] G. Egan, *Dichronauts*. Night Shade, 2017.

[11] I. I. Smolyaninov and E. E. Narimanov, "Metric Signature Transitions in Optical Metamaterials," *Phys. Rev. Lett.* **105** (2010) 067402, arXiv:0908.2407 [physics.optics].

[12] S. Weinfurtner, A. White, and M. Visser, "Trans-Planckian physics and signature change events in Bose gas hydrodynamics," *Phys. Rev. D* **76** (2007) 124008, arXiv:gr-qc/0703117.

[13] W. Craig and S. Weinstein, "On determinism and well-posedness in multiple time dimensions," *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* **465** no. 2110, (Jul, 2009) 3023–3046.

[14] S. Weinstein, "Multiple Time Dimensions," arXiv:0812.3869 [physics.gen-ph].

[15] J. B. Hartle and S. W. Hawking, "Wave Function of the Universe," *Phys. Rev. D* **28** (1983) 2960–2975.

[16] A. Vilenkin, "Boundary Conditions in Quantum Cosmology," *Phys. Rev. D* **33** (1986) 3560.

[17] A. Stern and C. Xu, "Signature change in matrix model solutions," *Phys. Rev. D* **98** no. 8, (2018) 086015, arXiv:1808.07963 [hep-th].

[18] K. Hatakeyama, K. Anagnostopoulos, T. Azuma, M. Hirasawa, Y. Ito, J. Nishimura, S. Papadoudis, and A. Tsuchiya, "Relationship between the Euclidean and Lorentzian versions of the type IIB matrix model," *PoS* **LATTICE2021** (2022) 341, arXiv:2112.15368 [hep-lat].

[19] J. Nishimura, "Signature change of the emergent space-time in the IKKT matrix model," *PoS* **CORFU2021** (2022) 255, arXiv:2205.04726 [hep-th].

[20] A. D. Sakharov, "Cosmological Transitions With a Change in Metric Signature," *Sov. Phys. JETP* **60** (1984) 214–218.

[21] I. Arefeva and I. Volovich, "Kaluza-Klein Theories and the signature of space-time," *Phys. Lett. B* **164** (1985) 287–292.

[22] I. Y. Arefeva, I. V. Volovich, and B. G. Dragovic, "ON VACUUM SOLUTIONS IN KALUZA-KLEIN THEORIES WITH ARBITRARY SIGNATURE," *Fizika* **18** (1986) 150–156.

[23] I. Y. Arefeva, I. V. Volovich, and B. G. Dragovic, "Spontaneous Reduction in Multidimensional ($D = 10$, 11) Supergravity Theories With Arbitrary Signature," *Theor. Math. Phys.* **70** (1987) 297–304.

[24] P. K. F. Kuhfittig, "Periodic signature change in spacetimes of embedding class one," *Int. J. Adv. Astron.* **12** (2022) 167–176, arXiv:2204.14222 [gr-qc].

[25] J. J. Heckman, A. Joyce, J. Sakstein, and M. Trodden, "Exploring $\mathbf{2+2}$ Answers to $\mathbf{3+1}$ Questions," arXiv:2208.02267 [hep-th].

[26] T. Adamo, "Lectures on twistor theory," *PoS* **Modave2017** (2018) 003, arXiv:1712.02196 [hep-th].

[27] R. Monteiro, D. O'Connell, D. Peinador Veiga, and M. Sergola, "Classical solutions and their double copy in split signature," *JHEP* **05** (2021) 268, arXiv:2012.11190 [hep-th].

[28] E. Crawley, A. Guevara, N. Miller, and A. Strominger, "Black holes in Klein space," *JHEP* **10** (2022) 135, arXiv:2112.03954 [hep-th].

[29] J. Greensite, "Dynamical origin of the Lorentzian signature of space-time," *Phys. Lett. B* **300** (1993) 34–37, arXiv:gr-qc/9210008.

[30] Y. Itin and F. W. Hehl, "Is the Lorentz signature of the metric of space-time electromagnetic in origin?," *Annals Phys.* **312** (2004) 60–83, arXiv:gr-qc/0401016.

[31] M. Tegmark, "On the dimensionality of space-time," *Class. Quant. Grav.* **14** (1997) L69–L75, arXiv:gr-qc/9702052.

[32] M. P. Blencowe and M. J. Duff, "Supermembranes and the Signature of Space-time," *Nucl. Phys. B* **310** (1988) 387–404.

[33] H. van Dam and Y. J. Ng, "Why 3+1 metric rather than 4+0 or 2+2?," *Phys. Lett. B* **520** (2001) 159–162, arXiv:hep-th/0108067.

[34] N. Mankoc Borstnik and H. B. Nielsen, "Why nature has made a choice of one time and three space coordinates?," *J. Phys. A* **35** (2002) 10563–10573, arXiv:hep-ph/0108269.

[35] M. Carfora and G. Ellis, "The Geometry of classical change of signature," *Int. J. Mod. Phys. D* **4** (1995) 175–188, arXiv:gr-qc/9406043.

[36] G. Ellis, A. Sumeruk, D. Coule, and C. Hellaby, "Change of signature in classical relativity," *Class. Quant. Grav.* **9** (1992) 1535–1554.

[37] C. Hellaby, A. Sumeruk, and G. F. R. Ellis, "Classical signature change in the black hole topology," *Int. J. Mod. Phys. D* **6** (1997) 211–238, arXiv:gr-qc/9907042.

[38] T. Dray, "Einstein's equations in the presence of signature change," *J. Math. Phys.* **37** (1996) 5627–5636, arXiv:gr-qc/9610064.

[39] T. Dray, C. A. Manogue, and R. W. Tucker, "Particle production from signature change," *Gen. Rel. Grav.* **23** (1991) 967–971.

[40] T. Dray, C. A. Manogue, and R. W. Tucker, "The Scalar field equation in the presence of signature change," *Phys. Rev. D* **48** (1993) 2587–2590, arXiv:gr-qc/9303002.

[41] C. Hellaby and T. Dray, "Failure of standard conservation laws at a classical change of signature," *Phys. Rev. D* **49** (1994) 5096–5104, arXiv:gr-qc/9404001.

[42] T. Dray, G. Ellis, C. Hellaby, and C. A. Manogue, "Gravity and signature change," *Gen. Rel. Grav.* **29** (1997) 591–597, arXiv:gr-qc/9610063.

[43] C. Hellaby and T. Dray, "Reply comment: Comparison of approaches to classical signature change," *Phys. Rev. D* **52** (1995) 7333–7339, arXiv:gr-qc/9601040.

[44] T. Dray, G. Ellis, and C. Hellaby, "Note on signature change and Colombeau theory," *Gen. Rel. Grav.* **33** (2001) 1041–1046, arXiv:gr-qc/0012047.

[45] T. Dray and C. Hellaby, "Comment on 'smooth and discontinuous signature type change in general relativity'," *Gen. Rel. Grav.* **28** (1996) 1401, arXiv:gr-qc/9610062.

[46] F. Zhang, "Alternative route towards the change of metric signature," *Phys. Rev. D* **100** no. 6, (2019) 064043, arXiv:1909.10669 [gr-qc].

[47] A. White, S. Weinfurtner, and M. Visser, "Signature change events: A Challenge for quantum gravity?," *Class. Quant. Grav.* **27** (2010) 045007, arXiv:0812.3744 [gr-qc].

[48] H. Ooguri and C. Vafa, "Selfduality and $N = 2$ String MAGIC," *Mod. Phys. Lett. A* **5** (1990) 1389–1398.

[49] H. Ooguri and C. Vafa, "Geometry of N=2 strings," *Nucl. Phys. B* **361** (1991) 469–518.

[50] H. Ooguri and C. Vafa, "N=2 heterotic strings," *Nucl. Phys. B* **367** (1991) 83–104.

[51] I. Bars, "Survey of two time physics," *Class. Quant. Grav.* **18** (2001) 3113–3130, arXiv:hep-th/0008164.

[52] M. Mars, J. M. M. Senovilla, and R. Vera, "Signature change on the brane," *Phys. Rev. Lett.* **86** (2001) 4219–4222, arXiv:gr-qc/0012099.

[53] C. M. Hull, "Duality and the signature of space-time," *JHEP* **11** (1998) 017, arXiv:hep-th/9807127.

[54] R. Dijkgraaf, B. Heidenreich, P. Jefferson, and C. Vafa, "Negative Branes, Supergroups and the Signature of Spacetime," *JHEP* **02** (2018) 050, arXiv:1603.05665 [hep-th].

[55] A. Keurentjes, "E(11): Sign of the times," *Nucl. Phys. B* **697** (2004) 302–318, arXiv:hep-th/0402090.

[56] A. Keurentjes, "Time-like T duality algebra," *JHEP* **11** (2004) 034, arXiv:hep-th/0404174.

[57] P. P. Cook and P. C. West, "M-theory solutions in multiple signatures from E(11)," *JHEP* **12** (2005) 041, arXiv:hep-th/0506122.

[58] P. C. West, "E(11) and M theory," *Class. Quant. Grav.* **18** (2001) 4443–4460, arXiv:hep-th/0104081.

[59] J. Louko and R. D. Sorkin, "Complex actions in two-dimensional topology change," *Class. Quant. Grav.* **14** (1997) 179–204, arXiv:gr-qc/9511023.

[60] E. Witten, "A Note On Complex Spacetime Metrics," arXiv:2111.06514 [hep-th].

[61] T. Mohaupt, *A Short Introduction to String Theory*. Cambridge University Press, 3, 2022.

[62] D. Tong, "String Theory," arXiv:0908.0333 [hep-th].

[63] K. Becker, M. Becker, and J. H. Schwarz, *String theory and M-theory: A modern introduction*. Cambridge University Press, 12, 2006.

[64] J. Dai, R. G. Leigh, and J. Polchinski, "New Connections Between String Theories," *Mod. Phys. Lett. A* **4** (1989) 2073–2083.

[65] P. Horava, "Background Duality of Open String Models," *Phys. Lett. B* **231** (1989) 251–257.

[66] J. Polchinski, "Tasi lectures on D-branes," in *Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 96): Fields, Strings, and Duality*, pp. 293–356. 11, 1996. arXiv:hep-th/9611050.

[67] C. V. Johnson, "D-brane primer," in *Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 99): Strings, Branes, and Gravity*, pp. 129–350. 7, 2000. arXiv:hep-th/0007170.

[68] S. Alexandrov, A. Sen, and B. Stefański, "D-instantons in Type IIA string theory on Calabi-Yau threefolds," *JHEP* **11** (2021) 018, arXiv:2108.04265 [hep-th].

[69] S. Alexandrov, A. Sen, and B. Stefański, "Euclidean D-branes in type IIB string theory on Calabi-Yau threefolds," *JHEP* **12** (2021) 044, arXiv:2110.06949 [hep-th].

[70] M. B. Green, J. H. Schwarz, and E. Witten, *SUPERSTRING THEORY. VOL. 1: INTRODUCTION*. Cambridge Monographs on Mathematical Physics. 7, 1988.

[71] Y. Okawa and R. Sakaguchi, "Closed string field theory without the level-matching condition," arXiv:2209.06173 [hep-th].

[72] A. Sen, "Tachyon dynamics in open string theory," *Int. J. Mod. Phys. A* **20** (2005) 5513–5656, arXiv:hep-th/0410103.

[73] S. M. Carroll, *Spacetime and Geometry*. Cambridge University Press, 7, 2019.

[74] J. Polchinski, "Scale and Conformal Invariance in Quantum Field Theory," *Nucl. Phys. B* **303** (1988) 226–236.

[75] N. Hitchin, "Lectures on generalized geometry," arXiv:1008.0973 [math.DG].

[76] G. Aldazabal, D. Marques, and C. Nunez, "Double Field Theory: A Pedagogical Review," *Class. Quant. Grav.* **30** (2013) 163001, arXiv:1305.1907 [hep-th].

[77] O. Hohm, D. Lüst, and B. Zwiebach, "The Spacetime of Double Field Theory: Review, Remarks, and Outlook," *Fortsch. Phys.* **61** (2013) 926–966, arXiv:1309.2977 [hep-th].

[78] R. Blumenhagen, D. Lüst, and S. Theisen, *Basic concepts of string theory*. Theoretical and Mathematical Physics. Springer, Heidelberg, Germany, 2013.

[79] J. Polchinski, *String theory. Vol. 2: Superstring theory and beyond*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 12, 2007.

[80] L. J. Romans, "Massive N=2a Supergravity in Ten-Dimensions," *Phys. Lett. B* **169** (1986) 374.

[81] C. Vafa, "Evidence for F theory," *Nucl. Phys. B* **469** (1996) 403–418, arXiv:hep-th/9602022.

[82] T. Weigand, "F-theory," *PoS* **TASI2017** (2018) 016, arXiv:1806.01854 [hep-th].

[83] E. Witten, "String theory dynamics in various dimensions," *Nucl. Phys. B* **443** (1995) 85–126, arXiv:hep-th/9503124.

[84] P. Horava and E. Witten, "Heterotic and type I string dynamics from eleven-dimensions," *Nucl. Phys. B* **460** (1996) 506–524, arXiv:hep-th/9510209.

[85] P. Horava and E. Witten, "Eleven-dimensional supergravity on a manifold with boundary," *Nucl. Phys. B* **475** (1996) 94–114, arXiv:hep-th/9603142.

[86] T. Banks, W. Fischler, S. H. Shenker, and L. Susskind, "M theory as a matrix model: A Conjecture," *Phys. Rev. D* **55** (1997) 5112–5128, arXiv:hep-th/9610043.

[87] N. Ishibashi, H. Kawai, Y. Kitazawa, and A. Tsuchiya, "A Large N reduced model as superstring," *Nucl. Phys. B* **498** (1997) 467–491, arXiv:hep-th/9612115.

[88] E. Bergshoeff, E. Sezgin, and P. K. Townsend, "Supermembranes and Eleven-Dimensional Supergravity," *Phys. Lett. B* **189** (1987) 75–78.

[89] P. K. Townsend, "The eleven-dimensional supermembrane revisited," *Phys. Lett. B* **350** (1995) 184–187, arXiv:hep-th/9501068.

[90] M. J. Duff, P. S. Howe, T. Inami, and K. S. Stelle, "Superstrings in D=10 from Supermembranes in D=11," *Phys. Lett. B* **191** (1987) 70.

[91] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity," *Adv. Theor. Math. Phys.* **2** (1998) 231–252, arXiv:hep-th/9711200.

[92] E. Palti, "The Swampland: Introduction and Review," *Fortsch. Phys.* **67** no. 6, (2019) 1900037, arXiv:1903.06239 [hep-th].

[93] L. Gall and T. Mohaupt, "Supersymmetry algebras in arbitrary signature and their R-symmetry groups," *JHEP* **10** (2021) 203, arXiv:2108.05109 [hep-th].

[94] D. Tong, "Supersymmetric field theory,".

[95] S. R. Coleman and J. Mandula, "All Possible Symmetries of the S Matrix," *Phys. Rev.* **159** (1967) 1251–1256.

[96] R. Haag, J. T. Lopuszanski, and M. Sohnius, "All Possible Generators of Supersymmetries of the s Matrix," *Nucl. Phys. B* **88** (1975) 257.

[97] D. Z. Freedman and A. Van Proeyen, *Supergravity*. Cambridge Univ. Press, Cambridge, UK, 5, 2012.

[98] F. Quevedo, S. Krippendorf, and O. Schlotterer, "Cambridge Lectures on Supersymmetry and Extra Dimensions," arXiv:1011.1491 [hep-th].

[99] M. J. Duff, B. E. W. Nilsson, and C. N. Pope, "Kaluza-Klein Supergravity," *Phys. Rept.* **130** (1986) 1–142.

[100] C. Pope, "Kaluza-Klein theory,".

[101] I. Quiros, "Time-like versus space-like extra dimensions," arXiv:0707.0714 [gr-qc].

[102] S.-T. Yau, "On the ricci curvature of a compact kähler manifold and the complex monge-ampére equation, I," *Commun. Pure Appl. Math.* **31** no. 3, (1978) 339–411.

[103] C. WALL, "Classification problems in differential topology. v. on certain 6-manifolds.," *Inventiones mathematicae* **1** (1966) 355–374. http://eudml.org/doc/141839.

[104] T. Hubsch, *Calabi-Yau manifolds: A Bestiary for physicists*. World Scientific, Singapore, 1994.

[105] A. Strominger, S.-T. Yau, and E. Zaslow, "Mirror symmetry is T duality," *Nucl. Phys. B* **479** (1996) 243–259, arXiv:hep-th/9606040.

[106] P. Candelas, X. C. De La Ossa, P. S. Green, and L. Parkes, "A Pair of Calabi-Yau manifolds as an exactly soluble superconformal theory," *Nucl. Phys. B* **359** (1991) 21–74.

[107] P. Candelas and X. de la Ossa, "Moduli Space of Calabi-Yau Manifolds," *Nucl. Phys. B* **355** (1991) 455–481.

[108] C. M. Hull, "Timelike T duality, de Sitter space, large N gauge theories and topological field theory," *JHEP* **07** (1998) 021, arXiv:hep-th/9806146.

[109] C. M. Hull and R. R. Khuri, "Branes, times and dualities," *Nucl. Phys. B* **536** (1998) 219–244, arXiv:hep-th/9808069.

[110] R. Blumenhagen, M. Brinkmann, A. Makridou, L. Schlechter, and M. Traube, "dS Spaces and Brane Worlds in Exotic String Theories," *JHEP* **06** (2020) 077, arXiv:2002.11746 [hep-th].

[111] B. de Wit and H. Nicolai, "Hidden Symmetry in $d = 11$ Supergravity," *Phys. Lett. B* **155** (1985) 47–53.

[112] G. W. Moore, "Finite in all directions," arXiv:hep-th/9305139.

[113] O. Hohm, S. K. Kwak, and B. Zwiebach, "Double Field Theory of Type II Strings," *JHEP* **09** (2011) 013, arXiv:1107.0008 [hep-th].

[114] O. Hohm and H. Samtleben, "The many facets of exceptional field theory," *PoS* **CORFU2018** (2019) 098, arXiv:1905.08312 [hep-th].

[115] L. Gall and T. Mohaupt, "Supersymmetry algebras in arbitrary signature and their R-symmetry groups," *JHEP* **10** (2021) 203, arXiv:2108.05109 [hep-th].

[116] E. Bergshoeff and A. Van Proeyen, "The Many faces of OSp(1—32)," *Class. Quant. Grav.* **17** (2000) 3277–3304, arXiv:hep-th/0003261.

[117] E. A. Bergshoeff, J. Hartong, A. Ploegh, J. Rosseel, and D. Van den Bleeken, "Pseudo-supersymmetry and a tale of alternate realities," *JHEP* **07** (2007) 067, arXiv:0704.3559 [hep-th].

[118] M. Kontsevich and G. Segal, "Wick Rotation and the Positivity of Energy in Quantum Field Theory," *Quart. J. Math. Oxford Ser.* **72** no. 1-2, (2021) 673–699, arXiv:2105.10161 [hep-th].

[119] J.-L. Lehners, "Allowable complex metrics in minisuperspace quantum cosmology," *Phys. Rev. D* **105** no. 2, (2022) 026022, arXiv:2111.07816 [hep-th].

[120] T. Mohaupt and K. Waite, "Euclidean Actions, Instantons, Solitons and Supersymmetry," *J. Phys. A* **44** (2011) 175403, arXiv:1011.6301 [hep-th].

[121] S. Ferrara and S. Sabharwal, "Dimensional Reduction of Type II Superstrings," *Class. Quant. Grav.* **6** (1989) L77.

[122] M. Bodner, A. C. Cadavid, and S. Ferrara, "(2,2) vacuum configurations for type IIA superstrings: N=2 supergravity Lagrangians and algebraic geometry," *Class. Quant. Grav.* **8** (1991) 789–808.

[123] R. Bohm, H. Gunther, C. Herrmann, and J. Louis, "Compactification of type IIB string theory on Calabi-Yau threefolds," *Nucl. Phys. B* **569** (2000) 229–246, arXiv:hep-th/9908007.

[124] W. Sabra and O. Vaughan, "10D to 4D Euclidean Supergravity over a Calabi-Yau three-fold," *Class. Quant. Grav.* **33** no. 1, (2016) 015010, arXiv:1503.05095 [hep-th].

[125] W. A. Sabra, "Special geometry and space–time signature," *Phys. Lett. B* **773** (2017) 191–195, arXiv:1706.05162 [hep-th].

[126] L. Andrianopoli, M. Bertolini, A. Ceresole, R. D'Auria, S. Ferrara, P. Fre, and T. Magri, "N=2 supergravity and N=2 superYang-Mills theory on general scalar manifolds: Symplectic covariance, gaugings and the momentum map," *J. Geom. Phys.* **23** (1997) 111–189, arXiv:hep-th/9605032.

[127] G. Lopes Cardoso and T. Mohaupt, "Special geometry, Hessian structures and applications," *Phys. Rept.* **855** (2020) 1–141, arXiv:1909.06240 [hep-th].

[128] E. Lauria and A. Van Proeyen, *$\mathcal{N} = 2$ Supergravity in $D = 4, 5, 6$ Dimensions*, vol. 966. 3, 2020. arXiv:2004.11433 [hep-th].

[129] V. Cortes, C. Mayer, T. Mohaupt, and F. Saueressig, "Special geometry of Euclidean supersymmetry. 1. Vector multiplets," *JHEP* **03** (2004) 028, arXiv:hep-th/0312001.

[130] V. Cortes and T. Mohaupt, "Special Geometry of Euclidean Supersymmetry III: The Local r-map, instantons and black holes," *JHEP* **07** (2009) 066, arXiv:0905.2844 [hep-th].

[131] V. Cortés, P. Dempster, T. Mohaupt, and O. Vaughan, "Special Geometry of Euclidean Supersymmetry IV: the local c-map," *JHEP* **10** (2015) 066, arXiv:1507.04620 [hep-th].

[132] V. Cortés, L. Gall, and T. Mohaupt, "Four-dimensional vector multiplets in arbitrary signature (I)," *Int. J. Geom. Meth. Mod. Phys.* **17** no. 10, (2020) 2050150, arXiv:1907.12067 [hep-th].

[133] S. Cecotti, S. Ferrara, and L. Girardello, "Geometry of Type II Superstrings and the Moduli of Superconformal Field Theories," *Int. J. Mod. Phys. A* **4** (1989) 2475.

[134] S. Ferrara and S. Sabharwal, "Quaternionic Manifolds for Type II Superstring Vacua of Calabi-Yau Spaces," *Nucl. Phys. B* **332** (1990) 317–332.

[135] J. Gutowski, T. Mohaupt, and G. Pope, "From static to cosmological solutions of $\mathcal{N} = 2$ supergravity," *JHEP* **08** (2019) 172, arXiv:1905.09167 [hep-th].

[136] J. Gutowski, T. Mohaupt, and G. Pope, "Cosmological Solutions, a New Wick-Rotation, and the First Law of Thermodynamics," *JHEP* **03** (2021) 293, arXiv:2008.06929 [hep-th].

[137] D. Klemm and M. Nozawa, "Geometry of Killing spinors in neutral signature," *Class. Quant. Grav.* **32** no. 18, (2015) 185012, arXiv:1504.02710 [hep-th].

[138] J. B. Gutowski and W. A. Sabra, "Real Killing Spinors in Neutral Signature," *JHEP* **11** (2019) 173, arXiv:1905.01910 [hep-th].

[139] W. A. Sabra, "Phantom Metrics With Killing Spinors," *Phys. Lett. B* **750** (2015) 237–241, arXiv:1507.04597 [hep-th].

[140] W. A. Sabra, "Kasner Branes with Arbitrary Signature," *Phys. Lett. B* **809** (2020) 135694, arXiv:2005.03953 [hep-th].

[141] W. A. Sabra, "Hypersymplectic geometry and supersymmetric solutions in (t,s) 5D supergravity," *Phys. Rev. D* **104** no. 4, (2021) 046012, arXiv:2105.00806 [hep-th].

[142] E. Witten, "THE SEARCH FOR HIGHER SYMMETRY IN STRING THEORY," *Phil. Trans. Roy. Soc. Lond. A* **329** (1989) 349–357.

[143] B. de Wit and A. Van Proeyen, "Potentials and Symmetries of General Gauged N=2 Supergravity: Yang-Mills Models," *Nucl. Phys. B* **245** (1984) 89–117.

[144] B. de Wit, P. G. Lauwers, and A. Van Proeyen, "Lagrangians of N=2 Supergravity - Matter Systems," *Nucl. Phys. B* **255** (1985) 569–608.

[145] W. A. Sabra, "Flow equations in arbitrary signature," *Phys. Lett. B* **829** (2022) 137084, arXiv:2111.05141 [hep-th].

[146] V. Cortes, C. Mayer, T. Mohaupt, and F. Saueressig, "Special geometry of euclidean supersymmetry. II. Hypermultiplets and the c-map," *JHEP* **06** (2005) 025, arXiv:hep-th/0503094.

[147] M. A. Lledo, O. Macia, A. Van Proeyen, and V. S. Varadarajan, "Special geometry for arbitrary signatures," *IRMA Lect. Math. Theor. Phys.* **16** (2010) 85–147, arXiv:hep-th/0612210.

[148] S. W. Hawking, "The Chronology protection conjecture," *Phys. Rev. D* **46** (1992) 603–611.

[149] T. Ortin, *Gravity and strings*. Cambridge Monographs on Mathematical Physics. Cambridge Univ. Press, 3, 2004.

[150] B. R. Greene, "String theory on Calabi-Yau manifolds," in *Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 96): Fields, Strings, and Duality*, pp. 543–726. 6, 1996. arXiv:hep-th/9702155.

[151] S. Bondarenko, "Dynamical Signature: Complex Manifolds, Gauge Fields and Non-Flat Tangent Space," *Universe* **8** no. 10, (2022) 497, arXiv:2111.06095 [gr-qc].

[152] M. Rocek and E. P. Verlinde, "Duality, quotients, and currents," *Nucl. Phys. B* **373** (1992) 630–646, arXiv:hep-th/9110053.

[153] A. Sen and C. Vafa, "Dual pairs of type II string compactification," *Nucl. Phys. B* **455** (1995) 165–187, arXiv:hep-th/9508064.

[154] D. D. K. Chow and G. Compère, "Black holes in N=8 supergravity from SO(4,4) hidden symmetries," *Phys. Rev. D* **90** no. 2, (2014) 025029, arXiv:1404.2602 [hep-th].

[155] V. Cortes, T. Mohaupt, and H. Xu, "Completeness in supergravity constructions," *Commun. Math. Phys.* **311** (2012) 191–213, arXiv:1101.5103 [hep-th].

[156] M. van Beest, J. Calderón-Infante, D. Mirfendereski, and I. Valenzuela, "Lectures on the Swampland Program in String Compactifications," *Phys. Rept.* **989** (2022) 1–50, arXiv:2102.01111 [hep-th].

[157] A. Sagnotti and M. Tsulaia, "On higher spins and the tensionless limit of string theory," *Nucl. Phys. B* **682** (2004) 83–116, arXiv:hep-th/0311257.

[158] G. L. Cardoso, B. de Wit, and S. Mahapatra, "Exact results for an STU-model," *JHEP* **02** (2020) 127, arXiv:1907.04077 [hep-th].

[159] K. Behrndt, I. Gaida, D. Lust, S. Mahapatra, and T. Mohaupt, "From type IIA black holes to T dual type IIB D instantons in N=2, D = 4 supergravity," *Nucl. Phys. B* **508** (1997) 659–699, arXiv:hep-th/9706096.

[160] C. Hull and B. Zwiebach, "Double Field Theory," *JHEP* **09** (2009) 099, arXiv:0904.4664 [hep-th].

[161] O. Hohm, D. Lüst, and B. Zwiebach, "The Spacetime of Double Field Theory: Review, Remarks, and Outlook," *Fortsch. Phys.* **61** (2013) 926–966, arXiv:1309.2977 [hep-th].

[162] D. S. Berman and D. C. Thompson, "Duality Symmetric String and M-Theory," *Phys. Rept.* **566** (2014) 1–60, arXiv:1306.2643 [hep-th].

[163] D. S. Berman and F. J. Rudolph, "Branes are Waves and Monopoles," *JHEP* **05** (2015) 015, arXiv:1409.6314 [hep-th].

[164] K. Morand, J.-H. Park, and M. Park, "Identifying Riemannian Singularities with Regular Non-Riemannian Geometry," *Phys. Rev. Lett.* **128** no. 4, (2022) 041602, arXiv:2106.01758 [hep-th].

[165] M. Nakahara, *Geometry, topology and physics*. 2003.

[166] M. Berger, "Sur les groupes d'holonomie homogènes de variétés à connexion affine et des variétés riemanniennes," *Bulletin de la Société Mathématique de France* **83** (1955) 279–330. http://www.numdam.org/articles/10.24033/bsmf.1464/.

[167] A. Kennon, "G_2-Manifolds and M-Theory Compactifications," arXiv:1810.12659 [hep-th].

[168] F. Bonetti, T. W. Grimm, E. Palti, and T. G. Pugh, "F-Theory on Spin(7) Manifolds: Weak-Coupling Limit," *JHEP* **02** (2014) 076, arXiv:1309.2287 [hep-th].