Optimal regulator for a class of nonlinear stochastic systems with random coefficients

Mashael Algoulity[1] and Bujar Gashi[2]

Abstract

We consider an optimal regulator problem for a class of nonlinear stochastic systems with a *square-root* nonlinearity and *random* coefficients, and using the quadratic-linear criterion. This represents a certain nonlinear generalisation of the stochastic linear-quadratic control problem with random coefficients. The solution if found in an explicit closed-form as an *affine state-feedback* control in terms of a Riccati and linear backward stochastic differential equations. As an application, we give the solution to an optimal investment problem in a market with random coefficients.

Keywords: Stochastic optimal control; Nonlinear systems; Riccati BSDEs; Optimal investment.

1. Introduction and problem formulation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a given complete probability space. Also let $(W_\ell(t), t \geq 0)$ be a one-dimensional standard Brownian motion defined on this space. The filtration $(\mathcal{F}_\ell(t), t \geq 0)$ is defined as the augmentation of $\sigma\{W_\ell(s) : 0 \leq s \leq t\}$ by all the $\mathbb{P}$-null sets of $\mathcal{F}$. Consider the *linear* scalar stochastic control system with random coefficients (for $t \in [0, T]$):

[1]Applied College, University of Tabuk, Tabuk 71491, Saudi Arabia; Email: m_alqolaity@ut.edu.sa.

[2]Department of Mathematical Sciences, The University of Liverpool, Peach Street, Liverpool, L69 7ZL, UK; Email: Bujar.Gashi@liverpool.ac.uk.

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$$
\begin{cases}
dx_\ell(t) = [a_\ell(t)x_\ell(t) + b_\ell(t)u_\ell(t) + c_\ell(t)]dt \\
\qquad\qquad +[d_\ell(t)x_\ell(t) + f_\ell(t)u_\ell(t) + g_\ell(t)]dW_\ell(t), \\
x_\ell(0) \in \mathbb{R}, \quad \text{is given,}
\end{cases} \tag{1}
$$

for some suitable adapted coefficient processes $a_\ell, b_\ell, c_\ell,\ d_\ell, f_\ell, g_\ell,$ and an adapted control process u_ℓ such that (1) has a unique strong solution. The one-dimensional stochastic *linear-quadratic (LQ)* control problem is the optimal control problem of minimizing the quadratic cost functional

$$
\mathbb{E}\left\{\int_0^T [x'_\ell(t)q_\ell(t)x_\ell(t)+u'_\ell(t)r_\ell(t)u_\ell(t)]dt+x'_\ell(T)s_\ell x_\ell(T)\right\},
$$

subject to (1), for some suitable adapted weight processes q_ℓ and r_ℓ, and a suitable $\mathcal{F}_\ell(T)$-measurable weight random variable s_ℓ. A *key* feature of this problem is that it admits an explicit-closed form solution as a *linear state-feedback* control the gain of which is given in terms of a Riccati differential equation. The LQ control problem has been studied extensively since its introduction by Kalman in [14] for deterministic systems (see, e.g., [1], [7], for a textbook account). One of the first solutions to the stochastic LQ problem with multiplicative noise was given by Wonham in [29], [30], for the case of deterministic coefficients (see, e.g., [33], [8], for a texbook account). A typical assumption in the LQ control problems is that the coefficients of the cost functional have certain definiteness properties. This assumption can be weakened further and it leads to indefinite LQ control, see, for example, [4], [20], [5], [24], [25], [23], [22], [19], [12], [18]. The stochastic LQ control problem with a fixed final state was solved in [9], in the setting of deterministic coefficients. For the systems of mean-field type see, for example, [31], [32]. The LQ control problem with random coefficients was considered by Bismut [2], and its solution is given in terms of the Riccati backward stochastic differential equation (BSDE). This case of the stochastic LQ control problem has been studied extensively since then (see, for example, [21], [27], [28]), and has more recently been generalised to criteria with state-dependent weights [10].

In this paper, we introduce a certain *nonlinear* generalisation to the stochastic LQ control problem with random coefficients by considering systems with *square-root* nonlinearity in the diffusion term as follows. Let $(W_1(t), t \geq 0)$ and $(W_2(t), t \geq 0)$ be two independent one-dimensional Brownian motions defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We consider the filtrations $(\mathcal{F}_2(t), t \geq 0)$ and $(\mathcal{F}(t), t \geq 0)$ defined as the augmentations of $\sigma\{W_2(s) : 0 \leq s \leq t\}$ and $\sigma\{W_1(s), W_2(s) : 0 \leq s \leq t\}$, respectively, by all the $\mathbb{P}$-null sets of $\mathcal{F}$. If E is an Euclidian space, then we denote by $L^\infty_{\mathcal{F}_2}(0, T; E)$ the set of E-valued $\mathcal{F}_2$-adapted uniformly bounded processes, by $L^2_{\mathcal{F}}(0, T; E)$ the set of E-valued $\mathcal{F}$-adapted square-integrable processes, $L^2_{\mathcal{F}_2}(0, T; E)$ the set of E-valued $\mathcal{F}_2$-adapted square-integrable processes, and by $L^\infty_{\mathcal{F}_2(T)}(E)$ the set of $\mathcal{F}_2(T)$-measurable bounded random variables. Consider the following *two* dimensional stochastic control system (for $t \in [0, T]$):

$$
\begin{cases}
dx_1(t) = [a_{11}(t)x_1(t) + a_{12}(t)x_2(t) + b_1(t)u(t) \\
\qquad + c_1(t)]dt + M(t, x(t), u(t))\, dW_1(t), \\
dx_2(t) = [a_{21}(t)x_1(t) + a_{22}(t)x_2(t) \\
\qquad + b_2(t)u(t) + c_2(t)]dt + [f_{21}(t)x_1(t) \\
\qquad + f_{22}(t)x_2(t) + h_2(t)u(t) + k_2(t)]dW_2(t), \\
M^2(t, x(t), u(t)) := x'(t)\tilde{Q}(t)x(t) + u(t)\tilde{k}'(t)x(t) \\
\qquad + \tilde{r}(t)u^2(t) + \tilde{a}'(t)x(t) + \tilde{b}(t)u(t) + \tilde{c}(t), \\
x_1(0), x_2(0) \in \mathbb{R} \quad \text{are given,}
\end{cases}
\tag{2}
$$

where $x(t) := [x_1(t) \quad x_2(t)]'$. Here x_1 and x_2 are the states of the system, u is the one-dimensional control process, whereas the remaining processes are given coefficients. If we define (for $t \in [0,T]$)

$$A(t) := \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix}, F(t) := \begin{bmatrix} 0 & 0 \\ f_{21}(t) & f_{22}(t) \end{bmatrix},$$

$$B(t) := \begin{bmatrix} b_1(t) \\ b_2(t) \end{bmatrix}, C(t) := \begin{bmatrix} c_1(t) \\ c_2(t) \end{bmatrix}, H(t) := \begin{bmatrix} 0 \\ h_2(t) \end{bmatrix},$$

$$K(t) := \begin{bmatrix} 0 \\ k_2(t) \end{bmatrix}, N(t, x(t), u(t)) := \begin{bmatrix} M(t, x(t), u(t)) \\ 0 \end{bmatrix},$$

then we can write the system (2) as:

$$\begin{cases} dx(t) = [A(t)x(t) + B(t)u + C(t)]dt \\ \qquad +N(t, x(t), u(t))dW_1(t) \\ \qquad +[F(t)x(t) + H(t)u(t) + K(t)]dW_2(t), \\ x(0) \in \mathbb{R}^2, \quad \text{is given.} \end{cases} \tag{3}$$

We associate with system (3) the following quadratic-linear cost functional:

$$\begin{aligned} J(u(\cdot)) \;\; &:= \;\; \mathbb{E}\Big\{ \int_0^T [x'(t)Q(t)x(t) + r(t)u^2(t) \\ &+ \;\; u(t)k'(t)x(t) + f(t)u(t) + h'(t)x(t)]dt \\ &+ \;\; x'(T)Sx(T) + v'x(T) \Big\}. \end{aligned} \tag{4}$$

We assume that all the given coefficients appearing in the system equation

(3) and in the cost functional (4) are random as follows:

$$A(\cdot) \in L^\infty_{\mathcal{F}_2}(0,T;\mathbb{R}^{2\times 2});\ B(\cdot) \in L^\infty_{\mathcal{F}_2}(0,T;\mathbb{R}^{2});$$

$$F(\cdot) \in L^\infty_{\mathcal{F}_2}(0,T;\mathbb{R}^{2\times 2});\ H(\cdot) \in L^\infty_{\mathcal{F}_2}(0,T;\mathbb{R}^{2});$$

$$C(\cdot) \in L^\infty_{\mathcal{F}_2}(0,T;\mathbb{R}^{2});\ K(\cdot) \in L^\infty_{\mathcal{F}_2}(0,T;\mathbb{R}^{2});$$

$$Q(\cdot),\ \tilde{Q}(\cdot) \in L^\infty_{\mathcal{F}_2}(0,T;\mathbb{S}^{2\times 2});$$

$$r(\cdot),\ \tilde{r}(\cdot),\ f(\cdot) \in L^\infty_{\mathcal{F}_2}(0,T;\mathbb{R});$$

$$h(\cdot),\ k(\cdot),\ \tilde{k}(\cdot),\ \tilde{a}(\cdot) \in L^\infty_{\mathcal{F}_2}(0,T;\mathbb{R}^{2});$$

$$\tilde{b}(\cdot),\ \tilde{c}(\cdot) \in L^\infty_{\mathcal{F}_2}(0,T;\mathbb{R});$$

$$S \in L^\infty_{\mathcal{F}_2(T)}(\mathbb{S}^{2\times 2});\ v \in L^\infty_{\mathcal{F}_2(T)}(\mathbb{R}^{2}),$$

where $\mathbb{S}^{2\times 2}$ is the set of real 2×2 symmetric matrices. We consider the following optimal stochastic regulator problem:

$$\begin{cases} \min\limits_{u(\cdot)\in\mathcal{A}} J(u(\cdot)) \\ \text{s. t. } (3), \end{cases} \tag{5}$$

where $\mathcal{A}$ is a suitable set of admissible controls to be defined in the next section. The considered system (2) is thus a generalisation of the linear stochastic control system (1), which corresponds to the state x_2 in (2), with another state x_1 the equation of which, due to the coefficient M, is nonlinear in general. Note that the coefficient M is such that its square is quadratic-linear in the state and control processes. A well-known example of the equation of state x_1 is the Cox-Ingersoll-Ross (CIR) model, which has wide applicability in modelling interest rates (see, for example, [6], [26]), and it corresponds to the coefficients a_{11}, c_1, being constant, a_{12}, b_1, $\tilde{Q}$, $\tilde{k}$, $\tilde{r}$, $\tilde{b}$, $\tilde{c}$, all being zero, and $\tilde{a} = [1 \quad 0]'$. Other examples of the equation of state x_1, that also admit an explicit solution, can be found in §4.4 of [16]. This type of equation with deterministic coefficients has been used to model nonlinear stochastic uncertainties in the control system model (see [3], [13]). In the special case of all coefficients being deterministic (and a slightly simpler nonlinearity), the problem (5) has been considered in [11] and an explicit-closed form solution found. The problem (5) is thus a generalisation of the nonlinear regulator problem of [11] to the case of random coefficients, and one motivation is the optimal investment problem in a market with random coefficients.

In §2, we give the solution to problem (5) in an explicit closed-form as a affine state-feedback control, the gains of which are given in terms of a *new* type of a Riccati BSDE and a linear BSDE. We use a completion of squares method which is more involved than in the stochastic LQ control due to the nonlinear system dynamics. The random coefficients have necessitated the use of BSDEs, which do not appear in [11] when considering only deterministic coefficients and where ordinary differential equations (ODEs) suffice to solve the problem. By their stochastic nature, the BSDEs are more general and thus more challenging to consider as compared to ODEs, and in particular the solution *pairs* appear here, a feature absent in ODEs. In Theorem 1 of §2 we show the solvability the Riccati type BSDE that appears in the solution to the optimal regulator problem (5). In section 3, we give an application of our results to an optimal investment problem in a market where in addition to a nonlinear model for the stochastic interest rate, which also appears in [11], the other market coefficients (the appreciation and the volatility of the sock) can also be random, and we thus generalise the result of [11].

2. Solution to the stochastic nonlinear regulator problem

In order to give a precise definition of the admissible set $\mathcal{A}$ and to state the solution to the problem (5), we introduce the following matrix-valued backward stochastic differential equation of Riccati type (the argument t

being suppressed):

$$
\begin{cases}
dP = P_1 \, dt + P_2 \, dW_2, \quad t \in [0,T], \\
P_1 := -(Q + A'P + PA + P_2F + F'P_2 + F'PF \\
\qquad +P_{11} \, \tilde{Q} - \hat{k} \, \hat{k}'/4\hat{r}), \\
\hat{r} := r + H'PH + P_{11} \, \tilde{r}, \\
\hat{k} := k + 2P \, B + 2 \, P_2 \, H + 2 \, F' \, P \, H + \tilde{k} \, P_{11}, \\
\hat{r} > 0 \;\; a.e. \;\; t \in [0,T] \;\; a.s., \\
P(T) = \; S \quad a.s.,
\end{cases} \tag{6}
$$

where P_{11} is the first element of the matrix P. We also introduce the following vector-valued linear backward stochastic differential equation:

$$
\begin{cases}
dY = Y_1 dt + Y_2 \, dW_2, \quad t \in [0,T], \\
Y_1 := -(h + 2PC + 2P_2K + 2FPK + \tilde{a} \; P_{11} \\
\qquad +A'Y + F'Y_2 - \hat{k}\hat{b}/2\hat{r}), \\
Y(T) = \; v \quad a.s..
\end{cases} \tag{7}
$$

Assumption 1. *There exist unique solution pairs* $(P(\cdot), P_2(\cdot))$ *and* $(Y(\cdot), Y_2(\cdot))$ *to equations (6) and (7), respectively.*

The Riccati BSDE (6) due to the term P_{11}, is not of the type that appears in the stochastic LQ control problem (compare, for example, with [27], [28]). However, our next result shows that the Riccati BSDE (6), in one case of coefficients, can be rewritten in a form to which the known results of Riccati BSDE of stochastic LQ control can be applied.

Theorem 1. *If* $k(t) = \tilde{k}(t) = 0$, $r(t) > 0$, $\tilde{r}(t) \geq 0$, $\tilde{Q}(t) \geq 0$, $Q(t) \geq 0$, *for all* $t \in [0,T]$ *a.s.,* $S \geq 0$ *a.s., then there exist unique solution pairs* $(P(\cdot), P_2(\cdot)) \in L^\infty_{\mathcal{F}_2}(0,T;\mathbb{S}^{2\times 2}) \times L^2_{\mathcal{F}_2}(0,T;\mathbb{S}^{2\times 2})$ *and* $(Y(\cdot), Y_2(\cdot)) \in L^2_{\mathcal{F}_2}(0,T;\mathbb{R}^2) \times$

$L^2_{\mathcal{F}_2}(0,T;\mathbb{R}^2)$ *to the Riccati BSDE* (6) *and linear BSDE* (7)*, respectively.*

Proof. Let $(W_i(t), t \geq 0)$, $i = 3, ..., 6$, be standard Brownian motions defined on the the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that Brownian motions $(W_i(t), t \geq 0)$, $i = 1, ..., 6$, are independent. The filtration $(\widetilde{\mathcal{F}}(t), t \geq 0)$, is defined as the augmentation of $\sigma\{W_i(s) : i = 1, \ldots, 6;\ \ 0 \leq s \leq t\}$ by all the $\mathbb{P}$-null sets of $\mathcal{F}$. Also let: $\tilde{C}_1 := T_1 \tilde{Q}^{1/2}$, $\tilde{C}_2 := F$, $\tilde{C}_3 := T_2 \tilde{Q}^{1/2}$, $\tilde{C}_4 := T_1$, $\tilde{C}_5 := T_2$, $\tilde{C}_6 := 0$, $\tilde{D}_1 := 0$, $\tilde{D}_2 := H$, $\tilde{D}_3 := 0$, $\tilde{D}_4 := 0$, $\tilde{D}_5 := 0$, $\tilde{D}_6 := T_3$, where

$$T_1 := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ T_2 := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ T_3 := \sqrt{\tilde{r}}\ \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

We further define the processes:

$$\begin{aligned}
\mathcal{N}(\tilde{K}) &:= r + \sum_{i=1}^{6} \tilde{D}_i' \tilde{K}\, \tilde{D}_i, \\
\mathcal{M}(\tilde{K}, \tilde{L}) &:= \tilde{K}\, B + \sum_{i=1}^{6} \tilde{C}_i' \tilde{K}\, \tilde{D}_i + \sum_{i=1}^{6} \tilde{L}_i\, \tilde{D}_i, \\
\mathcal{G}(\tilde{K}, \tilde{L}) &:= A'\, \tilde{K} + \tilde{K}\, A + Q + \sum_{i=1}^{6} \tilde{C}_i' \tilde{K}\, \tilde{C}_i \\
&\quad + \sum_{i=1}^{6} (\tilde{C}_i' \tilde{L}_i + \tilde{L}_i\, \tilde{C}_i) \\
&\quad - \mathcal{M}(\tilde{K}, \tilde{L}) \mathcal{N}^{-1}(\tilde{K}) \mathcal{M}'(\tilde{K}, \tilde{L}).
\end{aligned}$$

From [27], [28], it follows that the Riccati BSDE:

$$\begin{cases}
d\tilde{K} = -\mathcal{G}(\tilde{K}, \tilde{L})\, dt + \displaystyle\sum_{i=1}^{6} \tilde{L}_i\, dW_i, \quad t \in [0, T], \\
\tilde{K}(T) = S \quad a.s., \\
\mathcal{N}(\tilde{K}) > 0, \quad a.e.\ \ t \in [0, T]\ \ a.s.,
\end{cases} \tag{8}$$

has a unique solution $\tilde{K}(\cdot) \in L^{\infty}_{\widetilde{\mathcal{F}}}(0,T;\mathbb{S}^{2\times 2})$, $\tilde{L}_i(\cdot) \in L^2_{\widetilde{\mathcal{F}}}(0,T;\mathbb{S}^{2\times 2})$, $i = 1, ..., 6$. However, as all coefficients of (8) are $\mathcal{F}_2$-adapted, and S is $\mathcal{F}_2(T)$-measurable, then so must be the solution $(\tilde{K}(t), t \in [0,T])$. This implies that $(\tilde{L}_2(t), t \in [0,T])$ is $\mathcal{F}_2$-adapted and $\tilde{L}_1(t) = \tilde{L}_3(t) = \tilde{L}_4(t) = \tilde{L}_5(t) = \tilde{L}_6(t) =$

0 for $a.e.$ $t \in [0,T]$ $a.s.$. This further means that $(\tilde{K}(\cdot), \tilde{L}_2(\cdot))$ is the unique solution pair to the equation:

$$\begin{cases} d\tilde{K} = -[A'\,\tilde{K} + \tilde{K}\,A + Q + \tilde{L}_2\,F + F'\,\tilde{L}_2 \\ +F'\,\tilde{K}\,F + \tilde{K}_{11}\,\tilde{Q} \\ -(\tilde{K}\,B + \tilde{L}_2\,H + F'\tilde{K}\,H)\,(\tilde{K}\,B + \tilde{L}_2\,H \\ +F'\tilde{K}\,H)'(r + H'\,\tilde{K}\,H + \tilde{K}_{11}\,\tilde{r})^{-1}]dt \\ +\tilde{L}_2\,dW_2, \quad t \in [0,T], \\ r + H'\,\tilde{K}\,H + \tilde{K}_{11}\,\tilde{r} > 0, \quad a.e. \;\; t \in [0,T] \quad a.s., \\ \tilde{K}(T) = \; S \quad a.s., \end{cases} \tag{9}$$

where K_{11} is the first element of the matrix K. However, this is just equation (6), and thus its unique solvability is established. On the other hand, equation (7) is a linear BSDE with bounded coefficients, and it is well-known that it has a unique solution pair (see, for example, [33]). $\square$

We can now define the set of admissible controls $\mathcal{A}$ as the set of all $\mathcal{F}$-adapted one-dimensional processes u under which equation (3) has a unique real strong solution, and satisfies the following integrability requirements:

$$\mathbb{E} \int_0^T \left(x'\,P\,N + Y'\,N\right)\,dW_1 = 0, \tag{10}$$

$$\begin{aligned} &\mathbb{E} \int_0^T [(x'\,F' + uH' + K')Px + x'P_2x + x'P(Fx + Hu \\ &+K) + Y_2'x + Y'(FX + Hu + K)]dW_2 = 0. \end{aligned} \tag{11}$$

The control process u^* is defined as an affine state-feedback control law given as (for $t \in [0,T]$):

$$u^*(t) := -\frac{1}{2}\, \hat{r}^{-1}(t)\, [\hat{k}'(t)\, x(t) + \hat{b}(t)], \tag{12}$$

where $\hat{b}(t) := f(t) + P_{11}(t)\tilde{b}(t) + Y'(t)B(t) + Y_2'(t)\, H(t)$.

Theorem 2. *If $u^*(\cdot) \in \mathcal{A}$, then u^* the unique solution to the optimal stochastic control problem* (5)*. The corresponding optimal cost functional is: :*

$$\begin{aligned} J(u^*(\cdot)) \;&=\; x'(0)\; P(0)\, x(0) + Y'(0)\, x(0) + \mathbb{E}\int_0^T (2\, H'\, P\, K \\ &+\;\; K'\, P\, K + P_{11}\, \tilde{C} + Y'\, C + Y_2'\, K - \hat{r}^{-1}\hat{b}^2/4)\, dt. \end{aligned}$$

Proof. By Itô's product rule, we derive the following differentials:

$$d(P\, x) = (dP)\; x + P\; dx + (dP)\; dx$$

$$= P_1\; x\, dt + P_2\; x\; dW_2 + P\; (A\, x + B\, u + C)\, dt$$

$$+ P\, N\, dW_1 + P\; (F\, x + H\, u + K)\, dW_2$$

$$+ P_2\, (F x + H\, u + K)\; dt,$$

$$d(x'\, P\, x) = (dx')\, P\, x + x'\; d(Px) + (dx')\, d(Px)$$

$$= (x'\,A' + u\,B' + C')P\,x\,dt + N'\,P\,x\,dW_1 + (x'\,F'$$

$$+u\,H' + K')\,P\,x\,dW_2 + x'\,P_1\,x\;dt + x'\,P_2\,x\,dW_2$$

$$+x'\,P\,(Ax{+}Bu{+}C)\,dt{+}x'\,P\,N\,dW_1{+}x'P(Fx{+}K$$

$$+Hu)\,dW_2{+}x'\,P_2\,(Fx{+}Hu{+}K)\,dt{+}N'\,P\,N\,dt$$

$$+(x'\,F' + u\,H' + K')[P_2\,x + P(F\,x + H\,u + K)]\,dt$$

$$= [x'\,(A'\,P + P_1 + P\,A + P_2\,F + F'\,P_2 + F'\,P\,F$$

$$+P_{11}\,\tilde{Q})\,x + u(2\,B'\,P + 2\,H'\,P_2 + 2\,H P\,F$$

$$+P_{11}\,\tilde{k}')\,x + u^2(H'\,P\,H + P_{11}\,\tilde{r})$$

$$+(2\,C'\,P + 2\,K'\,P_2\; + 2\,K'\,P\,F + P_{11}\,\tilde{a}')\,x$$

$$+P_{11}\,\tilde{b}\,u + 2\,H'\,P\,K + K'\,P\,K + P_{11}\,\tilde{C}]\;\,dt$$

$$+x'\,P\,N\,dW_1 + [(x'\,F' + u\,H' + K')\,P\,x$$

$$+x'\,P_2\,x + x'\,P(Fx + H\,u + K)]\,dW_2, \tag{13}$$

$$d(Y'x) = (dY')x + Y'dx + (dY')dx = [(Y_1' + Y'A$$

$$+Y_2'F)x + (Y'B + Y_2'H)u + Y'C + Y_2'K]dt$$

$$+\,Y'\,N\,dW_1 + [Y_2'\,x + Y'(F\,x + \;H\,u + K)]\,dW_2. \tag{14}$$

By integrating both sides of (13) and (14) from 0 to T, and then taking the expectation, we obtain the following for all $u(\cdot) \in \mathcal{A}$, respectively:

$$\mathbb{E}\big[x'(T)\, s\, x(T)\big] = x'(0)\, P(0)\, x(0) + \mathbb{E}\int_0^T \big[x'\,(A'\,P$$

$$+P_1 + P\,A + P_2\,F + F'\,P_2 + F'\,P\,F + P_{11}\,\tilde{Q})\,x$$

$$+u\,(2\,B'\,P + 2\,H'\,P_2 + 2\,H'\,P\,F + P_{11}\,\tilde{k}')\,x$$

$$+u^2(H'\,P\,H + P_{11}\,\tilde{r}) + (2\,C'\,P + 2\,K'\,P_2 + 2\,K'\,P\,F$$

$$+P_{11}\,\tilde{a}')\,x + P_{11}\,\tilde{b}\,u + 2\,H'\,P\,K + K'\,P\,K + P_{11}\,\tilde{C}\big]\,dt,$$

$$\mathbb{E}[\,v'\,x(T)] = Y'(0)\,x(0) + \mathbb{E}\int_0^T [(Y_1' + Y'\,A + Y_2'\,F)x$$

$$+(Y'\,B + Y_2'\,H\big)\,u + Y'\,C + Y_2'\,K]\,dt.$$

The cost functional $J(u(\cdot))$, for all $u(\cdot) \in \mathcal{A}$, can now be written as:

$$J(u(\cdot)) = x'(0)\; P(0)\, x(0) + Y'(0)\, x(0) + \mathbb{E}\int_0^T [\, x'\,(Q$$

$$+A'\,P + P_1 + P\,A + P_2\,F + F'\,P_2 + F'\,P\,F + P_{11}\,\tilde{Q})\,x$$

$$+u(k' + 2B'P + 2H'P_2 + 2H'PF + P_{11}\tilde{k}')\,x$$

$$+u^2\,(r + H'\,P\,H + P_{11}\,\tilde{r}) + (h' + 2\,C'\,P + 2\,K'\,P_2$$

$$+2\,K'\,P\,F + P_{11}\,\tilde{a}' + \,Y_1' + Y'\,A + Y_2'\,F)\,x$$

$$+(f + P_{11}\,\tilde{b} + Y'\,B + Y_2'\,H)\,u$$

$$+2\,H'\,P\,K + K'\,P\,K + P_{11}\,\tilde{C} + Y'\,C + Y_2'\,K]\,dt.$$

The terms in the above integrand that depend explicitly on the control process u can be put together, and then, by the completion of squares method, be written as:

$$u^2\,(r + H'\,P\,H + P_{11}\,\tilde{r}) + u\,(k' + 2\,B'\,P + 2\,H'\,P_2$$

$$+2\,H'\,P\,F + P_{11}\,\tilde{k}')\,x + u\,(f + P_{11}\,\tilde{b} + Y'\,B + Y_2'\,H)$$

$$= \hat{r}\,u^2\; + u\;(\hat{k}'\,x + \hat{b})$$

$$= \hat{r}\;\left(u + \frac{\hat{k}'\,x + \hat{b}}{2\hat{r}}\right)^2 - \frac{1}{4\,\hat{r}}(x'\,\hat{k}\,\hat{k}'\,x + 2\,\hat{b}\,\hat{k}'\,x + \hat{b}^2).$$

The cost functional $J(u(\cdot))$, for all $u(\cdot) \in \mathcal{A}$, can now be further written as:

$$
\begin{aligned}
& J(u(\cdot)) = x'(0)P(0)x(0) + Y'(0)x(0) + \mathbb{E}\int_0^T \{2H'PK \\
& + K'\,P\,K + P_{11}\,\tilde{C} + Y'\,C + Y_2'\,K - \hat{r}^{-1}(\hat{b}^2/4) \\
& + \hat{r}[u + (\hat{k}'\,x + \hat{b})/2\hat{r}]^2\}dt \\
& \geq x'(0)\;P(0)\,x(0) + Y'(0)\,x(0) + \mathbb{E}\int_0^T [2\,H'\,P\,K \\
& + K'\,P\,K + P_{11}\,\tilde{C} + Y'\,C + Y_2'\,K - \hat{r}^{-1}\hat{b}^2/4]dt
\end{aligned}
$$

This lower bound is achieved if and only if $u(t) = u^*(t)$ for *a.e.* $t \in [0, T]$ *a.s.*. $\square$

3. Application to optimal investment

In order to apply Theorem 2, it is required to check whether or not $u^*(\cdot) \in \mathcal{A}$. In this section, we give an application to an optimal investment problem, which generalises a result of [11] to the market with random coefficients, where this assumption is verified and hence Theorem 2 can be applied. Thus, consider a market of a bank account with price S_0 and of a stock with price S_1, that are solutions to the following equations (for $t \in [0, T]$):

$$
\begin{cases}
dS_0 = S_0 \rho dt, \\
dS_1 = S_1(\mu dt + \sigma dW_2), \\
S_0(0) > 0 \quad \text{and} \quad S_1(0) > 0 \quad \text{are given,}
\end{cases}
$$

where the interest rate ρ, the appreciation rate μ, and the volatility σ are given market coefficients. Further consider an investor with the initial wealth $y_0 > 0$ that holds $v_0(t)$ and $v_1(t)$ number of shares at time t in the bank account and in the stock, respectively. The value of investor's portfolio at time t is thus $y(t) := v_0(t)S_0(t) + v_1(t)S_1(t)$. If $u_1(t) := v_1(t)S_1(t)$ denotes the amount of investor's wealth invested in the stock, then the portfolio is said to be *self-financing* if (for $t \in [0,T]$):

$$\begin{cases} dy = (\rho y + \eta u_1)dt + \sigma u_1 W_2, \\ \\ y(0) = y_0, \end{cases} \tag{15}$$

where $\eta(t) := \mu(t) - r(t)$. The *optimal investment* problem with a *logarithmic utility* from terminal wealth is the following optimal control problem:

$$\begin{cases} \max\limits_{u_1(\cdot)\in\mathcal{A}_\ell} \mathbb{E}[\log(y(T))], \\ \text{s.t. } (15), \end{cases}, \tag{16}$$

where $\mathcal{A}_\ell$ is a suitable set of admissible controls, which in particular ensures that $y(t) > 0$ for all $t \in [0,T]$ (see, for example, [15], [17], for a textbook account of the optimal investment problems). In [11], the problem (16) was solved under the assumption of η and σ being deterministic, and the interest rate ρ being the solution to the CIR equation:

$$\begin{cases} d\rho = (\alpha\rho + \beta)dt + \sqrt{\rho}dW_1, \quad t \in [0,T], \\ \\ \rho(0) > 0, \quad \text{is given}, \end{cases} \tag{17}$$

for some constants α and β. In this section, as an application of our results, we solve the problem (16) by assuming that $\eta(\cdot) \in L^\infty_{\mathcal{F}_2}(0,T;\mathbb{R})$, $0 < \sigma(\cdot) \in L^\infty_{\mathcal{F}_2}(0,T;\mathbb{R})$, $\sigma^{-1}(\cdot) \in L^\infty_{\mathcal{F}_2}(0,T;\mathbb{R})$, and thus generalise the result of [11]. As in [11], we define the control variable and the two system states as (for $t \in [0,T]$):

$$\begin{aligned} u(t) &:= u_1(t)/y(t), \quad x_1(t) := \rho(t), \\ x_2(t) &:= \log[y(t)] + \int_0^t \frac{1}{2}\sigma^2 u^2(s)ds. \end{aligned} \tag{18}$$

The state equations are now obtained from (17) and (15) (by applying Itô's formula to find the differential of $\log[y(t)]$), respectively, as:

$$\begin{cases} dx_1 = (\alpha x_1 + \beta)dt + \sqrt{x_1}dW_1, \\ dx_2 = (x_1 + \eta u)dt + \sigma u dW_2, \\ x_1(0) = \rho(0), \quad x_2(0) = \log(y_0). \end{cases} \tag{19}$$

It follows from (18) that $-\mathbb{E}[\log(y(T))]$ can be written as:

$$\mathbb{E}[\log(y(T))] = \mathbb{E}\left[\int_0^T \frac{1}{2}\sigma^2 u^2(s)ds - x_2(T)\right]. \tag{20}$$

It is thus clear that the problem of minimizing (20) subject to (19), which is equivalent to (16), is an example of our optimal control problem (5), and thus can be solved by applying Theorem 2. In this case, we have:

$$\begin{aligned}
&\tilde{Q} = 0, \quad \tilde{k} = 0, \quad \tilde{r} = 0, \quad \tilde{a}' = [1 \quad 0], \quad \tilde{b} = 0, \\
&\tilde{c} = 0, \quad M = \sqrt{x_1}, \quad Q = 0, \quad k = 0, \quad r = \frac{\sigma^2}{2}, \\
&f = 0, \quad h = 0, \quad S = 0, \quad v' = [0 \quad -1], \\
&A = \begin{bmatrix} \alpha & 0 \\ 1 & 0 \end{bmatrix}, B := \begin{bmatrix} 0 \\ \eta \end{bmatrix}, C := \begin{bmatrix} \beta \\ 0 \end{bmatrix}, H := \begin{bmatrix} 0 \\ \sigma \end{bmatrix}, \\
&K := \begin{bmatrix} 0 \\ 0 \end{bmatrix}, N := \begin{bmatrix} \sqrt{x_1} \\ 0 \end{bmatrix}, F := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.
\end{aligned}$$

As Q and S are zero, the solution pair to the corresponding Riccati BSDE (6) is $(P(t), P_2(t)) = (0, 0)$, $t \in [0, T]$, and from Theorem 1 we know that this pair is unique. The corresponding linear BSDE (7) is:

$$\begin{cases} dY = -(A'Y + F'Y_2)\, dt + Y_2\, dW_2, \quad t \in [0, T], \\ Y(T) = v \quad a.s.. \end{cases} \tag{21}$$

As A, F and v are constant, the unique solution pair to this equation is $(Y(t), Y_2(t)) = (e^{A'(T-t)}v, 0)$, $t \in [0, T]$. We also have that $u^*(t) = -\sigma^{-2}(t)Y'(t)B(t)$, $t \in [0, T]$. By Theorem 2, for u^* to be the required optimal control, it must be an admissible control. The equation of the state x_1 in (19) is the CIR model, which it is known to have a unique strong solution (see, for example, [26]). Under control u^*, the equation of the state x_2 in (19) is a linear stochastic differential equation with bounded coefficients, and thus has a unique strong solution (see, for example, [33]). It remains to show that the corresponding integrability requirements (10) and (11) also hold. The requirement (10) is:

$$\mathbb{E}\int_0^T Y'(t)N(t)dW_1(t) = 0,$$

which holds as its integrand is a square-integrable process (as x_1 is positive with a finite expectation). The requirement (11) is:

$$\mathbb{E}\int_0^T Y'(t)H(t)u^*(t)dW_2(t) = 0.$$

which hold as its integrand is uniformly bounded. By Theorem 2, we conclude that u^* is the unique solution to the optimal investment problem of minimizing (20) subject to (19).

4. Conclusions

We have considered an optimal regulator problem for a class of stochastic control systems with random coefficients that contain square-root nonlinearities in their diffusion terms. An explicit closed-form solution to this problem is obtained as an affine state-feedback control, which is expressed in terms of the solution pairs to ceratin Riccati and linear BSDEs. The solvability of such equations is also established for a class of coefficients, and the results applied to an optimal investment problem in market with random coefficients. Although we have considered only the two-dimensional systems, it is evident that the multi-dimensional systems can be considered similarly. Two further generalisations that can be considered is the weakening of the positivity assumption on the coefficient $\hat{r}$ to its nonnegativity, which will lead to an indefinite optimal control problem, as well as more general random coefficients, e.g. the case when the coefficients are $\mathcal{F}$ adapted, rather than only $\mathcal{F}_2$ adapted.

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