# Tight Bounds For Quantum Phase Estimation and Related Problems

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## Abstract

Phase estimation, due to Kitaev [arXiv'95], is one of the most fundamental subroutines in quantum computing, used in Shor's factoring algorithm, optimization algorithms, quantum chemistry algorithms, and many others. In the basic scenario, one is given black-box access to a unitary $U$, and an eigenstate $|\psi\rangle$ of $U$ with unknown eigenvalue $e^{i\theta}$, and the task is to estimate the eigenphase $\theta$ within $\pm\delta$, with high probability. The repeated application of $U$ and $U^{-1}$ is typically the most expensive part of phase estimation, so for us the *cost* of an algorithm will be that number of applications.

Motivated by the "guided Hamiltonian problem" in quantum chemistry, we tightly characterize the cost of several variants of phase estimation where we are no longer given an arbitrary eigenstate, but are required to estimate the *maximum* eigenphase of $U$, aided by *advice* in the form of states (or a unitary preparing those states) which are promised to have at least a certain overlap $\gamma$ with the top eigenspace. We give algorithms and matching lower bounds (up to logarithmic factors) for all ranges of parameters. We show a crossover point below which advice is not helpful: $o(1/\gamma^2)$ copies of the advice state (or $o(1/\gamma)$ applications of an advice-preparing unitary) are not significantly better than having no advice at all. We also show that having knowledge of the eigenbasis of $U$ does not significantly reduce cost. Our upper bounds use the subroutine of *generalized maximum-finding* of van Apeldoorn, Gilyén, Gribling, and de Wolf [Quantum'20], the state-based Hamiltonian simulation of Lloyd, Mohseni, and Rebentrost [Nature Physics'13], and several other techniques. Our lower bounds follow by reductions from a *fractional* version of the Boolean OR function *with advice*, which we lower bound by a simple modification of the adversary method of Ambainis [JCSS'02]. As an immediate consequence we also obtain a lower bound on the complexity of the Unitary recurrence time problem, matching an upper bound of She and Yuen [ITCS'23] and resolving an open question posed by them.

Lastly, we study how efficiently one can reduce the error probability in the basic phase-estimation scenario. We show that an algorithm solving phase estimation to precision $\delta$ with error probability at most $\varepsilon$ must have cost $\Omega\left(\frac{1}{\delta}\log\frac{1}{\epsilon}\right)$, matching the obvious way to error-reduce the basic constant-error-probability phase estimation algorithm. This contrasts with some other scenarios in quantum computing (e.g. search) where error-reduction costs only a factor $O(\sqrt{\log(1/\epsilon)})$. Our lower bound technique uses a variant of the polynomial method with trigonometric polynomials.

**2012 ACM Subject Classification** Theory of computation → Oracles and decision trees; Theory of computation → Quantum complexity theory

**Keywords and phrases** Phase estimation, quantum computing

**Related Version** A full version of this paper is available at [25].
*Full Version*: https://arxiv.org/abs/2305.04908

**Funding** *Nikhil S. Mande*: Part of this work was done while the author was a postdoc at QuSoft and CWI, Amsterdam
*Ronald de Wolf*: Partially supported by the Dutch Research Council (NWO/OCW), as part of the Quantum Software Consortium programme (project number 024.003.037).

**Acknowledgements** We thank Jordi Weggemans for useful comments and for a pointer to [21].

# 1 Introduction

## 1.1 Phase estimation

Kitaev [19] gave an elegant and efficient quantum algorithm for the task of *phase estimation* nearly 30 years ago. The task is easy to state: given black-box access to a unitary and an eigenstate, estimate the phase of the associated eigenvalue. Roughly speaking, the standard algorithm for this task sets up a superposition involving many different powers of the unitary to extract many different powers of the eigenvalue, and then uses a quantum Fourier transform to turn that into an estimate of the eigenphase.[1] Many of the most prominent quantum algorithms can either be phrased as phase estimation, or use phase estimation as a crucial subroutine. Some examples are Shor's period-finding algorithm [30] as presented in [10]; approximate counting [6] can be done using phase estimation on the unitary of one iteration of Grover's search algorithm [16], which also recovers the $O(\sqrt{N})$ complexity for searching an $N$-element unordered search space; the HHL algorithm for solving linear systems of equations estimates eigenvalues in order to invert them [17]. Applications of phase estimation in quantum chemistry are also very prominent, as discussed below.

More precisely, we are given black-box access to an $N$-dimensional unitary $U$ (and a controlled version thereof) and a state $|\psi\rangle$ that satisfies $U|\psi\rangle = e^{i\theta}|\psi\rangle$. Our goal is to output (with probability at least $2/3$) a $\tilde{\theta} \in [0, 2\pi)$ such that $|\tilde{\theta} - \theta|$ is at most $\delta$ in $\mathbb{R}$ mod $2\pi$. In the basic scenario we are given access to one copy of $|\psi\rangle$, and are allowed to apply $U$ and its inverse. Since the repeated applications of $U$ and $U^{-1}$ are typically the most expensive parts of algorithms for phase estimation, the *cost* we wish to minimize is the number of applications of $U$ and $U^{-1}$. We are additionally allowed arbitrary unitaries that do not depend on $U$, at no cost. Kitaev's algorithm has cost $O(1/\delta)$.

## 1.2 Phase estimation with advice

One of the core problems in quantum chemistry is the following: given a classical description of some Hamiltonian $H$ (for instance an "electronic structure" Hamiltonian in the form of a small number of local terms), estimate its *ground state energy*, which is its smallest eigenvalue. If $H$ is normalized such that its eigenvalues are all in $[0, 2\pi)$ and we define the unitary $U = e^{iH}$ (which has the same eigenvectors as $H$, with an eigenvalue $\lambda$ of $H$ becoming the eigenvalue $e^{i\lambda}$ for $U$), then finding the ground state energy of $H$ is equivalent to finding the smallest eigenphase of $U$. If we are additionally given a *ground state* $|\psi\rangle$ (i.e., an eigenstate corresponding to the smallest eigenphase), then phase estimation is tailor-made to estimate the ground state energy. However, in quantum chemistry it is typically hard to prepare the ground state of $H$, or even something close to it. What can sometimes be done is the preparation of some quantum state that has some non-negligible "overlap" $\gamma$ with the ground space, for instance the "Hartree-Fock state". We will call such a state an *advice state*. In the complexity-theoretic context, this problem of ground state estimation for a local Hamiltonian given an advice state, is known as the "guided local Hamiltonian problem", and has received quite some attention recently [13, 8, 12, 32] because of its connections with quantum chemistry as well as deep complexity questions such as the PCP conjecture. These complexity-theoretic results typically focus on the BQP-completeness of certain special cases

[1] An added advantage of the standard algorithm for phase estimation is that it can also work with a quantum Fourier transform that is correct on average rather than in the worst case [23]. However, there are also approaches to phase estimation that avoid the QFT altogether, see e.g. [28].

of the guided local Hamiltonian problem, and don't care about polynomial overheads of the cost in the number of qubits $\log N$ and in the parameters $\delta$ and $\gamma$. In contrast, we care here about getting essentially optimal bounds on the cost of phase estimation in various scenarios.

To be more precise, suppose our input unitary is $U = \sum_{j=0}^{N-1} e^{i\theta_j}|u_j\rangle\langle u_j|$ with each $\theta_j \in [0, 2\pi)$. Let $\theta_{\max} = \max_{j\in\{0,1,\ldots,N-1\}} \theta_j$ denote the maximum eigenphase, and let $S$ denote the space spanned by all eigenstates with eigenphase $\theta_{\max}$, i.e., the "top eigenspace". Advice is given in the form of a state $|\alpha\rangle$ whose projection on $S$ has squared norm at least $\gamma^2$: $\|P_S|\alpha\rangle\|^2 \geq \gamma^2$. Note that if $S$ is spanned by a single eigenstate $|u_{\max}\rangle$, then this condition is the same as $|\langle\alpha|u_{\max}\rangle| \geq \gamma$, which is why we call $\gamma$ the *overlap* of the advice state with the target eigenspace. The task $\mathsf{maxQPE}_{N,\delta}$ is to output, with probability at least 2/3, a $\delta$-precise (in $\mathbb{R}$ mod $2\pi$) estimate of $\theta_{\max}$.[2]

We will distinguish between the setting where the advice is given in the form of a number of copies of the advice state $|\alpha\rangle$, or the potentially more powerful setting where we can apply (multiple times) a *unitary* $A$ that prepares $|\alpha\rangle$ from some easy-to-prepare initial state, say $|0\rangle$. We would have such a unitary $A$ for instance if we have a procedure to prepare $|\alpha\rangle$ ourselves in the lab. We can also distinguish between the situation where the eigenbasis $|u_0\rangle, \ldots, |u_N\rangle$ of $U$ is known (say, the computational basis where $|u_j\rangle = |j\rangle$) and the potentially harder situation where the eigenbasis is unknown. These two binary distinctions give us four different settings. For each of these settings we determine essentially optimal bounds on the cost of phase estimation, summarized in Table 1.

| Row | Basis | Access to advice | Number of accesses | Upper bound | Lower bound |
|---|---|---|---|---|---|
| 1 | known | state | $o\left(\frac{1}{\gamma^2}\right)$ | $\widetilde{O}\left(\frac{\sqrt{N}}{\delta}\right)$, Lemma 20 | $\Omega\left(\frac{\sqrt{N}}{\delta}\right)$, Lemma 13 |
| 2 | known | state | $\Omega\left(\frac{1}{\gamma^2}\right)$ | $\widetilde{O}\left(\frac{1}{\gamma\delta}\right)$, Lemma 22 | $\Omega\left(\frac{1}{\gamma\delta}\right)$, Lemma 14 |
| 3 | unknown | state | $o\left(\frac{1}{\gamma^2}\right)$ | $\widetilde{O}\left(\frac{\sqrt{N}}{\delta}\right)$, Lemma 20 | $\Omega\left(\frac{\sqrt{N}}{\delta}\right)$, Lemma 13 |
| 4 | unknown | state | $\Omega\left(\frac{1}{\gamma^2}\right)$ | $\widetilde{O}\left(\frac{1}{\gamma\delta}\right)$, Lemma 22 | $\Omega\left(\frac{1}{\gamma\delta}\right)$, Lemma 14 |
| 5 | known | unitary | $o\left(\frac{1}{\gamma}\right)$ | $\widetilde{O}\left(\frac{\sqrt{N}}{\delta}\right)$, Lemma 20 | $\Omega\left(\frac{\sqrt{N}}{\delta}\right)$, Lemma 15 |
| 6 | known | unitary | $\Omega\left(\frac{1}{\gamma}\right)$ | $\widetilde{O}\left(\frac{1}{\gamma\delta}\right)$, Lemma 21 | $\Omega\left(\frac{1}{\gamma\delta}\right)$, Lemma 16 |
| 7 | unknown | unitary | $o\left(\frac{1}{\gamma}\right)$ | $\widetilde{O}\left(\frac{\sqrt{N}}{\delta}\right)$, Lemma 20 | $\Omega\left(\frac{\sqrt{N}}{\delta}\right)$, Lemma 15 |
| 8 | unknown | unitary | $\Omega\left(\frac{1}{\gamma}\right)$ | $\widetilde{O}\left(\frac{1}{\gamma\delta}\right)$, Lemma 21 | $\Omega\left(\frac{1}{\gamma\delta}\right)$, Lemma 16 |

**Table 1** Our results for the cost of $\mathsf{maxQPE}_{N,\delta}$. We assume $\gamma > 1/\sqrt{N}$ since a random state has overlap $1/\sqrt{N}$ with the target eigenspace with high probability, and such a state can be prepared at no cost. The 'Basis' column indicates whether the eigenbasis of $U$ is known; 'Access to advice' indicates whether we get copies of the advice state or a unitary to prepare it; 'Number of accesses' refers to the number of accesses to advice that we have. The last two columns show our bounds with references to the lemmas where they are stated and proved. The $\widetilde{O}(\cdot)$ in the upper-bound column hides a factor $\log N$ for the odd-numbered rows, and $\log(1/\gamma)$ for the even-numbered rows.

Let us highlight some interesting consequences of our results. First, a little bit of advice is no better than no advice: the upper bounds in the odd-numbered rows of Table 1 are actually obtained by algorithms that don't use the given advice ($o(1/\gamma^2)$ copies of $|\alpha\rangle$ or $o(1/\gamma)$ applications of $A$ and $A^{-1}$) at all, yet their costs essentially match the lower bounds for algorithms that use advice.

[2] It doesn't really matter, but we focus on the *maximum* rather than minimum eigenphase of $U$ because eigenphase 0 (i.e., eigenvalue 1) is a natural baseline, and we are looking for the eigenphase furthest away from this baseline.

We remark here that the same proofs yield the same asymptotic lower bounds for algorithms with access to at most $c/\gamma^2$ advice states for Theorem 12, Rows 1 and 3 of Table 1, and for algorithms with access to at most $c/\gamma$ advice unitaries for Rows 5 and 7 of Table 1, where $c$ is a suitably small constant. We chose to use $o(\cdot)$ to avoid clutter.

A second interesting consequence is that too much advice is no better than a moderate amount of advice: the upper bounds in Rows 2 and 4 use $O(1/\gamma^2)$ advice states, and the upper bounds in Rows 6 and 8 use $O(1/\gamma)$ advice unitaries, and using more advice does not reduce the cost further. Thirdly, it turns out that knowledge of the eigenbasis of $U$ doesn't really help in reducing the cost: the costs in row 1 and row 3 are the same, and similarly for rows 2 vs. 4, 5 vs. 7 and 6 vs. 8.

Our upper bounds use the subroutine of *generalized maximum-finding* of van Apeldoorn, Gilyén, Gribling, and de Wolf [2] which allows us to find maximum values in the second register of a two-register superposition even when the first of these two registers has an unknown basis. We derive the upper bound of row 4 from the upper bound of row 8 by using roughly $1/\gamma$ copies of $|\alpha\rangle$ to simulate one reflection around the state $|\alpha\rangle = A|0\rangle$, using the techniques of Lloyd, Mohseni, and Rebentrost [24].[3] Our lower bounds follow from reductions from a fractional version of the Boolean OR function with advice. We show a lower bound for this by a simple modification of the adversary method [1] taking into account the input-dependent advice in the initial state.

## Comparison with related work

Some of the results in our table were already (partially) known. A cost-$\widetilde{O}(\sqrt{N}/\delta)$ algorithm for the adviceless setting with unknown eigenbasis (implying the upper bounds of rows 1, 3, 5, 7) was originally due to Poulin and Wocjan [27], and subsequently improved in the log-factors by van Apeldoorn et al. [2]; the latter algorithm is basically our proof of Lemma 20. Lin and Tong [21] (improving upon [11]) studied the situation with an advice-preparing unitary. Their setting is slightly different from ours, they focus on preparing the ground state[4] of a given Hamiltonian without a known bound on its spectrum, but [21, Theorem 8] implies a cost-$O(\log(1/\gamma)\log(1/\delta)\log\log(1/\delta)/\gamma\delta)$ algorithm for our row 8. Their follow-up paper [22] further reduces the number of auxiliary qubits with a view to near-term implementation, but does not reduce the cost further. Our cost-$O(\log(1/\gamma)/\gamma\delta)$ algorithm is slightly better in the log-factors than theirs, and uses quite different techniques ([21] uses quantum singular value transformation [15]).

On the lower-bound side, $\Omega(1/\delta)$ for the cost of phase estimation has long been known to hold when the success probability is required to be a constant, this follows for instance from the approximate counting lower bound of Nayak and Wu [26] (see also [4]). Lin and Tong [21, Theorem 10] proved lower bounds of $\Omega(1/\gamma)$ and $\Omega(1/\delta)$ on the cost for the setting with known eigenbasis and advice unitary (our row 6, and hence also row 8). This is subsumed by our stronger (and essentially optimal) $\Omega(1/\gamma\delta)$ lower bound in row 6. As far as we are aware, ours is the first paper to systematically tie together these different results and to complete the table with tight upper and lower bounds for the cost in all 8 cases.

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[3] We only stated the cost (number of applications of $U$ and $U^{-1}$) of our algorithms here in the upper-bound column of Table 1. However, one can verify that the gate-complexities of our algorithms are only worse by log-factors: they use three main subroutines, all of which have only small overheads in gate-complexity. These subroutines are basic quantum phase estimation [19], generalized maximum-finding [2], and the simulation of a unitary reflecting about the state $|\alpha\rangle$ given a small number of copies of $|\alpha\rangle$.

[4] Because generalized maximum-finding (Lemma 17) actually outputs a state in addition to an estimate, our algorithms can be modified to also output a state that is close to the top eigenspace of $U$.

Let us also mention some recent work that is not directly covered by our results. First, lower bounds for the slightly unusual small-success-probability regime were recently studied by Lin [20]. Second, there has been work to make phase estimation more efficient in the important special case where the unitary $U = e^{iH}$ is induced by a Hamiltonian $H$ given classically as the sum of relatively simple terms, when the cost of phase estimation interacts with the cost of Hamiltonian simulation. See for instance the recent paper by Wan, Berta, and Campbell [31] and references therein.

#### Application

She and Yuen [29, Theorems 1.6 and 1.7] studied the $(t, \delta)$-*Unitary recurrence time problem*, which is to distinguish whether an input unitary $U$ satisfies $U^t = I$ or $\|U^t - I\| \geq \delta$, promised that one of these is the case (see Definition 7). They proved non-matching upper and lower bounds for the cost of quantum algorithms for this problem (see Theorem 8 in this paper). As an immediate application of our lower bound for fractional OR with advice, we also obtain improved lower bounds for the unitary recurrence time problem that match the upper bound of She and Yuen and answer one of their open problems [29, Section 2].

▶ **Theorem 1** (Lower bound for Unitary recurrence time)**.** *Any quantum algorithm solving the $(t, \delta)$-recurrence time problem for $N$-dimensional unitaries has cost $\Omega(t\sqrt{N}/\delta)$.*

Interestingly, our lower bound uses the adversary method as opposed to their usage of the polynomial method.

### 1.3 Phase estimation with small error probability

For our results in this subsection we revert to the original scenario of phase estimation, where an algorithm is given the actual eigenstate $|\psi\rangle$ as input and the goal is to estimate its eigenphase $\theta$. However, we now consider the regime where we want small error probability $\varepsilon$ rather than constant error probability $1/3$. Let $\mathsf{QPE}_{N,\delta,\varepsilon}$ denote the task of computing, with error probability $\leq \varepsilon$, a $\delta$-approximation of $\theta$. By repeating Kitaev's $O(1/\delta)$-cost phase estimation algorithm $O(\log(1/\varepsilon))$ times and taking the median of the answers, we have the following $\varepsilon$-dependent upper bound.

▶ **Theorem 2** (Kitaev + standard error-reduction)**.** *For all integers $N \geq 2$ and all $\varepsilon \in (0, 1/2), \delta \in [0, 2\pi)$, there exists an algorithm that solves $\mathsf{QPE}_{N,\delta,\varepsilon}$ with cost $O\left(\frac{1}{\delta}\log\frac{1}{\varepsilon}\right)$.*

Grover's algorithm [16] can compute the $\mathsf{OR}_N$ function with error probability $\leq 1/3$ using $O(\sqrt{N})$ queries to its $N$ input bits. Interestingly, there exists an $\varepsilon$-error quantum algorithm for $\mathsf{OR}_N$ with only $O(\sqrt{N\log(1/\varepsilon)})$ queries, which is asymptotically optimal [7], and similarly one can reduce error from $1/3$ to $\varepsilon$ for all symmetric Boolean functions at the expense of only a factor $\sqrt{\log(1/\varepsilon)}$ in the query complexity [33]. This is a speed-up over the naive $O(\log(1/\varepsilon))$ multiplicative overhead. Since optimal quantum algorithms with error probability $1/3$ for $\mathsf{OR}_N$ and for all symmetric functions can be derived from phase estimation, one may ask if one can achieve such an efficient error-reduction for quantum phase estimation as well: is there an algorithm for $\mathsf{QPE}_{N,\delta,\varepsilon}$ of cost $O\left(\frac{1}{\delta}\sqrt{\log(1/\varepsilon)}\right)$? We answer this in the negative, showing Theorem 2 is tight.

▶ **Theorem 3.** *For integers $N \geq 2$ and $\varepsilon, \delta \in (0, 1/2)$,*[5] *every algorithm that solves* $\mathsf{QPE}_{N,\delta,\varepsilon}$ *has cost* $\Omega\left(\frac{1}{\delta}\log\frac{1}{\varepsilon}\right)$.

In particular, this means that the optimal complexity of $\mathsf{OR}_N$ with small error probability $\varepsilon$ of [7] can*not* be derived from a phase estimation routine, in contrast to the case of $\mathsf{OR}_N$ (and search) with constant error probability. To show Theorem 3 we first argue that a cost-$C$ algorithm for $\mathsf{QPE}_{N,\delta,\varepsilon}$ gives us a cost-$C$ algorithm that distinguishes $U = I$ versus $U = I - (1 - e^{i\theta})|0\rangle\langle 0|$ where $\theta \notin [-3\delta, 3\delta] \bmod 2\pi$. We then note that the acceptance probability of such an algorithm can be written as a degree-$2C$ *trigonometric* polynomial in $\theta$, and invoke a known upper bound on the growth of such trigonometric polynomials in order to lower bound their degree.

## 2 Preliminaries

We state the required preliminaries in this section. All logarithms are taken base 2. For a positive integer $N$, $U(N)$ denotes the space of $N$-dimensional unitaries, and $I$ denote the $N$-dimensional Identity matrix (we drop the subscript if the dimension is clear from context).

For a positive integer $N \geq 2$ and a value $\theta \in [0, 2\pi)$, define the $N$-dimensional unitary $U_\theta$ as $U_\theta = I - (1 - e^{i\theta})|0\rangle\langle 0|$. In other words, $U_\theta$ is the diagonal matrix with all 1's except the first entry, which is $e^{i\theta}$. For an integer $j \in \{0, 1, \ldots, N-1\}$ and $\delta \in [0, 2\pi)$, define $M_{j,\delta} = I - (1 - e^{i\delta})|j\rangle\langle j|$.

### 2.1 Model of computation

Here we give a description of our model of computation for all tasks considered in this paper. All problems considered in this paper have the following properties:

- **Input:** An $N$-dimensional unitary $U$. We have access to the input as described below.
- **State space:** The state space of an algorithm comprises two registers: the first register is $N$-dimensional, and the second register is an arbitrarily large workspace.
- **Access to input and allowed operations:** An algorithm $\mathcal{A}$ may apply $U$ and $U^{-1}$ to the first register, and unitaries independent of $U$ to the whole space. It performs a POVM at the end to determine the classical output.
- **Cost of an algorithm:** Total number of applications of $U$ and $U^{-1}$.

Depending on the specific problem under consideration, the following properties are variable.

- **Initial state:** The initial state is assumed to be $|0\rangle|0\rangle$ unless mentioned otherwise.
- **Input promise:** The subset of $U(N)$ (possibly the full set) from which the input is taken.
- **Output:** The output requirement.
- **Advice:** We may be given access to a specific number of "advice states" $|\alpha\rangle$, or access to a specific number of applications of a unitary $A$ that prepares an advice state (e.g., $A|0\rangle = |\alpha\rangle$).

### 2.2 Problems of interest

We list our problems of interest here. All problems fit in the framework of the previous subsection, so we skip descriptions of the input, access to the input and allowed operations, and the workspace.

[5] We require $\delta < 2\pi/5$ for our proof of Claim 23 to work. This requirement can be strengthened a little to $\delta < 2\pi/3$, but we state our theorem with $\delta < 1/2$ for ease of notation.

▶ **Definition 4** (Phase Estimation). *Let $N \geq 2$ be an integer and $\varepsilon, \delta > 0$. The task* $\mathsf{QPE}_{N,\delta,\varepsilon}$ *is:*

- ***Advice:*** *We are given a single state $|\psi\rangle$ (in other words, our starting state is $|\psi\rangle|0\rangle$) with the promise that $U|\psi\rangle = e^{i\theta}|\psi\rangle$.*
- ***Output:*** *With probability at least $1-\varepsilon$, output $\tilde{\theta} \in [0, 2\pi)$ such that $|\tilde{\theta} - \theta| \leq \delta \mod 2\pi$.*

▶ **Definition 5.** *Let $N \geq 2$ be an integer and $\varepsilon, \delta \in (0,1)$. The task* $\mathsf{dist}_{N,\delta,\varepsilon}$ *is:*

- ***Input promise:*** *$U \in \{I, \{U_\theta : \theta \notin [\delta, \delta] \mod 2\pi\}\}$.*
- ***Output:*** *With probability at least $1-\varepsilon$, output 1 if $U = I$, and output 0 otherwise.*

We next define the natural variant of phase estimation that we consider when an algorithm need not be given a state from the target eigenspace.

▶ **Definition 6** (Maximum phase estimation). *Let $N \geq 2$ be an integer and $\delta > 0$. The task* $\mathsf{maxQPE}_{N,\delta}$ *is:*

- ***Input promise:*** *We consider two cases: one where the eigenbasis of $U$ is known, and the other where it is unknown. In the former case, we may assume $U = \sum_{j=0}^{N-1} e^{i\theta_j}|j\rangle\langle j|$. Define $\theta_{\max} = \max_{j \in \{0,1,\ldots,N-1\}} \theta_j \in [0, 2\pi)$.*
- ***Advice:*** *We consider two cases:*
  - *In one case we are given access to advice in the form of a state $|\alpha\rangle$ such that $\|P_S|\alpha\rangle\|^2 \geq \gamma^2$, where $P_S$ denotes the projection on $S$, the space of all eigenstates with eigenphase $\theta_{\max}$. If $S$ is spanned by one $|u_{\max}\rangle$, this requirement is the same as $|\langle\alpha|u_{\max}\rangle| \geq \gamma$.*
  - *In the other case, we have black-box access to a unitary $A$ that prepares such a state $|\alpha\rangle$. We can apply $A$ and $A^{-1}$. As before, $\gamma$ is the* overlap *of $|\alpha\rangle$ with the target eigenspace.*
- ***Number of accesses to advice:*** *We either have 'few' accesses to advice as defined above ($o(1/\gamma^2)$ advice states or $o(1/\gamma)$ advice unitaries), or 'many' accesses to advice ($\Omega(1/\gamma^2)$ advice states or $\Omega(1/\gamma)$ advice unitaries).*
- ***Output:*** *With probability at least $2/3$, output a value in $[\theta_{\max} - \delta, \theta_{\max} + \delta] \mod 2\pi$.*

▶ **Definition 7** (Unitary recurrence time, [29, Definition 1.5]). *For integers $N \geq 2, t \geq 1$ and $\delta \in (0,1)$, the $(t,\delta)$-recurrence time problem is:*

- ***Input promise:*** *Either $U = I$, or $\|U^t - I\| \geq \delta$ in spectral norm.*
- ***Output:*** *With probability at least $2/3$: output 1 if $U = I$, and 0 otherwise.*

The following are the non-matching upper and lower bounds for this problem of She and Yuen [29].

▶ **Theorem 8** ([29, Theorems 1.6 and 1.7]). *Let $\delta \leq \frac{1}{2\pi}$. Every quantum algorithm solving the $(t,\delta)$-recurrence time problem for $d$-dimensional unitaries has cost $\Omega\left(\max\left(t/\delta, \sqrt{d}\right)\right)$. The $(t,\delta)$-recurrence time problem can be solved with cost $O(t\sqrt{d}/\delta)$.*

## 2.3 Trigonometric polynomials and their growth

▶ **Definition 9** (Trigonometric Polynomials). *A function $p : \mathbb{R} \to$ is said to be a* trigonometric polynomial *of degree $d$ if there exist complex numbers $\{a_k : k \in \{-d, \ldots, d\}\}$ such that for all $\theta \in \mathbb{R}$,*

$$p(\theta) = \sum_{k=-d}^{d} a_k e^{ik\theta}.$$

▶ **Theorem 10** ([5, Theorem 5.1.2]). *Let $t$ be a degree-$n$ real-valued trigonometric polynomial and $s \in (0, \pi/2]$ be such that $\mu(\{\theta \in [-\pi, \pi) : |t(\theta)| \leq 1\}) \geq 2\pi - s$, where $\mu$ denotes the Lebesgue measure on $\mathbb{R}$. Then, $\sup_{x \in \mathbb{R}} |t(x)| \leq \exp(4ns)$.*

## 3 Lower bounds for maximum phase estimation and Unitary recurrence time

In this section we show lower bounds on the quantum complexity of maximum phase estimation obtained by varying all its parameters (see Section 2.1 and Definition 6). In this section and the next, we refer to the row numbers of Table 1 when stating and proving our bounds.

Recall that for an integer $j \in \{0, 1, \ldots, N-1\}$ and $\delta \in [0, 2\pi)$ we define $M_{j,\delta} = I - (1 - e^{i\delta})|j\rangle\langle j|$. Our lower bounds will be by reduction from the following "Fractional OR with advice" problem, which fits in the framework of the model described in Section 2.1.

▶ **Definition 11** (Fractional OR with advice). *Let $N \geq 2$ be integer, $\delta > 0$. The task* $\mathsf{frOR}_{N,\delta,t}$ *is:*

- ***Input promise:*** *$U \in \{I, \{M_{j,\delta} : j \in \{1, 2, \ldots, N-1\}\}\}$.*
- ***Advice:*** *When $U = I$ we are given $t$ copies of $|0\rangle$ as advice. When $U = M_{j,\delta}$, we are given $t$ copies of the state $\gamma|j\rangle + \sqrt{1-\gamma^2}|0\rangle$, i.e., part of our starting state is $(\gamma|j\rangle + \sqrt{1-\gamma^2}|0\rangle)^{\otimes t}$.*
- ***Output:*** *With probability at least $2/3$: output 1 if $U = I$, and 0 if $U \neq I$.*

We first show a lower bound on the cost of computing $\mathsf{frOR}_{N,\delta,t}$ when $t = o(1/\gamma^2)$. All of our lower bounds in Table 1 as well as our lower bound for the Unitary recurrence time problem will use this lower bound. We refer the reader to the full version of the paper [25, Appendix A] for the proof. The proof follows along the same lines as Ambainis' adversary lower bound [1, Theorem 4.1] of $\Omega(\sqrt{N})$ queries for the $N$-bit Search problem, but now we additionally take into account the initial advice states and the fact that our input unitaries are only *fractional* versions of phase queries.

▶ **Theorem 12.** *For an integer $N \geq 2$, real numbers $\gamma \geq 1/\sqrt{N}$, $\delta \in [0, \pi]$ and $t = o(1/\gamma^2)$, every algorithm solving* $\mathsf{frOR}_{N,\delta,t}$ *has cost $\Omega(\sqrt{N}/\delta)$.*

▶ **Lemma 13** (Lower bound for Rows 1,3). *Row 1 (and hence Row 3) has a lower bound of $\Omega(\sqrt{N}/\delta)$.*

**Proof.** A cost-$C$ algorithm $\mathcal{A}$ for $\mathsf{maxQPE}_{N,\delta}$ with $t$ advice states and known eigenbasis of $U$ immediately yields a cost-$C$ algorithm $\mathcal{A}'$ for $\mathsf{frOR}_{N,3\delta,t}$: run $\mathcal{A}$ on the input unitary, output 1 if the output phase is in $[-\delta, \delta]$ modulo $2\pi$, and output 0 otherwise. When $U = I$, the correctness of $\mathcal{A}$ guarantees that with probability at least $2/3$, the value output by $\mathcal{A}$ is in $[-\delta, \delta] \mod 2\pi$. When $U = M_{j,3\delta}$, the correctness of $\mathcal{A}$ guarantees that with probability at least $2/3$, the value output by $\mathcal{A}$ is in $[2\delta, 4\delta]$. For $\delta < 2\pi/5$, we have $[-\delta, \delta] \mod 2\pi \cap [2\delta, 4\delta] \mod 2\pi = \emptyset$. Thus, $\mathcal{A}'$ solves $\mathsf{frOR}_{N,3\delta,t}$ and has cost $C$. Theorem 12 yields the bound $C = \Omega\left(\sqrt{N}/\delta\right)$ when $t = o(1/\gamma^2)$, giving the desired result. ◀

▶ **Lemma 14** (Lower bound for Rows 2,4). *Row 2 (and hence Row 4) has a lower bound of $\Omega(1/\gamma\delta)$.*

**Proof.** We prove the required lower bound for $\mathsf{maxQPE}_{N,\delta}$ with inputs satisfying the promise that $U \in \left\{I_N, \left\{M_{j,3\delta} : j \in \left\{1, 2, \ldots, 1/\gamma^2 - 1\right\}\right\}\right\}$. Because of this assumption, we may take

the uniform superposition over the first $1/\gamma^2$ computational basis states as our advice state: the algorithm should work with such an advice state, since it has overlap $\gamma$ with the top eigenspace for each of the possible $U$. However, an algorithm can prepare such advice states at no cost, so we may assume that the algorithm has no access to advice at all. As in the previous proof, this gives an algorithm of the same cost for $\mathsf{frOR}_{1/\gamma^2,3\delta,0}$ (ignoring all other dimensions). Theorem 12 with $N = 1/\gamma^2$ and $t = 0$ yields the required lower bound of $\Omega(1/\gamma\delta)$. ◀

▶ **Lemma 15** (Lower bound for Rows 5,7). *Row 5 (and hence Row 7) has a lower bound of* $\Omega(\sqrt{N}/\delta)$.

**Proof.** Towards the required lower bound, consider a cost-$C$ algorithm $\mathcal{A}$ solving $\mathsf{maxQPE}_{N,\delta}$ with inputs satisfying the promise $U \in \{I_N, \{M_{j,3\delta} : j \in \{1, 2, \ldots, N-1\}\}\}$, and with $t = o(1/\gamma)$ accesses to a unitary that prepares an advice state that has overlap at least $\gamma$ with the target eigenspace. We want to construct an algorithm $\mathcal{A}'$ for $\mathsf{maxQPE}_{N,\delta}$ with the same promised inputs that uses *no* advice, and with cost not much larger than that of $\mathcal{A}$. Note that we may assume $\gamma = o(1)$, since otherwise $t = 0$, so then $\mathcal{A}$ itself already uses no advice.

We first show how an algorithm can itself implement a good-enough advice unitary $A$ quite cheaply. Assuming without loss of generality that $1/3\delta$ is an integer, $U^{1/3\delta}$ is actually a "phase query": if $U = M_{j,3\delta}$, then we have $U^{1/3\delta} = I - 2|j\rangle\langle j|$, which is the diagonal matrix with 1's everywhere except a $-1$ in the $j$th entry; and if $U = I$ then $U^{1/3\delta} = I$. Thus $A$ can start by mapping $|0\rangle$ to a uniform superposition over all indices, and then use Grover's algorithm with $U^{1/3\delta}$ as our query operator to amplify the amplitude of $|j\rangle$ to $\geq \gamma$. We know that $O(\gamma\sqrt{N})$ "Grover iterations" suffice for this (see, for example, [34, Section 7.2] for details). Each Grover iteration would use one phase-query $U^{1/3\delta}$, so the overall cost (number of applications of $U$ and $U^{-1}$) of this advice unitary is $O(\gamma\sqrt{N}/\delta)$. If $U = I$, the state just remains the uniform superposition.

We now have all components to describe $\mathcal{A}'$: Run $\mathcal{A}$, and whenever $\mathcal{A}$ invokes an advice unitary, use the above $A$. Since $\mathcal{A}$ uses at most $t$ advice unitaries, the cost of $\mathcal{A}'$ is at most $C + t \cdot O(\gamma\sqrt{N}/\delta)$. Note that $\mathcal{A}'$ uses no advice at all anymore, and solves $\mathsf{maxQPE}_{N,\delta}$ under the promise that the input unitary satisfies $U \in \{I_N, \{M_{j,3\delta} : j \in \{1, 2, \ldots, N-1\}\}\}$. Again, this immediately yields an algorithm of the same cost for $\mathsf{frOR}_{N,3\delta,0}$ as in the previous two proofs. Theorem 12 now implies

$$C + O(t\gamma\sqrt{N}/\delta) = \Omega(\sqrt{N}/\delta),$$

and hence $C = \Omega(\sqrt{N}/\delta)$ since $t = o(1/\gamma)$ ($t \leq c/\gamma$ for sufficiently small constant $c$ also suffices). ◀

▶ **Lemma 16** (Lower bound for Rows 6,8). *Row 6 (and hence Row 8) has a lower bound of* $\Omega\left(1/\gamma\delta\right)$.

**Proof.** Just as in the proof of Lemma 14, we may assume $N = 1/\gamma^2$ by only allowing input unitaries of the form $U \in \left\{I_N, \left\{M_{j,3\delta} : j \in \left\{1, 2, \ldots, 1/\gamma^2 - 1\right\}\right\}\right\}$. With this assumption, we may assume that we have no access to advice (i.e., $t = 0$) since an algorithm can prepare a good-enough advice state (namely the uniform superposition over all $1/\gamma^2$ basis states) at no cost. This yields the required lower bound of $\Omega(1/\gamma\delta)$ by Lemma 15. ◀

Finally we prove an optimal lower bound for the Unitary recurrence time problem, matching She and Yuen's upper bound (Theorem 8) and resolving one of their open problems [29, Section 2].

**Proof of Theorem 1.** Consider an algorithm $\mathcal{A}$ solving the $(t,\delta)$-recurrence time problem. Restrict to inputs of the form $U \in \{I_N, \{M_{j,3\delta/t} : j \in \{1,2,\ldots,N-1\}\}\}$. When $U = I$ we have $U^t = I$. When $U = M_{j,3\delta/t}$, we have $\|U^t - I\| = |1 - e^{3i\delta}| \geq \delta$ for all $\delta \in [0,1]$. Thus, $\mathcal{A}$ solves $\mathsf{frOR}_{N,3\delta/t,0}$. Theorem 12 yields the required lower bound of $\Omega(t\sqrt{N}/\delta)$. ◀

## 4 Upper bounds for maximum phase estimation

In this section we show upper bounds on the quantum complexity of our 8 variants of maximum phase estimation (see Section 2.1, Definition 6 and Table 1). We require the following generalized maximum-finding procedure, adapted from [2, Lemma 48]; we changed their wording a bit and modified it from minimum-finding to maximum-finding.

▶ **Lemma 17** ([2, Lemma 48])**.** *There exists a quantum algorithm $\mathcal{M}$ and constant $C > 0$ such that the following holds. Suppose we have a $q$-qubit unitary $V$ such that*

$$V|0\rangle = \sum_{k=0}^{K-1} |\psi_k\rangle|x_k\rangle,$$

*where $x_0 > x_1 > \cdots > x_{K-1}$ are distinct real numbers (written down in finite precision), and the $|\psi_k\rangle$ are unnormalized states. Let $X$ be the random variable obtained if we were to measure the last register, so $\Pr[X = x_k] = \||\psi_k\rangle\|^2$. Let $M \geq C/\sqrt{\Pr[X \geq x_j]}$ for some $j$. Then $\mathcal{M}$ uses at most $M$ applications of $V$ and $V^{-1}$, and $O(qM)$ other gates, and outputs an $x_i \geq x_j$ with probability at least $3/4$ (in particular, if $j = 0$ then $\mathcal{M}$ outputs the maximum).*

▶ Remark 18. It may be verified by going through [2, Lemma 48] that the only applications of $V$ and $V^{-1}$ used by $\mathcal{M}$ are to prepare $V|0\rangle$ starting from $|0\rangle$, and to reflect about $V|0\rangle$.

We can use generalized maximum-finding to approximate the largest eigenphase starting from the ability to prepare a superposition of eigenstates (possibly with some additional workspace qubits):

▶ **Lemma 19.** *There exists a quantum algorithm $\mathcal{B}$ such that the following holds. Suppose we have an $N$-dimensional unitary $U$ with (unknown) eigenstates $|u_0\rangle, \ldots, |u_{N-1}\rangle$ and associated eigenphases $\theta_0, \ldots, \theta_{N-1} \in [0, 2\pi)$. Suppose we also have a unitary $A$ such that*

$$A|0\rangle = \sum_{j=0}^{N-1} \alpha_j |u_j\rangle|\phi_j\rangle,$$

*where $\sum_{j:\theta_j=\theta_{\max}} |\alpha_j|^2 \geq \gamma^2$ and the $|\phi_j\rangle$ are arbitrary (normalized) states. Then $\mathcal{B}$ uses at most $O(1/\gamma)$ applications of $A$ and $A^{-1}$, and $O(\log(1/\gamma)/\gamma\delta)$ applications of $U$ and $U^{-1}$, and with probability at least $2/3$ outputs a number $\theta \in [\theta_{\max} - \delta, \theta_{\max} + \delta] \mod 2\pi$.*

**Proof.** Let $\tilde{V}$ be the unitary that applies phase estimation with unitary $U$, precision $\delta$, and small error probability $\eta$ (to be determined later), on the first register of the state $A|0\rangle$, writing the estimates of the phase in a third register. Then

$$\tilde{V}|0\rangle = \sum_{j=0}^{N-1} \alpha_j |u_j\rangle|\phi_j\rangle|\tilde{\theta_j}\rangle,$$

where, for each $j$, $|\tilde{\theta_j}\rangle$ is a superposition over estimates of $\theta_j$, most of which are $\delta$-close to $\theta_j$.

For the purposes of analysis, we would like to define a "cleaned up" unitary $V$ (very close to $\tilde{V}$) that doesn't have any estimates with error $> \delta$. Let $|\tilde{\theta_j}'\rangle$ be the state obtained from $|\tilde{\theta_j}\rangle$

by removing the estimates that are more than $\delta$-far from $\theta_j$, and renormalizing. Because we ran phase estimation with error probability $\leq \eta$, it is easy to show that $\left\| |\tilde{\theta_j}'\rangle - |\tilde{\theta_j}\rangle \right\| = O(\sqrt{\eta})$. Then there exists[6] a unitary $V$ such that $\left\|\tilde{V} - V\right\| = O(\sqrt{\eta})$ and

$$V|0\rangle = \sum_{j=0}^{N-1} \alpha_j |u_j\rangle|\phi_j\rangle|\tilde{\theta_j}'\rangle = \sum_{k=0}^{K-1} |\psi_k\rangle|x_k\rangle,$$

where the $x_k$ are the distinct estimates that have support in the last register, and the $|\psi_k\rangle$ are (unnormalized) superpositions of the $|u_j\rangle|\phi_j\rangle$'s that are associated with those estimates.

The largest $x_k$'s are good estimates of $\theta_{\max}$. Algorithm $\mathcal{B}$ now applies the maximum-finding algorithm $\mathcal{M}$ of Lemma 17 with the unitary $\tilde{V}$. Let us first analyze what would happen if $\mathcal{B}$ used the cleaned-up $V$ instead of $\tilde{V}$. Let $X$ denote the random variable obtained if we measure the last register, and note that $\Pr[X \geq \theta_{\max} - \delta] \geq \sum_{j:\theta_j=\theta_{\max}} |\alpha_j|^2 \geq \gamma^2$ because all estimates in $V|0\rangle$ have error $\leq \delta$. Hence $\mathcal{B}$ would use $O(1/\gamma)$ applications of $V$ and $V^{-1}$ to find a $\theta \in [\theta_{\max} - \delta, \theta_{\max} + \delta]$ with success probability $\geq 3/4$. Algorithm $\mathcal{B}$ will actually use $\tilde{V}$ and $\tilde{V}^{-1}$ instead of $V$ and $V^{-1}$, which (because errors in quantum circuits add at most linearly) incurs an overall error in operator norm of $\leq O(\sqrt{\eta}) \cdot O(1/\gamma)$. Choosing $\eta \ll \gamma^2$, this overall error can be made an arbitrarily small constant. The success probability of the algorithm can drop slightly below 3/4 now, but is still $\geq 2/3$.

It remains to analyze the cost of $\mathcal{B}$. Each $\tilde{V}$ uses 1 application of $A$, and $O(\log(1/\eta)/\delta) = O(\log(1/\gamma)/\delta)$ applications of $U$ and $U^{-1}$ for phase estimation (Theorem 2), so $\mathcal{B}$ uses $O(1/\gamma)$ applications of $A$ and $A^{-1}$, and $O(\log(1/\gamma)/\gamma\delta)$ applications of $U$ and $U^{-1}$ in total. ◀

The upper bounds for our 8 variants of phase estimation (see Table 1) will all follow from this. We start with the 4 odd-numbered rows, where it turns out the advice is not actually needed to meet our earlier lower bounds. The next proof is basically the same as [2, Lemma 50] about estimating the minimal eigenvalue of a Hamiltonian (this improved slightly upon [27]; see also [14, Lemma 3.A.4]).

▶ **Lemma 20** (Upper bound for Rows 1, 3, 5, 7)**.** *There is an algorithm that uses no advice and solves the case in Row 3 (and hence in Rows 1, 5, and 7 as well) with cost $O(\sqrt{N}\log(N)/\delta)$.*

**Proof.** Let $A$ be the unitary that maps $|0\rangle$ to the maximally entangled state in $N$ dimensions. This state can be written in any orthonormal basis, including the (unknown) eigenbasis of $U$:

$$A|0\rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} |j\rangle|j\rangle = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} |u_j\rangle|\overline{u_j}\rangle,$$

where $|\overline{u_j}\rangle$ denotes the entry-wise conjugated version of $|u_j\rangle$. Applying Lemma 19 with this $A$, $|\phi_j\rangle = |\overline{u_j}\rangle$, and $\gamma = 1/\sqrt{N}$ gives the result. ◀

The next two lemmas cover the 4 cases where the advice states/unitaries *are* helpful.

▶ **Lemma 21** (Upper bound for Rows 6, 8)**.** *There is a quantum algorithm that uses $O(1/\gamma)$ applications of the advice unitary (and its inverse) and solves the case in Row 8 (and hence the case in Row 6 as well) with cost $O(\log(1/\gamma)/\gamma\delta)$.*

**Proof.** Apply Lemma 19 with the unitary $A$ that maps $|0\rangle$ to $|\alpha\rangle$, with empty states $|\phi_j\rangle$. ◀

[6] This is fairly easy to show, see e.g. [9, proof of Theorem 2.4 in Appendix A].

▶ **Lemma 22** (Upper bound for Rows 2, 4). *There is a quantum algorithm that uses $O(1/\gamma^2)$ copies of the advice state and solves the case in Row 4 (and hence in Row 2) with cost $O(\log(1/\gamma)/\gamma\delta)$.*

**Proof.** We will build upon the algorithm for Row 8 of Lemma 21. By Remark 18 and the algorithm in Lemma 21, its $O(1/\gamma)$ applications of the advice unitary $A$ and its inverse $A^{-1}$ are only used there for two purposes: (1) to prepare a copy of the advice state $A|0\rangle = |\alpha\rangle$, and (2) to reflect about $|\alpha\rangle$. We now want to replace these applications of $A$ by using copies of the advice state. For (1) this is obvious. Assume the algorithm for Row 8 uses (2) at most $C/\gamma$ times, for some constant $C$. To implement these reflections, we will invoke the result of Lloyd, Mohseni, and Rebentrost [24] (see also [18]), who showed that given a number $t > 0$ and $O(t^2/\eta)$ copies of a mixed quantum state $\rho$, one can implement the unitary $e^{it\rho}$ up to error $\eta$ (in diamond-norm difference between the intended unitary and the actually-implemented channel). We will use this result with $\rho = |\alpha\rangle\langle\alpha|$, $t = \pi$, $\eta = \gamma/(100C)$, noting that the implemented unitary $e^{i\pi|\alpha\rangle\langle\alpha|} = I - 2|\alpha\rangle\langle\alpha|$ is a reflection about $|\alpha\rangle$ (up to a global minus sign that doesn't matter).

Accordingly, we can implement the $\leq C/\gamma$ reflections used by the algorithm for Row 8 using $O(1/\gamma^2)$ copies of $|\alpha\rangle$, each reflection implemented with error $\leq \eta$. Because errors in quantum circuits add at most linearly, the overall error between the algorithm of Row 8 and our simulation of it (using copies of $|\alpha\rangle$) is at most $\eta \cdot C/\gamma \leq 1/100$. Hence we obtain an algorithm for Row 4 that uses $O(1/\gamma^2)$ copies of $|\alpha\rangle$ and has the same cost $O(\log(1/\gamma)/\gamma\delta)$ as the algorithm of Row 8. ◀

## 5 Tight bounds for phase estimation with small error probability

Here we prove our lower bound for quantum algorithms solving phase estimation with precision $\delta$ and error probability at most $\varepsilon$, Theorem 3, which follows from Claims 23 and 24 below.

▷ Claim 23. For all integers $N \geq 2$ and all $\varepsilon, \delta \in (0, 1/2)$, if there is a cost-$d$ algorithm solving $\mathsf{QPE}_{N,\delta,\varepsilon}$, then there is a cost-$d$ algorithm solving $\mathsf{dist}_{N,\delta,\varepsilon}$.

**Proof.** Consider an algorithm $\mathcal{A}$ of cost $d$ that solves $\mathsf{QPE}_{N,\delta,\varepsilon}$. We construct below an algorithm $\mathcal{A}'$ of cost $d$ solving $\mathsf{dist}_{N,\delta,\varepsilon}$. Let $U \in U(N)$ be the input. The following is the description of $\mathcal{A}'$:

1. Run $\mathcal{A}$ with inputs $U$ and $|0\rangle$.
2. Output 1 if the output of $\mathcal{A}$ is in $[-\delta, \delta] \mod 2\pi$, and output 0 otherwise.

Clearly $\mathcal{A}'$ is a valid algorithm, as far as access to input and allowed operations are concerned, since its initial state is $|0\rangle$, it applies $U, U^{-1}$, some unitaries independent of $U$, and finally performs a two-outcome projective measurement to determine the output bit. The cost of $\mathcal{A}'$ is $d$.

The correctness follows along the same lines as the proofs in Section 3. We prove correctness here for completeness. First note that $|0\rangle$ is an eigenstate of all $U \in \{I\} \cup \{U_\theta : \theta \notin [-3\delta, 3\delta] \mod 2\pi\}$. When $U = I$, the correctness of $\mathcal{A}$ guarantees that with probability at least $1 - \varepsilon$, the value output by $\mathcal{A}$ is in $[-\delta, \delta] \mod 2\pi$. When $U = U_\theta$, the correctness of $\mathcal{A}$ guarantees that with probability at least $1 - \varepsilon$, the value output by $\mathcal{A}$ is in $[\theta - \delta, \theta + \delta] \mod 2\pi$. For $\theta \notin [-3\delta, 3\delta] \mod 2\pi$ we have $[-\delta, \delta] \mod 2\pi \cap [\theta - \delta, \theta + \delta] \mod 2\pi = \emptyset$ since $\delta < 1/2 < 2\pi/5$, and hence $\mathcal{A}'$ solves $\mathsf{dist}_{N,\delta,\varepsilon}$. ◀

We next show a lower bound for the cost of algorithms computing $\mathsf{dist}_{N,\delta,\varepsilon}$.

▷ Claim 24. For all integers $N \geq 2$, $\varepsilon, \delta \in (0, 1/2)$, every algorithm for $\mathsf{dist}_{N,\delta,\varepsilon}$ has cost $\Omega\left(\frac{1}{\delta}\log\frac{1}{\varepsilon}\right)$.

In order to prove Claim 24, we first show that amplitudes of basis states in low-cost algorithms that run on $U_\theta$ are low-degree trigonometric polynomials in $\theta$. This is analogous to the fact that amplitudes of basis states in query algorithms for Boolean functions are low-degree (algebraic) polynomials in the input variables [3, Lemma 4.1], and our proof is inspired by theirs.

▷ Claim 25. Let $t > 0$ be a positive integer and let $\theta \in [0, 2\pi]$. Consider a quantum circuit that has starting state $|0\rangle$, uses an arbitrary number of $\theta$-independent unitaries, uses $t$ applications of controlled-$U_\theta$ and controlled-$U_\theta^{-1}$ in total, and performs no intermediate measurements. Then the amplitudes of basis states before the final measurement are degree-$t$ trigonometric polynomials in $\theta$.

**Proof.** We prove this by induction on $t$. The claim is clearly true when $t = 0$ since all amplitudes are constants in this case. For the inductive step, suppose the claim is true for $t = d$. Let $|\psi_d\rangle$ denote the state of the circuit just before the application of the $(d+1)$th application of $U_\theta$ (the argument for $U_\theta^{-1}$ is similar, and we skip it). By the inductive hypothesis, we have

$$|\psi_d\rangle = \sum_w \sum_{b\in\{0,1\}} \sum_{j=0}^{N-1} p_{j,b,w}(\theta)|j\rangle|b\rangle|w\rangle,$$

where the first register is where $U_\theta$ and $U_\theta^{-1}$ act, the second register is the control qubit, and the last register represents the workspace (i.e., $U_\theta$ and $U_\theta^{-1}$ do not act on this register), and each $p_{j,b,w}$ is a trigonometric polynomial of degree at most $d$ in $\theta$. For a basis state $|j\rangle|b\rangle|w\rangle$, we have

$$U_\theta|j\rangle|b\rangle|w\rangle = \begin{cases} e^{i\theta}|0\rangle|b\rangle|w\rangle & \text{if } j = 0 \text{ and } b = 1 \\ |j\rangle|b\rangle|w\rangle & \text{otherwise.} \end{cases}$$

In both cases, the amplitudes of the basis states after the application of $U_\theta$ are degree-$(d+1)$ trigonometric polynomials in $\theta$. After the last application of $U_\theta$ the algorithm will apply an input-independent unitary. The amplitudes after that unitary are linear combinations of the amplitudes before, which won't increase degree. This concludes the inductive step, and hence the theorem. ◀

**Proof of Claim 24.** Consider a cost-$t$ algorithm $\mathcal{A}'$ solving $\mathsf{dist}_{N,\delta,\varepsilon}$. Claim 25 implies that on input $U_\theta$, the amplitudes of the basis states before the final measurement are degree-$t$ trigonometric polynomials in $\theta$. The acceptance-probability polynomial $p : \mathbb{R} \to \mathbb{R}$ given by $p(\theta) := \Pr[\mathcal{A}'(U_\theta) = 1]$ is a degree-$2t$ trigonometric polynomial, because it is the sum of squares of moduli of certain amplitudes, and each of these squares is a degree-$2t$ trigonometric polynomial. The correctness of the algorithm ensures that $p(0) \in [1-\varepsilon, 1]$ and $p(\theta) \in [0, \varepsilon]$ for all $\theta \notin [-3\delta, 3\delta] \mod 2\pi$. See Figure 1 for a visual depiction of the behaviour of $p$ for $\theta \in [-\pi, \pi)$.

Scaling by a global factor of $1/\varepsilon$, we obtain a trigonometric polynomial $q$ of degree $2t$ satisfying:

- $q(0) \geq (1-\varepsilon)/\varepsilon > 1/(2\varepsilon)$, and
- $q(\theta) \in [0, 1]$ for all $\theta \in [-\pi, \pi) \setminus [-3\delta, 3\delta]$.

Thus, Theorem 10 is applicable with $s = 6\delta$, which implies $1/(2\varepsilon) \leq \sup_{x\in\mathbb{R}} |q(x)| \leq \exp(24t\delta)$. By taking logarithms and rearranging we get $t = \Omega\left(\frac{1}{\delta}\log\frac{1}{\varepsilon}\right)$, proving the theorem. ◀

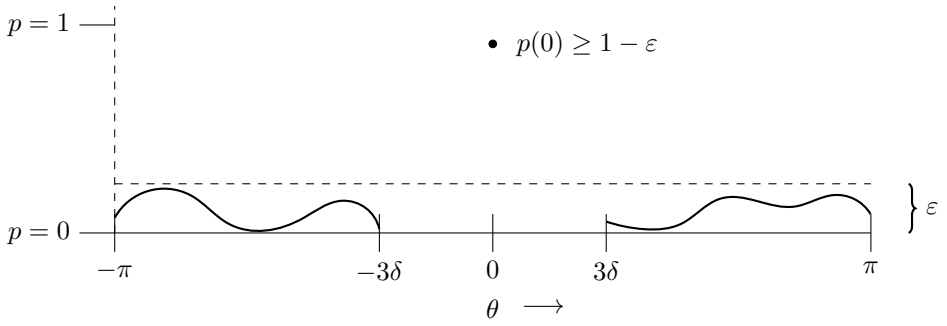


**Figure 1** Acceptance probability $p$ of $\mathcal{A}'$ as a function of $\theta$ in the proof of Claim 24

## 6 Conclusion

In this paper we considered several natural variants of the fundamental phase estimation problem in quantum computing, and proved essentially tight bounds on their cost in each setting. As an immediate application of one of our bounds, we resolved an open question of [29, Section 2].

We mention two interesting questions in the first variant of phase estimation we considered, where an algorithm is given a number of copies of advice states/unitaries instead of black-box access to a perfect eigenstate as in the basic phase estimation setup. First, are the logarithmic overheads in the input dimension $N$ and the inverse of the overlap $\gamma$ in our upper bounds (see Table 1) necessary, or can we give tighter upper bounds? Second, can we show the $\log(1/\varepsilon)$-dependence on the error probability also in the advice-guided case, like we did for basic phase estimation (Theorem 3)?

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