The Contest Game for Crowdsourcing Reviews

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Abstract

We consider a *contest game* modelling a contest where reviews for a *proposal* are crowdsourced from *n players.* Player i has a *skill* s_i, strategically chooses a *quality* $q \in \{1, 2, \ldots, Q\}$ for her review and pays an *effort* $\mathsf{f}_q \geq 0$, strictly increasing with q. Under *voluntary participation,* a player may opt to not write a review, paying zero effort; *mandatory participation* does not provide this option. For her effort, she is awarded a *payment* per her *payment function*, which is either *player-invariant*, like, e.g., the popular *proportional allocation*, or *player-specific*; it is *oblivious* when it does not depend on the numbers of players choosing a different quality. The *utility* to player i is the difference between her payment and her *cost,* calculated by a *skill-effort* function $\Lambda(s_i, \mathsf{f}_q)$. Skills may vary for *arbitrary players*; *anonymous players* means $s_i = 1$ for all players i. In a *pure Nash equilibrium,* no player could unilaterally increase her utility by switching to a different quality. We show the following results about the existence and the computation of a pure Nash equilibrium:

- We present an exact potential to show the existence of a pure Nash equilibrium for the contest game with arbitrary players and player-invariant and oblivious payments. A particular case of this result provides an answer to an open question from [6]. In contrast, a pure Nash equilibrium might not exist *(i)* for player-invariant payments, even if players are anonymous, *(ii)* for proportional allocation payments and arbitrary players, and *(iii)* for player-specific payments, even if players are anonymous; in the last case, it is $\mathcal{NP}$-hard to tell. These counterexamples prove the tightness of our existence result.
- We show that the contest game with proportional allocation, voluntary participation and anonymous players has the *Finite Improvement Property*, or *FIP*; this yields two pure Nash equilibria. The *FIP* carries over to mandatory participation, except that there is now a single pure Nash equilibrium. For arbitrary players, we determine a simple sufficient condition for the *FIP* in the special case where the skill-effort function has the product form $\Lambda(s_i, \mathsf{f}_q) = s_i\, \mathsf{f}_q$.
- We introduce a novel, discrete concavity property of player-specific payments, namely *three-discrete-concavity,* which we exploit to devise, for constant Q, a polynomial-time $\Theta(n^Q)$ algorithm to compute a pure Nash equilibrium in the contest game with arbitrary players; it is a special case of a $\Theta\left(n\, Q^2 \binom{n+Q-1}{Q-1}\right)$ algorithm for arbitrary Q that we present. This settles the parameterized complexity of the problem with respect to the parameter Q. The computed equilibrium is *contiguous*: players with higher skills are contiguously assigned to lower qualities. Both three-discrete-concavity and the algorithm extend naturally to player-invariant payments.

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1 Introduction

Contests [39] are modelled as games where strategic contestants, or *players*, invest efforts in competitions to win valuable prizes, such as monetary awards, scientific credit or social reputation. Such competitions are ubiquitous in contexts such as promotion tournaments in organizations, allocation of campaign resources, content curation and selection in online platforms, financial support of scientific research by governmental institutions and question-and-answer forums. This work joins an active research thread on the existence, computation and efficiency of (pure) Nash equilibria in games for crowdsourcing, content curation, information aggregation and other relative tasks [1, 2, 4, 5, 6, 12, 14, 15, 16, 17, 18, 21, 25, 40].

In a *crowdsourcing contest* (see, e.g., [8, 13, 33]), solutions to a certain task are solicited. When the task is the evaluation of proposals requesting funding, a set of expert advisors, or *reviewers,* file peer-reviews of the proposals. We shall consider a contest game for crowdsourcing reviews, embracing and wide-extending a corresponding game from [6, Section 2] that was motivated by issues in the design of blockchains and cryptocurrencies. In the contest game, funding agencies wish to collect peer-reviews of esteem *quality*. *Costs* are incurred to reviewers; they reflect various overheads, such as time, participation cost or reputational loss, and are supposed to increase with the reviewers' *skills* and *efforts.*[1] Both skills and efforts are modelled as discrete; such modelling is natural since, for example, monetary expenditure, the time to spend on projects, and man-power are usually measured in discrete units. Naturally, efforts increase with the achieved qualities of the reviews. Efforts map collectively into *payments* rewarded to the reviewers to counterbalance their costs. We proceed to formalize these considerations.

1.1 The Contest Game for Crowdsourcing Reviews

We assume familiarity with the basics of finite games, as articulated, e.g., in [24]; we shall restrict attention to finite games. In the *contest game for crowdsourcing reviews,* henceforth abbreviated as the *contest game,* there are n *players* $1, 2, \ldots, n$, with $n \geq 2$, simultaneously writing reviews for a *proposal*. Each player $i \in [n]$ has a *skill* $s_i > 0$. Players are *anonymous* if their skills are the same; then, take $s_i = 1$ for all $i \in [n]$. Else they are *arbitrary.*

The *strategy* q_i of a player $i \in [n]$ is the *quality* of the review she writes; she chooses it from a finite set $\{1, 2, \ldots, Q\}$, with $Q \geq 2$. For a given *quality vector* $\mathbf{q} = \langle q_1, \ldots, q_n \rangle$, the *load* on quality q, denoted as $\mathsf{N}_{\mathbf{q}}(q)$, is the number of players choosing quality q; so $\sum_{q \in [Q]} \mathsf{N}_{\mathbf{q}}(q) = n$. A partial quality vector $\mathbf{q}_{-i}$ results by excluding q_i from $\mathbf{q}$, for some player $i \in [n]$. $\mathsf{Players}_{\mathbf{q}}(q)$ is the set of players choosing quality q in $\mathbf{q}$. f_q is the *effort* paid by a player writing a review of quality q; it is an increasing function of q with $\mathsf{f}_1 < \mathsf{f}_2 < \ldots < \mathsf{f}_Q$. *Mandatory participation* is modeled by setting $\mathsf{f}_1 > 0$; under *voluntary participation,* modeled by setting $\mathsf{f}_1 = 0$, a player may choose not to write a review and save effort.

Given a quality vector $\mathbf{q}$ and a player $i \in [n]$, the *payment* awarded to player $i \in [n]$ for her review is the value $\mathsf{P}_i(\mathbf{q})$ determined by her *payment function* P_i, obeying the *normalization condition* $\sum_{k \in [n]} \mathsf{P}_k(\mathbf{q}) \leq 1$. Payments are *oblivious* if for any player $i \in [n]$ and quality vector $\mathbf{q}$, $\mathsf{P}_i(\mathbf{q}) = \mathsf{P}_i(\mathsf{N}_{\mathbf{q}}(q_i), \mathsf{f}_{q_i})$; that is, $\mathsf{P}_i(\mathbf{q})$ depends only on the quality q_i chosen by player i and the load on it. Note that oblivious payments are not necessarily player-invariant as for

[1] One might argue that the cost of a reviewer for writing a review of a given quality decreases with her skill and claim that skill is a misnomer; however, it can also be argued that skilled players are incurred higher costs upon drawing more skills than necessary for writing a decent review. For consistency, we chose to keep using skills in the same way as in [6].

different players $i, k \in [n]$, it is not necessary that $\mathsf{P}_i = \mathsf{P}_k$. Payments are *player-invariant* if for every quality vector $\mathbf{q}$, for any players $i, k \in [n]$ with $q_i = q_k$, $\mathsf{P}_i(\mathbf{q}) = \mathsf{P}_k(\mathbf{q})$; thus, players choosing the same quality are awarded the same payment. A player-invariant payment function $\mathsf{P}_i(\mathbf{q})$ can be represented by a two-argument payment function $\mathsf{P}_i(q, \mathbf{q}_{-i})$, for a quality $q \in [Q]$ and a partial quality vector $\mathbf{q}_{-i}$, for a player $i \in [n]$. We consider the following player-invariant payments:

- The *proportional allocation* $\mathsf{PA}_i(\mathbf{q}) = \dfrac{\mathsf{f}_{q_i}}{\sum_{k \in [n]} \mathsf{f}_{q_k}}$; thus, $\sum_{i \in [n]} \mathsf{PA}_i(\mathsf{q}) = \dfrac{\sum_{i \in [n]} \mathsf{f}_{q_i}}{\sum_{i \in [n]} \mathsf{f}_{q_i}} = 1$. Proportional allocation is widely studied in the context of contests with smooth allocation of prizes (cf. [39, Section 4.4]). For proportional allocation with voluntary participation (by which $\mathsf{f}_1 = 0$), in the scenario where all players choose quality 1, the payment to any player becomes $\frac{0}{0}$, so it is indeterminate.[2] To remove indeterminacy and make payments well-defined, we define the payment to any player choosing quality 1 in the case where all players choose 1 to be 0. Note that proportional allocation is not oblivious.
- The *equal sharing per quality* $\mathsf{ES}_i(\mathbf{q}) = \mathsf{C}_{\mathsf{ES}} \cdot \dfrac{\mathsf{f}_{q_i}}{\mathsf{N}_{\mathbf{q}}(q_i)}$; so f_{q_i} is shared evenly by players choosing q_i. Since $\sum_{i \in [n]} \mathsf{ES}_i(\mathbf{q}) = \mathsf{C}_{\mathsf{ES}} \cdot \sum_{i \in [n]} \dfrac{\mathsf{f}_{q_i}}{\mathsf{N}_{\mathbf{q}}(q_i)}$, we take $\mathsf{C}_{\mathsf{ES}} = \left(\max_{\mathbf{q}} \sum_{i \in [n]} \dfrac{\mathsf{f}_{q_i}}{\mathsf{N}_{\mathbf{q}}(q_i)} \right)^{-1}$. Note that the equal sharing per quality is different from the standard equal sharing, by which *all* players choosing quality at least some $q \in [Q]$ share f_q equally. Thus, standard equal sharing is not oblivious, while the equal sharing per quality is. Both the equal sharing per quality and the equal sharing allow for a player's payment to decrease with an increase in quality; this happens, for example, in standard equal sharing when a player switches from a lower quality with very high load to a higher quality with a significantly smaller total load on qualities at least the higher quality.
- The $K\mathsf{Top}$ *allocation* $K\mathsf{Top}_i(\mathbf{q}) = \mathsf{C}_{K\mathsf{Top}} \cdot \begin{cases} 0\,, & \text{if } q_i \leq Q - K \\ \dfrac{\mathsf{f}_{q_i}}{\mathsf{N}_{\mathbf{q}}(q_i)}\,, & \text{if } q_i > Q - K \end{cases}$; so players choosing a quality q higher than a certain quality $Q-K$ share f_q evenly. Since $\sum_{i \in [n]} K\mathsf{Top}_i(\mathbf{q}^{\ell}) = \mathsf{C}_{K\mathsf{Top}} \sum_{q_i > Q-K} \dfrac{\mathsf{f}_{q_i}}{\mathsf{N}_{\mathbf{q}}(q_i)}$, we take $\mathsf{C}_{K\mathsf{Top}} = \left(\max_{\mathbf{q}^{\ell}} \sum_{q_i > Q-K} \dfrac{\mathsf{f}_{q_i}}{\mathsf{N}_{\mathbf{q}}(q_i)} \right)^{-1}$. Note that the $K\mathsf{Top}$ allocation is different from the standard $K\mathsf{Top}$ allocation, considered in, e.g., [14, 22, 40], by which *all* players choosing quality higher than $Q - K$ share f_q equally; so the utility of a player i choosing a quality $q_i > Q - K$ in $\mathbf{q}$ is $\dfrac{\mathsf{f}_{q_i}}{\sum_{q > Q-K} \mathsf{N}_{\mathbf{q}}(q)}$. Thus, the standard $K\mathsf{Top}$ allocation is not oblivious, while the $K\mathsf{Top}$ allocation is.

A generalization of a player-invariant payment function results by allowing the payment to player $i \in [n]$ to be a function $\mathsf{P}_i(i, \mathbf{q})$ of both i and $\mathbf{q}$; it is called a *player-specific* payment function. The *cost* or *skill-effort function* $\Lambda : \mathbb{R}_{\geq 1} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$, with $\Lambda(\cdot, 0) = 0$, is a monotonically increasing, polynomial-time computable function in both skill and effort.

For a quality vector $\mathbf{q}$, the *utility function* is assumed to be of quasi-linear form with respect to payment and cost and is defined as $\mathsf{U}_i(\mathbf{q}) = \mathsf{P}_i(\mathbf{q}) - \Lambda(s_i, \mathsf{f}_{q_i})$, for each player $i \in [n]$. In a *pure Nash equilibrium* $\mathbf{q}$, for every player $i \in [n]$ and deviation of her to strategy $q \in [Q]$, $q \neq q_i$, $\mathsf{U}_i(\mathbf{q}) \geq \mathsf{U}_i(q, \mathbf{q}_{-i})$; so no player could increase her utility by unilaterally switching to a different quality. We consider the following problems for deciding the existence of a pure Nash equilibrium and computing one if there is one:

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[2] This means that all values c satisfy $0 = 0 \cdot c$.

- ∃PNE with Player-Invariant and Oblivious Payments
- ∃PNE with Player-Invariant Payments
- ∃PNE with Proportional Allocation and Arbitrary Players
- ∃PNE with Proportional Allocation and Anonymous Players
- ∃PNE with Player-Specific Payments

The most significant difference between the contest game and the contest games traditionally considered in Contest Theory [39] is that the it adopts players with a *discrete* action space, choosing over a finite number of qualities, while the latter focus on players with a *continuous* one. (See [11] for an exception.) Alas, the contest game is comparable to classes of contests studied in Contest Theory [39] with respect to several characteristics:

- Casting qualities as individual contests, the contest game resembles *simultaneous contests* (cf. [39, Section 5]), in which players simultaneously invest efforts across the set of contests.
- While in an *all-pay contest* (cf. [39, Chapter 2]) all players competing for a non-splittable prize must pay for their bid and the winner takes all of it, all players are awarded payments, summing up to at most 1, in the contest game.
- The utility $\mathsf{U}_i(\mathbf{q}) = \mathsf{P}_i(\mathbf{q}) - \Lambda(s_i, \mathsf{f}_{q_i})$ in the contest game can be cast as *smooth* (cf. [39, Chapter 4]): *(i)* each player receives a portion $\mathsf{P}_i(\mathbf{q})$ of the prize according to an allocation mechanism that is a smooth function of the invested efforts $\{\mathsf{f}_q\}_{q\in[Q]}$ (except when all players invest zero effort (cf. [39, start of Section 4], which may happen under proportional allocation with voluntary participation) and *(ii)* utilities are quasilinear in payment and cost; in this respect, U_i corresponds to a *contest success function* [37].

We shall need some definitions from Game Theory, applying to finite games with players i maximizing utility U_i. All types of potentials map profiles to numbers. A game is an *(exact) potential game* [27] if it admits a *exact potential* Φ: for each player $i \in [n]$, for any pair q_i and q'_i of her strategies and for any partial profile $\mathbf{q}_{-i}$, $\mathsf{U}_i(q'_i, \mathbf{q}_{-i}) - \mathsf{U}_i(q_i, \mathbf{q}_{-i}) = \Phi(q'_i, \mathbf{q}_{-i}) - \Phi(q_i, \mathbf{q}_{-i})$. A game is an *ordinal potential game* [27] if it admits a *ordinal potential* Φ: for each player $i \in [n]$, for any pair q_i and q'_i of her strategies and for any partial profile $\mathbf{q}_{-i}$, $\mathsf{U}_i(q'_i, \mathbf{q}_{-i}) > \mathsf{U}_i(q_i, \mathbf{q}_{-i})$ if and only if $\Phi(q'_i, \mathbf{q}_{-i}) > \Phi(q_i, \mathbf{q}_{-i})$. A game is a *generalized ordinal potential game* [27] if it admits a *generalized ordinal potential* Φ: for each player $i \in [n]$, for any pair q_i and q'_i of her strategies, and for any partial profile $\mathbf{q}_{-i}$, $\mathsf{U}_i(q_i, \mathbf{q}_{-i}) > \mathsf{U}_i(q'_i, \mathbf{q}_{-i})$ implies $\Phi(q_i, \mathbf{q}_{-i}) > \Phi(q'_i, \mathbf{q}_{-i})$. So a potential game is a strengthening of an ordinal potential game, which is a strengthening of a generalized ordinal potential game. Every generalized ordinal potential game has at least one pure Nash equilibrium [27, Corollary 2.2].

We recast some definitions from Game Theory in the context of the contest game. An *improvement step* out of the quality vector $\mathbf{q}$ and into the $\mathbf{q}'$ occurs when there is a unique player $i \in [n]$ with $q_i \neq q'_i$ such that $\mathsf{U}_i(\mathbf{q}) < \mathsf{U}_i(\mathbf{q}')$; so it is profitable for player i to switch from q_i to q'_i. An *improvement path* is a sequence $\mathbf{q}^{(1)}, \mathbf{q}^{(2)}, \ldots$, such that for each quality vector $\mathbf{q}^{(\rho)}$ in the sequence, where $\rho \geq 1$, there occurs an improvement step out of $\mathbf{q}^{\rho}$ and into $\mathbf{q}^{(\rho+1)}$. A *finite* improvement path has finite length. The *Finite Improvement Property,* abbreviated as *FIP,* requires that all improvement paths are finite; that is, there are no cycles in the directed *quality improvement graph,* whose vertices are the quality vectors and there is an edge from quality vector $\mathbf{q}^{(1)}$ to $\mathbf{q}^{(2)}$ if and only if an improvement step occurs from $\mathbf{q}^{(1)}$ to $\mathbf{q}^{(2)}$. Every game with the *FIP* has a pure Nash equilibrium: a *sink* in the quality improvement graph; there are games without the *FIP* that also have [27]. By [27, Lemma 2.5], a game has a generalized ordinal potential if and only if it has the *FIP*.

1.2 Results

We study the existence and the computation of pure Nash equilibria for the contest game. *When do pure Nash equilibria exist for arbitrary players, player-invariant or player-specific payments and for arbitrary n and Q?* For the special case of the contest game with proportional allocation payments and a skill-effort function $\Lambda(s_i, \mathsf{f}_q) = s_i \mathsf{f}_q$, this has been advocated as a significant open problem in [6, Section 6]. *What is the time complexity of deciding the existence of a pure Nash equilibrium and computing one in case there exists one? Is this complexity affected by properties of the payment or the skill-effort function, or by numerical properties of skills and efforts, and how?* We shall present three major results:

- Every contest game with arbitrary players and player-invariant and oblivious payments has a pure Nash equilibrium, for any values of n and Q and any skill-effort function Λ (Theorem 1). We devise an *exact potential* [27] for the contest game and resort to the fact that every *potential game* has a pure Nash equilibrium [27, Corollary 2.2]. By Theorem 1, the contest game with equal sharing per quality and $K\mathsf{Top}$ allocation has a pure Nash equilibrium. However, existence does not extend beyond player-invariant *and* oblivious payments: We prove the tightness of our existence result (Theorem 1) by exhibiting simple contest games with no pure Nash equilibrium when:
  - Payments are player-invariant but not oblivious, even if players are anonymous (Proposition 3).
  - Payments are proportionally allocated and players are arbitrary (Proposition 4).
  - Payments are player-specific, even if players are anonymous (Proposition 6). The $\mathcal{NP}$-completeness of deciding the existence of a pure Nash equilibrium follows by a simple reduction from the problem of deciding the existence of a pure Nash equilibrium in a succinctly represented strategic game [32, Theorem 2.4.1] (Theorem 7).
- We show that the contest game with proportional allocation, voluntary participation and anonymous players has the *FIP* (Theorem 8). The contest game is found to have two pure Nash equilibria in this case. A simplification of the proof for voluntary participation establishes the *FIP* for mandatory participation (Theorem 10); the number of pure Nash equilibria drops to one. As the key to establish these results, we show the *No Switch from Lower Quality to Higher Quality* Lemma: in an improvement step, a player necessarily switches from a higher quality to a lower quality (Lemma 9).
  These results are complemented with a very simple, $\Theta(1)$ algorithm that works under proportional allocation, for arbitrary players, with $\Lambda(s_i, \mathsf{f}_q) = s_i\, \mathsf{f}_q$ and making stronger assumptions on skills and efforts to compute a pure Nash equilibrium (Theorem 12). The algorithm simply assigns all players to quality 1; so it runs in optimal time $\Theta(1)$.
- Finally, we consider a player-specific payment function that is also *three-discrete-concave*: for any triple of qualities q_i, q_k and q, the difference between the payments when incrementing the load on q and decrementing the load on q_i is at most the difference between the payments when incrementing the load on q_k and decrementing the load on q. Three-discrete-concave functions make a new class of discrete-concave functions that we introduce; similar classes of discrete-concave functions, such as *L-concave*, are extensively discussed in the excellent monograph by Murota [28]. We present a $\Theta\left(n \cdot Q^2 \binom{n+Q-1}{Q-1}\right)$ algorithm to decide the existence of and compute a pure Nash equilibrium for three-discrete-concave player-specific payments and arbitrary players (Theorem 14).
  Exhaustive enumeration of *all* quality vector incurs an *exponential* $\Theta(Q^n)$ time complexity. To bypass the intractability, we focus on *contiguous* profiles, where any players i and k,

with $s_i \geq s_k$, are assigned to qualities q and q', respectively, with $q \leq q'$; they offer a significant advantage: the cost for their exhaustive enumeration drops to $\Theta\left(\binom{n+Q-1}{Q-1}\right)$. We prove the *Contigufication Lemma*: any pure Nash equilibrium for the contest game can be transformed into a contiguous one (Proposition 15). So, it suffices to search for a contiguous, pure Nash equilibrium. The algorithm is polynomial-time $\Theta(n^Q)$ for *constant* Q, settling the parameterised complexity of the problem when payments are player-specific.

We extend the algorithm for three-discrete-concave player-specific payments to obtain a $\Theta\left(\max\{n, Q^2\} \cdot \binom{n+Q-1}{Q-1}\right)$ algorithm for three-discrete-concave player-invariant payments (Theorem 20). The improved time complexity for arbitrary Q in comparison to the case of three-discrete-concave player-specific payments is due to the fact that the player-invariant property allows dealing with the payment of only one, instead of all, of the players choosing the same quality.

1.3 Related Work and Comparison

The contest game studied here is inspired by, embraces and extends in two significant ways an interesting contest game introduced in [6]. First, we consider an arbitrary payment function, whereas [6] focuses on proportional allocation. Second, we consider a cost function that is an arbitrary function of skill and effort, whereas [6] focuses on the product of skill and effort. Although we have considered a single proposal in our contest game, multiple proposals can also be accommodated, as in [6].

Casting qualities as *resources,* the contest game resembles *unweighted congestion games* [31]; adopting their original definition in [31], there are, though, two significant differences: *(i)* players choose sets of resources in a (weighted or unweighted) congestion game while they choose a single quality in a contest game, and *(ii)* the utilities (specifically, their payment part) depend on the loads on *all* qualities in a contest game, while costs on a resource depend only on the load on the resource in an congestion game. However, their dissimilarity is trimmed down when restricting the comparison to contest games with an oblivious payment function, where a payment depends only on the load on the particular quality, and to *singleton* (unweighted) congestion games, first introduced in [30], where each player chooses a single resource. Note that the payments in a contest game with an oblivious payment function may be player-specific, while, in general, costs in a singleton congestion game are not.

Congestion games with player-specific payoffs were introduced by Milchtaich [26] as singleton congestion games where the payoff to a player choosing a resource is given by a player-specific payoff function. (In fact, player-specific payments in this paper have been inspired by player-specific payoffs in [26].) In [26, Theorem 2], it is shown that, under a standard monotonicity assumption on the payoff function, these games always have a pure Nash equilibrium. An example is provided in [26, Section 5] of a congestion game with player-specific payoffs that lacks the *Finite Improvement Property (FIP).* In contrast, Theorem 1 shows that the contest game with a player-invariant and oblivious payment function, a special case of a congestion game with player-specific payoffs, has a potential function; thus, it identifies a subclass of congestion games with player-specific payoffs that does have the stronger *FIP*.

Gairing *et al.* [20] consider cost-minimizing players and *non-singleton* congestion games with player-specific costs; [20, Theorem 3.1], shows that there is a potential for the strict subclass of congestion games with *linear* player-specific costs of the form $\mathsf{f}_{ie}(\delta) = \alpha_{ie} \cdot \delta$, where $\alpha_{ie} \geq 0$, for a player i and a resource e; δ is the number of players choosing resource e.

For the potential function result (Theorem 1) for the contest game with a player-invariant and oblivious payment function, we consider *general* player-specific utilities of the form $\mathsf{U}_i(q) = \mathsf{P}_i(i, \mathsf{N}(q)) - \Lambda(s_i, \mathsf{f}_q)$, where $\mathsf{P}_i(i, \mathsf{N}(q)) \geq 0$ is not necessarily linear and Λ is an arbitrary non-negative function, which is independent of $\mathsf{N}(q)$ and could be non-monotone. Theorem 1 is a significant generalization of [20, Theorem 3.1], which assumed *linear* player-specific costs, and an extention of it, due to the subtracted term $\Lambda(s_i, \mathsf{f}_q)$. however, it is also a restriction of [20, Theorem 3.1], since the contest game is singleton and P_i is assumed player-invariant.

The contest games considered in the proofs of the existence of pure Nash equilibria for [6, Theorems 1 and 3] assume $Q = 3$ and $Q = 2$, respectively, and deal with proportional allocation, voluntary participation and a skill-effort function $\Lambda(s_i, \mathsf{f}_q) = s_i \mathsf{f}_q$, for any player $i \in [n]$ and quality $q \in [Q]$. Pure Nash equilibria are ill-defined in all considered cases of voluntary participation as they ignore the indeterminacy arising in case all players choose quality 1. Putting aside this correctness issue, Theorem 1 generalizes the context of [6, Theorem 3] from the case $Q = 2$ to arbitrary Q, for *any* player-invariant and oblivious payment function and *any* skill-effort function; Theorems 8 and 10 generalize the context of [6, Theorem 1] from $Q = 3$ to arbitrary Q, while they significantly strengthen the claimed results for these ill-defined cases, since *(i)* they establish the *FIP,* which is a property stronger than the existence of a pure Nash equilibrium, *(ii)* they cover together both voluntary and mandatory participation, and *(iii)* they explicitly determine the pure Nash equilibria and their number, while the outlined convergence arguments for claiming [6, Theorem 1] do not.

The contest game is related to *project games* [5], where each *weighted* player i selects a single *project* $\sigma_i \in S_i$ among those available to him, where several players may select the same project. Weights w_{i,σ_i} are *project-specific*; they are called *universal* when they are fixed for the same project and *identical* when the fixed weights are the same over all projects. The utility of player i is a fraction r_{σ_i} of the proportional allocation of weights on the project σ_i. Projects can be considered to correspond to qualities in the contest game, which, in contrast, has, in general, neither weights nor fractions but has the extra term $\Lambda(s_i, \mathsf{f}_q)$ for the cost.

For the contest game in [16], there are m *activities* and player $i \in [n]$ chooses an *output vector* $\mathbf{b}_i = \langle b_{i1}, \ldots, b_{im} \rangle$, with $b_{i\ell} \in \mathbb{R}_{\geq 0}$, $\ell \in [m]$; the case $b_{i\ell} = 0$ corresponds to voluntary participation. In contrast, there are no activities in the contest game; but one may view the single proposal and quality vectors in it (as well as in the contest game in [6]) as an activity and output vectors, respectively. There are $C \geq 1$ *contests* awarding prizes to the players based on their output vectors; allocation is equal sharing in [16], by which players receiving a prize share are "filtered" using a function f_c associated with contest c. The special case of the contest game in [16] with $C = 1$ can be seen to correspond to a contest game in our context; nevertheless, to the best of our understanding, no results transfer between the contest games in [16] and in this paper, as their definitions are different; for example, we do not see how to embed output vectors in our contest game, or skill-effort costs in the contest game in [16].

Listed in [39, Section 6.1.3] are more examples of player-invariant payments, including *proportional-to-marginal contribution* (motivated by the marginal contribution condition in *(monotone) valid utility games* [38]) and *Shapley-Shubick* [34, 35]. Games employing proportional allocation, equal sharing and K-Top allocation have been studied, for example, in [5, 10, 18, 29, 41], in [16, 25] and in [14, 22, 40], respectively. Accounts on proportional allocation and equal sharing in simultaneous contests appear in [39, Section 5.4 & Section 5.5], respectively. Player-invariant payments enhance *Anonymous Independent Reward Schemes (AIRS)* [9], where payments, termed as *rewards,* are only allowed to depend on the quality of the individual review, or *content* in the context of user-generated content platforms.

A plethora of results in Contest Theory establish the inexistence of pure Nash equilibria in contests with continuous strategy spaces; see, e.g, [3] or [33, Example 1.1]. Still for continuous strategy spaces, for proportional allocation, existence, uniqueness and characterization of pure Nash equilibria is established in [39, Theorem 4.9] for two-player contests and in [23] for contests with an arbitrary number of players, assuming additional conditions on the utility functions. All-pay contests with discrete action spaces were considered in [11]. In our view, the analysis of contest games with discrete action spaces is more challenging; it requires combinatorial arguments, instead of concavity and continuity arguments, typically employed for contests with continuous action spaces.

2 (In)Existence of a Pure Nash Equilibrium

We show:

▶ **Theorem 1.** *The contest game with arbitrary players and player-invariant and oblivious payments has an exact potential and a pure Nash equilibrium.*

Proof. Define the function $\Phi : \{\mathbf{q}\} \rightarrow \mathbb{R}$ as

$$\Phi(\mathbf{q}) \quad = \quad \sum_{q\in[Q]} \Gamma(\mathsf{N}_{\mathbf{q}}(q)) - \sum_{k\in[n]} \Lambda(s_k, \mathsf{f}_{q_k})\,,$$

where the function $\Gamma : \mathbb{N}\cup\{0\} \rightarrow \mathbb{R}$ will be defined later. We prove that Φ is an exact potential.

Consider a player $i \in [n]$ switching from strategy q_i, to strategy $\widehat{q}_i$, while other players do not change strategies. So the quality vector $\mathbf{q} = \langle q_1, \ldots, q_{(i-1)}, q_i, q_{i+1}, \ldots, q_n\rangle$ is transformed into $\widehat{\mathbf{q}} := \langle q_1, \ldots, q_{i-1}, \widehat{q}_i, q_{i+1}, \ldots, q_n\rangle$; thus, $\mathsf{N}_{\widehat{\mathbf{q}}}(q_i) = \mathsf{N}_{\mathbf{q}}(q_i) - 1$, $\mathsf{N}_{\widehat{\mathbf{q}}}(\widehat{q}_i) = \mathsf{N}_{\mathbf{q}}(\widehat{q}_i) + 1$ and $\mathsf{N}_{\widehat{\mathbf{q}}}(\widetilde{q}) = \mathsf{N}_{\mathbf{q}}(\widetilde{q})$ for each quality $\widetilde{q} \neq q_i, \widehat{q}_i$. To simplify notation, denote q_i and $\widehat{q}_i$ as q and $\widehat{q}$, respectively. So,

$$\mathsf{U}_i(\mathbf{q}) - \mathsf{U}_i(\widehat{\mathbf{q}}) \quad = \quad [\mathsf{P}_i(\mathbf{q})]_{\left[\mathsf{N}_{\mathbf{q}}(q),\mathsf{N}_{\mathbf{q}}(\widehat{q})\right]} - [\mathsf{P}_i(\widehat{\mathbf{q}})]_{\left[\mathsf{N}_{\mathbf{q}}(q)-1,\mathsf{N}_{\mathbf{q}}(\widehat{q})+1\right]} + \Lambda(s_i, \mathsf{f}_{\widehat{q}}) - \Lambda(s_i, \mathsf{f}_q)\,,$$

where $[\mathsf{P}_i(\mathbf{q})]_{\left[\mathsf{N}_{\mathbf{q}}(q),\mathsf{N}_{\mathbf{q}}(\widehat{q})\right]}$ and $[\mathsf{P}_i(\mathbf{q})]_{\left[\mathsf{N}_{\mathbf{q}}(q)-1,\mathsf{N}_{\mathbf{q}}(\widehat{q})+1\right]}$ denote the payments awarded to i when the loads on qualities q and $\widehat{q}$ are $(\mathsf{N}_{\mathbf{q}}(q), \mathsf{N}_{\mathbf{q}}(\widehat{q}))$ and $(\mathsf{N}_{\mathbf{q}}(q) - 1, \mathsf{N}_{\mathbf{q}}(\widehat{q}) + 1)$, respectively, while loads on other qualities remain unchanged. So $[\mathsf{P}_i(\mathbf{q})]_{\left[\mathsf{N}_{\mathbf{q}}(q),\mathsf{N}_{\mathbf{q}}(\widehat{q})\right]} = \mathsf{P}_i(\mathbf{q})$ and $[\mathsf{P}_i(\mathbf{q})]_{\left[\mathsf{N}_{\mathbf{q}}(q)-1,\mathsf{N}_{\mathbf{q}}(\widehat{q})+1\right]} = \mathsf{P}_i(\widehat{\mathbf{q}})$. Clearly,

$$\begin{aligned}\Phi(\mathbf{q}) - \Phi(\widehat{\mathbf{q}}) \quad &= \quad \Gamma(\mathsf{N}_{\mathbf{q}}(q)) + \Gamma(\mathsf{N}_{\mathbf{q}}(\widehat{q})) - \Lambda(s_i, \mathsf{f}_q) - \Big(\Gamma(\mathsf{N}_{\mathbf{q}}(q) - 1) + \Gamma(\mathsf{N}_{\mathbf{q}}(\widehat{q}) + 1) - \Lambda(s_i, \mathsf{f}_{\widehat{q}})\Big) \\ &= \quad \Gamma(\mathsf{N}_{\mathbf{q}}(q)) - \Gamma(\mathsf{N}_{\mathbf{q}}(q) - 1) - (\Gamma(\mathsf{N}_{\mathbf{q}}(\widehat{q}) + 1) - \Gamma(\mathsf{N}_{\mathbf{q}}(\widehat{q}))) + \Lambda(s_i, \mathsf{f}_{\widehat{q}}) - \Lambda(s_i, \mathsf{f}_q)\,.\end{aligned}$$

Now define the function Γ such that for a quality vector $\mathbf{q}$, for each quality $q \in [Q]$,

$$\Gamma(\mathsf{N}_{\mathbf{q}}(q)) - \Gamma(\mathsf{N}_{\mathbf{q}}(q) - 1) \quad = \quad [\mathsf{P}_i(\mathbf{q})]_{\left[\mathsf{N}_{\mathbf{q}}(q),\mathsf{N}_{\mathbf{q}}(\widehat{q})\right]}\;,$$

We set $\widehat{q}$ for q and $\mathsf{N}_{\mathbf{q}}(\widehat{q}) + 1$ for $\mathsf{N}_{\mathbf{q}}(q)$ to obtain

$$\Gamma(\mathsf{N}_{\widehat{\mathbf{q}}}(\widehat{q}) + 1) - \Gamma(\mathsf{N}_{\widehat{\mathbf{q}}}(\widehat{q})) \quad = \quad [\mathsf{P}_i(\widehat{\mathbf{q}})]_{\left[\mathsf{N}_{\mathbf{q}}(q)-1,\mathsf{N}_{\mathbf{q}}(\widehat{q})+1\right]}\;,$$

if $\mathsf{N}_{\mathbf{q}}(q) \geq 1$, and $\Gamma(0) = 0$. Note that Γ is well-defined: the left-hand side is a function of $\mathsf{N}_{\mathbf{q}}$ only, as also is the right-hand side since $\mathsf{P}_i(\mathbf{q})$ is independent of *(i)* i, since P is player-invariant, and *(ii)* the loads on qualities other than q, since P is oblivious. An explicit formula for $\Gamma(\mathsf{N}_{\mathbf{q}}(q)$ follows from its definition:

$$\begin{aligned} \Gamma(\mathsf{N}_{\mathbf{q}}(q)) &= \left(\Gamma(\mathsf{N}_{\mathbf{q}}(q)-2) + [\mathsf{P}_i(\mathbf{q})]_{\left[\mathsf{N}_{\mathbf{q}}(q)-1,\mathsf{N}_{\mathbf{q}}(\widehat{q})+1\right]}\right) + [\mathsf{P}_i(\mathbf{q})]_{\left[\mathsf{N}_{\mathbf{q}}(q),\mathsf{N}_{\mathbf{q}}(\widehat{q})\right]} = \ldots \\ &= [\mathsf{P}_i(\mathbf{q})]_{\left[1,\mathsf{N}_{\mathbf{q}}(q)+\mathsf{N}_{\mathbf{q}}(\widehat{q})-1\right]} + [\mathsf{P}_i(\mathbf{q})]_{\left[2,\mathsf{N}_{\mathbf{q}}(q)+\mathsf{N}_{\mathbf{q}}(\widehat{q})-2\right]} + \ldots + [\mathsf{P}_i(\mathbf{q})]_{\left[\mathsf{N}_{\mathbf{q}}(q),\mathsf{N}_{\mathbf{q}}(\widehat{q})\right]} \end{aligned}$$

Hence, by definition of Γ,

$$\Phi(\mathbf{q}) - \Phi(\widehat{\mathbf{q}}) = [\mathsf{P}_i(\mathbf{q})]_{\left[\mathsf{N}_{\mathbf{q}}(q),\mathsf{N}_{\mathbf{q}}(\widehat{q})\right]} - [\mathsf{P}_i(\widehat{\mathbf{q}})]_{\left[\mathsf{N}_{\mathbf{q}}(q)-1,\mathsf{N}_{\mathbf{q}}(\widehat{q})+1\right]} + \Lambda(s_i, \mathsf{f}_{\widehat{q}}) - \Lambda(s_i, \mathsf{f}_q)\,.$$

Hence, $\Phi(\mathbf{q}) - \Phi(\widehat{\mathbf{q}}) = \mathsf{U}_i(\mathbf{q}) - \mathsf{U}_i(\widehat{\mathbf{q}})$, Φ is an exact potential and a pure Nash equilibrium exists. ◀

Since Γ, P and Λ are poly-time computable, so is also the exact potential Φ used for the proof of Theorem 1 since it involves summations of values of Γ, P and Λ. Hence, ∃PNE WITH PLAYER-INVARIANT AND OBLIVIOUS PAYMENTS $\in \mathcal{PLS}$.

▷ **Open Problem** 2. Determine the precise complexity of ∃PNE WITH PLAYER-INVARIANT AND OBLIVIOUS PAYMENTS. We remark that no $\mathcal{PLS}$-hardness results for computing pure Nash equilibria are known for either singleton congestion games [26] or for project games [5], which, in some sense, are also singleton as the contest game is; moreover, all known $\mathcal{PLS}$-hardness results for computing pure Nash equilibria in congestion games apply to congestion games that are not singleton. These remarks appear to speak against $\mathcal{PLS}$-hardness.

We next show that existence of pure Nash equilibria is not guaranteed if P is not player-invariant and oblivious simultaneously. We start by showing:

▶ **Proposition 3.** *There is a contest game with mandatory participation, player-invariant payments and anonymous players that has neither the FIP nor a pure Nash equilibrium.*

Proof. Consider the contest game with two players 1 and 2 with skill $\frac{1}{3}$ and three qualities 1, 2 and 3, with $\mathsf{f}_q = q$ for $q \in [3]$. So participation is mandatory. Assume a product skill-effort function $\Lambda(\frac{1}{3}, \mathsf{f}_q) = \frac{1}{3}\mathsf{f}_q$, $q \in [3]$; so $\Lambda(\frac{1}{3}, \mathsf{f}_1)) = \frac{1}{3}$, $\Lambda(\frac{1}{3}, \mathsf{f}_2) = \frac{2}{3}$ and $\Lambda(\frac{1}{3}, \mathsf{f}_3) = 1$. The payment function P gives payment 1 to the player, if any, choosing the strictly highest quality, or gives payment $\frac{1}{2}$ to each player in case of a tie; so $\mathsf{P}_i(1,1) = \mathsf{P}_i(2,2) = \mathsf{P}_i(3,3) = \frac{1}{2}$ for each player $i \in [2]$, $\mathsf{P}_1(2,1) = \mathsf{P}_2(1,2) = \mathsf{P}_1(3,1) = \mathsf{P}_2(1,3) = \mathsf{P}_1(3,1) = \mathsf{P}_2(1,3) = 1$ and $\mathsf{P}_2(2,1) = \mathsf{P}_1(1,2) = \mathsf{P}_2(3,1) = \mathsf{P}_1(1,3) = \mathsf{P}_2(3,1) = \mathsf{P}_1(1,3) = 0$. Note that these payment functions are not oblivious as the payment to a player choosing a particular quality depends on the numbers of players choosing higher qualities. We check that the game neither has the *FIP* nor a pure Nash equilibrium:

- If player 1 chooses 1, then player 2 gets utility $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ when choosing 1, $1 - \frac{2}{3} = \frac{1}{3}$ when choosing 2, and $1 - 1 = 0$ when choosing 3. So player 2 chooses 2.
- If player 1 chooses 2, then player 2 gets utility $0 - \frac{1}{3} = -\frac{1}{3}$ when choosing 1, $\frac{1}{2} - \frac{2}{3} = -\frac{1}{6}$ when choosing 2, and $1 - 1 = 0$ when choosing 3. So player 2 chooses 3.
- If player 1 chooses 3, then player 2 gets utility $0 - \frac{1}{3} = -\frac{1}{3}$ when choosing 1, $0 - \frac{2}{3} = -\frac{2}{3}$ when choosing 2, and $\frac{1}{2} - 1 = -\frac{1}{2}$ when choosing 3. So player 2 chooses 1.

Since players are anonymous and payments are player-invariant, player 1 best-responds to player 2 in an identical way. Now note that the best-responses form the cycle $\langle 1,2\rangle \rightsquigarrow \langle 3,2\rangle \rightsquigarrow \langle 3,1\rangle \rightsquigarrow \langle 2,1\rangle \rightsquigarrow \langle 2,3\rangle \rightsquigarrow \langle 1,3\rangle \rightsquigarrow \langle 1,2\rangle$, while quality vectors outside the cycle are not pure Nash equilibria. Hence, there is no pure Nash equilibrium. ◀

We continue to prove:

▶ **Proposition 4.** *There is a contest game with mandatory participation, proportional allocation and arbitrary players that has neither the FIP nor a pure Nash equilibrium.*

Proof. Fix an integer parameter $k \geq 2$. Consider the contest game with players 1 and 2 qualities $1, 2, \ldots, Q$ with $\mathsf{f}_q = q$ for each $q \in [Q]$, where $Q = k+1$, and $s_1 = \dfrac{1}{4k-2+\dfrac{1}{k+1}}$ and $s_2 = \dfrac{1}{4k+2+\dfrac{1}{k+1}}$. Consider a quality vector (q_1, q_2). Then,

$$\mathsf{U}_1(q_1,q_2) = \frac{q_1}{q_1+q_2} - \frac{1}{4k-2+\frac{1}{k+1}} q_1$$

and

$$\mathsf{U}_2(q_1,q_2) = \frac{q_2}{q_1+q_2} - \frac{1}{4k+2+\frac{1}{k+1}} q_2 \, .$$

We check that a best-response cycle is possible. Consider a unilateral deviation of player 1 to quality $q_1' > q_1$. Then,

$$\begin{aligned}
\mathsf{U}_1(q_1',q_2) - \mathsf{U}_1(q_1,q_2) &= \frac{q_1'}{q_1'+q_2} - \frac{q_1}{q_1+q_2} - (q_1'-q_1)\frac{1}{4k-2+\frac{1}{k+1}} \\
&= \frac{(q_1'-q_1)q_2}{(q_1'+q_2)(q_1+q_2)} - (q_1'-q_1)\frac{1}{4k-2+\frac{1}{k+1}} \\
&= (q_1'-q_1)\left(\frac{q_2}{(q_1'+q_2)(q_1+q_2)} - \frac{1}{4k-2+\frac{1}{k+1}}\right) .
\end{aligned}$$

Similarly, for a unilateral deviation of player 2 to quality q_2',

$$\mathsf{U}_2(q_1,q_2') - \mathsf{U}_1(q_1,q_2) = (q_2'-q_2)\left(\frac{q_1}{(q_1+q_2')(q_1+q_2)} - \frac{1}{4k+2+\frac{1}{k+1}}\right) .$$

Consider the sequence of deviations $(1,1) \rightsquigarrow (1,2) \rightsquigarrow (2,2) \rightsquigarrow \ldots \rightsquigarrow (k-1,k) \rightsquigarrow (k,k) \rightsquigarrow (k,k+1)$, where players 2 and 1 alternate in taking steps. We prove that these steps are improvements:

- Consider first the step $(\kappa,\kappa) \rightsquigarrow (\kappa,\kappa+1)$, taken by player 2, where $1 \leq \kappa \leq k$. Then,

$$\begin{aligned}
\mathsf{U}_2(\kappa,\kappa+1) - \mathsf{U}_2(\kappa,\kappa) &= \frac{\kappa}{(\kappa+(\kappa+1))(\kappa+\kappa)} - \frac{1}{4k+2+\frac{1}{k+1}} \\
&= \frac{1}{2(2\kappa+1)} - \frac{1}{2(2k+1)+\frac{1}{k+1}} \\
&\geq \frac{1}{2(2k+1)} - \frac{1}{2(2k+1)+\frac{1}{k+1}} \\
&> 0 \, .
\end{aligned}$$

  So the step $(\kappa,\kappa) \rightsquigarrow (\kappa,\kappa+1)$ is an improvement for player 2.

- Consider now the step $(\kappa-1,\kappa) \rightsquigarrow (\kappa,\kappa)$, taken by player 1, where $1 \leq \kappa \leq k$. Then,

$$\begin{aligned}
\mathsf{U}_1(\kappa,\kappa) - \mathsf{U}_1(\kappa-1,\kappa) &= \frac{\kappa}{(\kappa+\kappa)((\kappa-1+\kappa)} - \frac{1}{2(2k-1)+\frac{1}{k+1}} \\
&= \frac{1}{2(2\kappa-1)} - \frac{1}{2(2k-1)+\frac{1}{k+1}} \\
&\geq \frac{1}{2(2k-1)} - \frac{1}{2(2k-1)+\frac{1}{k+1}} \\
&> 0\,.
\end{aligned}$$

So the step $(\kappa-1,\kappa) \rightsquigarrow (\kappa,\kappa+1)$ is an improvement for player 1.

So a unilateral deviation to the immediately higher quality by a player is an improvement. We can similarly prove that a unilateral deviation to a higher quality by either player is an improvement. In particular, no quality vector (q_1,q_2) with $q_1 \leq k$ and $q_2 \leq k+1$ is a pure Nash equilibrium. We will prove that there is an improvement cycle starting with the quality vector $(k,k+1)$.

- Consider first the unilateral deviation $(k,k+1) \rightsquigarrow (k-1,k+1)$ by player 1 to quality $k-1$. Then,

$$\begin{aligned}
&\mathsf{U}_1(k-1,k+1) - \mathsf{U}_1(k,k+1) \\
&= -\left(\frac{k+1}{((k-1)+(k+1))(k+k+1)} - \frac{1}{2(2k-1)+\frac{1}{k+1}}\right) \\
&= -\frac{k+1}{2k(2k+1)} + \frac{1}{2(2k-1)+\frac{1}{k+1}}\,.
\end{aligned}$$

Thus, $\mathsf{U}_1(k-1,k+1) > \mathsf{U}_1(k,k+1) > 0$ if and only if

$$(k+1)\left[2(2k-1)+\frac{1}{k+1}\right] \quad < \quad 2k(2k+1)$$

or

$$2(k+1)(2k-1)+1 \quad < \quad 2k(2k+1)$$

which is verified directly. Hence, the unilateral deviation $(k,k+1) \rightsquigarrow (k-1,k+1)$ by player 1 is an improvement.

- Consider now the unilateral deviation $(k-1,k+1) \rightsquigarrow (k-1,k)$ by player 2 to quality k. Then,

$$\begin{aligned}
&\mathsf{U}_2(k-1,k) - \mathsf{U}_2(k-1,k+1) \\
&= -\left(\frac{k-1}{((k-1)+(k+1))(k+(k-1))} - \frac{1}{2(2k+1)+\frac{1}{k+1}}\right) \\
&= -\frac{k-1}{2k(2k-1)} + \frac{1}{2(2k+1)+\frac{1}{k+1}}\,.
\end{aligned}$$

Thus, $\mathsf{U}_2(k-1,k) > \mathsf{U}_2(k-1,k+1) > 0$ if and only if

$$(k-1)\left[2(2k+1)+\frac{1}{k+1}\right] \quad < \quad 2k(2k-1)$$

or

$$2(k-1)(2k+1)+\frac{k-1}{k+1} \quad < \quad 2k(2k-1)$$

which is verified directly. Hence, the unilateral deviation $(k-1,k+1) \rightsquigarrow (k-1,k)$ by player 2 is an improvement.

- Now the unilateral deviation $(k-1,k) \rightsquigarrow (k,k)$ by player 1 is an improvement as it is a deviation from a lower quality to a higher. The unilateral deviation $(k,k) \rightsquigarrow (k,k+1)$ by player 1 is an improvement for the same reason. Thus, we get the improvement cycle $(k,k+1) \rightsquigarrow (k-1,k+1) \rightsquigarrow (k-1,k) \rightsquigarrow (k,k) \rightsquigarrow (k,k+1)$.

Finally, note that $(k+1,k+1)$ is not a pure Nash equilibrium since the unilateral deviation of player 1 to strategy k is an improvement:

$$\begin{aligned}
\mathsf{U}_1(k,k+1)-\mathsf{U}_1(k+1,k+1) &= -\left(\frac{k+1}{(k+k+1)2\,(k+1)}-\frac{1}{2(2k-1)+\frac{1}{k+1}}\right)\\
&= -\left(\frac{1}{2\,(2k+1)}-\frac{1}{2(2k-1)+\frac{1}{k+1}}\right)\\
&= \frac{1}{2(2k-1)+\frac{1}{k+1}}-\frac{1}{2\,(2k+1)}\\
&> 0\,,
\end{aligned}$$

since $2(2k-1)+\frac{1}{k+1}<2(2k+1)$. The claim follows. ◀

> ▷ **Open Problem** 5. Determine the precise complexity of ∃PNE WITH PLAYER-INVARIANT PAYMENTS and ∃PNE WITH PROPORTIONAL ALLOCATION AND ARBITRARY PLAYERS. We are tempted to conjecture that both are $\mathcal{NP}$-complete.

We now turn to player-specific payments. We show:

▶ **Proposition 6.** *There is a contest game with player-specific payments and anonymous players that has neither the FIP nor a pure Nash equilibrium.*

Proof. Consider the contest game with two players 1 and 2, and two qualities 1 and 2 with $\mathsf{f}_1=1$ and $\mathsf{f}_2=2$. Assume a skill-effort function $\Lambda(1,\mathsf{f}_q)=\mathsf{f}_q$ for all qualities $q\in[Q]$; so $\Lambda(1,\mathsf{f}_1)=1$ and $\Lambda(1,\mathsf{f}_2)=2$. Similarly to *Matching Pennies,* player 1 has big payment when alone on a quality, else very small, and player 2 has big payment when not alone, else very small. Formally, define $\mathsf{P}_1(1,1)=\mathsf{P}_1(2,2)=10^3$ $\mathsf{P}_1(1,2)=\mathsf{P}_1(2,1)=10$, $\mathsf{P}_2(1,2)=\mathsf{P}_2(2,1)=10^3$ and $\mathsf{P}_2(1,1)=\mathsf{P}_2(2,2)=10$. We check that there is no pure Nash equilibrium:

- If player 1 chooses 1, then player 2 gets utility 10^3-1 when choosing 1, and $10-2=8$ when choosing 2. So player 2 chooses 2.
- If player 1 chooses 2, then player 2 gets utility 10^3-1 when choosing 1, and $10-1=9$ when choosing 2. So player 2 chooses 1.
- If player 2 chooses 1, then player 1 gets utility $10-1=9$ when choosing 1, and 10^3-2 when choosing 2. So player 1 chooses 2.
- If player 2 chooses 2, then player 1 gets utility 10^3-1 when choosing 1, and $10-2=8$ when choosing 2. So player 1 chooses 1.

Now note that the best-responses form the cycle $\langle 1,2\rangle \rightsquigarrow \langle 1,1\rangle \rightsquigarrow \langle 2,1\rangle \rightsquigarrow \langle 2,2\rangle \rightsquigarrow \langle 1,2\rangle$, while quality vectors outside the cycle are not Nash equilibria. Hence, there is no pure Nash equilibrium. ◀

We continue to show:

▶ **Theorem 7.** ∃PNE WITH PLAYER-SPECIFIC PAYMENTS *is $\mathcal{NP}$-complete, even if players are anonymous.*

Proof. ∃PNE WITH PLAYER-SPECIFIC PAYMENTS $\in \mathcal{NP}$ since one can guess a quality vector and verify the conditions for a pure Nash equilibrium. To prove $\mathcal{NP}$-hardness, we reduce from the $\mathcal{NP}$-complete problem of deciding the existence of a pure Nash equilibrium in a (finite) succinctly represented strategic game [32, Theorem 2.4.1]. So consider such a game with n players, m strategies and payoff functions $\{\mathsf{F}_i\}_{i\in[n]}$ represented by a poly-time algorithm computing, for a pair of a profile $\mathbf{s}$ and a player $i \in [n]$, the payoff $\mathsf{F}(i,\mathbf{s})$ of player i in $\mathbf{s}$. Construct a contest game with n players, $Q = m$, so that the quality vectors coincide with pure profiles of the strategic game. Define the payment function as $\mathsf{P}_i(i,\mathbf{q}) = \mathsf{F}_i(i,\mathbf{s}) + \Lambda(s_i,\mathsf{f}_q)$ for a player i and a strategy vector $\mathbf{q}$; thus, $\mathsf{U}_i(\mathbf{q}) = \mathsf{F}_i(i,\mathbf{s})$. $\mathcal{NP}$-hardness follows. ◀

3 Proportional Allocation

3.1 Anonymous Players

We show:

▶ **Theorem 8.** *The contest game with proportional allocation, voluntary participation and anonymous players has the FIP and two pure Nash equilibria.*

Proof. It suffices to prove that there is no cycle in the quality improvement graph. Recall that voluntary participation means $\mathsf{f}_1 = 0$. We prove that improvement is possible only if, subject to an exception, the deviating player is switching from a higher quality to a lower quality:

▶ **Lemma 9** (No Switch from Lower Quality to Higher Quality). *Fix a quality vector $\mathbf{q}$ and two distinct qualities $\widetilde{q},\widehat{q} \in [Q]$ with $\widetilde{q} < \widehat{q}$. In an improvement step of a player out of $\mathbf{q}$, $\mathsf{N}_{\mathbf{q}}(\widetilde{q})$ increases and and $\mathsf{N}_{\mathbf{q}}(\widehat{q})$ decreases.*

Proof. Denote $\mathsf{f}_{\widetilde{q}} = \beta$, $\mathsf{f}_{\widehat{q}} = \gamma > \beta$, $\chi = \sum_{q\in[Q]\setminus\{\widetilde{q},\widehat{q}\}} \mathsf{N}_{\mathbf{q}}(q) \geq 0$ and $\mathsf{A} = \sum_{q\in[Q]\setminus\{\widetilde{q},\widehat{q}\}} \mathsf{N}_{\mathbf{q}}(q)\,\mathsf{f}_q \geq 0$. Denote the loads on qualities $\widetilde{q}$ and $\widehat{q}$ as x and y, respectively; thus, $y = n - \chi - x$ We shall abuse notation to denote the quality vector $\mathbf{q}$ as (x,y).

(D1) A deviation of a player from $\widehat{q}$ to $\widetilde{q}$ will be depicted as
$$\begin{array}{c}(x,y)\\ \searrow\\ (x+1,y-1)\end{array}$$
with $x \geq 0$ and $y \geq 1$, so as to guarantee the existence of at least one player $i \in \mathsf{Players}_{\mathbf{q}}(\widehat{q})$. Call such a deviation *rightward&downward.*

(D2) A deviation of a player from $\widetilde{q}$ to $\widehat{q}$ will be depicted as
$$\begin{array}{c}(x-1,y+1)\\ \nwarrow\\ (x,y)\end{array}$$
with $y \geq 0$ and $x \geq 1$, so as to guarantee the existence of at least one player $i \in \mathsf{Players}_{\mathbf{q}}(\widetilde{q})$. Call such a deviation *leftward&upward.*

Note that a rightward&downward deviation is an improvement for the deviating player if and only if the reverse leftward&upward improvement step is not an improvement for her. We shall prove that a rightward&downward deviation $\overset{(x,y)}{\rightsquigarrow}(x+1, y-1)$, with $x \geq 0$ and $y \geq 1$, is an improvement unless $(x, y) = (n-1, 1)$. Consider a player $i \in \mathsf{Players}_{(x,y)}(\widehat{q})$. We proceed by case analysis.

1. Assume first that $\widetilde{q} \neq 1$, so that $\mathsf{f}_{\widetilde{q}} > 0$, implying that $\mathsf{f}_{\widehat{q}} > 0$ as well. So, in this case, denominators in proportional allocation fractions are always strictly positive; as we shall see in the analysis for the case $\widetilde{q}$, this is a crucial property. We have that

$$\mathsf{U}_i((x,y)) = \frac{\gamma}{\mathsf{A} + x\beta + (n-\chi-x)\gamma} - \gamma$$

and

$$\mathsf{U}_i((x+1,y-1)) = \frac{\beta}{\mathsf{A} + (x+1)\beta + (n-\chi-x-1)\cdot\gamma} - \beta\,.$$

$\overset{(x,y)}{\rightsquigarrow}(x+1, y-1)$ is an improvement when $\mathsf{U}_i((x+1,y-1)) > \mathsf{U}_i((x,y))$, or

$$\frac{\beta}{\mathsf{A} + (x+1)\beta + (n-\chi-x-1)\gamma} - \beta > \frac{\gamma}{\mathsf{A} + x\beta + (n-\chi-x)\gamma} - \gamma\,,$$

or

$$-\beta\,\frac{\mathsf{A} + x\beta + (n-\chi-x-1)\gamma}{\mathsf{A} + (x+1)\beta + (n-\chi-x-1)\gamma} > -\gamma\,\frac{\mathsf{A} + (x-1)\beta + (n-\chi-x)\gamma}{\mathsf{A} + x\beta + (n-\chi-x)\gamma}\,.$$

Since both denominators are strictly positive for every quality vector (x, y), the last is equivalent to

$$\begin{aligned} &\beta\,[\mathsf{A} + x\beta + (n-\chi-x)\gamma - \gamma][\mathsf{A} + x\beta + (n-\chi-x)\gamma] \\ <\;& \gamma\,[\mathsf{A} + (x-1)\beta + (n-x-y)\gamma][\mathsf{A} + (x+1)\beta + (n-\chi-x-1)\gamma] \end{aligned}$$

or

$$\begin{aligned} &\beta\,[\mathsf{A} + x\beta + (n-\chi-x)\gamma]^2 - \beta\gamma[\mathsf{A} + x\beta + (n-\chi-x)\gamma] \\ <\;& \gamma\,[\mathsf{A} + x\beta + (n-\chi-x)\gamma - 1][\mathsf{A} + x\beta + (n-\chi-x)\gamma + \beta - \gamma] \\ =\;& \gamma\,\big([\mathsf{A} + x\beta + (n-\chi-x)\gamma]^2 - [\mathsf{A} + x\beta + (n-\chi-x)\gamma] + (\beta-\gamma)[\mathsf{A} + x\beta + (n-\chi-x)\gamma] - (\beta-\gamma)\big) \\ =\;& \gamma\,\big([\mathsf{A} + x\beta + (n-\chi-x)\gamma]^2 - \gamma[\mathsf{A} + x\beta + (n-\chi-x)\gamma] + (\gamma-\beta)\big) \\ =\;& \gamma[\mathsf{A} + x\beta + (n-\chi-x)\gamma]^2 - \gamma^2[\mathsf{A} + x\beta + (n-\chi-x)\gamma] + \gamma(\gamma-\beta) \end{aligned}$$

or

$$(\gamma-\beta)[\mathsf{A} + x\beta + (n-\chi-x)\gamma]^2 - \gamma(\gamma-\beta)[\mathsf{A} + x\beta + (n-\chi-x)\gamma] + \gamma(\gamma-\beta) > 0\,.$$

Since $\gamma > \beta$, the last inequality is equivalent to

$$[\mathsf{A} + x\beta + (n-\chi-x)\gamma]^2 - \gamma[\mathsf{A} + x\beta + (n-\chi-x)\gamma] + \gamma > 0$$

or

$$[\mathsf{A} + x\beta + (n-\chi-x)\gamma]^2 > \gamma\,([\mathsf{A} + x\beta + (n-\chi-x)\gamma] - \beta)\,.$$

Since $n - \chi - x \geq 1$, it follows that $\mathsf{A} + x\beta + (n-\chi-x)\gamma \geq \gamma$, which implies

$$\begin{aligned} [\mathsf{A} + x\beta + (n-\chi-x)\gamma]^2 &\geq \gamma[\mathsf{A} + x\beta + (n-\chi-x)\gamma] \\ &> \gamma[\mathsf{A} + x\beta + (n-\chi-x)\gamma - \beta]\,, \end{aligned}$$

since $\beta > 0$. It follows that $\overset{(x,y)}{\searrow}(x+1, y-1)$ is an improvement, implying that $\overset{(x-1,y+1)}{\nwarrow}(x, y)$, with $x \geq 0$ and $y > 0$, is not.

2. Assume now that $\widetilde{q} = 1$, so that $\mathsf{f}_{\widetilde{q}} = 0$. Then, it is no longer the case that denominators in proportional allocation fractions are always strictly positive. Specifically, when $x = n-1$ and $y = 1$, some denominator becomes 0 as we shall see. So the case $x = n-1$ and $y = 1$ will require special handling. We proceed with the details. In all cases, we have that

$$\mathsf{U}_i((x, y)) = \frac{\beta}{\mathsf{A} + x \cdot 0 + + (n - \chi - x) \cdot \gamma} - \beta = \beta \left(\frac{1}{\mathsf{A} + (n - \chi - x)\gamma} - 1 \right),$$

and

$$\mathsf{U}_i((x+1, y-1)) = \frac{0}{\mathsf{A} + (x+1) \cdot 0 + (n - \chi - x - 1) \cdot \gamma} - 0 = \frac{0}{\mathsf{A} + (n - \chi - x - 1)\gamma}$$

Note that if $y = 1$ and $x = n-1$, then in $\mathsf{U}_i(x+1, y-1)$, $\mathsf{A} = 0$, $\chi = 0$ and $n - \chi - x - 1 = 0$, so that the denominator in the fraction of $\mathsf{U}_i(x+1, y-1)$ becomes also 0, making the fraction indeterminate; in this case, $\mathsf{U}_i((x+1, y-1))$ is 0 by the way indeterminacy is removed. In all other cases, the denominator is strictly positive, which results again in $\mathsf{U}_i((x+1, y-1)) = 0$. So, $\mathsf{U}_i((x+1, y-1)) = 0$ in every case. $\overset{(x,y)}{\searrow}(x+1, y-1)$ is an improvement when $\mathsf{U}_i((x+1, y-1)) > \mathsf{U}_i((x, y))$ or

$$\frac{1}{\mathsf{A} + (n - \chi - x)\gamma} < 1\,.$$

- $\underline{(x, y) = (n-1, 1)}$: Then, the denominator in $\mathsf{U}_i((x, y)$ becomes 1, resulting to $\mathsf{U}_i((x, y))$ is also 0, implying that neither the rightward&downward deviation $\overset{(x,y)}{\searrow}(x+1, y-1)$ nor the leftward&upward deviation $\overset{(x-1,y+1)}{\nwarrow}(x, y)$ is an improvement.
- $\underline{(x, y) \neq (n-1, 1)}$: Thus, either $x = n$ or $x \leq n-2$. We proceed by case analysis.
  - $\underline{x = n}$: Then, $y = 0$ and there can be no $\overset{(x,y)}{\searrow}(x+1, y-1)$ deviation out of $(n, 0)$.
  - $\underline{x \leq n-2}$: Then, $n - \chi - x \geq 2$ and $\mathsf{A} + (n - \chi - x)\gamma \geq 2\gamma > 2$. It follows that the necessary and sufficient condition for an improvement holds.

  It follows that, unless $(x, y) = (n-1, 1)$, the rightward&downward deviation $\overset{(x,y)}{\searrow}(x+1, y-1)$ is an improvement, implying that the leftward&upward deviation $\overset{(x-1,y+1)}{\nwarrow}(x, y)$ is not.

Hence, rightward&downward deviations are improvements except when $(x, y) = (n-1, 1)$.

◀

It follows that the quality improvement graph has two sinks, representing two pure Nash equilibria:

- The node $(n-1, 1)$, corresponding to $\mathsf{N}_{\mathbf{q}}(1) = n-1$, $\mathsf{N}_{\mathbf{q}}(2) = 1$ and $\mathsf{N}_{\mathbf{q}}(q) = 0$ for each quality $q \in [Q]$ with $q > 2$.
- The node $(n, 0)$, corresponding to $\mathsf{N}_{\mathbf{q}}(1) = n$ and $\mathsf{N}_{\mathbf{q}}(q) = 0$ for each quality $q \in [Q]$ with $q > 1$. This node is unreachable by improvement steps.

◀

Under mandatory participation, it no longer holds that $\mathsf{f}_1 = 0$, and Case **2.** in the proof of Lemma 9 does not arise; as a result, the node $(n-1, 1)$, corresponding to $\mathsf{N}_{\mathbf{q}}(1) = n-1$, $\mathsf{N}_{\mathbf{q}}(2) = 1$ and $\mathsf{N}_{\mathbf{q}}(q) = 0$ for each quality $q \in [Q]$ with $q > 2$, is not a sink anymore since the unilateral deviation of a player from quality 2 to quality 1 is now an improvement since $\mathsf{f}_1 > 0$. So we have now a unique pure Nash equilibrium, where all players choose quality 1. The rest of the proof of Theorem 8 transfers over. Hence, we have:

▶ **Theorem 10.** *The contest game with proportional allocation, mandatory participation and anonymous players has the FIP and a unique pure Nash equilibrium.*

Given the counter-example contest game in Proposition 4, Theorem 10 establishes a *separation* with respect to the *FIP* property and the existence of a pure Nash equilibrium between arbitrary players and anonymous players, under mandatory participation and proportional allocation. Theorems 8 and 10 imply:

▶ **Corollary 11.** *The contest game with proportional allocation and anonymous players has a generalized ordinal potential.*

3.2 Mandatory Participation

We show:

▶ **Theorem 12.** *There is a* $\Theta(1)$ *algorithm that solves* ∃PNE with Proportional Allocation and Arbitrary Players *with lower-bounded skills* $\min_{i\in[n]} s_i \geq \dfrac{\mathsf{f}_2}{\mathsf{f}_2 - \mathsf{f}_1}$ *and skill-effort functions* $\Lambda(s_i, \mathsf{f}_q) = s_i \mathsf{f}_q$*, for all players* $i \in [n]$ *and qualities* $q \in [Q]$*.*

Proof. By definition of utility and mandatory participation, the utility of each player $i \in [n]$ is more than $-s_i\mathsf{f}_1$. If player i deviates to 2, its utility will be less than $\mathsf{f}_2 - \mathsf{f}_2 s_i = -\mathsf{f}_2(s_i - 1)$. The assumption implies that $-\mathsf{f}_2(s_i - 1) \leq -\mathsf{f}_1 s_i$ for all players $i \in [n]$. So player i does not want to switch to quality 2. Since efforts are increasing, for all qualities q with $2 < q \leq Q$, the utility of player i when she deviates to q will be less than $-\mathsf{f}_q(s_i - 1) < -\mathsf{f}_2(s_i - 1) \leq -\mathsf{f}_1 s_i$, by the assumption. So player i does not want to switch to any quality $q > 2$ either. Hence, assigning all players to quality 1 is a pure Nash equilibrium. ◀

Since $\dfrac{\mathsf{f}_2}{\mathsf{f}_2 - \mathsf{f}_1} > 1$, the assumption made for Theorem 12 that all skills are lower-bounded by $\dfrac{\mathsf{f}_2}{\mathsf{f}_2 - \mathsf{f}_1}$ in Theorem 12 cannot hold for anonymous players where $s_i = 1$ for all players $i \in [n]$. This assumption is reasonable for real contests for crowdsourcing reviews where a minimum skill is required for reviewers in order to eliminate the risk of receiving inferior solutions of low quality. Indeed, crowdsourcing firms can target crowd contributors based on exhibiting skills, like performance in prior contests. Clearly, the assumption made for Theorem 12, enabling the existence of a pure Nash equilibrium, could *not* hold for the counter-example contest game in Proposition 4.

4 Three-Discrete-Concave Payments and Contiguity

Say that the load vector $\mathsf{N}_{\mathbf{q}}$ is *contiguous* if players 1 to $\mathsf{N}_{\mathbf{q}}(1)$ choose quality 1, players $\mathsf{N}_{\mathbf{q}}(1)+1$ to $\mathsf{N}_{\mathbf{q}}(1)+\mathsf{N}_{\mathbf{q}}(2)$ choose quality 2, and so on till players $\sum_{q\in[Q-1]}\mathsf{N}_{\mathbf{q}}(q)+1$ to n choose quality $q_{\mathsf{last}} \leq Q$ such that for each quality $\widehat{q} > q_{\mathsf{last}}$, $\mathsf{N}_{\mathbf{q}}(\widehat{q}) = 0$; so for any players i and k, with $i < k$, choosing distinct qualities q and q', respectively, we have $q < q'$. Clearly, a contiguous load vector determines by itself which $\mathsf{N}_{\mathbf{q}}(q)$ players choose each quality $q \in [Q]$.

Say that an *inversion* occurs in a load vector $\mathsf{N}_{\mathbf{q}}$ if there are players i and k with $i < k$ choosing qualities q_i and q_k, respectively, with $q_i > q_k$; thus, $s_i \geq s_k$ while $\mathsf{f}_{q_i} > \mathsf{f}_{q_k}$. Call i an *inversion witness*; call i and k an *inversion pair*. Clearly, no inversion occurs in a load vector $\mathsf{N}_{\mathbf{q}}$ if and only if $\mathsf{N}_{\mathbf{q}}$ is contiguous.

Given a contiguous load vector $\mathsf{N}_{\mathbf{q}}$, denote, for each quality $q \in [Q]$ such that $\mathsf{Players}_{\mathbf{q}}(q) \neq \emptyset$, the minimum and the maximum, respectively, player index $i \in \mathsf{Players}_{\mathbf{q}}(q)$ as $\mathsf{first}_{\mathbf{q}}(q)$ and $\mathsf{last}_{\mathbf{q}}(q)$, respectively. Clearly, $\mathsf{first}_{\mathbf{q}}(q) = \sum_{\widehat{q}<q}\mathsf{N}_{\mathbf{q}}(\widehat{q})+1$ and $\mathsf{last}_{\mathbf{q}}(q) = \sum_{\widehat{q}\leq q}\mathsf{N}_{\mathbf{x}}(\widehat{q})$; so $\mathsf{first}_{\mathbf{q}}(1) = 1$ for $\mathsf{N}_{\mathbf{q}}(1) > 0$ and $\mathsf{last}_{\mathbf{q}}(Q) = n$ for $\mathsf{N}_{\mathbf{q}}(Q) > 0$.

Order the players so that $s_1 \geq s_2 \geq \ldots \geq s_n$. Recall that $\mathsf{f}_1 < \mathsf{f}_2 < \ldots < \mathsf{f}_Q$. Represent a quality vector $\mathbf{q}$ as follows:

- Use a *load vector* $\mathsf{N}_{\mathbf{q}} = \langle \mathsf{N}_{\mathbf{q}}(1), \mathsf{N}_{\mathbf{q}}(2), \ldots, \mathsf{N}_{\mathbf{q}}(Q)\rangle$.
- Specify which $\mathsf{N}_{\mathbf{q}}(q)$ players choose each quality $q \in [Q]$.

To simplify notation, we shall often omit to specify the players choosing each quality $q \in [Q]$. Thus, we shall represent a quality vector $\mathbf{q}$ by the load vector $\mathsf{N}_{\mathbf{q}}$.

4.1 Player-Specific Payments

Recall that a player-specific payment function $\mathsf{P}_i(\mathbf{q})$ can be represented by a two-argument payment function $\mathsf{P}_i(i, \mathbf{q})$, where $i \in [n]$ and $\mathbf{q}$ is a quality vector. We start by defining:

▶ **Definition 13.** *A player-specific payment function* P *is three-discrete-concave if for every player* $i \in [n]$*, for every load vector* $\mathsf{N}_{\mathbf{q}}$ *and for every triple of qualities* q_i*,* q_k*,* $q \in [Q]$*,*

$$
\begin{aligned}
& \mathsf{P}_i(i, (\mathsf{N}_{\mathbf{q}}(1), \ldots, \mathsf{N}_{\mathbf{q}}(q_i), \ldots, \mathsf{N}_{\mathbf{q}}(q_k)-1, \ldots, \mathsf{N}'_{\mathbf{q}}(q)+1, \ldots, \mathsf{N}'_{\mathbf{q}}(Q))) + \\
& \mathsf{P}_i(i, (\mathsf{N}_{\mathbf{q}}(1), \ldots, \mathsf{N}_{\mathbf{q}}(q_i)-1, \ldots, \mathsf{N}_{\mathbf{q}}(q_k), \ldots, \mathsf{N}_{\mathbf{q}}(q)+1, \ldots, \mathsf{N}_{\mathbf{x}}(Q))) \\
\leq\ & 2\,\mathsf{P}_i(i, (\mathsf{N}_{\mathbf{q}}(1), \ldots, \mathsf{N}_{\mathbf{q}}(q_i), \ldots, \mathsf{N}_{\mathbf{q}}(q_k), \ldots, \mathsf{N}_{\mathbf{q}}(q), \ldots, \mathsf{N}_{\mathbf{q}}(Q)))\,.
\end{aligned}
$$

The inequality in Definition 13 may be rewritten as

$$
\begin{aligned}
& \mathsf{P}_i(i, (\mathsf{N}_{\mathbf{q}}(1), \ldots, \mathsf{N}_{\mathbf{q}}(q_i)-1, \ldots, \mathsf{N}_{\mathbf{q}}(q_k), \ldots, \mathsf{N}_{\mathbf{q}}(q)+1, \ldots, \mathsf{N}_{\mathbf{q}}(Q))) - \\
& \mathsf{P}_i(i, (\mathsf{N}_{\mathbf{q}}(1), \ldots, \mathsf{N}_{\mathbf{q}}(q_i), \ldots, \mathsf{N}_{\mathbf{q}}(q_k), \ldots, \mathsf{N}_{\mathbf{q}}(q), \ldots, \mathsf{N}_{\mathbf{q}}(Q))) \\
\leq\ & \mathsf{P}_i(i, (\mathsf{N}_{\mathbf{q}}(1), \ldots, \mathsf{N}_{\mathbf{q}}(q_i), \ldots, \mathsf{N}_{\mathbf{q}}(q_k), \ldots, \mathsf{N}_{\mathbf{q}}(q), \ldots, \mathsf{N}_{\mathbf{q}}(Q))) - \\
& \mathsf{P}_i(i, (\mathsf{N}_{\mathbf{q}}(1), \ldots, \mathsf{N}_{\mathbf{q}}(q_i), \ldots, \mathsf{N}_{\mathbf{q}}(q_k)-1, \ldots, \mathsf{N}_{\mathbf{q}}(q)+1, \ldots, \mathsf{N}_{\mathbf{q}}(Q)))\,.
\end{aligned}
$$

We show:

▶ **Theorem 14.** *There is a* $\Theta\left(n \cdot Q^2 \binom{n+Q-1}{Q-1}\right)$ *algorithm that solves* ∃PNE WITH PLAYER-SPECIFIC PAYMENTS *for arbitrary players and three-discrete-concave player-specific payments; for constant* Q*, it is a* $\Theta(n^Q)$ *polynomial algorithm.*

Proof. We start by proving:

▶ **Proposition 15 (Contigufication Lemma for Player-Specific Payments).** *For three-discrete-concave player-specific payments, any pair of (i) a pure Nash equilibrium* $\mathsf{N}_{\mathbf{q}} = \langle \mathsf{N}_{\mathbf{q}}(1), \ldots, \mathsf{N}_{\mathbf{q}}(Q)\rangle$ *and (ii) player sets* $\mathsf{Players}_{\mathbf{q}}(q)$ *for each quality* $q \in [Q]$*, can be transformed into a contiguous pure Nash equilibrium.*

Proof. If no inversion occurs in $\mathsf{N}_{\mathbf{q}}$, then $\mathsf{N}_{\mathbf{q}}$ is contiguous and we are done. Else take the earliest inversion witness i, together with the earliest player k such that i and k make an inversion. We shall also consider a player $\iota \in [n] \setminus \{i, k\}$. Since payments are player-specific,

$$\mathsf{U}_i(\mathsf{N}_{\mathbf{q}}) \quad = \quad \mathsf{P}_i(i, \mathsf{N}_{\mathbf{q}}) - \Lambda(s_i, \mathsf{f}_{q_i})$$

and

$$\mathsf{U}_k(\mathsf{N}_{\mathbf{q}}) \quad = \quad \mathsf{P}_k(k, \mathsf{N}_{\mathbf{q}}) - \Lambda(s_k, \mathsf{f}_{q_k})\,.$$

1. Player i does not want to switch to quality $q \neq q_i$ if and only if

$$\begin{aligned} & \mathsf{P}_i(i, (\mathsf{N}_{\mathbf{q}}(1), \ldots \mathsf{N}_{\mathbf{q}}(q_k), \ldots, \mathsf{N}_{\mathbf{q}}(q_i), \ldots, \mathsf{N}_{\mathbf{q}}(Q))) - \Lambda(s_i, \mathsf{f}_{q_i}) \\ \geq\; & \mathsf{P}_i(i, (\mathsf{N}_{\mathbf{q}}(1), \ldots \mathsf{N}_{\mathbf{q}}(q) + 1, \ldots, \mathsf{N}_{\mathbf{q}}(q_i) - 1, \ldots, \mathsf{N}_{\mathbf{q}}(Q))) - \Lambda(s_i, \mathsf{f}_q)\,, \end{aligned}$$

or

$$\begin{aligned} \Lambda(s_i, \mathsf{f}_{q_i}) - \Lambda(s_i, \mathsf{f}_q) \quad \leq \quad & \mathsf{P}_i(i, (\mathsf{N}_{\mathbf{q}}(1), \ldots \mathsf{N}_{\mathbf{q}}(q), \ldots, \mathsf{N}_{\mathbf{q}}(q_i), \ldots, \mathsf{N}_{\mathbf{q}}(Q))) - \\ & \mathsf{P}_i(i, (\mathsf{N}_{\mathbf{q}}(1), \ldots, \mathsf{N}_{\mathbf{q}}(q) + 1, \ldots, \mathsf{N}_{\mathbf{q}}(q_i) - 1, \ldots, \mathsf{N}_{\mathbf{q}}(Q))) \qquad \text{(C.1)}\,. \end{aligned}$$

2. Player k does not want to switch to quality $q \neq q_k$ if and only if

$$\begin{aligned} & \mathsf{P}_k(k, (\mathsf{N}_{\mathbf{q}}(1), \ldots \mathsf{N}_{\mathbf{q}}(q_k), \ldots, \mathsf{N}_{\mathbf{q}}(q), \ldots, \mathsf{N}_{\mathbf{q}}(Q))) - \Lambda(s_k, \mathsf{f}_{q_k}) \\ \geq\; & \mathsf{P}_k(k, (\mathsf{N}_{\mathbf{q}}(1), \ldots \mathsf{N}_{\mathbf{q}}(q_k) - 1, \ldots, \mathsf{N}_{\mathbf{q}}(q) + 1, \ldots, \mathsf{N}_{\mathbf{q}}(Q))) - \Lambda(s_k, \mathsf{f}_q)\,, \end{aligned}$$

or

$$\begin{aligned} \Lambda(s_k, \mathsf{f}_{q_k}) - \Lambda(s_k, \mathsf{f}_q) \quad \leq \quad & \mathsf{P}_k(k, (\mathsf{N}_{\mathbf{q}}(1), \ldots \mathsf{N}_{\mathbf{q}}(q_k), \ldots, \mathsf{N}_{\mathbf{q}}(q), \ldots, \mathsf{N}_{\mathbf{q}}(Q))) - \\ & \mathsf{P}_k(k, (\mathsf{N}_{\mathbf{q}}(1), \ldots, \mathsf{N}_{\mathbf{q}}(q_k) - 1, \ldots, \mathsf{N}_{\mathbf{q}}(q) + 1, \ldots, \mathsf{N}_{\mathbf{x}}(Q))) \qquad \text{(C.2)}\,. \end{aligned}$$

3. Player ι does not want to switch to quality $q \neq q_\iota$ if and only if

$$\begin{aligned} & \mathsf{P}_\iota(\iota, (\mathsf{N}_{\mathbf{q}}(1), \ldots \mathsf{N}_{\mathbf{q}}(q), \ldots, \mathsf{N}_{\mathbf{q}}(q_\iota), \ldots, \mathsf{N}_{\mathbf{q}}(Q))) - \Lambda(s_\iota, \mathsf{f}_{q_\iota}) \\ \geq\; & \mathsf{P}_\iota(\iota, (\mathsf{N}_{\mathbf{q}}(1), \ldots \mathsf{N}_{\mathbf{q}}(q) + 1, \ldots, \mathsf{N}_{\mathbf{q}}(q_\iota) - 1, \ldots, \mathsf{N}_{\mathbf{q}}(Q))) - \Lambda(s_\iota, \mathsf{f}_q)\,, \end{aligned}$$

or

$$\begin{aligned} \Lambda(s_\iota, \mathsf{f}_{q_\iota}) - \Lambda(s_\iota, \mathsf{f}_q) \quad \leq \quad & \mathsf{P}_\iota(\iota, (\mathsf{N}_{\mathbf{q}}(1), \ldots \mathsf{N}_{\mathbf{q}}(q), \ldots, \mathsf{N}_{\mathbf{q}}(q_\iota), \ldots, \mathsf{N}_{\mathbf{q}}(Q))) - \\ & \mathsf{P}_\iota(\iota, (\mathsf{N}_{\mathbf{q}}(1), \ldots, \mathsf{N}_{\mathbf{q}}(q) + 1, \ldots, \mathsf{N}_{\mathbf{q}}(q_\iota) - 1, \ldots, \mathsf{N}_{\mathbf{q}}(Q))) \qquad \text{(C.3)}\,. \end{aligned}$$

> Swap the qualities chosen by players i and k; so they now choose q_k and q_i, respectively. Choices of other players are preserved.

Denote as $\mathsf{N}_{\mathbf{q}'}$ the resulting load vector; clearly, for each $\widehat{q} \in [Q]$, $\mathsf{N}_{\mathbf{q}'}(\widehat{q}) = \mathsf{N}_{\mathbf{q}}(\widehat{q})$. We prove:

▶ **Lemma 16.** *The earliest inversion witness in $\mathbf{q}'$ is either i or some player $\widehat{i} > i$.*

Proof. Assume, by way of contradiction, that the earliest inversion witness in $\mathbf{q}'$ is a player $j < i$. Since the earliest inversion witness in $\mathbf{q}$ is i, j is not an inversion witness in $\mathbf{q}$. Let $\widehat{q}$ be the quality chosen by j in $\mathbf{q}$ and $\mathbf{q}'$. Since players other than i and k do not change qualities in $\mathbf{q}'$, j makes an inversion pair with either i or k in $\mathbf{q}'$. There are two cases.

- j makes an inversion pair with i in $\mathbf{q}'$: Since i chooses quality q_k in $\mathbf{q}'$, it follows that $\widehat{q} > q'$. Since $k > j$ and k chooses quality q_k in $\mathbf{q}$, this implies that j and k make an inversion pair in $\mathbf{q}$.

- j makes an inversion pair with k in $\mathbf{q}'$: Since k chooses quality q_i in $\mathbf{q}'$, it follows that $\widehat{q} > q_i$. Since $i > j$ and i chooses quality q_i in $\mathbf{q}$, this implies that j and i make an inversion pair in $\mathbf{q}$.

In either case, since $i > j$, i is not the earliest witness of inversion in $\mathbf{q}$. A contradiction. ◀

We continue to prove:

▶ **Lemma 17.** $\mathsf{N}_{\mathbf{q}'}$ *is a pure Nash equilibrium if and only if* $\mathsf{N}_{\mathbf{q}}$ *is.*

Proof. We consider the following cases:

1. Player i does not want to switch to quality $q \neq q_k$ if and only if

$$\begin{aligned} & \mathsf{P}_i(i, (\mathsf{N}_{\mathbf{q}}(1), \ldots, \mathsf{N}_{\mathbf{q}}(q_i), \ldots, \mathsf{N}_{\mathbf{q}}(q_k), \ldots, \mathsf{N}_{\mathbf{q}}(q), \ldots, \mathsf{N}_{\mathbf{q}}(Q))) - \Lambda(s_i, \mathsf{f}_{q_k}) \\ \geq\ & \mathsf{P}_i(i, (\mathsf{N}_{\mathbf{q}}(1), \ldots, \mathsf{N}_{\mathbf{q}}(q_i), \ldots \mathsf{N}_{\mathbf{q}}(q_k) - 1, \ldots, \mathsf{N}_{\mathbf{q}}(q) + 1, \ldots, \mathsf{N}_{\mathbf{q}}(Q))) - \Lambda(s_i, \mathsf{f}_q) \end{aligned}$$

or

$$\begin{aligned} \Lambda(s_i, \mathsf{f}_{q_k}) - \Lambda(s_i, \mathsf{f}_q) \ \leq\ & \mathsf{P}_i(i, (\mathsf{N}_{\mathbf{q}}(1), \ldots, \mathsf{N}_{\mathbf{q}}(q_i), \ldots, \mathsf{N}_{\mathbf{q}}(q_k), \ldots, \mathsf{N}_{\mathbf{q}}(q), \ldots, \mathsf{N}_{\mathbf{q}}(Q))) - \\ & \mathsf{P}_i(i, (\mathsf{N}'_{\mathbf{q}}(1), \ldots, \mathsf{N}_{\mathbf{q}}(q_i), \ldots, \mathsf{N}_{\mathbf{q}}(q_k) - 1, \ldots, \mathsf{N}_{\mathbf{q}}(q) + 1, \ldots, \mathsf{N}_{\mathbf{q}}(Q))) \,. \quad \text{(C.4)} \end{aligned}$$

2. Player k does not want to switch to quality $q \neq q_i$ if and only if

$$\begin{aligned} & \mathsf{P}_k(k, (\mathsf{N}_{\mathbf{q}}(1), \ldots, \mathsf{N}_{\mathbf{q}}(q_i), \ldots, \mathsf{N}_{\mathbf{q}}(q_k), \ldots, \mathsf{N}_{\mathbf{q}}(q), \ldots, \mathsf{N}_{\mathbf{q}}(Q))) - \Lambda(s_k, \mathsf{f}_{q_i}) \\ \geq\ & \mathsf{P}_k(k, (\mathsf{N}_{\mathbf{q}}(1), \ldots, \mathsf{N}_{\mathbf{q}}(q_i) - 1, \ldots, \mathsf{N}_{\mathbf{q}}(q_k), \ldots, \mathsf{N}_{\mathbf{q}}(q) + 1, \ldots, \mathsf{N}_{\mathbf{q}}(Q))) - \Lambda(s_k, \mathsf{f}_q) \end{aligned}$$

or

$$\begin{aligned} \Lambda(s_k, \mathsf{f}_{q_i}) - \Lambda(s_k, \mathsf{f}_q) \ \leq\ & \mathsf{P}_k(k, (\mathsf{N}_{\mathbf{q}}(1), \ldots, \mathsf{N}_{\mathbf{q}}(q_i), \ldots, \mathsf{N}_{\mathbf{q}}(q_k), \ldots, \mathsf{N}_{\mathbf{q}}(q), \ldots, \mathsf{N}_{\mathbf{q}}(Q))) - \\ & \mathsf{P}_k(k, (\mathsf{N}_{\mathbf{q}}(1), \ldots, \mathsf{N}_{\mathbf{q}}(q_i) - 1, \ldots, \mathsf{N}_{\mathbf{q}}(q_k), \ldots, \mathsf{N}_{\mathbf{q}}(q) + 1, \ldots, \mathsf{N}_{\mathbf{q}}(Q))) \,. \quad \text{(C.5)} \end{aligned}$$

3. Player ι does not want to switch to quality $q_\kappa \in [Q] \setminus \{q_\iota\}$ in $\mathbf{q}'$ if and only if

$$\begin{aligned} & \mathsf{P}_\iota(\iota, (\mathsf{N}_{\mathbf{q}}(1), \ldots, \mathsf{N}_{\mathbf{q}}(q_\iota), \ldots, \mathsf{N}_{\mathbf{q}}(q_\kappa), \ldots, \mathsf{N}_{\mathbf{q}}(Q))) - \Lambda(s_\iota, \mathsf{f}_{q_\iota}) \\ \geq\ & \mathsf{P}_\iota(\iota, (\mathsf{N}_{\mathbf{q}}(1), \ldots, \mathsf{N}_{\mathbf{q}}(q_\iota) - 1, \ldots, \mathsf{N}_{\mathbf{q}}(q_\kappa) + 1, \ldots, \mathsf{N}_{\mathbf{q}}(Q))) - \Lambda(s_\iota, \mathsf{f}_{q_\kappa}) \end{aligned}$$

or

$$\begin{aligned} \Lambda(s_\iota, \mathsf{f}_{q_\iota}) - \Lambda(s_\iota, \mathsf{f}_{q_\kappa}) \ \leq\ & \mathsf{P}_\iota(\iota, (\mathsf{N}_{\mathbf{q}}(1), \ldots, \mathsf{N}_{\mathbf{q}}(q_\iota), \ldots, \mathsf{N}_{\mathbf{q}}(q_\kappa), \ldots, \mathsf{N}_{\mathbf{q}}(Q))) - \\ & \mathsf{P}_\iota(\iota, (\mathsf{N}_{\mathbf{q}}(1), \ldots, \mathsf{N}_{\mathbf{q}}(q_\iota) - 1, \ldots, \mathsf{N}_{\mathbf{q}}(q_\kappa) + 1, \ldots, \mathsf{N}_{\mathbf{q}}(Q))) \,. \quad \text{(C.6)} \end{aligned}$$

Hence, we conclude:

1. From the rewriting of the inequality for player i in Definition 13,

$$\begin{aligned} & \mathsf{P}_i(i, (\mathsf{N}_{\mathbf{q}'}(1), \ldots, \mathsf{N}_{\mathbf{q}}(q_i) - 1, \ldots, \mathsf{N}_{\mathbf{q}}(q_k), \ldots, \mathsf{N}_{\mathbf{q}}(q) + 1, \ldots, \mathsf{N}_{\mathbf{q}}(Q))) - \\ & \mathsf{P}_i(i, (\mathsf{N}_{\mathbf{q}}(1), \ldots, \mathsf{N}_{\mathbf{q}}(q_i), \ldots, \mathsf{N}_{\mathbf{q}}(q_k), \ldots, \mathsf{N}_{\mathbf{q}}(q), \ldots, \mathsf{N}_{\mathbf{q}}(Q))) \\ \leq\ & \mathsf{P}_i(i, (\mathsf{N}_{\mathbf{q}}(1), \ldots \mathsf{N}_{\mathbf{q}}(q_i), \ldots, \mathsf{N}_{\mathbf{q}}(q_k), \ldots, \mathsf{N}_{\mathbf{q}}(q), \ldots, \mathsf{N}_{\mathbf{q}}(Q))) - \\ & \mathsf{P}_i(i, (\mathsf{N}_{\mathbf{q}}(1), \ldots, \mathsf{N}_{\mathbf{q}}(q_i), \ldots, \mathsf{N}_{\mathbf{q}}(q_k) - 1, \ldots, \mathsf{N}_{\mathbf{q}}(q) + 1, \ldots, \mathsf{N}_{\mathbf{q}}(Q))) \,. \end{aligned}$$

Hence,

$$\begin{aligned} & \mathsf{P}_i(i, (\mathsf{N}_{\mathbf{q}}(1), \ldots, \mathsf{N}_{\mathbf{q}}(q_i) - 1, \ldots, \mathsf{N}_{\mathbf{q}}(q_k), \ldots, \mathsf{N}_{\mathbf{q}}(q) + 1, \ldots, \mathsf{N}_{\mathbf{q}}(Q))) - \\ & \mathsf{P}_i(i, (\mathsf{N}_{\mathbf{q}}(1), \ldots, \mathsf{N}_{\mathbf{q}}(q_i), \ldots, \mathsf{N}_{\mathbf{q}}(q_k), \ldots, \mathsf{N}_{\mathbf{q}}(q), \ldots, \mathsf{N}'_{\mathbf{q}}(Q))) \\ \leq\ & \Lambda(s_i, \mathsf{f}_q) - \Lambda(s_i, \mathsf{f}_{q_i}), \Lambda(s_i, \mathsf{f}_{q_k}) - \Lambda(s_i, \mathsf{f}_q) \\ \leq\ & \mathsf{P}_i(i, (\mathsf{N}_{\mathbf{q}}(1), \ldots \mathsf{N}_{\mathbf{q}}(q_i), \ldots, \mathsf{N}_{\mathbf{q}}(q_k), \ldots, \mathsf{N}_{\mathbf{q}}(q), \ldots, \mathsf{N}_{\mathbf{q}}(Q))) - \\ & \mathsf{P}_i(i, (\mathsf{N}_{\mathbf{q}}(1), \ldots, \mathsf{N}_{\mathbf{q}}(q_i), \ldots, \mathsf{N}_{\mathbf{q}}(q_k) - 1, \ldots, \mathsf{N}_{\mathbf{q}}(q) + 1, \ldots, \mathsf{N}_{\mathbf{q}}(Q))) \end{aligned}$$

if and only if both (C.1) and (C.4) hold.

2. From the rewriting of the inequality for player k in Definition 13,

$$\begin{aligned} & \mathsf{P}_k(k, (\mathsf{N}_{\mathbf{q}'}(1), \ldots, \mathsf{N}_{\mathbf{q}}(q_i), \ldots, \mathsf{N}_{\mathbf{q}}(q_k) - 1, \ldots, \mathsf{N}_{\mathbf{q}}(q) + 1, \ldots, \mathsf{N}_{\mathbf{q}}(Q))) - \\ & \mathsf{P}_k(k, (\mathsf{N}_{\mathbf{q}}(1), \ldots, \mathsf{N}_{\mathbf{q}}(q_i), \ldots, \mathsf{N}_{\mathbf{q}}(q_k), \ldots, \mathsf{N}_{\mathbf{q}}(q), \ldots, \mathsf{N}_{\mathbf{q}}(Q))) \\ \leq\ & \mathsf{P}_k(k, (\mathsf{N}_{\mathbf{q}}(1), \ldots \mathsf{N}_{\mathbf{q}}(q_i), \ldots, \mathsf{N}_{\mathbf{q}}(q_k), \ldots, \mathsf{N}_{\mathbf{q}}(q), \ldots, \mathsf{N}_{\mathbf{q}}(Q))) - \\ & \mathsf{P}_k(k, (\mathsf{N}_{\mathbf{q}}(1), \ldots, \mathsf{N}_{\mathbf{q}}(q_i) - 1, \ldots, \mathsf{N}_{\mathbf{q}}(q_k), \ldots, \mathsf{N}_{\mathbf{q}}(q) + 1, \ldots, \mathsf{N}_{\mathbf{q}}(Q)))\,. \end{aligned}$$

Hence,

$$\begin{aligned} & \mathsf{P}_k(k, (\mathsf{N}_{\mathbf{q}}(1), \ldots, \mathsf{N}_{\mathbf{q}}(q_i), \ldots, \mathsf{N}_{\mathbf{q}}(q_k) - 1, \ldots, \mathsf{N}_{\mathbf{q}}(q) + 1, \ldots, \mathsf{N}_{\mathbf{q}}(Q))) - \\ & \mathsf{P}_i(i, (\mathsf{N}_{\mathbf{q}}(1), \ldots, \mathsf{N}_{\mathbf{q}}(q_i), \ldots, \mathsf{N}_{\mathbf{q}}(q_k), \ldots, \mathsf{N}_{\mathbf{q}}(q), \ldots, \mathsf{N}_{\mathbf{q}}(Q))) \\ \leq\ & \Lambda(s_k, \mathsf{f}_q) - \Lambda(s_k, \mathsf{f}_{q_i}), \Lambda(s_k, \mathsf{f}_{q_k}) - \Lambda(s_k, \mathsf{f}_q) \\ \leq\ & \mathsf{P}_k(k, (\mathsf{N}_{\mathbf{q}}(1), \ldots \mathsf{N}_{\mathbf{q}}(q_i), \ldots, \mathsf{N}_{\mathbf{q}}(q_k), \ldots, \mathsf{N}_{\mathbf{q}}(q), \ldots, \mathsf{N}_{\mathbf{q}}(Q))) - \\ & \mathsf{P}_k(k, (\mathsf{N}_{\mathbf{q}}(1), \ldots, \mathsf{N}_{\mathbf{q}}(q_i) - 1, \ldots, \mathsf{N}_{\mathbf{q}}(q_k), \ldots, \mathsf{N}_{\mathbf{q}}(q) + 1, \ldots, \mathsf{N}_{\mathbf{q}}(Q))) \end{aligned}$$

if and only if both (C.2) and (C.5) hold.

3. Since (C.3) and (C.6) are identical, it follows that player ι does not want to switch to a quality $q_\kappa \neq q_\iota$ in $\mathbf{q}$ if and only if she does not want to switch to the quality q_κ in $\mathbf{q}'$.

The conclusions imply that no player wants to switch qualities in $\mathbf{q}$ if and only if she does not want to switch qualities in $\mathbf{q}'$. The claim follows. ◀

Now the earliest inversion witness, if any, in $\mathbf{q}'$ is either i, the earliest witness of inversion in $\mathbf{q}$, making an inversion pair with a player $\widehat{k} > k$, or greater than i. It follows inductively that a pure Nash equilibrium exists if and only if a contiguous pure Nash equilibrium exists. ◀

By Proposition 15, it suffices to search over contiguous load vectors. Fix a load vector $\mathsf{N}_{\mathbf{q}}$ and a quality $q \in [Q]$ such that $\mathsf{Players}_{\mathbf{q}}(q) \neq \emptyset$. No player choosing quality q wants to switch to the quality $q' \neq q$ if and only if for all players $i \in \mathsf{Players}_{\mathbf{q}}(q)$,

$$\mathsf{P}_i(i, \mathsf{N}_{\mathbf{q}}) - \Lambda(s_i, \mathsf{f}_q) \quad \geq \quad \mathsf{P}_i(i, (\mathsf{N}_{\mathbf{q}}(1), \ldots, \mathsf{N}_{\mathbf{q}}(q) - 1, \ldots, \mathsf{N}_{\mathbf{q}}(q') + 1, \ldots, \mathsf{N}_{\mathbf{q}}(Q))) - \Lambda(s_i, \mathsf{f}_{q'})$$

or

$$\Lambda(s_i, \mathsf{f}_q) - \Lambda(s_i, \mathsf{f}_{q'}) \quad \leq \quad \mathsf{P}_i(i, \mathsf{N}_{\mathbf{q}}) - \mathsf{P}_i(i, (\mathsf{N}_{\mathbf{q}}(1), \ldots, \mathsf{N}_{\mathbf{q}}(q) - 1, \ldots, \mathsf{N}_{\mathbf{q}}(q') + 1, \ldots, \mathsf{N}_{\mathbf{q}}(Q)))\,. \tag{C.7}$$

Since P is player-specific, $\mathsf{P}_i(i, \mathsf{N}_{\mathbf{q}})$ and $\mathsf{P}_i(i, (\mathsf{N}_{\mathbf{q}}(1), \ldots, \mathsf{N}_{\mathbf{q}}(q)-1, \ldots, \mathsf{N}_{\mathbf{q}}(q')+1, \ldots, \mathsf{N}_{\mathbf{q}}(Q)))$ are not constant over all players choosing quality q in $\mathsf{N}_{\mathbf{q}}$ and switching to quality q' in $(\mathsf{N}_{\mathbf{q}}(1), \ldots, \mathsf{N}_{\mathbf{q}}(q) - 1, \ldots, \mathsf{N}_{\mathbf{q}}(q') + 1, \ldots, \mathsf{N}_{\mathbf{q}}(Q))$, respectively. Hence, no player choosing quality $q \in [Q]$ wants to switch to a quality $q' \neq q$ if and only if (C.4) holds for all players $i \in \mathsf{Players}_{\mathbf{q}}(q)$.

To compute a pure Nash equilibrium, we enumerate all contiguous load vectors $\mathsf{N}_{\mathbf{q}} = \langle \mathsf{N}_{\mathbf{q}}(1), \mathsf{N}_{\mathbf{q}}(2), \ldots, \mathsf{N}_{\mathbf{q}}(Q) \rangle$, searching for one that satisfies (C.7), for each quality $q \in [Q]$ and for all players $i \in \mathsf{Players}_{\mathbf{q}}(q)$; clearly, there are $\binom{n+Q-1}{Q-1}$ contiguous load vectors (cf. [7, Section 2.6]). For a player-specific payment function, checking (C.7) for a quality $q \in [Q]$ entails no minimum computation but must be repeated n times for all players $i \in [n]$; checking that the inequality holds for a particular $q' \neq q$ takes time $\Theta(1)$, so checking that it holds for all qualities $q' \neq q$ takes time $\Theta(Q)$, and checking that it holds for all $q \in [Q]$ takes time $\Theta(Q^2)$. Thus, the total time is $\Theta\left(n \cdot Q^2 \cdot \binom{n+Q-1}{Q-1}\right)$. For constant Q, this is a polynomial $\Theta\left(n^Q\right)$ algorithm.

By Proposition 15, a contiguous load vector satisfying (C.7) for each quality $q \in [Q]$ exists if and only if it will be found by the algorithm enumerating all contiguous load vectors. Hence, the algorithm solves ∃PNE WITH PLAYER-SPECIFIC PAYMENTS. ◀

4.2 Player-Invariant Payments

Recall that a player-invariant payment function $\mathsf{P}_i(\mathbf{q})$ can be represented by a two-argument payment function $\mathsf{P}_i(q, \mathbf{q}_{-i})$, where $q \in [Q]$ and $\mathbf{q}_{-i}$ is a partial quality vector, for some player $i \in [n]$. In correspondence to three-discrete-concave player-specific payments, we define:

▶ **Definition 18.** *A player-invariant payment function* P *is three-discrete-concave if for every player* $i \in [n]$*, for every load vector* $\mathsf{N}_\mathbf{q}$ *and for every triple of qualities* q_i, q_k, $q \in [Q]$,

$$\begin{aligned} & \mathsf{P}_i(q, (\mathsf{N}_\mathbf{q}(1), \ldots, \mathsf{N}_\mathbf{q}(q_i), \ldots, \mathsf{N}_\mathbf{q}(q_k) - 1, \ldots, \mathsf{N}_\mathbf{q}(q) + 1, \ldots, \mathsf{N}_\mathbf{q}(Q))) + \\ & \mathsf{P}_i(q, (\mathsf{N}_\mathbf{q}(1), \ldots, \mathsf{N}_\mathbf{q}(q_i) - 1, \ldots, \mathsf{N}_\mathbf{q}(q_k), \ldots, \mathsf{N}_\mathbf{q}(q) + 1, \ldots, \mathsf{N}_\mathbf{q}(Q))) \\ \leq\ & 2\,\mathsf{P}_i(q_i, (\mathsf{N}_\mathbf{q}(1), \ldots, \mathsf{N}_\mathbf{q}(q_i), \ldots, \mathsf{N}_\mathbf{q}(q_k), \ldots, \mathsf{N}_\mathbf{q}(q), \ldots, \mathsf{N}_\mathbf{q}(Q)))\,. \end{aligned}$$

In correspondence to Proposition 15, we prove a Contigufication Lemma for three-discrete-concave player-invariant payment functions:

▶ **Proposition 19 (Contigufication Lemma for Player-Invariant Payments).** *For three-discrete-concave player-invariant payments, any pair of (i) a pure Nash equilibrium* $\mathsf{N}_\mathbf{q} = \langle \mathsf{N}_\mathbf{q}(1), \ldots, \mathsf{N}_\mathbf{q}(Q) \rangle$ *and (ii) player sets* $\mathsf{Players}_\mathbf{q}(q)$ *for each quality* $q \in [Q]$*, can be transformed into a contiguous pure Nash equilibrium.*

By Proposition 19, it suffices to search over contiguous load vectors. Fix a load vector $\mathsf{N}_\mathbf{q}$ and a quality $q \in [Q]$ such that $\mathsf{Players}_\mathbf{q}(q) \neq \emptyset$. No player choosing quality q wants to switch to the quality $q' \neq q$ if and only if for all players $i \in \mathsf{Players}_\mathbf{q}(q)$,

$$\mathsf{P}_i(q, \mathsf{N}_\mathbf{q}) - \Lambda(s_i, \mathsf{f}_q) \quad \geq \quad \mathsf{P}_i(q', (\mathsf{N}_\mathbf{q}(1), \ldots, \mathsf{N}_\mathbf{q}(q) - 1, \ldots, \mathsf{N}_\mathbf{q}(q') + 1, \ldots, \mathsf{N}_\mathbf{q}(Q))) - \Lambda(s_i, \mathsf{f}_{q'})$$

or

$$\Lambda(s_i, \mathsf{f}_q) - \Lambda(s_i, \mathsf{f}_{q'}) \quad \leq \quad \mathsf{P}_i(q, \mathsf{N}_\mathbf{q}) - \mathsf{P}_i(q', (\mathsf{N}_\mathbf{q}(1), \ldots, \mathsf{N}_\mathbf{q}(q) - 1, \ldots, \mathsf{N}_\mathbf{q}(q') + 1, \ldots, \mathsf{N}_\mathbf{q}(Q)))\,. \quad \text{(C.8)}$$

Since P is player-invariant, $\mathsf{P}_i(q, \mathsf{N}_\mathbf{q})$ and $\mathsf{P}_i(q', (\mathsf{N}_\mathbf{q}(1), \ldots, \mathsf{N}_\mathbf{q}(q){-}1, \ldots, \mathsf{N}_\mathbf{q}(q'){+}1, \ldots, \mathsf{N}_\mathbf{q}(Q)))$ are constant over all players choosing quality q in $\mathsf{N}_\mathbf{q}$ and switching to quality q' in $(\mathsf{N}_\mathbf{q}(1), \ldots, \mathsf{N}_\mathbf{q}(q) - 1, \ldots, \mathsf{N}_\mathbf{q}(q') + 1, \ldots, \mathsf{N}_\mathbf{q}(Q))$, respectively. Hence, no player $\widehat{i}$ choosing quality $q \in [Q]$ wants to switch to a quality $q' \neq q$ if and only if (C.8) holds for each quality $q' \neq q$, where $\widehat{i} \in \mathsf{Players}_\mathbf{q}(q)$ is arbitrarily chosen.

To compute a pure Nash equilibrium, we enumerate all contiguous load vectors $\mathsf{N}_\mathbf{q} = \langle \mathsf{N}_\mathbf{q}(1), \mathsf{N}_\mathbf{q}(2), \ldots, \mathsf{N}_\mathbf{q}(Q) \rangle$, searching for one that satisfies (C.8), for each quality $q \in [Q]$ and for a player $\widehat{i} \in \mathsf{Players}_\mathbf{q}(q)$; clearly, there are $\binom{n+Q-1}{Q-1}$ contiguous load vectors (cf. [7, Section 2.6]. For player-invariant payments, checking (C.8) for a quality $q \in [Q]$ entails the computation of the minimum of a function on a set of size $\mathsf{N}_\mathbf{q}(q)$; computation of the minima for all qualities $q \in [Q]$ takes time $\sum_{q \in [Q]} \Theta(\mathsf{N}_\mathbf{q}(q)) = \Theta\left(\sum_{q \in [Q]} \mathsf{N}_\mathbf{q}(q)\right) = \Theta(n)$. Thus, the total time is $\binom{n+Q-1}{Q-1} \cdot (\Theta(n) + \Theta(Q^2)) = \Theta\left(\max\{n, Q^2\} \cdot \binom{n+Q-1}{Q-1}\right)$.

By Proposition 15, a contiguous load vector satisfying (C.8) for each quality $q \in [Q]$ exists if and only if it will be found by the algorithm enumerating all contiguous load vectors. Hence, it follows:

▶ **Theorem 20.** *There is a* $\Theta\left(\max\{n, Q^2\} \cdot \binom{n+Q-1}{Q-1}\right)$ *algorithm that solves* ∃PNE WITH PLAYER-INVARIANT PAYMENTS *for arbitrary players and three-discrete-concave player-invariant payments; for constant* Q*, it is a* $\Theta(n^Q)$ *polynomial algorithm.*

▷ **Open Problem** 21. Investigate the possibility of improving the time complexities of the algorithms in Theorems 14 and 20. For constant Q, this means reducing the exponent Q of n. Assumptions stronger than three-discrete-concavity on the payments might be required.

5 Open Problems and Directions for Further Research

This work poses far more challenging problems and research directions about the contest game than it answers. To close we list a few open research directions.

1. Study the computation of *mixed* Nash equilibria. Work in progress confirms the existence of contest games with $Q = 3$ and $n = 3$ that have only one mixed Nash equilibrium, which is irrational. We conjecture that the problem is $\mathcal{PPAD}$-complete for $n = 2$.
2. Determine the complexity of computing *best-responses* for the contest game. We conjecture $\mathcal{NP}$-hardness; techniques similar to those used in [16, Section 3] could be useful.
3. Formulate incomplete information contest games with discrete strategy spaces and study their Bayes-Nash equilibria. Ideas from Bayesian congestion games [19] will very likely be helpful. Study existence and complexity properties of pure Bayes-Nash equilibria.
4. In analogy to weighted congestion games [26, 31], formulate the *weighted* contest game with discrete strategy spaces, where reviewers have *weights,* and study its pure Nash equilibria.

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