The Gravity of Warped Throats

de Sitter vacua and Gravitational Waves from Type IIB string theory

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Abstract

In this thesis we discuss challenges and opportunities arising from warping the extra dimensions of string theory. After reviewing the required background (including the essentials of Type IIB string theory; flux compactifications; conifolds, warping, and the Klebanov-Strassler and GKP solutions; and the KKLT and LVS proposals) we will discuss de Sitter solutions in warped flux compactifications. We revisit some strongly-warped solutions, present a new solution in a weakly-warped regime and discuss the advantages of weak warping. We then consider the robustness of the new solution in the presence of subleading corrections to the scalar potential. We also explore the difficulties of realising alternative quintessence models as quasi-de Sitter solutions, showing in particular that the generic behaviour of the (single field) scalar potential arising for different types of string theory moduli does not allow for a slow-roll accelerated expansion at the tail of a runaway.

We then take the first steps in understanding the effects of warping in gravitational wave signatures of extra dimensions. By considering the tower of Kaluza-Klein spin-2 states arising from a warped compactification of Type IIB string theory, we study the effects of warping on their masses and wavefunction profiles, which we then use to compute corrections to the Newtonian potential that one can compare with current constraints on fifth forces. This allows us to combine theoretical consistency constraints on the parameter space with the range of parameters experimentally excluded, thereby providing a direct connection between string theory quantities and observations. Although a careful study of gravitational wave signals is left for future work, we briefly outline how these results directly apply in that context and suggest which sources might be more promising for future detection.

Declaration

I hereby declare that the material presented in this doctoral thesis is the result of my own research activity together with my collaborators. All references to other people's work are cited explicitly. All my work has been carried out in the String and Beyond the Standard Model Phenomenology group in the Department of Mathematical Sciences at the University of Liverpool during my doctoral studies.

The results presented in this thesis are based on the following publications

1. Bruno Valeixo Bento, D. Chakraborty, S. Parameswaran, I. Zavala,
   Dark Energy in String Theory
   PoS CORFU2019 (2020) 123 [arXiv:2005.10168]
2. Bruno Valeixo Bento, D. Chakraborty, S. Parameswaran, I. Zavala,
   A new de Sitter solution with a weakly warped deformed conifold
   JHEP 12 (2021) 124 [arXiv:2105.03370]
3. Bruno Valeixo Bento, D. Chakraborty, S. Parameswaran, I. Zavala,
   Gravity at the tip of the throat
   JHEP 09 (2022) 208 [arXiv:2204.02086]
4. Bruno Valeixo Bento, D. Chakraborty, S. Parameswaran, I. Zavala,
   A guide to frames, 2π's, scales and corrections in string compactifications
   Submitted for publication [arXiv:2301.05178]
5. Bruno Valeixo Bento, D. Chakraborty, S. Parameswaran, I. Zavala,
   de Sitter vacua – when are 'subleading corrections' really subleading?
   Submitted for publication [arXiv:2306.07332]

The following, also published during my doctoral studies, will not be discussed in this thesis

6. Bruno Valeixo Bento, F. Dowker, S. Zalel,
   If time had no beginning: growth dynamics for past-infinite causal sets
   Class.Quant.Grav. 39 (2022) 4, 045002 [arXiv:2109.10749]

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[1] I am sorry my first gesture was to kick you out of my desk.

assure you I am. You were there the whole time and only made it better — I *did* honestly enjoy being your flatmate for nothing less than 3 years and, believe it or not, you made it feel like home. I enjoyed our movie nights, our discussions, our games and our *house rules*. And thanks for getting me into Twitter (I think). Now trust me on this: you are smarter than you think, funnier than you think, nicer than you think and that's why people like you (even though you're sometimes clueless; and French). I think you grew a lot these past few years and I am proud of you — I'm sure you'll be okay, cRazy as it might sound. *Who would have thought, huh?*

Ana Sofia and Pedro Gonçalo[2] — friendship tends to walk the land of love in paved roads; ours took a run for the fields and ran free. It is hard to thank you enough for being there every single day, even harder to thank you for who you are — Friends that capitals and qualifiers will never be able to capture.

Finally, to my home — my mum, my sister, and my grandma. Whatever changes in my life, wherever I end up and however long I stay away, you are and will always be my home.

[2]Tu é que és prigui!

Abbreviations

GR	General Relativity
UV	Ultraviolet (high-energy)
IR	Infrared (low-energy)
EFT	Effective Field Theory
NS	Neveu-Schwarz (sector; usually NSNS)
R	Ramond (sector; usually RR)
KK	Kaluza-Klein (tower/states)
$\mathrm{CY}_{\mathbf{3}}$	Calabi-Yau 3-fold
KS	Klebanov-Strassler (solution)
GKP	Gidding-Kachru-Polchinski (solution)
GVW	Gukov-Vafa-Witten (superpotential)
KKLT	Kachru-Kallosh-Linde-Trivedi (mechanism)
LVS	Large Volume Scenario
$\overline{\mathbf{D3}}$	Anti-D3 brane
NPNS	Non-perturbative no-scale (structure)
GW	Gravitational Wave

Contents

1. Introduction

> There is no direct experimental evidence for string theory.
>
> Joseph Conlon, *Why String Theory?*

And yet the birth of string theory could not have been more experimental in nature. What later gave rise to a theory of strings, started as nothing more than a *formula* that nicely incorporated both empirical evidence and mathematical properties expected of strongly interacting systems. Nowhere in Veneziano's dual resonance model [1] could a string be found, let alone a theory of quantum gravity — or, at least, so it seemed. The development of string theory seems to be tied to a series of "accidental discoveries" that would lead to its current form.[1]

Shortly after Veneziano's proposal,[2] it was understood that a possible origin of his formula was a tower of harmonic oscillators [6, 7] whose properties strongly suggested an interpretation in terms of strings [8–10]. The oscillator formalism (even without the string interpretation) revealed many of the "accidents" that would become some of string theory's most famous properties — among them the need for extra dimensions [11–15]. Even the massless spin-2 state that inevitably appeared in the dual resonance models might be seen as an accident, not immediately associated with a graviton (it was then called *pomeron*) and considered a shortcoming more than a miracle.

> (...) at some point in our deliberations [with Scherk] we said, "Just for the fun of it, let's see whether this massless spin-2 particle behaves in the right way to give the standard gravitational force of the Einstein theory of general relativity." (...) And it became clear to both of us, immediately, that this was the way to make a consistent quantum theory for gravity.[a]
>
> [a]Schwarz, John H. (2002), in *Interview with John H. Schwarz*, Caltech Oral Histories

It was the shift from a theory of hadrons to a theory of quantum gravity, and the jump in energy scales that it entailed, that pulled string theory away from experimental reach — once it became a theory of gravity its "natural" scale went from the scale of the strong interactions (at

[1]See [2, 3] for a broad and historical account of string theory's development.
[2]And its generalisations such as [4, 5].

around 1 GeV) to that of the gravitational coupling, $M_{\rm Pl} \sim 10^{19}$ GeV. The lack of experimental evidence for string theory is therefore not a characteristic of strings but rather one of quantum gravity theories and the large scales at which they are needed — it tells us more about the questions we are asking than the answers we are getting.

Nevertheless, this makes the attempt to connect string theory's rich and highly constrained structure to current and future experimental observations one of the main challenges of current research. Despite the highly theoretical and mathematical tools that boosted the progress in string theory for decades, this goal was never abandoned — trying to reproduce a 4-dimensional theory with gravity, a particle content that is consistent with the Standard Model and a cosmological history that agrees with experimental data, was the driving force behind many of the theory's breakthroughs. Among them, flux compactifications of Type IIB string theory[3] gained a prominent role in the context of cosmology and high-energy physics, due to its range of ingredients that allow us to construct realistic models and address some of the "inconveniences" of string theory.

One such inconvenience is the presence of extra dimensions and arguably the first step towards string phenomenology is addressing their fate. Usually this requires 6 of the 10 dimensions in which the superstring propagates to be compactified in such a way as to hide their effects at low energies. However, their shapes and sizes still play a role in the low-energy physics and the need to reproduce current observations can be used as a guide for what geometries one should use to describe them. In particular, the leftover supersymmetry of compactified Type IIB leads us to consider Calabi-Yau manifolds as attractive candidates. While most of the extra-dimensional physics can be pushed to extremely high-energies by making those dimensions small enough, thereby effectively hiding them from our experiments, a residual clue to their existence is left in the form of a number of massless scalar fields in the 4d theory. Explaining why these scalars have never been seen is one of the challenges caused by the extra dimensions and constitutes the problem of moduli stabilisation. A partial solution to this problem comes in the form of non-trivial fluxes through the compact space that can generate a potential for some of the massless scalars and give them a mass.

When one puts together one of the most generic features of Calabi-Yau manifolds — conifold singularities — and the presence of these fluxes, a rather interesting structure arises which became know as a *warped throat*. This warping turns out to be particularly useful in addressing hierarchy problems, suppressing extremely high energies down to much lower scales. For that very reason, warped throats have become a crucial ingredient in string constructions that attempt to address a very phenomenological conundrum — the current accelerated expansion of the Universe and the cosmological constant problem. While the simplest solution would be to find a de Sitter vacuum in the theory with a very low vacuum energy, the status of de Sitter vacua

[3] Interestingly, it took some time for Type IIB to gain its current status. In its early stages, it was thought to be a theory of gravity only, and it was the Heterotic string that was seen as the most promising avenue towards realising the Standard Model of particle physics. This changed with the introduction of D-branes as a way to realise the gauge groups required by the Standard Model.

in string theory has been heavily debated for the past two decades and a consensus is yet to be found. Not only does a deeper understanding of some of the best proposals require a better understanding of the dynamics of warped throats and the nature of warping, this would also lead to a better understanding of their phenomenology and potential signatures. These may include gravitational signatures that one might be able to probe in the near future through gravitational waves, using an increasing network of detectors that cover a wide range of frequencies.

The work presented in this thesis focuses on both the challenges and the opportunities arising from warped extra dimensions in string theory constructions. We begin by discussing moduli stabilisation in Type IIB supergravity and presenting different proposals for obtaining controlled de Sitter vacua at low-energies, where different regimes of warping can be exploited in order to face the challenges of flux compactifications. We also discuss briefly runaway quintessence models as alternatives to de Sitter vacua. Then we explore the opportunities that might arise from strong warping when taken together with the advent of gravitational wave astronomy, by studying the effects of warping on the Kaluza-Klein tower of spin-2 states and its phenomenology. Below we give a summary of the main points and results discussed in each chapter.

Warped throats have become major ingredients in string phenomenology about which there is still a lot to learn. The usefulness of warping in addressing concrete problems, such as the cosmological constant problem, together with its potential to connect string theory effects to direct observations make warped throats worthy of attention and a well-motivated direction for future research.

Summary of Chapter 2

In Chapter 2 we review the essentials of Type IIB string theory, starting with the structure of the bosonic string, comparing it with the new features of the superstring and showing how it leads to Type IIB strings. We focus on the crucial role played by symmetries and boundary conditions, obtain the spectrum of closed and open strings, and introduce the GSO projection that results in the Type IIB theory. We then explain how Type IIB supergravity gives a low-energy description of this theory, discuss some of its features and introduce the low-energy actions of localised objects — such as D-branes and O-planes — and their role in tadpole cancellation.

Summary of Chapter 3

In Chapter 3 we introduce the background on flux compactifications that will be crucial for the following chapters. We start with a general discussion of compactification and how it leads to discrete infinite (Kaluza-Klein) towers of states, how massless states in these towers are determined by the topology of the compact space and how the compactified theory can be seen as a low-energy effective description of the higher-dimensional one. We then motivate the use of Calabi-Yau manifolds, describe the dimensional reduction of Type IIB supergravity and introduce moduli stabilisation with fluxes. Finally, we introduce conifold singularities, review the warped deformed conifold and the Klebanov-Strassler solution, and finish with the GKP solution whose use of fluxes can create large hierarchies in the low-energy theory.

Summary of Chapter 4

In Chapter 4 we discuss de Sitter solutions in the context of Type IIB flux compactifications. After reviewing the cosmological constant problem and dark energy, we introduce the KKLT and LVS proposals for Kähler moduli stabilisation and the uplift mechanism relying on $\overline{\text{D3}}$-branes to obtain de Sitter vacua. We then focus on the conifold modulus and how its stabilisation depends on the warping regime we consider — we revisit the strongly-warped solutions and outline a new weakly-warped solution that might help us avoid potential dangers related to the D3-tadpole constraint. We also discuss the effect of subleading corrections on this weakly-warped solution. We finish the chapter by considering the alternative to de Sitter vacua provided by slow-roll quintessence models, focusing in particular on the difficulties to find accelerated expansion at the tail of (single-field) runaway potentials, which seem ubiquitous in string theory.

Summary of Chapter 5

In Chapter 5 we turn our attention to the opportunities created by strong warping when put together with the growing investment in gravitational wave astronomy. We start by studying more carefully the spin-2 tower of Kaluza-Klein states that arises in warped compatifications, paying careful attention to the interplay between the warping and the size of the compact space, as well as the masses and wavefunction profiles of the tower of states. We then show how this can be applied to the study of fifth forces arising from the spin-2 tower and how we can compare the parameter space of warped string compactifications to the parameter space excluded by fifth force constraints. We end the chapter with an outline of how the same ingredients can be used to make predictions for gravitational wave signatures of warped extra dimensions.

2. Essentials of Type IIB

Beginnings are such delicate times.

Frank Herbert

2.1 String Theory

Our current view of string theory started with an interpretation of the dual-resonance model of Veneziano [1], which originally had nothing to do with oscillating strings. It was a model whose sole purpose was to explain the Regge trajectories observed in experimental data of hadronic scattering and it did so by providing a formula for the amplitudes that had the right symmetries and reproduced observations. One might say that string theory could not have started more experimentally oriented than it did. Only later was the structure of this amplitude connected with oscillators [6] and strings [8–10], and the properties of the dual-resonance models derived from these structures. While nowadays one always starts with a string whose quantisation gives an infinite tower of states, it is interesting to note that historically it was quite the opposite — the Veneziano amplitude [1] described an infinite tower of states associated with an infinite family of oscillators [6] that ultimately suggested it was arising from an oscillating string [8–10].

Just as the action for a relativistic particle is given by the length of its world-line, also the action for a relativistic string is the area of its world-sheet Σ. The action one obtains in this straightforward way is known as the Nambu-Goto action [10, 16] and does indeed allow us to study the dynamics of a string as it propagates in spacetime. However, its form is rather difficult to work with (especially when quantising the string) due to the square-root that inevitably appears in the action. A much simpler (classically equivalent) action is obtained by introducing an additional auxiliary metric field $h_{\alpha\beta}(\sigma, \tau)$ with signature $(-, +)$ on the world-sheet. The resulting action, known as the Polyakov action[1] [17–19], describes d massless scalars X^μ coupled

[1]This form of the action was originally found by L. Brink, P. Di Vecchia and P. Howe [17], and independently by S. Deser and B. Zumino [18] (as a generalisation including local world-sheet supersymmetry), but it was named after Polyakov after his application to the path integral quantisation fuelled the development of string theory — it not only allowed field theoretic tools to be employed, but also revealed the connection between string perturbation

to 2d gravity,

$$S_{\mathrm{P}} = -\frac{1}{4\pi\alpha'}\int_\Sigma d^2\sigma\sqrt{-h}\; h^{\alpha\beta}(\partial_\alpha X^\mu)(\partial_\beta X^\nu)\eta_{\mu\nu}\,, \tag{2.1}$$

where $h = \det h_{\alpha\beta}$, (σ, τ) are the coordinates on the world-sheet and $\eta_{\mu\nu}$ is the d-dimensional spacetime metric. The world-sheet scalars X^μ tell us how the string is embedded in spacetime — they are maps from the string world-sheet into the d-dimensional spacetime in which the string propagates. The action (2.1) is written in terms of the pulled-back metric

$$G_{\alpha\beta} = \frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta}\eta_{\mu\nu}\,. \tag{2.2}$$

The tension of the string is given by $T = \frac{1}{2\pi\alpha'}$, so that α' sets the scale associated with string dynamics. It is important to distinguish the spacetime theory (whose metric is $\eta_{\mu\nu}$) and the world-sheet theory (whose metric is $h_{\alpha\beta}$) — from the spacetime viewpoint, we have a single propagating string; from the world-sheet point of view, we have d free scalars[2] propagating in 2d coupled to a metric $h_{\alpha\beta}$ which is not dynamical. One should keep this in mind, for example when discussing symmetries, as some will be spacetime symmetries, while others will be world-sheet symmetries.

The dynamics of the string and, ultimately, its spectrum is mostly determined by two things:

`symmetries` $\oplus$ `boundary conditions`

The symmetries of the action will not only constrain the possible solutions, but also determine in which situations one can consistently quantise the string (e.g. critical dimensions of the bosonic and supersymmetric string); and the choice of boundary conditions will determine what states will arise from this quantisation. The Polyakov action (2.1) is invariant under

- → Spacetime Poincaré transformations of X^μ — this is a global symmetry;
- → Reparametrisations of (σ, τ) — these are diffeomorphisms on the world-sheet;
- → Weyl rescallings of $h_{\alpha\beta}$.

The fact that the action is invariant under Weyl rescallings is of crucial importance — it tells us that (2.1) describes a conformal field theory (CFT), which is central for a plethora of methods used to quantise the string and study its interactions. It is also, together with world-sheet diffeomorphisms, what allows us to choose the convenient conformal gauge in which the world-sheet metric is flat, $h_{\alpha\beta} = \eta_{\alpha\beta}$. This gauge is unique to 2d and therefore gravity is completely gauged away only in $d \leq 2$. Although the metric $h_{\alpha\beta}$ is flat and not dynamical, its equation of

expansions and Riemann surfaces, and could be used to study string theory in curved spaces [2]. Ultimately, it connects all the way back to Liouville, a point which is made very clear in Polyakov's paper [19].

[2]This can be generalised by taking a metric $G_{\mu\nu}(X)$ rather than the flat metric $\eta_{\mu\nu}$, which results in a non-linear sigma-model (an interacting theory).

motion is still important, as it effectively implements a constraint on the scalar field theory,[3]

$$T_{\alpha\beta} = \frac{4\pi}{\sqrt{-h}}\frac{\delta S_{\rm P}}{\delta h^{\alpha\beta}} = -\frac{1}{\alpha'}\left(\partial_\alpha X^\mu \partial_\beta X^\nu - \frac{1}{2}\eta_{\alpha\beta}\partial_\gamma X^\mu \partial^\gamma X^\nu\right) = 0\,. \tag{2.3}$$

Due to differomorphism invariance on the world-sheet we also have energy-momentum conservation, $\nabla^\alpha T_{\alpha\beta} = 0$. Together with $T_{\alpha\beta} = 0$, this implies the existence of infinitely many conserved charges L_m corresponding to the generators of the Virasoro algebra, which is expressing the conformal invariance of the theory. The generators are defined by $T_{\alpha\beta}$ and therefore X^μ, and we can think of $L_m = 0$ as imposing the constraint $T_{\alpha\beta} = 0$.

It is also very convenient to work with conformal (or light-cone) coordinates $\sigma^\pm = \tau \pm \sigma$, in terms of which the equations of motion for X^μ are simply

$$\partial_+\partial_- X^\mu = 0\,, \tag{2.4}$$

which is nothing but a massless wave equation with general solution

$$X^\mu(\sigma,\tau) = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-)\,. \tag{2.5}$$

Since these correspond, respectively, to left- and right-moving waves, we refer to X^μ_L and X^μ_R as the left- and right-moving modes of the string. While these are totally independent for a closed string, they are mixed for an open string — this is where the boundary conditions first come into play. The set of allowed boundary conditions follows from the need to cancel the boundary terms one would get from varying the action (2.1) with $\delta X^\mu(\tau_0) = \delta X^\mu(\tau_1) = 0$,

$$\delta S_{\rm P}^{\rm boundary} = -T\int_{\tau_0}^{\tau_1} d\tau\, \Big[(\partial_\sigma X_\mu)\;\delta X^\mu\Big]_{\sigma=0}^{\sigma=\ell}\,, \tag{2.6}$$

for a string of length ℓ. We see that the boundary term vanishes if

`Closed string` $\quad X^\mu(\sigma+\ell) = X^\mu(\sigma)$

`Open string`
$$\partial_\sigma X^\mu\big|_{\sigma=0,\ell} = 0 \quad \texttt{(Neumann)}$$
$$\delta X^\mu\big|_{\sigma=0,\ell} = 0 \quad \texttt{(Dirichlet)}$$

for closed strings[4] and open strings, respectively. For open strings, either Neumann or Dirichlet boundary conditions must be imposed at each end of the string ($\sigma = 0, \ell$) for each X^μ ($\mu = 0, ..., d-1$), for a total of $2d$ independent choices. Whereas a Dirichlet boundary condition (D) fixes the end-point of the open string in the corresponding spacetime direction, a Neumann

[3]This constraint is what makes the Polyakov action classically equivalent to the Nambu-Goto action.

[4]These are the only periodic boundary conditions which are Poincaré invariant. However, if we consider strings propagating on a background which is not Poincaré invariant (e.g. in string compactifications), more general periodic conditions $X^\mu(\sigma+\ell) = M^\mu{}_\nu X^\mu(\sigma)$, with constant $M \in O(1, d-1)$, are allowed for some X^μ. Non-trivial M leads to the so-called twisted states [20].

boundary condition (N) forbids the string from moving in the σ-direction at the end-point. Different boundary conditions can be chosen in different directions; for example, one can choose (N) boundary conditions along $(p+1)$ directions and (D) boundary conditions along the remaining $(d-p-1)$ directions, so that the end point of the string is confined to a $(p+1)$-dimensional subspace.

This leads to a very satisfying connection between boundary conditions and symmetries. Since a Dirichlet boundary condition fixes the end-point of a string, it spontaneously breaks translation invariance, which implies that the momentum carried by the string is not conserved. However, spacetime Poincaré invariance requires the total momentum to be conserved, which forces the string to exchange momentum with the $(p+1)$-dimensional subspace on which it ends. Hence this subspace must be a dynamical object extending in p spatial directions — these objects are known as D-branes and are important ingredients in modern string phenomenology.[5] The transverse fluctuations of such a D-brane correspond to massless scalar fields living on the world-volume of the brane, which are the Goldstone bosons of the spontaneously broken symmetry [20].

Let us see how all this structure leads to the spectrum of the string — the spacetime states arising from its quantisation. Interesting and important as it is, it could hardly be justified including an exhaustive discussion of all cases here.[6] We will focus on the specific case of the open string with (N) boundary conditions along $(p+1)$ directions x^a and (D) boundary conditions along $(d-p-1)$ directions x^i on both ends, for which the general solution of the wave equation is[7]

$$\text{(NN)} \qquad X^a(\sigma,\tau) = x^a + \frac{2\pi\alpha'}{\ell} p^a \tau + i\sqrt{2\alpha'} \sum_{n\in\mathbb{Z}/\{0\}} \frac{1}{n}\alpha^a_n \, e^{-i\frac{n\pi}{\ell}\tau} \cos\left(\frac{n\pi\sigma}{\ell}\right), \tag{2.7}$$

$$\text{(DD)} \qquad X^i(\sigma,\tau) = x^i_0 + \frac{1}{\ell}(x^i_\ell - x^i_0)\sigma + \sqrt{2\alpha'} \sum_{n\in\mathbb{Z}/\{0\}} \frac{1}{n}\alpha^i_n \, e^{-i\frac{n\pi}{\ell}\tau} \sin\left(\frac{n\pi\sigma}{\ell}\right), \tag{2.8}$$

when the end-points are fixed at x^i_0 and x^i_ℓ; x^a and p^a are the centre-of-mass position and momentum of the string along the (N) directions. The α^μ_n are the Fourier modes of oscillation and encode eigenmodes of vibration of the string. Because X^μ is also a coordinate in spacetime, it must be real, which implies the relation

$$\alpha^\mu_{-n} = (\alpha^\mu_n)^*\,, \tag{2.9}$$

[5] D-branes were proposed as dynamical objects by Polchinski in 1995 [21] and were key ingredients in the second string revolution that followed.

[6] For this we refer to [22, 23] for a short and approachable introduction to the bosonic string, including the essentials of CFT and the occasional nod to the superstring. Although somewhat longer, [24] is also a good place to start and covers more advanced topics as well. For a more in depth dive into the topic, where the bosonic and supersymmetric case can be nicely followed in parallel, [20] is a good place to go; it is also a great resource for the CFT methods so relevant to string theory and string interactions. Of course, the classic textbooks [25–29] are also valuable references.

[7] The centre-of-mass momentum along the $(d-p-1)$ directions is zero and the centre-of-mass position $x^\mu_{\text{C.M.}} = \frac{x^\mu_0 + x^\mu_\ell}{2}$ is fixed.

and so positive frequency oscillator modes ($n < 0$) are not independent of negative frequency modes ($n > 0$). From the classical Poisson brackets $\{X^\mu(\sigma,\tau), \dot{X}^\mu(\sigma',\tau)\}_{\text{P.B.}}$ we can derive

$$\{\alpha^\mu_m, \alpha^\nu_n\}_{\text{P.B.}} = -im\delta_{m+n}\eta^{\mu\nu}\,, \tag{2.10}$$

which will become commutation relations upon quantisation. The Virasoro generators, can then be defined as

$$L_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} \alpha_{m-n}\cdot\alpha_n\,, \tag{2.11}$$

as long as we define $\alpha^a_0 = \sqrt{2\alpha'}p^\mu$ and $\alpha^i_0 = \frac{\Delta x^i}{\sqrt{2\alpha'}\pi}$, with $\Delta x^i = x^i_\ell - x^i_0$, so that the Poisson bracket between two such generators becomes

$$\{L_m, L_n\}_{\text{P.B.}} = -i(m-n)L_{m+n}\,. \tag{2.12}$$

We include these to emphasise a point that will be relevant in a moment: although not everything about the quantum string is present in its classical description, a lot of its consistency is tied to it. So let us jump to the quantum world and see what exactly we mean by this — it is time to quantise the string.

We will work in terms of operators.[8] That means that the functions $X^\mu(\sigma,\tau)$ are now quantum mechanical operators and the Poisson brackets become commutators,

$$\{\ \cdot\ ,\ \cdot\ \}_{\text{P.B.}} \to \frac{1}{i}[\ \cdot\ ,\ \cdot\ ]\,. \tag{2.13}$$

This imediately leads to

$$[\alpha^\mu_m, \alpha^\nu_n] = m\ \delta_{m+n}\eta^{\mu\nu}\,, \tag{2.14}$$

while the reality condition becomes $(\alpha^\mu_n)^\dagger = \alpha^\mu_{-n}$. This leads us to the following conclusion: the oscillator modes of the vibrating string, once quantised (and rescalled), behave as familiar quantum harmonic oscillators with creation operators α^μ_{-n} and annihilation operators α^μ_n, for $n > 0$. Its ground state is defined as the state for which $\alpha^\mu_n\,|0\rangle$, $\forall n > 0$, and we create states by acting on $|0\rangle$ with the operators α^μ_{-n}.

It turns out that there is some residual freedom in our choice of conformal gauge. The specific gauge fixing which allows us to work with physical degrees of freedom only is known as the light-cone gauge[9] and it is chosen such that $X^+ = \frac{2\pi\alpha'}{\ell}\tau$, in terms of spacetime light-cone coordinates

[8] Polchinski would surely be the first to point out the usefulness of working with the path integral instead and it is indeed a powerful approach. For our purposes though it will suffice to take the operator approach, which is easier and more direct.

[9] The gauge fixing is performed through a specific world-sheet coordinate transformation. This gauge provides unique representatives for physical states and makes unitarity explicit at the expense of manifest Lorentz covariance [23].

(X^+, X^-, X^i), $i = 2, ..., d-1$, with

$$X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^1)\,. \tag{2.15}$$

On the one hand this means that the only non-zero oscillator mode in X^+ is α_0^+. On the other, in the light-cone gauge, the constraints alone fix all α_n^- in terms of the transverse oscillators α_n^i, which are therefore the only independent variables [23].

Great care must be taken in bringing the Virasoro generators L_m in (2.12) to the quantum theory, due to its definition in terms of *products* of α_n^μ that are now non-commuting operators. These products are only well-defined once we specify their appropriate ordering, and the classical theory does not necessarily tell us anything with regard to this. Let us define the L_m in terms of normal-ordered products,

$$L_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} :\alpha_{m-n}\cdot\alpha_n: \,. \tag{2.16}$$

Since $\alpha_n^+ = 0\,,\ \forall n \neq 0$, leaving out the $n = 0$ term in the sum (after all, α_0 commutes with all α_m^μ), we can write

$$\begin{aligned}\sum_{n\neq 0}\alpha_{m-n}\cdot\alpha_n &= \sum_{n\neq 0}[-\alpha_{m-n}^+\alpha_n^- - \alpha_{m-n}^-\cancel{\alpha_n^+} + \delta_{ij}\alpha_{m-n}^i\alpha_n^j] \\ &= -\alpha_0^+\alpha_m^- + \delta_{ij}\sum_{n\neq 0}\alpha_{m-n}^i\alpha_n^j\,,\end{aligned} \tag{2.17}$$

and working out the last term in terms of normal-ordered products using the commutation relations (2.14),

$$\begin{aligned}\sum_{n\neq 0}\alpha_{m-n}^i\alpha_n^j &= \sum_{n<0}\alpha_{m-n}^i\alpha_n^j + \sum_{n>0}\alpha_{m-n}^i\alpha_n^j \\ &= \sum_{n<0}(\alpha_n^i\alpha_{m-n}^j + [\alpha_{m-n}^i, \alpha_n^j]) + \sum_{n>0} :\alpha_{m-n}^i\alpha_n^j: \\ &= \sum_{n<0} :\alpha_n^i\alpha_{m-n}^j: + \sum_{n>0} :\alpha_{m-n}^i\alpha_n^j: + \sum_{n<0}(m-n)\delta_{m,0}\ \delta^{ij} \\ &= \sum_{n\neq 0} :\alpha_n^i\alpha_{m-n}^j: + \delta^{ij}\sum_{n=1}^{\infty}(m+n)\delta_{m,0}\,,\end{aligned} \tag{2.18}$$

we see that the normal ordered definition matches the classical definition for all L_m apart from

$$L_0 = \frac{1}{2}\sum_{n\in\mathbb{Z}} :\alpha_n\cdot\alpha_{-n}: + \frac{d-2}{2}\sum_{n=1}^{\infty} n\,. \tag{2.19}$$

This looks like trouble — the last term is a divergent series! At face value, having a divergent series to which no value can be assigned would mean that our classical definition of L_0 simply does not define its quantum version. However, this divergent series can in fact be regularised and uniquely assigned the value

$$\sum_{n=1}^{\infty} n = -\frac{1}{12}\,, \tag{2.20}$$

whose appearance in string theory has by now become famous, so that L_0 is uniquely defined as

$$\begin{aligned} L_0 &= \frac{1}{2}\sum_{n\in\mathbb{Z}} :\alpha_n \cdot \alpha_{-n}: -\frac{d-2}{24} \\ &= \alpha'\, p^a p_a + \alpha' \left(\frac{\Delta x^i}{2\pi\alpha'}\right)^2 - \frac{d-2}{24} + N_{\text{tr}}\,, \end{aligned} \tag{2.21}$$

where the first two terms correspond to α_0^μ for the (NN) x^a and the (D) x^i directions, respectively, and N_{tr} is the transverse number operator

$$N_{\text{tr}} = \delta_{ij} \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^j\,, \tag{2.22}$$

counting the total number of transverse oscillators in a state. Having properly defined the generators L_m in terms of the oscillators, one can use the commutations relations (2.14) to find

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{d}{12}m(m^2-1)\delta_{m+n}\,. \tag{2.23}$$

We immediately see that there is an extra term compared to (2.12), which followed from the conformal invariance of the classical theory. The extra term seems to break this conformal invariance and is therefore known as the conformal anomaly. The constraints are imposed on physical states as

$$L_m\,|\text{phys}\rangle = 0, \quad m \geq 0\,, \tag{2.24}$$

which for L_0 implies that a physical open string state will have a mass ($m^2 = -p^a p_a$)

$$\alpha' m^2 = N_{\text{tr}} + \alpha' \left(\frac{\Delta x^i}{2\pi\alpha'}\right)^2 - \frac{d-2}{24}\,, \tag{2.25}$$

determined by the number of transverse oscillators N_{tr}, the separation between the Dp-branes along which the ends of the string can propagate Δx^i and the number of spacetime dimensions d. It shows that a string that oscillates more (i.e. containing more modes of vibration) results in more massive states and that stretching a string a distance Δx^i costs energy, which can be seen as potential energy trying to pull the D-branes together.

We can now determine the spectrum of the open bosonic string. Let us specify to the simpler case in which both ends of the string end on the same Dp-brane, i.e. $\Delta x^i = 0$. The mass relation

(2.25) tells us that the ground state with no oscillators has a mass

$$\alpha' m^2 |0\rangle = \left(N_{\rm tr} - \frac{d-2}{24}\right)|0\rangle = -\frac{d-2}{24}|0\rangle \ , \tag{2.26}$$

determined by the number of spacetime dimensions d — for any $d > 2$ the ground state is a tachyon with $m^2 < 0$. The first excited states with $N_{\rm tr} = 1$ are $\alpha^a_{-1}|0\rangle\ ,\ a = 2, ..., p$ (NN) and $\alpha^i_{-1}|0\rangle\ ,\ i = p+1, d-1$ (DD), with

$$\alpha' m^2 |0\rangle = \left(N_{\rm tr} - \frac{d-2}{24}\right)|0\rangle = \frac{26-d}{24}|0\rangle \ . \tag{2.27}$$

But here is the key point: the (D) boundary conditions along $(d-p-1)$-directions broke the theory's d-dimensional Poincaré invariance down to a $(p+1)$-dimensional one — any state will only carry momentum along the $(p+1)$-directions parallel to the brane and can never be translated along the transverse directions. Since we can make Lorentz boosts along the $(p+1)$-directions preserving the symmetry, we can always choose a frame in which

$$\texttt{(massive)} \quad p^\mu = (m, \underbrace{0, ..., 0, 0}_{p}, 0, ..., 0)\,, \tag{2.28}$$

$$\texttt{(massless)} \quad p^\mu = (E, \underbrace{0, ..., 0}_{p-1}, E, 0, ..., 0)\,, \tag{2.29}$$

whose invariance subgroups (or little groups) are SO(p) and SO($p-1$), respectively, corresponding to the allowed rotations that leave the momentum unchanged. The upshot is that any state preserving covariance must fall into representations of one of these groups, depending on its mass.

Since the state $\alpha^a_{-1}|0\rangle$ transforms as a vector of SO($p-1$) (while the $\alpha^i_{-1}|0\rangle$ transform as scalars), it must be massless, which requires

$$d = 26\,. \tag{2.30}$$

Any other spacetime dimension would not be compatible with a covariant theory. The bosonic string must therefore propagate in 26 dimensions.[10]

At this point, we can create more states by acting on the vacuum with different combinations of oscillators. All other states will have $N_{\rm tr} > 1$ and therefore $\alpha' m^2 = N_{\rm tr} - 1 > 0$. In particular all those states will have masses proportional to the scale $1/\sqrt{\alpha'}$ and will fall into irreducible representations of SO(p).[11] We show the first 2 levels in Table 2.1.

[10]This critical dimension can be found in a number of different ways, for example by requiring that the commutation relations for the Lorentz generators are preserved or, using the path integral approach, by requiring that the conformal anomaly in (2.23) is cancelled by the ghost system (which is ultimately determined by the local symmetries of the theory, nicely tying the critical dimension with the symmetries on the world-sheet).

[11]Combinations of oscillators $\alpha^{i_1}_{-m}...\alpha^j_{-n}|0\rangle$ will form representations of the transverse group SO($p-1$) which will combine into representations of SO(p).

$\alpha' m^2$	States	Little group rep.
-1	$\|0\rangle$	(1)
0	$\alpha^a_{-1}\|0\rangle$ $\alpha^i_{-1}\|0\rangle$	$(p+1)$ $\{d-p-1\}\times\ (1)$
1	$\alpha^a_{-2}\|0\rangle \quad \alpha^i_{-2}\|0\rangle$ $\alpha^a_{-1}\alpha^j_{-1}\|0\rangle \quad \alpha^i_{-1}\alpha^a_{-1}\|0\rangle$ $\alpha^a_{-1}\alpha^b_{-1}\|0\rangle \quad \alpha^i_{-1}\alpha^j_{-1}\|0\rangle$	$\left(\frac{(p-1)(p-2)}{2}\right)$ $2\times\{d-p-1\}\times\ (p+1)$ $\oplus\ \{d-p-1\}\times\{d-p-2\}\times\ (1)$

Table 2.1: Spectrum of the open bosonic string up to level 2, with (NN) boundary conditions along $(p+1)$ directions X^a and (DD) boundary condition along the remaining $(d-p-1)$ directions X^i.

Now it is fairly simple to get the closed string spectrum. Since for a closed string the left- and right-moving modes are independent, we now have two sets of oscillators, which we can call α^μ_{-m} and $\overline{\alpha}^\mu_{-m}$. Because the closed string is free to move in all directions, these states will simply correspond to tensor products of the states of an open string with only (NN) boundary conditions. The only extra constraint we have is known as level-matching, $N_{\text{tr}} = \overline{N}_{\text{tr}}$. This condition is rooted in a translation invariance along the string, following from the periodic boundary conditions.[12] Acting with the creation operators and respecting level matching, we find for the first levels the states in Table 2.2.

$\alpha' m^2$	States	Little group rep.
-4	$\|0\rangle$	(1)
0	$\alpha^i_{-1}\overline{\alpha}^j_{-1}\|0\rangle$	$(299)\oplus(276)\oplus(1)$
4	$\alpha^i_{-2}\overline{\alpha}^j_{-2}\|0\rangle \quad \alpha^i_{-1}\alpha^j_{-1}\overline{\alpha}^k_{-1}\overline{\alpha}^l_{-1}\|0\rangle$ $\alpha^i_{-2}\overline{\alpha}^j_{-1}\overline{\alpha}^k_{-1}\|0\rangle \quad \alpha^i_{-1}\alpha^j_{-1}\overline{\alpha}^k_{-2}\|0\rangle$	$(20150)\oplus(32175)\oplus(52026)$ $\oplus(324)\oplus(300)\oplus(1)$

Table 2.2: Spectrum of the close bosonic string up to level 2 — it can be obtain from tensor products of open string states at the same mass level (due to level matching, i.e. $N_{\text{tr}} = \overline{N}_{\text{tr}}$).

The ground state of the closed string is still a tachyon. The massless states are a symmetric traceless rank-2 tensor (graviton), an anti-symmetric rank-2 tensor (Kalb-Ramond 2-form) and a scalar (dilaton). Just by including these first few levels we see that the number of states and the size of their representations grow quickly with N_{tr}.

The presence of the tachyon seems discouraging. In practice, it tells us that the ground state is not stable, which usually means that we are expanding the theory around the wrong vacuum. Although the bosonic string already includes several of the important features for string phenomenology — such as the graviton, vector fields and gauge groups that can arise from different brane configurations — it does not contain any fermions. This together with the presence of the

[12]See however [30] for a study of closed string field without level-matching.

tachyon leads us to consider a generalisation of string theory which not only includes fermionic states as part of its spectrum, but that we can also hope will cure our tachyonic problem.

2.2 Superstrings

2.2.1 Strings with fermions and supersymmetry

The upgrade of the bosonic string that includes spacetime fermions and will ultimately give us a way to get rid of the tachyon (resulting in more than one consistent theory and allowing us to study phenomenology) is a descendent of the fermionic dual resonance models proposed by Ramond [12], Neveu and Schwarz [14]. Already in [14] it was noted that some new *supergauge operator* would be the "nicest" way to get a "reasonable" spectrum of states. In the authors' own words, the proposal that certain states in the spectrum obtained without these supergauge operators should be absent was called *"a conjecture because we have not yet fully proved it, but we have no doubt about its truth"* [14]. It was only in later work [31, 32] that these supergauges were better understood, with invariance under supergauge transformations being closely connected to the absence of ghost states in the two dimensional field theory that described the fermionic models [32], and included in what would become the superstring action [17, 18].

This new symmetry, which became known as supersymmetry,[13] is also strong enough to guarantee the absence of the worrisome tachyon when imposed as a spacetime symmetry (i.e. as a symmetry of the spacetime spectrum of the string). To find this spectrum, let us introduce the generalisation of the Polyakov action that includes bosons and fermions related by supersymmetry,

$$S = -\frac{1}{8\pi}\int d^2\sigma\sqrt{-h}\Bigg(\frac{2}{\alpha'}h^{\alpha\beta}(\partial_\alpha X^\mu)(\partial_\beta X^\nu)\eta_{\mu\nu} + 2i(\overline{\psi}^\mu\rho^\alpha\partial_\alpha\psi^\nu)\eta_{\mu\nu} - i\overline{\chi}_\alpha\rho^\beta\rho^\alpha\psi^\mu\Big(\sqrt{\frac{2}{\alpha'}}(\partial_\beta X^\nu) - \frac{i}{4}\overline{\chi}_\beta\psi^\nu\Big)\Bigg)\,. \tag{2.31}$$

Just as for the bosonic string, a great deal of the properties of the superstring spectrum and the ability to quantise the string follow from the symmetries of this action. Compared to the bosonic version, the action (2.31) is invariant under a greater number of symmetries,

→ Spacetime Poincaré transformations — global symmetry;

→ Local supersymmetry transformations of $(X^\mu, h^{\alpha\beta}, \psi^\mu, \chi_\alpha)$;

→ World-sheet Lorentz transformations;

[13] Supersymmetry was also introduced in [33] as an extension of the algebra of generators of the Poincaré group before its appearance in the fermionic dual models.

→ Reparametrisations of (σ, τ) — diffeomorphisms on the world-sheet;

→ Weyl rescallings of $(h_{\alpha\beta}, \psi^\mu, \chi_\alpha)$;

→ Super-Weyl rescallings of χ_a.

The action (2.31) describes a super-conformal field theory (SCFT) in 2d, due to the Weyl and super-Weyl rescalings, which we can again use together with the remaining symmetries to choose a superconformal gauge that fixes both $h_{\alpha\beta} = \eta_{\alpha\beta}$ and $\chi_\alpha = 0$ [20]. These fields are therefore not dynamical, just as $h_{\alpha\beta}$ was non-dynamical for the bosonic string. Yet, as before, their equations of motion are still implementing powerful constraints on the remaining degrees of freedom,[14]

$$T_{\alpha\beta} = \frac{4\pi}{\sqrt{-h}} \frac{\delta S}{\delta h^{\alpha\beta}} = -\frac{1}{\alpha'}\Big(\partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2}\eta_{\alpha\beta}\partial^\gamma X^\mu \partial_\gamma X_\mu\Big) - \frac{i}{2}\Big(\overline{\psi}^\mu \rho_{(\alpha}\partial_{\beta)}\psi_\mu\Big) = 0\,, \tag{2.32}$$

$$T^{\rm F}_\alpha = -i\frac{2\pi}{\sqrt{-h}} \frac{\delta S}{\delta \overline{\chi}^\alpha} = -\frac{1}{4}\sqrt{\frac{2}{\alpha'}}\rho^\beta \rho_\alpha \psi^\mu \partial_\beta X_\mu = 0\,. \tag{2.33}$$

It is worth taking a step back at this point and recall what the bosonic string has taught us. We saw the crucial role the constraint $T_{\alpha\beta} = 0$ played after giving rise to the Virasoro generators L_m, with L_0 being ultimately responsible for determining the spectrum of the string and its critical dimension. We now have two key differences — on the one hand $T_{\alpha\beta}$ no longer depends only on the bosonic oscillators α^μ_m through the fields X^μ, but also on the fermionic degrees of freedom ψ^μ; on the other hand we now have a new constraint altogether involving both bosons and fermions, $T^{\rm F}_\alpha = 0$ (known as the supercurrent). We shall soon see how these come into play to determine our new spectrum of states.

In terms of light-cone coordinates $\sigma^\pm$, the equations of motion for X^μ and ψ^μ become[15]

$$\partial_+\partial_- X^\mu = 0\,, \tag{2.34}$$

$$\partial_+\psi^\mu_- = \partial_-\psi^\mu_+ = 0\,, \tag{2.35}$$

and thus X^μ again splits into left- and right-movers (2.5), while $\psi^\mu_+ = \psi^\mu_+(\sigma^+)$ and $\psi^\mu_- = \psi^\mu_-(\sigma^-)$. What about the boundary conditions? For the bosonic fields, we find the same options as for (2.6) — periodic, Neumann or Dirichlet. On the other hand, the boundary term of the fermionic fields, after varying the action (2.31) with $\delta\psi^\mu(\tau_0) = \delta\psi^\mu(\tau_1) = 0$, is

$$\delta S^{\rm boundary} = -\frac{1}{2\pi}\int_{\tau_0}^{\tau_1} d\tau \Big[\eta_{\mu\nu}(\psi^\mu_+\delta\psi^\nu_+ - \psi^\mu_-\delta\psi^\nu_-)\Big]^{\sigma=\ell}_{\sigma=0}\,. \tag{2.36}$$

This boundary term vanishes if

`Closed string`	$\psi^\mu_\pm(\sigma+\ell) = \pm\psi^\mu_\pm(\sigma)$
`Open string`	$\psi^\mu_+\big\rvert_{\sigma=0,\ell} = \pm\psi^\mu_-\big\rvert_{\sigma=0,\ell}$

[14] More precisely, for theories with fermions one must work with the zweibein e^a_α in terms of which $h_{\alpha\beta} = e^a_\alpha e^b_\beta \eta_{ab}$.
[15] The $\pm$ in $\psi^\mu_\pm$ denote spinor components, which we have left out until now for convenience.

We now have two choices for closed string boundary conditions for each component of the fermions — the periodic one (+) is known as Ramond (R) boundary condition, while the anti-periodic (−) is called Neuveu-Schwarz (NS) boundary condition. Because of Poincaré invariance the choice must be the same for all directions μ, but can be different for the two components ψ_+^μ and ψ_-^μ, resulting in a total of 4 different possibilities: (R,R), (R,NS), (NS,R) and (NS,NS). For open strings, preserving Poincaré invariance in $(p+1)$ directions requires us to impose the same boundary conditions in all those directions, although as before we are allowed to choose the sign independently on each end of the string — choosing the same sign on both ends defines a Ramond (R) boundary condition, while choosing different signs defines a Neveu-Schwarz (NS) boundary condition.

Since we will be mostly working with states arising from the closed string, let us focus on that case. The general solution to the equations of motion for a closed string is

$$X_R^\mu(\tau-\sigma) = \frac{1}{2}x^\mu + \frac{\pi\alpha'}{\ell}p^\mu(\tau-\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu\, e^{\frac{2\pi}{\ell}in(\tau-\sigma)}\,, \tag{2.37}$$

$$X_L^\mu(\tau+\sigma) = \frac{1}{2}x^\mu + \frac{\pi\alpha'}{\ell}p^\mu(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\bar{\alpha}_n^\mu\, e^{\frac{2\pi}{\ell}in(\tau+\sigma)}\,, \tag{2.38}$$

$$\psi_-^\mu(\sigma,\tau) = \sqrt{\frac{2\pi}{\ell}}\sum_{r\in\mathbb{Z}+\phi} b_r^\mu\, e^{-\frac{2\pi}{\ell}ir(\tau-\sigma)}\,, \tag{2.39}$$

$$\psi_+^\mu(\sigma,\tau) = \sqrt{\frac{2\pi}{\ell}}\sum_{r\in\mathbb{Z}+\phi} \bar{b}_r^\mu\, e^{-\frac{2\pi}{\ell}ir(\tau+\sigma)}\,, \tag{2.40}$$

where $\phi = 0, \frac{1}{2}$ for (R) and (NS) boundary conditions, respectively, which can be chosen independently for ψ_-^μ and ψ_+^μ. As before, the α_n^μ, $\bar{\alpha}_n^\mu$, b_r^μ, $\bar{b}_r^\mu$ are the Fourier modes of oscillation encoding the different eigenmodes of vibration of the string, except that we now have both bosonic and fermionic modes of vibration. Since the X^μ and ψ^μ are real, these modes must satisfy

$$\alpha_{-n}^\mu = (\alpha_n^\mu)^*\,, \qquad \bar{\alpha}_{-n}^\mu = (\bar{\alpha}_n^\mu)^*\,,$$
$$b_{-r}^\mu = (b_r^\mu)^*\,, \qquad \bar{b}_{-r}^\mu = (\bar{b}_r^\mu)^*\,,$$

so that positive frequency modes $(n, r < 0)$ are related to negative frequency modes $(n, r > 0)$. While the bosonic modes satisfy the same algebra as in (2.10) in terms of Poisson brackets, for the fermionic modes Poisson brackets must be replaced by Dirac brackets.[16] The algebra

[16] This can be understood in terms of different kinds of constraints in a system. While for the bosonic modes we are dealing with first class constraints, for the fermionic modes we have second class constraints, which force us to replace the Poisson brackets by Dirac brackets [20].

satisfied by the modes is then

$$\begin{aligned} \{\alpha^\mu_m, \alpha^\nu_n\}_{\text{P.B.}} &= -im\ \eta^{\mu\nu}\delta_{m+n} & \{b^\mu_r, b^\nu_s\}_{\text{D.B.}} &= -i\eta^{\mu\nu}\delta_{r+s}\,, \\ \{\bar\alpha^\mu_m, \bar\alpha^\nu_n\}_{\text{P.B.}} &= -im\ \eta^{\mu\nu}\delta_{m+n}\,, & \{\bar b^\mu_r, \bar b^\nu_s\}_{\text{D.B.}} &= -i\eta^{\mu\nu}\delta_{r+s}\,, \\ \{\alpha^\mu_m, \bar\alpha^\nu_n\}_{\text{P.B.}} &= 0\,, & \{b^\mu_r, \bar b^\nu_s\}_{\text{D.B.}} &= 0\,. \end{aligned} \tag{2.41}$$

The bosonic Poisson brackets will become commutation relations in the quantum theory, but the Dirac brackets involving fermionic modes will give anti-commutation relations instead. The Virasoro operators L_m and the new generators G_r of the superconformal algebra (following from T^{F}_α) can be decomposed in terms of the oscillator modes as

$$L_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} \alpha_{-n}\cdot\alpha_{m+n} + \frac{1}{2}\sum_{r\in\mathbb{Z}+\phi}\left(r+\frac{m}{2}\right) b_{-r}\cdot b_{m+r}\,, \tag{2.42}$$

$$G_r = \sum_n \alpha_{-n}\cdot b_{r+n}\,, \tag{2.43}$$

so that classically the generators of the conformal and superconformal symmetries satisfy the algebra

$$\{L_m, L_n\}_{\text{D.B.}} = -i(m-n)L_{m+n}\,, \tag{2.44}$$

$$\{L_m, G_r\}_{\text{D.B.}} = -i\left(\frac{m}{2}-r\right)G_{m+r}\,, \tag{2.45}$$

$$\{G_r, G_s\}_{\text{D.B.}} = -2iL_{r+s}\,, \tag{2.46}$$

with an exact copy for the barred operators.

We are now ready to quantise the superstring. As we did for the bosonic string, we work with operators and replace the Poisson bracket and the Dirac bracket with commutators and anti-commutators respectively,

$$\{\ \cdot\ ,\ \cdot\ \}_{\text{P.B.}} \to \frac{1}{i}[\ \cdot\ ,\ \cdot\ ]\,, \qquad \{\ \cdot\ ,\ \cdot\ \}_{\text{D.B.}} \to \frac{1}{i}\{\ \cdot\ ,\ \cdot\ \}\,, \tag{2.47}$$

which results in the algebra[17]

$$\begin{aligned} [\alpha^\mu_m, \alpha^\nu_n] &= m\ \eta^{\mu\nu}\delta_{m+n} & \{b^\mu_r, b^\nu_s\} &= \eta^{\mu\nu}\delta_{r+s}\,, \\ [\bar\alpha^\mu_m, \bar\alpha^\nu_n] &= m\ \eta^{\mu\nu}\delta_{m+n}\,, & \{\bar b^\mu_r, \bar b^\nu_s\} &= \eta^{\mu\nu}\delta_{r+s}\,, \\ [\alpha^\mu_m, \bar\alpha^\nu_n] &= 0\,, & \{b^\mu_r, \bar b^\nu_s\} &= 0\,. \end{aligned} \tag{2.48}$$

[17] It is interesting to keep in mind that, historically, the theory that would later become superstring theory started with this algebra and only later was it interpreted in terms of oscillation modes of a vibrating string, analogously to what happened with the bosonic string before it.

We are still allowed to choose the light-cone gauge for the superstring, with spacetime light-cone coordinates $(X^+, X^-, X^i),\ i = 2, ..., d-1$,

$$X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^1)\,, \qquad \psi^\pm = \frac{1}{\sqrt{2}}(\psi^0 \pm \psi^1)\,. \tag{2.49}$$

This again fixes $\alpha_n^+ = 0\,, \forall n \neq 0$, and α_n^- in terms of the transverse oscillator α_n^i, but we now also find $\psi^+ = 0$ (which can be eliminated by residual supersymmetry transformations) so that $b_r^+ = 0\,, \forall r$, and b_r^- is also fixed by the transverse oscillators.

We now need to revisit the discussion surrounding the quantum definition of the Virasoro generators and again prescribe an ordering. As before, we define the L_m operators in the quantum theory in terms of normal ordered products,

$$L_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} : \alpha_{-n}\cdot\alpha_{m+n} : + \frac{1}{2}\sum_{r\in\mathbb{Z}+\phi}\left(r + \frac{m}{2}\right) : b_{-r}\cdot b_{m+r} :\,, \tag{2.50}$$

and note that the G_r operators have no ambiguity since the α_m^μ and b_r^μ commute. We know from our computation in the bosonic string case that the bosonic modes will only have a subtlety for the L_0 operator (2.19), which we saw was the root of the critical dimension $d = 26$. However the operators L_m have now a contribution from the fermionic operators b_r^μ as well that can affect the bosonic result for the critical dimension. From the fermionic modes we get

$$\begin{aligned}\sum_{r\neq 0}\left(r + \frac{m}{2}\right) b_{-r}\cdot b_{m+r} &= \sum_{r\neq 0}\left(r + \frac{m}{2}\right)[-\cancel{b_{-r}^+} b_{m+r}^- - b_{-r}^-\cancel{b_{m+r}^+} + \delta_{ij} b_{-r}^i b_{m+r}^j] \\ &= \delta_{ij}\sum_{r\neq 0}\left(r + \frac{m}{2}\right) b_{-r}^i b_{m+r}^j\,,\end{aligned} \tag{2.51}$$

which we rewrite in terms of normal ordered products using the commutation relations (2.48),

$$\begin{aligned}\sum_{r\neq 0}\left(r + \frac{m}{2}\right) b_{-r}^i b_{m+r}^j &= \sum_{r<0}\left(r + \frac{m}{2}\right) b_{-r}^i b_{m+r}^j + \sum_{r>0}\left(r + \frac{m}{2}\right) b_{-r}^i b_{m+r}^j \\ &= \sum_{r<0}\left(r + \frac{m}{2}\right)[-b_{m+r}^i b_{-r}^j + \{b_{-r}^i, b_{m+r}^j\}] + \sum_{r>0}\left(r + \frac{m}{2}\right) : b_{-r}^i b_{m+r}^j : \\ &= -\sum_{r<0}\left(r + \frac{m}{2}\right) : b_{m+r}^i b_{-r}^j : + \sum_{r>0}\left(r + \frac{m}{2}\right) : b_{-r}^i b_{m+r}^j : \\ &\quad + \sum_{r<0}\left(r + \frac{m}{2}\right)\delta^{ij}\delta_{m,0} \\ &= \sum_{r\neq 0}\left(r + \frac{m}{2}\right) : b_{-r}^i b_{m+r}^j : - \delta_{m,0}\delta^{ij}\sum_{r>0} r\,.\end{aligned} \tag{2.52}$$

We can therefore make two immediate remarks. Firstly, we still find that only the definition of L_0 is affected, with all other operators being unambiguously defined by their classical expressions and L_0 differing by a naïvely divergent series. However, in contrast with the bosonic contribution, the fermionic modes contribute with a negative term. Recalling that the range of r depends on the choice of boundary condition, i.e. $r \in \mathbb{Z}$ for (R) boundary conditions and $r \in \mathbb{Z}+\frac{1}{2}$ for (NS) boundary conditions, this contribution will be different for the two sectors,

$$\texttt{(R)} \qquad \sum_{r=1}^{\infty} r = -\frac{1}{12}\,, \tag{2.53}$$

$$\texttt{(NS)} \qquad \sum_{r=\frac{1}{2}}^{\infty} r = \sum_{\tilde{r}=1}^{\infty} \frac{\tilde{r}}{2} = -\frac{1}{24}\,, \tag{2.54}$$

where we again use the regularised value of the series (2.20). The definition of L_0 for the superstring is therefore

$$L_0 = \frac{1}{2}\sum_{n\in\mathbb{Z}} :\alpha_n\cdot\alpha_{-n}: + \frac{1}{2}\sum_{r\in\mathbb{Z}+\phi} r :b_{-r}\cdot b_r: - \frac{d-2}{8}\phi\,. \tag{2.55}$$

We see that for (R) boundary conditions ($\phi = 0$) the bosonic and fermionic contributions completely cancel, while for (NS) boundary conditions ($\phi = \frac{1}{2}$) our previous result $-\frac{d-2}{24}$ (2.21) changes to $-\frac{d-2}{16}$ due to the fermionic modes — this will affect the critical dimension of the string, which will no longer be $d = 26$. We can rewrite L_0 as

$$L_0 = \alpha' p^\mu p_\mu - \frac{d-2}{8}\phi + N_{\text{tr}}^{(\alpha)} + N_{\text{tr}}^{(b)}\,, \tag{2.56}$$

in terms of the transverse number operators

$$N_{\text{tr}}^{(\alpha)} = \delta_{ij}\sum_{n=1}^{\infty}\alpha_{-n}^i\alpha_n^j\,, \qquad N_{\text{tr}}^{(b)} = \delta_{ij}\sum_{r\in\mathbb{Z}_+ +\phi} r\, b_{-r}^i b_r^j\,, \tag{2.57}$$

whose eigenvalues count the number of each type of oscillators in a generic state. Similarly to the bosonic case, these definitions allow us to determine the algebra satisfied by the operators L_m and G_r, and we again find conformal anomalies signalling the (potential) breaking of conformal and super-conformal symmetries. Imposing all constraints on physical states,

$$L_m\,|\text{phys}\rangle = \bar{L}_m\,|\text{phys}\rangle = 0\,,\ \forall\ m>0\,, \qquad G_r\,|\text{phys}\rangle = \bar{G}_r\,|\text{phys}\rangle = 0\,,\ \forall\ r>0\,, \tag{2.58}$$

and in particular focusing on the L_0 and $\bar{L}_0$ constraints, we find that a physical state in the closed superstring spectrum must satisfy

$$\alpha' m^2 = N_{\text{tr}}^{(\alpha)} + N_{\text{tr}}^{(b)} + \bar{N}_{\text{tr}}^{(\alpha)} + \bar{N}_{\text{tr}}^{(b)} - \frac{d-2}{8}\phi - \frac{d-2}{8}\bar{\phi}\,, \tag{2.59}$$

where we must take into account both left- and right-moving modes. The ground states are again defined as the states annihilated by all annihilation operators,

$$\alpha_m^\mu\,|0\rangle = b_r^\mu\,|0\rangle = 0\,. \tag{2.60}$$

However, unlike a ground state of the bosonic operators only, the fermionic operator b_0^μ makes this ground state more subtle. The reason for this is that

$$b_r^\mu(b_0^\nu\,|0\rangle) = (-b_0^\nu b_r^\mu + \{b_r^\mu, b_0^\nu\})\,|0\rangle = -b_0^\nu \cancel{b_r^\mu\,|0\rangle} + \eta^{\mu\nu}\delta_{r,0} = 0\,, \tag{2.61}$$

for all annihilation operators b_r^μ , $r > 0$, *all* of which annihilate the state $b_0^\mu\,|0\rangle$ — this makes $b_0^\mu\,|0\rangle$ a ground state itself. Note that b_0^μ is an oscillator associated with (R) boundary conditions, for which $r \in \mathbb{Z}$, and hence this defines the (R) sector ground state. Moreover, $N_{\text{tr}}^{(b)} b_0^\mu\,|0\rangle = 0$ due to the definition of the number operator (2.57) and therefore $\alpha' m^2(b_0^\mu\,|0\rangle) = 0$, so that $b_0^\mu\,|0\rangle$ indeed gives degenerate (in mass) states that we identify as the ground state. Since the operators b_0^μ satisfy the Clifford algebra $\{b_0^\mu, b_0^\nu\} = \eta^{\mu\nu}$, these degenerate ground states form a spinor representation of SO($d-1$,1) [20]. The result is a single (NS) ground state $|0\rangle_{\text{NS}}$ (a spacetime scalar) and a (R) ground state which is a spacetime spinor $|s\rangle_{\text{R}}$. The fact that all oscillators are spacetime vectors means that whether a state built out of the ground state (by acting with a given combination of creation operators on it) is a boson or a fermion is completely determined by the ground state itself — states built from $|s\rangle_{\text{R}}$ will all be fermions and states built from $|0\rangle_{\text{NS}}$ will all be bosons.[18]

We are finally ready to build the superstring spectrum. Recalling that for the closed string we must impose level matching, any state will have the same number of left- and right-moving oscillators. Since we can choose to start with $|s\rangle_{\text{R}}$ or $|0\rangle_{\text{NS}}$ for the left- and right-moving excitations, we now have 4 different sectors (R–R), (NS–R), (R–NS) and (NS–NS). From now on, we drop the labels (R) and (NS) in the ground states, so that we can label them as left (L) and right (R) instead.

In the (NS–NS) sector, the ground state is the oscillator vacuum $|0\rangle_L \times |0\rangle_R$. Having no oscillators, $N_{\text{tr}}^{(\alpha)} = N_{\text{tr}}^{(b)} = \bar{N}_{\text{tr}}^{(\alpha)} = \bar{N}_{\text{tr}}^{(b)} = 0$, it has a mass

$$\alpha' m^2\,|0\rangle_L \times |0\rangle_R = -\frac{d-2}{16}\,|0\rangle_L \times |0\rangle_R\;, \tag{2.62}$$

resulting in a tachyon for $d > 2$. At this level it seems like the superstring is not helping us deal with the tachyon problem we found in the bosonic case. The first excited state in this sector is $\bar{b}_{-1/2}^i\,|0\rangle_L \times b_{-1/2}^j\,|0\rangle_R$, since the half-integer oscillators of the (NS) sector give a lower mass contribution than any integer oscillators. This can be decomposed into irreducible representations of SO($d-2$), resulting in a symmetric traceless tensor (graviton), an antisymmetric tensor (2-form) and a scalar (dilaton). The same reasoning that led us to conclude that the first excited state for the bosonic string was massless, once again requires these states to be massless, so that

$$\alpha' m^2\,|0\rangle_L \times |0\rangle_R = 2\left(\frac{1}{2} - \frac{d-2}{16}\right)|0\rangle_L \times |0\rangle_R \overset{!}{=} 0\,, \tag{2.63}$$

[18] This is true for the closed string, but only in part for the open string — open strings with mixed (ND) or (DN) boundary conditions actually interchange the (R) and (NS) sectors, with a single (R) ground state and a degenerate (NS) ground state [20].

fixing the critical dimension of the superstring to be

$$d = 10\,. \tag{2.64}$$

Any other spacetime dimension would be inconsistent with covariance and hence the superstring must propagate in 10 dimensions. Note how crucial the contribution from the fermions was in changing the critical dimension of the string, which we can track back to the new definition of L_0 (2.55) and even further back to the appearance of the fermionic fields in the energy-momentum tensor $T_{\alpha\beta}$ (2.32). Although less obvious, supersymmetry on the worldsheet was also crucial, allowing us to decouple ghost states and guaranteeing the consistency of the theory — in our approach, this is reflected in the possibility to use supersymmetry transformations to fix ψ^+ when going to light-cone gauge.[19]

There is one final subtlety regarding the (R) ground state. Although it can be described by 16 states, we can split it into two possible definite chiralities $|+\rangle$ and $|-\rangle$, each of which describing half of the states, and build our states from each of these chirality-definite (R) ground states. These states all have

$$\alpha' m^2 \left|\pm\right\rangle_L \times \left|\pm\right\rangle_R = 0\,, \tag{2.65}$$

since in the (R) sector $\phi = \bar{\phi} = 0$ in (2.59). The massless states in the remaining (R–NS) and (NS–R) sectors are then built from the (R) ground states $|\pm\rangle_{L,R}$ and the states $\bar{b}^i_{-1/2}\left|0\right\rangle_L$, $b^i_{-1/2}\left|0\right\rangle_R$. We summarise all states up to the massless level in Table 2.3.

There are a couple of things to note in the closed superstring spectrum. First, we still have a tachyonic state that appears to signal some sort of instability. On the other hand, the spectrum with all states included is not spacetime supersymmetric. It turns out that the spectrum of the superstring including all the states in all the sectors is inconsistent and it must be truncated to a subset of states in order to be compatible with worldsheet modular invariance. Using partition functions to study string scattering amplitudes, one can show that this modular invariance indeed truncates the spectrum in terms of the so-called GSO projection [37, 38], resulting in a spacetime supersymmetric spectrum. The GSO projection makes use of the operator $(-1)^F$, where F is the worldsheet fermion number, effectively counting the number of fermionic oscillators in a given state minus 1 (this is so that $(-1)^F \left|0\right\rangle_{\rm NS} = -\left|0\right\rangle_{\rm NS}$). We then require all physical states in the (NS) sector to have $(-1)^F = +1$, *projecting out* all other states. For the (R) sector, we include in the definition of $(-1)^F$ the chirality operator, so that the definite-chirality ground states have opposite eigenvalues, $(-1)^F \left|+\right\rangle_{\rm R} = +\left|+\right\rangle_{\rm R}$ and $(-1)^F \left|-\right\rangle_{\rm R} = -\left|-\right\rangle_{\rm R}$, with the GSO projection keeping only states with eigenvalue $(-1)^F = +1$ or states with eigenvalue $(-1)^F = -1$ and projecting out all others.[20] Note that this truncation is only consistent provided

[19] The critical dimension $d = 10$ and its relation to the absence of ghosts was first discussed in [34–36].

[20] When presented at this level, the GSO projection seems extremely ad-hoc and unnatural. Going slightly deeper into string amplitude computations with partition functions, however, one can actually see it pop out from consistency considerations. In summary, when writing down the one-loop partition function, preserving the modular invariance of the torus (which is the Riemann surface formed by the string worldsheet associated with one-loop amplitudes) requires us to sum over different spin structures. This is because modular transformations

$\alpha' m^2$	States	$(-1)^{\bar{F}}$	$(-1)^F$	Reps.
(NS–NS) sector				
-2	$\|0\rangle_L \times \|0\rangle_R$	-1	-1	(1)
0	$\bar{b}^i_{-1/2}\|0\rangle_L \times b^j_{-1/2}\|0\rangle_R$	$+1$	$+1$	$(1)+(28)+(35)_v$
(R–R) sector				
0	$\|+\rangle_L \times \|+\rangle_R$	$+1$	$+1$	$(1)+(28)+(35)_s$
	$\|-\rangle_L \times \|-\rangle_R$	-1	-1	$(1)+(28)+(35)_c$
	$\|-\rangle_L \times \|+\rangle_R$	-1	$+1$	$(8)_v+(56)_v$
	$\|+\rangle_L \times \|-\rangle_R$	$+1$	-1	$(8)_v+(56)_v$
(R–NS) sector				
0	$\|+\rangle_L \times b^i_{-1/2}\|0\rangle_R$	$+1$	$+1$	$(8)_c+(56)_c$
	$\|-\rangle_L \times b^i_{-1/2}\|0\rangle_R$	-1	$+1$	$(8)_s+(56)_s$
(NS–R) sector				
0	$\bar{b}^i_{-1/2}\|0\rangle_L \times \|+\rangle_R$	$+1$	$+1$	$(8)_c+(56)_c$
	$\bar{b}^i_{-1/2}\|0\rangle_L \times \|-\rangle_R$	$+1$	-1	$(8)_s+(56)_s$

Table 2.3: Spectrum of the closed superstring up to level $\alpha' m^2 = 0$, for all sectors. We highlight the states surviving the choice of GSO projection that results in the Type IIB string theory.

string interactions between the surviving states do not produce any of the states that have been projected out, which can be shown to follow from locality of the algebra of vertex operators [20]. We include in Table 2.3 the eigenvalues of these operators for each row of states.

We immediately find that the tachyon gets projected out and we no longer have a state signalling an instability of the theory. The massless level of the (NS–NS) sector survives, providing us with the graviton, 2-form and dilaton states. Since we can choose to keep either $(-1)^F = +1$ or $(-1)^F = -1$ states in the (R) sector, there are two inequivalent but consistent truncations of the spectrum, $(-1)^{\bar{F}} = (-1)^F$ or $(-1)^{\bar{F}} = -(-1)^F$. In practice, these correspond to left- and right-moving ground states having the same or opposite chirality, respectively. If we choose to keep ground states with the same chirality, the (R–R) sector after the GSO projection contains a scalar, a 2-form and a self-dual 4-form. The (R–NS) and (NS–R) sectors are left with one gravitino and one spinor (dilatino) each sharing the same chirality. This is the massless spectrum of Type IIB string theory, a chiral theory with $\mathcal{N} = 2$ supersymmetry. It is the theory whose phenomenology we are going to explore and we therefore highlight its spectrum in Table 2.3.

It is worth mentioning that choosing the left- and right-moving ground states to have opposite

can change the spin structure required by the fermion fields and the partition function can only be modular invariant if it is written in terms of the right combination of spin structures. This singles out the GSO projection appearing in the partition function as $\frac{1}{2}(1-(-1)^F)$ for the (NS) sector and $\frac{1}{2}(1 \pm (-1)^F)$ for the (R) sector.

chirality gives us Type IIA string theory, which also has $\mathcal{N} = 2$ supersymmetry. Contrary to Type IIB, this theory is not chiral as it contains a gravitino of each chirality. The (R–R) sector is also different, containing a 1-form and a 3-form. Although we will not explore Type IIA much further, this theory is also widely studied not only in the context of moduli stabilisation and phenomenology, but also in relation to the string theory duality web.

2.2.2 Worldsheet parity, Type I string theory and Orientifolds

Let us consider the following operation on the string worldsheet

$$\Omega : \sigma \to \ell - \sigma\,. \tag{2.66}$$

This is known as a parity operation and it effectively reverses the string. Focusing on the bosonic fields X^μ for a closed string, $X^\mu(\sigma+\ell,\tau) = X^\mu(\sigma,\tau)$, the solution (2.5) is such that acting with Ω gives

$$\Omega X^\mu(\sigma,\tau) = X^\mu(\ell-\sigma,\tau) = X^\mu(-\sigma,\tau) = X^\mu_L(\tau-\sigma) + X^\mu_R(\tau+\sigma)\,, \tag{2.67}$$

which has the effect of making the left-movers X^μ_L right-movers $(\tau-\sigma)$ and the right-movers X^μ_R left-movers $(\tau+\sigma)$. Worldsheet parity therefore exchanges left-moving waves with right-moving waves and consequently it will exchange left- and right-moving oscillators[21] $\bar{\alpha}^\mu_m \leftrightarrow \alpha^\mu_m$, $\bar{b}^\mu_{-r} \leftrightarrow b^\mu_{-r}$. This is not entirely surprising as our parity operation is precisely to swap what one would call the "left" and the "right" ends of a string. From the worldsheet perspective (i.e. from the solution for X^μ) this is a global symmetry, since swapping left- and right-movers does not change the solution. However, that is not the case from the spacetime perspective (i.e. from the states build out of the individual oscillators), since not all of them are invariant under this parity transformation. For example, the anti-symmetric 2-form that corresponds to the anti-symmetric combination $\bar{b}^{[i}_{-1/2}b^{j]}_{-1/2}$ (see Table 2.3) is not invariant under the exchange $\bar{b}^i_{-1/2} \leftrightarrow b^j_{-1/2}$, since it picks up a minus sign.

One may however require that the string spectrum itself respects this symmetry, which leads to the construction of the unoriented string. In order to do that, one considers the projector $P_\Omega = \frac{1}{2}(1+\Omega)$ that selects the parity invariant states (e.g. while the NS–NS 2-form is *projected out* of the spectrum, the graviton and the dilaton survive the projection). As with many other steps in the construction of string theories, we must take into account any consistency constraints that would forbid us from performing this projection. An important example of this is the different GSO projections that lead to Type IIA and Type IIB strings — while the Type IIB projection is left-right symmetric (in the sense that the left- and right-movers are treated identically), that is not the case for Type IIA where $(-1)^{\bar{F}} \neq (-1)^F$. Hence it seems we can only define a theory of

[21] We restrict our discussion to the bosonic coordinates, but the same applies to the fermionic oscillators. One should keep in mind however that some extra care is required when defining the action of Ω on worldsheet fermionic fields.

unoriented strings starting from Type IIB, which is the Type I theory. Importantly, the Type I theory only has half of the supersymmetry, i.e. it is an $\mathcal{N} = 1$ theory (indeed, only an invariant combination of the 2 gravitinos in Type IIB survives the projection).

There is however one powerful property of Type II string theories — they are related by a duality known as T-duality. This is a duality between two apparently different theories defined on backgrounds with a direction compactified on a circle, $M_{10} = M_9 \times S^1$. T-duality relates a theory on such a background with a circle of radius R to a theory on a similar background but with a circle of radius $\sqrt{\alpha'}/R$. This is one of the surprising dualities that lead to the realisation that all (supersymmetric) string theories are connected to each other. For our current purposes, the crucial observation is that T-duality flips the sign of the right-moving circle coordinate X^i_R in such a way that along this direction in the T-dual theory (here the Type IIA theory) worldsheet parity acts as

$$X^i_{\mathrm{IIA}}(t,\sigma) \to -X^i_{\mathrm{IIA}}(t,\ell-\sigma)\,. \tag{2.68}$$

This combines our worldsheet parity with a spacetime reflection $\mathcal{R}$ along the i^{th} direction. Only when this combined action leaves the states invariant do we have a truly unoriented theory — since this happens at the fixed point of (2.68), there is an 8-dimensional surface where the theory is unoriented. This surface is known as an orientifold plane, which we call O8-plane. Away from the O-plane, the theory is not unoriented but instead related to its mirror image with respect to this plane.[22] Under this new perspective, we can think of the Type I theory as having an O9-plane which fills the whole of spacetime.

In fact, this is not just a matter of interpretation. Just as D-branes are physical objects which must be accounted for in the theory (as they couple to fields in the spectrum), also O-planes are physical objects and need to be included in a consistent description. By studying string scattering amplitudes, one finds the contribution of both D-branes and O-planes, determining their effective tensions and charges, and how they couple to the different states in the spectrum. Unlike D-branes though, O-planes are not dynamical objects in the sense that there are no modes associated to fluctuations of these surfaces. Taking this into account, one finds that the Type I theory is only consistent with its O9-plane if there are also 32 D9-branes present (this guarantees the cancellation of a tadpole[23] generated by the O9-plane) [20]. The presence of the D-branes means that Type I superstring theory necessarily contains open strings and the fact that the D9-branes are necessarily coincident results in an SO(32) gauge group which is crucial for anomaly cancelation and the consistency of the theory.

Repeatedly applying T-duality along different directions, one can hop between Type IIB and Type IIA, and find lower-dimensional O-planes. We find that Type IIB has Op-planes with p odd

[22] In a truly unoriented theory, projection removes half the states locally; in this case, it relates the amplitude to find a string at some point to the amplitude to find it at the image point with respect to the orientifold plane [39].

[23] We will later apply this logic in reverse: we will use O-planes to cancel tadpole contributions due to fluxes and D-branes once 6 of the 10 dimensions are compactified. We therefore postpone any more detail about tadpole cancellation to that discussion.

and Type IIA has Op-planes with p even. Just as the Type I theory with its O9-plane had half the supersymmetry of Type IIB, any O-plane will break half the supersymmetry of the original theory, with different O-planes breaking different combinations of the original supersymmetry. These different types of O-planes will also couple to different states and are important ingredients in IIA and IIB phenomenology. We will find in particular that O-planes are extremely useful for cancelling tadpole charges in a way which preserves $\mathcal{N} = 1$ supersymmetry in the compactified theory.

2.2.3 Multiple string theories united by duality

While introducing orientifold planes, we briefly mentioned the T-duality between Type IIA and Type IIB as part of a larger web of dualities relating all consistent supersymmetric string theories. Let us here say a few more words regarding this powerful structure.

In order to do that, we need to mention two more consistent string theories, apart from the ones we already encountered. These are the Heterotic theories, which divide into two types depending on their gauges groups — for consistency, these can only be SO(32) and $E_8 \times E_8$ [40]. The Heterotic theories are a consistent combination of the bosonic string and the superstring[24] that exploits the independence between left-movers and right-movers. It gives fermionic partners to the right-movers making them supersymmetric, but not to the left-movers, so that one can think of them as belonging to a bosonic string. The result is that the critical dimension is $d = 10$ for the left-movers but $d = 26$ for the left-movers, and a consistent 10d theory therefore requires us to compactify these 16 extra dimensions. Requiring the absence of anomalies fixes the compact space in such a way that generates either an SO(32) or $E_8 \times E_8$ gauge group.[25]

The construction of the Heterotic string was actually motivated by two important observations. The first was the anomaly cancellation result that selected SO(32) and $E_8 \times E_8$ as the unique gauge groups that would lead to supersymmetric Yang-Mills theory coupled to $\mathcal{N} = 1$, $d = 10$ supergravity without gauge and gravitational anomalies [41]. The other was the connection between the low-energy limit of string theories and anomaly-free $d = 10$ supergravity theories. Although the Type I theory had an SO(32) gauge group, there was no known string theory with an $E_8 \times E_8$ gauge group.

These five supersymmetric string theories were shown to all be connected by dualities, changing their status as independent fundamental theories into a new picture where they are all different

[24] The name heterotic was chosen precisely because of this hybrid combination. Although the greek word "heterosis" only means alteration (following from "heteros", meaning different), the word was actually chosen because of its use in genetics/biology, where it describes the "increased vigour displayed by crossbred animals or plants" [40].

[25] The compact space must be a torus of common radius $R = \sqrt{\alpha'}$, corresponding to a lattice which must be integer, even and self-dual. Since there are only two such lattices, the one corresponding to the weigths of $\mathrm{Spin}(32)/\mathbb{Z}_2$ and the direct product of two lattices corresponding to the weights of E_8, this leads to the unique identification of $\mathrm{SO}(32) \cong \mathrm{Spin}(32)/\mathbb{Z}_2$ and $E_8 \times E_8$ as the gauge groups of the 10d theory.

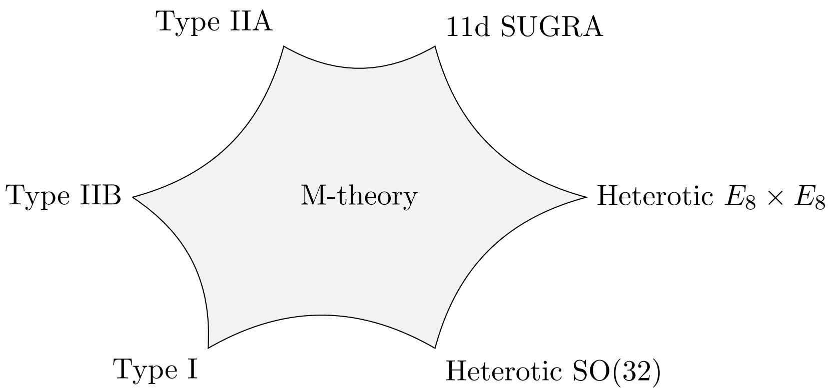


Figure 2.1: All consistent supersymmetric string theories were shown to be connected by a web of dualities — this was a catalyst to the second superstring revolution that happened in the 90's. Not only do the dualities relate theories at different couplings and energies, they also relate theories defined in different dimensions. They are all now seen as different limits of a more fundamental theory which became known as M-theory.

limits of a common more fundamental one (Fig. 2.1), which became know as M-theory and is, as of yet, not fully known or understood. The duality web also includes 11-dimensional supergravity, which suggested that this fundamental theory was actually an 11-dimensional theory whose low-energy description was 11d supergravity[26] [42]. Although we will not explore these dualities further, they have been widely explored in the string theory literature and are seen as an important step towards a non-perturbative and more fundamental theory.

2.3 Type IIB supergravity

2.3.1 Type IIB at low energies

The spectrum of Type IIB string theory includes an infinite tower of massive states — in fact, the higher one goes in mass, the more states each level possesses. The mass scale of this tower is

$$M_s = \frac{1}{\sqrt{\alpha'}}\,, \tag{2.69}$$

defining what we call the string scale. The theory itself does not tell us what this scale should be and only observations could give its actual value. It therefore depends on what physics string theory is actually describing — in its original form, it was a theory of hadrons and so this scale was thought to be around the nuclear scale ($M_s \sim 1$ GeV) [43]; once it became a theory of (quantum) gravity the "natural" scale became that of the gravitational coupling ($M_{\rm Pl} \sim 10^{19}$ GeV). We will see that we can indeed relate the string scale of the fundamental string theory to the Planck mass of a low-energy 4-dimensional description through the coupling

[26]For this reason, M-theory and 11d supergravity are commonly interchanged, although one is the fundamental theory of which the other is simply a low-energy description.

of string interactions and the volume of the extra dimensions.

A string scale which is near the Planck scale is many orders of magnitude above any energy we can currently (and most likely for a very long time) probe with experiments. Although cosmological observations may help bridge the gap between our detectors and these extremely high energies, it is still reasonable to assume that most of the physics we are interested in describing happens at energies $E \ll M_s$. One would therefore expect the high-energy states not to contribute significantly — from the effective field theory point of view, we could integrate out the high-energy modes and work with a theory describing the physics at low energies. In the low-energy effective theory only the massless states of the spectrum are propagating degrees of freedom (since these are the only ones which do not require $E \gtrsim M_s$) and interactions with the massive tower will appear as low-energy interactions between massless states suppressed by powers of E/M_s (these can be thought of as corrections to the theory of massless states and are suppressed by this small ratio). The massless states of Type IIB string theory correspond to the field content of a 10-dimensional chiral $\mathcal{N} = 2$ supergravity theory, which was named Type IIB supergravity [44–48].

One way of constructing the low-energy action of Type IIB string theory is to take the massless states and build the most general Lagrangian compatible with the symmetries of the theory (e.g. diffeomorphisms, gauge invariance and supersymmetry) up to some order in a derivative expansion. The derivative expansion should be controlled by the scale at which new degrees of freedom would appear, i.e. by the small ratio ∂^2/M_s^2 so that higher-derivative operators will be less important at low energies.[27] We can then compute different scattering amplitudes within this EFT involving the generic couplings in terms of which we wrote the interactions. By comparing these amplitudes with the scattering amplitudes of the same states in the full theory (which are computed as correlation functions of physical state vertex operators [20]), we can match the EFT couplings with the string theory result at the same order in the E/M_s expansion. On the string theory side, the amplitudes are computed in terms of a loop-expansion that turns out to be controlled by the string coupling g_s, which is determined by the vacuum expectation value of the dilaton state — the strings are weakly-coupled when this value is small, and only then is this loop-expansion consistent. In this way, the EFT is constructed such that the low-energy limit of the string theory amplitudes at a given order in the loop-expansion is reproduced.

Once all this matching is done and the couplings determined, the action describing the massless bosonic states of Type IIB supergravity at leading order in g_s and α' becomes[28]

[27] Recall that derivatives in the action are associated with momenta in the scattering amplitudes, so that ∂^2 really represents $p^2 \sim E \ll M_s$.

[28] See [49] for the dilaton dependence of the RR sector.

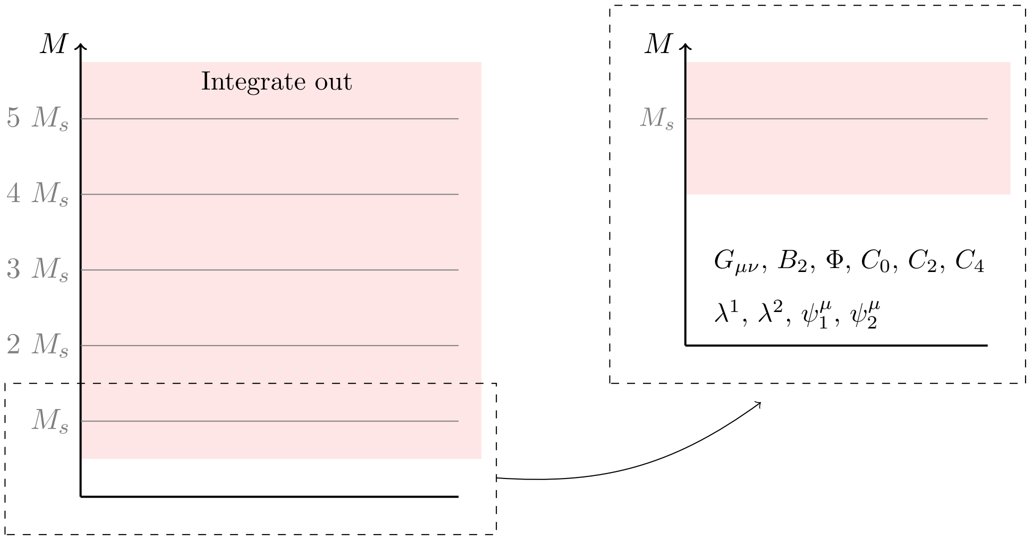


Figure 2.2: At energies much $E \ll M_s$, the physics of Type IIB string theory can be described by an effective field theory including only its massless states $G_{\mu\nu}$, B_2, Φ, C_0, C_2, C_4 (bosonic) and λ^1, λ^2, ψ_1^μ, ψ_2^μ (fermionic). The massive string states whose masses are well above the energies we want to describe are integrated out and only contribute the low-energy theory through effective interactions between the massless states which are suppressed by the small ratio $E/M_s \ll 1$.

$$S^{\mathrm{S}}_{\mathrm{IIB}} = \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-G^S}\Big\{e^{-2\Phi}\left(R + 4(\partial_\mu\Phi)(\partial^\mu\Phi) - \frac{1}{2}|H_3|^2\right) - \left(\frac{1}{2}|F_1|^2 + \frac{1}{2}|\tilde{F}_3|^2 + \frac{1}{4}|\tilde{F}_5|^2\right)\Big\}$$
$$- \frac{1}{4\kappa_{10}^2}\int C_4 \wedge H_3 \wedge F_3 \,, \tag{2.70}$$

where R is the Ricci scalar for the metric G^S_{MN}, Φ is the dilaton, H_3 is the field-strength of the NS–NS 2-form B_2 and F_p is the field-strength of the RR $(p-1)$-form C_{p-1},

$$H_3 = \mathrm{d}B_2 \,, \qquad F_p = \mathrm{d}C_{p-1} \,, \quad p = 1, 3, 5 \,.$$

The RR field strengths F_3 and F_5 appear in the action through the gauge invariant combinations

$$\tilde{F}_3 = F_3 - C_0 H_3 \,, \qquad \tilde{F}_5 = F_5 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3 \,,$$

and we define the contractions as

$$|F_p|^2 = \frac{1}{p!}F_{\mu_1\ldots\mu_p}F^{\mu_1\ldots\mu_p} \,.$$

Note that these definitions lead to the non-standard Bianchi identities

$$\mathrm{d}\tilde{F}_3 = H_3 \wedge F_1 \tag{2.71}$$

$$\mathrm{d}\tilde{F}_5 = H_3 \wedge F_3 \tag{2.72}$$

which follow from $\mathrm{d}H_3 = \mathrm{d}F_1 = \mathrm{d}F_3 = \mathrm{d}F_5 = 0$. Moreover, the type IIB action must be supplemented with the self-duality condition[29]

$$\tilde{F}_5 = \star \tilde{F}_5 \,. \tag{2.73}$$

Apart from the bosonic fields in (2.70) there are also two gravitinos (ψ^1_M, ψ^2_M) and two dilatinos (λ^1, λ^2), whose action is rarely shown. The reason is that for most purposes the work can be done by dealing with the bosonic action only and occasionally referring to the supersymmetry transformations for the fermions. This work is no exception.

Having matched the amplitudes computed using the low-energy EFT with the same amplitudes computed using the UV (string) theory, we know precisely how the couplings in Type IIB supergravity are related to the couplings (and therefore fundamental scales) of the underlying Type IIB string theory. The relation between the string scale α' and the 10d string frame gravitational coupling κ_{10} is

$$2\kappa_{10}^2 = (2\pi)^7 \alpha'^4 \,. \tag{2.74}$$

A common convention for the string length[30] l_s, which we use below, is

$$(2\pi)^2 \alpha' = l_s^2 \,, \tag{2.75}$$

although sometimes $\alpha' = l_s^2$ is used instead.

The Einstein frame

Here is an important observation — the low-energy supergravity action (2.70) shows that the way the graviton and the dilaton states interact is such that, from the field theory point of view, 10d gravity is not in the canonical Einstein-Hilbert form, but rather more akin to a scalar-tensor modified theory of gravity. Physically this means that the graviton does not have canonical kinetic terms and the string-frame metric G^S_{MN} does not directly correspond to the (field theory) propagating graviton. Only once we write the action in canonical form can we read off the gravitational interactions of all fields in the way we are used to in GR. The frame in which the action takes the canonical Einstein-Hilbert form — i.e. the Ricci scalar does not couple to anything other than $\sqrt{-G^E}$ — is known as the Einstein frame, which we can choose by performing a conformal transformation of the 10d metric $G^S \to G^E = e^{2\Upsilon} G^S$ (see Appendix

[29]Notice the factor of $\frac{1}{4}$ rather than $\frac{1}{2}$ in the kinetic term, which accounts for the fact that only half the degrees of freedom should be present.

[30]Note that the string scale (which we defined as the mass of the tower of string states) is $M_s = \frac{1}{\sqrt{\alpha'}} = \frac{2\pi}{l_s}$ for this choice of conventions. The notation $m_s = \frac{1}{l_s}$ is commonly used with this convention, with the relation $M_s = 2\pi m_s$.

A.1). The action in Einstein frame becomes

$$S_{\rm IIB}^{\rm E} = \frac{1}{2\kappa^2}\Bigg\{ \int d^{10}x\sqrt{-G}\left(R - \frac{1}{2}(\partial_M\Phi)(\partial^M\Phi) - \frac{g_s}{2}e^{-\Phi}|H_3|^2\right) \\ - \int d^{10}x\sqrt{-G}\left(\frac{e^{2\Phi}}{2}|F_1|^2 + \frac{g_s}{2}e^{\Phi}|\tilde{F}_3|^2 + \frac{g_s^2}{4}|\tilde{F}_5|^2\right) - \frac{g_s^2}{2}\int C_4\wedge H_3\wedge F_3\Bigg\}\,, \qquad (2.76)$$

after fixing our conventions such that the metric in the string frame and the metric in the Einstein frame are the same at the vacuum,[31] allowing us to discuss quantities in a frame-independent way *at the vacuum*. For that choice, the Einstein frame gravitational coupling is related to the string scale as

$$2\kappa^2 = 2\kappa_{10}^2 g_s^2 = (2\pi)^7 g_s^2\alpha'^4 \qquad \text{or} \qquad 2\kappa^2 = \frac{g_s^2\, l_s^8}{2\pi}\,. \qquad (2.77)$$

The constant g_s that now appears in the action and physical quantities is defined by the vacuum expectation value of the dilaton field, $g_s \equiv e^{\langle\Phi\rangle}$. Let us emphasise that this *does not* mean that the vev of Φ is determined by some constant g_s which relates the physical scales; it means that the constant g_s that relates these physical scales and couplings is only fixed by the vev of the dilaton. In particular, it implies that these couplings are not constants in the theory, but instead must be dynamically fixed by giving the dilaton an expectation value.

This reflects the fact that the UV theory which completes this low-energy description is a theory of strings whose only scale is the string scale M_s, meaning that *everything* else in the theory is dynamically determined through strings interacting with each other — this unique scale is simply giving us a ruler with which we can measure different quantities. Hence, string theory comes with no free parameters and all phenomena will, deep down, be entirely determined by its internal structure and dynamics. On the other hand, this also means that the dilaton is not allowed to remain a flat direction. In some way or another, it must develop a potential which will fix its vev and make it massive.[32] In order for the perturbative expansion used in the computation of string amplitudes to remain valid, this should happen at small values of g_s.

It is convenient to write the equations of motion for the scalars and form fields in differential form language (see Appendix B),

$$\mathrm{d}\star\mathrm{d}\Phi = e^{2\Phi}|F_1|^2 + \frac{g_s}{2}\left(e^{\Phi}|\tilde{F}_3|^2 - e^{-\Phi}|H_3|^2\right) \qquad (2.78a)$$

$$d(e^{2\Phi}\star F_1) = -g_s e^{\Phi}\tilde{F}_3\wedge\star H_3 \qquad (2.78b)$$

$$\mathrm{d}(e^{-\Phi}\star H_3 - e^{\Phi}C_0\star\tilde{F}_3) = g_s F_3\wedge\tilde{F}_5 \qquad (2.78c)$$

$$\mathrm{d}(e^{\Phi}\star\tilde{F}_3) = -g_s H_3\wedge\tilde{F}_5 \qquad (2.78d)$$

[31] This can be seen by setting the dilaton Φ to its constant vev in the string frame action (2.70), which leaves a canonical Einstein-Hilbert term with a new gravitational coupling involving this vev (2.77).

[32] We will see later how this is related to observational constraints on fifth forces.

$$\mathrm{d} \star \tilde{F}_5 = H_3 \wedge F_3 \tag{2.78e}$$

and the Einstein equations in component form,

$$\begin{aligned} R_{MN} = &\frac{1}{2}(\partial_M \Phi)(\partial_N \Phi) + \frac{e^{2\Phi}}{2}(\partial_M C_0)(\partial_N C_0) + \frac{g_s^2}{4 \times 4!}(\tilde{F}_5)_{MPQRS}(\tilde{F}_5)_N{}^{PQRS} \\ &+ \frac{g_s}{4}\Big(e^{\Phi}(\tilde{F}_3)_{MPQ}(\tilde{F}_3)_N{}^{PQ} + e^{-\Phi}(H_3)_{MPQ}(H_3)_N{}^{PQ}\Big) \\ &- \frac{g_s}{8} G_{MN}\Big(e^{\Phi}|\tilde{F}_3|^2 + e^{-\Phi}|H_3|^2\Big) . \end{aligned} \tag{2.79}$$

These must be supplemented by the self-duality condition (2.73), which is compatible with the equation of motion and Bianchi identity for $\tilde{F}_5$, but is not implied by them.

The way the 3-forms H_3 and F_3 appear in the action (2.76) and equations of motion appears to have some underlying structure — it almost seems as if there is a symmetry between them. In fact, there is indeed a symmetry relating the two fields B_2 and C_2 (which as one might guess from the equations of motion will necessarily also involve the scalars Φ and C_0), which we now briefly discuss.

The $\mathbf{SL}(2, \mathbb{Z})$ invariance

There is no shortage of examples in physics where symmetries opened the door to a deeper understanding of theories, phenomena and apparently accidental connections. From understanding the deeply symmetric structure of Maxwell's equations as a consequence of their underlying gauge symmetry, to formulating the Higgs mechanism by which massless particles acquire a mass due to the *breaking* of a gauge symmetry, there is a lot to be gained from making symmetries as manifest as possible.[33]

It turns out that the Type IIB supergravity action in Einstein frame (2.76) does have a symmetry which is not obvious from the way it is currently written. While choosing the right field redefinitions that makes this symmetry manifest requires some hindsight, once these are performed the symmetry becomes easier to spot. Here is the clever field redefinition [50–53]

$$\tau = C_0 + i e^{-\Phi} , \tag{2.80a}$$

$$G_3 = \tilde{F}_3 - i e^{-\Phi} H_3 = F_3 - \tau H_3 , \tag{2.80b}$$

[33] This is not necessarily true when performing explicit computations, as it is possible that keeping symmetries manifest throughout the computation might make it more technically involved and cumbersome. A nice example is the derivation of the string spectrum, which is most easily obtained by working in a non-manifestly Lorentz invariant way and *then* imposing this invariance on the resulting spectrum. However, when studying string interactions, a covariant approach *is* the best way of making progress.

with which we can write (2.76) as

$$S_{\mathrm{IIB}}^{\mathrm{E}} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-G}\left(R - \frac{(\partial_\mu\tau)(\partial^\mu\bar{\tau})}{2(\mathrm{Im}\,\tau)^2} - \frac{g_s}{2(\mathrm{Im}\,\tau)}|G_3|^2 - \frac{g_s^2}{4}|\tilde{F}_5|^2\right)$$
$$- \frac{ig_s^2}{8(\mathrm{Im}\,\tau)\kappa^2}\int C_4\wedge G_3\wedge\overline{G}_3\,. \tag{2.81}$$

Both τ and G_3 are complex fields: τ is known as the axio-dilaton (since it encodes both the dilaton and the RR scalar C_0 which appears in the action (2.76) with an axion shift symmetry) and G_3 is often simply called the 3-form field strength (since it encodes both 3-form field strengths in (2.76)). Written in this form, a symmetry associated with $\mathrm{SL}(2,\mathbb{R})$ transformations becomes manifest in the Type IIB action.[34] Such a transformation leaves the metric and 4-form (C_4) invariant and acts on the remaining fields as

$$\begin{pmatrix} C_2 \\ B_2 \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} C_2 \\ B_2 \end{pmatrix}, \qquad \tau \to \frac{a\tau+b}{c\tau+d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})\,, \tag{2.82}$$

that is, $ad - bc = 1$. This symmetry allows us to mix the two forms, B_2 and C_2, as long as we transform the axio-dilaton τ accordingly and choose an $\mathrm{SL}(2,\mathbb{R})$ transformation — and because this is a symmetry, any such choice of 2-forms is equivalent to any other. An interesting special case is the choice

$$\begin{pmatrix} C_2 \\ B_2 \end{pmatrix} \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} C_2 \\ B_2 \end{pmatrix} = \begin{pmatrix} B_2 \\ -C_2 \end{pmatrix}, \qquad \tau \to -\frac{1}{\tau}\,. \tag{2.83}$$

To see why this is interesting, let us set $C_0 = 0$ in order to make the following relation explicit

$$\frac{1}{g_s} \equiv \langle \mathrm{Im}\,\tau\rangle \to \frac{1}{\langle \mathrm{Im}\,\tau\rangle} \equiv g_s\,. \tag{2.84}$$

This means that the $\mathrm{SL}(2,\mathbb{R})$ transformation which exchanges $C_2 \leftrightarrow B_2$ also inverts the string coupling g_s, turning a weakly-coupled theory into a strongly-coupled one or vice-versa. This type of weak-strong mapping between theories is called S-duality [54, 55] and it was one of the crucial tools to understand the web of dualities between 10d superstring theories.

It turns out that this $\mathrm{SL}(2,\mathbb{R})$ symmetry does not fully survive in the Type IIB *string theory*, because the 2-forms B_2 and C_2 couple to fundamental strings (F1) and D-strings (D1) respectively. These strings are therefore charged under their respective 2-forms and this charge must be quantised such that

$$\int_\gamma H_3 \in \mathbb{Z}\,, \qquad \int_\gamma F_3 \in \mathbb{Z}\,. \tag{2.85}$$

[34]Indeed, once the action is written in terms of τ and G_3, it is possible to start from a generic linear transformation for (C_2 B_2) and determine, not only which subgroup – $\mathrm{SL}(2,\mathbb{R})$ – leaves the action invariant, but also how τ must transform in a non-linear way.

If, for example, the RR 2-form transforms as $C_2 \to aC_2 + bB_2$ the quantisation condition becomes

$$\int_\gamma F_3 \to a \int_\gamma F_3 + b \int_\gamma H_3 = a \cdot \mathbb{Z} + b \cdot \mathbb{Z} \overset{!}{\in} \mathbb{Z}\,, \tag{2.86}$$

so that maintaining charge quantisation requires the transformation group to be restricted to the discrete subgroup $\mathrm{SL}(2,\mathbb{Z})$. Note that the S-duality transformation above belongs to this subgroup and is therefore preserved in string theory (the S-duality group is itself a $\mathbb{Z}_2$ subgroup of $\mathrm{SL}(2,\mathbb{Z})$).[35] Since S-duality exchanges B_2 and C_2, it also exchanges the fundamental string and the D-string which are charged under these fields — therefore, in Type IIB, the F1 is S-dual to the D1. This also makes their magnetic duals, the NS5 and D5 branes respectively, S-dual to each other, while the D3-brane is self-dual (which follows from the fact that C_4 does not transform under S-duality).

The equations of motion for the scalars and form fields are now

$$\mathrm{d} \star \mathrm{d}\tau = \frac{\mathrm{d}\tau \wedge \star \mathrm{d}\tau}{i(\mathrm{Im}\,\tau)} - i\frac{g_s}{2} G_3 \wedge \star G_3\,, \tag{2.87a}$$

$$\mathrm{d} \star G_3 = \frac{\mathrm{d}\bar{\tau} \wedge \mathrm{Re}\, G_3}{i(\mathrm{Im}\,\tau)} - i g_s G_3 \wedge \tilde{F}_5 \tag{2.87b}$$

$$\mathrm{d} \star \tilde{F}_5 = \frac{i}{2(\mathrm{Im}\,\tau)} G_3 \wedge \overline{G}_3\,, \tag{2.87c}$$

and the Einstein equations

$$R_{MN} = \frac{1}{2}\frac{(\partial_M \tau)(\partial_N \bar{\tau})}{(\mathrm{Im}\,\tau)^2} + \frac{g_s^2}{4 \times 4!} (\tilde{F}_5)_{MPQRS} (\tilde{F}_5)_N{}^{PQRS}$$
$$+ \frac{g_s}{4} \frac{(G_3)_{MPQ} (\overline{G}_3)_N{}^{PQ}}{(\mathrm{Im}\,\tau)} - G_{MN} \frac{g_s}{8} \frac{|G_3|^2}{(\mathrm{Im}\,\tau)}\,, \tag{2.88}$$

and as before these must be supplemented by the self-duality condition (2.73). The Bianchi identity for G_3 becomes

$$\mathrm{d}G_3 = -\mathrm{d}\tau \wedge H_3\,. \tag{2.89}$$

Although we have a low-energy description of the closed string sector of Type IIB, there is more to string theory than just its closed strings — in fact, localised objects such as D-branes and O-planes play crucial roles in modern string phenomenology, and we will therefore close this chapter by briefly introducing them.

[35] In fact, the statement that $\mathrm{SL}(2,\mathbb{Z})$ is a symmetry of Type IIB string theory is a conjecture and has not been proven (see chapter 18.6 of [20] for a discussion of S-duality and other non-perturbative dualities). If true, it implies the existence of (p,q)-strings carrying p units of B_2 charge and q units of C_2 charge, which would be stable against decay into two strings when p, q are relatively prime — restricting to transformations in $\mathrm{SL}(2,\mathbb{Z})$ guarantees that a, b are relatively prime and therefore there is always a stable (p,q)-string associated with the transformation (2.86), which is described at weak coupling as a bound state of p fundamental strings and q D-strings [26].

2.3.2 Localised objects: D-branes, O-planes and tadpole cancellation

The action (2.81) gives an effective description of the massless closed string states of Type IIB string theory. However one should also include in this description the localised objects that we encountered in the previous sections — Dp-branes and Op-planes. These objects have both tensions and charges, and are therefore crucial for the study of string theory vacua. Both D-branes and O-planes couple to fields in the (NS-NS) and (R-R) sectors, whose couplings at leading order in the string coupling are contained in the so-called Dirac-Born-Infled (DBI) and Chern-Simons (CS) or Wess-Zumino (WZ) actions respectively [39]. These effective actions can be obtained in a similar way as (2.81) by computing string scattering amplitudes in a loop expansion and matching the couplings that appear in the EFT.

For a Dp-brane, we find in the string frame[36] [26, 39]

$$S_{\rm DBI}^{\rm Dp} = -T_{{\rm D}p} \int_{\mathcal{W}} d^{p+1}\xi \; e^{-\Phi} \sqrt{-\det\left(P[G+B_2] - 2\pi\alpha' \hat{F}_2\right)} \tag{2.90}$$

$$S_{\rm CS/WZ}^{\rm Dp} = \pm T_{{\rm D}p} \int_{\mathcal{W}} \left\{ \sum_n P[C_n e^{-B_2}] e^{2\pi\alpha' \hat{F}_2} \right\}, \tag{2.91}$$

where $\mathcal{W}$ is the world-volume of the brane, with coordinates $\xi^a\,,\ a \in \{0, ..., p\}$, $T_{{\rm D}p} = \frac{2\pi}{l_s^{p+1}}$ the brane tension, $\hat{F}_2$ is the field-strength of the gauge field A^a which is part of the open string spectrum, and $P[G+B_2]$ denotes the pull-back of the graviton and (NS-NS) 2-form onto $\mathcal{W}$,

$$P[G+B_2] = g_{ab} + B_{ab} = \frac{\partial X^M}{\partial \xi^a} \frac{\partial X^N}{\partial \xi^b} (G_{MN} + B_{MN})\,. \tag{2.92}$$

As for the Polyakov action for the string, $X^M(\xi^a)$ gives the embedding of $\mathcal{W}$ into spacetime. We can choose the so-called static gauge to align $p+1$ of the X^M coordinates with the $p+1$ directions on $\mathcal{W}$, so that $X^a = \xi^a$ and $X^i\,,\ i \in \{p+1, ..., 9\}$, are the coordinates transverse to the brane. As we discussed in Section 2.1, Dp-branes are dynamical objects — their positions along X^i can fluctuate,

$$X^i = X^i_0 + 2\pi\alpha' \phi^i(\xi^a) + \ldots\,, \tag{2.93}$$

and these fluctuations are encoded by $(9-p)$ scalar fields ϕ^i that are precisely the remaining scalars in the massless spectrum of the open string.

On the other hand, the CS/WZ part of the action describes the coupling of the Dp-brane to the (R-R) fields and is written as a formal sum where one should only pick out the forms with the

[36] We omit a factor $\sqrt{\frac{\hat{A}\left((2\pi)^2\alpha' R_T\right)}{\hat{A}\left((2\pi)^2\alpha' R_N\right)}}$ (where $\hat{A}$ denotes the A-roof genus and R_T/R_N is the tangent/normal curvature) in the CS/WZ action whose leading term is 1 [20, 56]; it would be important if one wanted to take into account curvature corrections to this action.

correct dimension under the integral,

$$S^{\rm Dp}_{\rm CS/WZ} = \pm T_{\rm Dp} \int_{\mathcal{W}} [C_{p+1} - C_{p-1} \wedge (B_2 - 2\pi\alpha' \hat{F}_2) + \ldots\,]\,, \tag{2.94}$$

with C_p and B_2 properly pulled-back onto $\mathcal{W}$. The $\pm$ in front of the action corresponds to Dp/$\overline{\text{D}p}$-branes, respectively, whose charges have opposite signs but are always the same as the tension in magnitude. The CS/WZ action identifies Dp-branes as the objects charged under C_{p+1} so that they act as sources to these fields. After compactification, a Dp-brane with the right configuration of world-volume flux can also look like an object charged under other $C_{p'}$ fields — in (2.94) we see the term that can give rise to an effective charge under C_{p-1}.

The action for O-planes is very similar to the one for D-branes, but importantly it does not include any world-volume fields. That is because O-planes are non-dynamical and have no degrees of freedom associated with them (their positions cannot fluctuate away from the fixed points that define them). The DBI and CS/WZ actions for an Op-plane are given by[37]

$$S^{\rm Op}_{\rm DBI} = -T_{\rm Op} \int_{\mathcal{W}} d^{p+1}\xi \; e^{-\Phi} \sqrt{-g}\,, \tag{2.95}$$

$$S^{\rm Op}_{\rm CS/WZ} = T_{\rm Op} \int_{\mathcal{W}} C_{p+1}\,, \tag{2.96}$$

where $T_{\rm Op} = -2^{p-5}\, T_{\rm Dp}$ is the orientifold tension, from which we can see that an Op-plane has *negative* tension and couples to C_{p+1} with a charge opposite to the one of a Dp-brane. These properties, together with the way orientifolds break supersymmetry, make O-planes extremely important in string compactifications involving fluxes and D-branes. We shall further expand on this point when we discuss flux compactifications in Chapter 3.

If we expand the determinant in the DBI action of D-branes and O-planes, we find a vacuum energy

$$S^{\rm Dp/Op}_{\rm DBI,\,vac} = -T_p \int d^{p+1}\xi \; \sqrt{-g}\; e^{-\Phi}\,, \tag{2.97}$$

with T_p being the tension of a Dp-brane/Op-plane, that will contribute to the Einstein equations with an energy-momentum tensor

$$T^{\rm loc}_{MN} = -\frac{\partial \xi^a}{\partial X^M} \frac{\partial \xi^b}{\partial X^N} g_{ab}\; T_p\; e^{-\Phi}\; \delta^{9-p}(X^i)\,. \tag{2.98}$$

D-branes and O-planes are therefore acting as sources of vacuum energy with different signs. We will see that the negative tension of the O-planes in particular will be important for flux compactifications.

We can now go back to a point that we mentioned in the discussion of the Type I theory —

[37] Once again we omit a term $\sqrt{\frac{L\left(\pi^2\alpha' R_T\right)}{L\left(\pi^2\alpha' R_N\right)}}$ (where L is the Hirzebruch polynomial) depending on the curvature. As before, its contribution is 1 at leading order and all other terms involve α'-suppressed curvature corrections.

the notion of tadpole cancellation and the contribution of D-branes and O-planes. From their coupling to C_{p+1} in the CS/WZ action, we find that these localised objects contribute to the equations of motion of the (R-R) form field. In order to see their contribution, it helps rewriting the action in terms of the Poincaré form dual to $\mathcal{W}$ (see Appendix B) for a generic object with charge Q_p,

$$S^{\text{Dp/Op}}_{\text{CS/WZ}} = Q_p \int_{\mathcal{W}} C_{p+1} = Q_p \int_{\text{spacetime}} C_{p+1} \wedge \delta_{\mathcal{W}} \,, \tag{2.99}$$

with $\delta_{\mathcal{W}} = \delta^{9-p}(X^i)\, \mathrm{d}X^{p+1} \wedge \ldots \wedge \mathrm{d}X^9$ in the static gauge where X^i correspond to the transverse coordinates. Hence there is a contribution to the equation of motion[38]

$$\mathrm{d} \star \mathrm{d}C_{p+1} = \text{non-localised} + 2\kappa^2 \sum_i Q_p^{(i)} \; \delta_{\mathcal{W}} \,, \tag{2.100}$$

with the non-localised term referring to any contribution coming from the closed-string effective action and $Q_p^{(i)}$ being the charge of each localised object. If we integrate this equation over the transverse space $\mathcal{W}^{\perp}$ we find

$$\int_{\delta\mathcal{W}^{\perp}} \star \mathrm{d}C_{p+1} \overset{\text{Stokes}}{=} \int_{\mathcal{W}^{\perp}} \mathrm{d} \star \mathrm{d}C_{p+1} = \int_{\mathcal{W}^{\perp}} \text{non-localised} + 2\kappa^2 \sum_i Q_p^{(i)} \,. \tag{2.101}$$

This is the origin of the tadpole cancellation condition. When $\mathcal{W}^{\perp}$ is a compact space without a boundary ($\delta\mathcal{W}^{\perp} = 0$), this must be zero. Tadpole cancellation is simply the requirement that the equations of motion for the (R-R) fields are satisfied once all contributions are taken into account — it is called tadpole cancellation because a non-zero value leads to non-zero one-point correlation functions whose Feynman diagrams are known as tadpoles. If any tadpoles are left uncancelled the low-energy EFT becomes anomalous and therefore inconsistent. As we will see in Section 3.4, all charges (including fluxes) must be quantised, which means that the tadpole constraint is an integer condition. It implies that when we consider compactified solutions, there are consistency constraints relating the flux numbers and the number of D-branes and O-planes, and we are not allowed to have an arbitrary combination of these ingredients.[39]

With Type IIB supergravity and the effective actions of D-branes and O-planes, we have a field theory description of the low-energy dynamics of Type IIB string theory. Not only does this make it easier to explore the phenomenology of Type IIB, in the sense that one can use the vast set of tools and intuition developed over the years, but it also makes it easier to connect with known (and observationally tested) results which are mostly formulated in terms of field theories. Nevertheless, if our goal is to connect string theory to phenomenological observations,

[38]We assume here that all localised objects share the same world-volume, i.e. that they are on top of each other. In general, there may be other localised objects oriented along different directions. The space over which we integrate the equation will determine which of these contribute to the tadpole constraint.

[39]Note that the story is slightly different in the Type I construction where we first encountered O-planes. There the O9-plane couples to a C_{10} form whose field-strength tensor must vanish in 10d, $\mathrm{d}C_{10} = 0$, so that the integral leading to the tadpole cancellation condition vanishes without need for a compact space with no boundary. Note also that there are no non-localised terms in that case, so that we simply find $(N_{\text{D9}} - 2^4)T_{\text{D9}} = 0$. This gives 16 D9-branes plus their images under the orientifold, recovering the 32 D9-branes required for consistency.

we still have to deal with an important aspect of Type IIB: the fact that it is a 10-dimensional theory, in contrast to the 4 dimensions that we observe. In the next chapter we will explore the process of compactification, understand how phenomenological requirements can restrict the set of possible compact spaces and encounter a number of scalar fields in the 4d theory which naïvely appear as flat directions and thus are not only completely arbitrary but also massless. We will find that some of these scalars can get a potential from non-trivial background fluxes, while the others remain as flat directions, whose stabilisation will be addressed in Chapter 4.

3. String compactifications

> (...) a realm where everything is very hopeful,
> very beautiful, exceedingly promising, (...)
> free of parameters—but with no obvious
> connection with experiment!
>
> Murray Gell-Mann

3.1 Extra dimensions as UV physics

As we have seen, Type IIB string theory is a theory of supersymmetric strings propagating in 10 spacetime dimensions. At low enough energies ($E \ll M_S$) their inherent "stringyness" is too detailed to resolve, a statement which becomes more precise from an EFT perspective — not having enough energy to excite higher vibrational modes of the strings, physics at low energies is well described by a field theory which encodes only the massless degrees of freedom and their interactions. For Type IIB string theory, this low-energy EFT is precisely the Type IIB supergravity we briefly discussed in the previous chapter.

Despite seeming well-equipped to address our phenomenological needs, with its graviton, fermions, gauge fields and symmetries, this 10-dimensional supergravity still comes with a slight inconvenience — the fact that it is, after all, a 10-dimensional theory. All our observations so far point to the conclusion that we live in a 4-dimensional spacetime and thus, if Type IIB string theory really underlies our reality, our 4-dimensional physics must arise from this 10-dimensional description.

Before the rise of D-branes and their world-volume theories, the only viable route appeared to be a compactification of the extra dimensions. Although we will shortly dive deeper on what it actually means, in practice a compactification of a higher-dimensional theory is a lower-dimensional description of its low-energy physics. With the possibility of confining open string states on lower-dimensional subspaces (D-branes), an alternative becomes possible in which our experience would be confined to these branes, effectively forbidding us to explore the extra dimensions — this is the braneworld approach [57, 58].

Nowadays, Type IIB phenomenology commonly incorporates aspects of both approaches, rather than choosing one and only one strategy. Yet, it is the compactification concepts and machinery that will be key to the phenomenology we will explore and thus we will turn to them and highlight their main features.

3.1.1 Compact dimensions

As the name suggests, the first step towards compactifying a theory is to consider compact dimensions. Generically, all 9 spatial directions of spacetime in Type IIB can be infinite in extent, just as our observed 3 spatial directions seem to be,

$$-\infty < x^M < \infty\,, \quad \forall M \in \{1, 9\}\,. \tag{3.1}$$

Naïvely, such an infinite extra dimension would have easily observable effects and should have been detected already.[1] Although it does not become immediately clear how big the extra dimensions can be while still remaining hidden, they should at least be finite,

$$-L < x^M < L\,, \quad \forall M \in \{4, 9\}\,. \tag{3.2}$$

To see what this implies, let us take the simplest case of the dilaton of Type IIB (in Einstein frame and ignoring its couplings to other fields),

$$\begin{aligned} S_{\text{IIB}} &\supset \frac{1}{4\kappa^2}\int d^{10}x\sqrt{-G}\; G^{MN}(\partial_M\Phi)(\partial_N\Phi) \\ &= \frac{1}{4\kappa^2}\int d^{10}x\sqrt{-G}\;\Delta_{10}\Phi\cdot\Phi + \sum_{N=0}^{9}\left[\sqrt{-G}\; G^{MN}(\partial_M\Phi)\cdot\Phi\right]_{x^N_{\text{min}}}^{x^N_{\text{max}}}\,, \end{aligned} \tag{3.3}$$

where we can define the 10d Laplacian operator as

$$\Delta_{10}\Phi = \frac{1}{\sqrt{-G}}\partial_N\left(\sqrt{-G}\; G^{MN}\partial_M\Phi\right)\,. \tag{3.4}$$

Equation (3.3) highlights the role of boundary terms in deriving the equation of motion for the (free) dilaton, $\Delta_{10}\Phi = 0$. After reviewing the construction of string theories, we are now very used to having boundary conditions determine a lot of the physical properties of the theory. There are 3 possibilities,

`(periodic)` $\quad G^{MN}(\partial_M\Phi)\cdot\Phi\big|_{x^N=L} = G^{MN}(\partial_M\Phi)\cdot\Phi\big|_{x^N=-L}$

`(vanishing)` $\quad \Phi\big|_{x^N=\pm L} = 0$

[1] The caveat is that certain properties of both the theory and the solutions may in fact hide an extra dimension even if it is infinite, as in the case of certain braneworld scenarios [58].

$$\texttt{(static)} \qquad\qquad (\partial_M \Phi)\big|_{x^N=\pm L} = 0$$

analogous to the periodic, Dirichlet and Neumann boundary conditions for the string. While for infinite dimensions ($L \to \infty$) one will typically discard the boundary term by saying that "the field goes to zero at infinity" (i.e. by choosing vanishing boundary conditions infinitely far away), compact dimensions require us to dispose of the boundary terms by choosing boundary conditions at some finite values of x^N. The most common choice for compact dimensions is to make them periodic, but it is worth highlighting the alternatives which still arise in many instances.

For this choice of boundary conditions for non-compact $\{x^\mu\}$ and compact $\{y^m\}$ directions, the dynamics of the dilaton is determined by

$$\begin{cases} \Delta_{10}\Phi = 0 \\ \Phi(x^\mu, y^m) \to 0 \quad \text{as } x^\mu \to \pm\infty \\ G^{Mn}(\partial_M \Phi)\cdot \Phi\big|_{y^n=L} = G^{Mn}(\partial_M \Phi)\cdot \Phi\big|_{y^n=-L} \end{cases} \tag{3.5}$$

Crucially, the dilaton is still a 10-dimensional field. Simply compactifying the extra dimensions did not make the problem 4-dimensional. It is now time for symmetry to be taken into account, in particular maximal symmetry in 4d, which requires the spacetime metric to factorise as

$$ds^2_{10} = f^{-1}(y) g_{\mu\nu}(x) dx^\mu dx^\nu + f(y) g_{mn}(y) dy^m dy^n \,, \tag{3.6}$$

splitting the Laplacian and hence the equation of motion as

$$\Delta_{10}\Phi = f\cdot \Delta_4\Phi + f^{-1}\cdot\Delta_6\Phi = 0 \implies \Delta_4\Phi = -f^{-2}\cdot\Delta_6\Phi \,, \tag{3.7}$$

where Δ_4 and Δ_6 only depend on non-compact $\{x^\mu\}$ and compact $\{y^m\}$ coordinates, respectively. This is extremely useful, as it makes the equation of motion separable and we can therefore focus on solutions of the form

$$\Phi(x^\mu, y^m) = \Phi_x(x^\mu)\cdot\Phi_y(y^m) \,, \tag{3.8}$$

for which the equation reduces to[2]

$$\frac{\Delta_4\Phi_x}{\Phi_x} = -f^{-2}\frac{\Delta_6\Phi_y}{\Phi_y} = \texttt{const} \,. \tag{3.9}$$

Solving the equation of motion for the 10d dilaton is thus equivalent to solving

$$\begin{cases} \Delta_4\Phi_x = -k^2\cdot\Phi_x \\ \Delta_6\Phi_y = k^2\cdot f^2\Phi_y \end{cases} \,. \tag{3.10}$$

[2]This is because the left-hand side only depends on x^μ while the right-hand side only depends on y^m, so that the equation is only satisfied if these are both constants.

The first thing we notice is that the 4d equation resembles the equation of motion of a 4d scalar field of mass k. We also find that the 6d equation is an eigenvalue problem with eigenvalue k — to solve it we need the metric on the 6d extra dimensions, which represents local information, and the boundary conditions, which encode global information about the compact space. For a solution $\Phi_y^{(k)}$ depending on k, the boundary conditions will determine which values of k are allowed, not only constraining the set of solutions $\Phi_y^{(k)}$ but also determining which equation its pair $\Phi_x^{(k)}$ will satisfy. Since the Laplace operator is linear, the most general solution is a sum over all allowed solutions,

$$\Phi(x^\mu, y^m) = \sum\!\!\!\!\!\!\int_k \Phi_x^{(k)}(x^\mu) \cdot \Phi_y^{(k)}(y^m)\,, \tag{3.11}$$

where we emphasise the fact that it is *a priori* unclear whether we have a discrete sum or a continuous range for k, until the boundary conditions are imposed.

Example: T^6 compactification

Let us consider a flat g_{mn} metric and take $f(y) = 1$ for simplicity,

$$ds_6^2 = \delta_{mn} dy^m dy^n\,. \tag{3.12}$$

The 6d torus T^6 is obtained by identifying the coordinates as $y^m \sim y^m + 2\pi R$, which makes them periodic with period $2\pi R$. This fixes periodic boundary conditions along $\{y^m\}$, which become compact directions with $L = \pi R$. However, before imposing any boundary conditions, the equation for Φ_y becomes

$$\delta^{mn}\partial_m\partial_n\Phi_y = -k^2 \cdot \Phi_y\,, \tag{3.13}$$

whose general solution is given by

$$\Phi_y(y^m) = \prod_{m=4}^{9} \begin{cases} c_1 + c_2 y^m\,, & k = 0 \\ c_3 e^{ik^{\underline{m}} y^{\underline{m}}} + c_4 e^{-ik^{\underline{m}} y^{\underline{m}}}\,, & k > 0 \end{cases} \tag{3.14}$$

where the index $\underline{m}$ is not summed. This is a solution of (3.13) for any values of k^m such that $k^2 = \delta_{mn} k^m k^n$, i.e. we do not yet have a discrete set of solutions. It is the boundary conditions (periodic, $y^m \sim y^m + 2\pi R$, for the T^6) that impose the constraints

$$c_2 = 0\,, \quad k^m = \frac{n}{R}\,, \quad n \in \mathbb{N}\,, \tag{3.15}$$

finally discretising the set of solutions. Since writing $\Phi_y(y^m)$ in momentum space would identify the k^m with the momentum p^m along the y^m direction, this is equivalent to the statement that the momentum gets quantised by these boundary conditions. A general solution for Φ is then of the form

$$\Phi(x^\mu, y^m) = \sum_{\vec{n}} \Phi_x^{(\vec{n})}(x^\mu) \cdot e^{i\frac{\vec{n}\cdot\vec{y}}{R}} \,. \tag{3.16}$$

The reason why we reviewed this standard procedure in detail is to emphasise the following: it is *not* the periodicity that discretises the set of solutions, but instead the *compactness* of the $\{y^m\}$ directions. Indeed, had we chosen for example to put Φ in a 6-dimensional box with boundary conditions $\Phi(x^\mu, y^m = \pm L) = 0$, the solution (3.14) would be constrained by

$$c_1 = c_2 = 0\,, \quad k^m = \frac{n\pi}{2L}\,, \quad n \in \mathbb{N}\,, \tag{3.17}$$

again restricting k^m to integer values. In fact, one can see the discreteness disappear without compactness by sending $R \to \infty$ in (3.15) and noting that in this limit k^m is allowed to take a continuum of values.

3.1.2 The 4d EFT: Integrating out high-momentum modes

Let us simplify our notation by defining $\Phi_x^{(k)}(x^\mu) = \varphi_k(x^\mu)$, $\Phi_y^{(k)}(y^m) = u_k(y^m)$. From our discussion above, if the extra dimensions $\{y^m\}$ are compact, the solution for the 10d dilaton field $\Phi(x^\mu, y^m)$ takes the general form

$$\Phi(x^\mu, y^m) = \sum_k \varphi_k(x^\mu) \cdot u_k(y^m)\,, \tag{3.18}$$

where $u_k(y^m)$ are eigenfunctions of the operator Δ_6 associated with an eigenvalue k which can only take a discrete set of values. One way to read (3.18) is that Φ is a superposition of modes labelled by k, each one of which forced to satisfy the equation

$$(\Delta_4 + k^2)\ \varphi_k = 0\,. \tag{3.19}$$

We could interpret it from two different perspectives. From the 10d point-of-view, the discretisation of the momentum along the compact directions gives a clear separation between momentum modes, since the momentum along $\{y^m\}$ can only take a discrete set of values. On the other hand, we could interpret (3.19) as giving an equation of motion for each of an infinite number of 4d scalar fields $\varphi_k(x^\mu)$ of increasing mass k, which gives rise to a tower of massive states usually called the Kaluza-Klein tower. In both scenarios, if we were only able to probe energies much lower than the smallest momentum/mass scale ($E \ll k_{\rm min}$), we could integrate out all modes with $k > 0$. We could then write the action as

$$\begin{aligned} S_{\rm IIB} &\supset \frac{1}{4\kappa^2} \int d^{10}x\sqrt{-G}\ \Delta_{10}\Phi \cdot \Phi \\ &= \frac{1}{4\kappa^2} \int d^4x\sqrt{-g_4} \int d^6y\sqrt{g_6}\ f^2 \sum_{k,\ell} [\Delta_4\varphi_k + k^2\varphi_k] \cdot u_k\ \varphi_\ell \cdot u_\ell \end{aligned}$$

$$= \frac{1}{4\kappa^2} \int d^4x \sqrt{-g_4} \sum_{k,\ell} [\Delta_4 \varphi_k + k^2 \varphi_k] \; \varphi_\ell \int d^6y \sqrt{g_6} \; f^2 \; u_k \cdot u_\ell$$
$$= \frac{1}{4\kappa^2} \int d^4x \sqrt{-g_4} \sum_{k,\ell} [\Delta_4 \varphi_k + k^2 \varphi_k] \; \varphi_\ell \; \delta_{k\ell}$$
$$= \frac{1}{4\kappa^2} \int d^4x \sqrt{-g_4} \sum_{k} [\Delta_4 \varphi_k + k^2 \varphi_k] \; \varphi_k \,, \tag{3.20}$$

where we used the orthogonality[3] of the eigenfunctions $u_k(y)$,

$$\int d^6y \sqrt{g_6} \; f^2 \; u_k \cdot u_\ell = \delta_{k\ell} \,, \quad \forall \; k, \ell \in \mathbb{Z} \,. \tag{3.21}$$

Therefore our 10d theory of a single free and massless dilaton Φ is equivalent to a 4d theory describing an infinite tower of free scalar fields $\varphi_k(x^\mu)$ of mass k. At energies much smaller than the minimum scale ($E \ll k_{\text{min}}$), we can integrate out the infinite tower of states with $k > 0$ and accurately describe the physics with the massless state alone,

$$S_{\text{EFT}} = \frac{1}{4\kappa^2} \int d^4x \sqrt{-g_4} \; \Delta_4 \varphi_0 \cdot \varphi_0 + \mathcal{O}\Big(\frac{E}{k_{\text{min}}}\Big) \,, \tag{3.22}$$

assuming $k = 0$ is allowed (this is the case for the T^6 example, but not for the 6-dimensional box). We conclude that the low-energy description of a 10d theory for which 6 dimensions are compact is a 4d theory describing the solutions for which

$$\Delta_6 u_0 = 0 \,, \tag{3.23}$$

i.e. modes associated with harmonic functions on the compact space. Although for a function (0-form) there is a unique solution to this equation (which is in fact a constant), this is not always the case, as for example when we look at the form fields in Type IIB. It therefore becomes useful to generalise the discussion to any differential form.

3.1.3 Connection with cohomology

In Appendix B we include a summary of the relevant concepts and theorems that allow us to relate the set of solutions to (3.23) to the topology of the compact space $\mathcal{M}$. The upshot is that the space of harmonic p-forms, $\text{Harm}^p(\mathcal{M})$, is isomorphic to the homology group $H_p(\mathcal{M})$ of cycles of $\mathcal{M}$ defined up to boundaries, and so the number of solutions to $\Delta_6 u_{p,0} = 0$ (where we generalise to harmonic p-forms $u_{p,0}$) is given by the Betti numbers b_p.

[3] Here we assume that the eigenfunctions $u_k(y)$ are already properly normalised. Later on we will see that it is important to keep track of this normalisation.

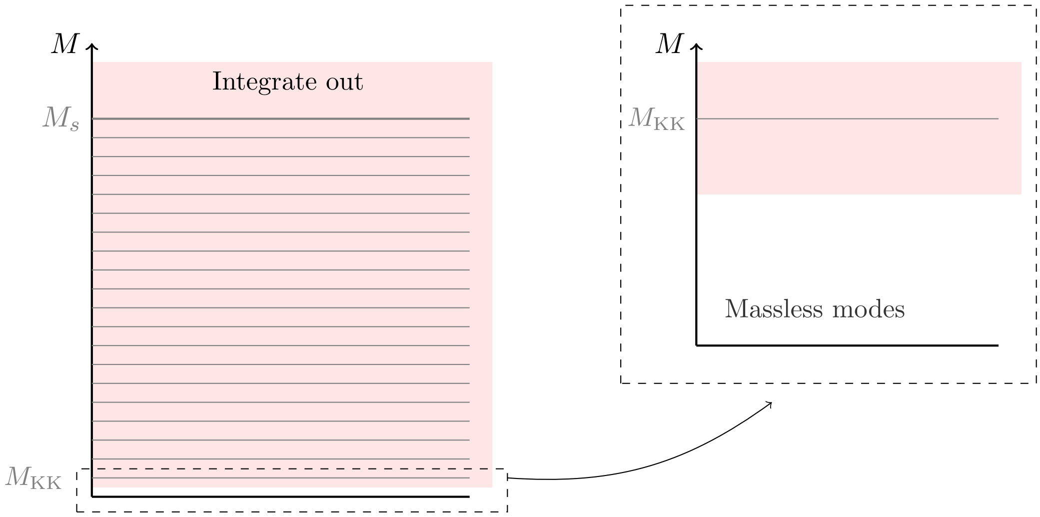


Figure 3.1: Compactifying Type IIB supergravity (which was itself an effective description of Type IIB string theory only valid at energies $E \ll M_s$) gives a discrete set of internal-momentum modes whose scale is given by multiples of $M_{\rm KK}$. At energies $E \ll M_{\rm KK}$, we can integrate out all modes with non-zero internal momentum and be left with a theory describing only the modes with zero momentum — this is effectively a 4d supergravity theory. Since the 10d theory only included massless fields, these zero-momentum modes are massless in the 4d effective theory. We will soon see that adding non-trivial fluxes to the compactifications generates a potential for some of these fields, giving them a mass (cf. section 3.4 and Fig. 4.1).

Let us bring back the T^6 example. Since $\dim(T^6) = 6$, there are 7 homology groups on T^6, with

$$b_p = \binom{6}{p} . \tag{3.24}$$

In particular, the number of harmonic 0-forms (functions) on T^6 is 1 and therefore dimensionally reducing the action for a free dilaton on T^6 would result in a single massless 4d scalar field ϕ_0. If instead we were trying to dimensionally reduce the 2-form B_2 we would face 3 distinct cases,

$$\begin{aligned} &\Delta_6 B_{\mu\nu} && \texttt{(0-form in 6d)}\,,\\ &\Delta_6 B_{m\nu} && \texttt{(1-form in 6d)}\,,\\ &\Delta_6 B_{mn} && \texttt{(2-form in 6d)}\,, \end{aligned}$$

depending on the number of components along $\{y^m\}$. Since $b_1 = 6$ and $b_2 = 15$, B_2 can be decomposed as

$$B_2(x^\mu, y^m) = \mathcal{B}_2(x^\mu) + \mathcal{B}_1^i(x^\mu)\cdot\omega_1^i(y^m) + \mathcal{B}_0^j(x^\mu)\cdot\omega_2^j(y^m)\,, \tag{3.25}$$

with $i \in \{1, ..., 6\}$, $j \in \{1, ..., 15\}$. Hence, the low-energy description of a free 10-dimensional 2-form on a T^6 is given in terms of a 4-dimensional 2-form $\mathcal{B}_2$ (independently of the compact space position), 6 1-form (vector) fields $\mathcal{B}_1$ and 15 scalar fields $\mathcal{B}_0$ (both weighed by a so-called wavefunction that depends on the compact space position y^m).

We conclude that the topology of the compact space determines how many and which (massless) fields remain after we integrate out the high-momentum modes of the tower. The properties of the low-energy description, i.e. the 4d theory whose phenomenology we want to explore, are therefore tightly connected to the choice of compact space. As we shall see in the next section, preserving $\mathcal{N} = 1$ supersymmetry will pick a specific type of spaces called Calabi-Yau manifolds.

3.1.4 A comment on strings and winding

Our discussion above was purely field theoretical and did not take into account the stringy nature of Type IIB. However, the fact that the fundamental degrees of freedom are strings rather than particles has important consequences when some spacetime directions are periodic. Recall that a closed string is such that

$$X^\mu(\sigma + \ell) = X^\mu(\sigma)\,, \tag{3.26}$$

where ℓ is the length of the string. When a certain direction is periodic, i.e. $X^i \sim X^i + 2\pi R$, the string can *wind* around it a certain number of times before closing,

$$X^\mu(\sigma + \ell) = X^\mu(\sigma) + 2\pi R \cdot w \sim X^\mu(\sigma)\,,\; w \in \mathbb{Z}\,. \tag{3.27}$$

Since the string has tension, winding it around the periodic direction will cost some energy, which is reflected in a contribution to the mass of the string proportional to the winding number w and the size of the periodic dimension,

$$\Delta m \propto w\frac{R}{\alpha'} \propto \frac{wR}{l_s} M_s\,. \tag{3.28}$$

This gives a discrete tower of modes labelled by $w \in \mathbb{Z}$ and whose mass grows with w. It is in fact through the connection between Kaluza-Klein modes and these winding modes, that T-duality reveals itself [59].

3.2 Why Calabi-Yau manifolds?

In Appendix C we briefly describe the necessary concepts and tools related to Calabi-Yau manifolds. In this section we will explain why these spaces are so useful in string phenomenology and how their topology determines several properties of the low-energy description of Type IIB supergravity.

As we saw in the previous section, the choice of compact space is tightly connected to the 4d phenomenology one obtains upon compactifying the 10-dimensional theory — not only does it determine the number of massless fields that are left, but it also constrains the symmetries that survive. Rather than blindly studying different solutions of the 10d theory on different compact spaces and asking which physics will arise in 4d, one may turn the logic around and specify the 4d physics one is looking for in the first place, asking instead which (if any) compact spaces give solutions of the 10d theory with these properties.

We are typically interested in solutions for which the 10d spacetime factorises into a 4d spacetime and a 6d internal space,

$$M_{10} = M_4 \times M_6 \,. \tag{3.29}$$

We could then require M_4 to be homogeneous and isotropic, which is equivalent to saying it should be maximally symmetric — either Minkowski ($\mathbb{M}$), de Sitter (dS) or anti-de Sitter (AdS). A direct consequence of this choice is that, not only must any form field either have a spacetime filling background or only internal components, but also any vacuum expectation value (vev) must be constant. The most general 10-dimensional metric compatible with 4d maximal symmetry is

$$ds^2 = e^{2A(y)} g_{\mu\nu}(x) \mathrm{d}x^\mu \mathrm{d}x^\nu + g_{mn}(y) \mathrm{d}y^m \mathrm{d}y^n \,, \tag{3.30}$$

with $g_{\mu\nu}$ being the metric on one of the maximally symmetric spacetimes, g_{mn} any 6d metric on the internal space and $e^{2A(y)}$ a function known as the warp factor that only depends on the internal space coordinates.

What about supersymmetry? Our 10d theory, Type IIB supergravity, has $\mathcal{N} = 2$ local supersymmetry, meaning that at every point there are two operators Q_α that can act on bosons and fermions, and exchange them in such a way that the action remains invariant. This is a symmetry of the action, but is not necessarily a symmetry of an arbitrary vacuum solution. Having the vacuum break some or all of this symmetry is analogous to how the Higgs potential gives a vacuum which breaks electroweak symmetry down to electromagnetism, $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \to \mathrm{U}(1)_{\mathrm{em}}$. In the Higgs case, $\mathrm{U}(1)_{\mathrm{em}}$ is the residual group of transformations which do not change the vacuum configuration of the Higgs field (which is the only scalar in the Standard Model and therefore the only field which can have a non-vanishing vev).

When the symmetry in question is supersymmetry, the logic is the same — the residual set of supercharges which do not change the vacuum configuration will tell us how much supersymmetry remains in our 4d theory. Since a supersymmetry transformation of a bosonic field is proportional to fermionic fields — whose expectation value vanishes in a maximally symmetric solution — bosonic field vevs are automatically invariant. On the other hand, the 4 fermions transform under supersymmetry as

$$\delta_\epsilon \psi^a_M = \nabla_M \epsilon^a - \frac{1}{4}(\not{H}_3)_M (\sigma^3 \epsilon)^a + \frac{1}{8} e^\Phi \tilde{\not{F}}_3 \Gamma_M (\sigma^1 \epsilon)^2 + \frac{1}{8} e^\Phi (\not{\partial} C_0 + \tilde{\not{F}}_5) \Gamma_M (i\sigma^2 \epsilon)^a \,, \tag{3.31}$$

$$\delta_\epsilon \lambda^a = \not{\partial}\Phi \, \epsilon^a - e^\Phi \not{\partial} C_0 (i\sigma^2 \epsilon)^a - \frac{1}{2} \not{H}_3 (\sigma^3 \epsilon)^a - \frac{1}{2} e^\Phi \not{F}_3 (\sigma^1 \epsilon)^a \,, \tag{3.32}$$

where $a = 1, 2$ labels each gravitino ψ^a_M and dilatino λ^a, as well as supersymmetry parameters

ϵ^a. The supersymmetry transformations ϵ^a that leave the vacuum invariant are then given by non-trivial solutions of $\langle \delta_\epsilon \psi^a_M \rangle = \langle \delta_\epsilon \lambda^a \rangle = 0$. The fully general equations are hard to solve and a proper treatment requires the tools of generalised geometry, which we will not go into here (see [60, 61] for an introduction). We will instead consider the simpler case where the vev of the internal components of form fields — associated with the presence of non-trivial fluxes — are set to zero, in which case the transformations simplify to

$$\delta_\epsilon \psi^a_M = \nabla_M \epsilon^a + \frac{1}{8} e^\Phi \partial\!\!\!/ C_0 \Gamma_M (i\sigma^2 \epsilon)^a \,, \tag{3.33}$$

$$\delta_\epsilon \lambda^a = \partial\!\!\!/ \Phi \ \epsilon^a - e^\Phi \partial\!\!\!/ C_0 (i\sigma^2 \epsilon)^a \,. \tag{3.34}$$

Using also the equations of motion for the scalars (2.78a–2.78b) and the fact that their vevs must be constant in 4d, $\langle \delta_\epsilon \lambda^a \rangle = 0$ is automatically satisfied and the remaining condition becomes

$$\nabla_M \epsilon^a = 0 \,. \tag{3.35}$$

We conclude that without fluxes the supersymmetry transformations that leave the vacuum invariant are the ones parametrised by a covariantly constant spinor ϵ^a. From the 4d components ($M = \mu$) of this equation [60],

$$\nabla_\mu \epsilon^a + \frac{1}{2} (\gamma_\mu \gamma_5 \otimes \nabla\!\!\!\!/ A) \epsilon^a = 0 \,, \tag{3.36}$$

and the relation

$$[\nabla_\mu, \nabla_\nu] \epsilon^a = \frac{1}{4} R_{\mu\nu\rho\sigma} \gamma^{\rho\sigma} \epsilon^a = \frac{R}{2} \gamma_{\mu\nu} \epsilon^a \,, \tag{3.37}$$

we find the condition[4]

$$\Big(R + (\nabla A)^2 \Big) \gamma_{\mu\nu} \epsilon^a = 0 \,. \tag{3.38}$$

This implies that $(\nabla A)^2$ is constant, which for a compact space means that the function $A(y)$ must be constant and $(\nabla A)^2$ vanishes. Hence, we also have $R = 0$, so that a supersymmetric solution with maximal symmetry and no fluxes can only be unwarped Minkowski spacetime.

On the other hand, the supersymmetry spinors ϵ^a will decompose into 4d and 6d spinors,

$$\epsilon^a (x^\mu, y^m) = \zeta^a (x^\mu) \otimes \eta^a (y^m) \,. \tag{3.39}$$

The condition (3.36) with constant $A(y)$ implies that ζ^a is actually constant over M_4 and parametrises global supersymmetry transformations in 4d. In turn, the internal components

[4] This follows from the relation $R_{\mu\nu\rho\sigma} = \frac{R}{2}(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho})$ valid for maximally symmetric spacetimes together with properties of the $\gamma^{\rho\sigma}$ matrices.

($M = m$) of (3.35) require the internal space to be Ricci flat,[5]

$$\nabla_m \epsilon^a = 0 \implies [\nabla_m, \nabla_n]\eta^a = \frac{1}{4} R_{mnpq}\gamma^{pq}\eta^a = 0 \quad \implies \quad R_{mn} = 0\,. \tag{3.40}$$

So the upshot is that without fluxes we can only preserve supersymmetry on a flat unwarped 4d background with a Ricci-flat compact 6d space.

However, requiring the existence[6] of a covariantly constant spinor on M_6 is a stronger condition than just asking M_6 to be Ricci-flat. The condition $\nabla_m \eta^a = 0$ tells us that the manifold must be such that it is possible to define a spinor which never changes when it is parallel transported, in particular when it is transported around a closed loop. Generically, on a 6d manifold a spinor can be transported around a loop and end up rotated by a Spin(6) $\cong$ SU(4) transformation, which defines the generic holonomy group. However, manifolds with more structure can have smaller holonomy groups (subgroups of SU(4)) — for example, the T^6 we considered before has trivial holonomy group, since its flatness means that a spinor will never be rotated. As a consequence, compactifying the 10d theory on T^6 actually preserves all the supersymmetry.

How much supersymmetry is that? A supercharge is a definite chirality Weyl spinor. In 10d such a spinor has 16 components (supercharges) and Type IIB has 2 independent such spinors, which determines its 10d $\mathcal{N} = 2$ supersymmetry. In 4d a Weyl spinor of definite chirality has only 4 components, which means that the 16 supercharges must redistribute between 4 different spinors. Thus we end up with a total of 8 different spinors with 4 supercharges each. Type IIB compactified on a T^6 has 4d $\mathcal{N} = 8$ supersymmetry — this is definitely not what we are looking for.

In order to get the minimum amount of supersymmetry, M_6 must allow for no more than 1 covariantly constant spinor. Since this spinor belongs to the fundamental representation of SU(4), we can write it as [28]

$$\eta^a = \begin{pmatrix} \eta_0^a \\ \eta_1^a \\ \eta_2^a \\ \eta_3^a \end{pmatrix} \xrightarrow{\text{SU(4)}} \begin{pmatrix} \eta_0^a \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{3.41}$$

after performing an SU(4) rotation. The subgroup of SU(4) which leaves η^a invariant is SU(3) — if the holonomy group of M_6 is precisely SU(3), one and only one covariantly constant spinor can be defined on it. Manifolds with SU(3) holonomy — know as Calabi-Yau manifolds — will take central stage in our discussion of Type IIB compactifications.

One can show this more rigorously by examining how the spin representations of positive and

[5] A proof can be found in Exercises 9.6 and 9.7 of [29].

[6] The fact that there must exist a nowhere vanishing spinor is in itself a topological constraint on M_6. It reduces the structure group of the manifold from Spin(6) $\cong$ SU(4) to SU(3), in order to allow for a spinor which does not transform under the transition functions [61].

negative chirality in 10d decompose into 4d and 6d representations, and by looking for the largest subgroup of SU(4) for which a spinor of definite chirality can be invariant, leading us to a singlet of SU(3). Since neither the group theoretic tools nor the argument itself will play a crucial role in the work presented here, we simply refer to [20, 29, 61] for more details.

One thing worth emphasising is that Type IIB on a Calabi-Yau manifold still has $\mathcal{N} = 2$ supersymmetry in 4d, since each of its original supersymmetries ϵ^a will give an independent ζ^a. Hence this is still not suitable for phenomenological applications.[7] Yet the structure of Calabi-Yau manifolds and the amount of supersymmetry are so powerful that it is still extremely useful to work with them — fortunately there is a way to take the $\mathcal{N} = 2$ theory and project out one of the supersymmetries.

3.3 Counting moduli

3.3.1 Decomposing 10d fields

Putting together the previous two subsections, we learn two important lessons.

1. Preserving the least amount of supersymmetry in 4d requires us to compactify the 10d theory on Calabi-Yau manifolds;
2. The topological information about the chosen Calabi-Yau tells us how to decompose the 10d fields in a basis of harmonic forms — this gives us the number of massless fields of each type in the low-energy description of the theory.

The bosonic form fields of Type IIB supergravity are then decomposed as[8] (see Appendix C for details on the basis elements)

$$\begin{aligned}
\Phi(x^\mu, y^m) &= \Phi(x^\mu) \cdot \mathbf{1}\,, \\
B_2(x^\mu, y^m) &= B_2(x^\mu) \cdot \mathbf{1} + b^a(x^\mu) \cdot \omega_a(y^m)\,, \\
C_0(x^\mu, y^m) &= C_0(x^\mu) \cdot \mathbf{1}\,, \\
C_2(x^\mu, y^m) &= C_2(x) \cdot \mathbf{1} + c^a(x^\mu) \cdot \omega_a(y^m)\,,
\end{aligned}$$

[7] As an interesting remark, Calabi-Yau manifolds were originally identified as promising internal spaces in the context of the heterotic string [62] — a crucial difference with respect to Type IIB is that heterotic theories have $\mathcal{N} = 1$ supersymmetry, so that choosing M_6 to be Calabi-Yau does indeed provide an $\mathcal{N} = 1$ 4d theory that is suitable for phenomenological applications.

[8] A 4-form would in general be expanded as

$$C_4(x^\mu, y^m) = D_2^a(x^\mu) \cdot \omega_a + V_1^K(x^\mu) \cdot \alpha_K(y^m) - \tilde{V}_{1,K}(x^\mu) \cdot \beta^K(y^m) + \rho_a(x^\mu) \cdot \tilde{\omega}^a(y^m)\,.$$

However, the self-duality of C_4 relates the terms expanded in α_K to the ones expanded in β^K, as well as the terms expanded in $\tilde{\omega}^a$ to the ones expanded in ω_a, such that only half of them are true degrees of freedom. We can therefore choose to keep the V_1^K vectors and the ρ_a scalars.

$$C_4(x^\mu, y^m) = V_1^K(x^\mu) \cdot \alpha_K(y^m) + \rho_a(x^\mu) \cdot \tilde{\omega}^a(y^m)\,, \tag{3.42}$$

where $a = 1, ..., h^{1,1}$ and $K = 0, ..., h^{2,1}$. There is, however, one last bosonic field which is not a differential form: the metric G_{MN}. In terms of its tensorial structure, we can decompose the metric as

$$G_{MN}(x^\mu, y^m) = g_{\mu\nu}(x^\mu) \cdot \varpi(y^m) + A_\mu^j(x^\mu) \cdot \varpi_q^j(y^m) + \xi^i(x^\mu) \cdot \varpi_{pq}^i(y^m)\,. \tag{3.43}$$

Since $\varpi(y^m)$ is just a function, it is by definition a 0-form — $h^{0,0} = 1$ therefore tells us that there is only one 4d massless graviton $g_{\mu\nu}(x^\mu)$. On the other hand, any $\varpi_q(y^m)$ defines a 1-form and hence $h^{1,0} = h^{0,1} = 0$ tells us that there are no massless vector fields arising from G_{MN}. The only tricky case is $\varpi_{pq}(y^m)$, since this is a symmetric tensor rather than a 2-form. By analysing the graviton equation of motion a bit more carefully, we will not only understand how to count the number of scalars $\xi^i(y^m)$ in terms of the Calabi-Yau Hodge numbers $(h^{1,1}, h^{2,1})$, but also set the stage for our discussion of gravitational waves in chapter 5.

3.3.2 The graviton equation of motion

The whole reason we were looking for harmonic forms in the first place was to find a basis of solutions to the equations of motion along the internal directions in terms of which the 10d fields could be decomposed (3.18). The equation of motion for the metric G_{MN} is the Einstein equation

$$R_{MN} = T_{MN} - \frac{1}{8} T^P{}_P\, G_{MN}\,, \tag{3.44}$$

where T_{MN} is the energy-momentum tensor including all other fields (cf. (2.76)). Unlike the equations for the remaining fields, however, the kinetic term for the propagating graviton does not appear explicitly in (3.44) — in order to find it, one must perturb the metric $G_{MN} \to \overline{G}_{MN} + h_{MN}$ and expand (3.44) to first order in perturbations. In particular,

$$R_{MN} = \bar{R}_{MN} + R^{(1)}_{MN} + \mathcal{O}(h^2)\,, \tag{3.45}$$

$$T_{MN} = \bar{T}_{MN} + T^{(1)}_{MN} + \mathcal{O}(h^2)\,, \tag{3.46}$$

$$T^P{}_P = \bar{T}^P{}_P - h^{MN}\bar{T}_{MN} + {T^{(1)}}^P{}_P + \mathcal{O}(h^2)\,, \tag{3.47}$$

where background quantities are denoted with a bar (e.g. $\bar{R}_{MN}$) and terms linear in h_{MN} with a superscript (1) (e.g. $R^{(1)}_{MN}$). Let us focus on the Ricci tensor, which at linear order takes the form

$$R^{(1)}_{MN} = -\frac{1}{2}\left(\Box_{10} h_{MN} - 2\bar{g}^{PQ}\nabla_P \nabla_{(M} h_{N)Q} + \nabla_M \nabla_N h_{10}\right)\,, \tag{3.48}$$

where the covariant derivatives ∇_M are with respect to the background metric $\bar{G}_{MN}$, $\Box_{10} = \bar{G}^{PQ}\nabla_P\nabla_Q$ and $h_{10} = \bar{G}^{PQ}h_{PQ}$. Commuting the covariant derivatives using

$$\nabla_P\nabla_M h_{NQ} = \nabla_M\nabla_P h_{NQ} + \bar{R}_{PMNA}h_Q{}^A + \bar{R}_{PMQA}h_N{}^A \tag{3.49}$$

and contracting the PQ indices with $\bar{G}^{PQ}$,

$$\bar{G}^{PQ}\nabla_P\nabla_M h_{NQ} = \bar{G}^{PQ}\nabla_M\nabla_P h_{NQ} + \bar{R}^P{}_{MNA}h_P{}^A + \bar{R}_{MA}h_N{}^A\,, \tag{3.50}$$

we can write $R^{(1)}_{MN}$ as

$$R^{(1)}_{MN} = -\frac{1}{2}\Box_{10}h_{MN} + \bar{R}_{P(M}h_{N)}{}^P + \bar{R}^P{}_{MNQ}h_P{}^Q + \nabla_{(M}\nabla^P\left(h_{N)P} - \frac{1}{2}\bar{G}_{N)P}h_{10}\right)\,. \tag{3.51}$$

Now there are two ways in which this expression can be simplified. Firstly, we notice that the background equations (i.e. the equations at zeroth order in h_{MN}) can be used to write

$$\bar{R}_{P(M}h_{N)}{}^P = \bar{T}_{P(M}h_{N)}{}^P - \frac{1}{8}\bar{T}^Q{}_Q\bar{G}_{P(M}h_{N)}{}^P\,, \tag{3.52}$$

in terms of the background energy-momentum tensor. Secondly, due to differomorphism invariance not all fluctuations h_{MN} are actually physical — some can be undone by a coordinate transformation. We can thus choose a gauge, known as the harmonic gauge, in which

$$\nabla^P\left(h_{NP} - \frac{1}{2}\bar{G}_{NP}h_{10}\right) = 0\,. \tag{3.53}$$

The equation of motion for the fluctuations h_{MN} then becomes

$$\begin{aligned}
-\frac{1}{2}\Box_{10}h_{MN} + \bar{R}_M{}^P{}_N{}^Q h_{PQ} = \bar{T}_{MN} - \frac{1}{8}\bar{T}^P{}_P\bar{G}_{MN}& \\
+ T^{(1)}_{MN} - \frac{1}{8}\left(h^{PQ}\bar{T}_{PQ} + \bar{G}^{PQ}T^{(1)}_{PQ}\right)\bar{G}_{MN} - \frac{1}{8}\bar{T}^P{}_P h_{MN}& \\
- \left(\bar{T}_{P(M} - \frac{1}{8}\bar{T}^Q{}_Q\bar{G}_{P(M}\right)h_{N)}{}^P\,.&
\end{aligned} \tag{3.54}$$

This is the equation of motion for the propagating graviton (at linear order) — it is also the equation for 10d gravitational waves. Note that every term on the right-hand side contains the energy-momentum tensor, including all other Type IIB fields. Although this may seem unnecessarily complicated,[9] not only does it highlight the fact that the right expansion follows from the graviton's equation of motion, just as it did for every other field, but it also sets up the stage for our later discussion of gravitational waves.

Focusing on the internal components, in the absence of fluxes and for the product spacetime we

[9] Indeed, most textbooks introduce it as a discussion of deformations of the internal metric which preserve its Ricci-flatness, i.e. for which the manifold stays a Calabi-Yau. The equation one ends up analysing is, of course, the same, but the picture one gets is slightly different.

are considering, the equation greatly simplifies to

$$\Box_4 h_{mn} + \nabla^p \nabla_p h_{mn} + 2\bar{R}_m{}^p{}_n{}^q h_{pq} = 0\,. \tag{3.55}$$

Hence, the internal tensors ϖ^i_{mn} in which we want to expand the metric (3.43) must satisfy the equation

$$\nabla^p \nabla_p \varpi_{mn} + 2\bar{R}_m{}^p{}_n{}^q \varpi_{pq} = 0\,. \tag{3.56}$$

It is important to remember that the background metric is Kähler (see Appendix C) because this means that the only non-vanishing components of the Riemann tensor are the ones with index structure $R_{i\bar{j}k\bar{l}}$. This implies that the $\varpi_{i\bar{j}}$ and $\varpi_{\bar{i}\bar{j}}$ components completely decouple and can be discussed (and counted) separately,

$$\nabla^p \nabla_p \varpi_{i\bar{j}} + 2\bar{R}_i{}^k{}_{\bar{j}}{}^{\bar{l}} \varpi_{k\bar{l}} = 0 \qquad \overset{\text{(B.8)}}{\Longrightarrow} \qquad \Delta_6\, \omega_{i\bar{j}} = 0\,, \tag{3.57}$$

$$\nabla^p \nabla_p \varpi_{\bar{i}\bar{j}} + 2\bar{R}_{\bar{i}}{}^{\bar{k}}{}_{\bar{j}}{}^{\bar{l}} \varpi_{\bar{k}\bar{l}} = 0 \qquad \Longrightarrow \qquad \Delta_6\, \chi_{ij\bar{l}} = 0\,. \tag{3.58}$$

Although ϖ_{mn} is not a form, the equation for the components $\varpi_{i\bar{j}}$ is the same as the component form of the equation $\Delta_6 \omega_{i\bar{j}} = 0$ for an actual $(1,1)$-form ω (B.8). Therefore, solutions to the equations of motion for $\varpi_{i\bar{j}}$ define harmonic $(1,1)$-forms and we know that there are $h^{1,1}$ independent such forms, ω_a.

The story is similar for the $\varpi_{\bar{i}\bar{j}}$ components. We can use the unique holomorphic 3-form on the Calabi-Yau to define $(2,1)$-forms as

$$\chi = \frac{1}{2!} \cdot \frac{1}{2} \Omega_{ij}{}^{\bar{j}} \varpi_{\bar{j}\bar{l}}\, \mathrm{d}z^i \wedge \mathrm{d}z^j \wedge \mathrm{d}\bar{z}^{\bar{l}}\,, \tag{3.59}$$

which are harmonic precisely when the equation of motion for $\varpi_{\bar{i}\bar{j}}$ is satisfied. Hence, there are $h^{2,1}$ independent solutions defining the harmonic $(2,1)$-forms χ^k.

Due to the way the metric is connected to the Kähler form J on the Calabi-Yau (C.4), fluctuations of the form $h_{i\bar{j}}$ correspond to fluctuations of the Kähler structure itself — for this reason, these are known as Kähler deformations and the associated scalars will be called Kähler moduli. On the other hand, a change of the form $h_{\bar{i}\bar{j}}$ is not holomorphic, which means that the result can only remain a Calabi-Yau with respect to a *different* complex structure — these are therefore known as complex structure deformations and the associated scalars will be called complex structure moduli.

We can finally decompose the metric as[10]

$$G_{MN}(x^\mu, y^m) = g_{\mu\nu}(x^\mu) \cdot \mathbf{1} + v^a(x^\mu) \cdot (\omega_a)^a_{i\bar{j}}(y^m) + z^k(x^\mu) \cdot \frac{1}{||\Omega||^2} \overline{\Omega}_{\bar{i}}{}^{ab} (\chi_k)_{ab\bar{j}}(y^m)\,. \tag{3.61}$$

An important point to make is that harmonic $(1,1)$-forms are real, which in turn means that the $h^{1,1}$ scalars v^a are also real. In the low-energy description they actually combine with the $h^{1,1}$ real scalars b^a arising from the B_2 field to form $h^{1,1}$ complex scalars, $t^a = b^a + iv^a$ — these are known as complexified Kähler moduli. On the other hand, the $h^{2,1}$ scalars z^k are complex and have 2 real degrees of freedom.

Now that we know how all the bosonic fields of Type IIB supergravity decompose in terms of harmonic functions on a Calabi-Yau, we can plug this back into the action (2.76), integrate over the compact space M_6 and integrate out the massive towers only keeping the massless modes of every field. This process results in an $\mathcal{N} = 2$ theory in 4d, in which the massless fields fill several supergravity multiplets. More precisely, the compactification gives rise to 1 gravity multiplet, $h^{2,1}$ vector multiplets, $h^{1,1}$ hypermultiplets and 1 tensor multiplet. This precise structure will not be crucial for us, so we do not go into the details regarding the action and the multiplets (see [60, 61] for more details).

3.4 Moduli stabilisation with fluxes

So far we have seen how to find a 4-dimensional low-energy description of Type IIB supergravity, which itself was already a low-energy description of Type IIB string theory. We found that breaking most (but not all) of the supersymmetry selected a very special type of compact manifolds known as Calabi-Yau. On these backgrounds, the 4d theory is an $\mathcal{N} = 2$ supergravity theory describing a number of massless fields determined by the topology of the Calabi-Yau. Nevertheless, these results followed from the assumption that the solution of interest had no fluxes, i.e. that the background values of the form fields actually vanished. Even if curiosity alone were not enough[11] to explore the case where the solution *does* have fluxes, there are at least two phenomenologically motivated reasons that push us in that very direction.

1. A theory with $\mathcal{N} = 2$ supersymmetry cannot contain chiral matter, which is essential to describe the known particles and forces of the Standard Model — an $\mathcal{N} = 2$ theory simply does not fit the data. The only acceptable amount of supersymmetry for a theory which could include the Standard Model is $\mathcal{N} = 1$.

[10] Note that we had to invert the definition of the $(2,1)$-forms,

$$\chi_{ab\bar{j}} = \frac{1}{2!} \cdot \frac{1}{2} \Omega_{ij}{}^{\bar{j}} \varpi_{\bar{i}\bar{j}} \implies \varpi_{\bar{i}\bar{j}} = \frac{1}{||\Omega||^2} \overline{\Omega}_{\bar{i}}{}^{ab} \chi_{ab\bar{j}}\,, \tag{3.60}$$

where $||\Omega||^2 = \frac{1}{3!}\Omega_{abc}\overline{\Omega}^{abc}$ and it is useful to use the relation $\overline{\Omega}^{a_1 a_2 b}\Omega_{a_1 a_2 c} = 2! \cdot ||\Omega||^2 \delta^b_c$ [20].

[11] Observations suggest that curiosity alone is typically enough among physicists.

2. The massless spectrum we found includes several massless moduli. On the one hand, since 4d masses and couplings depend on the vacuum expectation value of these fields, they are completely arbitrary unless these moduli have a potential that determines their vev, which is not the case when the moduli are massless. On the other hand, massless fields with gravitational-strength couplings violate the equivalence principle and mediate fifth forces that have so far never been observed [63].

A very particular example of how moduli vevs determine couplings comes from the dilaton, whose expectation value controls the string perturbative expansion itself. As we shall see, it also appears in the relation between the 4d Planck scale (i.e. the interaction strength of 4d gravity in our low-energy description) and the fundamental string scale $1/\alpha'$. Yet, while it might be unsatisfying to leave this degree of arbitrariness in the 4d theory, the strongest reason to fix these moduli is observational: we simply do not see them.

Crucial for us is the fact that fluxes will allow us to have warping and consider warped compactifications, which are the central focus of our work.

3.4.1 $\mathcal{N} = 1$ solutions from fluxes

So what happens when we include non-vanishing fluxes? First, everything becomes extremely more complicated. For one, the supersymmetry variation (3.31) no longer gives simple conditions on spinors and as a direct consequence the solution with fluxes is likely not to have a covariantly constant spinor. This effectively removes the immediate constraint on the holonomy of the compact space. However, the mere existence of a globally defined spinor on this space such that we could have a globally defined solution to the more general equation restricts the structure group of the manifold (the group of transformations required to glue together different patches of the manifold) to be contained in SU(3).[12] The right framework to discuss these flux backgrounds is generalised complex geometry,[13] which gives a unified description of complex and symplectic geometries [61]. Since the formalism itself is rather complicated, let us simply state the conclusion: with these tools, flux backgrounds can be found (configurations which solve the supersymmetry conditions and consequently the equations of motion) in which supersymmetry is broken down to $\mathcal{N} = 1$. These correspond to generalised Calabi-Yau manifolds and the breaking of supersymmetry can be understood as an obstruction to finding a second independent spinor arising from the fluxes.

[12] Note that this is *not* the SU(3) holonomy group we found before, but an SU(3) *structure group* — it therefore does not define a Calabi-Yau in general. The globally defined spinor can be used to construct a real 2-form J and a complex 3-form Ω, which for a Calabi-Yau manifold are both closed due to the fact that the spinor is covariantly constant. In the more general case, however, none of them needs to be closed and the manifold will not necessarily be a Calabi-Yau [61].

[13] Generalised geometry gives a manifestly SO(d, d) covariant description, making the symmetries of string theory (most easily found in the context of torus compactifications) a symmetry of a tangent space at each point of a generic manifold (in a way analogous to general relativity) [61].

In particular, reducing the supersymmetry from $\mathcal{N}=2$ to $\mathcal{N}=1$ means that the two 4d spinors ζ^a in (3.39) must be related to each other. Because of the 4d maximal symmetry this relation can only be of proportionality and therefore the 10d spinor can be decomposed as

$$\eta^a(x^\mu, y^m) = \zeta(x^\mu) \otimes c_a \eta(y^m)\,, \tag{3.62}$$

for complex functions c_a, $a = 1, 2$, which must be linked in order to give $\mathcal{N}=1$ supersymmetry. This is done by the fluxes, whose specific configuration results in only one specific combination of the 2 supersymmetries being preserved.

Moreover, the SU(3) structure on its own is already a powerful constraint — in fact, on such a manifold one can always find a connection, possibly with torsion[14], such that $\nabla'_m \eta^a = 0$. With respect to this connection, the manifold does have SU(3) holonomy. In the absence of fluxes, the connection is torsionless and the result is a Calabi-Yau; but once we do have non-vanishing fluxes, torsion is generated by the flux configuration. One can decompose a generic torsion into 5 different classes, W_i, $i = 1, ..., 5$, and classify the resulting manifold according to the set of non-vanishing components — Calabi-Yau manifolds are the special case for which all torsion classes vanish [60].

When one puts all this together, a classification of possible $\mathcal{N}=1$ vacua arising in Type IIB supergravity with fluxes is obtained. Although the full set of possible configurations is much richer [64], we will focus on a specific type of solutions which dominate a large portion of the Type IIB phenomenology literature, belonging to the set of solutions known as type B. Their spinors are related as $c_1 = \pm i c_2$ and they have both H_3 and F_n non-zero fluxes, related in such a way that the complex 3-form flux G_3 (2.80b) is imaginary self-dual and has no $(0, 3)$ component,

$$\star G_3 = i G_3 \quad \text{and} \quad G_{(0,3)} = 0\,. \tag{3.63}$$

Out of these type B solutions, our focus is the set with $W_{1,2,3} = 0$ and $2W_5 = 3W_4 = -6\overline{\partial} A$, where A is precisely the warp factor in our warped metric ansatz (3.30) and is determined by the $\tilde{F}_5$ flux. These correspond to a conformal Calabi-Yau metric, whose conformal factor is the inverse of the warp factor,

$$g_{mn} = e^{-2A(y)} (g_{\text{CY}})_{mn}\,. \tag{3.64}$$

One advantage of this class of solutions is that we can use our knowledge of the properties of Calabi-Yau manifolds to study them.

It is important to note that these solutions preserve a specific combination of supersymmetries and therefore we can only add objects that preserve the same supersymmetry combination. For type B solutions, the allowed objects are D3/D7-branes and O3/O7-planes,[15] as well as

[14] A connection ∇'_m with torsion $T_{mn}{}^q$ generates an extra term in the relation $[\nabla'_m, \nabla'_n]V_p = R_{mn}{}^q{}_p V_q - 2T_{mn}{}^q \nabla'_q V_p$.

[15] We have seen before that orientifold planes break half of the supersymmetry, since one (combination) of the supercharges gets projected out. In order to preserve the $\mathcal{N}=1$ supersymmetry left over by the fluxes, the

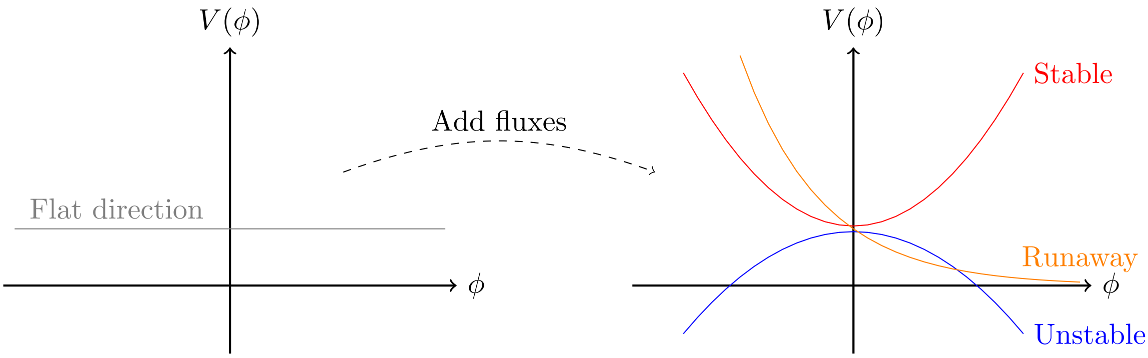


Figure 3.2: The low energy theory arising from compactifying Type IIB supergravity on a Calabi-Yau 3-fold includes a set of scalar fields without a potential, which are therefore massless — these are the moduli of the compactification. Adding non-trivial fluxes generates a potential for some of these moduli (in particular $h^{2,1}$ complex structure moduli and the dilaton) that may stabilise them at a specific value with a given mass. For each scalar (or more precisely, each direction in the moduli space), the flux-generated potential may locally have a (metastable) minimum, (unstable) maximum, or even have no critical point and develop a runaway towards some limit in the moduli space.

D5-branes wrapped on collapsed 2-cycles (which look effectively like D3-branes) [60]. Adding O-planes has another consequence on the resulting EFT — since an orientifold projects out some of the states in the theory, in the presence of O3/O7-planes only a subset of the moduli survives the projection. The cohomology groups $H^{p,q}$ and hence the set of harmonic forms on a given Calabi-Yau (see Appendix C) will split between those that are even $H^{p,q}_+$ and those that are odd $H^{p,q}_-$ under the orientifold involution (i.e. the spacetime reflection defining the orientifold), such that $h^{p,q} = h^{p,q}_+ + h^{p,q}_-$, and depending on how each field transforms under a world-sheet parity transformation only one of these sets remains in the decomposition into harmonic forms (see [60, 61, 65] for details).

3.4.2 Moduli stabilisation

In the presence of fluxes, a potential is generated for some of the moduli and those will no longer be arbitrary and massless. Generically, this potential may or may not have stable minima and the scalars could either be fixed at a finite value or be sent to either zero or infinity (Fig. 3.2) — in any case, this process is dynamical (it follows from the fluxes that are present in the solution) and some of the previously flat directions are lifted. These moduli are then said to be stabilised and get a mass, through which one can hope to avoid the undetected long-range forces.

In order to discuss the flux generated potential, one should be clear regarding what exactly is meant by *fluxes*. So far we have referred to fluxes as the vacuum expectation values of the (internal) field-strengths H_3 and F_n — in fact, this is not totally accurate. What we call flux is a non-trivial configuration of the field strengths, where *non-trivial* refers to the integrals of

O-planes must project out the right supercharges and leave the surviving one untouched. Conversely, we could think of breaking supersymmetry with orientifolds and then restrict to flux configurations that preserve the same supercharge.

these field-strength configurations over certain cycles being non-zero,[16]

$$\texttt{(Fluxes)} \qquad \int_{\Sigma_3} H_3 \neq 0\,, \quad \int_{\Sigma_n} F_n \neq 0\,. \tag{3.65}$$

Since a cycle is a manifold with no boundary, Stokes theorem tells us that the fluxes will vanish whenever the field-strengths are globally exact forms,

$$\int_{\Sigma_3} H_3 = \int_{\Sigma_3} \mathrm{d}B_2 \overset{\text{Stokes}}{=} \int_{\delta\Sigma_3} B_2 \overset{\text{cycle}}{=} 0\,, \tag{3.66}$$

which is what one would expect from the definitions of the field-strengths as $H_3 = \mathrm{d}B_2\,, F_n = \mathrm{d}C_{n-1}$. If however there are sources that obstruct the "exactness" of the field-strengths (i.e. it becomes impossible to globally define form potentials B_2 and C_{n-1} in terms of which the field-strengths can be expressed as exact forms), the fluxes might no longer vanish. Although in supergravity these non-zero fluxes could take any real value, in string theory fluxes must be quantised (analogously to how electric and magnetic charges must be quantised in electromagnetism),

$$\texttt{(Fluxes in string theory)} \qquad \frac{1}{(2\pi)^2\alpha'}\int_{\Sigma_3} H_3 \in \mathbb{Z}\,, \qquad \frac{1}{(2\pi\sqrt{\alpha'})^{n-1}}\int_{\Sigma_n} F_n \in \mathbb{Z}\,. \tag{3.67}$$

In the context of Calabi-Yau manifolds, we know how to count the number of cycles and how to define a basis of forms which are dual to these cycles (see Appendix C). We can therefore define electric and magnetic fluxes as

$$\frac{1}{(2\pi)^2\alpha'}\int_{A_K} H_3 = m^K\,, \qquad \frac{1}{(2\pi)^2\alpha'}\int_{B^K} H_3 = e_K\,, \tag{3.68}$$

$$\frac{1}{(2\pi)^2\alpha'}\int_{A_K} F_3 = m^K_{RR}\,, \qquad \frac{1}{(2\pi)^2\alpha'}\int_{B^K} F_3 = (e_{RR})_K\,, \tag{3.69}$$

where $K = 1, ..., h^{2,1}+1$ and (A_K, B^K) are pairs of 3-cycles on a Calabi-Yau, related by Hodge duality. There are no F_1 and F_5 fluxes because there are no non-trivial 1-and 5-cycles (or, in other words, no closed 1-and 5-forms which are not exact). By considering the basis of Poincaré dual forms (α_K, β^K) defined as

$$\int_{A_L} \alpha_K = \int_{\mathrm{CY}} \alpha_K \wedge \beta^L = \delta^L_K\,, \qquad \int_{B^K} \beta^L = \int_{\mathrm{CY}} \beta^L \wedge \alpha_K = -\delta^L_K\,, \tag{3.70}$$

we can expand the field-strengths as

$$H_3 = \mathrm{d}B_2 + (2\pi)^2\alpha'\big(m^K\alpha_K - e_K\beta^K\big)\,, \tag{3.71}$$

$$F_3 = \mathrm{d}C_2 + (2\pi)^2\alpha'\big((m_{RR})^K\alpha_K - (e_{RR})_K\beta^K\big)\,. \tag{3.72}$$

[16] It is always useful to have in mind the example of electromagnetism, where one could compute the flux of the field-strength $F_{\mathrm{e.m.}}$ for example *through* a sphere (and indeed the sphere S^2 is a cycle, as it has no boundary — see Appendix B).

Since α_K and β^K are related to each other through Hodge duality, the electric and magnetic fluxes are swapped in $\star H_3$ and $\star F_3$ compared to H_3 and F_3, motivating this identification.[17] We can also use the basis of 3-cycles A_K to define the complex structure moduli z^k through the periods of the holomorphic form Ω,[18]

$$z^k = \frac{Z^K}{Z^0}\,, \quad \text{where} \quad Z^K = \int_{A_K} \Omega\,. \tag{3.73}$$

It is also useful to define the periods of Ω through the dual B^K cycles as

$$\mathcal{F}_K = \int_{B^K} \Omega\,, \tag{3.74}$$

which can be expressed as functions of z^k, $\mathcal{F}_K = \mathcal{F}_K(z^J)$.

Dimensionally reducing the Type IIB action (2.76) with non-zero fluxes (3.71) and (3.72), one gets extra terms in the 4d supergravity action that generate a potential for the complex structure moduli z^k and the dilaton [60]. This can be understood as follows. As we have discussed, the fluxes change the equations of motion in such a way that a Calabi-Yau manifold is no longer a solution. However, it turns out that it is consistent to assume that the resulting spectrum is the same as the one without fluxes, except that some of the massless modes become massive, as long as the scale of the fluxes is much smaller than the compactification scale — one could think of the fluxes as giving corrections to the solution in which they vanish.[19] This truncation of the spectrum was shown to give a consistent gauged supergravity action, in which some of the axionic symmetries become gauged and a potential is generated for a subset of the moduli. In Type IIB compactifications this subset includes the complex structure moduli and the axio-dilaton, but not the Kähler moduli — since there is always at least one Kähler modulus ($h^{1,1} \geq 1$), this means that fluxes are never sufficient to fully stabilise all moduli in Type IIB and extra contributions need to be included.[20]

For the solutions of interest with $\mathcal{N} = 1$ supersymmetry, this scalar potential can be conveniently encoded in terms of a holomorphic superpotential W and the Kähler potential that determines

[17] In other words, a *magnetic* flux through B^K, $\int_{B^K} H_3$, is equivalent to an *electric* flux through A_K, $\int_{A_K} \star H_3$.

[18] Note that there are $h^{2,1}$ complex structure moduli but a total of $(h^{2,1}+1)$ 3-cycles A_K. The periods themselves define a set of projective coordinates Z^K, in terms of which one can define the actual moduli as the special coordinates $z^k = Z^K/Z^0$ [60]. The coordinates Z^K are projective because complex rescalings of Ω do not change the complex structure and should therefore not count as an actual degree of freedom in the moduli space of complex structures [66].

[19] Another way of thinking about it is that, since the KK scale is much bigger than the flux contribution, at this scale the fluxes appear to be essentially zero. It is only at the massless level that the effect of the fluxes is felt, so that the net effect is the same spectrum as in the zero-flux solution plus a potential encoding the flux contribution to the action.

[20] In Type IIA compactifications, the subset of moduli fixed by the fluxes includes instead the Kähler moduli, but not the complex structure moduli. However, there are manifolds with $h^{2,1} = 0$ (known as rigid manifolds) and it could be that fluxes are the only necessary ingredient for full moduli stabilisation in Type IIA (e.g. [67]).

the metric on the moduli (field) space,[21] which is given in terms of J and Ω as $K = K_{\rm cs} + K_{\rm Kähler}$,

$$K_{\rm cs} = -\log\left(i \int \Omega \wedge \overline{\Omega}\right) = -\log\left(\bar{Z}^K \mathcal{F}_K - Z^K \bar{\mathcal{F}}_K\right) \tag{3.75}$$

$$K_{\rm Kähler} = -\log\left(\frac{4}{3} \int J \wedge J \wedge J\right) = -\log\left(\frac{i}{6}\mathcal{K}_{abc}(t-\bar{t})^a (t-\bar{t})^b (t-\bar{t})^c\right), \tag{3.76}$$

where the triple intersection numbers $\mathcal{K}_{abc}$ are defined as

$$\mathcal{K}_{abc} = \int \omega_a \wedge \omega_b \wedge \omega_c\,. \tag{3.77}$$

Using the properties of Calabi-Yau manifolds, one can also write $K_{\rm Kähler} = -2\log\mathcal{V}$. These definitions follow from the dimensional reduction of the Type IIB action (2.76), but do not take warping into account — we will see in particular that the warp factor appears in the Kähler potential, but not the superpotential. In terms of W and K, the scalar potential for the moduli takes the form

$$V = e^K \left(K^{I\bar{J}} (D_I W)(D_{\bar{J}}\overline{W}) - 3|W|^2\right), \tag{3.78}$$

where the covariant (or Kähler) derivative is defined as

$$D_I W = \partial_I W + (\partial_I K) W\,. \tag{3.79}$$

The superpotential that reproduces the scalar potential generated by the fluxes and found by dimensional reduction of the action (2.76) was shown to take the Gukov-Vafa-Witten form [71]

$$W/M_{\rm Pl}^3 = \frac{g_s^{3/2}}{\sqrt{4\pi}\cdot l_s^5} \int G_3 \wedge \Omega\,, \tag{3.80}$$

in terms of which the dependence on the complex structure moduli (through Ω) and the axio-dilaton (through G_3) becomes clear. A derivation of this superpotential from the scalar potential generated by the fluxes, including the overall factor, is shown in Appendix D.2. One can also write it explicitly in terms of the fluxes and $(Z^K, \mathcal{F}_K)$ as

$$W/M_{\rm Pl}^3 = \frac{g_s^{3/2}}{\sqrt{4\pi}\cdot l_s^5} \left\{\left((e_{RR})_K - e_K \tau\right) Z^K - \left((m_{RR})^K - m^K \tau\right)\mathcal{F}_K\right\}\,. \tag{3.81}$$

In this $\mathcal{N} = 1$ supergravity framework, a solution is supersymmetric if [72–74]

$$D_I W = 0 \tag{3.82}$$

[21] The moduli field space itself has a rich geometric structure — while the complex structure moduli span a special Kähler manifold of complex dimension $h^{2,1}$ [68], the remaining scalars (complexified Kähler moduli, the scalars arising from C_0, C_2, C_4 and the dilaton, and the scalars dual to the 2-forms arising from B_2 and C_2) span a quaternionic Kähler manifold of quaternionic dimension $h^{1,1}+1$ [69]. The latter has a special Kähler base spanned by the complexified Kähler moduli for which we can write the Kähler potential $K_{\rm Kähler}$. This rich structure is interesting for example in the study of dualities and the signature of spacetime [70].

for all fields, in which case the scalar potential $V_{\text{SUSY}} = -3e^K|W|^2$ can only be negative (AdS) or zero (Minkowski). Moreover, at this level (without additional corrections to the potential), there is no superpotential generated for the Kähler moduli and therefore $D_aW = (\partial_a K)W$, $a \in$ Kähler. As a consequence, a solution can only be supersymmetric if $W = 0$. On the other hand, $K_{\text{Kähler}}$ (3.76) satisfies a so-called no-scale structure giving $K^{a\bar{b}}(\partial_a K)(\partial_b K) = -3$, so that the Kähler moduli contribution to the first term in the scalar potential (3.78) perfectly cancels the second term,

$$V_{\text{no-scale}} = e^K \; K^{\bar{\imath}j}(D_iW)(D_{\bar{\jmath}}\overline{W}) \,, \qquad i,j \in \text{complex structure} \,. \tag{3.83}$$

While a supersymmetric solution for the complex structure moduli will then leave all Kähler moduli as flat directions, a non-supersymmetric solution will leave a runaway for the volume modulus (since $e^K \sim 1/\mathcal{V}^2$). A big part of Type IIB phenomenology involves tweaking these properties in such a way as to change some or all of these conclusions. In particular, we need extra contributions arising from quantum corrections to the leading scalar potential in order to stabilise the Kähler moduli that include at the very least the volume modulus of the compactification. Taking into account different contributions leads us to different proposals such as the KKLT [75] and LVS [76, 77] constructions that we will discuss in Chapter 4.

Before moving on, we should also emphasise that the tadpole cancellation condition (2.101) associated with the $\tilde{F}_5$ equation of motion (2.78e) (which is the one involving H_3 and F_3 fluxes) should also be satisfied in a consistent solution. In terms of their expansion in the basis of 3-cycles, the constraint reads

$$\frac{g_s^2}{2\kappa^2} \cdot (2\pi)^4 \alpha'^2 \{-m^K (e_{RR})_K + (m_{RR})^K e_K\} + \sum_i Q_3^{(i)} \,. \tag{3.84}$$

Replacing $Q_3^{(i)} = q_3^{(i)} T_{\text{D3}}$, with $q_3^{(i)} = \pm 1$ for D3/$\overline{\text{D3}}$-branes and $q_3^{(i)} = -\frac{1}{4}$ for O3-planes, and using $T_{\text{D3}} = \frac{2\pi}{l_s^4} = \frac{2\pi}{(2\pi)^4 \alpha'^4}$ together with (2.77), we can write it as

$$(m_{RR})^K e_K - m^K (e_{RR})_K + N_{\text{D3}} - N_{\overline{\text{D3}}} - \frac{N_{\text{O3}}}{4} = 0 \,, \tag{3.85}$$

in terms of the number of D3/$\overline{\text{D3}}$-branes and O3-planes. Note that negative contributions to the tadpole can come from both $\overline{\text{D3}}$-branes and O3-planes, but there is an important difference between them — while O3-planes preserve the same supersymmetry as the fluxes in the Type B solutions, the $\overline{\text{D3}}$ will break it. Therefore, if one wants to cancel positive contributions to the D3-charge tadpole using localised objects, while preserving $\mathcal{N} = 1$ supersymmetry, one must use O3-planes.

3.5 The warped deformed conifold

3.5.1 Conifold singularities

We have seen that the topology and geometry of the compact space determine many of the properties of the low-energy theory, not only telling us the number of fields, but also defining the field space metrics and couplings for these fields. Before moduli stabilisation, the massless scalars in our 4d theory are unfixed and their background values are completely arbitrary. Since some of these fields parametrise the geometry of the compact space (controlling the sizes of cycles and the complex structure of the manifold), moving around in moduli space corresponds to exploring different geometries for the compact manifold. An interesting question to ask is whether moving in moduli space would also allow us to explore different topologies: this would be much more surprising, as we are no longer thinking of cycles of different sizes, but instead of different cycles altogether. This would have to correspond not to moving within one moduli space, but to moving *between* moduli spaces that could in particular have different dimensions.

The result would resemble a connected web of Calabi-Yau moduli spaces and is exactly what was found in multiple works [78–80]. The boundaries separating these connecting spaces were shown to correspond to conifolds, spaces that are smooth apart from a number of isolated conical singularities[22] [80]. The key point is that these conifolds can be seen as the singular limit of non-singular Calabi-Yau manifolds of different topologies, with each node being either the result of an S^3 or an S^2 that collapses to zero size (see Fig. 3.3). One can also go in the opposite direction — from the singular conifold with nodes to a non-singular manifold where each node was either replaced by an S^3 (deformation) or by an S^2 (small resolution) [80, 81].

The study of conifold singularities was motivated in part by efforts to find generic features of string theory compactifications that could give model-independent predictions and allow us to test the fundamental principles and their consequences in terms of low-energy observables. The fact that these singularities arise in most Calabi-Yau compactifications made them a natural candidate as one of these generic features. In the low-energy EFT, these conifold singularities manifest themselves in the form of singularities of the moduli space metric, signalling a breakdown of the effective description. To see this, recall the definition of the coordinates z^K in terms of the periods over the cycles A^K (3.73). If the 3-cycle A^1 shrinks to zero size (in which case it is called a vanishing cycle), the corresponding coordinate z^1 will go to zero,

$$z^1 = \frac{1}{Z^0} \int_{A^1} \Omega \longrightarrow 0 \,. \tag{3.86}$$

Hence we can think of the point $z^1 = 0$ in moduli space as the conifold point where a 3-cycle degenerates to zero size and the manifold becomes singular — this is the boundary of moduli

[22] Conifolds are nodal varieties, spaces which have only isolated double points (nodes) where the constraint defining the space as $C(x) = 0$ also has a vanishing gradient $\mathrm{d}C(x) = 0$, but a regular hessian matrix [80].

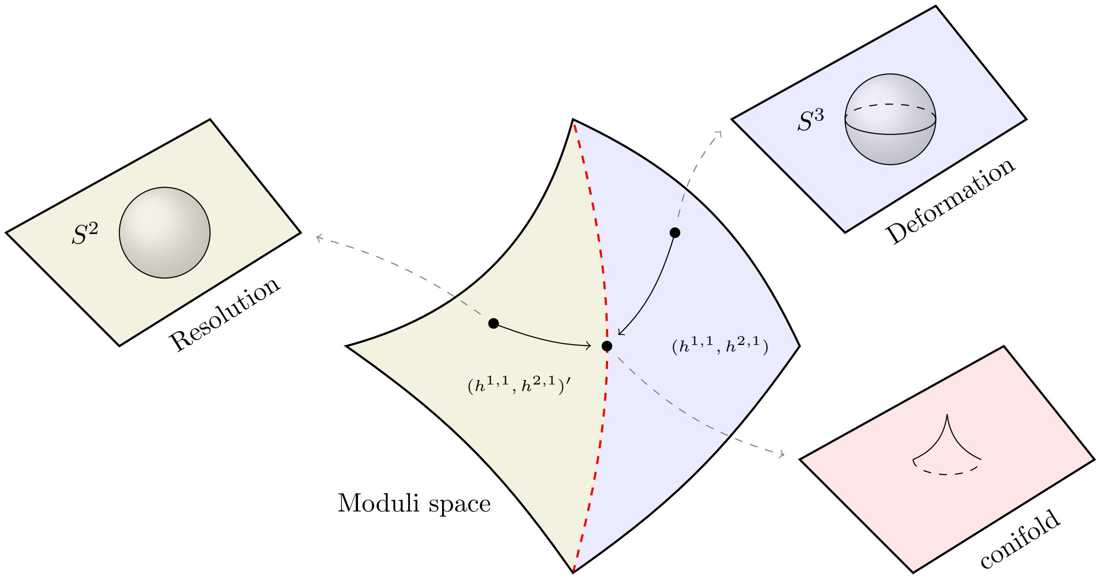


Figure 3.3: Conifolds can be seen as the singular limit of non-singular Calabi-Yau manifolds of different topologies, with each node being either the result of an S^3 or an S^2 in the non-singular spaces collapsing to zero size.

space represented in Fig. 3.3. While the cycle A^1 is well-defined as the 3-cycle that vanishes at $z^1 = 0$, the dual cycle B_1 can now go around the singularity and is no longer uniquely defined. Generically, going once around the singular point can shift the B_1 cycle by A^1 (notice that this still satisfies the orthogonality conditions defining the B_K as dual to A^K),

$$B_1 \longrightarrow B_1 + A^1\,, \tag{3.87}$$

$$\Longrightarrow\; \mathcal{F}_1(z^k) = \int_{B_1} \Omega \longrightarrow \int_{B_1} \Omega + \int_{A^1} \Omega = \mathcal{F}_1(z^k) + z^1\,. \tag{3.88}$$

This is a strong constraint on the form of $\mathcal{F}_1$ and in fact restricts it to

$$\mathcal{F}_1(z^k) = \frac{1}{2\pi i} z^1 \log z^1 + \Pi_0 + \mathcal{O}(z^1)\,, \tag{3.89}$$

such that the first term guarantees (3.88) and the remaining terms are single valued when going around the singularity. Since the Kähler potential on the complex structure moduli space is given by (3.75) and determines the metric as[23] $G_{I\bar{J}} = \partial_I \partial_{\bar{J}} K_{\rm cs}$, we find

$$G_{1\bar{1}} = -\frac{1}{2\pi i\; \Pi_0} \log z^1 \bar{z}^1 + \text{const.} + \mathcal{O}(z^1)^2\,, \tag{3.90}$$

which becomes singular at $z^1 = 0$. This singularity in the complex structure moduli space is telling us that the corresponding manifold also becomes singular at $z^1 = 0$ and is now a conifold, but it also means that the effective theory is breaking down at this point (i.e. the effective description stops being valid when the compact space becomes a singular conifold).

[23] Here the derivatives $\partial_I \equiv \frac{\partial}{\partial z^I}$ are taken with respect to the moduli space coordinates z^K.

3.5.2 Why is there a singularity?

Typically an effective theory breaks down when the physics we are trying to describe requires states that are not taken into account by our description. For example, if we are using our 4d EFT obtained from integrating out the massive towers of KK modes, we can only work at energies below M_{KK} — that is because at higher energies, the tower of states will start contributing to the physics in important ways and an effective theory without it will no longer be a good approximation (Fig. 3.1). Could the breakdown of our EFT near the conifold point also be associated with a state that we have unknowingly integrated out? In other words, is there a state not taken into account in the EFT which becomes important near the conifold point? In [82], it was argued that this is indeed the case and that the missing states are actually black holes, so that making sense of (and removing) conifold singularities[24] in the low-energy effective theory required that black holes were treated as fundamental particles, in the sense that they can appear in virtual loops in a way that depends on their mass and charges.

In contrast with our KK tower example, these black hole states do not become important because we are exploring higher energies, but rather because the states themselves become lighter as we approach the conifold point, eventually becoming massless at $z^1 = 0$ (Fig. 3.4). The 10-dimensional theory has 3-brane solutions[25] [83] that wrap a 3-cycle in the compact space, which in the 4d compactified theory look like charged black holes whose mass depends on the charges[26] (m^K, n_K) and periods,

$$M_{\rm BH} \propto |m^K \mathcal{F}_K - n_K Z^K|\,. \tag{3.91}$$

In the simplest case[27] when the black brane wraps a single cycle A^1, the only non-zero charge is n_1 and the mass is proportional to the complex structure modulus z^1,

$$M_{\rm BH} \propto |z^1|\,. \tag{3.92}$$

Therefore when A^1 is a vanishing cycle, these black hole states become massless as we approach the conifold point at $z^1 = 0$. Once that happens, the effective theory breaks down because we are no longer allowed to integrate them out. When the black hole state of mass $M_{\rm BH} \propto |z^1|$ is

[24] We should keep in mind that there are two types of singularity involved. On the one hand, we have the singular points of a conifold, which are singularities on the 6d space describing the internal geometry of spacetime at these special points in moduli space. On the other, there is the singularity in moduli space itself at the conifold point $z^1 = 0$, which we can see in (3.90). They are of course related to each other, but many discussions surrounding the conifold become much clearer when one remembers the distinction (cf. Fig. 3.3). Here we mean the singularity in the moduli space.

[25] These extended black hole solutions were originally not associated with D-branes. Only after D-branes were understood as dynamical objects in their own right [21], were these solutions interpreted as D3-brane solutions and the conifold singularity resolution rephrased in terms of D3-branes wrapping vanishing cycles.

[26] Note that the $h^{2,1}$ complex structure moduli belong to $\mathcal{N} = 2$ vector multiplets containing also a vector field each. These charges are the black hole electric and magnetic charges under these $h^{2,1}$ vector fields.

[27] The discussion can be generalised to include P degenerating 3-cycles related to each other through R homology relations. Such a conifold point connects a Calabi-Yau with Hodge numbers $(h^{1,1}, h^{2,1})$ to a Calabi-Yau with Hodge numbers $(h^{1,1} + R, h^{2,1} + R - P)$ [84]. Note that the simple case with only one vanishing cycle simply changes $h^{2,1} \to h^{2,1} - 1$.

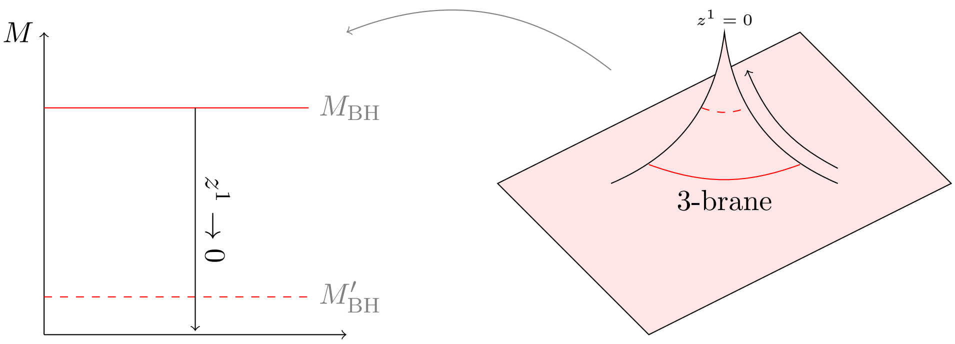


Figure 3.4: A 3-brane wrapping a 3-cycle in the compact space gives rise to a black hole state in the low-energy theory whose mass is proportional to the charges and periods (3.91). When the 3-brane is wrapping a vanishing cycle whose corresponding period goes to zero at the conifold point, the black hole state in the EFT becomes massless and our theory with this state integrated out is no longer consistent.

integrated out, the gauge coupling $g_{1\bar{1}}$ associated with the vector in the same vector multiplet as z^1 receives a one-loop correction depending on the mass as $g_{1\bar{1}} \sim g^{(0)}_{1\bar{1}} + \log{(M_{\rm BH})}$ (which is why these states must be allowed to run in loops in order to resolve conifold singularities). Since the gauge coupling in the supersymmetric theory is related to the periods as $g_{1\bar{1}} \sim \partial_{\bar{1}}\mathcal{F}_1$, the result is the log-term in (3.88) that diverges as $z^1 \to 0$ [82, 84]. If instead we do not integrate out the black hole states and the one-loop correction to the coupling is absent, so is the divergent behaviour of $\mathcal{F}_1$ and $G_{1\bar{1}}$ — the effective theory that includes these states is well-behaved even at the conifold point.

There are (at least) two key points to take from this discussion. The first is that string theory compactifications can be well-defined even when the compact space is a conifold with isolated singularities as long as we include all relevant states in the effective theory. The second is simply the converse statement — if we want an effective theory that is well-defined without these D-brane massless states, we cannot reach the conifold point.

3.5.3 Deforming the conifold

Let us now describe the geometry near a conifold singularity. Generically, a conifold as originally defined [80] may have a number of isolated singularities, for each of which we can give a local description of the geometry, although for the phenomenological applications we are interested in only one such singularity is relevant. Locally they can be defined by the equation

$$\sum_{i=1}^{4} w_i^2 = 0\,, \tag{3.93}$$

where $w_i \in \mathbb{C}^4$. That this describes a cone can be seen from the fact that when w_i satisfies this condition, so does λw_i for any $\lambda \in \mathbb{C}$ (we could think of w_i as coordinates in $\mathbb{CP}^4$). To determine the base of the cone, we intersect it with a sphere in $\mathbb{C}^4$ of radius r,

$$\sum_{i=1}^{4} |w_i|^2 = \vec{x}\cdot\vec{x} + \vec{y}\cdot\vec{y} = r^2\,, \tag{3.94}$$

with $\vec{w} = \vec{x} + i\vec{y} = (w_1, w_2, w_3, w_4)$. In terms of $\vec{x}$ and $\vec{y}$, the two conditions intersect at

$$\vec{x}\cdot\vec{x} = \frac{r^2}{2}\,,\quad \vec{y}\cdot\vec{y} = \frac{r^2}{2}\,,\quad \vec{x}\cdot\vec{y} = 0\,, \tag{3.95}$$

which describes an S^3 (first equation) with an S^2 fibered over it (remaining two equations). The base of the cone is therefore topologically $S^3 \times S^2$, which nicely connects with our previous discussion of how one can get a singular conifold either by collapsing an S^3 or an S^2 to zero size. Indeed, as $r \to 0$ the radius of the S^3 and S^2 in the base vanishes. More precisely, the base corresponds to the $T^{1,1} = \frac{\mathrm{SU(2)\times SU(2)}}{\mathrm{U(1)}}$ manifold for which we can write the Einstein metric [80, 81]

$$\begin{aligned} ds^2_{T^{1,1}} = {} & \frac{1}{9}(d\psi^2 + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2 \\ & + \frac{1}{6}(d\theta_1^2 + \sin^2\theta_1 d\psi_1^2) + \frac{1}{6}(d\theta_2^2 + \sin^2\theta_2 d\psi_2^2)\,, \end{aligned} \tag{3.96}$$

with $0 \le \theta_i < \pi$, $0 \le \phi_i < 2\pi$ and $0 \le \psi < 4\pi$. The metric on the conifold is therefore

$$ds^2_{\mathrm{con}} = dr^2 + r^2 ds^2_{T^{1,1}}\,, \tag{3.97}$$

which is Ricci-flat and becomes singular at $r = 0$. The metric on $T^{1,1}$ can be made diagonal

$$ds^2_{T^{1,1}} = \frac{1}{9}(g^5)^2 + \frac{1}{6}\sum_{i=1}^{4}(g^i)^2\,, \tag{3.98}$$

in terms of the 1-forms [85]

$$g^1 = \frac{e^1 - e^3}{\sqrt{2}}\,,\qquad g^2 = \frac{e^2 - e^4}{\sqrt{2}}\,,\qquad g^3 = \frac{e^1 + e^3}{\sqrt{2}}\,,\qquad g^4 = \frac{e^2 + e^4}{\sqrt{2}}\,,\qquad g^5 = e^5\,, \tag{3.99}$$

where

$$\begin{aligned} e^1 &= -\sin\theta_1 d\phi_1\,, \\ e^2 &= d\theta_1\,, \\ e^3 &= \cos\psi\sin\theta_2 d\phi_2 - \sin\psi d\theta_2\,, \\ e^4 &= \sin\psi\sin\theta_2 d\phi_2 + \cos\psi d\theta_2\,, \\ e^5 &= d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2\,. \end{aligned} \tag{3.100}$$

This basis is particularly useful to discuss the metric on the deformed conifold. Deforming the conifold corresponds to changing the defining equation (3.93) in such a way that the S^3 in the corresponding base always has a finite radius. This is achieved by including a parameter ϵ as

$$\sum_{i=1}^{4} w_i^2 = \epsilon^2 \,, \tag{3.101}$$

in which case the equations defining the base of the cone become

$$\vec{x}\cdot\vec{x} = \frac{r^2+\epsilon^2}{2}\,, \qquad \vec{y}\cdot\vec{y} = \frac{r^2-\epsilon^2}{2}\,, \qquad \vec{x}\cdot\vec{y} = 0\,, \tag{3.102}$$

so that $\vec{y}\cdot\vec{y} \geq 0$ requires $r \geq \epsilon$ and the minimum radius of the S^3 is now $r_{\text{min}} = \epsilon$. This effectively removes the conifold singularity by replacing it by an S^3 of finite size, which is exactly how we introduced the notion of deformation. It means in particular that the manifold defined through (3.101) is not technically a conifold, but instead a smooth manifold associated with a point in the moduli space on the right in Fig. 3.3. Since sending $\epsilon \to 0$ recovers the definition of a conifold, this parameter must be related to the complex structure modulus $z^1 \to 0$. The metric on the deformed conifold takes the form [81, 85, 86]

$$\begin{aligned} ds^2_{\text{def con}} = \frac{\epsilon^{4/3}}{2}\mathcal{K}(\tau)\Bigg(&\frac{1}{3\mathcal{K}^3(\tau)}\left(d\tau^2 + (g^5)^2\right) + \sinh^2(\tau/2)\left((g^1)^2+(g^2)^2\right) \\ &+ \cosh^2(\tau/2)\left((g^3)^2+(g^4)^2\right)\Bigg)\,, \end{aligned} \tag{3.103}$$

where the function $\mathcal{K}$ only depends on the radial coordinate τ and is defined as

$$\mathcal{K}(\tau) = \frac{(\sinh(2\tau)-2\tau)^{1/3}}{2^{1/3}\sinh\tau}\,. \tag{3.104}$$

The fact that the metric only depends on τ is reflecting the symmetries of the deformed conifold whose base is still topologically $S^3\times S^2$. For large τ, the metric (3.103) approaches the conifold metric (3.97), with $r^2 = \frac{3}{2^{5/3}}\epsilon^{4/3}e^{2\tau/3}$. On the other hand, for small τ the metric approaches

$$ds^2_{\text{def con}} = dr^2_{\ll} + \frac{r^2_{\ll}}{8}\big[\underbrace{(g^1)^2 \;+\; (g^2)^2}_{S^2}\big] + R^2_\epsilon\big[\underbrace{(g^3)^2+(g^4)^2+\tfrac{1}{2}(g^5)^2}_{S^3}\big]\,, \tag{3.105}$$

where $r^2_{\ll} = \frac{\epsilon^{4/3}}{4}\left(\frac{2}{3}\right)^{1/3}\tau^2$ and the radius of the S^3 is fixed by ϵ as $R^2_\epsilon = \frac{\epsilon^{4/3}}{2}\left(\frac{2}{3}\right)^{1/3}$. In principle, the metric (3.103) describes a non-compact Calabi-Yau manifold and one could ask whether it may arise as a background of Type IIB — this would require the deformed conifold metric to solve the equations of motion for all Type IIB fields for a given configuration of sources and/or fluxes. Such a solution was indeed found in [86], but what lead to its discovery was not the web of Calabi-Yau moduli spaces that originally motivated the study of conifolds.

Despite their origin in the context of topology change and boundaries in moduli space [66, 80, 81], what made conifolds explode in the string theory literature was the proposal of AdS/CFT [87], which originally described the duality between Type IIB string theory on $\mathrm{AdS}_5 \times S^5$ and the $\mathcal{N} = 4$ SU(N) theory which was the low-energy limit of the world-volume theory on a stack of N D3-branes. This gave an explicit realisation of earlier holographic proposals [88, 89] and kick-started a massive exploration of string theory backgrounds and their possible dual field theories. A natural generalisation was to replace S^5 by a 5d Einstein manifold X_5, which was argued to also be related to 4d conformal field theories. Requiring that this new background broke some (but not all) of the supersymmetries, a dual field theory was first identified precisely for the case where $X_5 = T^{1,1}$, by noting that this theory was the low-energy limit of the world-volume theory on a stack of D3-branes placed at the conifold singularity [90]. Further exploration of this duality, both on the supergravity side and from the field theory perspective, and in particular of the way the two were mapped onto each other, lead to the introduction of fractional D3-branes into the original background of N D3-branes at the singularity [91–93]. A fractional D3-brane is actually a D5-brane which wraps a collapsed 2-cycle in the internal space — on a conifold background, the D5-brane can wrap the S^2 which collapses to zero size at the singularity. In the field theory, adding these fractional branes breaks the conformal symmetry and the gauge couplings start to flow logarithmically with energy scale; this is related to the supergravity side through a logarithmic warping of the conifold, which would eventually reach a metric singularity. However, it was conjectured that this singularity could be resolved by the strong dynamics of the gauge theory. What happens, then, in the supergravity background? In [86], the conjecture was proven and in the supergravity side this was shown to correspond to the deformation of the conifold.

3.5.4 Klebanov-Strassler solution

The background found in [86] is the warped deformed conifold that will form the basis of our phenomenological explorations. It has a non-singular metric, whose warp factor is well behaved everywhere and leads to an exponential suppression of scales in the 4d theory. The setup consists of M D5-branes wrapping the S^2 of $T^{1,1}$ at the tip of the cone, which act as sources of F_3 flux,

$$\frac{1}{(2\pi)^2\alpha'}\int_{S^3_{\mathrm{tip}}} F_3 = M\,. \tag{3.106}$$

The flux through the 3-sphere created by the M D5-branes is what prevents it from collapsing. From the supergravity point of view, one may replace the D5-branes brane by M units of F_3 flux through the S^3. A solution can then be found for a metric that takes the conformal Calabi-Yau form

$$ds^2_{10} = H^{-1/2}(y)g_{\mu\nu}dx^\mu dx^\nu + H^{1/2}(y)ds^2_{\mathrm{def\ con}}\,, \tag{3.107}$$

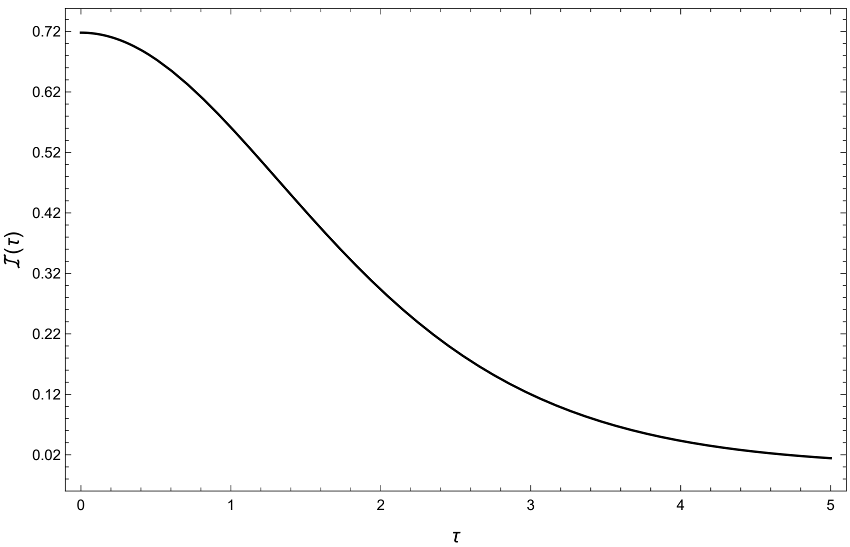


Figure 3.5: Plot of the function $I(\tau)$ in (3.108) controlling the behaviour of $H(\tau)$ for fixed values of ϵ and $g_s M$ — the warping will be stronger for larger values of M and smaller values of ϵ. In particular, we will rely on exponentially small values of ϵ later on in order to get exponentially large warping near the tip of the deformed conifold. Note that the warp factor is biggest at $\tau = 0$ (tip), where it is regular, and decays exponentially for large τ.

with a warp factor given by

$$H(\tau) = 2^{2/3} \frac{(\alpha' g_s M)^2}{\epsilon^{8/3}} I(\tau) \,, \qquad I(\tau) \equiv \int_\tau^\infty dx \frac{x \coth(x) - 1}{\sinh^2 x} (\sinh(2x) - 2x)^{1/3} \,. \tag{3.108}$$

This warp factor depends on the F_3 flux and the deformation parameter ϵ, but for fixed non-zero values of these parameters it is regular at $\tau = 0$ (Fig. 3.5). The dilaton is assumed to be constant, $e^\phi = g_s$, and H_3 is related to F_3 as

$$H_3 = -g_s \star_6 F_3 \,, \tag{3.109}$$

in a way which is compatible with a self-dual $(2,1)$ G_3 flux (3.63) of an $\mathcal{N} = 1$ theory with a conformal Calabi-Yau background.

Note that the physical size of the S^3 at the tip of the throat is $R^2_{S^3} = H^{1/2} R^2_\epsilon$, which for this solution does not depend on the deformation parameter ϵ,

$$R^2_{S^3} \left(\frac{2^{2/3}}{3^{1/3}} I^{1/2}(\tau = 0) \right) (g_s M) \alpha' \approx (g_s M) \alpha' \,, \tag{3.110}$$

from which it follows that a well under control supergravity approximation (i.e. such that the α'-expansion with which the Type IIB low-energy supergravity was derived is under control) requires $g_s M \gg 1$. This consistency condition will have an important role in both chapter 4 on de Sitter constructions and chapter 5 on gravitational signatures of warped compactifications.

The warped deformed conifold in the Klebanov-Strassler (KS) solution [86] is non-compact.

From the theoretical point of view, this would be perfectly acceptable and correspond to a 10d solution of Type IIB. For this 10d solution, ϵ is just the parameter fixing the complex structure of the deformed conifold. However, for phenomenology we typically require the 6d space to be compact[28] and therefore this geometry is assumed to give a local description of a small region of a compact space. This region is usually referred to as a *warped throat* within the compact space, whose warping creates a hierarchy of scales between the tip and the far away unwarped region typically called the bulk. A solution of this type was proposed in [94] where it was shown that the modulus associated with the complex structure deformation ϵ gets stabilised at exponentially small values, providing exponentially large hierarchies.

3.6 Hierarchies from fluxes

3.6.1 GKP solution

We can finally describe the warped solution on which later chapters are based. It became known as the GKP solution after Giddings, Kachru and Polchinski, who first proposed it [94], and consists of a flux compactification of Type IIB string theory, whose compact space contains a warped throat — a region which is well described by the warped deformed conifold and therefore the KS solution. Moreover the solution stabilises the complex structure modulus associated with the deformation of the conifold at exponentially small values with F_3 and H_3 fluxes through the S^3 at the tip.

The GKP solution was motivated by the hierarchy problem and was therefore attempting to provide a construction in which the Planck scale M_{Pl} was connected to the much lower electroweak scale M_{EW} in a dynamical way. It partly drew inspiration from earlier proposals where warping was responsible for this connection [57, 58] and where warped metrics would arise in the context of string compactifications [95], while going further in explaining how the moduli involved in warped solutions could be stabilised in such a way as to actually provide the large hierarchy they tried to achieve. Although some work on the stabilisation of these moduli existed [96, 97], GKP provided the first string compactification realisation of an exponential hierarchy with a stabilised modulus.

The starting point for the GKP solution is the Type IIB supergravity action (2.81) supplemented with the action for localised sources such as D-branes and O-planes that we introduced at the end of Section 2.3, together with a warped ansatz for the metric respecting 4d Poincaré invariance,[29]

$$ds^2 = e^{2A(y)}\eta_{\mu\nu}dx^\mu dx^\nu + e^{-2A(y)}\tilde{g}_{mn}dy^m dy^n \,. \tag{3.111}$$

[28] As mentioned before, braneworld theories could potentially lift the requirement that the 6d space is compact and small.

[29] For this discussion we will use the notation of [94] with the warp factor denoted as $H(y) = e^{-4A(y)}$ so as to facilitate comparison. We can then easily change notation back to $H(y)$.

Importantly, Poincaré invariance only allows the background G_3 flux to have internal components, while the most general form the $\tilde{F}_5$ background is allowed to take is

$$\tilde{F}_5 = (1+\star)\, \mathrm{d}\alpha \wedge \mathrm{vol}_4 \,, \tag{3.112}$$

with $\alpha = \alpha(y)$ a function of the compact space coordinates y^m only and vol_4 the volume form on the (unwarped) 4d spacetime. The axio-dilaton $\tau = \tau(y)$ will only be allowed to vary in the compact directions. Using these in the Einstein equations (2.88) supplemented with the contribution from the localised sources (2.98),[30] the 4d components become[31]

$$\begin{aligned} R_{\mu\nu} &= -g_{\mu\nu}\left(\frac{g_s}{8}\frac{|G_3|^2}{(\mathrm{Im}\,\tau)} + \frac{g_s^2}{4}e^{-8A}(\partial_m\alpha)(\partial^m\alpha)\right) + \kappa^2\Big(T^{\mathrm{loc}}_{\mu\nu} - \frac{1}{8}g_{\mu\nu}T^{\mathrm{loc}}\Big) \\ &\overset{!}{=} -\frac{1}{4}\eta_{\mu\nu}\Big(\tilde{\nabla}^2 e^{4A} - e^{-4A}(\partial_{\tilde{m}}e^{4A})(\partial^{\tilde{m}}e^{4A})\Big)\,, \end{aligned} \tag{3.113}$$

where the second line is $R_{\mu\nu}$ computed for the metric ansatz (3.111). Tracing the equation, this implies

$$\tilde{\nabla}^2 e^{4A} = e^{2A}\frac{g_s}{2}\frac{|G_3|^2}{(\mathrm{Im}\,\tau)} + e^{-6A}\big[g_s^2(\partial_m\alpha)^2 + (\partial_m e^{4A})^2\big] + \frac{\kappa^2}{2}e^{2A}(T^m{}_m - T^\mu{}_\mu)^{\mathrm{loc}}\,. \tag{3.114}$$

Since our internal space is compact, integrating the left-hand side over it gives zero. This simple fact happens to have important consequences for the allowed configurations of fluxes and localised sources that would give a warped Minkowski solution with a compact 6d space. In particular, since the flux and warping terms on the right-hand side are positive definite, they can only be non-zero if the localised source contribution is negative and able to cancel them. Therefore, in the absence of localised sources, fluxes must vanish and the warping is constant — this constitutes the no-go theorem of [98]. If we want to find a warped solution we need to include in the construction the right kind of localised object.

In order to preserve Poincaré invariance, these objects must extend along the 4d spacetime. Hence a Dp-brane or Op-plane will extend along these 4 directions and wrap $p-3$ directions in the compact space. Using (2.98) we find

$$(T^m{}_m - T^\mu{}_\mu)^{\mathrm{loc}} = (7-p)T_p\; e^{-\Phi}\delta^{9-p}(X^i)\,, \tag{3.115}$$

which tells us that a negative contribution can only come from objects with $p > 7$ or negative tension. Having a negative tension is precisely one of the interesting properties of O-planes. Due to the existence of orientifolds, string theory does have a way of evading the no-go theorem of [98]. From our discussion of supersymmetry in Section 3.4, we can start to see the similarities between the GKP solution and the type B $\mathcal{N}=1$ solutions. The objects that preserve the same

[30] Note that the Einstein equations (2.88) are in trace reversed form, so that this energy-momentum tensor will contribute with a term $\kappa^2\left(T_{MN} - \frac{1}{8}G_{MN}T^P{}_P\right)$.

[31] The extra factors of g_s follow from our choice of conventions for the Einstein frame metric with $\Phi_0 = \langle\Phi\rangle$, which differs from the one in [94], where $\Phi_0 = 0$ (see Appendix A.1).

supersymmetry as the flux configuration in these solutions are D3/D7-branes, O3/O7-planes and fractional D5-branes. We should therefore consider O3-planes as the objects providing negative tension. Note also that D7-branes do not contribute to this constraint, at least at leading order in α' [94].

Importantly, while (3.114) can be satisfied with the inclusion of O-planes and guarantee that the Einstein equations are solved, we must also solve the equations of motion for all other fields. In particular, we must solve the equation of motion for $\tilde{F}_5$ which receives contributions from D3-branes and O3-planes (2.100),

$$\mathrm{d} \star \tilde{F}_5 = \frac{i}{2} \frac{G_3 \wedge \overline{G}_3}{(\mathrm{Im}\, \tau)} + \frac{2\kappa^2}{g_s^2} \sum_i Q_3^{(i)} \; \delta_{\mathcal{W}} \, . \tag{3.116}$$

It will thus give rise to a tadpole cancellation condition (2.101),

$$\frac{i g_s^2}{4\kappa^2} \int_{M_6} \frac{G_3 \wedge \overline{G}_3}{(\mathrm{Im}\, \tau)} + \sum_i Q_3^{(i)} = 0 \, , \tag{3.117}$$

known as the D3-charge tadpole cancellation and often conveniently expressed in terms of H_3 and F_3 fluxes instead,

$$\frac{g_s^2}{2\kappa^2} \int_{M_6} H_3 \wedge F_3 + \sum_i Q_3^{(i)} = 0 \, . \tag{3.118}$$

Actually, D7-branes wrapping 4-cycles in the compact space can also contribute with an effective charge which in F-theory [99] is given by $Q_3^{\mathrm{eff}} = -\frac{\chi(\mathrm{CY}_4)}{24}$, in terms of the Euler-characteristic of the corresponding Calabi-Yau 4-fold [94]. Although these contributions can be invoked in order to cancel the tadpole, they come with constraints of their own related to the growing number of moduli one will need to stabilise (presumably using fluxes) when one tries to increase Q_3^{eff} [100–106].

The equation of motion itself is an equation for the function α in $\tilde{F}_5$, in terms of which it can be written as

$$\tilde{\nabla}^2 \alpha = e^{2A} \frac{i}{2} \frac{G_{mnp} \star_6 \overline{G}^{mnp}}{6(\mathrm{Im}\, \tau)} + 2 e^{-6A} (\partial_m \alpha)(\partial^m e^{4A}) + \frac{2\kappa^2}{g_s^2} e^{2A} \sum_i Q_3^{(i)} \; \delta_{\mathcal{W}} \, , \tag{3.119}$$

which can be subtracted from the Einstein equation constraint (3.114) to give[32]

$$\begin{aligned} \tilde{\nabla}^2 \big(e^{4A} - g_s \alpha\big) = {} & \frac{g_s}{6(\mathrm{Im}\, \tau)} e^{2A} |i G_3 - \star_6 G_3|^2 + e^{-6A} |e^{4A} - g_s \alpha|^2 \\ & + \frac{2\kappa^2}{g_s^2} e^{2A} \left[\frac{g_s}{4} (T^m{}_m - T^\mu{}_\mu)^{\mathrm{loc}} - T_3 \rho_3^{\mathrm{loc}} \right] \, , \end{aligned} \tag{3.120}$$

where we introduce $T_3 \rho_3^{\mathrm{loc}} = \sum_i Q_3^{(i)} \; \delta_{\mathcal{W}}$ in the notation of [94]. It was argued in [94] that

[32] Notice once again the factors of g_s differing from [94], which follow from the Einstein frame convention.

many localised objects satisfy the inequality $\frac{g_s}{4}(T^m{}_m - T^\mu{}_\mu)^{\rm loc} \geq T_3\rho_3^{\rm loc}$, so that the last term is always non-negative, just as the first two terms. This is true in particular for D3/D7-branes and O3-planes, which are the localised objects we are interested in. As a consequence, running the same argument as before and integrating (3.120) over the compact space, we find that

$$\star_6 G_3 = iG_3 \qquad\qquad e^{4A} = g_s\alpha\,. \tag{3.121}$$

The G_3 flux must be imaginary-self-dual and the warp factor is determined by the $\tilde{F}_5$ background — these are (some of) the same conditions satisfied by the type B solutions we discussed in Section 3.4. If the localised source inequality holds, a solution actually requires it to be saturated, which happens for D3/D7-branes and O3-planes. Note also that an imaginary-self-dual G_3 flux has important consequences, as it implies that the flux contribution to the D3-charge tadpole (3.117) is always positive. Hence we must use O3-planes to cancel it, since the alternative $\overline{\rm D3}$-branes would break supersymmetry.

The discussion so far was all about global properties of the flux compactification. To achieve large hierarchies, one can "zoom in" on a region of the compact space that can be described by the KS solution, with its deformed conifold and M units of F_3 flux. In contrast with the non-compact KS solution however, in GKP the deformed conifold in "glued" to a compact Calabi-Yau orientifold and therefore the cycle dual to the S^3 at the tip is now finite. The presence of the (unspecified) compact Calabi-Yau means that also the H_3 flux through this cycle can be generically given by K, such that overall we have

$$\frac{1}{(2\pi)^2\alpha'}\int_A F_3 = M \quad \text{and} \quad \frac{1}{(2\pi)^2\alpha'}\int_B H_3 = K\,. \tag{3.122}$$

We already know that the A cycle (the S^3) is associated with a complex structure modulus[33]

$$z = \frac{1}{Z_0}\int_A \Omega \propto \epsilon\,, \tag{3.123}$$

and we can use Z_0 to fix $|z| = \epsilon$. This makes explicit the fact that the deformation parameter ϵ is really a deformation modulus in the 4d theory and is therefore left undetermined unless a potential is generated by the fluxes. This is precisely what the non-trivial fluxes M and K through the A and B cycles will do. Applying our previous results (3.81) and (3.89) to this setup, we find

$$W = \frac{g_s^{3/2}}{\sqrt{4\pi}\cdot l_s^5}\cdot(2\pi)^2\alpha'\left(-\frac{M}{2\pi i}z\log z - K\tau z + \text{holomorphic}\right)\,. \tag{3.124}$$

Since we are not adding any corrections to the flux superpotential, the Kähler moduli remain as flat directions and satisfy the no-scale structure. Since the potential (3.83) is then positive-

[33]See Appendix B of [107] for the relation between ϵ and Ω that leads to this identification.

definite, the minimum occurs at $D_i W = 0$, which includes in particular

$$\begin{aligned} D_z W &\propto -\frac{M}{2\pi i}(1+\log z) - K\tau + \text{holomorphic} + (\partial_z K)W \\ &\approx i\left\{\frac{M}{2\pi}\log z - \frac{K}{g_s}\right\} + \mathcal{O}(1) \stackrel{!}{=} 0\,, \end{aligned} \tag{3.125}$$

if we assume that $\epsilon = |z| \ll 1$, which we want in order to get a large warp factor (3.108), and that $\langle\tau\rangle = i g_s^{-1}$ with $g_s \ll 1$. The potential is then minimised at

$$z \approx e^{-\frac{2\pi K}{g_s M}} \ll 1\,, \tag{3.126}$$

as long as the ratio of fluxes is large enough. This will indeed provide an exponentially large warp factor and therefore exponentially large hierarchies between different points in the compact space, and is at the basis of many attempts to obtain a de Sitter solution in Type IIB flux compactifications. These typically start as AdS solutions where only the complex structure moduli and the dilaton are stabilised, while the Kähler moduli are flat directions, and add certain corrections to the scalar potential in order to stabilise them. A de Sitter vacuum is then achieved by introducing an $\overline{\text{D3}}$-brane at the tip of the warped throat, which breaks supersymmetry and contributes with a warped down vacuum energy. A large hierarchy is usually required in order to preserve the minimum for the Kähler moduli once the brane is introduced.

3.6.2 The warped background metric

Before moving on, let us be more explicit with regard to the background metric and, in particular, the sense in which the deformed conifold is "glued" onto the compact Calabi-Yau. We will consider vacuum solutions whose background is a warped product spacetime $\mathcal{M}_{10} = \mathbb{R}^{1,3} \times_{\text{w}} X_6$, where $\mathbb{R}^{1,3}$ is a 4d Lorentzian spacetime and X_6 is a 6d compact space. We write the Einstein frame metric as[34]

$$ds_{10}^2 = H^{-1/2}(y)\; e^{2\omega(x)} g_{\mu\nu} dx^\mu dx^\nu + H^{1/2}(y)\; \mathcal{V}^{1/3} g_{mn} dy^m dy^n\,, \tag{3.127}$$

where x^μ ($\mu = 0, ..., 3$) are 4d spacetime coordinates and y^m ($m = 4, ..., 9$) are 6d coordinates on the compact space X_6, which we will take to be a Calabi-Yau orientifold (CY_3). The 6d metric $g_{mn} = (g_6)_{mn}$ is then Ricci-flat and normalised such that

$$\int d^6 y \sqrt{g_6} \equiv l_s^6,$$

[34]Since we start with Einstein frame metric (A.15) and action (A.17), the volumes $\mathcal{V}$ and $\mathcal{V}_{\text{w}}$ are Einstein frame volumes.

with $\mathcal{V} = \mathcal{V}_E(x)$ keeping track of the physical size of the compact space. We define the warp factor H as

$$H(y) \equiv 1 + \frac{e^{-4A_0(y)}}{\mathcal{V}^{2/3}}\,, \tag{3.128}$$

which is motivated as follows.

First, the background warp factor in (3.111) that solves the 10d Einstein equations in the presence of fluxes (3.114) is only fixed up to a constant shift,

$$e^{-4A(y)} = e^{-4A_0(y)} + c\,.$$

This can be seen as an integration constant depending on the boundary conditions — when we say that the warped deformed conifold is glued onto the compact (unwarped) CY_3, we mean that far away from the tip ($\tau \to \infty$), the warping should vanish or, in other words, the function $e^{-4A(y)} \to \texttt{const}$. In this limit, it becomes an overall rescalling of the compact CY_3 and must therefore be related to the overall volume modulus $\mathcal{V}$ [108]. Moreover, the fact that $g_{mn} \to \lambda g_{mn}$ together with $e^{2A} \to \lambda e^{2A}$ is a gauge redundancy of the metric (3.111) [94, 109, 110] allows us to choose $\lambda = c^{1/2}$ and rewrite $e^{-4A(y)} = 1 + \frac{e^{-4A_0(y)}}{c}$, which naturally recovers the unwarped case in the $c \to \infty$ limit and indeed relates $c = \mathcal{V}^{2/3}$ with the unwarped volume of the CY_3.

The factor $e^{2\omega(x)}$ is introduced to Weyl rescale to the 4d Einstein frame, with metric $g_{\mu\nu}$. In Appendix A.2, we explicitly go through this change of frames and discuss how the dimensional reduction relates the string scale to the Planck scale as

$$m_s \approx \frac{g_s}{\sqrt{4\pi\mathcal{V}}} M_{\mathrm{Pl}}\,. \tag{3.129}$$

We also introduce other important scales (e.g. M_{KK} and $M_{\mathrm{KK}}^{\mathrm{tip}}$) and highlight the convention independence of physical mass ratios.

The background metric (3.127) will form the basis of the main two chapters of this thesis. In chapter 4 we will discuss Type IIB de Sitter solutions relying on uplifts, introducing the KKLT [75] and LVS [76, 77] proposals, and presenting new work in the context of LVS where a weakly-warped solution can be found that seems to help us circumvent the Tadpole Problem [100–106]. We will explore this solution and discuss whether potentially dangerous corrections might prevent us from trusting it. Then, in chapter 5 we will explore the phenomenology of these warped backgrounds in the context of gravitational interactions. We will discuss their effects on fifth forces arising from KK towers of spin-2 states and begin an exploration of gravitational waves in the presence of strong warping.

4. Searching for de Sitter vacua

> Large numbers may intimidate, but they should not scare. What they should provoke is not flight, but instead reflection on what is the right question to ask.
>
> Joseph Conlon, *Why String Theory?*

4.1 Dark Energy: What is it made of?

Towards the end of the 20^{th} century, as astronomical observations brought more and better data regarding the evolution of the Universe, it was becoming increasingly clear that a cosmological constant capable of accelerating its expansion — if there was one — had to be rather small. So small, in fact, that it disagreed with estimates one could make within particle physics by many orders of magnitude. This discrepancy came to be known as the cosmological constant problem [111]. At the heart of this problem was the vacuum energy of quantum fields, which can contribute to the Einstein equations as a cosmological constant. At one loop, this contribution depends on the masses of the particles running in the loops and it gives a much larger contribution from the Standard Model particles alone than what would be allowed by observations — it would therefore require an extremely fine-tuned cancellation between this vacuum energy and a genuine cosmological constant Λ. Yet, it is not the fine-tuning *per se* that constitutes the cosmological constant problem, but rather its extreme sensitivity to UV physics reflected by a radiative instability — trying to improve the computation by including higher loops will bring important corrections and more extreme fine-tuning would be required at every order [111–113].

What if the vacuum energy of quantum fields could precisely cancel and the cosmological constant was exactly zero? Supersymmetry, for example, provides a mechanism to cancel the vacuum energy of bosons with that of their fermionic partners, resulting in a vanishing contribution. If the cosmological constant had to be so small, zero might have been the most practical and perhaps even natural value for Λ. Tempting as it may seem, nature had other ideas — shortly before the century came to an end, measurements of Type IA Supernovae (SNIa) showed that the expansion of the Universe is accelerating [114, 115]. Rather than zero, a cosmological constant (or something behaving very much like one) was required at a scale $10^{-120}M_{\rm Pl}^4 \sim (10^{-3}\ {\rm eV})^4$. Within the Standard Model of Cosmology (ΛCDM), such a cosmological constant — or, more generally, what one calls Dark Energy — should account for $\sim 70\%$ of the current energy budget of the Universe [116].

Its extreme sensitivity to the UV suggests that an explanation of Dark Energy might require a UV complete theory of gravity, such as string theory. Although not a cure to the cosmological constant problem on its own, supersymmetry which is an important part of string theory could prove useful in addressing the problem. The extremely small scale of Dark Energy compared to scales like $M_{\rm Pl}$, M_s or even the TeV scale that we can currently probe, makes this a problem of hierarchies where warped compactifications could be key. There are two questions one should answer with regard to the cosmological constant:

1. How does it get its measured (and extremely small) *positive* value, $\Lambda \sim +10^{-120}M_{\rm Pl}^4$?
2. How is it protected from large corrections?

Although specific approaches to the problem may link the two questions together, they are often addressed individually. In the work we shall present in this chapter, we will only touch on the first (and within it restrict our attention to the + sign) — we will explore the possibility of getting a de Sitter vacuum in string theory. At the level of a classical field theory description, this corresponds to a vacuum solution with a scalar potential which is positive at the minimum and can therefore contribute as a positive cosmological constant

$$S \sim -\int d^4x\ \sqrt{-g}\ V(\varphi^i)\big|_{\rm min} \sim -\int d^4x\ \sqrt{-g}\ \Lambda\,. \tag{4.1}$$

Such an understanding of Dark Energy in string theory therefore requires an understanding of moduli stabilisation. We have already encountered hints of no-go theorems [87] forbidding de Sitter supergravity solutions in the absence of negative tension objects and we have seen that in an $\mathcal{N}=1$ supergravity framework, a de Sitter vacuum must necessarily spontaneously break supersymmetry. This connects the problem of moduli stabilisation with the one of supersymmetry breaking, which is anyway necessary to reproduce observations up to the TeV scale. On its own, moduli stabilisation in string theory is already quite tricky — we have seen, for example, how fluxes can stabilise the complex structure moduli and the axio-dilaton in Type IIB compactifications, but leave all Kähler moduli unfixed and massless. Moreover, it was shown early on by Dine and Seiberg [117] that, in regimes where both the string-loop expansion (controlled by g_s) and the α' expansion (controlled by the volume of the compact space) are well under control, finding a local minimum for the dilaton and the volume modulus requires fine-tuning between terms appearing at different orders in these expansions.

On the other hand, the role of fluxes in string compactifications actually led to the picture of a landscape of vacua [118], so vast that one might hope would contain our own Universe within it — if one is allowed to explore this string Landscape, the odd fine-tuning of Λ might be explained through an anthropic reasoning, in terms of which we as observers (rather than the fine-tuned solution) are the oddity. These flux compactifications also provided ways to produce hierarchies [86, 94], which are crucial ingredients in some of the best proposals for de Sitter vacua in string theory [75–77].

Nevertheless, the proposed constructions are usually so close to the boundaries of control that consensus over their status within string phenomenology has yet to be obtained (see e.g. [119–121] for reviews). This lead to the proposal that de Sitter vacua might be in the so-called Swampland of effective theories[1] [125, 126], which despite looking healthy at low-energies do not admit a UV completion into a theory of quantum gravity [127]. These difficulties might tempt us to consider alternatives to exact de Sitter vacua — among them are quasi-de Sitter solutions described by scalars fields slowly-rolling in a positive region of the scalar potential. These so-called quintessence models would provide an effective cosmological constant slowly varying in time [128]. However, over the years more evidence was gathered suggesting that quintessence models are also extremely hard to realise in string theory [120, 129–132]. We shall have more to say about quintessence at the end of this chapter.

4.2 Kähler moduli stabilisation: KKLT and LVS

In order to look for de Sitter solutions, we will focus on Type IIB compactifications with fluxes and in particular the GKP type of solutions [94] with a region described by a deformed conifold [93]. As we have summarised in the previous chapter, the low-energy effective theory is $\mathcal{N} = 1$ supergravity and contains a large number of massless scalar fields, out of which only the complex structure moduli and the axio-dilaton can be stabilised by the fluxes through the GVW superpotential (3.124). Even if the stabilised moduli do not break supersymmetry, the Kähler moduli that are flat directions at this level will do so unless the superpotential vanishes and the solution is Minkowski.

Our task, if we want a vacuum with full moduli stabilisation, is to find a potential for these flat directions. Since the potential so far considered was just the leading term in different expansions, one may look for corrections at sub-leading order in these expansions which would automatically provide the dominant terms for Kähler moduli. There are indeed perturbative corrections to the Kähler potential that can break the no-scale structure and give a non-trivial contribution.[2] It can also in general receive non-perturbative corrections. In contrast, the holomorphic superpotential is protected from perturbative corrections and is therefore independent of the Kähler moduli to

[1] See [122–124] for reviews of the Swampland programme.

[2] Note that the Kähler potential for the moduli fields arises from dimensional reduction of the Type IIB supergravity action (2.76) which was itself obtained at leading order in g_s (string loop expansion) and in α' (curvature corrections).

all orders in perturbation theory [72–74]. However, it can receive non-perturbative corrections from effects such as gaugino condensation on D7-branes wrapping 4-cycles in the compact space or Euclidean D3-brane instantons [133] which depend on Kähler moduli. The full Kähler and super potentials including these corrections are given by

$$\mathcal{K} = K_{\text{c.s.}} + K_{\text{Kähler}} + K_\tau + K_{\text{p}}\,, \tag{4.2}$$

$$W = W_{\text{flux}} + W_{\text{np}}\,, \tag{4.3}$$

where the Kähler potential for the axio-dilaton is

$$K_\tau = -\log(\tau - \bar{\tau})\,. \tag{4.4}$$

The Kähler potential $K_{\text{Kähler}}$ can also be written in terms of the volume of the compactification $V = \mathcal{V} \cdot l_s^6$ as

$$K_{\text{Kähler}} = -2\log\mathcal{V}\,, \tag{4.5}$$

with the $\mathcal{V}$ being a function of the Kähler moduli. We are now in a position to introduce the KKLT [75] and LVS [76, 77] proposals. Although they rely on different combinations of ingredients, both achieve stabilisation in two steps, first fixing the complex structure moduli and axio-dilaton using fluxes, and integrating them out, and then taking into account corrections to the flux potential in order to stabilise the Kähler moduli at subleading order. This logic relies on the fact that often the masses generated by fluxes for the former are much larger than the scale at which the latter are stabilised, since this is done at subleading order in the scalar potential (Fig. 4.1). The difference in the two proposals is in the subleading corrections that generate a potential for the Kähler moduli.

4.2.1 KKLT

The KKLT construction[3] adds non-perturbative contributions to the superpotential and requires the vacuum expectation value of the flux term to be small enough so that it can compete (as a constant) with the non-perturbative correction, which requires a fine-tuning of $W_0 \equiv \langle W_{\text{flux}} \rangle \ll 1$.[4] In [75] the simple example of a single Kähler modulus T is given, for which a superpotential is generated by either gaugino condensation on D7-branes[5] ($a = \frac{2\pi}{N}$, with N the rank of the gauge group on the D7-branes) or Euclidean D3-brane instantons ($a = 2\pi$),

$$W_{\text{KKLT}} = W_0 + Ae^{-aT}\,, \tag{4.6}$$

[3] Named after S. Kachru, R. Kallosh, A. Linde and S. Trivedi, who first proposed it.

[4] See [134] for a proposal of a de Sitter construction that does not require supersymmetry-breaking fluxes W_0 other than those generated by the addition of $\overline{\text{D3}}$-branes at the tip of the KS throat.

[5] The supersymmetry-breaking fluxes generated by the addition of $\overline{\text{D3}}$-branes at the tip of the throat can also affect gaugino condensation, by giving rise to mass terms for the fermions on the D7-branes. A new bound relating various parameters of compactifications with strongly-warped throats was derived in [135] and it must be satisfied in order for Kähler moduli to be stabilised by gaugino condensation.

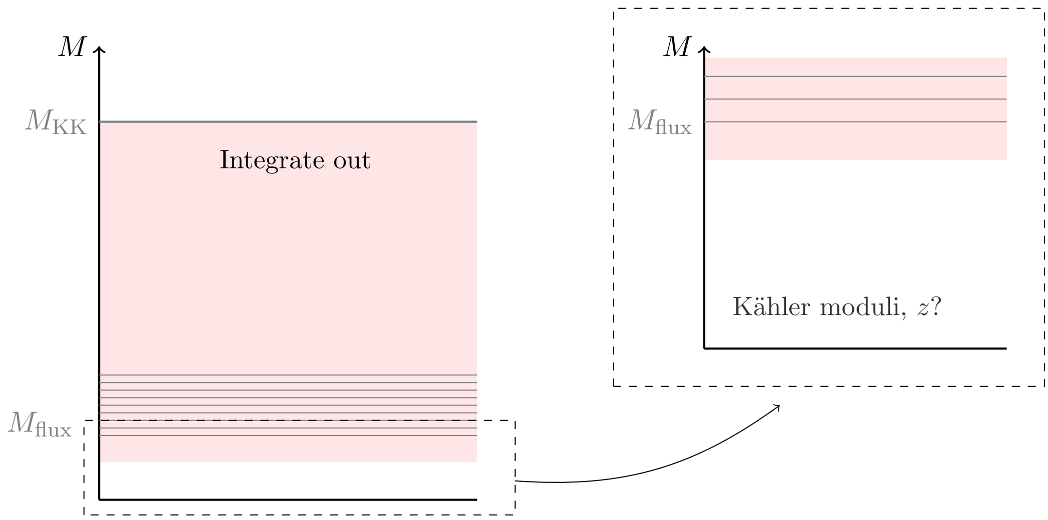


Figure 4.1: Adding fluxes to the compactification generates a scalar potential for the complex structure moduli and the axio-dilaton through the superpotential (3.80), but leaves the Kähler moduli as flat directions. Therefore, the former can be stabilised and get a mass of order $M_{\rm flux} \ll M_{\rm KK}$ which is often much larger than the scale of the subleading effects that eventually stabilise the Kähler moduli. When that happens, one can integrate out the flux-stabilised fields and leave only the ones untouched by the fluxes. Once warping is introduced in the picture, the warped-down masses of some complex structure moduli (e.g. the conifold modulus z) might be to small to be integrated out and should therefore be included in the low-energy theory for the Kähler moduli.

where $A = \langle A(z^i, \tau)\rangle$ generically depends on the stabilised values of the fields that were integrated out.[6] In terms of the modulus T we have $\mathcal{V} = (T + \bar{T})^{3/2}$, so that $K(T, \bar{T}) = -3\log\big(T + \bar{T}\big)$, and hence the supersymmetry condition that also gives a minimum for the potential is

$$D_T W = 0 \implies W_0 = -Ae^{-a\,\mathrm{Re}\,T}\left\{1 + \frac{2a}{3}\,\mathrm{Re}\,T\right\} \ll 1\,. \tag{4.7}$$

The minimum is supersymmetric and AdS with

$$V_{\rm AdS} \propto -\frac{a^2 A^2 e^{-2\,\mathrm{Re}\,T}}{6\,\mathrm{Re}\,T}\,. \tag{4.8}$$

The KKLT solution as been widely studied over the years and whether it provides a construction which can be realised explicitly and fully under control seems to still be up for debate [136–146]. Several of these discussions also include the final ingredient of this de Sitter proposal — the addition of an $\overline{\rm D3}$-brane whose positive potential is warped down so as to preserve the minimum for T while bringing it to positive values. This step is usually based on the GKP solution [94] and the warped deformed conifold [86]. Since it is common to both proposals and will be key to the work presented in this chapter, we shall come back to this point in further detail.

[6]Note that here we are sticking to the conventions used in [75], in which the 10d change of frames is performed with $\langle\Phi_0\rangle = 0$ (see Appendix A.1).

4.2.2 LVS

In the Large Volume Scenario [76, 77], rather than balancing $\langle W_{\text{flux}}\rangle$ against quantum corrections, we balance different quantum corrections against each other. It requires more than one Kähler modulus and works with a Calabi-Yau manifold of "Swiss cheese" form, corresponding to an overall large volume with other (blow-up) moduli being associated with geometric "holes". The case explicitly analysed in [76, 77] is that of a Calabi-Yau 3-fold (CY_3) with $h^{1,1}=2$ and a volume

$$\mathcal{V} = \kappa_b \tau_b^{3/2} - \kappa_s \tau_s^{3/2}\,, \tag{4.9}$$

where τ_i is the volume of an internal four-cycle and becomes the real part of $T_i = \tau_i + i\theta_i$ (the axion θ_i corresponds to deformations of the RR-form C_4). Through a suitable redefinition of τ_b, one can set $\kappa_b \equiv 1$. The LVS construction relies on both perturbative α' corrections to the Kähler potential and non-perturbative corrections to the superpotential,

$$K_{\text{Kähler}} = -2\log\left[\mathcal{V} + \frac{\xi}{2}\right]\,, \tag{4.10}$$

$$W = W_{\text{flux}} + \sum_i A_i e^{-\frac{a_i}{g_s}T_i}\,, \tag{4.11}$$

where $\xi = -\frac{\chi(\text{CY}_3)\zeta(3)}{2(2\pi)^3}$, $\chi(\text{CY}_3)$ being the Euler characteristic of the CY_3 and $\zeta(3)\approx 1.202$ [147]. The simplest case is typically studied in which only the leading non-perturbative effect is considered [76, 77, 148]

$$W_{\text{LVS}} = W_{\text{flux}} + Ae^{-\frac{a}{g_s}T_s}\,. \tag{4.12}$$

The scalar potential arising from these ingredients will have central stage in this chapter and we shall discuss it in much greater detail later on. At this level of the discussion, the result is that the Kähler moduli are stabilised at

$$\mathcal{V} \approx \tau_b^{3/2} \approx \frac{3W_0 g_s \kappa_s \sqrt{\tau_s}e^{\frac{a}{g_s}\tau_s}}{4aA}\,, \qquad \tau_s \approx \left(\frac{\xi}{2\kappa_s}\right)^{2/3}\,, \tag{4.13}$$

where again $W_0 = \langle W_{\text{flux}}\rangle$, so that the volume of the compact space $\mathcal{V}$ is stabilised at exponentially large values. At the minimum, the potential is negative and breaks supersymmetry, and thus the LVS vacuum is non-supersymmetric AdS.

Note that, although the superpotential for LVS (4.12) is formally the same as the one for KKLT (4.6) (with T_s replacing T), we have chosen to write it in a slightly different way. Firstly, we kept W_{flux} rather than W_0 — the reason for this is that, as we shall see in what follows, once the warped deformed conifold and its modulus z are taken into account, one might not be able to integrate out z with the rest of the flux stabilised moduli and should therefore include it in the analysis of the Kähler moduli. In practice, this means that $W_{\text{flux}} = W_0 + W(z)$ with W_0

not including z. A second difference is in the factor of g_s appearing in the non-perturbative term — this follows from our choice of conventions for the change of frames in 10d ($\Phi_0 = \langle\Phi\rangle$, see Appendix A.1), which is also reflected in the non-perturbative correction in $K_{\text{Kähler}}$ (4.10). In Appendix A.3, we give a brief overview of these corrections, highlighting the convention dependence.[7]

4.3 The Uplift

The two steps of the KKLT and LVS proposals provide a moduli stabilisation mechanism that results in an AdS vacuum. In order to get a de Sitter solution, both require a third step that takes this negative minimum and adds a positive contribution that turns it positive — this is the so-called uplift. There are several different proposals for the origin of this uplift term — e.g. $\overline{\text{D}p}$-branes [75], magnetised branes [149], T-branes [150], dilaton dependent non-perturbative effects [151], to mention a few (for more on this, see [121]).[8]

In what follows, we take an $\overline{\text{D3}}$-brane as the source of the uplift as is customary in both KKLT and LVS. The vacuum energy contribution is obtained from the brane action[9] (2.90), which at leading order reads[10]

$$S_{\overline{\text{D3}}} = -T_{\text{D3}} \int_{\mathcal{W}} d^4\xi \; g_s^{-1} \sqrt{-g_{\overline{\text{D3}}}} \; - T_{\text{D3}} \int_{\mathcal{W}} C_4 \,, \tag{4.14}$$

with $g_{\overline{\text{D3}}}$ being the pull-back of the 10d metric onto the world-volume $\mathcal{W}$ of the brane, which must extend along all non-compact directions (leaving the brane localised at a point in the 6d compact space). Aligning the world-volume coordinates ξ^a with the 4d coordinates x^μ of (3.127), we have that $g_{\overline{\text{D3}}} = H^{-1/2}(y_{\overline{\text{D3}}}) \cdot e^{2\omega} \cdot g_4$. On the other hand, the background solution for $\tilde{F}_5$ (3.112) together with the fact that the H_3 and F_3 backgrounds can only have internal components means that the background solution for C_4 along the non-compact directions (recall that we must pull-back C_4 onto $\mathcal{W}$ in the CS/WZ term) is

$$C_4 = \alpha \;\text{vol}_4 = g_s^{-1} H^{-1}(y_{\overline{\text{D3}}}) \;\text{vol}_4 \,. \tag{4.15}$$

[7] Both reasons would also apply to KKLT, so we could have just adapted the KKLT discussion in the same way. However, since we will not work with KKLT in what follows, we chose to present it in the original notation for easier comparison.

[8] There are also proposals that incorporate other effects directly in the Kähler moduli stabilisation, rather than adding an uplifting contribution at a later stage (see [121] for a review). See also [152] for an extended version of the KKLT scenario without a supersymmetric AdS vacuum.

[9] The $\overline{\text{D3}}$-brane can equivalently be described in a supersymmetric way within the low energy effective supergravity theory using constrained superfields [153–164], though this is not necessary in what follows.

[10] Notice that we have set $e^{-\Phi} = g_s^{-1}$, since the dilaton has already been stabilised.

Putting everything together, the action for the $\overline{\text{D3}}$-brane is (cf. Appendix A.2 for Weyl-rescaling)

$$S_{\overline{\text{D3}}} = -2T_{\text{D3}} g_s^{-1} \int d^4x \sqrt{-\det g_{\mu\nu}} \left(\frac{\mathcal{V}_{\text{w}}^0}{\mathcal{V}_{\text{w}}}\right)^2 H^{-1}(y_{\overline{\text{D3}}}) \,, \tag{4.16}$$

and using $T_{\text{D3}} = \frac{2\pi}{l_s^4}$, the ratio m_s/M_p (3.129) and $\mathcal{V}_{\text{w}} \approx \mathcal{V}$, this leads to the potential

$$V_{\overline{\text{D3}}} = \left(\frac{g_s^3}{8\pi}\right) \frac{2}{\mathcal{V}^2} H^{-1}(y_{\overline{\text{D3}}})\; M_{\text{Pl}}^4 \,. \tag{4.17}$$

This is where the hierarchy in the GKP solution arising from the warp factor on the deformed conifold comes into play.[11] The natural scale of the $\overline{\text{D3}}$ contribution is of order M_s (this is most easily seen from (4.16) and $T_{\text{D3}} \sim M_s^4$) and its volume dependence is such that whenever it dominates over all other terms it results in a runaway for $\mathcal{V}$ — if we leave it unsuppressed, the brane potential will destroy the minimum we started with and push us towards the decompactification limit ($\mathcal{V} \to \infty$). However, when there is a strongly warped region where $H(y) \gg 1$ the brane potential may be suppressed enough and uplift the solution without destroying it.

Recall that the position of the brane along the transverse directions (in the compact space), is associated with world-volume scalars $\phi^m(x^\mu)$ that make it dynamical — it is given by the vacuum expectation value $(y_{\overline{\text{D3}}})^m \sim \langle \phi^m \rangle$. A term like (4.16), which depends on the scalars through the warp factor, generates a potential for them whose minimum corresponds to the region of maximum warping. Hence, at least in the absence of competing effects, the brane is pulled towards the tip of the deformed conifold ($\tau = 0$). Since most discussions involve strong warping, at the tip $H(y_{\text{tip}}) \approx \frac{e^{-4A_0^{\text{tip}}}}{\mathcal{V}^{2/3}}$ and so $V_{\overline{\text{D3}}}$ becomes (cf. (3.108))

$$V_{\overline{\text{D3}}} = \left(\frac{g_s^3}{8\pi}\right) \frac{(2\pi)^4}{\mathcal{V}^{4/3}} c'' \frac{|z|^{4/3}}{(g_s M)^2} M_{\text{Pl}}^4 \,. \tag{4.18}$$

where $c'' = \frac{2^{1/3}}{I(0)} \approx 1.75$.

We will, however, be interested in solutions with weak warping where $H \sim 1$ throughout the whole compact space and will therefore use the exact expression (4.17). In this weakly-warped case, there is no large hierarchy suppressing the brane potential and one would expect it to keep its natural M_s scale. Yet, we will see that there are solutions in this regime that appear to be stable and positive — how can the $\overline{\text{D3}}$-brane not destroy the minimum for $\mathcal{V}$? Although we will analyse this in detail in what follows, the key is that the weakly-warped solution is found when the vev of the flux superpotential W_0 and the scale of the non-perturbative effects A are large enough to compete with $V_{\overline{\text{D3}}}$ and avoid destabilisation. Working with LVS, one can use the exponentially large volume to suppress the overall scale of the potential and guarantee that

[11] Interestingly, recent work as suggested that taking into account $(\alpha')^2$ curvature corrections to the brane action might be enough to lower the tension, providing an uplifting mechanism where the smallness of the uplift is achieved by tuning these corrections [165]. If confirmed, this would remove the need for strong warping and consequently large tadpole contributions.

it remains within the validity of the EFT.

In order to understand what motivates us to look for a weakly-warped de Sitter solution, we need to discuss the fate of the complex structure modulus z controlling the deformation of the conifold. Not only can the presence of warping in the background exponentially suppress the flux generated mass of z and make it comparable to the scale at which the Kähler moduli are stabilised (hence forbidding us from safely integrating it out), it also adds important corrections to its Kähler potential [166, 167]. These corrections and the fact that the brane potential (4.17) depends on z through the warp factor will make the brane contribution potentially dangerous, not only to $\mathcal{V}$, but to z itself — unless certain conditions are met, the $\overline{\text{D3}}$-brane uplift in the strongly-warped regime may push $z \to 0$ towards the singular conifold and destroy the solution [168]. Moreover, a large warping generated *a la* GKP requires a large flux contribution to the D3-charge tadpole (3.118) whenever the supergravity approximation near the tip is under control [86] (it may also be necessary to prevent the $\overline{\text{D3}}$-brane from annihilating with fluxes into a number of D3-branes [169]). Such a contribution can be extremely hard to cancel, which lead to further study of the usual ingredients that can help with the cancellation (in particular D7-branes in the context of F-theory) and eventually to the proposal of the Tadpole Conjecture [100–106].

4.4 Conifold modulus

Not long after the GKP proposal [94], the dimensional reduction of Type IIB was revisited in [170] with warping effects taking centre stage. By rederiving the scalar potential for the moduli on a warped background, one concludes that while the GVW superpotential (3.80) does not get corrected by the warping, the Kähler potential does,

$$K/M_{\text{Pl}}^2 = -2\log\mathcal{V}_{\text{w}} - \log(-i(\tau-\bar{\tau})) - \log\left(\frac{i}{l_s^6}\int H\ \Omega\wedge\overline{\Omega}\right)\,. \tag{4.19}$$

A more detailed reduction taking into account subtleties related to KK modes, compensator fields and warping contributions was given in [109], including a deeper understanding of how one should define the Kähler moduli in the presence of warping (this was revisited in [108] focusing on the universal Kähler modulus).

For us, the key point is already present in the corrected Kähler potential (4.19). Since the metric on moduli space is derived from the Kähler potential as $G_{I\bar{J}} = \partial_I\partial_J K$, warping corrections to K correspond to warping corrections to the moduli space metrics themselves. This includes, in particular, the metric associated with the complex structure modulus z, controlling the deformation of the conifold, which gets the following correction (cf. (3.90))

$$K_{z\bar{z}} = \frac{1}{\pi||\Omega||^2}\left(\log\frac{\Lambda_0^3}{|z|} + \frac{c'}{(2\pi)^4}\frac{1}{\mathcal{V}^{2/3}}\frac{(g_sM)^2}{|z|^{4/3}}\right)\,. \tag{4.20}$$

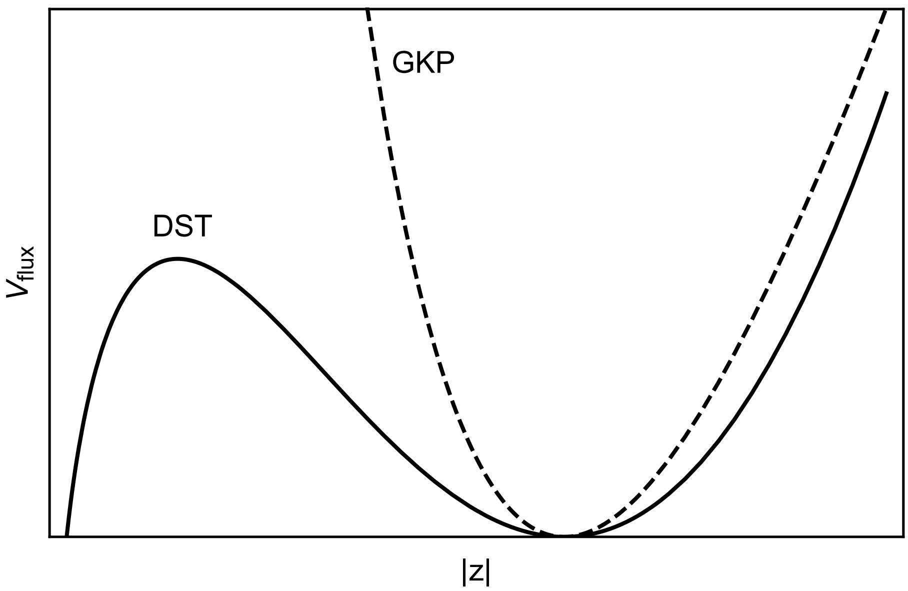


Figure 4.2: The stabilisation mechanism proposed in [94] for the deformation modulus z together with the metric (3.90) gives a scalar potential that only has the minimum at $z \sim e^{-\frac{2\pi K}{g_s M}}$ (3.126) (GKP). Once warping effects are taken into account [166], the Kähler potential receives corrections (4.21) that bring the potential down to zero as $z \to 0$ (DST). Although the corrected potential still has a minimum at (3.126), not only does it become susceptible to destabilisation by the $\overline{\text{D3}}$ uplift, but it also corresponds to a much smaller mass for the z modulus.

In Appendix D.3, we review the computation of $K_{z\bar{z}}$ performed in [166], making explicit the appearance of the volume modulus in the warping correction term. The final result is valid when $|z| \ll \Lambda_0^3$, where we define the dimensionless $\Lambda_0 = \Lambda_{\text{UV}}/l_s$ encoding the radial distance from the tip at which the deformed conifold is glued to the CY_3 and the constant $c' = 1.18$. This metric corresponds to a contribution to the Kähler potential of the form

$$\mathcal{K}(z,\bar{z}) = \frac{1}{\pi||\Omega||^2}\left[|z|^2\left(\log\frac{\Lambda_0^3}{|z|}+1\right)+\frac{9c'(g_sM)^2}{(2\pi)^4\mathcal{V}^{2/3}}|z|^{2/3}\right]. \tag{4.21}$$

Notice that the warping contribution to the Kähler potential mixes the deformation modulus z and the volume modulus $\mathcal{V}$, in such a way that large volumes suppress the effect of the warping. We can combine the Kähler potentials for the two moduli such that (see e.g. [171])

$$\mathcal{K} = -3\log\left(\mathcal{V}^{2/3} - \frac{1}{(2\pi)^4}\frac{3c'(g_sM)^2}{\pi||\Omega||^2}|z|^{2/3}\right) + \ldots \tag{4.22}$$

It is interesting to note that despite the mixing between z and $\mathcal{V}$, this correction preserves the no-scale structure of the Kähler potential [109, 142].

Notice that the correction term in (4.20) does indeed vanish when $M = 0$, in which case there is no warping, and becomes more relevant as $z \to 0$ — for small enough $|z|$, this term will start to dominate and bring the scalar potential for z down to zero[12] (Fig. 4.2)

[12] Recall that it is the *inverse* metric that appears in the scalar potential (3.78).

Let us now turn to the superpotential, which keeps its GVW form (3.80). We use the periods over 3-cycles (3.123) and (3.74), in particular

$$z/l_s^3 = \int_A \Omega\,, \qquad\qquad \mathcal{F}(z) = \Pi_0 + \frac{z\cdot l_s^3}{2\pi i}\left(\log\frac{\Lambda_0}{z}+1\right)\,, \tag{4.23}$$

and the 3-form fluxes on the 3-cycles (3.122) responsible for the deformation of the conifold. While the A-cycle is unambiguisly defined as the S^3 even when the conifold is glued to a compact Calabi-Yau, the B-cycle could in principle extend into the bulk and the H_3 flux K through this cycle could receive contributions from outside the conifold. However, let us assume that all flux comes from the conifold region,

$$\frac{1}{(2\pi)^2\alpha'}\int_B H_3 = \frac{1}{(2\pi)^2\alpha'}\int_{\tau\le\tau_\Lambda}\int_{S^2} H_3 = K\,, \tag{4.24}$$

with τ_Λ corresponding to the radial coordinate where we glue the deformed conifold to the compact Calabi-Yau. The integral in (4.24) can be computed for the conifold, using the approximation of the metric (3.103) in the limit $\tau\to\infty$ (3.97), with $r^2 = \frac{3}{2^{5/3}}\epsilon^{4/3}e^{2\tau/3}$ and the functions (D.41) (see Appendix D.3 for details on τ_Λ and Λ_0),

$$K = \frac{1}{(2\pi)^2\alpha'}\int_{\tau\le\tau_\Lambda}\int_{S^2} H_3 \approx \frac{g_s M}{2\pi}\tau_\Lambda = \frac{g_s M}{2\pi}\left(\log\frac{\Lambda_0^3}{|z|}+\frac{3}{2}\log\frac{2^{5/3}}{3}\right)\,. \tag{4.25}$$

Neglecting contributions from the bulk, this relates the parameters g_s, M, K, Λ_0 and the value of the deformation modulus z — in a solution where the bulk does not contribute to the flux, this topological relation must be satisfied. In the context of the deformed conifold solution of [86], this would indeed be the case, with the flux number M being the only free flux, since it is a solution for constant dilaton and therefore satisfies the relation $g_s^2|F_3|^2 = |H_3|^2$ (3.109). Interestingly, this relation between K and the cutoff scale Λ_0 takes the form of the GKP solution for the deformation modulus [94, 166]

$$|z| \approx \Lambda_0^3\,\exp\left\{-\frac{2\pi K}{g_s M}\right\}\,, \tag{4.26}$$

which is therefore consistent with an H_3 flux through the B-cycle dominated by the conifold contribution.

The superpotential (3.80) takes the form

$$W/M_p^3 = \frac{g_s^{3/2}}{\sqrt{4\pi}}\left[W_0 e^{i\sigma} - \frac{M}{2\pi i}z\left(\log\frac{\Lambda_0^3}{z}+1\right) - i\frac{K}{g_s}z\right]\,. \tag{4.27}$$

with a constant superpotential $W_0 e^{i\sigma}$ containing all z-independent terms, in particular the contribution from the integrated-out complex structure moduli and the dilaton which are stabilised by the remaining fluxes, with $\langle\tau\rangle = i g_s^{-1}$ — this is nothing but their flux superpotential evaluated at the vev, $\langle W_{\text{flux}}\rangle$.

Combining the Kähler potential (4.21) and superpotential (4.27), the resulting scalar potential (3.78) is

$$V_{\rm KS} = \left(\frac{g_s^3}{8\pi}\right)\frac{\pi g_s}{\mathcal{V}^2}\left(\log\frac{\Lambda_0^3}{|z|} + \frac{1}{(2\pi)^4}\frac{c'(g_sM)^2}{\mathcal{V}^{2/3}|z|^{4/3}}\right)^{-1}\left|\frac{M}{2\pi}\log\frac{\Lambda_0^3}{z} - \frac{K}{g_s}\right|^2 M_{\rm Pl}^4\,. \tag{4.28}$$

In addition to this scalar potential originating from the fluxes, we include the contribution from the $\overline{\rm D3}$-brane (4.17) at the tip of the deformed conifold,

$$V_{\overline{\rm D3}} = c_{\rm D3}\left(\frac{g_s^3}{8\pi}\right)\frac{2}{\mathcal{V}^2}\left\{1 + \frac{1}{(2\pi)^4}\frac{2}{c''}\frac{(g_sM)^2}{\mathcal{V}^{2/3}|z|^{4/3}}\right\}^{-1} M_p^4\,, \tag{4.29}$$

where $c_{\rm D3} = 1$ in the presence of the $\overline{\rm D3}$-brane and zero otherwise.[13] The deformation modulus appears in the brane potential through the warp factor of the metric, and we see that the suppression is provided by the vev of this modulus, through $|z|^{4/3}$. As we saw, this energy suppression ensures that the positive energy density from the probe $\overline{\rm D3}$-brane uplifts an otherwise AdS minimum for the volume modulus to a near Minkowki minimum, instead of dominating the potential and causing a runaway. How much suppression is required, and hence how large the hierarchy (3.128) needs to be, depends on the stabilisation mechanism of the volume modulus and, in particular, on the depth of the AdS minimum prior to the uplift.

It is useful to introduce the following constants [172]

$$\varepsilon = \frac{g_sM}{2\pi K}\,,\quad \delta_1 = \frac{g_s^3}{8}\times\frac{K^2}{g_s}\,,\quad \delta_2 = \frac{g_s^3}{8\pi}\times c''\frac{c'}{\delta_1} = \frac{1}{\pi}\times c''c'\frac{g_s}{K^2}\,,\quad \delta_3 = \frac{(2\pi)^4}{2}\frac{c''}{(g_sM)^2}\,, \tag{4.30}$$

as well as the parameter

$$\beta \equiv \frac{\mathcal{V}^{2/3}\log\frac{\Lambda_0^3}{\zeta}}{\frac{c'}{(2\pi)^4}\frac{(g_sM)^2}{\zeta^{4/3}}} = C\,\mathcal{V}^{2/3}\Lambda_0^4\;xe^{-\frac{4}{3}x}, \tag{4.31}$$

where we defined $z = \zeta e^{i\theta}$, with ζ de saxion and θ the axion of the deformation modulus z. We also introduced the constant $C = \frac{(2\pi)^4}{c'(g_sM)^2}$ and the variable $x \equiv \log\frac{\Lambda_0^3}{\zeta}$, which is useful when studying different regimes for this scalar potential. Using these parameters and constants, the potentials (4.28) and (4.29) become (in Planck units, $M_{\rm Pl} = 1$)

$$\begin{aligned} V &= V_{\rm KS} + V_{\overline{\rm D3}} \\ &= \frac{\delta_1 C}{\mathcal{V}^{4/3}}\Lambda_0^4 e^{-\frac{4}{3}x}\left[(1+\beta)^{-1}(1-\epsilon x)^2 + c_{\rm D3}\delta_2\Big(1+\delta_3\mathcal{V}^{2/3}\Lambda_0^4 e^{-\frac{4}{3}x}\Big)^{-1}\right]\,, \end{aligned} \tag{4.32}$$

where we assumed that the axion is stabilised at zero, $\langle\theta\rangle = 0$ (this will be confirmed below).

The parameter β plays a crucial role in our analysis — it measures the suppression of the warping

[13] More generally, we can also think of $c_{\rm D3}$ as the number of $\overline{\rm D3}$-branes at the tip of the deformed conifold.

contribution to the potential. Indeed, it is clear that the warping (3.128) is suppressed by large volumes, so that larger compact spaces will be less affected by warping effects than smaller ones. Our definition is such that when β is large the warping correction to $K^{z\bar{z}}$ in (4.28) becomes subdominant. Previous works have assumed the regime $\beta \ll 1$, where the warping completely dominates (this is even true in the LVS case studied in [148], where the large volume of the bulk is still not relevant for the stabilisation of the deformation modulus) — in [172] we explore the opposite regime $\beta \gg 1$ for which the warping is subdominant. From the definition (4.31),

$$\beta \approx \frac{\mathcal{V}^{2/3}}{e^{-4A_0^{tip}}} \log \frac{\Lambda_0^3}{\zeta} = (H-1)^{-1} \log \frac{\Lambda_0^3}{\zeta} \,, \tag{4.33}$$

and we see that leaving the $\beta \ll 1$ regime seems to require us to have weak warping and/or long conifolds. In fact, due to the solution for ζ, smaller values of $\varepsilon = \frac{g_s M}{2\pi K}$ (and hence larger values of z, compatible with smaller warping) rather than long conifolds will be helpful.

Before introducing the new weakly-warped solution of [172], let us review the strongly-warped case, which will give us the motivation to look for solutions at weak warping.

4.4.1 Strongly-warped solutions

In previous studies, it has been assumed that the warping term dominates over the volume term in (4.21) and (4.28), that is $\beta \ll 1$, in which case the potential becomes[14]

$$V \approx \frac{\delta_1 C}{\mathcal{V}^{4/3}} \Lambda_0^4 \, e^{-\frac{4}{3}x} \left[(1-\varepsilon x)^2 + c_{\rm D3}\delta_2 \right] \,, \tag{4.35}$$

and hence

$$V' = \frac{\delta_1}{\mathcal{V}^{4/3}} C \Lambda_0^4 \frac{4}{3} e^{-\frac{4}{3}x} \left[\left(1 + c_{\rm D3}\delta_2 + \frac{3}{2}\varepsilon\right) - \left(2 + \frac{3}{2}\varepsilon\right) \varepsilon x + (\varepsilon x)^2 \right] \,, \tag{4.36}$$

which shows a minimum at $\zeta = 0$ ($x \to \infty$) and may or may not have critical points for $\zeta > 0$ (x finite). Notice that, in the absence of the brane ($c_{\rm D3} = 0$) one immediately obtains the GKP solution (which corresponds to $\varepsilon x = 1$). Once the brane is introduced ($c_{\rm D3} = 1$), the condition that guarantees the existence of a non-trivial minimum is

$$\frac{\delta_2}{\varepsilon^2} = \frac{4\pi c' c''}{g_s M^2} \leq \frac{9}{16} \,, \tag{4.37}$$

[14] Note that

$$\delta_3 \mathcal{V}^{2/3} \Lambda_0^4 e^{-\frac{4}{3}x} = \frac{c' c''}{2} \frac{\beta}{x} \,. \tag{4.34}$$

Therefore, for $\beta \ll 1$ and as along as $x \gtrsim 1$, this term is subdominant compared to 1.

and when this is satisfied we obtain the solutions

$$\zeta = \Lambda_0^3 \, \exp\left\{ -\frac{2\pi K}{g_s M} - \left(\frac{3}{4} \pm \sqrt{\frac{9}{16} - \frac{4\pi c' c''}{g_s M^2}} \right) \right\} . \tag{4.38}$$

The bound (4.37) was first found in [168] and requires the F_3 flux M to be large enough in order for a solution to exist, $\sqrt{g_s}M \gtrsim 6.8$ (see also [173] for a 5d analysis of this bound). When this bound is violated, the potential for the deformation modulus ζ turns into a runaway that drives it towards $\zeta = 0$ where the singular conifold is recovered (Fig. 4.3). Together with the requirement that the suppression to the brane potential, through the deformation modulus

$$|z|^{4/3} \sim \exp\left\{ -\frac{8\pi K}{3 g_s M} \right\} = \exp\left\{ -\frac{8\pi (MK)}{3 g_s M^2} \right\} , \tag{4.39}$$

be large enough,[15] this translates into a lower bound on MK which contributes to the D3-tadpole cancellation condition (3.85) and makes it difficult to satisfy [168]. Although there is no demonstrated upper bound for the number of orientifold planes one can have on a given CY_3, all known examples on moduli stabilisation by fluxes have at most 64 O3-planes [168], which limits their negative contribution to the tadpole. In more generic F-theory constructions, D7-branes can be taken into account and naïvely bring a much larger negative contribution — this motivated further exploration of these ingredients in flux compactifications and eventually to the proposal of the Tadpole Conjecture [100–106]. The key point is that even in these F-theory setups, the tadpole can be extremely hard to cancel and a large contribution from MK might be problematic.

Note that one can trace this back to the warping correction to the deformation modulus metric (4.20) (or equivalently the Kähler potential (4.21)), which brings its scalar potential down to zero as $\zeta \to 0$. Therefore, the validity of the bound (4.37) is tied to the validity of this correction all the way down to $\zeta = 0$. Although the result [166] claims this to be the case, recent work as suggested that a more careful derivation of the Kähler potential gives a qualitatively different result, with a potential which no longer goes to zero at small z [107]. It was also argued that other consistency conditions such as the bound of [169] and the validity of the supergravity approximation used in [86] already put stronger constraints on M than (4.37). In any case, the large tadpole danger motivates us to look for weakly-warped solutions which might help avoid this problem — on the one hand, allowing for weak warping will immediately loosen the bound on MK that arises from imposing large hierarchies (4.39); on the other, in the absence of warping one expects the mass of z to take its natural flux value and this modulus to be much less sensitive to the low-energy ingredients, so that adding the $\overline{\mathrm{D3}}$-brane would no longer cause the runaway towards the singular conifold. Of course, this requires a controlled solution to exist at weak warping for which also the volume modulus stabilisation is safe from a weakly-warped brane contribution — we will see how this can be achieved in later sections.

[15] As we mentioned before, in this argument, "large enough" can only be made precise when one includes the stabilisation of the volume modulus explicitly.

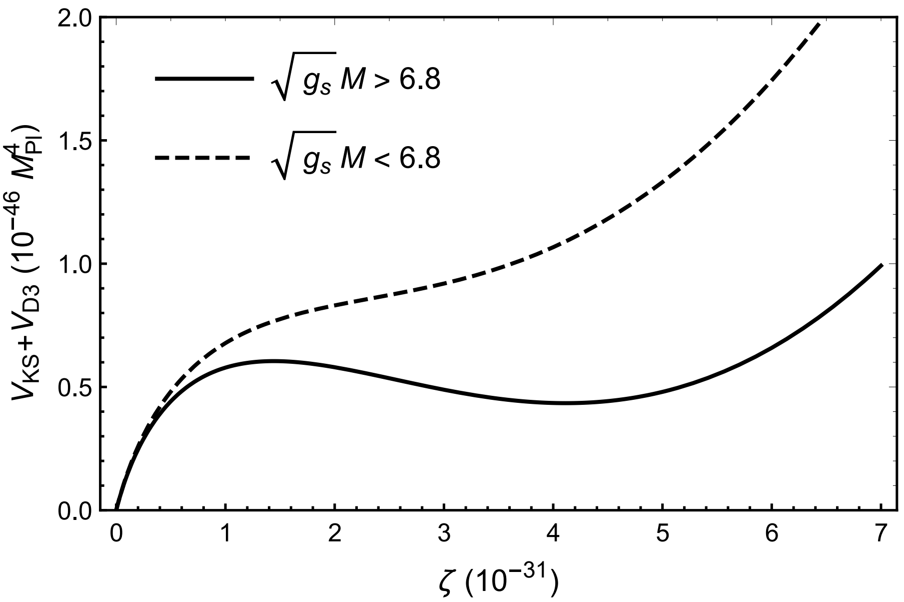


Figure 4.3: Comparison between two different choices of M ($M = 5, M = 25$ with $K = M$ in both cases) for the potential (4.32) for an example with $\beta \ll 1$, with the choice of parameters $\Lambda_0 = 0.1,\ g_s = 0.1,\ \mathcal{V} = 10^3$ and given $||\Omega||^2 = 8,\ c' = 1.18,\ c'' = 1.75$.

As a final remark, it is important that the solution is consistent with the approximation we started with, i.e. the non-trivial solution $\zeta_{\rm min}$ that follows from this analysis must be such that $\beta \ll 1$ given the vev for the volume once it is stabilised (which is verified for the parameters used in Fig. 4.3).

4.4.2 Weakly-warped solutions

Let us then consider the new regime, where the warping term in (4.21) is subdominant,[16] that is $\beta \gg 1$, so that the potential becomes

$$\begin{aligned} V &\approx \frac{\delta_1}{\mathcal{V}^2}\frac{1}{x}\left[\left(1-\frac{1}{\beta}\right)\left(1-\varepsilon x\right)^2 + \beta c_{\rm D3}\delta_2\left(1+\delta_3\mathcal{V}^{2/3}\Lambda_0^4 e^{-\frac{4}{3}x}\right)^{-1}\right] \\ &= \frac{\delta_1}{\mathcal{V}^2}\left[\frac{1}{x}\left(1-\frac{e^{\frac{4}{3}x}}{C\,\mathcal{V}^{2/3}\Lambda_0^4 x}\right)\left(1-\varepsilon x\right)^2 + c_{\rm D3}\delta_2\,\frac{C\,\mathcal{V}^{2/3}\Lambda_0^4}{e^{\frac{4}{3}x}+\delta_3\mathcal{V}^{2/3}\Lambda_0^4}\right] \end{aligned} \tag{4.40}$$

and thus

$$V' = \frac{\left(1-(\epsilon x)^2\right)}{x^2}\frac{\delta_1}{\zeta\mathcal{V}^2} - \frac{2(1-\epsilon x)(3-2x(1-\epsilon x))}{3x^3}\frac{\delta_1}{\zeta^{7/3}C\ \mathcal{V}^{8/3}} + \frac{4c_{\rm D3}\delta_1\delta_2 C\ \zeta^{1/3}}{3\mathcal{V}^{4/3}\left(1+\delta_3\mathcal{V}^{2/3}\zeta^{4/3}\right)^2}\,. \tag{4.41}$$

---

[16] To be precise, we are considering a "weakly-but-still-warped scenario", in which the warping-induced term in the deformation modulus metric [166] provides an important subleading correction to the scalar potential (4.40), even though the interplay of all the ingredients is such that the warping is small. Indeed the fluxes needed to stabilise the deformation modulus will still source some warping near the tip of the deformed conifold and it is only due to the balance between this effect and the overall volume of the compact space that the resulting warping is small.

Without further assumptions, it is difficult to solve $V' = 0$ analytically. Since we are interested in the regime where $\beta \gg 1$, we can instead solve the equation perturbatively in powers of $1/\beta$. Reintroducing β in V' and expanding for large β, we find

$$V' = \frac{\delta_1}{\zeta \mathcal{V}^2} \frac{(1-(\epsilon x)^2)}{x^2} + \frac{2\delta_1(2c_{\rm D3}\delta_2 C^2 x^3 + {\delta_3}^2(2x^3\epsilon^2 - 4x^2\epsilon + x(2+3\epsilon) - 3))}{3{\delta_3}^3 \zeta \mathcal{V}^2 x^2} \frac{1}{\beta} + \mathcal{O}\Big(\frac{1}{\beta^2}\Big) \,. \tag{4.42}$$

We see that at leading order in $1/\beta$ we obtain the usual GKP solution with $\epsilon x = 1$, as one would expect in the absence of warping. The next-to-leading order correction to this solution can be determined by writing

$$x_{\rm min} = x_0 + \frac{x_1}{\beta} + \mathcal{O}\Big(\frac{1}{\beta^2}\Big) \,, \tag{4.43}$$

plugging $x_{\rm min}$ into V' and solving at each order in $1/\beta$, which gives

$$x_{\rm min} = \frac{1}{\epsilon} + c_{\rm D3} \frac{2\delta_2 C^2}{3{\delta_3}^2 \epsilon^4} \frac{1}{\beta}\bigg|_{x=1/\epsilon} + \mathcal{O}\Big(\frac{1}{\beta^2}\Big) \,, \tag{4.44}$$

or in terms of ζ,

$$\zeta_{\rm min} \approx \Lambda_0^3 \exp\left\{ -\frac{2\pi K}{g_s M} - \frac{4K e^{\frac{8\pi K}{3 g_s M}} c_{\rm D3}}{3\pi^2 c' M \Lambda_0^4 \mathcal{V}^{2/3}} \right\} \,, \tag{4.45}$$

corresponding to a small shift of the GKP solution $\zeta_{\rm GKP} \approx \Lambda_0^3 \exp\left\{-\frac{2\pi K}{g_s M}\right\}$ towards smaller values of ζ. The correction depends on $c_{\rm D3}$, showing that it is coming from the brane contribution to the potential and is suppressed by $1/\beta$, so that if there was no warping whatsoever ($\beta \to \infty$) the correction would be absent. Importantly, we always find a solution provided β is large enough so that the expansion in (4.42) is valid — therefore in the weakly-warped limit there is no uplifting runaways bound on the fluxes (cf. 4.37) and the anti-brane does not destabilise the conifold modulus.

Since $V \to 0$ as $\zeta \to 0$ and we have a minimum, there must be a maximum at some $0 < \zeta_{\rm max} < \zeta_{\rm min}$. If $\zeta_{\rm max}$ is sufficiently smaller than $\zeta_{\rm min}$, we will have $\epsilon x_{\rm max} \gg 1$. Taking this limit in (4.41), we have

$$V' = \frac{\delta_1}{\mathcal{V}^{4/3}} \frac{4C}{3\zeta^{7/3}} \left\{ \frac{c_{\rm D3}\delta_2 \zeta^{8/3}}{\left(1+\delta_3 \mathcal{V}^{2/3}\zeta^{4/3}\right)^2} - \frac{3\epsilon^2}{4C\mathcal{V}^{2/3}}\zeta^{4/3} + \frac{\epsilon^2}{C^2\, \mathcal{V}^{4/3}} \right\} . \tag{4.46}$$

In terms of the variable $y \equiv H_{\rm tip} - 1 = \frac{2}{(2\pi)^4} \frac{(g_s M)^2}{c'' \mathcal{V}^{2/3} \zeta^{4/3}}$ and since $y \neq 0$, one can show that V' is proportional to

$$V' \propto -y^3 + \left(\frac{3}{2c'c''} - 2\right) y^2 + \left(\frac{3}{c'c''} - \frac{16\pi c_{\rm D3}}{c'c''(g_s M^2)} - 1\right) y + \frac{3}{2c'c''} \,. \tag{4.47}$$

The sign of the discriminant of this cubic equation is only a function of $g_s M^2$ and always negative — there can only be one real solution. On the other hand, the variable $y = H - 1 < 1$ and

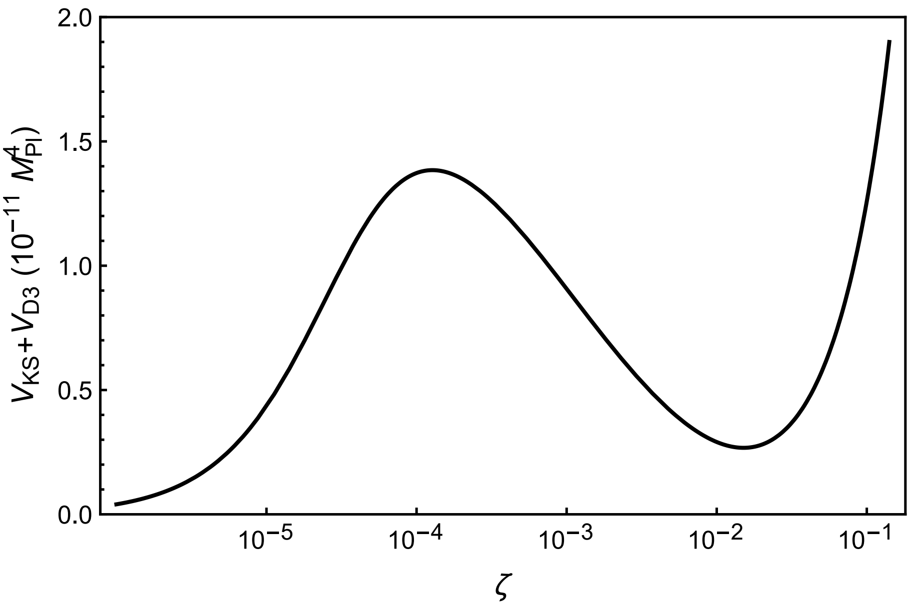


Figure 4.4: Potential (4.32) in the regime $\beta \ll 1$, with the choice of parameters $\Lambda_0 = 1,\ g_s = 0.15, M = 20, K = 2,\ \mathcal{V} = 10^4$ and given $||\Omega||^2 = 8,\ c' = 1.18,\ c'' = 1.75$. Note that (4.45) and (4.48) give respectively $\zeta_{\text{min}} \approx 1.51 \times 10^{-2}$ and $\zeta_{\text{max}} \approx 2.81 \times 10^{-4}$.

the coefficients are all $\mathcal{O}(1)$ or more. Hence, neglecting the cubic term, we find the approximate solution[17]

$$y_{\text{max}} \approx \frac{(3 - c'c'')(g_s M^2) - 16\pi c_{\text{D3}}}{(g_s M^2)(4c'c'' - 3)} + \frac{\sqrt{c'c''(c'c''+6)(g_s M^2)^2 - 32\pi c_{\text{D3}}(g_s M^2)(3 - c'c'') + 64(2\pi)^2 c_{\text{D3}}^2}}{(g_s M^2)(4c'c'' - 3)} . \tag{4.48}$$

In Fig. 4.4 we plot the potential (4.32) in the regime $\beta \gg 1$ for a specific set of parameters, which shows both the minimum near the GKP solution and a maximum at $\zeta_{\text{max}} \ll \zeta_{\text{min}}$ in contrast with the $\beta \ll 1$ regime in which these appear near each other (4.38). Note also that the value of ζ_{min} in the weakly-warped case should not be too small in order for β (4.31) to be large.

We can understand this result in terms of the subdominance of the warping-induced term in (4.20) — by suppressing this term, one is delaying the onset of its effect on the scalar potential pushing the maximum towards smaller values of ζ; this allows the potential to grow significantly before turning around towards zero at $\zeta = 0$, which is reflected in a larger mass for ζ that can now survive the introduction of the brane uplift.

One should still keep in mind that our weakly-warped solution can only be a true solution if we are also able to stabilise the Kähler moduli (including in particular the volume modulus). Since large volumes favour a regime of large β (4.31), we will embed our setup into the Large Volume

[17] The second solution of the quadratic equation is never positive and hence we discard it.

Scenario [76, 77], thereby treating the volume modulus stabilisation explicitly. For comparison, we will use a similar construction to the one in [148] and take the same "Swiss cheese" Calabi-Yau manifold for the bulk geometry as the one in the example therein.

4.5 de Sitter solutions in LVS

We now study the full moduli stabilisation problem including the coupled system of Kähler and complex structure moduli, by embedding the above system into the Large Volume Scenario. We therefore extend our bulk Kähler moduli field content to include the two moduli of a "Swiss cheese" Calabi-Yau. After working out the full scalar potential for this six real field system, we proceed to find metastable de Sitter vacua in the two regimes explored above, where the warping dominates or is suppressed in the deformation modulus dynamics. The strongly-warped regime ($\beta \ll 1$) was studied in [148], and we reproduce their results giving some further consistency checks, while the weakly-warped regime ($\beta \gg 1$) was first introduced in [172].

4.5.1 The scalar potential

Putting together our discussion of LVS and the setup outlined in the previous section, we consider the following Kähler and super potentials

$$\begin{aligned}\mathcal{K}/M_p^2 =& -2\log\left[\mathcal{V}+\frac{\xi}{2}\right]-\log\left(-i(\tau-\bar{\tau})\right)-\log\left(||\Omega||^2\right)\\ &+\frac{1}{\pi||\Omega||^2}\left(|z|^2\left(\log\frac{\Lambda_0^3}{|z|}+1\right)+\frac{1}{(2\pi)^4}\frac{9c'(g_sM)^2}{\mathcal{V}^{2/3}}|z|^{2/3}\right)\,, \end{aligned} \tag{4.49}$$

$$W/M_p^3 = \frac{g_s^{3/2}}{\sqrt{4\pi}}\left(W_0e^{i\sigma}+\left[-\frac{M}{2\pi i}z\left(\log\frac{\Lambda_0^3}{z}+1\right)-i\frac{K}{g_s}z\right]+Ae^{-\frac{a}{g_s}T_s}\right)\,, \tag{4.50}$$

where $z=\zeta e^{i\theta}$. Below we will use the expression for the gravitino mass that follows from these definitions[18]

$$m_{3/2}=e^{\mathcal{K}/2}|W|\approx\frac{1}{\sqrt{8\pi}||\Omega||}\cdot\frac{g_s^2W_0}{\mathcal{V}}M_p\,. \tag{4.51}$$

Using these, we can compute the scalar potential V in the limit $\mathcal{V}\gg 1$ (i.e. we use the

[18]Note that the exact factors of g_s in the gravitino mass are convention dependent when expressed in terms of the Einstein frame volume $\mathcal{V}$ (see Appendix D.2) [174].

supergravity formula for the scalar potential (3.78) and expand it around $1/\mathcal{V} = 0$) and $\zeta \ll 1$,[19]

$$V = \frac{g_s^4}{8\pi||\Omega||^2}\left(\frac{8a^2A^2\sqrt{\tau_s}e^{-2\frac{a}{g_s}\tau_s}}{3\kappa_s g_s^2\mathcal{V}} + \frac{4aA\tau_s e^{-\frac{a}{g_s}\tau_s}}{g_s\mathcal{V}^2}W_0\cos\left(\frac{a}{g_s}\theta_s+\sigma\right) + \frac{3W_0^2\xi}{4\mathcal{V}^3}\right.$$
$$\left. + \frac{\pi||\Omega||^2}{\mathcal{V}^2}\left(\log\frac{\Lambda_0^3}{\zeta} + \frac{1}{(2\pi)^4}\frac{c'(g_sM)^2}{\mathcal{V}^{2/3}\zeta^{4/3}}\right)^{-1}\left[\frac{M^2}{(2\pi)^2}\theta^2 + \left(\frac{M}{2\pi}\log\frac{\Lambda_0^3}{\zeta} - \frac{K}{g_s}\right)^2\right]\right). \tag{4.52}$$

Notice that the T_b axion, θ_b, remains a flat direction at leading order and would be stabilised by subleading non-perturbative effects. Looking at $\partial_\theta V = \partial_{\theta_s} V = 0$, we find the solutions for the remaining axions

$$\langle\theta\rangle = 0\,, \qquad \langle\theta_s\rangle = \frac{n\pi-\sigma}{a}g_s, \quad n\in\mathbb{Z}\,, \tag{4.53}$$

and choose $n = 1$, such that $\cos\left(\frac{a}{g_s}\theta_s + \sigma\right) = -1$. By inspecting the Hessian matrix in the axion directions, $\partial_i\partial_\theta V$ and $\partial_i\partial_{\theta_s} V$, where i runs through all fields, we conclude that these completely decouple from the other moduli and therefore we can fix the axions to their minima and then analyse the 3-field system $(\mathcal{V}, \tau_s, \zeta)$. In particular, the axion masses are always positive, making these stable directions. The potential then becomes

$$\begin{aligned} V =& \frac{g_s^4}{8\pi||\Omega||^2}\left(\frac{8a^2A^2\sqrt{\tau_s}e^{-2\frac{a}{g_s}\tau_s}}{3\kappa_s g_s^2\mathcal{V}} - \frac{4aAW_0\tau_s e^{-\frac{a}{g_s}\tau_s}}{g_s\mathcal{V}^2} + \frac{3W_0^2\xi}{4\mathcal{V}^3}\right.\\ &\left. + \frac{\pi||\Omega||^2}{c'}\frac{(2\pi)^4}{\mathcal{V}^{4/3}}\frac{\zeta^{4/3}}{(g_sM)^2}\Big(1+\beta\Big)^{-1}\left(\frac{M}{2\pi}\log\frac{\Lambda_0^3}{\zeta} - \frac{K}{g_s}\right)^2\right)\\ &+\left(\frac{g_s^3}{8\pi}\right)\frac{2}{\mathcal{V}^2}\left\{1+\frac{1}{(2\pi)^4}\frac{2}{c''}\frac{(g_sM)^2}{\mathcal{V}^{2/3}\zeta^{4/3}}\right\}^{-1}, \end{aligned} \tag{4.54}$$

where we introduced our variable β, defined in (4.31), and the brane potential (4.17).

The formal solution for τ_b does not depend on the choice of β regime, giving always

$$\mathcal{V} \approx \tau_b^{3/2} = \frac{3W_0 g_s\kappa_s\sqrt{\tau_s}e^{\frac{a}{g_s}\tau_s}}{aA}\frac{a\tau_s - g_s}{4a\tau_s - g_s}\,. \tag{4.55}$$

However, because it is implicitly given in terms of τ_s, there is a dependence on the choice of β hiding in the solution for τ_s. In turn, both ζ and τ_s will have different solutions depending on the regime of β that we look at. We now proceed to study the two regimes of strong warping, $\beta \ll 1$, and weak warping, $\beta \gg 1$.

[19] Notice that there is a competition between these two limits precisely in $K_{z\bar{z}}$ and therefore in the term $K^{z\bar{z}}(D_zW)(D_{\bar{z}}\bar{W})$, whose result depends on the β-regime one considers. Usually, it is assumed that the warp factor completely dominates in $K_{z\bar{z}}$ ($\beta \ll 1$), which is equivalent to taking the limit with the constraint $\zeta^{4/3}\mathcal{V}^{2/3} \ll 1$. In practice, $K_{z\bar{z}}$ can be written in terms of β and the limit taken without choosing the regime of interest.

4.5.2 Strongly-warped solutions

Let us start by reviewing the usual limit considered in the literature, $\beta \ll 1$. In this limit, the potential is[20]

$$V = \frac{g_s^4}{8\pi||\Omega||^2}\left(\frac{8a^2A^2\sqrt{\tau_s}e^{-2\frac{a}{g_s}\tau_s}}{3\kappa_s g_s^2\mathcal{V}} - \frac{4aAW_0\tau_s e^{-\frac{a}{g_s}\tau_s}}{g_s\mathcal{V}^2} + \frac{3W_0^2\xi}{4\mathcal{V}^3}\right.$$
$$\left. + \frac{\pi||\Omega||^2}{c'}\frac{(2\pi)^4}{\mathcal{V}^{4/3}}\frac{\zeta^{4/3}}{(g_sM)^2}\left[\frac{c'c''}{\pi g_s} + \left(\frac{M}{2\pi}\log\frac{\Lambda_0^3}{\zeta} - \frac{K}{g_s}\right)^2\right]\right), \tag{4.57}$$

and therefore the solution is [148]

$$\tau_s^{3/2}\frac{16a\tau_s(a\tau_s - g_s)}{(4a\tau_s - g_s)^2} = \frac{\xi}{2\kappa_s} + (2\pi)^4(\pi||\Omega||^2)\frac{8q_0\zeta^{4/3}\tau_b^{5/2}}{27g_s^2\kappa_s W_0^2}, \tag{4.58}$$

$$\zeta = \Lambda_0^3\, e^{-\frac{2\pi}{g_s}\frac{K}{M} - \left(\frac{3}{4}\pm\sqrt{\frac{9}{16}-\frac{4\pi c'c''}{g_sM^2}}\right)}, \tag{4.59}$$

with the constant $q_0 = \frac{3}{8\pi^2c'}\left(\frac{3}{4} - \sqrt{\frac{9}{16} - \frac{4\pi c'c''}{g_sM^2}}\right)$.

The value of the potential at the critical points is given by

$$V_{\text{crit}} \approx \frac{g_s^4}{8\pi||\Omega||^2}\cdot\frac{3g_sW_0^2\kappa_s\sqrt{\tau_s}}{4a\mathcal{V}^3}\left(-1+\alpha+\mathcal{O}(g_s)\right), \tag{4.60}$$

with the uplift parameter α defined as

$$\alpha \equiv \frac{(\pi||\Omega||^2)(2\pi)^4}{g_s}\cdot\frac{20aq_0\zeta^{4/3}\mathcal{V}^{5/3}}{27g_s^2W_0^2\kappa_s\sqrt{\tau_s}}. \tag{4.61}$$

In order to have a dS solution, we must necessarily have $\alpha > 1$. Whether this corresponds to a local minimum (rather than a maximum or a saddle point) is related to the masses of the three fields and is analysed in [148]. Two of the mass-eigenvalues are always positive, but a second

[20] Notice that our potential differs from the one in [148], apart from the overall factor $g_s^4/(8\pi||\Omega||^2)$, in several ways: (i) our warp factor (3.108) is $e^{-4A_0} \sim (g_sM)^2$ instead of $e^{-4A_0} \sim g_sM^2$ and (ii) the moduli τ_b and τ_s differ by a factor of g_s; both (i) and (ii) are due to our convention for the Einstein metric with the $g_s = e^{\langle\phi\rangle} = e^{\Phi_0}$ (see Appendix A.1); (iii) we have the factor $\pi||\Omega||^2$, which is coming from the metric $G_{z\bar{z}}$ and is therefore *not an overall factor*, which comes from the contribution of the remaining complex structure moduli stabilised in the bulk; (iv) a factor of $(2\pi)^4$ multiplying $\zeta^{4/3}$ which comes from defining ζ in units of l_s; and (v) Λ_0 is explicit as opposed to the potential used in [148]. There is, however, a simple way to map the two potentials. We simply remove the overall factor in (4.57) (which does not affect the stabilisation of the moduli in any case) and perform the following combined transformation

$$\mathcal{V}\to g_s^{3/2}\mathcal{V}\,,\qquad \tau_s\to g_s\tau_s\,,\qquad \zeta^{4/3}\to\frac{\zeta^{4/3}}{(\pi||\Omega||^2)(2\pi)^4}\,,\qquad \Lambda_0\to\left(\frac{1}{(\pi||\Omega||^2)(2\pi)^4}\right)^{1/4}\Lambda_0\,. \tag{4.56}$$

bound is derived from requiring

$$m_3^2 \approx \frac{g_s^4}{8\pi||\Omega||^2} \cdot \frac{3g_s W_0^2 \kappa_s \sqrt{\tau_s}}{4a\mathcal{V}^3} \left(\frac{9}{4} - \alpha + \mathcal{O}(g_s)\right) > 0\,. \tag{4.62}$$

Satisfying the two conditions $V_{\rm crit} > 0$ and $m_3^2 > 0$ constrains α to be in the range[21] $\alpha \in]1, \frac{9}{4}[$.

The hierarchy in this regime becomes (3.128)

$$\frac{H_{\rm IR}}{H_{\rm UV}} \sim 1 + \frac{e^{\frac{8\pi K}{3g_s M}}}{\Lambda_0^4\, \mathcal{V}^{2/3}}\,, \tag{4.63}$$

which depends not only on the flux numbers and string coupling, but also on the volume $\mathcal{V}$ (which is now exponentially large) and the length of the conifold Λ_0, larger values of which will decrease the hierarchy between the UV and the IR.

It is worth mentioning that recent works have been tightening the constraints on a fully under-control warped LVS construction and even questioning whether these solutions can be trusted at all [176–179]. Although we will not address all these potential dangers in detail, we will comment on some of them in the explicit example below and provide an analysis similar to [176] for the new weakly-warped solutions in section 4.6.

Example

Let us give an example to illustrate this solution. Using a set of parameters which corresponds to the example given in [148], we find the expected minimum and saddle point with one unstable direction. This is summarised in Table 4.1, together with the relevant physical scales[22], and Figs. 4.5 and 4.6.

W_0	σ	g_s	M	K	Λ_0	κ_s	χ	a	A
23	0	0.23	22	4	0.071	$\frac{\sqrt{2}}{9}$	-260	$\frac{\pi}{3}$	1

Solution	τ_s	τ_b	ζ	$m_1^2 \sim m_\zeta^2$	$m_2^2 \sim m_{\tau_s}^2$	$m_3^2 \sim m_{\tau_b}^2$
Minimum	1.73	151	2.1×10^{-6}	1.19×10^{-4}	1.66×10^{-7}	1.75×10^{-13}
Saddle	1.90	263	2.1×10^{-6}	6.82×10^{-5}	4.82×10^{-8}	-1.82×10^{-14}

$\mathcal{V}$	M_s	$m_{\rm KK}$	$m_{3/2}$	$M_s^{\rm w}$	$m_{\rm KK}^{\rm tip}$	$m_{3/2}^{\rm w}$
1850	9.48×10^{-3}	2.71×10^{-3}	4.64×10^{-5}	2.04×10^{-6}	9.08×10^{-7}	9.99×10^{-9}
4271	6.24×10^{-3}	1.55×10^{-3}	2.01×10^{-5}	2.35×10^{-6}	1.04×10^{-6}	7.56×10^{-9}

Table 4.1: Solution and masses for the fields (τ_s, τ_b, ζ), and physical scales associated with the solution, for a set of parameters with $\beta \ll 1$. The mass scales are expressed in units of $M_{\rm Pl}$.

[21] Note that this agrees with [175], which corrects the result reported in [148], and [176].

[22] The physical scales are $M_s = 2\pi m_s$ (3.129), $m_{\rm KK}$ (D.15), $m_{3/2}$ (D.29), $M_s^{\rm w} = 2\pi m_s^{\rm w}$ (A.30), $m_{\rm KK}^{\rm tip}$ (D.16) and $m_{3/2}^{\rm w} = H^{-1/4}(0) m_{3/2}$.

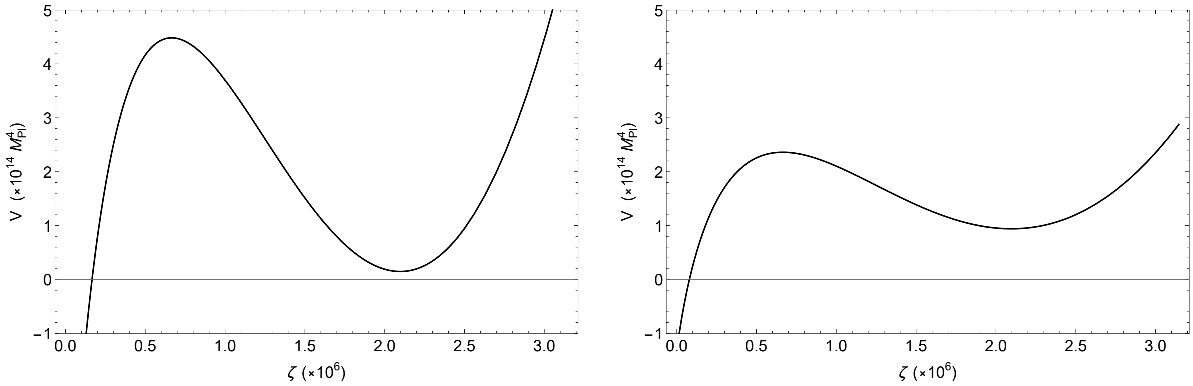


Figure 4.5: Plots of the potential (4.54) along the conifold modulus ζ direction near the minimum (left) and the saddle point (right), for the choice of parameters in Table 4.1.

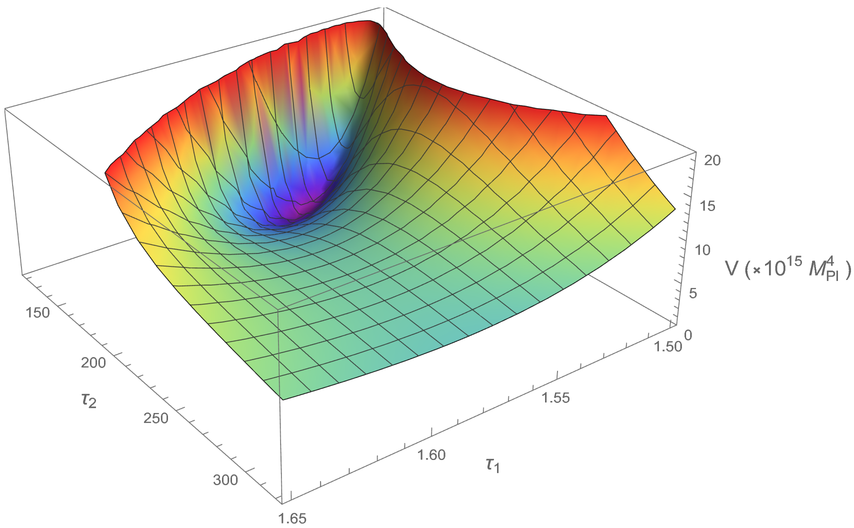


Figure 4.6: Plot of the potential (4.54) in the (τ_1, τ_2) plane (which is just a rotation of the (τ_b, τ_s) plane aligned with the eigenvectors of the Hessian matrix around the saddle point), for the choice of parameters in Table 4.1, where we can see both the minimum and saddle point solutions.

This solution gives the hierarchy of scales $m_{3/2}^{\mathrm{w}} < m_{\mathrm{KK}}^{\mathrm{tip}} < M_s^{\mathrm{w}} < m_{3/2} < m_{\mathrm{KK}} < M_s$, consistent with a 4d supergravity EFT description with cutoff $m_{\mathrm{KK}}^{\mathrm{tip}}$ (see also [180])[23]. We indeed find volumes which are exponentially large $\mathcal{V} \sim 10^3$ (note that, due to our choice of conventions, these are *both* Einstein frame *and* string frame volumes) and the volume modulus turns out to be the lightest, in particular much lighter than the deformation modulus.

Let us briefly address some possible control issues that have been raised in the recent literature. Two of them — namely the danger that the throat does not 'fit' into the bulk [140], and that singularities are induced in the bulk with no physical interpretation [144] — were both introduced in the context of KKLT, with LVS appearing to be safe due to its large volume. If the warped throat is to be glued to a compact CY_3, i.e. it describes a *sub*-region of the compact space (which is in particular well separated from the effects responsible for the non-perturbative

[23] If the masses of the other complex structure moduli and the dilaton are significantly lighter than this scale, then they should be integrated in.

contributions), then the (unwarped) radius of the throat should be smaller than the radius of the CY_3. In terms of the radial coordinate r, this means[24]

$$r_{\rm UV} = \Lambda_{\rm UV} < R_{\rm CY} = l_s\,, \tag{4.64}$$

which is equivalent to $\Lambda_0 < 1$. In [140], this inequality is expressed in terms of the flux numbers MK by choosing $r_{\rm UV}$ to be the point where the warping becomes negligible — this is indeed the minimum possible value of $r_{\rm UV}$. Since far away from the tip the warp factor (3.108) becomes [181]

$$e^{-4A_0} \approx \frac{L^4}{r^4}\left[1 + \frac{3g_sM}{8\pi K} + \frac{3g_sM}{2\pi K}\log\frac{r}{r_{\rm UV}}\right]\,, \tag{4.65}$$

where

$$L^4 \equiv \frac{27\pi}{4}\frac{g_sMK}{(2\pi)^4}\,l_s^4\,, \tag{4.66}$$

the warping becomes $\mathcal{O}(1)$ when $e^{-4A_0} \sim \mathcal{V}^{2/3} \implies r \sim L/\mathcal{V}^{1/6}$. Therefore, we must have[25]

$$\frac{L}{l_s\cdot\mathcal{V}^{1/6}} < \Lambda_0 < 1 \qquad \implies \qquad g_sMK < \frac{4}{27\pi}(2\pi)^4\mathcal{V}^{2/3}\,. \tag{4.67}$$

One can check using the parameters in Table 4.1 that this is indeed satisfied in our example, although it is worth emphasising that we *are not* cutting the deformed conifold at the point where the warping becomes negligible, but rather at a point farther away from the tip — it is important to check that the actual choice of Λ_0 (and not just L) satisfies this consistency condition. This presents a bigger danger for KKLT, where the volumes are much smaller than in LVS.

On the other hand, the singular-bulk problem [144] arises in KKLT because it is the volume modulus itself that fixes the size $\mathcal{V}_\Sigma$ of the 4-cycle Σ on which the non-perturbative effects that stabilise it are generated. In order for the $\overline{\rm D3}$-brane energy (fixed by the warp factor and thus the flux numbers at the tip) to uplift the KKLT AdS vacuum to almost Minkowski, we must have $V_{\rm AdS} \sim V_{\rm uplift}$ which translates into a relation between the 4-cycle volume and the flux numbers that source the warping. A second relation comes from the need to cancel the positive D3-brane charge sourced by the fluxes within the throat. Together, these relations imply that the warp factor on Σ satisfies

$$\left.\frac{|\partial H|}{H}\right|_\Sigma \gtrsim (g_sM^2)\frac{\mathcal{V}_\Sigma}{\mathcal{V}^{2/3}}\,, \tag{4.68}$$

which becomes large when $\mathcal{V}_\Sigma \sim \mathcal{V}^{2/3}$ as in KKLT, due to the requirement $g_sM \gg 1$ for a well under control supergravity approximation. This leads to a singularity that appears to be far from any sources that might resolve it. The LVS solution (and thus our current example)

[24] Note that in our background metric (3.127) $\mathcal{V}$ is factored out of the metric g_{mn}, which either describes the deformed conifold of size $r_{\rm UV}$ or the CY_3 whose coordinates were normalised such that $V_{\rm CY} = l_s^6$. One might want to include a factor of π from approximating the bulk with a torus, as pointed out in Appendix A of [144] and as consistent with the estimate of m_{KK} made in (D.15) — in that case the size of a circle in the torus $2\pi R_{\rm CY} > 2r_{\rm UV} \implies r_{\rm UV} < \pi R_{\rm CY}$.

[25] Note the factor of g_s, which is related to our choice of conventions for the change of frames (Appendix A.1).

circumvents this problem due to the hierarchy $\mathcal{V}_\Sigma = \tau_s \ll \tau_b \sim \mathcal{V}^{2/3}$.

There were however also some results specific to LVS. The analysis in [176] shows that several different corrections to the LVS potential can become relevant in different regions of parameter space, so that they cannot all be made sufficiently small at the same time. This was used to argue that the concrete example given in [148] and therefore our current example cannot be trusted on the basis of parametric suppression of these corrections alone, and one must instead check the specific coefficients of these corrections in order to judge whether or not they invalidate this solution. We will have more to say about this in section 4.6, where we perform a similar analysis for our weakly-warped solution.

Let us here focus on the concrete constraint on the tadpole contribution MK proposed in [178] based on two of these potentially dangerous corrections. It is shown that taking into account a warping correction arising from 10d R^4 terms and higher F-term corrections to the scalar potential, the negative D3-charge required to cancel the tadpole is bounded from below,

$$|Q_3| > MK = N_* \left(\frac{1}{3} \log N_* + \frac{5}{3} \log c_N + \log a - \frac{2}{3} \log \kappa_s + 8.2 + \mathcal{O}(\log(\log)) \right) , \tag{4.69}$$

where $N_* = \frac{9 g_s M^2}{16\pi}$ and $c_N \gg 1$ for the corrections to be parametrically suppressed. For the parameters used in our example and staying at the limit of control $c_N = 1$, one gets $|Q_3| \gtrsim 209$ (cf. $|Q_3| = 149$ for the explicit construction in [148] that we are using for this example). Note that for our choice of parameters $MK = 88$, which seems to be in contradiction with the above bound — this is because the corrections that are assumed to be under control when deriving (4.69) are actually not suppressed in this example [176]. Although this bound already puts some pressure on specific LVS constructions regarding the topological structure of the CY_3, it was argued in [179] that taking into account all potentially dangerous corrections greatly strengthens the bound and may require tadpole charges of up to $\mathcal{O}(10^6)$. This is, of course, problematic in the context of the Tadpole Conjecture [100].

4.5.3 Weakly-warped solutions

We now consider the new limit where the warping is subdominant in the Kähler metric for the conifold modulus (4.20), $\beta \gg 1$. In this case, the potential becomes[26]

$$V \approx \frac{g_s^4}{8\pi ||\Omega||^2} \left\{ \frac{8a^2 A^2 \sqrt{\tau_s} e^{-2\frac{a}{g_s}\tau_s}}{3\kappa_s g_s^2 \mathcal{V}} - \frac{4aAW_0 \tau_s e^{-\frac{a}{g_s}\tau_s}}{g_s \mathcal{V}^2} + \frac{3\xi W_0^2}{4\mathcal{V}^3} \right.$$

[26] Note that the warp factor in $V_{\overline{\mathrm{D3}}}$ can be written as

$$H = 1 + \frac{2}{c' c''} \frac{\log \frac{\Lambda_0^3}{\zeta}}{\beta} . \tag{4.70}$$

Here we assume that $\beta \gg \log \frac{\Lambda_0^3}{\zeta}$ and expand H accordingly.

$$+\frac{\pi||\Omega||^2}{g_s}\frac{1}{\mathcal{V}^2\log^2\frac{\Lambda_0^3}{\zeta}}\left(\log\frac{\Lambda_0^3}{\zeta}-\frac{1}{(2\pi)^4}\frac{c'(g_sM)^2}{\mathcal{V}^{2/3}\zeta^{4/3}}\right)\left(\frac{M}{2\pi}\log\frac{\Lambda_0^3}{\zeta}-\frac{k}{g_s}\right)^2$$
$$+\frac{||\Omega||^2}{g_s}\frac{2}{\mathcal{V}^2}\left(1-\frac{2}{c'c''}\frac{1}{(2\pi)^4}\frac{c'(g_sM)^2}{\mathcal{V}^{2/3}\zeta^{4/3}}\right)\Bigg\}\,, \tag{4.71}$$

which has a minimum at

$$\tau_s^{3/2}\frac{16a\tau_s(a\tau_s-g_s)}{(4a\tau_s-g_s)^2}\approx\frac{\xi}{2\kappa_s}+\frac{8||\Omega||^2\tau_b^{3/2}}{9g_s\kappa_sW_0^2}+\mathcal{O}\Big(\frac{1}{\beta}\Big)\,, \tag{4.72}$$

$$\zeta\approx\Lambda_0^3\exp\left\{-\frac{2\pi K}{g_sM}\right\}\left(1-\frac{4Ke^{\frac{8\pi K}{3g_sM}}c_{\rm D3}}{3\pi^2c'M\Lambda_0^4\,\tau_b}\right)\,, \tag{4.73}$$

together with the solution for τ_b (4.55). Notice that the solution for τ_s is implicit, since there is a dependence on τ_s in the second term through τ_b. If (4.72) is a solution, the function

$$F(\tau_s)=-\tau_s^{3/2}+\frac{\xi}{2\kappa_s}+\frac{2||\Omega||^2}{3aAW_0}\sqrt{\tau_s}\;e^{\frac{a}{g_s}\tau_s}\,, \tag{4.74}$$

where we have substituted τ_b (4.55) in the limit $a\tau_s\gg g_s$, must have a root. In particular, the exponential term cannot be too large, since it has to balance against the $\tau_s^{3/2}$, which suggests that the combination AW_0 must be exponentially large. We can make a better estimate by noting that $F(\tau_s)$ has a minimum, which must be non-positive for a root to exist. Again in the limit $a\tau_s\gg g_s$, this leads to the condition

$$AW_0>\frac{4||\Omega||^2}{9g_s}\exp\left\{\frac{a}{g_s}\Big(\frac{\xi}{2\kappa_s}\Big)^{2/3}\right\}\,, \tag{4.75}$$

which indeed corresponds to having AW_0 exponentially large. In terms of the volume modulus $\mathcal{V}$ vev this condition becomes

$$\frac{8||\Omega||^2\mathcal{V}}{9g_s\kappa_sW_0^2}<\frac{3g_s}{2}\left(\frac{\xi}{2\kappa_s}\right)^{1/3}\,, \tag{4.76}$$

so that the second term in (4.72) can be treated as a correction to the leading order solution $\tau_s^{(0)}\approx\Big(\frac{\xi}{2\kappa_s}\Big)^{2/3}$.

At this solution the vacuum energy is given by

$$V_{\rm crit}=\frac{g_s^4}{8\pi||\Omega||^2}\cdot\frac{3g_s\kappa_sW_0^2\sqrt{\tau_s}}{4a\mathcal{V}^3}\Big(-1+\tilde{\alpha}+\mathcal{O}(g_s)+\mathcal{O}(1/\beta)\Big)\,, \tag{4.77}$$

where we define the uplift parameter $\tilde{\alpha}$ as

$$\tilde{\alpha}=c_{\rm D3}\frac{\pi||\Omega||^2}{g_s}\cdot\frac{8a}{9\pi\kappa_s\sqrt{\tau_s}}\frac{\mathcal{V}}{g_sW_0^2}\,. \tag{4.78}$$

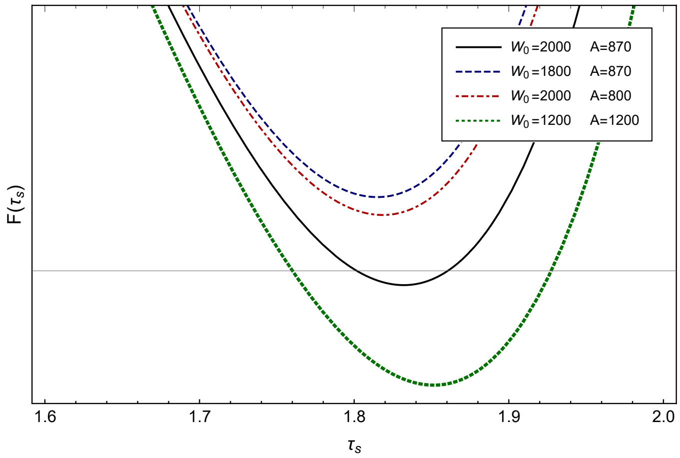


Figure 4.7: The plot shows (4.74) for different choices of the parameters W_0 and A, with all other parameters fixed. Critical points for the scalar potential (4.71) exist when $F(\tau_s) = 0$. We see that the product AW_0 needs to be large enough for a solution to exist (4.75).

Once again, a dS vacuum requires $\tilde{\alpha} > 1$. Although the definition of $\tilde{\alpha}$ differs from the one of α in the strongly-warped case (4.61), it still encodes the brane uplift and its back-reaction on the solution — this is clear from its dependence on c_{D3}. It is here simply adapted to the weakly-warped regime, making explicit the need for large W_0 and/or A (through the solution for $\mathcal{V}$, cf. 4.55) in order to suppress the effects of the brane. Keeping the solution consistent with the regime of validity of our 4d EFT requires $m_{3/2} < m_{\text{KK}}$, so that the gravitino mass remains below the cutoff and is not integrated out. Using the scales (D.15) and (4.51), together with the solution (4.55) and (4.72), and the condition (4.75) required for a solution to exist, we can write the ratio

$$\left(\frac{m_{3/2}}{m_{\text{KK}}}\right)^3 = \frac{g_s^3 W_0^3}{\sqrt{8}(2\pi)^3 ||\Omega||^3 \mathcal{V}} \gtrsim \frac{4 g_s W_0}{27(2\pi)^3 \xi^{1/3}} \frac{a}{2^{2/3} \kappa_s^{2/3}}\,, \tag{4.79}$$

which, together with large W_0, makes it difficult to find a region in parameter space for which the supergravity description remains valid. However, we show a working example below.

In order to determine whether this critical point is a minimum, we compute the eigenvalues of the matrix[27]

$$\mathbf{M} = h^{ab} \frac{\partial^2 V}{\partial \varphi^a \partial \varphi^b}\bigg|_{\text{crit}} \tag{4.80}$$

where $\varphi^a = \{\tau_b, \tau_s, \zeta\}$ and h_{ab} is the field space metric defined via $K_{i\bar{j}} \partial \Phi^i \partial \bar{\Phi}^{\bar{j}} = \frac{1}{2} h_{ab} \partial \varphi^a \partial \varphi^b$. Although the procedure is straightforward, we were not able to simplify the result enough to make the analytical expression useful — we will therefore confirm numerically that for the example provided below, all eigenvalues are stable and we do indeed have a minimum of the potential.

[27]Note that this is *not* the mass matrix, but rather a matrix with the same eigenvalues, i.e. whose eigenvalues correspond to the masses of the canonically normalised scalars (see e.g. Appendix C of [175]).

Example

We choose a set of parameters which guarantees both $V_{\text{crit}} > 0$ and that all eigenvalues of the mass matrix are positive, and thus a dS minimum. We have parameters associated with the fluxes for the remaining complex structure moduli and the axio-dilaton (W_0, g_s), with the conifold (M, K, Λ_0) and with the CY$_3$ and Kähler moduli (κ_s, ξ, a, A), where we recall $\xi = -\frac{\chi(\text{CY}_3)\zeta(3)}{2(2\pi)^3}$. The fixed parameters in the potential are $c' = 1.18$, $c'' = 1.75$, $||\Omega||^2 = 8$.

W_0	σ	g_s	M	K	Λ_0	κ_s	χ	a	A
2000	0	0.17	16	2	0.43	$\frac{\sqrt{2}}{9}$	-280	$\frac{\pi}{3}$	870

Table 4.2: Choice of parameters for the potential (4.54), with $\beta \gg 1$.

For the parameter set in Table 4.2, the critical points of the full potential (4.54) are summarised in Table 4.3. While one critical point is a minimum, the second is a saddle point with one unstable direction. Note that for both cases we indeed find $\beta > 1$ ($\beta_{\text{min}} \approx 7$ and $\beta_{\text{saddle}} \approx 15$).

τ_s	τ_b	ζ	V_{crit}	$m_1^2 \sim m_\zeta^2$	$m_2^2 \sim m_{\tau_s}^2$	m_3^2
1.80	239	4.17×10^{-4}	1.70×10^{-13}	9.23×10^{-5}	4.87×10^{-4}	4.32×10^{-11}
1.92	388	5.19×10^{-4}	1.13×10^{-12}	1.94×10^{-5}	1.59×10^{-4}	-5.01×10^{-12}

Table 4.3: Solutions and masses (in units of M_{Pl}) for the fields (τ_s, τ_b, ζ) for the parameter set in Table 4.2. Since all mass-squareds are positive, the first solution is metastable, whereas the second corresponds to a saddle point.

$\mathcal{V}$	M_s	$m_{\text{KK}}^{\text{bulk}}$	$m_{3/2}$	M_s^{w}	$m_{\text{KK}}^{\text{tip}}$	$m_{3/2}^{\text{w}}$
3684	4.96×10^{-3}	1.26×10^{-3}	1.11×10^{-3}	2.87×10^{-3}	1.74×10^{-3}	6.40×10^{-4}
7655	3.44×10^{-3}	7.76×10^{-3}	5.33×10^{-4}	2.58×10^{-3}	1.56×10^{-3}	3.99×10^{-4}

Table 4.4: Physical scales associated with the solutions in Table 4.3, for the parameter set in Table 4.2. The mass scales are expressed in units of M_{Pl}

From Table 4.4 we see that the cutoff for the EFT is now $m_{\text{KK}} \sim m_{\text{KK}}^{\text{tip}}$, which reflects the fact that this solution has weak warping. Note that $\frac{m_{3/2}}{m_{\text{KK}}} \approx 0.88$, which is marginally consistent with the 4d supergravity description (see also [180]). We can check that the conifold fits the bulk as in the strongly warped case, although it is longer in this example ($\Lambda_0 = 0.41$) [140, 144]. As expected from the fact that we have weak warping and the hierarchy $\tau_s \ll \tau_b \sim \mathcal{V}^{2/3}$, this solution avoids the singular-bulk problem [144, 145]. In section 4.6, we will analyse how robust this weakly-warped solution is when different corrections are taken into account, similarly to what was done in [176] for the strongly-warped case. Applying the LVS parametric tadpole constraint [178] to this example, again staying at the limit of control $c_N = 1$, we find $|Q_3| > 79$, which again is bigger than our $MK = 32$ — we will see in 4.6 that the correction arising from higher F-terms is in fact not suppressed in this weakly-warped regime, explaining the discrepancy.

In Fig. 4.8 we plot the potential along the ζ direction with (τ_s, τ_b) fixed at the minimum and the saddle point, while in Fig. 4.9 we plot the potential along the (τ_s, τ_b) directions with ζ fixed at the minimum, where we can clearly see both the minimum and the saddle point. We also see that the minimum for ζ does not change significantly between the minimum and the saddle point, although it becomes slightly lighter.

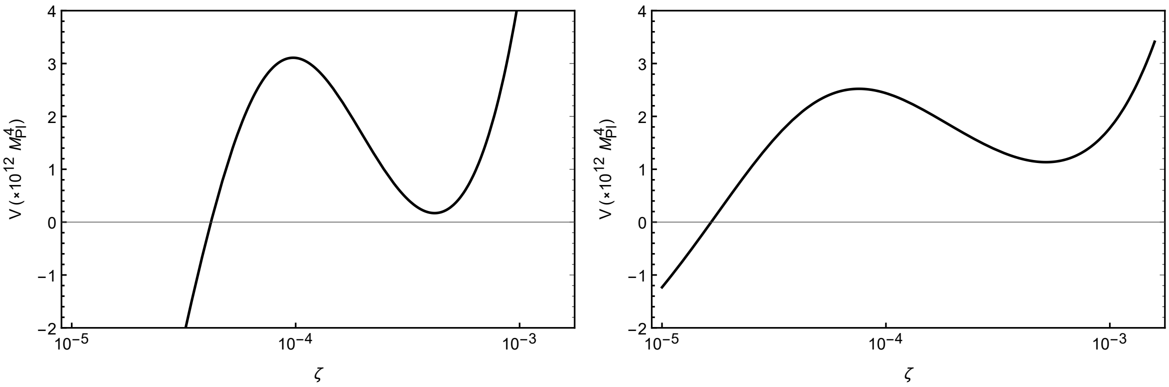


Figure 4.8: Plots of the potential (4.54) in each of the 3 directions (τ_s, τ_b, ζ), for the parameter set in Table 4.2, around the minimum (left) and the saddle (right).

4.6 Dangerous corrections?

Although there seem to exist explicit constructions of dS vacua with an $\overline{\text{D3}}$-brane uplift in the context of LVS, both in the weakly [172] and strongly-warped regimes [148], these implicitly rely on the assumption that all supposedly subleading corrections to the scalar potential are actually subleading and will not destabilise the solutions. The question of whether this assumption is justified was analysed in [176] in the strongly-warped regime studied in [148], by taking into account various types of corrections to the LVS potential (including two Kähler moduli, a deformation modulus and a nilpotent superfield describing the brane uplift). The conclusion for the strongly-warped regime was that some corrections do not have any parametric suppression, and moreover those that do are never suppressed simultaneously for any choice of parameters — they can therefore never be *all* self-consistently neglected.

It is worth fleshing out the main point in the argument of [176]. If a correction to the LVS potential *can* arise from some string theory effect (e.g. one-loop corrections), one of two things must happen — either this correction is parametrically suppressed for all quantities in the solution of interest (e.g. moduli vevs, masses, vacuum energy) and can therefore be neglected in the appropriate limit (e.g. large volume or weak coupling) within the EFT, or else one *must* not only compute this correction explicitly and include it in the analysis, but also worry about next-to-leading order corrections. While the result of [176] does not show that finding a stable

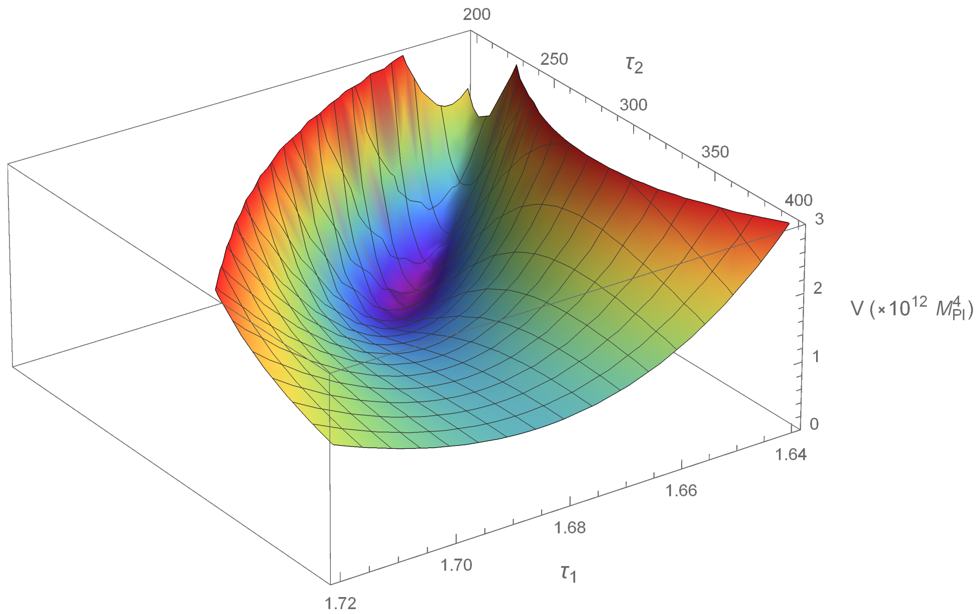


Figure 4.9: Plot of the potential (4.54) in the (τ_1, τ_2) plane (which is just a rotation of the (τ_b, τ_s) plane aligned with the eigenvectors of the Hessian matrix), for the parameters in Table 4.2 (in Planck units).

strongly-warped dS solution in LVS is impossible, it means that one must take at least some of these corrections into account, compute their coefficients explicitly and keep them in the EFT. This constitutes a challenge because, not only are these coefficients typically hard to compute explicitly, but also the known solutions in the LVS may be absent or at least get corrected in significant ways by these contributions.

It therefore becomes relevant to ask whether this is also the case in the weakly-warped regime [172]. In order to address this question, we will briefly outline the possible corrections (for more details see [176]) together with the corresponding terms correcting the scalar potential computed in the weakly-warped case and revisit the weakly-warped solution outlined in the previous section.

Curvature/loop corrections to the Kähler potential

String-loop corrections at order α'^2 from the exchange of KK modes between D7/D3 branes or O7/O3 planes [182–186] enter the Kähler potential (4.50) as [187–190]

$$\delta K = C_s^{\rm KK} \frac{g_s^2 \sqrt{\tau_s}}{\mathcal{V}} + C_b^{\rm KK} \frac{g_s^2 \sqrt{\tau_b}}{\mathcal{V}} \,, \tag{4.81}$$

and thus the scalar potential is corrected as

$$\delta V = C_s^{\rm KK} \cdot \frac{g_s^2}{6\kappa_s \tau_s} \cdot \frac{8a^2 A^2 \sqrt{\tau_s} e^{-2\frac{a\tau_s}{g_s}}}{3 g_s^2 \kappa_s \mathcal{V}} + C_s^{\rm KK} \cdot \frac{g_s^2}{3\kappa_s \tau_s} \cdot \frac{4aAW_0 \tau_s e^{-\frac{a\tau_s}{g_s}}}{g_s \mathcal{V}^2}$$

$$+\left\{ (C_s^{\rm KK})^2 \cdot \frac{2g_s^4}{9\kappa_s\sqrt{\tau_s}} + C_b^{\rm KK} \cdot \frac{8c' g_s^2 (g_s M)^2 \zeta^{2/3}}{\mathcal{V}^{1/3}} \right\} \cdot \frac{3W_0^2}{4\mathcal{V}^3}\,. \tag{4.82}$$

Note that in principle the coefficients $\{C_s^{\rm KK}, C_b^{\rm KK}\}$ depend on the complex structure moduli; however, as this dependence is unkown, we will simply assume that $C_s^{\rm KK}, C_b^{\rm KK} \sim \mathcal{O}(1)$ and neglect the derivative contributions.

We write each correction term such that its relation to the terms in the uncorrected LVS potential is manifest, isolating clearly each suppression factor. For example, all the terms above are suppressed by a factor of g_s^2 compared to their LVS counterparts. The $C_b^{\rm KK}$ term is also suppressed by the ratio $\zeta^{2/3}/\mathcal{V}^{1/3}$ but enhanced by $(g_s M)^2$.

On the other hand, threshold corrections to gauge couplings lead to the one-loop field redefinition of the Kähler modulus $\tau_s^{\rm new} = \tau_s^{\rm old} + C_s^{\rm log} \log \mathcal{V}$, which implies a correction to the Kähler potential once the physical volume is expressed in terms of the corrected modulus, $\mathcal{V} = \tau_b^{3/2} - \kappa_s(\tau_s - C_s^{\rm log} \log \mathcal{V})^{3/2}$ [191]. It contributes to the scalar potential as[28]

$$\delta V = C_s^{\rm log} \cdot \frac{g_s \log \mathcal{V}}{2\tau_s} \cdot \frac{8a^2A^2\sqrt{\tau_s}e^{-2\frac{a\tau_s}{g_s}}}{3g_s^2\kappa_s\mathcal{V}} + C_s^{\rm log} \cdot \frac{g_s \log \mathcal{V}}{\tau_s} \cdot \frac{4aAW_0\tau_s e^{-\frac{a\tau_s}{g_s}}}{g_s\mathcal{V}^2}$$
$$- C_s^{\rm log} \cdot 6 g_s \kappa_s \sqrt{\tau_s} \cdot \frac{3W_0^2}{4\mathcal{V}^3} \tag{4.83}$$

Recalling the LVS solution $\log \mathcal{V} \sim \frac{a\tau_s}{g_s}$, there seems to be no parametric suppression in the correction terms in the first line.

There are also corrections at order α'^3 from different sources [183–185, 192, 193], which contribute at leading order as a redefinition of $\xi \to \xi - \Delta\xi + C_1^\xi g_s \log \mathcal{V} + C_2^\xi g_s$ in K (4.50),

$$\delta V = (-\Delta\xi + C_1^\xi g_s \log \mathcal{V} + C_2^\xi g_s) \cdot \frac{3W_0^2}{4\mathcal{V}^3}\,. \tag{4.84}$$

Notice again that there seems to be no suppression for the $\Delta\xi$ and the C_1^ξ corrections.

Curvature corrections to the gauge-kinetic function

The non-perturbative superpotential which is a key ingredient in the stabilisation of the Kähler moduli arises from gaugino condensation on a stack of D7-branes [194–196]. Therefore curvature corrections to the D7-brane action [197–201], which correct the gauge-kinetic function of the D7-branes will affect this non-perturbative contribution. The gauge-kinetic function becomes

[28] This field redefinition leads to a non-linear relation between the volume $\mathcal{V}$ and the Kähler moduli (τ_b, τ_s). Treating this redefinition as a small correction, one can solve it perturbatively for $\mathcal{V} = \tau_b^{3/2} - \kappa_s\tau_s^{3/2} + \mathcal{O}(C_s^{\rm log})$ and use this leading order solution in the correction. Hence the volume becomes at leading order $\mathcal{V} \approx \tau_b^{3/2} - \kappa_s\left(\tau_s - C_s^{\rm log}\log\left(\tau_b^{3/2} - \kappa_s^{3/2}\right)\right)^{3/2}$.

$f_s = T_s - \frac{\chi_s}{24}\tau$, so that the generated superpotential gets corrected as

$$W_{np} = A \cdot e^{-\frac{a}{g_s}\left(\tau_s - \frac{\chi_s}{24 g_s}\right)} \,, \tag{4.85}$$

which is equivalent to a redefinition of $A \to A \cdot e^{\frac{a\chi_s}{24 g_s^2}}$. It is worth recalling that our weakly-warped solution required somewhat large values of the parameter A (4.75) and one might therefore expect that this correction in particular will benefit, rather than obstruct, these weakly-warped solutions.

Higher F-terms

There are also corrections arising from higher F-terms associated with 4 superspace derivatives in the 4d EFT [180, 202, 203]. These could come for example from integrating out KK modes [180] or from 10d 8-derivative terms with powers of G_3 [203], which would lead to such a contribution in the potential. The scalar potential gets corrected as

$$\delta V = C^F \cdot \frac{8 g_s^2 W_0^2}{3\mathcal{V}^{2/3}} \cdot \frac{3W_0^2}{4\mathcal{V}^3} \,. \tag{4.86}$$

This correction is suppressed relative to the LVS potential by the factor

$$\frac{m_{3/2}^2}{m_{KK}^2} \sim \frac{g_s^2 W_0^2}{\mathcal{V}^{2/3}} \,, \tag{4.87}$$

which one would generically want to be small in order to keep the gravitino in the EFT valid at energies $E \ll m_{KK}$ (and therefore preserve $\mathcal{N} = 1$ supersymmetry).

Notice that so far only one of the corrections is warping-related, the correction linear in C_b^{KK} which depends on the warping due to its dependence on ζ. It does indeed arise from the warping correction to the deformation modulus metric, which mixes this modulus with the volume $\mathcal{V}$ and was crucial in the discussion of our weakly-warped solutions. One would therefore not expect the effect of the other corrections discussed so far to be dependent on which warping regime we consider and, for those corrections, the only effect of having weak warping rather than strong warping is on the region of interest in parameter space. However, we will see that in order to find a solution in the weakly-warped regime, we are driven towards a region of parameter space where some of these corrections cannot be suppressed.

On the other hand, the two remaining corrections are intrinsically warping-related and one would therefore expect them to play a more important role in the distinction between the two regimes.

Curvature/warping corrections from conifold-flux backreaction

The charges responsible for the warping (such as branes, O-planes and fluxes) can have large contributions to the 10d curvature even at large volumes and result in large curvature corrections [140, 144]. The charge responsible for the warped KS throat is the flux contribution KM, which is large in the strongly-warped regime. This leads to corrections to the scalar potential, e.g. from dimensionally reducing the 10d R^4 terms of Type IIB, that take the form [176]

$$\delta V = C^{\text{flux}} \cdot \frac{5 g_s KM}{\mathcal{V}^{2/3}} \cdot \frac{3 W_0^2}{4\mathcal{V}^3} \,. \tag{4.88}$$

Since in the weakly-warped regime, one does not require very large charges KM, this correction is expected to be less dangerous than in the strongly-warped case.[29]

Curvature corrections in the conifold region

Since the radius of the S^3 at the tip of the conifold is $R^2_{S^3} \sim (g_s M)\alpha'$, one expects α' curvature corrections that are suppressed by $(g_s M)^{-1}$. E.g. the R^2 corrections to the DBI action of the $\overline{\text{D3}}$-brane in the warped deformed conifold background correct the brane tension with a term suppressed by $(g_s M)^{-2}$ [176].[30] In practice, this corresponds to the redefinition

$$c_{\text{D3}} \to c_{\text{D3}} \Big(1 + \frac{C^{\text{con}}}{(g_s M)^2}\Big) \,, \tag{4.89}$$

which is easily incorporated in both the potential and the solutions. Although the weakly-warped solutions do not require large KM, they still require the supergravity approximation to be under control near the tip of the conifold, which is controlled by the radius of the S^3 at the tip $R^2_{S^3} \gg 1$ — this means that even though the flux number M is in general lower in the weakly-warped case, it must still be large enough to make $g_s M \gg 1$ and keep this correction term suppressed in the brane action.

Let us now look at the solutions for the corrected weakly-warped LVS potential, as well as at the vacuum energy associated with these vevs, and examine the suppression of each correction on these quantities.

[29] In [178] it is argued that one should expect this warping correction to scale with the Euler number $\chi(\text{CY}_3)$, as long as one assumes a slowly varying warp factor such that cancellation between contributions from different regions of the compact space is not expected. The form of the correction used in [178] for the LVS parametric tadpole constraint therefore differs by a factor of ξ.

[30] These correspond to the $(\alpha')^2$ curvature corrections to the brane action studied in [165] that lower the tension of the brane and provide an alternative for warped throats.

4.6.1 Conifold modulus stabilisation

The only corrections involving the conifold modulus and therefore affecting its stabilisation are the ones associated with $\{C_b^{\rm KK}, C^{\rm con}\}$. While $C^{\rm con}$ is simply a shift of $c_{\rm D3}$ and so it can be trivially included in the solution (4.73), the correction $C_b^{\rm KK}$ can be taken into account by solving perturbatively in a similar way to how the $1/\beta$ corrections were obtained. We find that the vev of ζ gets corrected as

$$\zeta \approx \zeta_{\rm GKP}\left(1 - c_{\rm D3}\left(1+\frac{C^{\rm con}}{(g_sM)^2}\right)\frac{4K}{3\pi^2 c'' M\ \zeta_{\rm GKP}^{4/3}\mathcal{V}^{2/3}} - C_b^{\rm KK}\cdot\frac{2c'(2\pi)^2}{\pi||\Omega||^2}\frac{2\pi K}{g_sM}\frac{g_s^4\zeta_{\rm GKP}^{2/3}W_0^2}{\mathcal{V}^{4/3}}\right), \tag{4.90}$$

where $\zeta_{\rm GKP} = \Lambda_0^3\ e^{-\frac{2\pi K}{g_sM}}$. Notice that the $C^{\rm con}$ correction is not only suppressed by its own control parameter $(g_sM)^{-1}$, but also by $1/\beta$ since it arises from the $\overline{\rm D3}$-brane backreaction which itself contributes at $\mathcal{O}(1/\beta)$. As for $C_b^{\rm KK}$, it appears to be heavily suppressed by the factor $\frac{\zeta_{\rm GKP}^{2/3}}{\mathcal{V}^{2/3}}\cdot\frac{g_s^4W_0^2}{\mathcal{V}^{2/3}}$.

In the strongly-warped regime [176], the corrections change the bound on g_sM^2 required to prevent the brane uplift from causing a runaway for the deformation modulus (4.37) and thus have a direct effect on the allowed values of M. Since at weak warping the $\overline{\rm D3}$-brane backreaction does not lead to a tadpole bound on the fluxes and this remains true once the corrections are taken into account, they have no effect on the range of the flux M.

4.6.2 Kähler moduli

Since the formal solution for $\mathcal{V}$ follows directly from $\partial_{\tau_s}V = 0$, one can also infer that only a few of the corrections will appear — in particular, it will receive explicit corrections from $\{C_s^{\rm KK}, C_s^{\log}, \chi_s\}$ only. Of course, due to its dependence on τ_s, which will receive corrections from all terms, $\mathcal{V}$ also receives implicit corrections from every term. This is true for ζ as well, since the subleading terms in its vev (4.90) depend on $\mathcal{V}$.

Solving perturbatively to leading order in all corrections, we find

$$\mathcal{V} \approx \left(\frac{3(a\tau_s - g_s)}{4a\tau_s - g_s} + C_s^{\rm KK}\cdot\frac{g_s^2}{8\kappa_s\tau_s} - C_s^{\log}\cdot\frac{9a}{8}\right)\frac{g_sW_0\kappa_s\sqrt{\tau_s}}{aA}\cdot e^{\frac{a}{g_s}\left(\tau_s - \frac{\chi_s}{24g_s}\right)} \tag{4.91}$$

$$\begin{aligned}\tau_s \approx\ & \frac{\hat{\xi}^{2/3}}{(2\kappa_s)^{2/3}} + \frac{(1+2\tilde{\alpha})g_s}{3a} + \mathcal{O}\left(\frac{1}{\beta}\right) + \mathcal{O}(g_s^2) \\ & - C_s^{\rm KK}\cdot\frac{g_s^2}{3\kappa_s} + C_s^{\log}\cdot\frac{5a}{3}\left(\frac{\hat{\xi}}{2\kappa_s}\right)^{2/3} + C_1^{\xi}\cdot\frac{a}{3\kappa_s}\left(\frac{\hat{\xi}}{2\kappa_s}\right)^{1/3} + C_2^{\xi}\cdot\frac{g_s}{3\kappa_s}\left(\frac{\hat{\xi}}{2\kappa_s}\right)^{-1/3} \\ & + C^{\rm flux}\cdot\frac{55}{27\kappa_s}\frac{g_sKM}{\mathcal{V}^{2/3}}\left(\frac{\hat{\xi}}{2\kappa_s}\right)^{-1/3} + C^F\cdot\frac{88}{81\kappa_s}\frac{g_s^2W_0^2}{\mathcal{V}^{2/3}}\left(\frac{\hat{\xi}}{2\kappa_s}\right)^{-1/3}\end{aligned} \tag{4.92}$$

$$- C^{\rm con} \cdot \frac{2g_s\tilde{\alpha}}{3a(g_sM)^2} + C_b^{\rm KK} \cdot \frac{80c'(g_sM)^2}{27\kappa_s} \cdot \frac{g_s^2\zeta^{2/3}}{\mathcal{V}^{1/3}} \left(\frac{\hat{\xi}}{2\kappa_s}\right)^{-1/3}$$

with $\tilde{\alpha}$ given by (4.78) and $\hat{\xi} = \xi - \Delta\xi$. The solution for $\mathcal{V}$ (4.91) matches[31] the one found in [176] up to factors of g_s — this agrees with our expectation that the explicit form of the $\{C_s^{\rm KK}, C_s^{\rm log}, \chi_s\}$ corrections is independent of the warping regime. One can immediately see that, although the $C_s^{\rm KK}$ correction is suppressed by g_s^2, neither the $C_s^{\rm log}$ nor the χ_s corrections are suppressed — following the logic presented above, one must therefore include these corrections in the EFT analysis. Although the χ_s correction is not only not suppressed but actually enhanced by a factor of $1/g_s$, the fact that we know its explicit form means that it can be taken into account when looking for a solution and, as we saw in (4.85), it can be seen as a rescalling of the parameter A in the non-perturbative contribution to the superpotential. In fact, given the need for exponentially large AW_0 (4.75), this seems to help rather than hurt weakly-warped solutions.[32] The $C_s^{\rm log}$ correction, on the other hand, is not generically known explicitly and therefore one cannot follow the same procedure — this coefficient would have to be computed for the compact geometry of interest in order to be included in the EFT analysis.

The different g_s factors in the solution with respect to [176] arise from our choice of frame conventions, with τ_b and τ_s differing from the ones in [176] by a factor of g_s, i.e. $\tau_i^{\rm (ours)} = g_s\tau_i^{[176]}$ and $\mathcal{V}^{\rm (ours)} = g_s^{3/2}\mathcal{V}^{[176]}$ (see Appendix A.1). Recall that in these conventions, the Einstein-frame volumes and the string-frame volumes are the same at the vev, so that deciding e.g. whether α'-corrections are under control for a given solution can be done directly in terms of the Einstein-frame volumes.[33] It also makes explicit the $1/g_s$ factor in the non-perturbative superpotential, rather than it appearing in the solution for τ_s — both pictures are ultimately consistent since this factor of $1/g_s$ will appear in the potential one way or another. Because of this, however, our solution for τ_s (4.92) differs from the one in [176] by an overall factor of g_s, as can be most easily seen by looking at the terms that behave similarly in both warping regimes. One should therefore not think of this extra factor of g_s as extra suppression of any of the corrections, since it is really their relative size compared to the leading other terms that is relevant for control.

The first line of (4.92) shows the expected $\mathcal{O}(g_s)$ correction, with the $\overline{\rm D3}$-brane backreaction on τ_s encoded by the uplift parameter $\tilde{\alpha}$. The corrections on the second and third lines are suppressed in the same way as in the strongly-warped regime,[34] as expected from the fact that they do not involve the interplay between ζ and $\mathcal{V}$. While the $\{C_s^{\rm KK}, C^{\rm flux}, C^F\}$ terms are suppressed by powers of g_s and/or $\mathcal{V}$, the $\{C_s^{\rm log}, C_1^{\xi}, C_2^{\xi}\}$ are dangerous since they either have

[31] Apart from a factor of 3 in the $C_s^{\rm log}$ term.

[32] Indeed, for the CY$_3$ of [148], we have $\chi_s = 3$ and hence the correction represents a rescalling of A by a factor of 12, which would naïvely allow us to choose the much lower value $A \sim 70$.

[33] Note that checking whether the α'-corrections are under control, i.e. whether a given solution is consistent with the supergravity description being used, should be done using string-frame volumes, which are parametrically (in g_s) smaller than the Einstein-frame volumes when the convention $\tau_i^{\rm (S)} = g_s\tau_i^{\rm (E)}$ is used for the 10d change of frames, rather the one we use for which these volumes are the same at the vev. Therefore, one may naively think that the volumes are large "enough" in Einstein-frame, while the string-frame volumes are actually inconsistent with the α'-expansion.

[34] The $C_s^{\rm log}$ corrections again differs by a factor of 5/3.

no suppression whatsoever, $\{C_s^{\log}, C_1^\xi\}$; or are only suppressed by one power of g_s, $\{C_2^\xi\}$ — this is still dangerous because of the way the vev of the volume $\mathcal{V}$ depends on τ_s, i.e.

$$\mathcal{V} \propto e^{\frac{a}{g_s}\tau_s} \approx e^{\frac{a}{g_s}\tau_s^{(0)}} \left(1 + C_2^\xi \frac{a}{3\kappa_s}\left(\frac{\hat{\xi}}{2\kappa_s}\right)^{-1/3} + \ldots\right) . \tag{4.93}$$

It also follows that the $\{C_s^{\log}, C_1^\xi\}$ terms will both appear with a $1/g_s$ power and therefore grow at small couplings.

Finally, the $\{C^{\rm con}, C_b^{\rm KK}\}$ terms explicitly depend on the strength of the warping. While the $C^{\rm con}$ correction is suppressed by g_s and $(g_s M)^{-2}$, in the $C_b^{\rm KK}$ term one should pay closer attention to the $g_s M \gg 1$ factor possibly enhancing it.

It is worth noting that in this weakly-warped regime one never finds the tadpole bounds identified in [168], which were required in order to avoid a runaway towards the singular conifold at $\zeta = 0$ caused by the brane backreaction. The reason why we do not have it in the weakly-warped regime is that if the warping is subdominant, the potential for ζ grows more and more towards $\zeta = 0$ before ultimately dropping to zero (cf. Fig. 4.2) — in practice, at least around the minimum, it is as if the potential simply grows for small ζ. It is interesting to compare this with the potential found in [107], where it was argued that the ζ potential does indeed grow towards smaller ζ even in the strongly-warped case, in contrast with the behaviour proposed in [166]. Although this would invalidate the tadpole bound of [168] and change the exact form of the $\{C^{\rm con}\}$ correction in [176], one expects it to still be parametrically suppressed by $\frac{\alpha}{(g_s M)^2}$ [176] in the strongly-warped case.

4.6.3 Vacuum Energy

With the vevs of the moduli in (4.90–4.92), we can compute the vacuum energy given by the potential V at the minimum and look at the effect of each correction,

$$V_{\rm min} = \frac{g_s^4}{8\pi||\Omega||^2} \cdot \frac{3 g_s \kappa_s W_0^2 \sqrt{\tau_s}}{4a\mathcal{V}^3}\rho \tag{4.94}$$

with ρ defined as

$$\begin{aligned}\rho = {}& \tilde{\alpha} - 1 + \mathcal{O}(g_s) + \mathcal{O}(1/\beta) \\ & - C_s^{\rm KK} \cdot \frac{g_s^2}{6\kappa_s}\left(\frac{\hat{\xi}}{2\kappa_s}\right)^{-2/3} + C_s^{\log} \cdot \frac{a}{6}(11 + 12\log\nu) + C_1^\xi \cdot \frac{a\log\nu}{\kappa_s}\left(\frac{\hat{\xi}}{2\kappa_s}\right)^{-1/3} - C_2^\xi \cdot \frac{g_s}{3\hat{\xi}} \\ & - C^{\rm flux} \cdot \frac{10a}{9\kappa_s}\frac{KM}{\mathcal{V}^{2/3}}\left(\frac{\hat{\xi}}{2\kappa_s}\right)^{-1/3} - C^F \cdot \frac{16a}{27\kappa_s}\frac{g_s W_0^2}{\mathcal{V}^{2/3}}\left(\frac{\hat{\xi}}{2\kappa_s}\right)^{-1/3}\end{aligned} \tag{4.95}$$

$$- C^{\text{con}} \cdot \frac{5\tilde{\alpha}}{(g_s M)^2} - C_b^{\text{KK}} \cdot \frac{8c' a (g_s M)^2}{9\kappa_s} \frac{g_s \zeta^{2/3}}{\mathcal{V}^{1/3}} \Big(\frac{\hat{\xi}}{2\kappa_s} \Big)^{-1/3} .$$

where

$$\nu = \frac{3 g_s \kappa_s W_0}{8aA} \left(\frac{\hat{\xi}}{2\kappa_s} \right)^{1/3} . \tag{4.96}$$

The first line reproduces the leading order result (4.77) and requires $\tilde{\alpha} > 1$ provided all corrections can be safely neglected. The corrections appears in the vacuum energy with the same parametric suppression as in the strongly-warped case. In particular, the terms $\{C_s^{\log}, C_1^{\xi}\}$ appear unsuppressed and are thus the most dangerous.

Note that the non-perturbative no-scale behaviour [176] is manifesting itself in the way the factors $\{C^{\text{flux}}, C^F\}$ are suppressed *relative to the leading LVS solution* in ρ compared to their suppression in the off-shell potential, (4.86) and (4.88) respectively, having acquired a $1/g_s$ enhancement in the vacuum energy suppression factors (see discussion below).

It is worth pointing out that the unsuppressed corrections $\{C_s^{\log}, C_1^{\xi}, C_2^{\xi}\}$ are independent of the presence of the conifold modulus and the $\overline{\text{D3}}$-brane — these appear to be dangerous corrections for pure LVS, even before the uplift is considered. Supposing it is possible to choose a specific model (i.e. a compact space geometry) for which these unsuppressed corrections to the vevs and vacuum energy are not present, one can check whether the remaining corrections are all consistently suppressed and the solution is parametrically under control. Note that both C_s^{KK} and C^{con} are automatically suppressed as long as the string loop expansion ($g_s \ll 1$) and the supergravity approximation ($g_s M \gg 1$) are under control, which we must require for consistency of our analysis. In order to suppress the remaining corrections $\{C_b^{\text{KK}}, C^{\text{flux}}, C^F\}$ we require

$$(g_s M)^2 \cdot \frac{g_s \zeta^{2/3}}{\mathcal{V}^{1/3}} \, , \; \frac{KM}{\mathcal{V}^{2/3}} \, , \; \frac{g_s W_0^2}{\mathcal{V}^{2/3}} \ll 1 \, , \tag{4.97}$$

simultaneously. While the first two corrections are less dangerous for weakly-warped solutions that do not require large flux contributions, recall that a solution in the weakly-warped regime must also satisfy (4.76)

$$\frac{\mathcal{V}}{g_s^2 W_0^2} < \mathcal{O}(1) \, . \tag{4.98}$$

This condition, together with the one following from C^F, implies

$$\mathcal{V}^{1/3} \ll g_s \, , \tag{4.99}$$

which is never consistent. One can check explicitly using the parameter set of Table 4.2 that taking into account all numerical coefficients $\{a, \kappa_s, \hat{\xi}, ||\Omega||^2\}$ does not change the conclusion. Therefore the C^F corrections is unsuppressed unless $C^F \ll 1$.

It is worth noting that already in the strongly-warped case, the C^F correction seemed to be the main obstruction to smaller values of g_s, which may not be obvious from the suppression factor $\propto g_s W_0^2/\mathcal{V}^{2/3}$. One would naively expect that since $\mathcal{V} \propto g_s \cdot e^{1/g_s}$, smaller values of g_s would suppress this correction the strongest. However, it is important to note that one is not free to choose W_0 independently — in particular, in the strongly-warped case [176] (adapted to our conventions)

$$W_0^2 \propto \frac{\zeta^{2/3}\mathcal{V}^{5/3}}{\alpha\, g_s^3} \implies W_0^{1/3} \propto \frac{\zeta^{2/3}}{\alpha g_s^{4/3}} \cdot e^{\frac{5}{3}\frac{a}{g_s}\tau_s}\,, \tag{4.100}$$

with $\alpha \sim \mathcal{O}(1)$. Plugging this into the C^F suppression factor,

$$\frac{g_s W_0^2}{\mathcal{V}^{2/3}} \sim \frac{\zeta^{8/3}}{\alpha^4 g_s^5} e^{6\cdot\frac{a}{g_s}\tau_s}\,, \tag{4.101}$$

shows that indeed it grows for small g_s contrary to our initial expectation.

In summary, although the warping corrections are alleviated in the weakly-warped solutions, as one would expect, both the pure LVS corrections $\{C_s^{\log}, C_1^{\xi}, C_2^{\xi}\}$ and the F-term corrections present a danger (see Table 4.5). While the former behave in the same way in both warping regimes, the F-term corrections cannot be suppressed in the weakly-warped regime due to the requirement of large values of W_0. This is consistent with the discrepancy between the flux contribution MK following from the parameter set in our example and the LVS parametric tadpole bound that we found in the previous section, since the bound assumes higher F-term corrections to be parametrically suppressed.

4.6.4 Non-perturbative no-scale structure

In [176], a property of the LVS potential called the non-perturbative no-scale (NPNS) structure is emphasised due to its effect on the balance between leading and subleading (correction) terms for a given solution. The danger is that corrections to the off-shell scalar potential suppressed by what might be a small factor ϵ relative to the LVS terms, but which do not themselves satisfy the NPNS structure will appear in quantities such as $V_{\min}$ only suppressed by a factor ϵ/g_s.

The root of the NPNS is the fact that, due to the structure of the LVS potential, the minimum only appears at subleading order in g_s. Consider the change of variables $\mathcal{V} = \nu \cdot g_s \cdot e^{t/g_s}$ and $\tau_s = t/a$, in terms of which the LVS potential can be written as

$$V = g_s \cdot e^{-\frac{3}{g_s}t} f(t,\nu)\,, \tag{4.102}$$

Correction	Parametric suppression	Origin	Dangerous?	c.f. strong warping
$C_s^{\log}$	$g_s \log \mathcal{V}$	threshold corrections to gauge couplings	yes	similar
C_1^{ξ}	$g_s \log \mathcal{V}$	backreaction of 7-branes and exchange of KK modes with D7/O7 planes	yes	similar
C_2^{ξ}	g_s		yes	similar
C^F	$\frac{g_s^2 W_0^2}{\mathcal{V}^{2/3}}$	integrating out KK modes yielding 4 superspace derivatives in EFT	yes	similar
C_b^{KK}	$\frac{g_s^2 (g_s M)^2 \zeta^{2/3}}{\mathcal{V}^{1/3}}$	exchange of KK modes between D7/D3 branes or O7/O3 planes	can be	similar
C^{flux}	$\frac{g_s K M}{\mathcal{V}^{2/3}}$	backreaction from fluxes, branes and O-planes that source warping	can be	better
C_s^{KK}	g_s^2	exchange of KK modes between D7/D3 branes or O7/O3 planes	no	similar
C^{con}	$\frac{1}{(g_s M)^2}$	curvature correction in conifold region	no	better
χ_s	$A e^{-\frac{a}{g_s}(\tau_s - \frac{\chi_s}{24 g_s})}$	curvature corrections to D7 gauge kinetic function	no – it helps!	better

Table 4.5: Summary of subleading corrections considered, their origin and their parametric suppression (or not) in the off-shell scalar potential for the weakly-warped de Sitter vacuum. Note that the LVS volume stabilisation satisfies $\log \mathcal{V} \sim 1/g_s$.

where $f(t,\nu)$ does not depend on g_s.[35] A minimum of this potential therefore requires

$$\partial_t V = -\frac{3}{g_s} \cdot g_s \cdot e^{-\frac{3}{g_s}t} f(t,\nu) + g_s \cdot e^{-\frac{3}{g_s}t}(\partial_t f) \tag{4.105}$$

$$= -\frac{3}{g_s} V + g_s \cdot e^{-\frac{3}{g_s}t}(\partial_t f) \overset{!}{=} 0\,, \tag{4.106}$$

[35]Explicitly, we have

$$f(t,\nu) = \frac{8a^{3/2}A^2\sqrt{t}}{3\kappa_s \nu} - \frac{4AW_0 t}{\nu^2} + \frac{3W_0^2}{4\nu^3}\xi\,. \tag{4.103}$$

One might wonder whether W_0 and A in particular can be important for the argument we outline below. Note however that rescalling $\nu \to \frac{W_0}{A} \cdot \nu$, the function becomes

$$f(t,\nu) = \frac{A^3}{W_0}\left\{\frac{8a^{3/2}\sqrt{t}}{3\kappa_s \nu} - \frac{4t}{\nu^2} + \frac{3\xi}{4\nu^3}\right\}\,. \tag{4.104}$$

Since the ratio A^3/W_0 only appears as an overall factor, these parameters will not affect the argument for the NPNS.

which implies that the scalar potential at the minimum is suppressed by a factor of g_s compared to the off-shell potential (4.102),

$$V_{\text{min}} = \frac{g_s}{3} \cdot g_s \cdot e^{-\frac{3}{g_s}t}(\partial_t f)\,. \tag{4.107}$$

In other words, if one solves equation (4.105) perturbatively in g_s, i.e. $t = t_0 + t_1 g_s$, the leading order solution t_0 would result in a vanishing potential (which would leave ν a flat direction).

Now consider adding a correction suppressed by $\epsilon \cdot e^{-\frac{\lambda}{g_s}t}$,

$$V = g_s \cdot e^{-\frac{3}{g_s}t} f(t,\nu) + g_s \cdot \epsilon \cdot e^{-\frac{\lambda}{g_s}t} g(t,\nu)\,, \tag{4.108}$$

where $g(t,\nu)$ does not contain g_s or any other small parameters, so that

$$\begin{aligned}
\partial_t V =& -\frac{3}{g_s} \cdot g_s \cdot e^{-\frac{3}{g_s}t} f(t,\nu) + g_s \cdot e^{-\frac{3}{g_s}t}(\partial_t f) \qquad (4.109)\\
&- \frac{\lambda}{g_s} \cdot g_s \cdot \epsilon \cdot e^{-\frac{\lambda}{g_s}t} g(t,\nu) + g_s \cdot \epsilon \cdot e^{-\frac{\lambda}{g_s}t}(\partial_t g)\\
=& -\frac{3}{g_s} \cdot \left\{ g_s \cdot e^{-\frac{3}{g_s}t} f(t,\nu) + g_s \cdot \epsilon \cdot e^{-\frac{\lambda}{g_s}t} g(t,\nu) \right\} \qquad (4.110)\\
&+ g_s \cdot e^{-\frac{3}{g_s}t}(\partial_t f) + g_s \cdot \epsilon \cdot e^{-\frac{\lambda}{g_s}t}(\partial_t g) + (3-\lambda) \cdot \epsilon \cdot e^{-\frac{\lambda}{g_s}t} \cdot g(t,\nu)\\
=& -\frac{3}{g_s} V + g_s \cdot e^{-\frac{3}{g_s}t}(\partial_t f) + g_s \cdot \epsilon \cdot e^{-\frac{\lambda}{g_s}t}(\partial_t g)\\
&+ (3-\lambda) \cdot \epsilon \cdot e^{-\frac{\lambda}{g_s}t} \cdot g(t,\nu) \overset{!}{=} 0\,, \qquad (4.111)
\end{aligned}$$

which now implies for the vacuum energy

$$V_{\text{min}} = \frac{g_s}{3} \cdot g_s \cdot e^{-\frac{3}{g_s}t}(\partial_t f) + \frac{g_s}{3} \cdot g_s \cdot \epsilon \cdot e^{-\frac{\lambda}{g_s}t}(\partial_t g) + \frac{3-\lambda}{3} \cdot g_s \cdot \epsilon \cdot e^{-\frac{\lambda}{g_s}t} \cdot g(t,\nu)\,. \tag{4.112}$$

Note that, unless $\lambda = 3$ (i.e. the correction satisfies the NPNS structure), the last term is the leading correction contribution and *it is not suppressed by the extra factor of* g_s that appears in the first two terms — the g_s suppression characteristic of LVS (4.102). Therefore, to safely neglect this contribution from the correction one should have[36]

$$g_s \cdot \epsilon \cdot e^{-\frac{\lambda}{g_s}t} \ll g_s^2 \cdot e^{-\frac{3}{g_s}t} \quad \Rightarrow \quad \frac{\epsilon}{g_s} \cdot e^{-\frac{(\lambda-3)}{g_s}t} \ll 1\,, \tag{4.113}$$

rather than the naïvely expected $\epsilon \cdot e^{-\frac{(\lambda-3)}{g_s}t} \ll 1$. To compare with (4.95) one should look at ρ, by taking out the suppression factors $g_s^2 \cdot e^{-\frac{3}{g_s}t}$ as in (4.77),

$$\rho \propto -\frac{1}{3}(\partial_t f) - \frac{1}{3} \cdot \epsilon \cdot e^{-\frac{(\lambda-3)}{g_s}t}(\partial_t g) + \frac{(3-\lambda)}{3} \cdot \frac{\epsilon}{g_s} \cdot e^{-\frac{(\lambda-3)}{g_s}t} \cdot g(t,\nu)\,, \tag{4.114}$$

[36]Note that we must include the exponential $e^{-\frac{\lambda-3}{g_s}t}$, which corresponds to the volume suppression (whose power is given by $\lambda - 3$). In practice, ϵ counts all factors apart from powers of the volume.

where we can explicitly see the factor of (4.113).

This is why terms like $\{C^F, C^{\rm flux}\}$, which do not satisfy the NPNS structure ($\lambda \neq 3$), appear to be enhanced by $1/g_s$ in (4.95) compared to their suppression factors in the off-shell potential (4.86) and (4.88), while terms like $\{C_s^{\log}, C_1^\xi, C_2^\xi\}$, which do satisfy the NPNS ($\lambda = 3$), show the same suppression in $V_{\rm min}$ as they did in the off-shell potential (4.83) and (4.84). As an explicit example, for $C^{\rm flux}$ we have

$$\lambda = 3 + \frac{2}{3}\,, \qquad \epsilon \cdot e^{-\frac{\lambda-3}{g_s}t} = C^{\rm flux}\frac{5 g_s K M}{\mathcal{V}^{2/3}}\,, \qquad g(t,\nu) = \frac{3W_0^2}{4\nu}\,, \tag{4.115}$$

breaking the NPNS, while for C_2^ξ

$$\lambda = 3\,, \qquad \epsilon \cdot e^{-\frac{\lambda-3}{g_s}t} = C_2^\xi g_s\,, \qquad g(t,\nu) = \frac{3W_0^2}{4\nu}\,. \tag{4.116}$$

As a final remark, note that whether a correction breaks the NPNS structure or not, does not determine how dangerous it will be — indeed $\{C_s^{\rm KK}, C_s^{\log}\}$ both have an NPNS structure, but only $C_s^{\log}$ turns out to be dangerous; conversely, $\{C^{\rm flux}, C^F\}$ both break the NPNS structure, but only C^F appears to be dangerous. While the NPNS has interesting implications for the string loop expansion, one ultimately needs to check the way each correction is suppressed.

4.7 Runaway Quintessence

Given the challenges in constructing metastable de Sitter solutions in string theory, and the number of scalar fields whose potentials often develop runaway directions in moduli space [117], it is very natural to suppose that Dark Energy might be due to a slowly rolling or frozen runaway quintessence field. In quintessence models, one or more scalar fields have a scalar potential that is flat enough to allow them to either be frozen in place away from a minimum or to slowly-roll in such a way that their potential energy closely mimics a cosmological constant (Fig. 4.10).

The equation of motion for a single scalar field ϕ in an expanding Universe described by the flat FLRW metric,

$$ds^2_{\rm FLRW} = -dt^2 + a^2(t) d\vec{x}^2\,, \tag{4.117}$$

with scale factor $a(t)$ encoding the time evolution of the spacetime, is given by

$$\ddot{\phi} + 3H\dot{\phi} + g^{\phi\phi}V'(\phi) = 0\,. \tag{4.118}$$

The Hubble parameter $H \equiv \dot{a}/a$ provides a friction term (proportional to $\dot{\phi}$) that can slow down the field in its trajectory along the potential $V(\phi)$. The energy density of such a scalar is the sum of its kinetic and potential energies,

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)\,, \tag{4.119}$$

with $\dot{\phi}^2 \equiv g^{\phi\phi}\dot{\phi}\dot{\phi}$ depending on the normalisation of the kinetic term for ϕ (i.e. it will generically depend on the metric in moduli space). In the slow-roll approximation, $\dot{\phi}^2 \ll 2V(\phi)$ and $\ddot{\phi} \ll V'(\phi)$, so that the scalar field equations are approximately

$$\dot{\phi} \approx -\frac{V'(\phi)}{3H} \quad \text{and} \quad \rho_\phi \approx V(\phi)\,. \tag{4.120}$$

Therefore, in this approximation, the scalar field energy density enters the first Friedmann equation as an effective cosmological constant, in the same way a vacuum energy would. If we also assume that it dominates over other contributions, we find

$$H^2 = \frac{1}{3M_{\rm Pl}^2}\Big(\rho_\phi + \text{other contributions}\Big) \approx \frac{V(\phi)}{3M_{\rm Pl}^2}\,. \tag{4.121}$$

However, having the scalar field contributing with an approximately constant and positive energy density is not enough to guarantee an accelerated expansion. In fact, it follows from the definition of H that

$$\frac{\ddot{a}}{a} = H^2\Big(1 + \frac{\dot{H}}{H^2}\Big)\,, \tag{4.122}$$

and thus accelerated expansion ($\ddot{a} > 0$) requires

$$\epsilon \equiv -\frac{\dot{H}}{H^2} < 1\,. \tag{4.123}$$

One can use the expansion parameter ϵ to determine whether and when a certain scalar field system that dominates the energy density of the Universe will source an accelerated expansion and explain Dark Energy. In general, this requires us to know the dynamical evolution of the scalar field system and to take into account their kinetic energy as they move along the potential — one does not necessarily need slow-roll in order to have accelerated expansion.[37] Nonetheless, *if* one does have slow-roll, the expansion parameter is approximately given by the *slow-roll* parameter ϵ_V that is only a function of the scalar potential,

$$\epsilon_V = \frac{M_{\rm Pl}^2}{2} g^{\phi\phi}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 < 1\,. \tag{4.124}$$

We conclude that a quintessence field that slowly rolls along a runaway tail of its potential must satisfy both $V(\phi) > 0$ and $\epsilon_V < 1$ in order to give accelerated expansion.

Moreover, not only must this field result in accelerated expansion, it must also do so with an effective energy density that agrees with current observations $\rho_\phi \sim \rho_\Lambda \sim (10^{-3}\ \text{eV})^4$, which

[37]For example, for scaling solutions, the slow-roll approximation is generically not valid and the slow-roll parameter ϵ_V should not be used to determine whether there is accelerated expansion as was emphasised in [204].

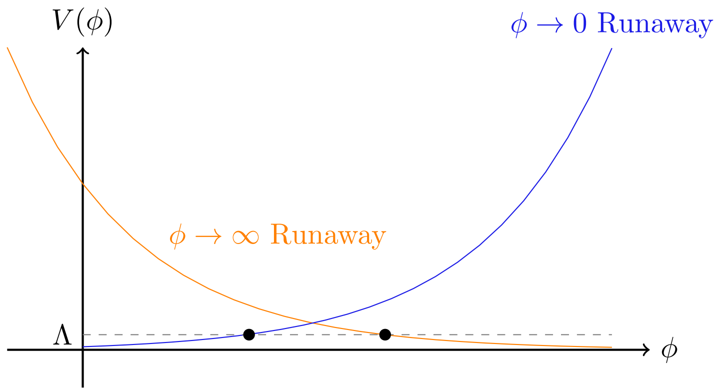


Figure 4.10: In runaway quintessence models a scalar field can either be frozen or slowly-roll along a flat region of a potential which is at the tail of a runaway. This is an expected behaviour for the asymptotic regions of moduli space and provides an effective cosmological constant.

generically forces the quintessence field to be extremely light, with a mass $m_\phi \lesssim 10^{-33}$ eV [38]—as we will see in much more detail in chapter 5, such light fields will mediate fifth forces unless their couplings to the Standard Model fields are somehow suppressed so as to evade current observational constraints. On the other hand, the usual string theory relation between scalar fields and couplings in the low-energy EFT means that a rolling quintessence field can lead to time variation of fundamental constants, which is also highly constrained by current data. These problems may be addressed if the quintessence field comes from a hidden sector that is sequestered from the visible sector (e.g. due to a separation in the compact space; we will see a concrete example of suppressed couplings in chapter 5), although there are as yet no explicit constructions that realise the degree of sequestering that would be necessary (see [205] for some challenges, [206] for a more optimistic point of view, and [207] for recent work in this direction).

Arguably the simplest class of runaway moduli are those originating from supersymmetric flat directions. However, it is impossible for such a runaway tail to play the role of Dark Energy and source an accelerated expansion. To see why, suppose some early Universe scenario, such as inflation, which ends in a supersymmetric Minkowski minimum in which most of the string moduli, Φ_i, are stabilised and heavy,

$$\langle D_i W_{\rm susy} \rangle = 0 \quad \text{and} \quad \langle W_{\rm susy} \rangle = 0 \,, \tag{4.125}$$

and we can safely integrate them out, leaving only an EFT describing the remaining flat directions. Assume for simplicity a single flat direction, corresponding to the chiral superfield Φ with scalar component also labelled $\Phi = \phi + i\theta$, with saxion ϕ and axion θ. As we have seen, the scalar potential in a supergravity framework (3.78) will depend on both the Kähler potential K

[38] This follows from requiring the second slow-roll parameter

$$\eta_V \equiv M_{\rm Pl}^2 \left| g^{\phi\phi} \frac{V''(\phi)}{V(\phi)} \right| \lesssim 1$$

in order to maintain slow-roll, together with $V(\phi) \sim (10^{-3}\ \text{eV})^4$ and $m_\phi^2 \equiv g^{\phi\phi} V''(\phi)$.

and superpotential W responsible for lifting this flat direction. These will in turn depend on the origin of the modulus Φ and one might hope that among the many moduli of string theory compactifications, there will be some that allow for slow-roll quintessence at the runaway tail. Let us therefore consider different types of moduli and explore their ability to provide accelerated expansion.

Bulk moduli

A bulk modulus may have the leading Kähler potential

$$K = -n \ln\big(\Phi + \bar{\Phi}\big)\,, \tag{4.126}$$

with e.g. $n = 3$ for the overall volume modulus, or $n = 1$ for another Kähler modulus, a complex structure or dilaton, or some other $n < 3$ for a fibre modulus.

As we are in a supersymmetric Minkowski background, the flat direction is protected to all finite orders by the non-renormalisation theorem — the axion shift symmetry forbids dependence on θ, and holomorphicity of the superpotential implies a dependence on ϕ alone is also forbidden [208] (see [209] and [161] for interesting generalisations). Even if K can receive perturbative corrections, the flat direction cannot be lifted so long as $W = 0$. However, as we saw in the previous sections, the axion shift symmetry is broken by non-perturbative effects, so that at some scale a leading order non-perturbative superpotential can be generated

$$W = Ae^{-a\Phi}\,, \tag{4.127}$$

which leads to a scalar potential

$$V = \frac{A^2}{2^n n\, M_{\rm Pl}^2} e^{-2a\phi}\phi^{-n}\left(n^2 + 4a^2\phi^2 + n\,(-3 + 4\,a\,\phi)\right)\,. \tag{4.128}$$

Thus the flat direction for ϕ is lifted. The axion, θ, at this leading order remains a flat direction, but can also be lifted by subleading corrections. The overall scale of the potential energy is fixed by the constant A, and the exponential suppression in ϕ. In a complete string theory model, $A = \langle A(\Phi_i)\rangle$ and could itself be exponentially suppressed in the heavy moduli Φ_i. For example, when the superpotential is generated through gaugino condensation in a hidden sector, $W = \mu^2 e^{-af}$, where μ is the scale at which the gauginos condense, a is determined from the hidden sector beta function coefficient [210], and the gauge kinetic function is given by $f = \Phi + \Delta_{1-{\rm loop}}(\Phi_i)$, with one-loop threshold corrections depending on heavy moduli. Thus one would expect to be able to obtain a non-perturbatively generated potential with energy density of order the observed cosmological constant $\Lambda \sim e^{-280} M_{\rm Pl}^4$.

However, it is very simple to show that such a scalar potential cannot source an accelerated

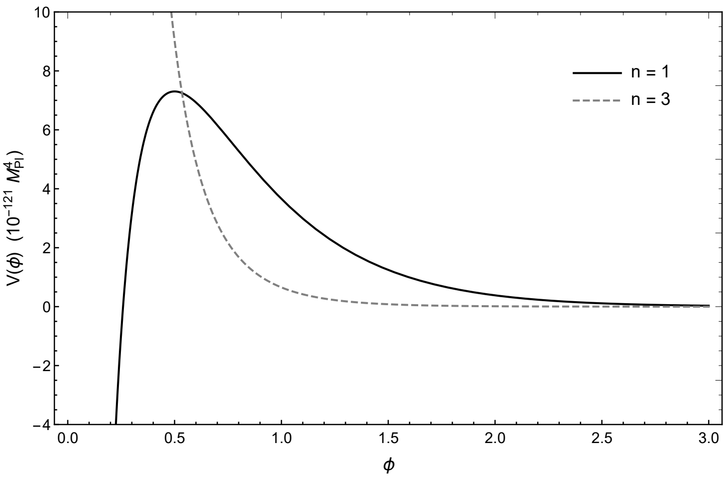


Figure 4.11: Potential (4.128) for $n = 1$ (left) and $n = 3$ (right) with $a = \sqrt{2}$ and $A = e^{-1105/8}$ in Planck units.

expansion at its tail. Note that so long as the field is frozen by Hubble friction, it mimics effectively a cosmological constant, but as the Hubble parameter falls the field will eventually begin to roll. As we have just reviewed, the Friedmann and Klein-Gordon equations imply that in order for a frozen or slowly rolling field to dominate the energy density of the Universe (over matter and radiation) and drive an accelerated expansion, the slow-roll condition (4.124) must be satisfied. Plugging the potential (4.128) into (4.124) and considering the tail of the runaway, one finds

$$\epsilon_V \to \frac{4}{n} a^2 \phi^2 \quad \text{as} \quad \phi \to \infty\,. \tag{4.129}$$

Therefore it is impossible to satisfy the slow-roll condition $\epsilon_V < 1$ (4.124) and drive an accelerated expansion at the tail of the non-perturbative runaway. Another way to state this is that the scalar potential for the canonically normalised field, $\varphi = M_{\text{Pl}}\sqrt{\frac{n}{2}}\log\phi$,

$$V(\varphi) \approx \frac{4A^2a^2}{2^n n\, M_{\text{Pl}}^2} e^{-2ae^{\sqrt{\frac{2}{n}}\frac{\varphi}{M_{\text{Pl}}}}} \left(e^{\sqrt{\frac{2}{n}}\frac{\varphi}{M_{\text{Pl}}}}\right)^{2-n} \quad \text{at large } \varphi, \tag{4.130}$$

with its double-exponential dependence, is too steep to allow a slow-roll accelerated expansion. Note that with suitably fine-tuned initial conditions, it is possible to obtain viable models of frozen quintessence at the hilltop of the potential $V(\phi)$ with $n = 1$ plotted in Fig. 4.11 (a fine-tuning of around 4% for the parameter space studied in [211] is sufficient). However, it is difficult to find an explanation for such special initial conditions, even an anthropic one.

One may be tempted to think that the failure to get accelerated expansion at the tail of the runaway is caused by the non-perturbative nature of the superpotential. Rather than considering a non-perturbative correction to W, one could try combining the Kähler potential (4.126) with a perturbative superpotential (later we will extend the superpotential to $W = W_0 + A\Phi^p$),

$$K = -n\ln\big(\Phi + \bar{\Phi}\big) \quad \text{and} \quad W = A\Phi^p\,, \tag{4.131}$$

for which

$$V(\phi) = \frac{A^2}{2^n n\ M_{\rm Pl}^2}\phi^{-n}(\phi^2+\theta^2)^{-1+p}\left(\left((-3+n)\,n - 4np + 4p^2\right)\phi^2 + (-3+n)n\theta^2\right) . \qquad (4.132)$$

If we consider the potential around $\theta \approx 0$,

$$\begin{aligned} V(\phi) \;\; &= \;\; \frac{A^2}{2^n n\ M_{\rm Pl}^2}\left((-3+n)n - 4np + 4p^2\right)\phi^{-n+2p} \\ &\quad + \frac{A^2}{2^n n\ M_{\rm Pl}^2} p\left((-3+n)n - 4np + 4p^2 + 4(n-p)\right)\phi^{-2-n+2p}\theta^2 + \mathcal{O}(\theta^3)\,, \end{aligned} \qquad (4.133)$$

it is easy to see that $\theta = 0$ corresponds to a metastable minimum as long as

$$p < \frac{n+1}{2} - \frac{\sqrt{n+1}}{2} \quad \text{or} \quad p > \frac{n+1}{2} + \frac{\sqrt{n+1}}{2} \qquad (4.134)$$

while the scalar potential is positive, $V(\phi) > 0$, at $\theta = 0$ provided

$$p < \frac{n}{2} - \frac{\sqrt{3n}}{2} \quad \text{or} \quad p > \frac{n}{2} + \frac{\sqrt{3n}}{2} \qquad (4.135)$$

For example, if $n = 1$ and $p \geq 2$, then θ is stabilised at $\theta = 0$, and the potential for the canonically normalised φ becomes $V(\varphi) = \frac{A^2}{2M_{\rm Pl}^2}(-2-4p+4p^2)e^{(-1+2p)\sqrt{2}\varphi/M_{\rm Pl}} > 0$ — however, the slow-roll parameter $\epsilon_V = (1-2p)^2$ is always greater than one. For general $n > 0$ and p, assuming $\theta = 0$, the slow-roll parameter is

$$\epsilon_V = \frac{(n-2p)^2}{n}\,, \qquad (4.136)$$

such that it is impossible to have simultaneously $\epsilon_V < 1$ and $V(\phi) > 0$.

If we relax somewhat the constraints from a supergravity description of the action, a Kähler potential (4.126) implies that any power-law scalar potential for ϕ will lead to a standard exponential scalar potential for the canonically normalised field

$$V(\phi) = A\phi^{-p} \quad \Longrightarrow \quad V(\varphi) = Ae^{-\sqrt{\frac{2p^2}{n}}\frac{\varphi}{M_{\rm Pl}}}\,. \qquad (4.137)$$

We have seen above that the structure of supergravity scalar potentials imposes relations between its coefficients, such that it is impossible to have both $\epsilon_V < 1$ and $V(\varphi) > 0$. However, for (4.137), a slow-roll accelerated expansion is possible for[39]

$$\frac{p^2}{n} \lesssim 1\,. \qquad (4.138)$$

[39]See [212, 213] for a dynamical systems analysis of such potentials. In [214], observational constraints on the dark energy equation of state $w(z)$ were used to constrain the constant in the de Sitter swampland conjecture $|\nabla V| \gtrsim cV$ to $c \lesssim 0.6$. Here, we consider the simplest scenario of a frozen field mimicking a cosmological constant, $w = -1$, for most of the cosmological history.

For the value $n = 1$ often seen for bulk string moduli, then we would need $p \lesssim 1$. A fibre modulus with e.g. $n = 2$ [215] would need $p \lesssim \sqrt{2}$. It would be very interesting to identify such perturbative runaways in explicit, well-under-control string constructions, though the lack of supersymmetry might make this particularly difficult. If successful, one would then have to furthermore explain how the hierarchy in the vacuum energy and mass are stable with respect to the ultraviolet cutoff (the second part of the cosmological constant problem as we reviewed in section 4.1), and how to avoid fifth forces (note that, even if not sequestered, fundamental constants would not vary with time so long as the quintessence field is frozen by Hubble friction).

Local moduli

Bearing in mind the need to suppress fifth forces, it is interesting to consider a local modulus, which may be sequestered from the Standard Model using geometric separation within the extra dimensions. However, once again the simplest models within supergravity do not allow for slow-roll quintessence. Consider for example a blow-up modulus with Kähler potential (see e.g. [216] for explicit string examples of such moduli)

$$K = k_0 - 2\ln\Big(k_1 - k_2(\Phi+\bar{\Phi})^{3/2}\Big) \tag{4.139}$$

$$\approx k_0 - 2\ln(k_1) + 2\frac{k_2}{k_1}(\Phi+\bar{\Phi})^{\frac{3}{2}}, \tag{4.140}$$

where in the second line we assumed small values of the blow-up modulus, $\frac{k_2}{k_1}\phi^{3/2} \ll 1$. Then, the canonically normalised field is

$$\varphi = \frac{2^{7/4}}{\sqrt{3}}\sqrt{\frac{k_2}{k_1}}\phi^{\frac{3}{4}}\,. \tag{4.141}$$

As before, consider a non-perturbative superpotential, $W = Ae^{-a\Phi}$. The full scalar potential becomes

$$V(\phi) = \frac{A^2}{3k_1^2 M_{\rm Pl}^2} e^{k_0 - 2a\phi + 2\frac{k_2}{k_1}(2\phi)^{3/2}} \left(-9 + 8(a\phi)\frac{a\phi}{\frac{k_2}{k_1}(2\phi)^{3/2}} - 24(a\phi) + 18\frac{k_2}{k_1}(2\phi)^{3/2} \right) \tag{4.142}$$

When $a\phi < \frac{k_2}{k_1}(2\phi)^{\frac{3}{2}} \ll 1$ or $\frac{k_2}{k_1}(2\phi)^{\frac{3}{2}} < a\phi \ll 1$ this leads to negative potential energy

$$V(\phi) \approx -\frac{3A^2 e^{k_0}}{k_1^2\, M_{\rm Pl}^2}\,. \tag{4.143}$$

When instead $a\phi \gtrsim 1$, the potential has an exponential dependence in ϕ

$$V(\phi) \approx \frac{8A^2}{3k_1^2 M_{\rm Pl}^2} e^{k_0 - 2a\phi} \frac{k_1}{k_2(2\phi)^{3/2}} a^2\phi^2\,, \tag{4.144}$$

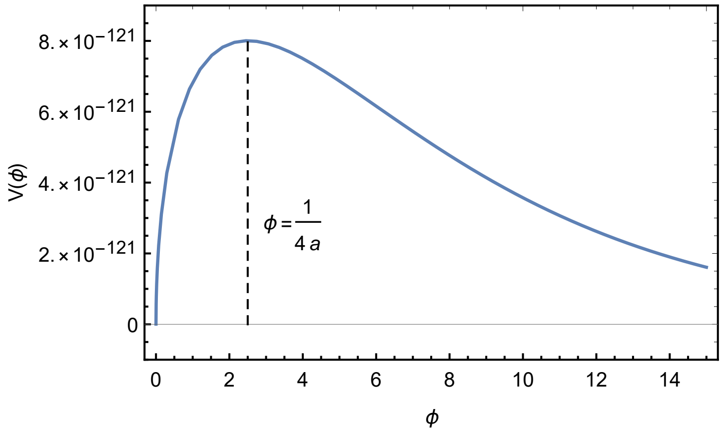


Figure 4.12: Potential arising from blow up modulus with K given in (4.140) and a non-perturbative superpotential, $W = Ae^{-a\Phi}$, for $k_0 = -265$, $k_1 = 2075$, $k_2 = 1$, $A = 1.5$ and $a = 0.1$, in Planck units.

which gives the slow-roll parameter

$$\epsilon_V = \frac{k_1}{2k_2(2\phi)^{3/2}}\frac{2}{3}(1-4a\phi)^2 \tag{4.145}$$

and thus $\epsilon_V > 1$ (so that slow-roll quintessence is impossible) unless $\phi \sim \frac{1}{4a}$, which corresponds to fine-tuning the initial value of ϕ to the hilltop (Fig. 4.12).

Another candidate amenable to sequestering is the complex structure modulus responsible for the deformation of a conifold, which has been one of the main characters of the de Sitter constructions discussed in the previous sections. Its Kähler potential takes the form (cf. (4.21))

$$K = k_0 + k_1|\Phi|^2\left(\ln\left(\frac{k_2}{|\Phi|}\right)+1\right) + k_3|\Phi|^{\frac{2}{3}}\,, \tag{4.146}$$

and, in the presence of fluxes, a superpotential is generated which is given by (cf. (4.27))

$$W = -iw_1\Phi\left(\ln\left(\frac{k_2}{\Phi}\right)+1\right) + iw_2\Phi\,. \tag{4.147}$$

Assuming a strongly-warped scenario with $|\Phi| \ll l_s^3$, the k_3 term dominates over the k_1 term in (4.146), and the slow-roll parameter becomes

$$\epsilon_V \approx \frac{16}{3}\left(1+\frac{3}{4k_3|\Phi|^{\frac{2}{3}}}\right) > 1\,, \tag{4.148}$$

not suitable for slow-roll quintessence.

More generally, a local modulus with Kähler potential of the form

$$K = k_0 + \frac{|\Phi|^{2n}}{k_1} \quad \text{or} \quad K = k_0 + \frac{(\Phi+\bar{\Phi})^{2n}}{k_1} \tag{4.149}$$

with a non-perturbative superpotential $W = Ae^{-a\Phi}$ (below we will extend this to $W = W_0 + Ae^{-a\Phi}$) leads, respectively, to the scalar potentials

$$\begin{aligned} V(\phi) &= \frac{A^2}{n^2\ M_{\rm Pl}^2} e^{k_0 - 2a\phi + \frac{\phi^{2n}}{k_1}} \left((a\phi)^2 \frac{k_1}{\phi^{2n}} - 3n^2 - 2n(a\phi) + n^2 \frac{\phi^{2n}}{k_1} \right) \quad \text{for } \theta = 0 \\ &\approx \frac{A^2}{n^2\ M_{\rm Pl}^2} e^{k_0 - 2a\phi} \left(a^2\phi^2\right) \frac{k_1}{\phi^{2n}} \quad \text{when } \phi^{2n} \ll k_1 \text{ and } a\phi \gtrsim \frac{\phi^{2n}}{k_1} \end{aligned} \tag{4.150}$$

and

$$\begin{aligned} V(\phi) &= \frac{A^2}{(2n-1)2n\ M_{\rm Pl}^2} e^{k_0 - 2a\phi + \frac{(2\phi)^{2n}}{k_1}} \left(4(a\phi)^2 \frac{k_1}{(2\phi)^{2n}} - 6n(2n-1) - 8n(a\phi) + 4n^2 \frac{(2\phi)^{2n}}{k_1} \right) \\ &\approx \frac{A^2}{(2n-1)2n\ M_{\rm Pl}^2} e^{k_0 - 2a\phi} \left(4a^2\phi^2\right) \frac{k_1}{(2\phi)^{2n}} \quad \text{when } (2\phi)^{2n} \ll k_1 \text{ and } a\phi \gtrsim \frac{(2\phi)^{2n}}{k_1}\,, \end{aligned} \tag{4.151}$$

(for $a\phi < \frac{\phi^n}{k_1} \ll 1$, we instead have a negative potential energy $V(\phi) \approx -\frac{3A^2 e^{k_0}}{M_{\rm Pl}^2}$ for both potentials). The corresponding slow-roll parameters are approximately

$$\epsilon_V \approx \frac{(n-1+a\phi)^2}{n^2} \frac{k_1}{\phi^{2n}} > 1 \tag{4.152}$$

and

$$\epsilon_V \approx \frac{2(n-1+2a\phi)^2}{(2n-1)n} \frac{k_1}{(2\phi)^{2n}} > 1\,. \tag{4.153}$$

Combining instead the Kähler potentials (4.149) with a perturbative superpotential $W = A\Phi^p$ (extended to $W = W_0 + A\Phi^p$ below) leads, respectively, to the power-law scalar potentials

$$\begin{aligned} V(\phi) &= \frac{A^2}{n^2\ M_{\rm Pl}^2} e^{k_0 + \frac{\phi^{2n}}{k_1}} \phi^{2p} \frac{k_1}{\phi^{2n}} \left(p^2 - n(3n-2p) \frac{\phi^{2n}}{k_1} + n^2 \frac{\phi^{4n}}{k_1^2} \right) \quad \text{for } \theta = 0 \\ &\approx \frac{A^2 e^{k_0}}{n^2\ M_{\rm Pl}^2} \frac{k_1}{\phi^{2n}} \phi^{2p} p^2 \quad \text{when } \phi^{2n} \ll k_1 \end{aligned} \tag{4.154}$$

and

$$\begin{aligned} V(\phi) &= \frac{A^2}{(2n-1)2n\ M_{\rm Pl}^2} e^{k_0 + \frac{(2\phi)^{2n}}{k_1}} \phi^{2p} \frac{k_1}{(2\phi)^{2n}} \left(4p^2 + 2n(3-6n+4p) \frac{(2\phi)^{2n}}{k_1} + 4n^2 \frac{(2\phi)^{4n}}{k_1^2} \right) \\ &\approx \frac{A^2 e^{k_0}}{(2n-1)2n\ M_{\rm Pl}^2} \frac{k_1}{(2\phi)^{2n}} \phi^{2p} (4p^2) \quad \text{when } (2\phi)^{2n} \ll k_1\,. \end{aligned} \tag{4.155}$$

In the first case, the deformation-like modulus, the slow-roll parameter is

$$\begin{aligned} \epsilon_V &= \frac{k_1}{\phi^{2n}} \frac{\left(p^2(p-n) + 3n(p-n)\frac{\phi^{2n}}{k_1} - n^2(2n-3p)\frac{\phi^{4n}}{k_1^2} + n^3 \frac{\phi^{6n}}{k_1^3} \right)^2}{n^2 \left(p^2 - n(3n-2p)\frac{\phi^{2n}}{k_1} + n^2 \frac{\phi^{4n}}{k_1^2} \right)^2} \\ &= \frac{(n-p)^2}{n^2} \frac{k_1}{\phi^{2n}} + \mathcal{O}\left(\frac{\phi^{2n}}{k_1}\right) \quad \text{or} \quad \frac{\phi^{6n}}{k_1^3} + \mathcal{O}\left(\frac{\phi^{8n}}{k_1^4}\right) \quad \text{for } n = p \end{aligned} \tag{4.156}$$

and $\epsilon_V < 1$ is only possible for $p = n$, where $V(\phi) = \frac{A^2 k_1}{M_{\rm Pl}^2} e^{k_0 + \frac{\phi^{2n}}{k_1}} \left(1 - \frac{\phi^{2n}}{k_1} + \frac{\phi^{4n}}{k_1^2}\right) > 0$. In this example, θ remains a flat direction. Unfortunately, for the well-known string theory example of the deformation modulus of the deformed conifold, $n = 1/3$ and $p = 1$, so slow-roll is not possible in agreement with our previous conclusion. Again, it would be extremely interesting to identify string theory constructions where $p = n$ (e.g. by generating a contribution to the superpotential $W \sim \Phi^{1/3}$, which could dominate at small $|\Phi|$).

In the second case, the blow-up-like modulus, the slow-roll parameter is

$$\begin{aligned} \epsilon_V &= \frac{k_1}{(2\phi)^{2n}} \frac{\left(8p^2(n-p) + 12np(2n-1-2p)\frac{(2\phi)^{2n}}{k_1} + n^2(2n-3-6p)\frac{(2\phi)^{4n}}{k_1^2} - n^3\frac{(2\phi)^{6n}}{k_1^3}\right)^2}{2n(2n-1)\left(4p^2 - 2n(6n-3-4p)\frac{(2\phi)^{2n}}{k_1} + 4n^2\frac{(2\phi)^{4n}}{k_1^2}\right)^2} \\ &= \frac{4(n-p)^2}{(2n-1)2n} \frac{k_1}{(2\phi)^{2n}} + \mathcal{O}\left(\frac{(2\phi)^{2n}}{k_1}\right) \qquad \text{for } n \neq p\,, \end{aligned} \tag{4.157}$$

or

$$\epsilon_V = \frac{(2\phi)^{2n}}{k_1} \frac{9}{2n(2n-1)} + \mathcal{O}\left(\frac{(2\phi)^{4n}}{k_1^2}\right) \qquad \text{for } n = p\,, \tag{4.158}$$

so that $\epsilon_V < 1$ is only possible for $p = n$, for which

$$V(\phi) = \frac{A^2 k_1}{2^n(n-1)\, M_{\rm Pl}^2} e^{k_0 + \frac{(2\phi)^n}{k_1}} \left(n - (n-3)\frac{(2\phi)^n}{k_1} + n\frac{(2\phi)^{2n}}{k_1^2}\right) > 0\,.$$

We must moreover have $n > 1$ in order for the axion value $\theta = 0$ to be metastable.

Generalising the analysis

So far we have assumed that the light quintessence field starts as a flat direction, and that along the runaway direction ($\phi \to \infty$ for a bulk or fibre modulus, and $\phi \to 0$ for a local modulus) $W \to 0$. We may extend the analysis to include a non-vanishing constant term, W_0, in the superpotential originating from the stabilisation of the heavy moduli, e.g. from fluxes $W_0 = \langle W_{\rm flux}\rangle$. Motivated by the simplicity of a runaway tail, we will assume in this analysis that there is no particular fine-tuning between the different ingredients. We will also not consider the possibility of fine-tuning initial values of ϕ to hilltops, but focus on sourcing quintessence along the runaway tail. The axion θ will be set to zero throughout. It will be helpful to introduce the following dimensionless variables

$$x = \frac{W - W_0}{W_0}\,, \qquad y = K - k_0 \ll 1\,, \qquad z = a\phi\,, \tag{4.159}$$

where y only applies to the local moduli ($y = \frac{\phi^{2n}}{k_1}$ for a deformation modulus and $y = \frac{(2\phi)^{2n}}{k_1}$ for a blow-up modulus; we always assume $y \ll 1$ for consistency) and z only applies to the non-perturbative superpotential (in actual string theory constructions, one usually needs $z > 1$ in order to neglect higher order non-perturbative effects), while x always measures the hierarchy between the terms in the superpotential ($x = \frac{Ae^{-a\phi}}{W_0}$ for the non-perturbative superpotential and $x = \frac{A\phi^p}{W_0}$ for the perturbative one; see [217] and [152] for recent work on hierarchically small and large W_0). We can then explore the parameter space for regions that allow, simultaneously, $V(\phi) > 0$ and $\epsilon_V < 1$. The results are summarised in Tables 4.6-4.7.

$$K = -n\log\left(\Phi + \bar{\Phi}\right), \qquad W = W_0 + Ae^{-a\Phi}$$

$$V = \frac{W_0^2}{M_{\rm Pl}^2}\frac{n(n-3)(1+x)^2+4nx(1+x)z+4x^2z^2}{n2^n\phi^n}$$

$$\epsilon_V = \frac{(n(1+x)+2xz)^2(n(n-3)(1+x)+4xz(n-1+z))^2}{n(n(n-3)(1+x)^2+4nx(1+x)z+4x^2z^2)^2}$$

Parameters		$V \to$	$\epsilon_V \to$	$V>0 \quad \epsilon_V<1$
$x \gg 1$	$z \gg 1$	$\frac{W_0^2}{M_{\rm Pl}^2}\frac{4x^2z^2}{n2^n\phi^n} > 0$	$\frac{4z^2}{n} > 1$	No-go
	$z \ll 1$	$\frac{W_0^2}{M_{\rm Pl}^2}\frac{(n-3)x^2}{2^n\phi^n}$	$n \geq 1$	No-go
$x \ll 1$	$xz \gg 1$	$\frac{W_0^2}{M_{\rm Pl}^2}\frac{4x^2z^2}{n2^n\phi^n} > 0$	$\frac{4z^2}{n} > 1$	No-go
	$xz \ll 1$	$\frac{W_0^2}{M_{\rm Pl}^2}\frac{(n-3)}{2^n\phi^n}$	$\frac{(n(n-3)+4xz^2)^2}{n(n-3)^2} \geq 1$	No-go

$$K = -n\log\left(\Phi + \bar{\Phi}\right), \qquad W = W_0 + A\Phi^p$$

$$V = \frac{W_0^2}{M_{\rm Pl}^2}\frac{(n(x+1)-2px)^2-3n(x+1)^2}{n2^n\phi^n}$$

$$\epsilon_V = \frac{(n(1+x)-2px)^2(n(n-3)+n(n-3)x-4px(n-p))^2}{n(4p^2x^2+n^2(1+x)^2-n(1+x)(3(1+x)+4px))^2}$$

Parameters	$V \to$	$\epsilon_V \to$	$V>0 \quad \epsilon_V<1$
$x \gg 1$	$\frac{W_0^2}{M_{\rm Pl}^2}\frac{x^2((n-2p)^2-3n)}{n2^n\phi^n}$	$\frac{(n-2p)^2}{n}$	No-go
$x \ll 1$	$\frac{W_0^2}{M_{\rm Pl}^2}\frac{(n-3)}{2^n\phi^n}$	n	No-go

Table 4.6: Summary of interesting parameter space for string inspired supergravity models of runaway quintessence with a bulk like or fibre like modulus. The parameters x, y, z are defined and discussed in and around eq. (4.159). Note that when $W_0 = 0$, one should take the limit $W_0x^2 \to Ae^{-a\phi}$ or $W_0x^2 \to A\phi^p$ respectively. In actual string compactifications, one usually requires $z = a\phi > 1$, to be able to neglect higher order non-perturbative terms.

This analysis shows that non-perturbative runaway potentials for bulk-like and fibre-like moduli with $K = -n\ln\left(\Phi + \bar{\Phi}\right)$, which contain exponentials of exponentials in the canonically normalised saxion, are too steep to source slow-roll quintessence along their tails. Although one might have expected that a bulk-like modulus with a perturbative runaway $W(\Phi) = A\Phi^p$ could lead to an exponential-like quintessence model for the canonically normalised saxion, we find that it is impossible to satisfy simultaneously $\epsilon_V < 1$ and $V(\phi) > 0$ (Table 4.6). The same is true for a local modulus with $K = k_0 + \frac{(\Phi+\bar{\Phi})^{2n}}{k_1}$ or $K = k_0 + \frac{|\Phi|^{2n}}{k_1}$ with a non-perturbative runaway, which despite having an exponential potential can never realise slow-roll quintessence (Tables 4.7 and 4.8). However, if a local modulus develops a perturbative runaway, it can source slow-roll quintessence within supergravity in very special cases, where the leading power in the superpotential, p, is equal to the leading power in the Kähler potential, n. As we explicitly checked, the conifold deformation modulus does not satisfy this relation ($p = 1, n = 1/3$). It would therefore be very interesting to find concrete string theory realisations of this scenario.

Different approaches to realise quintessence in string theory, including generalisations to multi-field models, seem to point in the same direction — just as with de Sitter vacua, one seems to be pushed towards the boundaries of control and require a large degree of fine-tuning (e.g. [120, 130–132, 204, 205, 207, 218–224]; see [121] for a review). This is in line with the swampland de Sitter conjecture [125, 126] which constrains the behaviour of any scalar potential arising from a consistent UV completion and translates into a lower bound on the slow-roll parameter ϵ_V (making quintessence difficult to achieve, *at least* in the slow-roll regime). The conjecture has since motivated both scenarios that embrace its consequences (combined with other conjectures, such as the distance conjecture) and explore the resulting phenomenology — e.g. the Dark Dimension proposal [225–230] — and those that try to evade these constraints altogether — e.g. expanding bubble braneworlds on non-SUSY AdS vacua [231–237], or coherent states over a supersymmetric Minkowski vacuum [238, 239]. At the moment, it is probably fair to say that more work is needed in order to obtain a fully under control scenario of accelerated expansion within string theory (see [121, 240] for reviews).

$$K = k_0 + \frac{|\Phi|^{2n}}{k_1}\,, \qquad W = W_0 + Ae^{-a\Phi}$$

$$V = \frac{W_0^2}{M_{\rm Pl}^2}\frac{e^{k_0+y}}{n^2 y}\left(n^2(1+x)^2(y-3)y - 2nx(1+x)yz + x^2z^2\right)$$

$$\epsilon_V = \frac{(n(1+x)y-xz)^2(n^2(1+x)(y-2)y+nx(1-2y)z+(z-1)xz)^2}{n^2y(n^2(1+x)^2(y-3)y-2nx(1+x)yz+x^2z^2)^2}$$

Parameters		$V \to$	$\epsilon_V \to$	$V>0 \quad \epsilon_V<1$
$x \gg 1$	$z \gg y$	$\frac{W_0^2}{M_{\rm Pl}^2}\frac{e^{k_0}}{n^2}x^2\left(\frac{z^2}{y}-3n^2\right)$	$\frac{(n-1+z)^2}{n^2(\frac{z^2}{y}-3n^2)}\frac{z^2}{y}\left(\frac{z}{y}\right)^2$	No-go
	$z \ll y$	$-\frac{W_0^2}{M_{\rm Pl}^2}\frac{e^{k_0}}{n}x^2\,(3n+2z)<0$	$\frac{4y}{9}$	No-go
$x \ll 1$	$xz \gg y$	$\frac{W_0^2}{M_{\rm Pl}^2}\frac{e^{k_0}}{n^2}\left(\frac{(xz)^2}{y}-3n^2\right)$	$\frac{(n-1+z)^2}{n^2(3n^2-\frac{(xz)^2}{y})^2}\frac{(xz)^2}{y}\left(\frac{xz}{y}\right)^2$	No-go
	$xz \ll y$	$-\frac{W_0^2}{M_{\rm Pl}^2}\frac{e^{k_0}}{n}\,(3n+2xz)<0$	$\frac{y}{9n^4}\left(\frac{xz^2}{y}-2n^2\right)^2$	No-go

$$K = k_0 + \frac{|\Phi|^{2n}}{k_1}\,, \qquad W = W_0 + A\Phi^p$$

$$V = \frac{W_0^2}{M_{\rm Pl}^2}\frac{e^{k_0+y}}{n^2 y}\left((px+n(1+x)y)^2 - 3n^2(1+x)^2y\right)$$

$$\epsilon_V = \frac{(p^3x^2+3n^2px(1+x)(y-1)y+n^3(1+x)^2(y-2)y^2+np^2x(y+x(3y-1)))^2}{n^2y(p^2x^2+2npx(1+x)y+n^2(1+x)^2(y-3)y)^2}$$

Parameters			$V \to$	$\epsilon_V \to$	$V>0 \quad \epsilon_V<1$
$p \neq n$	$x \gg 1$		$\frac{W_0^2}{M_{\rm Pl}^2}e^{k_0}\frac{p^2}{n^2}\frac{x^2}{y}>0$	$\frac{(p-n)^2}{n^2}\frac{1}{y}>1$	No-go
	$x \ll 1$	$x \gg y$	$\frac{W_0^2}{M_{\rm Pl}^2}\frac{p^2e^{k_0}}{n^2}\left(\frac{x^2}{y}-3\frac{n^2}{p^2}\right)$	$\frac{(n-p)^2}{n^2\left(\frac{x^2}{y}-3\frac{n^2}{p^2}\right)^2}\frac{x^2}{y}\left(\frac{x}{y}\right)^2>1$	No-go
		$x \ll y$	$-\frac{3W_0^2e^{k_0}}{M_{\rm Pl}^2}<0$	$\frac{4y}{9}<1$	No-go
Parameters			$V \to$	$\epsilon_V \to$	$V>0 \quad \epsilon_V<1$
$p = n$	$x \gg 1$		$\frac{W_0^2}{M_{\rm Pl}^2}e^{k_0}\frac{x^2}{y}>0$	$(1+xy)^2\frac{y}{x^2}<1$	Yes
	$x \ll 1$	$x^2 \gg y$	$\frac{W_0^2}{M_{\rm Pl}^2}e^{k_0}\frac{x^2}{y}>0$	$\frac{4y}{x^2}<1$	Yes
		$x^2 \ll y$	$-\frac{3W_0^2e^{k_0}}{M_{\rm Pl}^2}<0$	$\frac{4y}{9}\left(1+\frac{x}{y}\right)^2<1$	No-go

Table 4.7: Summary of interesting parameter space for string inspired supergravity models of runaway quintessence with a deformation like modulus. The parameters x, y, z are defined and discussed in and around eq. (4.159). Note that when $W_0 = 0$, one should take the limit $W_0x^2 \to Ae^{-a\phi}$ or $W_0x^2 \to A\phi^p$ respectively. We always assume $y \ll 1$ for consistency. In actual string compactifications, one also usually requires $z = a\phi > 1$, to be able to neglect higher order non-perturbative terms.

$$K = k_0 + \frac{(\Phi+\bar{\Phi})^{2n}}{k_1}\,, \qquad W = W_0 + Ae^{-a\Phi}$$

$$V = \frac{W_0^2}{M_{\rm Pl}^2}\frac{e^{k_0+y}}{n(2n-1)y}(2n^2(1+x)^2(y-3)y + 2x^2z^2 + n(1+x)y(3-x(3-4z)))$$

$$\epsilon_V = \frac{2(2n^3(1+x)^2(y-2)y^2-2x^2(z-1)z^2-3n^2(1+x)y(-2xz+y(-1+x(2z-1)))+nxz(-2xz+y(-5+2z+x(-5+6z))))^2}{n(2n-1)y(2n^2(1+x)^2(y-3)y+2x^2z^2-n(1+x)y(-3+x(4z-3)))^2}$$

Parameters		$V \to$	$\epsilon_V \to$	$V>0 \quad \epsilon_V<1$
$x \gg 1$	$z \gg y$	$\frac{W_0^2}{M_{\rm Pl}^2}\frac{2e^{k_0}}{n(2n-1)}x^2\left(\frac{z^2}{y} - \frac{3n(2n-1)}{2}\right)$	$\frac{2(n-1+z)}{n(2n-1)\left(\frac{z^2}{y}-\frac{3n(2n-1)}{2}\right)^2}\frac{z^2}{y}\left(\frac{z}{y}\right)^2$	No-go
	$z \ll y$	$-\frac{3W_0^2e^{k_0}}{M_{\rm Pl}^2}x^2 < 0$	$\frac{2n(4n-3)^2}{9(2n-1)^3}y < 1$	No-go
$x \ll 1$	$xz \gg y$	$\frac{W_0^2}{M_{\rm Pl}^2}\frac{2e^{k_0}}{n(2n-1)}\left(\frac{(xz)^2}{y} - \frac{3n(2n-1)}{2}\right)$	$\frac{2(n-1+z)^2}{n(2n-1)\left(\frac{(xz)^2}{y}-\frac{3n(2n-1)}{2}\right)^2}\frac{(xz)^2}{y}\left(\frac{xz}{y}\right)^2$	No-go
	$xz \ll y$	$-\frac{3W_0^2e^{k_0}}{M_{\rm Pl}^2} < 0$	$\frac{2y}{9n}\frac{(\frac{xz^2}{y}-n(4n-3))^2}{(2n-1)^3}$	No-go

$$K = k_0 + \frac{(\Phi+\bar{\Phi})^{2n}}{k_1}\,, \qquad W = W_0 + A\Phi^p$$

$$V = \frac{W_0^2}{M_{\rm Pl}^2}\frac{e^{k_0+y}}{n(2n-1)y}(2p^2x^2 + 4npx(1+x)y + n(1+x)^2(3+2n(y-3))y)$$

$$\epsilon_V = \frac{2(2p^3x^2+2n^3(1+x)^2(y-2)y^2+3n^2(1+x)y(2px(y-1)+(1+x)y)+npx(3(1+x)y+2p(y+x(3y-1))))^2}{n(2n-1)y(2p^2x^2+4npx(1+x)y+n(1+x)^2(3+2n(y-3))y)^2}$$

Parameters			$V \to$	$\epsilon_V \to$	$V>0 \quad \epsilon_V<1$
$p \neq n$	$x \gg 1$		$\frac{W_0^2}{M_{\rm Pl}^2}\frac{2p^2e^{k_0}}{n(2n-1)}\frac{x^2}{y} > 0$	$\frac{2(n-p)^2}{n(2n-1)}\frac{1}{y} > 1$	No-go
	$x \ll 1$	$x \gg y$	$\frac{W_0^2}{M_{\rm Pl}^2}\frac{2p^2e^{k_0}}{n(2n-1)}\left(\frac{x^2}{y} - \frac{3n(2n-1)}{2p^2}\right)$	$\frac{2(n-p)^2}{n(2n-1)\left(\frac{x^2}{y}-\frac{3n(2n-1)}{2p^2}\right)^2}\frac{x^2}{y}\left(\frac{x}{y}\right)^2 > 1$	No-go
		$x \ll y$	$-\frac{3W_0^2e^{k_0}}{M_{\rm Pl}^2} < 0$	$\frac{2n(4n-3)^2}{9(2n-1)^3}y < 1$	No-go
Parameters			$V \to$	$\epsilon_V \to$	$V>0 \quad \epsilon_V<1$
$p = n$	$x \gg 1$		$\frac{W_0^2}{M_{\rm Pl}^2}e^{k_0}\frac{2n}{2n-1}\frac{x^2}{y} > 0$	$\frac{9y}{2n(2n-1)} < 1$	Yes
	$x \ll 1$	$x^2 \gg y$	$\frac{W_0^2}{M_{\rm Pl}^2}e^{k_0}\frac{2n}{2n-1}\frac{x^2}{y} > 0$	$\frac{(4n-3)^2}{2n(2n-1)}\frac{y}{x^2} < 1$	Yes
		$x^2 \ll y$	$-\frac{3W_0^2e^{k_0}}{M_{\rm Pl}^2} < 0$	$\frac{2n}{9}\frac{(4n-3)^2}{(2n-1)^3}\left(1+\frac{x}{y}\right)^2 y$	No-go

Table 4.8: Summary of interesting parameter space for string inspired supergravity models of runaway quintessence with a blow-up like modulus. The parameters x, y, z are defined and discussed in and around eq. (4.159). Note that when $W_0 = 0$, one should take the limit $W_0x^2 \to Ae^{-a\phi}$ or $W_0x^2 \to A\phi^p$ respectively. We always assume $y \ll 1$ for consistency. In actual string compactifications, one also usually requires $z = a\phi > 1$, to be able to neglect higher order non-perturbative terms.

5. Gravitational signatures

> The problem is nature, not string theory! What I mean is that nature (...) seems to predict that the fundamental length is far beyond experiment.
>
> — Joseph Polchinski

The recent success of the LIGO and Virgo collaboration in directly observing gravitational waves (GW) from the merger of two black holes [241] kick-started the era of GW astronomy. Since then several signals were detected, not only originating from black hole mergers [242–245], but also from black hole-neutron star [246] and binary neutron star [247] mergers, with associated electromagnetic signals which can be used to extract more information from these events. In particular, the study of GW signals can be used to test General Relativity (GR) in an unprecedented way [248–254], constraining deviations from GR and therefore alternative theories of gravity and quantum gravity completions, such as string theory. With several ground and space-based experiments, such as the Einstein Telescope (ET) [255] and LISA [256, 257], planned for the near future, and interest in GW searches at ultra-high frequencies (UHF) in the range MHz–GHz [258] not covered by these experiments, the gravitational signals of modifications to GR will have the potential to test any theory (such as a UV completion) in which they arise.

This exciting progress in gravitational wave detection is complemented by a host of other diverse experiments and observations. From torsion table-top experiments, astronomical tests [259] and atom interferometry [260], to the Event Horizon telescope [261, 262] and collider searches [263], GR is being tested in all possible regimes, with strong and weak field tests. One should ultimately combine all these results, looking for how they match, differ or complement each other, in order to know what kind of deviations of GR are still possible and which are excluded [264]. A useful way to combine some of these tests is by choosing a common parameterisation, e.g. expressing the results in terms of a correction to the Newtonian potential [263] — in that way one can compare a specific string theory compactification setup with several experimental and observational results by looking at such corrections in the form of a single Yukawa interaction.

Has we have seen in the previous chapters, one of the main features of string theory is the presence of extra dimensions that need to be made compatible with all available observations — in particular, a typical step in a string theory construction is to *hide* the extra dimensions which have so far never been observed. As reviewed in chapter 3, one usually considers the compactification of the extra dimensions onto an internal compact space, which results in a 4d EFT in which each higher-dimensional field gives rise to an infinite KK tower of massive modes[1] (cf. Fig. 3.1), whose UV cutoff is the mass of the lightest KK state (or lower if lighter moduli have been integrated out, cf. Fig. 4.1). Since in string theory gravity is described by the ten-dimensional graviton, the KK towers include a tower of massive graviton KK modes which might have direct effects on gravitational waves and other gravitational effects [265–272]. These massive graviton states are usually integrated out as their masses are much higher than the energy scale of interest, such that the low-energy 4d theory gives simply GR. However, for high enough energies, i.e. for energies close to the masses of these states, the first KK modes will start contributing with corrections to the effective theory and, in particular, with corrections to the Newtonian potential[2].

On the other hand, within string theory, our Universe could be confined to a (3+1)-dimensional brane (or stack of branes), since the states giving rise to the Standard Model can come from open strings which end on different types of branes (cf. chapter 2) — these states are then confined to live on the brane and cannot directly probe the extra dimensions. The brane itself could be located at the tip of a warped throat in the internal compact space — the warped throat allows the natural high scale of the higher-dimensional theory, typically the string scale, to be suppressed on the brane, helping to bridge the gap between the UV scales considered in string theory and the observed IR scales of the 4d theory — this was precisely what motivated the GKP solution of section 3.6. Chapter 4 was all about applying these warped throat constructions in the search for de Sitter solutions — in proposals such as KKLT [75] and LVS [76, 77], strong warping is invoked in order to suppress the naturally high scale of an $\overline{\text{D3}}$-brane responsible for uplifting an AdS minimum into dS. Moreover, the Klebanov-Strassler (KS) solution [86] gives an explicit construction of such a background using a warped deformed conifold (cf. section 3.5) — this is a non-compact solution, but one usually considers smoothly gluing a finite portion of this solution to a compact Calabi-Yau 3-fold CY_3, such that the internal space is compact.

With this in mind, in [273] we started exploring the following question: how does the strong warping in these string compactifications affect the gravitational signatures of extra dimensions?

[1]Recall that the existence of such a discrete tower of states relies on the compactness of the internal space — if the extra dimensions are not compact, the spectrum will be a continuum of states (e.g. the spectrum of [58] is continuous whereas the very similar setup in [57] gives a discrete spectrum because the extra dimension is now compact). This has a direct impact on the form of the corrections to the Newtonian potential that arise from these extra-dimensional models, which will take a Yukawa-type form for compact cases such as [57] but a power-law form for non-compact cases such as [58].

[2]In principle, light moduli and higher string modes could also contribute with corrections to the Newtonian potential. We will show below that the higher string modes are typically heavier than the KK modes for our constructions, and so present subleading corrections. We will further assume no light moduli; alternatively their contributions could also be worked out in concrete constructions (e.g. taking into account the deformation modulus or volume modulus).

By considering a flux compactification of Type IIB supergravity in the presence of a warped throat, we studied its effects on the gravitational sector of the 4d EFT. Although this work was motivated by the prospects of GW astronomy, our main focus in [273] were the corrections to the Newtonian potential which can be compared to observations across diverse scales [263] — while the ingredients entering this analysis are also required for the prediction of GW signals in these models, this provides a simpler but important comparison between this setup and observational data.

The effects of extra dimensions in gravitational wave signals were also studied in [269], where the dimensional reduction of a D-dimensional gravitational theory down to 4d was performed, with warping taken into account. The effects of the warping were further explored in [270, 271] in the context of warped toroidal backgrounds. Here we will focus specifically on Type IIB supergravity and dimensionally reduce the 10d action down to 4d by considering the warped background to be described by a compact CY_3 with a warped throat described by the KS solution. The effects of this warped geometry on the tower of KK states were previously considered in [274] (see also [275]), where the mass spectrum of graviton KK modes was obtained. We reproduce the results found in [274], paying careful attention to the normalisation of the graviton KK mode wavefunctions which provide the couplings to other modes in the theory. The importance of this normalisation was already emphasised in [276], where it was noted that higher KK modes have stronger couplings when considering a KS warped throat rather than the Randall-Sundrum model (RSI) [57], the latter giving a good approximation only away from the tip of the throat.

We will consider a braneworld model within this warped Type IIB setup and study the corrections to the Newtonian potential between masses living on the brane due to the presence of the KK tower (this was done in the context of RSI in [277, 278]). We obtain direct relations between the phenomenological parameters characterising a Yukawa-type correction to the Newtonian potential and the parameters defining the string compactification. By combining this with consistency conditions on the compactification, we identify the exclusion region in the parameter space for a Yukawa-type correction to the Newtonian potential arising from a Type IIB brane model in which the Standard Model hierarchy is achieved by placing the brane somewhere along a KS warped throat. We will end the chapter with some implications of warped throats for gravitational wave experiments, identifying points in the parameter space of the KS solution which bring gravitational wave frequencies down to observable scales.

5.1 Dimensional reduction of Gravitational Waves

We have already encountered the equation describing 10-dimensional GWs (3.54), when we studied the equation of motion for the 10d graviton of Type IIB supergravity in section 3.3.

This wave equation can be conveniently rewritten as [273]

$$\Box_{10} h_{MN} + 2\bar{R}_M{}^P{}_N{}^Q h_{PQ} = 2\mathcal{T}^{(1)}_{MN} + 2h_{P(M}\bar{\mathcal{T}}_{N)}{}^P + \frac{1}{4}(\bar{G}^{PQ}\mathcal{T}^{(1)}_{PQ})G_{MN}\,, \tag{5.1}$$

in terms of

$$\mathcal{T}_{MN} \equiv -\frac{\delta\mathcal{L}}{\delta G^{MN}}\,, \tag{5.2}$$

such that the energy-momentum tensor is given by $T_{MN} = 2\mathcal{T}_{MN} + G_{MN}\mathcal{L}$, with $\mathcal{L}$ being the Type IIB Lagrangian that couples to the metric (i.e. not including the Chern-Simons topological term), which gives

$$\begin{aligned}\mathcal{T}_{MN} = &\frac{1}{2}(\partial_M\Phi)(\partial_N\Phi) + \frac{e^{2\Phi}}{2}(\partial_M C_0)(\partial_N C_0) + \frac{g_s^2}{4\times 4!}(\tilde{F}_5)_{MPQRS}(\tilde{F}_5)_N{}^{PQRS}\\ &+ \frac{g_s}{4}\Big(e^{\Phi}(\tilde{F}_3)_{MPQ}(\tilde{F}_3)_N{}^{PQ} + e^{-\Phi}(H_3)_{MPQ}(H_3)_N{}^{PQ}\Big)\,,\end{aligned} \tag{5.3}$$

whose linear order perturbation in h_{MN} is

$$\begin{aligned}\mathcal{T}^{(1)}_{MN} = -\Big(&\frac{g_s}{2}\Big(e^{\Phi}(\tilde{F}_3)_M{}^{RS}(\tilde{F}_3)_{NRP} + e^{-\Phi}(H_3)_M{}^{RS}(H_3)_{NRP}\Big)\\ &+ \frac{g_s^2}{4!}(\tilde{F}_5)_M{}^{SIJR}(\tilde{F}_5)_{NPIJR}\Big)\bar{G}^{PQ}h_{SQ}\,.\end{aligned} \tag{5.4}$$

Although in 10d the fluctuation h_{MN} described by these equations is indeed a GW, it does not look like one from the point of view of our 4d EFT — at low energies, when a 4-dimensional theory is a good description of the physical phenomena, only the $h_{\mu\nu}$ components correspond to GWs; the others will instead manifest as vectors or scalars (some of these scalar degrees of freedom were precisely the geometric moduli that we encountered in the previous sections). Using the background metric (3.127)[3] we can rewrite the wave equation (5.1) in terms of 4d and 6d operators, as equations for the different components $h_{\mu\nu}$, $h_{\mu n}$ and h_{mn}. It is also important to recall that each of these 10d degrees of freedom gives rise to an infinite tower of KK modes, including the spin-2 fields $h_{\mu\nu}$ that do describe 4d GWs. We will come back to this point in the next section.

On the other hand, the requirement that the 4d spacetime in (3.127) is maximally symmetric implies that the background solutions for H_3, F_3 and F_5 take the form

$$H_3 = \frac{1}{3!}H_{mnp}\,\mathrm{d}y^m\wedge\mathrm{d}y^n\wedge\mathrm{d}y^p\,, \tag{5.5}$$

$$F_3 = \frac{1}{3!}F_{mnp}\,\mathrm{d}y^m\wedge\mathrm{d}y^n\wedge\mathrm{d}y^p\,, \tag{5.6}$$

$$F_5 = \frac{1}{5!}(1+\star_{10})\sqrt{-g_4}\,\mathrm{d}\alpha\wedge\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\wedge\mathrm{d}x^\rho\wedge\mathrm{d}x^\sigma\,, \tag{5.7}$$

[3] In this chapter we slightly change notation in order to match the one used in [273], and will use $c = \langle\mathcal{V}\rangle^{2/3}$. Since we are working on a background solution, we will also set the 4d Weyl rescaling $e^{2\omega(x)} = 1$ in accordance with our conventions (cf. Appendix A.2).

where we also use the self-duality condition that must be imposed on F_5 (2.73). Since we are interested in the type B solutions of Type IIB flux compactifications that preserve $\mathcal{N} = 1$ supersymmetry (cf. section 3.4), the function $\alpha = \alpha(y)$ in the F_5 background is related to the warping, $g_s\alpha = H^{-1}$ (3.121). We can write these background fields in component form as

$$(H_3)_{MNP} = \delta^m_M \delta^n_N \delta^p_P H_{mnp} \tag{5.8}$$

$$(F_3)_{MNP} = \delta^m_M \delta^n_N \delta^p_P F_{mnp} \tag{5.9}$$

$$\begin{aligned}(F_5)_{MPQRS} &= \delta^m_M \delta^\pi_P \delta^\eta_Q \delta^\rho_R \delta^\sigma_S \sqrt{-g_4}(\partial_m\alpha)\epsilon_{\pi\eta\rho\sigma} \\ &\quad + \delta^a_M \delta^p_P \delta^q_Q \delta^r_R \delta^s_S \frac{\sqrt{-G}}{5!}\sqrt{-g_4}(\partial_m\alpha)\epsilon_{\pi\eta\rho\sigma}\epsilon^{m\pi\eta\rho\sigma}{}_{apqrs}\,. \end{aligned} \tag{5.10}$$

Similarly, the background values of $\partial_M\Phi$, $\partial_M C_0$ are only non-zero for internal components. We can therefore compute the different components of $\bar{\mathcal{T}}_{MN}$ (5.3) and $\mathcal{T}^{(1)}_{MN}$ (5.4),

$$\bar{\mathcal{T}}_{\mu\nu} = \frac{g_s^2}{4c^{1/2}H^5}(\partial H)^2 g_{\mu\nu}\,, \tag{5.11}$$

$$\bar{\mathcal{T}}_{\mu n} = 0\,, \tag{5.12}$$

$$\begin{aligned}\bar{\mathcal{T}}_{mn} &= \frac{1}{2}(\partial_m\Phi)(\partial_n\Phi) + \frac{e^{2\Phi}}{2}(\partial_m C_0)(\partial_n C_0) + \frac{g_s}{4cH}e^{-\Phi}(H_3)_{mpq}(H_3)_n{}^{pq} \\ &\quad + \frac{g_s}{4cH}e^{\Phi}(\tilde{F}_3)_{mpq}(\tilde{F}_3)_n{}^{pq} - \frac{g_s^2}{4H^4}(\partial_m H)(\partial_n H)\,, \end{aligned} \tag{5.13}$$

$$\mathcal{T}^{(1)}_{\mu\nu} = \frac{1}{4c^{1/2}H^{9/2}}(\partial H)^2(h_{\mu\nu} - h^\rho{}_\rho g_{\mu\nu}) - \frac{1}{4cH^{11/2}}\{g^{mp}g^{nq}(\partial_p H)(\partial_q H)h_{mn}\}g_{\mu\nu}\,, \tag{5.14}$$

$$\mathcal{T}^{(1)}_{\mu n} = -\frac{1}{4c^{1/2}H^{9/2}}h_{\mu m}g^{mp}(\partial_p H)(\partial_n H) \tag{5.15}$$

$$\mathcal{T}^{(1)}_{mn} = \frac{g_s}{2c^{3/2}H^{3/2}}(e^{-\Phi}H_m{}^{rs}H_{nrp} + e^{\Phi}F_m{}^{rs}F_{nrp})g^{pq}h_{sq} - \frac{1}{4H^{7/2}}h^\rho{}_\rho(\partial_m H)(\partial_n H)\,, \tag{5.16}$$

where $(\partial H)^2 \equiv g^{pq}(\partial_p H)(\partial_q H)$, from which follows the trace

$$\begin{aligned}G^{PQ}\mathcal{T}^{(1)}_{PQ} &= -\frac{g_s^2}{c^{1/2}H^4}h^\rho{}_\rho(\partial H)^2 - \frac{g_s^2}{cH^5}g^{mp}g^{nq}(\partial_p H)(\partial_q H)h_{mn} \\ &\quad + \frac{g_s}{2c^2H^2}(e^{-\Phi}H_{mrs}H^{mrq} + e^{\Phi}F_{mrs}F^{mrq})g^{sp}h_{pq}\,. \end{aligned} \tag{5.17}$$

When decomposing the wave equation (5.1) into the three equations describing the 4d dynamics of the tensor, vector and scalar modes, we are only taking into account fluctuations of h_{MN}, whereas all other field fluctuations are set to zero for simplicity. One should ultimately check that this is consistent — e.g. because the fields are heavier than the first KK modes of the graviton

— or take into account such fluctuations. Under this assumption, the equations become

$$\begin{aligned}
\mu\nu:\quad & H^{1/2}\Box_4 h_{\mu\nu} + \frac{\Delta_{\mathcal{M}} h_{\mu\nu}}{c^{1/2}H^{1/2}} + \frac{h_{\mu\nu}\Delta_M H}{2c^{1/2}H^{3/2}} - 2H h_{\rho\alpha} g^{\alpha\sigma} R^{\rho}{}_{\mu\nu\sigma} - \frac{g^{pq}\nabla_{(\mu}h_{\nu)p}\partial_q H}{2c^{1/2}H^{3/2}} \\
& + \frac{g^{pq}\partial_p h_{\mu\nu}\partial_q H}{c^{1/2}H^{3/2}} - \frac{1}{4c^{1/2}H^{5/2}}(\partial H)^2 h_{\mu\nu} - \frac{h^{\rho}{}_{\rho}}{8c^{1/2}H^{5/2}} h_{\mu\nu}(\partial H)^2 \\
& + g_{\mu\nu}\frac{g^{pq}g^{rs}}{2cH^{7/2}}\Big(h_{pr}(H\nabla_q\partial_s H - \partial_q H\partial_s H) + \frac{1}{4}h_{pq}\partial_r H\partial_s H\Big) \\
& - \frac{g_s}{8c^2H^{5/2}} g_{\mu\nu}(e^{-\Phi}H_{mrs}H^{mrq} + e^{\Phi}F_{mrs}F^{mrq})g^{sp}h_{pq} = 0\,,
\end{aligned} \tag{5.18}$$

$$\begin{aligned}
\mu n:\quad & H^{1/2}\Box_4 h_{\mu n} + \frac{\Delta_{\mathcal{M}} h_{\mu n}}{c^{1/2}H^{1/2}} - \frac{g^{pq}h_{\mu p}\nabla_p\partial_n H}{2c^{1/2}H^{3/2}} - \frac{1}{c^{1/2}H^{5/2}}(\partial H)^2 h_{\mu n} \\
& - \frac{g^{pq}\nabla_\mu h_{pn}\partial_q H}{2c^{1/2}H^{3/2}} + \frac{1}{2}H^{-1/2}g^{\rho\sigma}\nabla_\rho h_{\mu\sigma}\partial_n H + \frac{g^{pq}\partial_{[q}H\ \nabla_{n]}h_{\mu p}}{c^{1/2}H^{3/2}} \\
& - \frac{1}{c^{1/2}H^{1/2}}g^{pq}h_{\mu q}\tau^{(0)}_{np} - \frac{g_s^2}{4c^{1/2}H^{9/2}}(\partial H)^2 h_{\mu n} - \frac{g_s^2}{2c^{1/2}H^{9/2}}h_{\mu m}g^{mp}(\partial_p H)(\partial_n H) = 0\,,
\end{aligned} \tag{5.19}$$

$$\begin{aligned}
mn:\quad & H^{1/2}\Box_4 h_{mn} + \frac{\Delta_{\mathcal{M}} h_{mn}}{c^{1/2}H^{1/2}} - \frac{h_{mn}\Delta_{\mathcal{M}} H}{2c^{1/2}H^{3/2}} - \frac{2g^{rs}h_{pr}R^{p}{}_{mns}}{c^{1/2}H^{1/2}} \\
& \frac{g^{\rho\sigma}\nabla_\rho h_{\sigma(m}\partial_{n)}H}{H^{1/2}} - \frac{g^{pq}\nabla_p h_{q(m}\partial_{n)}H}{c^{1/2}H^{3/2}} + \frac{g^{pq}\nabla_{(n}h_{m)p}\partial_q H}{c^{1/2}H^{3/2}} \\
& - \frac{g^{pq}\nabla_p h_{mn}\partial_q H}{c^{1/2}H^{3/2}} + \frac{1}{c^{1/2}H^{5/2}}(\partial H)^2 h_{mn} - \frac{5g^{pq}h_{p(n}\partial_{m)}H\partial_q H}{4c^{1/2}H^{5/2}} \\
& + \frac{h^{\rho}{}_{\rho}}{2H^{3/2}}\Big(H\nabla_m\partial_n H - \partial_m H\partial_n H + \frac{1}{4}g_{mn}g^{pq}\partial_p H\partial_q H\Big) \\
& + \frac{h_{rs}\partial_p H\partial_q H}{8c^{1/2}H^{5/2}}\Big(\delta^{p}{}_{m}\delta^{q}{}_{n}g^{rs} + g_{mn}(g^{pr}g^{qs} - g^{rs}g^{pq})\Big) \\
& \Big(\frac{1}{4H^{7/2}}h^{\rho}{}_{\rho}(\partial H)^2 + \frac{1}{4c^{1/2}H^{9/2}}g^{mp}g^{nq}(\partial_p H)(\partial_q H)h_{mn} \\
& - \frac{g_s}{2c^{3/2}H^{3/2}}(e^{-\Phi}H_{mrs}H^{mrq} + e^{\Phi}F_{mrs}F^{mrq})g^{sp}h_{pq}\Big)g_{mn} - \frac{2}{c^{1/2}H^{1/2}}g^{pq}h_{q(m}\bar{\mathcal{T}}_{n)p} \\
& + \frac{g_s}{c^{3/2}H^{3/2}}(e^{-\Phi}H_m{}^{rs}H_{nrp} + e^{\Phi}F_m{}^{rs}F_{nrp})g^{pq}h_{sq} - \frac{1}{2H^{7/2}}h^{\rho}{}_{\rho}(\partial_m H)(\partial_n H) = 0\,,
\end{aligned} \tag{5.20}$$

where $\Box_4 = g^{\mu\nu}\nabla_\mu\nabla_\nu$, $\Delta_{\mathcal{M}} = g^{pq}\nabla_p\nabla_q$ and the covariant derivative ∇_ρ (∇_p) is with respect to the 4d metric $g_{\mu\nu}$ (6d metric g_{pq}).

The resulting equations (5.18–5.20) couple $h_{\mu\nu}$, $h_{\mu n}$ and h_{mn}, which makes it hard to find a general solution. Since we are most interested in the 4d gravitational modes $h_{\mu\nu}$, we may consider the simpler case where $h_{\mu n} = h_{mn} = 0$, i.e. a solution for which the vector and scalar modes vanish.[4] We also make the field redefinition $h_{\mu\nu} \to H^{-1/2}h_{\mu\nu}$, so that (5.18) describes

[4]Note that by choosing this simple solution, we are missing some possibly interesting effects, e.g. the breathing mode identified in [269] will not be present. We should also recall the relation between h_{mn} and the complex structure and Kähler moduli. See also [275] for an analysis of the spin-0 modes on a Klebanov-Strassler background.

fluctuations of the 4d (unwarped) metric $g_{\mu\nu}$. The equations become

$$\mu\nu: \quad \Box_4 h_{\mu\nu} + \frac{\Delta_{\mathcal{M}} h_{\mu\nu}}{c^{1/2}H} - 2h_{\rho\alpha}g^{\alpha\sigma}R^{\rho}{}_{\mu\nu\sigma} - \frac{h_{\mu\nu}h^{\rho}{}_{\rho}}{8c^{1/2}H^{5/2}}(g^{pq}\partial_p H\partial_q H) = 0\,, \tag{5.21}$$

$$\mu n: \quad \partial_n H(g^{\rho\sigma}\nabla_\rho h_{\mu\sigma}) = 0\,, \tag{5.22}$$

$$mn: \quad h^{\rho}{}_{\rho}\Big(H\nabla_m\partial_n H - \partial_m H\partial_n H + \frac{1}{4}g_{mn}g^{pq}\partial_p H\partial_q H\Big) + \frac{1}{2H^2}h^{\rho}{}_{\rho}\big((\partial H)^2 g_{mn} - 2(\partial_m H)(\partial_n H)\big) = 0\,. \tag{5.23}$$

Although we set $h_{\mu n} = h_{mn} = 0$, the equations describing these fluctuations must still be satisfied to ensure that this is indeed a solution. For a non-constant warp factor, which is the case we want to study, the vector equation implies that $g^{\rho\sigma}\nabla_\rho h_{\mu\sigma}$ vanishes. Tracing the scalar equation we find

$$h^{\rho}{}_{\rho}(H\Delta_{\mathcal{M}}H + 2(\partial \ln H)^2) = 0\,, \tag{5.24}$$

which gives $h^{\rho}{}_{\rho} = 0$. Therefore, the vector and scalar equations impose conditions on $h_{\mu\nu}$ which correspond to the 4d transverse-traceless gauge [269],

$$g^{\rho\sigma}\nabla_\rho h_{\mu\sigma} = 0\,, \qquad h^{\rho}{}_{\rho} = 0\,. \tag{5.25}$$

The equation for $h_{\mu\nu}$ is then

$$\Box_4 h_{\mu\nu} + \frac{\Delta_{\mathcal{M}} h_{\mu\nu}}{c^{1/2}H} - 2h_{\rho\alpha}g^{\alpha\sigma}R^{\rho}{}_{\mu\nu\sigma} = 0\,. \tag{5.26}$$

On the other hand, for a 4d spacetime that is maximally symmetric, we have the following relation

$$R^{\rho}{}_{\mu\nu\sigma} = \frac{\Lambda_4}{3}(\delta^{\rho}_{\nu}g_{\mu\sigma} - \delta^{\rho}_{\sigma}g_{\mu\nu})\,, \tag{5.27}$$

where $\Lambda_4 = \mathcal{R}_4/4$, which gives

$$R^{\rho}{}_{\mu\nu\sigma}g^{\sigma\alpha}h_{\alpha\rho} = \frac{\Lambda_4}{3}(h_{\mu\nu} - g_{\mu\nu}h^{\rho}{}_{\rho})\,, \tag{5.28}$$

and using $h^{\rho}{}_{\rho} = 0$ the wave equation becomes

$$\Box_4 h_{\mu\nu} + \frac{\Delta_{\mathcal{M}} h_{\mu\nu}}{c^{1/2}H} - \frac{2}{3}\Lambda_4 h_{\mu\nu} = 0\,. \tag{5.29}$$

This equation describes the 4d tensor components of h_{MN}, but we should remember that $h_{\mu\nu}(x^\mu, y^m)$ is still a function of both external and internal coordinates, which means it propagates in the 10-dimensional spacetime. In the 4d EFT, we must express it in terms of modes which are only functions of external coordinates $h_{\mu\nu}(x^\mu)$ and therefore only propagate in the 4d spacetime. More precisely, equation (5.29) corresponds to an infinite tower of 4d spin-2 modes $h^k_{\mu\nu}(x^\mu)$, where k labels each mode in the tower (cf. (3.18)). We should also note that (5.29) only describes the propagation of each mode in this tower, but not how it interacts with other degrees of freedom (including other modes in the same tower) and therefore how gravitational waves are sourced. It does however tell us the properties of the propagating degrees of freedom which carry information about how gravitational waves propagate in this warped background and the way they will couple to sources once these are taken into account. In the next section we will study these properties, which include the masses of each mode, their wavefunctions in the extra dimensions and the respective normalisations.

5.2 Warping the graviton tower

5.2.1 KK mode equation

Let us rewrite (5.29) in the following suggestive way

$$\Box_4 h_{\mu\nu} + \mathcal{O}_{\mathcal{M}} h_{\mu\nu} = 0\,, \tag{5.30}$$

with $\mathcal{O}_{\mathcal{M}} \equiv \frac{\Delta_{\mathcal{M}}}{c^{1/2}H} - \frac{2}{3}\Lambda_4$. In the 4d EFT, we must express $h_{\mu\nu}(x^\mu, y^p)$ in terms of modes which are only functions of external coordinates $h^k_{\mu\nu}(x^\mu)$, where k labels each mode in an infinite tower, and a complete basis Φ^k of eigenmodes of $\mathcal{O}_{\mathcal{M}}$,

$$h_{\mu\nu}(x^\mu, y^p) = \sum_k h^k_{\mu\nu}(x^\mu)\Phi_k(y^p)\,. \tag{5.31}$$

The decomposition is such that

$$\mathcal{O}_{\mathcal{M}} h_{\mu\nu} = \mathcal{O}_{\mathcal{M}}(h^k_{\mu\nu}(x^\mu)\Phi_k(y^p)) = h^k_{\mu\nu}\mathcal{O}_{\mathcal{M}}\Phi_k(y^p) = -m_k^2 h^k_{\mu\nu}\Phi_k(y^p) \tag{5.32}$$

where there is an implicit sum in k, following from the eigenvalue equation $\mathcal{O}_{\mathcal{M}}\Phi_k(y^p) = -m_k^2\Phi(y^p)$. Since Φ_k is a scalar,[5]

$$\Delta_{\mathcal{M}}\Phi_k = \frac{1}{\sqrt{g}}\partial_p(\sqrt{g}g^{pq}\partial_q\Phi_k)\,, \tag{5.33}$$

and hence the eigenvalue equation can be written as

$$\frac{1}{\sqrt{g}}\partial_p(\sqrt{g}g^{pq}\partial_q\Phi_k) + c^{1/2}H\Big(m_k^2 - \frac{2}{3}\Lambda_4\Big)\Phi_k = 0\,. \tag{5.34}$$

We should remember that the compact space contains two pieces glued together — a warped throat described by the Klebanov-Strassler solution and a compact bulk, which is usually chosen to be a Calabi-Yau whose metric we do not explicitly know. Therefore the 6d metric g_{pq} which

[5]Notice the difference in the form of the Laplacian $\Delta_{\mathcal{M}}$ compared to [269, 279], which follows from the different choice of metric conventions in (3.127).

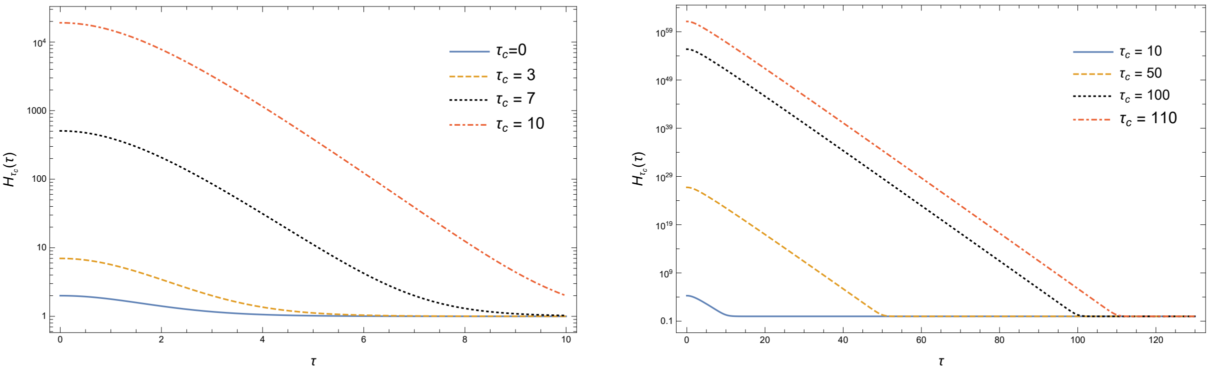


Figure 5.1: Warp factor $H(\tau)$ (in log scale) for different choices of τ_c. We can see how the choice of τ_c is directly related to the strength of the warping and the size of the warped throat (defined as the region with non-trivial warping).

appears in (5.34) will be different inside the throat and in the bulk — it corresponds to the metric of the warped deformed conifold in the throat region and (usually) to the unknown metric of a compact CY_3 in the bulk. However, we can still solve the equation in the region where the metric is unkown if $\Phi_k = 0$ in the CY_3 bulk, which for consistency implies that the modes must vanish at the point where these two regions meet. We will motivate further this boundary condition below.

Importantly, the question of whether a region is warped, through $H(\tau)$ (3.128), depends not only on the solution $e^{-4A_0(\tau)}$, but also on the size of the bulk to which the throat is glued. The warping will dominate when $e^{-4A_0(\tau)} \gg c$ and be negligible when $e^{-4A_0(\tau)} \ll c$, so that effectively the *warped throat* ends at $e^{-4A_0(\tau)} \sim c$ (recall the change in notation, $c = \langle \mathcal{V} \rangle^{2/3}$). This interplay between $e^{-4A_0(\tau)}$ and c gives an interesting intermediate regime of weak warping which was a key point of our discussion in chapter 4 of the new weakly-warped solutions [172]. For concreteness, we can define the gluing point τ_c such that $e^{-4A_0(\tau_c)} = c$, keeping in mind that the gluing must involve a smooth transition between the warped throat and the bulk around τ_c. Therefore τ_c can be defined implicitly in terms of the conifold parameters (cf. (3.108))

$$\frac{e^{-4A_0(\tau_c)}}{c} = 1 \implies 2^{2/3}\frac{(\alpha' g_s M)^2}{c\,\epsilon^{8/3}} = \frac{1}{I(\tau_c)}\,, \tag{5.35}$$

which allows us to rewrite the warp factor as

$$H_{\tau_c}(\tau) = 1 + \frac{I(\tau)}{I(\tau_c)}\,. \tag{5.36}$$

This highlights the fact that the warp factor depends on one parameter only, τ_c. Even though it is implicitly determined by a specific combination of the more familiar parameters through (5.35), it is a convenient parametrisation since a choice of τ_c has a clear physical interpretation. In Fig. 5.1 we show how the warp-factor $H(\tau)$ behaves for different choices of τ_c.

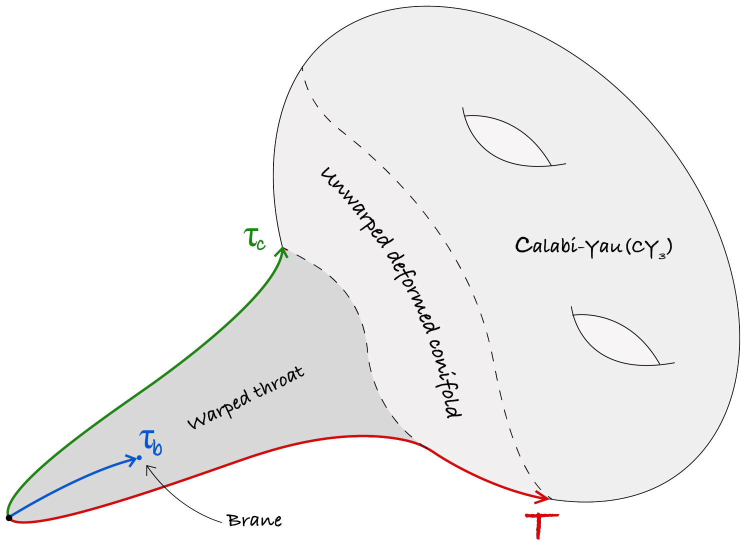


Figure 5.2: The internal space consists of a 6d compact manifold with a warped region described by the Klebanov-Strassler solution (i.e. a warped deformed conifold). We split the bulk into two different pieces: one piece is a generic CY_3 and the other takes the metric of an *unwarped* deformed conifold and serves as a transition between the warped throat and the CY_3 (with $\tau_c < \tau < T$, where τ is the radial coordinate in the deformed conifold metric (3.103)).

However, gluing the *warped throat* directly onto a compact CY_3, would not let us take the interesting limit where there is no warping and the whole internal space is the CY_3 whose metric we do not know, since the wavefunctions will be identically zero. In order to consider this regime, we may split the bulk region into two different pieces: one piece is the generic CY_3 and the other takes the metric of an *unwarped* deformed conifold and serves as a transition (with $\tau_c < \tau < T$) between the warped throat and the CY_3 (see Fig. 5.2). While we still solve the equation in the CY_3 with $\Phi_k = 0$, we can now have a non-vanishing wavefunction in the piece of the bulk described by the unwarped deformed conifold. Whereas τ_c determines the size of the warped throat, T determines the portion of the bulk in which the wavefunctions do not vanish[6] (more precisely, $T - \tau_c$ determines the extension of the wavefunction into the bulk). Notice that a fully warped conifold corresponds to the limit $\tau_c \to T$ and an unwarped conifold corresponds, roughly, to $\tau_c \to 0$.[7]

Therefore we will write (5.34) explicitly using the metric (3.103) in terms of the warp factor $H_{\tau_c}(\tau)$. We consider splitting the 6d coordinates y^m into a radial coordinate τ and angular coordinates θ^a, $a = 1, ..., 5$ (these are related to the 1-forms g^i in the conifold metric (3.99)). This will split the Laplacian into two pieces, one for τ and one along the angular coordinates.

[6] Note that T is nothing but the cutoff τ_Λ (related to the scale Λ_0) where the deformed conifold metric was glued to the compact CY_3 in the notation used in chapter 4. Also there, in the context of the new weakly-warped solutions, this cutoff was not necessarily determined by the warping and could be chosen such that at least part of the deformed conifold was effectively *unwarped*.

[7] Strictly speaking, the unwarped limit corresponds to $H(\tau) = 1$ for all τ, whereas when $\tau_c = 0$, $H(0) = 2$. At $\tau_c = 0$, the second term in (3.128) becomes of the same order as the first term, marking the boundary between a warped and an unwarped regime.

With this in mind, we can decompose the functions $\Phi_k(\tau, \theta^a)$ as (see [280])

$$\Phi_k(\tau, \theta^a) = G(\tau)^{-1/2} B_k(\tau)\varphi_k(\theta^a)\,, \tag{5.37}$$

such that (5.34) becomes a Schrödinger equation for $B_k(\tau)$

$$B_k'' - V_{\text{eff}} B_k = 0\,, \tag{5.38}$$

with an effective potential given by

$$V_{\text{eff}} = -g_{\tau\tau}\left(c^{1/2} H\left(m_k^2 - \frac{2}{3}\Lambda_4\right) + \frac{\mathcal{O}_{\text{ang}}\Phi_k}{\Phi_k}\right) + \frac{(G^{1/2})''}{G^{1/2}}\,, \tag{5.39}$$

where we factorise $(g_{\text{con}})_{mn} = K_{mn}(\tau)\gamma_{mn}(\theta)$ (with no summation) and define

$$G(\tau) = \frac{2^4}{\epsilon^{8/3}} g^{\tau\tau}\sqrt{g_{\text{con}}} = \mathcal{K}(\tau)^2 \sinh^2(\tau)\,, \tag{5.40}$$

$$\mathcal{O}_{\text{ang}}\varphi_k \equiv K^{ab}(\tau)\frac{1}{\sqrt{\gamma}}\partial_{\theta^a}\left(\sqrt{\gamma}\gamma^{ab}(\theta)\partial_{\theta^b}\varphi_k\right)\,, \tag{5.41}$$

with $g_{\tau\tau} = \frac{\epsilon^{4/3}}{6\mathcal{K}(\tau)^2}$ following directly from the metric (3.103), the operator $\mathcal{O}_{\text{ang}}$ containing the angular information of the metric and the sum in a, b implied. Since the contribution from the angular coordinates is more complicated, we look at modes with[8] $\mathcal{O}_{\text{ang}}\varphi_k = 0$ (usually known as the *s-orbital*). In this case, we have $\varphi_k(\theta^a) = \text{const.}$ and we can absorb it into the overall normalisation of the wavefunction. Notice that trying a constant wavefunction (i.e. with no dependence on the internal coordinates) requires $B_k(\tau) = G(\tau)^{1/2}$, in which case (5.38) reduces to

$$g_{\tau\tau}\left(m_k^2 - \frac{2}{3}\Lambda_4\right) = 0\,, \tag{5.42}$$

from which it follows that $m_k^2 = \frac{2}{3}\Lambda_4$. We see that for a flat Minkowski background, i.e. $\Lambda_4 = 0$, the constant wavefunction Φ_0 corresponds to a massless 4d graviton, which no longer exists if $\Lambda_4 \neq 0$.[9] In what follows we assume a Minkowski background and set $\Lambda_4 = 0$.

Defining also $\hat{E}_k = \epsilon^{2/3}c^{1/4}m_k$ the effective potential can be written as

$$V_{\text{eff}} = -\frac{H_{\tau_c}(\tau)}{6K(\tau)^2}\hat{E}_k^2 + \frac{(G^{1/2})''}{G^{1/2}}\,, \tag{5.43}$$

which we can think of as a family of potentials, each member of which is determined by a choice of τ_c and is only a function of τ (Fig. 5.3). There is no analytical solution for the corresponding Schrödinger equations — it can either be solved numerically (Section 5.2.4) or we can consider

[8] These s-orbital modes will be the lightest ones [280, 281]. The isometries in the KS throat imply that KK modes carry conserved angular momenta, which can have interesting cosmological consequences as explored in [281–283]. Although gluing to a compact CY breaks the isometries, as the wavefunctions of the KK modes are localised at the tip (or even vanish in the bulk), the effects of this isometry breaking should be small [281].

[9] Interestingly, if we choose a de Sitter background the zero mode mass matches the Higuchi bound [284, 285]. Note that this is consistent with current constraints on the graviton mass [286].

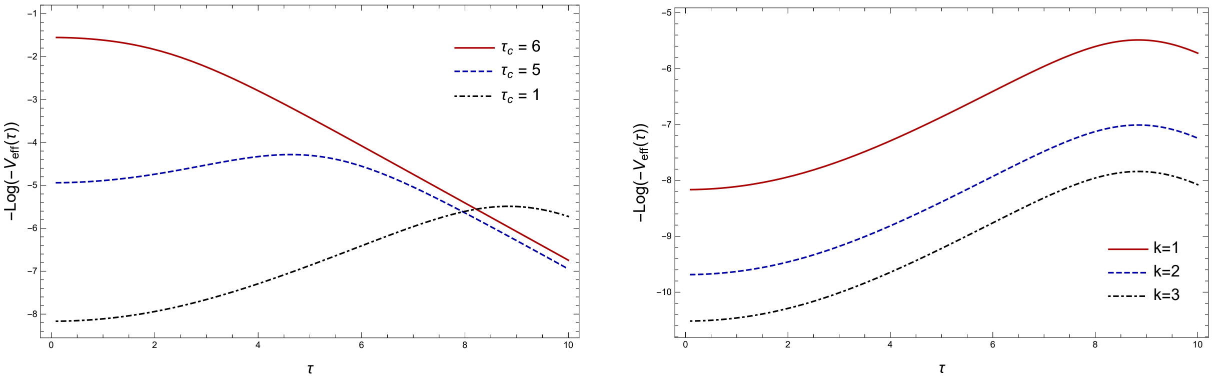


Figure 5.3: Effective potential V_{eff} corresponding to the first mode for different choices of τ_c (left) and for the first 3 modes when $\tau_c = 9$ (right), both with $T = 10$.

approximations to this potential (Section 5.2.5).

5.2.2 Boundary conditions

In order to obtain the wavefunctions, we must also impose boundary conditions, which we choose as follows.

1. The wavefunction is finite at $\tau = 0$, i.e.

$$\lim_{\tau\to 0} \Phi_k(\tau) < \infty \implies \lim_{\tau\to 0} B_k(\tau) = 0\,;$$

2. The wavefunctions vanish as they approach the CY_3 region, i.e.

$$\Phi_k(T) = 0 \implies B_k(T) = 0\,.$$

As we discussed below (5.34), the boundary condition at $\tau = T$ is useful if one wants to remain agnostic with respect to the geometry of the bulk beyond the conifold. Moreover, it is a well-motivated approximation thanks to the following arguments. Locally, a vanishing[10,11] wavefunction in the bulk is certainly a solution to the KK wave equations (5.34) irrespective of the metric. With limited knowledge of the compact CY region and gluing, we could not exclude a global obstruction to such a solution. However, the localisation of the KK modes at the tip of the throat where they minimise their energy à la Randall-Sundrum [57, 58] suggests that the solution is a good approximation, at least in the warped case. Moreover, it connects to the infinite throat limit ($\tau_c \to T \to \infty$), for which the wavefunction must decay towards zero in order to be normalisable. Note that in the unwarped case an alternative boundary condition would

[10]Note that a non-vanishing constant wavefunction would not be a solution to equation (5.34) for the massive modes in the tower.
[11]Although other KK modes will be present that are not suppressed in the bulk, we would expect these to be heavier than the modes considered here [170].

be $\partial_\tau \Phi_k(T) = 0$, motivated e.g. by the conifold ending on an O3-plane [280]. We would not expect the mass spectrum to change qualitatively and the precise predictions could be worked out easily following the same methods used here. However, we choose to explore the boundary condition $\Phi_k(T) = 0$ also for the unwarped case as it clearly demonstrates that the localisation of the wavefunctions in the warped case is a consequence of the warping and not the boundary conditions.

5.2.3 Normalisation of the wavefunctions

Although the normalisation of the eigenfunctions $\Phi_k(y)$ does not affect the propagation of each mode, it does encode the strength of their gravitational couplings. In order to fix the normalisation, we canonically normalise the kinetic terms of each mode (even though the kinetic term for $h_{\mu\nu}$ is a sum of terms involving the different possible index contractions, it is sufficient to look at one such term; the full kinetic term can be conveniently encoded in the so-called Lichnerowicz operator [287]),

$$\begin{aligned} S &= \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-G}\,\mathcal{R}_{10} \\ &= \frac{1}{2\kappa^2}\int d^4x\sqrt{-g_4}\,c^{3/2}\int d^6y\sqrt{g_6}\,H\mathcal{R}_4 + \ldots \\ &= \bar{S} + \frac{1}{2\kappa^2}\int d^4x\sqrt{-g_4}\left\{c^{3/2}\int d^6y\sqrt{g_6}H\;\nabla_\rho h_{\mu\nu}\nabla^\rho h^{\mu\nu}\right\} + \ldots \\ &= \bar{S} + \int d^4x\sqrt{-g_4}\left\{\frac{1}{2}\nabla_\rho h^k_{\mu\nu}\nabla^\rho h^{k',\mu\nu}\Big(c^{3/2}\int d^6y\sqrt{g_6}\;H\;\Phi_k(y)\Phi_{k'}(y)\Big) + \ldots\right\}. \end{aligned} \tag{5.44}$$

In the last line, we make the usual field redefinition $h_{\mu\nu} \to \kappa\, h_{\mu\nu}$, which gives the standard mass dimension of 4 for a 10d bosonic field $h_{\mu\nu}$ (notice that this means $\delta g_{\mu\nu} = \kappa\, h_{\mu\nu} = \frac{\sqrt{V_{\rm w}}}{M_{\rm Pl}} h_{\mu\nu}$), and substitute the decomposition (5.31), with an implicit sum in the indices k and k'. Being the eigenmodes of the operator $\mathcal{O}_{\mathcal{M}}$, the functions $\Phi_k \equiv \Phi_k(y^p)$ form an orthogonal basis[12] under the inner product weighted by H,

$$\int d^6y\sqrt{g_6}\;H\;\Phi_k(y)\Phi_{k'}(y) = \delta_{kk'}\,. \tag{5.45}$$

In order to have canonical kinetic terms for each spin-2 mode in 4d, $h^k_{\mu\nu}$, we include a normalisation constant in each wavefunction $\Phi_k = N_{(k)}\tilde{\Phi}_k$, with $N_{(k)}$ defined as

$$N_{(k)}^{-2} = c^{3/2}\int d^6y\sqrt{g_6}\;H\;|\tilde{\Phi}_k(y)|^2\,, \tag{5.46}$$

[12] One can show this by noting that (5.38) is a regular Sturm-Liouville problem, which guarantees (i) real eigenvalues, (ii) a unique (up to normalisation) eigenfunction for each eigenvalue and (iii) normalised eigenfunctions which form an orthonormal basis under the inner product $\int\sqrt{g_6}\;H\;\Phi_k(y)\Phi_{k'}(y)$. This is equivalent to verifying that the Hamiltonian for the corresponding Schrödinger equation is hermitian.

where $\tilde{\Phi}_k(y) = G(\tau)^{-1/2} B_k(\tau)$ is the wavefunction obtained by solving (5.38). This gives a canonically normalised action (at this order in perturbations)

$$S = \int d^4x \sqrt{-g_4} \left\{ \frac{1}{2} \nabla_\rho h^k_{\mu\nu} \nabla^\rho h^{\mu\nu}_k - \frac{1}{2} m_k^2 h^k_{\mu\nu} h^{\mu\nu}_k \right\} , \tag{5.47}$$

from which the decoupled equations of motion for each $h^k_{\mu\nu}$ could be derived,

$$\Box_4 h^k_{\mu\nu} - m_k^2 h^k_{\mu\nu} = 0 \, . \tag{5.48}$$

This is an infinite set of equations describing spin-2 modes of mass m_k in 4d. For the zero mode, $\tilde{\Phi}_0(\tau) = \texttt{const}$, which we can set to one by absorbing the constant in the normalisation $N_{(0)}$,

$$\Phi_0(\tau) = N_{(0)} \tilde{\Phi}_0(\tau) = N_{(0)} \, , \tag{5.49}$$

so that we have

$$N_{(0)}^{-2} = c^{3/2} \int d^6y \sqrt{g_6} \; H = V_{\rm w} \, . \tag{5.50}$$

Hence the graviton zero mode wavefunction is simply

$$\Phi_0(\tau) = \frac{1}{\sqrt{V_{\rm w}}} \, , \tag{5.51}$$

constant over the compact dimensions — the graviton zero mode does not localise.

We are interested in the contributions from higher modes with $m_{k\neq 0} \neq 0$. For these modes we find

$$\begin{aligned} N_{(k)}^{-2} &= c^{3/2} \int d^6y \sqrt{g_6} \; H |\tilde{\Phi}_k(y)|^2 & (5.52) \\ &= \frac{c^{3/2}\epsilon^4}{2^5 \cdot 3} \left(\int \prod_i g^i \right) \int_0^T d\tau \frac{\sinh^2(\tau)}{G(\tau)} H_{\tau_c}(\tau) |B_k(\tau)|^2 & (5.53) \\ &= \frac{2\pi^3}{3} c^{3/2} \epsilon^4 \mathcal{N}_{(k)}^{-2}(\tau_c, T) \, , & (5.54) \end{aligned}$$

where we used $\int \prod_i g^i = 64\pi^3$ and the fact that $\tilde{\Phi}_k = 0$ in the CY_3, and with τ_c given by (5.35). We also define $\mathcal{N}_{(k)}(\tau_c, T)$ as

$$\mathcal{N}_{(k)}^{-2}(\tau_c, T) \equiv \int_0^T d\tau \frac{H_{\tau_c}(\tau)}{\mathcal{K}(\tau)^2} |B_k(\tau)|^2 \, , \tag{5.55}$$

which only depends on the pair (τ_c, T), the known functions $I(\tau), G(\tau)$ and the wavefunctions $B_k(\tau)$. Note that one can easily take the limits $\tau_c \to 0$ (unwarped conifold) and $\tau_c \to T$ (fully warped conifold). In particular,

$$\mathcal{N}_{(k)}^{-2} \approx \frac{1}{I(\tau_c)} \int_0^{\tau_c} d\tau \frac{I(\tau)}{\mathcal{K}(\tau)^2} |B_k(\tau)|^2 + \int_{\tau_c}^{T} d\tau \frac{|B_k(\tau)|^2}{\mathcal{K}(\tau)^2} \,, \tag{5.56}$$

where we split the integral between the warped throat ($\tau < \tau_c$) and the unwarped portion of the deformed conifold ($\tau_c < \tau < T$).

As we mentioned before, the Schrödinger equation (5.38) that determines the masses and wavefunctions of the modes $h^k_{\mu\nu}$ does not admit an exact analytical solution. Therefore one can either solve it numerically or study some analytical approximations in order to get some insight into the behaviour of the solutions. We will start by presenting some numerical solutions, which will qualitatively set our expectations for the different limits, and will then give analytical approximations for the fully warped and unwarped conifold limits.

5.2.4 Numerical solutions

We can solve (5.38) with the effective potential (5.43) and the boundary conditions chosen above, for different values of (τ_c, T), which may be interpreted as the strength of the warping, given by τ_c, and the proportion of the conifold that is warped, given by τ_c/T.

In Fig. 5.4 we plot the eigenvalues $\hat{E}_k$ as a function of the ratio τ_c/T for $T = 10$ and $k = 1, 2, 3$ (left), and for $T = 150$ and $k = 1$ (right). We can see from the left plot that the effect of the warping only starts influencing the masses once the warped throat dominates over the (unwarped conifold) bulk $\tau_c/T \gtrsim 1/2$, with higher modes starting to feel the effects of warping for smaller values τ_c/T. In the right plot we clearly identify the dominant behaviour of the masses in the different regimes (throat dominated vs bulk dominated) — they are suppressed by the warping when the throat dominates and by the volume when the bulk dominates.

In Fig. 5.5 we plot the wavefunctions $\tilde{\Phi}_k(\tau)$ prior to normalisation (left) and the normalised wavefunctions $\Phi_k(\tau)$ (right) for the first three modes ($k = 1, 2, 3$), when $T = 30$ and $\tau_c = 0, 15, 30$, representing the unwarped, partially warped and fully warped regimes, respectively. Without the warping, $\tau_c = 0$, the wavefunctions will spread throughout the internal space, much like the zero mode, only being forced to go to zero by our choice of boundary conditions. When we increase the warping by setting $\tau_c/T = 1/2$, we see the wavefunction starting to localise near the tip, but reaching a plateau in the bulk — this is a transition between a warping dominated and a bulk dominated regime. When the warping completely dominates, the wavefunctions localise at the tip and quickly decrease as they approach the bulk. This illustrates how the balance between a strong warping and a large bulk may influence the localisation of the modes, i.e. the profile of their wavefunctions — this is reflected in the couplings to other modes, as we discuss in Section 5.3. Finally, we see how the normalisation affects the wavefunctions, with higher modes having larger amplitudes than lower modes — this will translate into stronger couplings

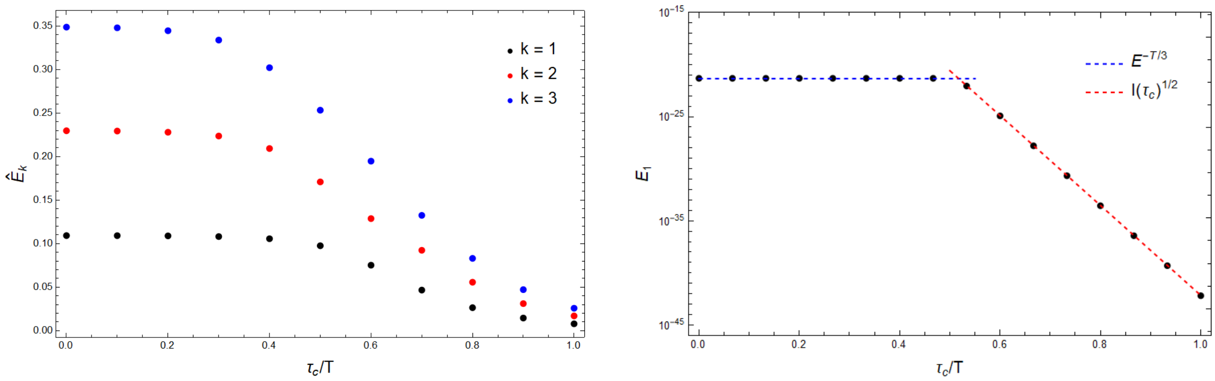


Figure 5.4: (Left) Eigenvalues $\hat{E}_k$ for the first 3 modes obtained by solving the Schrödinger equation numerically with $V_{\rm eff}$ (5.43), for fixed $T = 10$ and different values of τ_c; (Right) Eigenvalues $\hat{E}_1$ for the first excited mode obtained by solving the Schrödinger equation numerically with $V_{\rm eff}$ (5.43), for fixed $T = 150$ and different values of τ_c.

for higher modes, which was also found in [276]. We also see that the wavefunction amplitudes will be smaller for weaker warping and larger volumes, which can be understood by noting that in the absence of warping the wavefunctions will spread through a larger region and therefore have smaller overall amplitudes — this translates into weaker couplings.

5.2.5 Analytical approximation

Now that we have a qualitative expectation of what the solutions should look like in different regimes, let us study some analytical approximations. In order to find an approximate analytical solution for the Schrödinger equation (5.38), we split the potential into two pieces, the warped region when $\tau < \tau_c$ and the unwarped region when $\tau > \tau_c$, and approximate each region using the corresponding dominant term in $H_{\tau_c}(\tau)$ (5.36). The asymptotic behaviours of the functions appearing in $V_{\rm eff}$ (5.43), as $\tau \to \infty$, are given by

$$\frac{I(\tau)}{\mathcal{K}(\tau)^2} \to \frac{3}{2}\tau e^{-2\tau/3}\,, \tag{5.57}$$

$$\frac{(G^{1/2})''}{G^{1/2}} \to \frac{4}{9} - \frac{16}{3}\tau e^{-2\tau} \tag{5.58}$$

and $\mathcal{K}(\tau) \to 2^{1/3}e^{-\tau/3}$. Notice that the subleading term in the second line goes as $\tau(e^{-2\tau/3})^3$, which is subdominant compared to $\tau e^{-2\tau/3}$ in the first line. In this limit the effective potential becomes

$$V_{\rm eff} \overset{\tau\to\infty}{\approx} V_{\rm asym} \equiv \begin{cases} -\dfrac{1}{4}\dfrac{\hat{E}_k^2}{I(\tau_c)}\,\tau e^{-2\tau/3} + \dfrac{4}{9}\,, & \tau < \tau_c \\[2ex] -\dfrac{1}{6\cdot 2^{2/3}}\hat{E}_k^2 e^{2\tau/3} + \dfrac{4}{9} & \tau_c < \tau < T \end{cases} \tag{5.59}$$

We plot $V_{\rm asym}$ in Fig. 5.6 together with the exact form of $V_{\rm eff}$. We can see that this is a good

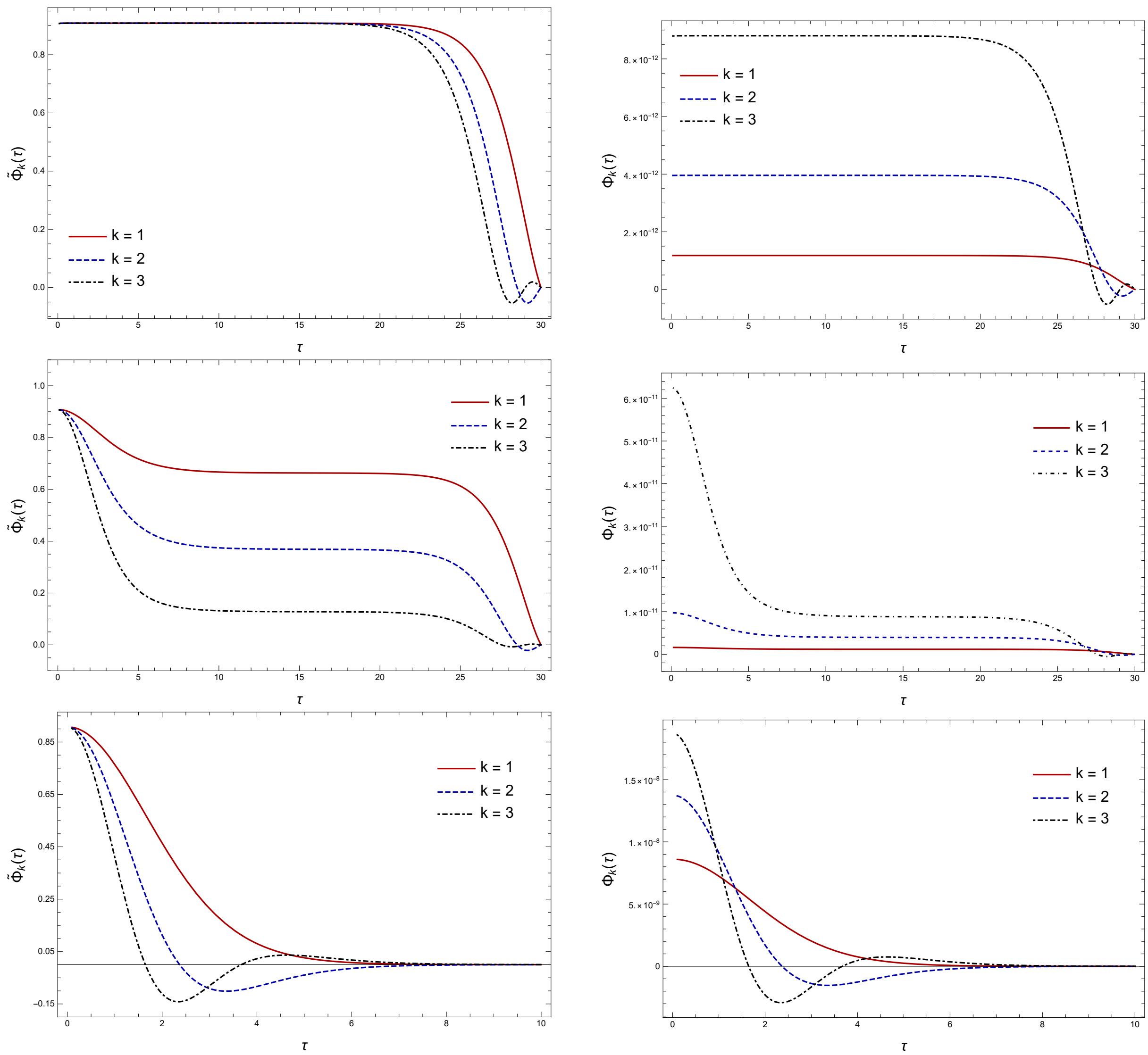


Figure 5.5: Plot of $\tilde{\Phi}_k(\tau)$ (left) and $\Phi_k(\tau)$ (right) with $k = 1, 2, 3$, for different values of $\tau_c = 0, 15, 30$ and fixed $T = 30$. When $\tau_c = 0$ (unwarped case) the wavefunctions spread as much as possible (given our boundary conditions) through the conifold, being constant for most of values of τ, and when $\tau_c = T$ they localise at the tip $\tau = 0$. Wavefunctions for modes with higher k always have a larger amplitude than lower modes.

approximation for large τ, as expected, but it behaves rather differently at small τ. This can be problematic since we are dealing with an eigenvalue problem, which depends crucially on this behaviour. For small τ, $\tau e^{-\frac{2\tau}{3}} \sim \tau$, so that (5.59) will approach the positive constant 4/9 as $\tau \to 0$. If instead we have only the exponential behaviour, the negative contribution at small τ remains, which means we can better approximate this regime (Fig. 5.6). For larger τ, however, this makes the approximation less accurate. We can modify this approximation by including a couple of free parameters, as suggested in [280], to obtain a V_{approx} that looks more like V_{eff} at

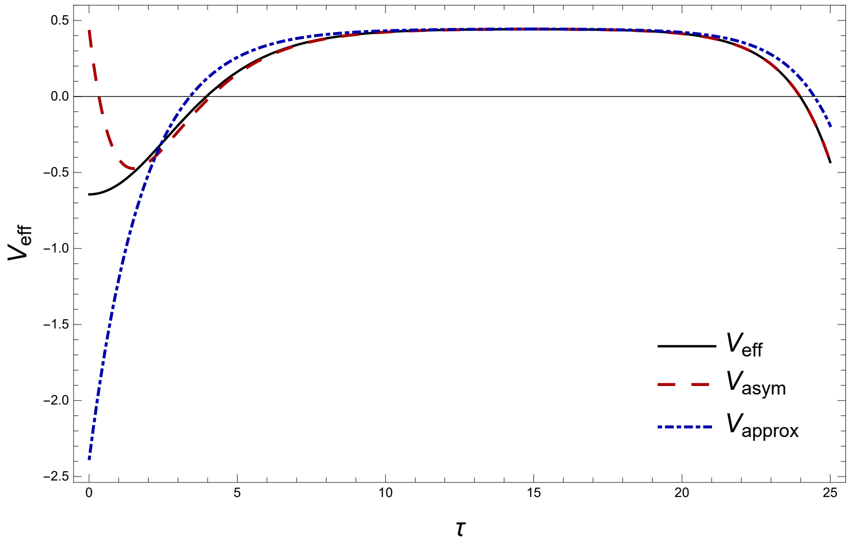


Figure 5.6: Effective potential V_{eff} (5.43) for the first excited mode, obtained numerically for $\tau_c = 15, T = 25$, together with V_{asym} (5.59) and V_{approx} (5.60) for the best fit values $(a, \nu) = (1.96, 2.45)$. We see that for small τ, V_{asym} this behaves rather differently from V_{eff}. Using V_{approx} (5.60) we can better approximate this regime. Notice that a better fit could be achieved by introducing further terms, but this would not allow for an analytical solution, which is the main goal of this approximation.

small τ. Using the notation in [280], we write the potential as

$$V_{\text{approx}} \equiv \begin{cases} (-\lambda_1^2 e^{-\frac{2}{\nu}\frac{2\tau}{3}} + 4)/9\,, & \tau < \tau_c \\ (-\lambda_2^2 e^{\frac{2\tau}{3}} + 4)/9\,, & \tau_c < \tau < T \end{cases} \tag{5.60}$$

where $\lambda_1 = a\hat{E}_k / I(\tau_c)^{1/2}$, with free parameters a and ν, and $\lambda_2 = \lambda_1 e^{-(1+\frac{2}{\nu})\frac{\tau_c}{3}}$ such that V_{approx} is continuous at τ_c (in the absence of the throat region, we would have $\lambda_2 = \frac{3^{1/2}}{2^{5/6}}\hat{E}_k$, which is what we use in the limit $\tau_c \to 0$). These free parameters should be thought of as a compensation for changing the asymptotic ($\tau \to \infty$) form of the potential. They should be chosen by comparing the analytical solution obtained with V_{approx} (5.60) with the numerical result obtained with V_{eff} (5.43).

The wavefunction will have a profile $\Phi_k^{(1)}(\tau)$ inside the throat ($\tau < \tau_c$), a profile $\Phi_k^{(2)}$ in the unwarped piece of the conifold ($\tau_c < \tau < T$), and will vanish $\Phi_k^{(3)}(\tau) = 0$ in the CY$_3$ ($\tau > T$). We therefore choose the following boundary conditions

1. The wavefunction is finite at $\tau = 0$, i.e.

$$\lim_{\tau\to 0} \Phi_k^{(1)}(\tau) < \infty \implies \lim_{\tau \to 0} B_k^{(1)}(\tau) = 0\,;$$

2. The wavefunctions match at $\tau = \tau_c$, i.e.

$$\Phi_k^{(1)}(\tau_c) = \Phi_k^{(2)}(\tau_c) \quad \text{and} \quad \partial_\tau \Phi_k^{(1)}(\tau_c) = \partial_\tau \Phi_k^{(2)}(\tau_c)\,;$$

3. The wavefunctions vanish as they approach the CY_3 region, i.e. $\Phi_k^{(2)}(T) = 0$.

The Schrödinger equation in the warped region is

$$(B_k^{(1)})'' + \left[\frac{\lambda_1^2}{9} e^{-4\tau/3\nu} - \frac{4}{9}\right] B_k^{(1)} = 0\,, \tag{5.61}$$

whose solution is a linear combination of Bessel functions

$$B_k^{(1)}(\tau) = C_1 J_\nu\left(\frac{\nu\lambda_1}{2} e^{-2\tau/3\nu}\right) + D_1 Y_\nu\left(\frac{\nu\lambda_1}{2} e^{-2\tau/3\nu}\right)\,, \quad 0 \le \tau \le \tau_c\,, \tag{5.62}$$

and the Schrödinger equation in the unwarped region is

$$(B_k^{(2)})'' + \left[\frac{\lambda_2^2}{9} e^{2\tau/3} - \frac{4}{9}\right] B_k^{(2)} = 0\,, \tag{5.63}$$

with solution

$$B_k^{(2)}(\tau) = C_2 J_2\left(\lambda_2 e^{\tau/3}\right) + D_2 Y_2\left(\lambda_2 e^{\tau/3}\right)\,, \quad \tau_c < \tau \le T\,, \tag{5.64}$$

where $C_{1,2}, D_{1,2}$ are integration constants that are fixed using the boundary conditions 1–3 as follows

1. The wavefunction is finite at $\tau = 0$ (hence we use $B^{(1)}(\tau)$),

$$C_1 J_\nu\left(\frac{\nu\lambda_1}{2}\right) + D_1 Y_\nu\left(\frac{\nu\lambda_1}{2}\right) = 0 \implies D_1 = -\frac{J_\nu(x_1)}{Y_\nu(x_1)} C_1\,, \tag{5.65}$$

   with $x_1 = \frac{\nu\lambda_1}{2}$.

2. The wavefunctions match at $\tau = \tau_c$. The first condition $\Phi_k^{(1)}(\tau_c) = \Phi_k^{(2)}(\tau_c) \implies B_k^{(1)}(\tau_c) = B_k^{(2)}(\tau_c)$ (cf. (5.37) with trivial angular dependence), which gives

$$C_1 J_\nu(x_2) + D_1 Y_\nu(x_2) = C_2 J_2(x_2) + D_2 Y_2(x_2)\,, \tag{5.66}$$

   with $x_2 = \frac{\nu\lambda_1}{2} e^{-2\tau_c/3\nu}$. The second condition $\partial_\tau \Phi_k^{(1)}(\tau_c) = \partial_\tau \Phi_k^{(2)}(\tau_c)$, after using

$$\partial_\tau \Phi^{(i)}(\tau) = -\frac{1}{2}\frac{G'(\tau)}{G(\tau)} \Phi^{(i)}(\tau) + G(\tau)^{-1/2} \partial_\tau B^{(i)}(\tau)\,, \quad i = 1, 2\,, \tag{5.67}$$

   and $\Phi_k^{(1)}(\tau_c) = \Phi_k^{(2)}(\tau_c)$, implies $\partial_\tau B^{(1)}(\tau_c) = \partial_\tau B^{(2)}(\tau_c)$. Moreover, we can use the relation

$$Z'_\alpha(x) = \frac{\alpha}{x} Z_\alpha(x) - Z_{\alpha+1}(x)\,, \tag{5.68}$$

   for a Bessel function $Z_\alpha(x)$, to rewrite $\partial_\tau B^{(1)}(\tau_c) = \partial_\tau B^{(2)}(\tau_c)$ as

$$\lambda_1\{C_1 J_{\nu+1}(x_2) + D_1 J_{\nu+1}(x_2)\} = -\lambda_2\{C_2 J_3(x_2) + D_2 Y_3(x_2)\}\,. \tag{5.69}$$

3. The wavefunctions vanish at $\tau = T$, i.e. $\Phi_k^{(2)}(T) = 0$, which implies

$$C_2 J_2(\lambda_2 e^{T/3}) + D_2 Y_2(\lambda_2 e^{T/3}) = 0 \implies D_2 = -\frac{J_2(x_3)}{Y_2(x_3)} C_2 \,, \tag{5.70}$$

with $x_3 = \lambda_2 e^{T/3}$.

Putting all these together, we find

$$D_1 = -\frac{J_\nu(x_1)}{Y_\nu(x_1)} C_1 \tag{5.71}$$

$$C_2 = \frac{Y_2(x_3)}{Y_\nu(x_1)} \frac{J_\nu(x_2)Y_\nu(x_1) - J_\nu(x_1)Y_\nu(x_2)}{J_2(x_2)Y_2(x_3) - J_2(x_3)Y_2(x_2)} C_1 \,, \tag{5.72}$$

$$D_2 = -\frac{J_2(x_3)}{Y_2(x_3)} \frac{Y_2(x_3)}{Y_\nu(x_1)} \frac{J_\nu(x_2)Y_\nu(x_1) - J_\nu(x_1)Y_\nu(x_2)}{J_2(x_2)Y_2(x_3) - J_2(x_3)Y_2(x_2)} C_1 \,, \tag{5.73}$$

as well as the quantisation condition (cf. [280])

$$e^{(1+\frac{2}{\nu})\frac{\tau_c}{3}} \frac{J_2(x_2)Y_2(x_3) - J_2(x_3)Y_2(x_2)}{J_\nu(x_2)Y_\nu(x_1) - J_\nu(x_1)Y_\nu(x_2)} = \frac{J_2(x_3)Y_3(x_2) - J_3(x_2)Y_2(x_3)}{J_{\nu+1}(x_2)Y_\nu(x_1) - J_\nu(x_1)Y_{\nu+1}(x_2)} \,. \tag{5.74}$$

Notice that this is an equation for $\lambda_1 = a\hat{E}_k$ depending on the choices of τ_c and T. Once more, this cannot be solved analytically. Instead, we can look at the different limits $\tau_c \to T$ (fully warped conifold) and $\tau_c \to 0$ (unwarped conifold) for which there are analytical solutions.

Fully warped conifold ($\tau_c \to T$)

Let us start by exploring the limit $\tau_c \to T$. This corresponds to a fully warped deformed conifold, so that the whole region where the wavefunctions are not forced to vanish is significantly warped. In practice, one should think of this as the limiting case of a warped throat that dominates over the bulk contribution (cf. Fig. 5.4).

In this limit $x_2 \to x_3$, so that $J_2(x_2)Y_2(x_3) - J_2(x_3)Y_2(x_2) \to 0$ which, together with the Bessel function property $J_2(x_3)Y_3(x_3) - J_3(x_3)Y_2(x_3) = -\frac{2}{\pi x_2} \neq 0$, implies

$$J_\nu(x_2)Y_\nu(x_1) - J_\nu(x_1)Y_\nu(x_2) \stackrel{!}{=} 0 \implies J_\nu\left(\frac{\nu\lambda_1}{2}\right) = 0 \,, \tag{5.75}$$

where we assume $x_2 \ll 1$. For large z, $J_\nu(z)$ can be approximated by

$$J_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \,, \tag{5.76}$$

whose roots are

$$z = k\pi + \left(\nu - \frac{1}{2}\right)\frac{\pi}{2}. \tag{5.77}$$

Hence, for $z = \frac{\nu\lambda_1}{2}$ and recalling the definitions of λ_1 and $\hat{E}_k$, we have

$$\hat{E}_k = I(\tau_c)^{1/2}\left\{\frac{2\pi}{a\nu}k + \left(\nu - \frac{1}{2}\right)\frac{\pi}{a\nu}\right\}, \tag{5.78}$$

or in terms of the masses m_k,

$$m_k = H_{\text{tip}}^{-1/4}\frac{M_s}{\sqrt{g_s M}} \cdot \left\{\frac{2\pi}{a\nu}k + \left(\nu - \frac{1}{2}\right)\frac{\pi}{a\nu}\right\}. \tag{5.79}$$

We can fit to a numerical solution with $\tau_c = T \gg 1$ to find the best values of a and ν for this analytical approximation,[13] which gives $\nu \sim 2.45$ and $a \sim 1.96$. Notice that the masses (5.79) are not only suppressed by the warping, as we would expect for a warped throat, but also depend on the characteristic length scale at the tip of the throat $R_{S^3} \approx \sqrt{\alpha' g_s M}$ (3.110), setting the KK scale associated with this tower (cf. (D.16)),

$$m_{\text{KK}} \equiv m_1 \approx H_{\text{tip}}^{-1/4}\frac{M_s}{\sqrt{g_s M}}. \tag{5.80}$$

The quantisation condition also implies $D_1 \approx 0$, so that in this limit the wavefunction takes the simpler form

$$\Phi_k(\tau) \approx \frac{N_{(k)}}{(\sinh(2\tau) - 2\tau)^{1/3}} J_\nu\left(\frac{\nu\lambda_1}{2}e^{-2\tau/3\nu}\right), \tag{5.81}$$

which peaks near the tip of the throat and quickly decays towards the bulk (Fig. 5.7), in good agreement with the corresponding regime in our numerical solutions (bottom plots in Fig. 5.5).

Notice that, while the wavefunction profiles constitute a local property of the solution, i.e. the way $h^k_{\mu\nu}$ is weighted by $\Phi_k(y)$ (5.31) depends on the position in the extra dimensions, the mode masses only depend on the strength of the warping and are not influenced by the internal space coordinates. Since the peak near the tip of the throat corresponds to a peak in the coupling strength of these modes, as we shall see in the next section, we say that the warped throat localises the fields at the tip and warps down their masses.

Unwarped conifold ($\tau_c \to 0$)

The opposite limit, $\tau_c \to 0$, corresponds to a fully unwarped conifold. Again, we should think of this as the limiting case of a warped throat which is dominated by a large bulk (cf. Fig. 5.4).

In this limit $x_2 \to x_1$, so that $J_\nu(x_2)Y_\nu(x_1) - J_\nu(x_1)Y_\nu(x_2) \to 0$. Together with the Bessel function property $J_{\nu+1}(x_1)Y_\nu(x_1) - J_\nu(x_1)Y_{\nu+1}(x_1) = \frac{2}{\pi x_1} \neq 0$, this implies

$$J_2(x_1)Y_2(x_3) - J_2(x_3)Y_2(x_1) \stackrel{!}{=} 0 \implies J_2(\lambda_2 e^{T/3}) = 0, \tag{5.82}$$

[13] The difference between our result and the one in [280] comes from an extra factor of $2^{1/3}$ in our $\hat{E}_k$.

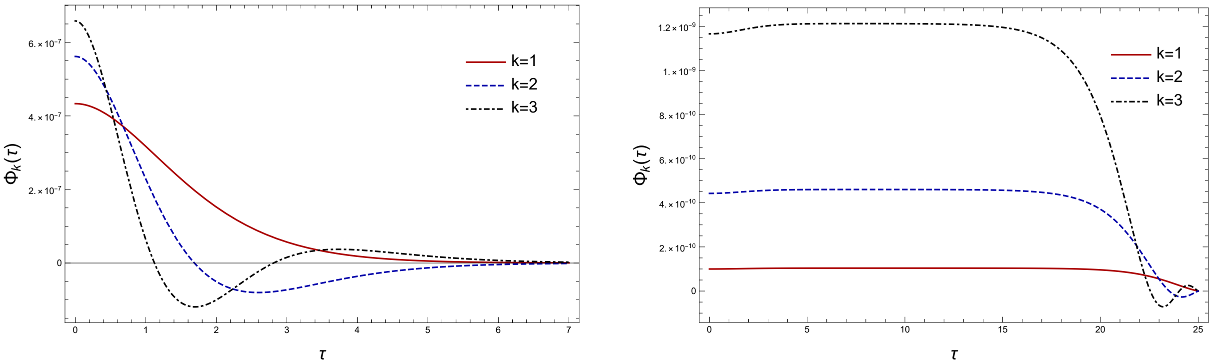


Figure 5.7: Wavefunctions obtained analytically in the limits $\tau_c \to T$ (5.81) and $\tau_c \to 0$, with $T = 25$. Notice the localisation near the tip in the limit $\tau_c \to T$ and the spreading of the wavefunctions in the limit $\tau_c \to 0$.

where we assume $x_1 \ll 1$. Using the same approximation for $J_\nu(z)$ (5.76), these roots are given approximately by

$$z = k\pi + \frac{3\pi}{4}\,. \tag{5.83}$$

Hence, for $z = \lambda_2 e^{T/3}$ and recalling that in the $\tau_c \to 0$ limit we have $\lambda_2 = \frac{3^{1/2}}{2^{5/6}}\hat{E}_k$, we find

$$\hat{E}_k = \frac{2^{5/6}}{3^{1/2}} e^{-T/3} \Big\{ \pi k + \frac{3\pi}{4} \Big\}\,, \tag{5.84}$$

or in terms of the masses m_k,

$$m_k = \frac{1}{R_{\text{con}}} \Big\{ \pi k + \frac{3\pi}{4} \Big\}\,, \tag{5.85}$$

where $R_{\text{con}} = c^{1/4} r_T$ is the physical size of the conifold, with r_T the value of the radial coordinate $r = \frac{3^{1/2}}{2^{5/6}} \epsilon^{2/3} e^{\tau/3}$ at which the conifold meets the CY_3, i.e. at $\tau = T$. Notice that the masses are now suppressed by the size of the conifold, R_{con}, which is the overall volume of the compact space felt by these modes (recall that their wavefunctions vanish in the CY_3 portion) — this corresponds to the usual definition of m_{KK} fixed by the characteristic scale of the compact space. This quantisation condition also implies $D_2 \approx 0$, so that in this limit the wavefunctions take the form[14]

$$\Phi_k(\tau) \approx \frac{N_{(k)}}{(\sinh(2\tau) - 2\tau)^{1/3}} J_2\left(\lambda_2 e^{\tau/3}\right)\,. \tag{5.86}$$

In Fig. 5.7 we plot the approximate wavefunctions in the two limits described above. We can clearly see the localisation near the tip in the limit $\tau_c \to T$ and the spreading of the wavefunctions in the limit $\tau_c \to 0$, as one would expect from the physical interpretation of the two regimes. These should be compared with the wavefunctions obtained numerically (Fig. 5.5).

Now that we have a full description of the tower of spin-2 KK modes, which are the 4d manifestation of the extra dimensions, we may study its effects on 4d gravitational interactions. As

[14] In this limit, it is actually important to keep the D_2 term in the wavefunction at small values of τ, otherwise it will not be regular at $\tau = 0$.

we mentioned at the beginning of the chapter, an interesting application is to the prediction of gravitational wave signals of these warped extra dimensions. We will have more to say about this in section 5.4. As a starting point, we will study the new interactions mediated by the massive modes in the KK tower, which contribute at low-energies as corrections to the Newtonian potential between two masses. This will allow us to introduce and explore several important features that will also be useful in the study of GWs, and compare quantities derived from this string compactification with current observational constrains on fifth forces.

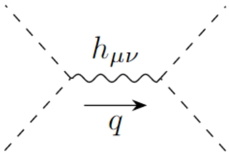
5.3 Fifth forces

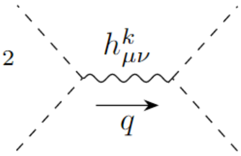
In Type IIB constructions, the Standard Model degrees of freedom typically arise from open string states which are confined on some intersection of D-branes. As a consequence, these states are not allowed to travel in all directions and directly probe the compact space, but will rather be localised in some specific region. Depending on its position, the coupling between Standard Model particles and the tower of KK states that we studied in the previous section may be either suppressed or enhanced. Let us see how this works in practice.

We will consider the corrections to the Newtonian gravitational potential between two point masses living on a (3+1)-dimensional brane sitting somewhere along the deformed conifold region of the compact space and compare with experimental and observational constraints. In Section 5.3.1 we derive the modifications to the Newtonian potential induced by the KK tower of spin-2 modes, which can be parameterised by a Yukawa-type interaction with a single-massive field and parameters (α, λ). In Section 5.3.2, we relate these phenomenological parameters to the parameters of the Klebanov-Strassler solution. By fixing the warped-down scale at the brane to be $\sim$ TeV scale, and bearing in mind the need for D3-tadpole cancellation (3.85), we explore the predictions for (α, λ) for a range of string parameters and compare with current constraints. Then, in Sections 5.3.3 and 5.3.4, we restrict to the fully warped and unwarped cases, respectively, where we can obtain clear theoretical bounds on the (α, λ) parameter space depending on where the brane lies along the conifold.

5.3.1 Modified Newtonian potential

In order to study the effects of the KK tower of spin-2 modes obtained in Section 5.2, we picture a braneworld scenario, with the Standard Model fields localised on a (3+1)-dimensional brane (stack/intersection) at y_{brane} in the compact space. We want to consider the corrections to the Newtonian potential between masses living on the brane due to the presence of the infinite tower of massive KK gravitons. The (Fourier transformed) potential is obtained by looking at a scattering diagram where two particles interact through the exchange of a virtual graviton, in the limit where the energy of the graviton goes to zero [288], For this we need both the graviton

$V(q) = \lim_{q^0 \to 0}$ 

$= \lim_{q^0 \to 0} \sum_k |\Phi_k(y)|^2$ 

propagator and the interactions with matter on the brane. Using the decomposition (5.31) and normalising the wavefunctions such that (5.45) is satisfied, each propagator is simply the 4d propagator for a spin-2 field. Assuming the spacetime is approximatly flat Minkowski, we have [289]

$$D^{(4,m_k)}_{\mu\nu\alpha\beta}(x,x') = \int \frac{d^4q}{(2\pi)^4} \frac{P^{(m_k)}_{\mu\nu\alpha\beta}(q)}{q^2 - m_k^2 + i\varepsilon} e^{-iq\cdot(x-x')} \,, \tag{5.87}$$

where $D^{(4,m_k)}_{\mu\nu\alpha\beta}(x,x')$ is the 4d propagator of the k^{th} mode (with mass m_k) and with the polarisation tensor

$$P^{(m=0)}_{\mu\nu\alpha\beta}(q) = \frac{1}{2}(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta}) \,, \tag{5.88}$$

$$\begin{aligned} P^{(m>0)}_{\mu\nu\alpha\beta}(q) = &\frac{1}{2}(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta}) - \frac{1}{2m^2}(\eta_{\mu\alpha}q_\nu q_\beta + \eta_{\mu\beta}q_\nu q_\alpha + \eta_{\mu\beta}q_\mu q_\alpha) \\ &+ \frac{1}{6}\left(\eta_{\mu\nu} + \frac{2}{m^2}q_\mu q_\nu\right)\left(\eta_{\alpha\beta} + \frac{2}{m^2}q_\alpha q_\beta\right) . \end{aligned} \tag{5.89}$$

We now consider the brane action, which contains the fields living in the world-volume of the brane at y_b and their interactions with the graviton,

$$S_{\text{brane}} = \int d^4x \sqrt{-\mathfrak{g}}\ \mathcal{L}_M \,, \tag{5.90}$$

where $\mathfrak{g}_{\mu\nu}$ is the pullback of the metric G_{MN} onto the (3+1)-dimensional brane (cf. (2.92))

$$\mathfrak{g}_{\mu\nu} = G_{MN} \frac{\partial X^M}{\partial x^\mu} \frac{\partial X^N}{\partial x^\nu} \,, \tag{5.91}$$

for 10d coordinates X^M and 4d coordinates on the brane $\xi^\mu = x^\mu$, which we align with the 4d coordinates x^μ of (3.127). Recalling that the brane positions can fluctuate, we have $\partial_\mu X^M = \delta^M_\mu + (2\pi\alpha')\partial_\mu\phi^M$, with brane fluctuations $\phi^M(x^\mu)$ (2.93), which gives for the pulled-back metric

$$\mathfrak{g}_{\mu\nu} = G_{\mu\nu} + 2(2\pi\alpha')G_{M(\mu}\partial_{\nu)}\phi^M + (2\pi\alpha')^2 G_{MN}\partial_\mu\phi^M\partial_\nu\phi^N \,. \tag{5.92}$$

We may therefore expand $\sqrt{-\mathfrak{g}}$ in the action[15] and express it in terms of G_{MN},

$$S_{\text{brane}} = \int d^4x \sqrt{-\det G_{\mu\nu}}\ \mathcal{L}_M \left(1 + (2\pi\alpha')G_M{}^\mu\partial_\mu\phi^M + \frac{1}{2}(2\pi\alpha')^2 G_{MN}\partial_\mu\phi^M\partial^\mu\phi^N\right) . \tag{5.93}$$

We will neglect the fluctuations of the brane and study the interactions between graviton KK

[15] We use the known result $\det(1+\epsilon A) = 1 + \epsilon \text{Tr}(A) + \mathcal{O}(()\epsilon^2)$, when $\epsilon \ll 1$.

modes and matter fields only, so that the action is simply

$$S_{\text{brane}} = \int d^4x \sqrt{-\det G_{\mu\nu}}\; \mathcal{L}_M \,. \tag{5.94}$$

When we perturb the bulk metric $G_{MN} \to \bar{G}_{MN} + \delta G_{MN}$, the action becomes

$$S_{\text{brane}} = \bar{S}_{\text{brane}} - \frac{1}{2}\int d^4x \sqrt{-\det \bar{G}_{\mu\nu}}\; \tilde{T}^{\mu\nu}\delta G_{\mu\nu} \,, \tag{5.95}$$

where $\tilde{T}_{\mu\nu}$ is the energy-momentum tensor with respect to $G_{\mu\nu}$. Using the background metric (3.127) and the fluctuations in terms of (canonically normalised) perturbations of $g_{\mu\nu}$, with $\delta G_{\mu\nu} = \kappa\; H^{-1/2}\; h_{\mu\nu}$,

$$S_{\text{brane}} = \bar{S}_{\text{brane}} - \frac{1}{2}\int d^4x \sqrt{-\det g_{\mu\nu}}\; \sum_k \left(\kappa\; \Phi_k(y_b)\right) T^{\mu\nu} h^k_{\mu\nu} \,, \tag{5.96}$$

where $T_{\mu\nu}$ is now defined with respect to $g_{\mu\nu}$,

$$\tilde{T}_{\mu\nu} = -\frac{2}{\sqrt{-\det g_{\mu\nu}}} \frac{\delta(\sqrt{-\det G_{\mu\nu}}\mathcal{L}_M)}{\delta g^{\mu\nu}} H(y_b)^{3/2} = T_{\mu\nu} H(y_b)^{3/2} \,. \tag{5.97}$$

From this action we conclude that the k^{th} mode couples to matter with a coupling $(\kappa\; \Phi_k(y_b))$ that depends on the 10d gravitational coupling $\kappa = \sqrt{V_{\text{w}}}/M_{\text{Pl}}$ and its wavefunction evaluated at the position of the brane y_b. For the zero mode, with constant wavefunction (5.51), the coupling is therefore the usual $1/M_{\text{Pl}}$ no matter where in the compact space the brane is located.

Using $\bar{S}_{\text{brane}}$, we can also see how the warping affects the energy scales on the braneworld theory. Taking $\mathcal{L}_M$ to include a single scalar field φ with a Higgs-like potential [57],

$$\begin{aligned} \bar{S}_{\text{brane}} &= \int d^4x \sqrt{-\det G_{\mu\nu}}\; \{G^{\mu\nu}(D_\mu\varphi)^\dagger(D_\nu\varphi) - \lambda(|\varphi|^2 - v_0^2)^2\} \\ &= \int d^4x \sqrt{-\det g_{\mu\nu}}\; \{H(y_b)^{-1/2} g^{\mu\nu}(D_\mu\varphi)^\dagger(D_\nu\varphi) - \lambda(H(y_b)^{-1/2}|\varphi|^2 - H(y_b)^{-1/2}v_0^2)^2\} \\ &\to \int d^4x \sqrt{-\det g_{\mu\nu}}\; \{g^{\mu\nu}(D_\mu\varphi)^\dagger(D_\nu\varphi) - \lambda(|\varphi|^2 - v^2)^2\} \,, \end{aligned} \tag{5.98}$$

with the field redefinition $\varphi \to H(y_b)^{1/4}\varphi$, so that the field is canonically normalised. The mass scales are then warped down as $v = H(y_b)^{-1/4} v_0$, which again depends on the position of the brane in the compact space — the biggest hierarchy is achieved by placing the brane at the tip of the throat ($y_b = 0$) where the warping is maximised.

Putting everything together, the gravitational potential in momentum space is given by

$$V(q) = \frac{V_{\text{w}}}{M_{\text{Pl}}^2} \sum_k |\Phi_k(y_b)|^2 \left.\frac{T_1^{\mu\nu} P^{(m_k)}_{\mu\nu\alpha\beta} T_2^{\alpha\beta}}{|q^2 - m^2|}\right|_{q^0\to 0} \,, \tag{5.99}$$

where $T_1^{\mu\nu} = m_1\delta_0^\mu\delta_0^\nu$ and $T_2^{\alpha\beta} = m_2\delta_0^\alpha\delta_0^\beta$ are the energy-momentum tensors of two point particles of masses m_1 and m_2 at rest, so that only $P_{0000}^{(m)}$ is relevant, which in the $q^0 \to 0$ limit simply gives

$$P_{0000}^{(m)}(q) \begin{cases} \frac{1}{2} & m = 0\,, \\ \frac{2}{3} & m > 0\,. \end{cases} \tag{5.100}$$

Inserting this in the potential, we obtain

$$V(q) = \frac{m_1 m_2 V_{\rm w}}{M_P^2}\left\{\frac{1}{2}\frac{|\Phi_0(y_b)|^2}{\mathbf{q}^2} + \frac{2}{3}\sum_{k>0}\frac{|\Phi_k(y_b)|^2}{\mathbf{q}^2 + m_k^2}\right\}\,, \tag{5.101}$$

or in position space

$$V(r) = G_N\frac{m_1 m_2}{r}V_{\rm w}\left\{|\Phi_0(y_b)|^2 + \frac{4}{3}\sum_{k>0}|\Phi_k(y_b)|^2 e^{-m_k r}\right\}\,, \tag{5.102}$$

where we used $M_{\rm Pl}^{-2} = 8\pi G_N$, together with the factor of 4π coming from the Fourier transform, to write the potential in terms of Newton's constant G_N. The first term gives the contribution of the massless graviton, which is independent of the position of the brane due to (5.51) and reproduces the Newtonian gravitational potential. The second term contains the contribution from the tower of massive KK-modes, weighed by their respective wavefunctions and suppressed by the exponential $e^{-m_k r}$, which corrects the Newtonian potential and becomes negligible for large distances and larger KK masses. The range of these new interactions is therefore determined by the masses of the KK modes.

Using the zero mode wavefunction (5.51) we have

$$V(r) = G_N\frac{m_1 m_2}{r}\left\{1 + \frac{4}{3}V_{\rm w}\sum_{k>0}|\Phi_k(y_b)|^2 e^{-m_k r}\right\}\,. \tag{5.103}$$

Figure 5.8 shows the experimental bounds on deviations from the Newtonian gravitational potential — from laboratory, geophysical, astrophysical and collider constraints[16] [263, 292] — parametrised as $V(r) = G_N\frac{m_1 m_2}{r}(1 + \delta V)$ with

$$\delta V = \alpha\, e^{-r/\lambda}\,, \tag{5.104}$$

where α is a dimensionless parameter describing the strength of the interaction and λ has dimensions of length and is given in meters (m). This parametrisation arises from considering the correction coming from a Yukawa-type interaction involving a single massive field, which takes the exponential form above. Since we have an infinite tower of massive scalars, we will have an infinite sum of such Yukawa terms. We must, however, either keep only the first mode, which is the dominant contribution due to the exponential suppression for larger masses, or

[16]See [290, 291] for cosmological constraints from overclosure and the diffuse cosmic gamma ray background, which however assume Planckian couplings with the KK tower.

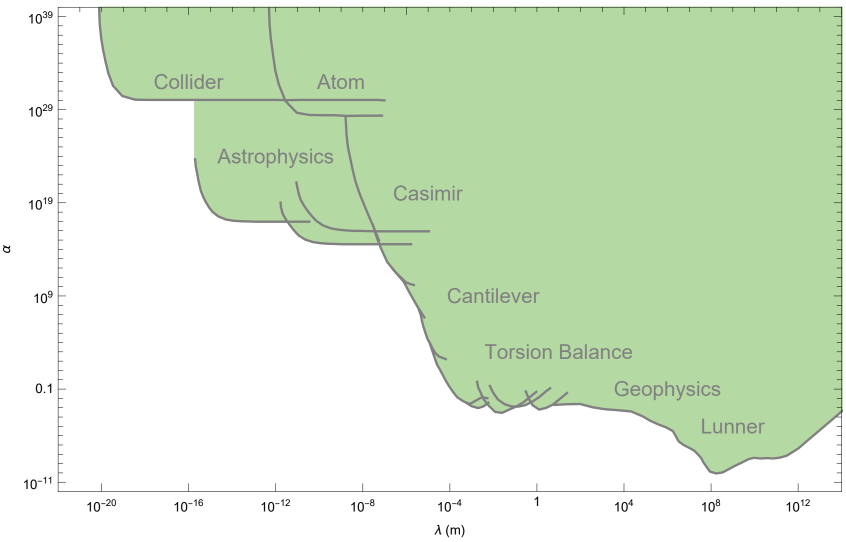


Figure 5.8: Experimental constraints on the parameters α (coupling strength) and λ (range) of a Yukawa-type interaction, with the shaded area indicating the excluded region of parameter space at 95% confidence level. Figure adapted from [263, 292]. See [263, 292] for details and further references.

rewrite the sum of Yukawa terms in the form of δV given above if we want to compare our predictions with the experimental constraints.

After exploring the general case, we will consider in detail the fully warped ($\tau_c \to T$) and unwarped ($\tau_c \to 0$) limits, with the brane located at different points in the compact space — either at the tip ($\tau = 0$) or away from the tip ($\tau \gg 1$). In these limits we are able to exclude large portions of parameter space by taking into account the usual control requirements in string compactifications: $g_s < 1$, $g_s M > 1$, and well-motivated upper bounds on the flux number M coming from the flux contribution to the D3-tadpole.

5.3.2 General case

As discussed in Section 5.2.4, solving numerically the Schrödinger equation (5.38) with the potential[17] (5.43) for a given choice of (τ_c, T) we find a set $(\hat{E}_k, B_k(\tau))$ of eigenvalues and eigenfunctions. We then substitute

$$\Phi_k(\tau) = N_{(k)} \tilde{\Phi}_k(\tau)\,, \tag{5.105}$$

$$N_{(k)}^{-2} = \frac{2\pi^3}{3} c^{3/2} \epsilon^4 \mathcal{N}_{(k)}^{-2}(\tau_c, T)\,, \tag{5.106}$$

$$G(\tau) = (\sinh(2\tau) - 2\tau)^{2/3}\,, \tag{5.107}$$

$$m_k = \hat{E}_k / (\epsilon^{2/3} c^{1/4})\,, \tag{5.108}$$

[17] With trivial angular dependence, i.e. s-orbital.

with $\mathcal{N}^{-2}_{(k)}(\tau_c, T)$ as defined in (5.55) and $\tilde{\Phi}_k(\tau) = G(\tau)^{-1/2} B_k(\tau)$, into (5.103) to obtain

$$\delta V = \frac{2}{\pi^3} \frac{V_{\rm w}}{c^{3/2}\epsilon^4} \sum_{k>0} \mathcal{N}^2_{(k)}(\tau_c, T) |\tilde{\Phi}_k(\tau_b)|^2 e^{-\hat{E}_k \frac{r}{\epsilon^{2/3} c^{1/4}}} . \tag{5.109}$$

Defining $A(\tau_b, \tau_c, T)$ and $\mu(\tau_c, T)$ such that

$$A(\tau_b, \tau_c, T) e^{-\mu(\tau_c, T) \frac{r}{\epsilon^{2/3} c^{1/4}}} \equiv \sum_{k>0}^{N} \mathcal{N}^2_{(k)}(\tau_c, T) |\tilde{\Phi}_k(\tau_b)|^2 e^{-\hat{E}_k \frac{r}{\epsilon^{2/3} c^{1/4}}} , \tag{5.110}$$

we can write the parameters (α, λ) in (5.104) as

$$\alpha = \frac{2}{\pi^3} \frac{V_{\rm w}}{c^{3/2}\epsilon^4} A(\tau_b, \tau_c, T) \qquad\qquad \lambda^{-1} = \frac{\mu(\tau_c, T)}{\epsilon^{2/3} c^{1/4}} . \tag{5.111}$$

It will be useful to replace $V_{\rm w}$ with the hierarchy between the brane and the bulk, which has a direct physical interpretation. From (5.98) we know that the hierarchy between the fundamental scale and the world-volume theory on the brane is given by the warp factor (5.36) evaluated at the position of the brane,

$$H_{\tau_c}(\tau_b) = 1 + \frac{I(\tau_b)}{I(\tau_c)} . \tag{5.112}$$

For string theory, the natural scale in the UV is the string scale m_s. If M_b is the scale on the brane, using (3.129), we can write it in terms of the known Planck scale $M_{\rm Pl}$,

$$H_{\tau_c}(\tau_b)^{-1/4} = \frac{v}{v_0} = \frac{M_b}{m_s} = \frac{M_b}{M_{\rm Pl}} \frac{\sqrt{4\pi \mathcal{V}_{\rm w}}}{g_s} , \tag{5.113}$$

where $M_{\rm Pl} = 2.14 \times 10^{18}$ GeV. This implies that the hierarchy between the known scales M_b and $M_{\rm Pl}$,

$$\mathcal{H} \equiv \frac{M_b}{M_{\rm Pl}} = H_{\tau_c}(\tau_b)^{-1/4} \frac{g_s}{\sqrt{4\pi \mathcal{V}_{\rm w}}} , \tag{5.114}$$

depends on the volume and string coupling, as well as the warp factor $H_{\tau_c}(\tau_b)$. If we choose $M_b = 1$ TeV, trying to solve the hierarchy problem, the hierarchy takes the value $\mathcal{H} \sim 10^{-15}$. Although we will use this as a concrete example in what follows, one could consider more general cases where either the hierarchy problem is not explained by the warping or the UV scale is not necessarily the string scale m_s. Our analysis can be easily adapted to these cases.

Using this in (5.111), we find

$$\alpha = \frac{(2\pi)^2}{(g_s M)^3} \frac{2A(\tau_b, \tau_c, T)}{I(\tau_c)^{3/2}} H_{\tau_c}(\tau_b)^{-1/2} \frac{g_s^2}{\mathcal{H}^2} , \tag{5.115a}$$

$$\lambda^{-1} = \frac{\mathcal{H}}{2^{1/6}} \frac{2\pi}{\sqrt{g_s M}} \frac{H_{\tau_c}(\tau_b)^{1/4}}{I(\tau_c)^{1/4}} \frac{\mu(\tau_c, T)}{l_p} . \tag{5.115b}$$

The free parameters in (5.115) are six in total $(\tau_c, T, g_s, M, \mathcal{H}, \tau_b)$. We should remember that

	M	τ_b	$\mathcal{V}_{\mathrm{w}}$	$m_{\mathrm{KK}}^{\mathrm{bulk}}$	$H(\tau_b)^{-1/4} m_{\mathrm{KK}}^{\mathrm{bulk}}$	m_{KK}	MK
A_1	20	0	9.6×10^{16}	600	3×10^{-3}	14	755
A_2	75					7.3	10382
A_3	20	15	3.0×10^{20}	3	8×10^{-4}	0.253	806

Table 5.1: All points A_i have $\tau_c = T = 40$ (corresponding to the fully warped conifold) and fixed $\mathcal{H} = 10^{-15}$, $g_s = 0.2$. Masses are given in TeV and volumes are in string units. See main text for discussion.

$\mathcal{H}$ is keeping the dependence on $\mathcal{V}_{\mathrm{w}}$, which is fully determined by the choice of $(\tau_c, \tau_b, g_s, \mathcal{H})$ through (5.114) — one should check that a choice of these parameters is consistent with the supergravity requirement $\mathcal{V}_{\mathrm{w}} \gg 1$. Note also that τ_c is determined by the deformation modulus $|z| = \epsilon^2/l_s^3$, which means we can think of a choice of τ_c as representing a choice of $|z|$ (which in turn depends on the flux parameter K via $z \sim e^{-\frac{2\pi K}{g_s M}}$ [94, 142, 148, 166, 168, 172]). We will fix the hierarchy $\mathcal{H} = 10^{-15}$ as explained above and give two examples for the position of the brane τ_b, which leaves four free parameters. In principle, the position of the brane should also be determined dynamically, since it becomes a modulus that experiences a potential due to several different ingredients [293–295]. In this work we assume that the position can be fixed to a certain value due to the balance between these ingredients, without addressing the issue explicitly.

Notice that generically

$$\mathcal{V}_{\mathrm{w}} \approx \mathcal{V}_{\mathrm{bulk}} + \mathcal{V}_{\mathrm{conifold}} \,, \tag{5.116}$$

so that the value we end up fixing for $\mathcal{V}_{\mathrm{w}}$ includes both contributions. On the other hand, we are choosing T independently of $\mathcal{V}_{\mathrm{w}}$, which means that in general $\mathcal{V}_{\mathrm{w}} \geq \mathcal{V}_{\mathrm{conifold}}$. Due to our boundary conditions, the scale m_{KK} associated with our graviton KK tower will not feel the $\mathcal{V}_{\mathrm{bulk}}$ contribution and thus it will be different from the scale $m_{\mathrm{KK}}^{\mathrm{bulk}}$ one would find from the size of the whole compact space (D.15). Moreover, the characteristic size setting this scale will also depend on the warping regime, as we explicitly saw in the previous section. Therefore, in the examples below we show both scales for comparison, although only m_{KK} corresponds to a physical scale in our (albeit restricted) case. We also show the value of a warped down $m_{\mathrm{KK}}^{\mathrm{bulk}}$ (when warping is significant) which can be compared with the actual value m_{KK}.

In Fig. 5.9 we show a sample of predictions (λ_i, α_i) for different choices of the parameters, divided in three main groups: the fully warped limit, with $\tau_c = T$; the unwarped conifold limit, with $\tau_c = 0$; and a mid-regime with $\tau_c = T/2$ — in this regime we see the competition between the throat trying to localise the modes and the bulk trying to spread them evenly throughout the compact space (see discussion on boundary conditions in Section 5.2). The string coupling is fixed to $g_s = 0.2$ in all parameter sets. Tables 5.1–5.3 summarize the parameter choices and the relevant quantities for each set of examples.

The first thing to note is that none of these examples lies within the excluded region of parameter space, both due to the small couplings and small length scales — this means that none of

	M	τ_b	$\mathcal{V}_{\rm w}$	$m_{\rm KK}^{\rm bulk}$	$m_{\rm KK}$	$R_{\rm con}$	MK
B_1	20	0	2.3×10^{27}	7×10^{-5}	0.275	7.0×10^{3}	556
B_2	75				0.142	1.4×10^{4}	7585
B_3	20	15	3.2×10^{27}	6×10^{-5}	0.232	7.0×10^{3}	558

Table 5.2: All points B_i have $\tau_c = 0, T = 30$ (corresponding to the unwarped conifold) and fixed $\mathcal{H} = 10^{-15}$, $g_s = 0.2$. Masses are given in TeV and volumes are in string units. See main text for discussion.

	M	τ_b	$\mathcal{V}_{\rm w}$	$m_{\rm KK}^{\rm bulk}$	$H(\tau_b)^{-1/4}m_{\rm KK}^{\rm bulk}$	$m_{\rm KK}$	$R_{\rm con}$	MK
C_1	20	0	1.0×10^{24}	1×10^{-2}	2×10^{-4}	4.9	125	546
C_2	75					2.5	243	7442
C_3	20	15	2.3×10^{24}	7×10^{-5}	6×10^{-5}	0.104	125	595

Table 5.3: All points C_i have $\tau_c = 15, T = 30$ (corresponding a partially warped conifold) and fixed $\mathcal{H} = 10^{-15}$, $g_s = 0.2$. See main text for discussion.

them can be excluded from this large set of gravitational experiments and observations. The most likely case to be probed in the near future is the fully warped limit, which is not far from the collider experiments — these need to go slightly up in energy, but mostly be able to probe smaller couplings, which requires an increase in the statistics (i.e. higher luminosity).[18] By contrast, the case that seems harder to probe is the unwarped conifold (representing an unwarped compactification),[19] which must have all its scale suppression coming from a large compactification volume, $\mathcal{V}$.

We can see that in all cases, increasing the flux number M slightly lowers both the masses and the couplings of the graviton KK modes. It also leads to a larger tadpole charge MK, which is determined through the choice of $(\tau_c, g_s, M) \to c|z|^{4/3} \to MK$ from (5.35) and $z \sim e^{-\frac{2\pi K}{g_s M}}$, with $\mathcal{V}_{\rm w}$ being determined through (5.114) by further fixing $(\mathcal{H}, \tau_b)$. In fact, we can put this together to find

$$MK \sim -\frac{3(g_s M^2)}{2\pi} \log\left\{\frac{2^{1/3}}{(2\pi)^{5/6}}\frac{\sqrt{g_s M}}{g_s^{1/3}} I(\tau_c)^{1/6} I(\tau_b)^{1/12}\mathcal{H}^{1/3}\right\}. \tag{5.117}$$

From our previous discussion on the tadpole cancellation condition and how it lead to the Tapole Conjecture [101, 102, 105, 106], we know that large values of MK may be difficult to accomodate in these string flux compactifications. If we restrict to O3-planes, all examples in the literature have a number of O3-planes less than or equal to 64, which gives a tadpole charge contribution[20] $|Q_{\rm O3}| \leq 32$. With no other ingredients, tadpole cancellation would require $MK \leq 32$. Note however that, even staying close to the boundary of control (e.g. $g_s = 0.5, M = 4 \implies g_s M = 2$)

[18] Of course, the fact that we are close to the collider energies is not fully surprising, since the hierarchy $\mathcal{H}$ plays a crucial role in determining the masses and roughly matches the current reach of colliders (1 TeV). What is much less obvious is that the couplings are enhanced by the warping in such a way that brings these predictions closer to the parameter space reached by the colliders.

[19] This can be compared with the ADD models [296, 297], with $n = 6$ large extra dimensions. Note that we cannot model $n = 1, 2$ ADD models with the conifold.

[20] There is an extra factor of 2 arising from the $\mathbb{Z}_2$ action of the orientifold when the fluxes are those of the un-orientifolded space [168].

and with very small warping $\tau_c = 1$, (5.117) gives $MK \sim 47 > 32$. Relaxing the hierarchy such that it connects the string scale with e.g. 10^3 TeV rather than 1 TeV still gives $MK \sim 38 > 32$.

If one doubles $g_s M$ for better control of the supergravity approximation (e.g. $g_s = 0.25, M = 16 \implies g_s M = 4$), (5.117) gives $MK \sim 360$. For the choice of parameters in A_1 we find $MK \sim 755$ as reported in Table 5.1 (in choosing the parameters for the sample points A_i we also require $m_{\text{KK}} < m_{\text{KK}}^{\text{bulk}}$ and $r_{\text{UV}} > l_s$, which is why we actually use different values of g_s and M giving the same $g_s M = 4$). Just asking for the hierarchy to come (at least partially) from the warping requires a large tadpole contribution MK, requiring us to take the Tadpole Conjecture into account when judging the vality of our setup.

We also see that moving the brane away from the tip of the throat has a big effect on the gravitational corrections. For A_3 the effect is more intuitive, being due to the localisation of the modes, since the couplings depend explicitly on the wavefunction profile (Fig. 5.5). Graviton KK modes will therefore have much weaker couplings to modes living away from the tip of the throat. However, (5.115a) tells us that the coupling strength also depends on the warp factor, with weaker warping giving stronger couplings, which explains why the coupling actually *increases* for C_3, rather than decrease — in this mid-regime the wavefunctions reach a constant plateau rather than quickly decaying towards zero (see Fig. 5.5), so that the biggest effect will come from the decrease in the warp factor and the coupling increases.

Although the A_i have strong warping, one still requires a large volume $\mathcal{V}_{\text{w}}$ to meet the fixed hierarchy, even if significantly smaller than in the unwarped examples B_i. Having lower values of $\mathcal{V}_{\text{w}}$ would require stronger warping and therefore larger tadpole contributions $MK > \mathcal{O}(10^3)$. Note also that in C_i a big part of the hierarchy is coming from the volume rather than the warp factor, which is why the volume is much larger than in A_i (thereby allowing for smaller tadpoles MK) and why it does not change a lot from C_1 to C_3 when moving the brane away from the tip.

In all cases, the scale m_{KK} associated with the graviton tower is around the TeV scale, which is reflecting our choice of hierarchy. One might expect that this alone would be enough to make these modes detectable since we are able to access these energies at colliders like the LHC. However, as is well-known, the energy scale (masses of the modes) alone is not enough to determine whether new effects are detectable. The way these extra modes couple to Standard Model particles (which in our setup are confined on the brane) is extremely important — the KK gravitons might be extremely light and yet couple so weakly to Standard Model particles that they are still undetectable with current experiments and observations. This makes the details of the compactification crucial when studying these effects, not only because they determine the masses of the modes, but also because they will affect the profile of the wavefunctions over the extra dimensions and consequently the couplings to other states.

When looking for examples to show in Fig. 5.9 we could not find any points within the excluded

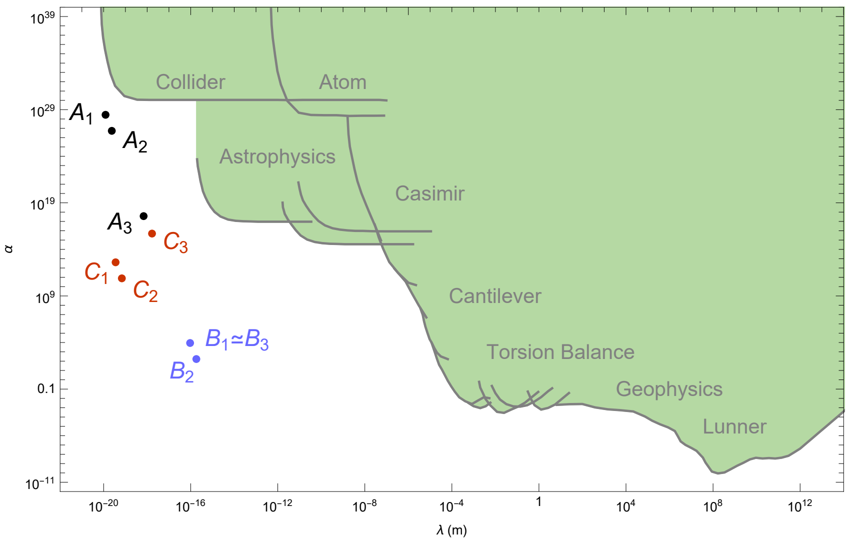


Figure 5.9: A few examples of the corrections δV (5.104) to the Newtonian potential for different choices of the free parameters in (5.115). The hierarchy is fixed to $\mathcal{H} = 10^{-15}$ and $g_s = 0.2$ in all parameter sets. All points A_i have $\tau_c = T = 40$, corresponding to the fully warped conifold limit. All points B_i have $\tau_c = 0$ and $T = 30$, corresponding to the unwarped conifold limit. All points C_i have $\tau_c = T/2 = 15$, corresponding to a mid-regime with a warped throat and a piece of the bulk described by an unwarped conifold. We also choose $M = 20$ and 75 for $i = 1, 3$ and $i = 2$, respectively, and $\tau_b = 0$ and 15 for $i = 1, 2$ and $i = 3$, respectively. We notice in particular that all predictions are outside the excluded region, with the couplings being too small to be probed at colliders and the length scales too small to be probed by large scale experiments. See main text for detailed discussion. Figure adapted from [263, 292], with the shaded area indicating the excluded region of parameter space at 95% confidence level.

region, despite apparently having a lot of freedom.[21] In the next two sections, we will focus on the fully warped and unwarped limits, through which we can gain further insight into the allowed region of parameter space.

5.3.3 Fully warped deformed conifold

The fully warped deformed conifold corresponds to the limit $\tau_c \to T$. In this limit, the solution pair $(\hat{E}_k, \Phi_k)$ only depends on τ_c, so that $A(\tau_b, \tau_c, T) = A(\tau_b, \tau_c)$ and $\mu(\tau_c, T) = \mu(\tau_c)$. In particular, we know from the analytical approximations (confirmed using the numerical solutions) that in this limit

$$\hat{E}_k \approx I(\tau_c)^{1/2} e_k \,, \qquad e_k = \frac{2\pi}{a\nu} k + \left(\nu - \frac{1}{2}\right) \frac{\pi}{a\nu} \,, \tag{5.118}$$

with $\nu \sim 2.45, a \sim 1.96$. Notice that we must always have $\tau_b < \tau_c$ since our boundary condition $\Phi_k(T = \tau_c) = 0$ would imply vanishing contributions to a brane at $\tau_b \geq \tau_c$, and hence

$$H_{\tau_c}(\tau_b) \approx \frac{I(\tau_b)}{I(\tau_c)} \,. \tag{5.119}$$

[21]The conifold background we are considering does not allow for an anisotropic compactification, which would be required to realise an unwarped ADD model [296, 297] with less than 6 large extra dimensions.

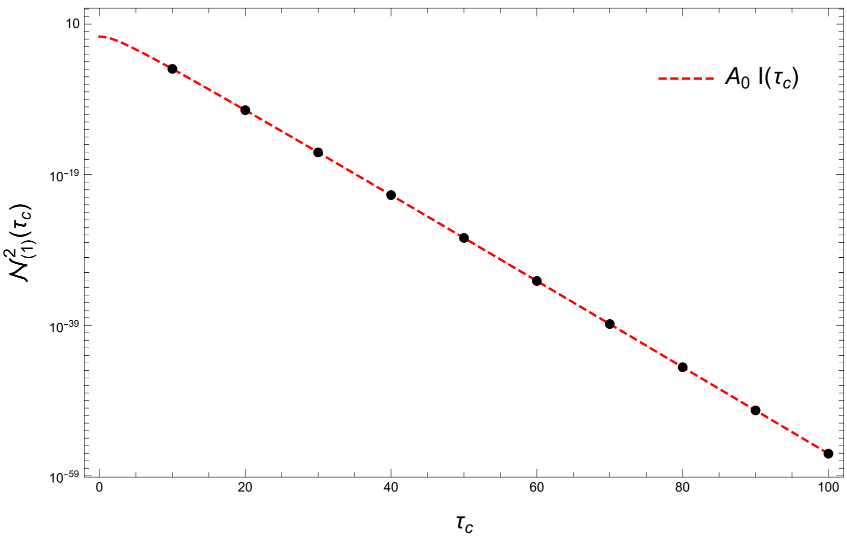


Figure 5.10: Plot of $\mathcal{N}^2_{(1)}(\tau_c)$ in the fully warped limit $\tau_c \to T$. We see that the normalisations fall with $A_0 I(\tau_c)$, with $A_0 \approx 0.3$.

Assuming that only the first massive mode has a relevant contribution,[22] we find $A(\tau_b, \tau_c) \approx \mathcal{N}^2_{(1)}(\tau_c)|\tilde{\Phi}_1(\tau_b)|^2$ and $\mu(\tau_c) \approx I(\tau_c)^{1/2} e_1$, where $\mathcal{N}^2_{(1)}(\tau_c) \approx A_0 I(\tau_c)$ and $A_0 \approx 0.3$ (see Fig. 5.10) — physically this implies smaller masses for modes living in longer warped throats (i.e. with stronger warping); as for the couplings, they will depend strongly on τ_b (cf. left panel of Fig. 5.7). Notice that taking the leading contribution to be the first KK mode rather than the massive string states is well-justified as, in this strongly warped limit, we have (5.80)

$$m_{\text{KK}} \approx \frac{M_s^{\text{w}}}{\sqrt{g_s M}}\,, \tag{5.120}$$

that is, the first KK mode is suppressed with respect to the warped string scale by the factor $\sqrt{g_s M}$. Since the supergravity approximation requires $g_s M \gg 1$ and masses appear exponentially in the corrections δV (5.109), effects from the warped down tower of string states should be suppressed with respect to leading effects from the first KK mode on which we focus here.

The parameters (α, λ) become

$$\alpha \approx \frac{(2\pi)^2}{(g_s M)^3}\frac{2|\tilde{\Phi}_1(\tau_b)|^2}{I(\tau_b)^{1/2}}\frac{g_s^2}{\mathcal{H}^2}\,, \tag{5.121a}$$

$$\lambda^{-1} \approx \frac{\mathcal{H}}{2^{1/6}}\frac{2\pi}{\sqrt{g_s M}} I(\tau_b)^{1/4}\frac{e_1}{l_p}\,. \tag{5.121b}$$

For fixed $(\tau_b, \mathcal{H})$, the bounds $g_s < 1$ and $g_s M > 1$, required for control of the string loop

[22] Although higher modes have larger couplings at the tip, as emphasised in [276], the exponential suppression from higher masses will dominate, so that the net result is an exponential suppression of the contributions from higher modes to the gravitational potential compared to the first mode in the tower.

expansion and supergravity approximation, respectively, translate into direct bounds on (α, λ),

$$\alpha < \frac{(2\pi)^2}{\mathcal{H}^2} \frac{4|\tilde{\Phi}_1(\tau_b)|^2}{I(\tau_b)^{1/2}}, \tag{5.122a}$$

$$\lambda^{-1} < \frac{\mathcal{H}}{2^{1/6}} (2\pi) I(\tau_b)^{1/4} \frac{e_1}{l_p}. \tag{5.122b}$$

However, there is a stronger bound on α given by the combination

$$\alpha\lambda^6 = \frac{1}{(2\pi)^4} \frac{4|\tilde{\Phi}_1(\tau_b)|^2}{I(\tau_b)^2} \frac{g_s^2}{\mathcal{H}^8} \frac{l_p^6}{e_1^6}, \tag{5.123}$$

which is bounded from above and is a diagonal line with fixed negative slope in Fig. 5.11.

Finally, one can find a lower bound on the combination

$$\alpha\lambda^2 = \frac{2^{1/3}}{\mathcal{H}^4} \frac{2|\tilde{\Phi}_1(\tau_b)|^2}{I(\tau_b)} \frac{l_p^2}{e_1^2} \frac{1}{M^2}, \tag{5.124}$$

by putting an upper bound on the flux number M. Thinking in terms of the tadpole contribution MK and since the flux numbers are both positive and no smaller than 1, we have $MK < (MK)_{\max} \implies M < (MK)_{\max}$. Imposing the bound $M < (MK)_{\max}$ allows us to connect with the tapole cancellation discussion. Inspired by the examples given below (5.117) we choose three reference bounds, $M < (MK)_{\max} = 32, 100, 1000$. Note that all of these will include point A_1 in Fig. 5.9 ($M = 20$), although $MK \sim 755$ — for a given choice of bound $M < (MK)_{\max}$, all points with $MK < (MK)_{\max}$ clearly lie within the allowed region $M < (MK)_{\max}$, even though there are also points inside that region with $MK > (MK)_{\max}$, such as A_1.

The allowed regions of parameter space are shown in Fig. 5.11. The points (A, B, C) correspond to the specific choice $g_s = 0.2$ and $M = 20$, for each choice of $\tau_b = 0, 20, 40$ (note that α and λ in (5.121) are independent of τ_c, though recall that a combination of τ_c and $\mathcal{V}_{\rm w}$ is fixed by our choice of hierarchy via (5.114)). Putting the brane away from the tip allows the KK graviton modes to have lower masses due to stronger warping without affecting the hierarchy between the bulk and the brane (which we keep fixed), while at the same time suppressing their couplings to matter on the brane, which depend on their wavefunctions (see Fig. 5.5). This means that we can move to regions of larger λ but at the expense of also moving to lower α — this gives rise to the diagonal dashed line in Fig. 5.11, which is an upper bound for the allowed regions which never crosses the excluded region of parameter space.

The upper right bounds on each region follow from (5.123) and $g_s < 1$, while the left bounds follow from (5.121b) and $g_s M > 1$. The lower lines in each triangle represent the lower bound on (5.124) with $M < 32, 100, 1000$ (with larger values giving weaker bounds, i.e. lower lines).

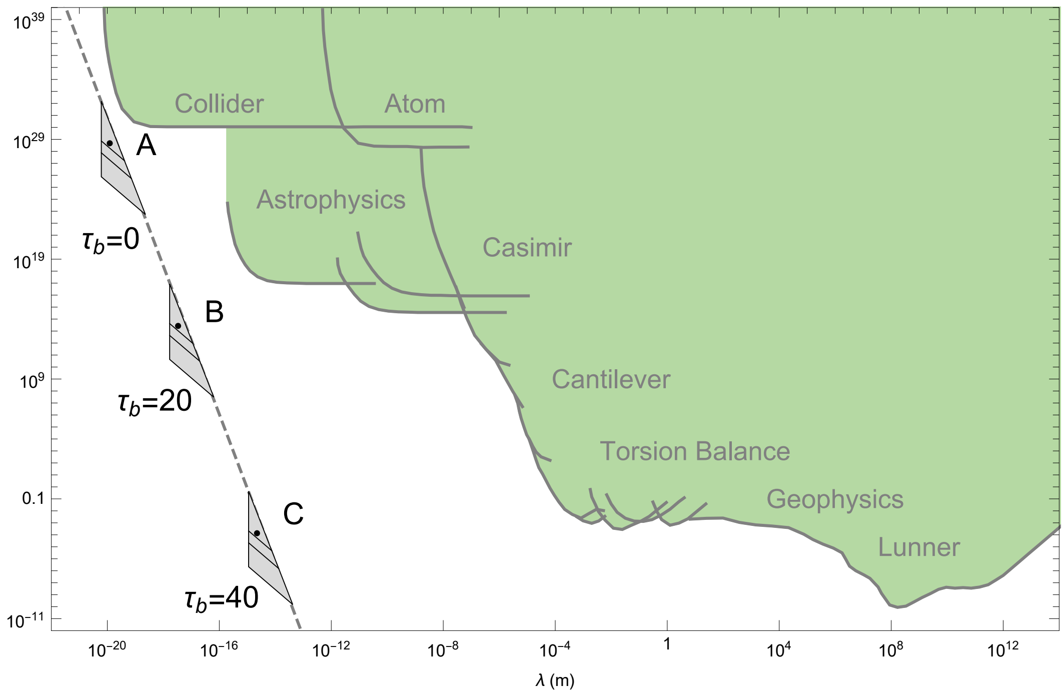


Figure 5.11: The shaded triangles correspond to the allowed regions of parameter space for the fully warped limit $\tau_c \to T$, for different choices of τ_b and fixed $\mathcal{H} = 10^{-15}$. All points (A, B, C) have $g_s = 0.2, M = 20$. The upper right bounds on each region follow from (5.123) and $g_s < 1$, while the left bounds follow from (5.121b) and $g_s M > 1$. The lower lines in each triangle represent the lower bound on (5.124) with $M < 32, 100, 1000$ (with larger values giving weaker bounds, i.e. lower lines). Figure adapted from [263, 292], with the shaded area indicating the excluded region of parameter space at 95% confidence level.

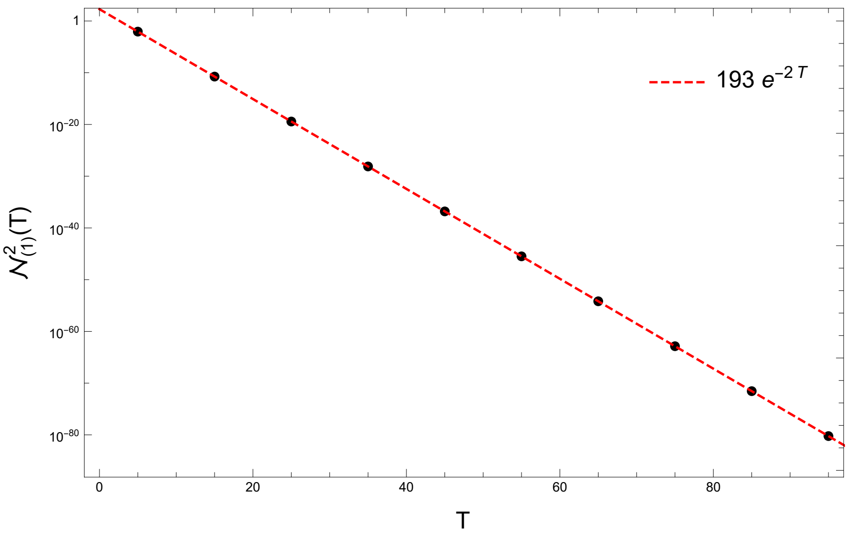


Figure 5.12: Plot of $\mathcal{N}^2_{(1)}(T)$ in the unwarped limit $\tau_c \to 0$. We see that the normalisations fall with T as χe^{-aT}, with $\chi \approx 193$ and $a \approx 2$. See main text for further discussion.

5.3.4 Unwarped deformed conifold

The unwarped deformed conifold corresponds to the limit $\tau_c \to 0$. In this limit, the solution pair $(\hat{E}_k, \Phi_k)$ only depends on T, so that $A(\tau_b, \tau_c, T) = A(\tau_b, T)$ and $\mu(\tau_c, T) = \mu(T)$. In particular, we know from the analytical approximations (confirmed using the numerical solutions) that

$$\hat{E}_k \approx e^{-T/3} e_k\,, \qquad e_k = \frac{2^{5/6}}{3^{1/2}} \left\{ \pi k + \frac{3\pi}{4} \right\}. \tag{5.125}$$

Again notice that we must always have $\tau_b < T$ since our boundary condition $\Phi_k(T) = 0$ would imply vanishing countributions to a brane at $\tau_b \geq T$. In this limit

$$\mathcal{H} \approx \frac{g_s}{\sqrt{4\pi \mathcal{V}_{\rm w}}}\,. \tag{5.126}$$

The general result (5.115) is expressed in terms of τ_c, whose (implicit) definition is (5.35). However, this follows from the condition $H(\tau_c) - 1 = 1$, whereas a fully unwarped conifold would have $H(\tau) = 1$ for all τ. This is not possible for the deformed conifold, since the presence of fluxes necessary to deform the conifold will automatically source some warping, i.e. $H(\tau) \neq 1$ for any finite τ. Nevertheless, we could still have a deformed conifold for which

$$H(\tau) - 1 = 2^{2/3} \frac{(\alpha' g_s M)^2}{c\ \epsilon^{8/3}} I(\tau) \ll 1 \quad \forall\ \tau\,, \tag{5.127}$$

which gives $H(\tau) \approx 1$ as we would expect from an unwarped compactification. In this case, the definition of τ_c (5.35) is no longer useful — one should instead substitute back in (5.115) the parameters of the conifold and remove τ_c completely. This gives

$$\alpha = \frac{1}{(2\pi)^4}\frac{8A(\tau_b,T)}{c^{3/2}|z|^2}\frac{g_s^2}{\mathcal{H}^2}\,, \tag{5.128a}$$

$$\lambda^{-1} = \frac{\mathcal{H}}{c^{1/4}|z|^{1/3}}\frac{\mu(T)}{l_p}\,. \tag{5.128b}$$

One can now relate $c^{1/4}|z|^{1/3}$ to the physical size of the conifold,

$$R_{\text{con}} = \frac{3^{1/2}}{2^{5/6}}(c^{1/4}|z|^{1/3}l_s)e^{T/3}\,, \tag{5.129}$$

such that the parameters become

$$\alpha = \frac{1}{(2\pi)^4}\frac{27A(\tau_b,T)}{8}e^{2T}\left(\frac{l_s}{R_{\text{con}}}\right)^6\frac{g_s^2}{\mathcal{H}^2}\,, \tag{5.130a}$$

$$\lambda^{-1} = \frac{3^{1/2}}{2^{5/6}}\mathcal{H}\left(\frac{l_s}{R_{\text{con}}}\right)e^{T/3}\frac{\mu(T)}{l_p}\,. \tag{5.130b}$$

Assuming that only the first massive mode has a relevant contribution (as all other modes will be exponentially suppressed), we find $A(\tau_b,T) \approx \mathcal{N}_{(1)}^2(T)|\tilde{\Phi}_1(\tau_b)|^2$ and $\mathcal{N}_{(1)}^2 \approx \chi e^{-2T}$, with $\chi \approx 193$ (see Fig. 5.12), and $\mu(\tau_c) \approx e^{-T/3}e_1$ — we can interpret this physically by noting that larger conifolds (larger $R_{\text{con}} \sim e^{T/3} \implies V_{\text{con}} \sim R_{\text{con}}^6 \sim e^{2T}$) will lead to mode wavefunctions spreading over a larger volume, and since they spread evenly throughout the internal space, they consequently have a smaller amplitude at each point. This means that modes living on larger unwarped conifolds have weaker couplings to fields living on the brane. Again, neglecting contributions from massive string states is well-justified as now

$$m_{\text{KK}} \approx \frac{M_s}{(R_{\text{con}}/l_s)}\,, \tag{5.131}$$

which gives the usual hierarchy between the KK and string scales in the absence of warping, with the characteristic scale of the compact space given by R_{con}.

With these simplifications we find

$$\alpha \approx \frac{1}{(2\pi)^4}\frac{27\chi|\tilde{\Phi}(\tau_b)|^2}{8}\left(\frac{l_s}{R_{\text{con}}}\right)^6\frac{g_s^2}{\mathcal{H}^2}\,, \tag{5.132a}$$

$$\lambda^{-1} \approx \frac{3^{1/2}}{2^{5/6}}\mathcal{H}\left(\frac{l_s}{R_{\text{con}}}\right)\frac{e_1}{l_p}\,. \tag{5.132b}$$

Once again we can put bounds on these parameters by using the consistency conditions that must be satisfied in our setup. First, we note that the conifold cannot be arbitrarily small, since the validity of our supergravity approximation requires any region of interest to be at

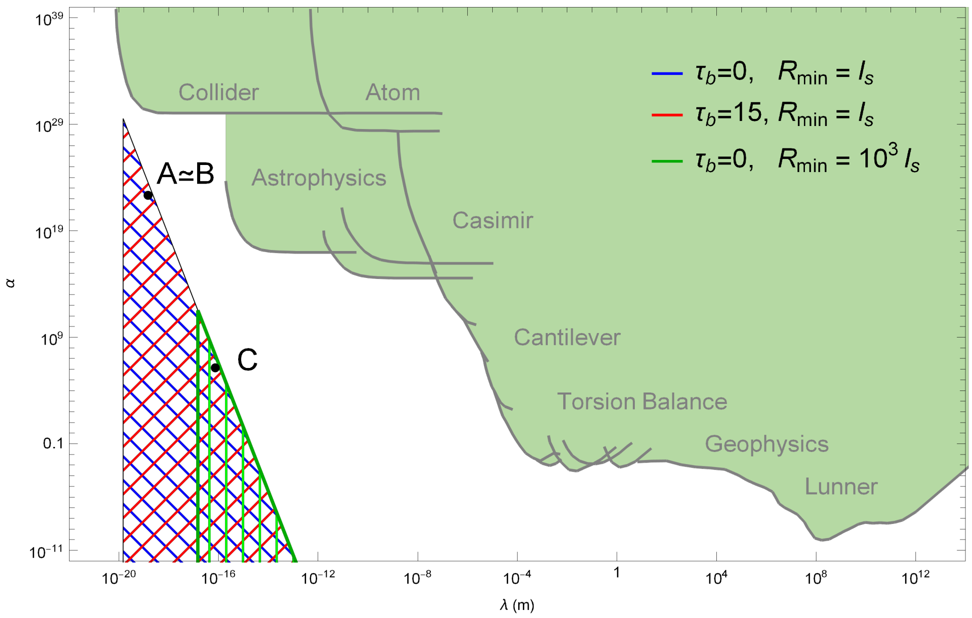


Figure 5.13: Allowed regions of parameter space for the unwarped limit $\tau_c \to 0$, for different choices of the parameters $(\tau_b, R_{min}/l_s)$ and fixed $\mathcal{H} = 10^{-15}$. See main text for further discussion. Figure adapted from [263, 292], with the shaded area indicating the excluded region of parameter space at 95% confidence level.

least bigger that the string size l_s. This gives us the absolute minimum value that $R_{\rm con}$ could take, but for better control of the approximations one may actually want to have larger lower bounds $R_{\rm min} < R_{\rm con}$, with $R_{\rm min} = l_s$ corresponding to the boundary of control. Each choice of $R_{\rm min}$ directly puts a bound on λ. On the other hand, the strongest bound on α comes from eliminating $R_{\rm con}$ through the combination

$$\alpha\lambda^6 = \frac{4\chi|\tilde{\Phi}(\tau_b)|^2}{(2\pi)^4}\left(\frac{l_p}{e_1}\right)^6 \frac{g_s^2}{\mathcal{H}^8}\,, \tag{5.133}$$

which is bounded from above by $g_s < 1$.

With these bounds we can plot the allowed region of parameter space over the experimental and observational constraints (see Fig. 5.13). The points (A, B, C) correspond to the specific choice $g_s = 0.2, M = 20$, with $R_{\rm con} = 10\ l_s$ for A and B, and $R_{\rm con} = 10^4\ l_s$ for C. Notice that the upper limit following from (5.133) is only a function of τ_b (once the hierarchy $\mathcal{H}$ has been fixed) and in the unwarped limit the wavefunctions are constant for most values of τ (see Fig. 5.5). Therefore the allowed regions in this case are only functions of $R_{\rm min}$, larger values of which decrease their size within the region with $R_{\rm min} = l_s$ (bigger triangle). One should remember that the example points in Fig. 5.9 have $c^{1/4}|z|^{1/3}$ fixed by $(g_s, M, \tau_c = 0)$ through (5.35) — for $T = 30$ as in Fig. 5.9, this implies $R_{\rm con} \approx 5 \times 10^3\ l_s$. There is again no overlap between the theoretically allowed regions of parameter space and the experimentally excluded regions.

5.4 4d Gravitational Waves

All the available and planned GW experiments give us the opportunity to probe regimes of strong gravity, which could potentially give interesting constraints to explicit models [265, 266, 268, 269, 272]. Although [273] did not include a careful study of GW signatures, we do work out how a warped throat affects the scales involved in GW detection. In particular, we identify points in the Klebanov-Strassler parameter space where the hierarchy problem is solved by warping and the gravitational wave frequencies corresponding to the graviton KK tower reach LISA, LIGO-Virgo/ET and UHF windows.

From (5.48) we obtain a wave equation for each mode k in an infinite tower, whose frequency is given by[23]

$$\omega_k^2 = m_k^2 + |\vec{p}_k|^2 \,, \tag{5.135}$$

where $\vec{p}_k$ is the momentum of the wave. When an event (e.g. a black hole merger) generates gravitational waves it does so by exciting $h_{\mu\nu}(x^\mu, y^p)$, which is decomposed into an infinite tower of modes in the 4d EFT (5.31), all of which are excited. For the zero mode we have $m_0 = 0$, $h^0_{\mu\nu}$ corresponds to the massless graviton and the gravitational wave will have a frequency associated to the source event. Higher modes $h^{k>0}_{\mu\nu}$ belong to the tower of massive spin-2 states, which have $m_k > 0$. If $m_k \gg |\vec{p}_k|$, the frequency of the wave is given by the mass of the respective mode,

$$\omega_k \approx m_k \,. \tag{5.136}$$

When these frequencies are much higher than the frequency range covered by GW experiments, we can only probe the zero mode, while the massive tower remains out of reach together with the extra dimensions that it encodes. If on the other hand the masses of the KK modes are low enough to give frequencies covered by current or future experiments, then one might hope to detect a characteristic signature of the extra dimensions — a tower of signals whose frequencies are separated by a constant gap $\Delta\omega = m_1 \equiv m_{\rm KK}$. For reference, we give in Table 5.4 the frequency ranges and corresponding $m_{\rm KK}$ for some current and future GW detectors.[24]

We focus on the fully warped case studied in Section 5.3.3. Since $m_k = \lambda^{-1}$, (5.115) is fixed by choosing a value for m_k that could potentially be detected by one of these GW experiments. In

[23]This follows from

$$(p_k)^\mu (p_k)_\mu = -m_k^2 \implies -E_k^2 + |\vec{p}_k|^2 = -m_k^2 \,, \tag{5.134}$$

and by noting that for a wave $E_k = \hbar\omega_k$ (with $\hbar = 1$ and $c = 1$ in all equations).

[24]We obtain $f_{\rm GW}$ by reintroducing factors of c and $\hbar$, which were set to 1,

$$f_{\rm GW} = \frac{m_{\rm KK} c^2}{2\pi\hbar} \,, \tag{5.137}$$

where $m_{\rm KK}$ must be in (kg) such that the result is expressed in (Hz).

	$f_{\rm GW}$ (Hz)	$m_{\rm KK}$ (eV)
LISA	$10^{-4} - 10^{0}$	$10^{-31} - 10^{-27}$
LIGO-Virgo/ET	$10^{1} - 10^{4}$	$10^{-26} - 10^{-23}$
UHF	$10^{6} - 10^{9}$	$10^{-21} - 10^{-18}$

Table 5.4: Frequency range of different gravitational wave experiments and the associated mass ranges for the case in which the wave corresponds to the excitation of a massive spin-2 mode of mass $m_{\rm KK}$.

	$f_{\rm GW}$ (Hz)	$m_{\rm KK}$ (eV)	τ_b	τ_c	$z^{1/3}$	r_{UV}	$\mathcal{V}_{th}$	MK
LISA	10^{0}	10^{-27}	195	239	1.51×10^{-47}	1.70	290	3259
LIGO-Virgo/ET	10^{4}	10^{-23}	168	211	1.51×10^{-43}	1.64	240	2906
UHF	10^{9}	10^{-18}	133	176	1.51×10^{-38}	1.57	183	2464

Table 5.5: Explicit examples of parameters and scales associated with GW signals covered by LISA, LIGO-Virgo/ET and UHF. We fix $\mathcal{H} = 10^{-15}$ and $\mathcal{V}_w = 10^{15}$ in all cases. Lengths and volumes are given in string units.

this limit m_k is given by (5.121b) and we can fix the ratio

$$\frac{m_{\rm KK}}{\mathcal{H}} = \frac{I(\tau_b)^{1/4}}{2^{1/6}} \frac{2\pi}{\sqrt{g_s M}} \frac{e_1}{l_p} , \tag{5.138}$$

by fixing both $m_{\rm KK} \equiv m_1$ and $\mathcal{H}$, and choosing the values of the parameters g_s and M. In what follows we choose $g_s = 0.2$ and $M = 20$ as before. This determines the position of the brane τ_b. One can then use (5.114) to determine $|z|/\mathcal{V}_{\rm w}$, which emphasises the fact that only a combination of these parameters is fixed by our choices. To have a concrete example, we fix the volume to $\mathcal{V}_{\rm w} \sim 10^{15}$, which uniquely fixes $|z|$ and therefore MK. In Table 5.5 we give three examples by taking the upper bounds on the frequency ranges for LISA, LIGO-Virgo/ET and UHF. We see that all these examples require tadpole contributions of $\mathcal{O}(10^3)$, which brings us back to considering the Tadpole Conjecture [100]. The lower the frequency, the larger the warping must be to suppress $m_{\rm KK}$, which we can see by the larger τ_c that results in longer throats. Since we are fixing the volume to $\mathcal{V}_{\rm w} \sim 10^{15}$ in all cases, this also requires a larger τ_b in order to keep the hierarchy fixed — if the warping is stronger, the brane must be farther away from the tip. Different values of $\mathcal{V}_{\rm w}$ will require different values of MK, since the strength of the warping depends on the combination $\mathcal{V}_{\rm w}^{1/6}|z|^{1/3}$ (5.35). For the UHF case, choosing $\mathcal{V}_{\rm w} \sim 10^2$ would require $MK \sim 2021$, while $\mathcal{V}_{\rm w} \sim 10^{30}$ would require $MK \sim 2847$, which is always $\mathcal{O}(10^3)$.

It is important to note that even if the warping is such that $m_{\rm KK}$ is low enough to give frequencies that lie in the ranges of any of these GW experiments, one is not guaranteed to make a detection — we must also take into account the amplitude of the waves. This requires a more careful analysis of the wave equations, which includes in particular the GW source,

$$\Box_4 h^k_{\mu\nu} - m_k^2 h^k_{\mu\nu} = c^{3/2} \int d^6y \sqrt{g_6}\ H\ \Phi_k(y) \mathrm{T}_{\mu\nu}(x,y) \,. \tag{5.139}$$

It is clear that how strongly a given mode is sourced depends on the overlap of its wavefunction $\Phi_k(y)$ and the energy-momentum tensor $T_{\mu\nu}(x,y)$ of the source. Hence, most of the lessons we learn from our (simpler) analysis of fifth force signatures — including the wavefunction profiles of the KK modes in different limits and the dependence of both masses and wavefunctions on the parameter space of the compactification setup — are crucial in understanding GW signatures as well.

For example, one would expect a source localised on the Standard Model brane, $T_{\mu\nu}(x,y) \sim T_{\mu\nu}(x)\delta^{(6)}(y_b)$ (e.g. a neutron star merger) to have very small couplings (cf. (5.96)), due to the large suppression of the KK mode wavefunctions away from the tip of the throat (where the hierarchy $\mathcal{H}$ would not be inconsistently large even when the warping is strong enough to significantly lower the GW frequencies). Even if such an event was energetic enough to source higher KK modes, the amplitudes of those signals would likely be too small for detection. Conversely, we would expect a source localised on a brane at the tip of the throat, $T_{\mu\nu}(x,y) \sim T_{\mu\nu}(x)\delta(y_{\rm tip})$, to have its signals enhanced by the peaked wavefunctions of the KK modes. Interestingly, whereas the former example with suppressed amplitudes could be accompanied by an electromagnetic signal carrying more detailed information about the source, the latter with enhanced amplitudes would not contribute with this companion radiation. On the other hand, a source which is higher dimensional (e.g. a higher dimensional black hole) might be able to couple strongly to the KK modes, not being confined to the brane far away from the tip. Bulk moduli (e.g. complex structure and Kähler moduli) might also source the wave equation and these could have wavefunctions with big overlaps with the KK modes, and may therefore provide signals with higher amplitudes. It would also be interesting to understand if any resonance effects, which are not obvious at this level of our discussion, can enhance the wave amplitude in either case.

One could also consider events in the very early Universe, where it is natural to consider higher energy sources, for which large redshifts might help lower the high frequencies and thus somewhat relax the need for strong warping. This might be particularly relevant in relation to the large tadpole contributions and the Tadpole Conjecture [100]. As a final remark we also note that there are proposals of experimental setups capable of detecting frequencies much higher than the ones covered by the detectors we have been considering [298–301].

6. Conclusions

> Deep in the human unconscious is a pervasive need for a logical Universe that makes sense. But the real Universe is always one step beyond logic.
>
> ---
>
> Frank Herbert

And the way we make sense of the Universe has become increasingly more abstract over the course of history. Partly because of that, string theory often faces criticism for its reliance on mathematical consistency and apparent detachment from experience and the real world. Despite its recurrent serendipity, it is likely that full consensus will only be reached upon experimental verification, in line with the traditional view of empirical science — more importantly, experiments and observations might be the only way to take that one step beyond the logic of the equations and make sense of *our* Universe.

At the start of this thesis, we highlighted the reason why one should not be surprised that there is no direct experimental evidence for string theory. Let us reiterate: the lack of experimental evidence is not a characteristic of strings, but one of quantum gravity theories and the large scales at which they are needed. While these scales remain out of reach of our detectors (potentially by many orders of magnitude), so will the stringy effects inherent to string theory. Although one can not rush the technological development that might slowly close this gap, there are structures within the framework itself that naturally arise when addressing concrete problems (such as the hierarchy and cosmological constant problems) and could help us do just that — warped extra dimensions are a perfect example and were the focus of our work.

We have seen how "warped throats" naturally arise in the context of flux compactifications of Type IIB string theory on Calabi-Yau manifolds, which constitute a promising arena for phenomenological studies of high-energy particle physics and cosmology. Moreover, we have studied how warped throats play a crucial role in some of the best proposals for de Sitter vacuum solutions, whose status in string theory has been heavily debated for over two decades and has recently been put into question in the context of the Swampland programme. Motivated by recent works regarding the (in)consistency of these constructions (e.g. the Tadpole Conjecture), we introduced a new de Sitter solution on the same deformed conifold background as in previous proposals, but in a region of parameter space where the warping is weak due to the interplay between the conifold dynamics and the overall size of the compact space. While this new solution evades several potential issues (e.g. large tadpoles, the singular-bulk problem, conifolds that do not fit in the compact space), we have also found that it cannot be fully trusted without further investigation that takes into account potentially dangerous corrections to the scalar potential (this is similar to what was previously shown for the strongly-warped solutions). These results suggest that one should take a closer look at these corrections and study how they affect current de Sitter constructions — more than just preserving the solutions, these may even benefit them or replace some of their (potentially problematic) ingredients.

By studying warped solutions in different regimes — strong warping vs weak warping — one can also gain some insight into what ingredients are actually playing crucial roles. It is commonly believed that strong warping is necessary in order to suppress the $\overline{\text{D3}}$-brane uplift in the KKLT and LVS proposals, and avoid destabilising the moduli. However, our weakly-warped solutions show that the minimum can be maintained even in a weakly-warped regime, by balancing the unsuppressed brane contribution with a large scale in the superpotential. Despite looking promising, as it does not require strongly-warped throats and the consequent large tadpoles, it is precisely this large superpotential scale that makes higher corrections to the potential important. Motivated by the difficulty to find a de Sitter vacuum, we briefly discussed runaway quintessence models as an alternative explanation of dark energy, concluding that these are (at least) equally hard to achieve.

Warped throats also present opportunities for experimental searches due to their power to bring down unaccessible large scales. Given the current growth of gravitational wave astronomy, it is natural to ask what signatures these warped extra dimensions would leave on gravitational waves. Since the extra-dimensional effects would naïvely give rise to frequencies that are extremely large (in contrast with the range covered by many of current and future GW experiments), warped throats might just be the mechanism that makes these effects observable. In the second part of this thesis, we started an exploration of these signatures.

By determining the masses and wavefunction profiles of the tower of spin-2 KK modes in the presence of a warped throat, we studied their effects on the ranges and coupling strengths of fifth forces mediated by these modes. This allowed us to compare concrete predictions based on the parameter space of consistent Type IIB warped compactifications to current observational constraints, finding that none of it can be excluded — despite the smaller masses of the KK modes achieved through warping — once one takes into account the effect of warping on the KK mode couplings. This teaches us important lessons for the prediction of gravitational wave signatures. Although we do not go into the details of GW predictions, we ended with a brief comparison between the scales covered by the LVK/LISA/ET detectors and the parameter space that allows the KK mode masses to be suppressed enough for detection and a discussion of how the location of both sources and detectors within the compact space affects the amplitude of

GWs. This suggests different scenarios (e.g. sources localised at the tip of a warped throat with the Standard Model living away from it) as the most promising for detection, which motivates future work in this direction.

Warping has become one of the main characters in the string phenomenology endeavour of connecting string theory to the Universe we observe. One can either start with the data that needs to be accommodated by the theory — such as the observation that the expansion of the Universe is currently accelerating — or, conversely, with the existing warped constructions whose signatures we might try to observe — e.g. using gravitational waves. Explaining the nature of Dark Energy and exploiting our recent ability to detect gravitational waves have become major goals in high-energy physics, which makes warped throats extremely relevant and interesting objects among those that string theory has to offer. One may hope that by understanding them better, one might not only be able to solve long-standing problems but also finally find some evidence for string theory, getting us one step closer to decoding the logic of our own Universe.

A. Conventions

A.1 Changing frames in 10d

In the string frame, the gravitational part of the action is not in the canonical Einstein-Hilbert form. Physically this means that the graviton does not have canonical kinetic terms and the string-frame metric G^S does not directly correspond to the propagating graviton — only once we write the action in canonical form can we read off the gravitational interactions of all fields in the way we are used to in GR. The frame in which the action takes the canonical Einstein-Hilbert form – i.e. the Ricci scalar does not couple to anything other than $\sqrt{-G^E}$ — is known as the Einstein frame, which we can choose by performing a conformal transformation of the 10d metric $G^S \to G^E = e^{2\Upsilon} G^S$, with a conformal factor

$$\Upsilon = -\frac{\Phi - \Phi_0}{4}\,, \tag{A.1}$$

where the constant Φ_0 is a choice of convention.[1] Hence the two metrics are related by

$$G^E_{MN} = e^{-\frac{\Phi-\Phi_0}{2}} G^S_{MN}\,, \tag{A.2}$$

and the first term in the action (2.70) becomes, in the Einstein frame,

$$S^{\mathrm{E}}_{\mathrm{grav}} = \frac{1}{2\kappa^2}\int d^{10}x\,\sqrt{-G^E}\Big\{R^E - \frac{9}{2}(G^E)^{\mu\nu}(\partial_\mu\Phi)(\partial_\nu\Phi)\Big\}\,, \tag{A.3}$$

where $\kappa \equiv e^{\Phi_0}\kappa_{10}$ is the rescaled coupling. Including the contribution from the kinetic term of Φ, which also transforms under this conformal transformation, the Einstein frame gravitational plus dilaton action becomes

$$S^{\mathrm{E}}_{\mathrm{grav}+\Phi} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-G^E}\,\Big\{R^E - \frac{1}{2}(\partial_\mu\Phi)(\partial^\mu\Phi)\Big\}\,. \tag{A.4}$$

Note that the dilaton is canonically normalised in Einstein frame.

[1] Choosing the Einstein frame fixes the required conformal transformation up to a constant, which is a simple rescaling of the coupling constant κ, so that one still obtains the Einstein frame — this constant therefore is a matter of convention.

The kinetic terms of the NS and RR form fields include an implicit metric associated to the index contraction of the forms. For a generic p-form η we have

$$
\begin{aligned}
|\eta|_S^2 &= \frac{1}{p!}(G^S)^{\mu_1\nu_1}...(G^S)^{\mu_p\nu_p}\eta_{\mu_1...\mu_p}\eta_{\nu_1...\nu_p} \\
&= \frac{1}{p!}(e^{2\Upsilon})^p(G^E)^{\mu_1\nu_1}...(G^E)^{\mu_p\nu_p}\eta_{\mu_1...\mu_p}\eta_{\nu_1...\nu_p} = e^{2\Upsilon\cdot p}\ |\eta|_E^2\,.
\end{aligned}
\tag{A.5}
$$

Putting everything together, with the appropriate choice (A.1), the action (2.70) in Einstein frame becomes

$$
\begin{aligned}
S_{\rm IIB}^{\rm E} = \frac{1}{2\kappa^2}\Bigg\{ &\int d^{10}x\sqrt{-G^E}\left(R^E - \frac{1}{2}(\partial_\mu\Phi)(\partial^\mu\Phi) - \frac{e^{\Phi_0}}{2}e^{-\Phi}|H_3|_E^2\right) \\
&- \int d^{10}x\sqrt{-G^E}\left(\frac{e^{2\Phi}}{2}|F_1|_E^2 + \frac{e^{\Phi_0}}{2}e^{\Phi}|\tilde{F}_3|_E^2 + \frac{e^{2\Phi_0}}{4}|\tilde{F}_5|_E^2\right) - \frac{e^{2\Phi_0}}{2}\int C_4\wedge H_3\wedge F_3\Bigg\}\,.
\end{aligned}
\tag{A.6}
$$

Note that the Chern-Simons term in the action does not transform, apart from via the constant relating κ and κ_{10}, as it is a topological term, independent of the metric. A common choice of Φ_0 is such that the metric in the string frame and the metric in the Einstein frame are the same at the vacuum, i.e. $\Phi_0 = \langle\Phi\rangle$ – this allows us to discuss quantities in a frame-independent way *at the vacuum*. For that choice, the action in Einstein frame reads

$$
\begin{aligned}
S_{\rm IIB}^{\rm E} =& \frac{1}{2\kappa^2}\Bigg\{ \int d^{10}x\sqrt{-G}\left(R - \frac{1}{2}(\partial_\mu\Phi)(\partial^\mu\Phi) - \frac{g_s}{2}e^{-\Phi}|H_3|^2\right) \\
&- \int d^{10}x\sqrt{-G}\left(\frac{e^{2\Phi}}{2}|F_1|^2 + \frac{g_s}{2}e^{\Phi}|\tilde{F}_3|^2 + \frac{g_s^2}{4}|\tilde{F}_5|^2\right) - \frac{g_s^2}{2}\int C_4\wedge H_3\wedge F_3\Bigg\},
\end{aligned}
\tag{A.7}
$$

where we dropped the E, as all metrics are in Einstein frame. With this choice, the gravitational coupling is related to the string scale as

$$
2\kappa^2 = 2\kappa_{10}^2 g_s^2 = (2\pi)^7 g_s^2\alpha'^4 \qquad \text{or} \qquad 2\kappa^2 = \frac{g_s^2\ l_s^8}{2\pi}\,. \tag{A.8}
$$

Another common choice of convention is $\Phi_0 = 0$. In this case, volumes are frame-dependent in the vacuum (A.14) and one needs to be careful in using the right frame, e.g. when checking whether the α'-expansion is under control for a certain vacuum, which should be done using the string frame volume. For this choice the gravitational coupling is related to the string scale as

$$
2\kappa^2 = 2\kappa_{10}^2 = (2\pi)^7\alpha'^4 \qquad \text{or} \qquad 2\kappa^2 = \frac{l_s^8}{2\pi}\,. \tag{A.9}
$$

In terms of the fields G_3 and τ, i.e. in its manifestly $\mathrm{SL}(2,\mathbb{R})$-invariant form, the Einstein frame action (2.70) takes the form

$$S^{\rm E}_{\rm IIB} = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-G}\left(R - \frac{(\partial_\mu\tau)(\partial^\mu\bar{\tau})}{2({\rm Im}\,\tau)^2} - \frac{e^{\Phi_0}}{2({\rm Im}\,\tau)}|G_3|^2 - \frac{e^{2\Phi_0}}{4}|\tilde{F}_5|^2\right)$$
$$- \frac{1}{2\kappa^2}\frac{ie^{2\Phi_0}}{4}\int\frac{1}{({\rm Im}\,\tau)}C_4\wedge G_3\wedge\overline{G}_3\,. \tag{A.10}$$

Another occasionally used convention (see e.g. [164, 302–304]) is to redefine the RR forms in Einstein frame as $C^E_p = e^{\Phi_0}C^S_p$, so that the action becomes

$$S^{\rm E}_{\rm IIB} = \frac{1}{2\kappa^2}\int\left(R\star 1 - \frac{d\tau\wedge\star d\tau}{2({\rm Im}\,\tau)^2} - \frac{G_3\wedge\star\overline{G}_3}{2({\rm Im}\,\tau)} - \frac{1}{4}\tilde{F}_5\wedge\star\tilde{F}_5 - \frac{i}{4({\rm Im}\,\tau)}C_4\wedge G_3\wedge\overline{G}_3\right), \tag{A.11}$$

where the axio-dilaton was also redefined as $\tau^E = e^{\Phi_0}\tau^S = e^{\Phi_0}C^S_0 + ie^{-\varphi}$, with $e^{-\varphi} = e^{-(\Phi-\Phi_0)}$, and $G^E_3 = e^{\Phi_0}G^S_3 = F^E_3 - \tau^E H_3$. Note that in terms of τ^E we have $\langle {\rm Im}\,\tau^E\rangle = 1$, rather than $\langle {\rm Im}\,\tau^S\rangle = g_s^{-1}$, and flux quantisation takes the form

$$\frac{1}{(2\pi)^2\alpha'}\int_{\mathcal{C}_p} F^E_p \in e^{\Phi_0}\mathbb{Z}\,. \tag{A.12}$$

Although with this field redefinition the action looks the same regardless of the choice of convention, one must keep in mind both of these differences and the fact that also the gravitational coupling κ will depend on the convention in use.

It is worth emphasising that volumes measured using a string frame metric may differ from the ones measured using an Einstein frame metric, depending on the convention used, i.e. on the choice of Φ_0. To see this, recall that the two metrics are related by $G^E_{MN} = e^{-\frac{\Phi-\Phi_0}{2}}G^S_{MN}$. Therefore, a generic d-dimensional volume can be written as

$$V^E_d = \int d^dy\sqrt{g^E_d}\ f(y) = \int d^dy\sqrt{g^S_d}\ e^{-\frac{d}{4}(\Phi-\Phi_0)}f(y)\,, \tag{A.13}$$

where we allow for some function $f(y)$, such as $H(y)$ in the definition of $V_{\rm w}$ (A.20). Therefore, the volumes in the two frames (assuming Φ is stabilised) are related as $V^S_d = \left(e^{\langle\Phi\rangle-\Phi_0}\right)^{\frac{d}{4}}V^E_d$, which for the volume of the 6d compact space in string compactifications means

$$\mathcal{V}_S = e^{\frac{3}{2}(\langle\Phi\rangle-\Phi_0)}\mathcal{V}_E\,. \tag{A.14}$$

Hence, it is important to note the convention being used for the Einstein frame metric and how it relates to quantities that are obtained in either string frame or Einstein frame (e.g. perturbative and non-perturbative corrections).

A.2 Changing frames in 4d

In order to obtain a 4d EFT at low energies, we consider a compactification (or dimensional reduction) of the 10d theory down to 4 dimensions. The 4d theory describes perturbations around a 10d vacuum solution and is valid for energies much lower than the compactification scale[2]. We consider a vacuum solution which corresponds to a warped product spacetime $\mathcal{M}_{10} = \mathbb{R}^{1,3} \times_{\mathrm{w}} X_6$, where $\mathbb{R}^{1,3}$ is a 4d Lorentzian spacetime and X_6 is a 6d compact space. The Einstein frame metric takes the form[3]

$$ds_{10}^2 = H^{-1/2}(y)\; e^{2\omega(x)} g_{\mu\nu} dx^\mu dx^\nu + H^{1/2}(y)\; \mathcal{V}^{1/3} g_{mn} dy^m dy^n \,, \tag{A.15}$$

where x^μ ($\mu = 0, ..., 3$) are 4d coordinates and y^m ($m = 4, ..., 9$) are 6d coordinates on the compact space X_6. The metric $g_{mn} = (g_6)_{mn}$ is the 6d metric of a Calabi-Yau (Ricci flat) manifold normalised such that

$$\int d^6y \sqrt{g_6} \equiv l_s^6,$$

with $\mathcal{V} = \mathcal{V}_E(x)$ keeping track of the physical size of the compact space, and the warp factor H is defined as

$$H(y) \equiv 1 + \frac{e^{-4A_0(y)}}{\mathcal{V}^{2/3}} \,. \tag{A.16}$$

The factor $e^{2\omega(x)}$ is introduced to Weyl rescale to the 4d Einstein frame, with metric $g_{\mu\nu}$, as we now describe.

Dimensionally reducing the 10d Einstein-Hilbert term (in Einstein frame)

$$S_{\mathrm{IIB}}^{\mathrm{E}} = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-G}\; R_{10} \tag{A.17}$$

down to 4d using the ansatz (A.15) gives, among other contributions, the term

$$S_{4\mathrm{d}} \supset \frac{1}{2\kappa^2} \int d^4x \sqrt{-g_4} \cdot e^{2\omega(x)} \Big(\mathcal{V} \int d^6y \sqrt{g_6} \cdot H(y) \Big) R_4 \,. \tag{A.18}$$

Any choice of $e^{2\omega(x)}$ that leaves a non-canonical coupling of the volume modulus $\mathcal{V}$ to R_4 is said to be in the Jordan frame. Requiring a canonical form for the Einstein-Hilbert term instead — which defines the 4d Einstein frame — fixes the Weyl rescaling $e^{2\omega(x)}$, up to a constant factor $e^{2\omega_0}$, as

$$e^{2\omega(x)} = \frac{e^{2\omega_0} \cdot l_s^6}{\mathcal{V} \int d^6y \sqrt{g_6} \cdot H(y)} \equiv \frac{e^{2\omega_0} \cdot l_s^6}{V_{\mathrm{w}}} = \frac{e^{2\omega_0}}{\mathcal{V}_{\mathrm{w}}} \,, \tag{A.19}$$

[2]Depending on the details of the compactification, this scale could correspond to different scales.
[3]Since we start with Einstein frame metric (A.15) and action (A.17), the volumes $\mathcal{V}$ and $\mathcal{V}_{\mathrm{w}}$ are Einstein frame volumes.

where we defined the *warped volume* $V_{\rm w} = \mathcal{V}_{\rm w} \cdot l_s^6$ as[4]

$$V_{\rm w} \equiv \mathcal{V} \underbrace{\int d^6y\sqrt{g_6} \cdot H(y)}_{\langle H\rangle_{\rm av}\cdot\ l_s^6} . \tag{A.20}$$

This definition of $V_{\rm w}$ only differs from $\mathcal{V} \cdot l_s^6$ by the factor $\langle H\rangle_{\rm av}$, the average of the warp factor over the compact space. If the integral is dominated by the unwarped bulk, then $\langle H\rangle_{\rm av} \approx 1$ and $V_{\rm w} \approx \mathcal{V} \cdot l_s^6$.

Note the similarities with the conformal transformation in 10d to go from string frame to Einstein frame, where we also had some freedom in the form of a constant. There a convenient choice was the one for which the two metrics matched *at the vacuum*. Here we are going from the Jordan frame, in which some scalars couple to the Ricci scalar in the action, to the 4d Einstein frame, in which we recover the canonical Einstein-Hilbert term. The two metrics will match *at the vacuum* if we choose $e^{2\omega_0} = \langle \mathcal{V}_{\rm w}\rangle$. The action in Einstein frame for general ω_0 becomes

$$S_{\rm 4d}^{\rm E} \supset \frac{e^{2\omega_0} \cdot l_s^6}{2\kappa^2} \int d^4x\sqrt{-g_4} \cdot R_4 \equiv \frac{M_{\rm Pl}^2}{2} \int d^4x\sqrt{-g_4} \cdot R_4 \,, \tag{A.21}$$

which defines the relation between the string scale[5] ($m_s = 1/l_s$) and the Planck scale as

$$m_s = \frac{e^{\Phi_0}}{\sqrt{4\pi e^{2\omega_0}}} M_{\rm Pl} \,. \tag{A.22}$$

For the convenient choice $e^{\Phi_0} = g_s$ and $e^{2\omega_0} = \langle \mathcal{V}_{\rm w}\rangle$, this relation becomes

$$m_s = \frac{g_s}{\sqrt{4\pi \mathcal{V}_{\rm w}}} M_{\rm Pl} = \langle H\rangle_{\rm av}^{-1/2} \frac{g_s}{\sqrt{4\pi\mathcal{V}}} M_{\rm Pl} \,. \tag{A.23}$$

Note also that in the unwarped limit the warped volume tends to the volume modulus of the compactification, $\mathcal{V}_{\rm w} \to \mathcal{V}$, and – with these choices of convention for the Weyl rescalings – we recover the common expression for the ratio $m_s/M_{\rm Pl}$. If instead we choose conventions $\Phi_0 = 0 = \omega_0$, then $m_s = M_{\rm Pl}/\sqrt{4\pi}$. This convention dependence arises from the fact that we are comparing quantities measured in different frames, as m_s is the string mass measured in the string frame. This can be immediately seen for string states localised in regions of constant warp factor, $H(y_0)$ (these states could arise, for example, from open string states localised on a brane at the tip of a warped throat). The mass of a string state with momentum p^μ measured

[4]Note that this differs from the volume of the 6d compact space in the ansatz (A.15), which is

$$\mathcal{V} \int d^6y\sqrt{g_6}\ H^{3/2}(y) \,.$$

[5]See footnote 30.

in Einstein frame would then be

$$\begin{aligned}\left((m_s^{\mathrm{w}})^E\right)^2 &= g_E^{\mu\nu}p_\mu p_\nu = H(y_0)^{-1/2}\cdot\frac{e^{2\omega_0}}{\mathcal{V}_E}\cdot G_E^{\mu\nu}p_\mu p_\nu \\ &= H(y_0)^{-1/2}\cdot\frac{e^{2\omega_0}}{\mathcal{V}_S\, e^{\frac{3}{2}(\Phi-\Phi_0)}}\cdot e^{\frac{\Phi-\Phi_0}{2}}\cdot G_S^{\mu\nu}p_\mu p_\nu \\ &= H(y_0)^{-1/2}\cdot\frac{e^{2\Phi}}{\mathcal{V}_S}\cdot\frac{e^{2\omega_0}}{e^{2\Phi_0}}\cdot(m_s^S)^2\,, \end{aligned} \tag{A.24}$$

from which follows a *warped* string scale given by

$$(m_s^{\mathrm{w}})^E \equiv H^{-1/4}(y_0)\cdot\frac{e^{\Phi}}{\sqrt{4\pi\mathcal{V}_S}}M_{\mathrm{Pl}}\,, \tag{A.25}$$

whose relation to M_{Pl} is fully convention independent in terms of string frame volumes. For closed string states that are allowed to propagate along directions of varying warp factor, one needs to take into account their dimensional reduction (see section D.1 for details). Let us take the scalar field ψ to represent a closed string state of mass m_s^S measured in the string frame,

$$\begin{aligned} S &\propto \int d^{10}x\sqrt{-G^S}\Big\{-\frac{1}{2}G_S^{MN}(\partial_M\psi)(\partial_N\psi)+(m_s^S)^2\psi^2\Big\} \\ &\propto \int d^{10}x\sqrt{-G^E}\Big\{-\frac{1}{2}e^{\frac{\Phi-\Phi_0}{2}}G_E^{MN}(\partial_M\psi)(\partial_N\psi)+(m_s^S)^2\psi^2\Big\} \\ &\propto \int d^4x\; e^{4\omega(x)}\sqrt{-g_4}\int d^6y\sqrt{g_6}\;H^{1/2}(y)\mathcal{V}\Big\{-\frac{1}{2}e^{\frac{\Phi-\Phi_0}{2}}H^{1/2}(y)e^{-2\omega(x)}g^{\mu\nu}(\partial_\mu\psi)(\partial_\nu\psi) \\ &\qquad\qquad + \text{internal momentum} + (m_s^S)^2\psi^2\Big\} \\ &\propto \int d^4x\sqrt{-g_4}\;e^{\frac{\Phi-\Phi_0}{2}}e^{2\omega(x)}\mathcal{V}\int d^6y\sqrt{g_6}\;H(y)\Big\{-\frac{1}{2}g^{\mu\nu}(\partial_\mu\psi)(\partial_\nu\psi)+\text{internal momentum} \\ &\qquad\qquad + e^{-\frac{\Phi-\Phi_0}{2}}e^{2\omega(x)}H^{-1/2}(y)(m_s^S)^2\psi^2\Big\} \end{aligned} \tag{A.26}$$

Following the same steps as for the KK modes and using (A.19), we find for the lowest mode ψ_0 in the tower

$$\begin{aligned} S &\propto \int d^4x\sqrt{-g_4}\Big\{-\frac{1}{2}g^{\mu\nu}(\partial_\mu\psi_0)(\partial_\nu\psi_0)\cdot\int d^6y\sqrt{g_6}\cdot H(y)\;(\xi^0)^2 \\ &\qquad + e^{-\frac{\Phi-\Phi_0}{2}}\cdot\frac{e^{2\omega_0}}{\mathcal{V}_E}\cdot(m_s^S)^2\psi_0^2\cdot\int d^6y\sqrt{g_6}\cdot H(y)\;(\xi^0)^2\cdot H^{-1/2}(y)\Big\}\,, \\ &= \int d^4x\sqrt{-g_4}\Big\{-\frac{1}{2}g^{\mu\nu}(\partial_\mu\psi_0)(\partial_\nu\psi_0)+(m_s^E)^2\psi_0^2\Big\}\,, \end{aligned} \tag{A.27}$$

and we identify the Einstein frame mass as

$$(m_s^E)^2 = \frac{e^{2\Phi}}{\mathcal{V}_S}\cdot\frac{e^{2\omega_0}}{e^{2\Phi_0}}\cdot\Big\{\int d^6y\sqrt{g_6}\cdot H(y)\;(\xi^0)^2\cdot H^{-1/2}(y)\Big\}\cdot(m_s^S)^2\,. \tag{A.28}$$

Therefore, assuming again that most of the compact space is unwarped (or in the absence of warping, $H = 1$) and using (A.23) we find for the Einstein frame string scale

$$m_s^E = \frac{e^{\Phi}}{\sqrt{4\pi \mathcal{V}_s}} M_{\rm Pl}\,, \tag{A.29}$$

so that its relation to $M_{\rm Pl}$ is fully convention independent in terms of string frame volumes. Finally, we recover the standard relation between the warped string scale for localised states (A.25) and the bulk string scale (A.29) as

$$(m_s^{\rm w})^E = H^{-1/4}(y_0) \cdot m_s^E\,. \tag{A.30}$$

A.3 Quantum corrections in different conventions

Since the flux superpotential leaves all Kähler moduli unstabilised, either leaving them as flat directions or generating runaways, one must resort to higher-order corrections to the EFT in order to stabilise them. Both perturbative and non-perturbative corrections have been considered in the literature — while the former are computed at the level of the 10d EFT and in string frame, the latter are obtained directly at the level of the 4d EFT and are computed in Einstein frame. One must therefore be careful with the conventions being used to change frames, i.e. the choice of Φ_0, in order to remain consistent.

Perturbative corrections

In [147], it was shown that α'-corrections to the Type IIB effective action (A.6) manifest as corrections to the 4d volume modulus Kähler potential and spoil the no-scale structure of its scalar potential. These corrections arise from higher-derivative terms at order $(\alpha')^3$ appearing in the type IIB effective action,

$$S_{IIB}^S = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G^S}\; e^{-2\Phi} \Big(R^S + 4(\partial\Phi)_S^2 + (\alpha')^3 \cdot \frac{\zeta(3)}{3 \cdot 2^{11}} \cdot J_0 \Big)\,, \tag{A.31}$$

where the higher-order term is schematically given by

$$J_0 \sim (R_{MNPQ})^4\,. \tag{A.32}$$

One must also add a term

$$\delta S_\Phi^S \sim \int d^{10}x \sqrt{-G^S} e^{-2\Phi} (\alpha')^3 (\nabla^2 \Phi)\; Q\,, \tag{A.33}$$

where $Q \sim (R_{MNPQ})^3$ is a generalisation of the 6d Euler integrand $\int_{X_6} d^6y \sqrt{g_6}\; Q = \chi$, with χ the Euler characteristic of X_6 [147]. This term corrects the 10d solution to the equation of motion for Φ, such that $\Phi = \Phi_{10} + \frac{\zeta(3)}{16} Q$. It is then shown in [147] that this leads to a correction to the Kähler potential of the form

$$K = -2\log\left(\mathcal{V}_S + \frac{\xi}{2}\right) = -2\log\left(\mathcal{V}_E \, e^{\frac{3}{2}(\Phi-\Phi_0)} + \frac{\xi}{2}\right) \tag{A.34}$$

$$= -2\log\left(\mathcal{V}_E + \frac{\xi}{2} e^{-\frac{3}{2}(\Phi-\Phi_0)}\right) + \dots \,, \tag{A.35}$$

where we have used (A.14), keeping Φ_0 unspecified, and ξ is defined as[6]

$$\xi = -\frac{\zeta(3)\chi}{2(2\pi)^3} \,. \tag{A.36}$$

Note in particular that the correction expressed in Einstein frame depends on the convention one chooses for Φ_0,

$$\Phi_0 = 0 \implies K = -2\log\left(\mathcal{V} + \frac{\xi}{2 g_s^{3/2}}\right) \,, \tag{A.37}$$

$$\Phi_0 = \langle\Phi\rangle \implies K = -2\log\left(\mathcal{V} + \frac{\xi}{2}\right) \,, \tag{A.38}$$

where we have assumed as usual that the dilaton has been stabilised by fluxes.

Non-perturbative corrections

Although the superpotential W does not receive perturbative corrections, it may receive non-perturbative corrections from either instantons arising from Euclidean D3-branes wrapping 4-cycles or gaugino condensation on the world-volume theory of D7-branes wrapped around internal 4-cycles. Let us consider the latter case in some detail. In what follows, $T_p = \frac{2\pi}{l_s^{p+1}}$ is the brane tension. The DBI action for a Dp-brane, in the string frame, is given by (2.90) [20, 26, 29, 39]

$$S_{\mathrm{DBI}}^{\mathrm{Dp}} = -T_{\mathrm{D}p} \int d^{p+1}\sigma \; e^{-\Phi} \sqrt{-\det\left(g^S + B + \frac{l_s^2}{2\pi} F\right)} \,, \tag{A.39}$$

where g^S and B refer to the pull-back of the string frame metric $(G^S)_{MN}$ and 2-form B_{MN} onto the world-volume of the brane and F to the field-strength F_{ab} of the brane gauge fields. Rewriting the action in terms of the Einstein frame metric,

$$S_{\mathrm{Dp}}^{\mathrm{DBI}} = -T_p \int d^{p+1}\sigma \; e^{-\Phi} \sqrt{-\det g^S} \sqrt{\det\left(\mathbf{1} + (g^S)^{-1}\left(B + \frac{l_s^2}{2\pi} F\right)\right)} \tag{A.40}$$

[6] In [147], we find the definition $\xi = -\frac{\zeta(3)\chi(X_6)}{2}$. The missing factor of $(2\pi)^3$ comes from their conventions for the volume, $\mathcal{V}_{[147]} = V_6/(2\pi\alpha')^3$, whereas we are using $\mathcal{V} = V_6/l_s^6 = (2\pi)^{-3}\, V_6/(2\pi\alpha')^3$, with the convention $(2\pi)^2\alpha' = l_s^2$. There are also instances in the literature where the factor of 1/2 is absorbed into the definition of ξ.

$$\supset -T_p \int d^{p+1}\sigma\ e^{-\Phi}\sqrt{-\det g^S}\ \frac{1}{4}\left(\frac{l_s^2}{2\pi}\right)^2 (g^S)^{ac}(g^S)^{bd}F_{ab}F_{cd} \tag{A.41}$$

$$= -\frac{T_p}{4}\frac{l_s^4}{(2\pi)^2}\int d^{p+1}\sigma\ \sqrt{-\det g^E}\ e^{\frac{p-3}{4}(\Phi-\Phi_0)}e^{-\Phi}F_{ab}F^{ab}\,, \tag{A.42}$$

where the indices in $F_{ab}F^{ab}$ are contracted with Einstein frame metrics. This is the kinetic term for the brane gauge bosons (see Appendix A.2. of [163]) and tells us the gauge coupling of the corresponding theory, which is a key parameter for gaugino condensation. If the brane is wrapping a $(p-3)$–cycle Σ_{p-3}, we find the corresponding 4d term (assuming that Φ is constant over the cycle)

$$S^{\text{4d}}_{\text{Dp}} \supset -\frac{1}{8\pi l_s^{p-3}}\int d^4x\ \sqrt{-\det g_4^E}\ e^{\frac{p-3}{4}(\Phi-\Phi_0)}e^{-\Phi}\underbrace{\left(\int d^{p-3}\sigma\sqrt{g^E_{p-3}}\right)}_{\tau^E_{\Sigma_{p-3}}l_s^{p-3}}F_{ab}F^{ab} \tag{A.43}$$

$$= -\int d^4x\ \sqrt{-\det g_4^E}\left\{\frac{\tau^E_{\Sigma_{p-3}}}{8\pi e^{\Phi}}\ e^{\frac{p-3}{4}(\Phi-\Phi_0)}\right\}F_{ab}F^{ab}\,, \tag{A.44}$$

and we can read off the gauge coupling g_c,

$$\frac{1}{g_c^2} = \frac{\tau^E_{\Sigma_{p-3}}}{4\pi e^{\langle\Phi\rangle}}\ e^{\frac{p-3}{4}(\langle\Phi\rangle-\Phi_0)}\,, \tag{A.45}$$

where we have assumed as usual that the dilaton has been stabilised by fluxes at some higher scale. Gaugino condensation on the world-volume theory of D7-branes will then give a non-perturbative contribution to the superpotential [210],[7]

$$W_{np} \sim e^{-\frac{8\pi^2}{g_c^2}\frac{1}{N}} = e^{-\frac{2\pi}{N}\frac{\tau^E}{g_s}e^{\langle\Phi\rangle-\Phi_0}}\,, \tag{A.46}$$

where we used $e^{\langle\Phi\rangle} = g_s$. Holomorphicity of W then leads to the general contribution

$$W_{np} = \sum_i A_i e^{i\frac{a_i}{g_s}e^{\langle\Phi\rangle-\Phi_0}T_i^E}\,, \tag{A.47}$$

where the sum is over the contributing cycles, $a_i = \frac{2\pi}{N_i}$ and the fields $T_i = b_i + i\tau_i$ are the complexified Kähler moduli. Hence, we can compare the two most common conventions for Φ_0,

$$\Phi_0 = 0 \implies W_{np} = \sum_i A_i e^{ia_iT_i^E}\,, \tag{A.48}$$

$$\Phi_0 = \langle\Phi\rangle \implies W_{np} = \sum_i A_i e^{i\frac{a_i}{g_s}T_i^E}\,. \tag{A.49}$$

[7] Here N is the number of branes stacked on top of each other, responsible for the gauge group. It appears through the beta-function coefficient [210].

The superpotential as written, in terms of Einstein frame 4-cycle volumes, appears to depend on the choice of convention for Φ_0, but one should recall that the 4-cycle volumes also depend on this choice of convention. The convention-independence becomes manifest if we express the 4-cycle volumes in terms of the string frame metric (A.13),

$$\tau_i^S = e^{\langle\Phi\rangle - \Phi_0} \tau_i^E \,. \tag{A.50}$$

B. Differential forms and Cohomology

The spectrum of string theory includes several differential form fields, such as B_2 already present in the bosonic string and (C_2, C_4) that arise in Type IIB. Strictly speaking, also the scalars (Φ, C_0) are differential 0-forms. The machinery of differential geometry is therefore extremely useful when discussing the physics of string theory[1]. We will therefore compile here for easy reference some of the concepts and results which are crucial for string compactifications, briefly describing the map between harmonic forms on a compact space and cohomology and homology groups. We leave to Appendix C an equally brief discussion of Calabi-Yau manifolds and Dolbeault-cohomology, crucial for Calabi-Yau compactifications of Type IIB. A detailed presentation of these and many other topics in differential geometry can be found in [305, 306].

A differential r-form ω_r on a manifold $\mathcal{M}$ is a totally antisymmetric tensor of type $(0, r)$ and can be written as

$$\omega_r = \frac{1}{r!}\omega_{\mu_1\mu_2\ldots\mu_r} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_r} \,. \tag{B.1}$$

The vector space of r-forms at a point $p \in \mathcal{M}$ is denoted $\Omega^r_p(\mathcal{M})$. The exterior product between two forms ξ_q and η_r gives the $(q+r)$-form

$$\xi_q \wedge \eta_r = \frac{1}{q!r!}\xi_{\mu_1\ldots\mu_q}\eta_{\nu_1\ldots\nu_r} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_q} \wedge dx^{\nu_1} \wedge \ldots \wedge dx^{\nu_r} \,, \tag{B.2}$$

which is only non-zero if $q+r \leq m = \dim(\mathcal{M})$. A form $\omega_m \in \Omega^m_p(\mathcal{M})$ is known as a top form and is the only type of form that can be integrated over $\mathcal{M}$. We can also think of *smooth* r-forms on $\mathcal{M}$, $\omega_r \in \Omega^r(\mathcal{M})$, for which the exterior derivative operator $\mathrm{d} : \Omega^r(\mathcal{M}) \to \Omega^{r+1}(\mathcal{M})$ can be defined as

$$\mathrm{d}\omega = \frac{1}{(r+1)!}\partial_\nu \omega_{\mu_1\ldots\mu_r} dx^\nu \wedge dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_r} \,. \tag{B.3}$$

If we have a metric g defined on $\mathcal{M}$, we can also define the Hodge star operator as

$$\star\omega = \frac{\sqrt{|g|}}{(m-r)!}\left(\frac{1}{r!}\omega_{\mu_1\ldots\mu_r}\right)\epsilon^{\mu_1\ldots\mu_r}{}_{\nu_{r+1}\ldots\nu_m} dx^{\nu_{r+1}} \wedge \ldots \wedge dx^{\nu_m} \,, \tag{B.4}$$

mapping an r-form to an $(m-r)$-form with the help of the totally antisymmetric tensor $\epsilon^{\mu_1\cdots\mu_m} = g^{\mu_1\nu_1}\ldots g^{\mu_m\nu_m}\epsilon_{\nu_1\ldots\nu_m} = g^{-1}\epsilon_{\mu_1..\mu_m}$. This gives us a natural way to define an inner product of two

[1] In fact, differential geometry is an extremely power tool to study almost any physics.

r-forms $\omega_r, \eta_r \in \Omega^r(\mathcal{M})$ as the integral over a top form (the only forms which one can integrate over $\mathcal{M}$),

$$(\omega, \eta) \equiv \int \omega \wedge \star\eta = \frac{1}{r!} \int \omega_{\mu_1 \ldots \mu_r} \eta^{\mu_1 \ldots \mu_r} \sqrt{|g|}\, dx^1 \ldots dx^m \tag{B.5}$$

The inner product is symmetric and, if the metric is Riemannian, positive definite, i.e. $(\omega, \omega) \geq 0$ and only 0 for $\omega = 0$. The adjoint exterior derivative, $\mathrm{d}^\dagger : \Omega^r(\mathcal{M}) \to \Omega^{r-1}(\mathcal{M})$, is then defined with respect to the inner product such that

$$(\mathrm{d}\omega, \eta) = (\omega, \mathrm{d}^\dagger \eta) \implies \mathrm{d}^\dagger \equiv (-1)^{1+m(r+1)} \star \mathrm{d} \star . \tag{B.6}$$

Finally, we can define the Laplacian, $\Delta : \Omega^r(\mathcal{M}) \to \Omega^r(\mathcal{M})$, in terms of d and $\mathrm{d}^\dagger$ as

$$\Delta = \mathrm{d}\mathrm{d}^\dagger + \mathrm{d}^\dagger \mathrm{d} . \tag{B.7}$$

The Laplacian acting on the components of p-form ω can be expressed as [20]

$$\Delta\omega_{m_1 \ldots m_p} = -\nabla^q \nabla_q \omega_{m_1 \ldots m_p} + pR_{q[m_1} \omega^q{}_{m_2 \ldots m_p} - \frac{1}{2} p(p-1) R_{qr[m_1 m_2} \omega^{qr}{}_{m_3 \ldots m_p]} , \tag{B.8}$$

in terms of covariant derivatives, the Riemann and the Ricci tensors. This expression is useful in section 3.3.

Differential forms can then be classified in terms of how the operators $(\mathrm{d}, \mathrm{d}^\dagger, \Delta)$ act on them,

`(closed)`	$\mathrm{d}\omega = 0$	`(co-closed)`	$\mathrm{d}^\dagger \omega = 0$
`(exact)`	$\omega = \mathrm{d}\eta$	`(co-exact)`	$\omega = \mathrm{d}^\dagger \xi$
`(harmonic)`	$\Delta\omega = 0\,.$		

One can show that a form ω is harmonic if and only if it is both closed and co-closed[2].

$$\Delta\omega = 0 \quad \iff \quad \mathrm{d}\omega = 0 \text{ and } \mathrm{d}^\dagger \omega = 0\,. \tag{B.9}$$

It also follows from $\mathrm{d}^2 = 0 = (\mathrm{d}^\dagger)^2$ that any (co-)exact form is also (co-)closed. The converse statement is not true in general (at least globally[3]) and there may be closed forms $\mathrm{d}\omega = 0$ which are not exact $\omega \neq \mathrm{d}\eta$. Let us define the equivalence class

$$[\omega_p] = \{\tilde{\omega}_p : \tilde{\omega}_p = \omega_p + \mathrm{d}\eta_{p-1}\} \in H^p(\mathcal{M})\,, \tag{B.10}$$

by identifying all closed forms ω which differ by an exact form. These equivalence classes form the so-called de Rham cohomology group $H^r(\mathcal{M})$.

[2] See section 7.9 of [305].

[3] Poincaré's lemma determines that on any contractible region of $\mathcal{M}$, any closed form is also exact, which means that locally all closed forms are also exact.

On the other hand, the homology group is defined in a very similar way for manifolds $\mathcal{C}^p$ and submanifolds $\mathcal{C}^{p-1}$ through the boundary operator $\delta : \mathcal{C}^p \to \mathcal{C}^{p-1}$. We can again define

`(closed = cycles)`	$\delta\mathcal{C}^p = \{\}$
`(exact = boundary)`	$\mathcal{C}^p = \delta\mathcal{C}^{p-1}\,.$

Since the boundary operator is also nilpotent, $\delta^2 = 0$, all boundaries are cycles, but not all cycles are necessarily boundaries. Homology equivalence classes are defined by identifying all cycles (closed manifolds) which differ by a boundary (exact submanifold),

$$[\mathcal{C}^p] = \{\tilde{\mathcal{C}}^p : \mathcal{C}^p + \delta\mathcal{C}^{p+1}\} \in H_p\,. \tag{B.11}$$

A p-form ω_p and a p-dimensional submanifold $\mathcal{C}^p$ can be paired by integrating ω_p over $\mathcal{C}^p$,

$$(\mathcal{C}^p, \omega_p) = \int_{\mathcal{C}^p} \omega_p\,, \tag{B.12}$$

which relates the exterior derivative d and the boundary operator δ as dual operators through Stokes theorem. Precisely because of Stokes theorem, this inner product is well-defined in $H_p(\mathcal{M}) \times H^p(\mathcal{M})$, since the elements differing by exact components integrate to the same value. This leads to the important result of de Rham's theorem.

de Rham's theorem

If $\mathcal{M}$ is a compact manifold, then $H^p(\mathcal{M})$ and $H_p(\mathcal{M})$ are finite dimensional and the dual vector space of each other. They are therefore isomorphic, $H^p(\mathcal{M}) \cong H_{m-p}(\mathcal{M})$.

This means in particular that we can learn about cohomology classes $[\omega_p]$ by thinking about homology classes $[\mathcal{C}^p]$, i.e. from the topology of the compact manifold. Another important connection comes from Hodge's decomposition theorem.

Hodge decomposition theorem

Let $(\mathcal{M}, g)$ be a compact orientable Riemannian manifold without a boundary. Then any r-form $\omega \in \Omega^r(\mathcal{M})$ can be (globally) uniquely decomposed into

$$\omega = \mathrm{d}\eta + \mathrm{d}^\dagger\xi + \gamma\,, \quad \text{with } \gamma \text{ a harmonic form.} \tag{B.13}$$

This theorem is useful because it leads to the following connection between the space of harmonic forms on $\mathcal{M}$ and the cohomology group $H^p(\mathcal{M})$.

Hodge's theorem

On a compact orientable Riemannian manifold $(\mathcal{M}, g)$, $H^p(\mathcal{M})$ is isomorphic to the space of harmonic p-forms, $\text{Harm}^p(\mathcal{M})$.

Hence, the set of harmonic forms, cohomology groups and homology groups are all isomorphic

$$\text{Harm}^p(\mathcal{M}) \cong H^p(\mathcal{M}) \cong H_{m-p}(\mathcal{M}) \,. \tag{B.14}$$

The reason why this result is so useful is that it relates the number of independent solutions of a differential equation to the topology of the underlying manifold,

$$\Delta\omega_p = 0 \longleftrightarrow \dim\, H_{d-p}(\mathcal{M}) \equiv b_{d-p} \,, \tag{B.15}$$

in particular the number of cycles which cannot be deformed into one another. The numbers b_p are called Betti numbers and are defined as the dimensions[4] of the homology groups $H_p(\mathcal{M})$. They are related to a topological invariant known as the Euler characteristic by

$$\chi(\mathcal{M}) = \sum_{p=0}^{m} (-1)^p \; b_p(\mathcal{M}) \,. \tag{B.16}$$

For completeness, let us also mention Poincaré duality, which relates cohomology classes in $H^p(\mathcal{M})$ with cohomology classes in $H^{m-r}(\mathcal{M})$ through the inner product

$$\langle \omega, \eta \rangle = \int_{\mathcal{M}} \omega \wedge \eta \,. \tag{B.17}$$

This establishes the isomorphism $H^p(\mathcal{M}) \cong H^{m-p}(\mathcal{M})$ and, consequently, the identity $b_p = b_{m-p}$. Thus, the number of harmonic p-forms on $\mathcal{M}$ is b_p.

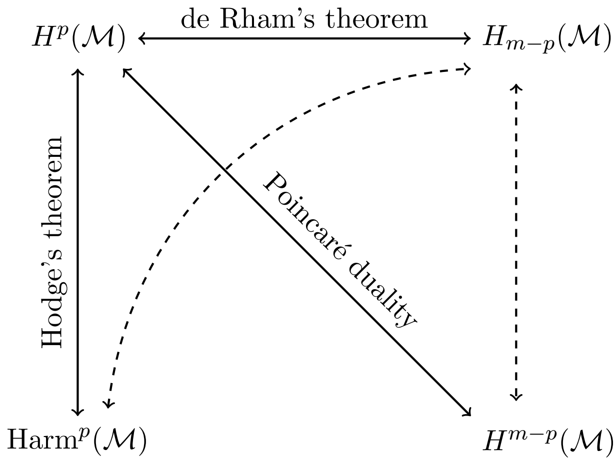
Combining the two inner-products (B.12) and (B.17), we can also define a duality between $(m-p)$-forms and p-cycles, which we also call Poincaré duality.

Poincaré duality (between forms and cycles)

An $(m-p)$-form ω_{m-p} is Poincaré dual to a p-cycle $\mathcal{C}^p$ if $\langle \eta_p, \omega_{m-p} \rangle = (\mathcal{C}^p, \eta_p)$, for any p-form η_p, i.e.

$$\int_{\mathcal{C}^p} \eta_p = \int_{\mathcal{M}} \eta_p \wedge \omega_{m-p} \,. \tag{B.18}$$

[4] More precisely, they are defined as the rank of the homology *groups*, but the distinction will not be important for our purposes.


$H^p(\mathcal{M})$
de Rham's theorem
$H_{m-p}(\mathcal{M})$
Hodge's theorem
Poincaré duality
$\mathrm{Harm}^p(\mathcal{M})$
$H^{m-p}(\mathcal{M})$

C. Calabi-Yau manifolds

Requiring that some supersymmetry is preserved in a compactification of the 10d theory puts strong constraints on the allowed compact spaces. In particular, preserving the minimum amount of supersymmetry fixes the holonomy group of the compact space to be SU(3), i.e. the space should be a Calabi-Yau manifold. In this appendix we briefly outline the characteristics of a Calabi-Yau manifold, including the structures that determine its holonomy and the counting of harmonic forms through Dolbeault cohomology classes.

Calabi-Yau manifolds

Calabi-Yau manifolds are Kähler manifolds (complex and symplectic) on which the first Chern class is trivial ($c_1 = 0$). These manifolds always admit a Ricci-flat metric.

Let us define each of these terms. One point worth emphasising is that each of them is adding structure to the manifold and therefore restricting the group of allowed transformations upon parallel transport — this is what reduces the holonomy group and in turn the number of supersymmetries that remain in the compactified solution.

A complex manifold is a ($2n$)-dimensional manifold that admits a complex structure, which is perhaps not extremely enlightening. A complex structure is a map I on the tangent space such that $I^2 = -1$, which is also integrable. The map is called a(n almost) complex structure because its n ($+i$) and n ($-i$) eigenvalues allow us to define holomorphic and anti-holomorphic vectors, such as

$$\frac{\partial}{\partial z} \equiv \frac{\partial}{\partial x^1} + i\frac{\partial}{\partial x^2}\,, \quad \frac{\partial}{\partial \bar{z}} \equiv \frac{\partial}{\partial x^1} - i\frac{\partial}{\partial x^2}\,, \quad ... \tag{C.1}$$

This effectively splits the tangent space into two pieces, a holomorphic space and an anti-holomorphic space. However, it does not guarantee that there are globally defined holomorphic coordinates z^i, unless the (almost) complex structure is also integrable so that holomorphic one-forms dz^i give holomorphic coordinates z^i.

Complex manifolds look locally like $\mathbb{C}^n$. They have holonomy (at most) GL($n,\mathbb{C}$) because parallel transfort must preserve its complex structure. Any r-form on a complex manifold can be decomposed into its holomorphic and anti-holomorphic components,

$$\omega_{p,q} = \frac{1}{p!q!}\omega_{i_1 \dots i_p \bar{\jmath}_1 \dots \bar{\jmath}_q}\, \mathrm{d}z^{i_1} \wedge \dots \wedge \mathrm{d}z^{i_p} \wedge \mathrm{d}\bar{z}^{\bar{\jmath}^1} \wedge \dots \wedge \mathrm{d}\bar{z}^{\bar{\jmath}_q}\,, \tag{C.2}$$

as can the exterior derivative $\mathrm{d} = \partial + \bar{\partial}$, with $\partial : (p,q) \to (p+1,q)$ and $\bar{\partial} : (p,q) \to (p,q+1)$.

On the other hand, a sympletic manifold is a $(2n)$-dimensional manifold on which a nowhere-vanishing 2-form J can be defined, such that

$$\mathrm{d}J = 0 \quad \text{and} \quad \underbrace{J \wedge \dots \wedge J}_{n} \neq 0\,. \tag{C.3}$$

As before, preserving this extra structure reduces the set of allowed transformation upon parallel transport, which is this case gives a holonomy group (at most) $\mathrm{Sp}(2n)$.

Kähler manifolds are manifolds for which both structures are compatible, i.e. complex and sympletic manifolds on which

$$J = J_{i\bar{\jmath}}\, \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^{\bar{\jmath}}\,. = i g_{i\bar{\jmath}}\, \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^{\bar{\jmath}}\,. \tag{C.4}$$

Having to preserve both structures at the same time, the holonomy group must be contained in the intersection of $\mathrm{GL}(n,\mathbb{C})$ and $\mathrm{Sp}(2n)$, which is $\mathrm{U}(3)$. This is not yet the $\mathrm{SU}(3)$ holonomy we are after, so we will require one more condition to get there. The metric on a Kähler manifold can be derived from a real scalar function K, the Kähler potential,

$$g_{i\bar{\jmath}} = \partial_i \bar{\partial}_{\bar{\jmath}} K\,. \tag{C.5}$$

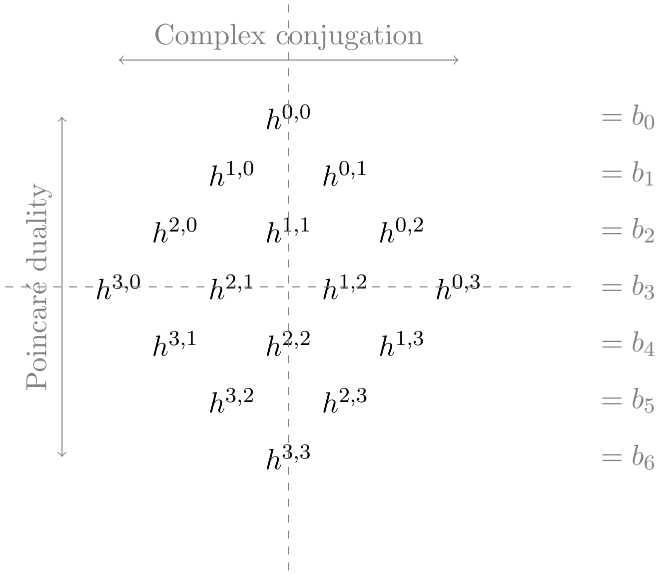
Not only can de Rham cohomology classes be defined in terms of d-closed and d-exact (p,q)-forms (cf. Appendix B), we can also define Dolbeault cohomology classes in terms of ∂-closed and ∂-exact (p,q)-forms. On Kähler manifolds,

$$H^{p,q}_{\mathrm{d}} = H^{p,q}_{\partial} = H^{p,q}_{\bar{\partial}}\,, \tag{C.6}$$

so that the cohomology groups coincide. The numbers $h^{p,q} \equiv \dim H^{p,q}_{*}$ satisfy the conditions

$$\sum_{k=0}^{p} h^{p-k,k} = b_p\,, \quad h^{p,q} = h^{q,p} = h^{n-p,n-q}\,. \tag{C.7}$$

The Hodge numbers are commonly presented in the form of a Hodge diamond, which highlights their several symmetries.



Finally, we can define the Ricci 2-form,

$$\mathcal{R} = iR_{i\bar{\jmath}}\, \mathrm{d}z^i \wedge \mathrm{d}\bar{z}^{\bar{\jmath}}, \tag{C.8}$$

which is closed, $\mathrm{d}\mathcal{R} = 0$, on a Kähler manifold with metric (C.5). Being a closed form, it must belong to some cohomology class — the class that includes $\mathcal{R}$ is called the first Chern class

$$c_1 = \frac{1}{2\pi}[\mathcal{R}]\,. \tag{C.9}$$

Calabi-Yau manifolds are precisely those Kähler manifolds on which the first Chern class is trivial, i.e. for which the Ricci 2-form belongs to the class of exact forms and is therefore a *globally* exact form itself. This means that on a Calabi-Yau manifold there exists a globally well-defined one-form $\mathcal{A}$ such that $\mathcal{R} = \mathrm{d}\mathcal{A}$. Note that this would not necessarily imply that the metric is Ricci-flat, $R_{mn} = 0$ — however, it was conjectured by Eugenio Calabi [307] and later proved by Shing-Tung Yau [308] that these manifolds do indeed always admit a Ricci-flat metric.[1]

On a $(2n)$-dimensional Calabi-Yau manifold there exists a unique covariantly constant holomorphic $(n, 0)$-form Ω, which must be preserved under parallel transport around closed loops. This restricts the holonomy further down to $\mathrm{SU}(n)$. The Hodge numbers are also further constrained [20],

$$\begin{aligned} h^{n,0} &= 1 \\ h^{p,0} &= 0\,, \quad \text{for} \quad 0 < p < n\,, \end{aligned}$$

[1] In fact, the theorem is more general and does not restrict to manifolds whose first Chern class is zero — these simply turn out to be the immediate application of interest to us. The construction of Calabi-Yau manifolds was also generalised by Yau, shortly after his original proof, to non-compact manifolds.

together with $h^{0,0} = 1$ for connected manifolds. From $h^{1,0} = 0$, it follows that there are no continuous isometries on Calabi-Yau manifolds. The Hodge triangle on a 6d Calabi-Yau takes the simple form

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

with only 2 free Hodge numbers, $h^{1,1}$ and $h^{2,1}$, characterising the topology of the manifold. They are related to the Euler characteristic as

$$\chi(\mathcal{M}) = 2(h^{1,1} - h^{2,1})\,. \tag{C.10}$$

Here is the upshot of all this: on a Calabi-Yau manifold we know a lot more about the forms that enter the decomposing of harmonic forms. For example, in the decomposition of B_2 (3.25) in section 3.1, we included harmonic 1-forms $\omega_1^i(y^m)$, but since a Calabi-Yau has $b_1 = h^{1,0}+h^{0,1} = 0$, there are no harmonic 1-forms in which we could expand. Hence, B_2 will not give rise to massless spacetime vectors.

The elements of $H^{1,1}$ (i.e. the representatives of the independent cohomology classes in $H^{1,1}$) are commonly denoted as ω_a, $a = 1, ..., h^{1,1}$, while the elements of $H^{2,2}$ are denoted as $\tilde{\omega}^a$, and these can be chosen such that

$$\int \omega_a \wedge \tilde{\omega}^b = \delta_a^b\,. \tag{C.11}$$

Likewise, α_K and β^K, $K = 0, ..., h^{2,1}$, are commonly used for the real 3-form elements of H^3, which can also be chosen such that

$$\int \alpha_K \wedge \beta^L = \delta_K^L\,, \tag{C.12}$$

while χ_k denotes the basis of complex elements of $H^{2,1}$ alone (with $\overline{\chi}^k$, Ω and $\overline{\Omega}$ completing the $2(h^{2,1} + 1)$ elements of H^3).

D. Details of Flux Compactifications

D.1 Kaluza-Klein (KK) scale(s)

Let us determine the Kaluza-Klein (KK) scale at which the towers of massive states associated with the compact dimensions appear. Considering the simple case of a 10d scalar field ρ,

$$\begin{aligned} S &= \int d^{10}x\sqrt{-G}\left\{-\frac{1}{2}G^{MN}(\partial_M\rho)(\partial_N\rho)\right\} && \text{(D.1)} \\ &= \int d^4x \int d^6y \cdot H^{-1}(y)e^{2\omega}\sqrt{-g_4}\cdot H^{3/2}(y)\ \mathcal{V}\sqrt{g_6} \\ &\qquad \left\{-\frac{1}{2}H^{1/2}(y)e^{-2\omega}g^{\mu\nu}(\partial_\mu\rho)(\partial_\nu\rho) - \frac{1}{2}H^{-1/2}(y)\mathcal{V}^{-1/3}g^{mn}(\partial_m\rho)(\partial_n\rho)\right\} && \text{(D.2)} \\ &= \int d^4x\sqrt{-g_4}\ \mathcal{V}\int d^6y\sqrt{g_6}\left\{-\frac{1}{2}H(y)g^{\mu\nu}(\partial_\mu\rho)(\partial_\nu\rho) - \frac{1}{2}\frac{e^{2\omega}}{\mathcal{V}^{1/3}}g^{mn}(\partial_m\rho)(\partial_n\rho)\right\} && \text{(D.3)} \\ &= \int d^4x\sqrt{-g_4}\ \mathcal{V}\int d^6y\sqrt{g_6}\left\{-\frac{1}{2}H(y)g^{\mu\nu}(\partial_\mu\rho)(\partial_\nu\rho) + \frac{1}{2}\frac{e^{2\omega}}{\mathcal{V}^{1/3}}H(y)(\Delta_6\rho)\cdot\rho\right\}, && \text{(D.4)} \end{aligned}$$

where in the last step we integrated the second term by parts and defined the internal space Laplacian operator

$$\Delta_6\rho \equiv \frac{H^{-1}(y)}{\sqrt{g_6}}\partial_m(\sqrt{g_6}\ g^{mn}\partial_n\rho)\,. \tag{D.5}$$

Decomposing the field $\rho(x,y)$ in a basis of eigenfunctions of Δ_6 (i.e. $\Delta_6\xi^k = -\lambda_k^2\xi^k$, with no sum over k),

$$\rho(x,y) = \sum_k \varrho_k(x)\xi^k(y)\,, \tag{D.6}$$

the action for the 10d scalar ρ becomes an action for infinitely many 4d scalars $\varrho_k(x)$,

$$\begin{aligned} S = \int d^4x\sqrt{-g_4}\ \mathcal{V}\int d^6y\sqrt{g_6}\sum_{k,l}\Bigg\{ &-\frac{1}{2}H(y)g^{\mu\nu}(\partial_\mu\varrho_k)(\partial_\nu\varrho_l)(\xi^k\xi^l) \\ &-\frac{1}{2}\frac{e^{2\omega}}{\mathcal{V}^{1/3}}H(y)\lambda_k^2\ \varrho_k\varrho_l\ (\xi^k\xi^l)\Bigg\} \end{aligned} \tag{D.7}$$

$$= \int d^4x\sqrt{-g_4}\;\mathcal{V}\sum_{k,l}\Bigg\{-\frac{1}{2}g^{\mu\nu}(\partial_\mu\varrho_k)(\partial_\nu\varrho_l)\cdot\int d^6y\sqrt{g_6}\cdot H(y)\;\xi^k\xi^l$$
$$-\frac{1}{2}\frac{e^{2\omega}}{\mathcal{V}^{1/3}}\lambda_k^2\varrho_k\varrho_l\cdot\int d^6y\sqrt{g_6}\cdot H(y)\;\xi^k\xi^l\Bigg\}. \tag{D.8}$$

Since the eigenmodes ξ^k satisfy the orthogonality condition

$$\int d^6y\sqrt{g_6}\cdot H(y)\;\xi^k\xi^l = c_1\delta^{kl}\,, \tag{D.9}$$

the KK modes decouple and the action reduces to

$$S = \sum_k\int d^4x\sqrt{-g_4}\;\Bigg\{-\frac{1}{2}g^{\mu\nu}(\partial_\mu\varrho_k)(\partial_\nu\varrho_k)(c_1\mathcal{V}) + \frac{1}{2}\frac{e^{2\omega}}{\mathcal{V}^{1/3}}\lambda_k^2\varrho_k^2(c_1\mathcal{V})\Bigg\} \tag{D.10}$$

$$= \sum_k\int d^4x\sqrt{-g_4}\;\Bigg\{-\frac{1}{2}g^{\mu\nu}(\partial_\mu\varrho_k^c)(\partial_\nu\varrho_k^c) + \frac{1}{2}\cdot\frac{\lambda_k^2}{\mathcal{V}^{1/3}}\frac{e^{2\omega_0}}{\mathcal{V}_{\rm w}}\cdot(\varrho_k^c)^2\Bigg\}, \tag{D.11}$$

where in the second line we canonically normalise the fields ϱ_k. Therefore, the mass of ϱ_k^c is

$$m_k = \frac{\lambda_k}{\mathcal{V}^{1/6}}\left(\frac{e^{2\omega_0}}{\mathcal{V}_{\rm w}}\right)^{1/2} \qquad\Longrightarrow\qquad m_{\rm KK} = \frac{\lambda_1}{\mathcal{V}^{1/6}}\left(\frac{e^{2\omega_0}}{\mathcal{V}_{\rm w}}\right)^{1/2}, \tag{D.12}$$

where we identify the KK scale, $m_{\rm KK}$, with the mass of the lightest mode m_1.

To determine λ_k (and the eigenfunctions[1]) one must solve the eigenvalue equation

$$\frac{1}{\sqrt{g_6}}\partial_m(\sqrt{g_6}\;g^{mn}\partial_n\xi^k) + H(y)\cdot\lambda_k^2\cdot\xi^k = 0\,, \tag{D.13}$$

together with appropriate boundary conditions. It is therefore not possible to give a fully generic expression for λ_k, and thus $m_{\rm KK}$, as it depends on the details of the compactification. If we consider the case of a torus with a single common radius as the prototypical example of an isotropic compact space[2] with characteristic scale l_s and constant warp factor[3] $H = 1$, the eigenvalue is $\lambda_1 = (2\pi)\cdot m_s$. Then the KK scale becomes

$$m_{\rm KK} = \left(\frac{e^{2\omega_0}}{\mathcal{V}_{\rm w}}\right)^{1/2}\cdot\frac{2\pi}{\mathcal{V}^{1/6}}m_s = \frac{2\pi}{\mathcal{V}^{1/6}}\cdot\frac{e^{\Phi_0}}{\sqrt{4\pi}\mathcal{V}_{\rm w}^{1/2}}M_{\rm Pl}\,. \tag{D.14}$$

[1] These are commonly referred to as the wavefunctions of the modes ϱ_k.

[2] More generically one could consider different scales in different directions, which would result in different KK scales.

[3] This is consistent with our normalisation for the coordinates y^m, such that $\int d^6y\sqrt{g_6} = l_s^6$ — it corresponds to an identification of the normalised coordinates $y^m \sim y^m + 1$ (with $ds_6^2 = l_s^2\;dy^m dy_m$), rather than $y^m \sim y^m + 2\pi$. Moreover, with a constant warp factor, $H(y) \equiv H_0$, the eigenfunctions respecting the (periodic) boundary conditions on the torus would be $\xi^{\vec{k}} \propto e^{2\pi i\;\vec{k}\cdot\vec{y}}$, with a vector of integers $\vec{k}$ labeling the modes, and hence $\Delta_6\xi^{\vec{k}} = -H_0^{-1}\cdot(2\pi)^2k^2\cdot m_s\cdot\xi^{\vec{k}}$, giving $\lambda_{\vec{k}} = H_0^{-1/2}\cdot(2\pi|\vec{k}|)\cdot m_s$. On the other hand, a constant warp factor can always be absorbed in the definition of the overall volume and so we can set it to $H = 1$.

We should note that generally there are two distinct volumes appearing in $m_{\rm KK}$ (D.12). One usually takes $\mathcal{V}_{\rm w} \approx \mathcal{V}$, i.e. one assumes that the unwarped region of the compact space dominates the volume integral, in which case we recover the well-known volume suppression $m_{\rm KK} \sim M_{\rm Pl}/\mathcal{V}^{2/3}$ (of course, this is not an issue for the constant warp factor case $H = 1$ where $\mathcal{V} = \mathcal{V}_{\rm w}$). For the convenient choice $e^{\Phi_0} = g_s$ and $e^{2\omega_0} = \langle \mathcal{V}_{\rm w} \rangle$, and approximating $\mathcal{V}_{\rm w} \approx \mathcal{V}$, we have

$$m_{\rm KK} = \frac{2\pi}{\mathcal{V}^{1/6}} m_s = \frac{2\pi g_s}{\sqrt{4\pi}\mathcal{V}^{2/3}} M_{\rm Pl}\,, \tag{D.15}$$

while for the common alternative choice $\Phi_0 = 0$, the factor of g_s will be absent.

When the warp factor is not trivial, its functional form will qualitatively change the eigenvalue equation (D.13) and the solutions λ_k will depend, in particular, on the balance between warped and unwarped regions of the compact space. When the warping dominates over the unwarped bulk, the eigenfunctions will localise near the region of maximum warping (typically the tip of a warped throat) and the KK scale will not only be warped down, but also depend on the characteristic scale of the tip geometry rather than that of the overall compact space,

$$m_{\rm KK} \sim H_{\rm tip}^{-1/4} \cdot \frac{2\pi}{\mathcal{V}_{\rm tip}^{1/6}} \cdot m_s\,. \tag{D.16}$$

In the classic example of a Klebanov-Strassler throat (cf. section 3.5), the characteristic scale is the size of the S^3 at the tip, $R_{S^3} \approx \sqrt{\alpha' g_s M}$, and hence this is what sets the KK scale for towers whose eigenfunctions localise near the tip. In section 5.2 we find this explicitly by solving the eigenvalue equation in the fully warped limit, which represents the regime where the warping dominates over the bulk (cf. (5.80)).

One also often finds in the literature another scale

$$m_{\rm KK}^{\rm w} \equiv H^{-1/4}(y_0)\; m_{\rm KK}^{\rm loc}\,, \tag{D.17}$$

where $m_{\rm KK}^{\rm loc}$ corresponds to a KK scale associated with modes localised on a subspace at some fixed $y = y_0$ (e.g. the tower of states associated with fields living on the world-volume of a brane wrapping an internal cycle of the compact space).

Finally, it is interesting to note that the ratio $m_{\rm KK}/m_s$ is manifestly independent of the choice for e^{Φ_0}, i.e. on the convention used in changing from string frame to Einstein frame, whereas the ratio $m_{\rm KK}/M_{\rm Pl}$ is manifestly independent of the choice for e^{ω_0}, i.e. on the convention used in going to 4d Einstein frame. After taking into account the dependence of the Einstein frame volume on the frame convention, the ratio $m_{\rm KK}/M_{\rm Pl}$ is also actually independent of the choice for e^{Φ_0}, as it must be. Approximating $\mathcal{V}_{\rm w} \approx \mathcal{V}$ and expressing the ratio in terms of the string-frame volume, the convention-dependent factors fall out,

$$\frac{m_{\rm KK}}{M_{\rm Pl}} = \frac{2\pi g_s}{\sqrt{4\pi}\mathcal{V}_S^{2/3}}\,. \tag{D.18}$$

D.2 Flux scalar potential

The scalar potential for the moduli fields and the dilaton comes from the terms R, $|G_3|^2$ and $|\tilde{F}_5|^2$ in the action (2.76), after dimensional reduction to 4d. The contributions from the R and $\tilde{F}_5$ terms can be shown to give (see section 5.3 of [170])

$$\frac{1}{2\kappa^2}\int\left(R\star 1-\frac{e^{2\Phi_0}}{4}\tilde{F}_5\wedge\star\tilde{F}_5\right)=\frac{e^{\Phi_0}}{2\kappa^2}\int d^4x\sqrt{-g_4}\cdot e^{4\omega(x)}\int H^{-1}\frac{G_3\wedge i\overline{G}_3}{2(\mathrm{Im}\tau)}\,, \tag{D.19}$$

which we can put together with the $G_3\wedge\star\overline{G}_3$ term to give in total

$$S_{\mathrm{IIB}}^{\mathrm{E}}\supset\frac{e^{\Phi_0}}{2\kappa^2}\int d^4x\sqrt{-g_4}\cdot e^{4\omega(x)}\int\frac{H^{-1}}{2(\mathrm{Im}\tau)}G_3\wedge(i\overline{G}_3+\star_6\overline{G}_3) \tag{D.20}$$

$$=\frac{e^{\Phi_0}}{2\kappa^2}\int d^4x\sqrt{-g_4}\cdot e^{4\omega(x)}\int\frac{H^{-1}}{(\mathrm{Im}\tau)}G_3^+\wedge\star_6\overline{G}_3^+\,, \tag{D.21}$$

with $G_3^+=\frac{1}{2}(G_3+i\star_6 G_3)$ such that $\star_6 G_3^+=-iG_3^+$ [170]. Using the metric (3.127) we can rewrite this action in terms of g_{mn},

$$S_{\mathrm{IIB}}^{\mathrm{E}}\supset\int d^4x\sqrt{-g_4}\left\{\frac{e^{\Phi_0}}{2\kappa^2}\,e^{4\omega(x)}\int H^{-1}\,\frac{G_3^+\wedge\star_{g6}\overline{G}_3^+}{(\mathrm{Im}\tau)}\right\}\equiv-\int d^4x\sqrt{-g_4}\,V\,, \tag{D.22}$$

which defines the 4d scalar potential V as

$$V=-i\,\frac{e^{\Phi_0}}{2\kappa^2}\,e^{4\omega(x)}\int H\,\frac{(H^{-1}G_3^+)\wedge(H^{-1}\overline{G}_3^+)}{(\mathrm{Im}\tau)}\,. \tag{D.23}$$

It is now possible to rewrite this potential in an $\mathcal{N}=1$ supergravity form, by defining [71]

$$W=\frac{1}{a}\int G_3\wedge\Omega\,, \tag{D.24}$$

where a is a normalisation constant to be determined below, and using that

$$\int H\,(H^{-1}G_3^+)\wedge(H^{-1}\overline{G}_3^+)=\frac{a^2}{\int H\,\Omega\wedge\overline{\Omega}}G^{\alpha\bar{\beta}}(D_\alpha W)(D_{\bar{\beta}}\overline{W}) \tag{D.25}$$

where α,β run over the complex structure moduli and the axio-dilaton. Using this, the scalar potential becomes

$$V=-i\,\frac{e^{\Phi_0}}{2\kappa^2}\,\frac{e^{4\omega(x)}}{(\mathrm{Im}\,\tau)}\frac{a^2}{\int H\,\Omega\wedge\overline{\Omega}}\left(G^{i\bar{\jmath}}(D_iW)(D_{\bar{\jmath}}\overline{W})-3|W|^2\right)$$

$$=\frac{e^{\Phi_0}}{2\kappa^2}\left(\frac{e^{2\omega_0}\cdot l_s^6}{V_{\mathrm{w}}}\right)^2\frac{1}{(\mathrm{Im}\,\tau)}\frac{a^2}{l_s^6}\frac{l_s^6}{i\int H\,\Omega\wedge\overline{\Omega}}\left(G^{i\bar{\jmath}}(D_iW)(D_{\bar{\jmath}}\overline{W})-3|W|^2\right)$$

$$= \frac{2\pi}{e^{\Phi_0}\cdot l_s^8}\Big(\frac{e^{2\Phi_0}M_{\rm Pl}^2\cdot l_s^2\cdot l_s^6}{4\pi V_{\rm w}}\Big)^2\frac{1}{({\rm Im}\,\tau)}\frac{a^2}{l_s^6}\frac{l_s^6}{i\int H\ \Omega\wedge\overline{\Omega}}\left(G^{i\bar{\jmath}}(D_iW)(D_{\bar{\jmath}}\overline{W})-3|W|^2\right)$$

$$= \frac{e^{3\Phi_0}}{4\pi\cdot l_s^4}M_{\rm Pl}^4\frac{a^2}{l_s^6}\cdot\left(\frac{l_s^6}{V_{\rm w}}\right)^2\frac{1}{2({\rm Im}\,\tau)}\frac{l_s^6}{i\int H\ \Omega\wedge\overline{\Omega}}\cdot M_{\rm Pl}^2\left(\frac{G^{i\bar{\jmath}}}{M_{\rm Pl}^2}(D_iW)(D_{\bar{\jmath}}\overline{W})-\frac{3}{M_{\rm Pl}^2}|W|^2\right)$$

$$= \left\{\frac{e^{3\Phi_0}}{4\pi\cdot l_s^{10}}M_{\rm Pl}^6\right\}e^{K/M_{\rm Pl}^2}\left(K^{i\bar{\jmath}}(D_iW)(D_{\bar{\jmath}}\overline{W})-\frac{3}{M_{\rm Pl}^2}|W|^2\right), \tag{D.26}$$

where now i, j run over complex structure moduli, Kähler moduli and the axio-dilaton, the Kähler potential K is given by

$$K/M_{\rm Pl}^2 = -2\log\mathcal{V}_{\rm w}-\log(-i(\tau-\bar{\tau}))-\log\left(\frac{i}{l_s^6}\int H\ \Omega\wedge\overline{\Omega}\right) \tag{D.27}$$

and $K^{i\bar{\jmath}}$ is the inverse field space metric that follows from $K_{i\bar{\jmath}} = \partial_i\partial_{\bar{\jmath}}K$. Note that the volume term in K includes not only the overall volume modulus $\mathcal{V}$, but also the other Kähler moduli. This scalar potential leads to the normalisation

$$W/M_{\rm Pl}^3 = \frac{e^{\frac{3}{2}\Phi_0}}{\sqrt{4\pi}\cdot l_s^5}\int G_3\wedge\Omega\,. \tag{D.28}$$

We can see that the normalisation constant, a, is convention-dependent through the choice of e^{Φ_0}.

An important scale in string flux compactifications is the gravitino mass, which for this super and Kähler potentials is given by

$$m_{3/2} = e^{\frac{K}{2M_{\rm Pl}^2}}\frac{|W|}{M_{\rm Pl}^2} = \frac{e^{\frac{1}{2}\langle\Phi\rangle}}{\mathcal{V}_{\rm w}\ ||\Omega||_{\rm w}}\frac{e^{\frac{3}{2}\Phi_0}W_0}{\sqrt{8\pi}}M_{\rm Pl}\,, \tag{D.29}$$

where $e^{\frac{1}{2}\langle\Phi\rangle}$ comes from $\langle{\rm Im}\,\tau\rangle$, $||\Omega||_{\rm w}^2\cdot l_s^6 = i\int H\ \Omega\wedge\overline{\Omega}$ and we define

$$W_0/M_{\rm Pl}^3 \equiv \left\langle\frac{1}{l_s^5}\int G_3\wedge\Omega\right\rangle. \tag{D.30}$$

A consistent 4d supergravity description requires that the gravitino remains in the theory, i.e. its mass is not above the EFT cutoff — typically $m_{\rm KK}$ — and therefore integrated out.[4] It follows from (D.29) and (D.14) that the important ratio (assuming the bulk dominates all the integrals, so that $\mathcal{V}_{\rm w}\approx\mathcal{V}$ and $||\Omega||_{\rm w}\approx||\Omega||$)

$$\frac{m_{3/2}}{m_{\rm KK}} = H_0^{1/2}\frac{e^{\frac{1}{2}(\langle\Phi\rangle+\Phi_0)}}{\mathcal{V}_E^{1/3}}\frac{W_0}{\sqrt{2}(2\pi)||\Omega||}\,, \tag{D.31}$$

where we highlight the fact that the volume being used is the Einstein frame volume, $\mathcal{V}_E$. Note

[4] It was shown that the ratio $m_{3/2}/m_{\rm KK}$ also serves as a control parameter for certain corrections to the scalar potential, e.g. from higher F-terms [180].

that this mass ratio, as written in terms of the Einstein frame volume, seems to depend on the convention used for the 10d change of frames, i.e. the choice of Φ_0. It is however convention-independent, as it must be, since the Einstein frame volume also depends on the choice of Φ_0. If we express the mass ratio in terms of the string frame volume instead, which corresponds to the volume perceived by the string itself, using (A.14) we find

$$\frac{m_{3/2}}{m_{\rm KK}} = H_0^{1/2} \, \frac{e^{\langle \Phi \rangle}}{\mathcal{V}_S^{1/3}} \frac{W_0}{\sqrt{2}(2\pi)||\Omega||} \,, \tag{D.32}$$

which is manifestly independent of conventions[5].

D.3 Conifold modulus metric on moduli space

Here we will review the computation of the metric component $G_{z\bar{z}}$ for the conifold deformation modulus, z, following closely [166] and making explicit the appearance of the volume modulus in the warping correction. Let us save the notation z to denote the dimensionless complex structure modulus that we will introduce below and refer to the coordinate defined through the periods (3.123) as S (this is also the notation used in [166], which will hopefully facilitate comparison).

The metric in the complex structure moduli space can be computed using

$$G_{\alpha\bar{\beta}} = \frac{i\int H \; \chi_\alpha \wedge \chi_{\bar{\beta}}}{i\int H \; \Omega \wedge \bar{\Omega}}, \tag{D.33}$$

with the warp factor defined in (3.128), which corresponds to the warping corrected Kähler potential [170]

$$\mathcal{K}_{\rm c.s.} = -\log\left(\frac{i}{l_s^6}\int H \; \Omega\wedge\bar{\Omega}\right). \tag{D.34}$$

We assume that all complex structure moduli are stabilised in the UV, i.e. in the bulk, except for the deformation modulus S that governs the Klebanov-Strassler geometry and lives in the highly-warped region. In particular, this means we can split the Kähler potential into two different contributions

$$\begin{aligned}\mathcal{K}_{\rm c.s.} &= -\log\left(\frac{i}{l_s^6}\int H \; \Omega\wedge\bar{\Omega}\right)\\ &= -\log\left(\frac{i}{l_s^6}\int_{\rm bulk} H \; \Omega\wedge\bar{\Omega} + \frac{i}{l_s^6}\int_{\rm conifold} H \; \Omega\wedge\bar{\Omega}\right)\end{aligned}$$

[5] Note that we give mass ratios for canonically normalised fields defined in the Einstein frame. Whilst these mass ratios must be invariant under change of conventions, a change in frame would come with field redefinitions, and new masses and couplings. In a setup in which all couplings, including the gravitational coupling, are constant (e.g. assuming that the dilaton and volume modulus are stabilised and integrated out), the change of frames becomes a change of convention from one Einstein frame to another Einstein frame, and the mass ratios would be invariant.

$$
\begin{aligned}
&\approx -\log\left(\frac{i}{l_s^6}\int_{\text{bulk}}\Omega\wedge\bar\Omega+\frac{i}{l_s^6}\int_{\text{conifold}}H\ \Omega\wedge\bar\Omega\right)\\
&=-\log\left(\frac{i}{l_s^6}\int_{\text{bulk}}\Omega\wedge\bar\Omega\left(1+\frac{i\int_{\text{conifold}}H\ \Omega\wedge\bar\Omega}{i\int_{\text{bulk}}\Omega\wedge\bar\Omega}\right)\right)\\
&=-\log\left(\frac{i}{l_s^6}\int_{\text{bulk}}\Omega\wedge\bar\Omega\right)-\log\left(1+\frac{i\int_{\text{conifold}}H\ \Omega\wedge\bar\Omega}{i\int_{\text{bulk}}\Omega\wedge\bar\Omega}\right)\\
&\approx \mathcal{K}^{\text{UV}}_{\text{c.s.}}+\frac{e^{\mathcal{K}^{\text{UV}}_{\text{c.s.}}}}{l_s^6}i\int_{\text{conifold}}h\ \Omega\wedge\bar\Omega\\
&=\mathcal{K}^{\text{UV}}_{\text{c.s.}}+\mathcal{K}(S,\bar S)\,,
\end{aligned}
\tag{D.35}
$$

where the first approximation follows from $H\sim 1$ in the bulk and the second assumes the contribution from the bulk is much bigger than the one from the conifold. Since the UV contribution only depends on moduli that were integrated out, it is simply a constant in our EFT,

$$
\mathcal{K}^{\text{UV}}_{\text{c.s.}}=-\log\left(\frac{i}{l_s^6}\int_{\text{bulk}}\Omega\wedge\bar\Omega\right)=-\log\left(\frac{||\Omega||^2V_6}{l_s^6}\right)\,,
\tag{D.36}
$$

and we should recall that $||\Omega||^2=\frac{1}{3!}\Omega_{mnp}\Omega^{mnp}=8$ is fixed by the normalisation of the globally defined covariant spinor which is a requirement for preserving $\mathcal{N}=1$ supersymmetry in 4d [309].

We can now find the conifold contribution to the metric following the computations in [166]

$$
G_{S\bar S}=\frac{e^{\mathcal{K}^{\text{UV}}_{\text{c.s.}}}}{l_s^6}i\int_{\text{conifold}}H\ \chi_S\wedge\chi_{\bar S}.
\tag{D.37}
$$

For the deformed conifold (3.103), the (2,1)-form χ_S is given by

$$
\chi_S=g^3\wedge g^4\wedge g^5+d[F(\tau)(g^1\wedge g^3+g^2\wedge g^4)]-id[f(\tau)(g^1\wedge g^2)+k(\tau)(g^3\wedge g^4)],
\tag{D.38}
$$

where the functions F,f,k were computed in [86]

$$
F(\tau)=\frac{\sinh\tau-\tau}{2\sinh\tau}\,,
\tag{D.39}
$$

$$
f(\tau)=\frac{\tau\coth\tau-1}{2\sinh\tau}(\cosh\tau-1)\,,
\tag{D.40}
$$

$$
k(\tau)=\frac{\tau\coth\tau-1}{2\sinh\tau}(\cosh\tau+1)\,.
\tag{D.41}
$$

It will be useful to look at the limits $\tau\to 0$ and $\tau\to\infty$ of these functions,

$$
F(\tau\to 0)=\frac{\tau^2}{12}\,,\qquad f(\tau\to 0)=\frac{\tau^3}{12}\,,\qquad k(\tau\to 0)=\frac{\tau}{3}\,,
\tag{D.42}
$$

$$
F(\tau\to\infty)=\frac{1}{2}-\tau e^{-\tau}\,,\qquad f(\tau\to\infty)=\frac{\tau}{2}\,,\qquad k(\tau\to\infty)=\frac{\tau}{2}\,.
\tag{D.43}
$$

Surprisingly the combination we need, $\chi_S \wedge \chi_{\bar{S}}$, is a total τ-derivative

$$\chi_S \wedge \chi_{\bar{S}} = -\frac{2i}{64\pi^4} d\tau \wedge \left(\prod_i g^i\right) \frac{d}{d\tau}[f + F(k-f)], \tag{D.44}$$

from which we find for $G_{S\bar{S}}$ (with all integrals performed over the conifold region)

$$\begin{aligned}
G_{S\bar{S}} &= \frac{e^{\mathcal{K}^{\rm UV}_{\rm c.s.}}}{l_s^6} i \int H \ \chi_S \wedge \chi_{\bar{S}} \\
&= \frac{i}{||\Omega||^2 V_6} \int H \ \chi_S \wedge \chi_{\bar{S}} \\
&= \frac{1}{||\Omega||^2 V_6} \frac{2}{64\pi^4} \int H \ d\tau \wedge \left(\prod_i g^i\right) \frac{d}{d\tau}[f + F(k-f)] \\
&= \frac{2}{64\pi^4 ||\Omega||^2 V_6} \left(\int \prod_i g^i\right) \int d\tau \ H \ \frac{d}{d\tau}[f + F(k-f)] \\
&\overset{\rm b.p.}{=} \frac{2}{\pi ||\Omega||^2 V_6} \left(\int d\tau \frac{d}{d\tau}[H \ \{f + F(k-f)\}] - \int d\tau \frac{dH}{d\tau} \cdot \{f + F(k-f)\}\right),
\end{aligned} \tag{D.45}$$

where in the last step we integrate by parts and use $\int \prod_i g^i = 64\pi^3$. We are left with an integral in τ, whose domain corresponds to the conifold region only, which is glued to the bulk at some finite value τ_Λ that constitutes the integral upper bound. In terms of the coordinate defined in the limit $\tau \to \infty$,

$$r^2 = \frac{3}{2^{5/3}} s^{2/3} e^{2\tau/3}, \tag{D.46}$$

this corresponds to a cutoff scale $\Lambda_{\rm UV}$,

$$\Lambda_{\rm UV}^2 \equiv r_{\rm UV}^2 = \frac{3}{2^{5/3}} s^{2/3} e^{2\tau_\Lambda/3} \implies \tau_\Lambda = \frac{3}{2}\log\frac{2^{5/3}}{3} + \log\frac{\Lambda_{\rm UV}^3}{s}, \tag{D.47}$$

We see that the first term in the integral is just a boundary term, so it suffices to evaluate $H\ \{f+F(k-f)\}$ at $\tau \to 0$ and $\tau \to \tau_\Lambda$ (where we can think of $\tau_\Lambda \gg 1$ and use the approximations for $\tau \to \infty$). It is useful to recall the warp factor for the deformed conifold, now written in terms of the complex structure $s = |S| = \epsilon^2$

$$e^{-4A_0(y)} = 2^{2/3} \frac{(\alpha' g_s M)^2}{s^{4/3}} I(\tau), \qquad I(\tau) \equiv \int_\tau^\infty dx \ \frac{x \coth(x) - 1}{\sinh^2 x} (\sinh(2x) - 2x)^{1/3}, \tag{D.48}$$

At $\tau \to 0$, $I(0) \approx 0.718$ and $f + F(k-f) = 0$, and at $\tau \to \tau_\Lambda$ (in the bulk), $H \approx 1$ and

$$f + F(k-f) \approx \frac{\tau_\Lambda}{2} = \frac{3}{4}\log\frac{2^{5/3}}{3} + \frac{1}{2}\log\frac{\Lambda_{\rm UV}^3}{s}. \tag{D.49}$$

As for the second term, all we need is the derivative of H

$$\frac{dH}{d\tau} = \frac{1}{\mathcal{V}^{2/3}}\frac{de^{-4A_0(y)}}{d\tau} = 2^{2/3}\frac{(\alpha' g_s M)^2}{s^{4/3}}\frac{1}{\mathcal{V}^{2/3}}\frac{dI(\tau)}{d\tau}$$
$$= -4\times 2^{2/3}\frac{(\alpha' g_s M)^2}{s^{4/3}}\frac{1}{\mathcal{V}^{2/3}}\frac{f+F(k-f)}{(\sinh(2\tau)-2\tau)^{2/3}}\,, \tag{D.50}$$

and so the integral (which is well approximated by taking $\tau_\Lambda \to \infty$) gives

$$-4\times 2^{2/3}\frac{(\alpha' g_s M)^2}{s^{4/3}}\frac{1}{\mathcal{V}^{2/3}}\int_0^{\tau_\Lambda}\frac{\{f+F(f-k)\}^2}{(\sinh(2\tau)-2\tau)^{2/3}} \approx 0.093\times(-4\times 2^{2/3})\frac{(\alpha' g_s M)^2}{s^{4/3}}\frac{1}{\mathcal{V}^{2/3}}. \tag{D.51}$$

Putting everything together, the metric $G_{S\bar{S}}$ becomes

$$G_{S\bar{S}} = \frac{1}{\pi||\Omega||^2 V_6}\left(\frac{3}{2}\log\frac{2^{5/3}}{3} + \log\frac{\Lambda^3_{\rm UV}}{s} + 0.093\times 8\times 2^{2/3}\times\frac{(\alpha' g_s M)^2}{s^{4/3}}\frac{1}{\mathcal{V}^{2/3}}\right). \tag{D.52}$$

Defining the constant $c' \approx 0.093\times 8\times 2^{2/3} \approx 1.18$ and for $s \ll \Lambda^3_{\rm UV}$ (which is equivalent to the assumption $\tau_\Lambda \gg 1$), we can neglect the first term

$$G_{S\bar{S}} = \frac{1}{\pi||\Omega||^2 V_6}\left(\log\frac{\Lambda^3_{\rm UV}}{s} + c'\frac{(\alpha' g_s M)^2}{s^{4/3}}\frac{1}{\mathcal{V}^{2/3}}\right). \tag{D.53}$$

This metric corresponds to the Kähler potential

$$\mathcal{K}(S,\bar{S}) = \frac{1}{\pi||\Omega||^2 V_6}\left(|S|^2\left(\log\frac{\Lambda^3_{\rm UV}}{|S|}+1\right) + \frac{9c'(\alpha' g_s M)^2}{\mathcal{V}^{2/3}}|S|^{2/3}\right). \tag{D.54}$$

We can now make a field redefinition, introducing a dimensionless deformation modulus $z = S/l_s^3$ in terms of which the Kähler potential becomes

$$\mathcal{K}(z,\bar{z}) = \frac{l_s^6}{\pi||\Omega||^2 V_6}\left(|z|^2\left(\log\frac{\Lambda_0^3}{|z|}+1\right) + \frac{9c'(g_s M)^2}{(2\pi)^4\mathcal{V}^{2/3}}|z|^{2/3}\right), \tag{D.55}$$

where now $\Lambda_0 = \Lambda_{\rm UV}/l_s$ is expressed in string units of l_s. With this redefinition we can say that "z is small", i.e. the dimensionless quantity $|S|/l_s^3 \ll 1$ or $|S|$ is small in string units.

Finally, we can make the choice $V_6 = l_s^6$. Keeping it allows us to keep track of V_6 and remember where this factor comes from but, being just a volume integral, we can always choose to normalize it in this way and let the volume modulus keep the overall volume dependence.

References

[1] G. Veneziano, "Construction of a crossing - symmetric, Regge behaved amplitude for linearly rising trajectories", *Nuovo Cim. A* **57** (1968) 190–197.

[2] D. Rickles, *A brief history of string theory: From dual models to M-theory*. The frontiers collection. Springer, Heidelberg, Germany, 2014.

[3] J. Conlon, *Why string theory?* CRC Pr., Boca Raton, 2016.

[4] M. A. Virasoro, "Alternative constructions of crossing-symmetric amplitudes with regge behavior", *Phys. Rev.* **177** (1969) 2309–2311.

[5] J. A. Shapiro, "Electrostatic analogue for the virasoro model", *Physics Letters B* **33** (1970) no. 5, 361–362.

[6] S. Fubini, D. Gordon, and G. Veneziano, "A general treatment of factorization in dual resonance models", *Phys. Lett. B* **29** (1969) 679–682.

[7] Y. Nambu, "Quark model and the factorization of the Veneziano amplitude", in *International Conference on Symmetries and Quark Models, Wayne State U., Detroit*. 1997.

[8] L. Susskind, "Structure of hadrons implied by duality", *Phys. Rev. D* **1** (1970) 1182–1186.

[9] H. Nielsen, "An almost physical interpretation of the integrand of the n-point veneziano integrand", in *the 15th International Conference on High Energy Physics (Kiev)*. 1970.

[10] T. Goto, "Relativistic quantum mechanics of one-dimensional mechanical continuum and subsidiary condition of dual resonance model", *Prog. Theor. Phys.* **46** (1971) 1560–1569.

[11] C. Lovelace, "Pomeron form factors and dual regge cuts", *Physics Letters B* **34** (1971) no. 6, 500–506.

[12] P. Ramond, "Dual Theory for Free Fermions", *Phys. Rev. D* **3** (1971) 2415–2418.

[13] A. Neveu and J. H. Schwarz, "Quark Model of Dual Pions", *Phys. Rev. D* **4** (1971) 1109–1111.

[14] A. Neveu and J. H. Schwarz, "Factorizable dual model of pions", *Nucl. Phys. B* **31** (1971) 86–112.

[15] J. Scherk and J. H. Schwarz, "Dual Models for Nonhadrons", *Nucl. Phys. B* **81** (1974) 118–144.

[16] Y. Nambu, "Strings, Monopoles and Gauge Fields", *Phys. Rev. D* **10** (1974) 4262.

[17] S. Deser and B. Zumino, "A Complete Action for the Spinning String", *Phys. Lett. B* **65** (1976) 369–373.

[18] L. Brink, P. Di Vecchia, and P. S. Howe, "A Locally Supersymmetric and Reparametrization Invariant Action for the Spinning String", *Phys. Lett. B* **65** (1976) 471–474.

[19] A. M. Polyakov, "Quantum Geometry of Bosonic Strings", *Phys. Lett. B* **103** (1981) 207–210.

[20] R. Blumenhagen, D. Lüst, and S. Theisen, *Basic concepts of string theory*. Theoretical and Mathematical Physics. Springer, Heidelberg, Germany, 2013.

[21] J. Polchinski, "Dirichlet Branes and Ramond-Ramond charges", *Phys. Rev. Lett.* **75** (1995) 4724–4727, `arXiv:hep-th/9510017`.

[22] D. Tong, "String Theory", `arXiv:0908.0333 [hep-th]`.

[23] T. Mohaupt, *A Short Introduction to String Theory*. Cambridge University Press, 3, 2022.

[24] B. Zwiebach, *A first course in string theory*. Cambridge University Press, 7, 2006.

[25] J. Polchinski, *String theory. Vol. 1: An introduction to the bosonic string*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 12, 2007.

[26] J. Polchinski, *String theory. Vol. 2: Superstring theory and beyond*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 12, 2007.

[27] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory Vol. 1: 25th Anniversary Edition*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 11, 2012.

[28] M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory Vol. 2: 25th Anniversary Edition*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 11, 2012.

[29] K. Becker, M. Becker, and J. Schwarz, *String theory and M-theory: A modern introduction*. Cambridge University Press, 12, 2006.

[30] H. Erbin and M. Médevielle, "Closed string theory without level-matching at the free level", *JHEP* **03** (2023) 091, arXiv:2209.05585 [hep-th].

[31] J.-L. Gervais and B. Sakita, "Field Theory Interpretation of Supergauges in Dual Models", *Nucl. Phys. B* **34** (1971) 632–639.

[32] J. Wess and B. Zumino, "Supergauge Transformations in Four-Dimensions", *Nucl. Phys. B* **70** (1974) 39–50.

[33] Y. A. Golfand and E. P. Likhtman, "Extension of the Algebra of Poincare Group Generators and Violation of p Invariance", *JETP Lett.* **13** (1971) 323–326.

[34] J. H. Schwarz, "Physical States and Pomeron Poles in the Dual Pion Model", *Nucl. Phys. B* **46** (1972) 61–74.

[35] P. Goddard and C. B. Thorn, "Compatibility of the Dual Pomeron with Unitarity and the Absence of Ghosts in the Dual Resonance Model", *Phys. Lett. B* **40** (1972) 235–238.

[36] R. C. Brower and K. A. Friedman, "Spectrum Generating Algebra and No Ghost Theorem for the Neveu-schwarz Model", *Phys. Rev. D* **7** (1973) 535–539.

[37] F. Gliozzi, J. Scherk, and D. I. Olive, "Supergravity and the Spinor Dual Model", *Phys. Lett. B* **65** (1976) 282–286.

[38] F. Gliozzi, J. Scherk, and D. I. Olive, "Supersymmetry, Supergravity Theories and the Dual Spinor Model", *Nucl. Phys. B* **122** (1977) 253–290.

[39] J. Polchinski, "Tasi lectures on D-branes", in *Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 96): Fields, Strings, and Duality*, pp. 293–356. 11, 1996. arXiv:hep-th/9611050.

[40] D. J. Gross, J. A. Harvey, E. J. Martinec, and R. Rohm, "The Heterotic String", *Phys. Rev. Lett.* **54** (1985) 502–505.

[41] M. B. Green and J. H. Schwarz, "Anomaly Cancellation in Supersymmetric D=10 Gauge Theory and Superstring Theory", *Phys. Lett. B* **149** (1984) 117–122.

[42] E. Witten, "String theory dynamics in various dimensions", *Nucl. Phys. B* **443** (1995) 85–126, arXiv:hep-th/9503124.

[43] L. Lusanna, "Extended Hadrons and Regge Slope", *Lett. Nuovo Cim.* **11** (1974) 213.

[44] M. B. Green and J. H. Schwarz, "Supersymmetrical Dual String Theory", *Nucl. Phys. B* **181** (1981) 502–530.

[45] M. B. Green and J. H. Schwarz, "Supersymmetrical Dual String Theory. 2. Vertices and Trees", *Nucl. Phys. B* **198** (1982) 252–268.

[46] M. B. Green and J. H. Schwarz, "Supersymmetrical Dual String Theory. 3. Loops and Renormalization", *Nucl. Phys. B* **198** (1982) 441–460.

[47] M. B. Green and J. H. Schwarz, "Supersymmetrical String Theories", *Phys. Lett. B* **109** (1982) 444–448.

[48] M. B. Green, J. H. Schwarz, and L. Brink, "Superfield Theory of Type II Superstrings", *Nucl. Phys. B* **219** (1983) 437–478.

[49] A. A. Tseytlin, "On dilaton dependence of type II superstring action", *Class. Quant. Grav.* **13** (1996) L81–L85, arXiv:hep-th/9601109.

[50] M. B. Green and J. H. Schwarz, "Extended Supergravity in Ten-Dimensions", *Phys. Lett. B* **122** (1983) 143–147.

[51] J. H. Schwarz and P. C. West, "Symmetries and Transformations of Chiral N=2 D=10 Supergravity", *Phys. Lett. B* **126** (1983) 301–304.

[52] J. H. Schwarz, "Covariant Field Equations of Chiral N=2 D=10 Supergravity", *Nucl. Phys. B* **226** (1983) 269.

[53] P. S. Howe and P. C. West, "The Complete N=2, D=10 Supergravity", *Nucl. Phys. B* **238** (1984) 181–220.

[54] A. Font, L. E. Ibanez, D. Lust, and F. Quevedo, "Strong - weak coupling duality and nonperturbative effects in string theory", *Phys. Lett. B* **249** (1990) 35–43.

[55] A. Sen, "Strong - weak coupling duality in four-dimensional string theory", *Int. J. Mod. Phys. A* **9** (1994) 3707–3750, arXiv:hep-th/9402002.

[56] T. Van Riet and G. Zoccarato, "Beginners lectures on flux compactifications and related Swampland topics", arXiv:2305.01722 [hep-th].

[57] L. Randall and R. Sundrum, "A Large mass hierarchy from a small extra dimension", *Phys. Rev. Lett.* **83** (1999) 3370–3373, arXiv:hep-ph/9905221.

[58] L. Randall and R. Sundrum, "An Alternative to compactification", *Phys. Rev. Lett.* **83** (1999) 4690–4693, arXiv:hep-th/9906064.

[59] B. Sathiapalan, "Duality in Statistical Mechanics and String Theory", *Phys. Rev. Lett.* **58** (1987) 1597.

[60] M. Grana, "Flux compactifications in string theory: A Comprehensive review", *Phys. Rept.* **423** (2006) 91–158, arXiv:hep-th/0509003.

[61] M. Graña, H. Triendl, M. Graña, and H. Triendl, *String theory compactifications*. Springer, 2017.

[62] P. Candelas, G. T. Horowitz, A. Strominger, and E. Witten, "Vacuum configurations for superstrings", *Nucl. Phys. B* **258** (1985) 46–74.

[63] T. Damour, "Questioning the equivalence principle", *Comptes Rendus de l'Académie des Sciences-Series IV-Physics* **2** (2001) no. 9, 1249–1256.

[64] M. Grana, R. Minasian, M. Petrini, and A. Tomasiello, "Supersymmetric backgrounds from generalized Calabi-Yau manifolds", *JHEP* **08** (2004) 046, arXiv:hep-th/0406137.

[65] T. W. Grimm and J. Louis, "The Effective action of N = 1 Calabi-Yau orientifolds", *Nucl. Phys. B* **699** (2004) 387–426, arXiv:hep-th/0403067.

[66] P. Candelas, X. C. De La Ossa, P. S. Green, and L. Parkes, "A Pair of Calabi-Yau manifolds as an exactly soluble superconformal theory", *Nucl. Phys. B* **359** (1991) 21–74.

[67] O. DeWolfe, A. Giryavets, S. Kachru, and W. Taylor, "Type IIA moduli stabilization", *JHEP* **07** (2005) 066, arXiv:hep-th/0505160.

[68] A. Strominger, "SPECIAL GEOMETRY", *Commun. Math. Phys.* **133** (1990) 163–180.

[69] J. Bagger and E. Witten, "Matter Couplings in N=2 Supergravity", *Nucl. Phys. B* **222** (1983) 1–10.

[70] M. Medevielle, T. Mohaupt, and G. Pope, "Type-II Calabi-Yau compactifications, T-duality and special geometry in general spacetime signature", *JHEP* **02** (2022) 048, arXiv:2111.09017 [hep-th].

[71] S. Gukov, C. Vafa, and E. Witten, "CFT's from Calabi-Yau four folds", *Nucl. Phys. B* **584** (2000) 69–108, arXiv:hep-th/9906070. [Erratum: Nucl.Phys.B 608, 477–478 (2001)].

[72] J. Wess and J. Bagger, *Supersymmetry and supergravity.* Princeton University Press, Princeton, NJ, USA, 1992.

[73] D. Z. Freedman and A. Van Proeyen, *Supergravity.* Cambridge Univ. Press, Cambridge, UK, 5, 2012.

[74] F. Quevedo, S. Krippendorf, and O. Schlotterer, "Cambridge lectures on supersymmetry and extra dimensions",.

[75] S. Kachru, R. Kallosh, A. D. Linde, and S. P. Trivedi, "De Sitter vacua in string theory", *Phys. Rev. D* **68** (2003) 046005, arXiv:hep-th/0301240.

[76] V. Balasubramanian, P. Berglund, J. P. Conlon, and F. Quevedo, "Systematics of moduli stabilisation in Calabi-Yau flux compactifications", *JHEP* **03** (2005) 007, arXiv:hep-th/0502058.

[77] J. P. Conlon, F. Quevedo, and K. Suruliz, "Large-volume flux compactifications: Moduli spectrum and D3/D7 soft supersymmetry breaking", *JHEP* **08** (2005) 007, arXiv:hep-th/0505076.

[78] P. S. Green and T. Hubsch, "Possible Phase Transitions among Calabi-Yau Compactifications", *Phys. Rev. Lett.* **61** (1988) 1163.

[79] P. S. Green and T. Hubsch, "Connecting Moduli Spaces of Calabi-yau Threefolds", *Commun. Math. Phys.* **119** (1988) 431–441.

[80] P. Candelas, P. S. Green, and T. Hubsch, "Rolling Among Calabi-Yau Vacua", *Nucl. Phys. B* **330** (1990) 49.

[81] P. Candelas and X. C. de la Ossa, "Comments on Conifolds", *Nucl. Phys. B* **342** (1990) 246–268.

[82] A. Strominger, "Massless black holes and conifolds in string theory", *Nucl. Phys. B* **451** (1995) 96–108, arXiv:hep-th/9504090.

[83] G. T. Horowitz and A. Strominger, "Black strings and P-branes", *Nucl. Phys. B* **360** (1991) 197–209.

[84] B. R. Greene, "String theory on Calabi-Yau manifolds", in *Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 96): Fields, Strings, and Duality*, pp. 543–726. 6, 1996. arXiv:hep-th/9702155.

[85] R. Minasian and D. Tsimpis, "On the geometry of nontrivially embedded branes", *Nucl. Phys. B* **572** (2000) 499–513, arXiv:hep-th/9911042.

[86] I. R. Klebanov and M. J. Strassler, "Supergravity and a confining gauge theory: Duality cascades and chi SB resolution of naked singularities", *JHEP* **08** (2000) 052, arXiv:hep-th/0007191.

[87] J. M. Maldacena, "The Large N limit of superconformal field theories and supergravity", *Adv. Theor. Math. Phys.* **2** (1998) 231–252, arXiv:hep-th/9711200.

[88] G. 't Hooft, "Dimensional reduction in quantum gravity", *Conf. Proc. C* **930308** (1993) 284–296, arXiv:gr-qc/9310026.

[89] L. Susskind, "The World as a hologram", *J. Math. Phys.* **36** (1995) 6377–6396, arXiv:hep-th/9409089.

[90] I. R. Klebanov and E. Witten, "Superconformal field theory on three-branes at a Calabi-Yau singularity", *Nucl. Phys. B* **536** (1998) 199–218, arXiv:hep-th/9807080.

[91] I. R. Klebanov and N. A. Nekrasov, "Gravity duals of fractional branes and logarithmic RG flow", *Nucl. Phys. B* **574** (2000) 263–274, arXiv:hep-th/9911096.

[92] S. S. Gubser and I. R. Klebanov, “Baryons and domain walls in an N=1 superconformal gauge theory”, *Phys. Rev. D* **58** (1998) 125025, arXiv:hep-th/9808075.

[93] I. R. Klebanov and A. A. Tseytlin, “Gravity duals of supersymmetric SU(N) x SU(N+M) gauge theories”, *Nucl. Phys. B* **578** (2000) 123–138, arXiv:hep-th/0002159.

[94] S. B. Giddings, S. Kachru, and J. Polchinski, “Hierarchies from fluxes in string compactifications”, *Phys. Rev. D* **66** (2002) 106006, arXiv:hep-th/0105097.

[95] H. L. Verlinde, “Holography and compactification”, *Nucl. Phys. B* **580** (2000) 264–274, arXiv:hep-th/9906182.

[96] W. D. Goldberger and M. B. Wise, “Modulus stabilization with bulk fields”, *Phys. Rev. Lett.* **83** (1999) 4922–4925, arXiv:hep-ph/9907447.

[97] W. D. Goldberger and M. B. Wise, “Phenomenology of a stabilized modulus”, *Phys. Lett. B* **475** (2000) 275–279, arXiv:hep-ph/9911457.

[98] J. M. Maldacena and C. Nunez, “Supergravity description of field theories on curved manifolds and a no go theorem”, *Int. J. Mod. Phys. A* **16** (2001) 822–855, arXiv:hep-th/0007018.

[99] C. Vafa, “Evidence for F theory”, *Nucl. Phys. B* **469** (1996) 403–418, arXiv:hep-th/9602022.

[100] I. Bena, J. Blåbäck, M. Graña, and S. Lüst, “The tadpole problem”, *JHEP* **11** (2021) 223, arXiv:2010.10519 [hep-th].

[101] E. Plauschinn, “The tadpole conjecture at large complex-structure”, *JHEP* **02** (2022) 206, arXiv:2109.00029 [hep-th].

[102] S. Lüst, “Large complex structure flux vacua of IIB and the Tadpole Conjecture”, arXiv:2109.05033 [hep-th].

[103] I. Bena, J. Blåbäck, M. Graña, and S. Lüst, “Algorithmically Solving the Tadpole Problem”, *Adv. Appl. Clifford Algebras* **32** (2022) no. 1, 7, arXiv:2103.03250 [hep-th].

[104] I. Bena, C. Brodie, and M. Graña, “D7 moduli stabilization: the tadpole menace”, *JHEP* **01** (2022) 138, arXiv:2112.00013 [hep-th].

[105] M. Graña, T. W. Grimm, D. van de Heisteeg, A. Herraez, and E. Plauschinn, “The tadpole conjecture in asymptotic limits”, *JHEP* **08** (2022) 237, arXiv:2204.05331 [hep-th].

[106] S. Lüst and M. Wiesner, “The Tadpole Conjecture in the Interior of Moduli Space”, arXiv:2211.05128 [hep-th].

[107] S. Lüst and L. Randall, "Effective Theory of Warped Compactifications and the Implications for KKLT", *Fortsch. Phys.* **70** (2022) no. 7-8, 2200103, arXiv:2206.04708 [hep-th].

[108] A. R. Frey, G. Torroba, B. Underwood, and M. R. Douglas, "The Universal Kahler Modulus in Warped Compactifications", *JHEP* **01** (2009) 036, arXiv:0810.5768 [hep-th].

[109] S. B. Giddings and A. Maharana, "Dynamics of warped compactifications and the shape of the warped landscape", *Phys. Rev. D* **73** (2006) 126003, arXiv:hep-th/0507158.

[110] L. Aparicio, F. Quevedo, and R. Valandro, "Moduli Stabilisation with Nilpotent Goldstino: Vacuum Structure and SUSY Breaking", *JHEP* **03** (2016) 036, arXiv:1511.08105 [hep-th].

[111] S. Weinberg, "The Cosmological Constant Problem", *Rev. Mod. Phys.* **61** (1989) 1–23.

[112] J. Martin, "Everything You Always Wanted To Know About The Cosmological Constant Problem (But Were Afraid To Ask)", *Comptes Rendus Physique* **13** (2012) 566–665, arXiv:1205.3365 [astro-ph.CO].

[113] A. Padilla, "Lectures on the Cosmological Constant Problem", arXiv:1502.05296 [hep-th].

[114] **Supernova Search Team** Collaboration, A. G. Riess *et al.*, "Observational evidence from supernovae for an accelerating universe and a cosmological constant", *Astron. J.* **116** (1998) 1009–1038, arXiv:astro-ph/9805201.

[115] **Supernova Cosmology Project** Collaboration, S. Perlmutter *et al.*, "Measurements of Ω and Λ from 42 high redshift supernovae", *Astrophys. J.* **517** (1999) 565–586, arXiv:astro-ph/9812133.

[116] **Planck** Collaboration, N. Aghanim *et al.*, "Planck 2018 results. VI. Cosmological parameters", *Astron. Astrophys.* **641** (2020) A6, arXiv:1807.06209 [astro-ph.CO]. [Erratum: Astron.Astrophys. 652, C4 (2021)].

[117] M. Dine and N. Seiberg, "Is the Superstring Weakly Coupled?", *Phys. Lett. B* **162** (1985) 299–302.

[118] L. Susskind, "The Anthropic landscape of string theory", arXiv:hep-th/0302219.

[119] U. H. Danielsson and T. Van Riet, "What if string theory has no de Sitter vacua?", *Int. J. Mod. Phys. D* **27** (2018) no. 12, 1830007, arXiv:1804.01120 [hep-th].

[120] M. Cicoli, S. De Alwis, A. Maharana, F. Muia, and F. Quevedo, "De Sitter vs Quintessence in String Theory", *Fortsch. Phys.* **67** (2019) no. 1-2, 1800079, arXiv:1808.08967 [hep-th].

[121] M. Cicoli, J. P. Conlon, A. Maharana, S. Parameswaran, F. Quevedo, and I. Zavala, "String Cosmology: from the Early Universe to Today", arXiv:2303.04819 [hep-th].

[122] T. D. Brennan, F. Carta, and C. Vafa, "The String Landscape, the Swampland, and the Missing Corner", *PoS* **TASI2017** (2017) 015, arXiv:1711.00864 [hep-th].

[123] E. Palti, "The Swampland: Introduction and Review", *Fortsch. Phys.* **67** (2019) no. 6, 1900037, arXiv:1903.06239 [hep-th].

[124] M. van Beest, J. Calderón-Infante, D. Mirfendereski, and I. Valenzuela, "Lectures on the Swampland Program in String Compactifications", *Phys. Rept.* **989** (2022) 1–50, arXiv:2102.01111 [hep-th].

[125] G. Obied, H. Ooguri, L. Spodyneiko, and C. Vafa, "De Sitter Space and the Swampland", arXiv:1806.08362 [hep-th].

[126] H. Ooguri, E. Palti, G. Shiu, and C. Vafa, "Distance and de Sitter Conjectures on the Swampland", *Phys. Lett. B* **788** (2019) 180–184, arXiv:1810.05506 [hep-th].

[127] C. Vafa, "The String landscape and the swampland", arXiv:hep-th/0509212.

[128] S. Tsujikawa, "Quintessence: A Review", *Class. Quant. Grav.* **30** (2013) 214003, arXiv:1304.1961 [gr-qc].

[129] B. Valeixo Bento, D. Chakraborty, S. L. Parameswaran, and I. Zavala, "Dark Energy in String Theory", *PoS* **CORFU2019** (2020) 123, arXiv:2005.10168 [hep-th].

[130] M. Cicoli, F. Cunillera, A. Padilla, and F. G. Pedro, "Quintessence and the Swampland: The Numerically Controlled Regime of Moduli Space", *Fortsch. Phys.* **70** (2022) no. 4, 2200008, arXiv:2112.10783 [hep-th].

[131] M. Cicoli, F. Cunillera, A. Padilla, and F. G. Pedro, "Quintessence and the Swampland: The Parametrically Controlled Regime of Moduli Space", *Fortsch. Phys.* **70** (2022) no. 4, 2200009, arXiv:2112.10779 [hep-th].

[132] M. Brinkmann, M. Cicoli, G. Dibitetto, and F. G. Pedro, "Stringy multifield quintessence and the Swampland", *JHEP* **11** (2022) 044, arXiv:2206.10649 [hep-th].

[133] R. Flauger, V. Gorbenko, A. Joyce, L. McAllister, G. Shiu, and E. Silverstein, "Snowmass White Paper: Cosmology at the Theory Frontier", in *Snowmass 2021*. 3, 2022. arXiv:2203.07629 [hep-th].

[134] I. Bena, E. Dudas, M. Graña, G. L. Monaco, and D. Toulikas, "Bare-Bones de Sitter", arXiv:2202.02327 [hep-th].

[135] I. Bena, E. Dudas, M. Graña, G. Lo Monaco, and D. Toulikas, "Anti D3-branes and gaugino condensation", arXiv:2211.14381 [hep-th].

[136] J. Polchinski, "Brane/antibrane dynamics and KKLT stability", arXiv:1509.05710 [hep-th].

[137] J. Moritz, A. Retolaza, and A. Westphal, "Toward de Sitter space from ten dimensions", *Phys. Rev. D* **97** (2018) no. 4, 046010, arXiv:1707.08678 [hep-th].

[138] F. F. Gautason, V. Van Hemelryck, and T. Van Riet, "The Tension between 10D Supergravity and dS Uplifts", *Fortsch. Phys.* **67** (2019) no. 1-2, 1800091, arXiv:1810.08518 [hep-th].

[139] F. F. Gautason, V. Van Hemelryck, T. Van Riet, and G. Venken, "A 10d view on the KKLT AdS vacuum and uplifting", *JHEP* **06** (2020) 074, arXiv:1902.01415 [hep-th].

[140] F. Carta, J. Moritz, and A. Westphal, "Gaugino condensation and small uplifts in KKLT", *JHEP* **08** (2019) 141, arXiv:1902.01412 [hep-th].

[141] Y. Hamada, A. Hebecker, G. Shiu, and P. Soler, "Understanding KKLT from a 10d perspective", *JHEP* **06** (2019) 019, arXiv:1902.01410 [hep-th].

[142] R. Blumenhagen, D. Kläwer, and L. Schlechter, "Swampland Variations on a Theme by KKLT", *JHEP* **05** (2019) 152, arXiv:1902.07724 [hep-th].

[143] R. Blumenhagen, M. Brinkmann, D. Klaewer, A. Makridou, and L. Schlechter, "KKLT and the Swampland Conjectures", *PoS* **CORFU2019** (2020) 158, arXiv:2004.09285 [hep-th].

[144] X. Gao, A. Hebecker, and D. Junghans, "Control issues of KKLT", *Fortsch. Phys.* **68** (2020) 2000089, arXiv:2009.03914 [hep-th].

[145] F. Carta and J. Moritz, "Resolving spacetime singularities in flux compactifications & KKLT", *JHEP* **08** (2021) 093, arXiv:2101.05281 [hep-th].

[146] S. Lüst, C. Vafa, M. Wiesner, and K. Xu, "Holography and the KKLT scenario", *JHEP* **10** (2022) 188, arXiv:2204.07171 [hep-th].

[147] K. Becker, M. Becker, M. Haack, and J. Louis, "Supersymmetry breaking and alpha-prime corrections to flux induced potentials", *JHEP* **06** (2002) 060, arXiv:hep-th/0204254.

[148] C. Crinò, F. Quevedo, and R. Valandro, "On de Sitter String Vacua from Anti-D3-Branes in the Large Volume Scenario", *JHEP* **03** (2021) 258, arXiv:2010.15903 [hep-th].

[149] C. P. Burgess, R. Kallosh, and F. Quevedo, "De Sitter string vacua from supersymmetric D terms", *JHEP* **10** (2003) 056, arXiv:hep-th/0309187.

[150] M. Cicoli, F. Quevedo, and R. Valandro, "De Sitter from T-branes", *JHEP* **03** (2016) 141, arXiv:1512.04558 [hep-th].

[151] M. Cicoli, A. Maharana, F. Quevedo, and C. P. Burgess, "De Sitter String Vacua from Dilaton-dependent Non-perturbative Effects", *JHEP* **06** (2012) 011, arXiv:1203.1750 [hep-th].

[152] A. Linde, "KKLT without AdS", *JHEP* **05** (2020) 076, arXiv:2002.01500 [hep-th].

[153] R. Kallosh and T. Wrase, "Emergence of Spontaneously Broken Supersymmetry on an Anti-D3-Brane in KKLT dS Vacua", *JHEP* **12** (2014) 117, arXiv:1411.1121 [hep-th].

[154] E. A. Bergshoeff, K. Dasgupta, R. Kallosh, A. Van Proeyen, and T. Wrase, "$\overline{\text{D3}}$ and dS", *JHEP* **05** (2015) 058, arXiv:1502.07627 [hep-th].

[155] R. Kallosh, F. Quevedo, and A. M. Uranga, "String Theory Realizations of the Nilpotent Goldstino", *JHEP* **12** (2015) 039, arXiv:1507.07556 [hep-th].

[156] I. n. García-Etxebarria, F. Quevedo, and R. Valandro, "Global String Embeddings for the Nilpotent Goldstino", *JHEP* **02** (2016) 148, arXiv:1512.06926 [hep-th].

[157] K. Dasgupta, M. Emelin, and E. McDonough, "Fermions on the antibrane: Higher order interactions and spontaneously broken supersymmetry", *Phys. Rev. D* **95** (2017) no. 2, 026003, arXiv:1601.03409 [hep-th].

[158] B. Vercnocke and T. Wrase, "Constrained superfields from an anti-D3-brane in KKLT", *JHEP* **08** (2016) 132, arXiv:1605.03961 [hep-th].

[159] R. Kallosh, B. Vercnocke, and T. Wrase, "String Theory Origin of Constrained Multiplets", *JHEP* **09** (2016) 063, arXiv:1606.09245 [hep-th].

[160] L. Aalsma, J. P. van der Schaar, and B. Vercnocke, "Constrained superfields on metastable anti-D3-branes", *JHEP* **05** (2017) 089, arXiv:1703.05771 [hep-th].

[161] M. P. Garcia del Moral, S. Parameswaran, N. Quiroz, and I. Zavala, "Anti-D3 branes and moduli in non-linear supergravity", *JHEP* **10** (2017) 185, arXiv:1707.07059 [hep-th].

[162] N. Cribiori, C. Roupec, T. Wrase, and Y. Yamada, "Supersymmetric anti-D3-brane action in the Kachru-Kallosh-Linde-Trivedi setup", *Phys. Rev. D* **100** (2019) no. 6, 066001, arXiv:1906.07727 [hep-th].

[163] S. Parameswaran and F. Tonioni, "Non-supersymmetric String Models from Anti-D3-/D7-branes in Strongly Warped Throats", *JHEP* **12** (2020) 174, arXiv:2007.11333 [hep-th].

[164] F. Tonioni, *Fundamental and Phenomenological Aspects of Anti-D-Brane Supersymmetry Breaking*. PhD thesis, U. Liverpool, 2022.

[165] A. Hebecker, S. Schreyer, and G. Venken, "Curvature corrections to KPV: do we need deep throats?", *JHEP* **10** (2022) 166, arXiv:2208.02826 [hep-th].

[166] M. R. Douglas, J. Shelton, and G. Torroba, "Warping and supersymmetry breaking", arXiv:0704.4001 [hep-th].

[167] M. R. Douglas and G. Torroba, "Kinetic terms in warped compactifications", *JHEP* **05** (2009) 013, arXiv:0805.3700 [hep-th].

[168] I. Bena, E. Dudas, M. Graña, and S. Lüst, "Uplifting Runaways", *Fortsch. Phys.* **67** (2019) no. 1-2, 1800100, arXiv:1809.06861 [hep-th].

[169] S. Kachru, J. Pearson, and H. L. Verlinde, "Brane / flux annihilation and the string dual of a nonsupersymmetric field theory", *JHEP* **06** (2002) 021, arXiv:hep-th/0112197.

[170] O. DeWolfe and S. B. Giddings, "Scales and hierarchies in warped compactifications and brane worlds", *Phys. Rev. D* **67** (2003) 066008, arXiv:hep-th/0208123.

[171] E. Dudas and S. Lüst, "An update on moduli stabilization with antibrane uplift", *JHEP* **03** (2021) 107, arXiv:1912.09948 [hep-th].

[172] B. V. Bento, D. Chakraborty, S. L. Parameswaran, and I. Zavala, "A new de Sitter solution with a weakly warped deformed conifold", *JHEP* **12** (2021) 124, arXiv:2105.03370 [hep-th].

[173] L. Randall, "The Boundaries of KKLT", *Fortsch. Phys.* **68** (2020) no. 3-4, 1900105, arXiv:1912.06693 [hep-th].

[174] B. Valeixo Bento, D. Chakraborty, S. Parameswaran, and I. Zavala, "A guide to frames, 2π's, scales and corrections in string compactifications", arXiv:2301.05178 [hep-th].

[175] C. Crinò, *On de Sitter string vacua in the Large Volume Scenario*. PhD thesis, Trieste U., 2022.

[176] D. Junghans, "LVS de Sitter vacua are probably in the swampland", *Nucl. Phys. B* **990** (2023) 116179, arXiv:2201.03572 [hep-th].

[177] X. Gao, A. Hebecker, S. Schreyer, and G. Venken, "Loops, local corrections and warping in the LVS and other type IIB models", *JHEP* **09** (2022) 091, arXiv:2204.06009 [hep-th].

[178] X. Gao, A. Hebecker, S. Schreyer, and G. Venken, "The LVS parametric tadpole constraint", *JHEP* **07** (2022) 056, arXiv:2202.04087 [hep-th].

[179] D. Junghans, "Topological constraints in the LARGE-volume scenario", *JHEP* **08** (2022) 226, arXiv:2205.02856 [hep-th].

[180] M. Cicoli, J. P. Conlon, A. Maharana, and F. Quevedo, "A Note on the Magnitude of the Flux Superpotential", *JHEP* **01** (2014) 027, arXiv:1310.6694 [hep-th].

[181] D. Baumann and L. McAllister, *Inflation and String Theory*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 5, 2015. arXiv:1404.2601 [hep-th].

[182] M. Berg, M. Haack, and B. Kors, "String loop corrections to Kahler potentials in orientifolds", *JHEP* **11** (2005) 030, arXiv:hep-th/0508043.

[183] M. Berg, M. Haack, J. U. Kang, and S. Sjörs, "Towards the one-loop Kähler metric of Calabi-Yau orientifolds", *JHEP* **12** (2014) 077, arXiv:1407.0027 [hep-th].

[184] M. Haack and J. U. Kang, "One-loop Einstein-Hilbert term in minimally supersymmetric type IIB orientifolds", *JHEP* **02** (2016) 160, arXiv:1511.03957 [hep-th].

[185] M. Haack and J. U. Kang, "Field redefinitions and Kähler potential in string theory at 1-loop", *JHEP* **08** (2018) 019, arXiv:1805.00817 [hep-th].

[186] I. Garcia-Etxebarria, H. Hayashi, R. Savelli, and G. Shiu, "On quantum corrected Kahler potentials in F-theory", *JHEP* **03** (2013) 005, arXiv:1212.4831 [hep-th].

[187] M. Cicoli, J. P. Conlon, and F. Quevedo, "Systematics of String Loop Corrections in Type IIB Calabi-Yau Flux Compactifications", *JHEP* **01** (2008) 052, arXiv:0708.1873 [hep-th].

[188] M. Berg, M. Haack, and E. Pajer, "Jumping Through Loops: On Soft Terms from Large Volume Compactifications", *JHEP* **09** (2007) 031, arXiv:0704.0737 [hep-th].

[189] G. von Gersdorff and A. Hebecker, "Kahler corrections for the volume modulus of flux compactifications", *Phys. Lett. B* **624** (2005) 270–274, arXiv:hep-th/0507131.

[190] M. Berg, M. Haack, and B. Kors, "On volume stabilization by quantum corrections", *Phys. Rev. Lett.* **96** (2006) 021601, arXiv:hep-th/0508171.

[191] J. P. Conlon and F. G. Pedro, "Moduli Redefinitions and Moduli Stabilisation", *JHEP* **06** (2010) 082, arXiv:1003.0388 [hep-th].

[192] R. Minasian, T. G. Pugh, and R. Savelli, "F-theory at order α'^3", *JHEP* **10** (2015) 050, arXiv:1506.06756 [hep-th].

[193] I. Antoniadis, Y. Chen, and G. K. Leontaris, "Logarithmic loop corrections, moduli stabilisation and de Sitter vacua in string theory", *JHEP* **01** (2020) 149, arXiv:1909.10525 [hep-th].

[194] H. Jockers and J. Louis, "The Effective action of D7-branes in N = 1 Calabi-Yau orientifolds", *Nucl. Phys. B* **705** (2005) 167–211, arXiv:hep-th/0409098.

[195] H. Jockers and J. Louis, "D-terms and F-terms from D7-brane fluxes", *Nucl. Phys. B* **718** (2005) 203–246, arXiv:hep-th/0502059.

[196] M. Haack, D. Krefl, D. Lust, A. Van Proeyen, and M. Zagermann, "Gaugino Condensates and D-terms from D7-branes", *JHEP* **01** (2007) 078, arXiv:hep-th/0609211.

[197] C. P. Bachas, P. Bain, and M. B. Green, "Curvature terms in D-brane actions and their M theory origin", *JHEP* **05** (1999) 011, arXiv:hep-th/9903210.

[198] M. Bershadsky, C. Vafa, and V. Sadov, "D-branes and topological field theories", *Nucl. Phys. B* **463** (1996) 420–434, arXiv:hep-th/9511222.

[199] M. B. Green, J. A. Harvey, and G. W. Moore, "I-brane inflow and anomalous couplings on d-branes", *Class. Quant. Grav.* **14** (1997) 47–52, arXiv:hep-th/9605033.

[200] Y.-K. E. Cheung and Z. Yin, "Anomalies, branes, and currents", *Nucl. Phys. B* **517** (1998) 69–91, arXiv:hep-th/9710206.

[201] R. Minasian and G. W. Moore, "K theory and Ramond-Ramond charge", *JHEP* **11** (1997) 002, arXiv:hep-th/9710230.

[202] S. P. de Alwis, "Constraints on LVS Compactifications of IIB String Theory", *JHEP* **05** (2012) 026, arXiv:1202.1546 [hep-th].

[203] D. Ciupke, J. Louis, and A. Westphal, "Higher-Derivative Supergravity and Moduli Stabilization", *JHEP* **10** (2015) 094, arXiv:1505.03092 [hep-th].

[204] G. Shiu, F. Tonioni, and H. V. Tran, "Accelerating universe at the end of time", arXiv:2303.03418 [hep-th].

[205] M. Berg, D. Marsh, L. McAllister, and E. Pajer, "Sequestering in String Compactifications", *JHEP* **06** (2011) 134, arXiv:1012.1858 [hep-th].

[206] L. Aparicio, M. Cicoli, S. Krippendorf, A. Maharana, F. Muia, and F. Quevedo, "Sequestered de Sitter String Scenarios: Soft-terms", *JHEP* **11** (2014) 071, arXiv:1409.1931 [hep-th].

[207] B. S. Acharya, A. Maharana, and F. Muia, "Hidden Sectors in String Theory: Kinetic Mixings, Fifth Forces and Quintessence", *JHEP* **03** (2019) 048, arXiv:1811.10633 [hep-th].

[208] M. Dine and N. Seiberg, "Nonrenormalization Theorems in Superstring Theory", *Phys. Rev. Lett.* **57** (1986) 2625.

[209] C. P. Burgess, C. Escoda, and F. Quevedo, "Nonrenormalization of flux superpotentials in string theory", *JHEP* **06** (2006) 044, arXiv:hep-th/0510213.

[210] A. Hebecker, "Lectures on Naturalness, String Landscape and Multiverse", arXiv:2008.10625 [hep-th].

[211] Y. Olguin-Trejo, S. L. Parameswaran, G. Tasinato, and I. Zavala, "Runaway Quintessence, Out of the Swampland", *JCAP* **01** (2019) 031, arXiv:1810.08634 [hep-th].

[212] E. J. Copeland, A. R. Liddle, and D. Wands, "Exponential potentials and cosmological scaling solutions", *Phys. Rev. D* **57** (1998) 4686–4690, arXiv:gr-qc/9711068.

[213] E. J. Copeland, M. Sami, and S. Tsujikawa, "Dynamics of dark energy", *Int. J. Mod. Phys. D* **15** (2006) 1753–1936, arXiv:hep-th/0603057.

[214] P. Agrawal, G. Obied, P. J. Steinhardt, and C. Vafa, "On the Cosmological Implications of the String Swampland", *Phys. Lett. B* **784** (2018) 271–276, arXiv:1806.09718 [hep-th].

[215] M. Cicoli, C. P. Burgess, and F. Quevedo, "Fibre Inflation: Observable Gravity Waves from IIB String Compactifications", *JCAP* **03** (2009) 013, arXiv:0808.0691 [hep-th].

[216] M. Cicoli, J. P. Conlon, and F. Quevedo, "General Analysis of LARGE Volume Scenarios with String Loop Moduli Stabilisation", *JHEP* **10** (2008) 105, arXiv:0805.1029 [hep-th].

[217] M. Demirtas, M. Kim, L. Mcallister, and J. Moritz, "Vacua with Small Flux Superpotential", *Phys. Rev. Lett.* **124** (2020) no. 21, 211603, arXiv:1912.10047 [hep-th].

[218] S. Hellerman, N. Kaloper, and L. Susskind, "String theory and quintessence", *JHEP* **06** (2001) 003, arXiv:hep-th/0104180.

[219] M. Cicoli, F. G. Pedro, and G. Tasinato, "Natural Quintessence in String Theory", *JCAP* **07** (2012) 044, arXiv:1203.6655 [hep-th].

[220] M. C. David Marsh, "The Swampland, Quintessence and the Vacuum Energy", *Phys. Lett. B* **789** (2019) 639–642, arXiv:1809.00726 [hep-th].

[221] A. Hebecker, T. Skrzypek, and M. Wittner, "The F-term Problem and other Challenges of Stringy Quintessence", *JHEP* **11** (2019) 134, arXiv:1909.08625 [hep-th].

[222] P. Marconnet and D. Tsimpis, "Universal accelerating cosmologies from 10d supergravity", *JHEP* **01** (2023) 033, arXiv:2210.10813 [hep-th].

[223] J. Calderón-Infante, I. Ruiz, and I. Valenzuela, "Asymptotic Accelerated Expansion in String Theory and the Swampland", arXiv:2209.11821 [hep-th].

[224] F. Apers, J. P. Conlon, M. Mosny, and F. Revello, "Kination, Meet Kasner: On The Asymptotic Cosmology of String Compactifications", arXiv:2212.10293 [hep-th].

[225] M. Montero, C. Vafa, and I. Valenzuela, "The dark dimension and the Swampland", *JHEP* **02** (2023) 022, arXiv:2205.12293 [hep-th].

[226] L. A. Anchordoqui, I. Antoniadis, and D. Lust, "Dark dimension, the swampland, and the dark matter fraction composed of primordial black holes", *Phys. Rev. D* **106** (2022) no. 8, 086001, arXiv:2206.07071 [hep-th].

[227] R. Blumenhagen, M. Brinkmann, and A. Makridou, "The dark dimension in a warped throat", *Phys. Lett. B* **838** (2023) 137699, arXiv:2208.01057 [hep-th].

[228] E. Gonzalo, M. Montero, G. Obied, and C. Vafa, "Dark Dimension Gravitons as Dark Matter", arXiv:2209.09249 [hep-ph].

[229] L. A. Anchordoqui, I. Antoniadis, and D. Lust, "Aspects of the dark dimension in cosmology", *Phys. Rev. D* **107** (2023) no. 8, 083530, arXiv:2212.08527 [hep-ph].

[230] L. A. Anchordoqui, I. Antoniadis, N. Cribiori, D. Lust, and M. Scalisi, "The Scale of Supersymmetry Breaking and the Dark Dimension", *JHEP* **05** (2023) 060, arXiv:2301.07719 [hep-th].

[231] S. Banerjee, U. Danielsson, G. Dibitetto, S. Giri, and M. Schillo, "Emergent de Sitter Cosmology from Decaying Anti–de Sitter Space", *Phys. Rev. Lett.* **121** (2018) no. 26, 261301, arXiv:1807.01570 [hep-th].

[232] S. Banerjee, U. Danielsson, G. Dibitetto, S. Giri, and M. Schillo, "de Sitter Cosmology on an expanding bubble", *JHEP* **10** (2019) 164, arXiv:1907.04268 [hep-th].

[233] S. Banerjee, U. Danielsson, and S. Giri, "Dark bubbles: decorating the wall", *JHEP* **04** (2020) 085, arXiv:2001.07433 [hep-th].

[234] S. Banerjee, U. Danielsson, and S. Giri, "Bubble needs strings", *JHEP* **21** (2020) 250, arXiv:2009.01597 [hep-th].

[235] S. Banerjee, U. Danielsson, and S. Giri, "Dark bubbles and black holes", *JHEP* **09** (2021) 158, arXiv:2102.02164 [hep-th].

[236] S. Banerjee, U. Danielsson, and S. Giri, "Curing with Hemlock: Escaping the swampland using instabilities from string theory", *Int. J. Mod. Phys. D* **30** (2021) no. 14, 2142029, arXiv:2103.17121 [hep-th].

[237] S. Banerjee, U. Danielsson, and S. Giri, "Dark Bubble: FAQs. Misconceptions, and why it is not Randall-Sundrum", arXiv:2212.14004 [hep-th].

[238] S. Brahma, K. Dasgupta, and R. Tatar, "Four-dimensional de Sitter space is a Glauber-Sudarshan state in string theory", *JHEP* **07** (2021) 114, arXiv:2007.00786 [hep-th].

[239] H. Bernardo, S. Brahma, K. Dasgupta, M.-M. Faruk, and R. Tatar, "de Sitter Space as a Glauber-Sudarshan State: II", *Fortsch. Phys.* **69** (2021) no. 11-12, 2100131, arXiv:2108.08365 [hep-th].

[240] J. Moritz, *On accelerated expansion in string theory*. PhD thesis, Hamburg U., 2019.

[241] **LIGO Scientific, Virgo** Collaboration, B. P. Abbott *et al.*, "Observation of Gravitational Waves from a Binary Black Hole Merger", *Phys. Rev. Lett.* **116** (2016) no. 6, 061102, arXiv:1602.03837 [gr-qc].

[242] **LIGO Scientific, Virgo** Collaboration, B. P. Abbott *et al.*, "GW151226: Observation of Gravitational Waves from a 22-Solar-Mass Binary Black Hole Coalescence", *Phys. Rev. Lett.* **116** (2016) no. 24, 241103, arXiv:1606.04855 [gr-qc].

[243] **LIGO Scientific, VIRGO** Collaboration, B. P. Abbott *et al.*, "GW170104: Observation of a 50-Solar-Mass Binary Black Hole Coalescence at Redshift 0.2", *Phys. Rev. Lett.* **118** (2017) no. 22, 221101, arXiv:1706.01812 [gr-qc]. [Erratum: Phys.Rev.Lett. 121, 129901 (2018)].

[244] **LIGO Scientific, VIRGO** Collaboration, F. Marion, "GW150914: Observation of gravitational waves from a binary black hole merger", *Nuovo Cim. C* **39** (2017) no. 4, 310.

[245] **LIGO Scientific, Virgo** Collaboration, B. P. Abbott *et al.*, "GW170814: A Three-Detector Observation of Gravitational Waves from a Binary Black Hole Coalescence", *Phys. Rev. Lett.* **119** (2017) no. 14, 141101, arXiv:1709.09660 [gr-qc].

[246] **LIGO Scientific, KAGRA, VIRGO** Collaboration, R. Abbott *et al.*, "Observation of Gravitational Waves from Two Neutron Star–Black Hole Coalescences", *Astrophys. J. Lett.* **915** (2021) no. 1, L5, arXiv:2106.15163 [astro-ph.HE].

[247] **LIGO Scientific, Virgo** Collaboration, B. P. Abbott *et al.*, "GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral", *Phys. Rev. Lett.* **119** (2017) no. 16, 161101, arXiv:1710.05832 [gr-qc].

[248] E. Berti *et al.*, "Testing General Relativity with Present and Future Astrophysical Observations", *Class. Quant. Grav.* **32** (2015) 243001, arXiv:1501.07274 [gr-qc].

[249] **LIGO Scientific, Virgo** Collaboration, B. P. Abbott *et al.*, "Tests of general relativity with GW150914", *Phys. Rev. Lett.* **116** (2016) no. 22, 221101, arXiv:1602.03841 [gr-qc]. [Erratum: Phys.Rev.Lett. 121, 129902 (2018)].

[250] **LIGO Scientific, Virgo** Collaboration, B. P. Abbott *et al.*, "Tests of General Relativity with GW170817", *Phys. Rev. Lett.* **123** (2019) no. 1, 011102, arXiv:1811.00364 [gr-qc].

[251] **LIGO Scientific, Virgo** Collaboration, B. P. Abbott *et al.*, "Tests of General Relativity with the Binary Black Hole Signals from the LIGO-Virgo Catalog GWTC-1", *Phys. Rev. D* **100** (2019) no. 10, 104036, arXiv:1903.04467 [gr-qc].

[252] **LIGO Scientific, Virgo** Collaboration, N. K. Johnson-McDaniel, "Summary of Tests of General Relativity with the Binary Black Hole Signals from the LIGO-Virgo Catalog GWTC-1", in *54th Rencontres de Moriond on Gravitation*. 5, 2019. arXiv:1905.05565 [gr-qc].

[253] **LIGO Scientific, Virgo** Collaboration, R. Abbott *et al.*, "Tests of general relativity with binary black holes from the second LIGO-Virgo gravitational-wave transient catalog", *Phys. Rev. D* **103** (2021) no. 12, 122002, arXiv:2010.14529 [gr-qc].

[254] **LIGO Scientific, VIRGO, KAGRA** Collaboration, R. Abbott *et al.*, "Tests of General Relativity with GWTC-3", arXiv:2112.06861 [gr-qc].

[255] M. Punturo *et al.*, "The Einstein Telescope: A third-generation gravitational wave observatory", *Class. Quant. Grav.* **27** (2010) 194002.

[256] **eLISA** Collaboration, P. A. Seoane *et al.*, "The Gravitational Universe", arXiv:1305.5720 [astro-ph.CO].

[257] **LISA** Collaboration, P. Amaro-Seoane *et al.*, "Laser Interferometer Space Antenna", arXiv:1702.00786 [astro-ph.IM].

[258] N. Aggarwal *et al.*, "Challenges and opportunities of gravitational-wave searches at MHz to GHz frequencies", *Living Rev. Rel.* **24** (2021) no. 1, 4, arXiv:2011.12414 [gr-qc].

[259] E. G. Adelberger, B. R. Heckel, and A. E. Nelson, "Tests of the gravitational inverse square law", *Ann. Rev. Nucl. Part. Sci.* **53** (2003) 77–121, arXiv:hep-ph/0307284.

[260] S. Dimopoulos, P. W. Graham, J. M. Hogan, and M. A. Kasevich, "Testing general relativity with atom interferometry", *Phys. Rev. Lett.* **98** (2007) 111102, arXiv:gr-qc/0610047.

[261] **Event Horizon Telescope** Collaboration, K. Akiyama *et al.*, "First M87 Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole", *Astrophys. J. Lett.* **875** (2019) L1, arXiv:1906.11238 [astro-ph.GA].

[262] D. Psaltis, "Testing General Relativity with the Event Horizon Telescope", *Gen. Rel. Grav.* **51** (2019) no. 10, 137, arXiv:1806.09740 [astro-ph.HE].

[263] J. Murata and S. Tanaka, "A review of short-range gravity experiments in the LHC era", *Class. Quant. Grav.* **32** (2015) no. 3, 033001, arXiv:1408.3588 [hep-ex].

[264] T. Baker, D. Psaltis, and C. Skordis, "Linking Tests of Gravity On All Scales: from the Strong-Field Regime to Cosmology", *Astrophys. J.* **802** (2015) 63, arXiv:1412.3455 [astro-ph.CO].

[265] S. S. Seahra, C. Clarkson, and R. Maartens, "Detecting extra dimensions with gravity wave spectroscopy: the black string brane-world", *Phys. Rev. Lett.* **94** (2005) 121302, arXiv:gr-qc/0408032.

[266] C. Clarkson and R. Maartens, "Gravity-wave detectors as probes of extra dimensions", *Gen. Rel. Grav.* **37** (2005) 1681–1687, arXiv:astro-ph/0505277.

[267] C. Clarkson and S. S. Seahra, "A gravitational wave window on extra dimensions", *Class. Quant. Grav.* **24** (2007) F33–F40, arXiv:astro-ph/0610470.

[268] S. Chakraborty, K. Chakravarti, S. Bose, and S. SenGupta, "Signatures of extra dimensions in gravitational waves from black hole quasinormal modes", *Phys. Rev. D* **97** (2018) no. 10, 104053, arXiv:1710.05188 [gr-qc].

[269] D. Andriot and G. Lucena Gómez, "Signatures of extra dimensions in gravitational waves", *JCAP* **06** (2017) 048, arXiv:1704.07392 [hep-th]. [Erratum: JCAP 05, E01 (2019)].

[270] D. Andriot and D. Tsimpis, "Gravitational waves in warped compactifications", *JHEP* **06** (2020) 100, arXiv:1911.01444 [hep-th].

[271] D. Andriot, P. Marconnet, and D. Tsimpis, "Warp factor and the gravitational wave spectrum", *JCAP* **07** (2021) 040, arXiv:2103.09240 [hep-th].

[272] Y. Du, S. Tahura, D. Vaman, and K. Yagi, "Probing Compactified Extra Dimensions with Gravitational Waves", *Phys. Rev. D* **103** (2021) no. 4, 044031, arXiv:2004.03051 [gr-qc].

[273] B. Valeixo Bento, D. Chakraborty, S. Parameswaran, and I. Zavala, "Gravity at the tip of the throat", *JHEP* **09** (2022) 208, arXiv:2204.02086 [hep-th].

[274] H. Firouzjahi and S. H. H. Tye, "The Shape of gravity in a warped deformed conifold", *JHEP* **01** (2006) 136, arXiv:hep-th/0512076.

[275] M. Berg, M. Haack, and W. Mueck, "Glueballs vs. Gluinoballs: Fluctuation Spectra in Non-AdS/Non-CFT", *Nucl. Phys. B* **789** (2008) 1–44, arXiv:hep-th/0612224.

[276] G. Shiu, B. Underwood, K. M. Zurek, and D. G. E. Walker, "Probing the geometry of warped string compactifications at the LHC", *Phys. Rev. Lett.* **100** (2008) 031601, arXiv:0705.4097 [hep-ph].

[277] P. Callin and F. Ravndal, "Higher order corrections to the Newtonian potential in the Randall-Sundrum model", *Phys. Rev. D* **70** (2004) 104009, arXiv:hep-ph/0403302.

[278] P. Callin, "Corrections to the Newtonian potential in the two-brane Randall-Sundrum model", arXiv:hep-ph/0407054.

[279] C. Bachas and J. Estes, "Spin-2 spectrum of defect theories", *JHEP* **06** (2011) 005, arXiv:1103.2800 [hep-th].

[280] H. Firouzjahi and S. H. H. Tye, "The Shape of gravity in a warped deformed conifold", *JHEP* **01** (2006) 136, arXiv:hep-th/0512076.

[281] J. F. Dufaux, L. Kofman, and M. Peloso, "Dangerous Angular KK/Glueball Relics in String Theory Cosmology", *Phys. Rev. D* **78** (2008) 023520, arXiv:0802.2958 [hep-th].

[282] L. Kofman and P. Yi, "Reheating the universe after string theory inflation", *Phys. Rev. D* **72** (2005) 106001, arXiv:hep-th/0507257.

[283] A. Berndsen, J. M. Cline, and H. Stoica, "Kaluza-Klein relics from warped reheating", *Phys. Rev. D* **77** (2008) 123522, arXiv:0710.1299 [hep-th].

[284] A. Higuchi, "Forbidden Mass Range for Spin-2 Field Theory in De Sitter Space-time", *Nucl. Phys. B* **282** (1987) 397–436.

[285] A. Higuchi, "Massive Symmetric Tensor Field in Space-times With a Positive Cosmological Constant", *Nucl. Phys. B* **325** (1989) 745–765.

[286] C. de Rham, J. T. Deskins, A. J. Tolley, and S.-Y. Zhou, "Graviton Mass Bounds", *Rev. Mod. Phys.* **89** (2017) no. 2, 025004, arXiv:1606.08462 [astro-ph.CO].

[287] C. de Rham, "Massive Gravity", *Living Rev. Rel.* **17** (2014) 7, arXiv:1401.4173 [hep-th].

[288] P. Callin and F. Ravndal, "Higher order corrections to the Newtonian potential in the Randall-Sundrum model", *Phys. Rev. D* **70** (2004) 104009, arXiv:hep-ph/0403302.

[289] G. F. Giudice, R. Rattazzi, and J. D. Wells, "Quantum gravity and extra dimensions at high-energy colliders", *Nucl. Phys. B* **544** (1999) 3–38, arXiv:hep-ph/9811291.

[290] L. J. Hall and D. Tucker-Smith, "Cosmological constraints on theories with large extra dimensions", *Phys. Rev. D* **60** (1999) 085008, arXiv:hep-ph/9904267.

[291] **Particle Data Group** Collaboration, R. L. Workman *et al.*, "Review of Particle Physics", *PTEP* **2022** (2022) 083C01.

[292] J. A. R. Cembranos, A. L. Maroto, and H. Villarrubia-Rojo, "Constraints on hidden gravitons from fifth-force experiments and stellar energy loss", *JHEP* **09** (2017) 104, arXiv:1706.07818 [hep-ph].

[293] D. Baumann, A. Dymarsky, I. R. Klebanov, J. M. Maldacena, L. P. McAllister, and A. Murugan, "On D3-brane Potentials in Compactifications with Fluxes and Wrapped D-branes", *JHEP* **11** (2006) 031, arXiv:hep-th/0607050.

[294] D. Baumann, A. Dymarsky, I. R. Klebanov, and L. McAllister, "Towards an Explicit Model of D-brane Inflation", *JCAP* **01** (2008) 024, arXiv:0706.0360 [hep-th].

[295] D. Baumann, A. Dymarsky, S. Kachru, I. R. Klebanov, and L. McAllister, "D3-brane Potentials from Fluxes in AdS/CFT", *JHEP* **06** (2010) 072, arXiv:1001.5028 [hep-th].

[296] N. Arkani-Hamed, S. Dimopoulos, and G. R. Dvali, "The Hierarchy problem and new dimensions at a millimeter", *Phys. Lett. B* **429** (1998) 263–272, arXiv:hep-ph/9803315.

[297] I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, and G. R. Dvali, "New dimensions at a millimeter to a Fermi and superstrings at a TeV", *Phys. Lett. B* **436** (1998) 257–263, arXiv:hep-ph/9804398.

[298] A. Ejlli, D. Ejlli, A. M. Cruise, G. Pisano, and H. Grote, "Upper limits on the amplitude of ultra-high-frequency gravitational waves from graviton to photon conversion", *Eur. Phys. J. C* **79** (2019) no. 12, 1032, arXiv:1908.00232 [gr-qc].

[299] A. Berlin, D. Blas, R. Tito D'Agnolo, S. A. R. Ellis, R. Harnik, Y. Kahn, and J. Schütte-Engel, "Detecting high-frequency gravitational waves with microwave cavities", *Phys. Rev. D* **105** (2022) no. 11, 116011, arXiv:2112.11465 [hep-ph].

[300] V. Domcke, C. Garcia-Cely, and N. L. Rodd, "Novel Search for High-Frequency Gravitational Waves with Low-Mass Axion Haloscopes", *Phys. Rev. Lett.* **129** (2022) no. 4, 041101, arXiv:2202.00695 [hep-ph].

[301] A. Ringwald and C. Tamarit, "Revealing the cosmic history with gravitational waves", *Phys. Rev. D* **106** (2022) no. 6, 063027, arXiv:2203.00621 [hep-ph].

[302] P. McGuirk, G. Shiu, and F. Ye, "Soft branes in supersymmetry-breaking backgrounds", *JHEP* **07** (2012) 188, arXiv:1206.0754 [hep-th].

[303] L. Martucci, J. Rosseel, D. Van den Bleeken, and A. Van Proeyen, "Dirac actions for D-branes on backgrounds with fluxes", *Class. Quant. Grav.* **22** (2005) 2745–2764, arXiv:hep-th/0504041.

[304] R. C. Myers, "Dielectric branes", *JHEP* **12** (1999) 022, arXiv:hep-th/9910053.

[305] M. Nakahara, *Geometry, topology and physics*. CRC Press, 2003.

[306] T. Frankel, *The geometry of physics: An introduction*. 1997.

[307] E. Calabi, *On Kähler Manifolds with Vanishing Canonical Class*, pp. 78–89. Princeton University Press, Princeton, 1957. https://doi.org/10.1515/9781400879915-006.

[308] S.-T. Yau, "Calabi's conjecture and some new results in algebraic geometry", *Proceedings of the National Academy of Sciences* **74** (1977) no. 5, 1798–1799.

[309] D. Lust and D. Tsimpis, "Supersymmetric AdS(4) compactifications of IIA supergravity", *JHEP* **02** (2005) 027, arXiv:hep-th/0412250.