Testing Granger non-causality in Expectiles*

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ABSTRACT

This paper aims to derive a consistent test of Granger causality at a given expectile. We also propose a sup-Wald test for jointly testing Granger causality at all expectiles that has the correct asymptotic size and power properties. Expectiles have the advantage of capturing similar information as quantiles, but they also have the merit of being much more straightforward to use than quantiles, since they are defined as least squares analogue of quantiles. Studying Granger causality in expectiles is practically simpler and allows us to examine the causality at all levels of the conditional distribution. Moreover, testing Granger causality at all expectiles provides a sufficient condition for testing Granger causality in distribution. A Monte Carlo simulation study reveals that our tests have good finite-sample size and power properties for a variety of data-generating processes and different sample sizes. Finally, we provide two empirical applications to illustrate the usefulness of the proposed tests.

Keywords: Granger causality in expectiles, Granger causality in distribution, expectile regression function, asymmetric loss function, sup-Wald test.

Journal of Economic Literature classification: C12, C22

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This paper aims to derive a consistent test of Granger causality at a given expectile. We also propose a sup-Wald test for jointly testing Granger causality at all expectiles that has the correct asymptotic size and power properties. Expectiles have the advantage of capturing similar information as quantiles, but they also have the merit of being much more straightforward to use than quantiles, since they are define as least squares analogue of quantiles. Studying Granger causality in expectiles is practically simpler and allows us to examine the causality at all levels of the conditional distribution. Moreover, testing Granger causality at all expectiles provides a sufficient condition for testing Granger causality in distribution. A Monte Carlo simulation study reveals that our tests have good finite-sample size and power properties for a variety of data-generating processes and different sample sizes. Finally, we provide two empirical applications to illustrate the usefulness of the proposed tests.

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1 Introduction

Causal relations between economic and/or financial time series are typically examined using the popular concept of Granger causality introduced by Wiener (1956) and Granger (1969). While the concept is naturally defined in terms of conditional distribution [see Granger (1980) and Granger and Newbold (1986)], early studies often focus on the conditional mean [see e.g. Granger (1969)] or the conditional variance [see e.g. Granger et al. (1986) and Cheung and Ng (1996)]. Causality in other aspects of the conditional distribution of the variables of interest (e.g. high-order conditional moments, quantiles/expectiles) has been less studied in practice, despite empirical evidence showing that for many economic and financial time series, e.g. returns and output, conditional quantiles/expectiles are predictable, but not the conditional mean; see Taylor, J. W. (2008), Lee and Yang (2012), Cenesizoglu and Timmermann (2008), and Chuang et al. (2009) among others. In this paper, we pay particular attention to the concept of expectiles introduced by Newey and Powell (1987), which provides a more complete picture of the conditional distribution of variable interest. In particular, we define Granger non-causality in expectiles and propose a parametric test for it. When all expectiles are considered jointly, the proposed test is equivalent to testing Granger non-causality in distribution. Rather than checking a necessary condition for Granger non-causality, our approach analyzes a continuous space of conditional expectile functions that fully characterizes non-causality in distribution.

The theory of Wiener-Granger causality has generated a considerable literature; for reviews, see Lütkepohl (2005), Boudjellaba, Dufour and Roy (1992, 1994), and Dufour and Taamouti (2010) and the references therein. However, to the best of our knowledge, no test has been proposed to test Granger non-causality in expectiles. Most of the existing tests focus on Granger causality in mean, thus they cannot be used in the presence of causality in expectiles. Several tests and measures have been proposed to detect Granger causality in distribution; see Su and White (2007, 2008), Taamouti et al. (2014), Bouezmarni and Taamouti (2014), Bouezmarni et al. (2012) and the references therein. However, these tests and measures are not informative about the level(s) (mean, variance, other high-order moments, expectiles) of distribution where the causality exists. To overcome this issue, Song and Taamouti (2020) consider measures and tests of Granger causality in quantiles; see also Jeong et al. (2012) and Troster (2018).

Another interesting way of looking at causality across different levels of conditional distribution, particularly the tails, is by using the concept of expectile. The latter dates back to Newey and Powell (1987), but recently re-gained a lot interest in both theoretical and applied work in econometrics and finance, see Taylor (2008), Sobotka and Kneib (2012), Holzmann and Klar (2016), Kratschmer and Zahle (2017), and Daouia et al. (2019) among others. Expectiles have the advantage of capturing similar information as quantiles, but they also have the merit of being much more straightforward to use than quantiles, since they are defined as least squares analogue of quantiles. For more details on the interpretation of expectiles and their relationship with quantiles see Philipps (2022). Studying Granger causality in expectiles is practically simpler and can help us examine the causality in distribution. Roughly speaking, expectiles can be seen as a mixture of mean and quantiles. On the one hand, they are determined by an asymmetrically weighted deviations criterion, where the L_1-norm of quantiles is replaced by the L_2-norm of expectation. On the other hand, they represent a generalization of the ordinary least square regression - 50% expectile regression is the classical mean regression. In addition, using expectiles instead of quantiles has the following advantages. First, expectiles lies in the computational expedience. Indeed, the estimation based on expectile regression reduces to weighted least squares fits since the optimality criterion is differentiable with respect to the regression effects, while linear programming routines have to be used in case of quantile regression. The second advantage is that inference on expectiles is much easier than inference on quantiles, and their estimation makes more efficient use of the available data since weighted least squares rely on the distance to data points, while empirical quantiles only utilize the information on whether an observation is below or above the predictor; see Newey and Powell (1987), Abdous and Remillard (1995), and Sobotka and Kneib (2012) among others. Finally, as a risk measure it has been shown that expectiles have the attractive property of coherence [see e.g. Bellini et al. (2014)], while quantiles suffer from the lack of subadditivity. Expectiles were shown to be the only coherent and elicitable risk measures in Ziegel (2016). Further discussion and application of expectiles as risk measures are given in Taylor (2008), Kuan et al. (2009), Emmer et al. (2015), Delbaen (2013), Bellini and Di Bernardino (2017), and Daouia et al. (2018).

In this paper, we propose a consistent parametric test of Granger causality in expectiles, which is based on expectile regressions. We first propose a Wald-type test for non-causality at a fixed τ-expectile, for $\tau \in (0, 1)$. Thereafter, we propose a sup-Wald test for testing Granger causality at all expectiles. The latter test checks for the joint statistical significance of the parameters of expectile regressions for all τ in $(0, 1)$, which is consistent against any deviation from non-causality in distribution, as opposed to the conventional tests of non-causality in moments. Testing Granger non-causality at all expectiles provides a sufficient condition for testing Granger non-causality in distribution. The proposed sup-Wald test statistic has the correct asymptotic size and power properties. A Monte Carlo simulation study reveals that our tests have good finite-sample size and power properties for a variety of data-generating processes and different sample sizes. We also compare the properties (size and power) of our *expectile-based* test with the existing *quantile-based* test for different values of τ, see e.g. Koenker and Machado (1999). The simulation results show that our test outperforms the test based on quantile regression even for weak degree of causality and for both small and large samples. Finally, we provide two empirical applications to illustrate the usefulness of our tests. In these applications, we re-examine the Granger causality from volume and exchange rates to stock returns using expectile.

The rest of this paper is organized as follows. In Section 2, we introduce the notations and define the concept of Granger non-causality in expectiles. In Section 3, we consider a Wald-type test and a sup-Wald test for testing Granger causality at a given expectile and all expectiles jointly, respectively. In Section 4, we use Monte Carlo simulations to investigate the finite sample size and power properties of our tests, and we compare their performance to those of quantile-based test. Section 5 contains two applications using economic and financial data and Section 6 concludes. The proofs of the theoretical results can be found in the Appendix.

2 Granger causality in expectiles

We consider two variables of interest Y and Z. Let $\{(Y_t, Z_t) : t \in \mathbb{Z}\}$ be the strictly stationary and ergodic time series process of Y and Z defined on some probability space (Ω, F, P). We denote $L^2 \equiv L^2(\Omega, F, P)$ a Hilbert space of real random variables with finite second moments, defined on the probability space (Ω, F, P). In the following, we are interested in testing Granger causality in expectiles between Y and Z.

We now need to define the information sets as this is important for causality analysis.

Hereafter, we consider a sequence I of "reference information sets" such that:

$$I = \{I(t) : t \in \mathbb{Z}\,,\, t > \omega\},\ \text{with } t < t' \Rightarrow I(t) \subseteq I(t') \text{ for all } t > \omega\,,$$

where $I(t)$ is an information set that represent a Hilbert subspace of L^2, $\omega \in \mathbb{Z} \cup \{-\infty\}$ represents a "starting point", and $\mathbb{Z}$ is the set of integers. The "starting point" ω is typically equal to a finite initial date (such as $\omega = -1$, 0 or 1) or to $-\infty$; in the latter case $I(t)$ is defined for all $t \in \mathbb{Z}$. The information set $I(t)$ could correspond to a (possibly empty) set, whose elements represent the information available at any point of time, such as time independent variables (e.g., the constant in a regression model) and deterministic processes (e.g., deterministic trends).

We denote $Y(\omega,\, t-1]$ the information set spanned by Y_s for $\omega < s \le t-1$, and similarly for $Z(\omega,\, t-1]$. That is, $Y(\omega,\, t-1]$ and $Z(\omega,\, t-1]$ represent the information contained in the history of the variables Y and Z up to time $t-1$, respectively. Furthermore, the information sets obtained by "adding" $Y(\omega,\, t-1]$ to $I(t-1)$ and $Z(\omega,\, t-1]$ to $I_Y(t-1)$ are defined as

$$I_Y(t-1) = I(t-1) + Y(\omega,\, t-1]\,,\ I_{YZ}(t-1) = I_Y(t-1) + Z(\omega,\, t-1].$$

We assume that Y and Z are Markovians of orders p and q, respectively. In this case, $Y(\omega,\, t-1] \equiv Y(t-p,\, t-1] \in \mathbb{R}^p$ and $Z(\omega,\, t-1] \equiv Z(t-q,\, t-1] \in \mathbb{R}^q$, and $I_{YZ}(t-1) \in \mathbb{R}^d$, for $d = p+q$. Furthermore, let $F_Y(y_t|I_{YZ}(t-1))$ and $F_Y(y_t|I_Y(t-1))$ be the conditional distribution functions of y_t given $I_{YZ}(t-1)$ and $I_Y(t-1)$, respectively, which we assume continuous for all $y_t \in \mathbb{R}$.

Before defining Granger causality in expectiles, we remind the reader how Granger non-causality in distribution and mean can be characterized in terms of restricted and unrestricted information sets. To avoid unnecessary repetitions, we treat only the causality from Z to Y. Following Granger (1969), causality from Z to Y one period ahead is defined as follows: Z causes Y if observations on Z up to time $t-1$ can help predict Y_t given the past of Y up to time $t-1$. Formally, Granger non-causality from Z to Y is characterized as follows:

$$F_Y(y_t|I_{YZ}(t-1)) = F_Y(y_t|I_Y(t-1)),\ \text{for all } y_t \ \in \mathbb{R}, \tag{1}$$

and will be referred to as Granger non-causality in *distribution.* Although the concept of Granger causality is naturally defined in terms of conditional distributions, applied work usually test for Granger non-causality in mean, which is only a necessary condition of Granger non-causality in distribution. In other words, the most used characterisation of Granger non-causality from Z to Y uses expectations:

$$E(y_t|I_{YZ}(t-1)) = E(y_t|I_Y(t-1)), \tag{2}$$

where $E(y_t|I_{YZ}(t-1))$ and $E(y_t|I_Y(t-1))$ denote the expectations of $F_Y(y_t|I_{YZ}(t-1))$ and $F_Y(y_t|I_Y(t-1))$, respectively. Conditional mean, however, only measures one aspect of the conditional distribution and helps detect the mean dependence only. Indeed, a causal effect that concerns the tail area of the distribution might significantly differ from a causal effect that takes place at the center of the distribution.

On the one hand, failing to reject non-causality in (2) is compatible with Granger non-causality in mean, but says nothing about Granger causality in other moments or other aspects (quantiles/expectiles) of conditional distribution. On the other hand, testing Granger non-causality in distribution using (1) will not be informative about the level(s) (mean, variance, other high-order moments, quantiles/expectiles) of distribution where the causality exists. To overcome this issue, we propose to use expectiles as a different way of looking at Granger causality between random variables. For more details on expectiles and their interpretations see Philipps (2022). Philipps (2022) discusses the relationship between expectiles and quantiles and argues that comparisons between expectiles and quantiles are inadequate to establish where expectiles reside on the horizon of statistics. Expectiles will allow us to determine the pattern of causality across the conditional distribution of the variable of interest. They also provide a sufficient condition for testing Granger non-causality in distribution as in (1), since expectiles are known to completely characterize the distribution.

Formally, let $\mu_\tau(.|I_S)$ denotes the τth-expectile of the conditional distribution $F_Y(.|I_S)$, for $I_S \equiv I_{YZ}(t-1)$, $I_Y(t-1)$, defined as:

$$\mu_\tau(y_t|I_S) = \underset{b(\tau)\in\mathbb{R}}{argmin}\ \mathbb{E}\left\{\psi_\tau(y_t - b(\tau))\left[|y_t - b(\tau)|^p - |y_t|^p\right]|\, I_S\right\}, \tag{3}$$

for $p = 2$ and $\psi_\tau(a) = |\tau - \mathbb{I}(a \leq 0)|$, where $\mathbb{I}(.)$ is an indicator function that takes the value one if $y_t - b(\tau) \leq 0$ and zero otherwise. Thus, an alternative way for testing Granger non-causality in distribution in (1) corresponds to testing

$$\mu_\tau(y_t|I_{YZ}(t-1)) = \mu_\tau(y_t|I_Y(t-1)),\ \text{for all}\, \tau \in T \subset (0,1), \tag{4}$$

where T is a subset of $(0, 1)$. If (4) holds, then we say that Z does not Granger cause Y in all expectiles, and therefore there is no causality in distribution from Z to Y. In the next section, we will also test (4) at a given τ, which corresponds to testing Granger non-causality at a given expectile τ. The need for working with expectiles is motivated by the limits of quantiles defined by Equation (3) for $p = 1$. Despite their strong intuitive appeal, quantiles are not always satisfactory. They can be criticized for being somewhat difficult to work with and compute as the corresponding loss function is not continuously differentiable, although modern efficient linear programming algorithms are available to help. Most importantly, quantiles are relatively inefficient against long-tailed distributions as they are based on absolute rather than squared loss minimization. However, as we will see in the next sections, it is much easier to work with expectiles both theoretically and computationally; see Efron (1991) and Sobotka et al. (2013), Schnabel and Eilers (2009); Sobotka and Kneib (2012) among others.

3 Testing non-causality in expectiles

In this section, we use expectile regressions to build statistical procedures for testing Granger non-causality in expectiles. To this end, we first consider parametric regression models to estimate the τth-expectile of $F_Y(.|I_t)$, for I_t some information set and $\tau \in (0, 1)$. We assume that the conditional function of the τth-expectile, say $\mu_\tau(y_t|.)$, is correctly specified and belongs to a family M of linear functions:

$$M = \{m(., \theta(\tau)) \,|\, \theta(.) : \tau \longrightarrow \theta(\tau) \in \Theta \subset \mathbb{R}^{d+1}, \text{ for } \tau \in \mathcal{T} \subset (0, 1)\},$$

where $\theta(\tau)$ is a vector of parameters that characterise the τth-expectile of Y.

Let $y_{t-1,p} = [y_{t-1}, ..., y_{t-p}]'$, $z_{t-1,q} = [z_{t-1}, ..., z_{t-q}]'$, $x_{t-1} = [1, y_{t-1,p}, z_{t-1,q}]$, and consider the following correctly specified τth-expectile regression:

$$\begin{aligned} y_t &= \mu_\tau(y_t|I_{YZ}(t-1)) + \epsilon_t^\tau = a(\tau) + y'_{t-1,p}\alpha(\tau) + z'_{t-1,q}\beta(\tau) + \epsilon_t^\tau \\ &= x'_{t-1}\theta(\tau) + \epsilon_t^\tau, \quad \text{for } \tau \in (0, 1)\,, \end{aligned} \tag{5}$$

where $\theta(\tau) = [a(\tau), \alpha(\tau)', \beta(\tau)']'$ is the $(d+1)$-dimensional parameter vector, with $d = p+q$. The conditional τth-expectile of the error term ϵ_t^τ is equal to zero as a result of correctly specified regression model. Using the parameters of regression (5), testing Granger non-causality from

Z to a given τth-expectile of Y is equivalent to testing the null hypothesis:

$$H_0^\tau : \beta(\tau) = 0, \text{ for a given } \tau \in (0,1)\,, \tag{6}$$

and if the null is rejected this would suggest that Z Granger causes Y at a given expectile τ.

In order to implement the test of causality in expectile, the vector of parameters $\theta(\tau)$ is estimated by minimizing the following asymmetrically weighted quadratic deviation function:

$$\hat{\theta}(\tau) = \underset{\theta(\tau)\in\mathbb{R}^{d+1}}{argmin} \sum_{t=1}^{T} \psi_\tau(\epsilon_t^\tau)\left(y_t - x_{t-1}'\theta(\tau)\right)^2, \tag{7}$$

where $\psi_\tau(\epsilon_t^\tau) = |\tau - \mathbb{I}(\epsilon_t^\tau \leq 0)|$, with $\mathbb{I}(.)$ is an indicator function that takes the value one if $\epsilon_t^\tau \leq 0$ and zero otherwise. The minimization problem in (7) is known as expectile regression and the estimator $\hat{\theta}(\tau)$ can also be obtained using a likelihood-based approach that uses Gaussian density with unequal weights placed on positive and negative disturbances. One of the advantages of expectile regression is that estimation basically reduces to (iteratively) weighted least squares fits since the optimality criterion is differentiable, while linear programming routines have to be used in the case of quantile regression. Sobotka et al. (2011) compare expectiles and quantiles using both theory and simulations and find that the estimation of expectiles may be more efficient that the estimation of quantiles for a number of distributions. They also show a smaller probability to obtain crossing expectile curves than crossing quantile curves. In addition, contrary to quantile regression estimator, expectile regression estimator has an explicit form, which is given by:

$$\hat{\theta}(\tau) = \left(\sum_{t=1}^{T} \psi_\tau(\hat{\epsilon}_t^\tau) x_{t-1} x_{t-1}'\right)^{-1} \left(\sum_{t=1}^{T} \psi_\tau(\hat{\epsilon}_t^\tau) x_{t-1} y_t\right), \tag{8}$$

where $\hat{\epsilon}_t^\tau = y_t - x_{t-1}'\hat{\theta}(\tau)$. Although the estimator in (8) is not of a closed form, it can be computed straightforwardly using iterated weighted least squares.

Newey and Powell (1987) have studied the properties of the estimation of expectile regression

$$y_t = x_{t-1}'\theta(\tau) + \epsilon_t^\tau$$

when the data $\{(x_{t-1}, y_t) \in \mathbb{R}^{d+1} \times \mathbb{R}\}$ are i.i.d. In particular, they showed that the least asymmetrically weighted squares estimate $\hat{\theta}(\tau)$ is consistent and asymptotically normally distributed

$$\hat{\theta}(\tau) \sim N(\theta(\tau), \Sigma^{-1} V \Sigma^{-1})$$

with

$$\Sigma = \mathbb{E}\left[\psi_\tau(\epsilon_t^\tau)x_{t-1}x_{t-1}'\right] \text{ and } V = \mathbb{E}\left[\psi_\tau^2(\epsilon_t^\tau)(\epsilon_t^\tau)^2 x_{t-1}x_{t-1}'\right].$$

To estimate the variance-covariance matrix of $\hat{\theta}(\tau)$, Newey and Powell (1987) proposed to estimate Σ and V as follows:

$$\hat{\Sigma} = \frac{1}{T}\sum_{t=1}^{T}\psi_\tau(\hat{\epsilon}_t^\tau)x_{t-1}x_{t-1}', \quad \hat{V} = \frac{1}{T}\sum_{t=1}^{T}x_{t-1}x_{t-1}'\psi_\tau(\hat{\epsilon}_t^\tau)^2 Var(\hat{\epsilon}_t^\tau),$$

where $\hat{\epsilon}_t^\tau = y_t - x_{t-1}'\hat{\theta}(\tau)$. They then extended this result to establish the asymptotic normality of an estimator $(\hat{\theta}(\tau_1), \ldots, \hat{\theta}(\tau_m))$ for a grid point $(\tau_1, \ldots, \tau_m)$, which was in turn used to build a test statistic for testing null hypotheses of the form $R\theta(\tau) = \beta(\tau)$, where R is a selection matrix.

However, the data in our expectile regression model (5) are not i.i.d, thus testing the null in (6) or more generally

$$H_0 : \beta(\tau) = 0, \ \forall \tau \in (0,1) \tag{9}$$

requires weak convergence of the process $\sqrt{T}(\hat{\theta}(\tau) - \theta(\tau))$. To overcome this issue, we establish the weak convergence of the process $\sqrt{T}(\hat{\theta}(\tau) - \theta(\tau))$ under time series data and we derive the asymptotic distribution of a test statistic that we use to test (9). The following assumptions are required to establish the asymptotic results:

Assumptions:

(A.1) $\{(Y_t, Z_t), t \geq 0\}$ is a strictly stationary process.

(A.2) ϵ_t^τ are random variables with mean zero and finite variance σ^2 and continuous distribution function.

(A.3) $E[\psi_{\tau_1}(\epsilon_t^{\tau_1})\psi_{\tau_2}(\epsilon_t^{\tau_2})\epsilon_t^{\tau_1}\epsilon_t^{\tau_2}x_{t-1}\,x_{t-1}']$ is nonsingular and $E[\psi_\tau(\epsilon_t^\tau)x_{t-1}x_{t-1}']$ is finite.

(A.4) $\mathbb{E}(||x_{t-1}||_2)^{2(1+\eta)}$, for some constant $\eta > 0$ is finite and where $||.||_2$ is L_2-norm.

All of the above assumptions are fairly common in the literature. Assumption (**A.2**) on homoscedasticity of the error term ϵ_t^τ will be relaxed in the Monte Carlo simulation study. Assumption (**A.4**) implies that $||x_{t-1}||_2 = o_p(T/\log(T))^{1/2}$, see Proposition 3.1 in Portnoy (1984). The following Theorem 1 establishes the asymptotic properties of the process $\sqrt{T}(\hat{\theta}(\tau) - \theta(\tau))$ when the above assumptions are satisfied [see the proof of Theorem 1 in the appendix].

Theorem 1 *Under Assumptions **(A.1)**-**(A.4)**, the empirical process* $\sqrt{T}(\hat{\theta}(\tau) - \theta(\tau))$ *converges to a Gaussian process* $\mathcal{Z}(\tau)$ *with covariance function*

$$Cov(\mathcal{Z}(\tau_1), \mathcal{Z}(\tau_2)) = \Psi_{\tau_1}^{-1} \Psi_{\tau_1,\tau_2} \Psi_{\tau_2}^{-1}, \tag{10}$$

where

$$\Psi_{\tau_i} = E[\psi_{\tau_i}(\epsilon_t^{\tau_i}) x_{t-1}\, x'_{t-1}], \text{ for } i = 1, 2, \text{ and } \Psi_{\tau_1,\tau_2} = E[\psi_{\tau_1}(\epsilon_t^{\tau_1}) \psi_{\tau_2}(\epsilon_t^{\tau_2}) \epsilon_t^{\tau_1} \epsilon_t^{\tau_2} x_{t-1}\, x'_{t-1}].$$

Theorem 1 will be needed to establish the asymptotic distribution of a Kolmogorov-Smirnov type-Wald test statistic that we are going to use when we test (9). However, notice that the terms Ψ_{τ_i}, for $i = 1, 2$, and Ψ_{τ_1,τ_2} in (10) are unknown, but they can be estimated as follows:

$$\widehat{\Psi}_{\tau_i} = \frac{1}{T} \sum_{t=1}^{T} \psi_{\tau_i}(\hat{\epsilon}_t^{\tau_i}) x_{t-1} x'_{t-1}, \text{ for } i = 1, 2, \text{ and } \widehat{\Psi}_{\tau_1,\tau_2} = \frac{1}{T} \sum_{t=1}^{T} x_{t-1} x'_{t-1} \psi_{\tau_1}(\hat{\epsilon}_t^{\tau_1}) \psi_{\tau_2}(\hat{\epsilon}_t^{\tau_2}) \hat{\epsilon}_t^{\tau_1} \hat{\epsilon}_t^{\tau_2}. \tag{11}$$

For i.i.d. data, Newey and Powell (1987) state that $\widehat{\Psi}_{\tau_1}^{-1} \widehat{\Psi}_{\tau_1,\tau_2} \widehat{\Psi}_{\tau_2}^{-1}$ converges in probability to $\Psi_{\tau_1}^{-1} \Psi_{\tau_1,\tau_2} \Psi_{\tau_2}^{-1}$. The proof of this convergence can be adapted to time series under Assumptions **(A.1)**-**(A.3)**.

Now, from Theorem 1 and by observing that $\sqrt{T}(\hat{\beta}(\tau) - \beta(\tau)) = \sqrt{T}(R\hat{\theta}(\tau) - R\theta(\tau))$, for $R = [0_{q,1+p}, I_{q,q}]$ a $q \times (d+1)$ matrix, we can deduce that the process $\sqrt{T}(\hat{\beta}(\tau) - \beta(\tau))$ converges to a Gaussian process $\mathcal{W}(\tau)$ with covariance function:

$$Cov(\mathcal{W}(\tau_1), \mathcal{W}(\tau_2)) = R \Psi_{\tau_1}^{-1} \Psi_{\tau_1,\tau_2} \Psi_{\tau_2}^{-1} R',$$

In Corollary 1 below, we derive the asymptotic normality of $\sqrt{T}(\hat{\beta}(\tau) - \beta(\tau))$ for fixed $\tau \in (0, 1)$ [see the proof of Corollary 1 in the appendix]. Moreover, the result in this corollary will be used to build a test for

$$H_0^\tau : \beta(\tau) = 0 \quad \text{versus} \quad H_1^\tau : \beta(\tau) \neq 0.$$

Corollary 1 *For a fixed* τ, *for* $\tau \in (0, 1)$, *and under Assumptions **(A.1)**-**(A.3)**, we have*

$$\sqrt{T} \left(R \widehat{\Psi}_{\tau}^{-1} \widehat{\Psi}_{\tau,\tau} \widehat{\Psi}_{\tau}^{-1} R' \right)^{-1/2} (\hat{\beta}(\tau) - \beta(\tau))$$

converges to $N(0, I_{q \times q})$, *where* $\widehat{\Psi}_{\tau}$ *and* $\widehat{\Psi}_{\tau,\tau}$ *are similarly defined by (11).*

To test the null hypothesis in (9), we first define the empirical process:

$$W_T(\tau) = T(\hat{\beta}(\tau) - \beta(\tau))' \left(R\widehat{\Psi}_{\tau}^{-1}\widehat{\Psi}_{\tau,\tau}\widehat{\Psi}_{\tau}^{-1}R' \right)^{-1} (\hat{\beta}(\tau) - \beta(\tau)).$$

Using the process $W_T(\tau)$, we consider a test statistic that is based on Kolmogorov-Smirnov type-Wald test, say KSW_T, which is defined as follows:

$$KSW_T = \sup_{\tau \in (0,1)} W_T(\tau).$$

Note that, an alternative way for testing the null hypothesis in (9) consists of constructing a Cramér–von Mises type test based on the process $\sqrt{T}\left(R\Psi_{\tau_1}^{-1}\Psi_{\tau_1,\tau_2}\Psi_{\tau_2}^{-1}R'\right)^{-1/2}(\hat{\beta}(\tau) - \beta(\tau))$. In this paper, however, we focus on Kolmogorov-Smirnov (KS) type test to allow for some comparison with the KS-type Wald test proposed by Koenker and Machado (1999) in the context of quantile regressions. The following proposition states the asymptotic distribution of the test statistic KSW_T for testing (9) [see the proof of Proposition 1 in the appendix].

Proposition 1 *Under Assumptions **(A.1)**-**(A.3)**, and the null hypothesis H_0, we have*

$$KSW_T \xrightarrow{d} \sup_{\tau} ||\mathcal{Z}(\tau)||_2^2,$$

where $\mathcal{Z}$ is a Gaussian process and $||.||_2$ is the Euclidean norm. For fixed τ, $||\mathcal{Z}(\tau)||_2^2$ is a centred Chi-square random variable with q degrees of freedom.

The result in Proposition 1 will be used to test Granger causality in distribution from Z to Y. We next use Monte Carlo simulations to assess the finite-sample properties (size and power) of the tests derived in Corollary 1 and Proposition 1 for a variety of data-generating processes and different sample sizes. We will also compare the properties of our expectiles-based test with the existing quantiles-based test, see e.g. Koenker and Machado (1999).

4 Monte Carlo simulations

We conduct a Monte Carlo simulation study to investigate the performance of the tests we proposed previously. Our primary interest is to assess the properties of the tests derived in Corollary 1 and Proposition 1. In particular, we examine their size and power properties using the data-generating processes (DGPs) presented in Table 1.

Table 1: DGPs used in the simulation study

DGPs	Variables	
	Y_t	Z_t
DGP1	ϵ_{1t}	ϵ_{2t}
DGP2	$Y_t = 0.5Y_{t-1} + \epsilon_{1t}$	$Z_t = 0.5Z_{t-1} + \epsilon_{2t}$
DGP3	$Y_t = (0.01 + 0.5Y_{t-1}^2)^{0.5}\epsilon_{1t}$	$Z_t = 0.5Z_{t-1} + \epsilon_{2t}$
DGP4	$Y_t = \sqrt{h_{1,t}}\epsilon_{1t}$, with $h_{1,t} = 0.01 + 0.9h_{1,t-1} + 0.05Y_{t-1}^2$	$Z_t = \sqrt{h_{2,t}}\epsilon_{2t}$, with $h_{2,t} = 0.01 + 0.9h_{2,t-1} + 0.05Z_{t-1}^2$
DGP5	$Y_t = 0.5Y_{t-1} + cZ_{t-1} + \epsilon_{1t}$	$Z_t = 0.5Z_{t-1} + \epsilon_{2t}$
DGP6	$Y_t = 0.5Y_{t-1} + cZ_{t-1} + \epsilon_{1t}$ $\epsilon_{1t} \sim N(0, t/T + 2)$	$Z_t = 0.5Z_{t-1} + \epsilon_{2t}$ $\epsilon_{2t} \sim N(0, t/T + 2)$
DGP7	$Y_t = 0.5Y_{t-1} + cZ_{t-1} + \epsilon_{1t}$ $\epsilon_{1t} \sim N(0, 1/t + 2)$	$Z_t = 0.5Z_{t-1} + \epsilon_{2t}$ $\epsilon_{2t} \sim N(0, 1/t + 2)$

Note: This table summarizes the DGPs that we consider in the simulation study to investigate the properties (size and power) of the tests of Granger causality in expectiles.

The first four DGPs [DGP1-DGP4] in Table 1 were used to evaluate the empirical size of the tests since the causality in these DGPs does not exist. DGP3 and DGP4 allow for heteroscedasticity, with the former model includes the ARCH effect of Engle (1982) and the latter incorporates the GARCH effect of Bollerslev (1986). The last three DGPs [DGP5-DGP7] of Table 1 allow for Granger causality in expectiles, thus they serve to illustrate the power of the tests. DGP5 exhibits Granger causality in the presence of homoscedastic errors, whereas DGP6 and DGP7 both alow for heteroskedastic errors.

These DGPs focus on testing Granger causality from Z to Y using the regression equation:

$$Y_t = \mu(\tau) + 0.5Y_{t-1} + c(\tau)Z_{t-1} + \epsilon_{1t},$$

where Z is generated using the processes in Table 1. We then test the null hypothesis H_0: $c(\tau) = 0$ against the alternative hypothesis H_1: $c(\tau) \neq 0$, for $\tau \in (0, 1)$. In our simulation and under the alternative hypothesis, $c(\tau)$ does not change with τ and takes one of the following values $c(\tau) = c = 0.1$, 0.5, and 0.9, which capture different degrees of causality from Z to Y. A higher (in absolute value) of $c(\tau)$ indicates a stronger causality from Z to Y.

Five sample sizes $T = 100$, 200, 500, 1000, and 2000 were considered. For each sample size T and DGP, 1000 (number of simulations) independent realisations of length T were obtained as follows: (i) we generate $T + 200$ i.i.d. noises ϵ_{1t} and ϵ_{2t} from $N(0, 1)$, except for DGP6 and DGP7 that are generated under normality with variances different from one, and (ii) each noise's sequence is used to generate Y_t and Z_t, for $t = 1 ... T+200$. The initial values of Y_t and Z_t were set to zero (resp. to one). To attenuate the impact of these initial values, the first 200 observations were discarded. Finally, for the implementation of the Kolmogorov-Smirnov type-Wald test for testing Granger non-causality in expectiles [similarly Granger non-causality in quantiles], we choose τs [either expectiles or quantiles] by setting $\tau_j = \frac{j}{100}$, for $j = 1, \ldots, n$, with $n = 99$. For example, the range of values for the whole set $\tau \in (0, 1)$ is $\{0.01, 0.02, 0.03....., 0.98, 0.99\}$. We then calculate the sup-Wald test as $KSW_T = \sup_{j=1,\ldots,n} W_T(\tau_j)$.

4.1 Size and power of expectile-based tests

The results for the empirical size of the test derived in Corollary 1 are presented in Table 2. From this, we see that for almost all DGPs and sample sizes under consideration, the empirical sizes of the test of Granger causality at a given expectile are very close to the nominal levels

Table 2: Empirical size of the proposed test of the Granger non-causality test in expectiles

	$\alpha = 5\%$				$\alpha = 10\%$			
DGPs	**DGP1**	**DGP2**	**DGP3**	**DGP4**	**DGP1**	**DGP2**	**DGP3**	**DGP4**
				$T = 100$				
$\tau = 0.10$	0.043	0.075	0.058	0.062	0.10	0.11	0.11	0.117
$\tau = 0.25$	0.054	0.061	0.053	0.052	0.11	0.114	0.109	0.107
$\tau = 0.50$	0.045	0.064	0.056	0.046	0.10	0.106	0.091	0.10
$\tau = 0.75$	0.059	0.065	0.067	0.053	0.09	0.099	0.112	0.115
$\tau = 0.90$	0.054	0.053	0.071	0.054	0.9	0.103	0.101	0.111
				$T = 200$				
$\tau = 0.10$	0.058	0.061	0.058	0.053	0.112	0.097	0.112	0.114
$\tau = 0.25$	0.046	0.065	0.045	0.059	0.095	0.097	0.106	0.108
$\tau = 0.50$	0.053	0.052	0.051	0.049	0.112	0.119	0.091	0.104
$\tau = 0.75$	0.055	0.059	0.065	0.048	0.111	0.094	0.100	0.101
$\tau = 0.90$	0.051	0.066	0.048	0.049	0.114	0.093	0.102	0.092
				$T = 500$				
$\tau = 0.10$	0.047	0.051	0.068	0.049	0.101	0.103	0.120	0.116
$\tau = 0.25$	0.051	0.054	0.055	0.056	0.091	0.095	0.104	0.115
$\tau = 0.50$	0.054	0.048	0.057	0.061	0.104	0.096	0.09	0.104
$\tau = 0.75$	0.055	0.052	0.064	0.053	0.104	0.082	0.10	0.104
$\tau = 0.90$	0.054	0.046	0.057	0.056	0.083	0.106	0.110	0.101
				$T = 1000$				
$\tau = 0.10$	0.055	0.05	0.057	0.06	0.113	0.097	0.110	0.095
$\tau = 0.25$	0.057	0.06	0.061	0.04	0.099	0.110	0.113	0.103
$\tau = 0.50$	0.051	0.057	0.048	0.052	0.104	0.089	0.102	0.105
$\tau = 0.75$	0.051	0.045	0.052	0.043	0.093	0.105	0.101	0.09
$\tau = 0.90$	0.045	0.056	0.065	0.046	0.102	0.120	0.110	0.120

Note: This table reports the empirical size of the test in Corollary 1 for testing the Granger non-causality at a given expectile from Z to Y at $\alpha = 5\%$ and 10% significance levels. The number of simulations is equal to 1000.

$\alpha = 5\%$ and $\alpha = 10\%$. The empirical size is well controlled in both small and large samples and in the presence and absence of heteroskedasticity. This result generally holds for extreme expectiles like for $\tau = 0.1$ and 0.9.

The results for the empirical power of the test in Corollary 1 are reported in Tables 3-5. The latter show that the proposed test has good power (very close to 1 or equal to 1) for fixed values of τ, for $\tau \in (0,1)$, and all DGPS and sample sizes under consideration, especially for moderate and high degrees of causality [$c = 0.5$ and $c = 0.9$]. When the degree of causality is low, say $c = 0.1$, then as one can expect, the power is low but it improves very significantly (equal to 1) once the sample size is increased to say $T = 2000$.

Finally, Tables 6 and 7 report the results of the empirical size and power of the Kolmogorov-Smirnov type-Wald test in Proposition 1, respectively. We provide the results for the whole set of τ, say (0.1), but also for the following subsets: $(0.10, 0.25)$, $(0.25, 0.75)$, and $(0.75, 0.90)$. These results show that the Kolmogorov-Smirnov test controls well its size and has a very good power for a variety of data-generating processes and different sample sizes. Its power reaches 1 even for small samples, e.g. $T = 100$. We remind the reader that, for $\tau \in (0.1)$, the Kolmogorov-Smirnov test can also be seen as a test of Granger causality in distribution.

4.2 Expectiles versus quantiles-based tests

We consider an additional simulation exercise to compare the size and power of the expectile-based test in Corollary 1 with those of quantile-based test [see e.g. Koenker and Machado (1999)] for different values of τ. We use the same simulation settings as in the previous subsection and for a fair comparison between the two tests, we consider the DGP5 of Table 1:

$$Y_t = \mu + 0.5Y_{t-1} + cZ_{t-1} + \epsilon_{1t},$$

where $Z_t = 0.5Z_{t-1} + \epsilon_{2t}$, with $\epsilon_{1t}, \epsilon_{2t} \sim N(0,1)$. Under DGP5, conditional expectiles and quantiles of Y_t can be expressed as follows:

$$\begin{aligned} \mu_\tau(Y_t|Y_{t-1}, Z_{t-1}) &= \mu + 0.5Y_{t-1} + cZ_{t-1} + \mu_\tau(\epsilon_{1t}|Y_{t-1}, Z_{t-1}) \\ q_\tau(Y_t|Y_{t-1}, Z_{t-1}) &= \mu + 0.5Y_{t-1} + cZ_{t-1} + q_\tau(\epsilon_{1t}|Y_{t-1}, Z_{t-1}), \end{aligned} \tag{12}$$

Table 3: Empirical power of the proposed test of the Granger non-causality test in expectiles

					DGP5					
Sample size	$T = 100$		$T = 200$		$T = 500$		$T = 1000$		$T = 2000$	
α	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%
					c=0.1					
$\tau = 0.10$	0.178	0.229	0.253	0.375	0.538	0.692	0.833	0.903	1	1
$\tau = 0.25$	0.194	0.263	0.31	0.448	0.670	0.766	0.927	0.962	1	1
$\tau = 0.50$	0.211	0.327	0.361	0.474	0.713	0.818	0.952	0.981	1	1
$\tau = 0.75$	0.202	0.290	0.345	0.440	0.695	0.775	0.938	0.961	1	1
$\tau = 0.90$	0.167	0.251	0.238	0.364	0.522	0.673	0.835	0.907	1	1
				c=0.5						
$\tau = 0.10$	0.975	0.996	1	1	1	1	1	1	1	1
$\tau = 0.25$	0.997	1	1	1	1	1	1	1	1	1
$\tau = 0.50$	1	1	1	1	1	1	1	1	1	1
$\tau = 0.75$	0.999	0.998	1	1	1	1	1	1	1	1
$\tau = 0.90$	0.977	0.987	1	1	1	1	1	1	1	1
				c=0.9						
$\tau = 0.10$	1	1	1	1	1	1	1	1	1	1
$\tau = 0.25$	1	1	1	1	1	1	1	1	1	1
$\tau = 0.50$	1	1	1	1	1	1	1	1	1	1
$\tau = 0.75$	1	1	1	1	1	1	1	1	1	1
$\tau = 0.90$	1	1	1	1	1	1	1	1	1	1

Note: This table reports the empirical power of the test in Corollary 1 for testing the Granger non-causality at a given expectile from Z to Y at $\alpha = 5\%$ and 10% significance levels. The number of simulations is equal to 1000.

Table 4: Empirical power of the proposed test of the Granger non-causality test in expectiles

DGP6										
Sample size	$T = 100$		$T = 200$		$T = 500$		$T = 1000$		$T = 2000$	
α	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%
c=0.1										
$\tau = 0.10$	0.152	0.260	0.277	0.345	0.539	0.661	0.829	0.883	1	1
$\tau = 0.25$	0.203	0.306	0.303	0.448	0.663	0.763	0.920	0.948	1	1
$\tau = 0.50$	0.201	0.284	0.354	0.478	0.691	0.802	0.952	0.963	1	1
$\tau = 0.75$	0.188	0.292	0.307	0.445	0.672	0.760	0.920	0.962	1	1
$\tau = 0.90$	0.147	0.257	0.253	0.341	0.529	0.634	0.838	0.885	1	1
c=0.5										
$\tau = 0.10$	0.968	0.989	0.998	1	1	1	1	1	1	1
$\tau = 0.25$	0.998	0.998	1	1	1	1	1	1	1	1
$\tau = 0.50$	0.999	1	1	1	1	1	1	1	1	1
$\tau = 0.75$	0.997	0.998	1	1	1	1	1	1	1	1
$\tau = 0.90$	0.965	0.991	1	1	1	1	1	1	1	1
c=0.9										
$\tau = 0.10$	1	1	1	1	1	1	1	1	1	1
$\tau = 0.25$	1	1	1	1	1	1	1	1	1	1
$\tau = 0.50$	1	1	1	1	1	1	1	1	1	1
$\tau = 0.75$	1	1	1	1	1	1	1	1	1	1
$\tau = 0.90$	1	1	1	1	1	1	1	1	1	1

Note: This table reports the empirical power of the test in Corollary 1 for testing the Granger non-causality at a given expectile from Z to Y at $\alpha = 5\%$ and 10% significance levels. The number of simulations is equal to 1000.

Table 5: Empirical power of the proposed test of the Granger non-causality test in expectiles

DGP7										
Sample size	$T = 100$		$T = 200$		$T = 500$		$T = 1000$		$T = 2000$	
α	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%
c=0.1										
$\tau = 0.10$	0.161	0.236	0.262	0.386	0.556	0.644	0.834	0.903	1	1
$\tau = 0.25$	0.176	0.288	0.337	0.423	0.660	0.781	0.927	0.963	1	1
$\tau = 0.50$	0.215	0.310	0.350	0.485	0.726	0.789	0.950	0.984	1	1
$\tau = 0.75$	0.215	0.309	0.326	0.437	0.693	0.768	0.941	0.970	1	1
$\tau = 0.90$	0.174	0.239	0.222	0. 361	0.533	0.687	0.839	0.904	1	1
c=0.5										
$\tau = 0.10$	0.976	0.991	1	1	1	1	1	1	1	1
$\tau = 0.25$	0.996	1	1	1	1	1	1	1	1	1
$\tau = 0.50$	1	1	1	1	1	1	1	1	1	1
$\tau = 0.75$	1	1	1	1	1	1	1	1	1	1
$\tau = 0.90$	0.978	0.991	1	1	1	1	1	1	1	1
c=0.9										
$\tau = 0.10$	1	1	1	1	1	1	1	1	1	1
$\tau = 0.25$	1	1	1	1	1	1	1	1	1	1
$\tau = 0.50$	1	1	1	1	1	1	1	1	1	1
$\tau = 0.75$	1	1	1	1	1	1	1	1	1	1
$\tau = 0.90$	1	1	1	1	1	1	1	1	1	1

Note: This table reports the empirical power of the test in Corollary 1 for testing the Granger non-causality at a given expectile from Z to Y at $\alpha = 5\%$ and 10% significance levels. The number of simulations is equal to 1000.

Table 6: Empirical size of the proposed sup-Wald test of Granger non-causality in expectiles

	$\alpha = 5\%$				$\alpha = 10\%$			
	DGP1	**DGP2**	**DGP3**	**DGP4**	**DGP1**	**DGP2**	**DGP3**	**DGP4**
				$\boldsymbol{\tau} \in (0.1)$				
$T = 100$	0.044	0.048	0.046	0.058	0.088	0.100	0.096	0.102
$T = 200$	0.048	0.046	0.054	0.064	0.082	0.088	0.114	0.110
$T = 500$	0.062	0.066	0.038	0.046	0.110	0.136	0.098	0.104
$T = 1000$	0.050	0.044	0.038	0.046	0.102	0.100	0.096	0.104
				$\boldsymbol{\tau} \in (0.10.0.25)$				
$T = 100$	0.047	0.044	0.047	0.054	0.089	0.105	0.097	0.103
$T = 200$	0.046	0.047	0.055	0.061	0.085	0.089	0.115	0.108
$T = 500$	0.058	0.062	0.042	0.048	0.105	0.126	0.097	0.103
$T = 1000$	0.051	0.048	0.041	0.047	0.101	0.99	0.094	0.102
				$\boldsymbol{\tau} \in (0.25.0.75)$				
$T = 100$	0.043	0.045	0.042	0.059	0.087	0.105	0.093	0.104
$T = 200$	0.049	0.044	0.058	0.065	0.081	0.086	0.115	0.111
$T = 500$	0.061	0.065	0.041	0.047	0.108	0.131	0.094	0.103
$T = 1000$	0.052	0.045	0.039	0.044	0.105	0.107	0.095	0.105
				$\boldsymbol{\tau} \in (0.75.0.90)$				
$T = 100$	0.041	0.047	0.045	0.055	0.089	0.106	0.093	0.105
$T = 200$	0.045	0.047	0.053	0.065	0.081	0.089	0.109	0.112
$T = 500$	0.060	0.064	0.039	0.041	0.112	0.129	0.095	0.107
$T = 1000$	0.052	0.043	0.041	0.045	0.104	0.103	0.094	0.103

Note: This table reports the empirical size of the test in Proposition 1 for testing Granger non-causality in expectiles from Z to Y at $\alpha = 5\%$ and 10% significance level. The number of simulations is equal to 1000. The results correspond to the whole set $\tau \in (0, 1)$, but also the subsets: $\tau \in (0.10, 0.25)$, $(0.25, 0.75)$, and $(0.75, 0.90)$.

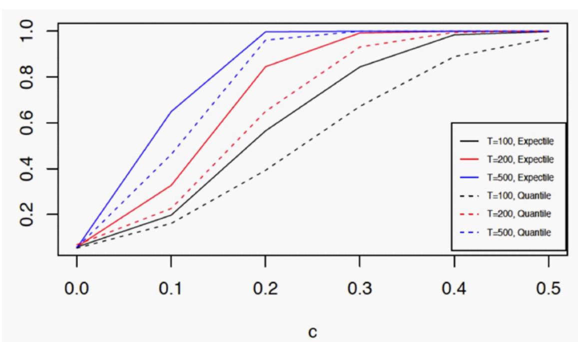
Table 7: Empirical power of our sup-Wald test of Granger non-causality in expectiles for all $\tau \in (0,1)$

	$\alpha = 5\%$			$\alpha = 10\%$		
	DGP5	**DGP6**	**DGP7**	**DGP5**	**DGP6**	**DGP7**
			$\boldsymbol{\tau} \in (0.1)$			
$T = 100$	0.982	0.984	0.986	0.996	0.998	0.998
$T = 200$	1.000	1.000	1.000	1.000	1.000	1.000
$T = 500$	1.000	1.000	1.000	1.000	1.000	1.000
$T = 1000$	1.000	1.000	1.000	1.000	1.000	1.000
			$\boldsymbol{\tau} \in (0.10.0.25)$			
$T = 100$	0.980	0.982	0.983	0.994	0.995	0.997
$T = 200$	1.000	1.000	1.000	1.000	1.000	1.000
$T = 500$	1.000	1.000	1.000	1.000	1.000	1.000
$T = 1000$	1.000	1.000	1.000	1.000	1.000	1.000
			$\boldsymbol{\tau} \in (0.25.0.75)$			
	$\alpha = 5\%$			$\alpha = 10\%$		
	DGP5	**DGP6**	**DGP7**	**DGP5**	**DGP6**	**DGP7**
$T = 100$	0.981	0.984	0.983	0.993	0.994	0.996
$T = 200$	1.000	1.000	1.000	1.000	1.000	1.000
$T = 500$	1.000	1.000	1.000	1.000	1.000	1.000
$T = 1000$	1.000	1.000	1.000	1.000	1.000	1.000
			$\boldsymbol{\tau} \in (0.75.0.90)$			
	$\alpha = 5\%$			$\alpha = 10\%$		
	DGP5	**DGP6**	**DGP7**	**DGP5**	**DGP6**	**DGP7**
$T = 100$	0.980	0.981	0.983	0.991	0.992	0.994
$T = 200$	1.000	1.000	1.000	1.000	1.000	1.000
$T = 500$	1.000	1.000	1.000	1.000	1.000	1.000
$T = 1000$	1.000	1.000	1.000	1.000	1.000	1.000

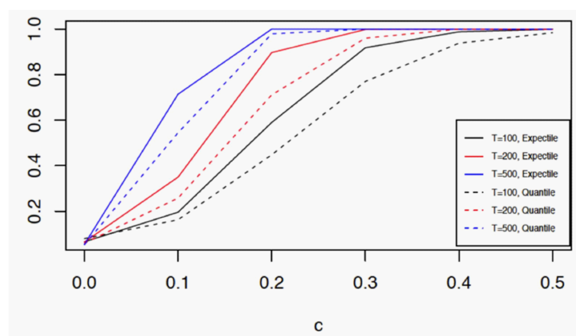
Note: This table reports the empirical power of the test in Proposition 1 for testing Granger non-causality in expectiles from Z to Y at $\alpha = 5\%$ and 10% significance level. The number of simulations is equal to 1000. The results correspond to the whole set $\tau \in (0,1)$, but also the subsets: $\tau \in (0.10, 0.25)$, $(0.25, 0.75)$, and $(0.75, 0.90)$.

for $\tau \in (0,1)$, where $\mu_\tau(\epsilon_{1t}|Y_{t-1}, Z_{t-1})$ and $q_\tau(\epsilon_{1t}|Y_{t-1}, Z_{t-1})$ are the τth-expectile and τth-quantile of the error term ϵ_{1t}. The two equations in (12) show that the degree of causality (measured by c) is the same for expectiles and quantiles and does not change with τ, which guarantees fair comparison of the properties of expectile- and quantile-based tests. The results are reported in Figure 1.

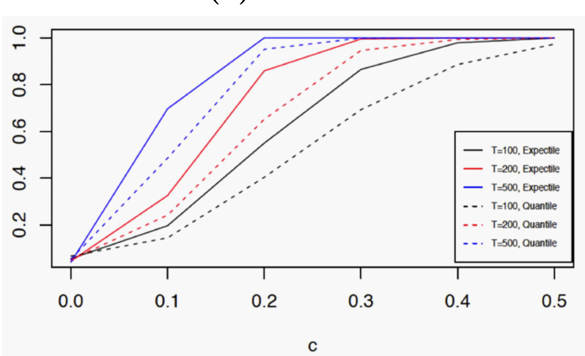
Figure 1 illustrates the powers of the above-mentioned tests for testing Granger non-causality based on expectile and quantile regressions for a range of values of τ. From this, we see that the test derived in Corollary 1 outperforms the test based on quantile regression even when the degree of causality is weak (when c is low). All the subfigures of Figure 1 show that our test not only outperforms the test based on quantiles for large samples such as $T = 500$, but it also provides reliable results for smaller samples such as $T = 100$. The results are consistent across all the values of τ under consideration.



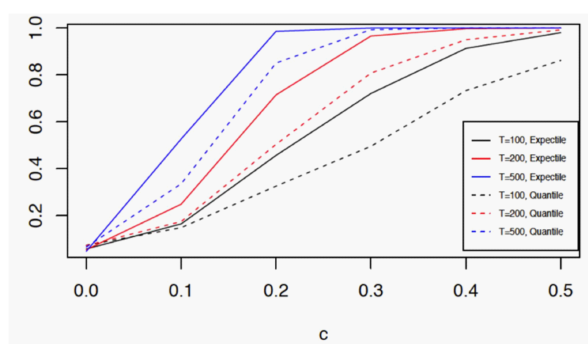
(a) $\tau = 75\%$



(b) $\tau = 50\%$



(c) $\tau = 25\%$

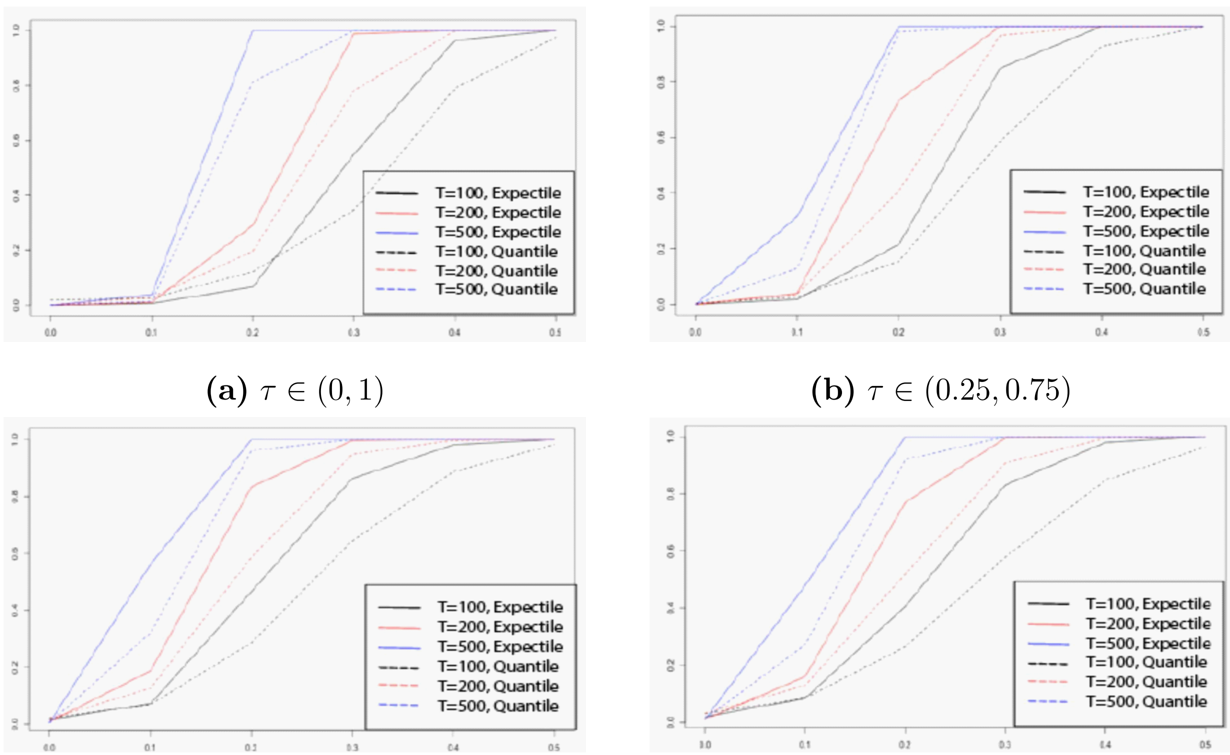


(d) $\tau = 10\%$

Figure 1: This figure compares the power of expectile-based test (Corollary 1) with the power of quantile-based test [see e.g. Koenker and Machado (1999)] for testing Granger non-causality from Z to Y at $\alpha = 5\%$ significance level and for a range of values of τ : $\tau = 75\%$, 50%, 25%,

and 10%. The number of simulations is equal to 1000.

Finally, we run additional simulations to compare the finite sample performance of our Kolmogorov-Smirnov type-Wald test in Proposition 1 and a similar quantile-based test [see Koenker and Machado (1999)] for the whole set of τ, say $(0, 1)$, and the following subsets: $(0.10, 0.25)$, $(0.25, 0.75)$, and $(0.75, 95)$. Specifically, we compare the powers of sup-Wald tests for testing Granger non-causality based on expectile and quantile regressions. Interestingly, Figure 2 shows that for all above-mentioned cases, we still find that the performance of our sup-Wald-expectile based test dominates that of sup-Wald-quantile based test.



(a) $\tau \in (0, 1)$ **(b)** $\tau \in (0.25, 0.75)$

(c) $\tau \in (0.75, 0.90)$ **(d)** $\tau \in (0.10, 0.25)$

Figure 2: This figure compares the power of expectile-based test (Corollary 1) with the power of quantile-based test [see e.g. Koenker and Machado (1999)] for testing Granger non-causality from Z to Y at $\alpha = 5\%$ significance level and for a range of values of $\tau : \tau = 75\%$, 50%, 25%, and 10%. The number of simulations is equal to 1000.

5 Empirical applications

5.1 Causality between returns and volume

As a first application of the tests proposed in Section (3), we consider the problem of testing Granger causality (hereafter predictability) in expectiles from volume to market returns. Predicting the conditional distribution of stock returns using mean and quantile regressions has been the focus of many empirical studies, see Fama and French (1988), Keim and Stambaugh (1986), Ülkü and Onishchenko (2019), Baur et al. (2012) among others. Chuang et al. (2009) have recently investigated the predictability of stock returns using volume based on parametric quantile regressions. They first define Granger non-causality in all quantiles and propose testing non-causality based on a sup-Wald type test. Using three major stock market indices, they find that the causal effects of volume on return are usually heterogeneous across quantiles. In particular, the quantile causal effects of volume on return exhibit a spectrum of V-shape relations so that the dispersion of return distribution increases with lagged volume.

There is, however, no study on the predictability of stock return distribution using volume based on expectile regressions. Expectiles have the advantage of capturing similar information as quantiles, but they also have the merit of being much more straightforward to use than quantiles, since expectiles are define as least squares analogue of quantiles. Furthermore, testing Granger causality (predictability) at all expectiles - as we do in this application and the next one - provides a sufficient condition for testing Granger causality in distribution. Thus, in this section we apply our tests to re-examine Granger causality in expectiles between returns and volume using daily data on three major stock market indices: NYSE, S&P 500 and FTSE 100.

5.1.1 Data description and results

The dataset that we use in this application comes from Yahoo Finance and consists of daily observations on NYSE, S&P 500 and FTSE 100 and their volumes. The sample runs from January 2010 to January 2020 for a total of 3032 observations. We compute the continuously compounded changes in prices (returns) and trading volume (volume growth rate) and we perform Augmented Dickey-Fuller tests (ADF-tests) for testing non-stationarity of the logarithmic price and volume and their first differences. Using ADF-tests with an intercept and trend, with an intercept and without trend, and without intercept and trend, the results show that all variables in logarithmic form are non-stationary, but their first differences are stationnary. Based on this outcome, we model the first difference of logarithmic price and volume rather than their levels. Consequently, the causality relations have to be interpreted in terms of growth rates.

To test the causality in expectiles from volume to stock market return, we use the expectile regression (ER):

$$r_t = \alpha_0(\tau) + \sum_{i=1}^{q} \alpha_i(\tau) r_{t-i} + \sum_{j=1}^{q} \beta_j(\tau) v_{t-j} + \epsilon_t^{\tau},$$

where r_{t-j} is the continuously compounded return from one of the stock market indices (NYSE, S&P 500 and FTSE 100) at time $t-j$, v_{t-j} is the corresponding continuously compounded volume at time $t-j$, and ϵ_t^{τ} is the error term with zero conditional expectile. Returns are calculated as $r_t = 100 \times (\ln(p_t) - \ln(p_{t-1}))$ and the growth rate of the volume as $v_t = 100 \times (\ln(V_t) - \ln(V_{t-1}))$, where p_t and V_t are the price and traded share volume of a given index at time t. We then test the null hypotheses:

$$H_0^{\tau} : \beta(\tau) = 0 \quad \text{versus} \quad H_1^{\tau} : \beta(\tau) \neq 0$$

and

$$H_0 : \beta(\tau) = 0, \ \forall \tau \in (0,1) \text{ versus } \quad H_1 : \beta(\tau) \neq 0 \text{ for some or all } \tau,$$

where $\beta(\tau) = \left(\beta_1(\tau), ..., \beta_q(\tau)\right)'$. The former null H_0^{τ} corresponds to testing non-causality (non-predictability) from volume to stock returns at a given expectile τ, whereas the latter null H_0 is about testing non-causality (non-predictability) at all expectiles $\tau \in (0,1)$, which is equivalent to testing independence between volume and stock returns.

The results for testing H_0^{τ} and H_0 for different lags q are reported in Table 8. Three different values of the number of lags are considered: $q = 3$, 10, and 12. In the absence of information criteria for the selection of number of lags for expectile autoregression models, we recommend to use a large number of lags. Table 8 shows that changes in volume are causing changes in stock returns. The causality happens generally at the upper expectiles (0.9 to 1) of stock returns, except for FTSE. The Kolmogorov-Smirnov type-Wald test of Proposition 1 also confirms the causality from volume to stock returns of NYSE, S&P 500 and FTSE 100, especially when the number lags is high ($q = 10$, 12).

Table 8: Granger-causality in expectiles between stock market return and volume

	Number of Lags = 3			Number of Lags = 10			Number of Lags = 12		
τ	NYSE	S&P	FTSE	NYSE	S&P	FTSE	NYSE	S&P	FTSE
0.01	1.132	3.865	3.606	9.716	13.01	**22.21****	11.41	10.99	**21.28****
0.05	0.921	1.401	2.564	6.313	6.287	7.820	8.141	8.37	9.070
0.10	0.874	1.475	2.745	6.882	7.036	7.613	8.552	8.31	10.27
0.20	1.187	1.934	3.277	5.841	7.991	10.82	7.809	8.72	11.55
0.30	1.826	2.342	3.191	6.114	8.004	12.75	8.169	10.05	13.36
0.40	2.401	2.693	3.004	6.637	7.913	14.08	8.641	10.04	14.59
0.50	3.056	3.163	2.703	7.069	8.303	14.86	9.424	10.54	15.46
0.60	3.890	3.776	2.251	7.648	9.070	15.00	10.445	11.58	15.56
0.70	4.768	4.354	1.800	8.520	10.07	15.21	11.76	13.02	15.86
0.80	5.523	5.334	1.445	10.29	12.05	15.27	15.31	15.15	17.02
0.90	**7.360***	**6.410***	1.638	14.62	15.96	14.12	**19.01***	17.16	14.42
0.95	**8.196****	**7.332***	2.654	**21.58****	**20.31****	15.11	**25.94****	18.24	15.14
0.99	**12.60*****	2.697	1.18	**51.74*****	**36.14*****	**19.18****	**40.95*****	**30.04*****	15.46
Sup-Wald	**12.60*****	**7.332***	3.606	**51.74*****	**36.14*****	**22.21****	**40.95*****	**30.04*****	**21.28****

Note: This table reports the results of Granger-causality in expectiles between stock market return (NYSE, S&P, FTSE) and volume. Each entry is a test for the null hypothesis of non-causality from volume to return at a given expectile τ, for $\tau \in (0,1)$. ***. **, and * denote significance at 1%, 5%, and 10% levels, respectively, and the corresponding critical values are 11.344, 7.814, and 6.251 for $\chi^2(3)$ [Number of Lags = 3], and 23.209, 18.307, and 15.987 for $\chi^2(10)$ [Number of Lags = 10], and 26.217, 21.026, and 18.549 for $\chi^2(12)$ [Number of Lags = 12].

Table 9: Granger-causality in quantiles between stock market return and volume

	Number of Lags = 3			Number of Lags = 10			Number of Lags = 12		
τ	NYSE	S&P	FTSE	NYSE	S&P	FTSE	NYSE	S&P	FTSE
0.01	1.349	4.673	4.880	7.519	11.31	**21.41****	6.898	10.15	12.96
0.05	0.285	0.392	0.165	2.809	2.801	2.561	2.112	4.538	2.112
0.10	0.074	0.266	0.173	2.613	2.446	3.297	3.107	3.335	2.576
0.20	0.012	0.119	0.286	0.520	0.724	2.050	1.551	1.043	2.111
0.30	0.111	0.017	0.218	0.846	0.440	1.229	1.099	0.782	1.049
0.40	0.189	0.150	0.234	0.644	0.726	0.720	0.775	0.984	0.890
0.50	0.104	0.165	0.150	1.100	0.627	0.756	0.766	0.684	0.698
0.60	0.105	0.190	0.030	0.443	0.656	1.040	0.434	0.628	1.023
0.70	0.055	0.018	0.099	0.578	1.004	0.971	0.634	1.771	1.254
0.80	0.096	0.405	0.090	0.763	1.541	1.542	1.122	1.481	1.856
0.90	1.682	1.196	0.014	2.214	3.824	2.319	2.785	1.066	3.497
0.95	2.443	2.316	0.546	5.364	7.460	5.057	5.200	4.873	7.267
0.99	4.084	2.600	0.355	14.537	**17.93***	6.368	12.22	14.43	5.491
Sup-Wald	4.664	4.673	4.880	14.537	**17.93***	**21.41****	12.22	14.43	12.96

Note: This table reports the results of Granger-causality in quantile between stock market return (NYSE, S&P, FTSE) and volume. Each entry is a test for the null hypothesis of non-causality from volume to return at a given quantile, for $\tau \in (0,1)$. ***. **, and * denote signi cance at 1%; 5%, and 10% levels, respectively, and the corresponding critical values are 11.344, 7.814, and 6.251 for $\chi^2(3)$ [Number of Lags = 3], and 23.209, 18.307, and 15.987 for $\chi^2(10)$ [Number of Lags = 10], and 26.217, 21.026, and 18.549 for $\chi^2(12)$ [Number of Lags = 12].

For a comparison, in Table 9 we report the results of testing Granger non-causality in quantiles from volume to stock returns at each given quantile τ and for all quantiles in $(0, 1)$ jointly using the Sup-Wald test of Koenker and Machado (1999). Comparing tables 8 and 9, we find that several causality patterns from volume to stock returns were detected by expectile-based Granger non-causality tests, but not by quantile-based Granger non-causality tests. For example, the tests of Granger non-causality in quantiles show that the volume does not cause FTSE's return, which contradicts the result obtained using the tests of Granger non-causality in expectiles.

5.2 Stock market return and exchange rates

Given its significance for the economy, the causal relationship between stock prices and exchange rates is of great importance for academics, policymakers and professionals. Thus, as a second application of our tests, we investigate the causality in expectiles from exchange rate to market returns.

In the literature, there is no academic consensus about this relationship and the results are somewhat mixed as to whether or not exchange rates affect the prices of stock indexes. On the one hand, Chow et al. (1997) find evidence that the link between changes in exchange rate and stock returns is stronger at long horizons, whereas Williamson (2001) find that this link has changed over time. Yang et al. (2014) have applied parametric regression models to investigate the causality between stock returns and exchange rate for nine Asian markets. Their empirical results show that there are more causal relations based on the quantile regression than the conventional mean regression. On the other hand, Griffin and Stulz (2001) conclude that this link is small and hardly significant. Most of the conclusions, however, were reached using mean regression-based tests. In this section, we apply the Granger causality in expectiles tests to examine the causal relationship between stock prices and exchange rates.

5.2.1 Data description and results

For this second application, we use data on S&P 500 Index and US/Canada, US/UK and US/Japen exchange rates. The data sets consist of monthly observations on S&P 500 Index and exchange rates and their obtained from St. Louis Fed and Yahoo Finance, respectively, and the sample runs from January 2004 to January 2020 for a total of 193 observations. As in the first application, we perform ADF-tests for testing non-stationarity of the logarithmic price and exchange rates and their first differences. The results show that all variables in logarithmic form are non-stationary, whereas their first differences are stationary. Thus, we model the first differences of logarithmic price and exchange rates rather than their level.

To test the causality in expectiles from exchange rates to stock market return, we use the expectile regression:

$$r_t = \alpha_0(\tau) + \sum_{j=1}^{q} \alpha_j(\tau) r_{t-j} + \sum_{j=1}^{q} \beta_j(\tau) ex_{t-j}^{\mathrm{US}/i} + \epsilon_t^{\tau},$$

where $ex_{t-j}^{\mathrm{US}/i}$, for $i =$Canada, UK and Japen, is the growth rate of exchange rate US/country i at time $t-j$, r_{t-j} is the continuously compounded return from the S&P 500 index at time $t-j$, and ϵ_t^{τ} is the error term with zero conditional expectile. As before, returns are calculated as $r_t = 100 \times (\ln(p_t) - \ln(p_{t-1}))$ and exchange rates as $ex_{t-j}^{\mathrm{US}/i} = 100 \times \left(\ln\left(EX_t^{\mathrm{US}/i}\right) - \ln\left(EX_{t-1}^{\mathrm{US}/i}\right)\right)$, where p_t and $EX_t^{\mathrm{US}/i}$ are the price of S&P 500 index and exchange rate US/country i at time t, respectively. We then test the null hypotheses:

$$H_0^{\tau} : \beta(\tau) = 0 \quad \text{versus} \quad H_1^{\tau} : \beta(\tau) \neq 0$$

and

$$H_0 : \beta(\tau) = 0,\ \forall \tau \in (0,1) \text{ versus } \quad H_1 : \beta(\tau) \neq 0 \text{ for some or all } \tau.$$

where $\beta(\tau) = \left(\beta_1(\tau), ..., \beta_q(\tau)\right)'$. The former null H_0^{τ} corresponds to testing non-causality from growth rate of exchange rate to stock return at a given expectile τ, whereas the latter null hypothesis H_0 is about testing non-causality at all expectiles $\tau \in (0,1)$, which is equivalent to testing independence between growth rate of exchange rate and stock return.

The results for different lags q are reported in Table 10. The latter shows that there is a strong Granger causality from US/UK exchange rate to S&P 500 index price at different expectiles, in particular the lower expectiles and up to 0.7 expectile. We also find some evidence of causality in expectiles from US/Canada and US/Japan exchange rates to S&P 500 index price, although it is less present compared to causality from US/UK exchange rate to S&P 500 index price. The Kolmogorov-Smirnov type-Wald test also confirms the causality from these exchange rates to stock returns, especially when the number lags is high ($q = 10$, 12).

Table 10: Granger-causality in expectiles between stock market return and exchange rates

	Number of Lags = 3			Number of Lags = 10			Number of Lags = 12		
τ	EX_{CAD}	EX_{UK}	EX_{JAP}	EX_{CAD}	EX_{UK}	EX_{JAP}	EX_{CAD}	EX_{UK}	EX_{JAP}
0.01	0.987	**8.271****	**7.562***	**30.67*****	5.091	13.16	16.64	**165.84*****	**22.59****
0.05	0.610	**6.298***	1.973	**26.73*****	12.28	10.74	**37.39*****	**63.03*****	11.10
0.10	0.913	**8.331****	1.404	12.12	**21.61****	8.635	**27.29*****	**49.02*****	10.47
0.20	1.143	**11.17****	1.202	11.62	**21.42****	8.293	**18.58***	**43.47*****	9.524
0.30	1.127	**9.474****	1.057	10.82	**18.49****	8.642	16.52	**31.88*****	9.763
0.40	1.114	**8.207****	0.927	11.12	**18.74****	9.633	15.14	**26.30*****	9.594
0.50	1.076	**7.508***	0.746	11.41	**16.72***	8.495	14.39	**21.90***	9.115
0.60	1.072	**6.861***	0.581	11.95	14.96	8.130	13.53	**19.11***	8.925
0.70	1.060	**6.387***	0.371	12.89	14.11	7.836	13.90	17.91	8.923
0.80	0.94	6.008	0.216	14.23	12.48	8.147	14.78	16.03	8.861
0.90	0.547	4.232	0.106	17.83	11.05	7.656	13.22	12.15	7.734
0.95	0.571	4.262	0.012	11.11	16.13	7.502	14.67	14.87	6.497
0.99	1.631	5.328	2.308	**22.37****	7.016	5.216	12.43	13.67	7.931
Sup-Wald	1.631	**11.33****	**7.562***	**30.67*****	**22.66****	14.99	**57.31*****	**165.84*****	**22.59****

Note: This table reports the results of Granger-causality in expectiles between stock market return and exchange rate (EX_{CAD}, EX_{UK}, EX_{JAP}). Each entry is a test for the null hypothesis of non-causality from exchange rate to return at a given expectile τ, for $\tau \in (0,1)$. ***. **, and * denote significance at 1%, 5%, and 10% levels, respectively, and the corresponding critical values are 11.344, 7.814, and 6.251 for $\chi^2(3)$ [Number of Lags = 3], and 23.209, 18.307, and 15.987 for $\chi^2(10)$ [Number of Lags = 10], and 26.217, 21.026, and 18.549 for $\chi^2(12)$ [Number of Lags = 12].

Table 11: Granger-causality in quantiles between stock market return and exchange rates

	Number of Lags = 3			Number of Lags = 10			Number of Lags = 12		
τ	EX_{CAD}	EX_{UK}	EX_{JAP}	EX_{CAD}	EX_{UK}	EX_{JAP}	EX_{CAD}	EX_{UK}	EX_{JAP}
0.01	1.714	3.214	**8.897****	14.54	**22.81****	8.492	**20.14***	15.57	13.34
0.05	0.773	5.003	2.109	8.520	14.48	6.962	15.47	12.47	13.77
0.10	0.921	3.771	1.362	5.872	8.951	4.029	12.04	10.54	6.615
0.20	0.569	1.464	0.958	9.094	5.393	1.637	7.030	7.108	3.021
0.30	0.101	0.722	0.337	1.966	3.972	1.833	3.028	4.396	2.976
0.40	0.271	0.760	0.197	1.620	3.030	1.53	2.992	2.502	1.648
0.50	0.125	0.506	0.254	1.507	2.020	1.783	3.271	1.837	1.164
0.60	0.310	0.315	0.277	1.440	0.680	1.482	4.478	0.832	1.281
0.70	0.663	0.284	0.201	1.710	1.091	0.844	2.972	1.548	0.817
0.80	0.033	0.342	0.266	1.658	1.196	1.481	2.747	1.511	1.356
0.90	0.158	1.024	0.245	2.492	1.827	0.988	2.592	2.408	1.289
0.95	0.459	1.627	0.110	2.792	3.980	1.463	2.526	4.842	1.234
0.99	2.696	0.703	1.423	2.935	3.612	2.098	2.924	5.083	2.913
Sup-Wald	2.69	**9.447****	**8.897****	14.54	**22.81****	8.492	**20.14***	15.57	13.77

Note: This table reports the results of Granger-causality in quantile between stock market return and exchange rate (EX_{CAD}, EX_{UK}, EX_{JAP}). Each entry is a test for the null hypothesis of non-causality from exchange rate to return at a given quantile, for $\tau \in (0,1)$. ***. **, and * denote signi cance at 1%; 5%, and 10% levels, respectively, and the corresponding critical values are 11.344, 7.814, and 6.251 for $\chi^2(3)$ [Number of Lags = 3], and 23.209, 18.307, and 15.987 for $\chi^2(10)$ [Number of Lags = 10], and 26.217, 21.026, and 18.549 for $\chi^2(12)$ [Number of Lags = 12].

Table 11 reports the results of testing Granger non-causality in quantiles from exchange rates to S&P 500 return at each given quantile τ and for all quantiles in $(0, 1)$. Comparing tables 10 and 11, we find that several causality patterns from exchange rates to stock return were undetected when we use quantile-based Granger causality tests. For example, the tests of Granger non-causality in quantiles (for many values of τ) show that the US/UK exchange rate does not cause S&P 500 return, which contradicts the result obtained using the tests of Granger non-causality in expectiles at different values of τ.

6 Conclusion

We proposed a consistent parametric test of Granger causality at a given expectile. We also derived a sup-Wald test for jointly testing Granger causality at all expectiles that has the correct asymptotic size and power properties. Working with expectiles has the advantage of capturing similar information as quantiles, but expectiles has the merit of being much more straightforward to implement than quantiles, since expectiles are define as least squares analogue of quantiles. In other words, studying Granger causality in expectiles is practically simpler and allows us to examine the causality at all levels of the conditional distribution. In addition, testing Granger causality at all expectiles provides a sufficient condition for testing Granger causality in distribution. A Monte Carlo simulation study revealed that our tests have good finite-sample size and power properties for a variety of data-generating processes and different sample sizes. We also compared the properties (size and power) of our expectile-based test with the existing quantile-based test for a range of values of τ. The simulation results showed that our test outperforms the test based on quantile regression even for weak degree of causality and in both small and large samples. Finally, we provided two empirical applications to illustrate the usefulness of our tests. In these applications, we re-examined the Granger causality from volume and exchange rates to stock returns using expectile regressions.

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7 Appendix A: Proofs

This appendix contains the proof of the main result in the text.

Proof of Theorem 1: Consider the check function:

$$\rho(u) = |\tau - \mathbb{I}(u \leq 0)|\, u^2 = \psi_\tau(u)\, u^2, \text{ for } \tau \in (0,1).$$

The derivative of the function ρ is given by $\rho'(u) = 2u\psi_\tau(u)$. The main idea of the proof of Theorem 1 is based on the following decomposition:

$$\rho(u-v) - \rho(u) = -2uv\psi_\tau(u) + v^2\psi_\tau(u) + (2\tau - 1)\, R(u,v)\,(u-v), \tag{13}$$

where $R(u,v) = (\mathbb{I}(u \leq 0) - \mathbb{I}(u \leq v))\,(u-v) = \int_0^v [\mathbb{I}(u \leq s) - \mathbb{I}(u \leq 0)]\, ds$. The above decomposition is obtained as follows. Note that:

$$
\begin{aligned}
|\tau - \mathbb{I}(u \le v)| - |\tau - \mathbb{I}(u \le 0)| &= \begin{cases} 0 & u \ge v, u \ge 0 \\ 2\tau - 1 & u \ge v, u \le 0 \\ 1 - 2\tau & u \le v, u \ge 0 \\ 0 & u \le v, u \le 0 \end{cases} \\
&= (2\tau - 1)(\mathbb{I}(u \le 0) - \mathbb{I}(u \le v))
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\rho(u - v) - \rho(u) &= |\tau - \mathbb{I}(u \le v)|(u - v)^2 - |\tau - \mathbb{I}(u \le 0)| u^2 \\
&= |\tau - \mathbb{I}(u \le v)|(u - v)^2 - |\tau - \mathbb{I}(u \le 0)|(u - v)^2 \\
&\quad + |\tau - \mathbb{I}(u \le 0)| v^2 - |\tau - \mathbb{I}(u \le 0)| 2uv \\
&= -2uv\psi_\tau(u) + v^2\psi_\tau(u) + (2\tau - 1)R(u, v)(u - v).
\end{aligned}
$$

Now, let $\epsilon_t^\tau = y_t - x_{t-1}'\theta(\tau)$ and $v_t = \frac{\delta' x_{t-1}}{\sqrt{T}}$, and denote by

$$
H_T(\delta) = \sum_{t=1}^{T} \left[\rho(\epsilon_t^\tau - v_t) - \rho(\epsilon_t^\tau)\right]. \tag{14}
$$

If we define

$$
\begin{aligned}
\widehat{\delta}(\tau) &= \arg\min_{\delta(\tau)} \sum_{t=1}^{\top} \rho\left(y_t - x_{t-1}'\theta(\tau) - \frac{x_{t-1}'\delta(\tau)}{\sqrt{T}}\right) \\
&= \arg\min_{\delta(\tau)} \sum_{t=1}^{\top} \rho\left(y_t - x_{t-1}'\left(\theta(\tau) + \frac{\delta(\tau)}{\sqrt{T}}\right)\right),
\end{aligned}
$$

then from (7), we have

$$
\widehat{\theta}(\tau) = \theta(\tau) + \frac{\widehat{\delta}(\tau)}{\sqrt{T}}
$$

and hence,

$$
\sqrt{T}\left(\widehat{\theta}(\tau) - \theta(\tau)\right) = \widehat{\delta}(\tau) \equiv \arg\min_{\delta \in \mathbb{R}^{d+1}} H_T(\delta). \tag{15}
$$

Observe that, using Equation (13), with $u = \epsilon_t^\tau = y_t - x_{t-1}'\theta(\tau)$ and $v = v_t, = \frac{\delta' x_{t-1}}{\sqrt{T}}$, we have

$$
\rho(\epsilon_t^\tau - v_t) - \rho(v_t) = \delta' \frac{-2x_{t-1}}{\sqrt{T}} \epsilon_t^\tau \psi_\tau(\epsilon_t^\tau) + \left(\frac{\delta' x_{t-1}}{\sqrt{T}}\right)^2 \psi_\tau(\epsilon_t^\tau) + (2\tau - 1)R(\epsilon_t^\tau, v_t)\left(\epsilon_t^\tau - \frac{\delta' x_{t-1}}{\sqrt{T}}\right).
$$

If we add the summation in the expression of the convex random objective function $H_T(\delta)$ in (14), we obtain the following decomposition:

$$H_T(\delta) = H_{1,T}(\delta) + H_{2,T}(\delta) + H_{3,T}(\delta),$$

where

$$\begin{aligned} H_{1,T}(\delta) &= -\delta' U_T = \tfrac{-2}{\sqrt{T}} \textstyle\sum_{t=1}^{T} \delta' x_{t-1} \epsilon_t^{\tau} \psi_{\tau}\left(\epsilon_t^{\tau}\right), \\ H_{2,T}(\delta) &= \tfrac{1}{T} \textstyle\sum_{t=1}^{T} \delta' x_{t-1} x_{t-1}' \delta \psi_{\tau}\left(\epsilon_t^{\tau}\right), \\ H_{3,T}(\delta) &= (2\tau - 1) \textstyle\sum_{t=1}^{T} R\left(\epsilon_t^{\tau}, v_t\right)\left(\epsilon_t^{\tau} - v_t\right). \end{aligned}$$

Now, first using the fact that $0 \leq R(\epsilon^{\tau}, v)$ and that $\left|R\left(\epsilon^{\tau}, v\right)\left(\epsilon^{\tau} - v\right)\right| = R\left(\epsilon^{\tau}, v\right)\left|\left(\epsilon^{\tau} - v\right)\right| \leq R\left(\epsilon^{\tau}, v\right)|v|$, we obtain

$$\left|H_{3,T}(\delta)\right| \leq |(2\tau - 1)| \frac{\left|\delta' x_{t-1}\right|}{\sqrt{T}} \sum_{t=1}^{T} R\left(\epsilon_t^{\tau}, v_t\right).$$

From Koenker (2005, pages 121-122), we have $\sum_{t=1}^{T} R\left(\epsilon_t^{\tau}, v_t\right) = O_p(1)$. Hence, from Assumption **(A.4)**,one can deduce that $H_{3,T}(\delta) = o_p(1)$. Second, under Assumptions (**A.1)** and (**A.2)** and the fact that $\Psi_{\tau} = E\left[\psi_{\tau}\left(\epsilon_t^{\tau}\right) x_{t-1} x_{t-1}'\right]$ is finite, we have:

$$H_{2,T}(\delta) = \frac{1}{T} \sum_{t=1}^{T} \delta' x_{t-1} x_{t-1}' \delta \psi_{\tau}\left(\epsilon_t^{\tau}\right) = \delta' E\left[x_{t-1} x_{t-1}' \psi_{\tau}\left(\epsilon_t^{\tau}\right)\right] \delta + o_p(1) = \delta' \Psi_{\tau} \delta + o_p(1). \quad (16)$$

Third, since $x_{t-1} \in I_{YZ}(t-1)$, then $E\left[\epsilon_t^{\tau} \psi_{\tau}\left(\epsilon_t^{\tau}\right) | I_{YZ}(t-1)\right] = 0$ and $x_{t-1} \epsilon_t^{\tau} \psi_{\tau}\left(\epsilon_t^{\tau}\right)$ is a martingale difference sequence. Hence $H_{1,T}(\delta)$ satisfies the conditions of central limit theorem. Furthermore, observe that the process

$$G(\tau) = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} x_{t-1} \epsilon_t^{\tau} \psi_{\tau}\left(\epsilon_t^{\tau}\right) = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} x_{t-1}(y_t - x_{t-1}'\theta(\tau)) \psi_{\tau}\left(y_t - x_{t-1}'\theta(\tau)\right)$$

can be rewritten as follows

$$\begin{aligned} G(\tau) \quad = \quad & \frac{\tau}{\sqrt{T}} \sum_{t=1}^{T} x_{t-1} y_t + \frac{\tau\theta(\tau)}{\sqrt{T}} \sum_{t=1}^{T} x_{t-1} x_{t-1}' + \frac{1-2\tau}{\sqrt{T}} \sum_{t=1}^{T} x_{t-1} y_t \mathbb{I}(y_t \leq x_{t-1}'\theta(\tau)) \\ & + \frac{(1-2\tau)\theta(\tau)}{\sqrt{T}} \sum_{t=1}^{T} x_{t-1} x_{t-1}' \mathbb{I}(y_t \leq x_{t-1}'\theta(\tau)). \end{aligned}$$

Thus, from Assumption **(A.4)** and the fact that conditional distribution of y_t given x_{t-1} has a continuous density, the four components in the above decomposition are stochastically continuous on each compact I in $(0, 1)$.

Therefore, we deduce the weak convergence of the empirical process $\frac{1}{\sqrt{T}}\sum_{t=1}^{T} x_{t-1}\epsilon_t^{\tau}\psi_{\tau}\left(\epsilon_t^{\tau}\right)$, i.e.,

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T} x_{t-1}\epsilon_t^{\tau}\psi_{\tau}\left(\epsilon_t^{\tau}\right) \Longrightarrow \mathcal{Z}\left(\tau\right),$$

where $\mathcal{Z}\left(\tau\right)$ is a Gaussian process with covariance function

$$Cov(\mathcal{Z}(\tau_1),\mathcal{Z}(\tau_2)) = E[\psi_{\tau_1}(\epsilon_t^{\tau_1})\psi_{\tau_2}(\epsilon_t^{\tau_2})\epsilon_t^{\tau_1}\epsilon_t^{\tau_2}x_{t-1}\,x_{t-1}'].$$

Hence,

$$H_T = \frac{-2}{\sqrt{T}}\sum_{t=1}^{T}\delta' x_{t-1}\epsilon_t^{\tau}\psi_{\tau}\left(\epsilon_t^{\tau}\right) + \frac{1}{T}\sum_{t=1}^{T}\delta' x_{t-1}x_{t-1}'\delta\psi_{\tau}\left(\epsilon_t^{\tau}\right) + o_p\left(1\right) \xrightarrow{P} -2\delta'\mathcal{Z}\left(\tau\right) + \delta'\Psi_{\tau}\delta,$$

we can deduce from Hjort and Pollard (1993) that:

$$\sqrt{T}\left(\hat{\theta}\left(\tau\right) - \theta\left(\tau\right)\right) \Longrightarrow \Psi_{\tau}^{-1}\mathcal{Z}\left(\tau\right).$$

Which concludes the proof of Theorem 1.

Proof of Corollary 1: Theorem 1 states that the empirical process $\sqrt{T}(\hat{\theta}(\tau)-\theta(\tau))$ converges to a Gaussian process $\mathcal{Z}(\tau)$ with the covariance function $Cov(\mathcal{Z}(\tau_1),\mathcal{Z}(\tau_2)) = \Psi_{\tau_1}^{-1}\Psi_{\tau_1,\tau_2}\Psi_{\tau_2}^{-1}$, where

$$\Psi_{\tau_i} = E[\psi_{\tau_i}(\epsilon_t^{\tau_i})x_{t-1}\,x_{t-1}'],\ \text{for } i=1,2, \text{ and } \Psi_{\tau_1,\tau_2} = E[\psi_{\tau_1}(\epsilon_t^{\tau_1})\psi_{\tau_2}(\epsilon_t^{\tau_2})\epsilon_t^{\tau_1}\epsilon_t^{\tau_2}x_{t-1}\,x_{t-1}'].$$

Thus, for a $q\times(d+1)$-matrix $R=[0_{q,1+p}, I_{q,q}]$ and from Theorem 1, the process $\sqrt{T}(\hat{\beta}(\tau)-\beta(\tau)) = \sqrt{T}(R\hat{\theta}(\tau)-R\theta(\tau))$, converges to a Gaussian process $\mathcal{W}(\tau)$ with the covariance matrix:

$$Cov(\mathcal{W}(\tau_1),\mathcal{W}(\tau_2)) = R\Psi_{\tau_1}^{-1}\Psi_{\tau_1,\tau_2}\Psi_{\tau_2}^{-1}R'.$$

Then, for a fixed τ, the process $\sqrt{T}\left(\Psi_{\tau}^{-1}\Psi_{\tau,\tau}R\Psi_{\tau}^{-1}R'\right)^{-1/2}(\hat{\beta}(\tau)-\beta(\tau))$ converges to $N(0, I_{q\times q})$. The result in Corollary 1 is obtained from the fact that

$$\widehat{\Psi}_{\tau_i} = \frac{1}{T}\sum_{t=1}^{T}\psi_{\tau}(\hat{\epsilon}_t^{\tau_i})x_{t-1}x_{t-1}',,\ \text{for } i=1,2, \text{ and } \widehat{\Psi}_{\tau_1,\tau_2} = \frac{1}{T}\sum_{t=1}^{T}x_{t-1}x_{t-1}'\psi_{\tau_1}(\hat{\epsilon}_t^{\tau_1})\psi_{\tau_2}(\hat{\epsilon}_t^{\tau_2})\hat{\epsilon}_t^{\tau_1}\hat{\epsilon}_t^{\tau_2}$$

are consistent estimators for $\Psi_{\tau_i}, i=1,2$ and $\Psi\tau_1,\tau_2$.

Proof of Proposition 1: From the proof of Corollary 1, the process $\sqrt{T}(\hat{\beta}(\tau) - \beta(\tau)) = \sqrt{T}(R\hat{\theta}(\tau) - R\theta(\tau))$ converges to a Gaussian process $\mathcal{W}(\tau)$ with the covariance matrix:

$$Cov(\mathcal{W}(\tau_1), \mathcal{W}(\tau_2)) = R\Psi_{\tau_1}^{-1}\Psi_{\tau_1,\tau_2}\Psi_{\tau_2}^{-1}R'.$$

By applying the continuous mapping theorem, we have

$$\sqrt{T}(\hat{\beta}(\tau) - \beta(\tau))' \left(R\Psi_{\tau}^{-1}\Psi_{\tau,\tau}\Psi_{\tau}^{-1}R'\right)^{-1} \sqrt{T}(\hat{\beta}(\tau) - \beta(\tau))$$

converge in distribution to

$$\mathcal{W}'(\tau) \left(R\Psi_{\tau}^{-1}\Psi_{\tau,\tau}\Psi_{\tau}^{-1}R'\right)^{-1} \mathcal{W}(\tau).$$

Thus, the result in Proposition 1 can be deduced from the fact that $\widehat{\Psi}_{\tau}$ and $\widehat{\Psi}_{\tau,\tau}$ are consistent estimators of Ψ_{τ} and $\Psi_{\tau,\tau}$ respectively.