Survival probability determination of nonlinear oscillators with fractional derivative elements under evolutionary stochastic excitation

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Abstract

An approximate analytical technique based on a combination of statistical linearization and stochastic averaging is developed for determining the survival probability of stochastically excited nonlinear/hysteretic oscillators with fractional derivative elements. Specifically, approximate closed form expressions are derived for the oscillator non-stationary marginal, transition, and joint response amplitude probability density functions (PDF) and, ultimately, for the time-dependent oscillator survival probability. Notably, the technique can treat a wide range of nonlinear/hysteretic response behaviors and can account even for excitation evolutionary power spectra with time-dependent frequency content. Further, the corresponding computational cost is kept at a minimum level since it relates, in essence, only to the numerical integration of a deterministic nonlinear differential equation governing approximately the evolution in time of the oscillator response variance. Overall, the developed technique can be construed as an extension of earlier efforts in the literature to account for fractional derivative terms in the equation of motion. The numerical examples include a hardening Duffing and a bilinear hysteretic nonlinear oscillators with fractional derivative terms. The accuracy degree of the technique is assessed by comparisons with pertinent Monte Carlo simulation data.

Keywords: Fractional derivative - Nonlinear System - Survival probability - First-passage time - Stochastic dynamics

1 Introduction

The first-passage time problem has been a persistent challenge in the field of stochastic engineering dynamics (e.g., [26, 28, 23]). It relates to the evaluation of the probability that the system response crosses a predetermined threshold for the first time over a given time interval. An alternative definition relates to determining the corresponding survival probability; that is, the probability that the system response stays below a prescribed barrier level over a given time interval. Clearly, knowledge of the time-dependent survival probability can be used for further development of reliability assessment and risk analysis procedures pertaining to diverse engineering systems.

Further, a large number of techniques have been developed over the last six decades for addressing the first-passage time problem. These range from semi-analytical approaches, valid for linear systems only and relying on Poisson distribution-based approximations (e.g., [49, 3]), to purely numerical Monte Carlo simulation (MCS) schemes (e.g., [18, 40, 2]). Furthermore,

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it has been shown that alternative methodologies based, indicatively, on stochastic averaging (e.g., [45, 50]), on probability density evolution equations (e.g., [23]), and on discrete versions of the Chapman-Kolmogorov equation (e.g., [19, 29, 21, 22, 6, 13, 5]) are capable of treating dynamic systems exhibiting diverse nonlinear/hysteretic behaviors, even when subjected to complex non-stationary and non-white/non-Gaussian stochastic excitations.

Nevertheless, in the aforementioned techniques the system equations of motion are modeled based on classical continuum (or discretized) mechanics theories. In this regard, generalizing standard stochastic dynamics methodologies to treat systems exhibiting time- and space-localized behaviors, described by operators based on wavelets and/or non-integer order derivatives, can be a quite challenging task (e.g., [44, 33, 32]). To elaborate further, the need for more accurate media behavior modeling has led recently to advanced mathematical tools such as fractional calculus (e.g., [31, 39, 38]). Fractional calculus can be construed as a generalization of ordinary calculus, and as such provides with enhanced modeling capabilities. In this context, it has been successfully employed in theoretical and applied mechanics for developing non-local continuum mechanics theories (e.g., [14, 48]), as well as for modeling viscoelastic materials (e.g., [15]).

In general, stochastic averaging relates to a Markovian approximation of an appropriately chosen amplitude of the system response, and to a dimension reduction of the original $2n$-dimensional problem to an n-dimensional problem; see [36, 51] for a broad perspective. Notably, various techniques relying on a stochastic averaging treatment of the governing equations of motion have been proposed for treating nonlinear/hysteretic systems endowed with fractional derivative elements, and for determining first-passage time statistics. Typically, these relate to semi-analytical or numerical schemes for solving the Backward-Kolmogorov (BK) equation for the survival probability of the oscillator response, or the associated Pontryagin equations for the first-passage time statistical moments (e.g., [8, 9, 24]). Indicatively, the derived BK equation was solved in [43, 12] by relying on a Galerkin scheme that utilizes a convenient set of confluent hypergeometric functions as orthogonal basis. Further, the Galerkin solution scheme was enhanced in [16] by resorting to a novel stochastic averaging treatment. It was shown that a Hilbert transform based definition of the oscillator response amplitude can circumvent the requirement of a priori determination of an equivalent natural frequency; thus, yielding flexibility in the ensuing analysis and potentially higher accuracy compared to a standard stochastic averaging treatment.

In this paper, an approximate analytical technique based on a combination of statistical linearization and stochastic averaging is developed for determining the survival probability of stochastically excited nonlinear/hysteretic oscillators with fractional derivative elements. Specifically, approximate closed form expressions are derived for the oscillator non-stationary marginal, transition, and joint response amplitude PDFs and, ultimately, for the time-dependent oscillator survival probability. Notably, the technique can treat a wide range of non-stationary excitations and can account even for evolutionary power spectra (EPS) with time-dependent frequency content. Further, the corresponding computational cost is kept to a minimum level since it relates, in essence, only to the numerical integration of a deterministic nonlinear differential equation governing approximately the evolution in time of the oscillator response variance. The numerical examples include a hardening Duffing and a bilinear hysteretic nonlinear oscillators with fractional derivative terms. The accuracy degree of the technique is assessed by comparisons with pertinent MCS data. Overall, the developed technique can be construed as an extension of the concepts and the results in [45] to account for fractional derivative terms in the equation of motion.

2 Mathematical formulation

2.1 Equivalent linear oscillator time-dependent elements and non-stationary marginal response amplitude PDF: A stochastic averaging solution treatment

In this section, the basic aspects of a recently developed approximate technique [17] for determining the stochastic response of nonlinear oscillators with fractional derivative elements are presented for completeness. The technique relies on a statistical linearization treatment [37] of the nonlinear oscillator equation of motion and yields an equivalent linear system (ELS) with time-dependent stiffness and damping elements. Next, resorting to stochastic averaging yields a deterministic nonlinear differential equation to be solved numerically for determining approximately the oscillator non-stationary response variance, and ultimately, the oscillator non-stationary marginal response amplitude PDF.

In this regard, consider a stochastically excited nonlinear oscillator with fractional derivative elements. Its equation of motion is given by

$$\ddot{x}(t) + \beta \mathcal{D}_{0,t}^{\alpha} x(t) + z(t, x, \dot{x}) = w(t), \tag{1}$$

where β is a constant coefficient, and $\mathcal{D}_{0,t}^{\alpha} x(t)$ represents the Caputo fractional derivative of order α ($0 < \alpha < 1$) defined as

$$\mathcal{D}_{0,t}^{\alpha} x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{x}(\tau)}{(t-\tau)^{\alpha}} \mathrm{d}\tau. \tag{2}$$

Further, $z(t, x, \dot{x})$ is an an arbitrary nonlinear function that can also account for hysteretic behaviors, $w(t)$ denotes a Gaussian, zero-mean non-stationary stochastic process with a broadband EPS $S(\omega, t)$.

Next, considering relatively light damping, it can be argued that the oscillator response exhibits a pseudo-harmonic behavior described by [17, 36]

$$x(t) = A(t) \cos\left(\omega(A)t + \psi(t)\right) \tag{3}$$

and

$$\dot{x}(t) = -\omega(A) A(t) \sin(\omega(A)t + \psi(t)), \tag{4}$$

where the response amplitude $A(t)$ and phase $\psi(t)$ are considered to be slowly-varying quantities with respect to time, and thus, approximately constant over one cycle of oscillation. Next, manipulating Eqs. (3) and (4) yields

$$A^2(t) = x^2(t) + \left(\frac{\dot{x}(t)}{\omega(A)}\right)^2. \tag{5}$$

Further, considering that

$$h(t, x, \mathcal{D}_{0,t}^{\alpha} x, \dot{x}) = \beta \mathcal{D}_{0,t}^{\alpha} x(t) + z(t, x, \dot{x}) - \beta_0 \dot{x}, \tag{6}$$

Eq. (1) is recast into [17, 12]

$$\ddot{x}(t) + \beta_0 \dot{x}(t) + h(t, x, \mathcal{D}_{0,t}^{\alpha} x, \dot{x}) = w(t), \tag{7}$$

where $\beta_0 = 2\zeta_0\omega_0$ represents a damping coefficient, with ω_0 and ζ_0 denoting, respectively, the natural frequency and damping ratio of the corresponding linear oscillator (i.e., $h(t, x, \mathcal{D}^{\alpha}_{0,t}x, \dot{x}) = \omega_0^2 x(t)$). Furthermore, an ELS for the oscillator in Eq. (7) is defined as [17]

$$\ddot{x}(t) + (\beta_0 + \beta(A))\,\dot{x}(t) + \omega^2(A)x(t) = w(t). \tag{8}$$

Next, applying a mean-square minimization between Eqs. (7) and (8), and approximating the involved fractional derivatives according to [24, 43], yields the ELS response amplitude-dependent damping and stiffness coefficients in the form [17]

$$\beta(A) = \frac{1}{A\omega(A)}S(A) + \frac{\beta}{\omega^{1-\alpha}(A)}\sin\left(\frac{\alpha\pi}{2}\right) - \beta_0 \tag{9}$$

and

$$\omega^2(A) = \frac{1}{A}F(A) + \beta\omega^{\alpha}(A)\cos\left(\frac{\alpha\pi}{2}\right), \tag{10}$$

where

$$S(A) = -\frac{1}{\pi}\int_0^{2\pi} z\left(A\cos\phi, -A\omega(A)\sin\phi\right)\sin\phi \mathrm{d}\phi, \tag{11}$$

$$F(A) = \frac{1}{\pi}\int_0^{2\pi} z\left(A\cos\phi, -A\omega(A)\sin\phi\right)\cos\phi \mathrm{d}\phi \tag{12}$$

and $\phi(t) = \omega(A)t + \psi(t)$.

Note that the equivalent elements $\omega(A)$ and $\beta(A)$ depend on the response non-stationary amplitude A to account for the nonlinearities and the fractional derivative terms of the original system. Thus, $\omega(A)$ and $\beta(A)$ can be construed as non-stationary stochastic processes, whose time-varying mean values are given by applying the expectation operator on Eqs. (9) and (10). This yields

$$\beta_{eq}(t) = \int_0^{\infty} \beta(A)p(A,t)\mathrm{d}A \tag{13}$$

and

$$\omega_{eq}^2(t) = \int_0^{\infty} \omega^2(A)p(A,t)\mathrm{d}A, \tag{14}$$

respectively. Further, Eqs. (13-14) can be associated with an alternative to Eq. (8) ELS of the form

$$\ddot{x}(t) + (\beta_0 + \beta_{eq}(t))\dot{x}(t) + \omega_{eq}^2(t)x(t) = w(t). \tag{15}$$

Next, it is readily seen that the evaluation of the ELS time-dependent damping $\beta_{eq}(t)$ and stiffness $\omega_{eq}(t)$ elements via Eqs. (13-14) requires knowledge of the non-stationary response amplitude PDF $p(A,t)$. In this regard, the stationary response amplitude PDF corresponding to a linear oscillator with fractional derivative terms and subjected to Gaussian white noise was obtained in closed-form in [46] based on stochastic averaging. Motivated by this analytical solution, a generalized form of this PDF was considered in [17] for modeling the non-stationary response amplitude PDF of the nonlinear oscillator governed by Eq. (1), or equivalently, by Eq. (7). This takes the form

$$p(A,t) = G\frac{A}{c(t)}\exp\left(-G\frac{A^2}{2c(t)}\right), \tag{16}$$

where

$$G = \frac{\sin\left(\frac{\alpha\pi}{2}\right)}{\omega_0^{1-\alpha}} \tag{17}$$

and $c(t)$ is a time-dependent coefficient to be determined. Further, based on a stochastic averaging solution treatment of Eq. (15), it was shown in [17] that substituting Eq. (16) for $p(A_2, t_2|A_1 = 0, t_1 = 0) = p(A, t)$ into the associated Fokker-Planck partial differential equation governing the evolution in time of the response amplitude PDF, i.e.,

$$\frac{\partial p(A_2,t_2|A_1,t_1)}{\partial t_2} = -\frac{\partial}{\partial A_2}\left\{\left(-\frac{1}{2}(\beta_0+\beta_{eq}(t_1,t_2))A_2 + \frac{\pi S(\omega_{eq}(t_1,t_2),t_2)}{2\omega_{eq}^2(t_1,t_2)A_2}\right) \times p(A_2,t_2|A_1,t_1)\right\} + \frac{1}{4}\frac{\partial}{\partial A_2}\left\{\frac{\pi S(\omega_{eq}(t_1,t_2),t_2)}{\omega_{eq}^2(t_1,t_2)}\frac{\partial p(A_2,t_2|A_1,t_1)}{\partial A_2} + \frac{\partial}{\partial A_2}\left[\frac{\pi S(\omega_{eq}(t_1,t_2),t_2)}{\omega_{eq}^2(t_1,t_2)}p(A_2,t_2|A_1,t_1)\right]\right\}, \tag{18}$$

and manipulating, leads to

$$\dot{c}(t) = -\left(\beta_0 + \beta_{eq}(t)\right)c(t) + \pi G\frac{S(\omega_{eq}(t),t)}{\omega_{eq}^2(t)}. \tag{19}$$

Eq. (19) constitutes a deterministic first-order nonlinear ordinary differential equation. This can be solved readily by any standard numerical integration scheme, such as the Runge–Kutta, for determining the time-dependent coefficient $c(t)$. Furthermore, $c(t)$ can be used for evaluating the ELS time-dependent damping and stiffness elements by employing Eqs. (13) and (14). Notably, it was shown in [17] that the scaled by G^{-1} time-dependent coefficient $c(t)$, where G is defined in Eq. (17), can approximate the oscillator non-stationary response variance, i.e.,

$$\mathrm{E}[x^2] = G^{-1}c(t). \tag{20}$$

2.2 Approximate closed form expressions for the oscillator non-stationary transition and joint response amplitude PDFs

In this section, novel approximate closed form expressions are derived for the oscillator non-stationary transition and joint response amplitude PDFs. They can be construed as generalizations of the results in [45] to account for fractional derivative elements in the governing equation of motion. The interested reader is also directed to [47, 42] for relevant earlier research efforts.

In this regard, following a similar analysis as in [45, 47] and motivated by the closed form expressions derived in [45] pertaining to oscillators with integer order derivative elements, the transition response amplitude PDF $p(A_2, t_2|A_1, t_1)$ corresponding to the oscillator governed by Eq. (1) is sought in the form

$$p(A_2,t_2|A_1,t_1) = G\frac{A_2}{c(t_1,t_2)}\exp\left(-G\frac{A_2^2+h^2(t_1,t_2)}{2c(t_1,t_2)}\right) I_0\left(G\frac{A_2 h(t_1,t_2)}{c(t_1,t_2)}\right), \tag{21}$$

where $c(t_1, t_2)$ and $h(t_1, t_2)$ denote time-dependent functions to be determined, and $I_0(\cdot)$ is the modified Bessel function of the first kind and of zero order.

Next, Eq. (21) is substituted into the Fokker-Planck Eq. (18). Specifically, denoting for simplicity $p = p(A_2, t_2|A_1, t_1)$, $c = c(t_1, t_2)$ and $h = h(t_1, t_2)$, Eq. (21) is differentiated first with respect to time. This yields

$$\frac{\partial p}{\partial t_2} = Gp\left[-G^{-1}\frac{\dot{c}}{c} - \frac{2ch\dot{h} - (A_2^2+h^2)\dot{c}}{2c^2} + \frac{I_1\left(G\frac{A_2 h}{c}\right)}{I_0\left(G\frac{A_2 h}{c}\right)}\frac{A_2\dot{h}c - A_2 h\dot{c}}{c^2}\right], \tag{22}$$

whereas differentiating with respect to the response amplitude leads to

$$\frac{\partial p}{\partial A_2} = Gp\left[-G^{-1}\frac{\dot{c}}{A_2} - \frac{A_2}{c} + \frac{I_1\left(G\frac{A_2 h}{c}\right)}{I_0\left(G\frac{A_2 h}{c}\right)}\frac{h}{c}\right]. \tag{23}$$

In Eqs. (22-23), $I_1(\cdot)$ denotes the Bessel function of the first kind and of order one. Note that the derivative of the Bessel function can be expressed as [1]

$$\frac{\partial I_1\left(G\frac{A_2 h}{c}\right)}{\partial A_2} = G\frac{h}{c}I_0\left(G\frac{A_2 h}{c}\right) - \frac{1}{A_2}I_1\left(G\frac{A_2 h}{c}\right). \tag{24}$$

Further, differentiating Eq. (23) with respect to A_2 and considering Eq. (24) yields

$$\frac{\partial^2 p}{\partial A_2^2} = \frac{\partial}{\partial A_2}\left(\frac{p}{A_2}\right) + G^2\frac{p}{c}\left[-2G^{-1} + \frac{A_2^2 + h^2}{c} - 2\frac{I_1\left(G\frac{A_2 h}{c}\right)}{I_0\left(G\frac{A_2 h}{c}\right)}\frac{A_2 h}{c}\right]. \tag{25}$$

Lastly, substituting Eqs. (21-23) and Eq. (25) into the Fokker-Planck Eq. (18), and manipulating, yields

$$\begin{aligned}
&\left(\dot{c} + (\beta_0 + \beta_{eq}(t_1,t_2))c - G\frac{\pi S(\omega_{eq}(t_1,t_2),t_2)}{\omega_{eq}^2(t_1,t_2)}\right) \\
&\times\left(-1 + G\frac{A_2^2 + h^2}{2c} - G\frac{I_1\left(G\frac{A_2 h}{c}\right)}{I_0\left(G\frac{A_2 h}{c}\right)}\frac{A_2 h}{c}\right) \\
&+ G\left(\dot{h} + \frac{1}{2}(\beta_0 + \beta_{eq}(t_1,t_2))h\right)\left(-h + \frac{I_1\left(G\frac{A_2 h}{c}\right)}{I_0\left(G\frac{A_2 h}{c}\right)}A_2\right) = 0.
\end{aligned} \tag{26}$$

Eq. (26) is satisfied if

$$\frac{dc(t_1,t_2)}{dt_2} + \left(\beta_0 + \beta_{eq}(t_1,t_2)\right)c(t_1,t_2) - \pi G\frac{S(\omega_{eq}(t_1,t_2),t_2)}{\omega_{eq}^2(t_1,t_2)} = 0 \tag{27}$$

and

$$\frac{dh(t_1,t_2)}{dt_2} + \frac{1}{2}\left(\beta_0 + \beta_{eq}(t_1,t_2)\right)h(t_1,t_2) = 0. \tag{28}$$

Note that Eqs. (27-28) are subject to the initial condition $p(A_2,t_1|A_1,t_1) = \delta(A_2 - A_1)$. Further, relying on the Markovian response assumption for the process A, the joint response amplitude PDF is given by

$$p(A_1,t_1;A_2,t_2) = p(A_1,t_1)p(A_2,t_2|A_1,t_1), \tag{29}$$

which, considering Eqs. (16) and (21), becomes

$$\begin{aligned}
p(A_1,t_1;A_2,t_2) = G^2&\frac{A_1 A_2}{c(t_1)c(t_1,t_2)}\, I_0\left(G\frac{A_2 h(t_1,t_2)}{c(t_1,t_2)}\right) \\
&\times \exp\left(-G\frac{A_2^2 c(t_1) + A_1^2 c(t_1,t_2) + h^2(t_1,t_2)c(t_1)}{2c(t_1,t_2)}\right).
\end{aligned} \tag{30}$$

2.3 Oscillator survival probability determination

In this section, the technique developed in [45] for determining approximately the survival probability of nonlinear oscillators subject to evolutionary stochastic excitation is extended to account for fractional derivative elements in the governing equation of motion.

In this regard, the survival probability $P_B(T)$ of the oscillator in Eq. (1) is defined as the probability that the response amplitude A stays below a given threshold B over the time interval $[t_0, T]$. Next, following [45] and taking into account that the response amplitude A varies slowly with respect to time, the time-domain is discretized into intervals of the form

$$[t_{i-1}, t_i], \quad i = 1, 2, \ldots, M, \quad t_0 = 0, \quad t_M = T \text{ and } t_i = t_{i-1} + qT_{eq}(t_{i-1}) \tag{31}$$

over which A is considered constant. Thus, $P_B(T)$ is also considered constant over $[t_{i-1}, t_i]$. In Eq. (31), the time intervals are defined by setting $t_0 = 0$, $t_M = T$ and using a time-step $\Delta t = t_i - t_{i-1} = qT_{eq}(t_{i-1})$ for $i = 1, 2, \ldots, M$, where $q \in (0, 1]$ and $T_{eq}(t)$ denotes the time-dependent equivalent natural period of the oscillator response given by $T_{eq}(t) = \frac{2\pi}{\omega_{eq}(t)}$. Note that the condition $q \in (0, 1]$ is consistent with the assumption adopted in section 2.1 that the response amplitude is approximately constant over one cycle of oscillation, i.e., over the instantaneous natural period $T_{eq}(t)$, and thus, over $qT_{eq}(t)$ as well.

Further, considering the discretization of the time domain shown in Eq. (31), the oscillator survival probability can be approximated by

$$P_B(T = t_M) = \prod_{i=1}^{M} \{1 - F_i\}, \tag{32}$$

where F_i is defined as the probability that A will cross the barrier B in the time interval $[t_{i-1}, t_i]$, given that no crossings have occurred prior to time t_{i-1}. Next, invoking the Markovian property for the process A and utilizing the definition of conditional probability yields

$$F_i = \frac{Prob\,\{(A(t_i) \geq B) \cap (A(t_{i-1}) < B)\}}{Prob\,\{A(t_{i-1}) < B\}} = \frac{Q_{i-1,i}}{H_{i-1}}, \tag{33}$$

where

$$Q_{i-1,i} = \int_B^0 \mathrm{d}A_i \int_0^B p(A_{i-1}, t_{i-1}; A_i, t_i)\mathrm{d}A_{i-1} \tag{34}$$

and

$$H_{i-1} = \int_0^B p(A_{i-1}, t_{i-1})\mathrm{d}A_{i-1}. \tag{35}$$

Attention is directed next to the efficient computation of the integrals in Eqs. (34-35). First, in agreement with their slowly varying behavior in time, the equivalent elements $\omega_{eq}(t)$ and $\beta_{eq}(t)$ are considered to be constant over $[t_{i-1}, t_i]$. In this regard, they are approximated as $\beta_{eq}(t) = \beta_{eq}(t_{i-1})$ and $\omega_{eq}(t) = \omega_{eq}(t_{i-1})$ over $[t_{i-1}, t_i]$ for $i = 1, 2, \ldots, M$. Further, taking into account Eq. (20) in conjunction with the condition $p(A_i, t_{i-1}|A_{i-1}, t_{i-1}) = \delta(A_i - A_{i-1})$, Eqs. (27) and (28) are solved over $[t_{i-1}, t_i]$ to yield

$$\begin{aligned} c(t_{i-1}, t_i) = {} & \frac{\pi G}{\omega_{eq}^2(t_{i-1})} \exp\left(-\left(\beta_0 + \beta_{eq}(t_{i-1})\right) t_i\right) \\ & \times \int_{t_{i-1}}^{t_i} \exp\left((\beta_0 + \beta_{eq}(t_{i-1}))z\right) S_w\left(\omega_{eq}(t_{i-1}), z\right) \mathrm{d}z \end{aligned} \tag{36}$$

and

$$h(t_{i-1}, t_i) = A_{i-1} \exp\left(-\frac{1}{2}\left(\beta_0 + \beta_{eq}(t_{i-1})\right)\tau_i\right), \tag{37}$$

respectively, where $\tau_i = t_i - t_{i-1}$. Furthermore, according to the theory of locally stationary processes (e.g., [34, 11, 30]), the excitation EPS is considered to vary slowly with time. Thus, employing the approximation $\exp\left(\left(\beta_0 + \beta_{eq}(t_{i-1})\right)t\right) S_w\left(\omega_{eq}(t_{i-1}), t\right) = \exp\left(\left(\beta_0 + \beta_{eq}(t_{i-1})\right)t_{i-1}\right)$ $\times S_w\left(\omega_{eq}(t_{i-1}), t_{i-1}\right)$ over $[t_{i-1}, t_i]$, Eq. (36) becomes

$$c(t_{i-1}, t_i) = \pi G \frac{S_w\left(\omega_{eq}(t_{i-1}), t_{i-1}\right)}{\omega_{eq}^2(t_{i-1})} \tau_i \, \exp\left(-(\beta_0 + \beta_{eq}(t_{i-1}))\tau_i\right). \tag{38}$$

Next, applying a Taylor series expansion around zero and considering the first order term only, the exponential on the right-hand side of Eq. (38) is approximated by $\exp\left(-(\beta_0 + \beta_{eq}(t_{i-1}))\tau_i\right) = 1 - (\beta_0 + \beta_{eq}(t_{i-1}))\tau_i$. Thus, Eq. (38) becomes

$$c(t_{i-1}, t_i) = \pi G \frac{S_w\left(\omega_{eq}(t_{i-1}), t_{i-1}\right)}{\omega_{eq}^2(t_{i-1})} \tau_i. \tag{39}$$

Similarly to the derivation of Eq. (39), Eq. (37) is approximated as

$$h(t_{i-1}, t_i) = A_{i-1}\sqrt{1 - (\beta_0 + \beta_{eq}(t_{i-1}))\tau_i}. \tag{40}$$

Further, applying a Taylor series expansion around $t = t_{i-1}$ in the time interval $[t_{i-1}, t_i]$, and considering the first order term only, yields

$$c(t_i) = c(t_{i-1}) + \left.\frac{\mathrm{d}c(t)}{\mathrm{d}t}\right|_{t=t_{i-1}} \tau_i. \tag{41}$$

Taking into account Eqs. (19) and (39), Eq. (41) becomes

$$c(t_i) = c(t_{i-1})(1 - (\beta_0 + \beta_{eq}(t_{i-1}))\tau_i) + c(t_{i-1}, t_i) \tag{42}$$

or, equivalently,

$$c(t_{i-1}, t_i) = c(t_i)\left(1 - r_i^2\right), \tag{43}$$

where

$$r_i^2 = \frac{c(t_{i-1})}{c(t_i)}\left(1 - (\beta_0 + \beta_{eq}(t_{i-1}))\tau_i\right). \tag{44}$$

The parameter r_i^2 can be construed as a measure of the correlation of the random variables A_{i-1} and A_i, since $r_i^2 \to 0$ for $\tau_i \to \infty$ and $r_i^2 \to 1$ for $\tau_i \to 0$; see also [45] for a relevant discussion. Next, considering Eqs. (39-40) and Eqs. (43-44), the joint response amplitude PDF of Eq. (30) becomes

$$\begin{aligned} p(A_{i-1}, t_{i-1}; A_i, t_i) = {} & \frac{G^2 A_{i-1} A_i}{c(t_{i-1})c(t_i)(1 - r_i)^2} I_0\left(\frac{G A_{i-1} A_i r_i}{\sqrt{c(t_{i-1})c(t_i)(1 - r_i^2)^2}}\right) \\ & \times \exp\left(-G\frac{A_i^2 c(t_{i-1}) + A_{i-1}^2 c(t_i)}{2c(t_{i-1})c(t_i)(1 - r_i^2)^2}\right). \end{aligned} \tag{45}$$

Further, the modified Bessel function of the first kind of order zero in Eq. (45) can be expanded in the form [1]

$$I_0(y) = 1 + \sum_{n=1}^{\infty} \frac{y^{2n}}{\prod_{k=1}^{n}(2k)^2}. \tag{46}$$

Furthermore, substituting Eqs. (45-46) into Eq. (34) yields

$$Q_{i-1,i} = D_0 + \sum_{n=1}^{N} D_n, \tag{47}$$

where

$$D_0 = \int_B^\infty \int_0^B \frac{G^2 A_{i-1} A_i}{c(t_{i-1})c(t_i)(1-r_i^2)} \exp\left(-G\frac{A_i^2 c(t_{i-1}) + A_{i-1}^2 c(t_i)}{2c(t_{i-1})c(t_i)(1-r_i^2)}\right) \mathrm{d}A_{i-1}\mathrm{d}A_i, \tag{48}$$

$$D_n = \frac{r_i^{2n} G^{2n+2}}{(c(t_{i-1})c(t_i))^{n+1}(1-r_i^2)^{2n+1}\prod_{k=1}^{N}(2k)^2} L_n, \tag{49}$$

and

$$L_n = \int_B^\infty \int_0^B (A_{i-1}A_i)^{2n+1} \exp\left(-G\frac{A_i^2 c(t_{i-1}) + A_{i-1}^2 c(t_i)}{2c(t_{i-1})c(t_i)(1-r_i^2)}\right) \mathrm{d}A_{i-1}\mathrm{d}A_i. \tag{50}$$

Note that Eqs. (48) and (50) can be integrated analytically to yield

$$D_0 = (1-r_i^2)\exp\left(-\frac{GB^2}{2c(t_i)(1-r_i^2)}\right)\left\{1 - \exp\left(-\frac{GB^2}{2c(t_{i-1})(1-r_i)^2}\right)\right\} \tag{51}$$

and

$$\begin{aligned} L_n = {} & 4^n(1-r_i^2)^{2n+2}c(t_{i-1})^{n+1}c(t_i)^{n+1}G^{-2n-2}\Gamma\left(n+1, \frac{GB^2}{2c(t_i)(1-r_i^2)}\right) \\ & \times \left\{\Gamma(n+1) - \Gamma\left(n+1, \frac{GB^2}{2c(t_{i-1})(1-r_i^2)}\right)\right\}, \end{aligned} \tag{52}$$

respectively. In Eq. (52), $\Gamma(z) = \int_0^\infty s^{z-1}\mathrm{e}^{-s}\mathrm{d}s$ and $\Gamma(\gamma, z) = \int_z^\infty s^{\gamma-1}\mathrm{e}^{-s}\mathrm{d}s$ denote the Gamma and upper incomplete Gamma functions, respectively (e.g., [1]). Lastly, considering Eq. (21), the integral of Eq. (35) is analytically calculated yielding

$$H_{i-1} = 1 - \exp\left(-\frac{GB^2}{2c(t_{i-1})}\right). \tag{53}$$

2.4 Mechanization of the technique

The mechanization of the herein developed approximate analytical technique for determining the survival probability of nonlinear/hysteretic oscillators with fractional derivative elements comprises the following steps:

i. Solve numerically the deterministic first-order nonlinear differential equation of Eq. (19) for determining the time-dependent coefficient $c(t)$. This can be done by employing standard integration schemes, such as the Runge-Kutta [41].

$ii.$ Determine the non-stationary response amplitude PDF $p(A,t)$ of Eq. (16) and the equivalent time-dependent damping $\beta_{eq}(t)$ and stiffness $\omega_{eq}(t)$ elements of Eqs. (13) and (14), respectively.

$iii.$ Consider a discretized time domain as $t_i = t_{i-1} + qT_{eq}(t_{i-1})$, where $q \in (0,1]$ and $T_{eq}(t) = \frac{2\pi}{\omega_{eq}(t)}$.

$iv.$ Use Eqs. (53) and (47) to determine H_{i-1} and $Q_{i-1,i}$, respectively.

$v.$ Determine the survival probability $P_B(T)$ based on Eq. (32).

Clearly, the herein developed approximate analytical technique for determining the oscillator survival probability is characterized by enhanced computational efficiency. Indeed, the computational cost is kept to a minimum level, since it corresponds, in essence, only to the numerical integration of the deterministic differential equation shown in Eq. (19). Further, the technique exhibits a quite high degree of versatility since it can account readily for diverse nonlinear/hysteretic response behaviors, and for excitation EPS of arbitrary form, even with time-varying frequency contents.

3 Numerical examples

The reliability of the herein developed approximate analytical technique is assessed by considering as numerical examples a hardening Duffing and a bilinear hysteretic nonlinear oscillators with fractional derivative elements. The oscillators are initially at rest and subjected to non-stationary stochastic excitation described by the non-separable EPS (e.g., [47, 17])

$$S_w(\omega, t) = S_0 \left(\frac{\omega}{5\pi}\right)^2 \exp(-b_0 t) t^2 \exp\left(-\left(\frac{\omega}{5\pi}\right)^2 t\right), \tag{54}$$

which is plotted in Fig. 1 for $S_0 = 0.5$ and $b_0 = 0.1$. Notably, the EPS of Eq. (54) comprises some of the main characteristics of seismic shaking, such as decreasing of the dominant frequency with time; see also [27, 4, 35, 10] for a broader perspective. In the ensuing analysis, the time-dependent survival probabilities determined by the approximate analytical technique are compared with MCS-based estimates. The latter are determined by utilizing the spectral representation scheme [25] to generate excitation realizations ($10,000$ samples) compatible with Eq. (54), in conjunction with an L1-algorithm [20] for integrating numerically Eq. (1) and for determining response realizations.

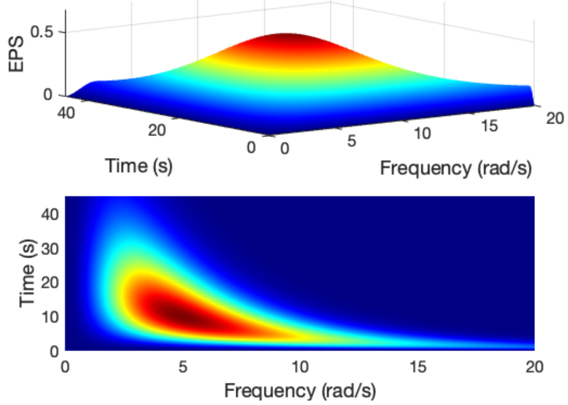


Fig. 1: Non-separable excitation evolutionary power spectrum.

3.1 Duffing nonlinear oscillator with fractional derivative elements

The equation of motion of a Duffing nonlinear oscillator with fractional derivative elements is given by Eq. (1) with

$$z(t, x, \dot{x}) = \omega_0^2 x(1 + \varepsilon x^2), \tag{55}$$

where the parameter $\varepsilon > 0$ controls the magnitude of the nonlinearity. Next, considering Eq. (55) in conjunction with Eq. (16) the time-dependent equivalent elements of Eqs. (13) and (14) become

$$\beta_{eq}(t) = -\beta_0 + \frac{\beta G \sin\left(\frac{\alpha\pi}{2}\right)}{c(t)} \int_0^\infty \frac{A}{\omega^{1-\alpha}(A)} \exp\left(-\frac{GA^2}{2c(t)}\right) \mathrm{d}A \tag{56}$$

and

$$\omega_{eq}^2(t) = \omega_0^2 + \frac{\beta G \cos\left(\frac{\alpha\pi}{2}\right)}{c(t)} \int_0^\infty \omega^\alpha(A) A \exp\left(-\frac{GA^2}{2c(t)}\right) \mathrm{d}A + \frac{3\varepsilon\omega_0^2 G}{4c(t)} \int_0^\infty A^3 \exp\left(-\frac{GA^2}{2c(t)}\right) \mathrm{d}A, \tag{57}$$

respectively. Note that for the special case of $\alpha = 1$ for which the fractional derivative degenerates to a standard first order derivative, Eq. (56) yields $\beta_{eq}(t) = -\beta_0 + \beta$. Thus, the effective damping coefficient of the ELS in Eq. (15) becomes $\beta_0 + \beta_{eq}(t) = \beta$. In other words, as anticipated for the limiting case $\alpha = 1$, the fractional derivative term in Eq. (1) becomes $\beta \mathcal{D}_{0,t}^\alpha x(t) = \beta \dot{x}(t)$ and contributes to damping only. Similarly, for $\alpha = 0$, Eq. (56) yields $\beta_{eq}(t) = -\beta_0$, and thus, the effective damping coefficient of the ELS in Eq. (15) becomes $\beta_0 + \beta_{eq}(t) = 0$. This is consistent with the original Eq. (1), where $\alpha = 0$ yields $\beta \mathcal{D}_{0,t}^\alpha x(t) = \beta x(t)$. That is, the fractional derivative term contributes to stiffness only.

Further, using the parameters values $\beta_0 = 0.07$, $\omega_0 = 3.6120$, $\beta = 0.07$, $b_0 = 0.15$, and $S_0 = 1$, Eq. (19) is solved numerically for determining $c(t)$. This is shown in Fig. 2 for an indicative fractional derivative order $\alpha = 0.75$ and nonlinearity magnitude $\varepsilon = 0.5$. Next, $c(t)$ is substituted into Eqs. (56-57) for evaluating $\omega_{eq}(t)$ and $\beta_{eq}(t) + \beta_0$, which are plotted in Fig. 3. Furthermore, setting $q = 0.1$ and $N = 60$, and following the steps outlined in section 2.4, the oscillator survival probability is determined by Eq. (32). This is plotted in Fig. 4 for various values of the fractional derivative order and in Fig. 5 for various barrier level values. Based on comparisons with MCS data (10,000 realizations), it is seen that the approximate technique exhibits a satisfactory degree of accuracy.

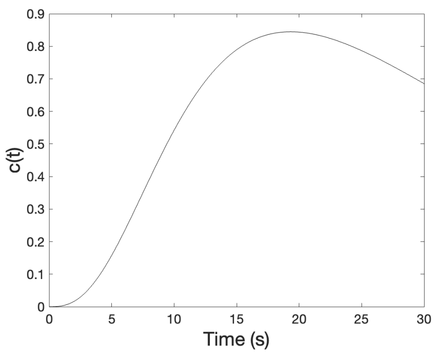
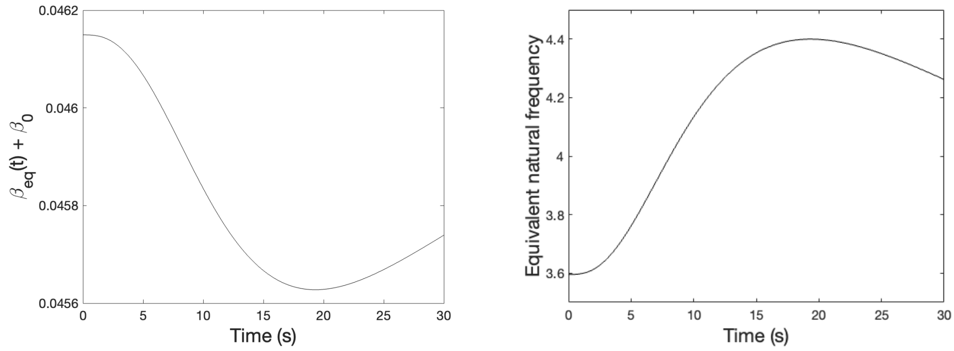


Fig. 2: Time-dependent coefficient $c(t)$ of Eq. (19) for the Duffing nonlinear oscillator described by Eqs. (1) and (55) with $\beta_0 = 0.07$, $\omega_0 = 3.6120$, and $\beta = 0.07$, fractional derivative order $\alpha = 0.75$ and nonlinearity magnitude $\varepsilon = 0.5$.



(a) (b)

Fig. 3: (a) Equivalent time-dependent damping $\beta_{eq}(t) + \beta_0$, and (b) equivalent time-dependent natural frequency $\omega_{eq}(t)$ corresponding to the oscillator described by Eqs. (1) and (55) with $\beta_0 = 0.07$, $\omega_0 = 3.6120$, and $\beta = 0.07$, fractional derivative order $\alpha = 0.75$ and nonlinearity magnitude $\varepsilon = 0.5$.

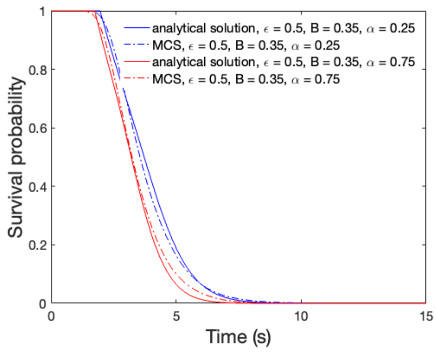


Fig. 4: Survival probability of a Duffing nonlinear oscillator with fractional derivative elements under evolutionary stochastic excitation for $\varepsilon = 0.5$, $B = 0.4$ and for various values of the fractional derivative order; comparison with MCS data $(10,000$ realizations).

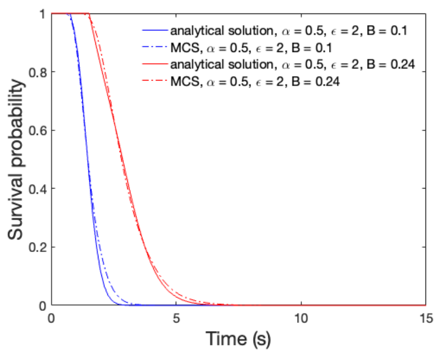


Fig. 5: Survival probability of a Duffing nonlinear oscillator with fractional derivative elements under evolutionary stochastic excitation for $\varepsilon = 2$, $\alpha = 0.5$ and for various values of the barrier level; comparison with MCS data $(10,000$ realizations).

3.2 Bilinear hysteretic oscillator with fractional derivative elements

Consider next a bilinear hysteretic oscillator with fractional derivative terms governed by Eq. (1) with

$$z(t,x,\dot{x}) = \gamma\omega_0^2 x + (1-\gamma)\omega_0^2 x_y z_0, \tag{58}$$

where γ denotes the post- to pre-yield stiffness ratio and x_y is the critical value of the displacement at which yielding occurs. Further, z_0 represents the hysteretic component corresponding to the elastoplastic characteristic. It is described by the differential equation (e.g., [37, 7])

$$x_y\dot{z}_0 = \dot{x}\left(1 - H(\dot{x})H(z_0-1) - H(-\dot{x})H(-z_0-1)\right), \tag{59}$$

where $H(\cdot)$ denotes the Heaviside step function.

Further, considering Eq. (58), Eq. (11) becomes

$$S(A) = \begin{cases} \frac{4x_y}{\pi}\left(1-\frac{x_y}{A}\right) & A > x_y \\ 0 & A \leq x_y \end{cases}, \tag{60}$$

whereas Eq. (12) takes the form

$$F(A) = \begin{cases} \frac{A}{\pi}\left(\Lambda - \frac{1}{2}\sin(2\Lambda)\right) & A > x_y \\ A & A \leq x_y \end{cases}, \tag{61}$$

with Λ given by

$$\Lambda = \arccos\left(1 - \frac{2x_y}{A}\right). \tag{62}$$

Furthermore, taking into account Eqs. (60-61) and Eq. (16), the ELS time-variant elements of Eqs. (13) and (14) become

$$\begin{aligned}\beta_{eq}(t) =& -\beta_0 + \frac{\beta G\sin\left(\frac{\alpha\pi}{2}\right)}{c(t)}\int_0^\infty \frac{A}{\omega^{1-\alpha}(A)}\exp\left(-\frac{GA^2}{2c(t)}\right)\mathrm{d}A \\ &+ \frac{4x_y\omega_0^2(1-\gamma)G}{\pi c(t)}\int_{x_y}^\infty \frac{1-\frac{x_y}{A}}{\omega(A)}\exp\left(-\frac{GA^2}{2c(t)}\right)\mathrm{d}A\end{aligned} \tag{63}$$

and

$$\begin{aligned}\omega_{eq}^2(t) = \omega_0^2 - (1-\gamma)\omega_0^2 &\left\{\exp\left(-\frac{Gx_y^2}{2c(t)}\right)\right. \\ &\left.-\frac{G}{\pi c(t)}\int_{x_y}^\infty\left(\Lambda - \frac{1}{2}\sin(2\Lambda)\right)A\exp\left(-\frac{GA^2}{2c(t)}\right)\mathrm{d}A\right\} \\ &+\frac{\beta G\cos\left(\frac{\alpha\pi}{2}\right)}{c(t)}\int_0^\infty \omega^\alpha(A)A\exp\left(-\frac{GA^2}{2c(t)}\right)\mathrm{d}A,\end{aligned} \tag{64}$$

respectively.

Next, applying the herein developed technique, whose steps are outlined in section 2.4, the survival probability of the bilinear hysteretic oscillator is determined. In particular, considering the parameters values $S_0 = 0.08$ and $b_0 = 0.12$ for the excitation EPS, and $\omega_0 = 5.47$, $\beta_0 = 0.1$ and $x_y = 0.07$ for the oscillator, Eq. (19) is solved numerically to compute the time-dependent function $c(t)$. This is shown in Fig. 6 for an indicative value of the fractional derivative order

$\alpha = 0.25$ and post- to pre-yield stiffness ratio $\gamma = 0.7$. Further, substituting $c(t)$ into Eqs. (63-64) the ELS damping and natural frequency elements are evaluated and plotted in Fig. 7. Furthermore, using the parameter values $q = 0.25$ and $N = 60$, the oscillator survival probability is determined by Eq. (32). This is plotted in Fig. 8 for various values of the fractional derivative order, and in Fig. 9 for various barrier level values. Comparisons with pertinent MCS-based estimates $(10,000$ samples) demonstrate a quite satisfactory degree of accuracy.

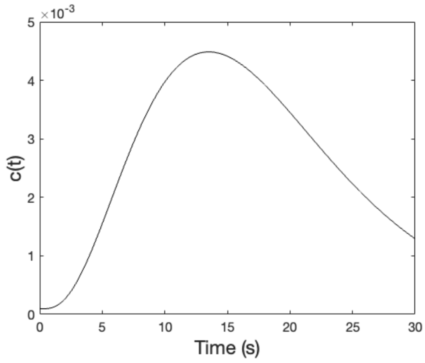
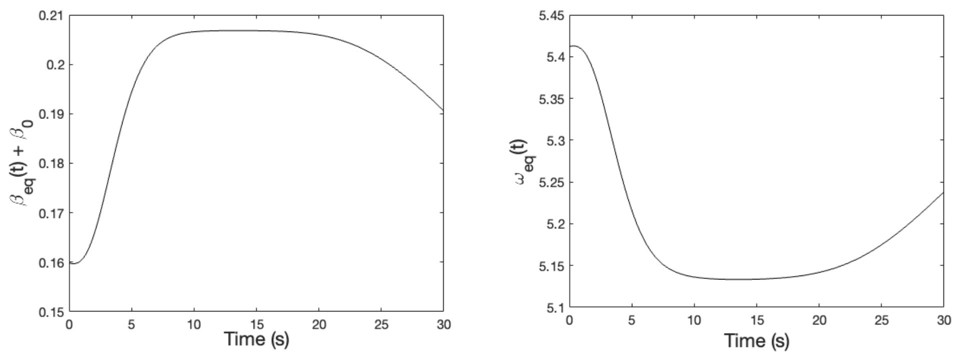


Fig. 6: Time-dependent coefficient $c(t)$ of Eq. (19) for the bilinear hysteretic oscillator described by Eqs. (1) and (58) with $\beta_0 = 0.1$, $\omega_0 = 5.47$ and $x_y = 0.07$, fractional derivative order $\alpha = 0.25$ and post- to pre-yield stiffness ratio $\gamma = 0.7$.



(a) (b)

Fig. 7: (a) Equivalent time-dependent damping $\beta_{eq}(t) + \beta_0$, and (b) equivalent time-dependent natural frequency $\omega_{eq}(t)$ corresponding to the oscillator described by Eqs. (1) and (58) with $\beta_0 = 0.1$, $\omega_0 = 5.47$ and $x_y = 0.07$, fractional derivative order $\alpha = 0.25$ and post-to pre-yield stiffness ratio $\gamma = 0.7$.

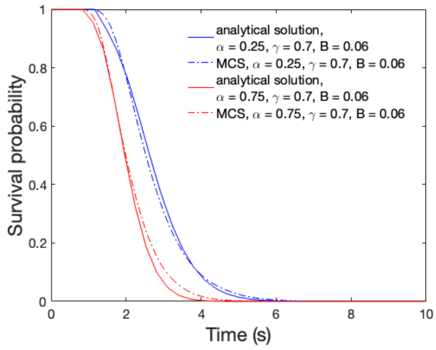


Fig. 8: Survival probability of a bilinear hysteretic oscillator with fractional derivative elements under evolutionary stochastic excitation for $\gamma = 0.7$, $B = 0.06$ and for various values of the fractional derivative order; comparison with MCS data $(10,000$ realizations).

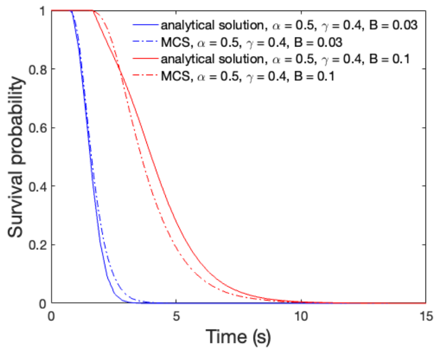


Fig. 9: Survival probability of a bilinear hysteretic oscillator with fractional derivative elements under evolutionary stochastic excitation for $\gamma = 0.4$, $\alpha = 0.5$ and for various values of the barrier level; comparison with MCS data $(10,000$ realizations).

4 Concluding remarks

In this paper, an approximate analytical technique has been developed for determining the survival probability of nonlinear/hysteretic oscillators with fractional derivative elements under evolutionary stochastic excitation. This has been done by relying on a combination of statistical linearization and stochastic averaging for deriving approximate closed form expressions for the oscillator non-stationary marginal, transition, and joint response amplitude PDFs and, ultimately, for the time-dependent oscillator survival probability. A significant advantage of the technique relates to the fact that it can account readily for a wide range of nonlinear/hysteretic response behaviors. Further, non-stationary excitation EPS, even of the non-separable kind with time-dependent frequency content, can be treated in a direct manner. Notably, the technique is characterized by significant efficiency. In fact, the associated computational cost relates, in essence, only to the numerical integration of a deterministic nonlinear differential equation governing approximately the evolution in time of the oscillator response variance. Overall, the developed technique can be construed as an extension of the concepts and the results in [45] to account for fractional derivative terms in the equation of motion. A hardening Duffing and a bilinear hysteretic nonlinear oscillators with fractional derivative elements have been considered as numerical examples. Comparisons with pertinent MCS data have shown that the approximate analytical technique exhibits a satisfactory degree of accuracy in determining the oscillator survival probability.

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