# Cops and Robbers on Multi-Layer Graphs

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**Abstract.** We generalise the popular *cops and robbers* game to multi-layer graphs, where each cop and the robber are restricted to a single layer (or set of edges). We show that initial intuition about the best way to allocate cops to layers is not always correct, and prove that the multi-layer cop number is neither bounded from above nor below by any function of the cop numbers of the individual layers. We determine that it is NP-hard to decide if $k$ cops are sufficient to catch the robber, even if each layer is a tree plus some isolated vertices. However, we give a polynomial time algorithm to determine if $k$ cops can win when the robber layer is a tree. Additionally, we investigate a question of worst-case division of a simple graph into layers: given a simple graph $G$, what is the maximum number of cops required to catch a robber over all multi-layer graphs where each edge of $G$ is in at least one layer and all layers are connected? For cliques, suitably dense random graphs, and graphs of bounded treewidth, we determine this parameter up to multiplicative constants. Lastly we consider a multi-layer variant of Meyniel's conjecture, and show the existence of an infinite family of graphs whose multi-layer cop number is bounded from below by a constant times $n/\log n$, where $n$ is the number of vertices in the graph.

**Keywords:** Cops and robbers, multi-layer graphs, pursuit–evasion games, Meyniel's conjecture

## 1 Introduction

We investigate the game of cops and robbers played on multi-layer graphs. Cops and robbers is a 2-player adversarial game played on a graph introduced independently by Nowakowski and Winkler [22], and Quilliot [25]. At the start of the game, the cop player chooses a starting vertex position for each of a specified number of cops, and the robber player then chooses a starting vertex position for the robber. Then in subsequent rounds, the cop player first chooses none, some, or all cops and moves them along exactly one edge to a new vertex. The robber player then either moves the robber along an edge, or leaves the robber on its current vertex. The cop player wins if after some finite number of rounds a cop occupies the same vertex as the robber, and the robber wins otherwise. Both players have perfect information about the graph and the locations of cops and robbers. Initially, research focussed on games with only one cop and one robber, and graphs on which the cop could win were classed as *copwin* graphs. Aigner and Fromme [1] introduced the idea of playing with multiple cops, and defined the *cop number* of a graph as the minimum number of cops required for the cop player to win on that graph. Many variants of the game have been studied, and for an in-depth background on cops and robbers, we direct the reader to [5].

In this paper, we play cops and robbers on multi-layer graphs where each cop and the robber will be associated with exactly one layer, and during their respective turns, will move only over the edges in their own layer. While we define multi-layer graphs formally in upcoming sections, roughly speaking, here a multi-layer graph is a single set of vertices with each layer being a different (though possibly overlapping) set of edges on those vertices. The variants we study could intuitively be based on the premise that the cops are assigned different modes of transport. For instance, a cop in a car may be able to move quickly down streets, while a cop on foot may be slower down a street, but be able to quickly cut between streets by moving through buildings or down narrow alleys.

Extending cops and robbers to multi-layer graphs creates some new variants, and generalises some existing variants. Fitzpatrick [14] introduced the *precinct* variant, which assigns to each cop a subset of the vertices (called their *beat*). In the precinct variant, a cop can never leave their beat. This can be modelled as multi-layer cops and robbers by restricting each layer to a given beat. Fitzpatrick [14] mainly considers the case were a beat is an isometric path, we allow more arbitrary (though usually spanning and connected) beats/layers. Clarke [11] studies the problem of covering a graph with a number of cop-win subgraphs to upper bound the cop number of a graph — again such constructions can be modelled as multi-layer graphs with the edges of each layer forming a cop-win graph. Another commonly studied variant of cops and robbers defines a speed $s$ (which may be infinite) such that the robber can move along a path of up to $s$ edges on their turn [6, Section 3.2]. These can also be modelled as multi-layer graphs by adding edges between any pair of vertices of distance at most $s$ that only belong to the layer the robber is occupying.

### 1.1 Further Related Work

Temporal graphs, in which edges are active only at certain time steps, are sometimes modelled as multi-layered graphs. There has been some work on cops and robbers on temporal graphs, though generally yielding quite a different game to the ones we consider here as a cop is not restricted to one layer. In particular, [3] considers cops and robbers on temporal graphs and when the full temporal graph is known they give a $O(n^3T)$ algorithm to determine the outcome of the game where $T$ is the number of timesteps.

Variants of cops and robbers are also studied for their relationships to other parameters of graphs. For instance, the cop number of a graph $G$ is at most one plus half the treewidth of $G$ [17]. And if one considers the “helicopter” variant of cops and robbers, the treewidth of a graph is strictly less than the helicopter cop number of the graph [29]. Toruńczyk [32] generalises many graph parameters, including treewidth, clique-width, degeneracy, rank-width, and twin-width, through the use of variants of cops and robbers. We introduce our multi-layer variants of cops and robbers partially in the hopes of spurring research towards multi-layer graph parameters using similar techniques.

Recently Lehner, resolving a conjecture by Schröeder [27], showed the cop number of a toroidal graph is at most three [19]. There is also an interesting connection between cop number and the genus of the host graph [1, 8, 26, 27]. It remains open whether any such connection can be made in the multi-layer setting.

### 1.2 Outline and Contributions

In Section 2 we define multi-layer graphs and multi-layer cops and robbers.

In Section 3 we develop several examples which highlight several counter-intuitive facts and properties of the multi-layer cops and robbers game. In particular, we show the multi-layer cop number is not bounded from above or below by a non-trivial function of the cop numbers of the individual cop layers.

In Section 4 we study the computational complexity of some multi-layer cops and robbers problems. We show that deciding if a given number of cops can catch a robber is $\mathsf{NP}$-hard even if each layer is a tree plus some isolated vertices, but that if only the robber layer is required to be a tree the problem is $\mathsf{FPT}$ in the number of cops and the number of layers of the graph.

In Section 5 we consider an extremal version of multi-layer cop number over all divisions into layers of a single-layer graph. In particular, for a given single-layer graph $G$ what is the maximum multi-layer cop number of any multi-layer graph $\mathcal{G}$ when all edges of $G$ are present in at least one layer of $\mathcal{G}$.

In Section 6 we consider Meyniel's conjecture, which states that the single-layer cop number is $\mathcal{O}(\sqrt{|V|})$ and is a central open question in cops and robbers. We investigate whether a multi-layer analogue of Meyniel's conjecture can hold and, determine the worst case multi-layer cop number up to a multiplicative $\mathcal{O}(\log n)$ factor. This contrasts with the situation on simple graphs, where the worst-case is only known up to a multiplicative $n^{1/2-o(1)}$ factor.

Finally in Section 7 we reflect and conclude with some open problems.

Due to space limitations, most proofs have been omitted. For a complete version with all proofs, please refer to [13].

## 2 Definitions and Notation

We write $[n]$ to mean the set of integers $\{1, \dots, n\}$, and given a set $V$ we write $\binom{V}{2}$ to mean all possible 2-element subsets (i.e., edges) of $V$. A simple graph is then defined as $G := (V, E)$ where $E \subseteq \binom{V}{2}$. For a vertex $v \in V$ we let $d_G(v) := |\{u : uv \in E\}|$ be the degree of vertex $v$ in $G$, and $\delta(G) := \min_{v \in V(G)} d_G(v)$ denote the minimum degree in a graph $G$. If, for all $v \in V$, $d_G(v) = r$ for some integer $r$, we say that $G$ is $r$-regular. If the exact value of $r$ is not important, we may just say that $G$ is regular. If, instead, for all $v \in V$, $d_G(v) \in \{r, r+1\}$ for some integer $r$, we say that $G$ is almost-regular. The *distance* between two vertices $u$ and $v$ in a graph is the length of a shortest path between $u$ and $v$.

A multi-layer graph $(V, \{E_1, \ldots, E_\tau\})$ consists of a vertex set $V$ and a collection $\{E_1, \ldots, E_\tau\}$, for some integer $\tau \geqslant 1$, of edge sets (or *layers*), where for each $i$, $E_i \subseteq \binom{V}{2}$. We often slightly abuse terminology and refer to a layer $E_i$ as a graph; when we do this, we specifically refer to the graph $(V, E_i)$ (i.e., we always include every vertex in the original multi-layer graph, even if such a vertex is isolated in $(V, E_i)$). For instance, we often restrict ourselves to multi-layer graphs where, for each $i \in [\tau]$, the simple graph $(V, E_i)$ is connected. We will say that each layer is *connected* to represent this notion. Given a multi-layer graph $(V, \{E_1, \ldots E_\tau\})$ let the *flattened* version of a multi-layer graph, written as $\mathsf{fl}(\mathcal{G})$, be the simple graph $G = (V, E_1 \cup \cdots \cup E_\tau)$.

Cops and robbers is typically played on a simple graph, with one player controlling some number of cops and the other player controlling the robber. On each turn, the cop player can move none, some, or all of the cops, however each cop can only move along a single edge incident to their current vertex. The robber player can then choose to move the robber along one edge, or have the robber stay still. The goal for the cop player is to end their turn with the robber on the same vertex as at least one cop, while the aim for the robber is to avoid capture indefinitely. If a cop player has a winning strategy on a graph $G$ with $k$ cops but not with $k-1$ cops, we say that the graph $G$ has cop number $k$, denoted $\mathsf{c}(G) = k$, and that $G$ is $k$-copwin. Given a multi-layer graph $(V, \{E_1, \ldots, E_\tau\})$, we will say the cop number of layer $E_i$ to mean the cop number of the graph $(V, E_i)$.

As this paper deals with both simple and multi-layer graphs, as well as cops and robbers variants played on these graphs, we will use *single-layer* as an adjective to denote when we are referring to either specifically a simple graph, or to cops and robbers played on a single-layer (i.e., simple) graph. This extends to parameters such as the cop number as well.

In this paper we consider the cops and robbers game on multi-layer graphs and so it will be convenient to define multi-layer graphs with a distinguished layer for the robber. More formally, for an integer $\tau \geqslant 1$, we use the notation $\mathcal{G} = (V, \{C_1, \ldots, C_\tau\}, R)$ to denote a multi-layer graph with vertex set $V$ and collection $\{C_1, \ldots, C_\tau, R\}$ of layers, where $\{C_1, \ldots, C_\tau\}$ are the cop layers and $R$ is the robber layer. In the cops and robber game on $\mathcal{G}$ each cop is allocated to a single-layer from $\{C_1, \ldots, C_\tau\}$, and the robber to $R$, and each cop (and the robber) will then only move along edges in their respective layer. We do not allow any cop or the robber to move between layers. We note that this is a slight abuse of notation, and that both $(V, \{C_1, \ldots, C_\tau, R\})$ and $(V, \{C_1, \ldots, C_\tau\}, R)$ both denote a multi-layer graphs with the the same collection $\{C_1, \ldots, C_\tau\} \cup \{R\}$ of edge sets, the latter has designated layers for the robber/cops whereas the former does not. We will use $E_i$ to denote edge sets in multi-layer graphs that do not have a cop or robber labels.

A setting that appears often is $R = C_1 \cup \cdots \cup C_\tau$, where the robber can use any edge that exists in a cop layer. This setting is given by the multi-layer graph $\mathcal{G} := (V, \{C_1, \ldots, C_\tau\}, C_1 \cup \cdots \cup C_\tau)$, but for readability we will instead use $\mathcal{G} := (V, \{C_1, \ldots, C_\tau\}, *)$ to denote this.

We define several variants of cops and robbers on multi-layer graphs, however in each of them we have an *allocation* $\mathbf{k} := (k_1, \ldots, k_\tau)$ of cops to layers, such that there are $k_i$ cops on layer $C_i$. We will often use $k := \sum_i k_i$ to refer to the total number of cops in a game.

We now define multi-layer cops and robbers: a two player game played with an allocation $\mathbf{k}$ on a multi-layer graph $\mathcal{G} = (V, \{C_1, \ldots, C_\tau\}, R)$. The two players are the cop player and the robber player. Each cop is assigned a layer such that there are exactly $k_i$ cops in layer $C_i$. The game begins with the cop player assigning each cop to some vertex, and then the robber player assigns the robber to some vertex. The game then continues with each player taking turns in sequence, beginning with the cop player. On the cop player's turn, the cop player may move each cop along one edge in that cop's layer. The cop player is allowed to move none, some, or all of the cops. The robber player then takes their turn, either moving the robber along one edge in the robber layer or letting the robber stay on its current vertex. This game ends as a victory for the cop player if, at any point during the game, the robber is on a vertex that is also occupied by one or more cops. The robber wins if they can evade capture indefinitely.

We can now begin defining our problems, starting with ALLOCATED MULTI-LAYER COPS AND ROBBER.

> ALLOCATED MULTI-LAYER COPS AND ROBBER
> *Input:* A tuple $(\mathcal{G}, \mathbf{k})$ where $\mathcal{G} = (V, \{C_1, \ldots, C_\tau\}, R)$ is a multi-layer graph and $\mathbf{k}$ is an allocation of cops to layers.
> *Question:* Does the cop player have a winning strategy when playing multi-layer cops and robbers on $\mathcal{G}$ with allocation $\mathbf{k}$?

We also consider a variant in which the cop player has a given number $k$ of cops, but gets to choose the layers to which the cops are allocated.

> MULTI-LAYER COPS AND ROBBER
> *Input:* A tuple $(\mathcal{G}, k)$ where $\mathcal{G} = (V, \{C_1, \ldots, C_\tau\}, R)$ is a multi-layer graph and $k \geqslant 1$ is an integer.
> *Question:* Is there an allocation $\mathbf{k}$ with $\sum_i k_i = k$ such that $(\mathcal{G}, \mathbf{k})$ is yes-instance for ALLOCATED MULTI-LAYER COPS AND ROBBER?

Lastly we consider MULTI-LAYER COPS AND ROBBER WITH FREE LAYER CHOICE, a variant of MULTI-LAYER COPS AND ROBBER in which, before the game is played, the layers in the multi-layer graph are not assigned to being either cop layers or robber layers. Instead the layers are simply labelled $E_1$ through $E_\tau$, and in this variant the cop player first allocates each cop to one layer, and then the robber player is free to allocate the robber to any layer.

Multi-layer cops and robber with free layer choice
*Input:* A tuple $(\mathcal{G}, k)$ where $\mathcal{G} = (V, \{E_1, \ldots, E_\tau\})$ is a multi-layer graph and $k \geqslant 1$ is an integer.
*Question:* Is there an allocation $\mathbf{k}$ with $\sum_i k_i = k$ such that for every $j$, $((V, \{E_1, \ldots, E_\tau\}, E_j), \mathbf{k})$ is a yes-instance for Multi-layer cops and robber?

We say that the multi-layer cop number of a multi-layer graph $\mathcal{G}$ is $k$ if $(\mathcal{G}, k)$ is a yes-instance for Multi-layer cops and robber but $(\mathcal{G}, k-1)$ is a no-instance for Multi-layer cops and robber. We will denote this with $\mathsf{mc}(\mathcal{G})$. We round out this section with a number of basic observations.

**Proposition 1.** *Let $\mathcal{G} = (V, \{C_1, \ldots, C_\tau\}, R)$ and $\mathcal{G}' = (V, \{C_1, \ldots, C_\tau\}, R')$ be any two multi-layer graphs where $R \subseteq R' \subseteq \binom{V}{2}$. If $(\mathcal{G}, k)$ is a no-instance to* Multi-layer cops and robber*, then $(\mathcal{G}', k)$ is a no-instance to* Multi-layer cops and robber*. Consequently, $\mathsf{mc}(\mathcal{G}) \leqslant \mathsf{mc}(\mathcal{G}')$.*

*Proof.* To win, the robber on $\mathcal{G}'$ uses the strategy from $\mathcal{G}$. The robber can execute this strategy as any edge in $R'$ is in $R$. Since the cop layers have no added edges, the strategy must be robber-win as else the cops would win on $\mathcal{G}$. □

**Proposition 2.** *Let $\mathcal{G} = (V, \{C_1, \ldots, C_\tau\}, R)$ and $\mathcal{G}' = (V, \{C'_1, \ldots, C'_\tau\}, R)$ be any two multi-layer graphs that satisfy $C_i \subseteq C'_i$ for every $i \in [\tau]$. If $(\mathcal{G}, k)$ is a yes-instance to* Multi-layer cops and robber*, then $(\mathcal{G}', k)$ is also a yes-instance to* Multi-layer cops and robber*.*

*Proof.* To win, the cops on $\mathcal{G}'$ use the strategy from $\mathcal{G}$. As no edge has been removed from $\mathcal{G}$ to create $\mathcal{G}'$, this must still result in the cops winning. □

**Proposition 3.** *Let $\mathcal{G} = (V, \{C_1, \ldots, C_\tau\}, *)$ be a multi-layer graph. If $(\mathcal{G}, k)$ is a yes-instance for* Multi-layer cops and robber*, then, letting $E_i = C_i$ for each $i \in [\tau]$, $((V, \{E_1, \ldots, E_\tau\}), k)$ is a yes-instance for* Multi-layer cops and robber with free layer choice*.*

*Proof.* Immediate from the problem definitions and Proposition 1. □

## 3 Counter Examples & Anti-Monotonicity Results

In this section we provide some concrete examples of cops and robbers on multi-layer graphs illustrating some peculiarities of the game that may seem counter-intuitive. We begin with the following that states that it is sometimes beneficial to put multiple cops on the same layer, and leave other layers empty.

**Theorem 1.** *For any $n \geqslant 4$ there exists a multi-layer graph $(V, \{C_H, C_V\}, *)$ on $n$ vertices such that a cop player can win with two cops if both cops are on $C_H$, or if both cops are on $C_V$, but the robber player can win if one cop is on $C_V$ and the other is on $C_H$.*

It is natural to ask if, given some multi-layer graph $\mathcal{G} = (V, \{C_1, \ldots, C_\tau\}, R)$, the multi-cop number of $\mathcal{G}$ is bounded from below by the minimum cop-number of a single cop layer; namely, does $\mathsf{mc}(\mathcal{G}) \geqslant \min_i \mathsf{c}((V, C_i))$ hold? Observe that, if $|V| = n$ and we let $S_n$ denote the star graph on $n$ vertices, any multi-layer graph $\mathcal{G} = (V, \{E(S_n), C_2, \ldots, C_\tau\}, R)$ has cop number 1, as the cop can start on the centre of the star and reach any other vertex in one move. This is not enough resolve the question directly, however in the next result we build on this idea to show a general bound of the form $\mathsf{mc}(\mathcal{G}) = \Omega(\min_i \mathsf{c}((V, C_i)))$ does not hold.

**Proposition 4.** *For any $c \geqslant 2$ there exist graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ such that $\mathsf{c}(G_1), \mathsf{c}(G_2) \geqslant c$ and $\mathsf{mc}((V, \{E_1, E_2\}, *)) = 2$.*

The idea of the proof is to take two $n$-vertex graphs with cop number $c$ and add a $n-1$ pendent vertices from a single vertex in each graph ($u_n$ and $v_n$ respectively). The graphs are then identified as cop layers in such a way that a cop at $u_n$ can police half the vertices, and a cop at $v_n$ can cover the other half. See Figure 1 for an illustration. In fact, in such a construction the two cops will catch the robber after at most one cop move. As a result, the edges present in the robber layer are irrelevant and we have the following corollary.

**Corollary 1.** *For any $c \geqslant 2$ there exist graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ such that $\mathsf{c}(G_1), \mathsf{c}(G_2) \geqslant c$, and for any set of edges $R \subseteq \binom{V}{2}$,*

$$\mathsf{mc}((V, \{E_1, E_2\}, R)) \leqslant 2.$$

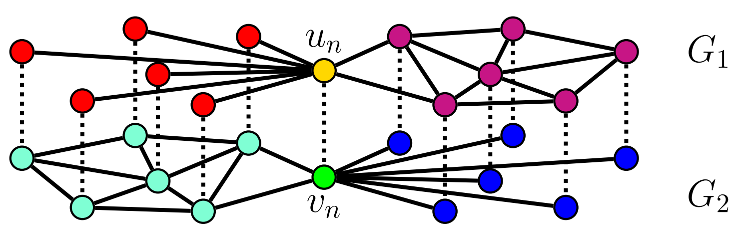


Fig. 1: Illustration of the construction in the proof of Proposition 4. The dotted edges signify an identification of the two end points of that edge.

We now consider the reverse inequality: is the multi-layer cop number bounded from above by a function of the cop numbers of the individual layers? If, in a multi-layer graph $\mathcal{G} = (V, \{C_1, \ldots, C_\tau\}, R)$, the robber layer is a subset of one of the cop layers, i.e. $R \subset C_i$ for some $i \in [\tau]$, then $\mathsf{mc}(\mathcal{G}) \leqslant \mathsf{c}((V, C_i))$ as the cop player can allocate $\mathsf{c}((V, C_i))$ cops to layer $i$, ignoring all other cop layers. The same reasoning gives an upper bound of $\sum_{i\in[\tau]} \mathsf{c}((V, C_i))$ on the cop number in the 'free choice layer' variant of the game. Thus, in this special case an upper bound that depends only on the cop numbers of individual layers does exist. However the next result shows that this is not the case in general.

**Theorem 2.** *For any positive integer $k$, there exists a multi-layer graph $\mathcal{G} = (V, \{C_1, C_2\}, R)$ on $O(k^3)$ vertices such that each of $(V, R)$, $(V, C_1)$, and $(V, C_2)$ are connected, $\mathsf{c}((V, R)) \leqslant 3$, $\mathsf{c}((V, C_i)) \leqslant 2$ for $i \in \{1, 2\}$, and $\mathsf{mc}(\mathcal{G}) \geqslant k$.*

## 4 Complexity Results

In this section we will examine multi-layer cops and robbers from a computational complexity viewpoint. For a background on computational complexity, we point the reader to [30]. First note that as determining the cop-number of a simple graph is EXPTIME-complete [18], Multi-layer cops and robber with free layer choice is also EXPTIME-complete by the obvious reduction that creates a multi-layer graph with one layer from a simple graph. The same reduction, and the fact that, unless the strong exponential time hypothesis fails[3], determining if a graph is $k$-copwin requires $\Omega(n^{k-o(1)})$ time [9], we also get that Multi-layer cops and robber with free layer choice requires $\Omega(n^{k-o(1)})$ time.

An algorithm that determines whether a simple graph $G$ is $k$-copwin in $O(kn^{k+2})$ time is given in [23]. Petr, Portier, and Versteegen show this by first constructing a *state graph* — a directed graph $H$ wherein each vertex of $H$ corresponds to a state of a game of cops and robbers played on the original graph $G$. They then give an $O(kn^{k+2})$ algorithm for finding all cop-win vertices of $H$, where a vertex is cop-win if the corresponding state either is a winning state for the cops, or can only lead to a winning state for the cops. We adapt their construction by only creating arcs of $H$ where the move of a given cop or robber is allowed (i.e., the edge in the multi-layer graph exists in the same layer as the cop or robber that is moving). By doing this we obtain the following.

**Theorem 3.** Allocated multi-layer cops and robber *can be solved in $O(k^2 n^{2k+2})$.*

Note that $\tau$, the number of layers, does not appear in the above as if $\tau \leqslant k$ then any dependence on $\tau$ is absorbed by the dependence on $k$, and if $k < \tau$ then at least $\tau - k$ layers must have zero cops allocated to them and can be ignored.

By taking an instance of dominating set $G = (V, E)$, and creating for each vertex $v \in V$ a layer $E_v$ containing every edge incident to $v$, we create an instance of Multi-layer cops and robber with free layer choice that has a winning strategy for $k$ cops if and only if $G$ has a dominating set of size $k$, leading to the following.

**Theorem 4.** Multi-layer cops and robber with free layer choice *is NP-hard, even if each layer is a tree plus some isolated vertices.*

Note that the input size to Multi-layer cops and robber with free layer choice is the number of bits required to represent both the underlying graph and each of the layers.

[3] See [12, Chapter 14] for background on the strong exponential time hypothesis.

If it is only the robber that is limited to a tree, however, then determining if $k$ cops can win is $\mathsf{FPT}$ in the number of cops and number of layers in the graph. In particular, this result applies even if the layers are not connected. We obtain this result by characterising whether the robber can win based on the existence of edges in the robber layer that cops can easily patrol.

**Theorem 5.** *Given a multi-layer graph $\mathcal{G} = (V, \{C_1, \ldots, C_\tau\}, R)$, if $R$ is a tree, then* Multi-layer cops and robber *on $\mathcal{G}$ can be solved in time $O(f(k, \tau) \cdot \mathsf{poly}(n))$, where $k$ is the number of cops, $\tau$ is the number of layers of $\mathcal{G}$, $f$ is a computable function independent of $n$, and $\mathsf{poly}(n)$ is a fixed polynomial in $n$.*

The next result follows immediately from the proof technique used to prove Theorem 5.

**Corollary 2.** *Given a multi-layer graph $\mathcal{G} = (V, \{C_1, \ldots, C_\tau\}, R)$, if $R$ is a tree, and each cop layer is connected, then $\mathsf{mc}(\mathcal{G}) \leqslant 2$.*

## 5 Extremal Multi-Layer Cop-Number

In this section we study, for a given simple connected graph $G = (V, E)$, the extremal multi-layer cop number of $G$. This is the multi-layer cop number maximised over the set of all multi-layer graphs with connected cop-layers, which when flattened give $G$. More formally, for given connected graph $G = (V, E)$, if we define the set

$$\begin{aligned}\mathcal{L}(G) = \{(V, \{C_1, \ldots, C_\tau\}, *) : E = C_1 \cup \cdots \cup C_\tau \\ \text{and for each } i \in [\tau], (V, C_i) \text{ is connected}\},\end{aligned}$$

then the extremal multi-layer cop-number of $G$ is given by

$$\mathsf{emc}_\tau(G) = \max_{\mathcal{G} \in \mathcal{L}} \mathsf{mc}(\mathcal{G})\,.$$

We generalise two tools for bounding the cop number of graphs to the setting of multi-layer graphs; $(1, k)$-existentially closed graphs [7] and bounds by domination number. See the arXiv version of this paper [13] for more details on the former method; here we will now outline our use of dominating sets.

Let $\mathcal{G} = (V, \{E_1, \ldots, E_\tau\})$ be a multi-layer graph (without designated layers). A *multi-layer dominating set* in $\mathcal{G}$ is a set $D \subseteq V \times \{1, \ldots, \tau\}$ of vertex-layer pairs such that for every $v \in V$, either $(v, i) \in D$ for some $i$, or there is a $(w, i) \in D$ such that $w \in V$ and $vw \in E_i$. We define the *domination number* $\gamma(\mathcal{G})$ of $\mathcal{G}$ to be the size of a smallest multi-layer dominating set in $\mathcal{G}$. Note that if $\mathcal{G}$ has a single-layer this definition aligns with the traditional notion of dominating set, which justifies the overloaded notation. It is a folklore result that the cop number is at most the size of any dominating set in the graph, this also holds in the multi-layer setting.

**Theorem 6.** *Let $\mathcal{G} := (V, \{E_1, \ldots, E_\tau\})$ be any multi-layer graph and $\mathcal{G}' := (V, \{E_1, \ldots, E_\tau\}, \binom{V}{2})$. Then, $\mathsf{mc}(\mathcal{G}') \leqslant \gamma(\mathcal{G})$.*

*Proof.* Let $D$ be any multi-layer dominating set of size $|D| = \gamma(\mathcal{G})$ and for each $(v, i) \in D$ place one cop in layer $i$ at the vertex $v$. The result now follows as if the robber is at an any vertex then they are adjacent to a cop in some layer and so the robber will be caught after the cops first move. $\square$

We now introduce the parameter $\delta(\mathcal{G})$ which is an analogue of minimum degree for a multi-layer graph $\mathcal{G} = (V, \{E_1, \ldots, E_\tau\})$. This is given by

$$\delta(\mathcal{G}) := \min_{v \in V} \sum_{i \in [\tau]} d_{(V,E_i)}(v). \tag{1}$$

Using this notion we prove a bound on the domination number of a multi-layer graph, the proof is based on a classic application of the probabilistic method [2, Theorem 1.2.2].

**Theorem 7.** *Let $\mathcal{G} = (V, \{E_1, \ldots, E_\tau\})$ be any multi-layer graph. Then,*

$$\gamma(\mathcal{G}) \leqslant \frac{n\tau}{\tau + \delta(\mathcal{G})} \cdot \left( \ln \left( \frac{\tau + \delta(\mathcal{G})}{\tau} \right) + 1 \right).$$

Note that there are least two other sensible definitions of 'multi-layer minimum degree', namely the minimum degree of each layer $\min_{i \in [\tau]} \min_{v \in V} d_{(V,E_i)}(v)$, and minimum number of neighbours within any layer $\delta(\mathsf{fl}(\mathcal{G}))$. Our definition of $\delta(\mathcal{G})$ above in (1) can be thought of as the 'minimum number of edges incident in any layer', this is arguably a less natural notion than $\delta(\mathsf{fl}(\mathcal{G}))$ however it gives a better bound in our application (Theorem 7), in particular.

**Proposition 5.** *For any multi-layer graph $\mathcal{G}$ we have $\delta(\mathsf{fl}(\mathcal{G})) \leqslant \delta(\mathcal{G})$.*

Returning to extremal multi-layer cop numbers, for a complete graph we obtain Theorem 8. The upper bound we arrive at by placing all $\tau$ cops on a single vertex $v$; as each edge of $K_n$ must be in some layer, there is no vertex that is not incident with $v$ in some layer. The lower bound requires more work and relies on constructing cop layers with no overlap by combining colour classes of an edge colouring of the clique due to Sofier [31].

**Theorem 8.** *Let $n \geqslant 1$, $1 \leqslant \tau < \lfloor \frac{n}{2} \rfloor$ be integers. Then, $\lceil \frac{\tau}{10} \rceil \leqslant \mathsf{emc}_\tau(K_n) \leqslant \tau$.*

We now consider the extremal multi-layer cop number of the binomial random graph $G_{n,p}$. For any integer $n \geqslant 1$, this is the probability distribution over all $n$-vertex simple graphs generated by sampling each possible edge independently with probability $0 < p = p(n) < 1$, see [4] for more details. The following result shows that, for a suitably dense binomial random graph $G_{n,p}$, with probability tending to 1 as $n \to \infty$, $\mathsf{emc}_\tau(G_{n,p}) = \Theta(\tau \log(n)/p)$. The single-layer cop number of $G_{n,p}$ in the same range is known to be $\Theta(\log(n)/p)$ [7], so in some sense our result generalises this result.

**Theorem 9.** *For $\varepsilon > 0$, if $n^{1/2+\varepsilon} \leqslant np = o(n)$, and $1 \leqslant \tau \leqslant n^\varepsilon$ then,*

$$\mathbb{P}\left( \frac{\varepsilon}{10} \cdot \frac{\tau \cdot \ln n}{p} \leqslant \mathsf{emc}_\tau(G_{n,p}) \leqslant 10 \cdot \frac{\tau \cdot \ln n}{p} \right) \geqslant 1 - e^{-\Omega(\sqrt{n})}.$$

The upper bound in the proof of Theorem 9 follows from Theorems 6 and 7, whereas the lower bound follows by independently choosing cop layers which are each distributed as a random graph with edge probability $\Theta(p/\tau)$ and then applying a generalised form of the existential closure technique developed in [7]. See [24] for results on the cop number of $G_{n,p}$ for other ranges of $p$.

The extremal multi-layer cop number of a graph $G$ is also bounded from above by the treewidth of $G$.

**Theorem 10.** *For any graph $G := (V, E)$, $\mathsf{emc}_\tau(G) \leqslant \mathsf{tw}(G)$. Furthermore, these cops can placed in any layers and still capture the robber.*

## 6 Multi-Layer Analogue of Meyniel's Conjecture

For the classical cop number, Meyniel's Conjecture [15] states that $\mathcal{O}(\sqrt{n})$ cops are sufficient to win cops and robbers on any graph $G$. After a sequence of results [10, 15, 16, 20, 28] the current best bound stands at $n \cdot 2^{-(1-o(1))\sqrt{\log_2 n}}$, see [5, Chapter 3] for a more detailed overview.

It is natural to explore analogues of Meyniel's Conjecture for the multi-layer cop number. Namely, what is the minimum number of cops needed to patrol any multi-layer graph with $\tau$ connected layers? Our results for the clique show that if $\tau$ is allowed to be arbitrary then no bound better than $\mathcal{O}(n)$ can hold. We conjecture that this is not the case when the number of layers is bounded.

*Conjecture 1.* For any fixed integer $\tau \geqslant 1$ and collection $(V, E_1), \ldots, (V, E_\tau)$ of connected graphs where $|V| = n$, we have

$$\mathsf{mc}((V, \{E_1, \ldots, E_\tau\}, *)) = o(n).$$

Observe that the connected assumption is necessary in Conjecture 1 if we do not have divergent minimum degree, as shown by the example with two cop layers given by two edge disjoint matchings who's union forms an even cycle. This conjecture might seem very modest in comparison to Meyniel's conjecture, however the following result shows that it would be almost tight.

**Theorem 11.** *For any positive integer $n$ there is a $n$-vertex multi-layer graph $\mathcal{G} = (V, \{C_1, C_2, C_3\}, *)$ such that $|V| = \Theta(n)$, each cop layer is connected and has cop-number* 2*, and*

$$\mathsf{mc}(\mathcal{G}) = \Omega\left(\frac{n}{\log n}\right).$$

The construction in Theorem 11 starts with a 3-edge coloured cubic expander graph $X$ on $N$ vertices, where each color class is a cop layer. The vertices of $X$ are then connected to the leaves of a star that has been subdivided $\Theta(\log N)$ many times — these can be used by all cops. The idea is that $k$ cops can police at most $2k$ vertices of $X$ within $\Theta(\log N)$ steps as it takes each cop this long to change location in $X$ (via the arms of the star). If $k = \Theta(N)$ is chosen to be a suitably small but constant fraction of $N$, then even with the vertices policed by the cops removed there is still an expander subgraph of $X$ not adjacent to any cops. The robber can then use this expander subgraph to change position before any cop can threaten them.

Many of the current approaches to Meyniel's Conjecture use some variation the fact that a single cop can guard any shortest path between any two vertices. For example the first step of the approach in [28] is to iteratively remove long geodesics until the graph has small diameter (following this a more sophisticated argument matching randomly placed cops to possible robber trajectories is applied). What makes Conjecture 1 difficult to approach is that, even for two layers, a shortest path in the flattened graph $\mathsf{fl}(\mathcal{G})$ may not live within a single cop layer. We note that [16] use a different approach based on expansion, their approach is more versatile however the authors were unable to apply it in the multi-layer setting.

This suggests a new or more refined approach is needed. However, by a direct application of Theorems 6 and 7, using a simple dominating set approach we can prove Conjecture 1 for multi-layer graphs with diverging minimum degree.

**Proposition 6.** *For any $n$-vertex multi-layer graph $\mathcal{G} = (V, \{E_1, \ldots, E_\tau\})$ satisfying $\delta(\mathcal{G})/\tau \to \infty$ as $n \to \infty$, we have* $\mathsf{mc}\Big(\big(V, \{E_1, \ldots, E_\tau\}, \binom{V}{2}\big)\Big) = o(n)$.

## 7 Conclusion and Open Problems

We studied the game of cops and robbers on multi-layer graphs, via several different approaches, including concrete strategies for certain graphs, the construction of counter-intuitive examples, algorithmic and hardness results, and the use of probabilistic methods and expanders for extremal constructions. We find that the multi-layer cop number cannot be bound from above or below by (non-constant) functions of the cop numbers of the individual layers. We bound an extremal variant for cliques and dense binomial random graphs (extending some tools from the single layer case along the way). We also find that a naive transfer of Meyniel's conjecture to the multi-layer setting is not true: there are multi-layer graphs which have multi-layer cop number in $\Omega(n/\log n)$. Algorithmically, we find that even if each layer is a tree plus some isolated vertices, the the free layer choice variant of the problem remains NP-hard. Positively, we find that the problem can be resolved by an algorithm that is FPT in the number of cops and layers if the layer the robber resides in is a tree.

We are hopeful that our contribution will spark future work in multi-layer variants of cops and robbers, and suggest a number of possible open questions:

We were not able to generalise some frequently used tools from single-layer cops and robbers: for example, we have no useful notion of a corner, or a retract, nor dismantleability - we are hopeful that such tools may exist.

We have made some progress on the parameterised complexity of our problems, but have only considered a limited set of parameters and have not considered any parameter that constraints the nature of interaction between the layers: if we, for example, require that the layers are very similar alongside other restrictions does that impact the computational complexity of our problems?

Single-layer cops and robbers has been very successful as a tool for defining useful graph parameters of simple graphs, and we ask whether multi-layer cops and robbers could be used to define algorithmically useful graph parameters.

Some of our bounds and extremal results are unlikely to be tight in number of layers or with respect to other graph characteristics: can they be improved?

Is the extremal multi-layer cop number of $G_{n,p}$ always $\Theta(\tau \cdot \mathsf{c}(G_{n,p}))$ w.h.p. for any $p$? Of particular is whether this holds even in the 'zig-zag' regime [21]?

While we showed that a naive adaptation of Meyniel's conjecture to our multi-layer setting fails, it is still possible that $o(|V|)$ cops are sufficient for a bounded number of connected layers. We have shown this for a special case related to degree: is it true in general?

Finally, while we introduced a particular notion of multi-layer dominating set for our use in proving other results (inspired by similar ideas in single-layer cops and robbers), we suggest that this multi-layer graph characteristic may also be interesting in its own right, in particular for algorithms for other problems on multi-layer graphs.

## Acknowledgements

This work was supported by the Engineering and Physical Sciences Research Council [EP/T004878/1].

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