

The Crewther relation, schemes, gauges and fixed points

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Abstract. We investigate the Crewther relation at high loop order in a variety of renormalization schemes and gauges. By examining the properties of the relation in schemes other than modified minimal subtraction ($\overline{\text{MS}}$) at the fixed points of Quantum Chromodynamics we propose a generalization of the Crewther relation that extends the $\overline{\text{MS}}$ construction of Broadhurst and Kataev. A derivation based on the properties of the renormalization group equation is provided for the generalization which is tested in various scenarios.

1 Introduction.

In [1] Crewther made an interesting and remarkable connection between the Adler D -function and the Bjorken sum rule in non-abelian gauge theories. Using the then available explicit perturbative expressions for those seemingly unrelated quantities Crewther showed that their product was a constant and there was no $O(a)$ correction where a is related to the strong coupling constant. When the higher order corrections subsequently became available in the modified minimal subtraction ($\overline{\text{MS}}$) scheme [2, 3, 4] it was apparent that Crewther's observation of constancy of the product was not retained at these orders. It was demonstrated in [5] however that the non-zero a dependence could be written as the product of two functions of a . One of the functions was the β -function of Quantum Chromodynamics (QCD). While the presence of a non-zero term appeared to undermine the ethos that Crewther's observation was an exact result, it was in fact in keeping with a more general property which is that of conformal symmetry, [6, 7, 8]. Indeed this non-zero $O(a^2)$ term is referred to as the conformal symmetry breaking term, [5], since the β -function vanishes when conformal symmetry is present. Later the improvement in loop technology meant the $O(a^4)$ terms of both the Adler D -function and Bjorken sum rules became available in [9]. It was shown in [9] that the resulting $O(a^4)$ correction to the conformal symmetry breaking term could also be accommodated within the product of the two loop β -function and a correction to the other function in the breaking term as a function of a . Thus the conformal aspect of the Crewther relation was reinforced.

In briefly summarizing the background to the development of Crewther's observation it is important to note that the so-called conformal symmetry breaking term was always determined in the $\overline{\text{MS}}$ scheme in the first instance, [5, 9]. More recently there have been several detailed studies of the Crewther construction in other renormalization schemes in [10, 11, 12]. One of the motivations in carrying out such investigations is to ascertain whether or not the conformal symmetry breaking term is always the product of the β -function and another function of a in schemes other than $\overline{\text{MS}}$. Amongst the schemes examined in [10, 11, 12] were the V scheme and the minimal momentum (mMOM) scheme. The former is based on the static quark potential, [13, 14], while the latter is defined with respect to the ghost-gluon vertex of QCD, [15]. In particular it preserves the non-renormalization property of that vertex in the Landau gauge, discovered by Taylor in [16], but extended to an arbitrary linear covariant gauge. The renormalization group functions of the mMOM scheme are known to five loops, [15, 17, 18, 19]. Although the Crewther relation mMOM scheme study only required four loop information one of its main conclusions was that at leading order the factorization of the conformal symmetry breaking term only occurred for specific values of the covariant gauge fixing parameter α , [12]. These were $\alpha = 0, -1$ and -3 . It was also recognized that the special cases of $\alpha = 0$ and -3 arose in other schemes, [12]. At the subsequent order the factorization only occurred in the Landau gauge where α vanishes. While disappointing, detailed investigations of the structure of where the factorization, [12], failed did provide significant insight into and key clues as to how to reconcile the hope that the Crewther construction could accommodate other schemes. Indeed this has to be the case if the Crewther relation is to be regarded as a fundamental property of a non-abelian gauge theory.

One interesting aspect of the emergence of the particular case of $\alpha = -3$ is that this value has already been noted as having special significance in situations primarily associated with phenomenology or infrared dynamics in QCD. For instance, one of the earliest occurrences was in [20] where it emerged in a study of the Wilson loop and its renormalization properties. This connection was noted again later in the case of Wilson operators in deep inelastic scattering in [21]. In a different context, [22], Schwinger-Dyson methods were used to examine models of chiral symmetry breaking and this choice of the gauge parameter was shown to be necessary for a scale invariant solution. In other words when the ansätze for the form factors have a power law behaviour then

the underlying equations can be solved to determine when chiral symmetry breaking occurs. Separately in [23, 24], where the worldline formalism was used to construct gauge invariant variables to study parton distribution functions, the choice of $\alpha = -3$ was required to ensure there were no divergences associated with rapidity. The same choice of α was also needed in the related area of deep inelastic scattering in [25] where renormalon chains of bubble graphs were examined in the large N_f expansion as that particular value of α identified an abelian characteristic of the underlying renormalization in a similar vein to [20, 21]. At leading order the approximation was relatively accurate for $\alpha = -3$ but corrections to this value were also established to preserve the abelian property, [26]. This leading order value also appeared in an exploration of Yang-Mills theory at low energy. In particular in [27, 28, 29] an effective Lagrangian was constructed in terms of dimension six gluonic operators that was a solution to the infrared Schwinger-Dyson equations. Indeed it was argued in [27, 28, 29] that to have a scaling solution for the gluon propagator which would lead to a linearly rising static quark potential associated with confinement required $\alpha = -3$. Similarly the role this particular value plays in the behaviour of the effective coupling constant derived in quark-quark scattering was discussed in [30]. Specifically at $\alpha = -3$ the effective coupling decreases as a function of the exchanged momentum consistent with asymptotic freedom. A more phenomenological origin for this gauge parameter choice was noted in [31]. Moreover the appearance of $\alpha = -3$ was not restricted to strong dynamics. It was shown in [32] that the Z boson is multiplicatively renormalizable for this gauge parameter choice in the electroweak sector of the Standard Model. In support of the infrared connection the critical properties of the QCD renormalization group equations in [33] indicated that there was a fixed point similar to that of Banks and Zaks, [34], but with a value of α close to -3 . That it was not exactly a negative integer was primarily because that critical point analysis was carried out at four loops which was an order much higher than the previous work on this special value. This would suggest that while the infrared dynamics observations are pointing to the same underlying property quantum corrections will necessarily have to be taken into account too. Our Crewther study will also lend weight to that point of view. Indeed the connection provided in [34] also directs our attention to perhaps the most significant clue to accommodate schemes for the Crewther relation. This is the fact that in examining the critical properties of the renormalization group functions of QCD treated in the (a, α) plane there is a fixed point in the neighbourhood of $\alpha = -3$ in the conformal window. While it is not precisely this value it is however suggestive of a connection to the Crewther relation with the discrepancy somehow being accounted for by higher order corrections lurking within a reconciliation of the factorization issue with the conformal symmetry property.

It is therefore the aim of this article to extend the Crewther relation in some way that it is inclusive of schemes beyond the $\overline{\text{MS}}$ one yet at the same time obeying the conformal aspects on top of respecting the renormalization group. In our investigation in order to gain as wide a picture as possible we will consider not only the mMOM scheme but a similar one used in lattice gauge theories, termed the modified Regularization Invariant (RI') scheme [35, 36], as well as the kinematic momentum subtraction (MOM) schemes developed by Celmaster and Gonsalves, [37, 38]. In addition we will examine the relation in two other gauges which are nonlinear covariant gauge fixings. The reason for this is that while the Lorenz suite of covariant gauges are canonically used in computations, with α arbitrary or zero, gauge independent results should be blind to the choice of the particular gauge fixing functional. Our aim is to construct a generalization of the conformal symmetry breaking term that is consistent with the properties of the renormalization group equation and preserves the conformal symmetry property as well as putting the origin of the special case of $\alpha \approx -3$ of the linear covariant gauge fixing on a solid footing. To achieve all these aims required both numerical examination of the perturbative expansions of the product of the Adler D -function and Bjorken sum rule as well as the analytic expressions. These will be studied at the highest available loop order in each of the different gauges and schemes. Indeed the way the

article is structured it will follow the same development ethos of [5] where numerical and analytic evidence was presented that led to and justified the factorization property of the original conformal symmetry breaking term of [5]. By assembling this evidence a clear picture emerges which allows us to formulate the generalization before providing an analytic justification of our observations. This is then robustly and satisfactorily tested across the various schemes and gauges. It would be remiss of us at this point not to mention the related work in this area provided in [39]. There the principle of maximal conformality was used to provide a way of setting the renormalization scale for phenomenological applications from the Crewther relation.

The article is organized as follows. We recall the core observations surrounding the Crewther relation in Section 2 as well as details of the factorization property in the mMOM scheme. The focus of Section 3 is to present numerical evidence that the conformal symmetry breaking term vanishes at a large number of fixed points of the QCD renormalization group equations for nonzero α . Based on those observations the consequences for the Crewther relation in the mMOM scheme are studied analytically in Section 4 where a generalization of [5] is introduced. The consistency of this generalization is then tested in other schemes and gauges in Section 5 as well as several other schemes in the linear covariant gauge in Section 6. The formalization of the generalized Crewther relation within the renormalization group is presented in Section 7. Section 8 by contrast is devoted to explaining the relation of the gauge parameter critical point values to the special cases of the gauge parameter that were indicated in the mMOM scheme in [12]. An overview of the field theory origin of certain properties of the Crewther relation that we observe is provided in Section 9. Conclusions are provided in Section 10 ahead of two appendices. Expressions illustrating the structure of key functions in the Crewther relation in a MOM scheme for a general Lie group are recorded in Appendix A. Finally Appendix B presents the renormalization group functions for a new renormalization scheme that is introduced in Section 6.

2 Background.

It is instructive to first review the construction given in [5] which revealed Crewther's observation that the product of the Bjorken sum rule and Adler D -function is not a constant. Instead there is a discrepancy which depends on the β -function in the $\overline{\text{MS}}$ scheme. We recall the sum rule and D -function to $O(a^3)$ are, [2, 3, 4],

$$\begin{aligned}
C_{\text{Bjr}}^{\overline{\text{MS}}}(a) &= 1 - 3C_F a + \left[\frac{21}{2}C_F^2 - 23C_F C_A + 8N_f T_F C_F \right] a^2 \\
&+ \left[\frac{440}{3}\zeta_5 C_F C_A^2 + \frac{1241}{9}C_F^2 C_A + \frac{7070}{27}N_f T_F C_F C_A + 48\zeta_3 N_f T_F C_F C_A \right. \\
&\quad - \frac{10874}{27}C_F C_A^2 - \frac{920}{27}N_f^2 T_F^2 C_F - \frac{176}{3}\zeta_3 C_F^2 C_A - \frac{160}{3}\zeta_5 N_f T_F C_F C_A \\
&\quad \left. - \frac{133}{9}N_f T_F C_F^2 - \frac{80}{3}\zeta_3 N_f T_F C_F^2 - \frac{3}{2}C_F^3 \right] a^3 + O(a^4) \tag{2.1}
\end{aligned}$$

and

$$\begin{aligned}
C_{\text{Adl}}^{\overline{\text{MS}}}(a) &= \left[1 + 3C_F a + \left[\frac{123}{2}C_F C_A - \frac{3}{2}C_F^2 - 44\zeta_3 C_F C_A - 22N_f T_F C_F + 16\zeta_3 N_f T_F C_F \right] a^2 \right. \\
&+ \left[\frac{160}{3}\zeta_5 N_f T_F C_F C_A + \frac{4832}{27}N_f^2 T_F^2 C_F + \frac{7168}{9}\zeta_3 N_f T_F C_F C_A + \frac{90445}{54}C_F C_A^2 \right. \\
&\quad - \frac{31040}{27}N_f T_F C_F C_A - \frac{10948}{9}\zeta_3 C_F C_A^2 - \frac{1216}{9}\zeta_3 N_f^2 T_F^2 C_F - \frac{440}{3}\zeta_5 C_F C_A^2 \\
&\quad \left. \left. - \frac{69}{2}C_F^3 - 572\zeta_3 C_F^2 C_A - 320\zeta_5 N_f T_F C_F^2 - 127C_F^2 C_A - 29N_f T_F C_F^2 \right] a^3 \right] a^2
\end{aligned}$$

$$+ 304\zeta_3 N_f T_F C_F^2 + 880\zeta_5 C_F^2 C_A] a^3 + O(a^4)] d_R \quad (2.2)$$

in the $\overline{\text{MS}}$ scheme where C_F , C_A and T_F are the usual colour group Casimirs, N_f is the number of (massless) quark flavours, d_R is the dimension of the fundamental representation and ζ_n is the Riemann zeta function. Although the $O(a^4)$ terms of both are available [9], and involve higher rank 4 colour Casimirs, the $O(a^3)$ expressions are sufficient for the moment to illustrate the properties of the discrepancy. We note that our focus will be on the flavour non-singlet construction throughout the article. For clarity we define the product in the same way as [5] through

$$C_{\text{Bjr}}^{\overline{\text{MS}}}(a) C_{\text{Adl}}^{\overline{\text{MS}}}(a) = d_R [1 + \Delta_{\text{csb}}^{\overline{\text{MS}}}(a)] \quad (2.3)$$

where csb on $\Delta_{\text{csb}}^{\overline{\text{MS}}}(a)$ labels the conformal symmetry breaking term, [5]. The scheme that the variables are in will be indicated by the superscript label throughout. It is clear from the product of (2.1) and (2.2) that $\Delta_{\text{csb}}^{\overline{\text{MS}}}(a)$ is $O(a^2)$. The subsequent $O(a^2)$ term is partly comprised of the sum of the two $O(a^2)$ terms of (2.1) and (2.2) which results in five terms where the coefficient of the C_F^2 term is 9. The remaining $O(a^2)$ part arises from the product of the one loop terms of each of (2.1) and (2.2) that precisely cancels the other C_F^2 contribution. In total four terms remain. One pair has rational coefficients and the other pair involves ζ_3 . While this summarizes the underlying algebra the major result of [5] was to observe that the remaining terms factorize with one of the factors equivalent to the one loop β -function coefficient of [40, 41]. Consequently it was proposed that $\Delta_{\text{csb}}^{\overline{\text{MS}}}(a)$ took the all orders form

$$\Delta_{\text{csb}}^{\overline{\text{MS}}}(a) = \frac{\beta^{\overline{\text{MS}}}(a)}{a} K_a^{\overline{\text{MS}}}(a). \quad (2.4)$$

We recall explicit computation produced, [5],

$$\begin{aligned} K_a^{\overline{\text{MS}}}(a) = & \left[12\zeta_3 C_F - \frac{21}{2} C_F \right] a \\ & + \left[\frac{326}{3} N_f T_F C_F + \frac{397}{6} C_F^2 - 240\zeta_5 C_F^2 + 136\zeta_3 C_F^2 + \frac{884}{3} \zeta_3 C_F C_A - \frac{629}{2} C_F C_A \right. \\ & \quad \left. - \frac{304}{3} \zeta_3 N_f T_F C_F \right] a^2 \\ & + \left[\frac{6496}{9} \zeta_3 N_f^2 T_F^2 C_F + \frac{11900}{3} \zeta_5 C_F C_A^2 - \frac{406043}{36} C_F C_A^2 - \frac{40336}{9} \zeta_3 N_f T_F C_F C_A \right. \\ & \quad - \frac{24880}{3} \zeta_5 C_F^2 C_A - \frac{9824}{9} N_f^2 T_F^2 C_F - \frac{8000}{3} \zeta_5 N_f T_F C_F C_A - \frac{7729}{18} N_f T_F C_F^2 \\ & \quad + \frac{2471}{12} C_F^3 + \frac{16570}{3} \zeta_3 C_F^2 C_A + \frac{67520}{9} N_f T_F C_F C_A + \frac{72028}{9} \zeta_3 C_F C_A^2 \\ & \quad + \frac{99757}{36} C_F^2 C_A - 5720\zeta_5 C_F^3 - 3668\zeta_3 N_f T_F C_F^2 - 1232\zeta_3^2 C_F C_A^2 - 840\zeta_7 C_F^2 C_A \\ & \quad - 128\zeta_3^2 N_f T_F C_F C_A + 320\zeta_5 N_f^2 T_F^2 C_F + 488\zeta_3 C_F^3 + 576\zeta_3^2 N_f T_F C_F^2 \\ & \quad \left. + 4000\zeta_5 N_f T_F C_F^2 + 5040\zeta_7 C_F^3 \right] a^3 + O(a^4) \end{aligned} \quad (2.5)$$

which we include as a reference point for later.

One subsequent question that was studied after the establishment of (2.3) and (2.4) was whether or not this relation took the same form in renormalization schemes other than $\overline{\text{MS}}$. This was examined at length in the mMOM scheme in several articles, [10, 11, 12]. The scheme is based on preserving the non-renormalization property of the ghost-gluon vertex in the Landau gauge, that Taylor observed in [16], in an arbitrary linear covariant gauge. QCD has been renormalized to

high loop order in mMOM in [15, 17, 18, 19]. In order to carry out a Crewther relation study the expressions for the Adler D -function and Bjorken sum rule had first to be established in the mMOM scheme, [12]. This was achieved by mapping the $\overline{\text{MS}}$ coupling constant dependence in $C_{\text{Bjr}}^{\overline{\text{MS}}}(a)$ and $C_{\text{Adl}}^{\overline{\text{MS}}}(a)$ to the mMOM coupling constant using the explicit relations between the variables in both schemes that are available in [15, 17] for instance. As this mapping is gauge parameter dependent it therefore produces expressions that are α dependent, [12]. In order to facilitate our subsequent analysis we note, [12],

$$\begin{aligned}
C_{\text{Bjr}}^{\text{mMOM}}(a, \alpha) &= 1 - 3C_F a + \left[\frac{3}{2}\alpha C_F C_A - \frac{107}{12}C_F C_A + \frac{3}{4}\alpha^2 C_F C_A + \frac{4}{3}N_f T_F C_F + \frac{21}{2}C_F^2 \right] a^2 \\
&+ \left[24\zeta_3 N_f T_F C_F C_A - \frac{20585}{144}C_F C_A^2 - \frac{208}{9}N_f T_F C_F^2 - \frac{176}{3}\zeta_3 C_F^2 C_A \right. \\
&\quad - \frac{160}{3}\zeta_5 N_f T_F C_F C_A - \frac{117}{8}\zeta_3 C_F C_A^2 - \frac{40}{3}N_f^2 T_F^2 C_F - \frac{21}{2}\alpha C_F^2 C_A \\
&\quad - \frac{21}{4}\alpha^2 C_F^2 C_A - \frac{9}{8}\alpha^2 \zeta_3 C_F C_A^2 - \frac{4}{3}\alpha N_f T_F C_F C_A - \frac{3}{2}C_F^3 \\
&\quad - \frac{2}{3}\alpha^2 N_f T_F C_F C_A + \frac{9}{16}\alpha^3 C_F C_A^2 + \frac{33}{4}\alpha \zeta_3 C_F C_A^2 + \frac{64}{3}\zeta_3 N_f T_F C_F^2 \\
&\quad + \frac{215}{48}\alpha C_F C_A^2 + \frac{349}{48}\alpha^2 C_F C_A^2 + \frac{440}{3}\zeta_5 C_F C_A^2 + \frac{832}{9}N_f T_F C_F C_A \\
&\quad \left. + \frac{1415}{36}C_F^2 C_A \right] a^3 + O(a^4) \tag{2.6}
\end{aligned}$$

and

$$\begin{aligned}
C_{\text{Adl}}^{\text{mMOM}}(a, \alpha) &= [1 + 3C_F a \\
&+ \left[\frac{569}{12}C_F C_A - \frac{46}{3}N_f T_F C_F - \frac{3}{2}C_F^2 - \frac{3}{2}\alpha C_F C_A - \frac{3}{4}\alpha^2 C_F C_A - 44\zeta_3 C_F C_A \right. \\
&\quad \left. + 16\zeta_3 N_f T_F C_F \right] a^2 \\
&+ \left[\frac{50575}{48}C_F C_A^2 - \frac{18929}{24}\zeta_3 C_F C_A^2 - \frac{2063}{48}\alpha C_F C_A^2 - \frac{2033}{3}N_f T_F C_F C_A \right. \\
&\quad - \frac{1355}{12}C_F^2 C_A - \frac{1273}{48}\alpha^2 C_F C_A^2 - \frac{440}{3}\zeta_5 C_F C_A^2 - \frac{69}{2}C_F^3 - \frac{9}{16}\alpha^3 C_F C_A^2 \\
&\quad + \frac{3}{2}\alpha C_F^2 C_A + \frac{3}{4}\alpha^2 C_F^2 C_A + \frac{23}{3}\alpha^2 N_f T_F C_F C_A + \frac{46}{3}\alpha N_f T_F C_F C_A \\
&\quad + \frac{58}{3}N_f T_F C_F^2 + \frac{143}{4}\alpha \zeta_3 C_F C_A^2 + \frac{160}{3}\zeta_5 N_f T_F C_F C_A + \frac{185}{8}\alpha^2 \zeta_3 C_F C_A^2 \\
&\quad + \frac{1424}{3}\zeta_3 N_f T_F C_F C_A - 572\zeta_3 C_F^2 C_A - 320\zeta_5 N_f T_F C_F^2 - 64\zeta_3 N_f^2 T_F^2 C_F \\
&\quad - 16\alpha \zeta_3 N_f T_F C_F C_A - 8\alpha^2 \zeta_3 N_f T_F C_F C_A + 96N_f^2 T_F^2 C_F + 256\zeta_3 N_f T_F C_F^2 \\
&\quad \left. + 880\zeta_5 C_F^2 C_A \right] a^3 + O(a^4) \tag{2.7}
\end{aligned}$$

where we include the explicit gauge parameter dependence in the mMOM scheme in the arguments of the functions. A general observation is that (2.3) has to be replaced by the more accommodating formal relation

$$C_{\text{Bjr}}^{\mathcal{S}}(a, \alpha) C_{\text{Adl}}^{\mathcal{S}}(a, \alpha) = d_R [1 + \Delta_{\text{csb}}^{\mathcal{S}}(a, \alpha)] \tag{2.8}$$

where \mathcal{S} labels the scheme which for the moment is mMOM. In [12] the objective was to write $a\Delta_{\text{csb}}^{\text{mMOM}}(a, \alpha)$ in the form $\beta^{\text{mMOM}}(a, \alpha)K_a^{\text{mMOM}}(a, \alpha)$ which was partially successful. By this we mean that while the coefficient of the leading term of $K_a^{\text{mMOM}}(a, \alpha)$ had the same value as the $\overline{\text{MS}}$ scheme the next order term, being α dependent, could not be accommodated within (2.4) except

for several specific values of α . These were $\alpha = 0, -1$ and -3 , [12]. At the subsequent loop order neither of the last two values preserved the structure of (2.4). Indeed the situation for the failure of $\alpha = -3$ was examined in depth. In particular where the breakdown arose was distilled to a set of terms with identifiable colour factors. In summarizing the state of play for the mMOM scheme, [12], it might appear that the relation (2.4) limits Crewther's original relation, [1], and its extension to include a term proportional to the β -function to a subset of schemes. However we do not believe this to be a satisfactory situation.

Instead while the establishment of (2.3) and (2.4) is an important observation, the properties of the $\overline{\text{MS}}$ scheme perhaps overlook aspects of the renormalization group functions in other schemes that actually gave a direction as to how to resolve the difficulty with the mMOM scheme. There will be several threads to our argument. One of the β -function properties has already been incorporated into (2.6) and (2.7) which is that in general the β -function is gauge dependent. In the $\overline{\text{MS}}$ scheme the β -function is independent of the gauge parameter, [42]. In addition the gauge parameter is in essence a second coupling constant in a gauge theory even though its origin is in the quadratic part of the Lagrangian rather than in a higher order interaction. The renormalization group function associated with the renormalization of α , that is not usually referred to as a β -function as such, is the anomalous dimension of the gauge parameter which we will denote by $\gamma_\alpha(a, \alpha)$ here. Indeed the relevance of this anomalous dimension can be seen in the renormalization group equation, derived from ensuring that a finite renormalized n -point Green's function $\Gamma_{(n)}(\mu, a, \alpha)$ is independent of the renormalization scale μ associated with a regularization, since

$$\left[\mu \frac{\partial}{\partial \mu} + \beta(a, \alpha) \frac{\partial}{\partial a} + \alpha \gamma_\alpha(a, \alpha) \frac{\partial}{\partial \alpha} - n \gamma_\phi(a, \alpha) \right] \Gamma_{(n)}(\mu, a, \alpha) = 0. \quad (2.9)$$

We have appended an α dependence in the β -function since in general it will depend on the gauge parameter. It is only in a subset of schemes such as $\overline{\text{MS}}$ and RI' that the β -function is independent of α . Here $\gamma_\phi(a, \alpha)$ represents the anomalous dimensions of all the fields of the n -function and we have simplified to the massless case. While this is a standard expression for a gauge theory if one interprets α as a second coupling constant then the second and third terms of (2.9) could be rewritten as the differential operator

$$\beta_1(g_i) \frac{\partial}{\partial g_1} + \beta_2(g_i) \frac{\partial}{\partial g_2} \quad (2.10)$$

where $g_1 = a, g_2 = \alpha$ and

$$\beta_1(a, \alpha) \equiv \beta(a, \alpha) \quad , \quad \beta_2(a, \alpha) \equiv \alpha \gamma_\alpha(a, \alpha). \quad (2.11)$$

The final thread of our argument rests in the observation of [5] that $\Delta_{\text{csb}}^{\overline{\text{MS}}}(a)$ vanishes at a fixed point of the underlying theory. This is clearly the case in $\overline{\text{MS}}$ as the β -function zeros determine the critical points of the renormalization group flow. In other schemes where the β -function is gauge parameter dependent the situation is more involved. For instance, in [43] the fixed point properties of QCD were studied to several loop orders in schemes other than $\overline{\text{MS}}$ such as the MOM schemes of Celmaster and Gonsalves, [37, 38]. In the analysis of [43] the gauge parameter was treated as a second coupling constant and the critical point values of a and α , where the functions of (2.11) are zero, were determined. On top of the Gaussian and Banks-Zaks critical points, [34, 44], that were derived from the vanishing of $\beta^{\overline{\text{MS}}}(a)$, there are fixed points for both a and α non-zero simultaneously. More recently the analysis of [43] has been extended to several further loop orders, [45], for the $\overline{\text{MS}}$, mMOM and the MOM schemes of [37, 38] in the linear covariant gauge as well as the nonlinear gauges considered here. Not only is the non-trivial infrared stable fixed point observed in [43] preserved to higher order it has analogues in the nonlinear gauges as well. Therefore considering all these ingredients together it should be the case that the product of $C_{\text{Bjr}}^{\text{mMOM}}(a, \alpha)$

and $C_{\text{Adl}}^{\text{mMOM}}(a, \alpha)$ gives d_R not only at the Gaussian and Banks-Zaks fixed points but also at the other non-trivial critical points.

mMOM 2 loop			
a_∞	α_∞	$O(a^3)$	$O(a^4)$
0.0033112583	0.0000000000	2.9999991596	3.0000039877
0.0032001941	-3.0301823312	2.9999982468	3.0000012469
9.1803474173	2.4636080795	1271156.8083213258	17202735.3015072510
mMOM 3 loop			
a_∞	α_∞	$O(a^3)$	$O(a^4)$
0.0031177883	0.0000000000	2.9999963264	3.0000001212
0.0031380724	-3.0274210489	2.9999973439	3.0000001217
0.1279084604	1.9051106246	6.2952539870	10.1893903424
mMOM 4 loop			
a_∞	α_∞	$O(a^3)$	$O(a^4)$
0.0031213518	0.0000000000	2.9999963720	3.0000001843
0.0031430130	-3.0273541344	2.9999974127	3.0000002080
0.1162651496	0.5286066929	5.3930704057	11.8942763573
0.1902883419	0.0000000000	13.5399867931	66.1969134786
mMOM 5 loop			
a_∞	α_∞	$O(a^3)$	$O(a^4)$
0.0031220809	0.0000000000	2.9999963814	3.0000001972
0.0031434144	-3.0273765993	2.9999974183	3.0000002151
0.0502252330	-3.8653031470	3.1912609578	3.2787374506
0.0577103776	0.0000000000	3.2818695828	3.7273436677

Table 1: Values of $C_{\text{Bjr}}^{\text{mMOM}}(a, \alpha)C_{\text{Adl}}^{\text{mMOM}}(a, \alpha)$ at critical points at successive loop orders for $N_f = 16$.

3 Numerical evidence.

In order to test the idea given in [5] that $C_{\text{Bjr}}^{\text{mMOM}}(a, \alpha)C_{\text{Adl}}^{\text{mMOM}}(a, \alpha)$ reflects some conformal property of the underlying field theory at a fixed point, we have carried out a numerical investigation in the first instance using the data given in [45]. In particular the critical point values of the coupling constant and gauge parameter are known to various loop orders in different renormalization schemes and gauges. The numerical analysis of this section will be for the $SU(3)$ colour group. Also as the Crewther relation is a purely four dimensional one the fixed points we consider include all real solutions with positive coupling constant where the functions of (2.11) vanish. To remain within the perturbative domain of applicability our analysis was restricted to values of N_f close to the top of the conformal window which is at $N_f = 16$ for $SU(3)$, [34, 44]. As a guide to this reasoning we have recorded the successive loop order evaluation of $C_{\text{Bjr}}^{\text{mMOM}}(a, \alpha)C_{\text{Adl}}^{\text{mMOM}}(a, \alpha)$ at criticality in Table 1 for $N_f = 16$. In the table the various three or four critical values of a and α , denoted by a_∞ and α_∞ in the same notation as [43], are given at two to five loops respectively in four sub-sectors as indicated by the sub-heading. The critical values are recorded to higher precision

$\overline{\text{MS}}$ 2 loop			
a_∞	α_∞	$O(a^3)$	$O(a^4)$
0.0033112583	0.0000000000	3.0000005633	3.0000037708
0.0033112583	-2.9529847269	3.0000005633	3.0000037708
0.0033112583	-203.8803486064	3.0000005633	3.0000037708
$\overline{\text{MS}}$ 3 loop			
a_∞	α_∞	$O(a^3)$	$O(a^4)$
0.0031618421	0.0000000000	2.9999981298	3.0000007963
0.0031618421	-2.9458392416	2.9999981298	3.0000007963
$\overline{\text{MS}}$ 4 loop			
a_∞	α_∞	$O(a^3)$	$O(a^4)$
0.0031699203	0.0000000000	2.9999982498	3.0000009437
0.0031699203	-2.9459051005	2.9999982498	3.0000009437
0.0031699203	-126.8989470199	2.9999982498	3.0000009437
0.0917000502	0.0000000000	4.1865556391	6.0730678402
0.0917000502	2.6061754737	4.1865556391	6.0730678402
0.0917000502	-4.8367657559	4.1865556391	6.0730678402
$\overline{\text{MS}}$ 5 loop			
a_∞	α_∞	$O(a^3)$	$O(a^4)$
0.0031708402	0.0000000000	2.9999982636	3.0000009605
0.0031708402	-2.9459382941	2.9999982636	3.0000009605
0.0468980276	0.0000000000	3.1531010299	3.2821634001
0.0468980276	-3.7013593081	3.1531010299	3.2821634001

Table 2: Values of $C_{\text{Bjr}}^{\overline{\text{MS}}}(a)C_{\text{Adl}}^{\overline{\text{MS}}}(a)$ at critical points at successive loop orders for $N_f = 16$.

than [45]. In each sub-sector the first fixed point is the Banks-Zaks solution, [34, 44], which is a saddle point in the (a, α) plane. The second fixed point is the infrared stable one whose critical coupling value tends to that of the Banks-Zaks one as the loop order increases. The remaining one or two fixed point solutions are artefacts of solving polynomial equations at higher loop order. While their critical coupling values are outside the range of perturbative validity they are included as they are relevant to subsequent arguments. The data in the final two columns are the $O(a^3)$ and $O(a^4)$ values of $C_{\text{Bjr}}^{\text{mMOM}}(a, \alpha)C_{\text{Adl}}^{\text{mMOM}}(a, \alpha)$ at criticality.

Several features emerge from the table. The first is that for the first two critical points the combination of the Adler function and Bjorken sum rule produces a value very close to the expected value of 3. Moreover there is a clear but slow convergence to 3 as the loop order increases. We qualify the lack of precise agreement by noting that in computing the product there will be errors from the truncation of the series. With the available $O(a^4)$ expressions for both series the product will have terms up to $O(a^8)$. However this would not be the true $O(a^8)$ expression for $C_{\text{Bjr}}^{\text{mMOM}}(a, \alpha)C_{\text{Adl}}^{\text{mMOM}}(a, \alpha)$ since the as yet undetermined $O(a^5)$ term of one factor would lead to an $O(a^8)$ contribution from the known $O(a^3)$ term of the other factor for instance. For the purposes of the tables we do not include such incomplete higher order contributions. Therefore the most reliable indication of the value of $C_{\text{Bjr}}^{\text{mMOM}}(a, \alpha)C_{\text{Adl}}^{\text{mMOM}}(a, \alpha)$ is that from the five loop

MOMg 3 loops			
a_∞	α_∞	$O(a^3)$	$O(a^4)$
0.0029141531	0.0000000000	2.9999877567	2.9999976812
0.0029387804	-3.0259847376	2.9999876182	2.9999981783
MOMc 3 loops			
a_∞	α_∞	$O(a^3)$	$O(a^4)$
0.0031361417	0.0000000000	2.9999980055	2.9999998120
0.0031893967	-3.0263835218	2.9999980117	3.0000010487
0.0108889865	-9.6139587420	3.0017410493	3.0006478718
MOMq 3 loops			
a_∞	α_∞	$O(a^3)$	$O(a^4)$
0.0031361417	0.0000000000	2.9999974139	2.9999999355
0.0031893967	-3.0263835218	3.0000000826	2.9999999820
0.0108889865	-9.6139587420	3.0015252800	3.0013631672

Table 3: Values of $C_{\text{Bjr}}^{\text{MOMi}}(a, \alpha)C_{\text{Adl}}^{\text{MOMi}}(a, \alpha)$ at critical points at successive loop orders for $N_f = 16$.

critical points at both $O(a^3)$ and $O(a^4)$. A guide to this is that the final two fixed points produce a value significantly closer to 3 than might have seemed possible from the corresponding lower loop data even at four loops. That the value for these two five loop fixed points is not as accurate as the Banks-Zaks or the infrared stable ones resides in the fact that one is still outside the domain of perturbative reliability. More importantly for the two that are clearly within the domain the critical value of $C_{\text{Bjr}}^{\text{mMOM}}(a, \alpha)C_{\text{Adl}}^{\text{mMOM}}(a, \alpha)$ remains close to 3. We have analysed the small deviation from 3 for these two fixed points in a variety of ways and concluded that the discrepancy is due to the terms beyond $O(a^4)$ that result from the product of two perturbative functions of a . Perhaps one clear indication of this is to repeat the same exercise that produced Table 1 but for the $\overline{\text{MS}}$ scheme as that case led to the expressions for $\Delta_{\text{csb}}^{\overline{\text{MS}}}(a)$ in (2.3). The results are recorded in Table 2 to the same loop order. What is apparent is that the same numerical accuracy emerges for $C_{\text{Bjr}}^{\overline{\text{MS}}}(a)C_{\text{Adl}}^{\overline{\text{MS}}}(a)$ at each loop order for the Banks-Zaks and infrared stable fixed points as the corresponding cases in the mMOM scheme. So the deviation from the value of 3 in the mMOM scheme is on a similar footing to $\overline{\text{MS}}$. While this reinforces the notion that the Crewther relation has a connection to conformal properties it does not resolve how the observations of [12] can be accommodated.

We conclude this section by recording the situation in other schemes as well as gauges. In [45] the fixed point properties of QCD in the MOM schemes of Celmaster and Gonsalves were studied in the linear covariant gauge as well as two nonlinear gauges. These were the Curci-Ferrari gauge and the Maximal Abelian Gauge (MAG) whose more technical properties we discuss later but note that for both gauges the perturbative renormalization group functions are only available to three loops. Our results for the linear gauge are given in Table 3 while those for the other gauges are noted in Tables 4 and 5 respectively where MOMg, MOMc and MOMq denote the three distinct MOM schemes based on the triple gluon, ghost-gluon and quark-gluon 3-point vertices respectively. For brevity we have solely recorded the situation at three loops. Again a very similar state to the five loop analysis arises. In each case and particularly at three loops the product of the Adler function and the Bjorken sum rule in effect evaluate to 3. Moreover it reinforces the observation that the Crewther relation has to accommodate properties consistent with the critical points of the

Curci-Ferrari MOMg 3 loops			
a_∞	α_∞	$O(a^3)$	$O(a^4)$
0.0029141531	0.0000000000	2.9999877567	2.9999976812
0.0031077787	-5.8860065418	2.9999745145	2.9999979271
0.0069944637	-6.2196239630	2.9999956606	3.0006212984
Curci-Ferrari MOMc 3 loops			
a_∞	α_∞	$O(a^3)$	$O(a^4)$
0.0031822955	0.0000000000	2.9999996960	3.0000000816
0.0033905650	-5.8484574473	3.0000096730	2.9999996739
0.0256527015	-17.6663992530	3.0598055895	2.4987327333
0.0776442642	2.1961076699	3.7711417659	2.5429487508
Curci-Ferrari MOMq 3 loops			
a_∞	α_∞	$O(a^3)$	$O(a^4)$
0.0031361417	0.0000000000	2.9999974139	2.9999999354
0.0032120881	-5.8504247987	3.0000013385	2.9999998776
0.0068658178	-12.1450763308	3.0002807117	3.0002920650

Table 4: Values of $C_{\text{Bjr}}^{\text{MOMi}}(a, \alpha)C_{\text{Adl}}^{\text{MOMi}}(a, \alpha)$ at critical points at successive loop orders for $N_f = 16$ in the Curci-Ferrari gauge.

underlying theory for a variety of schemes and gauges.

4 mMOM scheme.

In light of these arguments and numerical evidence as well as the association of $\beta(a, \alpha)$ and $\gamma_\alpha(a, \alpha)$ with the underlying β -functions of gauge fixed QCD we propose that the extension of the Crewther relation of [5] to accommodate renormalization schemes other than $\overline{\text{MS}}$ is

$$\Delta_{\text{csb}}^{\mathcal{S}}(a_{\mathcal{S}}, \alpha_{\mathcal{S}}) = \frac{\beta^{\mathcal{S}}(a_{\mathcal{S}}, \alpha_{\mathcal{S}})}{a_{\mathcal{S}}} K_a^{\mathcal{S}}(a_{\mathcal{S}}, \alpha_{\mathcal{S}}) + \alpha_{\mathcal{S}} \gamma_\alpha^{\mathcal{S}}(a_{\mathcal{S}}, \alpha_{\mathcal{S}}) K_\alpha^{\mathcal{S}}(a_{\mathcal{S}}, \alpha_{\mathcal{S}}) \quad (4.1)$$

where \mathcal{S} labels the scheme. Again we have included α in the argument of the β -function to be as general as possible. While (4.1) reflects the combination (2.10) with the identification of (2.11) in (2.9), unlike its $\overline{\text{MS}}$ counterpart two functions $K_a(a, \alpha)$ and $K_\alpha(a, \alpha)$ are now required to satisfy the vanishing of $\Delta_{\text{csb}}(a, \alpha)$ at all the critical points of theory in a scheme \mathcal{S} . In other words if the origin of the Crewther relation rests in the fact that it reflects conformal symmetry, which is related to vanishing β -functions, then this property should not be a pure $\overline{\text{MS}}$ one. The combination of the Bjorken sum rule and Adler D -function that is the Crewther relation should vanish at the fixed points of another scheme as suggested by our numerical investigation. So this can be resolved for other schemes provided the relevant β -functions are present as proposed in (4.1).

In keeping with the ethos of [5] where the coefficients of the renormalization group functions of (4.1) were established by explicit calculation we now focus in this section on the mMOM scheme to study the K -functions partly to be able to compare and contrast with [10, 11, 12]. Therefore we

MAG MOMg 3 loops			
a_∞	α_∞	$O(a^3)$	$O(a^4)$
0.0028654266	-0.4168022707	2.9999860489	2.9999968722
0.0029254124	-5.5625872573	2.9999889816	2.9999980333
MAG MOMc 3 loops			
a_∞	α_∞	$O(a^3)$	$O(a^4)$
0.0031683606	-0.4162766338	2.9999989914	3.0000000637
0.0031577946	-5.5686496127	2.9999979720	3.0000000132
0.0014328961	-60.7998590768	3.0000187570	2.9999749282
MAG MOMq 3 loops			
a_∞	α_∞	$O(a^3)$	$O(a^4)$
0.0031256674	-0.4162158544	2.9999968662	2.9999999505
0.0031921140	-5.5677230790	3.0000001817	3.0000000315

Table 5: Values of $C_{\text{Bjr}}^{\text{MOMi}}(a, \alpha)C_{\text{Adl}}^{\text{MOMi}}(a, \alpha)$ at critical points at successive loop orders for $N_f = 16$ in the MAG.

have constructed the following functions in the mMOM scheme

$$\begin{aligned}
K_a^{\text{mMOM}}(a, \alpha) = & \left[12\zeta_3 C_F - \frac{21}{2} C_F \right] a \\
& + \left[\frac{21}{4} C_F C_A \alpha^2 + \frac{21}{2} C_F C_A \alpha - \frac{2591}{12} C_F C_A + \frac{397}{6} C_F^2 - 240\zeta_5 C_F^2 \right. \\
& \quad \left. - 48\zeta_3 N_f T_F C_F - 12\zeta_3 C_F C_A \alpha - 6\zeta_3 C_F C_A \alpha^2 + 62N_f T_F C_F + 136\zeta_3 C_F^2 \right. \\
& \quad \left. + 182\zeta_3 C_F C_A \right] a^2 \\
& + \left[\frac{2471}{12} C_F^3 - \frac{1840145}{288} C_F C_A^2 - \frac{14740}{3} \zeta_5 C_F^2 C_A - \frac{8000}{3} \zeta_5 N_f T_F C_F C_A \right. \\
& \quad \left. - \frac{5092}{9} N_f T_F C_F^2 - \frac{4792}{3} \zeta_3 N_f T_F C_F^2 - \frac{4312}{3} \zeta_3 N_f T_F C_F C_A \right. \\
& \quad \left. - \frac{2961}{16} \zeta_3 C_F C_A^2 \alpha^2 - \frac{2113}{2} \zeta_3^2 C_F C_A^2 - \frac{1568}{3} N_f^2 T_F^2 C_F - \frac{397}{4} C_F^2 C_A \alpha \right. \\
& \quad \left. - \frac{397}{8} C_F^2 C_A \alpha^2 - \frac{63}{4} \zeta_3 C_F C_A^2 \alpha^3 - \frac{31}{2} N_f T_F C_F C_A \alpha^2 + \frac{27}{2} \zeta_3^2 C_F C_A^2 \alpha^2 \right. \\
& \quad \left. + \frac{155}{3} N_f T_F C_F C_A \alpha + \frac{391}{8} \zeta_3 C_F C_A^2 \alpha + \frac{561}{32} C_F C_A^2 \alpha^3 + \frac{4051}{96} C_F C_A^2 \alpha \right. \\
& \quad \left. + \frac{6251}{32} C_F C_A^2 \alpha^2 + \frac{11900}{3} \zeta_5 C_F C_A^2 + \frac{71251}{18} N_f T_F C_F C_A + \frac{132421}{72} C_F^2 C_A \right. \\
& \quad \left. + \frac{152329}{48} \zeta_3 C_F C_A^2 - 5720\zeta_5 C_F^3 - 840\zeta_7 C_F^2 C_A - 204\zeta_3 C_F^2 C_A \alpha \right. \\
& \quad \left. - 102\zeta_3 C_F^2 C_A \alpha^2 - 99\zeta_3^2 C_F C_A^2 \alpha - 40\zeta_3 N_f T_F C_F C_A \alpha \right. \\
& \quad \left. + 12\zeta_3 N_f T_F C_F C_A \alpha^2 + 160\zeta_3^2 N_f T_F C_F C_A + 180\zeta_5 C_F^2 C_A \alpha^2 \right. \\
& \quad \left. + 224\zeta_3 N_f^2 T_F^2 C_F + 320\zeta_5 N_f^2 T_F^2 C_F + 360\zeta_5 C_F^2 C_A \alpha + 488\zeta_3 C_F^3 \right. \\
& \quad \left. + 2400\zeta_5 N_f T_F C_F^2 + 3608\zeta_3 C_F^2 C_A + 5040\zeta_7 C_F^3 \right] a^3 + O(a^4) \tag{4.2}
\end{aligned}$$

and

$$\begin{aligned}
K_\alpha^{\text{mMOM}}(a, \alpha) = & \left[\frac{21}{4} C_F C_A + \frac{21}{4} C_F C_A \alpha - 6\zeta_3 C_F C_A - 6\zeta_3 C_F C_A \alpha \right] a^2 \\
& + \left[\frac{189}{32} \alpha^2 C_F C_A^2 + \frac{341}{3} N_f T_F C_F C_A + \frac{2035}{48} \alpha C_F C_A^2 + \frac{2109}{8} \zeta_3 C_F C_A^2 \right. \\
& + 9\zeta_3^2 C_F C_A^2 \alpha + 120\zeta_5 C_F^2 C_A + 120\zeta_5 C_F^2 C_A \alpha - \frac{9773}{32} C_F C_A^2 \\
& - \frac{443}{8} \zeta_3 C_F C_A^2 \alpha - \frac{397}{12} C_F^2 C_A - \frac{397}{12} C_F^2 C_A \alpha - \frac{27}{4} \zeta_3 C_F C_A^2 \alpha^2 \\
& \left. - 88\zeta_3 N_f T_F C_F C_A - 68\zeta_3 C_F^2 C_A - 68\zeta_3 C_F^2 C_A \alpha - 33\zeta_3^2 C_F C_A^2 \right] a^3 \\
& + O(a^4) . \tag{4.3}
\end{aligned}$$

These expressions were arrived at perturbatively by solving for the two unknown K -functions at each order in the coupling constant. This approach was taken from the point of view of seeing whether such an ansatz would admit an explicit solution and is a valid method to apply. What it did reveal was that one property of the K -functions is they are not strictly unique. This is in the sense that one could in principle arrange terms in one of the K -functions to be absorbed into the other in such a way that the vanishing of $\Delta_{\text{csb}}(a, \alpha)$ at all the fixed points is preserved. One noteworthy property of $K_\alpha^{\text{mMOM}}(a, \alpha)$ is that the leading term, which is always $O(a^2)$ for this function, vanishes at $\alpha = -1$ but not for the $O(a^3)$ term. This particular gauge parameter value was singled out for specific comment in [12] as being of special interest. Equally it was noted that its special status was not preserved at next order. Similar observations were also made for the case of $\alpha = -3$ and we will discuss both cases in a later section.

5 MOM schemes.

While the Crewther relation has been studied previously in various schemes but at length in the mMOM scheme in [12], it is instructive to study several specific schemes as well as gauges other than the commonly used linear covariant gauge. This is the topic for this section and we examine the MOM schemes of [37, 38]. In particular we will focus on the MOMg scheme associated with the triple gluon vertex as the mathematical content of the K -functions of each of the three MOM schemes is similar. The MOMg renormalization group functions of [37, 38] and the other two schemes were extended to three loops in [46] numerically as well as analytically for an arbitrary linear gauge in [47]. More recently the four loop Landau gauge renormalization group functions were determined in [48]. One reason we have not used the results of [48] in our calculations is partly as they are recorded for a specific gauge. However, another is that those higher order corrections will not contribute to the K -functions. This can be seen by simply examining the orders of a in the two terms of (4.1) and the fact that the Adler and Bjorken expressions are only available to $O(a^4)$. As the mapping between the coupling constants and gauge parameters in the $\overline{\text{MS}}$ and MOMg schemes are known, [47], we have found $C_{\text{Bjr}}^{\text{MOMg}}(a, \alpha)$ and $C_{\text{Adl}}^{\text{MOMg}}(a, \alpha)$ to $O(a^4)$. Consequently we have been able to determine the two K -functions parallel to those of the mMOM scheme. For instance the $SU(3)$ Yang-Mills expressions are

$$\begin{aligned}
K_a^{\text{MOMg}}(a, \alpha) \Big|_{N_f=0}^{SU(3)} = & [16\zeta_3 - 14] a \\
& + \left[\frac{6448}{9} \zeta_3 - \frac{14158}{27} - \frac{1280}{3} \zeta_5 - \frac{644}{27} \pi^2 - \frac{368}{9} \psi'(\tfrac{1}{3}) \zeta_3 - \frac{56}{9} \pi^2 \alpha^2 \right. \\
& \left. - \frac{32}{3} \psi'(\tfrac{1}{3}) \zeta_3 \alpha^2 + \frac{28}{3} \psi'(\tfrac{1}{3}) \alpha^2 + \frac{64}{9} \zeta_3 \pi^2 \alpha^2 + \frac{322}{9} \psi'(\tfrac{1}{3}) + \frac{736}{27} \zeta_3 \pi^2 \right]
\end{aligned}$$

$$\begin{aligned}
& -72\zeta_3\alpha - 42\alpha^2 - 42\psi'(\tfrac{1}{3})\alpha - 32\zeta_3\pi^2\alpha - 8\zeta_3\alpha^3 + 7\alpha^3 + 28\pi^2\alpha \\
& + 48\zeta_3\alpha^2 + 48\psi'(\tfrac{1}{3})\zeta_3\alpha + 63\alpha] a^2 \\
& + \left[\frac{1}{4}\psi'''(\tfrac{1}{3})\zeta_3\alpha^3 + \frac{7}{2}\psi'''(\tfrac{1}{3})\zeta_3\alpha^2 - \frac{127914947}{2592} - \frac{696830}{243}\pi^2 \right. \\
& - \frac{372568}{81}\zeta_3\pi^2\alpha - \frac{307715}{54}\psi'(\tfrac{1}{3})\alpha - \frac{210179}{324}\pi^2\alpha^2 \\
& - \frac{177919}{243}\psi'(\tfrac{1}{3})\pi^2\alpha - \frac{147991}{96}\alpha^2 - \frac{144856}{27}\psi'(\tfrac{1}{3})\zeta_3 - \frac{133352}{729}\zeta_3\pi^4\alpha \\
& - \frac{54362}{243}\zeta_3\pi^4 - \frac{50834}{81}\psi'(\tfrac{1}{3})^2\zeta_3\alpha - \frac{29440}{27}\zeta_5\pi^2 - \frac{25921}{216}\psi'(\tfrac{1}{3})^2 \\
& - \frac{24514}{3}\zeta_3\alpha - \frac{21805}{216}\psi'(\tfrac{1}{3})^2\alpha^2 - \frac{14812}{81}\psi'(\tfrac{1}{3})\zeta_3\pi^2 \\
& - \frac{12460}{81}\psi'(\tfrac{1}{3})\zeta_3\pi^2\alpha^2 - \frac{11663}{9}\psi'(\tfrac{1}{3})\zeta_3\alpha^2 - \frac{8918}{243}\pi^4\alpha^2 - \frac{5349}{32}\alpha^4 \\
& - \frac{4399}{72}\alpha^3 - \frac{2989}{32}\psi'''(\tfrac{1}{3}) - \frac{2560}{9}\zeta_5\pi^2\alpha^2 - \frac{1286}{81}\zeta_3\pi^4\alpha^3 - \frac{1205}{6}\zeta_3\alpha^3 \\
& - \frac{1115}{18}\pi^2\alpha^3 - \frac{1078}{27}\psi'(\tfrac{1}{3})\pi^2\alpha^3 - \frac{763}{12}\pi^2\alpha^4 - \frac{308}{9}\psi'(\tfrac{1}{3})^2\zeta_3\alpha^3 \\
& - \frac{250}{3}\psi'(\tfrac{1}{3})\zeta_3\alpha^3 - \frac{153}{2}\zeta_3\alpha^5 - \frac{147}{32}\alpha^6 - \frac{112}{9}\psi'(\tfrac{1}{3})\zeta_3\pi^2\alpha^4 - \frac{98}{27}\pi^4\alpha^4 \\
& - \frac{49}{4}\psi'(\tfrac{1}{3})\alpha^5 - \frac{49}{6}\psi'(\tfrac{1}{3})^2\alpha^4 - \frac{49}{16}\psi'''(\tfrac{1}{3})\alpha^2 - \frac{28}{3}\zeta_3\pi^2\alpha^5 \\
& - \frac{7}{32}\psi'''(\tfrac{1}{3})\alpha^3 + \frac{21}{4}\zeta_3\alpha^6 + \frac{28}{3}\psi'(\tfrac{1}{3})^2\zeta_3\alpha^4 + \frac{49}{6}\pi^2\alpha^5 + \frac{63}{2}\psi'''(\tfrac{1}{3})\alpha \\
& + \frac{98}{9}\psi'(\tfrac{1}{3})\pi^2\alpha^4 + \frac{112}{27}\zeta_3\pi^4\alpha^4 + \frac{218}{3}\zeta_3\pi^2\alpha^4 + \frac{427}{4}\psi'''(\tfrac{1}{3})\zeta_3 \\
& + \frac{500}{9}\zeta_3\pi^2\alpha^3 + \frac{539}{18}\psi'(\tfrac{1}{3})^2\alpha^3 + \frac{763}{8}\psi'(\tfrac{1}{3})\alpha^4 + \frac{867}{4}\zeta_3\alpha^4 + \frac{1071}{16}\alpha^5 \\
& + \frac{1115}{12}\psi'(\tfrac{1}{3})\alpha^3 + \frac{1232}{27}\psi'(\tfrac{1}{3})\zeta_3\pi^2\alpha^3 + \frac{1280}{3}\psi'(\tfrac{1}{3})\zeta_5\alpha^2 \\
& + \frac{3115}{27}\psi'(\tfrac{1}{3})^2\zeta_3\alpha^2 + \frac{3703}{27}\psi'(\tfrac{1}{3})^2\zeta_3 + \frac{4501}{324}\pi^4\alpha^3 + \frac{10192}{243}\zeta_3\pi^4\alpha^2 \\
& + \frac{14720}{9}\psi'(\tfrac{1}{3})\zeta_5 + \frac{19415}{8}\zeta_3\alpha^2 + \frac{21805}{162}\psi'(\tfrac{1}{3})\pi^2\alpha^2 + \frac{22400}{3}\zeta_7 \\
& + \frac{23326}{27}\zeta_3\pi^2\alpha^2 + \frac{25921}{162}\psi'(\tfrac{1}{3})\pi^2 + \frac{116683}{729}\pi^4\alpha + \frac{177919}{324}\psi'(\tfrac{1}{3})^2\alpha \\
& + \frac{186284}{27}\psi'(\tfrac{1}{3})\zeta_3\alpha + \frac{190267}{972}\pi^4 + \frac{203336}{243}\psi'(\tfrac{1}{3})\zeta_3\pi^2\alpha \\
& + \frac{210179}{216}\psi'(\tfrac{1}{3})\alpha^2 + \frac{289169}{48}\alpha + \frac{289712}{81}\zeta_3\pi^2 + \frac{307715}{81}\pi^2\alpha \\
& + \frac{348415}{81}\psi'(\tfrac{1}{3}) + \frac{485200}{27}\zeta_5 + \frac{9840343}{216}\zeta_3 - 23325\zeta_3^2 - 1920\zeta_5\alpha^2 \\
& - 1920\psi'(\tfrac{1}{3})\zeta_5\alpha - 109\psi'(\tfrac{1}{3})\zeta_3\alpha^4 - 36\zeta_3^2\alpha^3 - 36\psi'''(\tfrac{1}{3})\zeta_3\alpha + 9\zeta_3^2\alpha^2 \\
& + 14\psi'(\tfrac{1}{3})\zeta_3\alpha^5 + 320\zeta_5\alpha^3 + 1280\zeta_5\pi^2\alpha + 2052\zeta_3^2\alpha + 2880\zeta_5\alpha] a^3 \\
& + O(a^4) \tag{5.1}
\end{aligned}$$

and

$$\begin{aligned}
K_\alpha^{\text{MOMg}}(a, \alpha) \Big|_{N_f=0}^{SU(3)} &= \left[\frac{63}{2} - \frac{56}{9}\pi^2\alpha - \frac{32}{3}\psi'(\tfrac{1}{3})\zeta_3\alpha + \frac{21}{2}\alpha^2 + \frac{28}{3}\psi'(\tfrac{1}{3})\alpha + \frac{64}{9}\zeta_3\pi^2\alpha - 42\alpha \right. \\
& \left. - 36\zeta_3 - 21\psi'(\tfrac{1}{3}) - 16\zeta_3\pi^2 - 12\zeta_3\alpha^2 + 14\pi^2 + 24\psi'(\tfrac{1}{3})\zeta_3 + 48\zeta_3\alpha \right] a^2
\end{aligned}$$

$$\begin{aligned}
& + \left[684\zeta_3^2 + 960\zeta_5 - \frac{528991}{108}\psi'(\tfrac{1}{3}) - \frac{298948}{81}\zeta_3\pi^2 - \frac{207487}{324}\psi'(\tfrac{1}{3})\alpha \right. \\
& - \frac{179000}{729}\zeta_3\pi^4 - \frac{177037}{243}\psi'(\tfrac{1}{3})\pi^2 - \frac{50582}{81}\psi'(\tfrac{1}{3})^2\zeta_3 - \frac{26252}{81}\zeta_3\pi^2\alpha \\
& - \frac{12853}{3}\zeta_3 - \frac{5120}{27}\zeta_5\pi^2\alpha - \frac{2464}{27}\zeta_3\pi^4\alpha - \frac{2331}{8}\alpha^3 - \frac{2009}{9}\psi'(\tfrac{1}{3})\pi^2\alpha \\
& - \frac{1295}{6}\zeta_3\alpha^2 - \frac{917}{36}\alpha^2 - \frac{574}{3}\psi'(\tfrac{1}{3})^2\zeta_3\alpha - \frac{448}{27}\psi'(\tfrac{1}{3})\zeta_3\pi^2\alpha^3 \\
& - \frac{392}{81}\pi^4\alpha^3 - \frac{385}{3}\pi^2\alpha^3 - \frac{255}{2}\zeta_3\alpha^4 - \frac{245}{12}\psi'(\tfrac{1}{3})\alpha^4 - \frac{147}{16}\alpha^5 \\
& - \frac{140}{9}\zeta_3\pi^2\alpha^4 - \frac{98}{9}\psi'(\tfrac{1}{3})^2\alpha^3 - \frac{49}{24}\psi'''(\tfrac{1}{3})\alpha - \frac{35}{2}\pi^2\alpha^2 - \frac{7}{32}\psi'''(\tfrac{1}{3})\alpha^2 \\
& + \frac{1}{4}\psi'''(\tfrac{1}{3})\zeta_3\alpha^2 + \frac{7}{3}\psi'''(\tfrac{1}{3})\zeta_3\alpha + \frac{21}{2}\zeta_3\alpha^5 + \frac{21}{2}\psi'''(\tfrac{1}{3}) \\
& + \frac{70}{3}\psi'(\tfrac{1}{3})\zeta_3\alpha^4 + \frac{105}{4}\psi'(\tfrac{1}{3})\alpha^2 + \frac{112}{9}\psi'(\tfrac{1}{3})^2\zeta_3\alpha^3 + \frac{133}{4}\pi^4\alpha^2 \\
& + \frac{147}{2}\psi'(\tfrac{1}{3})^2\alpha^2 + \frac{245}{18}\pi^2\alpha^4 + \frac{385}{2}\psi'(\tfrac{1}{3})\alpha^3 + \frac{392}{27}\psi'(\tfrac{1}{3})\pi^2\alpha^3 \\
& + \frac{440}{3}\zeta_3\pi^2\alpha^3 + \frac{448}{81}\zeta_3\pi^4\alpha^3 + \frac{1280}{3}\zeta_5\pi^2 + \frac{1785}{16}\alpha^4 + \frac{2009}{12}\psi'(\tfrac{1}{3})^2\alpha \\
& + \frac{2156}{27}\pi^4\alpha + \frac{2296}{9}\psi'(\tfrac{1}{3})\zeta_3\pi^2\alpha + \frac{2560}{9}\psi'(\tfrac{1}{3})\zeta_5\alpha + \frac{11215}{12}\zeta_3\alpha \\
& + \frac{13126}{27}\psi'(\tfrac{1}{3})\zeta_3\alpha + \frac{20909}{144}\alpha + \frac{78711}{16} + \frac{149474}{27}\psi'(\tfrac{1}{3})\zeta_3 \\
& + \frac{156625}{729}\pi^4 + \frac{177037}{324}\psi'(\tfrac{1}{3})^2 + \frac{202328}{243}\psi'(\tfrac{1}{3})\zeta_3\pi^2 + \frac{207487}{486}\pi^2\alpha \\
& + \frac{528991}{162}\pi^2 - 1280\zeta_5\alpha - 640\psi'(\tfrac{1}{3})\zeta_5 - 220\psi'(\tfrac{1}{3})\zeta_3\alpha^3 \\
& - 98\psi'(\tfrac{1}{3})\pi^2\alpha^2 - 84\psi'(\tfrac{1}{3})^2\zeta_3\alpha^2 - 38\zeta_3\pi^4\alpha^2 - 36\zeta_3^2\alpha^2 \\
& - 30\psi'(\tfrac{1}{3})\zeta_3\alpha^2 - 12\psi'''(\tfrac{1}{3})\zeta_3 + 6\zeta_3^2\alpha + 20\zeta_3\pi^2\alpha^2 + 112\psi'(\tfrac{1}{3})\zeta_3\pi^2\alpha^2 \\
& \left. + 320\zeta_5\alpha^2 + 333\zeta_3\alpha^3 \right] a^3 + O(a^4) \tag{5.2}
\end{aligned}$$

where $\psi(z)$ is the Euler psi function. Its appearance is directly related to the symmetric point configuration of the triple gluon vertex and also arises in the other two MOM schemes based on the ghost-gluon (MOMc) and quark-gluon (MOMq) vertices. In [47] the expressions for the renormalization group functions additionally involved harmonic polynomials $H_i^{(2)}$ as well as

$$s_n(z) = \frac{1}{\sqrt{3}} \text{Im} \left[\text{Li}_n \left(\frac{e^{iz}}{\sqrt{3}} \right) \right] \tag{5.3}$$

where $\text{Li}_n(z)$ is the polylogarithm function. In [48] it was shown using the PSLQ algorithm that the actual number basis for MOM schemes was rationals, π , ζ_2 , ζ_3 , ζ_4 , ζ_5 , $\psi'(\tfrac{1}{3})$ and $\psi'''(\tfrac{1}{3})$. The absence of harmonic polylogarithms and the polylogarithm function was because they were not independent of this basis. Indeed we have demonstrated this by verifying that the relation

$$\begin{aligned}
s_2(\tfrac{\pi}{6}) & = \frac{1}{11664} \left[324\sqrt{3} \ln(3)\pi - 27\sqrt{3} \ln^2(3)\pi + 29\sqrt{3}\pi^3 - 1944\psi'(\tfrac{1}{3}) \right. \\
& \left. + 23328s_2(\tfrac{\pi}{2}) + 19440s_3(\tfrac{\pi}{6}) - 15552s_3(\tfrac{\pi}{2}) + 1296\pi^2 \right] \tag{5.4}
\end{aligned}$$

holds numerically. We have incorporated (5.4) in our computations and therefore any MOM scheme expressions will only involve the basis of [48]. We have also established relations similar to the MOMg ones of (5.2) for the other MOM schemes. Those results as well as the MOMg ones are

available for an arbitrary colour group and $N_f \neq 0$ in the data file associated with this article. As there was a special value for α in the mMOM case that means the K_α function was absent at leading order we have examined $K_\alpha^{\text{MOMg}}(a, \alpha)$ for $SU(3)$ to see if the same or another value of α arises to produce a zero leading order term of the MOMg K_α function. For $N_f \neq 0$ this term is N_f independent but unlike the mMOM case it is quadratic in α . Although there are analytic solutions for α when the leading term vanishes the numerical values are $\alpha = -1.617608$ and 2.492398 . The fact that neither values are integer is solely due to the presence of $\psi(\frac{1}{3})$, ζ_3 and π contributions which result from the renormalization prescription defining each MOM scheme and the particular external momentum configuration of the 3-point vertices where the renormalization constants are defined. For the MOMc and MOMq schemes the respective values of α are -2.358904 and -7.145900 as both leading order K_α functions are linear in the gauge parameter.

Having concentrated on the linear covariant gauge to this point we turn our attention to two nonlinear gauges which are the Curci-Ferrari gauge [49] and the MAG, [50, 51, 52]. However both gauges are not unrelated. The MAG is a gauge fixing where the gluon fields are partitioned into those whose associated group generator form an abelian subgroup of the non-abelian gauge group and the remainder. The former are referred to as the diagonal gluons and there are N_A^d such fields while there are N_A^o remaining off-diagonal fields where $N_A^d = N_c - 1$ and $N_A^o = N_c(N_c - 1)$ for the group $SU(N_c)$. The gauge fixing functional takes a different form in both sectors where the diagonal gluons are fixed in the Landau gauge but the N_A^o gluons have a nonlinear functional that involves the ghosts associated with the off-diagonal gluons. These ghost fields differ from those of the abelian subgroup gluons. The MAG has several similarities with the background field gauge in that the diagonal gluons behave like background field gluons. The Curci-Ferrari gauge is different from the MAG in that the gluons are not distinguished in the gauge fixing but do share a similar quadratic ghost term in the gauge fixing functional. Both gauges have quartic ghost interactions that have no analogue in the linear covariant gauge fixing. What does connect the MAG to the Curci-Ferrari gauge, however, is the fact that taking the formal $N_A^d \rightarrow 0$ limit produces the Curci-Ferrari gauge. This is of practical benefit as a check on any computations involving the MAG since the Curci-Ferrari gauge expressions have to be reproduced as $N_A^d \rightarrow 0$.

Although both the Bjorken sum rule and the Adler D -function were originally computed perturbatively to high loop order in the linear gauge, since the operators involved in the underlying field theory formalism are gauge independent the $\overline{\text{MS}}$ expressions for both quantities will be the same in either the Curci-Ferrari gauge or MAG. Therefore to determine the extension to the Crewther relation we have carried out the mapping of the $\overline{\text{MS}}$ coupling constant to the Curci-Ferrari coupling in the MOMg scheme as well as also the MOMc and MOMq schemes. As a check on this procedure, however, we carried out the explicit direct computation of the Adler D -function in both the Curci-Ferrari gauge and the MAG to high loop order. This could only be achieved by using the FORCER package, [53, 54], written in the symbolic manipulation language FORM, [55, 56]. The FORCER algorithm evaluates massless 2-point Feynman graphs up to four loops very efficiently. In carrying out this exercise it is important to recognize that in the Curci-Ferrari gauge there are several extra Feynman graphs to be evaluated for the D -function than the linear gauge due to the extra quartic ghost interaction. By contrast in the case of the MAG there are considerably more diagrams to evaluate compared to the other two gauges not only due to quartic ghost interactions but because there are twice as many gluon and ghost fields. These have intricately connected interactions. On top of this one has to implement a routine to handle the colour group factors associated with gluon and ghost fields as they lie in two sectors. So the group generators and structure functions obey a more complicated algebra. Useful in this context were the identities used in the three loop $\overline{\text{MS}}$ renormalization of the MAG, [57]. The overall outcome was that the $\overline{\text{MS}}$ Adler D -function resulted as expected. Once that was established for both gauges it was a simple exercise to change the renormalization of the D -function to another scheme such as the MOMg one for both gauges. Con-

sequently this checked that the direct application of the mapping from the linear gauge expression was consistent. The various coupling constant mappings were originally determined in [58].

Having summarized the background to both gauges we have managed to construct the respective K -functions in keeping with (4.1). The expressions for the Curci-Ferrari (CF) gauge are not dissimilar to those of the linear covariant gauge since

$$\begin{aligned}
K_{a\text{CF}}^{\text{MOMg}}(a, \alpha) \Big|_{N_f=0}^{SU(3)} &= [16\zeta_3 - 14] a \\
&+ \left[28\pi^2\alpha + 48\zeta_3\alpha^2 + 48\psi'(\tfrac{1}{3})\zeta_3\alpha - \frac{14158}{27} - \frac{1280}{3}\zeta_5 - \frac{644}{27}\pi^2 \right. \\
&\quad - \frac{368}{9}\psi'(\tfrac{1}{3})\zeta_3 - \frac{56}{9}\pi^2\alpha^2 - \frac{32}{3}\psi'(\tfrac{1}{3})\zeta_3\alpha^2 + \frac{28}{3}\psi'(\tfrac{1}{3})\alpha^2 + \frac{64}{9}\zeta_3\pi^2\alpha^2 \\
&\quad + \frac{322}{9}\psi'(\tfrac{1}{3}) + \frac{736}{27}\zeta_3\pi^2 + \frac{6448}{9}\zeta_3 - 72\zeta_3\alpha - 42\alpha^2 - 42\psi'(\tfrac{1}{3})\alpha \\
&\quad \left. - 32\zeta_3\pi^2\alpha - 8\zeta_3\alpha^3 + 7\alpha^3 + 63\alpha \right] a^2 \\
&+ \left[\frac{22400}{3}\zeta_7 - \frac{127914947}{2592} - \frac{696830}{243}\pi^2 - \frac{562472}{81}\zeta_3\pi^2\alpha \right. \\
&\quad - \frac{491041}{54}\psi'(\tfrac{1}{3})\alpha - \frac{402143}{243}\psi'(\tfrac{1}{3})\pi^2\alpha - \frac{389608}{729}\zeta_3\pi^4\alpha \\
&\quad - \frac{238097}{324}\pi^2\alpha^2 - \frac{164461}{96}\alpha^2 - \frac{144856}{27}\psi'(\tfrac{1}{3})\zeta_3 - \frac{114898}{81}\psi'(\tfrac{1}{3})^2\zeta_3\alpha \\
&\quad - \frac{54362}{243}\zeta_3\pi^4 - \frac{32389}{216}\psi'(\tfrac{1}{3})^2\alpha^2 - \frac{29948}{3}\zeta_3\alpha - \frac{29440}{27}\zeta_5\pi^2 \\
&\quad - \frac{25921}{216}\psi'(\tfrac{1}{3})^2 - \frac{18508}{81}\psi'(\tfrac{1}{3})\zeta_3\pi^2\alpha^2 - \frac{14812}{81}\psi'(\tfrac{1}{3})\zeta_3\pi^2 \\
&\quad - \frac{13643}{243}\pi^4\alpha^2 - \frac{12761}{9}\psi'(\tfrac{1}{3})\zeta_3\alpha^2 - \frac{5415}{32}\alpha^4 - \frac{2989}{32}\psi'''(\tfrac{1}{3}) \\
&\quad - \frac{2560}{9}\zeta_5\pi^2\alpha^2 - \frac{2236}{81}\zeta_3\pi^4\alpha^3 - \frac{1862}{27}\psi'(\tfrac{1}{3})\pi^2\alpha^3 - \frac{875}{12}\pi^2\alpha^4 \\
&\quad - \frac{841}{3}\zeta_3\alpha^3 - \frac{545}{12}\psi'(\tfrac{1}{3})\alpha^3 - \frac{532}{9}\psi'(\tfrac{1}{3})^2\zeta_3\alpha^3 - \frac{380}{9}\zeta_3\pi^2\alpha^3 \\
&\quad - \frac{153}{2}\zeta_3\alpha^5 - \frac{147}{32}\alpha^6 - \frac{112}{9}\psi'(\tfrac{1}{3})\zeta_3\pi^2\alpha^4 - \frac{98}{27}\pi^4\alpha^4 - \frac{63}{16}\psi'''(\tfrac{1}{3})\alpha^2 \\
&\quad - \frac{49}{4}\psi'(\tfrac{1}{3})\alpha^5 - \frac{49}{6}\psi'(\tfrac{1}{3})^2\alpha^4 - \frac{28}{3}\zeta_3\pi^2\alpha^5 - \frac{7}{16}\psi'''(\tfrac{1}{3})\alpha^3 \\
&\quad + \frac{1}{2}\psi'''(\tfrac{1}{3})\zeta_3\alpha^3 + \frac{9}{2}\psi'''(\tfrac{1}{3})\zeta_3\alpha^2 + \frac{21}{4}\zeta_3\alpha^6 + \frac{28}{3}\psi'(\tfrac{1}{3})^2\zeta_3\alpha^4 \\
&\quad + \frac{49}{6}\pi^2\alpha^5 + \frac{63}{2}\psi'''(\tfrac{1}{3})\alpha + \frac{98}{9}\psi'(\tfrac{1}{3})\pi^2\alpha^4 + \frac{112}{27}\zeta_3\pi^4\alpha^4 \\
&\quad + \frac{190}{3}\psi'(\tfrac{1}{3})\zeta_3\alpha^3 + \frac{250}{3}\zeta_3\pi^2\alpha^4 + \frac{427}{4}\psi'''(\tfrac{1}{3})\zeta_3 + \frac{545}{18}\pi^2\alpha^3 \\
&\quad + \frac{825}{4}\zeta_3\alpha^4 + \frac{875}{8}\psi'(\tfrac{1}{3})\alpha^4 + \frac{931}{18}\psi'(\tfrac{1}{3})^2\alpha^3 + \frac{1071}{16}\alpha^5 \\
&\quad + \frac{1280}{3}\psi'(\tfrac{1}{3})\zeta_5\alpha^2 + \frac{2128}{27}\psi'(\tfrac{1}{3})\zeta_3\pi^2\alpha^3 + \frac{2917}{36}\alpha^3 + \frac{3703}{27}\psi'(\tfrac{1}{3})^2\zeta_3 \\
&\quad + \frac{3913}{162}\pi^4\alpha^3 + \frac{4627}{27}\psi'(\tfrac{1}{3})^2\zeta_3\alpha^2 + \frac{5279}{2}\zeta_3\alpha^2 + \frac{14720}{9}\psi'(\tfrac{1}{3})\zeta_5 \\
&\quad + \frac{15592}{243}\zeta_3\pi^4\alpha^2 + \frac{25522}{27}\zeta_3\pi^2\alpha^2 + \frac{25921}{162}\psi'(\tfrac{1}{3})\pi^2 \\
&\quad \left. + \frac{32389}{162}\psi'(\tfrac{1}{3})\pi^2\alpha^2 + \frac{190267}{972}\pi^4 + \frac{238097}{216}\psi'(\tfrac{1}{3})\alpha^2 \right] a^3
\end{aligned}$$

$$\begin{aligned}
& + \frac{281236}{27} \psi'(\frac{1}{3}) \zeta_3 \alpha + \frac{289712}{81} \zeta_3 \pi^2 + \frac{340907}{729} \pi^4 \alpha + \frac{348415}{81} \psi'(\frac{1}{3}) \\
& + \frac{402143}{324} \psi'(\frac{1}{3})^2 \alpha + \frac{433885}{48} \alpha + \frac{459592}{243} \psi'(\frac{1}{3}) \zeta_3 \pi^2 \alpha + \frac{485200}{27} \zeta_5 \\
& + \frac{491041}{81} \pi^2 \alpha + \frac{9840343}{216} \zeta_3 - 23325 \zeta_3^2 - 1920 \zeta_5 \alpha^2 \\
& - 1920 \psi'(\frac{1}{3}) \zeta_5 \alpha - 125 \psi'(\frac{1}{3}) \zeta_3 \alpha^4 - 72 \zeta_3^2 \alpha^3 - 54 \zeta_3^2 \alpha^2 \\
& - 36 \psi'''(\frac{1}{3}) \zeta_3 \alpha + 14 \psi'(\frac{1}{3}) \zeta_3 \alpha^5 + 320 \zeta_5 \alpha^3 + 1280 \zeta_5 \pi^2 \alpha + 2052 \zeta_3^2 \alpha \\
& + 2880 \zeta_5 \alpha] a^3 + O(a^4) \tag{5.5}
\end{aligned}$$

and

$$\begin{aligned}
K_{\alpha \text{CF}}^{\text{MOMg}}(a, \alpha) \Big|_{N_f=0}^{SU(3)} &= \left[\frac{63}{2} + \frac{64}{9} \zeta_3 \pi^2 \alpha - \frac{56}{9} \pi^2 \alpha - \frac{32}{3} \psi'(\frac{1}{3}) \zeta_3 \alpha + \frac{21}{2} \alpha^2 + \frac{28}{3} \psi'(\frac{1}{3}) \alpha - 42 \alpha \right. \\
& \quad \left. - 36 \zeta_3 - 21 \psi'(\frac{1}{3}) - 16 \zeta_3 \pi^2 - 12 \zeta_3 \alpha^2 + 14 \pi^2 + 24 \psi'(\frac{1}{3}) \zeta_3 + 48 \zeta_3 \alpha \right] a^2 \\
& + \left[\frac{1}{2} \psi'''(\frac{1}{3}) \zeta_3 \alpha^2 - \frac{1149479}{108} \psi'(\frac{1}{3}) - \frac{620324}{81} \zeta_3 \pi^2 - \frac{612664}{729} \zeta_3 \pi^4 \right. \\
& \quad - \frac{556493}{243} \psi'(\frac{1}{3}) \pi^2 - \frac{158998}{81} \psi'(\frac{1}{3})^2 \zeta_3 - \frac{52297}{81} \psi'(\frac{1}{3}) \alpha \\
& \quad - \frac{26576}{81} \zeta_3 \pi^2 \alpha - \frac{22049}{3} \zeta_3 - \frac{5120}{27} \zeta_5 \pi^2 \alpha - \frac{2512}{27} \zeta_3 \pi^4 \alpha - \frac{2079}{8} \alpha^3 \\
& \quad - \frac{2009}{9} \psi'(\frac{1}{3}) \pi^2 \alpha - \frac{715}{3} \zeta_3 \alpha^2 - \frac{574}{3} \psi'(\frac{1}{3})^2 \zeta_3 \alpha - \frac{448}{27} \psi'(\frac{1}{3}) \zeta_3 \pi^2 \alpha^3 \\
& \quad - \frac{392}{81} \pi^4 \alpha^3 - \frac{255}{2} \zeta_3 \alpha^4 - \frac{245}{12} \psi'(\frac{1}{3}) \alpha^4 - \frac{147}{16} \alpha^5 - \frac{140}{9} \zeta_3 \pi^2 \alpha^4 \\
& \quad - \frac{116}{3} \zeta_3 \pi^4 \alpha^2 - \frac{98}{9} \psi'(\frac{1}{3})^2 \alpha^3 - \frac{21}{4} \psi'(\frac{1}{3}) \alpha^2 - \frac{21}{8} \psi'''(\frac{1}{3}) \alpha \\
& \quad - \frac{7}{16} \psi'''(\frac{1}{3}) \alpha^2 + \frac{7}{2} \pi^2 \alpha^2 + \frac{21}{2} \zeta_3 \alpha^5 + \frac{21}{2} \psi'''(\frac{1}{3}) + \frac{70}{3} \psi'(\frac{1}{3}) \zeta_3 \alpha^4 \\
& \quad + \frac{112}{9} \psi'(\frac{1}{3})^2 \zeta_3 \alpha^3 + \frac{147}{2} \psi'(\frac{1}{3})^2 \alpha^2 + \frac{196}{9} \alpha^2 + \frac{203}{6} \pi^4 \alpha^2 + \frac{245}{18} \pi^2 \alpha^4 \\
& \quad + \frac{357}{2} \psi'(\frac{1}{3}) \alpha^3 + \frac{392}{27} \psi'(\frac{1}{3}) \pi^2 \alpha^3 + \frac{448}{81} \zeta_3 \pi^4 \alpha^3 + \frac{1280}{3} \zeta_5 \pi^2 \\
& \quad + \frac{1785}{16} \alpha^4 + \frac{2009}{12} \psi'(\frac{1}{3})^2 \alpha + \frac{2198}{27} \pi^4 \alpha + \frac{2296}{9} \psi'(\frac{1}{3}) \zeta_3 \pi^2 \alpha \\
& \quad + \frac{2560}{9} \psi'(\frac{1}{3}) \zeta_5 \alpha + \frac{3076}{3} \zeta_3 \alpha + \frac{13288}{27} \psi'(\frac{1}{3}) \zeta_3 \alpha + \frac{14105}{144} \alpha \\
& \quad + \frac{104594}{243} \pi^2 \alpha + \frac{310162}{27} \psi'(\frac{1}{3}) \zeta_3 + \frac{481037}{48} + \frac{536081}{729} \pi^4 \\
& \quad + \frac{556493}{324} \psi'(\frac{1}{3})^2 + \frac{635992}{243} \psi'(\frac{1}{3}) \zeta_3 \pi^2 + \frac{1149479}{162} \pi^2 - 1280 \zeta_5 \alpha \\
& \quad - 640 \psi'(\frac{1}{3}) \zeta_5 - 204 \psi'(\frac{1}{3}) \zeta_3 \alpha^3 - 119 \pi^2 \alpha^3 - 98 \psi'(\frac{1}{3}) \pi^2 \alpha^2 \\
& \quad - 84 \psi'(\frac{1}{3})^2 \zeta_3 \alpha^2 - 72 \zeta_3^2 \alpha^2 - 36 \zeta_3^2 \alpha - 12 \psi'''(\frac{1}{3}) \zeta_3 - 4 \zeta_3 \pi^2 \alpha^2 \\
& \quad + 3 \psi'''(\frac{1}{3}) \zeta_3 \alpha + 6 \psi'(\frac{1}{3}) \zeta_3 \alpha^2 + 112 \psi'(\frac{1}{3}) \zeta_3 \pi^2 \alpha^2 + 136 \zeta_3 \pi^2 \alpha^3 \\
& \quad \left. + 297 \zeta_3 \alpha^3 + 320 \zeta_5 \alpha^2 + 684 \zeta_3^2 + 960 \zeta_5 \right] a^3 + O(a^4) \tag{5.6}
\end{aligned}$$

for $SU(3)$ Yang-Mills. For the MAG a similar picture emerges since

$$\begin{aligned}
K_{a \text{MAG}}^{\text{MOMg}}(a, \alpha) \Big|_{N_f=0}^{SU(3)} &= [16 \zeta_3 - 14] a \\
& + \left[\frac{6448}{9} \zeta_3 - \frac{14158}{27} - \frac{1280}{3} \zeta_5 - \frac{704}{27} \zeta_3 \pi^2 \alpha - \frac{512}{27} \zeta_3 \pi^2 - \frac{308}{9} \psi'(\frac{1}{3}) \alpha \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{224}{9}\psi'(\tfrac{1}{3}) - \frac{56}{27}\pi^2\alpha^2 - \frac{32}{9}\psi'(\tfrac{1}{3})\zeta_3\alpha^2 + \frac{28}{9}\psi'(\tfrac{1}{3})\alpha^2 + \frac{64}{27}\zeta_3\pi^2\alpha^2 \\
& + \frac{256}{9}\psi'(\tfrac{1}{3})\zeta_3 + \frac{352}{9}\psi'(\tfrac{1}{3})\zeta_3\alpha + \frac{448}{27}\pi^2 + \frac{616}{27}\pi^2\alpha - 48\zeta_3\alpha - 14\alpha^2 \\
& + 16\zeta_3\alpha^2 + 42\alpha] a^2 \\
& + \left[\frac{25291531}{540}\zeta_3 - \frac{640705981}{12960} - \frac{9335291}{3240}\psi'(\tfrac{1}{3})\alpha - \frac{5544823}{4860}\pi^2 \right. \\
& - \frac{1402261}{7290}\pi^4\alpha - \frac{448672}{1215}\psi'(\tfrac{1}{3})\zeta_3\pi^2\alpha - \frac{224123}{135}\psi'(\tfrac{1}{3})\zeta_3 \\
& - \frac{169595}{972}\pi^2\alpha^2 - \frac{151669}{30}\zeta_3\alpha - \frac{112718}{45}\zeta_3\pi^2\alpha - \frac{98147}{405}\psi'(\tfrac{1}{3})^2\alpha \\
& - \frac{22883}{36}\alpha^2 - \frac{14080}{9}\psi'(\tfrac{1}{3})\zeta_5\alpha - \frac{11896}{45}\zeta_3\pi^4 - \frac{10240}{9}\psi'(\tfrac{1}{3})\zeta_5 \\
& - \frac{9799}{27}\psi'(\tfrac{1}{3})\zeta_3\alpha^2 - \frac{8155}{243}\pi^4\alpha^2 - \frac{6944}{81}\psi'(\tfrac{1}{3})\zeta_3\pi^2\alpha^2 - \frac{5135}{216}\psi'(\tfrac{1}{3})\alpha^3 \\
& - \frac{2560}{27}\zeta_5\pi^2\alpha^2 - \frac{1940}{729}\zeta_3\pi^4\alpha^3 - \frac{1810}{81}\zeta_3\pi^2\alpha^3 - \frac{1792}{243}\psi'(\tfrac{1}{3})\pi^2\alpha^3 \\
& - \frac{1519}{27}\psi'(\tfrac{1}{3})^2\alpha^2 - \frac{987}{32}\alpha^4 - \frac{944}{15}\psi'(\tfrac{1}{3})\zeta_3\pi^2 - \frac{749}{8}\psi'''(\tfrac{1}{3}) \\
& - \frac{653}{18}\psi'''(\tfrac{1}{3})\zeta_3\alpha - \frac{512}{81}\psi'(\tfrac{1}{3})^2\zeta_3\alpha^3 - \frac{451}{48}\alpha^3 - \frac{413}{10}\psi'(\tfrac{1}{3})^2 \\
& - \frac{112}{81}\psi'(\tfrac{1}{3})\zeta_3\pi^2\alpha^4 - \frac{98}{243}\pi^4\alpha^4 - \frac{49}{54}\psi'(\tfrac{1}{3})^2\alpha^4 - \frac{11}{3}\psi'''(\tfrac{1}{3})\zeta_3\alpha^2 \\
& - \frac{1}{18}\psi'''(\tfrac{1}{3})\zeta_3\alpha^3 + \frac{7}{144}\psi'''(\tfrac{1}{3})\alpha^3 + \frac{17}{12}\zeta_3\alpha^3 + \frac{21}{2}\psi'(\tfrac{1}{3})\alpha^4 \\
& + \frac{28}{27}\psi'(\tfrac{1}{3})^2\zeta_3\alpha^4 + \frac{77}{24}\psi'''(\tfrac{1}{3})\alpha^2 + \frac{98}{81}\psi'(\tfrac{1}{3})\pi^2\alpha^4 + \frac{112}{243}\zeta_3\pi^4\alpha^4 \\
& + \frac{183}{4}\zeta_3\alpha^4 + \frac{236}{5}\psi'(\tfrac{1}{3})^2\zeta_3 + \frac{448}{81}\psi'(\tfrac{1}{3})^2\alpha^3 + \frac{826}{15}\psi'(\tfrac{1}{3})\pi^2 \\
& + \frac{905}{27}\psi'(\tfrac{1}{3})\zeta_3\alpha^3 + \frac{1280}{9}\psi'(\tfrac{1}{3})\zeta_5\alpha^2 + \frac{1736}{27}\psi'(\tfrac{1}{3})^2\zeta_3\alpha^2 \\
& + \frac{2048}{243}\psi'(\tfrac{1}{3})\zeta_3\pi^2\alpha^3 + \frac{3395}{1458}\pi^4\alpha^3 + \frac{4571}{144}\psi'''(\tfrac{1}{3})\alpha + \frac{5135}{324}\pi^2\alpha^3 \\
& + \frac{5311}{12}\zeta_3\alpha^2 + \frac{6076}{81}\psi'(\tfrac{1}{3})\pi^2\alpha^2 + \frac{9320}{243}\zeta_3\pi^4\alpha^2 + \frac{10409}{45}\pi^4 \\
& + \frac{19598}{81}\zeta_3\pi^2\alpha^2 + \frac{20480}{27}\zeta_5\pi^2 + \frac{22400}{3}\zeta_7 + \frac{28160}{27}\zeta_5\pi^2\alpha \\
& + \frac{56359}{15}\psi'(\tfrac{1}{3})\zeta_3\alpha + \frac{112168}{405}\psi'(\tfrac{1}{3})^2\zeta_3\alpha + \frac{169595}{648}\psi'(\tfrac{1}{3})\alpha^2 \\
& + \frac{392588}{1215}\psi'(\tfrac{1}{3})\pi^2\alpha + \frac{448246}{405}\zeta_3\pi^2 + \frac{485200}{27}\zeta_5 + \frac{695693}{240}\alpha \\
& + \frac{801292}{3645}\zeta_3\pi^4\alpha + \frac{5544823}{3240}\psi'(\tfrac{1}{3}) + \frac{9335291}{4860}\pi^2\alpha - 24900\zeta_3^2 \\
& - 640\zeta_5\alpha^2 - 22\zeta_3^2\alpha^3 - 12\zeta_3^2\alpha^4 - 12\psi'(\tfrac{1}{3})\zeta_3\alpha^4 - 7\pi^2\alpha^4 + 8\zeta_3\pi^2\alpha^4 \\
& \left. + 107\psi'''(\tfrac{1}{3})\zeta_3 + 522\zeta_3^2\alpha^2 + 1576\zeta_3^2\alpha + 1920\zeta_5\alpha \right] a^3 \tag{5.7}
\end{aligned}$$

and

$$\begin{aligned}
K_{\alpha}^{\text{MOMg}}(a, \alpha) \Big|_{N_f=0}^{SU(3)} &= \left[21 - \frac{352}{27}\zeta_3\pi^2 - \frac{154}{9}\psi'(\tfrac{1}{3}) - \frac{56}{27}\pi^2\alpha - \frac{32}{9}\psi'(\tfrac{1}{3})\zeta_3\alpha + \frac{28}{9}\psi'(\tfrac{1}{3})\alpha \right. \\
& \left. + \frac{64}{27}\zeta_3\pi^2\alpha + \frac{176}{9}\psi'(\tfrac{1}{3})\zeta_3 + \frac{308}{27}\pi^2 - 24\zeta_3 - 14\alpha + 16\zeta_3\alpha \right] a^2
\end{aligned}$$

$$\begin{aligned}
& + \left[640\zeta_5 - \frac{16773443}{9720}\psi'(\tfrac{1}{3}) - \frac{1763246}{1215}\zeta_3\pi^2 - \frac{775271}{7290}\pi^4 \right. \\
& - \frac{325472}{1215}\psi'(\tfrac{1}{3})\zeta_3\pi^2 - \frac{289513}{972}\psi'(\tfrac{1}{3})\alpha - \frac{71197}{405}\psi'(\tfrac{1}{3})^2 - \frac{64139}{30}\zeta_3 \\
& - \frac{41876}{243}\zeta_3\pi^2\alpha - \frac{14080}{27}\psi'(\tfrac{1}{3})\zeta_5 - \frac{12152}{243}\psi'(\tfrac{1}{3})\pi^2\alpha - \frac{9136}{729}\zeta_3\pi^4\alpha \\
& - \frac{5120}{81}\zeta_5\pi^2\alpha - \frac{3472}{81}\psi'(\tfrac{1}{3})^2\zeta_3\alpha - \frac{2140}{243}\zeta_3\pi^4\alpha^2 - \frac{1904}{81}\psi'(\tfrac{1}{3})\pi^2\alpha^2 \\
& - \frac{1280}{3}\zeta_5\alpha - \frac{665}{8}\psi'(\tfrac{1}{3})\alpha^2 - \frac{653}{54}\psi'''(\tfrac{1}{3})\zeta_3 - \frac{544}{27}\psi'(\tfrac{1}{3})^2\zeta_3\alpha^2 \\
& - \frac{448}{243}\psi'(\tfrac{1}{3})\zeta_3\pi^2\alpha^3 - \frac{392}{729}\pi^4\alpha^3 - \frac{329}{8}\alpha^3 - \frac{190}{3}\zeta_3\pi^2\alpha^2 - \frac{121}{4}\zeta_3\alpha^2 \\
& - \frac{98}{81}\psi'(\tfrac{1}{3})^2\alpha^3 - \frac{28}{3}\pi^2\alpha^3 - \frac{22}{9}\psi'''(\tfrac{1}{3})\zeta_3\alpha - \frac{1}{18}\psi'''(\tfrac{1}{3})\zeta_3\alpha^2 \\
& + \frac{7}{144}\psi'''(\tfrac{1}{3})\alpha^2 + \frac{32}{3}\zeta_3\pi^2\alpha^3 + \frac{77}{36}\psi'''(\tfrac{1}{3})\alpha + \frac{112}{81}\psi'(\tfrac{1}{3})^2\zeta_3\alpha^3 \\
& + \frac{392}{243}\psi'(\tfrac{1}{3})\pi^2\alpha^3 + \frac{448}{729}\zeta_3\pi^4\alpha^3 + \frac{476}{27}\psi'(\tfrac{1}{3})^2\alpha^2 + \frac{665}{12}\pi^2\alpha^2 \\
& + \frac{693}{16}\alpha^2 + \frac{707}{18}\zeta_3\alpha + \frac{1576}{3}\zeta_3^2 + \frac{2176}{81}\psi'(\tfrac{1}{3})\zeta_3\pi^2\alpha^2 + \frac{2345}{108}\alpha \\
& + \frac{2560}{27}\psi'(\tfrac{1}{3})\zeta_5\alpha + \frac{3038}{81}\psi'(\tfrac{1}{3})^2\alpha + \frac{3745}{486}\pi^4\alpha^2 + \frac{4571}{432}\psi'''(\tfrac{1}{3}) \\
& + \frac{7994}{729}\pi^4\alpha + \frac{13888}{243}\psi'(\tfrac{1}{3})\zeta_3\pi^2\alpha + \frac{20938}{81}\psi'(\tfrac{1}{3})\zeta_3\alpha + \frac{28160}{81}\zeta_5\pi^2 \\
& - 16\psi'(\tfrac{1}{3})\zeta_3\alpha^3 + \frac{81368}{405}\psi'(\tfrac{1}{3})^2\zeta_3 + \frac{284788}{1215}\psi'(\tfrac{1}{3})\pi^2 + \frac{289513}{1458}\pi^2\alpha \\
& + \frac{443012}{3645}\zeta_3\pi^4 + \frac{881623}{405}\psi'(\tfrac{1}{3})\zeta_3 + \frac{1405969}{720} + \frac{16773443}{14580}\pi^2 \\
& - 22\zeta_3^2\alpha^2 - 16\zeta_3^2\alpha^3 + 14\psi'(\tfrac{1}{3})\alpha^3 + 61\zeta_3\alpha^3 + 95\psi'(\tfrac{1}{3})\zeta_3\alpha^2 \\
& \left. + 348\zeta_3^2\alpha \right] a^3 + O(a^4) \tag{5.8}
\end{aligned}$$

for the same theory. What is evident in both cases is that the leading term of the MOMg K_a in both gauges, as well as the MOMc and MOMq schemes, is the same as that of the linear gauge in all schemes. In other words the remaining terms of K_a and all in K_α carry both the scheme and gauge dependence. Moreover by construction the Crewther relation vanishes at the corresponding fixed points of the Curci-Ferrari and MAG renormalization group functions. In these two nonlinear gauges $\gamma_\alpha(a, \alpha)$ is linearly independent of the gluon anomalous dimension $\gamma_A(a, \alpha)$ unlike the linear covariant gauge. In other words the critical points of both gauges are different.

What is not apparent from the $SU(3)$ expressions is the connection between the Curci-Ferrari gauge and MAG which requires the results for an arbitrary colour group. To assist with seeing how the $N_A^d \rightarrow 0$ limit takes effect we record the situation for the $O(a^2)$ K -functions in detail as the next order expressions are large but available in the associated data file. In the Curci-Ferrari gauge we have

$$\begin{aligned}
K_{a\text{CF}}^{\text{MOMg}}(a, \alpha) & = \left[12\zeta_3 C_F - \frac{21}{2} C_F \right] a \\
& + \left[136\zeta_3 C_F^2 - \frac{512}{27}\zeta_3\pi^2 N_f T_F C_F - \frac{321}{2} C_F C_A - \frac{224}{9}\psi'(\tfrac{1}{3}) N_f T_F C_F \right. \\
& - \frac{161}{27}\pi^2 C_F C_A - \frac{112}{3}\zeta_3 N_f T_F C_F - \frac{92}{9}\psi'(\tfrac{1}{3})\zeta_3 C_F C_A - \frac{21}{2} C_F C_A \alpha^2 \\
& \left. - \frac{21}{2}\psi'(\tfrac{1}{3}) C_F C_A \alpha - \frac{14}{9}\pi^2 C_F C_A \alpha^2 - \frac{8}{3}\psi'(\tfrac{1}{3})\zeta_3 C_F C_A \alpha^2 \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{7}{3}\psi'(\frac{1}{3})C_F C_A \alpha^2 + \frac{7}{4}C_F C_A \alpha^3 + \frac{16}{9}\zeta_3 \pi^2 C_F C_A \alpha^2 + \frac{63}{4}C_F C_A \alpha \\
& + \frac{158}{3}N_f T_F C_F + \frac{161}{18}\psi'(\frac{1}{3})C_F C_A + \frac{184}{27}\zeta_3 \pi^2 C_F C_A \\
& + \frac{256}{9}\psi'(\frac{1}{3})\zeta_3 N_f T_F C_F + \frac{356}{3}\zeta_3 C_F C_A + \frac{397}{6}C_F^2 + \frac{448}{27}\pi^2 N_f T_F C_F \\
& - 240\zeta_5 C_F^2 - 18\zeta_3 C_F C_A \alpha - 8\zeta_3 \pi^2 C_F C_A \alpha - 2\zeta_3 C_F C_A \alpha^3 \\
& + 7\pi^2 C_F C_A \alpha + 12\zeta_3 C_F C_A \alpha^2 + 12\psi'(\frac{1}{3})\zeta_3 C_F C_A \alpha \Big] a^2 + O(a^3) \quad (5.9)
\end{aligned}$$

and

$$\begin{aligned}
K_{\alpha \text{CF}}^{\text{MOMg}}(a, \alpha) &= \left[6\psi'(\frac{1}{3})\zeta_3 C_F C_A + 12\zeta_3 C_F C_A \alpha - \frac{21}{2}C_F C_A \alpha - \frac{21}{4}\psi'(\frac{1}{3})C_F C_A \right. \\
& - \frac{14}{9}\pi^2 C_F C_A \alpha - \frac{8}{3}\psi'(\frac{1}{3})\zeta_3 C_F C_A \alpha + \frac{7}{2}\pi^2 C_F C_A + \frac{7}{3}\psi'(\frac{1}{3})C_F C_A \alpha \\
& + \frac{16}{9}\zeta_3 \pi^2 C_F C_A \alpha + \frac{21}{8}C_F C_A \alpha^2 + \frac{63}{8}C_F C_A - 9\zeta_3 C_F C_A - 4\zeta_3 \pi^2 C_F C_A \\
& \left. - 3\zeta_3 C_F C_A \alpha^2 \right] a^2 + O(a^3) \quad (5.10)
\end{aligned}$$

while the MAG partners are

$$\begin{aligned}
K_{a \text{MAG}}^{\text{MOMg}}(a, \alpha) &= \left[12\zeta_3 C_F - \frac{21}{2}C_F \right] a \\
& + \left[\frac{7}{3}\psi'(\frac{1}{3})C_F C_A \alpha^2 - \frac{512}{27}\zeta_3 \pi^2 N_f T_F C_F - \frac{321}{2}C_F C_A - \frac{224}{9}\psi'(\frac{1}{3})N_f T_F C_F \right. \\
& - \frac{161}{27}\pi^2 C_F C_A - \frac{112}{3}\zeta_3 N_f T_F C_F - \frac{104}{3}\zeta_3 \pi^2 \frac{N_A^d}{N_A^o} C_F C_A \\
& - \frac{92}{9}\psi'(\frac{1}{3})\zeta_3 C_F C_A - \frac{91}{2}\psi'(\frac{1}{3})\frac{N_A^d}{N_A^o} C_F C_A - \frac{63}{4}\frac{N_A^d}{N_A^o} C_F C_A \alpha \\
& - \frac{35}{9}\pi^2 \frac{N_A^d}{N_A^o} C_F C_A \alpha - \frac{32}{9}\zeta_3 \pi^2 \frac{N_A^d}{N_A^o} C_F C_A \alpha^2 - \frac{21}{2}C_F C_A \alpha^2 \\
& - \frac{21}{2}\psi'(\frac{1}{3})C_F C_A \alpha - \frac{21}{4}\frac{N_A^d}{N_A^o} C_F C_A \alpha^3 - \frac{20}{3}\psi'(\frac{1}{3})\zeta_3 \frac{N_A^d}{N_A^o} C_F C_A \alpha \\
& - \frac{14}{3}\psi'(\frac{1}{3})\frac{N_A^d}{N_A^o} C_F C_A \alpha^2 - \frac{14}{9}\pi^2 C_F C_A \alpha^2 - \frac{8}{3}\psi'(\frac{1}{3})\zeta_3 C_F C_A \alpha^2 \\
& + \frac{7}{4}C_F C_A \alpha^3 + \frac{16}{3}\psi'(\frac{1}{3})\zeta_3 \frac{N_A^d}{N_A^o} C_F C_A \alpha^2 + \frac{16}{9}\zeta_3 \pi^2 C_F C_A \alpha^2 \\
& + \frac{28}{9}\pi^2 \frac{N_A^d}{N_A^o} C_F C_A \alpha^2 + \frac{35}{6}\psi'(\frac{1}{3})\frac{N_A^d}{N_A^o} C_F C_A \alpha + \frac{40}{9}\zeta_3 \pi^2 \frac{N_A^d}{N_A^o} C_F C_A \alpha \\
& + \frac{63}{4}C_F C_A \alpha + \frac{91}{3}\pi^2 \frac{N_A^d}{N_A^o} C_F C_A + \frac{158}{3}N_f T_F C_F + \frac{161}{18}\psi'(\frac{1}{3})C_F C_A \\
& + \frac{184}{27}\zeta_3 \pi^2 C_F C_A + \frac{256}{9}\psi'(\frac{1}{3})\zeta_3 N_f T_F C_F + \frac{356}{3}\zeta_3 C_F C_A + \frac{397}{6}C_F^2 \\
& + \frac{448}{27}\pi^2 N_f T_F C_F - 240\zeta_5 C_F^2 - 24\zeta_3 \frac{N_A^d}{N_A^o} C_F C_A \alpha^2 - 18\zeta_3 C_F C_A \alpha \\
& - 8\zeta_3 \pi^2 C_F C_A \alpha - 2\zeta_3 C_F C_A \alpha^3 + 6\zeta_3 \frac{N_A^d}{N_A^o} C_F C_A \alpha^3 + 7\pi^2 C_F C_A \alpha \\
& \left. + 12\zeta_3 C_F C_A \alpha^2 + 12\psi'(\frac{1}{3})\zeta_3 C_F C_A \alpha + 18\zeta_3 \frac{N_A^d}{N_A^o} C_F C_A \alpha \right]
\end{aligned}$$

$$+ 21 \frac{N_A^d}{N_A^o} C_F C_A \alpha^2 + 52 \psi'(\frac{1}{3}) \zeta_3 \frac{N_A^d}{N_A^o} C_F C_A + 136 \zeta_3 C_F^2 \Big] a^2 + O(a^3) \quad (5.11)$$

and

$$\begin{aligned} K_{\alpha}^{\text{MOMg}}(a, \alpha) = & \left[\frac{7}{2} \pi^2 C_F C_A - \frac{63}{8} \frac{N_A^d}{N_A^o} C_F C_A - \frac{63}{8} \frac{N_A^d}{N_A^o} C_F C_A \alpha^2 - \frac{35}{18} \pi^2 \frac{N_A^d}{N_A^o} C_F C_A \right. \\ & - \frac{32}{9} \zeta_3 \pi^2 \frac{N_A^d}{N_A^o} C_F C_A \alpha - \frac{21}{2} C_F C_A \alpha - \frac{21}{4} \psi'(\frac{1}{3}) C_F C_A \\ & - \frac{14}{3} \psi'(\frac{1}{3}) \frac{N_A^d}{N_A^o} C_F C_A \alpha - \frac{14}{9} \pi^2 C_F C_A \alpha - \frac{10}{3} \psi'(\frac{1}{3}) \zeta_3 \frac{N_A^d}{N_A^o} C_F C_A \\ & - \frac{8}{3} \psi'(\frac{1}{3}) \zeta_3 C_F C_A \alpha + \frac{7}{3} \psi'(\frac{1}{3}) C_F C_A \alpha + \frac{16}{3} \psi'(\frac{1}{3}) \zeta_3 \frac{N_A^d}{N_A^o} C_F C_A \alpha \\ & + \frac{16}{9} \zeta_3 \pi^2 C_F C_A \alpha + \frac{20}{9} \zeta_3 \pi^2 \frac{N_A^d}{N_A^o} C_F C_A + \frac{21}{8} C_F C_A \alpha^2 \\ & + \frac{28}{9} \pi^2 \frac{N_A^d}{N_A^o} C_F C_A \alpha + \frac{35}{12} \psi'(\frac{1}{3}) \frac{N_A^d}{N_A^o} C_F C_A + \frac{63}{8} C_F C_A \\ & - 24 \zeta_3 \frac{N_A^d}{N_A^o} C_F C_A \alpha - 9 \zeta_3 C_F C_A - 4 \zeta_3 \pi^2 C_F C_A - 3 \zeta_3 C_F C_A \alpha^2 \\ & + 6 \psi'(\frac{1}{3}) \zeta_3 C_F C_A + 9 \zeta_3 \frac{N_A^d}{N_A^o} C_F C_A + 9 \zeta_3 \frac{N_A^d}{N_A^o} C_F C_A \alpha^2 + 12 \zeta_3 C_F C_A \alpha \\ & \left. + 21 \frac{N_A^d}{N_A^o} C_F C_A \alpha \right] a^2 + O(a^3). \quad (5.12) \end{aligned}$$

Deleting the terms with a N_A^d/N_A^o factor in (5.11) and (5.12) reproduces (5.9) and (5.10) respectively. We have checked that this is also the case for the $O(a^3)$ terms.

6 Other schemes.

One question of interest concerns whether or not there are any schemes other than the $\overline{\text{MS}}$ one where $\Delta_{\text{csb}}(a, \alpha)$ involves only the β -function. It turns out in fact that, aside from the V scheme of [13, 14] whose β -function is independent of the gauge parameter by construction, there is at least one other such scheme which is the RI' scheme, [35, 36]. It was introduced in relation to lattice regularized QCD calculations and has a continuum spacetime analogue which has been used to renormalize the theory to high loop order [59, 60, 61, 62]. The prescription requires that the wave function renormalization constants of the gluon, ghost and quark fields are defined so that there are no $O(a)$ corrections to their 2-point functions at the subtraction point. In this sense it incorporates the approach of the MOM schemes of [37, 38]. The RI' scheme differs however in respect of the coupling constant renormalization. For each vertex function the coupling renormalization constant is defined according to the $\overline{\text{MS}}$ prescription at the subtraction point. So the RI' scheme can be regarded as a halfway house between the $\overline{\text{MS}}$ and the suite of MOM schemes of [37, 38]. In terms of the resulting renormalization group functions the relevant issue for this article is that the RI' β -function is formally equivalent to that of the $\overline{\text{MS}}$ β -function. By this we mean that although the coupling constant will be the variable in the respective schemes the actual coefficients of the coupling constant polynomial of the RI' scheme β -function are identically the same as their $\overline{\text{MS}}$ counterparts. We have checked explicitly that the same reasoning applies to the structure of $K_a^{\text{RI}'}(a)$

whose coefficients are precisely the same as those of $K_a^{\overline{\text{MS}}}(a)$ with in addition now

$$K_\alpha^{\text{RI}'}(a, \alpha) = 0. \quad (6.1)$$

In other words the ansatz of (2.3) holds primarily as a direct result of the RI' renormalization prescription of the vertex function.

Although this means that there is a scheme other than $\overline{\text{MS}}$ that takes the same form as (2.4) it is a relatively trivial observation. However the nature of the RI' scheme gives a more important insight into some of the underlying issues. In essence it indicates that the scheme the fields are renormalized in plays no role in whether a $K_\alpha(a, \alpha)$ function is needed. Instead it is the prescription for the coupling constant renormalization that is key. This is understandable since the $\overline{\text{MS}}$ Crewther construction is α independent. In light of this, one question that is of interest is what occurs if one considers a scheme that is opposite to the prescription of the RI' scheme. By this we mean one where the fields, gauge parameter and quark mass are renormalized in an $\overline{\text{MS}}$ way but the vertex functions, and thereby the coupling constant, are renormalized in a non- $\overline{\text{MS}}$ fashion. It is possible to explore this possibility given the four loop data that is available in [18, 62]. In this new scheme the gluon, ghost and quark 2-point functions as well as the quark 2-point function with the quark mass operator inserted, [62], are renormalized in the linear covariant gauge by including only the poles in ϵ into the respective renormalization constants. To determine the coupling constant renormalization we have chosen to use the ghost-gluon vertex in the configuration where the momentum of one ghost field is zero. This is the setup that is used to define the mMOM renormalization. The reason for choosing this vertex is that in the Landau gauge the vertex is finite according to Taylor's observation, [16]. Therefore in this new scheme the renormalization group functions ought to differ from the $\overline{\text{MS}}$ ones in a minimal way. We have labelled this scheme as $\widetilde{\text{RIc}}'$ reflecting that it is based on the ghost-gluon vertex, being a variation of the RI' scheme, and constructed the renormalization group equations to five loops. Expressions for them in an arbitrary gauge for a general colour group are given in the associated data file. To appreciate their structure we have recorded the $SU(3)$ expressions for the β -function and gauge parameter anomalous dimension in Appendix B as these are relevant to the current discussion. However it is worth noting that our expectation that the $\widetilde{\text{RIc}}'$ scheme has a minimal effect on the β -function structure is met in that the coefficients of the coupling constant in $\beta^{\widetilde{\text{RIc}}'}(a, 0)$ are precisely the same as those of $\beta^{\overline{\text{MS}}}(a)$ to five loops.

Equipped with the $\widetilde{\text{RIc}}'$ renormalization group functions we have determined the relation between the $\overline{\text{MS}}$ and $\widetilde{\text{RIc}}'$ coupling constant and gauge parameter that allows us to examine what $\Delta_{\text{csb}}^{\widetilde{\text{RIc}}'}(a, \alpha)$ is. By following the procedure used for other schemes it transpires that $K_\alpha^{\widetilde{\text{RIc}}'}(a, \alpha)$ has to be non-zero. More specifically we found

$$\begin{aligned} K_a^{\widetilde{\text{RIc}}'}(a, \alpha) = & \left[12\zeta_3 C_F - \frac{21}{2} C_F \right] a \\ & + \left[\frac{326}{3} N_f T_F C_F + \frac{397}{6} C_F^2 + \frac{884}{3} \zeta_3 C_F C_A - \frac{629}{2} C_F C_A - \frac{304}{3} \zeta_3 N_f T_F C_F \right. \\ & \quad \left. - 240\zeta_5 C_F^2 - 24\zeta_3 C_F C_A \alpha + 21 C_F C_A \alpha + 136\zeta_3 C_F^2 \right] a^2 \\ & + \left[\frac{81}{2} \zeta_3^2 C_F C_A^2 \alpha + \frac{85}{16} \zeta_3 C_F C_A^2 \alpha^2 + \frac{211}{16} C_F C_A^2 \alpha^2 - \frac{406043}{36} C_F C_A^2 \right. \\ & \quad - \frac{45517}{48} \zeta_3 C_F C_A^2 \alpha - \frac{40336}{9} \zeta_3 N_f T_F C_F C_A - \frac{24880}{3} \zeta_5 C_F^2 C_A \\ & \quad - \frac{9824}{9} N_f^2 T_F^2 C_F - \frac{8000}{3} \zeta_5 N_f T_F C_F C_A - \frac{7729}{18} N_f T_F C_F^2 - \frac{652}{3} N_f T_F C_F C_A \alpha \\ & \quad \left. - \frac{397}{2} C_F^2 C_A \alpha - \frac{27}{2} \zeta_3^2 C_F C_A^2 \alpha^2 + \frac{608}{3} \zeta_3 N_f T_F C_F C_A \alpha + \frac{2471}{12} C_F^3 \right] \end{aligned}$$

$$\begin{aligned}
& + \frac{6496}{9} \zeta_3 N_f^2 T_F^2 C_F + \frac{11900}{3} \zeta_5 C_F C_A^2 + \frac{16570}{3} \zeta_3 C_F^2 C_A + \frac{44939}{48} C_F C_A^2 \alpha \\
& + \frac{67520}{9} N_f T_F C_F C_A + \frac{72028}{9} \zeta_3 C_F C_A^2 + \frac{99757}{36} C_F^2 C_A - 5720 \zeta_5 C_F^3 \\
& - 3668 \zeta_3 N_f T_F C_F^2 - 1232 \zeta_3^2 C_F C_A^2 - 840 \zeta_7 C_F^2 C_A - 408 \zeta_3 C_F^2 C_A \alpha \\
& - 128 \zeta_3^2 N_f T_F C_F C_A + 320 \zeta_5 N_f^2 T_F^2 C_F + 488 \zeta_3 C_F^3 + 576 \zeta_3^2 N_f T_F C_F^2 \\
& + 720 \zeta_5 C_F^2 C_A \alpha + 4000 \zeta_5 N_f T_F C_F^2 + 5040 \zeta_7 C_F^3 \Big] a^3 + O(a^4) \tag{6.2}
\end{aligned}$$

and

$$\begin{aligned}
K_\alpha^{\widetilde{\text{RIc}'}}(a, \alpha) &= \left[\frac{21}{2} C_F - 12 \zeta_3 C_F \right] C_A a^2 \\
&+ \left[\frac{27}{2} \zeta_3^2 C_F C_A + \frac{231}{8} \zeta_3 C_F C_A \alpha + \frac{3461}{48} C_F C_A - \frac{1477}{16} \zeta_3 C_F C_A - \frac{397}{6} C_F^2 \right. \\
&\quad \left. - \frac{147}{8} C_F C_A \alpha - 136 \zeta_3 C_F^2 - 9 \zeta_3^2 C_F C_A \alpha + 240 \zeta_5 C_F^2 \right] C_A a^3 + O(a^4) . \tag{6.3}
\end{aligned}$$

Although this exercise did not produce another scheme whose Crewther structure was the same as those of $\overline{\text{MS}}$ and RI' $K_\alpha^{\widetilde{\text{RIc}'}}(a, \alpha)$ does have a different structure. For instance unlike the other schemes considered earlier the leading term is α independent and the linear terms in α at the next order are proportional to the leading term. Also like $K_\alpha^{\text{mMOM}}(a, \alpha)$ we note that $K_\alpha^{\widetilde{\text{RIc}'}}(a, \alpha)$ is proportional to C_A . These properties can be traced directly back to those of the vertex function that this scheme was based on.

7 Renormalization group connection.

Having accumulated evidence to support the ansatz (4.1) a natural question is whether it can be derived directly and in general from some renormalization group principle. This is important since our analysis so far indicating an extension to (2.4) is needed has been concerned with values of N_f in the conformal window. Any generalization should not be restricted to this range. It turns out the key lies in the way one maps the parameters of a theory in one scheme to the corresponding parameters in another scheme as well as the relation between the renormalization group functions. For instance if for the moment we regard a as the $\overline{\text{MS}}$ coupling constant and a_S and α_S as the respective coupling constant and gauge parameter in another scheme \mathcal{S} then we found the Adler D -function and the Bjorken sum rule using the relations

$$C_{\text{Bjr}}^{\overline{\text{MS}}}(a) = C_{\text{Bjr}}^{\mathcal{S}}(a_S, \alpha_S) \quad , \quad C_{\text{Adl}}^{\overline{\text{MS}}}(a) = C_{\text{Adl}}^{\mathcal{S}}(a_S, \alpha_S) \tag{7.1}$$

where

$$a \equiv a(a_S, \alpha_S) \tag{7.2}$$

expresses the $\overline{\text{MS}}$ coupling in terms of the coupling and gauge parameter in scheme \mathcal{S} and implies

$$\Delta_{\text{csb}}^{\overline{\text{MS}}}(a) = \Delta_{\text{csb}}^{\mathcal{S}}(a_S, \alpha_S) . \tag{7.3}$$

Using properties of the renormalization group equation the $\overline{\text{MS}}$ β -function is related to that in another scheme by

$$\beta^{\overline{\text{MS}}}(a) = \beta^{\mathcal{S}}(a_S, \alpha_S) \frac{\partial}{\partial a_S} a(a_S, \alpha_S) + \alpha_S \gamma_\alpha^{\mathcal{S}}(a_S, \alpha_S) \frac{\partial}{\partial \alpha_S} a(a_S, \alpha_S) \tag{7.4}$$

which recalling (2.4) implies $\Delta_{\text{csb}}^{\mathcal{S}}(a_{\mathcal{S}}, \alpha_{\mathcal{S}})$ can be written as

$$\Delta_{\text{csb}}^{\mathcal{S}}(a_{\mathcal{S}}, \alpha_{\mathcal{S}}) = \frac{\beta^{\mathcal{S}}(a_{\mathcal{S}}, \alpha_{\mathcal{S}})}{a_{\mathcal{S}}} \bar{K}_a^{\mathcal{S}}(a_{\mathcal{S}}, \alpha_{\mathcal{S}}) + \alpha_{\mathcal{S}} \gamma_{\alpha}^{\mathcal{S}}(a_{\mathcal{S}}, \alpha_{\mathcal{S}}) \bar{K}_{\alpha}^{\mathcal{S}}(a_{\mathcal{S}}, \alpha_{\mathcal{S}}) \quad (7.5)$$

where

$$\begin{aligned} \bar{K}_a^{\mathcal{S}}(a_{\mathcal{S}}, \alpha_{\mathcal{S}}) &= a_{\mathcal{S}} \left[\frac{1}{a} K_a^{\overline{\text{MS}}}(a) \right] \Big|_{\overline{\text{MS}} \rightarrow \mathcal{S}} \frac{\partial}{\partial a_{\mathcal{S}}} a(a_{\mathcal{S}}, \alpha_{\mathcal{S}}) \\ \bar{K}_{\alpha}^{\mathcal{S}}(a_{\mathcal{S}}, \alpha_{\mathcal{S}}) &= \left[\frac{1}{a} K_a^{\overline{\text{MS}}}(a) \right] \Big|_{\overline{\text{MS}} \rightarrow \mathcal{S}} \frac{\partial}{\partial \alpha_{\mathcal{S}}} a(a_{\mathcal{S}}, \alpha_{\mathcal{S}}) . \end{aligned} \quad (7.6)$$

The restriction on each appearance of $K_a^{\overline{\text{MS}}}(a)$ indicates that the $\overline{\text{MS}}$ coupling needs to be mapped to its \mathcal{S} scheme counterpart by inverting (7.2) so that the variables are in the \mathcal{S} scheme. By setting \mathcal{S} in (4.1) to be the $\overline{\text{MS}}$ scheme and comparing with (2.4) clearly implies

$$\bar{K}_{\alpha}^{\overline{\text{MS}}}(a, \alpha) = 0 . \quad (7.7)$$

However (7.6) demonstrates the consistency of this observation trivially. For instance if in the second relation of (7.6) we choose \mathcal{S} to be the $\overline{\text{MS}}$ scheme itself then $\frac{\partial a}{\partial \alpha}$ vanishes as a and α are independent in the $\overline{\text{MS}}$ scheme.

While the use of (7.4) formally justifies our ansatz of (4.1) it provides another way of constructing explicit expressions for the K -functions. However it turns out that the perturbative construction of the K -functions are not in agreement with those derived from (7.6). In other words for each of the schemes and gauges considered here we define the quantities

$$\Theta_a^{\mathcal{S}}(a, \alpha) = K_a^{\mathcal{S}}(a, \alpha) - \bar{K}_a^{\mathcal{S}}(a, \alpha) , \quad \Theta_{\alpha}^{\mathcal{S}}(a, \alpha) = K_{\alpha}^{\mathcal{S}}(a, \alpha) - \bar{K}_{\alpha}^{\mathcal{S}}(a, \alpha) \quad (7.8)$$

to illustrate the lack of uniqueness and note they are non-zero. This non-uniqueness of the perturbative expansion of the K -functions does not invalidate the observation that $\Delta_{\text{csb}}^{\mathcal{S}}(a_{\mathcal{S}}, \alpha_{\mathcal{S}})$ vanishes at critical points of the renormalization group equation. This still follows due to the coefficients of the K -functions of (7.5) being the quantities defining the fixed point locations. By way of example we record a few expressions for both $\Theta_i^{\mathcal{S}}(a, \alpha)$ in various schemes. First in the linear covariant gauge we have

$$\begin{aligned} \Theta_a^{\text{mMOM}}(a, \alpha) &= \left[\frac{434}{3} \alpha N_f T_F C_F C_A - \frac{2821}{12} \alpha C_F C_A^2 + 182 \alpha \zeta_3 C_F C_A^2 - 112 \alpha \zeta_3 N_f T_F C_F C_A \right. \\ &\quad + \frac{31}{8} \alpha^2 C_F C_A^2 + 31 \alpha^2 N_f T_F C_F C_A - 3 \alpha^2 \zeta_3 C_F C_A^2 - 24 \alpha^2 \zeta_3 N_f T_F C_F C_A \\ &\quad \left. + \frac{93}{8} \alpha^3 C_F C_A^2 - 9 \alpha^3 \zeta_3 C_F C_A^2 \right] a^3 + O(a^4) \\ \Theta_{\alpha}^{\text{mMOM}}(a, \alpha) &= \left[\frac{434}{3} N_f T_F C_F C_A - \frac{2387}{6} C_F C_A^2 + 308 \zeta_3 C_F C_A^2 - 112 \zeta_3 N_f T_F C_F C_A \right. \\ &\quad \left. - \frac{341}{4} \alpha C_F C_A^2 + 31 \alpha N_f T_F C_F C_A + 66 \alpha \zeta_3 C_F C_A^2 - 24 \alpha \zeta_3 N_f T_F C_F C_A \right] a^3 \\ &\quad + O(a^4) \end{aligned} \quad (7.9)$$

for the mMOM scheme. By contrast the partially related $\widetilde{\text{RIc'}}$ scheme expressions are simpler since

$$\begin{aligned} \Theta_a^{\widetilde{\text{RIc'}}}(a, \alpha) &= \left[\frac{326}{3} \alpha N_f T_F C_F C_A - \frac{2119}{12} \alpha C_F C_A^2 + \frac{494}{3} \alpha \zeta_3 C_F C_A^2 - \frac{304}{3} \alpha \zeta_3 N_f T_F C_F C_A \right. \\ &\quad \left. + \frac{163}{4} \alpha^2 C_F C_A^2 - 38 \alpha^2 \zeta_3 C_F C_A^2 \right] a^3 + O(a^4) \\ \Theta_{\alpha}^{\widetilde{\text{RIc'}}}(a, \alpha) &= \left[\frac{326}{3} N_f T_F C_F C_A - \frac{1793}{6} C_F C_A^2 + \frac{836}{3} \zeta_3 C_F C_A^2 - \frac{304}{3} \zeta_3 N_f T_F C_F C_A \right] a^3 \\ &\quad + O(a^4) . \end{aligned} \quad (7.10)$$

For balance we record the MOMg scheme $SU(3)$ Yang-Mills differences are

$$\begin{aligned}
\Theta_a^{\text{MOMg}}(a, \alpha) \Big|_{N_f=0}^{SU(3)} &= \left[\frac{98}{3} \pi^2 \alpha^4 + \frac{411}{8} \alpha^4 - \frac{238303}{108} \psi'(\tfrac{1}{3}) \alpha - \frac{128596}{81} \zeta_3 \pi^2 \alpha - \frac{99008}{729} \zeta_3 \pi^4 \alpha \right. \\
&\quad - \frac{86632}{243} \psi'(\tfrac{1}{3}) \pi^2 \alpha - \frac{24752}{81} \psi'(\tfrac{1}{3})^2 \zeta_3 \alpha - \frac{15680}{243} \zeta_3 \pi^4 \alpha^2 - \frac{15001}{36} \psi'(\tfrac{1}{3}) \alpha^2 \\
&\quad - \frac{13720}{81} \psi'(\tfrac{1}{3}) \pi^2 \alpha^2 - \frac{7372}{27} \zeta_3 \pi^2 \alpha^2 - \frac{3920}{27} \psi'(\tfrac{1}{3})^2 \zeta_3 \alpha^2 - \frac{3445}{3} \zeta_3 \alpha \\
&\quad - \frac{3299}{8} \alpha^3 - \frac{1792}{27} \psi'(\tfrac{1}{3}) \zeta_3 \pi^2 \alpha^3 - \frac{1568}{81} \pi^4 \alpha^3 - \frac{392}{9} \psi'(\tfrac{1}{3})^2 \alpha^3 \\
&\quad - \frac{112}{3} \zeta_3 \pi^2 \alpha^4 + \frac{448}{9} \psi'(\tfrac{1}{3})^2 \zeta_3 \alpha^3 + \frac{928}{3} \zeta_3 \pi^2 \alpha^3 + \frac{1568}{27} \psi'(\tfrac{1}{3}) \pi^2 \alpha^3 \\
&\quad + \frac{1792}{81} \zeta_3 \pi^4 \alpha^3 + \frac{2623}{8} \alpha^2 + \frac{3430}{27} \psi'(\tfrac{1}{3})^2 \alpha^2 + \frac{3686}{9} \psi'(\tfrac{1}{3}) \zeta_3 \alpha^2 \\
&\quad + \frac{13720}{243} \pi^4 \alpha^2 + \frac{15001}{54} \pi^2 \alpha^2 + \frac{15680}{81} \psi'(\tfrac{1}{3}) \zeta_3 \pi^2 \alpha^2 + \frac{21658}{81} \psi'(\tfrac{1}{3})^2 \alpha \\
&\quad + \frac{51415}{24} \alpha + \frac{64298}{27} \psi'(\tfrac{1}{3}) \zeta_3 \alpha + \frac{86632}{729} \pi^4 \alpha + \frac{99008}{243} \psi'(\tfrac{1}{3}) \zeta_3 \pi^2 \alpha \\
&\quad + \frac{238303}{162} \pi^2 \alpha - 464 \psi'(\tfrac{1}{3}) \zeta_3 \alpha^3 - 284 \pi^2 \alpha^3 - 229 \zeta_3 \alpha^2 - 49 \psi'(\tfrac{1}{3}) \alpha^4 \\
&\quad \left. - 33 \zeta_3 \alpha^4 + 56 \psi'(\tfrac{1}{3}) \zeta_3 \alpha^4 + 257 \zeta_3 \alpha^3 + 426 \psi'(\tfrac{1}{3}) \alpha^3 \right] a^3 + O(a^4) \\
\Theta_\alpha^{\text{MOMg}}(a, \alpha) \Big|_{N_f=0}^{SU(3)} &= \left[\frac{1078}{3} \psi'(\tfrac{1}{3}) \alpha^2 + \frac{2464}{9} \zeta_3 \pi^2 \alpha^2 - \frac{217624}{81} \zeta_3 \pi^2 - \frac{201641}{54} \psi'(\tfrac{1}{3}) \right. \\
&\quad - \frac{167552}{729} \zeta_3 \pi^4 - \frac{146608}{243} \psi'(\tfrac{1}{3}) \pi^2 - \frac{41888}{81} \psi'(\tfrac{1}{3})^2 \zeta_3 - \frac{39424}{243} \zeta_3 \pi^4 \alpha \\
&\quad - \frac{34496}{81} \psi'(\tfrac{1}{3}) \pi^2 \alpha - \frac{29216}{27} \zeta_3 \pi^2 \alpha - \frac{14102}{9} \psi'(\tfrac{1}{3}) \alpha - \frac{9856}{27} \psi'(\tfrac{1}{3})^2 \zeta_3 \alpha \\
&\quad - \frac{5830}{3} \zeta_3 - \frac{2156}{9} \pi^2 \alpha^2 - \frac{1507}{4} \alpha^2 - \frac{1232}{3} \psi'(\tfrac{1}{3}) \zeta_3 \alpha^2 + \frac{2783}{2} \alpha \\
&\quad + \frac{8624}{27} \psi'(\tfrac{1}{3})^2 \alpha + \frac{14608}{9} \psi'(\tfrac{1}{3}) \zeta_3 \alpha + \frac{28204}{27} \pi^2 \alpha + \frac{34496}{243} \pi^4 \alpha \\
&\quad + \frac{36652}{81} \psi'(\tfrac{1}{3})^2 + \frac{39424}{81} \psi'(\tfrac{1}{3}) \zeta_3 \pi^2 \alpha + \frac{43505}{12} + \frac{108812}{27} \psi'(\tfrac{1}{3}) \zeta_3 \\
&\quad \left. + \frac{146608}{729} \pi^4 + \frac{167552}{243} \psi'(\tfrac{1}{3}) \zeta_3 \pi^2 + \frac{201641}{81} \pi^2 - 836 \zeta_3 \alpha + 242 \zeta_3 \alpha^2 \right] a^3 \\
&\quad + O(a^4) \tag{7.11}
\end{aligned}$$

to illustrate the situation for a kinematic scheme. While the above linear covariant gauge expressions for $\Theta_a^S(a, \alpha)$ each vanish when $\alpha = 0$ this is not the case for the MAG as the α dependence of the respective $\gamma_\alpha(a, \alpha)$ leading terms differ, [57]. For instance in the MOMc scheme in that gauge we find

$$\begin{aligned}
\Theta_a^{\text{MOMc}}(a, \alpha) &= \left[\frac{2}{3} \pi^2 C_F C_A^2 \alpha^2 + \frac{2}{3} \psi'(\tfrac{1}{3}) \zeta_3 C_F C_A^2 \alpha^2 + \frac{8}{9} \zeta_3 \pi^2 \frac{N_A^d}{N_A^o} C_F C_A^2 \alpha^2 \right. \\
&\quad - \frac{3751}{3} \frac{N_A^o}{[N_A^o + 2N_A^d]} C_F C_A^2 \alpha - \frac{1425}{4} \frac{N_A^d}{N_A^o} C_F C_A^2 - \frac{1395}{8} \frac{N_A^d}{N_A^o} C_F C_A^2 \alpha \\
&\quad - \frac{1323}{8} \frac{N_A^{d^2}}{N_A^{o^2}} C_F C_A^2 \alpha^2 - \frac{1179}{4} \frac{N_A^{d^2}}{N_A^{o^2}} C_F C_A^2 \alpha - \frac{1023}{2} \frac{N_A^o}{[N_A^o + 2N_A^d]} C_F C_A^2 \\
&\quad \left. - \frac{775}{18} \psi'(\tfrac{1}{3}) \frac{N_A^d}{N_A^o} C_F C_A^2 \alpha - \frac{621}{4} \frac{N_A^{d^2}}{N_A^{o^2}} C_F C_A^2 - \frac{560}{27} \zeta_3 \pi^2 \frac{N_A^d}{N_A^o} C_F C_A^2 \alpha \right]
\end{aligned}$$

$$\begin{aligned}
& -\frac{475}{3}N_f T_F C_F C_A \alpha - \frac{344}{27}\pi^2 N_f \frac{N_A^d}{N_A^o} T_F C_F C_A \alpha - \frac{153}{2}\zeta_3 \frac{N_A^d}{N_A^o} C_F C_A^2 \alpha^2 \\
& - \frac{128}{9}\psi'(\frac{1}{3})\zeta_3 N_f \frac{N_A^d}{N_A^o} T_F C_F C_A \alpha - \frac{93}{4}\frac{N_A^{d^2}}{N_A^{o^2}} C_F C_A^2 \alpha^3 - \frac{86}{3}\pi^2 \frac{N_A^{d^2}}{N_A^{o^2}} C_F C_A^2 \\
& - \frac{86}{3}\pi^2 \frac{N_A^{d^2}}{N_A^{o^2}} C_F C_A^2 \alpha - \frac{64}{27}\zeta_3 \pi^2 N_f T_F C_F C_A \alpha - \frac{52}{9}\pi^2 C_F C_A^2 \alpha \\
& - \frac{52}{9}\psi'(\frac{1}{3})\zeta_3 C_F C_A^2 \alpha - \frac{43}{9}\pi^2 \frac{N_A^{d^2}}{N_A^{o^2}} C_F C_A^2 \alpha^2 - \frac{19}{18}\pi^2 \frac{N_A^d}{N_A^o} C_F C_A^2 \alpha^2 \\
& - \frac{16}{3}\zeta_3 \pi^2 \frac{N_A^d}{N_A^o} C_F C_A^2 - \frac{16}{3}\psi'(\frac{1}{3})N_f T_F C_F C_A \alpha - \frac{16}{3}\psi'(\frac{1}{3})\zeta_3 \frac{N_A^{d^2}}{N_A^{o^2}} C_F C_A^2 \alpha^2 \\
& - \frac{9}{2}\zeta_3 C_F C_A^2 \alpha^2 - \frac{9}{2}\zeta_3 C_F C_A^2 \alpha^3 - \frac{4}{3}\psi'(\frac{1}{3})\zeta_3 \frac{N_A^d}{N_A^o} C_F C_A^2 \alpha^2 \\
& - \frac{4}{9}\zeta_3 \pi^2 C_F C_A^2 \alpha^2 + \frac{19}{12}\psi'(\frac{1}{3})\frac{N_A^d}{N_A^o} C_F C_A^2 \alpha^2 + \frac{26}{3}\psi'(\frac{1}{3})C_F C_A^2 \alpha \\
& + \frac{32}{9}\pi^2 N_f T_F C_F C_A \alpha + \frac{32}{9}\zeta_3 \pi^2 \frac{N_A^{d^2}}{N_A^{o^2}} C_F C_A^2 \alpha^2 + \frac{32}{9}\psi'(\frac{1}{3})\zeta_3 N_f T_F C_F C_A \alpha \\
& + \frac{43}{6}\psi'(\frac{1}{3})\frac{N_A^{d^2}}{N_A^{o^2}} C_F C_A^2 \alpha^2 + \frac{64}{3}\zeta_3 \pi^2 \frac{N_A^{d^2}}{N_A^{o^2}} C_F C_A^2 + \frac{64}{3}\zeta_3 \pi^2 \frac{N_A^d}{N_A^o} C_F C_A^2 \alpha \\
& + \frac{83}{16}C_F C_A^2 \alpha^2 + \frac{93}{16}C_F C_A^2 \alpha^3 + \frac{104}{27}\zeta_3 \pi^2 C_F C_A^2 \alpha \\
& + \frac{172}{9}\psi'(\frac{1}{3})N_f \frac{N_A^d}{N_A^o} T_F C_F C_A \alpha + \frac{256}{27}\zeta_3 \pi^2 N_f \frac{N_A^d}{N_A^o} T_F C_F C_A \alpha \\
& + \frac{280}{9}\psi'(\frac{1}{3})\zeta_3 \frac{N_A^d}{N_A^o} C_F C_A^2 \alpha + \frac{775}{27}\pi^2 \frac{N_A^d}{N_A^o} C_F C_A^2 \alpha + \frac{1023}{2}C_F C_A^2 \\
& + \frac{1364}{3}N_f \frac{N_A^o}{[N_A^o + 2N_A^d]} T_F C_F C_A \alpha + \frac{1571}{16}\frac{N_A^d}{N_A^o} C_F C_A^2 \alpha^2 + \frac{18451}{24}C_F C_A^2 \alpha \\
& - 591\zeta_3 C_F C_A^2 \alpha - 396\zeta_3 C_F C_A^2 - 352\zeta_3 N_f \frac{N_A^o}{[N_A^o + 2N_A^d]} T_F C_F C_A \alpha \\
& - 69N_f \frac{N_A^d}{N_A^o} T_F C_F C_A \alpha - 62N_f \frac{N_A^d}{N_A^o} T_F C_F C_A \alpha^2 - 32\psi'(\frac{1}{3})\zeta_3 \frac{N_A^{d^2}}{N_A^{o^2}} C_F C_A^2 \\
& - 32\psi'(\frac{1}{3})\zeta_3 \frac{N_A^{d^2}}{N_A^{o^2}} C_F C_A^2 \alpha - 24\zeta_3 N_f T_F C_F C_A \alpha^2 - 12\psi'(\frac{1}{3})\frac{N_A^d}{N_A^o} C_F C_A^2 \\
& - \psi'(\frac{1}{3})C_F C_A^2 \alpha^2 + 8\pi^2 \frac{N_A^d}{N_A^o} C_F C_A^2 + 8\psi'(\frac{1}{3})\zeta_3 \frac{N_A^d}{N_A^o} C_F C_A^2 \\
& + 18\zeta_3 \frac{N_A^{d^2}}{N_A^{o^2}} C_F C_A^2 \alpha^3 + 31N_f T_F C_F C_A \alpha^2 + 43\psi'(\frac{1}{3})\frac{N_A^{d^2}}{N_A^{o^2}} C_F C_A^2 \\
& + 43\psi'(\frac{1}{3})\frac{N_A^{d^2}}{N_A^{o^2}} C_F C_A^2 \alpha + 48\zeta_3 N_f \frac{N_A^d}{N_A^o} T_F C_F C_A \alpha^2 + 56\zeta_3 N_f \frac{N_A^d}{N_A^o} T_F C_F C_A \alpha \\
& + 120\zeta_3 N_f T_F C_F C_A \alpha + 125\zeta_3 \frac{N_A^d}{N_A^o} C_F C_A^2 \alpha + 126\zeta_3 \frac{N_A^{d^2}}{N_A^{o^2}} C_F C_A^2 \\
& + 129\zeta_3 \frac{N_A^{d^2}}{N_A^{o^2}} C_F C_A^2 \alpha^2 + 234\zeta_3 \frac{N_A^{d^2}}{N_A^{o^2}} C_F C_A^2 \alpha + 270\zeta_3 \frac{N_A^d}{N_A^o} C_F C_A^2
\end{aligned}$$

$$\begin{aligned}
& + 396\zeta_3 \frac{N_A^o}{[N_A^o + 2N_A^d]} C_F C_A^2 + 968\zeta_3 \frac{N_A^o}{[N_A^o + 2N_A^d]} C_F C_A^2 \alpha \Big] a^3 + O(a^4) \\
\Theta_{\alpha \text{MAG}}^{\text{MOMc}}(a, \alpha) = & \left[\frac{32}{9} \pi^2 N_f T_F C_F C_A + \frac{32}{9} \psi'(\frac{1}{3}) \zeta_3 N_f T_F C_F C_A + \frac{44}{3} \psi'(\frac{1}{3}) C_F C_A^2 \right. \\
& - \frac{3751}{3} \frac{N_A^o}{[N_A^o + 2N_A^d]} C_F C_A^2 - \frac{704}{27} \zeta_3 \pi^2 \frac{N_A^d}{N_A^o} C_F C_A^2 - \frac{475}{3} N_f T_F C_F C_A \\
& - \frac{473}{9} \psi'(\frac{1}{3}) \frac{N_A^d}{N_A^o} C_F C_A^2 - \frac{344}{27} \pi^2 N_f \frac{N_A^d}{N_A^o} T_F C_F C_A - \frac{341}{4} C_F C_A^2 \alpha \\
& - \frac{128}{9} \psi'(\frac{1}{3}) \zeta_3 N_f \frac{N_A^d}{N_A^o} T_F C_F C_A - \frac{88}{9} \pi^2 C_F C_A^2 - \frac{88}{9} \psi'(\frac{1}{3}) \zeta_3 C_F C_A^2 \\
& - \frac{64}{27} \zeta_3 \pi^2 N_f T_F C_F C_A - \frac{16}{3} \psi'(\frac{1}{3}) N_f T_F C_F C_A + \frac{172}{9} \psi'(\frac{1}{3}) N_f \frac{N_A^d}{N_A^o} T_F C_F C_A \\
& + \frac{176}{27} \zeta_3 \pi^2 C_F C_A^2 + \frac{256}{27} \zeta_3 \pi^2 N_f \frac{N_A^d}{N_A^o} T_F C_F C_A + \frac{341}{2} \frac{N_A^d}{N_A^o} C_F C_A^2 \alpha \\
& + \frac{352}{9} \psi'(\frac{1}{3}) \zeta_3 \frac{N_A^d}{N_A^o} C_F C_A^2 + \frac{759}{4} \frac{N_A^d}{N_A^o} C_F C_A^2 + \frac{946}{27} \pi^2 \frac{N_A^d}{N_A^o} C_F C_A^2 \\
& + \frac{1364}{3} N_f \frac{N_A^o}{[N_A^o + 2N_A^d]} T_F C_F C_A + \frac{5225}{12} C_F C_A^2 \\
& - 352 \zeta_3 N_f \frac{N_A^o}{[N_A^o + 2N_A^d]} T_F C_F C_A - 330 \zeta_3 C_F C_A^2 - 154 \zeta_3 \frac{N_A^d}{N_A^o} C_F C_A^2 \\
& - 132 \zeta_3 \frac{N_A^d}{N_A^o} C_F C_A^2 \alpha - 69 N_f \frac{N_A^d}{N_A^o} T_F C_F C_A - 62 N_f \frac{N_A^d}{N_A^o} T_F C_F C_A \alpha \\
& - 24 \zeta_3 N_f T_F C_F C_A \alpha + 31 N_f T_F C_F C_A \alpha + 48 \zeta_3 N_f \frac{N_A^d}{N_A^o} T_F C_F C_A \alpha \\
& + 56 \zeta_3 N_f \frac{N_A^d}{N_A^o} T_F C_F C_A + 66 \zeta_3 C_F C_A^2 \alpha + 120 \zeta_3 N_f T_F C_F C_A \\
& \left. + 968 \zeta_3 \frac{N_A^o}{[N_A^o + 2N_A^d]} C_F C_A^2 \right] a^3 + O(a^4) . \tag{7.12}
\end{aligned}$$

By contrast the parallel expressions in the Curci-Ferrari gauge are

$$\begin{aligned}
\Theta_{\alpha \text{CF}}^{\text{MOMc}}(a, \alpha) = & \left[31 N_f T_F C_F C_A \alpha^2 + 377 \zeta_3 C_F C_A^2 \alpha - \frac{11557}{24} C_F C_A^2 \alpha - \frac{64}{27} \zeta_3 \pi^2 N_f T_F C_F C_A \alpha \right. \\
& - \frac{52}{9} \pi^2 C_F C_A^2 \alpha - \frac{52}{9} \psi'(\frac{1}{3}) \zeta_3 C_F C_A^2 \alpha - \frac{16}{3} \psi'(\frac{1}{3}) N_f T_F C_F C_A \alpha - \frac{9}{2} \zeta_3 C_F C_A^2 \alpha^2 \\
& - \frac{9}{2} \zeta_3 C_F C_A^2 \alpha^3 - \frac{4}{9} \zeta_3 \pi^2 C_F C_A^2 \alpha^2 + \frac{2}{3} \pi^2 C_F C_A^2 \alpha^2 + \frac{2}{3} \psi'(\frac{1}{3}) \zeta_3 C_F C_A^2 \alpha^2 \\
& + \frac{26}{3} \psi'(\frac{1}{3}) C_F C_A^2 \alpha + \frac{32}{9} \pi^2 N_f T_F C_F C_A \alpha + \frac{32}{9} \psi'(\frac{1}{3}) \zeta_3 N_f T_F C_F C_A \alpha \\
& + \frac{83}{16} C_F C_A^2 \alpha^2 + \frac{93}{16} C_F C_A^2 \alpha^3 + \frac{104}{27} \zeta_3 \pi^2 C_F C_A^2 \alpha + \frac{889}{3} N_f T_F C_F C_A \alpha \\
& \left. - 232 \zeta_3 N_f T_F C_F C_A \alpha - 24 \zeta_3 N_f T_F C_F C_A \alpha^2 - \psi'(\frac{1}{3}) C_F C_A^2 \alpha^2 \right] a^3 + O(a^4) \\
\Theta_{\alpha \text{CF}}^{\text{MOMc}}(a, \alpha) = & \left[31 N_f T_F C_F C_A \alpha + 66 \zeta_3 C_F C_A^2 \alpha + 638 \zeta_3 C_F C_A^2 - \frac{9779}{12} C_F C_A^2 - \frac{341}{4} C_F C_A^2 \alpha \right. \\
& \left. - \frac{88}{9} \pi^2 C_F C_A^2 - \frac{88}{9} \psi'(\frac{1}{3}) \zeta_3 C_F C_A^2 - \frac{64}{27} \zeta_3 \pi^2 N_f T_F C_F C_A \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{16}{3}\psi'(\frac{1}{3})N_fT_FC_FC_A + \frac{32}{9}\pi^2N_fT_FC_FC_A + \frac{32}{9}\psi'(\frac{1}{3})\zeta_3N_fT_FC_FC_A \\
& + \frac{44}{3}\psi'(\frac{1}{3})C_FC_A^2 + \frac{176}{27}\zeta_3\pi^2C_FC_A^2 + \frac{889}{3}N_fT_FC_FC_A - 232\zeta_3N_fT_FC_FC_A \\
& - 24\zeta_3N_fT_FC_FC_A\alpha]a^3 + O(a^4). \tag{7.13}
\end{aligned}$$

Again when $\alpha = 0$ for this and each of the other MOM schemes then $\Theta_{a\text{CF}}^S(a, \alpha)$ vanishes.

8 Another angle on Crewther's relation.

While the relation (4.1) appears to hold in all the schemes and gauges that we have considered so far it is instructive to return to Crewther's original relation (2.4) and consider gauges for which it is valid especially in light of the earlier analysis. In other words to determine some gauge $\bar{\alpha}$ for which

$$\Delta_{\text{csb}}(a, \bar{\alpha}) = \frac{\beta(a, \bar{\alpha})}{a} \hat{K}_a(a, \bar{\alpha}) \tag{8.1}$$

holds. This was the approach taken in [12] but here rather than assume the gauge parameter is a pure constant we will incorporate loop corrections. The motivation to do so is that various observations noted earlier had properties requiring a fixed value of $\alpha = -3$ at a particular loop order which turned out to be inconsistent when higher order effects were included. An approach in a similar vein to explore schemes where (8.1) holds was given recently in [63]. In particular the V scheme of [13, 14] was again studied since the coupling constant in that scheme is by construction gauge parameter independent being based on the static quark potential. The four loop renormalization group functions are known [64] and we note that the coefficients of the V scheme β -function are independent of the gauge parameter. The first step to achieve the form (8.1) is to recognize that for a gauge dependent scheme we have a two dimensional space parameterized by a and $\bar{\alpha}$. We can attempt to find a gauge for which (8.1) holds by considering curves $\bar{\alpha} = \bar{\alpha}(a)$ which are defined by some relation $F(a, \bar{\alpha}) = 0$. In particular, we are interested in curves that contain fixed points of the running coupling constant. A natural choice for this is the set of curves satisfying

$$\gamma_\alpha(a, \bar{\alpha}) + S\gamma_a(a, \bar{\alpha}) = 0 \tag{8.2}$$

where

$$\gamma_a(a, \alpha) \equiv \frac{\beta(a, \alpha)}{a} \tag{8.3}$$

since outside the Landau gauge our fixed points are defined by the requirement that $\gamma_\alpha(a, \alpha) = 0$ and $\gamma_a(a, \alpha) = 0$. The parameter S will be tuned later to examine various scenarios. Under these constraints (4.1) becomes

$$\Delta_{\text{csb}}(a, \bar{\alpha}) = [K_a(a, \bar{\alpha}) - SK_\alpha(a, \bar{\alpha})] \gamma_a(a, \bar{\alpha}) \tag{8.4}$$

meaning that

$$\hat{K}_a(a, \bar{\alpha}) = [K_a(a, \bar{\alpha}) - SK_\alpha(a, \bar{\alpha})] \tag{8.5}$$

when comparing with (8.1).

In keeping with our perturbative approach we will expand $\bar{\alpha}$ as a function of a via

$$\bar{\alpha}(a) = \sum_{n=0}^{\infty} \alpha^{(n)} a^n \tag{8.6}$$

as well as taking a similar expansion for the two renormalization group functions

$$\begin{aligned}\gamma_a(a, \alpha) &= -\beta_0 a - \sum_{n=1}^{\infty} \left(\sum_{m=0}^{P_m} \beta_n^{(m)} \alpha^m \right) a^{n+1} \\ \gamma_\alpha(a, \alpha) &= \sum_{n=1}^{\infty} \left(\sum_{m=0}^{Q_m} \gamma_n^{(m)} \alpha^m \right) a^n\end{aligned}\quad (8.7)$$

where $P_1 = 4$, $P_2 = 7$, $P_3 = 5$, $P_4 = 6$, $Q_1 = 1$, $Q_2 = 4$, $Q_3 = 7$, $Q_4 = 5$ and $Q_5 = 6$ cover all the schemes considered here. The specific values of P_m and Q_m were deduced from the α dependence in the MOMg scheme with the four and five loop ones corresponding to the mMOM scheme. With these two expressions and (8.6) solving (8.2) produces equations that determine $\alpha^{(m)}$. We find the formal solution is

$$\begin{aligned}\alpha^{(0)} \gamma_1^{(1)} &= S \beta_0 - \gamma_1^{(0)} \\ \alpha^{(1)} \gamma_1^{(1)} &= S \left[\beta_1^{(4)} \alpha^{(0)4} + \beta_1^{(3)} \alpha^{(0)3} + \beta_1^{(2)} \alpha^{(0)2} + \beta_1^{(1)} \alpha^{(0)} + \beta_1^{(0)} \right] \\ &\quad - \left[\gamma_2^{(4)} \alpha^{(0)4} + \gamma_2^{(3)} \alpha^{(0)3} + \gamma_2^{(2)} \alpha^{(0)2} + \gamma_2^{(1)} \alpha^{(0)} + \gamma_2^{(0)} \right] \\ \alpha^{(2)} \gamma_1^{(1)} &= S \left[\beta_2^{(7)} \alpha^{(0)7} + \beta_2^{(6)} \alpha^{(0)6} + \beta_2^{(5)} \alpha^{(0)5} + \beta_2^{(4)} \alpha^{(0)4} \right. \\ &\quad \left. + \beta_2^{(3)} \alpha^{(0)3} + \beta_2^{(2)} \alpha^{(0)2} + \beta_2^{(1)} \alpha^{(0)} + \beta_2^{(0)} \right] \\ &\quad - \left[\gamma_3^{(7)} \alpha^{(0)7} + \gamma_3^{(6)} \alpha^{(0)6} + \gamma_3^{(5)} \alpha^{(0)5} + \gamma_3^{(4)} \alpha^{(0)4} \right. \\ &\quad \left. + \gamma_3^{(3)} \alpha^{(0)3} + \gamma_3^{(2)} \alpha^{(0)2} + \gamma_3^{(1)} \alpha^{(0)} + \gamma_3^{(0)} \right] \\ &\quad + \alpha^{(1)} \left[S \left[4\beta_1^{(4)} \alpha^{(0)3} + 3\beta_1^{(3)} \alpha^{(0)2} + 2\beta_1^{(2)} \alpha^{(0)} + \beta_1^{(1)} \right] \right. \\ &\quad \left. - \left[4\gamma_2^{(4)} \alpha^{(0)3} + 3\gamma_2^{(3)} \alpha^{(0)2} + 2\gamma_2^{(2)} \alpha^{(0)} + \gamma_2^{(1)} \right] \right] \\ \alpha^{(3)} \gamma_1^{(1)} &= \sum_{n=0}^{P_4} S \beta_3^{(n)} \alpha^{(0)n} - \sum_{n=0}^{Q_4} \gamma_4^{(n)} \alpha^{(0)n} + \alpha^{(1)} \sum_{n=1}^{P_3} n S \beta_2^{(n)} \alpha^{(0)n-1} \\ &\quad - \sum_{n=0}^{Q_3} n \gamma_3^{(n)} \alpha^{(0)n-1} + 2\alpha^{(0)2} \left[2\alpha^{(0)} \alpha^{(2)} + 3\alpha^{(1)2} \right] \left[S \beta_1^{(4)} - \gamma_2^{(4)} \right] \\ &\quad + 3\alpha^{(0)} \left[\alpha^{(0)} \alpha^{(2)} + \alpha^{(1)2} \right] \left[S \beta_1^{(3)} - \gamma_2^{(3)} \right] \\ &\quad + \left[2\alpha^{(0)} \alpha^{(2)} + \alpha^{(1)2} \right] \left[S \beta_1^{(2)} - \gamma_2^{(2)} \right] + \alpha^{(2)} \left[S \beta_1^{(1)} - \gamma_2^{(1)} \right]\end{aligned}\quad (8.8)$$

for the first few orders. We recall that although our first interest at the moment is on the Crewther relation by finding the expression for $\bar{\alpha}$ that solves (8.2) the second component is to find the value of a_∞ that is the solution to $\gamma_a(a_\infty, \bar{\alpha}(a_\infty)) = 0$. In this way we will arrive at the fixed points of the running parameters of the theory. Taking this approach avoids the Banks-Zaks fixed point and will return a value of $\alpha \approx -3$. This particular integer value was discussed previously in [20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32]. In fact, varying the defining equation

$$F(a, \alpha, S) \equiv \gamma_\alpha(a, \alpha) + S \gamma_a(a, \alpha) \quad (8.9)$$

with respect to S results in

$$\frac{dF(a, \alpha, S)}{dS} = \gamma_a(a, \alpha) \quad (8.10)$$

which should evaluate to zero at a fixed point. So the positions of the fixed points determined in this way should be independent of S up to truncation. We will produce numerical data as evidence

of this. In particular choosing $S = 0$ results in a procedure analogous to the typical process for finding fixed points where the curves are defined such that the gauge parameter is stationary and the value of the coupling constant can then be chosen so that it is also stationary.

Before considering the details for a specific scheme it is instructive to examine the leading order equation for the various gauges. First while β_0 will be the same for all gauges since

$$\beta_0 = -\frac{4}{3}N_f T_F + \frac{11}{3}C_A \quad (8.11)$$

γ_1 is different in each of the three gauges. In a linear covariant gauge fixing we have

$$\gamma_1 = -\frac{1}{2}\alpha C_A + \frac{13}{6}C_A - \frac{4}{3}N_f T_F \quad (8.12)$$

which means we can write

$$\alpha^{(0)} = \frac{2}{C_A} \left[S \left[\frac{4}{3}N_f T_F - \frac{11}{3}C_A \right] - \frac{4}{3}N_f T_F + \frac{13}{6}C_A \right]. \quad (8.13)$$

As this would be singular in the abelian limit it is best to redefine the parameter S as $S = 1 + 3\sigma C_A$ which gives

$$\alpha^{(0)} = -3 + [8N_f T_F - 22C_A] \sigma. \quad (8.14)$$

So $S = 1$ produces $\alpha^{(0)} = -3$ corresponding to the solution pointed out in [12]. The higher order corrections will be scheme dependent. Repeating this leading order exercise produces

$$\alpha_{\text{CF}}^{(0)} = -6 + [16T_F N_f - 44C_A] \sigma \quad (8.15)$$

for the Curci-Ferrari gauge. To analyse the MAG the following modification has to be made to the defining equation

$$F_{\text{MAG}}(a, \alpha, S) = \bar{\alpha} \gamma_\alpha(a, \alpha) + \bar{\alpha} S \gamma_a(a, \alpha) \quad (8.16)$$

because the leading order term in $\gamma_\alpha^{\text{MAG}}(a, \alpha)$ is proportional to α^{-1} . So the equations for $\alpha^{(i)}$ then need to be modified in this case. The resulting leading order equation produces two solutions. For $S = 1$ the numerical values are $\alpha_{\text{MAG}}^{(0)} = -0.430953$ and -5.569047 . The first of these solutions corresponds to that termed as Banks-Zaks one found in [45]. For each of the three gauges the $S = 1$ solutions are independent of N_f unlike the $S = 0$ case.

This leading order analysis is scheme independent but this ceases to be the case when corrections are included. For instance, it was shown in [12] that for the linear covariant gauge the form of Crewther's original relation, given in (2.4) but in the mMOM scheme, was satisfied at next order for particular values of α which were 0, -1 and -3 . For $\alpha = 0$ this followed from the α dependence in $\gamma_\alpha(a, \alpha)$ whereas $\alpha = -1$ emerged as a special case since $K_\alpha \gamma_\alpha(a, \alpha)$ was proportional to $(\alpha + 1)$ at second order. The situation where $\alpha = -3$ is that value that solves $\gamma_a(a, \alpha) = -\gamma_\alpha(a, \alpha)$ at leading order which is suggestive of the $\alpha \approx -3$ infrared stable fixed point discussed earlier. It is worth examining whether this point of view has any credibility at higher order. Therefore we calculated higher order terms in the expansion in (8.6) for a variety of schemes and gauges. The full set of expressions for $\alpha^{(i)}$ are provided in the associated data file but we record the $SU(3)$ values for the first few terms are

$$\begin{aligned} \alpha_{\text{mMOM}}^{(0)} \Big|^{SU(3)} &= -3 + 4\sigma N_f - 66\sigma \\ \alpha_{\text{mMOM}}^{(1)} \Big|^{SU(3)} &= 725274\sigma^3 + 41283\sigma^2 + 3881196\sigma^4 - \frac{51}{2} - 96\sigma^3 N_f^3 + 5832\sigma^3 N_f^2 \\ &\quad - 114048\sigma^3 N_f + 132\sigma^2 N_f^2 - 4680\sigma^2 N_f - 864\sigma^4 N_f^3 + 42768\sigma^4 N_f^2 \end{aligned}$$

$$\begin{aligned}
& -705672\sigma^4 N_f + 31\sigma N_f + N_f + \frac{261}{2}\sigma \\
\alpha_{\text{mMOM}}^{(2)} \Big|^{SU(3)} = & -6562749600\sigma^6 N_f^2 + 59382298800\sigma^6 N_f + 103680\sigma^6 N_f^5 - 9720000\sigma^6 N_f^4 \\
& + 359251200\sigma^6 N_f^3 - \frac{542619}{2}\zeta_3\sigma^2 - \frac{1485}{2}\zeta_3\sigma - 4498659\zeta_3\sigma^3 \\
& - 25150150080\sigma^7 N_f^2 + 207488738160\sigma^7 N_f + 559872\sigma^7 N_f^5 - 46189440\sigma^7 N_f^4 \\
& + 1524251520\sigma^7 N_f^3 - 17465382\zeta_3\sigma^4 - 684712835928\sigma^7 - 27604036251\sigma^5 \\
& - 213252314220\sigma^6 + 6841137204\sigma^5 N_f - 12384\sigma^4 N_f^4 - 662256\sigma^5 N_f^4 \\
& + 30458592\sigma^5 N_f^3 - 658776888\sigma^5 N_f^2 + 5184\sigma^5 N_f^5 - \frac{78139809}{2}\sigma^3 \\
& + \frac{2146905}{4}\sigma^2 - \frac{3436534431}{2}\sigma^4 + 3408\sigma^3 N_f^3 - 233370\sigma^3 N_f^2 + 5290650\sigma^3 N_f \\
& + 7242\sigma^2 N_f^2 - \frac{271143}{2}\sigma^2 N_f - 72\sigma^2 N_f^3 - \frac{782}{9}\sigma N_f^2 + 954396\sigma^4 N_f^3 \\
& - 27691794\sigma^4 N_f^2 + 356848173\sigma^4 N_f - 6\zeta_3 N_f - \frac{10}{3}N_f^2 - \frac{9405}{8} - 702\zeta_3\sigma^2 N_f^2 \\
& + 28026\zeta_3\sigma^2 N_f + \frac{20}{3}\zeta_3\sigma N_f^2 - 65\zeta_3\sigma N_f + 432\zeta_3\sigma^3 N_f^3 - 30780\zeta_3\sigma^3 N_f^2 \\
& + 662904\zeta_3\sigma^3 N_f + 3888\zeta_3\sigma^4 N_f^3 - 192456\zeta_3\sigma^4 N_f^2 + 3175524\zeta_3\sigma^4 N_f \\
& + \frac{6517}{4}\sigma N_f + \frac{551}{4}N_f + 153\zeta_3 + \frac{909}{8}\sigma
\end{aligned} \tag{8.17}$$

in the mMOM scheme. Up to the truncation order this gives the solution to $\bar{\gamma}_a(a) = \bar{\gamma}_\alpha(a)$ allowing us to find a fixed point a_∞ with $\bar{\gamma}_a(a_\infty) = 0$. When $N_f = 16$ and $S = 1$ we have

$$\begin{aligned}
\alpha_{\text{mMOM}}^{(0)} \Big|_{N_f=16} &= -3 \\
\alpha_{\text{mMOM}}^{(1)} \Big|_{N_f=16} &= -9.5 \\
\alpha_{\text{mMOM}}^{(2)} \Big|_{N_f=16} &= 243.558910
\end{aligned} \tag{8.18}$$

which gives

$$\begin{aligned}
\gamma_a(a, \bar{\alpha}) = & -0.333333a + 103.666667a^2 + 152.000000a^3 + 1060.437500a^4 \\
& + 1929.093750a^5 + O(a^6)
\end{aligned} \tag{8.19}$$

to the two loop level. By this we mean the two loop solution of the equations $\bar{\gamma}_a(a) = 0$ and $\bar{\gamma}_\alpha(a) = 0$. These have a positive fixed point at $a_\infty = 0.0032000819$ implying $\alpha_\infty = -3.030400777$ at this approximation order. The respective two loop mMOM fixed point values from [45] are 0.0032001941 and -3.0301823312 which shows our $N_f = 16$ solution is in reasonable correspondence. We note that by solving this way we have introduced an additional truncation in the series for α . So it is not unexpected that the fixed point found in this way does not result in fully the same value. However by including higher order corrections we would expect that there would be a degree of convergence. Repeating the exercise at the three loop level we find two real fixed point solutions on the (a, α) plane which are $(0.0031380752, -3.0274132638)$ and $(0.1624143964, 1.8817663994)$. The first of these has a clear counterpart in the fixed point analysis of [45] which is $(0.0031380724, -3.0274210489)$ and is an infrared stable fixed point. The second solution is not as straightforward to place with a known fixed point in [45] but may map to the stable fixed point $(0.1279084064, 1.9051106246)$.

While this gives a flavour of the situation at the first few orders for $N_f = 16$ we have analysed the remaining values of N_f in the conformal window and the results are recorded in Table 6. In

N_f	Loop	$\bar{\alpha}$		[45]		Difference	
		a_∞	α_∞	a_∞	α_∞	δ_a	δ_α
8	2	0.049337	-3.863406	0.052219	-3.795565	-0.002882	-0.067841
	4	0.064323	-3.359243	0.060959	-3.485534	0.003364	0.126291
	5	0.046971	-2.951933	0.047937	-3.192836	-0.000966	0.240903
9	2	0.046323	-3.764323	0.048504	-3.705259	-0.002181	-0.059064
	4	0.056983	-2.993633	0.055612	-3.212828	0.001371	0.219195
	5	0.042375	-2.887326	0.043123	-3.003521	-0.000748	0.116195
10	2	0.042400	-3.657192	0.043898	-3.608171	-0.001498	-0.049021
	4	0.046825	-3.006516	0.046629	-3.124397	0.000196	0.117881
	5	0.038413	-2.930885	0.039060	-2.975110	-0.000647	0.044225
11	2	0.037453	-3.543073	0.038351	-3.505155	-0.000898	-0.037918
	3	0.043328	-3.551534	0.042908	-3.513882	0.000420	-0.037652
	4	0.038483	-3.055520	0.038443	-3.114044	0.000040	0.058524
	5	0.034366	-3.013171	0.034890	-3.025124	-0.000524	0.011953
12	2	0.031475	-3.424915	0.031919	-3.398484	-0.000444	-0.026431
	3	0.031275	-3.329545	0.031269	-3.323533	0.000006	-0.006012
	4	0.030877	-3.101381	0.030859	-3.126726	0.000018	0.025345
	5	0.029525	-3.092376	0.029873	-3.093637	-0.000348	0.001261
13	2	0.024644	-3.308050	0.024812	-3.292241	-0.000168	-0.015809
	3	0.023188	-3.213527	0.023188	-3.213214	0.000000	-0.000313
	4	0.023552	-3.122593	0.023543	-3.131342	0.000009	0.008749
	5	0.023430	-3.131575	0.023575	-3.130751	-0.000145	-0.000824
14	2	0.017346	-3.199477	0.017389	-3.191979	-0.000043	-0.007498
	3	0.016146	-3.138109	0.016146	-3.138534	0.000000	0.000425
	4	0.016447	-3.112174	0.016445	-3.114255	0.000002	0.002081
	5	0.016525	-3.119407	0.016555	-3.118834	-0.000030	-0.000573
15	2	0.010064	-3.105672	0.010070	-3.103310	-0.000006	-0.002362
	3	0.009537	-3.080460	0.009537	-3.080626	0.000000	0.000166
	4	0.009636	-3.076696	0.009636	-3.076936	0.000000	0.000240
	5	0.009658	-3.078272	0.009660	-3.078166	-0.000002	-0.000106
16	2	0.003200	-3.030401	0.003200	-3.030182	0.000000	-0.000219
	3	0.003138	-3.027413	0.003138	-3.027421	0.000000	0.000008
	4	0.003143	-3.027352	0.003143	-3.027354	0.000000	0.000002
	5	0.003143	-3.027378	0.003143	-3.027377	0.000000	-0.000001

Table 6: Fixed points solutions for a_∞ and α_∞ to the curve $\gamma_a(a, \bar{\alpha}) = -\gamma_\alpha(a, \bar{\alpha})$, in the columns headed with $\bar{\alpha}$ and the corresponding solutions for the fixed points in the (a, α) plane given in [45] together with the difference δ in values in the final pair of columns up to the five loop level in the mMOM scheme.

that table columns 3 and 4 record the solutions for all values of N_f in the conformal window for the $SU(3)$ group. The next pair of columns records the values of the critical point solutions of (2.11) from [45] for comparison. The final two columns record the difference δ_a and δ_α of the respective values of a_∞ and α_∞ . At the upper end of the conformal window, where perturbation theory is more reliable, there is a clear indication that the solution for \bar{a} is in good agreement with the fixed point of [45]. This continues to be the case for lower values of N_f but the agreement becomes less pronounced except at high loop order. Below around $N_f = 10$ one strays into territory where the critical coupling value ceases to be small and perturbation theory may be less reliable for drawing a precise conclusion. However in the region of validity it does support the premise that the Crewther relation reflects conformal properties. Indeed to complete this mMOM analysis we have calculated the value of $C_{\text{Bjr}}^{\text{mMOM}}(a, \alpha)C_{\text{Adl}}^{\text{mMOM}}(a, \alpha)$ for $N_f = 16$ at successive loop orders and present the outcome in Table 7. Again we observe that in effect the product evaluates to 3 similar to the earlier cases.

mMOM 2 loop			
a_∞	α_∞	$O(a^3)$	$O(a^4)$
0.0032000819	-3.0304007777	2.9999982454	3.0000012447
mMOM 3 loop			
a_∞	α_∞	$O(a^3)$	$O(a^4)$
0.0031380752	-3.0274132638	2.9999973440	3.0000001217
0.1624143964	1.8817663994	9.7789795843	20.2267682549
mMOM 4 loop			
a_∞	α_∞	$O(a^3)$	$O(a^4)$
0.0031430140	-3.0273515306	2.9999974127	3.0000002081
0.0922743518	0.7554173647	4.1917263936	6.5855502329
mMOM 5 loop			
a_∞	α_∞	$O(a^3)$	$O(a^4)$
0.0031434057	-3.0273781422	2.9999974182	3.0000002149
0.0498699873	-3.9546945229	3.1876456375	3.2609125084

Table 7: Values of $C_{\text{Bjr}}^{\text{mMOM}}(a, \alpha)C_{\text{Adl}}^{\text{mMOM}}(a, \alpha)$ from the solution for \bar{a} at successive loop orders for $N_f = 16$.

One question that arises is whether the choice of S affects the analysis and if so to what extent. While we could consider the numerical fixed points for an array of values of S , instead we will consider the fixed points in the $\overline{\text{MS}}$ and RI' schemes since the running of the coupling constant is α independent in both these schemes. This will allow us to calculate the fixed points as a function of S . The procedure is the same as for the mMOM scheme except that $\gamma_a(a_\infty) = 0$ immediately determines a_∞ . So solving $\gamma_a(a, \alpha) + S\gamma_a(a, \alpha) = 0$ means the S dependence appears solely in $\bar{a}(a_\infty)$ and allows us to study the relation to the loop order. Repeating the earlier example for $N_f = 16$ we find

$$\begin{aligned}
\alpha_{\overline{\text{MS}}}^{(0)} &= -2.777778 - 0.222222S \\
\alpha_{\overline{\text{MS}}}^{(1)} &= -52.851852 + 67.092593S - 0.074074S^2 \\
\alpha_{\overline{\text{MS}}}^{(2)} &= -78.713043 + 968.858403S + 45.225583S^2 - 0.006173S^3
\end{aligned}$$

Loop	a_∞	$\bar{\alpha}(a_\infty)$	$S = 1$	[45]
2	0.003311	$-2.952784 - 0.000061S - 0.000245S^2$	-2.953090	-2.952985
3	0.003162	$-2.945674 - 0.000400S + 0.000218S^2$ $-6.171139 \times 10^{-8}S^3$	-2.945856	-2.945839
4	0.003170	$-2.945920 - 0.000033S + 0.000016S^2$ $+1.111287 \times 10^{-7}S^3 - 2.848273 \times 10^{-10}S^4$	-2.945937	-2.945935
	0.091700	$-3.819444 + 8.65019S - 4.55718S^2$ $+0.00413990S^3 - 6.895175 \times 10^{-6}S^4$	0.2776970	-4.407403
5	0.003171	$-2.945966 + 1.06321 \times 10^{-6}S - 8.15566 \times 10^{-7}S^2$ $-3.72127 \times 10^{-8}S^3 + 8.257949 \times 10^{-10}S^4$ $-3.041220 \times 10^{-14}S^5$	-2.945966	-2.945965
	0.046999	$-4.691390 + 2.754610S - 1.377941S^2$ $-0.006615S^3 + 0.000053S^4 - 1.467961 \times 10^{-9}S^5$	-3.321283	-3.682521

Table 8: Values of $\bar{\alpha}(a_\infty)$ for $N_f = 16$ in the $\overline{\text{MS}}$ scheme as a function of S .

Loop	a_∞	$\bar{\alpha}(a_\infty)$	$S = 1$	[45]
2	0.003311	$-3.032990 - 0.005566S - 0.000559S^2$ $-0.000027S^3$	-3.039142	-3.040393
3	0.003162	$-3.028600 - 0.001502S + 0.000424S^2$ $+0.000045S^3 - 4.220754 \times 10^{-7}S^4 - 1.523738 \times 10^{-8}S^5$	-3.029633	-3.029389
4	0.003170	$-3.027400 - 0.000072S + 0.000095S^2$ $-0.000014S^3 + 1.097536 \times 10^{-6}S^4 + 5.156952 \times 10^{-8}S^5$ $-4.450271 \times 10^{-10}S^6 - 1.186739 \times 10^{-11}S^7$	-3.027390	-3.027317
	0.091700	$29.170852 + 37.509021S - 7.270245S^2$ $-1.363470S^3 + 0.036484S^4 + 0.001606S^5$ $-0.000011S^6 - 2.872889 \times 10^{-7}S^7$	58.084230	-7.001486
5	0.003171	$-3.027362 + 7.648957 \times 10^{-6}S + 0.000011S^2$ $-3.795632 \times 10^{-6}S^3 - 7.923119 \times 10^{-7}S^4$ $-5.731136 \times 10^{-8}S^5 + 1.945960 \times 10^{-9}S^6$ $-5.788853 \times 10^{-11}S^7 - 5.007868 \times 10^{-13}S^8$ $-1.052508 \times 10^{-14}S^9$	-3.027348	-3.027350
	0.046999	$3.423566 + 9.178962S - 4.964152S^2$ $+0.302490S^3 - 0.086406S^4 - 0.005043S^5$ $+0.000114S^6 + 3.328853 \times 10^{-6}S^7 - 2.417239 \times 10^{-8}S^8$ $-5.080334 \times 10^{-10}S^9$	7.849534	-6.454812

Table 9: Values of $\bar{\alpha}(a_\infty)$ for $N_f = 16$ in the RI' scheme as a function of S .

$$\begin{aligned}
\alpha_{\overline{\text{MS}}}^{(3)} &= 5792.714293 - 7038.053127S - 6394.380328S^2 + 5.436159S^3 - 0.008942S^4 \\
\alpha_{\overline{\text{MS}}}^{(4)} &= 29279.884317 - 325028.480677S - 166111.075407S^2 - 1468.595029S^3 \\
&\quad + 10.989272S^4 - 0.000301S^5
\end{aligned} \tag{8.20}$$

which lead to expressions for $\bar{\alpha}(a_\infty)$ given in Table 8 at successive loop order as a function of S . Included in the table are the evaluation of $\bar{\alpha}(a_\infty)$ at $S = 1$ and the critical point value from [45] to compare with. Clearly in the first instance the $S = 0$ values are an accurate estimate of the eventual $S = 1$ value. Equally as the loop order increases the dependence on S washes out quickly if one uses the coefficients of S itself as a guide. At four and five loops additional fixed points emerged. In each case the coefficients of S in the polynomial are relatively large for low powers which is due to the large critical coupling constant value.

While this analysis is in the $\overline{\text{MS}}$ scheme, similar to that of the original Crewther study [5], it is worth repeating the exercise in the RI' scheme to ascertain whether the same picture emerges. Therefore taking $N_f = 16$ again we find

$$\begin{aligned}
\alpha_{\text{RI}'}^{(0)} &= -2.777778 - 0.222222S \\
\alpha_{\text{RI}'}^{(1)} &= -77.074074 + 65.430041S - 0.168724S^2 - 0.008230S^3 \\
\alpha_{\text{RI}'}^{(2)} &= -712.810800 + 1384.485718S + 95.815127S^2 + 7.080171S^3 - 0.042219S^4 \\
&\quad - 0.001524S^5 \\
\alpha_{\text{RI}'}^{(3)} &= 58371.788435 + 26052.906414S - 10453.261109S^2 - 1844.453717S^3 \\
&\quad + 47.775386S^4 + 2.099823S^5 - 0.013971S^6 - 0.000373S^7 \\
\alpha_{\text{RI}'}^{(4)} &= 1094061.824587 + 115396.980509S - 836720.663333S^2 + 98112.588891S^3 \\
&\quad - 18705.923877S^4 - 1077.589340S^5 + 23.656656S^6 + 0.690162S^7 - 0.004954S^8 \\
&\quad - 0.000104S^9
\end{aligned} \tag{8.21}$$

with the subsequent implications for $\alpha(a_\infty)$ recorded in Table 9. We recall that as the RI' β -function has the same coefficients as those of the $\overline{\text{MS}}$ scheme the critical couplings will be the same and so there will also be four and five loop solutions. However comparing the solutions for each coefficient $\alpha_{\text{RI}'}^{(i)}$ with its $\overline{\text{MS}}$ partner we note that in the RI' scheme the degree of the polynomial solution in S at each loop order is higher. Despite this the same outcome emerges when S is set to unity in that the numerical value is in very good agreement with the fixed point values given in [45]. Again the convergence to a practically S independent solution for $\alpha(a_\infty)$ transpires. This is indicative of the idea argued earlier that the condition $\gamma_a(a, \alpha) + S\gamma_\alpha(a, \alpha) = 0$ can be used to find the stable infrared fixed point and also that the value of S is unimportant. This means that while the value of $S = 1$ has properties of interest specifically to studies of the Crewther relation the relation of this particular curve to the stable infrared fixed point is no more important than any similar curve generated from any other value of S .

To this point we have concentrated on the linear covariant gauge with this approach. However we have also examined the two nonlinear gauges in depth and for the kinematic schemes of [37, 38] although we are limited to three loops in this case. The outcome is in essence the same as the linear gauge case except that the leading value of $\bar{\alpha}$ is in the neighbourhood of $\alpha_{\text{CF}} = -6$ for the Curci-Ferrari gauge. Due to the nature of $\gamma_\alpha(a, \alpha)$ for the MAG the leading value of α is $\alpha_{\text{MAG}} = -5.56$. For both gauges the same property is present as the linear gauge in that over the range of the conformal window the higher order corrections do not unduly alter these respective α values. Equally for values of N_f down to around 12 the solutions for $\bar{\alpha}$ are in keeping with the fixed point solutions of [45]. We would expect this to be improved for lower values of N_f when higher

Curci-Ferrari Gauge					
N_f	Loop	$\bar{\alpha}$		[45]	
		a_∞	α_∞	a_∞	α_∞
8	3	0.116505	5.425313	0.116505	-0.525277
9	2	0.416667	7.333333	0.416667	-2.387888
	3	0.081803	1.386711	0.081803	-1.239118
10	2	0.175676	0.207207	0.175676	-2.981627
	3	0.060824	-0.762250	0.060824	-1.988405
11	2	0.098214	-2.202381	0.098214	-3.548192
	3	0.046039	-2.180305	0.046039	-2.744498
12	2	0.060000	-3.480000	0.060000	-4.078489
	3	0.034607	-3.235417	0.034607	-3.479125
13	2	0.037234	-4.312057	0.037234	-4.569040
	3	0.025191	-4.075904	0.025191	-4.169158
14	2	0.022124	-4.923304	0.022124	-5.020489
	3	0.017070	-4.768441	0.017070	-4.796907
15	2	0.011364	-5.409091	0.011364	-5.435844
	3	0.009818	-5.342638	0.009818	-5.347954
16	2	0.003311	-5.816777	0.003311	-5.819105
	3	0.003162	-5.808281	0.003162	-5.808458
MAG					
8	3	0.116505	-14.454928	0.116505	-14.732061
9	2	0.416667	-5.748610	0.416667	-1.775685
	3	0.081803	-9.342567	0.081803	-6.679483
10	2	0.175676	-5.644755	0.175676	-2.041475
	3	0.060824	-7.302268	0.060824	-9.374058
11	2	0.098214	-5.611372	0.098214	-2.385628
	3	0.046039	-6.350328	0.046039	-1.943195
12	2	0.060000	-5.594904	0.060000	-2.822900
	3	0.034607	-5.885931	0.034607	-2.691103
13	2	0.037234	-5.585093	0.037234	-3.356225
	3	0.025191	-5.668820	0.025191	-3.414438
14	2	0.022124	-5.578581	0.022124	-3.968278
	3	0.017070	-5.583155	0.017070	-4.074876
15	2	0.011364	-5.573944	0.011364	-4.620179
	3	0.009818	-5.563763	0.009818	-4.684705
16	2	0.003311	-5.570474	0.003311	-5.263519
	3	0.003162	-5.568155	0.003162	-5.273175

Table 10: Fixed points solutions for a_∞ and α_∞ to the curve $\gamma_a(a, \bar{\alpha}) = -\gamma_\alpha(a, \bar{\alpha})$ in the Curci-Ferrari gauge and MAG in the $\overline{\text{MS}}$ scheme.

order corrections become available. This viewpoint can be seen in Table 10. Similar tables are also available in the data file associated with this article but they are more comprehensive in that they include extra fixed points which have a mapping to the $\bar{\alpha}$ solution. In addition we include the results for all the schemes and gauges we have discussed in the article. Examining these wider tables it is evident that for certain schemes and gauges the connection with the fixed points of [45] and the solutions found by the method of this section is remarkably stable even at the lower end of the conformal window. Although for the MOM schemes of [37, 38] only a few perturbative orders are available. We note that in several schemes there were no fixed point solutions for various N_f values in [45]. In that instance there are no entries but we did find solutions for $\bar{\alpha}$. It is not clear whether this is indicating that there will be solutions for fixed points in these cases when higher order perturbative results become available in those schemes.

9 Wider perspective.

It is worth taking stock of the wider perspective that our investigations have arrived at. We begin by putting the choice of the left hand side of (8.2) in a field theory context. Clearly the combined renormalization group function

$$\hat{\gamma}_\alpha(a, \alpha) = \gamma_\alpha(a, \alpha) + \gamma_a(a, \alpha) \quad (9.1)$$

plays an important role in the analysis of the Crewther relation extension as well as being connected to the special choice of $\alpha = -3$ in the linear covariant gauge as illustrated in the previous section. To understand the origin of (9.1) we recall the purely gluonic sector of the Lagrangian of a non-abelian gauge theory is

$$L^{\text{gluonic}} = -\frac{1}{4}G_{\mu\nu}^a G^{a\mu\nu} - \frac{1}{2\alpha}(\partial^\mu A_\mu^a)^2 \quad (9.2)$$

for a linear covariant gauge fixing. The field strength tensor is given by

$$G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c \quad (9.3)$$

where g is the coupling and f^{abc} are the gauge group structure constants. While this is invariably the usual presentation of the gluonic sector and the one used for perturbative computations one can always redefine the fields and variables without altering any physical predictions. Therefore defining the rescaling

$$\hat{A}_\mu^a = gA_\mu^a \quad (9.4)$$

we have

$$L^{\text{gluonic}} = -\frac{1}{4a}\hat{G}_{\mu\nu}^a \hat{G}^{a\mu\nu} - \frac{1}{2\alpha a}(\partial^\mu \hat{A}_\mu^a)^2 \quad (9.5)$$

where $a = g^2/(16\pi^2)$ and $\hat{G}_{\mu\nu}^a$ corresponds to (9.3) at the formal value of $g = 1$. Formulating the gluonic sector in this way identifies the renormalization of the dimension four operator $\hat{G}_{\mu\nu}^a \hat{G}^{a\mu\nu}$ with its associated coupling which is $1/a$ and whose renormalization group function is related to the β -function. One can view the second term of (9.5) in the same light and regard the coupling associated with the linear gauge fixing term as αa . Clearly both operators are independent which can be seen by focussing on the quadratic part of (9.5). The field strength term is transverse while the other is purely longitudinal. Therefore the running of the associated couplings will be different with that of the latter operator being governed by (9.1). At a more formal level the treatment of the transverse and longitudinal components of the gauge field as separate entities within loop computations has been recognized earlier. See, for instance, [16, 65, 66, 67, 68, 69]. Indeed in

[68, 69] the concept of invariant charges of the two operators was coined and formalized within the Hopf algebra construct of renormalization theory. Therefore from the point of view of (9.5) a more appropriate formulation would be to treat αa as a second coupling constant in the renormalization group running in a gauge theory. Moreover while we concentrated on the linear covariant gauge repeating the rescaling argument for the Curci-Ferrari gauge and MAG will produce the same observation with regard to the definition of a second coupling from the two independent operators of the quadratic sector of L^{gluonic} . For completeness we note the first few terms of (9.1) for the three gauges are

$$\begin{aligned}
\hat{\gamma}_\alpha^{\text{lin},\overline{\text{MS}}}(a,\alpha) &= -[\alpha + 3] \frac{C_A a}{2} + [40N_f T_F - 6a^2 C_A - 33\alpha C_A - 95C_A] \frac{C_A a^2}{24} + O(a^3) \\
\hat{\gamma}_\alpha^{\text{CF},\overline{\text{MS}}}(a,\alpha) &= -[\alpha + 6] \frac{C_A a}{4} + [80N_f T_F - 3a^2 C_A - 51\alpha C_A - 190C_A] \frac{C_A a^2}{48} + O(a^3) \\
\hat{\gamma}_\alpha^{\text{MAG},\overline{\text{MS}}}(a,\alpha) &= -\left[2\alpha^2 N_A^d + \alpha^2 N_A^o + 12\alpha N_A^d + 6\alpha N_A^o + 12N_A^d\right] \frac{C_A a}{4\alpha N_A^o} \\
&+ \left[512N_A^d N_A^o N_f T_F + 80\alpha N_A^{o2} N_f T_F - 30\alpha^3 C_A N_A^{d2} - 27\alpha^3 C_A N_A^d N_A^o \right. \\
&- 3\alpha^3 C_A N_A^{o2} - 366\alpha^2 C_A N_A^{d2} - 339\alpha^2 C_A N_A^d N_A^o - 51\alpha^2 C_A N_A^{o2} \\
&+ 294\alpha C_A N_A^{d2} - 647\alpha C_A N_A^d N_A^o - 190\alpha C_A N_A^{o2} + 160\alpha N_A^d N_A^o N_f T_F \\
&\left. + 2016C_A N_A^{d2} - 928C_A N_A^d N_A^o\right] \frac{C_A a^2}{48\alpha N_A^{o2}} + O(a^3). \tag{9.6}
\end{aligned}$$

Clearly the leading terms of the first two gauges indicate the special gauge choices that emerged in the Crewther analysis. For the MAG the leading term vanishes at two values of α which are

$$\alpha_{\text{MAG}} = -3 \pm \sqrt{\frac{3[2N_A^d + 3N_A^o]}{[2N_A^d + N_A^o]}} \tag{9.7}$$

and reproduces the Curci-Ferrari values in the $N_A^d \rightarrow 0$ limit. Also for example

$$\alpha_{\text{MAG}}|^{SU(N_c)} = -3 \pm \sqrt{\frac{3[3N_c + 2]}{[N_c + 2]}}. \tag{9.8}$$

For $SU(3)$ the two values are $-3 \pm \sqrt{\frac{33}{5}}$ which equate to -0.43095348 and -5.56904651 respectively. It is important to note that these special gauge parameter values are constructed from the scheme independent terms of $\hat{\gamma}(a,\alpha)$ and therefore will be the foundation for the infrared stable fixed points in all schemes at higher orders.

It is clear the Crewther relation has provided another example where a special value of the gauge parameter plays a crucial role in a deeper property of QCD as observed in [12] in the mMOM scheme. The other situations where this occurred were in a variety of different problems and it was worth pausing to understand if there is any underlying connections as to when this arises. Moreover by doing so would provide a signpost where to expect the special case to occur in other computations. Examining the various articles where $\alpha = -3$ has been singled out, [20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32], several common themes emerge. For instance, in [23, 24, 25, 26] the quark 2-point function was the main topic. In [23] this involved solving the Schwinger-Dyson equation defining the quark 2-point function to examine the quark mass gap to ascertain the condition when chiral symmetry was broken. Although [24, 25] dealt with mesonic quark bound states in a gauge invariant setup the path integral and worldline approach effectively trade off gauge invariance for path dependence. This resulted in what was termed the

connector having a renormalization that at one loop depended on $(\alpha + 3)$. The special choice appeared to remove residual dependence on path and gauge at one loop. For the approaches in [25, 26, 30] the common thread was the resummation of a class of Feynman diagrams which invariably were one loop bubble graphs. In [25, 26] this naive non-abelianization was used to study the evolution of quark bilinear operators that underpin deep inelastic scattering processes. As the treatment of these operators was by insertion in a quark 2-point function the Green's function is α dependent. What was interesting is that the choice of $\alpha = -3$ provided a reliable estimate of the one loop N_f independent part of the operator anomalous dimension for a range of moments. This accuracy was not as reliable for the two loop case. Similarly in [30] a quark based Green's function was studied from which an effective coupling was constructed. Its partial resummation indicated that asymptotic freedom was conditional on the special gauge value. In [31] the same gauge parameter choice showed a remarkable connection between a set of gluonic bubble insertions in a quark current correlation function and the one loop gluonic contribution to the V scheme coupling constant derived from the static quark potential. By contrast the emergence of $\alpha = -3$ was not restricted to quark based quantities. In [27, 28, 29] a dimension six operator correction to the Yang-Mills action was used to probe infrared gluon dynamics to determine what conditions if any led to the gluon propagator having a double pole in the p^2 where p is the momentum. Such a behaviour would lead to a linear confining potential. Solving the Schwinger-Dyson equation for the ghost propagator with a massless double pole gluon propagator, that derived from the dimension six operator, the solution required $\alpha = -3$.

From an overall point of view the studies that have identified $\alpha = -3$ as having particular significance share several similar features. Aside from all using the linear covariant gauge the calculations were all carried out using the $\overline{\text{MS}}$ scheme. In addition most involved calculations of Green's functions which would introduce gauge parameter dependence. For instance, although the twist-2 operators used in the operator product expansion for deep inelastic scattering that were the focus of [25, 26] are gauge invariant, the quark 2-point function where they were inserted is gauge variant. Therefore the expression will be α dependent but the operator renormalization constant extracted from this Green's function will be gauge parameter independent in the $\overline{\text{MS}}$ scheme. By contrast if the 2-point correlation function for the twist-2 operator was evaluated then it would be gauge parameter independent in the $\overline{\text{MS}}$ scheme since the operator is gauge invariant. In other words the same result would ensue for the operator correlation function if it was evaluated in the Curci-Ferrari gauge or MAG in the $\overline{\text{MS}}$ scheme. However if the operator correlation function was evaluated in a scheme such as mMOM or the MOM schemes of [37, 38] then it would depend on the gauge parameter. This is similar to the situation of the anomalous dimension of a gauge invariant operator which depends on the gauge parameter in general but not in a scheme such as $\overline{\text{MS}}$. In other words the correlation function of gauge invariant operators is in general gauge parameter dependent but independent of it in a subset of renormalization schemes.

In distilling the essence of previous observations of a special gauge parameter value it is evident now where the Crewther relation rests within this framework. The two ingredients of the relation are correlation functions of gauge invariant operators which are the vector and axial vector quark currents. Therefore in the $\overline{\text{MS}}$ scheme each correlation function will be gauge parameter independent meaning the Crewther relation will be too. Equally applying the formalism to effect a change of scheme to say mMOM will produce gauge parameter dependence in the separate correlation functions and hence produce a similar dependence in $\Delta_{\text{csb}}(a, \alpha)$. This is evident in [12]. We have shown how that can be accommodated consistently using the renormalization group formalism in (7.5). A similar mapping that established (7.5) for the Crewther case could equally well be applied to the quantities computed in [22, 23, 24, 25, 26, 30, 31]. What is important to remember is that the gauge parameter dependence of any operator correlation function does not affect any prediction from its evaluation. The two underlying parameters, a and α , both run with respect to the

renormalization mass scale μ in a way derived from their definition in the renormalization group equation. The formal dependence of each on μ will be different in different schemes but ultimately within the domain of perturbative applicability the behaviour of the correlation function with respect to μ will be very similar since the scheme and gauge dependence will wash out as the loop order increases.

10 Discussion.

We have examined the Crewther relation at high loop order from a variety of different angles to first ascertain whether it is possible to consistently extend the observation of [5] to schemes other than $\overline{\text{MS}}$ such as mMOM and several kinematic schemes [37, 38] in different non-Lorenz covariant gauge fixings. In other words to see if the multiplicative form that the relation has in the $\overline{\text{MS}}$ and V schemes is retained universally in other schemes and gauges. While we have added the RI' scheme to that list, albeit for trivial reasons, the main conclusion is that the multiplicative form is not universal. In schemes where the β -function depends on the gauge parameter, which is not the case for the $\overline{\text{MS}}$, V and RI' schemes, the Crewther relation is already known to depend on the gauge parameter. This is despite the fact that the two correlation functions that lead to the relation involve gauge invariant operators. Instead in this situation the relation has to be extended to accommodate the gauge parameter dependence. Moreover we have extensively shown that this is fully consistent with quantum field theoretic considerations and primarily the properties of the renormalization group equation. Essentially the gauge parameter acts as a second coupling and therefore its associated β -function has to be included as indicated in (4.1). While we deduced this initially by probing the product of the Bjorken sum rule and Adler D -function using the fixed point properties of QCD, as the β -function will always be zero at such points, we showed how to effect scheme changes on the coupling and gauge parameters in general to verify (4.1) formally. Subsequently the robustness of this structure was probed from a different angle when we posed the problem of trying to find what conditions on the gauge parameter would preserve the multiplicative structure akin to that of (4.1). In doing so we showed that the infrared stable critical point value of the gauge parameter resulted, thereby completing the circle of our analysis. Equally by doing so we provided a concrete example of the invariant charge concept defined by the couplings associated with the transverse and longitudinal components of the gluon field. It would seem that this is not restricted to the Crewther relation since the gauge variant quantities studied in [22, 23, 24, 25, 26, 30, 31] could equally well be recast in the two coupling language whence the reasons for the emergence of a special gauge parameter value underlying key properties of each quantity would be apparent. If one took a wider viewpoint concerning where this study appears to point it might be that caution is advised when trying to adduce general properties of seemingly gauge independent quantities from one specific renormalization scheme such as $\overline{\text{MS}}$. The minimal way the underlying divergences are subtracted in divergent Green's function would appear to cloud more general considerations.

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A Results for a general colour group.

To illustrate aspects of the K -functions for an arbitrary colour group we record them for the MOMg scheme in a linear covariant by way of an example. We have

$$\begin{aligned}
K_a^{\text{MOMg}}(a, \alpha) = & \left[12\zeta_3 C_F - \frac{21}{2} C_F \right] a \\
& + \left[\frac{7}{3} \psi'(\frac{1}{3}) C_F C_A \alpha^2 - \frac{512}{27} \zeta_3 \pi^2 N_f T_F C_F - \frac{321}{2} C_F C_A - \frac{224}{9} \psi'(\frac{1}{3}) N_f T_F C_F \right. \\
& - \frac{161}{27} \pi^2 C_F C_A - \frac{112}{3} \zeta_3 N_f T_F C_F - \frac{92}{9} \psi'(\frac{1}{3}) \zeta_3 C_F C_A - \frac{21}{2} C_F C_A \alpha^2 \\
& - \frac{21}{2} \psi'(\frac{1}{3}) C_F C_A \alpha - \frac{14}{9} \pi^2 C_F C_A \alpha^2 - \frac{8}{3} \psi'(\frac{1}{3}) \zeta_3 C_F C_A \alpha^2 + \frac{7}{4} C_F C_A \alpha^3 \\
& + \frac{16}{9} \zeta_3 \pi^2 C_F C_A \alpha^2 + \frac{63}{4} C_F C_A \alpha + \frac{158}{3} N_f T_F C_F + \frac{161}{18} \psi'(\frac{1}{3}) C_F C_A \\
& + \frac{184}{27} \zeta_3 \pi^2 C_F C_A + \frac{256}{9} \psi'(\frac{1}{3}) \zeta_3 N_f T_F C_F + \frac{356}{3} \zeta_3 C_F C_A + \frac{397}{6} C_F^2 \\
& + \frac{448}{27} \pi^2 N_f T_F C_F - 240 \zeta_5 C_F^2 - 18 \zeta_3 C_F C_A \alpha - 8 \zeta_3 \pi^2 C_F C_A \alpha \\
& - 2 \zeta_3 C_F C_A \alpha^3 + 7 \pi^2 C_F C_A \alpha + 12 \zeta_3 C_F C_A \alpha^2 + 12 \psi'(\frac{1}{3}) \zeta_3 C_F C_A \alpha \\
& \left. + 136 \zeta_3 C_F^2 \right] a^2 \\
& + \left[\frac{1}{48} \psi'''(\frac{1}{3}) \zeta_3 C_F C_A^2 \alpha^3 - \frac{5457913}{1152} C_F C_A^2 - \frac{336299}{648} \psi'(\frac{1}{3}) C_F C_A^2 \alpha \right. \\
& - \frac{177919}{2916} \psi'(\frac{1}{3}) \pi^2 C_F C_A^2 \alpha - \frac{87808}{729} \psi'(\frac{1}{3}) \zeta_3 \pi^2 N_f T_F C_F C_A \alpha \\
& - \frac{78529}{1296} \pi^2 C_F C_A^2 \alpha^2 - \frac{78454}{243} \zeta_3 \pi^2 C_F C_A^2 \alpha - \frac{76832}{2187} \pi^4 N_f T_F C_F C_A \alpha \\
& - \frac{66269}{384} C_F C_A^2 \alpha^2 - \frac{47950}{243} \pi^2 N_f T_F C_F C_A \alpha - \frac{35689}{54} \psi'(\frac{1}{3}) N_f T_F C_F C_A \\
& - \frac{33338}{2187} \zeta_3 \pi^4 C_F C_A^2 \alpha - \frac{28672}{243} \psi'(\frac{1}{3}) \zeta_3 \pi^2 N_f T_F^2 C_F - \frac{27181}{1458} \zeta_3 \pi^4 C_F C_A^2 \\
& - \frac{26776}{81} \zeta_3 \pi^2 N_f T_F C_F C_A - \frac{25921}{2592} \psi'(\frac{1}{3})^2 C_F C_A^2 - \frac{25088}{729} \pi^4 N_f T_F^2 C_F \\
& - \frac{25417}{486} \psi'(\frac{1}{3})^2 \zeta_3 C_F C_A^2 \alpha - \frac{24040}{81} \psi'(\frac{1}{3}) \zeta_3 N_f T_F C_F C_A \alpha \\
& - \frac{21805}{2592} \psi'(\frac{1}{3})^2 C_F C_A^2 \alpha^2 - \frac{19208}{243} \psi'(\frac{1}{3})^2 N_f T_F C_F C_A \alpha \\
& - \frac{18032}{243} \psi'(\frac{1}{3}) \pi^2 N_f T_F C_F C_A - \frac{14257}{54} \pi^2 C_F C_A^2 - \frac{11008}{27} \zeta_3 \pi^2 N_f T_F C_F^2 \\
& - \frac{10240}{729} \zeta_3 \pi^4 N_f T_F C_F C_A - \frac{9986}{27} \psi'(\frac{1}{3}) \zeta_3 C_F C_A^2 - \frac{9850}{9} \zeta_3 N_f T_F C_F C_A \\
& - \frac{9809}{18} \zeta_3 C_F C_A^2 \alpha - \frac{9487}{108} \psi'(\frac{1}{3}) \zeta_3 C_F C_A^2 \alpha^2 - \frac{9131}{108} \psi'(\frac{1}{3}) C_F^2 C_A \\
& - \frac{9040}{3} \zeta_5 C_F^2 C_A - \frac{8768}{81} \pi^2 N_f T_F^2 C_F - \frac{8000}{3} \zeta_5 N_f T_F C_F C_A \\
& - \frac{7775}{4} \zeta_3^2 C_F C_A^2 - \frac{6656}{81} \pi^2 N_f T_F C_F^2 - \frac{6272}{81} \psi'(\frac{1}{3})^2 N_f T_F^2 C_F \\
& - \frac{5152}{81} \psi'(\frac{1}{3})^2 \zeta_3 N_f T_F C_F C_A - \frac{4459}{1458} \pi^4 C_F C_A^2 \alpha^2 - \frac{4058}{9} N_f T_F^2 C_F \\
& \left. - \frac{3920}{9} N_f T_F C_F^2 - \frac{3703}{243} \psi'(\frac{1}{3}) \zeta_3 \pi^2 C_F C_A^2 - \frac{3115}{243} \psi'(\frac{1}{3}) \zeta_3 \pi^2 C_F C_A^2 \alpha^2 \right]
\end{aligned}$$

$$\begin{aligned}
& -\frac{2989}{384}\psi'''(\frac{1}{3})C_FC_A^2 - \frac{2816}{27}\psi'(\frac{1}{3})\zeta_3N_fT_FT_F^2C_F - \frac{2560}{3}\psi'(\frac{1}{3})\zeta_5N_fT_FT_F^2C_F \\
& -\frac{2197}{9}N_fT_FT_FC_FC_A\alpha - \frac{1840}{9}\zeta_5\pi^2C_F^2C_A - \frac{1792}{729}\zeta_3\pi^4N_fT_FT_FC_FC_A\alpha^2 \\
& -\frac{1783}{128}C_FC_A^2\alpha^4 - \frac{1568}{243}\psi'(\frac{1}{3})\pi^2N_fT_FT_FC_FC_A\alpha^2 - \frac{1564}{9}\psi'(\frac{1}{3})\zeta_3C_F^2C_A \\
& -\frac{1328}{81}\zeta_3\pi^2N_fT_FT_FC_FC_A\alpha^2 - \frac{1191}{8}C_F^2C_A\alpha - \frac{1115}{216}\pi^2C_FC_A^2\alpha^3 \\
& -\frac{763}{144}\pi^2C_FC_A^2\alpha^4 - \frac{643}{486}\zeta_3\pi^4C_FC_A^2\alpha^3 - \frac{641}{27}\psi'(\frac{1}{3})N_fT_FT_FC_FC_A\alpha^2 \\
& -\frac{539}{162}\psi'(\frac{1}{3})\pi^2C_FC_A^2\alpha^3 - \frac{448}{81}\psi'(\frac{1}{3})^2\zeta_3N_fT_FT_FC_FC_A\alpha^2 - \frac{397}{6}\pi^2C_F^2C_A\alpha \\
& -\frac{397}{18}\psi'(\frac{1}{3})C_F^2C_A\alpha^2 - \frac{397}{24}C_F^2C_A\alpha^3 - \frac{160}{3}\zeta_5\pi^2C_F^2C_A\alpha^2 \\
& -\frac{136}{3}\psi'(\frac{1}{3})\zeta_3C_F^2C_A\alpha^2 - \frac{125}{18}\psi'(\frac{1}{3})\zeta_3C_FC_A^2\alpha^3 - \frac{109}{12}\psi'(\frac{1}{3})\zeta_3C_FC_A^2\alpha^4 \\
& -\frac{77}{27}\psi'(\frac{1}{3})^2\zeta_3C_FC_A^2\alpha^3 - \frac{51}{8}\zeta_3C_FC_A^2\alpha^5 - \frac{49}{48}\psi'(\frac{1}{3})C_FC_A^2\alpha^5 \\
& -\frac{49}{72}\psi'(\frac{1}{3})^2C_FC_A^2\alpha^4 - \frac{49}{128}C_FC_A^2\alpha^6 - \frac{49}{162}\pi^4C_FC_A^2\alpha^4 \\
& -\frac{49}{192}\psi'''(\frac{1}{3})C_FC_A^2\alpha^2 - \frac{38}{3}\zeta_3N_fT_FT_FC_FC_A\alpha^2 - \frac{28}{27}\psi'(\frac{1}{3})\zeta_3\pi^2C_FC_A^2\alpha^4 \\
& -\frac{16}{3}\psi'''(\frac{1}{3})\zeta_3N_fT_FT_FC_FC_A - \frac{13}{8}\zeta_3C_FC_A^2\alpha^3 - \frac{7}{9}\zeta_3\pi^2C_FC_A^2\alpha^5 \\
& -\frac{7}{384}\psi'''(\frac{1}{3})C_FC_A^2\alpha^3 + \frac{3}{4}\zeta_3^2C_FC_A^2\alpha^2 + \frac{7}{6}\psi'(\frac{1}{3})\zeta_3C_FC_A^2\alpha^5 \\
& +\frac{7}{9}\psi'(\frac{1}{3})^2\zeta_3C_FC_A^2\alpha^4 + \frac{7}{16}\zeta_3C_FC_A^2\alpha^6 + \frac{7}{24}\psi'''(\frac{1}{3})\zeta_3C_FC_A^2\alpha^2 \\
& +\frac{14}{3}\psi'''(\frac{1}{3})N_fT_FT_FC_FC_A + \frac{21}{8}\psi'''(\frac{1}{3})C_FC_A^2\alpha + \frac{28}{81}\zeta_3\pi^4C_FC_A^2\alpha^4 \\
& +\frac{49}{54}\psi'(\frac{1}{3})\pi^2C_FC_A^2\alpha^4 + \frac{49}{72}\pi^2C_FC_A^2\alpha^5 + \frac{109}{18}\zeta_3\pi^2C_FC_A^2\alpha^4 \\
& +\frac{125}{27}\zeta_3\pi^2C_FC_A^2\alpha^3 + \frac{217}{96}C_FC_A^2\alpha^3 + \frac{253}{12}N_fT_FT_FC_FC_A\alpha^2 \\
& +\frac{272}{9}\zeta_3\pi^2C_F^2C_A\alpha^2 + \frac{289}{16}\zeta_3C_FC_A^2\alpha^4 + \frac{308}{81}\psi'(\frac{1}{3})\zeta_3\pi^2C_FC_A^2\alpha^3 \\
& +\frac{357}{64}C_FC_A^2\alpha^5 + \frac{392}{81}\psi'(\frac{1}{3})^2N_fT_FT_FC_FC_A\alpha^2 + \frac{397}{4}C_F^2C_A\alpha^2 \\
& +\frac{397}{4}\psi'(\frac{1}{3})C_F^2C_A\alpha + \frac{397}{27}\pi^2C_F^2C_A\alpha^2 + \frac{427}{48}\psi'''(\frac{1}{3})\zeta_3C_FC_A^2 \\
& +\frac{539}{216}\psi'(\frac{1}{3})^2C_FC_A^2\alpha^3 + \frac{664}{27}\psi'(\frac{1}{3})\zeta_3N_fT_FT_FC_FC_A\alpha^2 \\
& +\frac{763}{96}\psi'(\frac{1}{3})C_FC_A^2\alpha^4 + \frac{920}{3}\psi'(\frac{1}{3})\zeta_5C_F^2C_A + \frac{1115}{144}\psi'(\frac{1}{3})C_FC_A^2\alpha^3 \\
& +\frac{1282}{81}\pi^2N_fT_FT_FC_FC_A\alpha^2 + \frac{1448}{9}\zeta_3N_fT_FT_FC_FC_A\alpha + \frac{1552}{9}\zeta_3N_f^2T_FT_FC_F \\
& +\frac{1568}{729}\pi^4N_fT_FT_FC_FC_A\alpha^2 + \frac{1792}{243}\psi'(\frac{1}{3})\zeta_3\pi^2N_fT_FT_FC_FC_A\alpha^2 + \frac{2471}{12}C_F^3 \\
& +\frac{2548}{729}\zeta_3\pi^4C_FC_A^2\alpha^2 + \frac{3115}{324}\psi'(\frac{1}{3})^2\zeta_3C_FC_A^2\alpha^2 + \frac{3128}{27}\zeta_3\pi^2C_F^2C_A \\
& +\frac{3328}{27}\psi'(\frac{1}{3})N_fT_FT_FC_F + \frac{3703}{324}\psi'(\frac{1}{3})^2\zeta_3C_FC_A^2 + \frac{4384}{27}\psi'(\frac{1}{3})N_f^2T_FT_FC_F
\end{aligned}$$

$$\begin{aligned}
& + \frac{4501}{3888} \pi^4 C_F C_A^2 \alpha^3 + \frac{4508}{81} \psi'(\frac{1}{3})^2 N_f T_F C_F C_A + \frac{5120}{9} \zeta_5 \pi^2 N_f T_F C_F^2 \\
& + \frac{5504}{9} \psi'(\frac{1}{3}) \zeta_3 N_f T_F C_F^2 + \frac{5632}{81} \zeta_3 \pi^2 N_f^2 T_F^2 C_F \\
& + \frac{7168}{81} \psi'(\frac{1}{3})^2 \zeta_3 N_f^2 T_F^2 C_F + \frac{7594}{3} \zeta_3 C_F^2 C_A + \frac{8960}{729} \pi^4 N_f T_F C_F C_A \\
& + \frac{9131}{162} \pi^2 C_F^2 C_A + \frac{9487}{162} \zeta_3 \pi^2 C_F C_A^2 \alpha^2 + \frac{10711}{96} \zeta_3 C_F C_A^2 \alpha^2 \\
& + \frac{11900}{3} \zeta_5 C_F C_A^2 + \frac{13388}{27} \psi'(\frac{1}{3}) \zeta_3 N_f T_F C_F C_A + \frac{14257}{36} \psi'(\frac{1}{3}) C_F C_A^2 \\
& + \frac{19972}{81} \zeta_3 \pi^2 C_F C_A^2 + \frac{20608}{243} \psi'(\frac{1}{3}) \zeta_3 \pi^2 N_f T_F C_F C_A \\
& + \frac{21805}{1944} \psi'(\frac{1}{3}) \pi^2 C_F C_A^2 \alpha^2 + \frac{21952}{243} \psi'(\frac{1}{3})^2 \zeta_3 N_f T_F C_F C_A \alpha \\
& + \frac{23975}{81} \psi'(\frac{1}{3}) N_f T_F C_F C_A \alpha + \frac{25088}{243} \psi'(\frac{1}{3}) \pi^2 N_f^2 T_F^2 C_F \\
& + \frac{25921}{1944} \psi'(\frac{1}{3}) \pi^2 C_F C_A^2 + \frac{28672}{729} \zeta_3 \pi^4 N_f^2 T_F^2 C_F + \frac{35689}{81} \pi^2 N_f T_F C_F C_A \\
& + \frac{39227}{81} \psi'(\frac{1}{3}) \zeta_3 C_F C_A^2 \alpha + \frac{47353}{36} C_F^2 C_A + \frac{48080}{243} \zeta_3 \pi^2 N_f T_F C_F C_A \alpha \\
& + \frac{50834}{729} \psi'(\frac{1}{3}) \zeta_3 \pi^2 C_F C_A^2 \alpha + \frac{76832}{729} \psi'(\frac{1}{3}) \pi^2 N_f T_F C_F C_A \alpha \\
& + \frac{78529}{864} \psi'(\frac{1}{3}) C_F C_A^2 \alpha^2 + \frac{87808}{2187} \zeta_3 \pi^4 N_f T_F C_F C_A \alpha \\
& + \frac{110051}{36} N_f T_F C_F C_A + \frac{116683}{8748} \pi^4 C_F C_A^2 \alpha + \frac{177919}{3888} \psi'(\frac{1}{3})^2 C_F C_A^2 \alpha \\
& + \frac{190267}{11664} \pi^4 C_F C_A^2 + \frac{327281}{576} C_F C_A^2 \alpha + \frac{336299}{972} \pi^2 C_F C_A^2 \alpha \\
& + \frac{741599}{288} \zeta_3 C_F C_A^2 - 5720 \zeta_5 C_F^3 - 1464 \zeta_3 N_f T_F C_F^2 - 840 \zeta_7 C_F^2 C_A \\
& - 360 \zeta_5 C_F^2 C_A \alpha^2 - 360 \psi'(\frac{1}{3}) \zeta_5 C_F^2 C_A \alpha - 306 \zeta_3 C_F^2 C_A \alpha \\
& - 136 \zeta_3 \pi^2 C_F^2 C_A \alpha - 34 \zeta_3 C_F^2 C_A \alpha^3 - 3 \zeta_3^2 C_F C_A^2 \alpha^3 - 3 \psi'''(\frac{1}{3}) \zeta_3 C_F C_A^2 \alpha \\
& + 60 \zeta_5 C_F^2 C_A \alpha^3 + 80 \psi'(\frac{1}{3}) \zeta_5 C_F^2 C_A \alpha^2 + 171 \zeta_3^2 C_F C_A^2 \alpha + 204 \zeta_3 C_F^2 C_A \alpha^2 \\
& + 204 \psi'(\frac{1}{3}) \zeta_3 C_F^2 C_A \alpha + 240 \zeta_5 \pi^2 C_F^2 C_A \alpha + 320 \zeta_5 N_f^2 T_F^2 C_F + 488 \zeta_3 C_F^3 \\
& + 540 \zeta_5 C_F^2 C_A \alpha + 640 \zeta_3^2 N_f T_F C_F C_A + 2080 \zeta_5 N_f T_F C_F^2 \\
& + 5040 \zeta_7 C_F^3 \Big] a^3 + O(a^4) \tag{A.1}
\end{aligned}$$

and

$$\begin{aligned}
K_\alpha^{\text{MOMg}}(a, \alpha) = & \left[12 \zeta_3 C_F C_A \alpha - \frac{21}{2} C_F C_A \alpha - \frac{21}{4} \psi'(\frac{1}{3}) C_F C_A - \frac{14}{9} \pi^2 C_F C_A \alpha \right. \\
& - \frac{8}{3} \psi'(\frac{1}{3}) \zeta_3 C_F C_A \alpha + \frac{7}{2} \pi^2 C_F C_A + \frac{7}{3} \psi'(\frac{1}{3}) C_F C_A \alpha + \frac{16}{9} \zeta_3 \pi^2 C_F C_A \alpha \\
& + \frac{21}{8} C_F C_A \alpha^2 + \frac{63}{8} C_F C_A - 9 \zeta_3 C_F C_A - 4 \zeta_3 \pi^2 C_F C_A - 3 \zeta_3 C_F C_A \alpha^2 \\
& \left. + 6 \psi'(\frac{1}{3}) \zeta_3 C_F C_A \right] a^2 \\
& + \left[\frac{1}{2} \zeta_3^2 C_F C_A^2 \alpha - \frac{548047}{1296} \psi'(\frac{1}{3}) C_F C_A^2 - \frac{177037}{2916} \psi'(\frac{1}{3}) \pi^2 C_F C_A^2 \right. \\
& - \frac{69841}{243} \zeta_3 \pi^2 C_F C_A^2 - \frac{44750}{2187} \zeta_3 \pi^4 C_F C_A^2 \\
& \left. - \frac{39424}{729} \psi'(\frac{1}{3}) \zeta_3 \pi^2 N_f T_F C_F C_A - \frac{34496}{2187} \pi^4 N_f T_F C_F C_A \right]
\end{aligned}$$

$$\begin{aligned}
& -\frac{28204}{243}\pi^2 N_f T_F C_F C_A - \frac{25291}{486}\psi'(\frac{1}{3})^2 \zeta_3 C_F C_A^2 \\
& -\frac{20231}{432}\psi'(\frac{1}{3})C_F C_A^2 \alpha - \frac{14608}{81}\psi'(\frac{1}{3})\zeta_3 N_f T_F C_F C_A - \frac{11221}{36}\zeta_3 C_F C_A^2 \\
& -\frac{8624}{243}\psi'(\frac{1}{3})^2 N_f T_F C_F C_A - \frac{3323}{192}C_F C_A^2 \alpha - \frac{2783}{18}N_f T_F C_F C_A \\
& -\frac{2009}{108}\psi'(\frac{1}{3})\pi^2 C_F C_A^2 \alpha - \frac{971}{27}\zeta_3 \pi^2 C_F C_A^2 \alpha - \frac{777}{32}C_F C_A^2 \alpha^3 \\
& -\frac{616}{81}\zeta_3 \pi^4 C_F C_A^2 \alpha - \frac{397}{8}C_F^2 C_A - \frac{397}{18}\pi^2 C_F^2 C_A - \frac{397}{24}C_F^2 C_A \alpha^2 \\
& -\frac{397}{27}\psi'(\frac{1}{3})C_F^2 C_A \alpha - \frac{385}{36}\pi^2 C_F C_A^2 \alpha^3 - \frac{320}{9}\zeta_5 \pi^2 C_F^2 C_A \alpha \\
& -\frac{287}{18}\psi'(\frac{1}{3})^2 \zeta_3 C_F C_A^2 \alpha - \frac{272}{9}\psi'(\frac{1}{3})\zeta_3 C_F^2 C_A \alpha - \frac{245}{144}\psi'(\frac{1}{3})C_F C_A^2 \alpha^4 \\
& -\frac{136}{3}\zeta_3 \pi^2 C_F^2 C_A - \frac{112}{81}\psi'(\frac{1}{3})\zeta_3 \pi^2 C_F C_A^2 \alpha^3 - \frac{98}{243}\pi^4 C_F C_A^2 \alpha^3 \\
& -\frac{85}{8}\zeta_3 C_F C_A^2 \alpha^4 - \frac{55}{3}\psi'(\frac{1}{3})\zeta_3 C_F C_A^2 \alpha^3 - \frac{49}{6}\psi'(\frac{1}{3})\pi^2 C_F C_A^2 \alpha^2 \\
& -\frac{49}{54}\psi'(\frac{1}{3})^2 C_F C_A^2 \alpha^3 - \frac{49}{64}C_F C_A^2 \alpha^5 - \frac{49}{288}\psi'''(\frac{1}{3})C_F C_A^2 \alpha \\
& -\frac{35}{24}\pi^2 C_F C_A^2 \alpha^2 - \frac{35}{27}\zeta_3 \pi^2 C_F C_A^2 \alpha^4 - \frac{23}{8}\zeta_3 C_F C_A^2 \alpha^2 \\
& -\frac{19}{6}\zeta_3 \pi^4 C_F C_A^2 \alpha^2 - \frac{7}{384}\psi'''(\frac{1}{3})C_F C_A^2 \alpha^2 - \frac{5}{2}\psi'(\frac{1}{3})\zeta_3 C_F C_A^2 \alpha^2 \\
& +\frac{1}{48}\psi'''(\frac{1}{3})\zeta_3 C_F C_A^2 \alpha^2 + \frac{5}{3}\zeta_3 \pi^2 C_F C_A^2 \alpha^2 + \frac{7}{8}\zeta_3 C_F C_A^2 \alpha^5 \\
& +\frac{7}{8}\psi'''(\frac{1}{3})C_F C_A^2 + \frac{7}{36}\psi'''(\frac{1}{3})\zeta_3 C_F C_A^2 \alpha + \frac{28}{3}\psi'(\frac{1}{3})\zeta_3 \pi^2 C_F C_A^2 \alpha^2 \\
& +\frac{28}{27}\psi'(\frac{1}{3})^2 \zeta_3 C_F C_A^2 \alpha^3 + \frac{35}{16}\psi'(\frac{1}{3})C_F C_A^2 \alpha^2 + \frac{35}{18}\psi'(\frac{1}{3})\zeta_3 C_F C_A^2 \alpha^4 \\
& +\frac{49}{8}\psi'(\frac{1}{3})^2 C_F C_A^2 \alpha^2 + \frac{98}{81}\psi'(\frac{1}{3})\pi^2 C_F C_A^2 \alpha^3 + \frac{110}{9}\zeta_3 \pi^2 C_F C_A^2 \alpha^3 \\
& +\frac{111}{4}\zeta_3 C_F C_A^2 \alpha^3 + \frac{112}{243}\zeta_3 \pi^4 C_F C_A^2 \alpha^3 + \frac{133}{48}\pi^4 C_F C_A^2 \alpha^2 \\
& +\frac{160}{3}\psi'(\frac{1}{3})\zeta_5 C_F^2 C_A \alpha + \frac{245}{216}\pi^2 C_F C_A^2 \alpha^4 + \frac{251}{48}C_F C_A^2 \alpha^2 \\
& +\frac{279}{16}\zeta_3 C_F C_A^2 \alpha + \frac{385}{24}\psi'(\frac{1}{3})C_F C_A^2 \alpha^3 + \frac{397}{6}C_F^2 C_A \alpha \\
& +\frac{397}{12}\psi'(\frac{1}{3})C_F^2 C_A + \frac{539}{81}\pi^4 C_F C_A^2 \alpha + \frac{544}{27}\zeta_3 \pi^2 C_F^2 C_A \alpha \\
& +\frac{574}{27}\psi'(\frac{1}{3})\zeta_3 \pi^2 C_F C_A^2 \alpha + \frac{595}{64}C_F C_A^2 \alpha^4 + \frac{794}{81}\pi^2 C_F^2 C_A \alpha \\
& +\frac{836}{9}\zeta_3 N_f T_F C_F C_A + \frac{971}{18}\psi'(\frac{1}{3})\zeta_3 C_F C_A^2 \alpha + \frac{2009}{144}\psi'(\frac{1}{3})^2 C_F C_A^2 \alpha \\
& +\frac{9856}{243}\psi'(\frac{1}{3})^2 \zeta_3 N_f T_F C_F C_A + \frac{14102}{81}\psi'(\frac{1}{3})N_f T_F C_F C_A \\
& +\frac{20231}{648}\pi^2 C_F C_A^2 \alpha + \frac{29216}{243}\zeta_3 \pi^2 N_f T_F C_F C_A \\
& +\frac{34496}{729}\psi'(\frac{1}{3})\pi^2 N_f T_F C_F C_A + \frac{39424}{2187}\zeta_3 \pi^4 N_f T_F C_F C_A \\
& +\frac{50582}{729}\psi'(\frac{1}{3})\zeta_3 \pi^2 C_F C_A^2 \alpha + \frac{69841}{162}\psi'(\frac{1}{3})\zeta_3 C_F C_A^2 \alpha + \frac{156625}{8748}\pi^4 C_F C_A^2
\end{aligned}$$

$$\begin{aligned}
& + \frac{177037}{3888} \psi' \left(\frac{1}{3}\right)^2 C_F C_A^2 + \frac{248837}{576} C_F C_A^2 + \frac{548047}{1944} \pi^2 C_F C_A^2 \\
& - 240 \zeta_5 C_F^2 C_A \alpha - 120 \psi' \left(\frac{1}{3}\right) \zeta_5 C_F^2 C_A - 102 \zeta_3 C_F^2 C_A - 34 \zeta_3 C_F^2 C_A \alpha^2 \\
& - 7 \psi' \left(\frac{1}{3}\right)^2 \zeta_3 C_F C_A^2 \alpha^2 - 3 \zeta_3^2 C_F C_A^2 \alpha^2 - \psi''' \left(\frac{1}{3}\right) \zeta_3 C_F C_A^2 + 57 \zeta_3^2 C_F C_A^2 \\
& + 60 \zeta_5 C_F^2 C_A \alpha^2 + 68 \psi' \left(\frac{1}{3}\right) \zeta_3 C_F^2 C_A + 80 \zeta_5 \pi^2 C_F^2 C_A + 136 \zeta_3 C_F^2 C_A \alpha \\
& + 180 \zeta_5 C_F^2 C_A \Big] a^3 + O(a^4) . \tag{A.2}
\end{aligned}$$

The corresponding MAG expression depends on N_A^d and N_A^o and is available in the data file.

B $\widetilde{\text{RIc}}'$ scheme β -function.

We record the $SU(3)$ expressions for the two key renormalization group equations that are central to the Crewther relation in the $\widetilde{\text{RIc}}'$ scheme. The five loop β -function is

$$\begin{aligned}
\beta^{\widetilde{\text{RIc}}'}(a, \alpha) \Big|^{SU(3)} &= \left[\frac{2}{3} N_f - 11 \right] a^2 + \left[\frac{38}{3} N_f + \frac{39}{2} \alpha - \frac{9}{2} \alpha^2 - 102 - 2 N_f \alpha \right] a^3 \\
&+ \left[\frac{21}{4} N_f \alpha^2 + \frac{27}{8} \alpha^3 - \frac{2857}{2} - \frac{2655}{16} \alpha^2 - \frac{325}{54} N_f^2 - \frac{137}{2} N_f \alpha - \frac{81}{8} \zeta_3 \alpha^3 \right. \\
&\quad \left. - \frac{9}{4} \zeta_3 N_f \alpha^2 + \frac{351}{16} \zeta_3 \alpha^2 + \frac{729}{16} \zeta_3 \alpha + \frac{4599}{16} \alpha + \frac{5033}{18} N_f \right] a^4 \\
&+ \left[\frac{29}{2} \zeta_3 N_f^2 \alpha + \frac{243}{8} \zeta_3 N_f \alpha^3 - \frac{1172325}{128} \alpha - \frac{717363}{128} \alpha^2 - \frac{149753}{6} \right. \\
&\quad \left. - \frac{50065}{162} N_f^2 - \frac{6472}{81} \zeta_3 N_f^2 - \frac{6321}{16} N_f \alpha - \frac{2025}{16} \zeta_5 \alpha^4 - \frac{1725}{4} \zeta_3 N_f \alpha \right. \\
&\quad \left. - \frac{1323}{2} \zeta_3 \alpha^3 - \frac{1093}{729} N_f^3 - \frac{621}{16} \zeta_3 \alpha^2 - \frac{567}{8} \alpha^4 - \frac{243}{2} \zeta_3 N_f \alpha^2 \right. \\
&\quad \left. - \frac{177}{8} N_f \alpha^3 - \frac{75}{4} \zeta_5 N_f \alpha^3 - \frac{45}{2} \zeta_5 N_f \alpha - \frac{13}{8} N_f^2 \alpha + \frac{3465}{16} \zeta_5 \alpha^3 \right. \\
&\quad \left. + \frac{3645}{4} \zeta_5 \alpha^2 + \frac{4131}{32} \zeta_3 \alpha^4 + \frac{4185}{8} \zeta_5 \alpha + \frac{6508}{27} \zeta_3 N_f + \frac{17169}{32} N_f \alpha^2 \right. \\
&\quad \left. + \frac{33705}{64} \alpha^3 + \frac{197307}{32} \zeta_3 \alpha + \frac{1078361}{162} N_f - 3564 \zeta_3 \right] a^5 \\
&+ \left[\frac{152}{81} \zeta_3 N_f^4 + \frac{159}{4} \zeta_3^2 N_f^2 \alpha - \frac{15970474577}{9216} \alpha - \frac{372908785}{3072} \alpha^2 \right. \\
&\quad \left. - \frac{207136629}{8192} \zeta_7 \alpha^2 - \frac{32567937}{512} \zeta_3 \alpha^2 - \frac{25960913}{1944} N_f^2 \right. \\
&\quad \left. - \frac{14959621}{144} \zeta_3 N_f \alpha - \frac{13960107}{8192} \zeta_7 \alpha^3 - \frac{9452457}{256} \zeta_3^2 \alpha - \frac{8157455}{16} \right. \\
&\quad \left. - \frac{5668191}{1024} \alpha^4 - \frac{3852207}{512} \zeta_5 \alpha^4 - \frac{3388995}{128} \zeta_3 \alpha^3 - \frac{3383613}{256} \zeta_4 \alpha \right. \\
&\quad \left. - \frac{2636597}{216} N_f^2 \alpha - \frac{1358995}{27} \zeta_5 N_f - \frac{1209579}{1024} \zeta_7 N_f \alpha - \frac{941891}{256} \zeta_5 N_f \alpha^2 \right. \\
&\quad \left. - \frac{886581}{128} \zeta_5 N_f \alpha - \frac{793143}{512} \zeta_5 \alpha^3 - \frac{698531}{81} \zeta_3 N_f^2 - \frac{621885}{2} \zeta_3 \right. \\
&\quad \left. - \frac{580203}{256} \zeta_4 \alpha^2 - \frac{573237}{2048} \zeta_7 \alpha^5 - \frac{476263}{1728} N_f^2 \alpha^2 - \frac{404607}{128} N_f \alpha^3 \right. \\
&\quad \left. - \frac{191727}{256} \zeta_4 \alpha^3 - \frac{151875}{128} \zeta_6 N_f \alpha - \frac{92097}{256} \zeta_3 \alpha^5 - \frac{63693}{2048} \zeta_7 N_f \alpha^4 \right. \\
&\quad \left. - \frac{43011}{256} \zeta_3^2 \alpha^4 - \frac{40365}{64} \zeta_3^2 N_f \alpha^2 - \frac{35559}{256} \zeta_4 \alpha^4 - \frac{34845}{32} \zeta_5 N_f \alpha^3 \right]
\end{aligned}$$

$$\begin{aligned}
& -\frac{33935}{6}\zeta_4 N_f - \frac{16235}{96}\zeta_3 N_f^2 \alpha^2 - \frac{14175}{256}\zeta_6 \alpha^5 - \frac{14175}{256}\zeta_6 N_f \alpha^2 \\
& -\frac{5895}{32}\zeta_3 N_f \alpha^4 - \frac{3105}{256}\zeta_5 \alpha^5 - \frac{1809}{16}\zeta_3^2 N_f \alpha^3 - \frac{1618}{27}\zeta_4 N_f^3 \\
& -\frac{1575}{256}\zeta_6 N_f \alpha^4 - \frac{1353}{8}\zeta_4 N_f^2 \alpha - \frac{1205}{2916}N_f^4 - \frac{460}{9}\zeta_5 N_f^3 - \frac{382}{9}\zeta_3 N_f^3 \alpha \\
& -\frac{153}{16}\zeta_4 N_f^2 \alpha^2 - \frac{100}{3}\zeta_5 N_f^3 \alpha + \frac{243}{32}\zeta_4 N_f \alpha^3 + \frac{459}{128}\zeta_4 N_f \alpha^4 \\
& +\frac{1701}{64}\zeta_3^2 N_f \alpha^4 + \frac{4131}{128}\zeta_4 \alpha^5 + \frac{5931}{8}\zeta_3^2 N_f \alpha + \frac{6025}{48}\zeta_5 N_f^2 \alpha^2 \\
& +\frac{6093}{64}N_f \alpha^4 + \frac{10526}{9}\zeta_4 N_f^2 + \frac{10935}{64}\zeta_3^2 \alpha^5 + \frac{13273}{16}\zeta_5 N_f^2 \alpha + \frac{16605}{128}\alpha^5 \\
& +\frac{19465}{81}N_f^3 \alpha + \frac{27285}{256}\zeta_5 N_f \alpha^4 + \frac{46143}{128}\zeta_4 N_f \alpha^2 + \frac{48722}{243}\zeta_3 N_f^3 \\
& +\frac{88209}{2}\zeta_4 + \frac{126891}{64}\zeta_4 N_f \alpha + \frac{188529}{64}\zeta_3 N_f \alpha^3 + \frac{207225}{1024}\zeta_6 \alpha^4 \\
& +\frac{324577}{128}\zeta_3 N_f \alpha^2 + \frac{381760}{81}\zeta_5 N_f^2 + \frac{502929}{256}\zeta_3^2 \alpha^3 + \frac{630559}{5832}N_f^3 \\
& +\frac{948933}{512}\zeta_5 \alpha + \frac{1075275}{1024}\zeta_6 \alpha^3 + \frac{1816857}{2048}\zeta_7 N_f \alpha^2 + \frac{2023353}{256}\zeta_3 \alpha^4 \\
& +\frac{2116639}{432}\zeta_3 N_f^2 \alpha + \frac{2480247}{256}\zeta_3^2 \alpha^2 + \frac{3067875}{1024}\zeta_6 \alpha^2 + \frac{3572667}{8192}\zeta_7 \alpha^4 \\
& +\frac{4811164}{81}\zeta_3 N_f + \frac{24148125}{1024}\zeta_6 \alpha + \frac{24208323}{512}\zeta_5 \alpha^2 + \frac{33423111}{1024}\alpha^3 \\
& +\frac{51960893}{2304}N_f \alpha^2 + \frac{192323061}{8192}\zeta_7 \alpha + \frac{336460813}{1944}N_f + \frac{435326091}{512}\zeta_3 \alpha \\
& +\frac{1723650635}{6912}N_f \alpha + 6\zeta_4 N_f^3 \alpha + 15\zeta_3^2 N_f^2 \alpha^2 + 288090\zeta_5 \Big] a^6 \\
& + O(a^7)
\end{aligned} \tag{B.1}$$

while the gauge parameter anomalous dimension also in the linear covariant gauge is

$$\begin{aligned}
\widetilde{\gamma}_\alpha^{\text{RIc}'}(a, \alpha) \Big|^{SU(3)} &= \left[\frac{13}{2} - \frac{3}{2}\alpha - \frac{2}{3}N_f \right] a + \left[\frac{9}{4}\alpha^2 + \frac{531}{8} + 2N_f \alpha - \frac{255}{8}\alpha - \frac{61}{6}N_f \right] a^2 \\
&+ \left[\frac{9}{4}\zeta_3 N_f \alpha^2 + \frac{81}{16}\zeta_3 \alpha^3 + \frac{215}{27}N_f^2 - \frac{27315}{32}\alpha - \frac{8155}{36}N_f - \frac{675}{16}\zeta_3 \alpha^2 \right. \\
&- \frac{243}{16}\zeta_3 - \frac{135}{32}\alpha^3 - \frac{27}{4}\zeta_3 N_f \alpha - \frac{21}{4}N_f \alpha^2 + \frac{409}{4}N_f \alpha + \frac{729}{16}\zeta_3 \alpha \\
&\left. + \frac{5445}{32}\alpha^2 + \frac{29895}{32} + 33\zeta_3 N_f \right] a^3 \\
&+ \left[\frac{17}{2}\zeta_3 N_f^2 \alpha + \frac{45}{4}\zeta_4 N_f \alpha + \frac{75}{4}\zeta_5 N_f \alpha^3 - \frac{23350603}{5184}N_f - \frac{21210899}{768}\alpha \right. \\
&- \frac{1012023}{256}\zeta_3 - \frac{184473}{64}\zeta_3 \alpha^2 - \frac{177021}{256}\alpha^3 - \frac{46755}{128}\zeta_5 \alpha^3 - \frac{44331}{64}N_f \alpha^2 \\
&- \frac{13781}{108}N_f^2 \alpha - \frac{13149}{256}\zeta_3 \alpha^4 - \frac{12549}{16}\zeta_3 N_f \alpha - \frac{8955}{16}\zeta_4 N_f - \frac{3355}{2}\zeta_5 N_f \\
&- \frac{255}{4}\zeta_5 N_f \alpha^2 - \frac{243}{8}\zeta_3 N_f \alpha^3 - \frac{45}{2}\zeta_5 N_f \alpha - \frac{27}{16}\zeta_4 N_f \alpha^2 - \frac{8}{3}\zeta_3 N_f^3 \\
&+ \frac{177}{8}N_f \alpha^3 + \frac{243}{64}\zeta_4 \alpha^3 + \frac{405}{8}\zeta_4 \alpha^2 + \frac{1809}{64}\alpha^4 + \frac{2017}{81}\zeta_3 N_f^2 \\
&\left. + \frac{4017}{16}\zeta_3 N_f \alpha^2 + \frac{4427}{1458}N_f^3 + \frac{4455}{128}\zeta_5 \alpha^4 + \frac{8019}{32}\zeta_4 + \frac{16443}{64}\zeta_4 \alpha \right]
\end{aligned}$$

$$\begin{aligned}
& + \frac{40905}{4} \zeta_5 + \frac{43033}{162} N_f^2 + \frac{68625}{128} \zeta_5 \alpha^2 + \frac{87831}{128} \zeta_3 \alpha^3 + \frac{198315}{128} \zeta_5 \alpha \\
& + \frac{311301}{128} \zeta_3 \alpha + \frac{387649}{216} \zeta_3 N_f + \frac{712315}{144} N_f \alpha + \frac{2137221}{256} \alpha^2 \\
& + \left. \frac{10596127}{768} + 33 \zeta_4 N_f^2 \right] a^4 \\
& + \left[\frac{3248889939}{4096} \zeta_5 - \frac{40020851929}{36864} \alpha - \frac{15059970043}{124416} N_f \right. \\
& - \frac{9151899939}{16384} \zeta_7 - \frac{514366083}{16384} \zeta_7 \alpha - \frac{322439503}{4608} \zeta_3 N_f \alpha \\
& - \frac{281043751}{4608} N_f \alpha^2 - \frac{169349259}{1024} \zeta_3 \alpha^2 - \frac{110282119}{864} \zeta_5 N_f \\
& - \frac{70747047}{8192} \zeta_7 \alpha^2 - \frac{65529567}{2048} \zeta_3^2 - \frac{46637683}{3456} N_f^2 \alpha - \frac{45332901}{2048} \zeta_5 \alpha^3 \\
& - \frac{44784819}{2048} \zeta_3^2 \alpha - \frac{32357475}{2048} \zeta_6 \alpha - \frac{31012901}{768} \zeta_4 N_f - \frac{17999415}{16384} \zeta_7 \alpha^4 \\
& - \frac{12215151}{2048} \zeta_3 \alpha^4 - \frac{3696633}{64} \alpha^3 - \frac{3378739}{1296} \zeta_5 N_f^2 - \frac{2259077}{11664} N_f^3 \\
& - \frac{2249775}{8} \zeta_6 - \frac{1905231}{512} \zeta_3 N_f \alpha^3 - \frac{1633959}{1024} \zeta_3^2 \alpha^3 - \frac{987899}{256} \zeta_5 N_f \alpha^2 \\
& - \frac{726489}{2048} \zeta_4 \alpha^3 - \frac{393579}{4096} \alpha^5 - \frac{370305}{4096} \zeta_5 \alpha^5 - \frac{257775}{512} \zeta_6 N_f \alpha \\
& - \frac{231399}{2048} \zeta_7 N_f \alpha^3 - \frac{123525}{1024} \zeta_6 \alpha^2 - \frac{105921}{128} \zeta_3^2 N_f \alpha^2 - \frac{94527}{2048} \zeta_3^2 \alpha^5 \\
& - \frac{80771}{243} \zeta_3 N_f^3 - \frac{45441}{4096} \zeta_4 \alpha^5 - \frac{40139}{96} \zeta_3 N_f^2 \alpha^2 - \frac{28575}{512} \zeta_6 N_f \alpha^3 \\
& - \frac{27285}{256} \zeta_5 N_f \alpha^4 - \frac{16775}{6} \zeta_6 N_f^2 - \frac{6093}{64} N_f \alpha^4 - \frac{2659}{3} \zeta_3^2 N_f^2 \\
& - \frac{1701}{64} \zeta_3^2 N_f \alpha^4 - \frac{1317}{4} \zeta_4 N_f^2 \alpha - \frac{729}{2} \zeta_4 N_f \alpha^2 - \frac{675}{8} \zeta_6 N_f \alpha^2 \\
& - \frac{459}{128} \zeta_4 N_f \alpha^4 - \frac{50}{3} \zeta_5 N_f^3 \alpha - \frac{32}{81} \zeta_3 N_f^4 - \frac{8}{3} \zeta_4 N_f^4 + \frac{99}{16} \zeta_4 N_f^2 \alpha^2 \\
& + \frac{139}{9} \zeta_3 N_f^3 \alpha + \frac{159}{8} \zeta_3^2 N_f^2 \alpha + \frac{1321}{27} \zeta_4 N_f^3 + \frac{1575}{256} \zeta_6 N_f \alpha^4 + \frac{2164}{9} \zeta_5 N_f^3 \\
& + \frac{2299}{48} \zeta_5 N_f^2 \alpha^2 + \frac{4715}{2916} N_f^4 + \frac{5895}{32} \zeta_3 N_f \alpha^4 + \frac{7425}{1024} \zeta_6 \alpha^4 + \frac{11045}{144} N_f^3 \alpha \\
& + \frac{14851}{32} \zeta_5 N_f^2 \alpha + \frac{16119}{256} \zeta_4 \alpha^4 + \frac{17469}{512} \zeta_4 N_f \alpha^3 + \frac{28305}{256} \zeta_3^2 N_f \alpha^3 \\
& + \frac{63693}{2048} \zeta_7 N_f \alpha^4 + \frac{65817}{16} N_f \alpha^3 + \frac{70875}{2048} \zeta_6 \alpha^5 + \frac{86103}{128} \zeta_4 \alpha^2 \\
& + \frac{114075}{128} \zeta_6 \alpha^3 + \frac{163139}{72} \zeta_4 N_f^2 + \frac{187509}{2048} \zeta_7 N_f \alpha + \frac{440809}{64} \zeta_3^2 N_f \\
& + \frac{543079}{432} N_f^2 \alpha^2 + \frac{638961}{256} \zeta_3^2 N_f \alpha + \frac{820773}{4096} \zeta_3 \alpha^5 + \frac{863541}{16384} \zeta_7 \alpha^5 \\
& + \frac{996423}{512} \zeta_5 N_f \alpha^3 + \frac{1164213}{2048} \zeta_3^2 \alpha^4 + \frac{1477789}{864} \zeta_3 N_f^2 \alpha + \frac{1816857}{2048} \zeta_7 N_f \alpha^2 \\
& + \frac{2670405}{512} \zeta_4 N_f \alpha + \frac{2988559}{128} \zeta_3 N_f \alpha^2 + \frac{3662827}{48} \zeta_7 N_f + \frac{4089225}{512} \zeta_5 N_f \alpha \\
& + \frac{5660469}{1024} \zeta_3^2 \alpha^2 + \frac{7740549}{2048} \alpha^4 + \frac{10890829}{2592} \zeta_3 N_f^2 + \frac{17186355}{4096} \zeta_5 \alpha^4 \\
& + \frac{24104115}{8192} \zeta_7 \alpha^3 + \frac{25440057}{2048} \zeta_3 + \frac{32297109}{2048} \zeta_5 \alpha^2 + \frac{39034359}{512} \zeta_4
\end{aligned}$$

$$\begin{aligned}
& + \frac{48001201}{3888} N_f^2 + \frac{61806555}{4096} \zeta_5 \alpha + \frac{77357835}{4096} \zeta_4 \alpha + \frac{97546581}{2048} \zeta_3 \alpha^3 \\
& + \frac{302449841}{20736} \zeta_3 N_f + \frac{612462769}{2048} + \frac{676597463}{1536} \alpha^2 + \frac{877406273}{3456} N_f \alpha \\
& + \frac{1067166537}{4096} \zeta_3 \alpha - 3 \zeta_4 N_f^3 \alpha + 15 \zeta_3^2 N_f^2 \alpha^2 + 63175 \zeta_6 N_f \Big] a^5 \\
& + O(a^6) .
\end{aligned} \tag{B.2}$$

The five loop expressions for these as well as the other renormalization group equations in an arbitrary Lie group are provided in the associated data file.

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