TENSOR ANALYSIS OF ELECTRICAL MACHINES AND POWER SYSTEMS

BY

J. W. LYNN

Tensor Analysis of Electrical Machines and Power Systems

Summary

J. W. Lynn

A unified approach to the analysis of a wide range of electrical machines is investigated. The method makes use of the transformation laws and properties of invariance of tensor equations. The equations of a primitive machine are set up, and these are transformed to give those of the required machine. The machine power input and torque are invariant under the transformation; ensuring that the identity of the machine being considered is not lost in the new equations. This method of analysis was first suggested by Kron, and has been fairly extensively applied by machine engineers. The papers now presented investigate the tensor character of various groups of machine parameters.

It is found that when the equations are set up in tensor form, the groups forming tensors have all got some physical significance. The equations in this form are independent of the reference axes chosen. In hunting analysis the tensor groups lead to equivalent circuits which give a somewhat clearer picture than previous circuits of the machines represented.

The idea of analysing a complex power system in the same way, namely by considering it as a transformed primitive system, was applied by Kron in the U.S.A., in his papers on Power System Loss Analysis. The Monograph 294S presented here gives an application of this work to a section of the British Network with a study of the effects of simplifying assumptions.

An indication is given of the way in which dynamical tensor equations (in Lagrangian form) can be associated with the electromagnetic field equations applied to machines. This approach appears to converge with recent work on the stability of high temperature arcs in thermonuclear reactors and with investigations of magnetohydrodynamic phenomena.

Tensor Analysis of Electrical Machines and Power Systems

1. Polyphase Systems and Transformations

Electrical power is generated on a large scale and transmitted by means of three-phase alternating current systems. This has meant that on a fairly extensive network it has been difficult to predict how the system would behave under conditions of faults or unbalanced loads, or following transient disturbances. The performance of the system generating machinery and industrial motor drives under such conditions has always received a great deal of attention.

In the solution of such problems it has been amply demonstrated that calculations are often greatly simplified by the use of substitute variables in place of the actual coil currents and voltages. Stationary networks are often analysed by means of three-phase symmetrical components. Rotating machinery is studied by using reference axes selected to reduce the number of variables in the machine equations. Electrical power engineers more than others have thus been accustomed to this principle of transforming actual quantities into more convenient components and to the idea of "transformation" of variables and reference systems. Concordia[1] has given a summary of the transformations commonly used in this field.

The advantages of a change of reference axes were clearly demonstrated by Blondel[2] in resolving three-phase alternator quantities along the direct and quadrature axes of the field structure. The extension of Blondel's ideas by Park,[3] Doherty and Nickle,[4] and the development of the concepts of operational machine-impedances which followed gave an intellectual impetus to the whole study of electrical machinery. An electrical machine became a complex arrangement of inductively coupled coils in relative motion, and the mathematically-minded power engineer had much scope for systematic application of the classical laws of Newton, Faraday and Maxwell.

While the three-phase synchronous machine was being elegantly and thoroughly investigated by the change from phase quantities to d- and q-axis quantities, other types of machines had still individual theories and each had its own physical description. One did not, for example, speak in terms of armature reaction or generated voltages in an induction motor or too often of flux linkages in a d.c. machine. In 1934 Kron[5] showed how the two-axis theory of Park could be used to give a unified theory embracing a wide range of machines. He described a primitive two-phase machine, wrote down its equations of performance and developed the transformations necessary to derive from these the equations of a given machine.

Several other writers have since given analyses of electrical machines in a generalised form. In 1939 Stanley[6] studied the polyphase induction motor by resolving the stator and rotor voltages, currents and flux linkages into axes in quadrature, similar to those used by Park. The resulting equations of the induction motor are found to be almost identical in form with the two-axis equations of the alternator.

In both cases the transformation leads to a set of linear differential equations with constant coefficients and solutions can be obtained by operational methods. The analysis is applicable to problems of variable speed and hunting.

In 1951 Sabbagh[7] applied the two-reaction theory to the analysis of several types of a.c. induction and commutator machines and showed how the vector diagrams for each machine could be drawn from the derived equations.

In 1952 Vowels[8] investigated the transient equations of synchronous machines, and showed that the two-axis equations of the alternator are identical in form with those of the cross-field metadyne.

Ku[9] has given a comprehensive survey of the unified theory of machines. He uses the relatively stationary axes of Park and Kron and also gives a detailed description of the use of axes rotating uniformly, independently of the rotating field structure. (Kron's analysis also embraces general rotating axes[10].)

In 1957 Ku and Shen[11] developed a two-reaction theory of induction motors having saliency and unsymmetrical windings on the rotor. Equivalent circuits are also given.

2. Network Transformations

Kron's method of machine analysis is based on the idea of transformation from a simple primitive system to a more complex derived system.

An early application of the ideas of groups of network transformations was that of Nathan Howitt.[12] He shows that static electrical networks can form groups with given functions invariant. For example, a linear transformation of currents can be used to derive a family of networks having the same operational driving-point impedance function.

If the elements of the network of Fig. 1a are written in the form of a matrix

$$Z = \begin{array}{c|c|c} & 1 & 2 \\ \hline 1 & 2p+2 & p+1 \\ \hline 2 & p+1 & p+1 \end{array} \qquad (p = d/dt)$$

a linear transformation of currents

$$i_1 = i_a$$
$$i_2 = 2i_a + 3i_b$$

gives a connection matrix

$$C = \begin{array}{c|c|c} & a & b \\ \hline 1 & 1 & \\ \hline 2 & 2 & 3 \end{array} \qquad C_{transpose} = \begin{array}{c|c|c} & 1 & 2 \\ \hline a & 1 & 2 \\ \hline b & & 3 \end{array}$$

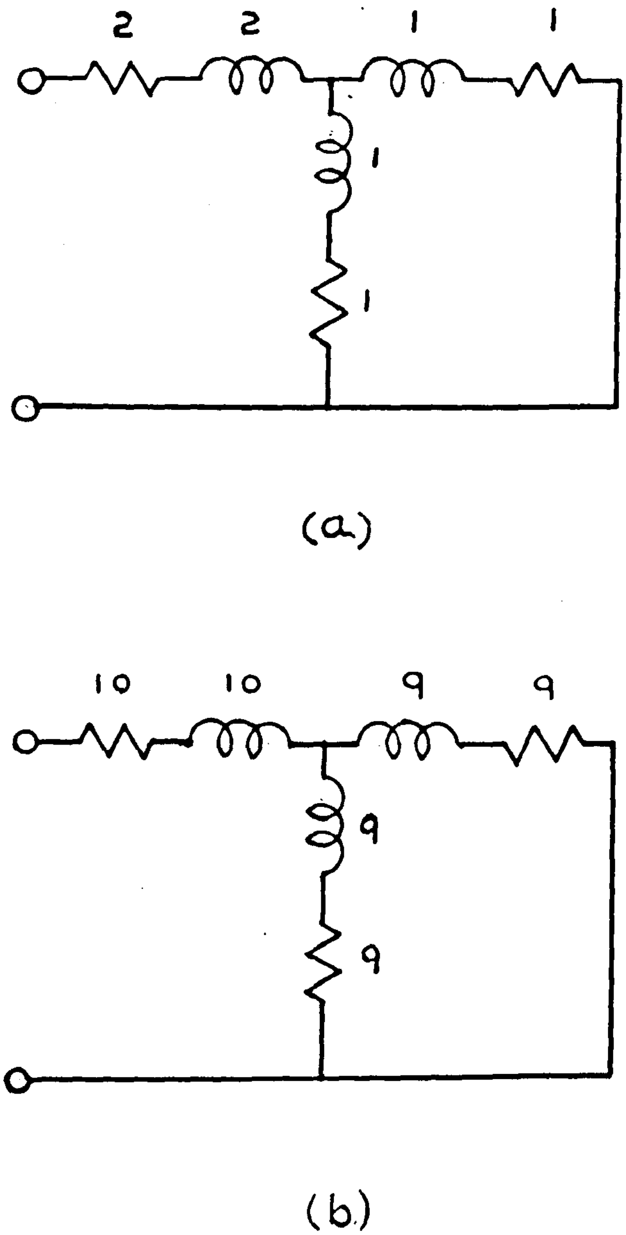


FIG. 1.

The operation $C_t.Z.C = Z'$ gives

$Z' =$

	1	2
a	$10p + 10$	$9p + 9$
b	$9p + 9$	$9p + 9$

The driving-point impedance function of each network is

$$Z_p = \frac{\text{Determinant of the network parameters}}{\text{Minor of the first row and column}}$$

and

$$Z_{(p)} = \frac{p^2 + 2p + 1}{p + 1} = Z'_{(p)}$$

This general principle is similar to that used by Kron[13] to simplify network analysis. The operation $C_t.Z.C$ can be carried out on the elements of any network or group of networks to give the impedance matrix of any complex interconnection of the coils. Kron has used this systematic approach in his analysis of interconnected power systems.[14] The application to a section of the British Grid System is given in I.E.E. Monograph 294S (attached). The implications of the interconnection of network elements by matrix operations (for example, the invariance of the network power) have been discussed by Gibbs[15] and Hoffmann.[16]

The above operation is in fact a tensor transformation. Kron extended this method to deal with coils in relative motion (machines).

The tensor operations can be applied to moving and rotating systems, but only by the introduction of the concepts of absolute or "covariant" differentiation, of the tensor calculus. These applications are investigated in Monographs 117S and 295S (attached).

3. Tensor Analysis of Machines

How physical phenomena appear to observers having different types of motion permeates the study of electrodynamics. It is important to separate those manifestations that are due to the point of view of the observer from those that are inherent in nature and not introduced by the interpretation of the observer.

Tensor analysis deals with transformations of sets of differential equations. The laws of transformation are such that a set of quantities or components cannot become zero if it transforms as a tensor and has non-zero values in any reference system. This means that tensors cannot arise simply because of the choice of reference axes. A single tensor may be made up of several component terms, some of which may individually become zero in a given reference system, the other terms changing accordingly. Tensor analysis became important in Relativity Theory because it enabled investigators to describe physical phenomena by equations which contained terms that were independent of the reference axes chosen. These terms, of course, would have different components in each system. Associated with the transformation laws and an inherent property of tensors is the fact that in any physical system the tensor equations will preserve the identity of quantities (such as energy or torque) that are unchanged in magnitude when different reference systems are used. That is, the tensor equations will give a property of invariance identifying the physical system in different co-ordinate axes. For example, the voltage equation of a single stationary coil may be written $e = Ri + Lpi$

When several coils are concerned this can be written as a matrix equation

$$e_c = R_{ca} i^a + L_{ca} p i^a \qquad (a).$$

A different interconnection of the coils would have the equation

$$e_r = R_{r\alpha} i^\alpha + L_{r\alpha} p i^\alpha$$

where

$$e_r = C^c_r e_c$$

$$R_{r\alpha} = R_{ca} C^c_r C^a_\alpha \qquad (\text{or } C_t \cdot R \cdot C) \qquad \text{etc.}$$

If the coils have relative angular velocity the equation becomes

$$e_c = R_{ca} i^a + L_{ca} p i^a + \frac{\partial L_{ca}}{\partial \theta} p\theta\, i^a$$

or

$$e_c = R_{ca} i^a + L_{ca} p i^a + G_{ca} p\theta\, i^a \qquad (b).$$

If the reference axes also rotate the equation becomes

$$e_r = R_{r\alpha} i^\alpha + L_{r\alpha} p i^\alpha + G_{r\alpha} p\theta\, i^\alpha + V_{r\alpha} p\theta' i^\alpha \qquad (c).$$

The machine torque

$$f = i^* \cdot \omega G \cdot i$$

is an absolute invariant.

The tensor form of all the above equations is

$$e_m = R_{mk} i^k + L_{mk} \frac{\delta i^k}{\delta t} \qquad (d).$$

where the covariant derivative

$$\frac{\delta i^k}{\delta t} = \frac{d i^k}{dt} + \Gamma_{uv}^{k} i^u i^v$$

The term $\Gamma_{uv}^{k} i^u i^v$ expands to give the equations (b) and (c) from equation (d). In the machine there are three voltages, which are all tensors,

(a) impressed voltage e

(b) resistance drop Ri

(c) "Flux voltage" ("Faraday" voltage) $L_{mk} \frac{\delta i^k}{\delta t}$

The third voltage is made up of voltages due to rate of change of linkages and rate of flux cutting. These components can change from one to the other as the reference system changes (as in equation (b) and (c)); but it is the total "Faraday" voltage, which is a tensor and it cannot be transformed to zero. It exists. Kron investigated electrical machines from this point of view. His analysis brings out the fact that there is the same magnetic structure for all machines and that the same physical phenomena occur. This method reduces the analysis to that of one representative member of a group. The others are found by routine transformations. Maxwell's equations in Lagrangian form can be applied directly to synchronous and induction machines in which the reference axes are fixed to the coils. However, when the reference axes are fixed to brushes then a modified form of Maxwell's equations must be used. Because of this complication graphical and vector techniques have been used in machine theory.

When the tensor form of the dynamical equations is used, groups of machine currents, voltages and flux linkages in any machine can be identified with the terms of electromagnetic field equations. The coexistence of electromagnetic and mechanical energy in the machine gives the equations a form similar to those of the Unified Field Theory where gravitational and electromagnetic fields are considered. Tensor analysis unifies the study of the whole group of electrical machines by investigating properties that are invariant and therefore independent of the type of machine or of the reference axes used.

Field Concepts in Electrical Machine Theory

Several publications appearing recently have covered the application of electromagnetic field theory to electrical machines.[17,18.] In 1938 Kron[19.] gave a comprehensive summary of the field concepts associated with the generalised, or primitive, machine and the corresponding quantities used in the coupled circuit approach. This paper has not received the attention it should, mainly because the equations are in tensor form with which engineers are not so familiar. Nevertheless, it would appear that when the tensor form is used, the field and rotating circuit relations can be more easily correlated. (The tensor analysis used in Kron's field theory is investigated in Monography 117S.)

This method of analysing electrical machines is very important. It appears to converge with recent work on the analysis of magneto-hydrodynamic systems and the stability of high-temperature arc plasma used in experiments on thermonuclear fusion.[20.] In both systems mechanical and electrical energies are interacting and the equations of both systems have to be formulated in several degrees of freedom. For this reason the salient points of Kron's paper are now summarised.

The symbols have the meaning used in the literature.

Electromagnetic Field Equations[21.]

$$\nabla \times H = J + \frac{\partial D}{\partial t} \quad \text{amperes/sq.m.}$$

$$\nabla \times E = -\frac{\partial B}{\partial t} \quad \text{volts/sq.m.}$$

$$\nabla \cdot D = \rho \quad \text{coulombs/c.m.}$$

$$\nabla \cdot B = 0$$

$$B = \text{curl } A \quad \text{webers/sq.m.}$$

$$A = \mu \iiint \frac{J}{r} \, dv$$

A conducting loop moving in a time-varying field,

$$\oint E\,dl = -\iint \left[\frac{\partial B}{\partial t} + u \,\text{div}\, B - \text{curl}\,(u \times B) \right] da \,.$$

$$\text{Curl } E = -\frac{\partial B}{\partial t} - u \,\text{div}\, B + \text{curl}\,(u \times B)$$

$$\text{Curl } H = \frac{\partial D}{\partial t} + u \,\text{div}\, D - \text{curl}\,(u \times D)$$

and

$$E = -\frac{\partial A}{\partial t} + (u \times B) - \text{grad}\, \psi \,.$$

Four dimensional form of field equations

$$\nabla \times H - \frac{\partial D}{\partial t} = J. \qquad \nabla \cdot D = \rho$$

$$\frac{\partial G_{jk}}{\partial x_k} = J_j$$

$$G_{jk} = \begin{pmatrix} 0 & H_3 & -H_2 & -icD_1 \\ -H_3 & 0 & H_1 & -icD_2 \\ H_2 & -H_1 & 0 & -icD_3 \\ icD_1 & icD_2 & icD_3 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \frac{\partial H_3}{\partial x_2} & -\frac{\partial H_2}{\partial x_3} & -ic\frac{\partial D_1}{\partial x_4} \\ -\frac{\partial H_3}{\partial x_1} & 0 & \frac{\partial H_1}{\partial x_3} & -ic\frac{\partial D_2}{\partial x_4} \\ \frac{\partial H_2}{\partial x_1} & -\frac{\partial H_1}{\partial x_2} & 0 & -ic\frac{\partial D_3}{\partial x_4} \\ ic\frac{\partial D_1}{\partial x_1} & ic\frac{\partial D_2}{\partial x_2} & ic\frac{\partial D_3}{\partial x_3} & 0 \end{pmatrix} = \begin{pmatrix} J_1 \\ J_2 \\ J_3 \\ ic\rho \end{pmatrix}$$

$J_1 = J_x$, $J_2 = J_y$, $J_3 = J_z$, $J_4 = ic$

$x_1 = x$, $x_2 = y$, $x_3 = z$, $x_4 = ict$

$$i = \sqrt{-1} \qquad c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$$

$$\nabla \times E + \frac{\partial B}{\partial t} = 0$$

$$\nabla \cdot B = 0$$

$$\frac{\partial F_{ij}}{\partial x_k} + \frac{\partial F_{ki}}{\partial x_j} + \frac{\partial F_{jk}}{\partial x_i} = 0$$

where F_{ij} =

0	B_3	$-B_2$	$-\frac{i}{c} E_1$
$-B_3$	0	B_1	$-\frac{i}{c} E_2$
B_2	$-B_1$	0	$-\frac{i}{c} E_3$
$\frac{i}{c} E_1$	$\frac{i}{c} E_2$	$\frac{i}{c} E_3$	0

Mechanical forces in the field are expressed by the stress tensor, derived from the field curl equations.

Force transmitted across a unit volume is

$$\int T\cdot n\, da = F_{elect} + F_{mag} + \epsilon_0 \frac{\partial}{\partial t}\int (E\times B)\, dv$$

= Electrostatic force + Magnetic force + rate of change of electromagnetic momentum.

$T = T_{e\text{-}static} + T_{e\text{-}mag.}$

$$\operatorname{div} T = E\rho + J\times B + \epsilon_0 \frac{\partial}{\partial t}(E\times B)$$

$$T_{e\text{-}stat} = \epsilon_0 \begin{bmatrix} E_x^2 - \frac{1}{2}E^2 & E_x E_y & E_x E_z \\ E_y E_x & E_y^2 - \frac{1}{2}E^2 & E_y E_z \\ E_z E_x & E_z E_y & E_z^2 - \frac{1}{2}E^2 \end{bmatrix}$$

$$T_{mag} = \frac{1}{\mu_0} \begin{array}{|c|c|c|} \hline B_x^2 - \frac{1}{2}B^2 & B_x B_y & B_x B_z \\ \hline B_y B_x & B_y^2 - \frac{1}{2}B^2 & B_y B_z \\ \hline B_z B_x & B_z B_y & B_z^2 - \frac{1}{2}B^2 \\ \hline \end{array}$$

Rainich[22] shows why the stress tensor can be expressed in the form used in relativity theory, in terms of the field tensor Fij, namely

$$T_{ij} = F_{ik}F_{kj} - \frac{1}{4}\delta_{ij}F_{sk}F_{ks}$$

$$F_{sk}F_{ks} = F_{12}F_{21} + F_{13}F_{31} + F_{14}F_{41} + F_{21}F_{12} + F_{23}F_{32} + \cdots$$

$$= 2\left(-L^2 - M^2 - N^2 + X^2 + Y^2 + Z^2\right)$$

$$F_{1k}F_{k1} = F_{12}F_{21} + F_{13}F_{31} + F_{14}F_{41} = -N^2 - M^2 - X^2$$

$$\therefore \quad T_{11} = F_{1k}F_{k1} - \frac{1}{4}F_{sk}F_{ks} \quad \text{etc}$$

In this notation Maxwell's curl equations have the form

$$\frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y} = \frac{\partial L}{\partial t} \qquad \frac{\partial M}{\partial z} - \frac{\partial N}{\partial y} = -\frac{\partial X}{\partial t}$$

$$\frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x} = \frac{\partial M}{\partial t} \qquad \frac{\partial N}{\partial x} - \frac{\partial L}{\partial z} = -\frac{\partial Y}{\partial t}$$

$$\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x} = \frac{\partial N}{\partial t} \qquad \frac{\partial L}{\partial y} - \frac{\partial M}{\partial x} = -\frac{\partial Z}{\partial t}$$

where X, Y, Z, and L, M, N are the components of the electric and magnetic fields respectively.

Rotating Machine Equations

Commutator machine with axes stationary

$$e = Ri + Lpi + Bp\theta$$

$$B = G \cdot i$$

G contains flux density terms.

When the current and flux waves are sinusoidal round the armature the terms of G can be obtained by transformation of the machine inductance matrix and

$$G = \gamma L$$

where γ =

	d	q
d		1
q	−1	

The machine equation can be written

$$e_r = R_{r\alpha} i^{\alpha} + L_{r\alpha} p i^{\alpha} + F_{rs} i^{s}$$

$$\text{where } i^{s} = \frac{dx^{s}}{dt} \cdot \frac{d\theta}{dt}$$

When the reference axes rotate independently of the rotor, then

$$e' = R' i' + L' p i' + G' i' p\theta + V' i' p\theta'$$

where $p\theta'$ is the angular velocity of the axes. These two equations come from a general tensor equation

$$e_\gamma = R_{\gamma\alpha} i^\alpha + L_{\gamma\alpha} \frac{\delta i^\alpha}{\delta t}$$

where

$$L_{\gamma\alpha} \frac{\delta i^\alpha}{\delta t} = L_{\gamma\alpha} \frac{d i^\alpha}{dt} + \Gamma_{\beta\pi,\gamma}\, i^\beta i^\pi$$

and

$$\Gamma_{\beta\pi,\gamma}\, i^\beta i^\pi = 2\,\Omega_{\gamma\beta,\pi}\, i^\beta i^\pi - 2\, S_{\gamma\beta\pi}\, i^\beta i^\pi$$

$$= \left(2\,\Omega^{\alpha}_{\gamma\beta}\, i^\beta i^\pi - 2\, S^{\alpha}_{\gamma\beta}\, i^\beta i^\pi\right) L_{\alpha\pi}$$

The link between circuit and field concepts is the vector potential. The machine has been idealised. The assumptions relevant to the vector potential are that the m.m.f. and flux density waves are sinusoidal round the armature and that the current density in the conductors is uniform. In this case the expressions[23.]

$$L = \frac{\int A \cdot J\, dv}{I^2} \quad \text{(inductance)}$$

and

$$\frac{d}{dt} \oint A \cdot dl = L \frac{dI}{dt}$$

give flux linkage L.i = A.

In this form the vector potential is given by the resultant flux linkage in each circuit.

$$A_\pi = L_{\pi\sigma} i^\sigma = \phi_\pi$$

In a 2-axis machine with an operational inductance matrix (field eliminated)

$$L_{\pi\sigma} = \begin{array}{c|c|c|} & d & q \\ \hline d & L_d & \\ \hline q & & L_q \\ \hline \end{array}$$

$$\phi_\pi = \begin{array}{|c|c|} \multicolumn{1}{c}{d} & \multicolumn{1}{c}{q} \\ \hline \phi_d & \phi_q \\ \hline \end{array} = \begin{array}{|c|c|} \multicolumn{1}{c}{d} & \multicolumn{1}{c}{q} \\ \hline L_d i^d & L_q i^q \\ \hline \end{array}$$

The field equations in covariant form become

B = Absolute Curl A

$$F_{\alpha\beta} = \frac{\delta\phi_\alpha}{\delta x^\beta} - \frac{\delta\phi_\beta}{\delta x^\alpha}$$

$$= \frac{\partial\phi_\alpha}{\partial x^\beta} - \frac{\partial\phi_\beta}{\partial x^\alpha} - 2 S_{\alpha\beta}^{\gamma}\phi_\gamma + 2\Omega_{\alpha\beta}^{\gamma}\phi_\gamma$$

In terms of time variations the index β takes the time value t, (an index s denotes a mechanical angle of rotation θ)

$$F_{\alpha t} = -E_\pi = \frac{\partial\phi_\pi}{\partial t} - 2 S_{\pi s\sigma} p\theta \cdot i^\sigma + 2\Omega_{\pi s,\sigma} p\theta' \cdot i^\sigma$$

corresponding to the equation

$$E = -\frac{dA}{dt} + B_1 \times v_1 + B_2 \times v_2$$

where v_1 is the angular velocity of the rotor.

v_2 is the angular velocity of the axes.

$$F_{\alpha\beta} = \begin{array}{c|c|c|c|c|} & d & q & s & t \\ \hline d & & & -B_d & -E_d \\ \hline q & & & -B_q & -E_q \\ \hline s & B_d & B_q & & \\ \hline t & E_d & E_q & & \\ \hline \end{array} = \begin{array}{c|c|c|c|c|} & d & q & s & t \\ \hline d & & & \phi_q & e_d \\ \hline q & & & \phi_d & e_q \\ \hline s & -\phi_q & -\phi_d & & \\ \hline t & -e_d & -e_q & & \\ \hline \end{array}$$

The absolute form of the symmetry equation can be written

$$\frac{\delta F_{\beta\gamma}}{\delta x^{\alpha}} + \frac{\delta F_{\gamma\alpha}}{\delta x^{\beta}} + \frac{\delta F_{\alpha\beta}}{\delta x^{\gamma}} = 0$$

This leads to

$$\frac{\delta F_{ms}}{\delta x^{t}} = -\frac{\delta B_n}{\delta t}$$

$$= \frac{\partial E_s}{\partial x^m} - \frac{\partial E_m}{\partial x^s} - 2 S_{sm}^{k} E_k + 2\,\Omega_{sm}^{k} E_k$$

and is equivalent to

$$\text{Abs. curl } E = -\frac{dB}{dt} - \text{abs. curl } (B \times v)$$

Also

$$\frac{\delta F_{ns}}{\delta x^m} + \frac{\delta F_{sm}}{\delta x^n} = 0$$

corresponding to

Abs. Div. A = 0

The stress tensor can be obtained from

$$T_\alpha^{\ \beta} = F_{\alpha\beta} H^{\beta\gamma} - \tfrac{1}{4}\,\delta_\alpha^{\beta} F_{\gamma\delta} H^{\gamma\delta}$$

Where $H^{\beta\gamma} =$

	d	q	s
d			i^q
q			i^d
s	$-i^q$	$-i^d$	

Then

$T_\alpha^{\ \beta} =$

	d	q	s	t
d	$B_d H^d - W$	$B_d H^q$		
q	$B_q H^d$	$B_q H^q - W$		
s			W	
t			$E_d H^d + E_q H^q$	$-W$

The instantaneous stored magnetic energy is

$$T_t^{\ t} = \tfrac{1}{2}\left(B_d H^d + B_q H^q\right) = -\tfrac{1}{2} B_\pi H^\pi = -W$$

Maxwell stresses

$$T_\pi^{\ \sigma} = B_\pi H^\sigma - \tfrac{1}{2}\,\delta_\pi^{\sigma} B_\epsilon H^\epsilon$$

Poynting Vector

$$T_{1}^{s} = e_d i^q - e_q i^d = E_\pi H^\pi$$

representing the power flowing into the machine.

The investigation of the above and other electromagnetic relationships is continuing.

References

1. Concordia, C. "Relations Among Transformations Used in Electrical Engineering Problems" General Electric Review, Vol. 41, 1938.

2. See I.E.E. Monograph 117S (attached) Ref. 18.

3. Monograph 117S Ref. 18.

4. Monograph 295S Ref. 11.

5. Monograph 117S Ref. 1.

6. Monograph 117S Ref. 9.

7. Monograph 117S Ref. 7.

8. Vowels, R.E. "Transient Analysis of Synchronous Machines" Proc. I.E.E. Part IV, Vol. 99, 1952.

9. Monograph 117S Ref. 23.

10. Monograph 117S Ref. 2.

11. Ku, Y.H., and Shen, D.W.C. "Two-reaction Theory of a General Induction Machine and its Equivalent Circuit" Trans. A.I.E.E., Power Apparatus and Systems, October 1957.

12. Howitt, N. "Group Theory and the Electric Circuit" Physical Review, Vol. 37, 1931.

13. Monograph 294S Ref. 25.

14. Monograph 294S Ref. 5.

15. Gibbs, W.J. "Power Invariance in Electrical Machines" Matrix and Tensor Quarterly, Vol. 8, 1957.

16. Hoffmann, B. "Nature of the Primitive System in Kron's Theory" Am. Journal of Physics, Vol. 23, 1955.

17. Mishkin, E. "Theory of the Squirrel-Cage Induction Motor Derived Directly from Maxwell's Field Equations" Quarterly Journal of Mechanics and Applied Mathematics, Vol. 7, 1954.

18. Cullen, Prof. A.C., and Barton, T.H.. "A Simplified Electromagnetic Theory of the Induction Motor using the Concept of Wave Impedance" Proc. I.E.E., Monograph 283.U, January 1958.

19. Kron, G. "Invariant Form of the Maxwell-Lorentz Field Equations for Accelerated Systems" Journal of Applied Physics, Vol. 9, 1938.

20. Cowling, T.G. "Magnetohydrodynamics" Interscience Publishers Inc., New York, 1957.

21. Stratton, J.A. "Electromagnetic Theory" McGraw Hill Book Co. Inc., 1941.

22. Rainich, G.Y. "The Mathematics of Relativity" John Wiley & Sons, Inc., 1950.

23. Kraus, J.D. "Electromagnetics" McGraw Hill Book Co. Inc., 1953.

Additional appendix to Monograph 295S

Derivation of equation 57 in Monograph 295S

(a) The first method uses equation 22. The synchronous machine impedance transformation is given by

$C =$

	ds	d	q	s
ds	1			
dr		$\cos\delta$	$-\sin\delta$	
qr		$\sin\delta$	$\cos\delta$	
s				1

$$(pC) = 0 \qquad p(\Delta C) = 0 \qquad p\,i' = 0$$

and therefore

$$Z' = C_t \cdot Z \cdot C\,(\Delta i' + \Delta\delta) + C_t \cdot Z \cdot \frac{\partial C}{\partial \delta} \cdot i' \cdot \Delta\delta$$

$$+ \left[\frac{\partial C_t}{\partial \delta} \cdot R \cdot C \cdot i' + \frac{\partial C_t}{\partial \delta} \cdot G \cdot C \cdot i' \cdot p\theta \right]$$

(b) In terms of transformed parameters equation 55 can be used. (See now equations 52 and 53.)

$C_t . Z . C = Z''$

(The transformation matrix is shown on the following page.)

$C_t \cdot Z \cdot C$

	ds	d	q	s
ds	$r_{ds} + L_{ds}\, p$	$M_d \cos\delta\, p$	$-M_d \sin\delta\, p$	
d	$M_d \cos\delta\, p$ $-M_d \sin\delta\, p\theta$	$r_{dr} \cos^2\delta$ $+L_{dr} \cos^2\delta\, p$ $-L_{dr} \sin\delta \cos\delta\, p\theta$ $+L_{qr} \cos\delta \sin\delta\, p\theta$ $+r_{qr} \sin^2\delta$ $+L_{qr} \sin^2\delta\, p$	$-r_{dr} \cos\delta \sin\delta$ $-L_{dr} \cos\delta \sin\delta\, p$ $+L_{dr} \sin^2\delta\, p\theta$ $+L_{qr} \cos^2\delta\, p\theta$ $+r_{qr} \sin\delta \cos\delta$ $+L_{qr} \sin\delta \cos\delta\, p$	$i^{qr} L_{qr} \cos\delta\, p$ $-i^{ds} M_d \sin\delta\, p$ $-i^{dr} L_{dr} \sin\delta\, p$
q	$-M_d \sin\delta\, p$ $-M_d \cos\delta\, p\theta$	$-r_{dr} \sin\delta \cos\delta$ $-L_{dr} \sin\delta \cos\delta\, p$ $-L_{dr} \cos^2\delta\, p\theta$ $-L_{qr} \sin^2\delta\, p\theta$ $+r_{qr} \sin\delta \cos\delta$ $+L_{qr} \cos\delta \sin\delta\, p$	$r_{dr} \sin^2\delta$ $+L_{dr} \sin^2\delta\, p$ $+L_{dr} \cos\delta \sin\delta\, p\theta$ $-L_{qr} \cos\delta \sin\delta\, p\theta$ $+r_{qr} \cos^2\delta$ $+L_{qr} \cos^2\delta\, p$	$-i^{qr} L_{qr} \sin\delta\, p$ $-i^{ds} M_d \cos\delta\, p$ $-i^{dr} L_{dr} \cos\delta\, p$
s	$i^{qr} M_d$	$i^{qr} L' \cos\delta$ $+i^{dr} L' \sin\delta$ $+i^{ds} M_d \sin\delta$	$-i^{qr} L' \sin\delta$ $+i^{dr} L' \cos\delta$ $+i^{ds} M_d \cos\delta$	$L p^2$

$$L' = (L_{qr} - L_{dr}).$$

$C_t \cdot Z \cdot \frac{\partial C}{\partial \delta}$.

	ds	d	q	s
ds		$-M_d \sin\delta\, p$	$-M_d \cos\delta\, p$	
d		$-T_{dr}\cos\delta\sin\delta$ $-L_{dr}\cos\delta\sin\delta\, p$ $+L_{dr}\sin^2\delta\, p\theta$ $+L_{qr}\cos^2\delta\, p\theta$ $+T_{qr}\cos\delta\sin\delta$ $+L_{qr}\sin\delta\cos\delta\, p$	$-T_{dr}\cos^2\delta$ $-L_{dr}\cos^2\delta\, p$ $+L_{dr}\sin\delta\cos\delta\, p\theta$ $-L_{qr}\cos\delta\sin\delta\, p\theta$ $-L_{qr}\sin^2\delta\, p$ $-T_{qr}\sin^2\delta$	
q		$T_{dr}\sin^2\delta$ $+L_{dr}\sin^2\delta\, p$ $+L_{dr}\cos\delta\sin\delta\, p\theta$ $-L_{qr}\sin\delta\cos\delta\, p\theta$ $+T_{qr}\cos^2\delta$ $+L_{qr}\cos^2\delta\, p$	$T_{dr}\sin\delta\cos\delta$ $+L_{dr}\sin\delta\cos\delta\, p$ $+L_{dr}\cos^2\delta\, p\theta$ $+L_{qr}\sin^2\delta\, p\theta$ $-T_{qr}\cos\delta\sin\delta$ $-L_{qr}\cos\delta\sin\delta\, p$	
s		$-i^{qr}L'\sin\delta$ $+i^{dr}L'\cos\delta$ $+i^{ds}M_d\cos\delta$	$-i^{qr}L'\cos\delta$ $-i^{dr}L'\sin\delta$ $-i^{ds}M_d\sin\delta$	

$\frac{\partial C_t}{\partial \delta} \cdot R \cdot C$

	ds	d	q
ds			
d			R
q		$-R$	

$\frac{\partial C_t}{\partial \delta} \cdot R \cdot C \cdot i'$

ds	
d	Ri^q
q	$-Ri^d$

As above, $p\theta$ terms + $\frac{\partial C_t}{\partial \delta} \cdot G \cdot C \cdot i' p\theta$

	s
ds	
d	$-2(L_{qr}-L_{dr})\cos\delta\sin\delta\, i^q p\theta - M_d i^{ds}\cos\delta\, p\theta$ $+(L_{qr}-L_{dr})(\cos^2\delta-\sin^2\delta)\, i^d p\theta$
q	$-2(L_{qr}-L_{dr})\cos\delta\sin\delta\, i^d p\theta + M_d i^{ds}\sin\delta\, p\theta$ $-(L_{qr}-L_{dr})(\cos^2\delta-\sin^2\delta)\, i^q p\theta$
s	

$C_t \cdot Z \cdot C$ 4th row

	ds	d	q
s	$M_d\sin\delta\, i^d$ $+M_d\cos\delta\, i^q$	$-2(L_{qr}-L_{dr})\cos\delta\sin\delta\, i^d$ $-(L_{qr}-L_{dr})(\cos^2\delta-\sin^2\delta)\, i^q$ $+i^{ds}M_d\sin\delta$	$i^{ds}M_d\cos\delta$ $-(L_{qr}-L_{dr})(\cos^2\delta-\sin^2\delta)\, i^d$ $+2(L_{qr}-L_{dr})\sin\delta\cos\delta\, i^q$

$$\frac{\partial C_t}{\partial \delta} \cdot G \cdot C$$

	ds	d	q
ds			
d	$-M_d \cos\delta$	$-L_{dr}\cos^2\delta$ $-L_{qr}\sin^2\delta$	$L_{dr}\cos\delta\sin\delta$ $-L_{qr}\sin\delta\cos\delta$
q	$M_d \sin\delta$	$L_{dr}\sin\delta\cos\delta$ $-L_{qr}\cos\delta\sin\delta$	$-L_{dr}\sin^2\delta$ $-L_{qr}\cos^2\delta$

$$C_t \cdot Z \cdot C \text{ 4th col.} + C_t \cdot Z \cdot \frac{\partial C}{\partial \delta} \cdot i'$$

Voltage equations, p terms only:

	s
ds	$-M_d \sin\delta\, i^d p - M_d \cos\delta\, i^q p$
d	$2(L_{qr}-L_{dr})\cos\delta\sin\delta\, i^d p - i^{ds} M_d \sin\delta\, p$ $+ (L_{qr}-L_{dr})(\cos^2\delta - \sin^2\delta)\, i^q p$
q	$-2(L_{qr}-L_{dr})\cos\delta\sin\delta\, i^q p - i^{ds} M_d \cos\delta\, p$ $+ (L_{qr}-L_{dr})(\cos^2\delta - \sin^2\delta)\, i^d p$
s	

$$\gamma = \frac{\partial C_t^{-1}}{\partial \delta} \cdot C_t =$$

	ds	d	q	s
ds				
d			-1	
q		1		
s				

L_{rd}

	ds	d	q
ds	L_{ds}	$M_d \cos\delta$	$-M_d \sin\delta$
d	$M_d \cos\delta$	$L_{dr} \cos^2\delta$ $+ L_{qr} \sin^2\delta$	$-L_{dr} \cos\delta \sin\delta$ $+ L_{qr} \sin\delta \cos\delta$
q	$-M_d \sin\delta$	$-L_{dr} \cos\delta \sin\delta$ $+ L_{qr} \cos\delta \sin\delta$	$L_{dr} \sin^2\delta$ $+ L_{qr} \cos^2\delta$

$$-\frac{\partial L_{rd}}{\partial \delta}\cdot i'$$

ds	d	q
$M_d \sin\delta\, i^d$ $+ M_d \cos\delta\, i^q$	$M_d \sin\delta\, i^{ds}$ $-2(L_{qr}-L_{dr})\cos\delta\sin\delta\, i^d$ $-(L_{qr}-L_{dr})(\cos^2\delta-\sin^2\delta)\, i^q$	$M_d \cos\delta\, i^{ds}$ $-(L_{qr}-L_{dr})(\cos^2\delta-\sin^2\delta)\, i^d$ $+2(L_{qr}-L_{dr})\cos\delta\sin\delta\, i^q$

$$-\frac{1}{2}\frac{\partial L_{rd}}{\partial \delta} = G' \qquad \text{(Ref. 2 Mon. 117S.)}$$

$$\frac{\partial G'}{\partial \delta}\cdot i' \cdot i'$$

	s
s	$M_d \cos\delta\, i^d i^{ds} - M_d \sin\delta\, i^q i^{ds}$ $-(L_{qr}-L_{dr})(\cos^2\delta-\sin^2\delta)\, i^d i^d$ $+4(L_{qr}-L_{dr})\cos\delta\sin\delta\, i^d i^q$ $+(L_{qr}-L_{dr})(\cos^2\delta-\sin^2\delta)\, i^q i^q$

Summation of the appropriate terms gives equation 57, Monograph 295S, derived via equations 22 and 55 respectively.

The Practical Application of Matrix Methods of Electrical Machine Analysis

BY J. W. LYNN, M.SC., A.M.I.E.E.,* AND A. S. ALDRED, M.SC.*

THE application of matrix algebra and tensor analysis to electrical circuits and machinery has to a large extent unified the theories of individual machine systems. Kron's methods of analysis have been clearly and simply presented by Dr. W. J. Gibbs both in his recent book[1] and in a series of articles.[2] In the latter, Gibbs has shown that for the analysis of most machines only the elementary parts of matrix theory are required. This form appears to be very suitable for presenting to engineering students a consistent general theory of electrical machines. As a step in this direction, the matrix equations of some well-known machines have been given by the authors to various groups of final-year students and the machines then investigated in the laboratory. The results are presented in this paper. The equations obtained by matrix methods are seen to agree in each case with those already very well known. However, many of the impedance matrices present the machine parameters in the form of open circuit self and mutual inductances of the windings, instead of the more familiar short-circuit values. This leads to difficulties of measurement since the latter are more or less constant whereas the former vary widely. For laboratory work it has often been found to be impracticable to measure the values of L and M directly, but using open and short-circuit tests for each pair of windings reasonable values of the parameters may be found by well-known techniques and involving only familiar assumptions. These are described in the appropriate sections. Six types of machine have been investigated. The equations for all of these have been obtained by transformation of the equations of the primitive machine[3] shown in Fig. 1, using Kron's connexion matrix C. For the amplidyne analysis the winding 5 in Fig. 1 is used, but for all other machines analyzed here this is not required and the appropriate row and column are then dropped from the matrices of Equations (1).

The primitive machine equations are,[3]

$$\begin{array}{c|c|}
 & \\
1 & e_1 \\
2 & e_2 \\
3 & e_3 \\
4 & e_4 \\
5 & e_5
\end{array}
=
\begin{array}{c|c|c|c|c|c|}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & r_1+L_1p & M_{12}p & & & M_{15}p \\
2 & M_{21}p & r_2+L_2p & L_3'p\theta & M_{34}'p\theta & M_{25}p \\
3 & -M_{21}'p\theta & -L_2'p\theta & r_3+L_3p & M_{34}p & -M_{25}'p\theta \\
4 & & & M_{43}p & r_4+L_4p & \\
5 & M_{51}p & M_{52}p & & & r_5+L_5p
\end{array}
\cdot
\begin{array}{c|c|}
 & \\
1 & i_1 \\
2 & i_2 \\
3 & i_3 \\
4 & i_4 \\
5 & i_5
\end{array}
\quad . \quad (1)$$

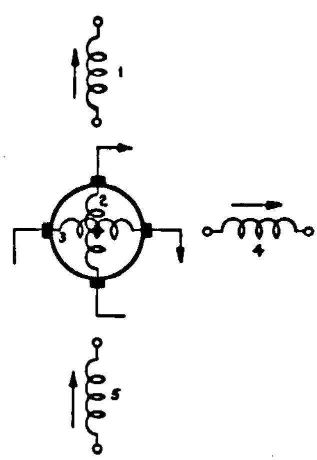


Fig. 1. The primitive machine

* Department of Electrical Engineering, the University of Liverpool.

Additional rows and columns are added when required to allow for the self impedances Z and Z_L in Fig. 2. The torque matrix G is[3]

$G =$

	1	2	3	4	5
1					
2			L_3'	M_{34}'	
3	$-M_{21}'$	$-L_2'$			$-M_{25}'$
4					
5					

(2)

The machine torque is given by $f = i^* . \omega G . i$ synchronous watts, where ω is the synchronous frequency; i is the current vector having five axis components; i^* is the conjugate of the current vector.

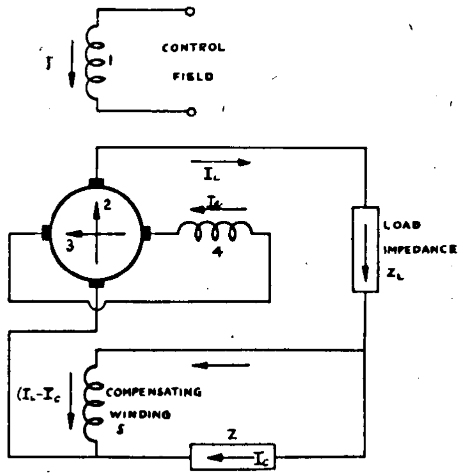


Fig. 2. The amplidyne generator

COMMUTATOR MACHINES

D.C. MACHINES

Amplidyne and Metadyne Generators

As examples of the more complex forms of d.c. machines the amplidyne and metadyne generators have been analyzed. The performance characteristics of these machines are now well known. In the following the equations of each machine have been obtained by comparison with the primitive machine shown in Fig. 1. By approaching the analysis in this way a single matrix may be set up which contains the equations of either machine. This matrix illustrates the points of similarity and also the differences between the two machines.

The amplidyne circuit is shown in Fig. 2. The compensating winding 5 is provided to neutralize flux set up by armature reaction due to current in winding 2. For full compensation the flux set up by the load current flowing in winding 5 must completely neutralize the armature flux set up by the load current in winding 2. When winding 5 over compensates, it is shown in the equations that the machine may be very nearly completely compensated by the addition of the external shunt impedance Z shown in Fig. 2. Using this shunt the current in winding 5 is reduced to the value, k times the load current, required for compensation. The required value of k is obtained from the equations.

In the metadyne generator the circuit connexions are identical with Fig. 2, but the winding 5 and shunt impedance Z are absent. This means that the metadyne makes use of the feedback voltage between windings 3 and 2. The effect is seen from the final equations, which show how this changes the characteristics of the machine from a constant (internal) voltage generator to an almost constant current generator.

In the analysis the parameters have been assumed constant over a given range of current values and iron loss is neglected. Hysteresis effects were minimized during testing by demagnetizing the machine before each test. In commutator machines of this type the flux distribution is obviously not sinusoidal. This gives rise to such terms as $M_{21}'p\theta$ in Equations (1), where

M_{21}' differs from M_{21}. This arises because of the difference between the average voltage generated by rotation in flux having space harmonics and the voltage induced by sinusoidal time rate of change of this flux linking a coil. In the machines considered here the complete terms, such as $i_1 M_{21}' p\theta$, etc., were measured during test.

The Equations (1) are then transformed by $C_t . e$ and $C_t . Z . C^{(3)}$ to give the amplidyne equations,

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{|c|} \hline V \\ \hline 0 \\ \hline 0 \\ \hline 0 \\ \hline \end{array} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{|c|c|c|c|} \multicolumn{1}{c}{a} & \multicolumn{1}{c}{b} & \multicolumn{1}{c}{c} & \multicolumn{1}{c}{d} \\ \hline Z_1 & 0 & (M_{15}-M_{12})p & -M_{15}p \\ \hline -M_{21}'p\theta & Z_3+Z_4+2M_{34}p & (-M_{25}'+L_2')p\theta & M_{25}'p\theta \\ \hline (M_{15}-M_{12})p & -(L_3'+M_{34}')p\theta & Z_5+Z_2+Z_L-2M_{25}p & -Z_5+M_{25}p \\ \hline -M_{15}p & 0 & -Z_5+M_{25}p & Z_5+Z \\ \hline \end{array} \cdot \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{|c|} \hline I \\ \hline I_a \\ \hline I_L \\ \hline I_C \\ \hline \end{array} \quad . \quad (4)$$

The impedances Z_1 etc. are in operational-form $(r_1 + L_1 p)$, etc.

The value of k can be found from the fourth of these equations

$$k = \frac{ZI_L + M_{25}pI_L - M_{15}pI}{(Z+Z_5)I_L}, \quad . \quad . \quad (5)$$

and since $M_{15}I$ is small compared with $M_{25}I_L$

$$k \text{ is approximately } \frac{Z+M_{25}p}{Z+Z_5}.$$

Now if the current in coil 5 be written kI_L the following connexion may be used:

$$C' = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{|c|c|c|} \multicolumn{1}{c}{a} & \multicolumn{1}{c}{b} & \multicolumn{1}{c}{c} \\ \hline 1 & & \\ \hline & 1 & \\ \hline & & 1 \\ \hline & & 1-k \\ \hline \end{array} \quad . \quad (6)$$

By carrying out the transformation $C_t' . Z' . C'$ the following impedance matrix is obtained for the amplidyne:

Impedance Matrices

The matrix connecting the amplidyne circuit of Fig. 2 with the primitive machine of Fig. 1 is

$$C = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ Z \\ Z_L \end{array}\begin{array}{|c|c|c|c|} \multicolumn{1}{c}{a} & \multicolumn{1}{c}{b} & \multicolumn{1}{c}{c} & \multicolumn{1}{c}{d} \\ \hline -1 & & & \\ \hline & & 1 & \\ \hline & -1 & & \\ \hline & -1 & & \\ \hline & & -1 & 1 \\ \hline & & & 1 \\ \hline & & 1 & \\ \hline \end{array} \quad . \quad (2)$$

$$\begin{array}{c} \\ a \\ b \\ c \end{array}\begin{array}{|c|c|c|} \multicolumn{1}{c}{a} & \multicolumn{1}{c}{b} & \multicolumn{1}{c}{c} \\ \hline r_1+L_1p & 0 & kM_{15}p-M_{12}p \\ \hline -M_{21}'p\theta & r_3+r_4+(L_3+L_4+2M_{34})p & L_2'p\theta-kM_{25}'p\theta \\ \hline kM_{15}p-M_{12}p & -(L_3'+M_{34}')p\theta & r_2+kr_5+r_L+(L_L+L_2+kL_5-kM_{25}-M_{25})p \\ \hline \end{array} \quad . \quad . \quad (7)$$

In the case of the metadyne with zero compensation the terms containing k and the terms M_{25} are zero since the compensating winding is absent. The impedance matrix of the metadyne is then

	a	b	c
a	r_1+L_1p	0	$-M_{12}p$
b	$-M_{21}'p\theta$	$r_3+r_4+(L_3+L_4+2M_{34})p$	$L_2'p\theta$
c	$-M_{12}p$	$-(L_3'+M_{34}')p\theta$	$r_2+r_L+(L_2+L_L)p$

. . . (8)

The matrix equation $e=Z.i$ for both machines may be written in general terms using time constants T and amplification factors μ as follows:[4]

$$\begin{array}{c|c|} a & V \\ b & 0 \\ c & 0 \end{array} = \begin{array}{c|c|c|c|} & a & b & c \\ a & 1+T_1p & 0 & -T_4p \\ b & -\mu_1 & 1+T_3p & \mu_3 \\ c & -T_2p & -\mu_2 & 1+T_5p \end{array} \cdot \begin{array}{c|c|} a & Ir_1 \\ b & I_q(r_3+r_4) \\ c & I_LR \end{array} \quad . \; . \; . \; . \quad (9)$$

where, for example

$$T_1\equiv\frac{L_1}{r_1}$$

$$\mu_1\equiv\frac{M_{21}'p\theta}{r_1}$$

$$T_3\equiv\frac{L_3+L_4+2M_{34}}{r_3+r_4},$$

and

$$R\equiv(r_2+r_L+kr_5).$$

By eliminating the rows and columns of axes a and b, a transfer function for each machine may be obtained, thus:

Metadyne

$$\frac{I_L}{V}=\frac{1}{R_1}\left[\frac{T_2p(1+T_3p)+\mu_1\mu_2}{(1+T_1p)(1+T_3p)(1+T_5p)-T_2T_4p^2(1+T_3p)-\mu_1\mu_2T_4p+\mu_2\mu_3(1+T_1p)}\right] \quad . \quad (10)$$

where

$$R_1=(r_2+r_L)$$

and the terms T and μ have values appropriate to terms of matrix 8.

Amplidyne

In the fully compensated amplidyne the terms T_2, T_4, and μ_3 from impedance Matrix 9 are zero and the equation becomes

$$\frac{I_L}{V}=\frac{1}{R_2}\left[\frac{\mu_1\mu_2}{(1+T_1p)(1+T_3p)(1+T_5p)}\right] \quad . \; . \; . \; . \; . \; . \quad (11)$$

where

$$R_2=r_2+kr_5+r_L$$

and the terms T and μ have values appropriate to terms of Matrix 7. The steady-state equations follow by putting $p=0$ in Equations (10) and (11).

Amplidyne

$$\frac{I_L(r_2+kr_5+r_L)}{V}=\mu_1\mu_2=\text{constant.} \quad (12)$$

This shows the constant internal generated voltage characteristic.

Metadyne

$$\frac{I_L}{V}=\frac{\mu_1\mu_2}{(r_2+r_L)(1+\mu_2\mu_3)} \doteq \frac{M_{12}'p\theta}{r_1L_2'p\theta} \doteq \text{constant.} \quad (13)$$

This neglects some small products of resistance terms and shows the approximately constant current nature of the metadyne.

For a machine with 80 to 90 per cent compensation, having a fairly constant current characteristic and the added advantage of greater power amplification, the analysis is similar.

Testing

(i) The resistances of the control field, the compensating and quadrature windings were measured using direct current. The effective armature resistance was determined by calculating it by adjusting the time constant of the load until the machine became unstable. All other circuit parameters were known at the point of oscillation. Routh's criterion was used. The method is suggested by Kron.(4) The effective resistance includes a back e.m.f. in the armature circuit due to slight displacement of flux during commutation. It is found to vary with the speed and operating conditions of the machine. The value of effective resistance is seen to be about six times the stationary d.c. value of resistance (see Table I).

(ii) The inductances of the compensating winding, quadrature winding, and armature were measured by impedance drop measurements at fifty cycles, but the inductance of the control field winding was too high for such tests. This field inductance was determined by measurement of the transient time constant using a unit function applied voltage.

In all cases it was found possible to choose a straight line magnetizing curve which was reasonably close over the proposed current range.

(iii) The 'generation' terms $M_{21}'p\theta$, etc., were obtained by energizing one field winding and measuring the voltage generated in the appropriate armature axis, with the machine driven at rated speed.

Table I gives the measured values of the parameters.

TABLE I

Resistance		Inductance		Rotation Voltage Coeff.	
Axis	Ohms	Axis	Henrys	Axis	Coeff.
r_1	1010	L_1	122	$L_2'p\theta$	42
r_2	8·42	L_2	0·1355	$L_3'p\theta$	42
r_3	8·42	L_3	0·1355	$M_{21}'p\theta$	1440
r_4	2·45	L_4	0·1177	$M_{25}'p\theta$	41·5
r_5	1	L_5	0·143	$M_{34}'p\theta$	42
r_Z	18	L_Z	0·169		
r_2*	1·3	M_{12}	3·82		
r_3*	1·3	M_{15}	3·76		
		M_{25}	0·126		
		M_{34}	0·1145		

* Stationary d.c. value.

Results

The equations of the machine, when connected first as an amplidyne and then as a metadyne, were obtained by substitution of the time constants obtained from Table I into Equations (10) to (13) as follows:

Amplidyne

Steady-state equation

$$\frac{I_L}{V}=\frac{11{\cdot}02}{9{\cdot}36+r_L}. \quad (14)$$

With 10 V applied to the control winding the load current was varied from 0 to 4·5 amps. The result is shown in Fig. 3 (*a*).

Transient equation

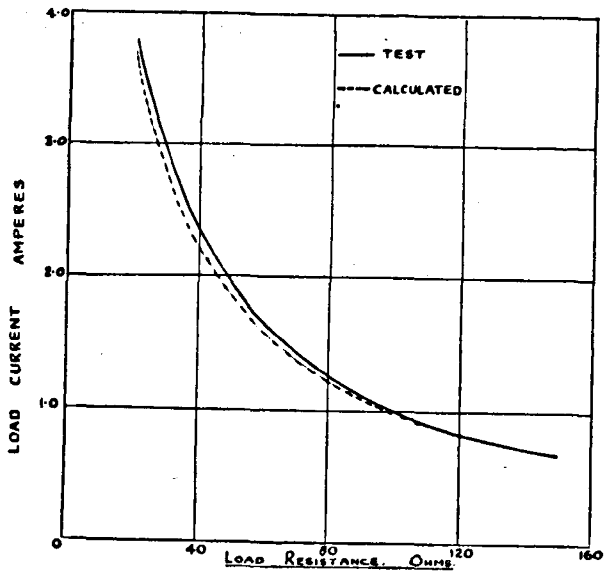
$$I_L=9{\cdot}9\,(0{\cdot}277-0{\cdot}439\,e^{-8{\cdot}26t}+0{\cdot}163\,e^{-22{\cdot}5t}). \quad (15)$$

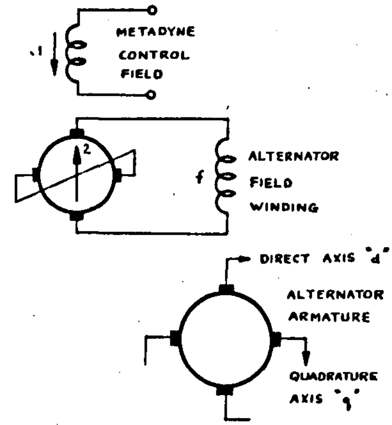
A step function of 9·9 V was applied and the result is shown in Fig. 4.

Metadyne

Steady-state equation

$$\frac{I_L}{V}=\frac{120}{3622+10{\cdot}87\,r_L}. \quad (16)$$





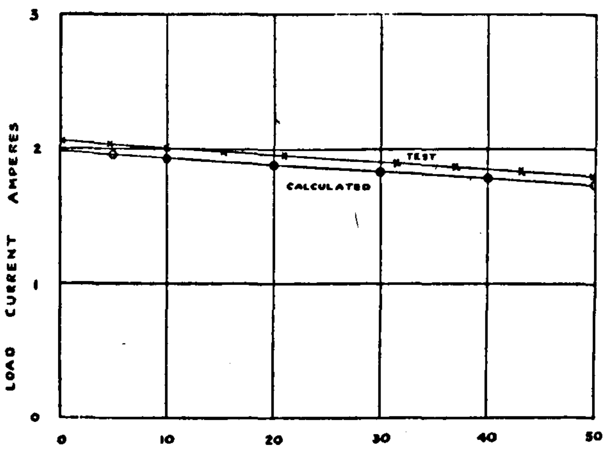


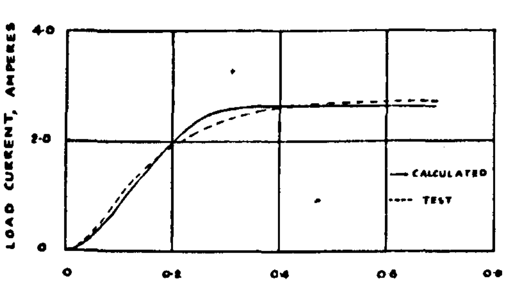
Fig. 3 (*a*) (*top left*). Amplidyne steady-state test

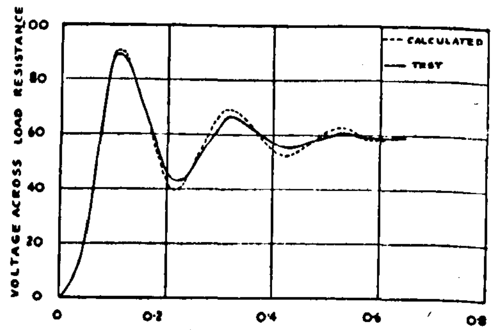
Fig. 3 (*b*) (*centre left*). Steady-state response of metadyne

Fig. 4 (*bottom left*). Transient response of amplidyne generator

Fig. 5 (*bottom right*). Transient response of metadyne

Fig. 6 (*top right*). Alternator with metadyne excitation





With 60 V applied to the control winding the load resistance was varied from 0 to 50 ohms. The result is shown in Fig. 3 (*b*).

Transient equation

$$I_L = \{1{\cdot}174 - 0{\cdot}0933\, e^{-111t} - 1{\cdot}223\, e^{-5{\cdot}5t} \sin(30{\cdot}6t + 1{\cdot}083)\}. \quad (17)$$

This is the solution of Equation (10) with load resistance 51·76 ohms in series with an inductance of 0·658 henry and a step-function voltage of 41 V applied to the control field. The result is shown in Fig. 5.

Metadyne Excitation of a Synchronous Machine

The advantage of setting up machine equations in matrix form is demonstrated by the following analysis. Operational equations of a synchronous machine are given below as used by Park, Crary, Concordia, and others.[5, 6]

It is seen that if the machine be excited from, say, a metadyne and the system analyzed in matrix form the operational form of the final equations is identical with those for direct excitation and that the various terms are obtained automatically. The analysis is given to show the method. The final equations are complicated but may be simplified by considering the Constant Flux Linkage Theorem.

Synchronous machine equations are expressed in terms of direct and quadrature axis quantities as follows [5]:

$$\left.\begin{array}{lll} \text{Field:} & & E_f = p\psi_f + r_f i_f \\ \text{Armature } D. \text{ Axis:} & & e_d = p\psi_d - \psi_q p\theta' - r i_d \\ \quad ,, \quad Q. \text{ Axis:} & & e_q = p\psi_q + \psi_d p\theta' - r i_q \end{array}\right\} \quad . \quad (18)$$

Where

E_f = excitation voltage.
ψ_f = resultant main field flux linkages.
ψ_d = direct axis resultant flux linkages.
ψ_q = quadrature axis resultant flux linkages.
r_f = field resistance.
r = armature resistance.
$p\theta'$ = is the machine rotor angular velocity.

In terms of inductances and currents the flux linkage equations may be written in matrix form.

$$\begin{array}{c|c|} f & E_f \\ d & -e_d \\ q & -e_q \end{array} = \begin{array}{c|c|c|c|} & f & d & q \\ \hline f & r_f + L_f p & M_d p & \\ d & M_d p & r + L_d p & -L_q p\theta' \\ q & M_d p\theta' & L_d p\theta' & r + L_q p \end{array} \cdot \begin{array}{c|c|} f & i_f \\ d & i_d \\ q & i_q \end{array} \quad . \quad . \quad . \quad . \quad (19)$$

where

$$\psi_d = -M_d i_f - L_d i_d$$
$$\psi_q = -L_q i_q$$
$$\psi_f = L_f i_f + M_d i_d.$$

Elimination of the field axes gives [7]

$$\begin{array}{c|c|} d & e_d - G(p)pE_f \\ q & e_q - G(p)p\theta' E_f \end{array} = \begin{array}{c|c|c|} & d & q \\ \hline d & -r - L_d(p)p & L_q(p)p\theta' \\ q & -L_d(p)p\theta' & -r - L_q(p)p \end{array} \cdot \begin{array}{c|c|} d & i_d \\ q & i_q \end{array} \quad . \quad . \quad (20)$$

where

$$G(p) = \frac{M_d}{r_f + L_f p} \cdot$$

$$L_d(p) = \frac{L_d r_f + L_d L_f p - M_d^2 p}{r_f + L_f p}$$

$$L_q(p) = L_q.$$

Now if the generator be excited from a metadyne the circuit is as shown in Fig. 6.

The metadyne matrix may be added to the alternator matrix thus

Met.	
	Alt.

. . (21)

If the quadrature axis in Matrix 8 be eliminated, Matrix 21 has the axes shown in Matrix 22.

	1	2	f	d	q
1					
2					
f					
d					
q					

. (22)

The connexion matrix combining the machines is

	1	2	d	q
1	1			
2		1		
$C=f$		-1		
d			1	
q				1

. (23)

The transformation $C_t.Z.C$ using this connexion and Matrix 22 gives a matrix of the form

	1	2	d	q
1				
2				
d				
q				

. (24)

Finally the axes 1 and 2 are eliminated, giving again the operational equations

$$\begin{array}{c|c|} d & e_d - G'(p)pV \\ q & e_q - G'(p)p\theta' V \end{array} \;\; \begin{array}{c} d \\ = \\ q \end{array} \;\; \begin{array}{|c|c|} \multicolumn{1}{c}{d} & \multicolumn{1}{c}{q} \\ -r-L_d'(p)p & L_q'(p)p\theta' \\ -L_d'(p)p\theta' & -r-L_q'(p)p \end{array} \cdot \begin{array}{c|c|} d & i_d \\ q & i_q \end{array} \quad . \quad . \quad . \quad (25)$$

where

$$G'(p)=M_d\left(M_{12}p+\frac{M_{21}'(L_3'+M_{34}')p\theta^2}{\Delta}\right)$$

$$L_d'(p)=\left\{L_d-\frac{M_d^2p(r_1+L_1p)}{\Delta}\right\}$$

$$L_q'(p)=L_q$$

$$\Delta=(r_1+L_1p)(r_2+r_f+L_2p+L_fp+A/C) - M_{12}p(M_{12}p+B/C)$$

$$A=L_3'(L_3'+M_{34}')p\theta^2$$

$$B=M_{21}'(L_3'+M_{34}')p\theta^2$$

$$C=r_3+r_4+(L_3+L_4+2M_{34})p$$

$p\theta$ = Electrical angular velocity of metadyne armature, radians/second

$p\theta'$ = Electrical angular velocity of alternator rotor, radians/second.

The Equations (25) would be simplified and used to analyse the transient behaviour of a synchronous machine in terms of the input voltage V to the metadyne control field.

A.C. MACHINES

The Single-Phase Series Motor

The circuit of the machine is shown in Figs. 7 (*a*) and 7 (*b*). The connexion matrix may be set up for (i) a motor with the compensating winding neutralizing armature reaction by being connected in series with the main field and in opposing series with the armature; (ii) compensation by short-circuited compensating winding, the transformer action between armature and quadrature stator winding setting up compensating flux. In each case the main field is in series with the armature quadrature axis brushes.

Series Compensation

Connexion matrix $C=$

	a
1	1
2	0
3	1
4	-1

. . . (26)

Operation on the primitive machine impedance matrix as before, *i.e.* $C_t.Z.C$, gives

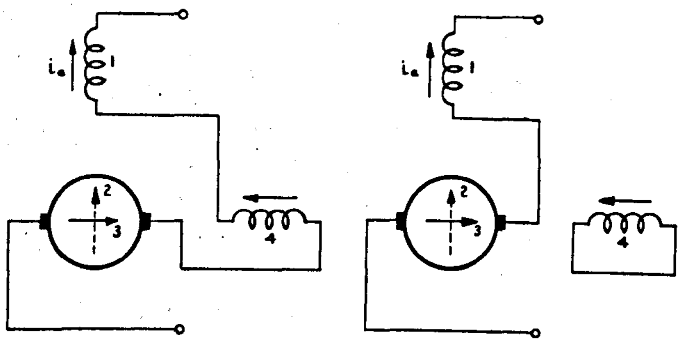
$$Z'=a\,\begin{array}{|c|} \multicolumn{1}{c}{a} \\ r_1+L_1p+M_{12}p\theta+r_3+L_3p-2M_{34}p \\ +r_4+L_4p \end{array} \qquad (27)$$

If compensated then

$$(L_3+L_4-2M_{34})=0.$$

Thus

$$V=(R+jX_1+vX_m)i \quad . \quad . \quad (28)$$



(a) (b)

Fig. 7 (*a*). Single-phase motor with series compensation

Fig. 7 (*b*). Single-phase motor with short-circuit compensation

where

$$V \equiv \text{applied r.m.s. voltage}$$
$$R = r_1 + r_3 + r_4$$
$$X_1 = \omega L_1$$
$$vX_m = M_{12} p\theta.$$

Where

$$p\theta = v\omega.$$

In the a.c. machine sinusoidal distribution of flux is assumed and the terms $M_{12}'p\theta$ arising in the d.c. machine may here be assumed equal to $M_{12}p\theta$. Saturation is not considered.

The torque matrix is, by $C_t . G . C$,

$$G' = \begin{array}{c|c|} & a \\ \hline a & M_{12} \\ \hline \end{array} \qquad . \quad . \quad (29)$$

and

$$f = i^* . \omega G' . i \text{ Synchronous watts,}$$

thus

$$f = \frac{V^2 X_m}{(R + vX_m)^2 + X_1^2} \qquad . \quad . \quad (30)$$

If

$$(L_3 + L_4 - 2M_{34}) \neq 0$$

then

$$f = \frac{V^2 X_m}{(R + vX_m)^2 + (X_1 + X_3 + X_4 - 2X_{34})^2} \quad . \quad (31)$$

Short-Circuit Compensation

The current in the compensating coil is i_s and the motor current is i_a. The connexion matrix is

$$C = \begin{array}{c|c|c|} & a & s \\ \hline 1 & 1 & \\ \hline 2 & & \\ \hline 3 & 1 & \\ \hline 4 & & -1 \\ \hline \end{array} \qquad . \quad . \quad (32)$$

$C_t . Z . C$ gives

$$Z' = \begin{array}{c|c|c|} & a & s \\ \hline a & r_1 + L_1 p + M_{12} p\theta + r_3 + L_3 p & -M_{34} p \\ \hline s & -M_{34} p & r_4 + L_4 p \\ \hline \end{array} \qquad (33)$$

Eliminating axis s,

$$V = \left(r_1 + L_1 p + M_{12} p\theta + r_3 + L_3 p - \frac{M_{34}^2 p^2}{r_4 + L_4 p} \right) i_a \qquad (34)$$

If fully compensated $L_3 L_4 = M_{34}^2$, an ideal case which neglects leakage. In this case, neglecting r_4 the equation is again

$$V = (R + jX_1 + vX_m) i_a. \qquad . \quad . \quad (35)$$

Torque is again $i_a^* . \omega G' . i_a$.

Consideration of the Coils undergoing Commutation

The commutation e.m.f. of self-induction is not included, but the short-circuit currents in the commutating coils, set up by transformer action and rotation can be simply considered. The coil is considered to be one or two turns of winding 2 of Fig. 1, short-circuited by the quadrature axis brushes. Thus the coil has voltages (*a*) induced by transformer action with the main field, and (*b*) rotation in quadrature axis flux.

The short-circuit current is i_d and for series compensation the connexion matrix is

$$C = \begin{array}{c|c|c|} & a & d \\ \hline 1 & 1 & \\ \hline 2 & & 1 \\ \hline 3 & 1 & \\ \hline 4 & -1 & \\ \hline \end{array} \qquad . \quad . \quad (36)$$

$C_t . Z . C$ gives

$$
\begin{array}{c|c|c|}
 & a & d \\
\hline
a & \begin{array}{c} r_1+r_3+r_4+ \\ (L_1+L_3+L_4-2M_{34})p \\ +M_{12}p\theta \end{array} & -M_{12}''p - L_2''p\theta \\
\hline
d & -M_{12}''p+(L_3-M_{34})p\theta & r_2''+L_2''p \\
\hline
\end{array}
\qquad (37)
$$

where r_2'' and L_2'' are the parameters of the short-circuited coil and M_{12}'' is the mutual inductance of the coil and main field winding. The equations are

$$
\left.\begin{array}{l}
V=[R+jX+vX_m]i_a-[jX_m''+vX_2'']i_d \\
0=[-jX_m''+v(X_3-X_{34})]i_a+[r_2''+jX_2'']i_d
\end{array}\right\} \qquad (38)
$$

The equations show that even with quadrature flux, i_d cannot be reduced to zero because of the quadrature time-phase difference between X_m'' and $v(X_3-X_{34})$. The e.m.f. of self-induction due to reversal of current may act with these two to give a smaller resultant and thus partial compensation of the instantaneous short-circuit current in the coils under the brushes.

The following equations show that the currents induced in the short-circuited coil give a positive additional torque by induction motor action.

$$
C_t . G . C = \begin{array}{c|c|c|}
 & a & d \\
\hline
a & M_{12} & -L_2'' \\
\hline
d & L_3-M_{34} & . \\
\hline
\end{array} \qquad (39)
$$

$$f = I^* . \omega G' . I$$

$$
I = \begin{array}{c|c|}
\hline
a & i_a \\
\hline
d & i_d \\
\hline
\end{array}
$$

$$f = i_a^* i_a X_{12} + i_a i_d (X_3 - X_{34}) - i_a i_d X_2'' \quad . \qquad (40)$$

The term $i_a i_d (X_3 - X_{34})$ is negligible since $(X_3 - X_{34})$ represents only leakage flux which is, comparatively, very small.

From Equation (38),

$$i_d = -\frac{jX_m''}{Z_2''}.$$

Thus the torque is the real part of

$$i_a^* \left(X_m + \frac{jX_m'' X_2''}{Z_2''} \right) i_a \quad . \quad . \qquad (41)$$

Circle diagrams for the machine may be obtained by writing the equation for input current in the form $i = Y . V$ where V is the constant applied terminal voltage. This equation can be written in the general form

$$i_P - j i_Q = \frac{(A+jB)+v(C+jD)}{(E+jF)+v(G+jH)} . V \quad . \qquad (42)$$

It has been shown [8] that the locus of the vector $i \equiv (i_P - j i_Q)$ in this case is a circle with coordinates

$$x = -\frac{V}{2} . \frac{(FD+CE-GA-BH)}{FG-EH}$$

$$y = -\frac{V}{2} . \frac{(HA+ED-CF-BG)}{FG-EH}$$

and radius

$$R = \left[\frac{V^2(DA-BC)}{FG-EH} + x^2 + y^2 \right]^{\frac{1}{2}} .$$

TESTING

The machine tested was rated at 7·5 h.p., 200 V, 25 c/s wound for six poles, with normal speed 960 r.p.m.

The resistances and reactances were measured by impedance-drop tests at rated frequency. Open and short-circuit tests

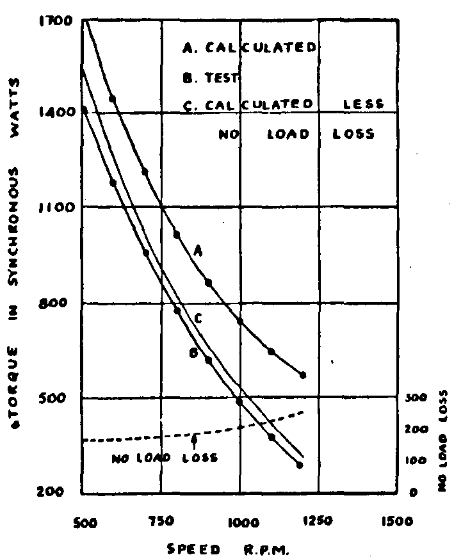


Fig. 8. Single-phase series motor

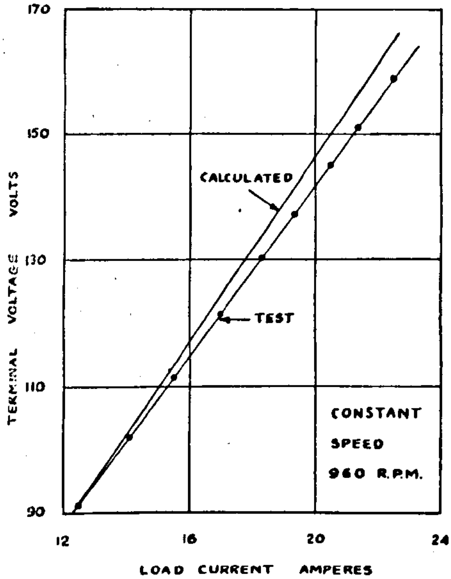


Fig. 9. Dynamic impedance of single-phase series motor

were carried out in each case and the resistances and reactances were referred to their respective windings. The internal connexions were brought to three sets of terminal pairs, namely, armature, main field, and quadrature windings. Since the equations were derived relative to such terminals on the primitive machine, the motor parameters were measured from these external terminal pairs. The parameters of simultaneously short-circuited coils were related to two fictitious direct axis terminals.

The measured parameters are shown in Table II:

TABLE II

Winding	Resistance ohms	Reactance ohms	Mutual reactance ohms
Main field	0·119	3·04	—
Compensating	0·279	3·49	—
Armature	0·277	3·97	—
Main field/arm.	—	—	3·04
Comp./arm.	—	—	3·49

Calculated and test results are shown in Figs. 8 and 9. These curves show that quite large no-load loss due to iron loss, friction, and windage must be included at some point in the analysis. Friction, especially in commutator machines with several sets of brushes, may absorb such a proportion of the input power that it cannot be neglected.

INDUCTION MACHINES

THREE-PHASE INDUCTION MOTOR

In order that the Equations (1) for the primitive machine may be used for a three-phase induction motor, the three-phase quantities must be resolved along stator and rotor axes stationary with respect to each other and each having direct axis and quadrature axis directions. This resolution has been used by Stanley.[9] The three-phase quantities are:

Stator voltages: e_a, e_b, e_c.
Rotor voltages: E_a, E_b, E_c.
Stator currents: i_a, i_b, i_c.
Rotor currents: I_a, I_b, I_c.

The stator currents are resolved into components similar to Clarke components α, β, 0.[10] In the following analysis of a balanced machine the '0' or zero-sequence components do not arise. The relation between α and β components in this case is also simplified.

The two-axis components of stator axis currents are defined as follows:

$$\left.\begin{aligned} i_1 &= \tfrac{2}{3}[i_a - \tfrac{1}{2}(i_b + i_c)] \\ i_4 &= \frac{1}{\sqrt{3}} \cdot (i_b - i_c) \end{aligned}\right\} \quad . \quad . \quad (43)$$

The two-axis components of rotor currents are defined

$$\left.\begin{aligned} i_2 = \tfrac{2}{3}[I_a \cos\theta + I_b \cos(\theta + 2\pi/3) \\ + I_c \cos(\theta - 2\pi/3)] \\ i_3 = \tfrac{2}{3}[I_a \sin\theta + I_b \sin(\theta + 2\pi/3) \\ + I_a \sin(\theta - 2\pi/3)] \end{aligned}\right\} \quad . \quad (44)$$

For the balanced case

$$\left.\begin{aligned} e_1 &= e_a & i_1 &= i_a \\ e_2 &= E_a & i_2 &= I_a \cos\theta \\ e_3 &= -jE_a = -je_2 & i_3 &= I_a \sin\theta = -ji_2 \\ e_4 &= -je_a & i_4 &= -ji_a \end{aligned}\right\} \quad (45)$$

The relationship between the three-phase machine parameters and their two-axes

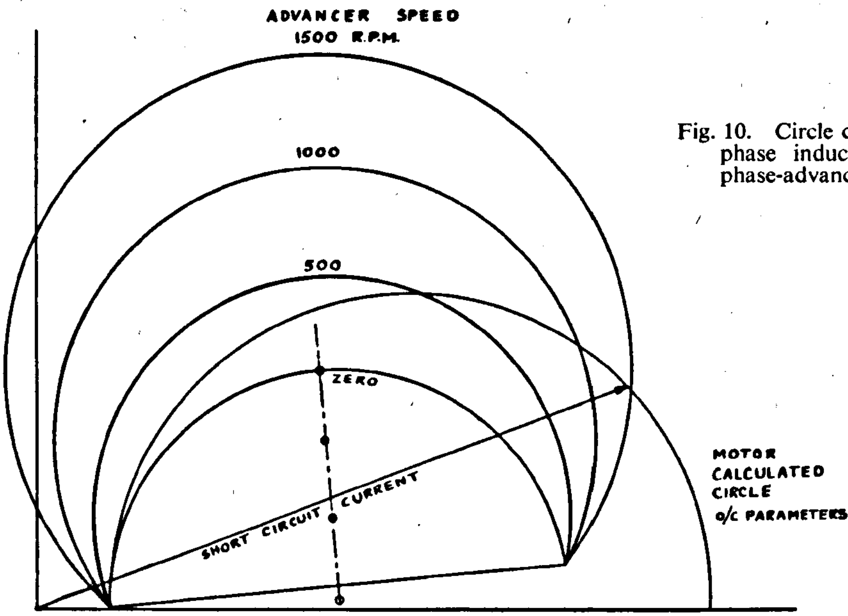


Fig. 10. Circle diagrams of three-phase induction motor with phase-advancer

components is explained below under 'Application of the induction motor equations'.

The 'connexion' relating the induction motor axis currents and the primitive machine is the unit matrix. The three-phase motor is thus analyzed as a two-phase machine whose currents, voltages, and parameters may be transformed by simple equations to give the three-phase quantities, and vice versa.

The Equations (1), therefore, are those of the two-axis form of the three-phase induction motor and also of the two-phase induction motor. Since no voltages are 'impressed' in the rotor axes of an ordinary induction motor, the voltages e_2 and e_3 are normally zero. These equations are seen to be identical with those derived by Stanley.

When the system is balanced the axes 3 and 4 may be eliminated[3] by the (non-invariant) transformation $1 . Z . P$ where

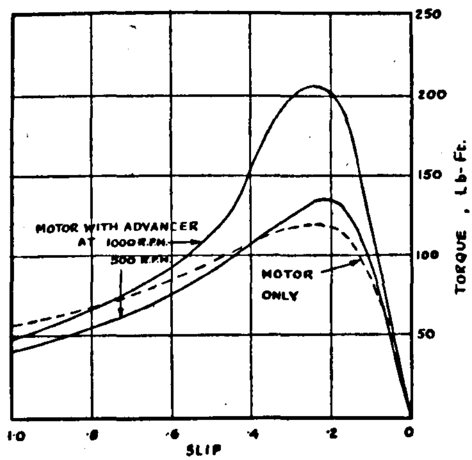
$$1 = \begin{array}{c|cccc|} & 1 & 2 & 3 & 4 \\ \hline 1 & 1 & & & \\ 2 & & 1 & & \\ 3 & & & 0 & \\ 4 & & & & 0 \\ \hline \end{array} \qquad P = \begin{array}{c|cc|} & 1 & 2 \\ \hline 1 & 1 & \\ 2 & & 1 \\ 3 & & -j \\ 4 & -j & \\ \hline \end{array} \quad . \quad . \quad . \quad (46)$$

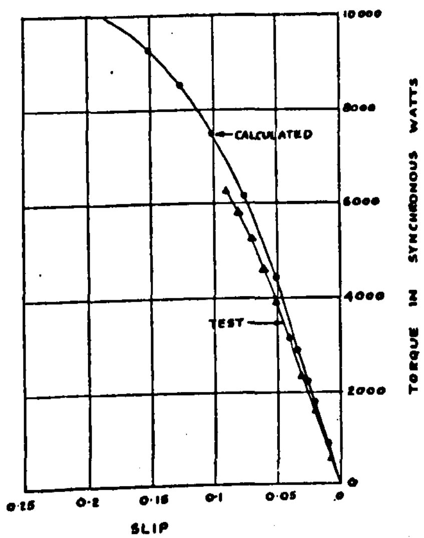
This operation gives for steady-state conditions, putting $p = j\omega$ and $p\theta = v\omega$ and $s = 1 - v$,

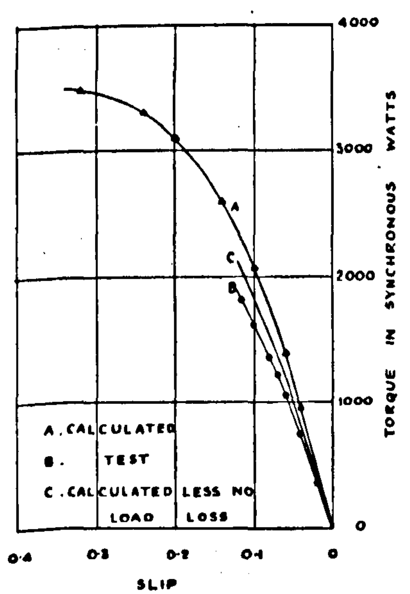
$$\begin{array}{c|c|} 1 & V \\ 2 & 0 \\ \end{array} = \begin{array}{c|c|c|} & 1 & 2 \\ \hline 1 & r_1 + jX_1 & jX_m \\ 2 & jsX_m & r_2 + jsX_2 \\ \hline \end{array} \cdot \begin{array}{c|c|} 1 & i_1 \\ 2 & i_2 \\ \end{array} \qquad (47)$$

Using the torque Matrix 2, the transformation $C_t . Z . C$ gives

$$G' = \begin{array}{c|cccc|} & 1 & 2 & 3 & 4 \\ \hline 1 & & & & \\ 2 & & & L_2 & M_{34} \\ 3 & -M_{12} & -L_2 & & \\ 4 & & & & \\ \hline \end{array} \quad . \qquad (48)$$







and the torque is $i^*.\omega G'.i$ where i is

1	i_1
2	i_2
3	i_3
4	i_4

. . . . (49)

For a balanced machine

$$i_3L_2i_2 = i_2L_3i_3$$

and

$$\text{torque} = -i^*_3X_mi_1 + i_2^*X_mi_4 \quad . \quad (50)$$

where

$$X_m = \omega M_{21} = \omega M_{34} = \omega M.$$

In Equations (47), V is the stator r.m.s. applied voltage per phase and i_1 and i_2 are the stator and rotor currents respectively. The equations are those of an equivalent two-phase motor. The total torque of the three-phase machine is then

$$f = -\tfrac{3}{2}i_3X_mi_1 + \tfrac{3}{2}i_2X_mi_4. \quad . \quad . \quad (51)$$

This equation may be verified by deriving the torque in terms of the three-phase current values.[9] For balanced conditions

$$i_3X_mi_1 = -i_2X_mi_4$$

from physical considerations and

$$\text{total torque three-phase} = -3i_3X_mi_1 = 3i_2^*.jX_mi_1 \quad . \quad (52)$$

$$\text{total torque two-phase} = -2i_3X_mi_1 = 2i_2^*.jX_mi_1 \quad . \quad (53)$$

Substituting the values of i_1 and i_2 obtained from Equations (47), the torque per phase of a three-phase or two-phase motor is given by

$$f = \frac{V^2sX_m^2r_2}{D^*D} \quad . \quad . \quad . \quad (54)$$

where

$$D = (r_1 + jX_1)(r_2 + jsX_2) + sX_m^2.$$

From Equations (47) the stator current is

$$i_1 = \frac{r_1 + jsX_1}{D} \cdot V \quad . \quad . \quad (55)$$

This may be written in the form of Equation (42) to give the circle diagram.

Fig. 11 (*top*). Torque speed curves for three-phase induction motor with phase-advancer
Fig. 12 (*centre*). Three-phase induction motor
Fig. 13 (*bottom*). Two-phase induction motor

Rotating Axes

The induction motor may also be analyzed from fixed stator axes together with axes rotating with the rotor. The transformation or 'connexion' matrix is, then, relative to the primitive machine of Fig. 1,

$C=$

	1	2r	3r	4
1	1			
2		$\cos\theta$	$-\sin\theta$	
3		$\sin\theta$	$\cos\theta$	
4				1

(56)

where $2r$ and $3r$ are rotor rotating axes.

The transformation of impedance to the required axes is given by [3]

$$Z' = C_t \cdot Z \cdot C + C_t \cdot L \cdot \frac{\partial C}{\partial \theta} \cdot p\theta. \qquad (57)$$

Where L is the primitive machine inductance matrix. For a balanced symmetrical machine the impedance matrix Z' may be simplified by elimination of the quadrature axes using the transformation $1 . Z' . P = Z''$ as before.

The equations then become

$$\begin{array}{c|c|} & \\ 1 & Ve^{j(\omega t+\phi)} \\ 2r & 0 \end{array} = \begin{array}{c|c|c|} & 1 & 2r \\ 1 & r_1+L_1p & Mpe^{j\theta} \\ 2r & Mpe^{-j\theta} & r_2+L_2p \end{array} \cdot \begin{array}{c|c|} & \\ 1 & i_1e^{j\omega t} \\ 2r & i_2e^{j\gamma} \end{array} \qquad (58)$$

where

$$\gamma = \omega t - \theta. \qquad (59)$$

and ϕ is the input power-factor angle.

Physically, this set of equations represents the machine currents and voltages in the form of vectors of constant magnitude rotating with angular velocities ω and $d\gamma/dt$ relative to their respective windings.

From Equation (59)

$$p\gamma = \omega - p\theta. \qquad (60)$$

For steady state

$$s\omega = \frac{d\gamma}{dt} = \omega - v\omega. \qquad (61)$$

Thus

$$Ve^{j(\omega t+\theta)} = i_1Z_1e^{j\omega t} + i_2M \cdot je^{j(\theta+\gamma)} \cdot \frac{d}{dt}(\theta+\gamma) \qquad (62)$$

or

$$Ve^{j\phi} = i_1Z_1 + i_2jX_m \qquad (63)$$

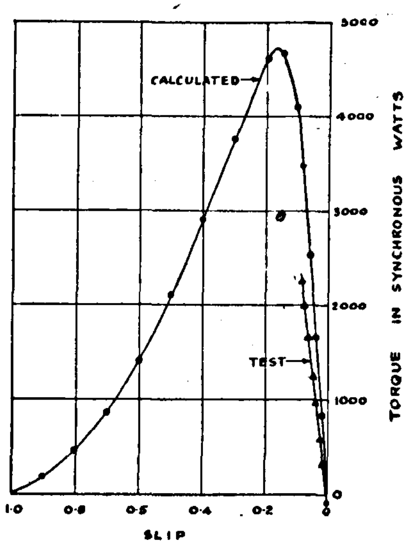


Fig. 14. Single-phase induction motor

and

$$0 = i_1Mpe^{j(\omega t-\theta)} + i_2\gamma_2e^{j\gamma} + i_2L_2pe^{j\gamma}. \qquad (64)$$

or

$$0 = i_1js\omega Me^{j\gamma} + (i_2r_2 + i_2js\omega L_2)e^{j\gamma} \qquad (65)$$

or

$$0 = i_1jsX_m + i_2(r_2 + jsX_2) \qquad (66)$$

These are seen to be identical with the equations obtained using stationary axes.

Simple Phase-advancer

The simplest type of phase-advancer consists of a wound rotor with a commutator, driven to rotate in an unwound stator, the stator merely completing the magnetic circuit. A two-pole armature will have three brushes set at 120° round the commutator. As with the induction motor, the three-phase currents in the rotor may be resolved into direct and quadrature axes relatively stationary.

The two-axis impedance matrix of the advancer may be written

$Z_a =$

	d	q
d	r_a+L_ap	$L_ap\theta'$
q	$-L_ap\theta'$	r_a+L_ap

(67)

where $p\theta'$ is the electrical angular velocity of the armature.

For balanced conditions

$$Z_a = \begin{array}{c|c|} & d \\ \hline d & r_a + js'X_a \\ \hline \end{array} \quad . \quad . \quad (68)$$

where s' is the slip of the armature current vector angular velocity relative to the rotating armature.

When used with an induction motor, the motor slip-rings on the rotor are connected to the advancer armature brushes.

In this case

$$s' = 1 - \nu'$$

where

$$\nu' = \frac{\text{Motor slip frequency angular velocity } \omega_s}{\text{advancer armature angular velocity } \omega_a}.$$

When Matrix 68 is combined with impedance matrix in Equation (47) for the induction motor, the overall motor impedance matrix becomes

$$\begin{array}{c|c|c|} & 1 & 2 \\ \hline 1 & r_1 + jX_1 & jX_m \\ \hline 2 & jsX_m & r_2 + r_a + jsX_2 + js'X_a \\ \hline \end{array} \quad . \quad (69)$$

It should be noted that when considering the interconnexion of the two machines, the reference axes must be the same in each case, if a simple connexion is to be used. In the above case stationary axes on each machine were used.

The effect of a phase-advancer on the motor performance is shown in Figs. 10 and 11.

SINGLE-PHASE INDUCTION MOTOR

The single-phase induction motor may be considered to be the same as the two-phase machine with zero quadrature stator current. The other three components are present since the single-phase motor has invariably a three-phase or two-phase rotor winding. The single-phase induction motor equations are thus

$$\begin{array}{c|c|} 1 & V \\ \hline 2 & 0 \\ \hline 3 & 0 \\ \hline \end{array} = \begin{array}{c|c|c|c|} & 1 & 2 & 3 \\ \hline 1 & r_1 + L_1 p & M_{12} p & \\ \hline 2 & M_{21} p & r_2 + L_2 p & L_3 p\theta \\ \hline 3 & -M_{21} p\theta & -L_2 p\theta & r_3 + L_3 p \\ \hline \end{array} \cdot \begin{array}{c|c|} 1 & i_1 \\ \hline 2 & i_2 \\ \hline 3 & i_3 \\ \hline \end{array} \quad . \quad . \quad . \quad . \quad (70)$$

Steady-state motor torque is again given by

$$-i_3^* \, . \, X_m \, . \, i_1.$$

Thus

$$f = \frac{V^2}{D^* D} [\nu X_{12}^2 r_2 \{ r_2^2 - X_2^2 (1 - \nu^2) \}] \quad . \quad (71)$$

where D is the determinant of the impedance matrix of Equation (70) and D^* is its conjugate.

Equation (71) is seen to give zero torque at standstill when $\nu = 0$ and a small negative torque at synchronous speed. It is seen to check identically with the equation given by Sabbagh.[11]

APPLICATION OF THE INDUCTION MOTOR EQUATIONS

The machine equations are relatively simple. For balanced operation it has been shown by Stanley and by Edith Clarke [8, 10] that the two axis parameters of a three-phase machine are identical with phase-to-neutral values measured with balanced three-phase applied voltages. For the three-phase machine, therefore, the parameters may be obtained from straightforward no-load and locked rotor tests. The machine resistances and inductances are referred to their respective windings. The two-phase and single-phase motors have usually a three-phase rotor winding. If, however, for a two-phase machine the stator parameters and the overall input leakage reactance be measured by balanced two-phase tests, it is possible to consider the rotor as an equivalent two-phase rotor carrying the same current and having fictitious values of parameters. The calculated values of reactance are referred to their respective windings, using the actual machine turns ratio phase to phase. If the rotor resistance be measured from the rotor terminals then this phase value is multiplied by 3/2 in order to keep the same value of rotor copper loss with the equivalent two-phase winding carrying the same current.

TABLE III

Data	Advancer	3-phase Motor	2-phase Motor
H.p.	—	15	5
Speed, r.p.m.	1500	945	1400
Voltage	15	440	200
Frequency c/s	50	50	50
Stator resistance (ohms)	—	0·562	0·98
Rotor resistance (ohms)	0·045	0·975	0·384*
Stator reactance (ohms)	—	32·8	48·2
Rotor reactance (ohms)	1·11	35·5	13·68*
Mutual reactance (ohms)	—	32·2	34·68

* Equivalent two-phase values.

The single-phase machine parameters may be determined in a similar manner.

If required, the derived equations may be used for more exact analysis by allowing for variations in the parameters due to carbon brushes, saturation, and skin effects. Table III gives the parameters of machines tested. Figs. 10 to 14 show the calculated and test results.

The two-phase machine was used for both two-phase and single-phase motor tests, in the latter case with one of the stator phases open-circuited.

CONCLUSION

The use of a primitive machine together with the application of matrix algebra is found to be a most powerful tool in the general analysis of electrical machinery. The fundamental concepts of inductively coupled windings remain unchanged throughout the analysis and the same unified approach and notation may be used. The only assumptions made are those already long established in the attempt to treat electrical machinery by linear analysis.

REFERENCES

1. Gibbs, *Tensors in Electrical Machine Theory* (London: Chapman and Hall, 1952).
2. Gibbs, *The Engineer*, **192**, 467, 485, 517, 546, 578, 1951.
3. Kron, *The Application of Tensors to the Analysis of Rotating Electrical Machinery*, 2nd Ed. (New York: *Gen. Elec. Rev.*, 1942).
4. Kron, *Short Course in Tensor Analysis*, p. 152 (New York: John Wiley, 1942).
5. Concordia, *Synchronous Machines* (New York: John Wiley, 1951).
6. Adkins, *J. Inst. Elect. Engrs*, Pt. II, **98**, 510, 1951.
7. Crary and Waring, *Gen. Elect. Rev.*, **35**, 578, 1932.
8. Liwschitz-Garik and Whipple, *Electric Machinery*, Vol. II, p. 184 (New York: Van Nostrand, 1946).
9. Stanley, *Trans. A. Inst. Elect. Engrs*, **57**, 751, 1938.
10. Clarke, *Circuit Analysis of A.c. Power Systems*, Vols. I and II (New York: John Wiley, 1943 and 1950).
11. Sabbagh, *Trans. A. Inst. Elect. Engrs*, Pt. II, **70**, 1748, 1951.

ACKNOWLEDGMENTS

The authors wish to thank Professor J. M. Meek, D.Eng., and Professor Emeritus F. J. Teago, D.Sc., of the University of Liverpool, for their interest and for the facilities granted in the laboratories of the Department of Electrical Engineering.

Reprinted from the BEAMA JOURNAL, *April and May* 1954

THE INSTITUTION OF ELECTRICAL ENGINEERS

FOUNDED 1871: INCORPORATED BY ROYAL CHARTER 1921

SAVOY PLACE, LONDON, W.C.2

THE TENSOR EQUATIONS OF ELECTRICAL MACHINES

By

J. W. LYNN, M.Sc., Associate Member

MONOGRAPH No. 117S

January, 1955

To be republished in

PART C OF THE PROCEEDINGS OF THE INSTITUTION

621.313.1.011

Monograph No. 117 S
Jan. 1955

THE TENSOR EQUATIONS OF ELECTRICAL MACHINES

By J. W. LYNN, M.Sc., Associate Member.

(*The paper was first received* 28*th December*, 1953, *and in revised form* 18*th August*, 1954. *It was published as an* INSTITUTION MONOGRAPH *in January*, 1955.)

SUMMARY

In the analysis of modern electrical machine systems there has been a trend towards the use of various components of the variables and parameters of the system, examples being the use of symmetrical components and direct- and quadrature-axis quantities. Such quantities may be considered to be different "reference frames" to which the various currents, fluxes, etc., are referred. Kron has shown that such transformations may be expressed in a general way when the equations of a network or machine are written in tensor form.

Transformations may then be carried out to any one of a number of reference axes, which may be stationary or rotating with respect to the windings of the machine or network. In particular it has been found that in certain cases there are advantages in analysing machines with respect to axes rotating with the flux.

The purpose of the present paper is to investigate the tensor form of the equations of electrical machines, to demonstrate the differences between tensor and non-tensor terms and to show how these terms are interpreted in application to simple cases. The equations of a primitive machine are examined in both stationary and rotating axes and the equations of a 3-phase series impedance and a 3-phase induction-motor are derived in both systems of reference.

LIST OF PRINCIPAL SYMBOLS

Tensor Notation

Indices:

a, b, c = Quantities in axes fixed to the machine stator and rotor windings.

k, n, m = Quantities in axes all relatively stationary.

α, β, γ = Quantities in axes fixed or free on the stator and rotating freely on the rotor.

s, t = Quantities associated with the mechanical rotational effects in the machine (e.g. generated voltages and torque).

u, v, w = Quantities in a general equation.

Electrical parts of the equations:

f_m, f_α, etc. = Electrical voltage vectors in axes denoted by indices.

x^k, x^α, etc. = Electric variables. The electrical charges in machine windings, referred to axes denoted by indices.

$\dot{x}^\alpha$ (equivalent to i^α) = Electric current vector, in axes denoted by indices.

$R_{\gamma\alpha}$ = Resistance matrix, in axes denoted by indices.

$L_{\gamma\alpha}$ = Inductance matrix, in axes denoted by indices.

$G_{\gamma\alpha}$ = Generated voltage coefficients, in axes denoted by indices.

Mechanical part of the equations:

f_s = Mechanical force.

x^s = Mechanical variable θ, the angular position of the machine rotor during rotation.

$\dot{x}^s$ (equivalent to i^s) = Angular velocity of machine rotor.

R_{st} = Mechanical friction coefficients.

L_{st} = Moment of inertia of machine rotor.

General symbols:

e = Generalized force vector (voltage or mechanical force).

R = Generalized dissipation matrix (resistance or friction).

L = Generalized inductance matrix (inductance or inertia).

i = Generalized current vector (electric current or angular velocity).

C^k_a = Connection matrix between quantities in axes denoted by indices.

C = Direct notation for C^k_a, etc.

C_t = Transpose of matrix C.

$\Omega^\delta_{\alpha\beta}$ = "Non-holonomic object" containing functions of C, in axes denoted by indices.

$[\alpha\beta, \gamma]$ = A "connection" term containing functions of $L_{\gamma\alpha}$ in axes denoted by indices.

$\Gamma_{\alpha\beta,\gamma}$ = A "connection" term containing both $[\alpha\beta, \gamma]$ and $\Omega_{\alpha\beta,\gamma}$ in axes denoted by indices.

T_{knm} = Rotational or torsion tensor containing the anti-symmetrical part of $\Gamma_{kn,m}$ in axes denoted by indices.

S_{knm} = Tensor components of T_{knm} giving the terms G_{mk}.

B = Flux-density matrix, a tensor.

φ = Flux-density matrix, a non-tensor.

Synchronous Machine Notation

e_f = Field voltage.

e_d = Direct-axis terminal voltage.

e_q = Quadrature-axis terminal voltage.

i^d = Direct-axis current.

i^q = Quadrature-axis current.

R_f = Field resistance.

R_d = Armature resistance in direct axis.

R_q = Armature resistance in quadrature axis.

L_f = Self-inductance of field winding.

L_d = Armature self-inductance in direct axis.

L_q = Armature self-inductance in quadrature axis.

M = Mutual inductance in direct axis (field-armature).

(1) INTRODUCTION

The recent increase in complexity of electrical power networks such as control systems and interconnected power-transmission systems has led to the introduction of systematic methods of analysing the behaviour of networks and machines. The powerful methods of symmetrical components brought about rapid methods of solution of problems associated with unbalanced polyphase circuits; the introduction of the two-reaction theory into synchronous-machine studies has simplified many aspects of the analysis of power-transmission systems. In dealing with complicated problems the trend has been towards the use of components of the quantities involved which, while they are entirely fictitious, lead to elegant solutions.[17,18] In such cases the actual variables and parameters of the system (currents, impedances, etc.) are transformed into the required components.

In 1934 Kron, in America, developed a technique[1] for dealing

Correspondence on Monographs is invited for consideration with a view to publication.
Mr. Lynn is in the Electrical Engineering Department, University of Liverpool.

systematically with such transformations. In his analysis Kron ensures that the electrical power in a network is invariant under a given transformation to new components. He does this by introducing the methods of the tensor calculus. Tensors are sets of quantities which are functions of a set of variables and are subject to certain laws of transformation when the variables are changed (see Appendix 10.1). The subject was developed in the field of the geometry of generalized spaces and consequently the terminology in the literature is largely geometrical; for example, the variables are referred to as co-ordinates. The following two properties of tensors are invaluable in application to any group of transformations:

(*a*) A set of functions usually written in the form g_{ab} is associated with tensor transformations, where a and b range over the n variables. This set of functions determines an invariant, which has the same magnitude in all reference frames.

The element of the invariant is usually written in the literature as dl and then

$$(dl)^2 = g_{ab}dx^a dx^b$$

where dx^a and dx^b are differentials of the variables. The use of this tensor is demonstrated in Appendix 10.1.

(*b*) If an entity is a tensor and exists with a non-zero value in one reference frame (or system of variables) then it has non-zero values, usually with different components, in all reference frames. Quantities which are not tensors may arise in one system of reference and become zero in another; in other words, they may arise because of the particular reference frame chosen. This effect occurs especially when there is relative motion between the reference axes.

Kron takes a primitive or elementary representative network which has comparatively simple equations; these are written in tensor form and transformed to give the equations of the required system.

A wide range of machines can be considered as a group of interconnections of the windings of the primitive machine shown in Fig. 2, and the equations of any of these may be obtained by transformation of the primitive-machine equations.

The expression "reference frame" denotes the system of measurement from which the variables and parameters are determined. In a static electrical network a change of components from branch to mesh currents is a change of reference frame.[2] In the case of a slip-ring induction motor, stator currents will be measured from stationary terminals and rotor currents from axes rotating with the rotor. In commutator machines the reference axes are relatively stationary. Park's application of Blondel's two-reaction theory simplified synchronous-machine theory by converting the stator and rotor quantities to relatively stationary axes. Kron's analysis deals systematically with equations of machine systems having either stationary or relatively moving reference axes, the transformations in all cases following the routine laws of matrix algebra and tensor calculus. His work has introduced to engineers wider concepts of transformation, invariance and theory of groups, all of which have been invaluable in analysis of complex systems in various branches of physics. The purpose of the present paper is twofold, namely to bring together some of the scattered works dealing with Kron's methods and present a continuous account of the development of the subject, and to analyse in detail the distinction between tensor and non-tensor terms in machine equations.

The analysis starts from the dynamical equation of Lagrange.[5] As is shown in Appendix 10.2, Lagrange's equation gives the relation between the potential and kinetic energies in a system and the applied forces, in terms of generalized co-ordinates. In electrical form the corresponding relations are those between the magnetic energy and the applied voltages, in terms of generalized variables which are the electrical charges in the network. This equation is very suitable for certain classes of transformations of co-ordinates, but it has been found that under the conditions of transformation obtaining in electrical machines a modified form of Lagrange's equation must be used. The modified equation was developed by Boltzmann and Hamel[13] to cover certain conditions of constraint in dynamical systems and, as Kron has shown,[6] the Boltzmann–Hamel equation can be used to form a basis for tensor analysis of electrical machines from the dynamical viewpoint.

(2) MACHINE EQUATIONS

The first type of primitive electrical machine to be considered is shown in Fig. 1. The rotor is assumed to be smooth and to

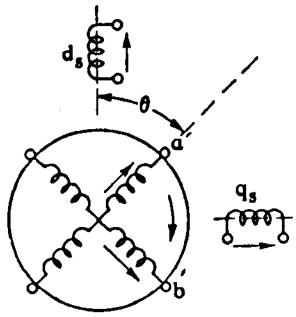


Fig. 1.—Primitive machine with axes fixed to windings.

have on it a symmetrical 2-phase winding sinusoidally distributed. The field is fixed in space and consists of windings *ds* and *qs* in the stator—direct and quadrature axes respectively. Iron loss and saturation are neglected. The armature axes a′ and b′ are fixed to the armature and rotate with it. Three-phase machines may be analysed by resolving the resultant armature current and flux vectors along two similar axes.[7]

The inductance of phase a′ of this machine may be written:[4]

$$\text{Phase a}' \text{ self-inductance} = L_A + L_B \cos 2\theta \quad . \quad (1)$$

where $L_A = (L_{dr} + L_{qr})/2$

$L_B = (L_{dr} - L_{qr})/2$

L_{dr} and L_{qr} are the self-inductances of rotor phase a′ when in the direct- and quadrature-axis positions respectively. The corresponding values of mutual inductance, rotor to stator, are M_d and M_q.

The equations of this machine may be derived from the dynamical equation of Lagrange:[4]

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^c}\right) - \frac{\partial T}{\partial x^c} + \frac{\partial F}{\partial \dot{x}^c} = f_c \quad . \quad . \quad . \quad . \quad (2)$$

In electrical machines the generalized variables x^c are the electrical charges in the circuits and the rotor angle θ. The quantities $\dot{x}^c$ therefore represent the currents i^c and the rotor angular velocity $d\theta/dt$. T, the stored magnetic energy, is given by

$$T = \tfrac{1}{2}L_{ab}\frac{dx^a}{dt}\frac{dx^b}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

F, the dissipative force, is given by

$$F = \tfrac{1}{2}R_{ab}\frac{dx^a}{dt}\frac{dx^b}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

For the above machine[4]

$$\begin{aligned} T = {} & \tfrac{1}{2}(L_A + L_B \cos 2\theta)(i^{a'})^2 + \tfrac{1}{2}(L_A - L_B \cos 2\theta)(i^{b'})^2 \\ & + \tfrac{1}{2}L_{ds}(i^{ds})^2 - L_B \sin 2\theta i^{a'} i^{b'} + M_d \cos \theta i^{ds} i^{a'} \\ & - M_d \sin \theta i^{ds} i^{b'} + M_q \sin \theta i^{qs} i^{b'} + M_q \cos \theta i^{qs} i^{b'} \\ & + \tfrac{1}{2}L_{qs}(i^{qs})^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5) \end{aligned}$$

$$F = \tfrac{1}{2}[R_{a'}(i^{a'})^2 + R_{b'}(i^{b'})^2 + R_{ds}(i^{ds})^2 + R_{qs}(i^{qs})^2] \quad (6)$$

Substituting eqns. (5) and (6) in eqn. (2), the complete set of equations for the machine may be written

$$e = Ri + pLi \quad \ldots \quad (7)$$

where $\left(p \equiv \frac{d}{dt}\right)$

In matrix form these are[2]

				ds	a′	b′	qs		
ds	e_{ds}		ds	$R_{ds} + pL_{ds}$	$pM_d \cos\theta$	$-pM_d \sin\theta$		ds	i^{ds}
a′	e_a	=	a′	$pM_d \cos\theta$	$R_{dr} + p(L_{dr}\cos^2\theta + L_{qr}\sin^2\theta)$	$p(L_{qr} - L_{dr})\sin\theta\cos\theta$	$pM_q \sin\theta$	a′	$i^{a'}$
b′	e_b		b′	$-pM_d \sin\theta$	$p(L_{qr} - L_{dr})\sin\theta\cos\theta$	$R_{qr} + p(L_{dr}\sin^2\theta + L_{qr}\cos^2\theta)$	$pM_q \cos\theta$	b′	$i^{b'}$
qs	e_{qs}		qs		$pM_q \sin\theta$	$pM_q \cos\theta$	$R_{qs} + pL_{qs}$	qs	i^{qs}

. . . . (8)

Note.—The index associated with each current value is written as a superscript since this is required by the index notation used in tensor calculus.

From eqn. (8) the inductance matrix may be written as below. An additional mechanical row and column s may be added to include the mechanical inertia.

The general inductance matrix is therefore:

	ds	a′	b′	qs	s
ds	L_{ds}	$-M_d \cos\theta$	$-M_d \sin\theta$		
a′	$M_d \cos\theta$	$L_{dr}\cos^2\theta + L_{qr}\sin^2\theta$	$(L_{qr} - L_{dr})\sin\theta\cos\theta$	$M_q \sin\theta$	
b′	$-M_d \sin\theta$	$(L_{qr} - L_{dr})\sin\theta\cos\theta$	$L_{dr}\sin^2\theta + L_{qr}\cos^2\theta$	$M_q \cos\theta$	
qs		$M_q \sin\theta$	$M_q \cos\theta$	L_{qs}	
s					L_{ss}

. . . . (9)

where L_{ss} is the moment of inertia of the rotor.

It is shown in Appendix 10.3 that in tensor form the equations of Lagrange are written

$$L_{cb}\frac{d^2x^b}{dt^2} + [ab,c]\frac{dx^a}{dt}\frac{dx^b}{dt} + R_{cb}\frac{dx^b}{dt} = f_c \quad \ldots \quad (10)$$

where
$$[ab,c] \equiv \frac{1}{2}\left(\frac{\partial L_{cb}}{\partial x^a} + \frac{\partial L_{ca}}{\partial x^b} - \frac{\partial L_{ab}}{\partial x^c}\right) \quad \ldots \quad (11)$$

The voltage equation is obtained by allowing the free index c to cover the electrical range of variables, the other indices ranging over the electrical and mechanical parts:[1]

$$e_c = L_{cb}\frac{d^2x^b}{dt^2} + [as,c]\frac{dx^a}{dt}\frac{dx^s}{dt} + [sb,c]\frac{dx^s}{dt}\frac{dx^b}{dt} + R_{cb}\frac{dx^b}{dt} \quad . \quad (12)$$

or
$$e_c = L_{cb}\frac{di^b}{dt} + [as,c]i^a i^s + [sb,c]i^s i^b + R_{cb}i^b \quad . \quad . \quad (13)$$

From eqn. (11),

$$[as,c] = \frac{1}{2}\left(\frac{\partial L_{cs}}{\partial x^a} + \frac{\partial L_{ca}}{\partial x^s} - \frac{\partial L_{as}}{\partial x^c}\right)$$

Since there is no mutual coupling between an electrical row and a mechanical column

$$\frac{\partial L_{cs}}{\partial x^a} = \frac{\partial L_{as}}{\partial x^c} = 0$$

$$[as,c]i^a i^s = \frac{1}{2}\frac{\partial L_{ca}}{\partial x^s} i^a \frac{dx^s}{dt} = \frac{1}{2}\frac{dL_{ca}}{dt}i^a \quad \ldots \quad (14)$$

$$[sb,c]i^s i^b = \frac{1}{2}\frac{dL_{cb}}{dt}i^b \quad \ldots \ldots \quad (15)$$

Thus Lagrange's equations give

$$e_c = R_{cb}i^b + L_{cb}\frac{di^b}{dt} + \frac{dL_{cb}}{dt}i^b \quad \ldots \quad (16)$$

or
$$e_c = R_{cb}i^b + \frac{d}{dt}(L_{cb}i^b) \quad \ldots \quad (17)$$

This is, of course, Maxwell's equation as shown in eqn. (7). The equation of torque is obtained by allowing the free index to cover only the mechanical part of the range of variables,[1] thus

$$f_s = R_{st}i^t + L_{st}\frac{d^2x^t}{dt^2} + [ab,s]i^a i^b \quad \ldots \quad (18)$$

where
$$[ab,s] = -\frac{1}{2}\frac{\partial L_{ab}}{\partial\theta}$$

Therefore
$$f_s = R_{st}\frac{d\theta}{dt} + L_{st}\frac{d^2\theta}{dt^2} - \frac{1}{2}\frac{\partial L_{ab}}{\partial\theta}i^a i^b \quad \ldots \quad (19)$$

$$R_{st}\frac{d\theta}{dt} = \text{frictional torque}$$

$$L_{st}\frac{d^2\theta}{dt^2} = \text{inertia torque}$$

and
$$-\frac{1}{2}\frac{\partial L_{ab}}{\partial\theta}i^a i^b = \text{electrical torque}$$

The second form of the primitive machine considered here is shown in Fig. 2. The rotating axes a' and b' of Fig. 1 are here

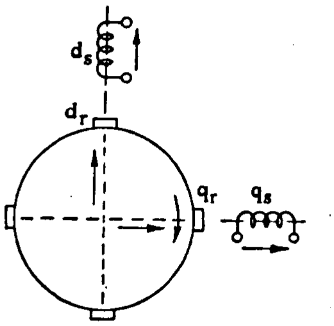


Fig. 2.—Primitive machine with stationary axes.

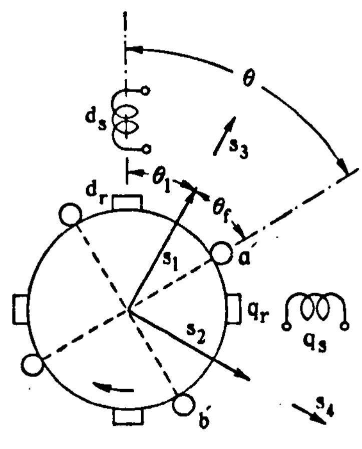


Fig. 3.—Primitive machine with axes rotating freely.

resolved along the direct and quadrature axes. All axes are now relatively stationary. This primitive machine has been used by Sabbagh, Stanley, Kron and others[7–11] as the basis of the analysis of many derived machines. The equations may be written down by inspection when the resistance voltage-drops and induced and generated voltages are considered. In matrix form these are written

$$\begin{array}{c} ds \\ dr \\ qr \\ qs \end{array}\begin{bmatrix} e_{ds} \\ e_{dr} \\ e_{qr} \\ e_{qs} \end{bmatrix} = \begin{array}{c} \\ ds \\ dr \\ qr \\ qs \end{array}\begin{array}{c} \begin{array}{cccc} ds & dr & qr & qs \end{array} \\ \begin{bmatrix} R_{ds}+L_{ds}p & M_dp & & \\ M_dp & R_{dr}+L_{dr}p & L'_{qr}p\theta & M'_qp\theta \\ -M'_dp\theta & -L'_{dr}p\theta & R_{qr}+L_{qr}p & M_qp \\ & & M_qp & R_{qs}+L_{qs}p \end{bmatrix}\end{array} \cdot \begin{array}{c} ds \\ dr \\ qr \\ qs \end{array}\begin{bmatrix} i^{ds} \\ i^{dr} \\ i^{qr} \\ i^{qs} \end{bmatrix} \quad . \ . \ . \ . \ . \ (20)$$

The voltage vector contains impressed voltages in all axes. The generated voltage coefficients are written, for example, $M'_dp\theta$ as compared with M_dp for the induced voltages since the flux waveform may not be sinusoidal. When the flux waveform is sinusoidal the maximum voltage generated by rotation of a coil in this flux at synchronous speed will be equal to the maximum voltage induced by this flux linking the coil and $M'_dp\theta i$ is equal to M_dpi and $M'_d = M_d$. Eqns. (20) are seen to include those for a synchronous machine according to Park's two-reaction theory.[18] These as used by Concordia[12] are written:

$$\left.\begin{array}{l} \text{Impressed field voltage, } e_f = i^fR_f + L_fpi^f + M_dpi^{dr} \\ \text{Generated voltage,} \\ \qquad e_d = -M_dpi^f - R_di^{dr} - L_dpi^{dr} + L_qp\theta i^{qr} \\ \text{Generated voltage,} \\ \qquad e_q = -M_dp\theta i^f - L_dp\theta i^{dr} - R_qi^{qr} - L_qpi^{qr} \end{array}\right\} \quad . \ (21)$$

In synchronous-machine studies the quadrature-axis stator (field) coils are omitted unless amortisseur windings are being considered. Here sinusoidal flux distribution is considered and $M'_d = M_d$, etc.

Eqns. (20) may be obtained from those of the previous form of primitive machine using the relationship

$$\left.\begin{array}{l} i^{a'} = i^{dr}\cos\theta + i^{qr}\sin\theta \\ i^{b'} = i^{qr}\cos\theta - i^{dr}\sin\theta \end{array}\right\} \quad . \ . \ . \ . \ (22)$$

In index notation the stationary-axis equations may be written[2]

$$\text{Voltage, } e_m = R_{mn}i^n + L_{mn}\frac{di^n}{dt} + G_{mn}i^np\theta \quad . \ . \ (23)$$

$$\text{Torque, } f_s = R_{st}i^t + L_{st}\frac{d^2\theta}{dt^2} - G_{mn}i^mi^n \quad . \ . \ (24)$$

Using the concepts of the tensor calculus these two equations are written by Kron[2] as one electro-mechanical equation,

$$e_w = R_{wv}i^v + L_{wv}\frac{di^v}{dt} + \Gamma_{uv,w}i^ui^v \quad . \ . \ . \ (25)$$

$$\text{voltage } e_m = R_{mn}i^n + L_{mn}\frac{di^n}{dt} + \Gamma_{ks,m}i^ki^s + \Gamma_{sn,m}i^si^n \ . \ (26)$$

$$\text{torque } f_s = R_{st}i^t + L_{st}\frac{di^t}{dt} + \Gamma_{kn,s}i^ki^n \quad . \ . \ (27)$$

The equation of Maxwell, eqn. (17), does not give these equations directly since it does not include generated voltages. The following Sections show that the connection $\Gamma_{uv,w}$ arises naturally because of the dynamical relationship between the two types of primitive machine, this being quasi-holonomic and non-integrable. Fig. 4 shows the form of the connection $\Gamma_{uv,w}$ when written as a matrix in the form of a cube, together with the arrangement of matrix multiplication leading from eqns. (26) and (27) to eqns. (23) and (24).

(3) NON-HOLONOMIC TRANSFORMATIONS

The currents in the armature axes a′ and b′ in Fig. 1 may be resolved along "d" and "q" axes shown in Fig. 2, the relationship being

$$\left.\begin{array}{l} i^{dr} = i^{a'}\cos\theta - i^{b'}\sin\theta \\ i^{qr} = i^{a'}\sin\theta + i^{b'}\cos\theta \end{array}\right\} \quad . \ . \ . \ . \ (28)$$

Since the variables in Lagrange's equations are the charges, x^a, eqns. (28) represent a transformation of differentials of the variables, where $dx^a/dt = i^a$, etc. These are equations of con-

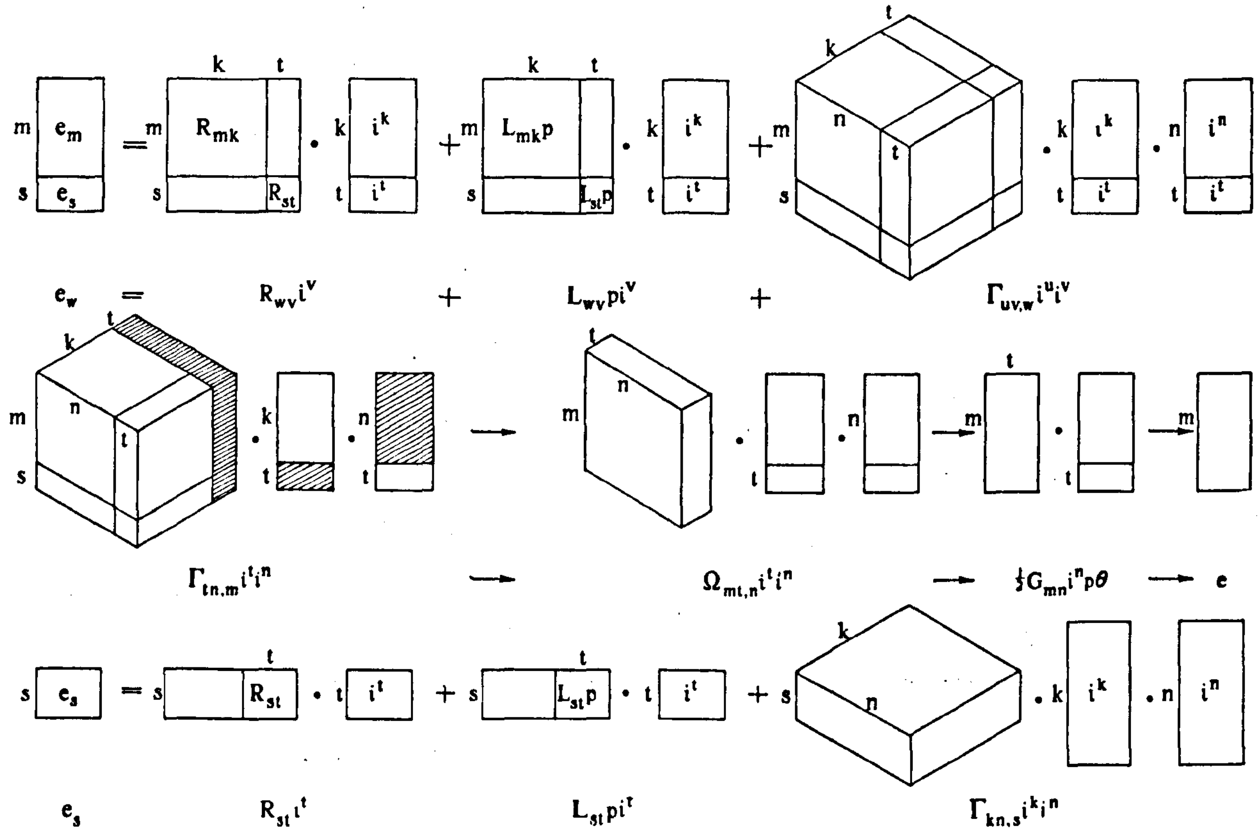


Fig. 4.—Machine tensor equation written in matrix form.

straint among the differentials of the variables x^k, and the transformation must therefore be written

$$\left.\begin{aligned} dx^{dr} &= dx^{a'}\cos\theta - dx^{b'}\sin\theta \\ dx^{qr} &= dx^{a'}\sin\theta + dx^{b'}\cos\theta \end{aligned}\right\} \quad . \quad . \quad . \quad (29)$$

They obtain at a given instant and cannot be integrated to give a relationship among the charges. There is no corresponding relationship

$$x^{dr} = f(x^{a'}, x^{b'}, \theta)$$

If such a function existed the following results would be obtained:[2]

$$dx^{dr} = \frac{\partial f^{dr}}{\partial x^{a'}}dx^{a'} - \frac{\partial f^{dr}}{\partial x^{b'}}dx^{b'}$$

Thus if

$$\frac{\partial f^{dr}}{\partial x^{a'}} = C^{dr}_{a'}$$

Then

$$\frac{\partial C^{dr}_{a'}}{\partial\theta} = -\sin\theta = \frac{\partial f^{dr}}{\partial\theta\partial x^{a'}} = \frac{\partial f^{dr}}{\partial x^{a'}\partial\theta} = \frac{\partial C^{dr}_{\theta}}{\partial x^{a'}} = 0$$

Such non-integrable relationship between sets of variables is fully treated in Reference 13.

In matrix form the transformation is written

$$dx^k = C^k_{.a}dx^a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

or

$$\begin{array}{c|c|} k\diagdown & \\ d & dx^{dr} \\ q & dx^{qr} \end{array} = \begin{array}{c|c|c|} k\diagdown a & a' & b' \\ d & \cos x^s & -\sin x^s \\ q & \sin x^s & \cos x^s \end{array} \begin{array}{c|c|} a\diagdown & \\ a' & dx^{a'} \\ b' & dx^{b'} \end{array} \quad (30a)$$

where x^s is the geometrical variable θ.

The dot in front of the index a indicates that a is a column index. The inverse transformation may be expressed

$$dx^a = C^a_{.k}dx^k$$

or

$$\begin{array}{c|c|} a\diagdown & \\ a' & dx^{a'} \\ b' & dx^{b'} \end{array} = \begin{array}{c|c|c|} a\diagdown k & d & q \\ a' & \cos x^s & \sin x^s \\ b' & -\sin x^s & \cos x^s \end{array} \begin{array}{c|c|} k\diagdown & \\ d & dx^{dr} \\ q & dx^{qr} \end{array} \quad (31)$$

Since the relationship is non-integrable and only the differentials of the charges x^k arise in the equations,

$$C^k_{.a} \neq \frac{\partial x^k}{\partial x^a} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32)$$

also, as shown in Appendix 10.3,

$$\frac{\partial C^a_m}{\partial x^n} \neq \frac{\partial C^a_n}{\partial x^m} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33)$$

Such a relationship is non-holonomic.[13] The non-holonomic form of Lagrange's equations was developed by Boltzmann and Hamel.[13] This is written

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^c}\right) - \frac{\partial T}{\partial x^c} + \frac{\partial T}{\partial \dot{x}^d}C^k_cC^n_a\left(\frac{\partial C^d_k}{\partial x^n} - \frac{\partial C^d_n}{\partial x^k}\right)\dot{x}^a + \frac{\partial F}{\partial \dot{x}^c} = f_c \quad . \quad (34)$$

This is seen to consist of Lagrange's eqn. (2) with the addition of terms containing the non-holonomic transformation and its inverse. If the relation between variables is holonomic,

$$\frac{\partial C^d_k}{\partial x^n} - \frac{\partial C^d_n}{\partial x^k} = 0 \quad . \quad . \quad . \quad . \quad . \quad (35)$$

Eqn. (34) may be written in tensor form in a manner similar to eqn. (10), and as shown by Kron[6] and Gibbs[5] it contains terms which retain the non-integrable relations of eqn. (33). The tensor form of the equation is

$$f_c = L_{ca}\frac{d^2x^a}{dt^2} + [ab,c]\frac{dx^a}{dt}\frac{dx^b}{dt} + R_{ca}\frac{dx^a}{dt} \quad . \quad . \quad (36)$$

where now

$$[ab,c] = \frac{1}{2}\left(\frac{\partial L_{cb}}{\partial x^a} + \frac{\partial L_{ca}}{\partial x^b} - \frac{\partial L_{ab}}{\partial x^c}\right) + \Omega_{cb,a} + \Omega_{ca,b} - \Omega_{ab,c} \quad . \quad (37)$$

where $$\Omega_{ab,c} = \Omega^d_{ab}L_{dc} \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

and $$\Omega^d_{ab} \equiv \tfrac{1}{2}C^k_aC^n_b\left(\frac{\partial C^d_k}{\partial x^n} - \frac{\partial C^d_n}{\partial x^k}\right) \quad . \quad . \quad . \quad . \quad (39)$$

(4) QUASI-HOLONOMIC SYSTEMS[6]

The machine axes transformations in Section 3 are not entirely non-holonomic because the angle θ, the mechanical part of the generalized variable x^k, arises in the analysis and transforms holonomically; i.e. the mechanical part of the transformation is one of variables and not of differentials. This simplifies the analysis considerably.

In transforming from the rotating-axis machine to that with stationary axes the transformation may be expressed

$$\left.\begin{aligned} dx^{ds} &= dx^{ds} \\ dx^{a'} &= \cos x^s\, dx^{dr} + \sin x^s\, dx^{qr} \\ dx^{b'} &= -\sin x^s dx^{dr} + \cos x^s dx^{qr} \\ dx^{qs} &= dx^{qs} \\ dx^s &= dx^s \end{aligned}\right\} \quad . \quad . \quad . \quad (40)$$

or $dx^a = C^a_{.k}dx^k$

$$\begin{array}{c|c|} a\diagdown & \\ \hline ds & dx^{ds} \\ a' & dx^{a'} \\ b' & dx^{b'} \\ qs & dx^{qs} \\ s & dx^s \end{array} = \begin{array}{c|c|c|c|c|c|} a\diagdown k & ds & dr & qr & qs & s \\ \hline ds & 1 & & & & \\ a' & & \cos x^s & \sin x^s & & \\ b' & & -\sin x^s & \cos x^s & & \\ qs & & & & 1 & \\ s & & & & & 1 \end{array} \quad \begin{array}{c|c|} k\diagdown & \\ \hline ds & dx^{ds} \\ dr & dx^{dr} \\ qr & dx^{qr} \\ qs & dx^{qs} \\ s & dx^s \end{array} \quad (41)$$

There are two distinct parts of the range of the old variables, namely x^P, the electrical variables, and x^Q, the geometrical variable θ. The L's and R's, whether electrical or mechanical, depend only on x^Q and do not contain the electrical co-ordinates x^P. Such an absence of specific co-ordinates in the components of geometric objects is called in geometry a "cylinder condition."[16]

In a holonomic reference frame the geometric objects are expressed as functions of the co-ordinates x^k, and in a new holonomic system they become functions of new co-ordinates. In a non-holonomic system, however, there exists instead of a set of new co-ordinates, a set of new differentials. Relative to a non-holonomic reference frame the geometric objects are expressed as functions of new differentials, and coefficients, which are functions of the old co-ordinates. In general, therefore, the new equations carry forward the original holonomic co-ordinates in these coefficients, along with the new differentials. In this analysis of machines, however, the geometric objects of the new frame may be expressed without carrying forward the old co-ordinates, because the coefficients are functions only of θ, which transforms holonomically to the same value in the new reference frame.

Thus, even though the new equations have been derived by non-holonomic transformation, there is nothing in their final form to characterize them as being in a non-holonomic reference frame. Therefore, as shown in Section 5, the final equations of the stationary-axis machine may be considered to be holonomic, with a connection $\Gamma_{kn,m}$ which differs in form from that of the first holonomic machine. It is for this reason that a further non-holonomic object may be set up between the machine in reference frame (iii) in Section 5 and that in frame (ii) (considered as holonomic).

The equations of the second primitive machine (with stationary axes) are now written

$$f_m = L_{mk}\frac{d^2x^k}{dt^2} + \left[\frac{1}{2}\left(\frac{\partial L_{mn}}{dx^k} + \frac{\partial L_{mk}}{dx^n} - \frac{\partial L_{kn}}{dx^m}\right) + \Omega_{mn,k} + \Omega_{mk,n} - \Omega_{kn,m}\right]\frac{dx^k}{dt}\frac{dx^n}{dt} + R_{mk}\frac{dx^k}{dt} \quad . \quad (42)$$

where $$L_{mk} = L_{ac}C^a_mC^c_k \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

$$dx^a = C^a_{.m}dx^m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (44)$$

and $$\Omega_{mn,k} = \tfrac{1}{2}C^a_mC^b_n\left(\frac{\partial C^h_a}{\partial x^b} - \frac{\partial C^h_b}{\partial x^a}\right)L_{hk} \quad . \quad . \quad . \quad (45)$$

From eqns. (41) and (43)

$$L_{mk} = \begin{array}{c|c|c|c|c|c|} m\diagdown k & ds & dr & qr & qs & s \\ \hline ds & L_{ds} & M_d & & & \\ dr & M_d & L_{dr} & & & \\ qr & & & L_{qr} & M_q & \\ qs & & & M_q & L_{qs} & \\ s & & & & & L_{ss} \end{array} \quad (46)$$

Thus the terms $\partial L_{kn}/\partial x_m$, etc., in eqn. (42) are zero since the inductances are not functions of either charges or angles.

It is obvious that the term C^h_a may be differentiated only with respect to the mechanical part of the range of variables, x^s, and the expansion of the non-holonomic object is therefore simplified. For example, in eqn. (45), either a or b must be x^s (or θ). Let the values of a, b, m, and n range from 1 to 5 to cover the five rows and columns of C^m_a. The only values of C^h_a and C^h_b inside the bracket of eqn. (45) which can be differentiated to give non-zero values are

$$a = 5, \quad b = 2 \text{ or } 3$$
$$b = 5, \quad a = 2 \text{ or } 3$$

Since either a or b must be 5, the following possible values of $C^a_mC^b_n$ may be written down

$$C^5_5C^2_2 \quad C^5_5C^3_2 \quad C^2_2C^5_5 \quad C^3_2C^5_5$$
$$C^5_5C^2_3 \quad C^5_5C^3_3 \quad C^2_3C^5_5 \quad C^3_3C^5_3$$

It is further seen by inspection that when a and b are both 5, Ω^h_{mn} is zero, and when a and b are unequal, one being 5, one term of the bracketed quantity is zero.

The general equations are therefore

$$2\Omega^h_{m(s)} = C^a_m C^5_5 \frac{\partial C^h_a}{\partial x^5} \quad . \quad . \quad . \quad . \quad . \quad (47)$$

$$2\Omega^h_{(s)n} = -\,C^5_5 C^b_n \frac{\partial C^h_b}{\partial x^5} \quad . \quad . \quad . \quad . \quad . \quad (48)$$

The possible non-zero values of Ω^h_{mn} may be written down by inspection and are shown in the following array:

$\Omega^2_{mn} =$ . . . (49)

$m \backslash n$	1	2	3	4	5
1					
2					Ω^2_{25}
3					Ω^2_{35}
4					
5		Ω^2_{52}	Ω^2_{53}		

A similar array may be written for Ω^3_{mn}. Also

$$2\Omega^2_{(5)3} = C^5_5 C^b_3 \frac{\partial C^2_b}{\partial x^5} \quad (b = 2, 3)$$

$$= \sin^2 x^s + \cos^2 x^s = 1 \quad . \quad . \quad . \quad (50)$$

Similarly

$$\left.\begin{array}{l} 2\Omega^2_{3(5)} = -\,1 \\ 2\Omega^3_{2(5)} = 1 \\ 2\Omega^3_{5(2)} = -\,1 \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad (51)$$

All the other values of Ω^h_{mn} are zero.

The resulting object is $\Omega^h_{m(s)}$ or $\Omega^h_{(s)n}$ according to whether m or n takes the geometrical angle co-ordinate value, x^s:

$2\Omega^h_{(s)n} =$. (52*a*)

$h \backslash n$	ds	dr	qr	qs	s
ds					
dr			1		
qr		-1			
qs					
s					

$2\Omega^h_{m(s)} =$. (52*b*)

$h \backslash m$	ds	dr	qr	qs	s
ds					
dr			-1		
qr		1			
qs					
s					

The object Ω^h_{mn} is seen to be skew-symmetric in the indices m and n.

In the equation of Lagrange the "Christoffel symbol" is written $[ab,c]$. In the non-holonomic form of the equation the corresponding connection is

$$[uv,w] + \Omega_{wv,u} + \Omega_{wu,v} - \Omega_{uv,w} \quad . \quad . \quad . \quad (53)$$

This is made up of the symmetrical part $[uv,w]$ and the skew-symmetric part $(\Omega_{wv,u} + \Omega_{wu,v} - \Omega_{uv,w})$, these together making an asymmetrical connection[2] written $\Gamma_{uv,w}$.

The non-holonomic equation of motion of the machine is therefore written as in eqn. (25):

$$e_w = R_{wv}\frac{dx^v}{dt} + L_{wv}\frac{d^2x^v}{dt^2} + \Gamma_{uv,w}\frac{dx^u}{dt}\frac{dx^v}{dt} \quad . \quad . \quad (54)$$

The electrical and mechanical parts of the equation are shown in Section 2, eqns. (26) and (27).

The symmetrical part is zero since L_{wv} is independent of x^s (which is equivalent to θ). The connection has then the following components:

$$\Gamma_{uv,w}i^u i^v \equiv \Gamma_{sn,m}i^s i^n + \Gamma_{ks,m}i^k i^s + \Gamma_{kn,s}i^k i^n. \quad . \quad (55)$$

where, for example,

$$\Gamma_{sn,m}i^s i^n \equiv (\Omega_{mn,s} + \Omega_{ms,n} - \Omega_{sn,m})i^s i^n \quad . \quad . \quad (56)$$

where $i^s \equiv \frac{d\theta}{dt}$

The term $\Omega_{mn,s}$ is seen from eqns. (45) and (48) to be zero, and since $\Omega_{sn,m}$ is skew symmetric in the indices s and n, the term $\Omega_{sn,m}i^s i^n$ is also zero.

Therefore

$$\Gamma_{sn,m}i^s i^n = \Omega_{ms,n}i^s i^n \quad . \quad . \quad . \quad . \quad . \quad (57)$$

$$\Gamma_{ks,m}i^k i^s = \Omega_{ms,k}i^k i^s \quad . \quad . \quad . \quad . \quad . \quad (58)$$

$$\Gamma_{kn,s}i^k i^n = 2\Omega_{sn,k}i^k i^n \quad . \quad . \quad . \quad . \quad . \quad (59)$$

Thus the electrical equation becomes

$$e_m = R_{mn}i^n + L_{mn}\frac{di^n}{dt} + 2\Omega_{ms,k}i^s i^k \quad . \quad . \quad . \quad (60)$$

and the mechanical equation is

$$f_s = R_{st}i^t + L_{st}\frac{di^t}{dt} + 2\Omega_{sn,k}i^n i^k \quad . \quad . \quad . \quad (61)$$

The last terms in eqns. (60) and (61) are found on expansion, using eqns. (46) and (52), to be

$2\Omega_{ms,k} =$ (table below) $= -\,2\Omega_{sn,k}$ (62)

$m \backslash k$	ds	dr	qr	qs	s
ds					
dr			L_{qr}	M_q	
qr	$-M_d$	$-L_{dr}$			
qs					
s					

Eqn. (60) is now seen to be identical with the matrix eqn. (20) written down by inspection in Section 2. The equations for the commutator or stationary-axis machine have been derived by transformation of the dynamical equation of motion of the holonomic rotating-axis or slip-ring machine.

If now a further transformation be carried out to axes rotating at any arbitrary angular velocity $p\theta_1$ it is found that some of the terms of the equation transform tensorially while additional voltage terms arise which are due only to the measurements from the new reference frame. Tensor analysis separates the tensor quantities which arise in all reference frames, from those quantities which may arise in one reference frame and disappear in others. The nature of the connection $\Gamma_{uv,w}$ is examined by considering the transformation to general rotating axes. The objects $\Gamma_{uv,w}$ and Ω^{w}_{uv} have the following laws of transformation:[2,5]

$$\Gamma_{u'v',w'} = \Gamma_{uv,w}C^{u}_{u'}C^{v}_{v'}C^{w}_{w'} + L_{wu}C^{w}_{w'}\frac{\partial C^{u}_{u'}}{\partial x^{v'}} \quad . \quad . \quad (63)$$

and $$\Omega^{w'}_{u'v'} = \Omega^{w}_{uv}C^{u}_{u'}C^{v}_{v'}C^{w'}_{w} + \frac{1}{2}\left(\frac{\partial C^{w'}_{u}}{\partial x^{v}} - \frac{\partial C^{w'}_{v}}{\partial x^{u}}\right)C^{u}_{u'}C^{v}_{v'} \quad . \quad (64)$$

The quantities $\Gamma_{ur,w}$ and $\Omega_{ur,w}$ are therefore not tensors.

The equations of the electrical machine with general rotating axes are derived in Section 5.

(5) THREE REFERENCE FRAMES

(5.1) Electrical Equations

(5.1.1) The Holonomic Frame. (i)

$$e_c = R_{ca}i^a + pL_{ca}i^a \quad . \quad . \quad . \quad . \quad . \quad . \quad (65)$$

$$e_c = R_{ca}i^a + L_{ca}pi^a + [ab,c]i^a i^b \quad . \quad . \quad . \quad (66)$$

(5.1.2) The Stationary Axis Frame. (ii)

From Fig. 2 it is seen that the transformation from frame (i) to frame (ii) is given by

$$dx^c = C^c_{.m}dx^m$$

where

$C^c_{.m} =$

c\m	ds	dr	qr	qs
ds	1			
a'		$\cos\theta$	$\sin\theta$	
b'		$-\sin\theta$	$\cos\theta$	
qs				1

. . (69)

The equations for frame (ii) are

$$e_m = R_{mk}i^k + L_{mk}pi^k + \Gamma_{kn,m}i^k i^n \quad . \quad . \quad . \quad (70)$$

where $$\Gamma_{kn,m}i^k i^n \equiv \Gamma_{sn,m}i^s i^n + \Gamma_{ks,m}i^k i^s \quad . \quad . \quad . \quad (71)$$

Eqn. (65) is shown in Section 2, eqn. (7). The inductance tensor is shown in matrix 9. From the inductance matrix and eqn. (11),

$$[ab,c]i^a i^b \equiv [sb,c]i^s i^b + [as,c]i^a i^s = 2[as,c]i^a i^s$$

and $$2[as,c]i^a i^s = \frac{\partial L_{ca}}{\partial\theta}\frac{d\theta}{dt}i^a$$

From matrix 9, $\frac{\partial L_{ca}}{\partial\theta}$ is

c\a	ds	a'	b'	qs
ds		$-M_d\sin\theta$	$-M_d\cos\theta$	
a'	$-M_d\sin\theta$	$2(L_{qr}-L_{dr})\sin\theta\cos\theta$	$(L_{qr}-L_{dr})(\cos^2\theta-\sin^2\theta)$	$M_q\cos\theta$
b'	$-M_d\cos\theta$	$(L_{qr}-L_{dr})(\cos^2\theta-\sin^2\theta)$	$-2(L_{qr}-L_{dr})\cos\theta\sin\theta$	$-M_q\sin\theta$
qs		$M_q\cos\theta$	$-M_q\sin\theta$	

. . . . (67)

The equation is therefore $$e_c = R_{ca}i^a + L_{ca}pi^a + \frac{\partial L_{ca}}{\partial\theta}\frac{d\theta}{dt}i^a \quad . \quad . \quad . \quad (68a)$$

This may be written $$e_c = R_{ca}i^a + L_{ca}pi^a + V_{ca}i^a p\theta + G_{ca}i^a p\theta \quad . \quad . \quad . \quad (68b)$$

where

$V_{ca} =$

c\a	ds	a'	b'	qs
ds		$-M_d\sin\theta$	$-M_d\cos\theta$	
a'		$(L_{qr}-L_{dr})\sin\theta\cos\theta$	$-L_{qr}\sin^2\theta - L_{dr}\cos^2\theta$	
b'		$L_{qr}\cos^2\theta + L_{dr}\sin^2\theta$	$-(L_{qr}-L_{dr})\sin\theta\cos\theta$	
qs		$M_q\cos\theta$	$-M_q\sin\theta$	

. . (68c)

$G_{ca} =$

c\a	ds	a'	b'	qs
ds				
a'	$-M_d\sin\theta$	$(L_{qr}-L_{dr})\sin\theta\cos\theta$	$L_{dr}\sin^2\theta + L_{qr}\cos^2\theta$	$M_q\cos\theta$
b'	$-M_d\cos\theta$	$-L_{dr}\cos^2\theta - L_{qr}\sin^2\theta$	$-(L_{qr}-L_{dr})\sin\theta\cos\theta$	$-M_q\sin\theta$
qs				

. . (68d)

Also $\Gamma_{kn,m} i^k i^n = [kn,m] i^k i^n + (\Omega_{mn,k} + \Omega_{mk,n} - \Omega_{kn,m}) i^k i^n$
$= 2\Omega_{ms,k} i^s i^k$. (72)

	ds	S_1	S_2	qs
ds	L_{ds}	$M_d \cos\theta_1$	$-M_d \sin\theta_1$	
S_1	$M_d \cos\theta_1$	$L_{dr}\cos^2\theta_1 + L_{qr}\sin^2\theta_1$	$(L_{qr} - L_{dr})\sin\theta_1\cos\theta_1$	$M_q \sin\theta_1$
S_2	$-M_q \sin\theta_1$	$(L_{qr} - L_{dr})\sin\theta_1\cos\theta_1$	$L_{dr}\sin^2\theta_1 + L_{qr}\cos^2\theta_1$	$M_q \cos\theta_1$
qs		$M_q \sin\theta_1$	$M_q \cos\theta_1$	L_{qs}

(80)

The terms L_{mk} and $2\Omega_{ms,k}$ are shown in Section 4, matrices 46 and 62. The equations may therefore be written

$$e_m = R_{mk} i^k + L_{mk} p i^k + G_{mk} i^k p\theta \quad . \quad . \quad . \quad (73)$$

(5.1.3) The Frame having Rotor Axes Rotating Freely. (iii)

The transformation matrix from frame (i) to frame (iii) is

$C^a_{\cdot\alpha} =$

$a \backslash \alpha$	ds	S_1	S_2	qs
ds	1			
a'		$\cos\theta_f$	$\sin\theta_f$	
b'		$-\sin\theta_f$	$\cos\theta_f$	
qs				1

. (74)

where, from Fig. 3, $\theta_f = \theta - \theta_1$.

and $$L_{\gamma\alpha} = L_{ca} C^c_\gamma C^a_\alpha \quad . \quad . \quad . \quad . \quad . \quad (79)$$

This becomes

The geometrical variable x^s transforming holonomically remains θ, from matrix eqn. (40).

Now $$[\alpha\beta,\gamma] = \frac{1}{2}\left(\frac{\partial L_{\gamma\beta}}{\partial x^\alpha} + \frac{\partial L_{\gamma\alpha}}{\partial x^\beta} - \frac{\partial L_{\alpha\beta}}{\partial x^\gamma}\right) \quad . \quad . \quad (81)$$

and $$[s\beta,\gamma] i^s i^\beta + [\alpha s,\gamma] i^\alpha i^s = 2[\alpha s,\gamma] i^\alpha i^s \quad . \quad . \quad . \quad (82)$$

and $$2[\alpha s,\gamma] i^\alpha i^s = \left(\frac{\partial L_{\gamma s}}{\partial x^\alpha} + \frac{\partial L_{\gamma\alpha}}{\partial x^s} - \frac{\partial L_{\alpha s}}{\partial x^\gamma}\right) i^\alpha p\theta$$

$$= \frac{\partial L_{\gamma\alpha}}{\partial\theta} i^\alpha p\theta$$

$$= \frac{\partial L_{\gamma\alpha}}{\partial\theta_1}\frac{d\theta_1}{d\theta}\frac{d\theta}{dt} i^\alpha$$

Therefore $$2[\alpha s,\gamma] i^\alpha i^s = \frac{\partial L_{\gamma\alpha}}{\partial\theta_1}\frac{d\theta_1}{dt} i^\alpha \quad . \quad . \quad . \quad . \quad (83)$$

$\frac{\partial L_{\gamma\alpha}}{\partial\theta_1} =$

$\gamma \backslash \alpha$	ds	S_1	S_2	qs
ds		$-M_d \sin\theta_1$	$-M_d \cos\theta_1$	
S_1	$-M_d \sin\theta_1$	$2(L_{qr} - L_{dr})\sin\theta_1\cos\theta_1$	$L_{qr}\cos^2\theta_1 - L_{qr}\sin^2\theta_1$ $-L_{dr}\cos^2\theta_1 + L_{dr}\sin^2\theta_1$	$M_q \cos\theta_1$
S_2	$-M_d \cos\theta_1$	$L_{qr}\cos^2\theta_1 - L_{qr}\sin^2\theta_1$ $-L_{dr}\cos^2\theta_1 + L_{dr}\sin^2\theta_1$	$-2(L_{qr} - L_{dr})\cos\theta_1\sin\theta_1$	$-M_q \sin\theta_1$
qs		$M_q \cos\theta_1$	$-M_q \sin\theta_1$	

(84)

The equations in frame (iii) are

$$e_\gamma = R_{\gamma\alpha} i^\alpha + L_{\gamma\alpha} p i^\alpha + \Gamma_{\alpha\beta,\gamma} i^\alpha i^\beta \quad . \quad . \quad . \quad (75)$$

where $$\Gamma_{\alpha\beta,\gamma} i^\alpha i^\beta \equiv \Gamma_{s\beta,\gamma} i^s i^\beta + \Gamma_{\alpha s,\gamma} i^\alpha i^s + \Gamma_{\alpha\beta,s} i^\alpha i^\beta \quad . \quad (76)$$

and $$\Gamma_{\alpha\beta,\gamma} i^\alpha i^\beta = \{[\alpha\beta,\gamma] + \Omega_{\gamma\beta,\alpha} + \Omega_{\gamma\alpha,\beta} - \Omega_{\alpha\beta,\gamma}\} i^\alpha i^\beta \quad . \quad (77)$$

The electrical part of the equation therefore becomes

$$e_\gamma = R_{\gamma\alpha} i^\alpha + L_{\gamma\alpha} p i^\alpha + \{2[\alpha s,\gamma] i^\alpha i^s + 2\Omega_{\gamma s,\alpha} i^\alpha i^s\} \quad . \quad (78)$$

From eqns. (38) and (47)

$$2\Omega_{\gamma s,\alpha} i^s i^\alpha = \frac{\partial C^{\cdot\delta}_c}{\partial\theta} C^{\cdot c}_\gamma L_{\delta\alpha}\frac{d\theta}{dt} i^\alpha$$

$$= \frac{\partial C_t^{-1}}{\partial\theta} C_t L_{\delta\alpha}\frac{d\theta}{dt} i^\alpha \quad . \quad . \quad . \quad . \quad (85)$$

Therefore $$2\Omega_{\gamma s,\alpha} i^s i^\alpha = \frac{\partial C_t^{-1}}{\partial\theta_f} C_t L_{\delta\alpha}\frac{d\theta_f}{dt} i^\alpha \quad . \quad . \quad . \quad . \quad (86)$$

where C_t is the transpose of matrix $C^{\cdot c}_\gamma$ (matrix 74). $2\Omega_{\gamma s,\alpha}$ is then

$\gamma \backslash \alpha$	ds	S_1	S_2	qs
ds				
S_1	$-M_d \sin\theta_1$	$(L_{qr} - L_{dr})\sin\theta_1\cos\theta_1$	$L_{dr}\sin^2\theta_1 + L_{qr}\cos^2\theta_1$	$M_q \cos\theta_1$
S_2	$-M_d \cos\theta_1$	$-L_{dr}\cos^2\theta_1 - L_{qr}\sin^2\theta_1$	$-(L_{qr} - L_{dr})\sin\theta_1\cos\theta_1$	$-M_q \sin\theta_1$
qs				

(87)

The equations may therefore be written

$$e_\gamma = R_{\gamma\alpha}i^\alpha + L_{\gamma\alpha}pi^\alpha + V_{\gamma\alpha}i^\alpha p\theta_1 + G_{\gamma\alpha}i^\alpha p\theta_1 + G_{\gamma\alpha}i^\alpha p\theta_f \quad . \quad (88)$$

or $$e_\gamma = R_{\gamma\alpha}i^\alpha + L_{\gamma\alpha}pi^\alpha + V_{\gamma\alpha}i^\alpha p\theta_1 + G_{\gamma\alpha}i^\alpha p\theta \quad . \quad . \quad (89)$$

The equations of the machine in reference frame (iii) may also be obtained by transformation of the reference frame (ii).

From Fig. 3 the transformation matrix is

$C^m_{.\gamma} =$ . . (90)

$m \diagdown \gamma$	ds	S_1	S_2	qs
ds	1			
dr		$\cos\theta_1$	$-\sin\theta_1$	
qr		$\sin\theta_1$	$\cos\theta_1$	
qs				1

$$L_{\gamma\alpha} = L_{mk}C^m_\gamma C^k_\alpha = \text{matrix } 80 \quad . \quad . \quad . \quad (91)$$

The equations may be written

$$e_\gamma = R_{\gamma\alpha}i^\alpha + L_{\gamma\alpha}pi^\alpha + \Gamma_{\alpha\beta,\gamma}i^\alpha i^\beta \quad . \quad . \quad . \quad (92)$$

where $$\Gamma_{\alpha\beta,\gamma} \equiv [\alpha\beta,\gamma] + \Omega_{\gamma\beta,\alpha} + \Omega_{\gamma\alpha,\beta} - \Omega_{\alpha\beta,\gamma} \quad . \quad (92a)$$

The terms $\Omega_{\alpha\beta,\gamma}$ are non-holonomic objects of the frame (iii) with respect to the holonomic frame (i), and

$$\Omega_{\alpha\beta,\gamma} = \Omega_{kn,m}C^k_\alpha C^n_\beta C^m_\gamma + \frac{1}{2}\left(\frac{\partial C^\delta_k}{\partial x^n} - \frac{\partial C^\delta_n}{\partial x^k}\right)C^k_\alpha C^n_\beta L_{\delta\gamma} \quad . \quad (93)$$

Now $$\Gamma_{\alpha\beta,\gamma}i^\alpha i^\beta = \{2[\alpha s,\gamma] + 2\Omega_{\gamma s,\alpha}\}i^\alpha p\theta \quad . \quad . \quad (94)$$

Also $$2\Omega_{\gamma s,\alpha} = 2\Omega_{ms,k}C^m_\gamma C^s_s C^k_\alpha + \frac{\partial C^\delta_k}{\partial x^s}C^k_\alpha C^s_s L_{\delta\gamma} \quad . \quad . \quad (95)$$

and $2\Omega'_{\gamma s,\alpha} =$ matrix 87

where $$\Omega'_{\gamma s,\alpha} \equiv \Omega_{ms,k}C^m_\gamma C^s_s C^k_\alpha \quad . \quad . \quad . \quad . \quad (96)$$

And $$\frac{\partial C^\delta_k}{\partial\theta}C^k_\alpha L_{\delta\gamma}p\theta = \frac{\partial C^\delta_k}{\partial\theta_1}C^k_\alpha L_{\delta\gamma}p\theta_1$$

$$\frac{\partial C^\delta_k}{\partial\theta_1}C^k_\alpha L_{\delta\gamma} = \text{minus matrix } 87 \quad . \quad . \quad . \quad (97)$$

$$2[\alpha s,\gamma] = \text{matrix } 84 \quad . \quad . \quad . \quad . \quad . \quad . \quad (98)$$

The equations therefore may be written

$$e_\gamma = R_{\gamma\alpha}i^\alpha + L_{\gamma\alpha}pi^\alpha + V_{\gamma\alpha}i^\alpha p\theta_1 + G_{\gamma\alpha}i^\alpha p\theta_1 + G_{\gamma\alpha}i^\alpha p\theta - G_{\gamma\alpha}i^\alpha p\theta_1 \quad . \quad (99)$$

or $$e_\gamma = R_{\gamma\alpha}i^\alpha + L_{\gamma\alpha}pi^\alpha + V_{\gamma\alpha}i^\alpha p\theta_1 + G_{\gamma\alpha}i^\alpha p\theta \quad . \quad (100)$$

(5.2) Equations of Torque

(5.2.1) Frame (i).

$$f_s = R_{st}\frac{d\theta}{dt} + L_{st}\frac{d^2\theta}{dt^2} + [ab,s]i^a i^b \quad . \quad . \quad (101)$$

or $$f_s = R_{st}\frac{d\theta}{dt} + L_{st}\frac{d^2\theta}{dt^2} - \frac{1}{2}\frac{\partial L_{ab}}{\partial\theta}i^a i^b \quad . \quad . \quad (102)$$

An asymmetrical matrix A may be written as the sum of symmetrical and skew-symmetrical parts, thus[2]

$$A = \frac{A + A_t}{2} + \frac{A - A_t}{2}$$

Using this relationship it may be shown that

$$\frac{1}{2}\frac{\partial L_{ab}}{\partial\theta}i^a i^b = G_{ab}i^a i^b \quad . \quad . \quad . \quad . \quad . \quad (103)$$

thus $$f_s = R_{st}\frac{d\theta}{dt} + L_{st}\frac{d^2\theta}{dt^2} - G_{ab}i^a i^b \quad . \quad . \quad . \quad (104)$$

(5.2.2) Frame (ii).

$$f_u = R_{uv}\frac{d\theta}{dt} + L_{uv}\frac{d^2\theta}{dt^2} - G_{nk}i^n i^k \quad . \quad . \quad . \quad (105)$$

(5.2.3) Frame (iii).

$$f_s = R_{st}\frac{d\theta}{dt} + L_{st}\frac{d^2\theta}{dt^2} + [\alpha\beta,s]i^\alpha i^\beta + 2\Omega_{s\beta,\alpha}i^\alpha i^\beta \quad . \quad (106)$$

$$f_s = R_{st}\frac{d\theta}{dt} + L_{st}\frac{d^2\theta}{dt^2} - \frac{1}{2}\frac{\partial L_{\alpha\beta}}{\partial\theta}i^\alpha i^\beta - G'_{\alpha\beta}i^\alpha i^\beta \quad . \quad (106a)$$

Now $$\frac{1}{2}\frac{\partial L_{\alpha\beta}}{\partial\theta}i^\alpha i^\beta = \frac{1}{2}\frac{\partial L_{\alpha\beta}}{\partial\theta_1}\frac{d\theta_1}{d\theta}i^\alpha i^\beta \quad . \quad . \quad . \quad (107)$$

and $$G'_{\alpha\beta} = \frac{\partial C_t^{-1}}{\partial\theta_f}C_t\frac{d\theta_f}{d\theta}L_{\delta\alpha} \quad . \quad . \quad . \quad . \quad (107a)$$

Thus $$f_s = R_{st}\frac{d\theta}{dt} + L_{st}\frac{d^2\theta}{dt^2} - G_{\alpha\beta}\left(\frac{d\theta_1}{d\theta} + \frac{d\theta_f}{d\theta}\right)i^\alpha i^\beta \quad . \quad (108)$$

or $$f_s = R_{st}\frac{d\theta}{dt} + L_{st}\frac{d^2\theta}{dt^2} - G_{\alpha\beta}i^\alpha i^\beta \quad . \quad . \quad . \quad (109)$$

(5.3) The Torque Tensor

It is seen from eqns. (62) and (87) that

$$G_{\gamma\alpha} = G_{mk}C^m_\gamma C^k_\alpha \quad . \quad . \quad . \quad . \quad . \quad (110)$$

It is also found on examination of the matrices in the previous Sections that

$$G_{mk} = G_{ca}C^c_m C^a_k \quad . \quad . \quad . \quad . \quad . \quad (111)$$

and $$G_{\gamma\alpha} = G_{ca}C^c_\gamma C^a_\alpha \quad . \quad . \quad . \quad . \quad . \quad (112)$$

The torque matrix therefore transforms as a tensor and is associated with the holonomic variable θ. The equations of frame (ii) may therefore be written[6]

$$e_m = R_{mk}i^k + L_{mk}pi^k + T_{msk}i^s i^k \quad . \quad . \quad (113)$$

where T_{msk} is a tensor; or

$$e_m = R_{mk}i^k + L_{mk}pi^k + (-S_{mnk} - S_{mkn} + S_{knm})i^k i^n \quad . \quad (114)$$

where S_{knm} is defined as a tensor having components equal to, but the negative of, $\Omega_{kn,m}$. The negative value is chosen in order that the tensor here defined will be that given by

$$\tfrac{1}{2}(\Gamma_{kn,m} - \Gamma_{nk,m}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (115)$$

the skew-symmetric part of $\Gamma_{kn,m}$. That this is a tensor may be proved by the equations of tensor calculus.[14] This is a well-known tensor quantity in geometry of n-dimensional spaces and is there termed the "torsion" tensor. In terms of the torsion tensor the equations of frame (iii), derived from frame (ii), may be written

$$e_\gamma = R_{\gamma\alpha}i^\alpha + L_{\gamma\alpha}pi^\alpha + \{[\alpha\beta,\gamma] - S_{\gamma\beta\alpha} - S_{\gamma\alpha\beta} + S_{\alpha\beta\gamma} + \Omega_{\gamma\beta,\alpha} + \Omega_{\gamma\alpha,\beta} - \Omega_{\alpha\beta,\gamma}\}i^\alpha i^\beta \quad . \quad (116)$$

where $$\Omega_{\alpha\beta,\gamma} = \left(\frac{\partial C^\delta_k}{\partial x^n} - \frac{\partial C^\delta_n}{\partial x^k}\right)C^k_\alpha C^n_\beta L_{\delta\gamma} \quad . \quad . \quad . \quad (117)$$

The terms $(-S_{\gamma\beta\alpha}S_{\gamma} -{}_{\alpha\beta} + S_{\alpha\beta\gamma})$ and $(\Omega_{\gamma\beta,\alpha} + \Omega_{\gamma\alpha,\beta} - \Omega_{\alpha\beta,\gamma})$ may be compared with eqn. (93) where on the right-hand side there are a tensor term and a non-holonomic object.

It is thus shown that in any of these three frames the following terms transform as tensors, e, i, L and G; also, G is associated with $p\theta$ the rotor speed. The generated voltage term due to V arises because of the choice of reference frame and is a function of the angular velocity of the frame with respect to the direct axis. If the angular velocity $p\theta_1$ becomes zero, θ_f becomes θ and the equations of frame (iii) give those of frame (ii). If $p\theta_1$ becomes $p\theta$, θ_f becomes zero and the frame (iii) equations give those of frame (i).

These relationships may also be derived by transforming Γ as a whole[2] using eqn. (63), instead of transforming the components of Γ as has been done here. This method, however, does not show so clearly the mechanism of the transformations.

(6) APPLICATION

The tensor equations of electrical machines may be used in two ways.[1] In the first, a comparison is made between the primitive machine and another type of machine whose equations are required; for example, a metadyne, Schrage motor, etc. This aspect of Kron's work has been extensively treated.[2,5,10,11] In such analysis the connection matrix C is set up between the currents in the primitive machine coils and those in the interconnected coils of the derived machine, and the equations of the primitive machine are then transformed as shown by Kron, Gibbs and others, to give the required equations. The second application has been used more recently.[20] This consists of transformations among the reference frames of a given machine.[15] A familiar example is that of the synchronous alternator which may be analysed by setting up the equations with respect to quantities appearing at stationary 3-phase armature terminals, or alternatively by using Park's equations which contain quantities arising with respect to rotating direct- and quadrature-field axes. The d- and q-axes quantities are, of course, fictitious, but this reference frame leads to linear differential equations, often with constant coefficients, and is therefore widely used. It has been found,[20,21] however, that Park's form of the equations becomes very complicated when rotor oscillations occur in the machine. In hunting studies a more suitable frame is one which rotates freely at synchronous angular velocity and is independent of the rotor oscillations. The equations in this form, of course, become identical with Park's equations when the rotor has a uniform angular velocity with no oscillations or acceleration. It is not proposed to discuss the oscillations of machines here, nor indeed to deal in detail with the use of the synchronous-machine equations, but simply to present the concepts of changes of reference frames using the tensor technique developed by Kron.

The reason for changes of reference frame is apparent from examination of Park's equations. In a rotating-field alternator the d- and q-axes of reference rotate synchronously with the field structure relative to the armature. Under balanced steady-state conditions the fluxes and current and voltage vectors along these axes will be constant in magnitude and rotating in space. Thus the steady-state equations of the alternator are obtained from eqns. (21) by putting terms such as $L_d p$ equal to zero, where p operates on a current component, retaining the p terms where p operates only on θ to give the angular velocity. The steady-state equations are then

$$\left.\begin{aligned} e_f &= R_f i^f \\ e_d &= -R_d i^{dr} + L_q p\theta i^{qr} \\ e_q &= -R_q i^{qr} - M_d p\theta i^f - L_d p\theta i^{dr} \end{aligned}\right\} \quad . \quad . \quad (118)$$

It is obvious that if differential equations are to be set up for any machine or network to which the alternator is connected, these must be expressed along the same reference frame, the operator p must have the same significance and the steady-state equations must therefore be obtained as before when terms such as Lp become zero. In most equations of a.c. machines and networks, with sinusoidal voltage and currents, the steady-state equations are obtained when Lp becomes $j\omega L$. In this case a transformation of reference frame is required if these machines and networks are to be analysed in conjunction with interconnected synchronous machines. Two very simple cases will illustrate the required transformations, namely the equations of a simple series impedance having resistance and inductive reactance, and those of a 3-phase induction motor. Both of these have been analysed by Kron, but the analysis as set out below demonstrates details of the general method of using tensor equations for this purpose. The transient equation of a simple RL series impedance may be written

$$e = Ri + Lpi$$

Under steady-state conditions, with sinusoidal voltage applied, the equation becomes

$$e = Ri + j\omega Li$$

which may be obtained from the transient equation by putting p equal to $j\omega$. When a 3-phase system is being considered the instantaneous line currents and phase voltages and impedances may be resolved into Clarke components.[22] In order to conform to the phase positions and direction of rotation shown in Fig. 1, the current components may be defined as follows:

$$i^{b'} = \tfrac{1}{3}(2i^A - i^B - i^C) \quad . \quad . \quad . \quad . \quad (119)$$

$$i^{a'} = \frac{1}{\sqrt{3}}(i^B - i^C) \quad . \quad . \quad . \quad . \quad . \quad (120)$$

$$i^0 = \tfrac{1}{3}(i^A + i^B + i^C) \quad . \quad . \quad . \quad . \quad (121)$$

where i^A, i^B, i^C are the instantaneous line currents (Miss Clarke uses indices α and β instead of b' and a' as written here). Zero-sequence currents i^0 are those residual currents in the neutral connection to an unbalanced load or point of fault. To simplify the analysis a balanced system will be considered with no zero-sequence currents. In a machine wound for three phases these instantaneous components $i^{b'}$ and $i^{a'}$ lie respectively along the axis of phase A and along the common axis of phases B and C in quadrature with phase A. These are the same as the axes b' and a' used in the holonomic primitive machine, Fig. 1, and are stationary with respect to the armature phase windings. The a' and b' components of an external 3-phase network would be connected to the machine axes as shown in Fig. 5(a). It is therefore possible to define for either a machine or stationary network a set of currents i^{S1} and i^{S2} expressed along axes rotating with uniform angular velocity with respect to the axes of the Clarke components. From Fig. 3 the relationships among such currents may be written

$$\left.\begin{aligned} i^{a'} &= i^{S1}\cos\theta_f + i^{S2}\sin\theta_f \\ i^{b'} &= -i^{S1}\sin\theta_f + i^{S2}\cos\theta_f \end{aligned}\right\} \quad . \quad . \quad . \quad (121)$$

For a stationary network the holonomic (Lagrangian) equations in terms of a' and b' components are

$$\begin{array}{c|c|} w\diagdown & \\ \hline a' & e_{a'} \\ \hline b' & e_{b'} \\ \hline \end{array} = \begin{array}{c|c|c|} w\diagdown v & a' & b' \\ \hline a' & R + Lp & \\ \hline b' & & R + Lp \\ \hline \end{array} \begin{array}{c|c|} & \\ \hline a' & i^{a'} \\ \hline b' & i^{b'} \\ \hline \end{array} \quad (122)$$

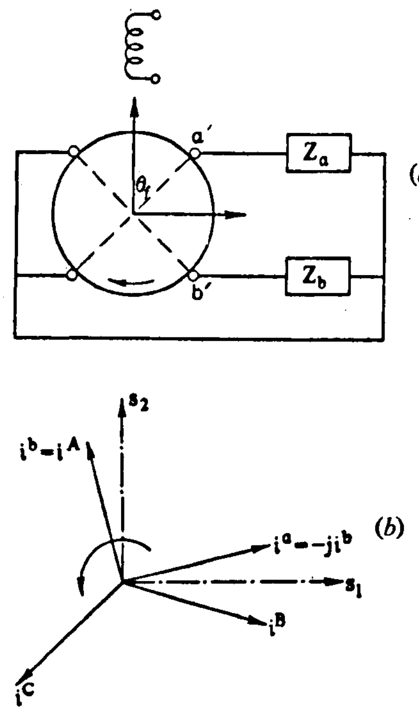


Fig. 5.—(a) Synchronous machine with external network. (b) Clarke components of a balanced 3-phase system.

or in index notation

$$e_w = R_{wv} i^v + L_{wv} p i^v$$

where

$R_{wv} =$

w\v	a′	b′
a′	R	
b′		R

and $L_{wv} =$

w\v	a′	b′
a′	L	
b′		L

In the free frame the equations become [see eqn. (78)]

$$e_\gamma = R_{\gamma\alpha} i^\alpha + L_{\gamma\alpha} p i^\alpha + 2[\alpha s,\gamma] i^s i^\alpha + 2\Omega_{\gamma s,\alpha} i^s i^\alpha \quad . \quad (123)$$

where [from eqn. (121)]

$C^v_{\cdot\alpha} =$ . . . . (124)

v\α	S_1	S_2
a′	$\cos\theta_f$	$\sin\theta_f$
b′	$-\sin\theta_f$	$\cos\theta_f$

and

$$i^v = C^v_{\cdot\alpha} i^\alpha$$

$$e_\alpha = C^{\cdot v}_{\alpha} e_v \quad (C^{\cdot v}_{\alpha} \text{ is the transpose of } C^v_{\cdot\alpha})$$

$$R_{\gamma\alpha} = R_{wv} C^w_\gamma C^v_\alpha$$

$$L_{\gamma\alpha} = L_{wv} C^w_\gamma C^v_\alpha$$

It is found that in this case

$$R_{\gamma\alpha} = R_{wv}$$

$$L_{\gamma\alpha} = L_{wv}$$

Thus $$2[\gamma s,\alpha] \equiv \left(\frac{\partial L_{\alpha s}}{\partial x^\gamma} + \frac{\partial L_{\alpha\gamma}}{\partial x^s} - \frac{\partial L_{\gamma s}}{\partial x^\alpha}\right) = 0$$

and therefore $$e_\gamma = R_{\gamma\alpha} i^\alpha + L_{\gamma\alpha} p i^\alpha + 2\Omega_{\gamma s,\alpha} i^s i^\alpha \quad . \quad . \quad (125)$$

where $$2\Omega_{\gamma s,\alpha} i^s i^\alpha = 2\Omega^\delta_{\gamma s} L_{\delta\alpha} i^\alpha p\theta = \frac{\partial C^{\cdot\delta}_{w}}{\partial\theta_f} C^{\cdot w}_{\gamma} L_{\delta\alpha} i^\alpha p\theta_f \quad . \quad (126)$$

In direct notation $$C^v_{\cdot\alpha} \equiv C \text{ and } 2\Omega^\delta_{\gamma s} = C_t \frac{\partial C_t^{-1}}{\partial\theta_f} \quad . \quad . \quad (127)$$

Thus $2\Omega^\delta_{\gamma s} =$ . . . . . (128)

γ\δ	S_1	S_2
S_1		+1
S_2	−1	

and $2\Omega_{\gamma s,\alpha} =$ (129)

γ\α	S_1	S_2
S_1		+L
S_2	−L	

$= V_{\gamma\alpha}$ . . (129)

Thus $$e_\gamma = R_{\gamma\alpha} i^\alpha + L_{\gamma\alpha} p i^\alpha + V_{\gamma\alpha} i^\alpha p\theta_f \quad . \quad . \quad . \quad (130)$$

A rotation matrix may be defined by[24]

$$V_{\gamma\alpha} = \rho^\delta_\gamma L_{\delta\alpha} \quad . \quad . \quad . \quad . \quad . \quad (131)$$

where $\rho^\delta_\gamma = 2\Omega^\delta_{\gamma s} =$ . . . (132)

γ\δ	S_1	S_2
S_1		+1
S_2	−1	

This is the rotation matrix used by Kron.[24]

In eqns. (132) above it is not a tensor, being part of the non-holonomic object $\Omega_{\gamma\beta\alpha}$. A similar matrix arises in connection with the formation of the torsion tensor defined in Section 5, and this is then referred to as the "rotation tensor" (there is an algebraic connection between this tensor and the coefficient of rotation defined by Ricci[14]).

The equations of the 3-phase network in terms of axes analogous to the rotating *d* and *q* axes of Park thus become

γ	
S_1	e_{S1}
S_2	e_{S2}

=

γ\α	S_1	S_2
S_1	$R + Lp$	$Lp\theta_f$
S_2	$-Lp\theta_f$	$R + Lp$

α	
S_1	i^{S1}
S_2	i^{S2}

(133)

where $$\left.\begin{aligned} e_{S1} &= e_{a'}\cos\theta_f - e_{b'}\sin\theta_f \\ e_{S2} &= e_{a'}\sin\theta_f + e_{b'}\cos\theta_f \end{aligned}\right\} \quad . \quad . \quad . \quad (134)$$

By trigonometrical substitution it may be shown that in the steady state, when $e_{b'} = E_A \cos\omega t$ and $e_{b'} = je_{a'}$,

$$\left.\begin{aligned} e_{S1} &= E_A \varepsilon^{j(\omega t - \theta_f)} \\ i^{S1} &= I^A \varepsilon^{j(\omega t - \theta_f)} \end{aligned}\right\} \quad . \quad (135) \qquad \left.\begin{aligned} e_{S2} &= je_{S1} \\ i^{S2} &= ji^{S1} \end{aligned}\right\} \quad . \quad (136)$$

When the rotating reference frame has synchronous angular velocity, $p\theta_f = \omega$,

$$\left.\begin{aligned} e_{S1} &= E_A \\ i^{S1} &= I^A \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (137)$$

The balanced steady-state equation may be written down by putting $p = 0$ in eqns. (133) and using eqns. (136). Thus

$$i_{S1} = Ri^{S1} + jv\omega Li^{S1} \quad . \quad . \quad . \quad . \quad (138)$$

And at synchronous angular velocity $v = 1$, and

$$e_{S1} = Ri^{S1} + jXi^{S1} \quad . \quad . \quad . \quad . \quad (139)$$

The operator p has therefore the same significance in the transient eqns. (133) as in Park's eqns. (21).

(6.1) The 3-Phase Induction Motor

A 3-phase induction motor has been analysed by Stanley[9] by expressing the equations in terms of an equivalent 2-phase machine. The 3-phase rotor and stator currents, voltages and flux linkages are resolved along two axes, namely the axis of the stator phase A and the common axis of the stator phases B and C in quadrature with phase A. Since both rotor and stator quantities are resolved along the same two axes these are relatively stationary and fixed on the stator as in Fig. 2.

The equations for a balanced symmetrical motor with voltages applied to the stator may therefore be written

	e
ds	e_{ds}
dr	0
qr	0
qs	e_{qs}

=

	ds	dr	qr	qs
ds	$R_1 + L_1p$	Mp		
dr	Mp	$R_2 + L_2p$	$L_2p\theta$	$Mp\theta$
qr	$-Mp\theta$	$-L_2p\theta$	$R_2 + L_2p$	Mp
qs			Mp	$R_1 + L_1p$

·

	i
ds	i^{ds}
dr	i^{dr}
qr	i^{qr}
qs	i^{qs}

(140)

or $e_m = R_{mk}i^k + L_{mk}pi^k + \Gamma_{kn,m}i^ki^n$.

where R_1 = Stator resistance per phase.
R_2 = Total rotor resistance per phase.
L_1 = Stator phase inductance.
L_2 = Rotor phase inductance.
M = Maximum value of mutual inductance stator/rotor phase.

In balanced steady state $i^{ds} = ji^{qs}$, $i^{dr} = ji^{qr}$ and p becomes $j\omega$, and $L_2(p - jp\theta)$ becomes $L_2(j\omega - jp\theta) = js\omega L_2$, where $s = 1 - v$

and $v = \dfrac{\text{angular velocity of rotor}}{\text{synchronous angular velocity}}$

Eqns. (140) therefore become, in balanced steady state,

$$e_{ds} = (R_1 + jX_1)i^{ds} + jX_m i^{dr} \quad . \quad . \quad . \quad (141)$$

$$0 = jsX_m i^{ds} + (R_2 + jsX_2)i^{dr} \quad . \quad . \quad . \quad (142)$$

When an induction motor is associated with a synchronous machine the motor equations may be written in terms of axes rotating with the flux as in Park's equations. In the balanced steady state, in this case, the vectors representing voltages, currents and flux linkages, are constant in magnitude and rotate synchronously in space around the stator of the machine. As shown in Fig. 3 there are two axes, S_3 and S_4, in quadrature on the stator, and two, S_1 and S_2, in quadrature on the rotor. The stator and rotor axes are both rotating synchronously and are again relatively stationary. One would therefore expect to find that the equations in this frame have a form similar to those of Stanley and Park. This is, in fact, the case. Both induced and generated voltages appear in the equations, but as in the synchronous-machine equations the steady-state equations of the motor are now obtained by putting such terms as Lp equal to zero, and only generated-voltage terms remain. It will be seen from the free-frame equations that, relative to these axes, the stator appears to rotate synchronously backwards and the rotor to rotate backwards at the angular velocity of slip.

The relationship among the currents in the stationary axes and those in the free frame may be written

$$\left.\begin{aligned} i^{ds} &= i^{S3}\cos\theta_1 - i^{S4}\sin\theta_1 \\ i^{dr} &= i^{S1}\cos\theta_1 - i^{S2}\sin\theta_1 \\ i^{qr} &= i^{S1}\sin\theta_1 + i^{S2}\cos\theta_1 \\ i^{qs} &= i^{S3}\sin\theta_1 + i^{S4}\cos\theta_1 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (143)$$

or

$$i^m = C^m_{\cdot\gamma} i^\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad (144)$$

where $C^m_{\cdot\gamma}$ =

$m \diagdown \gamma$	S_3	S_1	S_2	S_4
ds	$\cos\theta_1$			$-\sin\theta_1$
dr		$\cos\theta_1$	$-\sin\theta_1$	
qr		$\sin\theta_1$	$\cos\theta_1$	
qs	$\sin\theta_1$			$\cos\theta_1$

(145)

The equations in the free frame are, as before,

$$e_\gamma = R_{\gamma\alpha}i^\alpha + L_{\gamma\alpha}pi^\alpha + 2[\alpha s,\gamma]i^\alpha p\theta + 2\Omega_{\gamma s,\alpha}i^\alpha p\theta \quad . \quad (146)$$

where

$$e_\gamma = C^{\cdot m}_\gamma e_m$$

$$R_{\gamma\alpha} = R_{mk}C^m_\gamma C^k_\alpha$$

$$L_{\gamma\alpha} = L_{mk}C^m_\gamma C^k_\alpha$$

L_{mk} is shown in matrix 46.

It is found on carrying out the transformation that

$$L_{\gamma\alpha} = L_{mk}$$

$$R_{\gamma\alpha} = R_{mk}$$

therefore again $\quad 2[\alpha s,\gamma] = 0$

Now

$$2\Omega_{\gamma s,\alpha} = 2\Omega_{ms,k}C^m_\gamma C^s_s C^k_\alpha + \frac{\partial C^\delta_m}{\partial\theta}C^m_\gamma L_{\delta\alpha} \quad . \quad . \quad (147)$$

and [eqn. (62)]

$2\Omega_{ms,k}$ =

$m \diagdown k$	ds	dr	qr	qs
ds				
dr			L_2	M
qr	$-M$	$-L_2$		
qs				

. (148)

This term of the right-hand side of eqn. (147) is seen to transform tensorially (in this case to the same matrix)

$2\Omega_{ms,k}C^m_\gamma C^k_\alpha =$

$\gamma\backslash\alpha$	S_3	S_1	S_2	S_4
S_3				
S_1			L_2	M
S_2	$-M$	$-L_2$		
S_4				

$= 2\Omega'_{\gamma s,\alpha}$. (149)

This is the term referred to (by Kron) as the "torsion tensor," the name being taken from geometry.

Now $$\frac{\partial C^\delta_m}{\partial\theta}C^m_\gamma L_{\delta\alpha}p\theta = \frac{\partial C^\delta_m}{\partial\theta_1}C^m_\gamma L_{\delta\alpha}p\theta_1 \quad . \quad . \quad . \quad (150)$$

and

$\frac{\partial C^\delta_m}{\partial\theta_1}C^m_\gamma =$

$\gamma\backslash\delta$	S_3	S_1	S_2	S_4
S_3				-1
S_1			-1	
S_2		1		
S_4	1			

. (151)

Therefore

$\frac{\partial C^\delta_m}{\partial\theta_1}C^m_\gamma L_{\delta\alpha} =$

$\gamma\backslash\alpha$	S_3	S_1	S_2	S_4
S_3			$-M$	$-L_1$
S_1			$-L_2$	$-M$
S_2	M	L_2		
S_4	L_1	M		

. (152)

Thus $\left(2\Omega'_{\gamma s,\alpha}p\theta + \frac{\partial C^\delta_m}{\partial\theta_1}C^m_\gamma L_{\delta\alpha}p\theta_1\right) =$

$\gamma\backslash\alpha$	S_3	S_1	S_2	S_4
S_3			$-Mp\theta_1$	$-L_1p\theta_1$
S_1			$-L_2p\theta'$	$-Mp\theta'$
S_2	$Mp\theta'$	$L_2p\theta'$		
S_4	$L_1p\theta_1$	$Mp\theta_1$		

. (153)

where $p\theta'$ is the angular velocity of slip, namely $p\theta_1 - p\theta$ (which is equal to $-p\theta_f$). In the free frame the equations of the machine therefore become

	e			S_3	S_1	S_2	S_4			i
S_3	e_{S3}	=	S_3	$R_1 + L_1p$	Mp	$-Mp\theta_1$	$-L_1p\theta_1$	.	S_3	i^{S3}
S_1	0		S_1	Mp	$R_2 + L_2p$	$-L_2p\theta'$	$-Mp\theta'$		S_1	i^{S1}
S_2	0		S_2	$Mp\theta'$	$L_2p\theta'$	$R_2 + L_2p$	Mp		S_2	i^{S2}
S_4	e_{S4}		S_4	$L_1p\theta_1$	$Mp\theta_1$	Mp	$R_1 + L_1p$		S_4	i^{S4}

(154)

Eqn. (154) may also be derived by starting from the holonomic machine in frame (i), Section 5, the relationship among the currents is then (Fig. 3)

$$\left.\begin{aligned} i^{ds} &= i^{S3}\cos\theta_1 - i^{S4}\sin\theta_1 \\ i^{a'} &= i^{S1}\cos\theta_f + i^{S2}\sin\theta_f \\ i^{b'} &= -i^{S1}\sin\theta_f + i^{S2}\cos\theta_f \\ i^{qs} &= i^{S3}\sin\theta_1 + i^{S4}\cos\theta_1 \end{aligned}\right\} \quad . \quad . \quad . \quad (155)$$

or $i^a = C^a_\alpha i^\alpha$.

$C^a_\alpha =$

$a\backslash\alpha$	S_3	S_1	S_2	S_4
ds	$\cos\theta_1$			$-\sin\theta_1$
a'		$\cos\theta_f$	$\sin\theta_f$	
b'		$-\sin\theta_f$	$\cos\theta_f$	
qs	$\sin\theta_1$			$\cos\theta_1$

. (156)

The equations in the free frame, when transformed directly from the holonomic equations are [see Section 5, eqn. (78)]

$$e_\gamma = R_{\gamma\alpha}i^\alpha + L_{\gamma\alpha}pi^\alpha + 2[\alpha s,\gamma]i^\alpha i^s + 2\Omega_{\gamma s,\alpha}i^s i^\alpha$$

where in this case

$$2\Omega_{\gamma s,\alpha} = C^c_\gamma\frac{\partial C}{\partial\theta}L_{\delta\alpha} \quad . \quad . \quad . \quad . \quad (157)$$

This is the single non-holonomic object arising between the holonomic frame and the non-holonomic free frame.

Again $$L_{\gamma\alpha} = L_{ca}C^c_\gamma C^a_\alpha$$
$$R_{\gamma\alpha} = R_{ca}C^c_\gamma C^a_\alpha$$

For a symmetrical induction motor represented by Fig. 3, the inductance matrix 9 in Section 2 becomes

$L_{ca} =$

$c\backslash a$	ds	a'	b'	qs
ds	L_1	$M\cos\theta$	$-M\sin\theta$	
a'	$M\cos\theta$	L_2		$M\sin\theta$
b'	$-M\sin\theta$		L_2	$M\cos\theta$
qs		$M\sin\theta$	$M\cos\theta$	L_1

(158)

Using the relation $\theta = \theta_1 + \theta_f$ from Fig. 3, then

$L_{\gamma\alpha} =$

$\gamma\backslash\alpha$	S_3	S_1	S_2	S_4
S_3	L_1	M		
S_1	M	L_2		
S_2			L_2	M
S_4			M	L_1

. (159)

From matrix 156

$$C_{\gamma}^{\cdot c}\frac{\partial C_{c}^{\cdot \delta}}{\partial \theta} = \quad (160)$$

$\gamma \backslash \delta$	S_3	S_1	S_2	S_4
S_3				$-d\theta_1/d\theta$
S_1			$d\theta_f/d\theta$	
S_2		$-d\theta_f/d\theta$		
S_4	$d\theta_1/d\theta$			

And $C_{\gamma}^{\cdot c}\frac{\partial C_{c}^{\cdot \delta}}{\partial \theta}L_{\delta\alpha}\frac{d\theta}{dt}$ gives matrix 153.

It is seen that the induction-motor equations in a free frame rotating at uniform velocity, when derived from the stationary reference-frame equations, assume the form

$$e_{\gamma} = R_{\gamma\alpha}i^{\alpha} + L_{\gamma\alpha}pi^{\alpha} + \rho_1 L_{\gamma\alpha}i^{\alpha}p\theta + \rho_2 L_{\gamma\alpha}i^{\alpha}p\theta_1 \quad . \quad (161)$$

where $\rho_1 =$. (162)

	S_3	S_1	S_2	S_4
S_3				
S_1			-1	
S_2		1		
S_4				

and $\rho_2 =$. (163)

	S_3	S_1	S_2	S_4
S_3				-1
S_1			-1	
S_2		1		
S_4	1			

$\rho_1 L_{\gamma\alpha}$ is matrix 149 and $\rho L_{\gamma\alpha}$ is matrix 152.

Eqn. (161) may be written

$$e_{\gamma} = R_{\gamma\alpha}i^{\alpha} + L_{\gamma\alpha}pi^{\alpha} + G_{\gamma\alpha}i^{\alpha}p\theta + V_{\gamma\alpha}i^{\alpha}p\theta_1 \quad . \quad (164)$$

or in terms of flux vectors

$$\boldsymbol{e} = \boldsymbol{Ri} + p\boldsymbol{\psi} + \boldsymbol{B}p\theta + \boldsymbol{\varphi}p\theta_1 \quad . \quad . \quad . \quad (165)$$

Synchronous-machine equations in this form are discussed by Kron[15] and used in his hunting analysis.[20]

In balanced steady state

$$i^{S4} = ji^{S3} \qquad i^{S2} = ji^{S1} \quad . \quad . \quad . \quad . \quad (166)$$

The operator p is zero when this operates on the steady-state currents (but $p\theta_1 = v\omega$).

Thus, eqns. (154) become

$$\left.\begin{aligned} e_{S3} &= (R_1 + jv\omega L_1)i^{S3} + jv\omega Mi^{S1} \\ 0 &= js\omega Mi^{S3} + (R_2 + js\omega L_2)i^{S1} \end{aligned}\right\} \quad . \quad . \quad (167)$$

When the reference frame rotates at synchronous speed with respect to the stator winding, then $p\theta_1 = \omega$ and

$$\left.\begin{aligned} e_{S3} &= (R_1 + jX_1)i^{S3} + jX_m i^{S1} \\ 0 &= jsX_m i^{S3} + (R_2 + jsX_2)i^{S1} \end{aligned}\right\} \quad . \quad . \quad (168)$$

In this reference frame the operator p has the same significance as in the two-reaction theory of Park, and the free frame equations for a motor or network may be combined with those of synchronous machines when an interconnected network is being considered.

(6.2) Induction-Motor Torque Equations

The motor torque is given by eqn. (106).

Since $[\alpha\beta,s] = 0$, the equation (neglecting the mechanical friction term R_{st}) becomes:

$$\text{Impressed torque } f_s = L_{st}\frac{d^2\theta}{dt^2} + 2\Omega_{s\beta,\alpha}i^{\beta}i^{\alpha} \quad . \quad (169)$$

and at constant angular velocity

$$\text{Generated torque} = -2\Omega_{s\beta,\alpha}i^{\beta}i^{\alpha}$$

where, as in eqns. (62), $\Omega_{s\beta,\alpha}$ is the negative of $\Omega_{\gamma s,\alpha}$

$$\text{Thus } \Omega_{s\beta,\alpha} = -C_{\alpha}^{a}\frac{\partial C_{a}^{\delta}}{\partial \theta}L_{\delta\beta} = \quad . \quad (170)$$

$\alpha \backslash \beta$	S_3	S_1	S_2	S_4
S_3			$M\frac{d\theta_1}{d\theta}$	$L_1\frac{d\theta_1}{d\theta}$
S_1			$-L_2\frac{d\theta_f}{d\theta}$	$-M\frac{d\theta_f}{d\theta}$
S_2	$M\frac{d\theta_f}{d\theta}$	$L_2\frac{d\theta_f}{d\theta}$		
S_4	$-L_1\frac{d\theta_1}{d\theta}$	$-M\frac{d\theta_1}{d\theta}$		

$i^{\beta} =$ . . . . (171)

$\beta \backslash$	
S_3	i^{S3}
S_1	i^{S1}
S_2	i^{S2}
S_4	i^{S4}

$$\Omega_{s\beta,\alpha}i^{\beta}i^{\alpha} = i^{S1}i^{S4}M - i^{S2}i^{S3}M \quad . \quad . \quad . \quad (172)$$

or

$$\Omega_{s\beta,\alpha}i^{\beta}i^{\alpha} = \psi_d i^q - \psi_q i^d \quad . \quad . \quad . \quad . \quad (173)$$

where ψ_d and ψ_q are the flux linkages in axes S_1 and S_2 respectively.

The generated torque given by $\Omega_{s\beta,\alpha}i^{\beta}i^{\alpha}$ is seen to be that given by

$$f = G_{\gamma\alpha}i^{\gamma}i^{\alpha} \quad . \quad . \quad . \quad . \quad . \quad . \quad (174)$$

where $G_{\gamma\alpha} = \rho_1 L_{\gamma\alpha}$

or[8]

$$f = \boldsymbol{i}^{*}\boldsymbol{B} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (175)$$

where $\boldsymbol{i}^{*}$ is the conjugate of i^{γ}.

(7) CONCLUSION

The analysis of electrical machines may often be simplified by transforming the variables and parameters from the real phase reference frame to real or fictitious stationary or rotating reference axes. Changes of reference frames similar to those demonstrated in the paper have been used by Kron and Ku in the derivation of equivalent circuits for electrical machinery.[3,23] Tensor analysis is a most useful mathematical tool for handling such transformations. Tensor equations look complicated when first used because the notation is comparatively new to engineers. In practical application, however, when the equations of a machine are written in tensor form they are fairly simple and the technique of transformation and calculation of phenomena become a matter of routine procedure.

This method of handling machine problems ensures that the analysis is systematic and the equations are of a form that often leads to clearer concepts of the interactions of the various currents and fluxes in the system. It is possible to distinguish in the equations between terms that have existence in all reference frames and those which arise because of the reference frame chosen [compare for example the terms $G_{\gamma\alpha}i^{\alpha}p\theta$ and $V_{\gamma\alpha}i^{\alpha}p\theta_1$ in eqn. (164)].

It is intended that the foregoing presentation of Kron's work on the tensor equations of electrical machines should provide a groundwork on which may be built a more complete mathematical and physical analysis of machine stability problems by investigating the phenomena in various reference frames.

(8) ACKNOWLEDGMENTS

The author wishes to express his thanks to Prof. J. M. Meek of the University of Liverpool for his interest in the work, and to Dr. W. J. Gibbs of the British Thomson-Houston Co. Ltd., for discussions and advice during the preparation of the paper. He also wishes to express his appreciation of the encouragement he received in the earlier stages of the investigations from Prof. Emeritus F. J. Teago, formerly of the University of Liverpool, and Prof. P. P. Burns of the Queen's University of Belfast.

(9) REFERENCES

(1) KRON, G.: "Non-Riemannian Dynamics of Rotating Electrical Machinery," *Journal of Mathematics and Physics*, 1934, **13**, p. 103.
(2) KRON, G.: "The Application of Tensors to the Analysis of Rotating Electrical Machinery," *General Electric Review*, 1935 to 1938. Published in book form 1938 and 1942.
(3) KRON, G.: "Equivalent Circuits of Electric Machinery" (John Wiley, 1951).
(4) GIBBS, W. J.: "Limitations of Lagrangian Methods in Electrical Machine Theory," *Beama Journal*, 1950, **57**, pp. 243 and 382.
(5) GIBBS, W. J.: "Tensors in Electrical Machine Theory" (Chapman and Hall, 1952).
(6) KRON, G.: "Quasi-holonomic Dynamical Systems," *Journal of Applied Physics*, 1936, **7**, p. 143.
(7) SABBAGH, E. M.: "Application of 2-reaction Theory to Electric Motors," *Transactions of the American I.E.E.*, 1951, **70**, Part II, p. 1748.
(8) KRON, G.: "Short Course in Tensor Analysis" (John Wiley, 1942).
(9) STANLEY, H. C.: "An Analysis of the Induction Machine," *Transactions of the American I.E.E.*, 1938, **57**, p. 751.
(10) LYNN, J. W., and ALDRED, A. S.: "The Practical Application of Matrix Methods of Electrical Machine Analysis," *Beama Journal*, 1954, **61**, pp. 114 and 140.
(11) GIBBS, W. J.: "The Equations and Circle Diagram of the Schrage Motor," *Journal I.E.E.*, 1946, **93**, Part II, p. 621.
(12) CONCORDIA, C.: "Synchronous Machines" (John Wiley, 1951).
(13) WHITTAKER, E. T.: "Analytical Dynamics" (Cambridge University Press, 1937).
(14) SCHOUTEN, J., and STRUIK, D. J.: "Einfuhrung in die Neuren Methoden der Differentialgeometric," Band I (Springer, Berlin, 1935).
(15) KRON, G.: "Classification of the Reference Frames of a Synchronous Machine," *Transactions of the American I.E.E.*, 1950, **69**, Part II, p. 720.
(16) HOFFMANN, B.: "Kron's Non-Riemannian Electrodynamics," *Reviews of Modern Physics*, 1949, **21**, p. 535.
(17) WAGNER, C. F., and EVANS, R. D.: "Symmetrical Components" (McGraw-Hill Book Co., 1933).
(18) PARK, R. H.: "Two-Reaction Theory of Synchronous Machines," *Transactions of the American I.E.E.*, (i) 1929, **48**, p. 716, and (ii) 1933, **52**, p. 352.
(19) MCCONNELL, A. S.: "Applications of the Absolute Differential Calculus" (Blackie and Sons, London, 1931).
(20) KRON, G.: "A New Theory of Hunting," *Transactions of the American I.E.E.*, 1952, **71**, Part III, p. 859.
(21) HEFFRON, W. G., ROSENBERRY, G. M., and ROTHE, F. S.: "Generalized Hunting Equations of Power Systems," *ibid.*, p. 1095.
(22) CLARKE, E.: "Circuit Analysis of A.C. Power Systems" (John Wiley), Volumes I and II.
(23) KU, Y. H.: "Rotating-Field Theory and General Analysis of Synchronous and Induction Machines," *Proceedings I.E.E.*, Monograph No. 54 U, June, 1952 (**99**, Part IV, p. 410).
(24) KRON, G.: "Stationary Networks and Transmission Lines along Uniformly Rotating Reference Frames," *Transactions of the American I.E.E.*, 1949, **68**, Part II, p. 690.
(25) FORSYTH, A. R.: "A Treatise on Differential Equations" (Macmillan and Co., London, 1914. Fourth Edition), p. 309.

(10) APPENDIX

(10.1) Tensor Transformations[19]

Tensor analysis of electrical machines is largely concerned with transformations of machine equations.

Tensors are sets of quantities, often represented by matrices, which are (*a*) functions of a set of co-ordinates (variables) and (*b*) subject to certain conditions of transformation when the co-ordinates are changed. The basic laws of transformation are set out below.

Let P be a particular value given by n co-ordinates x^i $(i = 1, 2, \ldots n)$ in one reference system and by $\bar{x}^j (j = 1, 2, \ldots m)$ in another system. Let Q be a value close to P, given by $x^i + dx^i$ and $\bar{x}^j + d\bar{x}^j$. The two sets of differentials in the two co-ordinate systems are connected by the equations

$$d\bar{x}^j = \frac{\partial \bar{x}^j}{dx^i}dx^i \quad . \quad . \quad . \quad . \quad . \quad (176)$$

The infinitesimal displacement PQ gives an example of a "contravariant" transformation. The indices are written by convention, as superscripts.

Another form of transformation is given as follows. Consider a scalar A which is invariant in all co-ordinate systems. The partial derivatives of A with respect to the co-ordinates x^i in one reference frame are given by $A_i = \partial A/\partial x^i$. In another system with co-ordinates x^j the partial derivatives will be given by

$$\bar{A}_j = \frac{\partial A}{\partial x^i}\frac{\partial x^i}{\partial \bar{x}^j} = \frac{\partial \bar{A}}{\partial \bar{x}^j} \quad . \quad . \quad . \quad . \quad (177)$$

Consequently
$$\bar{A}_j = A_i\frac{\partial x^i}{\partial \bar{x}^j} \quad . \quad . \quad . \quad . \quad . \quad (178)$$

The vector whose components in the x's are partial derivatives A_i is the gradient of the scalar A (grad A). This is an example of a "covariant" vector, or tensor of the first order: it has one index written as a subscript. In general terms the two different forms of transformation may be written[19]

First-order tensors (vectors):

$$\text{Contravariant}\quad \bar{u}^j = u^i\frac{\partial \bar{x}^j}{\partial x^i} \quad . \quad . \quad . \quad (179)$$

$$\text{Covariant}\quad \bar{v}_j = v_\pi\frac{\partial x^\pi}{\partial \bar{x}^j} \quad . \quad . \quad . \quad . \quad (180)$$

Second- and higher-order tensors:

$$\text{Contravariant}\quad \bar{u}^{ab} = u^{\alpha\beta}\frac{\partial \bar{x}^a}{\partial x^\alpha}\frac{\partial \bar{x}^b}{\partial x^\beta} \quad . \quad . \quad (181)$$

$$\bar{u}^{abc} = u^{\alpha\beta\gamma}\frac{\partial \bar{x}^a}{\partial x^\alpha}\frac{\partial \bar{x}^b}{\partial x^\beta}\frac{\partial \bar{x}^c}{\partial x^\gamma} \quad . \quad . \quad . \quad . \quad (182)$$

$$\text{Covariant}\quad \bar{v}_{jk} = v_{\pi\rho}\frac{\partial x^\pi}{\partial \bar{x}^j}\frac{\partial x^\rho}{\partial \bar{x}^k} \quad . \quad . \quad . \quad (183)$$

$$\bar{v}_{jkm} = v_{\pi\kappa\mu}\frac{\partial x^\pi}{\partial \bar{x}^j}\frac{\partial x^\kappa}{\partial \bar{x}^k}\frac{\partial x^\mu}{\partial \bar{x}^m} \quad . \quad . \quad . \quad . \quad (184)$$

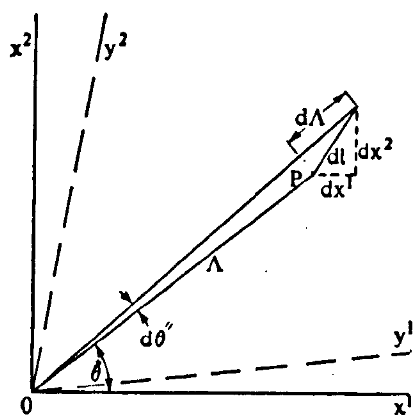


Fig. 6.—Cartesian, rectilinear and polar co-ordinates.

(10.1.1) Example of Simple Linear Transformation.

Let P be a point located by Cartesian co-ordinates x^1 and x^2 as in Fig. 6. A linear transformation to new rectilinear co-ordinates is given by the relation

$$\left.\begin{aligned} x^1 &= a^1_1 y^1 + a^1_2 y^2 \\ x^2 &= a^2_1 y^1 + a^2_2 y^2 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (185)$$

or

$$x^\alpha = C^\alpha_{\cdot u} y^u \quad . \quad . \quad . \quad . \quad . \quad . \quad (186)$$

where

$$x^\alpha = \begin{array}{c|c|} \alpha\backslash & \\ \hline 1 & x^1 \\ \hline 2 & x^2 \\ \hline \end{array} \quad . \quad . \quad . \quad . \quad . \quad (187)$$

$$C^\alpha_{\cdot u} = \begin{array}{c|c|c|} \alpha\backslash u & 1 & 2 \\ \hline 1 & a^1_1 & a^1_2 \\ \hline 2 & a^2_1 & a^2_2 \\ \hline \end{array} \quad . \quad . \quad . \quad . \quad (188)$$

$$y^u = \begin{array}{c|c|} u\backslash & \\ \hline 1 & y^1 \\ \hline 2 & y^2 \\ \hline \end{array} \quad . \quad . \quad . \quad . \quad . \quad . \quad (189)$$

(The column index of matrices such as $C^\alpha_{\cdot u}$ is preceded by a dot). In Cartesian co-ordinates the length of the vector OP is given by

$$(\Lambda)^2 = (x^1)^2 + (x^2)^2 \quad . \quad . \quad . \quad . \quad (190)$$

This length is invariant with respect to any change of co-ordinate system. In the new co-ordinate system, by substitution,

$$(\Lambda)^2 = y^1y^1(a^1_1a^1_1 + a^2_1a^2_1) + 2y^1y^2(a^2_2a^2_1 + a^1_2a^1_1) + y^2y^2(a^1_2a^1_2 + a^2_2a^2_2) \quad . \quad (191)$$

or
$$(\Lambda)^2 = g_{11}y^1y^1 + g_{12}y^1y^2 + g_{21}y^2y^1 + g_{22}y^2y^2 \quad . \quad (191a)$$

where $g_{11} = (a^1_1a^1_1 + a^2_1a^2_1)$, $g_{22} = (a^1_2a^1_2 + a^2_2a^2_2)$

and $g_{12} = g_{21} = (a^2_2a^2_1 + a^1_2a^1_1)$

In matrix notation
$$(\Lambda)^2 = g_{\alpha\beta}x^\alpha x^\beta \quad \left\{\begin{matrix}\alpha = 1, 2 \\ \beta = 1, 2\end{matrix}\right\} \quad . \quad (192)$$

In Cartesian co-ordinates

$$g_{\alpha\beta} = \begin{array}{c|c|c|} \alpha\backslash\beta & 1 & 2 \\ \hline 1 & 1 & \\ \hline 2 & & 1 \\ \hline \end{array} \quad . \quad . \quad . \quad . \quad (193)$$

and $g_{\alpha\beta}x^\alpha x^\beta$ is

$$\begin{array}{|c|c|} \backslash\alpha\ 1 & 2 \\ \hline x^1 & x^2 \\ \hline \end{array} \cdot \begin{array}{c|c|c|} \alpha\backslash\beta & 1 & 2 \\ \hline 1 & 1 & \\ \hline 2 & & 1 \\ \hline \end{array} \cdot \begin{array}{c|c|} \beta\backslash & \\ \hline 1 & x^1 \\ \hline 2 & x^2 \\ \hline \end{array} \quad (194)$$

The quantity $g_{\alpha\beta}$ is a covariant double tensor and transforms according to eqn. (183); thus from eqns. (183) and (188)

$$g_{uv} = g_{\alpha\beta}C^{\cdot\alpha}_{u}C^{\beta}_{\cdot v} \quad . \quad . \quad . \quad . \quad . \quad . \quad (195)$$

$$C^{\cdot\alpha}_{u} = \begin{array}{c|c|c|} u\backslash\alpha & 1 & 2 \\ \hline 1 & a^1_1 & a^2_1 \\ \hline 2 & a^1_2 & a^2_2 \\ \hline \end{array} \quad . \quad . \quad . \quad . \quad (196)$$

(the transpose of matrix 188)

and $g_{uv} =$

$$\begin{array}{c|c|c|} u\backslash\alpha & 1 & 2 \\ \hline 1 & a^1_1 & a^2_1 \\ \hline 2 & a^1_2 & a^2_2 \\ \hline \end{array} \cdot \begin{array}{c|c|c|} \alpha\backslash\beta & 1 & 2 \\ \hline 1 & 1 & \\ \hline 2 & & 1 \\ \hline \end{array} \cdot \begin{array}{c|c|c|} \beta\backslash v & 1 & 2 \\ \hline 1 & a^1_1 & a^1_2 \\ \hline 2 & a^2_1 & a^2_2 \\ \hline \end{array} \quad (197)$$

or

$$g_{uv} = \begin{array}{c|c|c|} u\backslash v & 1 & 2 \\ \hline 1 & a^1_1a^1_1 + a^2_1a^2_1 & a^2_2a^2_1 + a^1_2a^1_2 \\ \hline 2 & a^1_2a^1_1 + a^2_2a^2_1 & a^1_2a^1_2 + a^2_2a^2_2 \\ \hline \end{array} \quad . \quad . \quad . \quad (198)$$

which gives eqns. (191) when u and v each take the values 1 and 2.

The tensor g_{uv} or $g_{\alpha\beta}$ is called the "fundamental tensor" or "metric tensor" of the system. The metric tensor is necessary in order to calculate invariant properties of vectors or tensors, for example, the length of vectors, the angle between vectors and parallel displacements.

(10.1.2) Non-Linear Transformation.

An example of a non-linear transformation of variables is given by the change from Cartesian to polar co-ordinates. In this case the transformation must be one between differentials of the respective co-ordinate systems. From Fig. 6 it is seen that

$$\left.\begin{aligned} x^1 &= \Lambda \cos \theta'' \\ x^2 &= \Lambda \sin \theta'' \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (199)$$

$$\left.\begin{aligned} dx^1 &= -\Lambda \sin \theta'' d\theta'' + \cos \theta'' d\Lambda \\ dx^2 &= \Lambda \cos \theta'' d\theta'' + \sin \theta'' d\Lambda \end{aligned}\right\} \quad . \quad . \quad (200)$$

Now $$dx^\alpha = C^{\alpha}_{\cdot k} dx^k \quad . \quad . \quad . \quad . \quad . \quad (201)$$

where $dx^\alpha =$

α	
1	dx^1
2	dx^2

$dx^k =$

k	
1	$d\Lambda$
2	$d\theta''$

and $C^{\alpha}_{\cdot k} =$ . . . (202)

$\alpha \backslash k$	1	2
1	$\cos \theta''$	$-\Lambda \sin \theta''$
2	$\sin \theta''$	$\Lambda \cos \theta''$

Calling the invariant line element of length of any vector, in Cartesian co-ordinates, dl,

$$(dl)^2 = (dx^1)^2 + (dx^2)^2 = g_{\alpha\beta} dx^\alpha dx^\beta \quad . \quad . \quad (203)$$

where $g_{\alpha\beta}$ is given in matrix 193.

In polar co-ordinates

$$(dl)^2 = g_{kn} dx^k dx^n \quad . \quad . \quad . \quad . \quad . \quad . \quad (204)$$

where $$g_{kn} = g_{\alpha\beta} C^{\alpha}_{\cdot k} C^{\beta}_{\cdot n} \quad . \quad . \quad . \quad . \quad . \quad (205)$$

thus $g_{kn} =$ . . . . . . (206)

$k \backslash n$	1	2
1	1	
2		$(\Lambda)^2$

and $$g_{kn} dx^k dx^n = g_{11} d\Lambda d\Lambda + g_{22} d\theta'' d\theta'' \quad . \quad . \quad (207)$$

$$g_{12} = g_{21} = 0$$

therefore

$$(dl)^2 = (d\Lambda)^2 + (\Lambda)^2 (d\theta'')^2 = (dx^1)^2 + (dx^2)^2 \quad . \quad (208)$$

(10.2) Lagrange's Equation[8,13]

In dynamics the behaviour of a system may be calculated provided that the parameters of the system are known, i.e. the masses, inertias, etc., together with the forces and constraints acting on the system.

Certain properties of the system will be invariant under a transformation of co-ordinates, one of these invariants is the kinetic energy in the system.

Kinetic energy $T = \frac{1}{2} m v^2$.

Now $$(dl)^2 = g_{ab} dx^a dx^b$$

therefore $$\left(\frac{dl}{dt}\right)^2 = v^2 = g_{ab} \frac{dx^a}{dt} \frac{dx^b}{dt} = g_{ab} \dot{x}^a \dot{x}^b \quad . \quad . \quad (209)$$

Thus $$T = \tfrac{1}{2} m g_{ab} \dot{x}^a \dot{x}^b \quad . \quad . \quad . \quad . \quad . \quad . \quad (210)$$

Lagrange's dynamical equation in generalized co-ordinates x^c for a free system acted upon by forces f_c is written (neglecting potential energy in this case) thus:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^c}\right) - \frac{\partial T}{\partial x^c} = f_c$$

$$\begin{aligned} \frac{\partial T}{\partial \dot{x}^c} &= \tfrac{1}{2} m \frac{\partial (g_{ab} \dot{x}^a \dot{x}^b)}{\partial \dot{x}^c} \\ &= \tfrac{1}{2} m g_{ab} \dot{x}^b \frac{\partial \dot{x}^a}{\partial \dot{x}^c} + \tfrac{1}{2} m g_{ab} \dot{x}^a \frac{\partial \dot{x}^b}{\partial \dot{x}^c} \\ &= \tfrac{1}{2} m g_{ab} \dot{x}^b \delta^a_c + \tfrac{1}{2} m g_{ab} \dot{x}^a \delta^b_c \\ &= \tfrac{1}{2} m g_{cb} \dot{x}^b + \tfrac{1}{2} m g_{ac} \dot{x}^a \quad . \quad . \quad . \quad . \quad . \quad . \quad (211) \end{aligned}$$

where $\delta^a_c = 1, \ a = c$; $= 0, \ a \neq c$ $\qquad \delta^b_c = 1, \ b = c$; $= 0, \ b \neq c$

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^c}\right) &= \tfrac{1}{2} m \left(\frac{dg_{ab}}{dt} \dot{x}^b + \frac{dg_{ac}}{dt} \dot{x}^a + g_{cb} \ddot{x}^b + g_{ca} \ddot{x}^a \right) \\ &= \tfrac{1}{2} m \left(\frac{\partial g_{cb}}{\partial x^a} \frac{dx^a}{dt} \dot{x}^b + \frac{\partial g_{ac}}{\partial x^b} \frac{dx^b}{dt} \dot{x}^a \right) + \tfrac{1}{2} m g_{cb} \ddot{x}^b + \tfrac{1}{2} m g_{ca} \ddot{x}^a \\ &= \tfrac{1}{2} m \left(\frac{\partial g_{cb}}{\partial x^a} + \frac{\partial g_{ac}}{\partial x^b} \right) \dot{x}^a \dot{x}^b + m g_{cb} \ddot{x}^b \quad . \quad . \quad . \quad . \quad (212) \end{aligned}$$

$$\frac{\partial T}{\partial x^c} = \tfrac{1}{2} m \frac{\partial}{\partial x^c} (g_{ab} \dot{x}^a \dot{x}^b) = \tfrac{1}{2} m \frac{\partial g_{ab}}{\partial x^c} \dot{x}^a \dot{x}^b \quad . \quad . \quad (213)$$

Thus

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^c}\right) - \frac{\partial T}{\partial x^c} &= \tfrac{1}{2} m \left(\frac{\partial g_{cb}}{\partial x^a} + \frac{\partial g_{ca}}{\partial x^b} - \frac{\partial g_{ab}}{\partial x^c} \right) \dot{x}^a \dot{x}^b + m g_{cb} \ddot{x}^b \\ &= m [ab,c] \dot{x}^a \dot{x}^b + m g_{cb} \ddot{x}^b \\ &= m \{ [ab,c] \dot{x}^a \dot{x}^b + g_{cb} \ddot{x}^b \} \quad . \quad . \quad . \quad . \quad . \quad . \quad (214) \end{aligned}$$

The expression in compound brackets in eqn. (214) is that for the acceleration of a particle in terms of generalized variables or co-ordinates. The quantities $m g_{ab}$ define the metric tensor L_{ab} in dynamics. This term comprises the moments of inertia of the system.

Lagrange's equation may therefore be written as in Section 2, eqn. (10):

$$f_c = L_{ca} \ddot{x}^a + [ab,c] \dot{x}^a \dot{x}^b \quad . \quad . \quad . \quad . \quad (215)$$

While neither of the terms on the right-hand side is a tensor by itself, the expression on the right-hand side as a whole is a tensor. This is illustrated by a transformation to new co-ordinates,

$$f_\gamma = C^c_\gamma f_c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (216)$$

and $$f_\gamma = L_{\gamma\alpha} \ddot{x}^\alpha + [\alpha\beta,\gamma] \dot{x}^\alpha \dot{x}^\beta \quad . \quad . \quad . \quad . \quad (217)$$

where $$[L_{\gamma\alpha} \ddot{x}^\alpha + (\alpha\beta,\gamma) \dot{x}^\alpha \dot{x}^\beta] = [L_{ca} \ddot{x}^a + (ab,c) \dot{x}^a \dot{x}^b] C^c_\gamma \quad . \quad (218)$$

In an electrical network the equation corresponding to eqn. (215) becomes

$$e_w = L_{wr}\frac{d^2q^r}{dt^2} + [uv,w]\frac{dq^u}{dt}\frac{dq^r}{dt} \quad . \quad . \quad . \quad (219)$$

or

$$e_w = L_{wr}\frac{di^r}{dt} + [uv,w]i^u i^r \quad . \quad . \quad . \quad . \quad (220)$$

where the metric tensor L_{wr} comprises the self- and mutual inductances of the network elements. The voltage drop due to resistance may be added as an extra term $R_{wr}i^r$.

(10.3) Conditions necessary for an Equation to be Integrable[25]

If
$$Adx + Bdy + Cdz = 0 \quad . \quad . \quad . \quad . \quad (221)$$

has an integral $f(xyz) = K$

which on integration gives

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0$$

then
$$\frac{\partial f}{\partial x} = aA, \quad \frac{\partial f}{\partial y} = aB, \text{ and } \frac{\partial f}{\partial z} = aC$$

Hence
$$\frac{\partial}{\partial y}(aC) = \frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y} = \frac{\partial}{\partial z}(aB)$$

i.e.
$$a\left(\frac{\partial B}{\partial z} - \frac{\partial C}{\partial y}\right) + B\frac{\partial a}{\partial z} - C\frac{\partial a}{\partial y} = 0$$

and
$$\frac{\partial}{\partial y}(aC) = \frac{\partial a}{\partial y}C + a\frac{\partial C}{\partial y} = B\frac{\partial a}{\partial z} + a\frac{\partial B}{\partial z}$$

Therefore
$$a\left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) + B\frac{\partial a}{\partial z} - C\frac{\partial a}{\partial y} = 0 \quad . \quad . \quad [222(a)]$$

Similarly
$$a\left(\frac{\partial C}{\partial x} - \frac{\partial A}{\partial z}\right) + C\frac{\partial a}{\partial x} - A\frac{\partial a}{\partial z} = 0 \quad . \quad . \quad [222(b)]$$

and
$$a\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right) + A\frac{\partial a}{\partial y} - B\frac{\partial a}{\partial x} = 0 \quad . \quad . \quad [222(c)]$$

Multiplying eqns. [222(*a*)], [222(*b*)] and [222(*c*)] by A, B and C, respectively, and then adding,

$$A\left(\frac{\partial B}{\partial z} - \frac{\partial C}{\partial y}\right) + B\left(\frac{\partial C}{\partial x} - \frac{\partial A}{\partial z}\right) + C\left(\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right) = 0 \quad . \quad (223)$$

If eqn. (221) is integrable this condition must be satisfied. Eqn. (221) may be written

$$A_1dx^1 + A_2dx^2 + A_3dx^3 = 0$$

and the condition for integrability becomes

$$A_1\left(\frac{\partial A_2}{\partial x^3} - \frac{\partial A_3}{\partial x^2}\right) + A_2\left(\frac{\partial A_3}{\partial x^1} - \frac{\partial A_1}{\partial x^3}\right) + A_3\left(\frac{\partial A_1}{\partial x^2} - \frac{\partial A_2}{\partial x^1}\right) = 0 \quad . \quad . \quad . \quad . \quad (224)$$

or in general
$$a_{m,n}A_r + a_{n,r}A_m + a_{r,m}A_n = 0 \quad . \quad . \quad . \quad (225)$$

where
$$a_{m,n} = \frac{\partial A_m}{\partial x^n} - \frac{\partial A_n}{\partial x^m} \quad . \quad . \quad . \quad . \quad (226)$$

If, therefore, a set of equations, such as

$$dx^a = C^a_{\cdot m}dx^m$$

is not integrable, then
$$\frac{\partial C^h_a}{\partial x^b} \neq \frac{\partial C^h_b}{\partial x^a}$$

Printed in Great Britain by
UNWIN BROTHERS LIMITED, WOKING AND LONDON

THE INSTITUTION OF ELECTRICAL ENGINEERS

FOUNDED 1871: INCORPORATED BY ROYAL CHARTER 1921

SAVOY PLACE, LONDON, W.C.2

TENSOR ANALYSIS OF ELECTRICAL MACHINE HUNTING

By

J. W. LYNN, M.Sc., Associate Member.

MONOGRAPH No. 295 S

March, 1958

To be republished in

PART C OF THE PROCEEDINGS OF THE INSTITUTION

621.313.1

The Institution of Electrical Engineers
Monograph No. 295 S
Mar. 1958
©

TENSOR ANALYSIS OF ELECTRICAL MACHINE HUNTING

By J. W. LYNN, M.Sc., Associate Member.

(*The paper was first received 6th August, and in revised form 25th November,* 1957. *It was published as an* INSTITUTION MONOGRAPH *in March,* 1958.)

SUMMARY

The paper gives first a brief résumé of previous work on tensor analysis of electrical machines. The steady-state equations of the synchronous machine are written in Park's reference axes, and from these the hunting equations are derived. It is then shown that these equations are part of a general group of transformations of reference axes of the synchronous machines, all of which are embraced by the general tensor equations. The hunting equations are then derived in a freely-rotating reference system. These equations are rewritten in tensor form and the significance of the grouping of terms into tensors is discussed. The latter equations are shown to give a more realistic interpretation of the hunting equations and the equivalent circuit.

LIST OF PRINCIPAL SYMBOLS

Indices.

a, b, c = Quantities in axes fixed to the machine stator and rotor windings.
k, n, m = Quantities in axes all relatively stationary.
α, β, γ = Quantities in axes fixed or free on the stator and rotating freely on the rotor.
s, t = Quantities associated with the mechanical rotational effects in the machine (e.g. generated voltages and torque).
u, v, w = Quantities in a general equation.

Electrical parts of the equations.

V_m, V_α, etc. = Electrical voltage vectors in axes denoted by indices.
x^k, x^α, etc. = Electric variables. The electrical charges in machine windings, referred to axes denoted by indices.
$\dot{x}^\alpha (\equiv i^\alpha)$ = Electric current vector, in axes denoted by indices.
$R_{\gamma\alpha}$ = Resistance matrix, in axes denoted by indices.
$L_{\gamma\alpha}$ = Inductance matrix, in axes denoted by indices.
$G_{\gamma\alpha}$ = Generated voltage coefficients, in axes denoted by indices.

Mechanical part of the equations.

f_s = Mechanical force.
x^s = Mechanical variable θ, the angular position of the machine rotor during rotation.
$\dot{x}^s (\equiv i^s \equiv p\theta)$ = Angular velocity of machine rotor.
R_{ss} = Mechanical friction coefficients.
$L_{ss} (\equiv J)$ = Moment of inertia of machine rotor.

General symbols.

$\boldsymbol{v}$ = Generalized force vector (voltage or mechanical force).
$\boldsymbol{R}$ = Generalized dissipation matrix (resistance or friction).
$\boldsymbol{L}$ = Generalized inductance matrix (inductance or inertia).
$\boldsymbol{i}$ = Generalized current vector (electric current or angular velocity).
C^k_α = Connection matrix between quantities in axes denoted by indices.
$\boldsymbol{C}$ = Direct notation for C^k_γ, etc.
$\boldsymbol{C}_{(t)}$ = Transpose of matrix $\boldsymbol{C}$.
$\Omega^{\delta}_{\alpha\beta}$ = 'Non-holonomic object' containing functions of C, in axes denoted by indices.
$[\alpha\beta, \gamma]$ = A 'connection' term containing functions of $L_{\gamma\alpha}$ in axes denoted by indices.
$\Gamma_{\alpha\beta,\gamma}$ = A 'connection' term containing both $[\alpha\beta, \gamma]$ and $\Omega_{\alpha\beta,\gamma}$ in axes denoted by indices.
$S_{\alpha\beta\gamma}$ = Tensor giving the terms $G_{\gamma\alpha}$.

(1) INTRODUCTION

With rapid expansion in the field of control systems a fuller understanding of the dynamical behaviour of rotating electrical machines has become of increasing importance. For this reason, and also because of general developments in electrotechnics, the teaching of electrical machine theory from a generalized dynamical viewpoint is now being considered in universities and colleges. Brown, Kusko and White[1] give details of a laboratory machine for teaching purposes, the windings of which can be interconnected in a variety of ways to give the characteristics of a range of d.c. and a.c. machines.

One of the pioneers of this approach to electrical machine analysis was Gabriel Kron.[2,3,4] The matrix and tensor methods which he has developed since 1934 have led to a better understanding of the fundamental concepts underlying all machine systems. A survey of these methods is given in Reference 5. The above References show that the analysis of most types of electrical machines can be expressed by a single set of dynamical equations.

Recently the behaviour of oscillating-machine systems has been receiving a great deal of attention.[6,7,8] Transient and hunting conditions have been the subject of investigation throughout the history of machine analysis. The development of hunting analysis of synchronous machines can be indicated briefly by selection of one or two representative publications, as follows.

In 1929 Wennerberg[9] extended the early work of Kapp and Rosenberg. Starting with the design details of a 3-phase salient-pole machine he resolved the armature magnetomotive-force and flux waves into two axes in quadrature on the armature (stator). During hunting the field rotates and oscillates with respect to these axes. Wennerberg then derived equations for voltages, currents and torque during small oscillations of the rotor. The expressions are complicated because of the fixed armature axes chosen. All steady-state currents and voltages are functions of sin ωt and cos ωt (where ω is the synchronous angular velocity of the rotor), and the hunting equations are, of course, obtained by making small changes in steady-state values. In his expressions for hunting torque the trigonometrical terms of synchronous frequency ultimately disappear and the torque is expressed as

$$\Delta T = A'X \sin(h\omega)t + A''(h\omega)X \cos(h\omega)t$$

where $h\omega$ is the angular velocity of hunting. Inspection of this

Correspondence on Monographs is invited for consideration with a view to publication.
Mr Lynn is in the Department of Electrical Engineering, University of Liverpool.

expression yields the synchronizing and damping torque coefficients, A' and A''.

Prescott and Richardson,[10] using implicitly the same reference system as Wennerberg, derived a comprehensive set of equations giving the damping and synchronizing torque coefficients of a salient-pole alternator. They examined, in particular, the effects of armature resistance and armortisseur parameters on hunting. Curves are then given showing calculated and experimental results for a laboratory machine.

About this time the two-reaction theory of the salient-pole synchronous machine was developed. In 1926 Doherty and Nickle,[11] following Blondel, resolved the armature resultant m.m.f. and flux-linkage waves into axes along the field pole and in quadrature with it. These axes were considered to rotate synchronously along with the field structure. Both the armature and field quantities are therefore constant along these axes in the steady state, trigonometrical terms at synchronous frequency having been eliminated by transformation of the phase quantities into these axes. During hunting the reference axes rotate and oscillate with the field structure. In 1929 Park[12] expressed this theory in terms of transient or 'operational' impedances in these axes, which could be measured directly by tests on the machine. Park then extended his theory to include hunting conditions. His equations of hunting are physically the same as those of earlier investigators, but in terms of machine parameters they are more explicit. They are also more comprehensive in that the operational impedances incorporate the effects of the rotor (field) circuit parameters as well as those of armortisseurs. Park's equations have now become generally accepted in synchronous-machine theory.

Liwschitz,[13] using the latter reference axes, has analysed the hunting of a synchronous machine as a special case of the general problem of a doubly-fed machine. Concordia[14] has given a very comprehensive set of results of the application of Park's analysis to a particular machine.

In 1942 a.c. machines, including the salient-pole synchronous machine were described by a general theory of equivalent circuits, by Kron,[15] for both steady-state and hunting conditions. These circuits were subsequently used by Concordia, Kron and Crary.[16, 17]

The above work is confined to analysis of behaviour of a single machine synchronized with a large system. Kron[15] indicated that the analysis and equivalent circuits in Park's reference frame become complicated when external circuits are connected to the machine terminals. The synchronous-machine axes rotate and oscillate with the field structure, and the external network must also be analysed along oscillating axes. He then selected axes which are identical with those of Park in the steady state but which rotate uniformly and do not oscillate.[18] Equivalent hunting circuits were derived along these uniformly rotating axes, which could be interconnected to build up complete systems.[7, 19] These circuits are such that the resistance power loss in each mesh gives the damping torque. Synchronizing torque can also be read from the circuit.

The physical concepts arising in the new reference frame have been examined in detail.[19] In order to generalize his work on machines, Kron uses the methods of tensor calculus. Heffron, Rosenberry and Rothe[6, 7] have given an alternative, more conventional, analysis in which they compare hunting equations of an interconnected system in the reference frames of both Park and Kron, and point out the advantages of the uniformly rotating axes.

In the theory of relativity, Einstein's quest was for laws of nature that would hold irrespective of the reference frame chosen. Kron's approach to machine analysis has been from the same viewpoint. He looked for basic concepts which exist in all machines regardless of the reference axes and for equations expressing these concepts. He formulated these for his primitive machine and used already existing tensor analysis to deal with the transformation of these equations, to give those of any required machine with any chosen reference axes. As in relativity, it was found that the fundamental machine concepts having physical significance in steady-state, transient or hunting operation were all tensors.[18] Equivalent-circuit meshes were seen to yield groups of terms which constitute tensors. Conversely, if equivalent circuits were to be set up, the terms of the equations should be grouped into tensor quantities. Apart from equivalent circuits, this tensor grouping of the equations appears to give a better physical picture of the correlation of different forms of energy in any physical system. Prof. Kondo[20] has used tensor equations, identical in form with the machine equations, for the analysis of aircraft oscillations in which aerodynamic and other forces are considered.

The present paper shows how these tensor terms arise in hunting analysis of a synchronous machine and how the tensor grouping of terms is associated with the equivalent hunting circuit. The equations are particular cases of the general machine equations, and the same analysis is therefore directly applicable to many other types of d.c. and a.c. machines.

(2) TENSOR EQUATIONS OF ELECTRICAL MACHINES

(2.1) Matrix Equations

The voltage equations of the stationary-axis primitive machine shown in Fig. 1, with axes ds, dr, qr and qs, can be written down by inspection. Written in matrix form these are as follows:

$$\begin{array}{c|c|} & \\ \hline ds & v_{ds} \\ \hline dr & v_{dr} \\ \hline qr & v_{qr} \\ \hline qs & v_{qs} \\ \hline \end{array} = \begin{array}{c|c|c|c|c|} & ds & dr & qr & qs \\ \hline ds & r_{ds} + L_{ds}p & M_d p & & \\ \hline dr & M_d p & r_{dr} + L_{dr}p & L_{qr}p\theta & M_q p\theta \\ \hline qr & -M_d p\theta & -L_{dr}p\theta & r_{qr} + L_{qr}p & M_q p \\ \hline qs & & & M_q p & r_{qs} + L_{qs}p \\ \hline \end{array} \cdot \begin{array}{c|c|} & \\ \hline ds & i^{ds} \\ \hline dr & i^{dr} \\ \hline qr & i^{qr} \\ \hline qs & i^{qs} \\ \hline \end{array} \quad . \; . \; . \quad (1)$$

These are seen to include those derived by Park[12] in 1926 for the synchronous machine along direct and quadrature axes, usually written[21] as follows

Impressed field voltage.

$$v_{fd} = R_{fd}i^{fd} + L_{fd}pi^{fd} + M_d pi^{dr} \quad . \; . \; . \quad (2)$$

Generated voltage.

$$v_{dr} = - M_d pi^{fd} - R_{dr}i^{dr} - L_{dr}pi^{dr} + L_{qr}i^{qr}p\theta \quad . \quad (3)$$

$$v_{qr} = - M_d i^{fd}p\theta - L_{dr}i^{dr}p\theta - R_{qr}i^{qr} - L_{qr}pi^{qr} \quad . \quad (4)$$

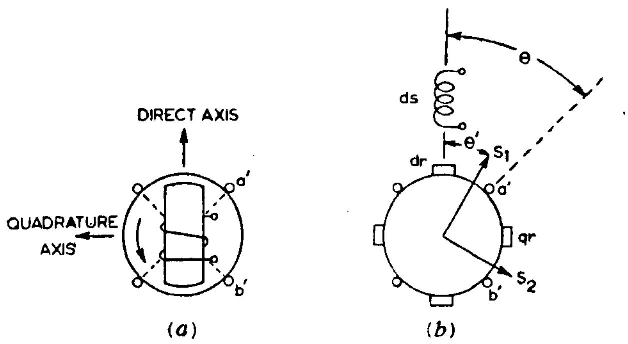


Fig. 1.—The primitive machine.
(*a*) 2-phase synchronous-machine field structure rotating.
(*b*) Primitive machine with armature rotating.

Generated electrical torque.

$$F = \psi_d i^{qr} - \psi_q i^{dr} \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where
$$\psi_d = - M_d i^{fd} - L_{dr} i^{dr} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and
$$\psi_q = - L_{qr} i^{qr} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

In eqns. (3) and (4) the quadrature field axis has been omitted for simplicity.

Equivalent circuits for the synchronous machine have been developed by Kron[15] and studied by Ku[22] by resolving the set of eqns. (1) into forward- and backward-rotating instantaneous symmetrical components.

Eqns. (1) are of the form

$$\boldsymbol{v} = \boldsymbol{Ri} + \boldsymbol{Lpi} + \boldsymbol{Gip\theta} \quad . \quad . \quad . \quad . \quad (8)$$

or
$$\boldsymbol{v} = \boldsymbol{Z} \, . \, \boldsymbol{i} \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

where
$$\boldsymbol{Z} = (\boldsymbol{R} + \boldsymbol{Lp} + \boldsymbol{Gp\theta}) \quad . \quad . \quad . \quad . \quad (10)$$

The torque is given by

$$\boldsymbol{f} = \boldsymbol{i}^* \, . \, \boldsymbol{G} \, . \, \boldsymbol{i} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

(the asterisk denoting conjugate values), the 'torque matrix' G being

	ds	*dr*	*qr*	*qs*
ds				
dr			L_{qr}	M_q
qr	$-M_d$	$-L_{dr}$		
qs				

. (12)

It is shown in Reference 3 that the impedance matrix of eqn. (1) can be transformed to give that for any other commutator machine, by the simple transformation

$$\boldsymbol{Z}' = \boldsymbol{C}_{(t)} \, . \, \boldsymbol{Z} \, . \, \boldsymbol{C} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where C is the matrix connecting mesh currents in the derived machine windings with the branch currents of the primitive machine coils. If the transformation is to be carried out to any reference axes which are rotating relative to the direct and quadrature axes, then, as shown in Reference 3, the transformation law is

$$\boldsymbol{Z}' = \boldsymbol{C}_{(t)} \, . \, \boldsymbol{Z} \, . \, \boldsymbol{C} + \boldsymbol{C}_{(t)} \, . \, \boldsymbol{L} \, . \, \frac{\partial \boldsymbol{C}}{\partial \theta'} p\theta' \quad . \quad . \quad . \quad (14)$$

where $p\theta'$ is the angular velocity of the reference frame with respect to the direct and quadrature axes system. This leads to an equation of the form

$$\boldsymbol{v}' = \boldsymbol{R}'\boldsymbol{i}' + \boldsymbol{L}'\boldsymbol{pi}' + \boldsymbol{G}'\boldsymbol{i}'\boldsymbol{p\theta} + \boldsymbol{V}'\boldsymbol{i}'\boldsymbol{p\theta}' \quad . \quad . \quad (15)$$

or
$$\boldsymbol{v}' = \boldsymbol{Z}' \, . \, \boldsymbol{i}' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

where
$$\boldsymbol{Z}' = (\boldsymbol{R}' + \boldsymbol{L}'\boldsymbol{p} + \boldsymbol{G}'\boldsymbol{p\theta} + \boldsymbol{V}'\boldsymbol{p\theta}') \quad . \quad . \quad . \quad (17)$$

If the direct- and quadrature-axis quantities are transformed to uniformly rotating axes S_1 and S_2, the connection matrix C is given by the relation between the currents in the two systems as shown in Fig. 1.

$$\begin{aligned} i^{ds} &= i^{ds} \\ i^{dr} &= i^{S1}\cos\theta' - i^{S2}\sin\theta' \\ i^{qr} &= i^{S1}\sin\theta' + i^{S2}\cos\theta' \\ i^{qs} &= i^{qs} \end{aligned} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

and

$C =$

	dr	S_1	S_2	*qs*
ds	1			
dr		$\cos\theta'$	$-\sin\theta'$	
qr		$\sin\theta'$	$\cos\theta'$	
qs				1

. (19)

The terms of expression (17) are given in full in Reference 5.

If the angle θ' is, in fact, the load angle λ, the axes S_1 and S_2 coincide with the voltage axes of the machine. The angle λ is constant in the steady state and the last term in eqn. (15) is zero. The impedance matrix is then given by the transformation (13), which gives (neglecting axis *qs*)

$$\boldsymbol{Z}' = \boldsymbol{C}_{(t)} \, . \, \boldsymbol{Z} \, . \, \boldsymbol{C} \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

$Z' =$

	ds	S_1	S_2
ds	$r_{ds} + L_{ds}p$	$M_d \cos\lambda p$	$-M_d \sin\lambda p$
S_1	$M_d\cos\lambda p$ $-M_d\sin\lambda p\theta$	$r_{dr} + L_{dr}\cos^2\lambda p$ $+ L_{qr}\sin^2\lambda p$ $+(L_{qr} - L_{dr})\sin\lambda\cos\lambda p\theta$	$(L_{qr} - L_{dr})\sin\lambda\cos\lambda p$ $+ L_{dr}\sin^2\lambda p\theta$ $+ L_{qr}\cos^2\lambda p\theta$
S_2	$-M_d\sin\lambda p$ $-M_d\cos\lambda p\theta$	$(L_{qr} - L_{dr})\sin\lambda\cos\lambda p$ $- L_{dr}\cos^2\lambda p\theta$ $- L_{qr}\sin^2\lambda p\theta$	$r_{qr} + L_{qr}\cos^2\lambda p$ $+ L_{dr}\sin^2\lambda p$ $-(L_{qr} - L_{dr})\cos\lambda\sin\lambda p\theta$

. . (21)

The matrix multiplication shown above, to give a change of reference frame, is extremely simple, and the obvious question that arises is whether any knowledge of tensors is required and whether there is any advantage in learning a new mathematical technique involving, among other things, a complicated index notation. The answer begins to appear when one looks at the general law of transformation of impedance during small disturbances of the machine voltages, currents and speed.[3] If the hunting impedance in Park's reference system is $\mathcal{Z}$, the hunting impedance matrix in a free frame is given by

$$\mathcal{Z}' = \left[C_{(t)}\,.\,\mathcal{Z}\,.\,C + C_{(t)}\,.\,L\,.\,\frac{\partial C}{\partial\theta'}p\theta'\right](\Delta i' + \Delta\theta')$$

$$+\left[C_{(t)}\,.\,\mathcal{Z}\,.\,\frac{\partial C}{\partial\theta'}i' + C_{(t)}\,.\,L\,.\,\frac{\partial C^2}{(\partial\theta')^2}(p\theta') + C_{(t)}\,.\,L\,.\,\frac{\partial C}{\partial\theta'}pi'\right.$$

$$\left. + \frac{\partial C_{(t)}}{\partial\theta'}\,.\,Z\,.\,C\,.\,i' - \frac{\partial V'}{\partial\theta'}\right]\Delta\theta' \quad . \quad (22)$$

The purpose of tensor analysis is to present all the transformations of machine reference frames in a consistent dynamical theory for steady-state and hunting analysis of all types of machines. The fundamental ideas underlying the tensor approach as developed by Kron have been examined in Reference 5. The salient points are now summarized.

(2.2) General Tensor Equations of Electrical Machines

In Reference 24 the different concepts of flux linkage and generated voltages arising under different transformations of machine reference axes have been classified. These components change with the angular velocity of the reference frames and some arise in one system of measurement and disappear in others. When machine equations are expressed in tensor form these various voltage components are part of a total 'tensor' voltage which includes both flux linkage and flux density. A tensor voltage cannot disappear under any transformation, and it is this, whatever its components, that is associated with the invariant power and stored electromagnetic energy in the machine. Resistance drop is also a tensor voltage.

Another advantage of using tensor equations is that there are available general routine laws covering all possible transformations of reference axes, the machine power remaining invariant with each change of reference system.

The tensor equation of the rotating electrical machine is an equation which has the same basic terms for all machines, these having different components for each machine or with each reference system chosen. The motion of the rotor is included in the form of rate of change of self and mutual coupling of the rotor and stator coils with changing rotor angle.

The voltage equation of a single coil having resistance and inductance is written

$$v = Ri + L\frac{di}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

The equation for a set of coils, some of which rotate with respect to the others, becomes

$$v_w = R_{wu}i^u + L_{wu}\frac{\delta i^u}{dt} \quad . \quad . \quad . \quad . \quad . \quad (24)$$

where $\frac{\delta i^u}{dt}$ is the 'absolute' derivative of the current with respect to time. Every term is then a tensor. As shown in Reference 5,

$$\frac{\delta i^u}{dt} = \frac{di^u}{dt} + \Gamma^u_{vw}i^v i^w \quad . \quad . \quad . \quad . \quad . \quad (25)$$

where Γ^u_{vw} contains functions of the matrix C which relates the currents in two different systems,

$$i^u = C^u_\alpha i^\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

and

$$\frac{\delta i^u}{dt} = \frac{\delta i^\alpha}{dt}C^u_\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

The indices range over the different machine variables, namely the electrical charges in each coil and the angle of rotation of the rotor. The coil currents and angular velocity of the rotor are then written, for example,

$$i^u = \frac{dq^u}{dt} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28)$$

and

$$i^s = \frac{d\theta}{dt} = p\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29)$$

The index s is used to denote the mechanical variable, the angle θ.

Also

$$L_{wu}\frac{\delta i^u}{dt} = L_{wu}\frac{di^u}{dt} + \Gamma_{uv,w}i^u i^v \quad . \quad . \quad . \quad . \quad (30)$$

The machine equation in its general form therefore becomes

$$v_w = R_{wu}i^u + L_{wu}pi^u + \Gamma_{uv,w}i^u i^v \quad . \quad . \quad . \quad . \quad (31)$$

The whole of tensor analysis of rotating electrical machinery is based on a knowledge of the components of the term $\Gamma_{uv,w}i^u i^v$ in eqn. (31). When these are understood and facility in manipulating the index notation has been attained, the group properties of electrical machines become clear and analysis of a wide range of machines under many different operating conditions becomes a matter of routine procedure. The study of the Γ terms and their expansion have been carried out in detail in Reference 5.

(2.3) Synchronous-Machine Systems

The analysis of synchronous machines can be carried out using any one of three reference systems shown in Fig. 1, namely

(a) Actual phase quantities, or 2-phase co-ordinates of these.

(b) The reference frame of Park. This is much more suitable for most cases, and is in general use for synchronous-machine studies at constant speed.

(c) The free frame. In this system the field quantities are referred to axes fixed on the field as in reference systems (a) and (b). Armature quantities are referred to axes which rotate uniformly with respect to the armature as in reference frame 2, but these can be chosen to have any uniform velocity, they can have any fixed position relative to the field structure, and the angular velocity of the reference frame is independent of any oscillations of the field structure. The simplest case is that in which the free frame coincides with that of Park under steady-state constant-speed conditions.

The machine equations in each frame have the same form and are identified by different systems of indices as follows:

Frame (a) $\quad v_c = R_{ca}i^a + L_{ca}pi^a + \Gamma_{ab,c}i^a i^b \quad . \quad . \quad . \quad (32)$

Frame (b) $\quad v_m = R_{mk}i^k + L_{mk}pi^k + \Gamma_{kn,m}i^k i^n \quad . \quad . \quad . \quad (33)$

Frame (c) $\quad v_\gamma = R_{\gamma\alpha}i^\alpha + L_{\gamma\alpha}pi^\alpha + \Gamma_{\alpha\beta,\gamma}i^\alpha i^\beta \quad . \quad . \quad . \quad (34)$

where $\quad v_m = C^c_m v_c \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$

$\quad L_{\gamma\alpha} = L_{mk}C^m_\gamma C^k_\alpha \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$

etc.

The most general form of the term $\Gamma_{uv,w}$ in eqn. (31) is, as shown in Reference 5,

$$\Gamma_{uv,w} \equiv \frac{1}{2}\left(\frac{\partial L_{wv}}{\partial x_u} + \frac{\partial L_{wu}}{\partial x_v} - \frac{\partial L_{uv}}{\partial x_w}\right)$$

$$- S_{wvu} - S_{wuv} + S_{uvw} + \Omega_{wv,u} + \Omega_{wu,v} - \Omega_{uv,w} \quad . \quad (37)$$

In the voltage equation

$$x^v = \theta \text{ and } i^v = p\theta \quad . \quad . \quad . \quad . \quad . \quad (38)$$

In the steady state

$$\Gamma_{uv,w}i^u i^v \equiv \left(\frac{\partial L_{wu}}{\partial x^v} - 2S_{wvu} + 2\Omega_{wv,u}\right)i^u i^v \quad . \quad . \quad (39)$$

$$-2S_{wvu} \equiv C_{(t)} . G . C \equiv G' \quad . \quad . \quad . \quad (40)$$

and the torque matrix in the new frame is

$$2\Omega_{wv,u} = \frac{\partial C_t^{-1}}{\partial \theta'} C_{(t)} L' = \rho L' \quad . \quad . \quad . \quad . \quad (41)$$

where L' is the inductance matrix in the new frame

and $\rho =$ (w \ u)

		−1	
	1		

. . . . . (42)

Thus $\Gamma_{uv,w}i^u i^v$ becomes, in the steady-state equations,

$$\frac{\partial L'}{\partial \theta'} . p\theta' . i' + G' . i' . \rho\theta + \rho L' . i' . p\theta' \quad . \quad . \quad (43)$$

where $p\theta$ is the angular velocity of the rotor and $p\theta'$ is the angular velocity of the reference frame with respect to the rotor.

In reference frame (*a*) the second and third terms of expression (43) do not arise. They appear only in transformations from reference frame (*a*) to any other reference frame.

In reference frame (*b*) the first and third terms are zero.

In reference frame (*c*) all three terms arise. An important point is that, while inductances along reference frame (*c*) can be constant in the steady state (if the axes rotate at synchronous speed or coincide with Park's axes), they are subject to incremental changes when the field structure oscillates. This means that a term such as $(\partial L'/\partial\theta')p\theta'$, which arises in the general equations, may be zero in the steady state, but under hunting conditions it becomes $(\partial L'/\partial\lambda)p(\Delta\lambda)$ and must be retained. The use of the general form of the term $\Gamma_{uv,w}$ in hunting equations ensures that all possible interactions of currents, fluxes and speed, together with their increments, are included in the final equations.

The disadvantage of Park's equations [frame (*b*)] for hunting analysis is that the armature reference frame is fixed along the field structure and oscillates with it. This means that circuits connected to the armature must also be analysed along the same oscillating reference system.[19] It is simpler and more realistic to analyse the machine in the first place along reference frame (*c*). External networks can then be more easily included. It is reference frame (*c*) that leads to the new equivalent circuit in which the system damping torque is given by the network mesh-resistance power loss.

(3) SYNCHRONOUS-MACHINE HUNTING EQUATIONS. PARK'S REFERENCE FRAME

The machine is that shown in Fig. 1, with armature axes *dr* and *qr*, stationary with respect to the field. The quadrature field circuit has been omitted in the following equations. The resistance, inductance, torque and impedance matrices are as follows:

$$v = Ri + Lpi + Gip\theta$$

or

$$v = Z . i$$

where

$Z =$

	ds	dr	qr
ds	$r_{ds} + L_{ds}p$	$M_d p$	
dr	$M_d p$	$r_{dr} + L_{dr}p$	$L_{qr}p\theta$
qr	$-M_d p\theta$	$-L_{dr}p\theta$	$r_{qr} + L_{qr}p$

. (44)

$R =$

	ds	dr	qr
ds	r_{ds}		
dr		r_{dr}	
qr			r_{qr}

. (45)

$L =$

	ds	dr	qr
ds	L_{ds}	M_d	
dr	M_d	L_{dr}	
qr			L_{qr}

. (46)

$G =$

	ds	dr	qr
ds			
dr			L_{qr}
qr	$-M_d$	$-L_{dr}$	

. . . (47)

The hunting equation may be found as in dynamics by considering small increments of the quantities in the steady-state equations (8) and (11).

Thus $$(v + \Delta v) = (Z + \Delta Z)(i + \Delta i) \quad . \quad . \quad . \quad (48)$$

or $$(v + \Delta v) = (R + \Delta R)(i + \Delta i) + (L + \Delta L)p(i + \Delta i) + (G + \Delta G)(i + \Delta i)(p\theta + \Delta p\theta) \quad . \quad (49)$$

Multiplying out and subtracting the original equation the hunting equation of voltage is found:

$$\Delta v = (R + Lp + Gp\theta)\Delta i + Gi(\Delta p\theta) \quad . \quad . \quad (50)$$

The torque equation becomes

$$\Delta f = i^* . G . \Delta i + \Delta i^* . G . i \quad . \quad . \quad . \quad (51)$$

The matrix form of eqns. (50) and (51) can now be written, including a mechanical row and column involving the mechanical variables.

The equation has the form

$$\Delta v = \mathbf{Z} . \Delta i \quad . \quad . \quad . \quad . \quad . \quad . \quad (52)$$

which, when written in full, expands to

	Δv	=		ds	dr	qr	s		Δi
ds	Δv_{ds}		ds	$r_{ds} + L_{ds}p$	$M_d p$			ds	Δi^{ds}
dr	Δv_{dr}		dr	$M_d p$	$r_{dr} + L_{dr}p$	$L_{qr}p\theta$	$i^{qr}L_{qr}p$	dr	Δi^{dr}
qr	Δv_{qr}	=	qr	$-M_d p\theta$	$-L_{dr}p\theta$	$r_{qr} + L_{qr}p$	$-i^{ds}M_d p$ $-i^{dr}L_{dr}p$	qr	Δi^{qr}
s	Δf		s	$i^{qr}M_d$	$2i^{qr}L_D$	$2i^{dr}L_D + i^{ds}M_d$	Jp^2	s	$\Delta\theta$

. . . (53)

where $$L_D = (L_{dr} - L_{qr})/2$$

and J is the rotor inertia constant.

These equations have been studied in detail both analytically and by means of equivalent circuits,[6, 22] and they will now be used only as a starting-point for the study of the hunting equations expressed in the free reference frame.

(4) HUNTING EQUATIONS IN THE FREE FRAME

In the free frame the reference axes do not oscillate with the rotor. The equations are first expressed along the axes d' and q' of Fig. 2, rotating synchronously with the terminal voltage vector position.

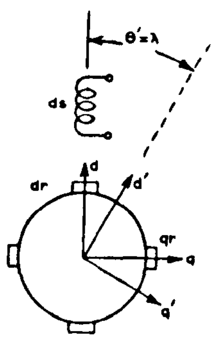


Fig. 2.—Free reference axes.
Axes d' and q' along terminal voltage axes.
Axes d and q coinciding with Park's axes.

This may be done in either of two ways:

(i) The hunting equations (53) can be transformed directly to those in the free frame using the law of transformation given by eqn. (22). In this equation the p-operator in $\mathbf{Z}$ refers only to $\Delta i'$ or $\Delta\theta'$ and not to C, and in Z; p refers only to C and i' and not to $\Delta\theta'$.

(ii) The more elegant method is to set up the steady-state equations in the new frame and from these to derive the hunting equation.

The steady-state equation is eqn. (34). Taking small increments of this as before, the hunting equations, as shown in detail in Appendix 11.1, are as follows:

Voltage equation.

$$\Delta v_\alpha + \frac{\partial v_\alpha}{\partial x^\beta}\Delta x^\beta =$$
$$(R_{\alpha\beta}\Delta i^\beta + L_{\alpha\beta}p\Delta i^\beta + \Gamma_{\beta s,\alpha}p\theta\Delta i^\beta + \Gamma_{s\gamma,\alpha}p\theta\Delta i^\gamma)$$
$$+\left[\Gamma_{\beta s,\alpha}i^\beta p + \Gamma_{s\gamma,\alpha}i^\gamma p + \frac{\partial\Gamma_{\beta s,\alpha}}{\partial\lambda}i^\beta p\theta + \frac{\partial\Gamma_{s\gamma,\alpha}}{\partial\lambda}i^\gamma p\theta + \frac{\partial L_{\alpha\beta}}{\partial\lambda}(pi^\beta)\right]\Delta\lambda \quad . \quad (54)$$

which becomes, for the synchronous machine,

$$\Delta \boldsymbol{v}' = (\boldsymbol{R}' + \boldsymbol{L}'p + \boldsymbol{G}'p\theta)\Delta\boldsymbol{i}' + \left(\frac{\partial \boldsymbol{L}'}{\partial\lambda}\boldsymbol{i}'p + \frac{\partial \boldsymbol{G}'}{\partial\lambda}\boldsymbol{i}'p\theta\right)\Delta\lambda \quad . \quad (55)$$

The torque equation is

$$\Delta f = \Delta\boldsymbol{i}'^{*}.\boldsymbol{G}'.\boldsymbol{i}' + \boldsymbol{i}'^{*}.\boldsymbol{G}'.\Delta\boldsymbol{i}' + \frac{\partial \boldsymbol{G}'}{\partial\lambda}\boldsymbol{i}'^{*}.\boldsymbol{i}'.\Delta\lambda + Jp^2\Delta\lambda \quad . \quad . \quad . \quad . \quad . \quad (56)$$

In matrix form these expand to

$$\begin{array}{c|c|} & \\ ds & \Delta v_{ds} \\ d' & \Delta v_{d'} \\ q' & \Delta v_{q'} \\ s & \Delta f \end{array} = \begin{array}{c|c|c|c|c|} & ds & d' & q' & s \\ ds & r_{ds} + L_d p & M_d\cos\lambda p & -M_d\sin\lambda p & \begin{array}{c}-M_d\sin\lambda i^d p\\ -M_d\cos\lambda i^q p\end{array} \\ d' & \begin{array}{c}M_d\cos\lambda p\\ -M_d\sin\lambda p\theta\end{array} & \begin{array}{c}r_{dr} + L_{dr}\cos^2\lambda p\\ +L_{qr}\sin^2\lambda p\\ +(L_{qr}-L_{dr})\sin\lambda\cos\lambda p\theta\end{array} & \begin{array}{c}(L_{qr}-L_{dr})\sin\lambda\cos\lambda p\\ +L_{dr}\sin^2\lambda p\theta\\ +L_{qr}\cos^2\lambda p\theta\end{array} & \begin{array}{c}2(L_{qr}-L_{dr})\cos\lambda\sin\lambda i^d p\\ +(L_{qr}-L_{dr})(\cos^2\lambda-\sin^2\lambda)i^q p\\ -i^{ds}M_d\sin\lambda p\\ -2(L_{qr}-L_{dr})\cos\lambda\sin\lambda i^q p\theta\\ +(L_{qr}-L_{dr})(\cos^2\lambda-\sin^2\lambda)i^d p\theta\\ -M_d\cos\lambda i^{ds}p\theta\end{array} \\ q' & \begin{array}{c}-M_d\sin\lambda p\\ -M_d\cos\lambda p\theta\end{array} & \begin{array}{c}(L_{qr}-L_{dr})\sin\lambda\cos\lambda p\\ -L_{dr}\cos^2\lambda p\theta\\ -L_{qr}\sin^2\lambda p\theta\end{array} & \begin{array}{c}r_{qr} + L_{qr}\cos^2\lambda p\\ +L_{dr}\sin^2\lambda p\\ -(L_{qr}-L_{dr})\cos\lambda\sin\lambda p\theta\end{array} & \begin{array}{c}-2(L_{qr}-L_{dr})\cos\lambda\sin\lambda i^q p\\ +(L_{qr}-L_{dr})(\cos^2\lambda-\sin^2\lambda)i^d p\\ -i^{ds}M_d\cos\lambda p\\ -2(L_{qr}-L_{dr})\cos\lambda\sin\lambda i^d p\theta\\ -(L_{qr}-L_{dr})(\cos^2\lambda-\sin^2\lambda)i^q p\theta\\ +M_d i^{ds}\sin\lambda p\theta\end{array} \\ s & \begin{array}{c}M_d\sin\lambda i^d\\ +M_d\cos\lambda i^q\end{array} & \begin{array}{c}i^{ds}M_d\sin\lambda\\ -(L_{qr}-L_{dr})(\cos^2\lambda-\sin^2\lambda)i^q\\ -2(L_{qr}-L_{dr})\cos\lambda\sin\lambda i^d\end{array} & \begin{array}{c}i^{ds}M_d\cos\lambda\\ -(L_{qr}-L_{dr})(\cos^2\lambda-\sin^2\lambda)i^d\\ +2(L_{qr}-L_{dr})\sin\lambda\cos\lambda i^q\end{array} & \begin{array}{c}Jp^2 + M_d\cos\lambda i^{ds}i^d\\ -M_d\sin\lambda i^{ds}i^q\\ -(L_{qr}-L_{dr})(\cos^2\lambda-\sin^2\lambda)i^d i^d\\ +(L_{qr}-L_{dr})(\cos^2\lambda-\sin^2\lambda)i^q i^q\\ +4(L_{qr}-L_{dr})\sin\lambda\cos\lambda i^d i^q\end{array} \end{array} \cdot \begin{array}{c|c|} & \\ ds & \Delta i^{ds} \\ d' & \Delta i^{d'} \\ q' & \Delta i^{q'} \\ s & \Delta\lambda \end{array} \quad . \quad . \quad . \quad . \quad . \quad (57)$$

These equations are very much simplified when the free axes are considered to coincide with Park's reference axes in the steady state. The angle λ then becomes zero. This does not imply that the machine load angle is zero, but only that the reference frame has been rotated. The load angle λ is still inherent in the equations in the computation of the steady-state currents involved in the hunting equations.

In the free frame along the field axes the machine-impedance matrix becomes

	ds	d	q	s
ds	$r_{ds} + L_{ds}p$	$M_d p$	.	$-M_d i^q p$
d	$M_d p$	$r_r + L_{dr}p$	$L_{qr}p\theta$	$(L_{qr} - L_{dr})i^q p$ $+(L_{qr} - L_{dr})i^d p\theta$ $-M_d i^{ds} p\theta$
q	$-M_d p\theta$	$-L_{qr}p\theta$	$r_r + L_{qr}p$	$(L_{qr} - L_{dr})i^d p$ $-(L_{qr} - L_{dr})i^q p\theta$ $-M_d i^{ds} p$
s	$M_d i^q$	$-(L_{qr} - L_{dr})i^q$	$M_d i^{ds}$ $-(L_{qr} - L_{dr})i^d$	$Jp^2 + M_d i^{ds} i^d$ $-(L_{qr} - L_{dr})i^d i^d$ $+(L_{qr} - L_{dr})i^q i^q$

. (58)

(5) THE TENSOR EQUATIONS OF HUNTING

The time rate of change of a vector with respect to axes fixed on the vector is written di^α/dt, where the components of the vector in the given reference frame are i^α. With respect to another co-ordinate system, e.g. one that rotates with respect to the original vector position, the time rate of change becomes the absolute derivative $\delta i^\alpha/dt$, where, in general terms,

$$\frac{\delta i^\alpha}{dt} = \frac{di^\alpha}{dt} + \Gamma^\alpha_{\beta\gamma} i^\beta \frac{dx^\gamma}{dt} \quad . \quad . \quad . \quad . \quad (60)$$

The absolute differential is written

$$\delta i^\alpha = di^\alpha + \Gamma^\alpha_{\beta\gamma} i^\beta dx^\gamma \quad . \quad . \quad . \quad . \quad (61)$$

In setting up the conventional equations of hunting, small changes Δi^α, etc., were considered in each of the terms of the steady-state equations. No consideration was given as to whether the resulting equations were tensor equations. In fact, as stated in Section 2.2, in general the ordinary differential of a tensor is not a tensor. The absolute differential shown in eqn. (61) is a tensor. The tensor equation of hunting has been developed by Kron in Reference 2, by taking absolute increments δi^α, etc., in each of the terms of the tensor steady-state equation.

Taking absolute increments of the steady-state eqn. (34) gives

$$(v_\gamma + \Delta v_\gamma) = (R_{\gamma\alpha} + \delta R_{\gamma\alpha})(i^\alpha + \delta i^\alpha) + (\delta L_{\gamma\alpha})\frac{\delta i^\alpha}{dt} + L_{\gamma\alpha}\delta\left(\frac{\delta i^\alpha}{dt}\right) \quad . \quad . \quad . \quad . \quad (62)$$

and the tensor equation of small oscillation becomes

$$\delta v_\gamma = R_{\gamma\alpha}\delta i^\alpha + L_{\gamma\alpha}\delta\left(\frac{\delta i^\alpha}{dt}\right) \quad . \quad . \quad . \quad . \quad (63)$$

It is now necessary to express this equation in terms of δi^α. This change introduces a new tensor which in geometry is called the Riemannian–Christoffel curvature tensor, because it gives a measure of the intrinsic curvature of any given space in Riemannian geometry. It arises from the fact that, as shown in Appendix 11.2,

$$\delta\left(\frac{\delta i^\alpha}{dt}\right) - \frac{\delta}{dt}(\delta i^\alpha) = K_{\delta\gamma\beta}^{\cdot\cdot\cdot\alpha} i^\beta i^\delta dx^\gamma \quad . \quad . \quad . \quad (64)$$

where the term on the right-hand side is the new tensor. Eqn. (63) now becomes

$$\delta v_\gamma = R_{\gamma\alpha}\delta i^\alpha + L_{\gamma\alpha}\frac{\delta}{dt}(\delta i^\alpha) + K_{\delta\pi\alpha\gamma} i^\alpha i^\delta dx^\pi \quad . \quad . \quad (65)$$

Appendix 11.2 shows how this equation is expanded to give the machine voltage and torque equations of hunting in tensor form. These are written by Kron in Reference 19 as follows:

Voltage equation.

$$\delta v_\gamma = \left\{R_{\gamma\alpha}\delta i^\alpha + L_{\gamma\alpha}\frac{\delta}{dt}(\delta i^\alpha)\right\} + K_{st\alpha\gamma} i^\alpha (p\theta)\Delta\lambda \quad . \quad (66)$$

or

$$\Delta \mathbf{v}' = \left\{[\mathbf{R}' + \mathbf{L}'p + \mathbf{G}'p\theta]\Delta\mathbf{i}' + \frac{\partial \mathbf{L}'}{\partial\lambda}\mathbf{i}'p(\Delta\lambda) + \mathbf{G}' . \rho\mathbf{i}' . p\theta . \Delta\lambda\right\} + \left[\frac{\partial \mathbf{G}'}{\partial\lambda}\mathbf{i}' - \mathbf{G}' . \rho\mathbf{i}'\right]p\theta . \Delta\lambda \quad . \quad (67)$$

Torque equation (neglecting the friction tensor $R_{\gamma\alpha}\delta i^\alpha$).

$$\delta f = \left\{J\frac{\delta}{dt}(\delta i^t)\right\} + K_{\delta t\alpha s} i^\delta i^\alpha \Delta\lambda \quad . \quad . \quad . \quad (68)$$

or

$$\Delta f = \left\{J\frac{d}{dt}(\Delta\omega) - [\Delta\mathbf{i}'^* . \mathbf{G}' . \mathbf{i}' + \mathbf{i}'^* . \mathbf{G}' . \Delta\mathbf{i}' + \mathbf{i}'^* . \mathbf{G}' . \rho\mathbf{i}' . \Delta\lambda]\right\} + \left[\mathbf{i}'^* . \frac{\partial \mathbf{G}'}{\partial\lambda} . \mathbf{i}' - \mathbf{i}'^* . \mathbf{G}' . \rho\mathbf{i}'\right]\Delta\lambda \quad . \quad (69)$$

The first set of terms in square brackets in eqn. (69) is part of the second term on the right-hand side of eqn. (65). It gives a complex quantity. The real part is in phase with the increment of angular velocity $\Delta\omega$, and is therefore in time quadrature with the displacement angle $\Delta\lambda$ and gives the damping torque. The imaginary part is in anti-phase with the displacement angle, and is counted a negative synchronizing torque. The second square-bracketed set of terms in eqn. (69), which is given by the new tensor term, is seen on inspection of its components to have only a positive real value, in phase with the displacement angle. It gives the machine positive synchronizing torque. The matrix components of these equations are examined in the following Section. Eqns. (67) and (69) are seen to consist of the non-tensor equations (55) and (56) with a term added and subtracted. Thus the tensor equations of hunting give the conventional equations, with an important difference, namely a regrouping of terms which leads to a change in interpretation.[7, 19] The mathematical implication of the regrouping, from non-tensor to tensor form, has been explained by Hoffmann.[25]

(6) EQUIVALENT CIRCUIT FOR HUNTING EQUATIONS IN THE FREE REFERENCE FRAME

Equivalent circuits for a.c. electrical machines can be obtained by operating upon the impedance matrices in such a way as to make them symmetrical.[4, 22] The primitive machine is an equivalent 2-phase machine, and resolution of the direct and quadrature quantities into 2-phase symmetrical co-ordinates leads to the required symmetry of the impedance matrix.

In the free frame the transformation is

$$\left.\begin{aligned} i^{ds} &= i^{ds} \\ i^{d} &= (i^f + i^b)/\sqrt{2} \\ i^{q} &= -j(i^f - i^b)/\sqrt{2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (70)$$

and

$$\left.\begin{aligned} i^{ds} &= i^{ds} \\ i^{f} &= (i^d + ji^q)/\sqrt{2} \\ i^{b} &= (i^d - ji^q)/\sqrt{2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (71)$$

and

$$\left.\begin{aligned} \mathbf{Z}'' &= \mathbf{C}'^{*}_{(t)} . \mathbf{Z}' . \mathbf{C}' \\ \mathbf{v}'' &= \mathbf{C}'_{(t)} . \mathbf{v}' \\ \mathbf{i}' &= \mathbf{C}' . \mathbf{i}'' \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (72)$$

the asterisk denoting conjugate values.

$C' = \frac{1}{\sqrt{2}} \times$ . . (73)

	ds	f	b	s
ds	$\sqrt{2}$			
d		1	1	
q		$-j$	j	
s				$\sqrt{2}$

The impedance matrix (59) now becomes

	ds	f	b	s
ds	$r_{ds} + L_{ds}p$	$\frac{1}{\sqrt{2}}M_d p$	$\frac{1}{\sqrt{2}}M_d p$	$\frac{1}{\sqrt{2}}jM_d(i^f - i^b)$
f	$\frac{1}{\sqrt{2}}M_d(p - jp\theta)$	$r_r + L_S(p - jp\theta)$	$L_D(p - jp\theta)$	$b_f(p - jp\theta)$
b	$\frac{1}{\sqrt{2}}M_d(p + jp\theta)$	$L_D(p + jp\theta)$	$r_r + L_S(p + jp\theta)$	$b_b(p + jp\theta)$
s	$-\frac{1}{\sqrt{2}}jM_d(i^f - i^b)$	$-b_b$	$-b_f$	$Jp^2 + jb_f i^b$ $-jb_b i^f$

. (74)

where $$L_S = \frac{L_{dr} + L_{qr}}{2} \text{ and } L_D = \frac{L_{dr} - L_{qr}}{2} \quad . \quad . \quad (74)$$

In this matrix,

$$\left.\begin{aligned} b_f &= (b_d + jb_q)/\sqrt{2} \\ b_b &= (b_d - jb_q)/\sqrt{2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (75)$$

where b_d and b_q are defined from matrix (58):

$$\left.\begin{aligned} b_d &= -B_d - L_{dr}i^q \\ b_q &= -B_q - L_{qr}i^d \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (76)$$

$$\left.\begin{aligned} B_d &= -L_{qr}i^q \\ B_q &= L_{dr}i^d + M_d i^{ds} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (77)$$

$$\left.\begin{aligned} B_f &= (B_d + jB_q)/\sqrt{2} = j\left(\frac{1}{\sqrt{2}}M_d i^{ds} + i^f L_S + i^b L_D\right) \\ B_b &= (B_d - jB_q)/\sqrt{2} = -j\left(\frac{1}{\sqrt{2}}M_d i^{ds} + i^f L_D + i^b L_S\right) \end{aligned}\right\} . \quad (78)$$

and

$$\begin{aligned} b_f &= -(B_f - i^f jL_S + i^b jL_D) \\ &= -j\left(\frac{1}{\sqrt{2}}M_d i^{ds} + 2i^b L_D\right) \quad . \quad . \quad . \quad (79) \end{aligned}$$

$$\begin{aligned} b_b &= -(B_b + i^b jL_S - i^f jL_D) \\ &= j\left(\frac{1}{\sqrt{2}}M_d i^{ds} + 2i^f L_D\right) \quad . \quad . \quad . \quad . \quad (80) \end{aligned}$$

The additional term in eqn. (67) which is added and subtracted is

$$G' . pi' . p\theta . \Delta\lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad (81)$$

This set of terms arises only in the last column of the voltage part of matrix (74). Thus the last column of this part of the matrix can be written

$$\frac{\partial L''}{\partial\lambda}i''p(\Delta\lambda) + \left\{[G'' . pi''] + \left[\frac{\partial G''}{\partial\lambda}i'' - G'' . pi''\right]\right\} p\theta\Delta\lambda \; . \; (82)$$

where G'' is now the 'symmetrical component' form of the matrix G. The terms of expression (82) in square brackets expand in matrix form to

$\frac{\partial G''}{\partial\lambda} =$. (83)

	ds	f	b
ds			
f	$-\frac{M_d}{\sqrt{2}}$		$-2L_D$
b	$-\frac{M_d}{\sqrt{2}}$	$-2L_D$	

$$G''\rho = \begin{array}{c|c|c|c|} & ds & f & b \\ \hline ds & & & \\ \hline f & & L_S & -L_D \\ \hline b & & -L_D & L_S \\ \hline \end{array} \quad . \quad (84)$$

$$K = \left(\frac{\partial G''}{\partial\lambda} - G''\rho\right) = \begin{array}{c|c|c|c|} & ds & f & b \\ \hline ds & & & \\ \hline f & -\frac{M_d}{\sqrt{2}} & -L_S & -L_D \\ \hline b & -\frac{M_d}{\sqrt{2}} & -L_D & -L_S \\ \hline \end{array} \quad . \quad (85)$$

Matrix (85) given by the tensor K is seen to comprise the quantities B_f and B_b, and expression (82) therefore divides the quantities b_f and b_b into two significant parts:

$$b_f = -B_f + (i^f jL_S - i^b jL_D) \quad . \quad . \quad . \quad . \quad (86)$$

$$b_b = -B_b - (i^b jL_S - i^f jL_D) \quad . \quad . \quad . \quad . \quad (87)$$

The equivalent circuit can now be drawn for steady hunting conditions at the hunting frequency $h\omega$. This is shown in Fig. 3.

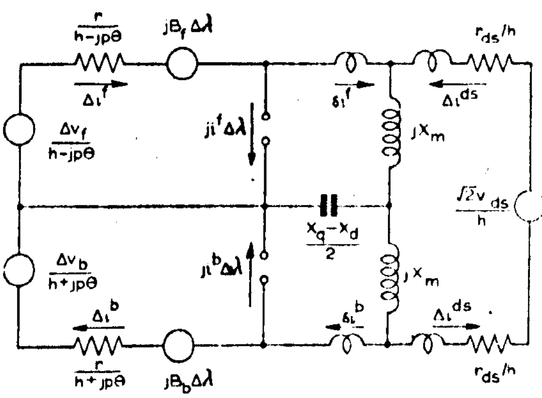


Fig. 3.—Equivalent circuit for salient-pole synchronous-machine hunting.

The addition of injected voltages and currents is indicated *a priori* by the grouping of terms shown in eqns. (86) and (87).

When the circuit is drawn in this manner the effects of increments of current Δi and the absolute changes δi become apparent.[19] In the symmetrical component form,

$$\delta i^\alpha = \Delta i^\alpha + \Gamma^\alpha_{\beta\gamma} i^\beta i^\gamma \quad . \quad . \quad . \quad . \quad . \quad (88)$$

becomes
$$\delta i^f = \Delta i^f - j i^f \Delta\lambda \quad . \quad . \quad . \quad . \quad . \quad (89)$$

and
$$\delta i^b = \Delta i^b + j i^b \Delta\lambda \quad . \quad . \quad . \quad . \quad . \quad (90)$$

(These are the 'absolute changes' used by Ku[22].)

The significance of active and reactive power in the equivalent circuit is discussed in the following Section.

(7) THE TORQUE EQUATION AND THE EQUIVALENT CIRCUIT

The machine torque equation can be written down from matrix (74), using the grouping of terms indicated by the tensor equation of torque [eqn. (69)],

$$\Delta f = \left\{\left[-j\frac{M_d}{\sqrt{2}}(i^f - i^b)\Delta i^{ds*} - b_f \Delta i^{f*} - b_b \Delta i^{b*}\right] + [i''^* . G''\rho . i''\Delta\lambda]\right\} + \left[i''^* . \frac{\partial G''}{\partial\lambda} . i'' - i''^* . G''\rho . i''\right]\Delta\lambda \quad . \quad (91)$$

Examination of the equivalent circuit meshes shows that the active and reactive power measured at the points indicated, namely $\Delta i V$, give real and imaginary parts of the first set of terms in square brackets in eqn. (91), with a time-quadrature difference. Under steady hunting conditions the total torque expression can be written, as in Park's reference frame, by

$$\Delta f = (T_{synch} + jT_{damp})\Delta\lambda \quad . \quad . \quad . \quad . \quad (92)$$

and the real component of power as measured from the equivalent circuit of Fig. 3, namely the resistance loss in each mesh, corresponds to the damping torque given by eqn. (92). The imaginary part gives the corresponding component of (negative) synchronizing torque.

The additional components of negative and positive synchronizing torque are given by the remaining two sets of terms in square brackets in eqn. (91). Components of these terms are included in the equivalent circuit, but they cannot be read off the circuit directly since the injected voltages and currents are already associated with the displacement angle $\Delta\lambda$. Fig. 4 shows

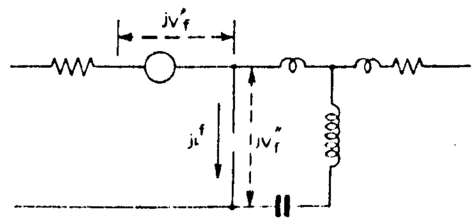


Fig. 4.—Contribution of forward armature mesh to synchronizing torque.

$i^f V_f'$ gives positive (impressed) synchronizing torque.
$i^f V_f''$ gives negative synchronizing torque.

how the circuit can be interpreted to indicate the contribution of each mesh to synchronizing torque.

Positive synchronizing torque given by the tensor K expands to

$$(-B_f i^f + B_b i^b)\Delta\lambda \quad . \quad . \quad . \quad . \quad . \quad (93)$$

This can be compared with the machine power output equation[22]

$$-B_f i^b + B_b i^f \quad . \quad . \quad . \quad . \quad . \quad (94)$$

and it is seen that the positive synchronizing torque at any load angle is, in fact, given by the reactive component of the machine power output. The simple equation for synchronizing power of a round-rotor machine given in textbooks[26] is usually written

$$\left(\frac{Ev}{x}\cos\lambda\right)\Delta\lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad (95)$$

and the expression in brackets is again the reactive component of the steady-state vector power output.

(8) CONCLUSION

The equations of performance of any conventional electrical machine can be derived by an automatic tensor transformation of those of the primitive machine. A similar type of transformation can be used to express the equations of a given machine in any one of several sets of reference axes. The transformation of Park's equations for the synchronous machine to those in Kron's freely rotating axes leads to simple overall equations for an interconnected system.

The complex interaction of currents and fluxes in a machine during hunting can be more easily followed in different reference systems when the corresponding transformation of equations is carried out using matrices and the routine methods of tensor analysis. The tensor form of the equations of hunting of a machine in the free frame has two advantages:

(*a*) Synchronizing torque terms are inherently grouped together in terms of the angle of oscillation $\Delta\lambda$. Positive and negative damping terms are inherently grouped in terms of the increment of angular velocity $\Delta\omega$ and give this component of torque as the real part of a complex expression. Damping torque is thus directly associated with the resistances in the electrical system.

(*b*) The terms of the tensor equations in symmetrical component form give, directly, the meshes in an equivalent circuit. This form of equivalent circuit can be interconnected with corresponding circuits for an external network and additional machines. The resistance power loss in each mesh gives the damping torque contributed by that part of the system.

The significance of tensor groups of terms in the steady-state equations of electrical machines has already been noted.[5] The tensor form of the hunting equations gives a physical picture of the hunting phenomena as represented by the equivalent circuit. It would appear that this method of formulating dynamical equations could be used to advantage in the analogue study of complex devices in which interchange of different forms of energy takes place, for example, aircraft, missiles and nuclear reactors.

(9) ACKNOWLEDGMENTS

The author wishes to express his appreciation of the support and encouragement of Prof. J. M. Meek of the University of Liverpool. He wishes to thank Mr. Gabriel Kron for his interest and valuable comments in correspondence during the preparation of the paper. The author is also indebted to his colleagues, Messrs. A. S. Aldred and C. V. Jones, for many stimulating discussions.

(10) REFERENCES

(1) BROWN, G. S., KUSKO, A., and WHITE, D. C.: 'A New Educational Programme in Energy Conversion', *Electrical Engineering*, 1956, **75**, p. 180.

(2) KRON, G.: 'Non-Riemannian Dynamics of Rotating Electrical Machinery', *Journal of Mathematics and Physics*, 1934, **13**, p. 103.

(3) KRON, G.: 'The Application of Tensor to the Analysis of Rotating Electrical Machinery', *General Electrical Review*, 1935–38. Published in book form 1938 and 1942.

(4) KRON, G.: 'Equivalent Circuits of Electric Machinery' (John Wiley, 1951).

(5) LYNN, J. W.: 'The Tensor Equations of Electrical Machines', *Proceedings I.E.E.*, Monograph No. 117 S, January, 1955 (**102 C**, p. 149).

(6) HEFFRON, W. G., ROSENBERRY, G. M., and ROTHE, F. S.: 'Generalized Hunting Equations of Power Systems', *Transactions of the American I.E.E.*, 1952, **71**, Part III, p. 1095.

(7) KRON, G.: 'A New Theory of Hunting', *ibid.*, 1952, **71**, Part III, p. 859. (See also contribution to discussion by Heffron.)

(8) MESSERLE, H. K., and BRUCK, R. W.: 'Steady-State Stability of Synchronous Generators as Affected by Regulators and Governors', *Proceedings I.E.E.*, Monograph No. 134 S, June, 1955 (**103 C**, p. 24).

(9) WENNERBERG, J.: 'Hunting Constants of Synchronous Machines for Oscillations of Small Amplitude', *Asea Journal*, 1929, **4**, p. 61.

(10) PRESCOTT, J. C., and RICHARDSON, J. E.: 'The Inherent Instability of Synchronous Machinery', *Journal I.E.E.*, 1934, **75**, p. 497.

(11) DOHERTY, R. E., and NICKLE, C. A.: 'Synchronous Machines', *Transactions of the American I.E.E.*, 1926, **45**, p. 912.

(12) PARK, R. H.: 'Two-Reaction Theory of Synchronous Machines', *ibid.*, 1929, **48**, p. 716, and 1933, **52**, p. 352.

(13) LIWSCHITZ, M. M.: 'Positive and Negative Damping in Synchronous Machines', *ibid.*, 1941, **60**, p. 210.

(14) CONCORDIA, C.: 'Synchronous Machine Damping and Synchronizing Torques', *ibid.*, 1951, **70**, Part I, p. 731.

(15) KRON, G.: 'Equivalent Circuits for the Hunting of Electrical Machinery', *ibid.*, 1942, **61**, p. 290.

(16) CONCORDIA, C., and KRON, G.: 'Damping and Synchronising Torques of Power Selsyns', *ibid.*, 1945, **64**, p. 367.

(17) KRON, G., CONCORDIA, C., and CRARY, S. B.: 'The Doubly Fed Machine', *ibid.*, 1942, **61**, p. 286.

(18) KRON, G.: 'Equivalent Circuits for Oscillating Systems and the Riemann–Christoffel Curvature Tensor', *ibid.*, 1943, **62**, p. 25.

(19) KRON, G.: 'A Physical Interpretation of the Riemann–Christoffel Curvature Tensor', *The Tensor* (New Series), 1955, **4**, p. 150.

(20) KONDO, K.: 'On the Dynamics of the Aeroplane and Non-Riemannian Geometry', Parts I and II, *Journal of the Japan Society of Aeronautical Engineering*, 1954, pp. 161–166 and pp. 193–196.

(21) CONCORDIA, C.: 'Synchronous Machines' (John Wiley, 1951).

(22) KU, Y. H.: 'Rotating-Field Theory and General Analysis of Synchronous and Induction Machines', *Proceedings I.E.E.*, Monograph No. 54 U, June, 1952 (**99**, Part IV, p. 410).

(23) GIBBS, W. J.: 'Tensors in Electrical Machine Theory' (Chapman and Hall, 1952).

(24) KRON, G.: 'Classification of the Reference Frames of a Synchronous Machine', *Transactions of the American I.E.E.*, 1950, **69**, Part II, p. 720.

(25) HOFFMANN, B.: 'Tensors and Equivalent Circuits', *Journal of Mathematics and Physics*, 1946, **25**, p. 21.

(26) SAY, M. G.: 'Performance and Design of Alternating Current Machines', 2nd Edition (Pitman, 1948).

(27) SCHOUTEN, J. A.: 'Ricci Calculus', 2nd Edition (Springer, Berlin, 1954).

(28) SYNGE, J. L.: 'On the Geometry of Dynamics', *Philosophical Transactions of the Royal Society*, A, 1927, **266**, p. 31.

(11) APPENDIX

(11.1) Small Oscillation Equation

The free-frame steady-state equation is

$$v_\alpha = R_{\alpha\beta}i^\beta + L_{\alpha\beta}pi^\beta + \Gamma_{\beta\gamma,\alpha}i^\beta i^\gamma \quad . \quad . \quad . \quad (96)$$

This is divided into voltage and torque equations.

Voltage equation.

The index α is electrical. The indices β and γ are electrical, indicating currents i^β or i^γ, or mechanical values s, indicating angular velocity $i^s \equiv p\theta$.

$$v_\alpha = R_{\alpha\beta}i^\beta + L_{\alpha\beta}pi^\beta + \Gamma_{s\gamma,\alpha}i^\gamma p\theta + \Gamma_{\beta s,\alpha}i^\beta p\theta \quad . \quad (97)$$

Torque equation.

The index α takes the mechanical part of the range. The indices β and γ have electrical values.

$$v_s = R_{ss}i^s + L_{ss}pi^s + \Gamma_{\beta\gamma,s}i^\beta i^\gamma \quad . \quad . \quad . \quad (98)$$

or

$$v_s = R_{ss}p\theta + Jp^2\theta + \Gamma_{\beta\gamma,s}i^\beta i^\gamma \quad . \quad . \quad . \quad (99)$$

The small-oscillation equation is obtained by taking small increments of values in eqn. (96). This gives

$$\Delta v_\alpha + \frac{\partial v_\alpha}{\partial x^\pi}\Delta x^\pi = R_{\alpha\beta}\Delta i^\beta + L_{\alpha\beta}p(\Delta i^\beta) + \Delta L_{\alpha\beta}pi^\beta$$
$$+ \Gamma_{\beta\gamma,\alpha}i^\beta\Delta i^\gamma + \Gamma_{\beta\gamma,\alpha}\Delta i^\beta i^\gamma + \Delta\Gamma_{\beta\gamma,\alpha}i^\beta i^\gamma \quad . \quad (100)$$

The corresponding voltage and torque equations are as follows:

Voltage equation.

$$\Delta v_\alpha + \frac{\partial v_\alpha}{\partial x^t}\Delta x^t = R_{\alpha\beta}\Delta i^\beta + L_{\alpha\beta}p(\Delta i^\beta) + \Gamma_{t\gamma,\alpha}\Delta(p\theta)i^\gamma$$
$$+ \Gamma_{\beta s,\alpha}\Delta i^\beta p\theta + \Gamma_{s\gamma,\alpha}\Delta i^\gamma p\theta + \Gamma_{\beta t,\alpha}i^\beta\Delta(p\theta)$$
$$+ \frac{\partial L_\beta}{\partial x^t}(pi^\beta)\Delta x^t + \frac{\partial\Gamma_{s\gamma,\alpha}}{\partial x^t}(p\theta)i^\gamma\Delta x^t + \frac{\partial\Gamma_{\beta s,\alpha}}{\partial x^t}i^\beta(p\theta)\Delta x^t \quad . \quad (101)$$

The index t denotes excursions of the rotor over the increment of speed,

$$\Delta(p\theta) = p(\Delta\theta) = p(\Delta\lambda)$$

The terms of the voltage equation are expanded in a manner indicated by the following example:[5]

$$\Gamma_{t\gamma,\alpha}i^\gamma\Delta i^t =$$
$$\{[t\gamma,\alpha] - S_{\alpha\gamma t} - S_{\alpha t\gamma} + S_{t\gamma\alpha} + \Omega_{\alpha\gamma,t} + \Omega_{\alpha t,\gamma} - \Omega_{t\gamma,\alpha}\}i^\gamma\Delta i^t$$
$$= \{[t\gamma,\alpha] - S_{\alpha t\gamma} + \Omega_{\alpha t,\gamma}\}i^\gamma\Delta x^t \quad . \quad (102)$$

Other terms are zero as shown in Sections 4 and 5 of Reference 5.

Now
$$-2S_{\alpha t,\gamma} = G' = C_{(t)}\,.\,G\,.\,C \quad . \quad . \quad . \quad (103)$$

where G' is the free-frame torque matrix and G is the torque matrix in Park's equations.

$$2\Omega_{\alpha t,\gamma} = \frac{dC_{(t)}^{-1}}{d\lambda}\,.\,C_{(t)}\,.\,L' = \rho L' = 2S_{\alpha t\gamma} \quad . \quad . \quad (104)$$

Also
$$[t\gamma,\alpha] \equiv \frac{1}{2}\left(\frac{\partial L_{\alpha\gamma}}{\partial x^t} + \frac{\partial L_{\alpha t}}{\partial x^\gamma} - \frac{\partial L_{t\gamma}}{\partial x^\alpha}\right) = \frac{1}{2}\frac{\partial L_{\alpha\gamma}}{\partial x^t} \quad . \quad (105)$$

The index s denotes the mechanical variable θ, and

$$\frac{\partial\Gamma_{\beta s,\alpha}}{\partial x^\pi}i^s i^\gamma\Delta x^\pi = \frac{\partial}{\partial x^\pi}\left(\frac{1}{2}\frac{\partial L_{\alpha\beta}}{\partial x^s} - S_{\alpha s\beta} + \Omega_{\alpha s,\beta}\right)i^s i^\beta\Delta x^\pi \quad . \quad (106)$$

$$\frac{\partial L_{\alpha\beta}}{\partial x^s}i^s \equiv \frac{\partial L_{\alpha\beta}}{\partial\theta}p\theta = 0 \quad . \quad . \quad . \quad . \quad (107)$$

$$\Omega_{\alpha s,\beta}i^s = \frac{dC_{(t)}^{-1}}{d\theta}\,.\,C_{(t)}\,.\,L'\,.\,p\theta = 0 \quad . \quad . \quad (108)$$

$$S_{\alpha s\beta}i^s = \tfrac{1}{2}G'p\theta \quad . \quad . \quad . \quad . \quad . \quad (109)$$

Thus
$$\frac{\partial\Gamma_{s\gamma,\alpha}}{\partial x^t}i^s i^\gamma\Delta x^t + \frac{\partial\Gamma_{\beta s,\alpha}}{\partial x^t}i^s i^\beta\Delta x^t = \frac{\partial G'}{\partial\lambda}i'\,.\,p\theta\,.\,\Delta\lambda \quad . \quad (110)$$

Therefore eqn. (101) becomes

$$\Delta\mathbf{v}' + \frac{d\mathbf{v}'}{d\boldsymbol{\lambda}}\Delta\boldsymbol{\lambda} = [\mathbf{R}' + \mathbf{L}'\mathbf{p} + \mathbf{G}'\mathbf{p}\boldsymbol{\theta}]\Delta\mathbf{i}'$$
$$+ \left[\frac{d\mathbf{L}'}{d\boldsymbol{\lambda}}\mathbf{i}'\mathbf{p} + \frac{d\mathbf{G}'}{d\boldsymbol{\lambda}}\mathbf{i}'\mathbf{p}\boldsymbol{\theta}\right]\Delta\boldsymbol{\lambda} \quad . \quad (111)$$

Torque equation.

$$\Delta F = Jp^2(\Delta\lambda) + \Gamma_{\beta\gamma,s}i^\beta\Delta i^\gamma + \Gamma_{\beta\gamma,s}\Delta i^\beta i^\gamma + \frac{\partial\Gamma_{\beta\gamma,s}}{\partial\lambda}i^\beta i^\gamma\Delta\lambda \quad . \quad . \quad . \quad . \quad (112)$$

$$\Gamma_{\beta\gamma,s}i^\beta\Delta i^\gamma = \{[\beta\gamma,s] - S_{s\gamma\beta} - S_{s\beta\gamma} + S_{\beta\gamma s}$$
$$+ \Omega_{s\gamma,\beta} + \Omega_{s\beta,\gamma} - \Omega_{\beta\gamma,s}\}i^\beta\Delta i^\gamma \quad . \quad (113)$$

$$[\beta\gamma,s]i^\beta\Delta i^\gamma = \frac{1}{2}\frac{d\mathbf{L}'}{d\boldsymbol{\theta}}\mathbf{i}'\Delta\mathbf{i}' = 0 \quad . \quad . \quad . \quad (114)$$

$$(\Omega_{s\gamma,\beta} + \Omega_{s\beta,\gamma} - \Omega_{\beta\gamma,s})i^\beta\Delta i^\gamma = \frac{d\mathbf{C}_{(t)}^{-1}}{d\boldsymbol{\theta}}\,.\,\mathbf{C}_{(t)}\,.\,\mathbf{L}'\,.\,\mathbf{i}'\,.\,\Delta\mathbf{i}' = 0 \quad . \quad . \quad . \quad . \quad (115)$$

$$(-S_{s\gamma\beta} - S_{s\beta\gamma} + S_{\beta\gamma s})i^\beta\Delta i^\gamma \text{ becomes}^5 \quad . \quad . \quad (116)$$

$$(-S_{s\gamma\beta} - S_{s\beta\gamma})i^\beta\Delta i^\gamma = \tfrac{1}{2}(\mathbf{i}'\,.\,\mathbf{G}'\,.\,\Delta\mathbf{i}' + \Delta\mathbf{i}'\,.\,\mathbf{G}'\,.\,\mathbf{i}') \quad . \quad (117)$$

Thus

$$\Gamma_{\beta\gamma,s}i^\beta\Delta i^\gamma + \Gamma_{\beta\gamma,s}\Delta i^\beta i^\gamma = \Delta\mathbf{i}'\,.\,\mathbf{G}'\,.\,\mathbf{i}' + \mathbf{i}'\,.\,\mathbf{G}'\,.\,\Delta\mathbf{i}' \quad . \quad (118)$$

$$\frac{\partial\Gamma_{\beta\gamma,s}}{\partial x^t}i^\beta i^\gamma\Delta x^t = \mathbf{i}'\,.\,\frac{d\mathbf{G}'}{d\boldsymbol{\lambda}}\,.\,\mathbf{i}'\,.\,\Delta\boldsymbol{\lambda} \quad . \quad . \quad . \quad (119)$$

Therefore eqn. (98) becomes $\left(\text{neglecting the friction term } R\frac{d\theta}{dt}\right)$

$$\Delta f = \Delta\mathbf{i}'^*\,.\,\mathbf{G}'\,.\,\mathbf{i}' + \mathbf{i}'^*\,.\,\mathbf{G}'\,.\,\Delta\mathbf{i}' + \mathbf{i}'^*\,.\,\frac{d\mathbf{G}'}{d\boldsymbol{\lambda}}\,.\,\mathbf{i}'\,.\,\Delta\boldsymbol{\lambda} + Jp^2(\Delta\boldsymbol{\lambda}) \quad . \quad . \quad . \quad . \quad (120)$$

(11.2) The Tensor $K_{\delta\gamma\beta\alpha}$

The absolute differential of a contravariant vector is

$$\delta\phi^l = d\phi^l + \Gamma^h_{kj}\phi^k dx^j \quad . \quad . \quad . \quad . \quad (121)$$

Therefore
$$\frac{\delta}{dt}(\delta i^h) = \frac{\delta}{dt}(di^h + \Gamma^h_{kj}i^k dx^j) \quad . \quad . \quad . \quad (122)$$

$$= \frac{d}{dt}(di^h + \Gamma^h_{kj}i^k dx^j) + \Gamma^h_{mn}(di^m + \Gamma^m_{pq}i^p dx^q)\frac{dx^n}{dt}$$
$$= \frac{d}{dt}(di^h) + \frac{d\Gamma^h_{kj}}{dt}i^k dx^j + \Gamma^h_{kj}\frac{di^k}{dt}dx^j + \Gamma^h_{kj}i^k\frac{d}{dt}(dx^j)$$
$$+ \Gamma^h_{mn}di^m i^n + \Gamma^h_{mn}\Gamma^m_{pq}i^p i^n dx^q \quad . \quad (123)$$

Similarly
$$\delta\left(\frac{\delta i^\beta}{dt}\right) = \delta\left(\frac{di^\beta}{dt} + \Gamma^\beta_{\gamma\delta}i^\gamma\frac{dx^\delta}{dt}\right) \quad . \quad . \quad . \quad (124)$$

$$= d\left(\frac{di^\beta}{dt}\right) + d\Gamma^\beta_{\gamma\delta}i^\gamma i^\delta + \Gamma^\beta_{\gamma\delta}di^\gamma i^\delta + \Gamma^\beta_{\gamma\delta}i^\gamma dx^\delta$$
$$+ \Gamma^\beta_{\pi\sigma}\frac{di^\pi}{dt}dx^\sigma + \Gamma^\beta_{\pi\sigma}\Gamma^\pi_{\alpha\beta}i^\alpha i^\beta dx^\sigma \quad . \quad (125)$$

Therefore $\delta\left(\frac{\delta i^{\beta}}{dt}\right) - \frac{\delta}{dt}(\delta i^{\beta})$

$$= \frac{\partial \Gamma^{\beta}_{\gamma\delta}}{\partial x^{\alpha}} i^{\gamma} i^{\delta} dx^{\alpha} - \frac{\partial \Gamma^{\beta}_{\pi\sigma}}{dx^{\lambda}} i^{\pi} i^{\lambda} dx^{\sigma} + \Gamma^{\beta}_{\sigma\pi}\Gamma^{\sigma}_{\alpha\beta} i^{\alpha} i^{\beta} dx^{\pi}$$

$$- \Gamma^{\beta}_{\lambda\epsilon}\Gamma^{\lambda}_{\gamma\theta} i^{\gamma} i^{\epsilon} dx^{\theta} + \Gamma^{\beta}_{\gamma\delta}\left[d\left(\frac{dx^{\delta}}{dt}\right) - \frac{d}{dt}(dx^{\delta})\right] i^{\gamma} \quad . \quad (126)$$

The bracketed difference in expression (126) is not zero because the electrical variables are non-holonomic, i.e. they are related only through non-integrable differentials,[5] and

$$\frac{\partial^2 x^{\delta}}{\partial x^{\gamma}\partial x^{\beta}} \neq \frac{\partial^2 x^{\delta}}{\partial x^{\beta}\partial x^{\gamma}} \quad . \quad . \quad . \quad . \quad . \quad (127)$$

This is shown in Reference 27, as follows:

$$\frac{\partial^2 P}{\partial x^i \partial x^j} - \frac{\partial^2 P}{\partial x^j \partial x^i} = C^{\mu}_j \frac{\partial C^{\lambda}_i}{\partial x^{\mu}}\frac{\partial P}{\partial x^{\lambda}} - C^{\mu}_i \frac{\partial C^{\lambda}_j}{\partial x^{\mu}}\frac{\partial P}{\partial x^{\lambda}}$$

$$= \frac{\partial P}{\partial x^h}\left(C^{\mu}_j C^h_{\lambda}\frac{\partial C^{\lambda}_i}{\partial x^{\mu}} - C^{\mu}_i C^h_{\lambda}\frac{\partial C^{\lambda}_j}{\partial x^{\mu}}\right)$$

$$= \frac{\partial P}{\partial x^h}\left[C^{j'}_j C^{i'}_i\left(\frac{\partial C^h_{i'}}{\partial x^{j'}} - \frac{\partial C^h_{j'}}{\partial x^{i'}}\right)\right]$$

$$= \frac{\partial P}{\partial x^h} . 2\Omega^h_{ij} = - \frac{\partial P}{\partial x^h} . 2\Omega^h_{ji} \quad . \quad . \quad . \quad . \quad (128)$$

where $\Omega^h_{ij} = \frac{1}{2} C^{j'}_j C^{i'}_i \left(\frac{\partial C^h_{i'}}{\partial x^{j'}} - \frac{\partial C^h_{j'}}{\partial x^{i'}}\right) \quad . \quad . \quad . \quad (129)$

and in expression (126)

$$\Gamma^{\beta}_{\gamma\delta}\left[d\left(\frac{dx^{\beta}}{dt}\right) - \frac{d}{dt}(dx^{\delta})\right] i^{\gamma} = - \Gamma^{\beta}_{\gamma\delta} i^{\gamma} . 2\Omega^{\delta}_{\pi\sigma} i^{\pi} dx^{\sigma} \quad . \quad (130)$$

With appropriate rearrangement of indices

$$K^{\alpha}_{\delta\gamma\beta} i^{\delta} i^{\beta} dx^{\gamma} = \delta\left(\frac{\delta i^{\alpha}}{dt}\right) - \frac{\delta}{dt}(\delta i^{\alpha}) \quad . \quad . \quad (131)$$

where

$$K^{\alpha}_{\delta\gamma\beta} = \frac{\partial \Gamma^{\alpha}_{\beta\delta}}{\partial x^{\gamma}} - \frac{\partial \Gamma^{\alpha}_{\beta\gamma}}{\partial x^{\delta}} + \Gamma^{\alpha}_{\lambda\gamma}\Gamma^{\lambda}_{\beta\delta} - \Gamma^{\alpha}_{\lambda\delta}\Gamma^{\lambda}_{\beta\gamma} + 2\Gamma^{\alpha}_{\beta\lambda}\Omega^{\lambda}_{\delta\gamma} \quad . \quad (132)$$

The tensor equation of small oscillations now becomes, as given by Kron,[18]

$$\Delta v_{\alpha} + \delta v_{\alpha} = \delta(R_{\alpha\beta} i^{\beta}) + L_{\alpha\beta}\frac{\delta}{\delta t}(\delta i^{\beta}) + K_{\delta\gamma\beta\alpha} i^{\beta} i^{\delta} dx^{\gamma} \quad . \quad (133)$$

which expands, as shown in References 2 and 18, giving

$$\Delta v_{\alpha} + \frac{\partial v_{\alpha}}{\partial x^{\beta}}\Delta x^{\beta} = - \Gamma_{\gamma\delta,\alpha}\frac{di^{\gamma}}{dt}\Delta x^{\delta} + R_{\alpha\beta}\Delta i^{\beta} + \Gamma_{\gamma\delta,\alpha}\frac{di^{\gamma}}{dt}\Delta x^{\delta}$$

$$+ L_{\alpha\beta}\frac{d}{dt}(\Delta i^{\beta}) - \frac{\partial L_{\alpha\beta}}{\partial x^{\gamma}}\frac{di^{\beta}}{dt}\Delta x^{\gamma} + \Gamma_{\beta\gamma,\alpha}\Delta i^{\beta} i^{\gamma} + \Gamma_{\beta\gamma,\alpha} i^{\beta}\Delta i^{\gamma}$$

$$+ \left(\frac{\partial \Gamma_{\beta\gamma,\alpha}}{\partial x^{\delta}} + \Gamma_{\lambda\delta,\alpha}\Gamma^{\lambda}_{\beta\gamma} - \Gamma_{\lambda\gamma,\alpha}\Gamma^{\lambda}_{\beta\delta} - 2\Gamma_{\beta\lambda,\alpha}\Omega^{\lambda}_{\delta\gamma}\right) i^{\beta} i^{\delta}\Delta x^{\gamma}$$

$$+ \left(\frac{\partial \Gamma_{\beta\delta,\alpha}}{\partial x^{\gamma}} - \frac{\partial \Gamma_{\beta\gamma,\alpha}}{\partial x^{\delta}} + \Gamma_{\lambda\gamma,\alpha}\Gamma^{\lambda}_{\beta\delta} - \Gamma_{\lambda\delta,\alpha}\Gamma^{\lambda}_{\beta\gamma} + 2\Gamma_{\beta\lambda,\alpha}\Omega^{\lambda}_{\delta\gamma}\right) i^{\beta} i^{\delta}\Delta x^{\gamma}$$

$$. \quad . \quad . \quad . \quad . \quad (134)$$

This is seen to be the same as the conventional equation with the following terms added and subtracted:

$$\Gamma_{\beta\gamma,\alpha}\frac{di^{\beta}}{dt}\Delta x^{\gamma} + \left(\Gamma_{\lambda\delta,\alpha}\Gamma^{\lambda}_{\beta\gamma} + 2\Gamma_{\beta\lambda,\alpha}\Omega^{\lambda}_{\delta\gamma} + \frac{\partial \Gamma_{\beta\gamma,\alpha}}{\partial x^{\delta}}\right) \quad . \quad (135)$$

In the synchronous-machine equations in the free frame the steady-state current is constant and

$$\Gamma_{\beta\gamma,\alpha}\frac{di^{\beta}}{dt}\Delta x^{\gamma} = 0 \quad . \quad . \quad . \quad . \quad . \quad (136)$$

The remaining terms expand in the voltage equation to

$$\left(\Gamma_{\lambda s,\alpha}\Gamma^{\lambda}_{\beta t} + 2\Gamma_{\beta\lambda,\alpha}\Omega^{\lambda}_{st} + \frac{\partial \Gamma_{\beta\gamma,\alpha}}{\partial x^{s}}\right) i^{\beta} i^{s}\Delta x^{\gamma}$$

$$+ \left(\Gamma_{\lambda\delta,\alpha}\Gamma^{\lambda}_{st} + 2\Gamma_{s\lambda,\alpha}\Omega^{\lambda}_{\delta t} + \frac{\partial \Gamma_{s\gamma,\alpha}}{\partial x^{\delta}}\right) i^{s} i^{\delta}\Delta x^{\gamma} \quad . \quad (137)$$

The index γ takes the value t indicating the mechanical variable which undergoes incremental changes, namely the load angle λ. The index s as usual denotes the holonomic variable, the angular position θ, of the rotor. Only the first and fifth terms have non-zero values. The additional terms in the voltage equation are therefore

$$(\Gamma_{\lambda s,\alpha}\Gamma^{\lambda}_{\delta t} + 2\Gamma_{s\lambda,\alpha}\Omega^{\lambda}_{\delta t}) i^{\delta} i^{s}\Delta\lambda \quad . \quad . \quad . \quad (138)$$

$$= (\Gamma_{\lambda s,\alpha} + \Gamma_{s\lambda,\alpha})\rho^{\lambda}_{\delta} i^{\delta} p\theta\Delta\lambda \quad . \quad . \quad . \quad (139)$$

$$= \mathbf{G}' . \rho \mathbf{i}' p\theta\Delta\lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (140)$$

The corresponding term in the torque equation is

$$(\Gamma_{\lambda\delta,s} + \Gamma_{\beta\lambda,s})\rho^{\lambda}_{\delta} i^{\beta} i^{\gamma}\Delta\lambda \quad . \quad . \quad . \quad . \quad (141)$$

$$= - \mathbf{i}' . \mathbf{G}' . \rho\mathbf{i}' . \Delta\lambda \quad . \quad . \quad . \quad . \quad (142)$$

The small oscillation equations are therefore as given by Kron in Reference 19, namely

Voltage equation.

$$\Delta\mathbf{v}' = [\mathbf{R}' + \mathbf{L}'p + \mathbf{G}'p\theta]\Delta\mathbf{i}' + \frac{\partial \mathbf{L}'}{\partial\lambda}\mathbf{i}'p(\Delta\lambda) + \mathbf{G}' . \rho\mathbf{i}' . p\theta . \Delta\lambda$$

$$+ \left[\frac{\partial \mathbf{G}'}{\partial\lambda}\mathbf{i}' - \mathbf{G}' . \rho\mathbf{i}'\right] p\theta . \Delta\lambda \quad . \quad (67)$$

Torque equation.

$$\Delta f = J\frac{d}{dt}(\Delta\omega) - [\Delta\mathbf{i}'^{*} . \mathbf{G}' . \mathbf{i}' + \mathbf{i}'^{*} . \mathbf{G}' . \Delta\mathbf{i}' + \mathbf{i}'^{*} . \mathbf{G}' . \rho\mathbf{i}'\Delta\lambda]$$

$$+ \left[\mathbf{i}'^{*} . \frac{\partial \mathbf{G}'}{\partial\lambda} . \mathbf{i}' - \mathbf{i}'^{*} . \mathbf{G}' . \rho\mathbf{i}'\right]\Delta\lambda \quad . \quad (69)$$

where $\Delta\lambda$ is the displacement angle and $\Delta\omega = p(\Delta\lambda)$ is the angular velocity of displacement.

Printed in Great Britain by
UNWIN BROTHERS LIMITED, WOKING AND LONDON

THE INSTITUTION OF ELECTRICAL ENGINEERS

FOUNDED 1871: INCORPORATED BY ROYAL CHARTER 1921

SAVOY PLACE, LONDON, W.C.2

THE ECONOMIC LOADING OF TRANSMISSION SYSTEMS

By

H. NICHOLSON, M.Eng., A.M.I.Mech.E., and J. W. LYNN, M.Sc., Associate Members.

MONOGRAPH No. 294 S

March, 1958

To be republished in

PART C OF THE PROCEEDINGS OF THE INSTITUTION

621.311.153:621.315.09

The Institution of Electrical Engineers
Monograph No. 294 S
Mar. 1958

THE ECONOMIC LOADING OF TRANSMISSION SYSTEMS

By H. NICHOLSON, M.Eng., A.M.I.Mech.E., and J. W. LYNN, M.Sc., Associate Members.

(*The paper was first received* 15*th May, and in revised form* 22*nd November*, 1957. *It was published as an* INSTITUTION MONOGRAPH *in March*, 1958.)

SUMMARY

In several publications Kron, Kirchmayer and others have outlined methods for transforming power-system operating data in terms of complex voltages and currents into information concerning real generator powers and transmission losses. These losses are related to the individual generator loadings by a set of constants which are dependent only upon operating conditions at normal load.

In the present paper Kron's methods for establishing a transmission loss equation for a hypothetical 3-generator, 2-load system are described and the analysis is extended for application to an actual section of the British network.

The coefficients of the loss equation are obtained from the basic impedance matrix of the system and various transformation matrices, which transform the basic matrix to the final loss matrix, based on certain operating assumptions.

A method is developed for combining the loss formula with station fuel costs for economic system operation. The resulting loading equations are illustrated in nomograph form as an aid to system load dispatching.

In the paper, owing to certain limitations of the required information regarding system loads, certain of the load-flow conditions have been estimated by calculation, and the accuracy of the loss analysis is correspondingly limited. However, the purpose of the paper is to illustrate available methods and not to formulate an accurate set of loading equations.

LIST OF PRINCIPAL SYMBOLS

I^{L1}, I^{L2}, etc. = Load currents.

I^{L} = Total system (or hypothetical) load current.

l_1, l_2, etc. = Ratios of individual to hypothetical load currents.

Z_{11}, Z_{22}, etc. = Impedance matrices for reference frames 1, 2, etc.

C_2^1, C_3^2, etc. = Transformation matrices between quantities in reference frames 1 and 2, 2 and 3, etc.

V_1, V_2, etc. = Terminal voltages of generators 1, 2, etc.

V_{L1}, V_{L2}, etc. = Voltages at loads L_1, L_2, etc.

$Z_{n.K}$ = Measured leakage impedance of system network between generator or load n and generator or load K.

$Z_{G.G}$ = A matrix of measured network self impedances seen from generating points of entry.

$Z_{L.L}$ = A matrix of measured network self impedances seen from load points of entry.

$Z_{L.G}$ = Measured mutual impedances with generator points energized.

$Z_{G.L}$ = Measured mutual impedances with load points energized.

I^1, I^2, etc. = Generator currents.

Z_{n-K} = Complex impedance components for impedance matrix Z_{33}.

θ_1, θ_2, etc. = Angles by which generator terminal voltages V_1, V_2, etc., are referred to a common axis.

V_{d_n}, V_{q_n} = Direct and quadrature components respectively of voltage vector V_4.

I^{dK}, I^{qK} = Direct and quadrature components respectively of current vector I^4.

$I^{d'K}$ = Components of current vector I^5.

Λ_n = Ratio of reactive to active power for generator n at normal load.

$|V_n|_0$ = Terminal voltage of generator n at normal load.

M_{nK} = General term of loss matrix Z_{66}.

B_{nK} = General term of loss matrix using simplified analysis.

R_{n-K} = Real components of impedance matrix Z_{33} used in final loss matrix.

L_n = Incremental transmission loss at generator n.

P_K, Q_K = Active and reactive power supplied by generator K.

P_L = Total system transmission losses $= \sum_n \sum_K P_n \, . \, B_{nK} \, . \, P_K$.

S_n = Cost of fuel input to station n, £/hour.

λ = Incremental cost of received power, £/MWh.

(1) INTRODUCTION

For optimum operating efficiency of large power systems it is necessary to co-ordinate generation on an equal incremental fuel-cost basis with the incremental cost of the transmission line losses. Extensive research has taken place in America into methods of combining these costs by the application of transmission-loss formulae. These express the transmission-line losses as functions of the generator and interconnector power and a set of constants. The loss equations, once obtained, are applicable for any condition of generation and load, if the simplifying assumptions made in deriving the equations are valid for all system conditions.

Following the Steinberg and Smith[1] publication in 1943 on the economic loading of power plants, a method for expressing total transmission losses in terms of generator power was pioneered by George[2] in the same year. Ward, Eaton and Hale[4] extended George's original methods, and provided a more generalized and analytical approach to the derivation of a loss formula, by use of the a.c. network analyser. George, Page and Ward[3] co-ordinated transmission losses and fuel costs by the application of a loss equation containing constants derived from a network analyser study.

Kron,[5] in 1951, derived a loss-equation using tensorial methods, requiring considerably fewer measurements and calculations. Kirchmayer and Stagg,[6] using Kron's methods, obtained a transmission-loss formula for the American Gas and Electric system and investigated the effects of Kron's simplifying assumptions on the accuracy of the loss formula. Kron,[9] in a second paper, considered the existence of off-nominal turn ratios and their representation on an a.c. network analyser with auto-transformers. He followed this by two further publications,[10,13] in which he considered the study and co-ordination of several interconnected transmission systems, by combining the solutions of small components by a series of transformations.

Correspondence on Monographs is invited for consideration with a view to publication.

Mr. Nicholson is with Lever Bros., Port Sunlight, Ltd., and Mr. Lynn is in the Electrical Engineering Department, University of Liverpool.

Imburgia, Kirchmayer, and Stagg[16] have described a computer for use in system load dispatching. The computer calculates, from the loss constants of a system, transmission loss penalty factors which are used in conjunction with an incremental fuel cost slide-rule for obtaining economic balance between generating stations. Operation of the slide-rule and the computer provides a method of combining transmission losses and generation costs, and of applying them to the loading of a system under rapidly changing conditions.

In Part 1 of Kron's work, six basic reference frames are established for solving steady-state power-system problems, and, in particular, for determining total and incremental transmission losses.

The transmission-loss formula to be derived, as in other methods, involves the generated power of all sources and a set of constants. These constants (self- and mutual-impedances) once established are suitable for use under any operating conditions, within the limits of the operating assumptions, unless a physical change in the system takes place. The constants are obtained from a.c. network-analyser data or by analytical methods. The use of tensor algebra provides a method of transforming operating data (complex voltages and currents) into information concerning real powers, losses and a set of constants by means of a series of operations called 'transformations of reference frames'.

The object of the present paper is to relate the performance of power systems to a set of linear equations, containing real generator powers P and incremental losses L incurred in the transmission system. These equations are of the form $L = M \cdot P$, where M in matrix form represents a set of real constants which are dependent only upon operating conditions at normal load.

(2) STUDY OF KRON'S ANALYSIS

(2.1) System Operating Assumptions

The methods outlined by Kron for determining a transmission-loss formula involve certain fundamental concepts of tensor analysis, and a number of assumptions concerning the operation of a power system. These assumptions are as follows:

(*a*) The ratio of each load current to the total load current of a system, at normal load, remains constant as the loads vary.

(*b*) The generator currents remain fixed in phase angle relative to each other as the generator loads vary.

(*c*) The generator voltage magnitudes remain constant.

(*d*) The ratio of reactive power to active power of each source remains constant.

(2.2) Reference Frames

This present power system study will involve the introduction of the following reference frames:

1. Measurement of leakage impedances Z_{11}.
2. Introduction of one hypothetical load I^L which replaces all system load currents, where $I^L = \sum_{1}^{n} I^{L_n}$.
3. Elimination of I^L, leaving generator currents only.
4. Generator currents and voltages are transformed into axes, in phase and in quadrature with the respective generator voltages. Matrices of complex quantities are changed into larger matrices containing only real quantities.
5. Using the assumption of constant ratio of generator active/reactive power, each generator current is replaced by its projection upon the terminal voltage existing at normal load.
6. Active components of generator currents are transformed into generator powers. Since voltage differences have been used in the previous reference frames, the transformation matrix in frame 6 yields generator power loss.

(2.2.1) Reference Frame 1.

Consider a hypothetical power system consisting of generators G_1, G_2, G_3 supplying loads L_1, L_2 via a transmission network as in Fig. 1.

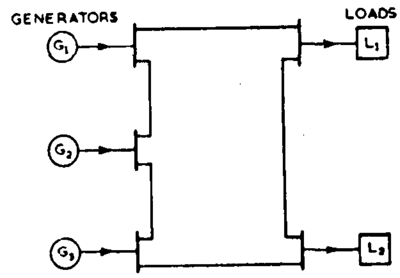


Fig. 1.—Hypothetical 3-generator 2-load power system.

The self and mutual leakage impedances of the system network are measured, usually from a network-analyser study, by:

(i) Disconnecting the generating plant and loads from the network.

(ii) Injecting unit current between each generator and a reference point in the system.

(iii) Measuring voltage differences between all generating and load points and the reference point.

This is repeated for measurement of load self-impedances by injecting unit current into the network at L_1 and L_2.

The impedances of the network from a generator point of entry are called generator self-impedances, and transfer or mutual impedances to other points. Impedances measured from load points are termed load self- and mutual impedances. The actual self-impedances of the generators and loads do not enter explicitly into the analysis.

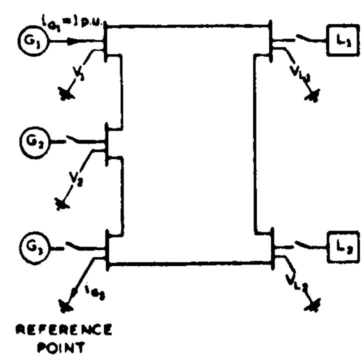


Fig. 2.—Reference frame 1: basic measurements.

Thus in Fig. 2, with unit current injected at G_1,

$$\left.\begin{array}{ll}\text{the generator self impedance,} & Z_{1.1} = V_1 \\ \text{and the mutual impedances,} & Z_{2.1} = V_2 \\ & Z_{3.1} = V_{L1} \\ & Z_{4.1} = V_{L2}\end{array}\right\} \quad \ldots \quad (1)$$

The impedance matrix for the system is therefore

$$Z_{11} = \begin{array}{c|cccc} & G_1 & G_2 & L_1 & L_2 \\ \hline G_1 & Z_{1.1} & Z_{1.2} & Z_{1.3} & Z_{1.4} \\ G_2 & Z_{2.1} & Z_{2.2} & Z_{2.3} & Z_{2.4} \\ L_1 & Z_{3.1} & Z_{3.2} & Z_{3.3} & Z_{3.4} \\ L_2 & Z_{4.1} & Z_{4.2} & Z_{4.3} & Z_{4.4} \end{array} = \begin{array}{c|cc} & G & L \\ \hline G & Z_{G.G} & Z_{G.L} \\ L & Z_{L.G} & Z_{L.L} \end{array} \quad (2)$$

Since the network mutual impedances form a symmetrical system,

$$Z_{G.L} = (Z_{L.G})_t$$

where $(Z_{L.G})_t$ is the transpose of matrix $(Z_{L.G})$.

That is, the mutual impedances with load points energized are equal to those with respect to the generator sources.

The equations for the system now take the form

$$\left.\begin{aligned} V_1 - V_R &= Z_{1.1}I^1 + Z_{1.2}I^2 + Z_{1.3}I^{L1} + Z_{1.4}I^{L2} \\ V_2 - V_R &= Z_{2.1}I^1 + Z_{2.2}I^2 + Z_{2.3}I^{L1} + Z_{2.4}I^{L2} \\ V_{L1} - V_R &= Z_{3.1}I^1 + Z_{3.2}I^2 + Z_{3.3}I^{L1} + Z_{3.4}I^{L2} \\ V_{L2} - V_R &= Z_{4.1}I^1 + Z_{4.2}I^2 + Z_{4.3}I^{L1} + Z_{4.4}I^{L2} \end{aligned}\right\} \quad . \quad (3)$$

If these system equations are solved by inverting the impedance matrix, the equivalent circuit will then be as shown in Fig. 3.

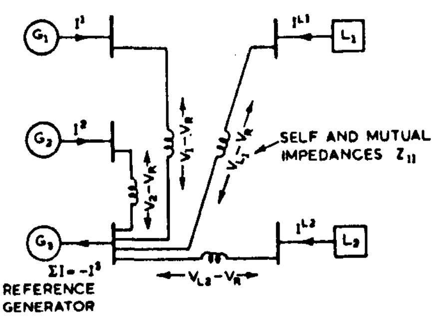


Fig. 3.—Equivalent circuit for measurement reference frame 1.

(2.2.2) Reference Frame 2.

Let the total system load current, during a normal load period, be given by

$$I^L = I^{L1} + I^{L2} \text{ (= hypothetical load)} \quad . \quad . \quad . \quad (4)$$

Then, from assumption (*a*),

$$\frac{I^{L1}}{I^L} = l_1, \quad \frac{I^{L2}}{I^L} = l_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where l_1, l_2 are constant complex ratios.

Thus

$$\left.\begin{aligned} I^{L1} &= l_1 I^L \\ I^{L2} &= l_2 I^L \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

or

$$I^{L_n} = C_L^{L_n} I^L \text{ where } C_L^{L_n} = \begin{array}{c|c|} & L \\ \hline 1 & l_1 \\ \hline 2 & l_2 \\ \hline \end{array} \quad . \quad . \quad . \quad (7)$$

The system with individual load currents may therefore be transformed into one which contains only the hypothetical total current I^L by the transformation matrix

$$C_2^1 = \begin{array}{c|c|c|c|} & 1 & 2 & L \\ \hline 1 & 1 & & \\ \hline 2 & & 1 & \\ \hline L_1 & & & l_1 \\ \hline L_2 & & & l_2 \\ \hline \end{array} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

(Note that the generator currents remain constant.)

and the currents and voltages in reference frame 2 are given by

$$I^1 = C_2^1 I^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$V_2 = (C_2^1)_t^* V_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

where $(C_2^1)_t^*$ is the conjugate of the matrix (C_2^1) transposed.

The system impedances in frame 2 are given by

$$Z_{22} = (C_2^1)_t^* Z_{11} C_2^1$$

$$= \begin{array}{c|c|c|c|} & 1 & 2 & L \\ \hline 1 & Z_{1.1} & Z_{1.2} & \begin{array}{c} Z_{1.3}l_1 \\ +Z_{1.4}l_2 \\ =(a_1) \end{array} \\ \hline 2 & Z_{2.1} & Z_{2.2} & \begin{array}{c} Z_{2.3}l_1 \\ +Z_{2.4}l_2 \\ =(a_2) \end{array} \\ \hline L & \begin{array}{c} Z_{3.1}l_1^* \\ +Z_{4.1}l_2^* \\ =(b_1) \end{array} & \begin{array}{c} Z_{3.2}l_1^* \\ +Z_{4.2}l_2^* \\ =(b_2) \end{array} & \begin{array}{c} Z_{3.3}l_1l_1^* \\ +Z_{3.4}l_2l_1^* \\ +Z_{4.3}l_1l_2^* \\ +Z_{4.4}l_2l_2^* \\ =(w) \end{array} \\ \hline \end{array} \quad . \quad (11)$$

The system performance is now represented by the equation

$$V_2 = Z_{22}I^2$$

i.e.

$$\left.\begin{aligned} V_1 - V_R &= Z_{1.1}I^1 + Z_{1.2}I^2 + a_1 I^L \\ V_2 - V_R &= Z_{2.1}I^1 + Z_{2.2}I^2 + a_2 I^L \\ V_L - V_R &= b_1 I^1 \quad + b_2 I^2 \quad + w I^L \end{aligned}\right\} \quad . \quad . \quad (12)$$

where in this example V_L, the hypothetical load voltage, is the weighted average of the 2-load voltages;

i.e.

$$V_L = l_1^* V_{L1} + l_2^* V_{L2} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and

$$\left.\begin{aligned} a_1 &= \sum_{K=3}^{4}\sum_{n=1}^{2} Z_{1.K}l_n a_2 = \sum_{K=3}^{4}\sum_{n=1}^{2} Z_{2.K}l_n \\ b_1 &= \sum_{K=3}^{4}\sum_{n=1}^{2} Z_{K.1}l_n^* b_2 = \sum_{K=3}^{4}\sum_{n=1}^{2} Z_{K.2}l_n^* \\ w &= \sum\sum\sum Z_{ll} \text{ as given above} \end{aligned}\right\} \quad . \quad . \quad (14)$$

The system may now be represented as in Fig. 4.

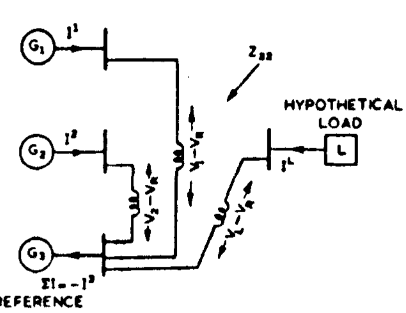


Fig. 4.—Equivalent circuit for reference frame 2 with one hypothetical load.

The system losses are now given by $I^{2*}Z_{22}I^2 = I^{2*}V_2$. However, this method of determining losses is not desirable since it involves the difference between large quantities, namely between generator and load powers.

(2.2.3) Reference Frame 3.

In this reference frame, the hypothetical load current is eliminated by introducing a transformation tensor C from a known equation of constraint. Now

$$I^1 + I^2 + I^3 + I^L = 0$$

Therefore $$I^L = -I^1 - I^2 - I^3 (= -\Sigma I^G) \quad . \quad . \quad . \quad (15)$$

Since the generator currents I^1 and I^2 remain unchanged in this reference frame, the required transformation matrix to replace the equation of constraint is

$C_3^2 =$ . . . . . (16)

	1	2	3
1	1		
2		1	
L	−1	−1	−1

The currents and voltages in this reference frame are now given by

$$I^2 = C_3^2 I^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

$$V_3 = (C_3^2)_t^* V_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

and the system impedances by

$$Z_{33} = (C_3^2)_t^* Z_{22} C_3^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

The impedance matrix Z_{33} is asymmetrical and complex.

The performance equations obtained from $V_3 = Z_{33}I^3$ are

$$\left.\begin{aligned} V_1 - V_L &= (Z_{1.1} - b_1 - a_1 + w)I^1 \\ &\quad + (Z_{1.2} - b_2 - a_1 + w)I^2 + (w - a_1)I^3 \\ V_2 - V_L &= (Z_{2.1} - b_1 - a_2 + w)I^1 \\ &\quad + (Z_{2.2} - b_2 - a_2 + w)I^2 + (w - a_2)I^3 \\ V_3 - V_L &= (w - b_1)I^1 + (w - b_2)I^2 + wI^3 \end{aligned}\right\} \quad . \quad (20)$$

The voltages V_3 represent the potential differences between generators and hypothetical load with currents I^3 entering and leaving at these points.

The components of the system equations are represented in Fig. 5. It is evident from this Figure that the load voltages and currents do not now exist in this reference frame 3.

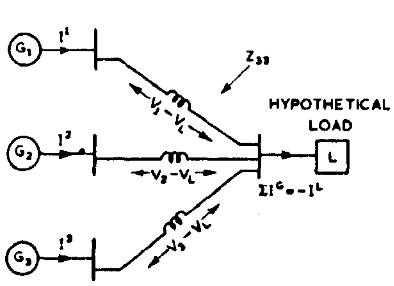


Fig. 5.—Equivalent circuit for reference frame 3.

The equation $I^{3*} . V_3$ represents the various losses in the system with reference to the generator currents and the voltage drops between the generators and hypothetical load. It does not involve the difference of large quantities.

(2.2.4) Reference Frame 4—Change of Axes.

For the hypothetical system under consideration, the generator and load terminal voltages and currents for a normal load period are illustrated in Fig. 6(*a*) and 6(*b*).

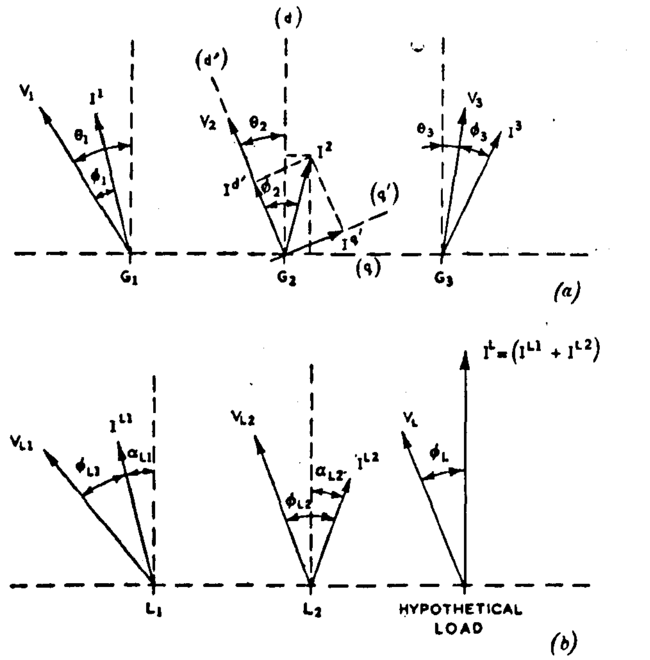


Fig. 6.—Vector diagrams for the hypothetical system at normal load.

(*a*) Normal load generator voltages and currents.
(*b*) Load voltages and currents at normal load.

The generator terminal voltages V_1, V_2 and V_3 are referred to a direct axis along the direction of the total load current, by angles θ_1, θ_2 and θ_3, respectively. The generator currents lag behind the terminal voltages by the respective angles ϕ_1, ϕ_2 and ϕ_3. The load voltage and current conditions at normal load are illustrated in Fig. 6(*b*).

A set of axes d' and q' is now introduced, along and at right angles to the generator terminal voltages. The generator currents projected upon the respective normal load terminal voltages will then represent the generator powers.

In this new reference frame, the active and reactive power components of each generator current I, will be given by,

$$I' = I^{d'} + jI^{q'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

The effect of rotating the axes d and q to new axes d' and q' is obtained by rotating the current vectors in the opposite direction, through an angle $-\theta$, so that

$$I = \varepsilon^{-j\theta}I' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Thus, for all generator currents, the transformation matrix required for quantities in reference frame 4 is

$C_4^3 =$ . . (23)

	1	2	3
1	$\varepsilon^{-j\theta_1}$		
2		$\varepsilon^{-j\theta_2}$	
3			$\varepsilon^{-j\theta_3}$

The new impedance matrix is now given by

$$Z_{44} = (C_4^3)_t^* Z_{33} C_4^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad (24)$$

and the system equations take the form

$$V_4 = Z_{44}I^4$$

or
$$|V_n| - V_L\varepsilon^{j\theta_n} = \sum_{K=1}^{3} Z_{n-K}\varepsilon^{j(\theta_n-\theta_K)}(I^K\varepsilon^{-j\theta_K}) \quad (n = 1, 2, 3) \qquad . \quad (25)$$

where Z_{n-K} is the general term of Z_{33};

or

$$|V_n| - V_L\varepsilon^{j\theta_n} = \sum_{K=1}^{3} \{[R_{n-K}\cos(\theta_n-\theta_K) - X_{n-K}\sin(\theta_n-\theta_K)] + j[X_{n-K}\cos(\theta_n-\theta_K) + R_{n-K}\sin(\theta_n-\theta_K)]\}(I^K\varepsilon^{-j\theta_K}) \quad . \quad . \quad . \quad . \quad (26)$$

where $(R_{n-K} + jX_{n-K})$ represents the components of Z_{n-K}. The voltage vector V_4 represents the voltage drop from generators to hypothetical load.

Any complex impedance, say $Z = R + jX$, may be replaced by a matrix containing only real numbers of the form

$$Z \doteq \begin{array}{c|c|c|} & d & q \\ \hline d & R & -X \\ \hline q & X & R \\ \hline \end{array} \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

This introduces direct and quadrature axes d and q, in which the voltage and current vectors are represented by in-phase and quadrature components.

Thus if $I = I^d + jI^q$ and $V = V_d + jV_q$, then, in matrix form,

$$I = \begin{array}{c|c|} d & I^d \\ \hline q & I^q \\ \end{array} \quad \text{and} \quad V = \begin{array}{c|c|} d & V_d \\ \hline q & V_q \\ \end{array} \quad . \quad . \quad . \quad (28)$$

Thus the complex equation $V_4 = Z_{44}I^4$ may be expressed in terms of real quantities by replacing the respective matrices by others containing twice as many equations and variables, but containing all real components.

For the 3-generator system, with the impedance Z_{44} expressed in real numbers, the general equations for system performance will take the form

$$\left.\begin{aligned} V_{d_n} &= \sum_{K=1}^{3} [R_{n-K}\cos(\theta_n-\theta_K) - X_{n-K}\sin(\theta_n-\theta_K)]I^{dK} \quad (n = 1, 2, 3) \\ &\quad - \sum_{K=1}^{3} [X_{n-K}\cos(\theta_n-\theta_K) + R_{n-K}\sin(\theta_n-\theta_K)]I^{qK} \\ V_{q_n} &= \sum_{K=1}^{3} [X_{n-K}\cos(\theta_n-\theta_K) + R_{n-K}\sin(\theta_n-\theta_K)]I^{dK} \quad (n = 1, 2, 3) \\ &\quad + \sum_{K=1}^{3} [R_{n-K}\cos(\theta_n-\theta_K) - X_{n-K}\sin(\theta_n-\theta_K)]I^{qK} \end{aligned}\right\} \quad . \quad (29)$$

(2.2.5) Reference Frame 5.

In this frame the reactive components of the generator currents are expressed in terms of the active components in phase with the normal load terminal voltages.

Using assumption (d) of Section 2.1, the components of the various generator currents may be expressed in the form

$$\left.\begin{aligned} I^d &= I^d \\ I^q &= \Lambda I^d \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

where the constant Λ is given by the ratio of the normal load measurements, $I^{q'}/I^{d'} = \tan\phi_n$ (in Fig. 6a) $= Q/P$, Q and P being the reactive and active power components respectively, of each generator at normal load.

Such a set of equations for the components of all generator currents in the 3-generator system will produce the transformation matrix

$$C_5^4 = \begin{array}{c|c|c|c|} & d_1 & d_2 & d_3 \\ \hline d_1 & 1 & & \\ \hline d_2 & & 1 & \\ \hline d_3 & & & 1 \\ \hline q_1 & \Lambda_1 & & \\ \hline q_2 & & \Lambda_2 & \\ \hline q_3 & & & \Lambda_3 \\ \hline \end{array}$$

i.e.
$$I^4 = C_5^4 I^5 \quad . \quad . \quad . \quad . \quad . \quad . \quad (31)$$

The new impedance matrix, containing only real numbers, is now given by $(C_5^4)_t Z_{44} C_5^4$, and the system equations are given by the general term

$$(V_{d_n} + \Lambda_n V_{q_n}) = \quad (n = 1, 2, 3)$$
$$\sum_{K=1}^{3} \{[R_{n-K}\cos(\theta_n-\theta_K) - X_{n-K}\sin(\theta_n-\theta_K)](1 + \Lambda_n\Lambda_K) + [X_{n-K}\cos(\theta_n-\theta_K) + R_{n-K}\sin(\theta_n-\theta_K)](\Lambda_n - \Lambda_K)\} \times (I^{dK} + \Lambda_K I^{qK}) \quad . \quad (32)$$

(2.2.6) Reference Frame 6—Transformation of Generator Current into Power.

By assuming that the generator terminal voltages remain practically constant in the region of the normal load values, the active power of each generator will be given by

$$\left.\begin{aligned} P_1 &= I^{d1'}|V_1|_0 \qquad & I^{d1'} &= 1/|V_1|_0 \quad P_1 \\ P_2 &= I^{d2'}|V_2|_0 \text{ or } & I^{d2'} &= 1/|V_2|_0 \quad P_2 \\ P_3 &= I^{d3'}|V_3|_0 & I^{d3'} &= 1/|V_3|_0 \quad P_3 \end{aligned}\right\} . \quad (33)$$

Thus the required transformation matrix for quantities in this reference frame is

$$C_6^5 = \begin{array}{c|c|c|c|} & 1 & 2 & 3 \\ \hline 1 & \frac{1}{|V_1|_0} & & \\ \hline 2 & & \frac{1}{|V_2|_0} & \\ \hline 3 & & & \frac{1}{|V_3|_0} \\ \hline \end{array} \quad . \quad . \quad (34)$$

The final impedance matrix is given by

$$Z_{66} = (C_6^5)_t Z_{55} C_6^5 \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

and the system equations are given by

$$\frac{V_{dn} + \Lambda_n V_{qn}}{|V_n|_0} = \sum_{K=1}^{3} \frac{1}{|V_n|_0 |V_K|_0}\{[R_{n-K}\cos(\theta_n - \theta_K) - X_{n-K}\sin(\theta_n - \theta_K)](1 + \Lambda_n\Lambda_K) + [X_{n-K}\cos(\theta_n - \theta_K) + R_{n-K}\sin(\theta_n - \theta_K)](\Lambda_n - \Lambda_K)\}P_K$$

or

$$= \sum_{K=1}^{3} M_{nK}P_K \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

where $P_K = (I^{dK} + \Lambda_K I^{qK})|V_K|_0$

is the real power supplied by generator K and M_{nK} represents a set of real constants.

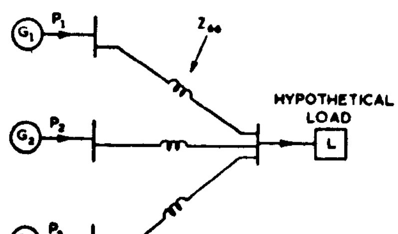


Fig. 7.—Equivalent circuit for the final reference frame 6 with active generator powers.

Fig. 7 illustrates the equivalent circuit with real impressed generator powers.

The final loss equation is now of the form

$$V_6 = Z_{66}I^6 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37)$$

The dimension of I^6 is that of power, and it represents the active power P supplied by each generator.

The term $V_6 = (C_6^5)_t V_5$ is dimensionless owing to the inverse voltage form of C_6^5. Its components are fractions, say L, and represent the incremental I^2R transmission losses of the system, ∂ loss/∂P.

Z_{66} represents the final 'loss' matrix containing real components M_{nK} of inverse power form Z/V^2.

Thus, in terms of generator powers and incremental I^2R losses, the system equations take the form

$$L_n = \sum_K M_{nK}P_K \quad . \quad . \quad . \quad . \quad . \quad . \quad (38)$$

which is equal to the incremental transmission loss at generator n, ∂ losses/∂P_n.

The total I^2R losses in the system are thus given by

$$P_{losses} = \sum_n \sum_K P_n M_{nK} P_K \quad . \quad . \quad . \quad . \quad . \quad (39)$$

(2.3) Simplification of the Analysis for Total and Incremental Loss Studies

For incremental loss calculations only the real parts of the differences of potential existing in the basic measurement reference frame 1 need to be measured. It is also sufficient to use only the real components of the matrix C_L for transformation from reference frame 1 to 2. This latter simplification will produce a symmetrical impedance matrix Z_{22} containing only real components, and the former a and b components change to $d = (a + b)/2$.

The total I^2R losses in the system may be found by using only the symmetrical part of Z_{66} containing coefficients R_{66}, say B. In this case, the total transmission losses are given by

$$P_{losses} = \sum_n \sum_K P_n B_{nK} P_K \text{ and } B_{nK} = B_{Kn} \quad . \quad (40)$$

Thus, for the 3-generator system,

$$P_{losses} = B_{11}P_1^2 + B_{22}P_2^2 + B_{33}P_3^2 + 2B_{12}P_1P_2 + 2B_{13}P_1P_3 + 2B_{23}P_2P_3 \quad . \quad (41)$$

The same result is also obtained if only the symmetrical part of R_{33} and the skew-symmetric part of X_{33} are used when transforming Z_{33} to Z_{66}, assuming Z_{33} has the form $R_{33} + jX_{33}$.

Using the above simplifications, the general loss coefficient takes the form

$$B_{nK} = \frac{R_{n-K}}{|V_n|_0 |V_K|_0}[\cos(\theta_n - \theta_K)(1 + \Lambda_n\Lambda_K) + \sin(\theta_n - \theta_K)(\Lambda_n - \Lambda_K)] \quad . \quad (42)$$

(3) STUDY OF THE LOADING CONDITIONS ON A SECTION OF THE BRITISH NETWORK

The Grid system for which a transmission-loss formula is to be determined consists of the 132 kV Warrington section of the North West, Merseyside and North Wales Divisional Network. This section comprises steam generating stations at Warrington, Percival Lane and Ince, and 33 kV Area Board load points at Warrington, Percival Lane, Ince, Crewe and Knutsford. Fig. 8 illustrates a simplified diagram of this system with the main generating, load and interconnection points. The available active-power generation exported at each busbar is given in this diagram, which also shows the contribution from Marchwiel and part of Maentwrog at the Crewe station. The diagram also illustrates the route length of each overhead line forming the network, and the per-unit impedances for a 100 MVA base at 132 kV.

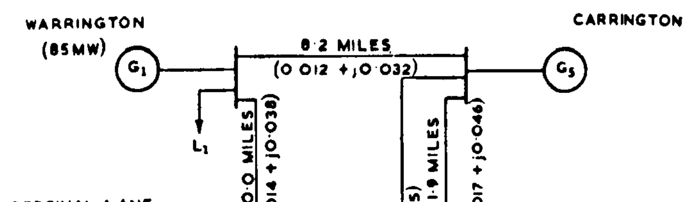


Fig. 8.—System diagram—Warrington Section.
Base apparent power—100 MVA.
Base voltage—132 kV.

(3.1) Load Flow Study

The study was carried out analytically in the absence of network-analyser facilities. The first part consisted in determining the approximate system load flows from the known generator loadings and the active-power demands at the load points. It was assumed that there was a normal 12% outage of the available generation at all stations, and the load demands used were those that existed in January, 1955, during the normal peak periods of 0700–1300 hours (see Fig. 10).

The power equations for a short line were then used for calculating the various terminal voltages and angles in the given system. Reactive line flows were also determined from these

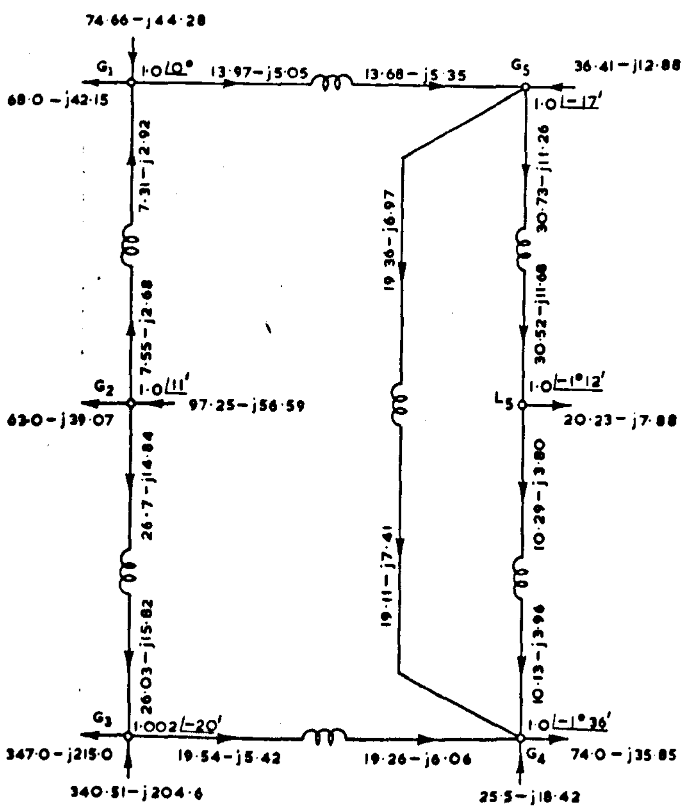


Fig. 9.—Load-flow data for a weekday peak period, including line flows, generator loadings, load demands and bus voltages.

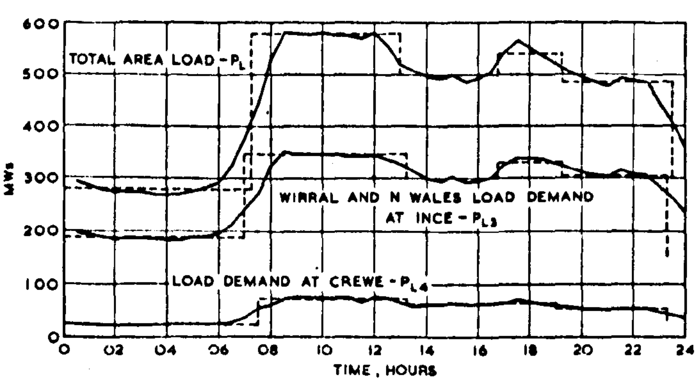


Fig. 10—System load demands—17th January, 1955.

(a) Total area load, P_L.
(b) Wirral and North Wales load demand at Ince, P_{L3}.
(c) Load demand at Crewe, P_{L4}.

equations, and balance of busbar conditions in the system loop was obtained by a series of successive approximations.

The results of the final approximation to system load flows to give the desired system balance and realistic generator and load reactive demands are illustrated in the load-flow diagram of Fig. 9.

(4) STUDY OF THE APPLICATION OF KRON'S METHODS TO A SECTION OF THE BRITISH NETWORK

(4.1) Reference Frame 1

For the measurement of the basic impedance matrix of the Warrington section network, illustrated in Fig. 8, the Carrington tie-line connection G_5 was chosen as reference point and was earthed.

With unit current impressed at the generator G_1 the various leakage impedances are given by

$$Z_{n,K} = \frac{V_{G_n}}{i_{G_K}} \qquad (43)$$

and in this case reference generator current I_{GR} = impressed current I_{G_K}.

Thus for generator points of entry,

$$\left.\begin{aligned} Z_{1.1} &= \frac{V_{G1}}{i_{G1}} = 0{\cdot}0105 + j0{\cdot}0280 \\ Z_{2.1} &= \frac{V_{G2}}{i_{G1}} = 0{\cdot}0087 + j0{\cdot}0231 \\ Z_{3.1} &= \frac{V_{G3}}{i_{G1}} = 0{\cdot}0073 + j0{\cdot}0195 \\ Z_{4.1} &= \frac{V_{G4}}{i_{G1}} = 0{\cdot}0025 + j0{\cdot}0067 \end{aligned}\right.$$

and for load points of entry,

$$\left.\begin{aligned} Z_{5.1} &= Z_{1.1} = 0{\cdot}0105 + j0{\cdot}0280 \\ Z_{6.1} &= Z_{2.1} = 0{\cdot}0087 + j0{\cdot}0231 \\ Z_{7.1} &= Z_{3.1} = 0{\cdot}0073 + j0{\cdot}0195 \\ Z_{8.1} &= Z_{4.1} = 0{\cdot}0025 + j0{\cdot}0067 \\ Z_{9.1} &= \frac{V_{L5}}{i_{G1}} = 0{\cdot}0011 + j0{\cdot}0029 \end{aligned}\right\} \qquad (44)$$

With unit current impressed at generator G_2,

$$\left.\begin{aligned} Z_{1.2} &= 0{\cdot}0087 + j0{\cdot}0231 = Z_{5.2} \\ Z_{2.2} &= 0{\cdot}0188 + j0{\cdot}0507 = Z_{6.2} \\ Z_{3.2} &= 0{\cdot}0160 + j0{\cdot}0428 = Z_{7.2} \\ Z_{4.2} &= 0{\cdot}0055 + j0{\cdot}0146 = Z_{8.2} \\ Z_{9.2} &= 0{\cdot}0023 + j0{\cdot}0063 \end{aligned}\right\} \qquad (45)$$

With unit current impressed at generator G_3

$$\left.\begin{aligned} Z_{1.3} &= 0{\cdot}0073 + j0{\cdot}0195 = Z_{5.3} \\ Z_{2.3} &= 0{\cdot}0160 + j0{\cdot}0428 = Z_{6.3} \\ Z_{3.3} &= 0{\cdot}0226 + j0{\cdot}0600 = Z_{7.3} \\ Z_{4.3} &= 0{\cdot}0078 + j0{\cdot}0206 = Z_{8.3} \\ Z_{9.3} &= 0{\cdot}0033 + j0{\cdot}0089 \end{aligned}\right\} \qquad (46)$$

With unit current impressed at generator G_4.

$$\left.\begin{aligned} Z_{1.4} &= 0{\cdot}0025 + j0{\cdot}0067 = Z_{5.4} \\ Z_{2.4} &= 0{\cdot}0055 + j0{\cdot}0146 = Z_{6.4} \\ Z_{3.4} &= 0{\cdot}0078 + j0{\cdot}0206 = Z_{7.4} \\ Z_{4.4} &= 0{\cdot}0158 + j0{\cdot}0419 = Z_{8.4} \\ Z_{9.4} &= 0{\cdot}0067 + j0{\cdot}0182 \end{aligned}\right\} \qquad (47)$$

With unit current impressed at the load L_1 the leakage impedances are equal to those obtained with unit current impressed at the generator G_1. Similarly, with impressed currents at L_2, L_3 and L_4 the leakage impedances are equal to those obtained with impressed currents at G_2, G_3 and G_4, respectively.

With unit current impressed at the load point L_5,

$$\left.\begin{aligned} Z_{1.9} &= 0{\cdot}0011 + j0{\cdot}0029 = Z_{5.9} \\ Z_{2.9} &= 0{\cdot}0023 + j0{\cdot}0063 = Z_{6.9} \\ Z_{3.9} &= 0{\cdot}0033 + j0{\cdot}0089 = Z_{7.9} \\ Z_{4.9} &= 0{\cdot}0067 + j0{\cdot}0182 = Z_{8.9} \\ Z_{9.9} &= 0{\cdot}0125 + j0{\cdot}0339 \end{aligned}\right\} \qquad (48)$$

The above impedances now form the components of the basic symmetrical leakage-impedance matrix Z_{11}.

(4.2) Reference Frame 2

In this analysis the hypothetical load for a normal load period will be given by

$$I^L = I^{L1} + I^{L2} + I^{L3} + I^{L4} + I^{L5} \quad . \quad . \quad . \quad (49)$$

and the components of the C_L matrix by

$$l_1 = \frac{I^{L1}}{I^L},\ l_2 = \frac{I^{L2}}{I^L},\ l_3 = \frac{I^{L3}}{I^L},\ l_4 = \frac{I^{L4}}{I^L},\ l_5 = \frac{I^{L5}}{I^L} \quad . \quad (50)$$

However, since available information for the given load points consists of the half-hourly active-power loading only, it is not possible to obtain the constant complex ratios, l_1, l_2, etc., in the above form. An assumption is thus made that the ratio of the active-power loading at the load points to the total active-power load remains constant, and approximates to the corresponding current ratios, as given above. Such an approximation assumes that all load voltages and power factors are equal, and that all load currents are displaced equally from the total load current. These assumptions are valid for the system under consideration owing to the small phase displacements between the terminal voltages, the previously assumed constant load power factors and the approximate nominal system voltages at the load points.

A similar assumption is made by George,[2] in which the average line voltage and average power factor for the heavily loaded portions of the systems are used in the analysis.

Fig. 10 illustrates the half-hourly integrated loadings on 17th January, 1955, for two of the five load points, and the total area load for the Warrington section network. To compare the general trend of each load to that of the total system load, the ratio of each half-hourly load to that of the total was calculated.

It was apparent from these ratios that the trend of the total load was reflected at all load points; that is, the pattern of demand was the same at all load points, as stated in assumption (*a*).

The loading ratios to be used in this analysis have been obtained by averaging the individual half hourly ratios, and are

$$l_1 = 0{\cdot}110,\ l_2 = 0{\cdot}108,\ l_3 = 0{\cdot}633,\ l_4 = 0{\cdot}111,\ l_5 = 0{\cdot}038 \quad . \quad . \quad . \quad . \quad (51)$$

The absence of reactive metering data at the load points prevented the use of complex ratios in this analysis. The leakage-impedance matrix for reference frame 2 is now given by

$$Z_{22} = (C_2^1)_t^* Z_{11} (C_2^1)$$

In this analysis $(C_2^1)^* = (C_2^1)$ and only the real components of Z_{11} are considered.

The components of Z_{22} are given and calculated as follows:

$$\left.\begin{aligned}
&Z_{1.1} = 0{\cdot}0105,\ Z_{1.2} = 0{\cdot}0087,\ Z_{1.3} = 0{\cdot}0073,\ Z_{1.4} = 0{\cdot}0025\\
&Z_{2.1} = 0{\cdot}0087,\ Z_{2.2} = 0{\cdot}0188,\ Z_{2.3} = 0{\cdot}0160,\ Z_{2.4} = 0{\cdot}0055\\
&Z_{3.1} = 0{\cdot}0073,\ Z_{3.2} = 0{\cdot}0160,\ Z_{3.3} = 0{\cdot}0226,\ Z_{3.4} = 0{\cdot}0078\\
&Z_{4.1} = 0{\cdot}0025,\ Z_{4.2} = 0{\cdot}0055,\ Z_{4.3} = 0{\cdot}0078,\ Z_{4.4} = 0{\cdot}0158\\
&d_1 = Z_{1.5}l_1 + Z_{1.6}l_2 + Z_{1.7}l_3 + Z_{1.8}l_4 + Z_{1.9}l_5 = 0{\cdot}00703\\
&d_2 = Z_{2.5}l_1 + Z_{2.6}l_2 + Z_{2.7}l_3 + Z_{2.8}l_4 + Z_{2.9}l_5 = 0{\cdot}01381\\
&d_3 = Z_{3.5}l_1 + Z_{3.6}l_2 + Z_{3.7}l_3 + Z_{3.8}l_4 + Z_{3.9}l_5 = 0{\cdot}01783\\
&d_4 = Z_{4.5}l_1 + Z_{4.6}l_2 + Z_{4.7}l_3 + Z_{4.8}l_4 + Z_{4.9}l_5 = 0{\cdot}00781\\
&\text{and } w \text{ is given by,}\\
&Z_{5.5}l_1l_1^* + Z_{5.6}l_2l_1^* + Z_{5.7}l_3l_1^* + Z_{5.8}l_4l_1^* + Z_{5.9}l_5l_1^* = 0{\cdot}000774\\
&Z_{6.5}l_1l_2^* + Z_{6.6}l_2l_2^* + Z_{6.7}l_3l_2^* + Z_{6.8}l_4l_2^* + Z_{6.9}l_5l_2^* = 0{\cdot}001492\\
&Z_{7.5}l_1l_3^* + Z_{7.6}l_2l_3^* + Z_{7.7}l_3l_3^* + Z_{7.8}l_4l_3^* + Z_{7.9}l_5l_3^* = 0{\cdot}011285\\
&Z_{8.5}l_1l_4^* + Z_{8.6}l_2l_4^* + Z_{8.7}l_3l_4^* + Z_{8.8}l_4l_4^* + Z_{8.9}l_5l_4^* = 0{\cdot}000867\\
&Z_{9.5}l_1l_5^* + Z_{9.6}l_2l_5^* + Z_{9.7}l_3l_5^* + Z_{9.8}l_4l_5^* + Z_{9.9}l_5l_5^* = 0{\cdot}000140\\
&\qquad 0{\cdot}014558 = w
\end{aligned}\right\} \quad . \quad (52)$$

(4.3) Reference Frame 3

In frame 3 the hypothetical load current I^L is replaced by the reference generator current $I^R = I^5$, using the equation of constraint given by

$$I^L = -I^1 - I^2 - I^3 - I^4 - I^5,\ \text{i.e.} = -\Sigma I^G \quad . \quad (53)$$

The impedance matrix is given by

$$Z_{33} = (C_3^2)_t^* Z_{22} (C_3^2)$$

and its components (R_{n-K}) are

$Z_{1.1} - 2d_1 + w$ $= +0{\cdot}011$	$Z_{1.2} - d_1 - d_2 + w$ $= +0{\cdot}00242$	$Z_{1.3} - d_1 - d_3 + w$ $= -0{\cdot}003$	$Z_{1.4} - d_1 - d_4 + w$ $= +0{\cdot}00222$	$w - d_1$ $= +0{\cdot}00753$
$Z_{2.1} - d_2 - d_1 + w$ $= +0{\cdot}00242$	$Z_{2.2} - 2d_2 + w$ $= +0{\cdot}00574$	$Z_{2.3} - d_2 - d_3 + w$ $= -0{\cdot}00108$	$Z_{2.4} - d_2 - d_4 + w$ $= -0{\cdot}00156$	$w - d_2$ $= +0{\cdot}00075$
$Z_{3.1} - d_3 - d_1 + w$ $= -0{\cdot}003$	$Z_{3.2} - d_3 - d_2 + w$ $= -0{\cdot}00108$	$Z_{3.3} - 2d_3 + w$ $= +0{\cdot}00150$	$Z_{3.4} - d_3 - d_4 + w$ $= -0{\cdot}00328$	$w - d_3$ $= -0{\cdot}00327$
$Z_{4.1} - d_4 - d_1 + w$ $= +0{\cdot}00222$	$Z_{4.2} - d_4 - d_2 + w$ $= -0{\cdot}00156$	$Z_{4.3} - d_4 - d_3 + w$ $= -0{\cdot}00328$	$Z_{4.4} - 2d_4 + w$ $= +0{\cdot}01474$	$w - d_4$ $= +0{\cdot}00675$
$w - d_1$ $= +0{\cdot}00753$	$w - d_2$ $= +0{\cdot}00075$	$w - d_3$ $= -0{\cdot}00327$	$w - d_4$ $= +0{\cdot}00675$	w $= +0{\cdot}01456$

. (54)

It will be noted that the components of Z_{33} are symmetric, and contain only real quantities similar to the real symmetrical components R_{33} of Kron's analysis,[5] and the *Kmn* coefficients of Ward, Eaton and Hale [Reference 4 eqn. (6)].

(4.4) Calculation of Loss Constants

The general term of the loss matrix is given by

$$B_{nK} = \frac{R_{n-K}}{|V_n|_0 |V_K|_0} [\cos(\theta_n - \theta_K)(1 + \Lambda_n\Lambda_K) + \sin(\theta_n - \theta_K)(\Lambda_n - \Lambda_K)] \quad . \quad . \quad . \quad . \quad (55)$$

The generator terminal voltage angles, θ, may be referred to any reference axis, and in the present study this has been taken along the terminal voltage of generator G_1.

The constant ratio ($\Lambda = Q/P$) for each generator, and the generator terminal voltages are those obtained from the normal load-flow study, and are illustrated in Table 1.

Table 1

GENERATOR DATA FROM LOAD-FLOW STUDY

Generator G_n	Generator output P (MW)	Generator output Q (MVAr)	Generator terminal voltage angle w.r.t. $G_1\theta$	Q/P ratio Λ_n	Per-unit bus voltage V_n
G_1	74·66	44·28	0	0·5931	1·0
G_2	97·25	56·59	+11′	0·5819	1·0
G_3	340·51	204·6	−20′	0·6009	1·0015
G_4	25·5	18·42	−1° 36′	0·7224	1·0
G_5	36·41	12·88	−17′	0·3537	1·0

From the foregoing general term,

$$B_{11} = \frac{1}{|V_1|_0^2} R_{1-1}(1 + \Lambda_1^2)$$
$$= 0{\cdot}011(1 + 0{\cdot}5931^2) = 0{\cdot}01487 \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)$$

$$B_{12} = \frac{R_{1-2}}{|V_1|_0 |V_2|_0}[\cos(\theta_1 - \theta_2)(1 + \Lambda_1\Lambda_2) + \sin(\theta_1 - \theta_2)(\Lambda_1 - \Lambda_2)]$$
$$= 0{\cdot}00242\,[\cos(-11')(1 + 0{\cdot}5931 \times 0{\cdot}5819) + \sin(-11')(0{\cdot}5931 - 0{\cdot}5819)] = 0{\cdot}003255 \quad . \quad (57)$$

All the B_{nK} coefficients obtained from the foregoing general term are given in Table 2.

Thus, for the 5-generator system, the total system per-unit loss is given by

$$P_L = 0{\cdot}0149P_1^2 + 0{\cdot}0077P_2^2 + 0{\cdot}002P_3^2 + 0{\cdot}0224P_4^2 + 0{\cdot}0164P_5^2 + 0{\cdot}0065P_1P_2 - 0{\cdot}0081P_1P_3 + 0{\cdot}0063P_1P_4 + 0{\cdot}0182P_1P_5 - 0{\cdot}0029P_2P_3 - 0{\cdot}0044P_2P_4 + 0{\cdot}0018P_2P_5 - 0{\cdot}0094P_3P_4 - 0{\cdot}0079P_3P_5 + 0{\cdot}0168P_4P_5 \quad . \quad . \quad . \quad . \quad . \quad (58)$$

where P_1, P_2, etc., are the per-unit station loadings.

For the station loadings obtained from the load-flow study, i.e. for

$P_1 = 74{\cdot}66$ MW, $P_2 = 97{\cdot}25$ MW, $P_3 = 340{\cdot}51$ MW, $P_4 = 25{\cdot}5$ MW, and $P_5 = 36{\cdot}41$ MW

the total system loss is, $P_L = 0{\cdot}006227$ per unit or 0·623 MW.

These losses do not compare favourably with those obtained from the load-flow study, the results of which are illustrated in Fig. 9. However, the main purpose of the flow study was for the determination of voltages, angles and Q/P ratios. Excessive errors in the losses estimated from Fig. 9 are the results of finding differences of large quantities.

(5) CO-ORDINATION OF INCREMENTAL FUEL COSTS AND TRANSMISSION LOSSES

In this Section the previously determined transmission losses of the system are combined with the incremental fuel costs of the generating stations, and a loading schedule is obtained which will give minimum operating costs for given total generation.

In the case of a number of generating stations supplying a power system and loaded on an equal incremental fuel cost basis, power will be transmitted from low- to high-cost regions due to the variation of fuel costs at different stations. For economic division of load between the stations it is thus necessary to consider the resulting transmission losses, and to amend the station operating costs accordingly.

(5.1) Methods of Co-ordination

The mathematical analysis for co-ordinating incremental fuel costs and transmission losses is based on the methods for determining the maxima and minima of a function of two variables, the latter also being related by an equation of constraint.

Using the methods of Lagrange's undetermined multipliers (Courant,[32] Kirchmayer and Stagg[8]), the condition for minimum fuel input is given by

$$\frac{dS_n}{dP_n} + \lambda\frac{\partial P_L}{\partial P_n} = \lambda \quad . \quad . \quad . \quad . \quad . \quad (59)$$

where dS_n/dP_n = incremental fuel cost (£/MWh) of station n.

and $\partial P_L/\partial P_n$ = incremental transmission loss (MW) for a megawatt change in generation (MW/MW) at station n.

Solution of the non-linear simultaneous equations obtained from eqn. (59) for each station, by variation of λ, will yield the plant schedules for different total system loadings.

If, in the general case, the fuel cost input curve for station n is assumed to be of the form

$$S_n = m_nP_n^2 + C_nP_n (\text{£/hr.}) \quad . \quad . \quad . \quad . \quad (60)$$

then

$$\frac{dS_n}{dP_n} = (2m_n)P_n + C_n = m_n'P_n + C_n \quad . \quad . \quad (61)$$

Table 2

COMPONENTS OF FINAL LOSS MATRIX

nK	11	22	33	44	55
B_{nK}	+0·01487	+0·00768	+0·00204	+0·02243	+0·01638
nK	12	13	14	15	23
B_{nK}	+0·00326	−0·00406	+0·00316	+0·00912	−0·00146
nK	24	25	34	35	45
B_{nK}	−0·00221	+0·00091	−0·00469	−0·00396	+0·00842

where $m_n' =$ slope of incremental cost curve (£/MWh/MW).

and $C_n =$ intercept on incremental cost scale (£/MWh).

Also
$$\frac{\partial P_L}{\partial P_n} = \sum_K 2B_{nk}P_K$$

Thus eqn. (59) becomes

$$(m_n'P_n + C_n) + \lambda \sum_K 2B_{nK}P_K = \lambda \quad . \quad . \quad . \quad (62)$$

For the five generator system under consideration, the co-ordination equations are

$$\left.\begin{aligned} m_1'P_1 + \lambda(2B_{11}P_1 + 2B_{12}P_2 + 2B_{13}P_3 + 2B_{14}P_4 \\ + 2B_{15}P_5) = \lambda - C_1 \\ m_2'P_2 + \lambda(2B_{12}P_1 + 2B_{22}P_2 + 2B_{23}P_3 + 2B_{24}P_4 \\ + 2B_{25}P_5) = \lambda - C_2 \\ \text{etc.} \end{aligned}\right\} \quad . \quad (63)$$

Available information, however, concerning station fuel costs will take the form

$$S_n = m_nP_n + C_n \quad . \quad . \quad . \quad . \quad . \quad . \quad (64)$$

where m_n and C_n now relate to the slope and intercept of the input/output curves.

Thus
$$\frac{dS_n}{dP_n} = m_n$$

In this case it is now possible to charge the system losses at the incremental rate of received power λ, and to obtain a loading schedule from the solution of linear equations.

(5.1.1) **Co-ordination Equations for a Five-Generator System.**

In terms of the actual B_{nK} constants and linear input-output curves, the co-ordination equations simplify to

$$\left.\begin{aligned} 0{\cdot}0298P_1 + 0{\cdot}0065P_2 - 0{\cdot}0081P_3 + 0{\cdot}0063P_4 \\ + 0{\cdot}0182P_5 = \frac{1}{100}\left(1 - \frac{m_1}{\lambda}\right) \\ 0{\cdot}0065P_1 + 0{\cdot}0154P_2 - 0{\cdot}0029P_3 - 0{\cdot}0044P_4 \\ + 0{\cdot}0018P_5 = \frac{1}{100}\left(1 - \frac{m_2}{\lambda}\right) \\ -0{\cdot}0081P_1 - 0{\cdot}0029P_2 + 0{\cdot}004P_3 - 0{\cdot}0094P_4 \\ - 0{\cdot}0079P_5 = \frac{1}{100}\left(1 - \frac{m_3}{\lambda}\right) \\ 0{\cdot}0063P_1 - 0{\cdot}0044P_2 - 0{\cdot}0094P_3 + 0{\cdot}0448P_4 \\ + 0{\cdot}0168P_5 = \frac{1}{100}\left(1 - \frac{m_4}{\lambda}\right) \\ 0{\cdot}0182P_1 + 0{\cdot}0018P_2 - 0{\cdot}0079P_3 + 0{\cdot}0168P_4 \\ + 0{\cdot}0328P_5 = \frac{1}{100}\left(1 - \frac{m_5}{\lambda}\right) \end{aligned}\right\} \quad . \quad (65)$$

(P_1, P_2, etc., are per-unit station loadings.)

or, in matrix form,
$$BP = \left(1 - \frac{m}{\lambda}\right)$$

The solution for the station loadings will now be obtained by inversion of the $5 \times 5B$ matrix. This is calculated using the methods of Kron,[25], p. 258, for the inverse of a two-row compound matrix.

Such an inversion yields the matrix equation

$$P = B^{-1}\left(1 - \frac{m}{\lambda}\right)$$

This gives the respective per-unit loadings for minimum total fuel cost as follows:

$$\left.\begin{aligned} P_1 &= 46{\cdot}922 - \frac{1}{\lambda}(5{\cdot}699m_1 + 4{\cdot}730m_2 + 29{\cdot}965m_3 \\ &\qquad + 5{\cdot}600m_4 + 0{\cdot}928m_5) \\ P_2 &= 45{\cdot}635 - \frac{1}{\lambda}(4{\cdot}730m_1 + 5{\cdot}486m_2 + 28{\cdot}758m_3 \\ &\qquad + 5{\cdot}452m_4 + 1{\cdot}209m_5) \\ P_3 &= 273{\cdot}446 - \frac{1}{\lambda}(29{\cdot}965m_1 + 28{\cdot}758m_2 + 174{\cdot}902m_3 \\ &\qquad + 32{\cdot}583m_4 + 7{\cdot}238m_5 \\ P_4 &= 51{\cdot}181 - \frac{1}{\lambda}(5{\cdot}600m_1 + 5{\cdot}452m_2 + 32{\cdot}583m_3 \\ &\qquad + 6{\cdot}362m_4 + 1{\cdot}184m_5) \\ P_5 &= 11{\cdot}420 - \frac{1}{\lambda}(0{\cdot}928m_1 + 1{\cdot}209m_2 + 7{\cdot}238m_3 \\ &\qquad + 1{\cdot}184m_4 + 0{\cdot}861m_5) \\ \Sigma P &= 428{\cdot}604 - \frac{1}{\lambda}(46{\cdot}922m_1 + 45{\cdot}635m_2 \\ &\qquad + 273{\cdot}446m_3 + 51{\cdot}181m_4 + 11{\cdot}420m_5) \end{aligned}\right\} \quad . \quad (66)$$

Loading Schedule.

From the weekly return of fuel costs at steam stations contained in Form CEA/G.S.-15 of the Central Electricity Authority, the following incremental costs were estimated, and they are used for illustrating the application of the above loading equations. These are assumed figures only, based on average fuel costs and existing orders of merit.

m_1	m_2	m_3	m_4	m_5
0·5	0·6	0·4	0·8	0·35 pence/kWh
2·08	2·50	1·67	3·33	1·46 £/MWh

Under these conditions the loading equations reduce to the following form:

$$\left.\begin{aligned} P_1 &= 46{\cdot}922 - \frac{93{\cdot}723}{\lambda} \\ P_2 &= 45{\cdot}635 - \frac{91{\cdot}500}{\lambda} \\ P_3 &= 273{\cdot}446 - \frac{545{\cdot}377}{\lambda} \\ P_4 &= 51{\cdot}181 - \frac{102{\cdot}606}{\lambda} \\ P_5 &= 11{\cdot}420 - \frac{22{\cdot}240}{\lambda} \\ \Sigma P &= 428{\cdot}604 - \frac{855{\cdot}446}{\lambda} \end{aligned}\right\} \quad . \quad . \quad . \quad (67)$$

The respective station loadings are now obtained by determining the value of λ for a summated generation (ΣP), and applying this same value of λ to each remaining equation.

For per-unit values of ΣP in excess of 4·316, corresponding to a value of λ equal to 2·0162£/MWh, it becomes necessary to base-load the available generation at G_4, say equal to the maximum per-unit value of 0·29.

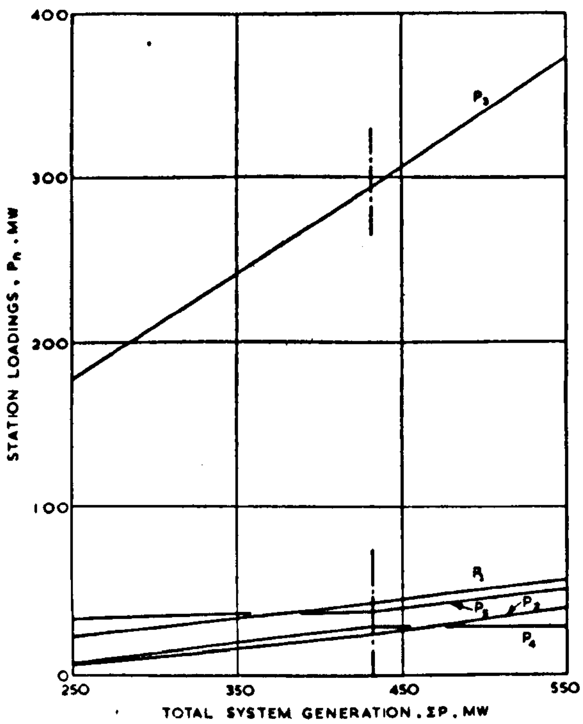


Fig. 11.—Generator loading schedule for optimum system economy.
G_4 base-loaded at $\Sigma P = 4 \cdot 316$.

(5.1.2) Co-ordination Equations with Generating Station G_4 (Crewe) Base-Loaded at its Maximum Available Generation of 29 MW.

With G_4 base-loaded at 29 MW, the co-ordination equations (65) can be reduced. Inversion of the B matrix will again give the solutions for generator loadings in the form

$$P = B^{-1}\left(C - \frac{m}{\lambda}\right)$$

Loading Schedule.

Using the previously assumed incremental fuel costs, and for values of ΣP in excess of $4 \cdot 316$, the loading equations now take the form

$$\left.\begin{aligned} P_1 &= 2\cdot1181 - \frac{3\cdot3918}{\lambda} \\ P_2 &= 2\cdot0178 - \frac{3\cdot5599}{\lambda} \\ P_3 &= 12\cdot7805 - \frac{19\cdot8330}{\lambda} \\ P_5 &= 1\cdot9493 - \frac{3\cdot1455}{\lambda} \\ \Sigma P \text{ (including } P_4 = 0\cdot29) &= 19\cdot1557 - \frac{29\cdot9302}{\lambda} \end{aligned}\right\} \quad . \quad (68)$$

A complete loading schedule obtained from these equations for system loads, ranging between 250 and 550 MW, is illustrated in Fig. 11.

The accuracy of the generator loading equations has been studied by checking the inversion of the B matrix, eqn. (65), from the product BB^{-1}. This was found to be of approximate unit matrix form, thus indicating that the inversion of B is correct.

An investigation has also been carried out to check the scaling of this matrix. From the study, which consisted of checking each step in the inversion of B, it is apparent that the expression $(B_{44}B_{55} - B_{45}{}^2)$ which appears throughout the inversion is having a powerful effect on the expressions for generator loadings, and in particular on the term $174 \cdot 902 m_3$ in eqn. (66).

$$\left.\begin{aligned} \text{Now} \qquad B_{44} &= \frac{R_{4-4}}{|V_4|_0^2}(1 + \Lambda_4^2) \\ \text{and} \qquad B_{55} &= \frac{R_{5-5}}{|V_5|_0^2}(1 + \Lambda_5^2) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (69)$$

Also, from the Z_{33} matrix, eqn. (54),

$$\left.\begin{aligned} R_{4-4} &= Z_{4.4} - 2d_4 + w \\ \text{and} \qquad R_{5-5} &= w \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (70)$$

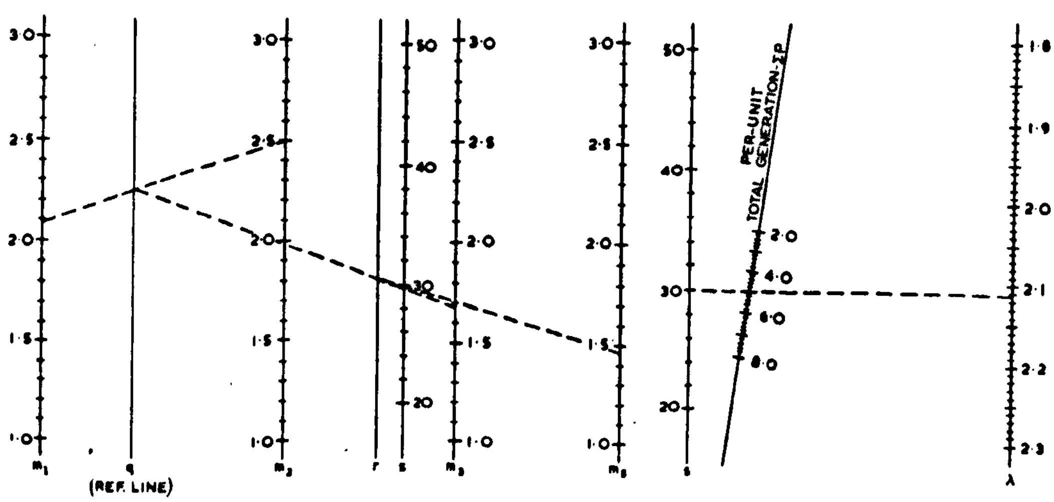


Fig. 12.—Nomogram for total generation $\sum_{1}^{5} P$.

Examples.

Left-hand Nomogram.

$m_1 = 2 \cdot 08$ £/MWh to $m_2 = 2 \cdot 50$ £/MWh gives q.
q to $m_3 = 1 \cdot 67$ £/MWh gives r.
r to $m_5 = 1 \cdot 46$ £/MWh gives s.

Right-hand Nomogram.

$s = 29 \cdot 8$ £/MWh to required total generation of 500 MW gives λ.

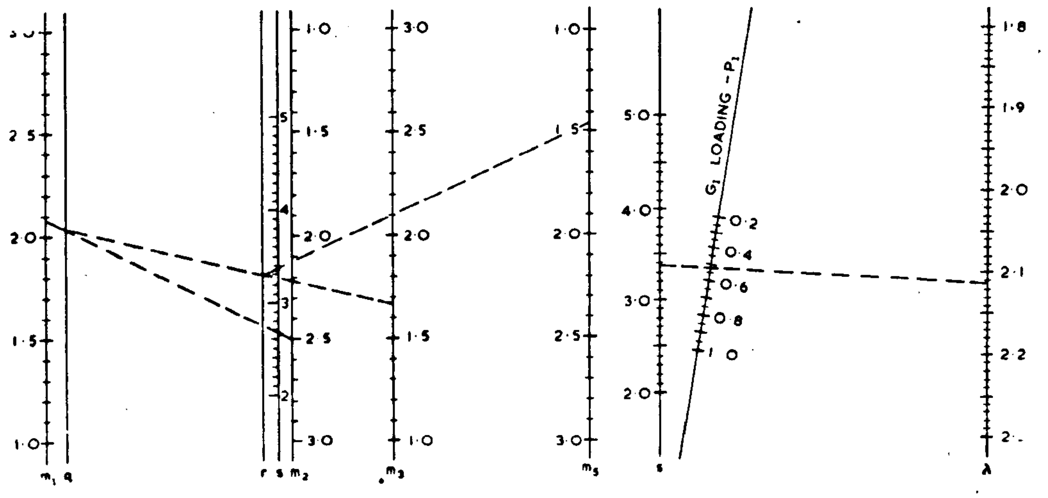


Fig. 13.—Nomogram for generator G_1 of loading P_1.

Example.
As in Fig. 12 to give *new* value of *s*.
$s = 3\cdot36$ £/MWh to $\lambda = 2\cdot113$ £/MWh (from Fig. 12) gives $P_1 = 52$ MW.

Now $$Z_{4.4} \simeq 2d_4 \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

Therefore the value of w appears to be influencing the components of B^{-1}.

From a study of the components of w ($=0\cdot014\,558$), it is found that the term $Z_{7.7}l_3l_3^* = Z_{3.3}l_3l_3^*$ ($=0\cdot009\,056$) is excessive in relation to all other quantities.

It can thus be concluded that the generator self-impedance $Z_{3.3}$ (with unit current injected at generator G_3), the loading ratio l_3 (i.e. load at Ince as a proportion of total load) and the ratio Λ_4 of reactive to active power for generator G_4 (Crewe), at normal load, are having powerful effects on individual generator loadings.

This investigation illustrates the fact that, in practice, considerable accuracy in computation will be required when the magnitudes of the system parameters and load ratios have wide differences.

(5.2) Loading Schedule Nomograms

The form of the final co-ordination equations enables a set of nomograms to be constructed to represent the various generator loadings for optimum operating efficiency.

The methods given in Reference 34 have been used for the five loading equations, including the one for total generation, and give rise to five nomograms, two of which are illustrated in Figs. 12 and 13.

Each diagram represents the respective loading equation and each variable is represented by a graduated line. Index lines drawn across the diagrams according to the values of the variables, in this case the station costs, will give a direct solution of the equations they represent. The various generator loadings are obtained by the use of the respective nomograms, using the known m values and the values of λ found from the total generation nomogram of Fig. 12.

The example given on the nomograms illustrates a loading schedule for a total system generation of 500 MW, and the various generator loadings compare with those obtained by calculation from the loading equations.

(6) CONCLUSION

In this present analysis, fuel costs and losses have been co-ordinated using the exact equations in conjunction with linear input-output characteristics for each station. With this assumption, the solutions obtained from one matrix inversion are valid for all variations in fuel costs, and the linear loading equations may be adapted for nomographic representation as illustrated in the paper. These charts will provide a method of developing rapidly the economic loading schedules as functions of total generation.

If the incremental fuel costs are assumed as functions of the station loadings, then each variation in station costs will necessitate a matrix inversion. In this case the rapid calculation of incremental transmission losses combined with generating costs, and the immediate application of these results to the system, will require the use of simplified network analysers or digital computers for the solution of the linear equations.

Comparing the economic loading schedule of Fig. 11 with the loadings obtained from the load flow study, it is apparent that for greater operating economy the generation at Percival Lane, G_2, and to a lesser extent that at Warrington, G_1, must be reduced and transferred to the more economic sources at Ince, G_3, and the Carrington interconnection, G_5—provided that adequate tie-line capacity is available.

A schedule for minimum loss can be obtained from the foregoing analysis, and if compared with that obtained by considering the effects of fuel costs, it becomes apparent that the value of the low-cost import at G_5 is much reduced due to the greater effects of line losses. Station loadings at G_1 and G_2 under these conditions are greater than the corresponding loadings obtained when considering fuel costs. Plant availability, provision for security of supplies and relative generation costs usually prevent the loading of stations on a minimum-transmission-loss basis. Such a condition, however, is of value in the planning of transmission systems and in comparing delivered costs from stations having equal incremental generation costs.

This analysis has been restricted to the determination of loading schedules for a small section of the North West, Merseyside and North Wales divisional network. The effects of incremental transmission losses on the selective loading of generating plant will become more apparent when the analysis is applied under light load conditions on this particular network, such as those obtaining at night and during the summer months, and when applied under more general conditions to larger sections of the Grid system.

(7) ACKNOWLEDGMENTS

The authors wish to express their thanks to Professor J. M. Meek, D.Eng., of Liverpool University, for his support and for the interest he has shown in the work. They are grateful to the Divisional Controller of the North West, Merseyside and North Wales Division of the Central Electricity Authority, for providing and giving permission to use technical data and for helpful discussions held with members of his staff.

One of the authors (H. N.) wishes to thank Lever Brothers, Port Sunlight, Ltd., for granting him permission to undertake these investigations.

(8) BIBLIOGRAPHY

(1) STEINBERG, M. S., and SMITH, T. H.: 'Economy Loading of Power Plants and Electric Systems' (John Wiley, New York, 1943).

(2) GEORGE, E. E.: 'Intrasystem Transmission Losses', *Transactions of the American I.E.E.*, 1943, **62**, p. 153.

(3) GEORGE, E. E., PAGE, H. W., and WARD, J. B.: 'Co-ordination of Fuel Cost and Transmission Loss by Use of the Network Analyser to Determine Plant Loading Schedules', *ibid.*, 1949, **68**, Part II, p. 1152.

(4) WARD, J. B., EATON, J. R., and HALE, H. W.: 'Total and Incremental Losses in Power Transmission Networks', *ibid.*, 1950, **69**, Part I, p. 626.

(5) KRON, G.: 'Tensorial Analysis of Integrated Transmission Systems, Part I—The Six Basic Reference Frames', *ibid.*, 1951, **70**, Part II, p. 1239.

(6) KIRCHMAYER, L. K., and STAGG, G. W.: 'Analysis of Total and Incremental Losses in Transmission Systems', *ibid.*, 1951, **70**, Part II, p. 1197.

(7) KIRCHMAYER, L. K., and MCDANIEL, G. H.: 'Transmission Losses and Economic Loading of Power Systems', *G.E.C. Review*, Oct., 1951, p. 39.

(8) KIRCHMAYER, L. K., and STAGG, G. W.: 'Evaluation of Methods of Co-ordinating Incremental Fuel Costs and Incremental Transmission Losses', *Transactions of the American I.E.E.*, 1952, **71**, Part III, p. 513.

(9) KRON, G.: 'Tensorial Analysis of Integrated Transmission Systems, Part II—Off-Nominal Turn Ratios', *ibid.*, 1952, **71**, Part III, p. 505.

(10) KRON, G.: 'Tensorial Analysis of Integrated Transmission Systems, Part III—The Primitive Division', *ibid.*, 1952, **71**, Part III, p. 814.

(11) GLIMN, A. F., KIRCHMAYER, L. K., and STAGG, G. W.: 'Analysis of Losses in Interconnected Systems', *ibid.*, 1952, **71**, Part III, p. 796.

(12) GLIMN, A. F., HABERMANN, R., KIRCHMAYER, L. K., and STAGG, G. W.: 'Loss Formulae Made Easy', *ibid.*, 1953, **72**, Part III, p. 730.

(13) KRON, G.: 'Tensorial Analysis of Integrated Transmission Systems, Part IV—The Interconnection of Transmission Systems', *ibid.*, 1953, **72**, Part III, p. 827.

(14) KIRCHMAYER, L. K., GLIMN, A. F., and STAGG, G. W.: 'Analysis of Losses in Loop-Interconnected Systems', *ibid.*, 1953, **72**, Part III, p. 944.

(15) BROWNLEE, W. R.: 'Co-ordination of Incremental Fuel Costs and Incremental Transmission Losses by Functions of Voltage Phase Angles', *ibid.*, 1954, **73**, Part III, p. 65.

(16) IMBURGIA, C. A., KIRCHMAYER, L. K., and STAGG, G. W.: 'A Transmission-Loss Penalty Factor Computer', *ibid.*, 1954, **73**, Part III, p. 567.

(17) HARKER, D. C., JACOBS, W. E., FERGUSON, R. W., and HARDER, E. L.: 'Loss Evaluation, Part I—Losses Associated with Sale Power—In-Phase Method', *ibid.*, 1954, **73**, Part III, p. 709.

(18) HARDER, E. L., FERGUSON, R. W., JACOBS, W. E., and HARKER, D. C.: 'Loss Evaluation, Part II—Current—and Power—Form Loss Formulae', *ibid.*, 1954, **73**, Part III, p. 716.

(19) TRAVERS, R. H., HARKER, D. C., LONG, R. W., and HARDER, E. L.: 'Loss Evaluation, Part III—Economic Dispatch Studies of Steam-Electric Generating Systems', *ibid.*, 1954, **73**, Part III, p. 1091.

(20) HARDER, E. L.: 'Economic Load Dispatching', *Westinghouse Engineering*, 1954, **6**, p. 194.

(21) GEORGE, E. E., and PIERCE, R. E.: 'Economics of Long-Distance Energy Transmission', *Transactions of the American I.E.E.*, 1948, **67**, Part III, p. 195.

(22) GEORGE, E. E.: 'Principles of Load Allocation Among Generating Units', *ibid.*, 1953, **72**, Part III, p. 49.

(23) WARD, J. B.: 'Economy Loading Simplified', *ibid.*, 1953, **73**, Part III, p. 1306.

(24) HALE, H. W.: 'Power Losses in Interconnected Transmission Networks', *ibid.*, 1952, **71**, Part III, p. 993.

(25) KRON, G.: 'Tensor Analysis of Networks' (John Wiley, New York, 1939).

(26) ZABORSZKY, J., and RITTENHOUSE, J. W.: 'Electric Power Transmission' (The Ronald Press Company, New York, 1954), p. 581.

(27) COOPER, A. R.: 'Load Dispatching and the Reasons for it, with Special Reference to the British Grid System', *Journal I.E.E.*, 1948, **95**, Part II, p. 713.

(28) OLDROYD, G.: 'Economy Loading of Generating Plant', *Electrical Review*, 9th December, 1949, p. 1095.

(29) DON, N.: 'Economical Loading', *ibid.*, 19th January, 1951, p. 112.

(30) TOMBS, F. L.: 'Economic Loading', *ibid.*, 23rd March, 1951, p. 581.

(31) PARSONS, L. J., and MARTENS, C.: 'Computer Matches Incremental Rates', *Power*, August, 1950, p. 114.

(32) COURANT, R.: 'Differential and Integral Calculus' (Interscience Publishers, New York, 1936), Vol. II, p. 188–211.

(33) LACOPO, M. J.: 'Interconnected System Energy Accounting Procedure and Related Operating Practices', *Transactions of the American I.E.E.*, 1953, **72**, Part III, p. 1196.

(34) ALLCOCK, H. J., and JONES, J. R.: 'The Nomogram' (Pitman, 1941).

(35) SEREBRENNIKOV, V. N.: 'Determination of Data for the Load Dispatcher's Selection of the Order of Loading Turbogenerator Sets', *Elektricheskie Stantsii*, 1954, **12**, p. 6.

(36) BILLAM, P. M.: 'Some Impressions of Electric Utility Practices in the U.S.A., *I.E.E. Student's Quarterly Journal*, 1954, **25**, p. 15.

Printed in Great Britain by
UNWIN BROTHERS LIMITED, WOKING AND LONDON