DIFFERENTIAL STRUCTURES IN ELECTROMAGNETIC FIELD AND CIRCUIT THEORY

by

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LIST OF SYMBOLS AND UNITS

(M.K.S. system of units is used throughout)

A	Vector potential (weber/metre)
"	Inverse transformation matrix
B	Magnetic flux density (weber/sq. metre)
C	Transformation matrix
c	Velocity of light (metre/second)
D	Electric flux density (coulomb/sq. metre)
d	Ordinary differential
$\bar{d}$	Exterior differential
E	Electric force (volt/metre)
e	Generalised skewsymmetric matrix whose components are ± 1
$\mathcal{E}$	Induced emf in the coil of an electric machine
F	First field tensor in four dimensions
G	Generated voltage and torque matrix of electric machine (henry or per-unit)
g	Metric tensor
H	Magnetic force (amp/metre)
"	Second field tensor in four dimensions

h	Ratio of hunting frequency to supply frequency
$\mathcal{H}$	MMF in field model of electric machine (amp)
I	Impressed node-pair current (amp)
i	Closed-mesh current (amp)
"	Current through electric machine windings (amp or per-unit)
$\mathcal{J}$	Linear current density in field model (amp/metre)
J	Current density (amp/sq. metre)
j	$\sqrt{-1}$
L	Inductance of electric machine windings (henry or per-unit)
l	Electric machine rotor length
M	Incidence matrix
"	Electric machine moment of inertia (Kg - m^2 or per-unit)
N	Number of turns of electric machine windings
P	Poynting's energy-flow vector (watt/sq. metre)
R	Resistance of electric machine windings (ohm or per-unit)
R_s	Frictional resistance of machine (Newton metre/second or per-unit)
r	Polar coordinate
"	Radius of electric machine rotor (metre)

S	Torsion tensor
s	Signature of metric
T	Stress-energy tensor
"	Shaft torque of electric machine (Newton metre or per-unit)
T_s	Synchronising torque coefficient
T_{de}	Electric damping torque coefficient
T_{dm}	Mechanical damping torque coefficient
t	Time (second)
ŧ	Co-time (= ct, metre)
V	Node-pair or open-path voltage (volt)
v	Impressed branch and mesh voltages (volt)
x, y, z	Coordinate axes
Y, y	Admittance
Z, z	Impedance

Greek letters

Γ	Affine connection, a function of metric tensor and non-holonomic object Ω
Δ	Small changes
δ	Covariant differential
$\bar{\delta}$	Divergence in exterior differential form $(= \pm * \bar{d} *)$
ε	Absolute tensor form of skewsymmetric e
ε_o	8.854×10^{-12} (coulomb/volt metre)
η_o	12.566×10^{-7} (weber/amp metre)
Θ	Rotation of electric machine rotor (radian)
θ	Polar coordinate
κ	Permittivity
λ	Wave length (metre)
"	Angular displacement of Kron's axes (radian)
μ	Permeability
π	3.1416
ρ	Charge density (coulomb/cu. metre)
"	Rotation tensor
σ	Conductivity (mho metre)
ϕ	Flux linkage of electric machine windings (weber or per-unit)

Ψ	Electric scalar potential (volt)
Ω	Magnetic scalar potential (amp)
"	Non-holonomic object, a function of transformation matrix C
ω	2π x rated frequency

Suffixes

t	transpose of
o	open-path
c	closed-path

Electric machine

ds, dr	Direct axis (field and armature)
qs, qr	Quadrature axis (field and armature)
s	Mechanical axis

Other symbols

$[\ ,]$ $\{\ \}$	Christoffel symbols, a function of metric tensor
*	Hodge (star) operator or dual
((,))	Inner product of exterior forms
∇	Gradient

CHAPTER 1

INTRODUCTION

With the advent of fast digital computers, the mathematical methods used for modelling, assessing and controlling large complex engineering systems are being strained to the utmost. For example, space and aircraft research, nuclear engineering and power systems now use advanced computer techniques for their design, construction and operation. Areas of analysis which were confined mainly to the realm of pure mathemeticians only two decades ago are now being utilised by engineers. Probability spaces, eigenvectors and state-spaces are some of the many concepts employed in such analyses. Multivariable engineering problems are now often expressed, mathematically, in terms of abstract spaces and subspaces. In terms of these, many of the physical variables, which exist in associated pairs, form orthogonal sub-spaces with p- and (n-p) dimensions, where n is the total number of variables. Examples of such quantities are voltage and current, electricity and magnetism, open-paths and closed-paths and matter and energy.

Much of the work done in this direction appears to be converging towards the matrix and tensor methods developed by Kron, during three decades, for large electric networks(1) and electrodynamic systems(2). Kron used electric network models for the study of a variety of systems of linear partial differential equations. Realizable physical networks were set up for nuclear reactor, turbine, elastic structure, transportation and other problems. An extension of this network to abstract spaces was his next step.

In this more general model, the previous electric currents and voltages were replaced by electromagnetic waves propagating across the structure. At this stage Kron found that magnetohydrodynamic waves could be theoretically propagated on two mutually orthogonal networks. The resulting wave network was found to have surprising adaptive characteristics with feedbacks and thresholds. So much so that Kron calls it a dynamo-type automaton(3). He has used this simulator to solve numerical problems in curve-fitting and is convinced that it has potentialities to recognise patterns, perform adaptive filtering, and in general to undertake multi-dimensional non-linear information processing. The number of adaptive parameters can be increased along several directions without any apparent limit. The prospects for such a device (purely a theoretical automaton, programmed on a computer) as seen by Kron are outlined in the following extracts from his own publications. Reference 4 (1958): "Such introduction (of Topology in the Calculus of Finite Differences) is accomplished by constructing, in a symbolic manner, magnetic and dielectric networks that have a sequence of one-, two-, three-, to k-dimensional network elements (instead of linear elements only).

In this more general model the previous electrical currents and voltages are replaced by electromagnetic waves propagating across the structure. 'The waves are always accompanied by free and bound electric charges as well as by magnetic poles. The sequence of alternating magnetic and dielectric networks may be called a "Wave-Model." The underlying geometrical configuration of 1-, to k-dimensional volume elements upon which the transverse and longitudinal electromagnetic waves are superimposed, is called a "polyhedron." A collection of two or more wave models (polyhedra) interconnected or inductively coupled will be called a Multi Dimensional Space Filter. In each 2-dimensional volume element not one value of a function (a scalar) is defined, but an entire field. For instance, with each scalar a probability distribution may be associated. The electric charges and magnetic poles may be associated with mass-particles and the latter endowed with elastic, hydrodynamic etc. properties. The author will not be surprised at all if eventually the entire paraphernalia of automic and nuclear physics will enter as modelling material to aid the mathematical unfolding of complicated problems.

As the electromagnetic waves propagate from a lower to a higher dimensional space, at the boundaries the waves must satisfy the generalised Stokes' theorems. An analogue of the steps to be taken at the multidimensional boundaries is given by a special linear electric circuit model, which the author developed fifteen years ago for the propagation of electromagnetic waves in the three dimensional physical space. The model already implies, in the linear network itself, the presence of two dimensional planes also, as indicated by the appearance of ideal transformers. The dielectric lines and magnetic planes of the Maxwell circuit form the first complete sequence in the k-dimensional space filter under study."

Reference 5 (1958): "It should be noted that during the simple wave propagation assumed for interpolation, numerical differentiation and integration etc. purposes, no feed-back was assumed to exist among the waves. As soon as more involved statistical or physical problems are investigated (such as curve-fitting, or smoothing), the waves in the various spaces begin to react upon each other. The author and his associates anticipate the appearance of linear and non-linear equations, boundary value and eigen value problems, as well as optimizations of various types, in which all spaces influence the answer to various degrees. In the solution of such extensive problems the method of tearing (Reference 3) promises to become a useful tool."

Reference 6 (1959): " The most obvious and simplest application of a polyhedran, or rather, a wave model, is to generalise the various formulas of the calculus of finite differences for non-uniform intervals assumed in a k-dimensional Euclidean space. This paper restricts itself to generalising the concept of divided differences of various orders. They are expected to be used for interpolation, curve fitting, smoothing, numerical differentiation and integration, and for other problems with which the calculus of finite differences deals.

It is anticipated that advanced problems in numerical analysis (Fourier transform, power density spectra), in operations research (quadratic and higher programming), in e conomics etc., will also offer opportunities for the use of space filters. The study of diffraction and scattering of electromagnetic waves in asymmetrical crystal and molecular structures would only be a first step into an ever widening field of physical application for multidimensional space filter concepts.

In order to use electric circuit theories for the analysis and solution of non-electrical problems (or for more advanced electrical problems) the author has developed, in references 1 and 2, a more general orthogonal theory that includes the mesh and junction-pair theories as special cases.

The same orthogonal point of view will now be applied to each multi-dimensional q-network separately; with some modification. The main change is that the networks will conduct not electric currents, but electromagnetic waves (and charges). In particular: 1. all odd dimensional spaces (1, 3, 5...) conduct two types of dielectric quantities: (a) solenoidal $\dot{d}^{\alpha}$, e_{α} and (b) lamellar $\dot{D}^{\alpha}$, E_{α}; and 2. all even dimensional spaces (0, 2, 4...) conduct two types of magnetic quantities (a) solenoidal h^{α}, $\dot{b}_{\alpha}$ and (b) lamellar H^{α}, $\dot{B}_{\alpha}$."

Reference 7 (1962): "The driving force behind the search for a new type of network organisation, not found in the literature, was the conviction that electricity demands its own highly specialised underlying topological structure. The electrical engineer must pick out from the available variety of topological structures, only those particular ones that actually fit the properties of the superimposed electric currents, or electromagnetic waves, and avoid all others.

With every n-dimensional polyhedron, there is associated a dual polyhedron whose p-simplexes, in general, are at right angles in space to the (n-p) simplexes of the primal polyhedron. It so happens that both polyhedra must be present to propagate an electromagnetic wave, as well as a magnetohydrodynamic wave. Thus, the " complete" 1-network, whose theory is ready to be generalised, does not consist of an isolated 1-network but of a 1-network with an environment ... The concept of environment implies also the immersion of a 1-network in a magnetic field, or in a plasma, with moving electric and magnetic charges and currents, etc.

In the 2-phase model of a polyhedron, the eight electromagnetic parameters appear, not in their vectorial form shown above, but in the tensorial form of four parameters $F_{\alpha\beta}$, $H^{\alpha\beta}$, s_{α} and s^{α}. Thus the three notions of (1) networks, (2) electricity, and (3) tensors are interwined into one structure, not only in conventional electric 1-networks, but also in multidimensional electromagnetic ones.

The electrical engineer will find an acquaintance with some of the topics treated in recently developed differential topology quite helpful in his eventual attempt to enlarge the practical use of solid-state network elements, or their theoretical use for model construction, from 1-dimensional electric to multidimensional electromagnetic networks. Mathematical texts with such titles as "Harmonic Integrals," "Fiber Bundles", "Grassman Algebra", etc. must be consulted for further study."

Reference 8 (1963): "Only a few years ago the author accidentally discovered that during those same three decades, while he was struggling to "invent" and use tensors in-the-large for the study of discrete global electrical 1-networks, a group of theoretical mathematicians did develop a systematic theory of "tensors in-the-large", in connection with the study of multiply connected multidimensional space structures. These disciplines are beginning to be grouped under the heading of "differential topology". It is also known as "differential geometry and topology in-the-large."

In addition to introducing tensors in-the-large, Cartan, De Rham, Hodge, Whitney, Eilenberg, Steenrod, Chern, Lichnerowicz, Chevally, Kodeira, and a host of other theoretical mathematicians have also superimposed upon the polyhedra - and other more general complexes - sets of "exterior" differential p-forms and their integrals. These integrals are expected to satisfy Stokes' theorem between two different dimensional p-networks. So are the electromagnetic and magnetohydrodynamic waves used by the author. He also employs in his polyhedral networks the incidence matrices to satisfy Stokes' theorem. (The connection matrices $C^{\alpha}_{\alpha'}$, serve to satisfy Kirchoff's laws).

The utilisation of "Atomic and Molecular Vibrations" as Electrodynamic Engineering:- The irregularly placed atoms in a polyatomic molecule may be considered to form the vertices of a polyhedron. When the atoms are excited by X-rays, the resulting dipole waves form a self sustained dynamical system. The self organising polyhedral waves used in multidimensional curve fitting are surprisingly complete analogues of the self sustaining, resonant crystal waves.

Thus the vibrating atoms in a polyatomic molecule and the various valence bonds between them may be looked upon as extremely complex two-phase, unbalanced magnetohydrodynamic generators, all connected together with a sequence of multi-dimensional transmission lines. Each p-simplex - defined by p+1 atoms - together with its dual n-p simplex, becomes an n-dimensional two phase "generalised" rotating electrical machine or transmission line. Both ray-optics and wave-optics may thus be generalised with their aid for the refraction and diffraction studies of polarized light waves propagating between distinct atoms through nonhomogeneous nonisotropic media. The overall molecular translational and rotational motions will also be eventually included in the study, which can assume either deterministic or probablistic interpretations.

It is within the bounds of possibility that the polyhedral waves under discussion will eventually be physically realised by means of crystals. The eight electrical parameters of each full wave in the sequence of waves can be increased by additional mechanical, fluid, thermodynamic, and other "adaptive" parameters that actually exist in a crystal. Since the concepts of "open" and "closed" (yes and no) permeates the organisation of the underlying and superimposed topological structures of both straight and curved polyhedra and their waves, some multi valued logic, to be yet developed, will eventually make it feasible to use crystals as analogue and digital computers. The computations will be based on the utilisation of as many parameters of the polyhedral waves existing in a crystal as possible.

It is well known that the neural net in the human body is energised by electromagnetic and more advanced chemical (ionic etc.) waves, rather than by more simple electrical impulses. As a result, a crystal "artificial brain" might simulate more closely many cognitive processes and other neural phenomena, than the 1-dimensional switching networks can ever hope to do. A multidimensional statistical information theory built around the polyhedral network will also have to be developed."

Reference 3 (1960): "The stage for all cognitive activities is an underlying n-dimensional field of magnetohydrodynamic plasma. Two spatially orthogonal polyhedra are immersed in the fluid, in order to crystallise the amorphous non linear field into a host of interconnected n-dimensional magnetohydrodynamic generators. The latter are straightforward generalisations of conventional rotating electrical machinery, and form the elementary "neurons" of the projected "artificial brain". The polyhedra also serve as a set of 2n non-holonomic reference frames (transmission networks) for the propagation of a sequence of magnetohydrodynamic waves across the automata, as well as a locus of tearing the overall field apart into a hierarchy of smaller component fields. The automata may be analysed and solved piecewise either as a continuous field structure with distributed constants, or a discrete polyhedral structure with lumped constants, or some combination of both.

The purpose of a projected series of papers will be to recapitulate and throw new light upon the tensorial and topological theories developed by the author for stationary and moving linear (1-dimensional) animated networks. Such a reinterpretation should facilitate the understanding of the rather striking "oscillatory" behaviour of stationary or moving multidimensional networks that form the backbones of the automata.

From an electrical engineering point of view, the overall automat~~i~~on consists of 2n distinct and different p-dimensional transmission networks, (p varies from 1 to n), all properly staggered within an n-dimensional space. At various points along two complementary (a p- and an n-p dimensional) transmission networks, a large number of n-dimensional generalised rotating electrical machines are connected, each excited with combined electrostatic and electromagnetic, as well as with other types of energies. In each n-dimensional rotating machine the stationary and mutually orthogonal p-simplexes and (n-p) simplexes take over the role of the stationary and mutually orthogonal direct-axis and quadrature-axis brushes or reference axes, that exist in a conventional two-dimensional rotating electrical machine."

Reference 9 (1965): "To represent an n-dimensional region of space, usually n reference axes are used, each axis being a 1-dimensional line. These are the conventional x, y, z ... axes. However another way to represent a region of an n-dimensional space is to assume two mutually orthogonal reference axes instead of n. But these reference axes now are not 1-dimensional lines, but multi-dimensional hyper planes or hyper surfaces. For instance if one of the axes (hypersurface) has p-dimensions, the other axis has n-p dimensions, so that the sum of their dimensions is n.

If several regions of n-space are to be so represented, (each region possessing two hyperplanes) the several p-hyper planes form a p-network and the same number of (n-p) hyperplanes form an (n-p) network. Thus an n-dimensional continuous region of space in general may possess a "primal" p-network plus a "dual" (n-p) network as its abstract reference frame. Such a representation may be looked upon as a straightforward generalisation of the "two-phase" reference frame concept used in rotating electrical machine studies, or in stationary power network studies.

During the last thirty years a number of new mathematical disciplines, known collectively today as the "topology of differentiable manifolds", has grown up, that deals with differentiable functions and their integrals superimposed upon multiply-connected networks of spaces, in which each of the interconnected spaces possesses different dimensions. The simplest of such interconnected space-structure is called a "polyhedron."

Although these topological texts do not deal explicitly with electromagnetic or any other physical waves propagating over polyhedral networks, nevertheless their concepts are much closer to the needs of the electrical engines than the concepts of old type algebraic topology are. In fact it was found that it is absolutely necessary to express the tensorial field-equations of

Maxwell in terms of "exterior differential" d and δ formalism used by the topology of differentiable manifolds, in order to set up a more complete topological structure for conventional electric networks."

This thesis is an attempt to establish the basic steps on which Kron builds the multidimensional adaptive wave automaton. The present author has searched, in vain, for some supporting work along the lines of Kron's wave structure. One of the difficulties encountered in such a study is the wide range of disciplines involved. The subject matter used includes network diakoptics, rotating electrical machine theory, tensor calculus, exterior differential forms and topology. It was felt that the wide field of application claimed by Kron warrants an examination of the mathematical and physical basis of his recent work. Many details have not been disclosed by Kron himself, even where he has published numerical results. A generalised approach to differential problems, particularly in field analysis, does not appear to be common in engineering literature. For example, the field tensors $F_{\alpha\beta}$ and $H^{\alpha\beta}$ used in classical and relativistic electromagnetism are not extensively used as such by engineers.

The present thesis does not reach as far as Kron's automaton. However, it is felt that some of the initial steps have been elucidated, some obstacles removed, some pitfalls have been uncovered and some interesting relationships found. These relations point to the correct interpretation of Kron's description in certain places.

After much thought, consultation and discussion, it was decided that the project should be based on the study of structure relations in electromagnetic field analysis and circuit theory. "Structure relations" are considered here to mean that the equations are expressed in an organised manner, related to the structure of a quasi-physical model. As a wide range of disciplines and mutually interrelated concepts is involved in Kron's work and few guide lines or boundaries in any direction were apparent, it seemed that a detailed study in the greatest depth of each step in turn would lead to frustration. The best course appeared to be (1) to examine the structure as a whole; (2) to decide on certain underlying essentials; (3) to investigate these and their mutual interrelations; (4) to carry the work to a stage at which future development could continue smoothly and continuously on a sound basis.

The elements of the system are

(a) Orthogonal networks

(b) Exterior differential forms

(c) Electromagnetic field networks

(d) Electrodynamis of machines

Essentially this project involves the examination of differential vector-relations in general and as applied to electromagnetic fields and machines. Some of the differential relations in fields can be interpreted as the algebraic relations on a network which has infinitesimal meshes. The following chapters describe a search for the structure relations amongst the elements described above. In Chapter 2, the search begins with an investigation of the algebraic quantities on networks as a preliminary to the study of equivalent network models of continuous fields.

CHAPTER 2

ALGEBRAIC STRUCTURAL RELATIONS IN ELECTRIC CIRCUIT THEORY

This chapter deals with algebraic relations in circuit theory. The mesh and node-pair analysis of circuits are shown to be special cases of the more general orthogonal network analysis first developed by Kron(10). A more general form of connection matrix is obtained using closed and open mesh concepts. The connection matrix, C_c, used in mesh analysis and A_o, used in node-pair analysis for the same network always satisfy the relation: $(A_o)_t \cdot C_c = 0$; hence the name orthogonal. It is later shown that, in geometric sense, the mesh currents, i, and node-pair voltages, V, form subspaces of a more general space representing the variables in an orthogonal network.(11) The algebraic relations can be represented by a flow diagram called the algebraic diagram. Kirchhoff's voltage and current laws can also be represented on this. The connection matrices between branches and meshes are shown to be different in nature to the incidence relations between lines and planes on the network. Connection matrices show the relationships of the branches to the paths of currents and voltages. Incidence matrices on the other hand show the structural construction of the branches(12).

2-1 Mesh Analysis

In a mesh network, the impressed voltages are in series with the impedance elements. The branches so formed are arranged together to form closed meshes. No currents are impressed across node-pairs. In fig. la, "primitive" branches are shown. In general these can be interconnected to form a network such as that given in fig. lb. This has a more general form than the "mesh"network defined above. In the primitve branches (fig. la), two types of voltages, v and V, as well as two types of currents, i and I, appear. The voltage, v, represents the impressed voltage from a source (generator) connected in series with the impedance element. The current, I, is an impressed current injected across the branches (such as by a constant current generator). The response voltage, V, is the voltage drop across the branch due to the two forms of excitation. The current, i,

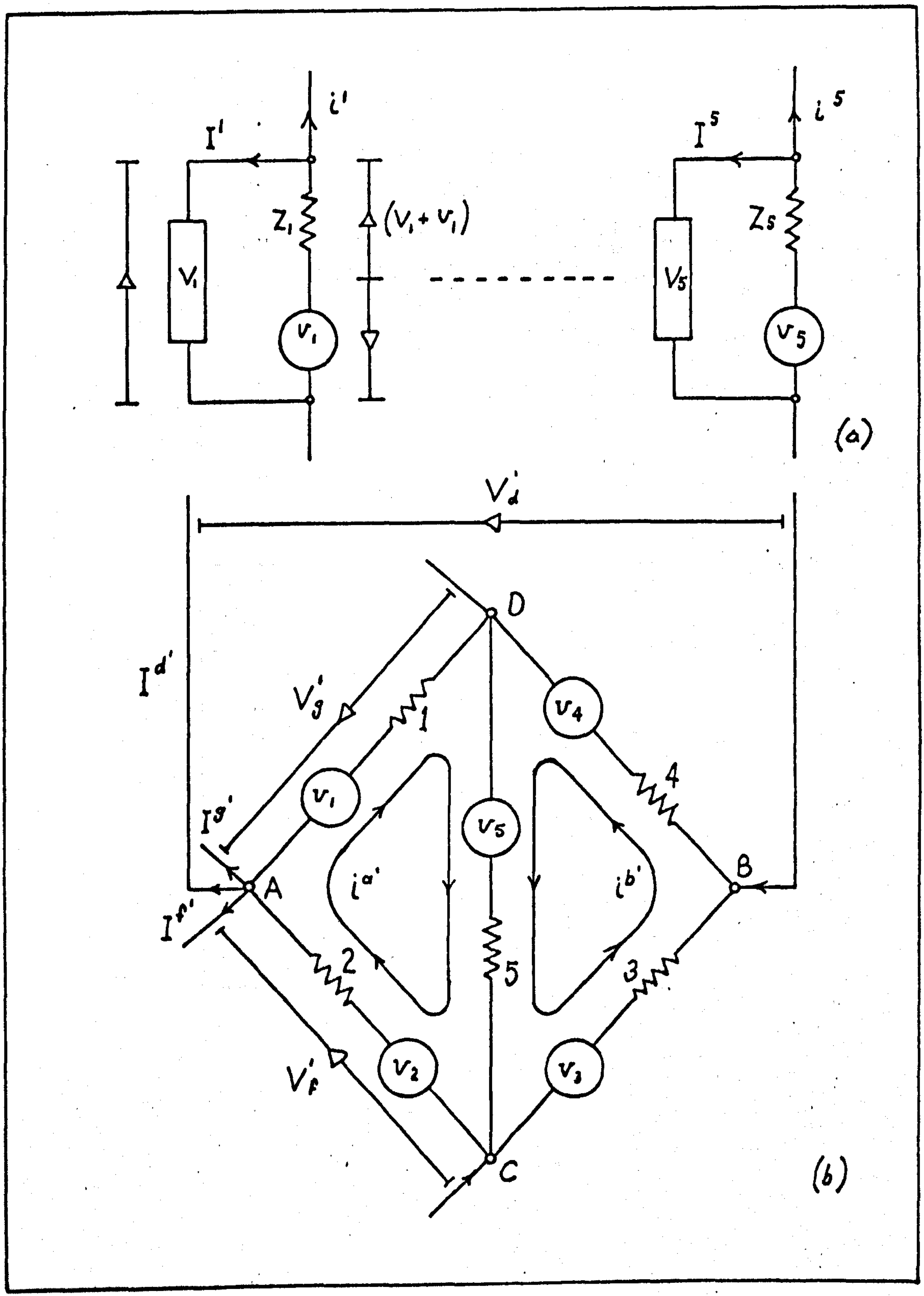


FIG 1 A NETWORK EXAMPLE

is similarly a response due to the two forms of excitation and flows through the closed meshes of the interconnected network.

No currents are impressed across the node-pairs in a "mesh" network. All the currents, I and I', are zero. The relation between branch and mesh currents in fig. 1. is given by:

$$\begin{array}{|c|} \hline i^1 \\ \hline i^2 \\ \hline i^3 \\ \hline i^4 \\ \hline i^5 \\ \hline \end{array} = \begin{array}{c|c|c|} & a' & b' \\ \hline 1 & 1 & \\ \hline 2 & 1 & \\ \hline 3 & & 1 \\ \hline 4 & & -1 \\ \hline 5 & 1 & 1 \\ \hline \end{array} \quad \begin{array}{|c|} \hline i^{a'} \\ \hline i^{b'} \\ \hline \end{array}$$

$$i = C_c \; . \; i^{c'} \qquad (2\text{-}1)$$

The primed quantities refer to the given network and the suffix c denotes closed mesh quantities. The notation is the one most recently employed by Kron$^{(1)}$. The connection matrix, C_c, is singular and cannot be inverted.

The mesh voltages are given by considering power invariance,

$$P = (v'_c + V'_c)_t \cdot i^{c'} = (v + V)_t \cdot i \tag{2-2}$$

Substitution of equation 2-1 in 2-2 yields:

$$(v'_c + V'_c)_t \cdot i^{c'} = (v + V)_t \cdot C_c \cdot i^{c'} \tag{2-3}$$

As this is true under all conditions and for all values of impedances,

$$(v'_c + V'_c)_t = (v + V)_t \cdot C_c$$

or

$$v'_c + V'_c = (C_c)_t \cdot (v + V) \tag{2-4}$$

In equation 2-4, V'_c is given by

$$(C_c)_t \cdot V$$

For the network of fig. 1b,

$$V'_a = V_1 + V_2 + V_5 = 0$$

$$V'_b = V_3 - V_4 + V_5 = 0$$

That is, V'_c sums up the voltage drops of the branches around closed meshes. It is zero by Kirchhoff's voltage law. Equation 2-4 now becomes:

$$v'_c = (C_c)_t \cdot v \qquad (2\text{-}5)$$

In Kron's original mesh analysis, the term, V, was ignored. The voltage drops in series with the impedance elements in the primitive network were shown as v. The currents in the primitive branches were equated to those in the corresponding elements in the given network. Hoffmann(13) defined additional voltages, u, added to the primitive network voltages, v, to constrain the currents to values equal to those in the corresponding elements of the given network. It is demonstrated there that these additional voltages cancel in the closed meshes of the given network - that is, $(C_c)_t . u=0$. Using the voltage V, shown in fig. la, the same result has been achieved here.

In the primitive network of fig. la, there is no constraint between the currents in the various elements.

With five primitive elements there are five degrees of freedom, as far as the currents are concerned. Hoffmann(11) has pointed out that the currents can be represented by the motion of a point in a 5-dimensional configuration space S, having the currents as coordinates. This path is known as the trajectory of the primitive network. For the given network (fig. 1b), there are three constraints as far as the currents, i, are concerned, thus losing three degrees of freedom. The trajectory now belongs to a 2-dimensional configuration space. Hoffmann calls this the subspace $\bar{S}$ of the given network. The mesh currents $i^{a'}$ and $i^{b'}$ can be represented by the motion of a point on this 2-dimensional surface. This path is known as the trajectory of the given network.

2-2- Node-Pair Analysis

In this analysis, currents, I, are impressed across the node-pairs. The impressed voltages, v, in series with the impedance elements are all zero. In the single-node concept of Maxwell, all injected currents entered (or departed) at several nodes and finally departed (or entered) at the ground node. But in the node-pair analysis, any pair of nodes can be used to impress the current. Open meshes can exist between any two nodes and currents can be assumed to flow along these. The suffix, o, will be used to describe the node-pair or open-mesh quantities. The relation between branch voltages, V, and the arbitrarily chosen (but independent) node-pair voltages, V'_o, of the network is:

$$V = A_o \cdot V'_o \qquad (2\text{-}6)$$

For the network of fig. 1b, the matrix relation 2-6 is:

$$\begin{array}{|c|} \hline V_1 \\ \hline V_2 \\ \hline V_3 \\ \hline V_4 \\ \hline V_5 \\ \hline \end{array} = \begin{array}{c|c|c|c|} & d' & f' & g' \\ \hline 1 & & & -1 \\ \hline 2 & & 1 & \\ \hline 3 & -1 & 1 & \\ \hline 4 & -1 & & 1 \\ \hline 5 & & -1 & 1 \\ \hline \end{array} \quad \begin{array}{|c|} \hline V'_d \\ \hline V'_f \\ \hline V'_g \\ \hline \end{array} \qquad (2\text{-}7)$$

The impressed currents, I'_o, for the network are given by considering power invariance,

$$P = (i^{o'} + I^{o'})_t \cdot V'_o = (i + I)_t \cdot V$$

$$= (i + I)_t \cdot A_o \cdot V'_o$$

or $$i^{o'} + I^{o'} = (A_o)_t \cdot (i + I) \qquad (2\text{-}8)$$

In equation 2-8, response mesh currents $i^{o'}$ are given by $(A_o)_t \cdot i$. For the network of fig. 1b, they are:

$$-i^3 - i^4 = 0$$

$$i^2 + i^3 - i^5 = 0$$

$$-i^1 + i^4 + i^5 = 0$$

Since currents, i, only circulate in the closed meshes, they add up to zero at the nodes by Kirchhoff's current law. Therefore, $i^{o'} = (A_o)_t \cdot i = 0$, and equation 2-8 now becomes:

$$I^{o'} = (A_o)_t \cdot I \qquad (2\text{-}9)$$

An interesting relation is established by mutliplying the connection matrix, C_c, by the transpose of A_o. For the network of fig. 1b,

$$
\begin{array}{c|c|c|c|c|c|}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
d' & & & -1 & -1 & \\
\hline
g' & & 1 & 1 & & -1 \\
\hline
f' & -1 & & & 1 & 1 \\
\hline
\end{array}
\quad
\begin{array}{|c|c|}
a' & b' \\
\hline
1 & \\
\hline
1 & \\
\hline
 & 1 \\
\hline
 & -1 \\
\hline
1 & 1 \\
\hline
\end{array}
\quad = \quad
\begin{array}{c|c|c|}
 & a' & b' \\
\hline
d' & 0 & 0 \\
\hline
g' & 0 & 0 \\
\hline
f' & 0 & 0 \\
\hline
\end{array}
$$

$$(A_{open})_t \cdot (C_{closed}) = 0 \qquad (2\text{-}10)$$

In section 2-1, it was mentioned that the given network (fig. 1b), has two degrees of freedom as far as closed-mesh currents are concerned. This was used to describe a 2-dimensional space $\bar{S}$. Three independent node-pair voltages can be taken for the same network and three independent open-path currents can be assumed. A 3-dimensional configuration space $\bar{\bar{S}}$ can be described using the three degrees of freedom of the open-path currents. In general if there are n branches and m closed meshes,

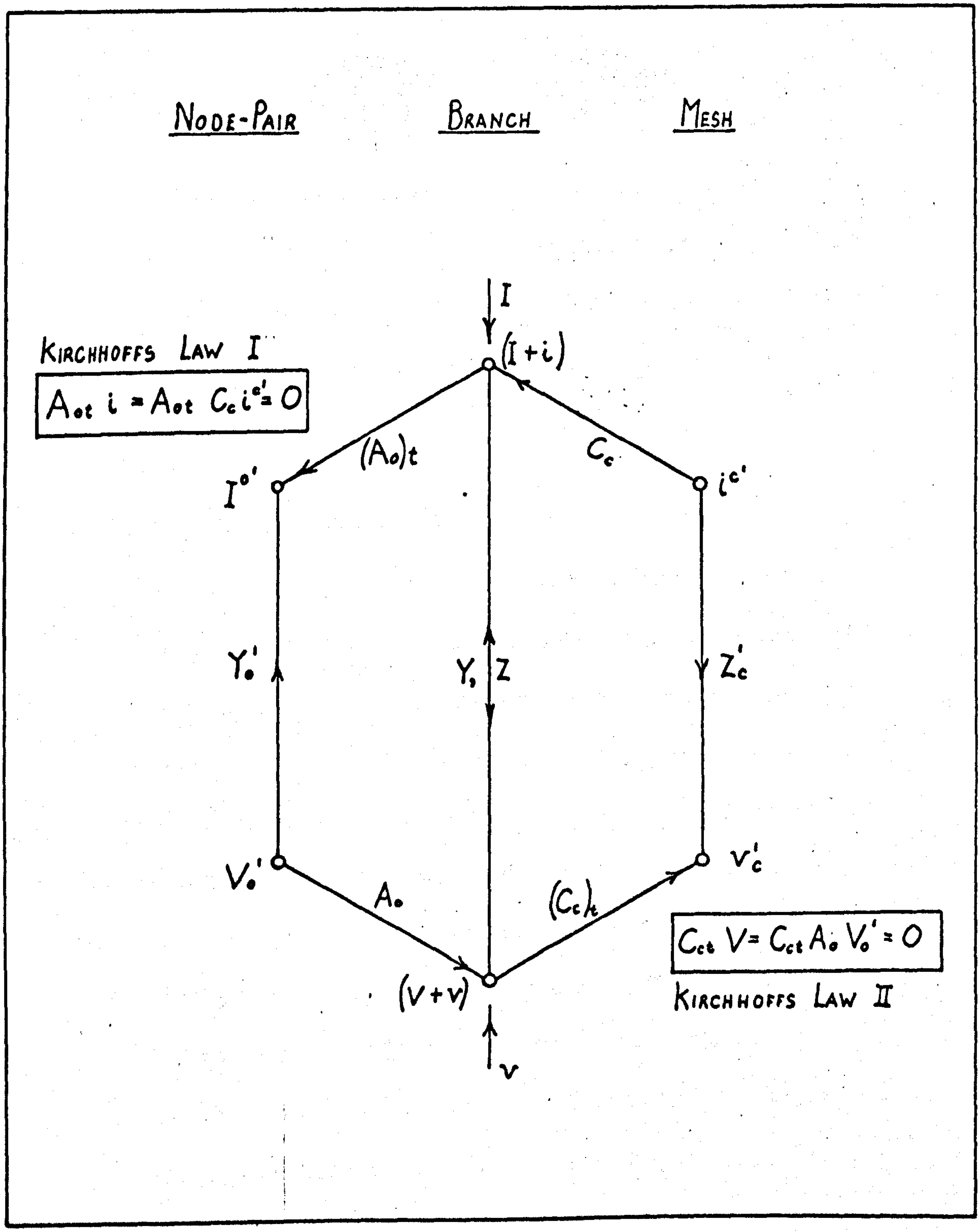


FIG. 2. ALGEBRAIC DIAGRAM FOR NETWOKS

there will be (n-m) node-pairs. The configuration space $\bar{S}$ will have m dimensions and $\bar{\bar{S}}$ will have (n-m) dimensions. The coordinates of the spaces are related to the primitive network quantities by C_c and A_o. They satisfy equation 2-10 and for this reason Kron suggests the name orthogonal subspace.

2-3 Algebraic Diagram

The algebraic structural relations expressed by equations 2-1, 2-5, 2-6, 2-9 and 2-10 can be summarised in the form of a flow diagram or algebraic diagram first suggested by Roth(14). The impressed quantities, I and v, are shown by the short vertical arrows. The long vertical lines give the impedances and admittances. An arrow connecting two dots gives a matrix relation. Each dot represents a row or a column matrix. Two or more arrows shunted by another give a matrix product relation. For example, in fig. 2, the arrow Z'_c connecting $i^{c'}$ and v'_c bridges the three arrows C_c, Z and $(C_c)_t$ giving:

$$Z'_c = (C_c)_t \cdot Z \cdot C_c \tag{2-11}$$

(The convention in taking products, is to go against the arrow).

The algebraic diagram also gives Kirchhoff's current and voltage laws as shown on fig. 2. The diagram, giving the algebraic structural relations of a network, will be developed further in a later section for field networks.

2-4 Orthogonal Network Analysis

Orthogonal networks are excited by a combination of mesh voltages and node-pair currents, and form the more general type in network analysis. They are solved by extending the concept of closed mesh to "open meshes"(10, 15). The open-mesh currents enter a node and after passing through one or more branches, depart at some other node. In the network of fig. 1b, a set of open-mesh currents $I^{d'}$, $I^{f'}$ and $I^{g'}$ can be assumed as follows:

$I^{d'}$: enters at B, flows through branches 3 and 2 and leaves at A

$I^{f'}$: enters at C, flows through 2 and leaves at A.

$I^{g'}$: enters at D, flows through 5 and 2, leaves at A

(The open-mesh currents need not all leave at a single node).

The relation between the five branch currents and the five mesh currents (two closed meshes, three open meshes) is now given by:

$$\begin{array}{|c|} \hline i^1 + I^1 \\ \hline i^2 + I^2 \\ \hline i^3 + I^3 \\ \hline i^4 + I^4 \\ \hline i^5 + I^5 \\ \hline \end{array} = \begin{array}{c|c|c|c|c|c|} & a' & b' & d' & f' & g' \\ \hline 1 & 1 & & & & \\ \hline 2 & 1 & & 1 & 1 & 1 \\ \hline 3 & & 1 & -1 & & \\ \hline 4 & & -1 & & & \\ \hline 5 & 1 & 1 & & & 1 \\ \hline \end{array} \quad \begin{array}{|c|} \hline i^{a'} \\ \hline i^{b'} \\ \hline I^{d'} \\ \hline I^{f'} \\ \hline I^{g'} \\ \hline \end{array}$$

(2-12)

In terms of partitioned sub-matrices,

$$\begin{array}{|c|}\hline \\ i + I \\ \\ \hline\end{array} = \begin{array}{|c|c|}\hline & \\ C_c & C_o \\ & \\ \hline\end{array} \begin{array}{|c|}\hline i^{c'} \\ \hline I^{o'} \\ \hline\end{array}$$

The connection matrix, C_o, gives the relation of the branches to the open meshes. Equation 2-12 contains the non-singular connection matrix C. This can be inverted and then used to express the given network currents in terms of the primitive network quantities. The open- and closed-mesh voltages are given, satisfying power invariance, by

$$\begin{array}{|c|}\hline (C_c)_t \\ \hline (C_o)_t \\ \hline\end{array} \begin{array}{|c|}\hline \\ v + V \\ \\ \hline\end{array} = \begin{array}{|c|}\hline (C_c)_t v + (C_c)_t V \\ \hline (C_o)_t v + (C_o)_t V \\ \hline\end{array}$$

$$= \begin{array}{|c|}\hline v'_c + V'_c \\ \hline v'_o + V'_o \\ \hline\end{array} \qquad (2\text{-}14)$$

The term $V'_c = (C_c)_t \cdot V$, sums up the voltage drops around closed meshes. It is zero by Kirchhoff's voltage law. Since the connection matrix, C, is non-singular, equation 2-14 can be rewritten as:

$$\boxed{v + V} = \boxed{C_t^{-1}} \quad \boxed{v' + V'} \qquad (2\text{-}15)$$

For the network of fig. 1b,

$$(C_t)^{-1} = \begin{array}{c|c|c|c|c|c|} & a' & b' & d' & f' & g' \\ \hline 1 & 1 & & & & -1 \\ \hline 2 & & & & 1 & \\ \hline 3 & & & -1 & 1 & \\ \hline 4 & & -1 & -1 & & 1 \\ \hline 5 & & & & -1 & 1 \\ \hline \end{array} \qquad (2\text{-}16)$$

Equation 2-15 is an extension of equation 2-6, $V = A_o V'_o$, used in nodal analysis of section 2-2. The partitioned matrix in equation 2-16, corresponding to open paths, can be recognised as matrix, A_o, in equation 2-7. Extending this concept to closed meshes as well,

$$A = (C_t)^{-1} = \left[\begin{array}{c|c} A_c & A_o \end{array}\right] \tag{2-17}$$

The impedance of the orthogonal network is given by the usual $C_t Z C$ transformation (reference 10). The unknowns in the orthogonal network analysis, usually, are V'_o and $i^{c'}$. In the analysis of mesh networks in section 2-1, $I^{o'}$ is zero. It is normally not required to calculate V'_o. In the analysis of nodal networks in section 2-2, v'_c is zero. It is not required to calculate $i^{c'}$ either. However, both V_o and i^c are present in all the networks.

2-5 Incidence Matrices

The incidence matrices connect points to lines, lines to planes, planes to cubes and so forth. For a given topological structure, there is only one set of incidence matrices. In fig. 3a, the points A, B, C and D are called 0-cells. They are said to be incident to the five 1-cells 1, 2, 3, 4 and 5. These in turn are incident to the 0-cells as well as the two triangular areas a and b (2- cells) they enclose. The circular arrows in fig. 3a give the orientation of the 2-cells. For 0-cells, an arrow entering a node has been taken as positive orientation. The incidence matrix M_2^1 relates the lines to planes. For fig. 3a,

$$M_2^1 = \begin{array}{c|c|c|} & a & b \\ \hline 1 & 1 & \\ \hline 2 & 1 & \\ \hline 3 & & -1 \\ \hline 4 & & 1 \\ \hline 5 & 1 & -1 \\ \hline \end{array} \tag{2-18}$$

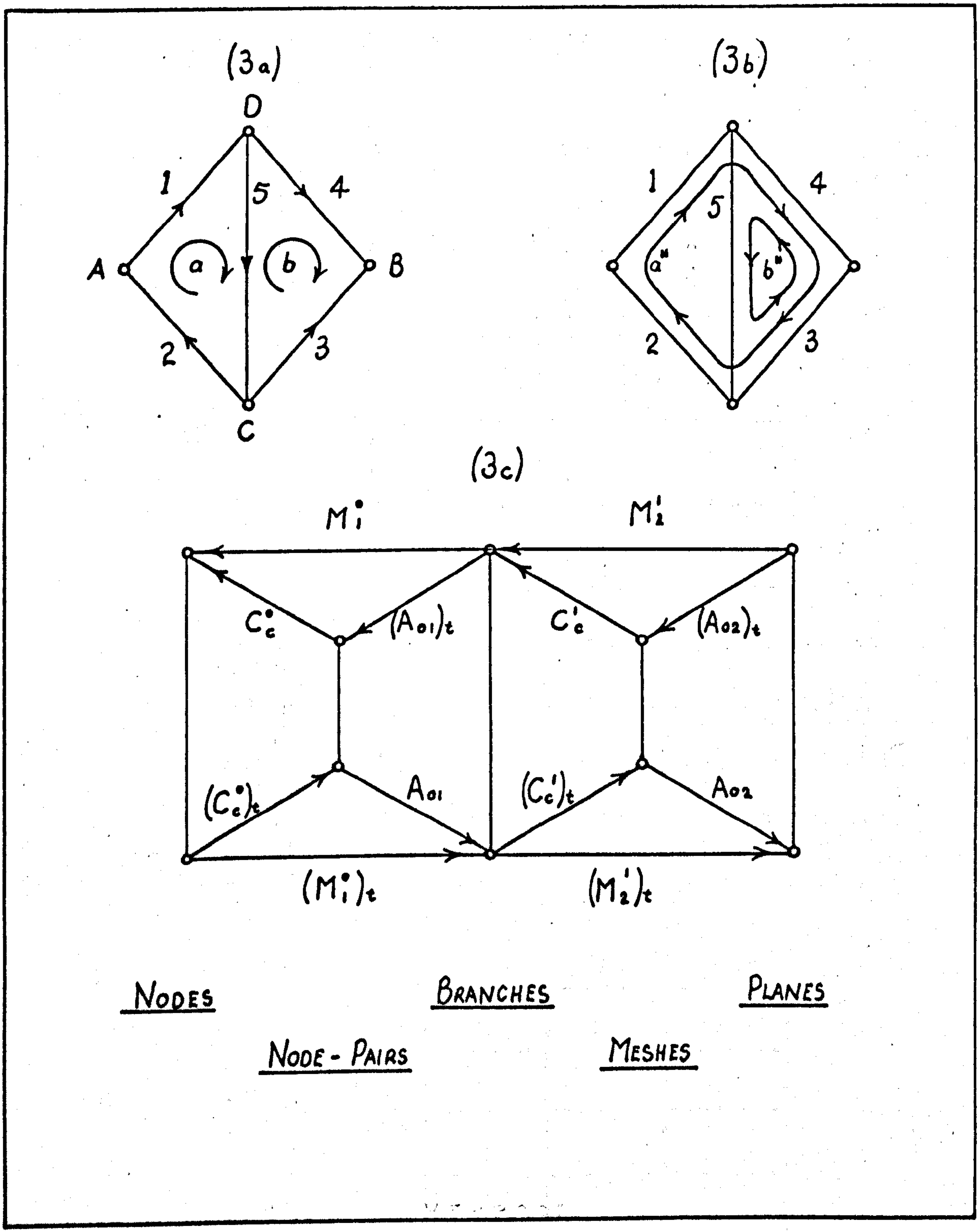


FIG. 3. EXTENDED ALGEBRAIC DIAGRAM

The incidence matrix M_1^o relates points to lines. For fig. 3a,

$$M_1^o = \begin{array}{c|c|c|c|c|c|}
 & 1 & 2 & 3 & 4 & 5 \\ \cline{2-6}
A & -1 & 1 & & & \\ \cline{2-6}
B & & & 1 & 1 & \\ \cline{2-6}
C & & -1 & -1 & & 1 \\ \cline{2-6}
D & 1 & & & -1 & -1 \\ \cline{2-6}
\end{array} \tag{2-19}$$

The incidence matrix M_1^o can be established in another way in two stages. First, from the nodes (0-cells) A, B, C and D, three independent pairs are chosen, say A and B, A and C, and A and D. The connection between the pairs and the 0-cells will be denoted by the matrix C_c^o.

$$C_c^o = \begin{array}{c|c|c|c|}
 & A \& B & A \& C & A \& D \\ \cline{2-4}
A & 1 & 1 & 1 \\ \cline{2-4}
B & -1 & & \\ \cline{2-4}
C & & -1 & \\ \cline{2-4}
D & & & -1 \\ \cline{2-4}
\end{array} \tag{2-20}$$

(this matrix is just an extension of the connection matrix idea used earlier in linking meshes and branches). The subscript, o, in C_c^o denotes connection between 0-cells.

Next, the connection matrix A_{ol} relating the node-pairs to branches is established. This has already been done in the nodal analysis of section 2-2. Using the matrix given by equation 2-7, the product:

$$C_c^o\,(A_{ol})_t = \begin{array}{c|c|c|c|} & A\&B & A\&C & A\&D \\ \hline A & 1 & 1 & 1 \\ \hline B & -1 & & \\ \hline C & & -1 & \\ \hline D & & & -1 \\ \hline \end{array} \quad \begin{array}{c|c|c|c|c|c|} & 1 & 2 & 3 & 4 & 5 \\ \hline A\&B & & & -1 & -1 & \\ \hline A\&C & & 1 & 1 & & -1 \\ \hline A\&D & -1 & & & 1 & 1 \\ \hline \end{array}$$

$$= \begin{array}{c|c|c|c|c|c|} & 1 & 2 & 3 & 4 & 5 \\ \hline A & -1 & 1 & & & \\ \hline B & & & 1 & 1 & \\ \hline C & & -1 & -1 & & 1 \\ \hline D & 1 & & & -1 & -1 \\ \hline \end{array} = M_1^o \qquad (2\text{-}21)$$

The result is shown on the algebraic diagram (fig. 3c). Suffix 1 has been added to the matrix, A_o, relating branches to the node-pairs. It implies the fact that a connection is made between 1-cells and their boundaries (bounding 0-circuit). The superscript 1 added to the matrix, C_c, relating branches to meshes imples that a connection is made between 1-cells and the 1-circuits. Next the connection matrix, A_{o2}, will be developed, and incidence matrix M_2^1 established in two stages.

Meshes a' and b' (fig. 1b) bound the areas a and b (fig. 3a) respectively. However, the direction of mesh b' is in an opposite sense to the orientation of area b. The connection between the meshes and the areas is:

$$A_{o2} = \begin{array}{c|c|c|} & a' & b' \\ \hline a & 1 & \\ \hline b & & -1 \\ \hline \end{array} \qquad (2\text{-}22)$$

The product C^1_c (obtained from equation 2-1) and $(A_{o2})_t$ can be seen to give incidence matrix M^1_2. This product relation is also represented on the algebraic diagram.

Another way of describing the electric network of fig. 1b is by using mesh currents $i^{a''}$ and $i^{b''}$ (fig. 3b). The connection, $\bar{C}^1_c$, between the mesh currents and branch currents of fig. 1b is:

$$\begin{array}{|c|} \hline i^1 \\ \hline i^2 \\ \hline i^3 \\ \hline i^4 \\ \hline i^5 \\ \hline \end{array} = \begin{array}{c|c|c|} & a'' & b'' \\ \hline 1 & 1 & \\ \hline 2 & 1 & \\ \hline 3 & -1 & 1 \\ \hline 4 & 1 & -1 \\ \hline 5 & & 1 \\ \hline \end{array} \quad \begin{array}{|c|} \hline i^{a''} \\ \hline i^{b''} \\ \hline \end{array} \qquad (2\text{-}23)$$

The connection between the meshes and the areas a and b is:

$$\bar{A}_{o2} = \begin{array}{c|c|c|} & a'' & b'' \\ \hline a & 1 & 1 \\ \hline b & & -1 \\ \hline \end{array}$$

The product $C_c^1 \; (A_{o2})_t$ can again be seen to result in M_2^1, the incidence matrix:

$$\begin{array}{c|c|c|} & a'' & b'' \\ \hline 1 & 1 & \\ \hline 2 & 1 & \\ \hline 3 & -1 & 1 \\ \hline 4 & 1 & -1 \\ \hline 5 & & 1 \\ \hline \end{array} \quad \begin{array}{c|c|c|} & a & b \\ \hline a'' & 1 & \\ \hline b'' & 1 & -1 \\ \hline \end{array} \quad = \quad \begin{array}{c|c|c|} & a & b \\ \hline 1 & 1 & \\ \hline 2 & 1 & \\ \hline 3 & & -1 \\ \hline 4 & & 1 \\ \hline 5 & 1 & -1 \\ \hline \end{array} \qquad (2\text{-}24)$$

The connection matrices used in electric circuit analysis are seen to give the relation between branches (1-cells), the node-pairs (bounding 0-circuits) and meshes (1-circuits). The closed- and open-mesh description can be varied. Two such descriptions have been given. The pairs of 0-cells chosen were A and B, A and C and A and D (fig. 3a). There are fifteen other possible ways of choosing three independent pairs of 0-cells. However, there is only one set of incidence matrices. Connection matrices relate the paths of voltages and currents to branches. Incidence matrices, on the other hand, show the structural construction of networks. Kirchhoff's voltage and current laws have been interpreted using connection matrices. Incidence matrices will be used in a later section in the application of Stokes' theorem.

In future sections, the field equations of Maxwell will be displayed as network equations relating to infinitesimal meshes. The differential structure of Maxwell's equations is there translated into an algebraic structure for the network, the field equations being given by a more general form of Kirchhoff's current and voltage laws. However, to arrive at a general network, the field equations must be expressed in a manner independent of the coordinate system. Exterior differential calculus has been found to be very convenient indeed in this respect.

CHAPTER 3

EXTERIOR DIFFERENTIAL STRUCTURES

This chapter deals with vectors, dyads and higher order forms in relation to exterior derivatives such as gradient, curl and divergence. In the study of exterior forms, differentials are so arranged as to generalise the concepts of curl, divergence, Stokes' theorem and Poincare Lemma. The relation between exterior products, exterior derivatives, star operator (dual) and the corresponding tensor operations are brought out in the concluding section. For the study of derivatives, it is supposed that the scalar, vector or tensor is given at every point of a region of space. Such an aggregate is called a scalar-, vector- or tensor-field.

3-1 Exterior 1-Forms

The differential of a scalar, f, which is a function of three Cartesian coordinates x, y and z is given by:

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \qquad (3\text{-}1)$$

In equation 3-1, if dx, dy and dz are now replaced by base vectors $\vec{dx}$, $\vec{dy}$ and $\vec{dz}$ along the coordinate axes, the gradient of f, $\vec{\nabla} f$ results.

$$\vec{\nabla} f = \frac{\partial f}{\partial x} \vec{dx} + \frac{\partial f}{\partial y} \vec{dy} + \frac{\partial f}{\partial z} \vec{dz}$$

(the symbols $\vec{i}$, $\vec{j}$ and $\vec{k}$ are used to denote base vectors in many books on vector algebra. In Cartesian coordinate system they are called unit vectors, since they are of unit length. In the more general curvilinear coordinates, they are not necessarily of unit length. When exterior differentials are studied, it is found to be convenient to use $\vec{dx}$, $\vec{dy}$ and $\vec{dz}$).

In the language of exterior algebra, the gradient $\vec{\nabla} f$ is still written as "df" but the "d" is often written with a bold face(16). In this thesis, a stroke above the "d" will distinguish it from ordinary differentials. With this notation,

$$\bar{d}f = \frac{\partial f}{\partial x} \vec{dx} + \frac{\partial f}{\partial y} \vec{dy} + \frac{\partial f}{\partial z} \vec{dz} \qquad (3\text{-}2)$$

The vector $\bar{d}f$ is an example of exterior 1-forms. In general, a vector need not be the gradient of a scalar. The more general expression for an exterior 1-form in an n-dimensional space is:

$$\omega = A_1 \vec{dx}^1 + A_2 \vec{dx}^2 + A_3 \vec{dx}^3 + \ldots + A_n \vec{dx}^n \qquad (3\text{-}3)$$

The exterior 1-form is an extension of the vector idea. The 1-forms are added and subtracted by adding and subtracting the corresponding coefficients of $\vec{dx}^k$ (k = 1, 2, . . . , n). In subsequent analysis, the arrows above $\vec{dx}^k$ terms will be omitted.

3-2 Exterior Products

Exterior product of two 1-forms is an extension of the cross-product idea of vectors. If two 1-forms are given, in Cartesian coordinates x, y and z, as:

$$\omega = A_1\, dx + A_2\, dy + A_3\, dz \qquad \text{and}$$

$$\lambda = B_1\, dx + B_2\, dy + B_3\, dz,$$

then the product

$$\begin{aligned}\omega \wedge \lambda = {} & (A_1 B_2 - A_2 B_1)\, dx \wedge dy \quad + \\ & (A_2 B_3 - A_3 B_2)\, dy \wedge dz \quad + \\ & (A_3 B_1 - A_1 B_3)\, dz \wedge dx \end{aligned} \tag{3-4}$$

The wedge ($\wedge$) denotes the "exterior" product. Three rules have been used to arrive at this result.

(i) The distributive law

$$(A_1\, dx + A_2\, dy) \wedge dz = A_1\, dx \wedge dz + A_2\, dy \wedge dz$$

(ii) The alternation rule

$$dx \wedge dz = - dz \wedge dx$$

(iii) $dz \wedge dz = 0$, a consequence of the alternation rule

(3-5)

The exterior products are not commutative. The three rules also serve to generalise exterior products of higher order differential forms in an n-dimensional space. (17)

An important deviation from the vector cross-product idea is observed from equation 3-4. The cross-product of two vectors is a third vector. However, the exterior product of two 1-forms results in a 2-form. The 2-form $\omega \wedge \lambda$ can be represented by an area (fig. 4a). The circular arrow in fig. 4a gives the "orientation" of the area. The components $(A_1B_2-A_2B_1)$, $(A_2B_3-A_3B_2)$ and $(A_3B_1-A_1B_3)$ given in equation 3-4, represent the projections of the shaded area on the three coordinate planes. If the order of exterior product is changed, then $\lambda \wedge \omega$ is still given by the same area; but the orientation (direction of the circular arrow) will be reversed. A vector $\vec{\mu}$ perpendicular to the shaded area (fig. 4a) corresponds to the vector cross product of the two vectors $\vec{\omega}$ and $\vec{\lambda}$ In vector notation, it is given by:

$$\vec{\mu} = \vec{\omega} \times \vec{\lambda}$$

The significance of the star-operator in fig. 4a will be explained in a later section. The 2-forms $dx \wedge dy$, $dy \wedge dz$ and $dz \wedge dx$ give unit areas on the three coordinate planes.

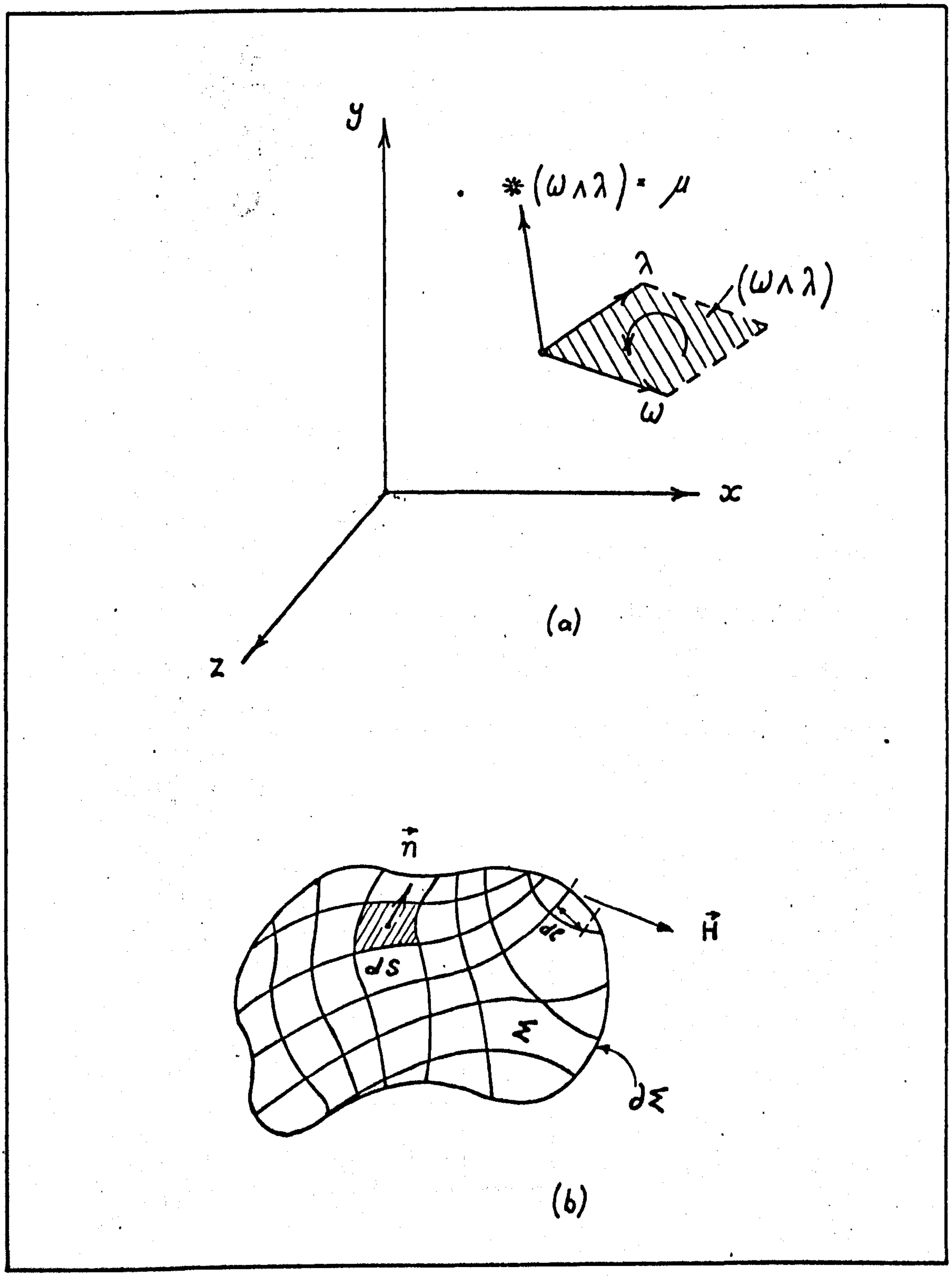


FIG 4 EXTERIOR PRODUCTS, DUALS AND INTEGRATION OF EXTERIOR FORMS

In a 4-dimensional space, there are four base vectors dx^1, dx^2, dx^3 and dx^4; but there are six unit areas $dx^1 \wedge dx^2$, $dx^1 \wedge dx^3$, $dx^1 \wedge dx^4$, $dx^2 \wedge dx^3$, $dx^2 \wedge dx^4$ and $dx^3 \wedge dx^4$.

3-3 Higher Order Differential Forms

These result from the exterior multiplication of 1-forms. In equation 3-4, two important properties of the 2-form emerge: (i) the coefficients of the $dx \wedge dx$ terms are zero; (ii) the coefficients of the $dy \wedge dx$ terms will be the negative of the coefficients of the $dx \wedge dy$ terms. A matrix representing the coefficients will, therefore, be skewsymmetric. For example, if

$$A_{ab} = \begin{array}{cc} & \longrightarrow b \\ \begin{array}{c}\downarrow \\ a\end{array} & \begin{array}{|c|c|c|}\hline & A_{12} & A_{31} \\ \hline -A_{12} & & A_{23} \\ \hline -A_{31} & -A_{23} & \\ \hline \end{array} \end{array}$$

$$\begin{aligned} \mu &= \tfrac{1}{2} \sum_{\substack{a=1\\b=1}}^{3} A_{ab} \; dx^a \wedge dx^b \qquad (3\text{-}6) \\ &= A_{12} \; dx^1 \wedge dx^2 + A_{23} \; dx^2 \wedge dx^3 \\ &\quad + A_{31} \, dx^3 \wedge dx^1 \end{aligned}$$

A p-form can be expressed in terms of the components of a skewsymmetric tensor of rank p as:

$$\mu = \frac{1}{p!} \sum (A_{\alpha\beta \ldots \pi}) \, dx^{\alpha} \wedge dx^{\beta} \wedge \ldots \wedge dx^{\pi} \qquad (3\text{-}7)$$

An alternative formulation of p-forms is in terms of "ordered p tuples"$^{(18)}$. In equation 3-6, the integers a and b can be arranged in an ascending order as:

a	b
1	2
1	3
2	3

$(1 \leq a < b \leq 3)$

In terms of these "ordered doubles",

$$\mu = \sum_{\langle a,b \rangle} A_{ab} \; dx^{a} \wedge dx^{b}$$

In this notation $\sum\limits_{a,b}$ stands for sum over all p-tuples; but when a and b are enclosed in brackets, $\sum\limits_{\langle a,b \rangle}$ now stands for sum over all "ordered" p-tuples. A p-form in an n-dimensional space, in terms of this arrangement, is:

$$\mu = \sum_{\langle H \rangle} (A_{h_1 h_2 h_3 \ldots h_p}) \, dx^{h_1} \wedge dx^{h_2} \wedge \ldots \wedge dx^{h_p} \qquad (3\text{-}8)$$

where h_1, h_2, ... h_p are all integers, no two of them being equal and arranged in an ascending order as:

$$1 \leqslant h_1 < h_2 < h_3 \ldots < h_p \leqslant n$$

The letter H denotes the p-tuple of integers and (H) denotes that they are "ordered". In a 4-dimensional space, the ordered triples are $dx^1 \wedge dx^2 \wedge dx^3$, $dx^1 \wedge dx^2 \wedge dx^4$, $dx^1 \wedge dx^3 \wedge dx^4$ and $dx^2 \wedge dx^3 \wedge dx^4$.

Two p-forms expressed as equation 3-8 can be added and subtracted by adding and subtracting the corresponding coefficients. A p-form ϖ and a q-form λ, in an n-space, can be multiplied satisfying the rules given by equation 3-5 as well as the associative law. In particular,

$$\begin{aligned} \varpi \wedge \lambda &= (-1)^{pq} \; \lambda \wedge \varpi \qquad \text{if } p+q \leqslant n \\ &= 0 \qquad \text{if } p+q > n \\ \varpi \wedge (\lambda \wedge \mu) &= (\varpi \wedge \lambda) \wedge \mu \quad \text{- associative law} \end{aligned} \qquad (3\text{-}9)$$

The exterior product is seen to generalise the vector cross-product idea to more than 3-dimensions and to higher order dyads, tensors etc. Whereas cross-product of two vectors is a third vector, exterior products of differential forms result in higher order forms. Cross-products are not associative, but exterior products are. In Enclidean 3-dimensional space, the exterior product in conjunction with a star operator (to be explained later) gives the vector cross-product (fig. 4a).

3-4 Inner Product

This operation is the extension of the vector dot-product idea. For two 1-forms ω and λ, given in Cartesian coordinate system, as:

$$\omega = A_1\, dx^1 + A_2\, dx^2 + A_3\, dx^3$$

$$\lambda = B_1\, dx^1 + B_2\, dx^2 + B_3\, dx^3$$

the inner product (dot product) is:

$$((\omega , \lambda)) = A_1 B_1 + A_2 B_2 + A_3 B_3$$

(The double brackets ((. . .)) will be used in this thesis to denote inner products).

For two p-forms given as:

$$\alpha = \omega_1 \wedge \omega_2 \wedge \cdots\cdots \wedge \omega_p$$

$$\beta = \lambda_1 \wedge \lambda_2 \wedge \cdots\cdots \wedge \lambda_p$$

The inner product is the determinant:

$$((\alpha,\beta)) = \begin{vmatrix} ((\omega_1, \lambda_1)) & \cdots\cdots\cdots\cdots & ((\omega_p, \lambda_1)) \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ ((\omega_1, \lambda_p)) & \cdots\cdots\cdots\cdots & ((\omega_p, \lambda_p)) \end{vmatrix} \qquad (3\text{-}10)$$

In other words, the inner product of two p-forms is the determinant of a matrix whose elements are the inner products $((\omega_i , \lambda_j))$. It follows that the inner product of a p-form and a q-form ($p \neq q$) is non-existant, for the matrix in equation 3-10 would then be singular, and will not have a determinant. The inner products are distributive and commutative.

$$((\omega , \lambda + \mu)) = ((\omega , \lambda)) + ((\omega , \mu))$$

$$((\omega , \lambda)) = ((\lambda , \omega))$$

In a general curvilinear coordinate system the base vectors dx^1, dx^2, . . . , dx^n are not necessarily orthogonal. Moreover, they may not be of unit length. In such cases the inner products of the base vectors (i. e. the dot-products) are expressed by a matrix g:

$$((dx^i , dx^j)) = g^{ij} \qquad (3\text{-}12)$$

The name orthonormal system is given to coordinate systems in which the elements of the matrix g are:

$$\pm 1 (i = j), \quad 0 \; (i \neq j)$$

In the space-time coordinate system used in relativity, the proper distance between two neighbouring events is(16):

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 - c^2 (dt)^2$$

where c is the velocity of light and t is time in seconds. The term ct has the unit of length and is known as cotime. It will be denoted by ŧ in this thesis. In terms of cotime, the proper distance:

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 - (dŧ)^2$$

In terms of inner product then:

$$((dŧ\ ,\ dŧ)) = -1$$

The number of negative signs associated with the inner products of base vectors is known as signature.

3-5 Hodge (Star) Operator

In Euclidean space, the 2-form resulting from a product of two 1-forms, ω and λ, can be represented by the shaded area in fig. 4a. The circular arrow gives the orientation of the area. The Hodge (star) operator(19) on this 2-form results in a 1-form, μ, perpendicular to the shaded area (fig. 4a). The 1-form μ is also known as the dual of the 2-form $(\omega \wedge \lambda)$.

The direction of this vector is in a right handed screw sense to the circular arrow.

Flanders(17) generalises the star operator for a p-form in an n-space in terms of inner and exterior products:

$$\omega \wedge \lambda = \langle\langle * \omega , \lambda \rangle\rangle \, d\sigma \qquad (3\text{-}14)$$

Here, ω is a p-form. The star operation on ω gives the dual (n-p) form $*\omega$, and

$$d\sigma = dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n,$$

for orthonormal systems (3-14a)

$$\lambda = \text{any (n-p) form}$$

In equation 3-14, $d\sigma$ is the unit n-dimensional volume, the inner product gives a scalar and $\omega \wedge \lambda$ results in an n-form (or an n-volume).

Equation 3-14 is best illustrated by some examples.

3-5-1

In the Cartesian coordinate system, if

$$\omega = A\,dx + B\,dy + C\,dz,$$

the dual of ω will be a 2-form. Let

$$*\omega = D\, dx \wedge dy + E\, dx \wedge dz + F\, dy \wedge dz$$

As λ can be any 2-form, let

$$\lambda = G\, dx \wedge dy + H\, dx \wedge dz + K\, dy \wedge dz$$

Equation 3-14 must be valid for all values of G, H and K. In this equation,

$$\omega \wedge \lambda = A\, G\, dx \wedge dx \wedge dy + A\, H\, dx \wedge dx \wedge dz$$

$$+ A\, K\, dx \wedge dy \wedge dz + \ldots$$

$$+ C\, K\, dz \wedge dy \wedge dz$$

$$= A\, K\, dx \wedge dy \wedge dz + B\, H\, dy \wedge dx \wedge dz$$

$$+ C\, G\, dz \wedge dx \wedge dy$$

(since $dx \wedge dx$ etc. terms are all zero)

$$= (A\, K - B\, H + C\, G)\, dx \wedge dy \wedge dz \qquad (3\text{-}15)$$

(since $dy \wedge dx = -\, dx \wedge dy$)

$$((*\omega, \lambda)) = D\, G\, ((dx \wedge dy,\ dx \wedge dy)) + \ldots$$

$$= D\, G \begin{vmatrix} ((dx,\ dx)) & ((dy,\ dx)) \\ ((dx,\ dy)) & ((dy,\ dy)) \end{vmatrix} + \ldots$$

$$= D\, G \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \ldots$$

$$= D\, G + E\, H + F\, K \qquad (3\text{-}16)$$

Substitution of equations 3-15 and 3-16 in 3-14 yields:

$$AK - BH + CG = DG + EH + FK$$

As this equation is valid for all values of G, H and K,

$$F = A; \quad E = -B; \quad D = C$$

so that,

$$\begin{aligned} *\omega &= A\, dy \wedge dz - B\, dx \wedge dz + C\, dx \wedge dy \\ &= A\, dy \wedge dz + B\, dz \wedge dx + C\, dx \wedge dy \end{aligned}$$

In this case the 1-form ω is a vector and the 2-form, $*\omega$, represents an area. The coefficients A, B and C give the projections of the area on the three coordinate planes. The area itself is normal to the vector ω.

3-5-2-

An example in space-time coordinate system will be considered next. The cotime, ct, will be used in the analysis (t is the time in seconds, c is the velocity of light and ct has the dimensions of length expressed by the symbol ŧ). The inner product has been seen in Section 3-4 to be:

$$((d\text{ŧ},\ d\text{ŧ})) = -1$$

A 2-form ω in a general case will have six terms. Consider the special case when the coefficient of $dy \wedge dz$ term is unity and all others are zero.

$$\text{i.e. } \omega = dy \wedge dz$$

The dual of ω will be a 2-form normal to this. It can only contain the term $dx \wedge dt$, since $*.\omega$ is normal to both dy and dz. Let

$$* \omega = K\,(dx \wedge dt)$$

If the 2-form λ in equation 3-14 be taken as $(dx \wedge dt)$, then

$$\begin{aligned} \omega \wedge \lambda &= dy \wedge dz \wedge dx \wedge dt \\ &= -dy \wedge dx \wedge dz \wedge dt \\ &= dx \wedge dy \wedge dz \wedge dt \qquad (3\text{-}17) \end{aligned}$$

$$\begin{aligned} ((*\omega\,,\ \lambda\ )) &= ((K\, dx \wedge dt,\ dx \wedge dt)) \\ &= K \begin{vmatrix} ((dx,dx)) & ((dt,\ dx)) \\ ((dx,\ dt)) & ((dt,\ dt)) \end{vmatrix} \\ &= K \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} = -K \qquad (3\text{-}18) \end{aligned}$$

Substitution of equations 3-17 and 3-18 in 3-14 yields:

$$1 = -K, \quad \text{or}$$

$$*(dy \wedge dz) = -dx \wedge dt \tag{3-19}$$

A similar procedure for $dx \wedge dt$ results in:

$$*(dx \wedge dt) = dy \wedge dz \tag{3-20}$$

That is:

$$**(dy \wedge dz) = -*(dx \wedge dt)$$
$$= -dy \wedge dz$$

This can be compared with the j-operator in a 2-dimensional plane ($j = \sqrt{-1}$). The dual of a vector $\vec{A}$ is $j\vec{A}$, and the dual of this is:

$$j.\ j.\ \vec{A} = -\vec{A}$$

Flanders(17) has shown that, for the general case:

$$**\omega = (-1)^{np+p+s}\ \omega \tag{3-21}$$

Here, p = order of the differential form ω

n = number of dimensions of the space

s = number of negative signs in the inner product of the base vectors

The dual of a 1-form in a 3-space is an area normal to the vector. The dual of a 2-form is a vector normal to the area represented by the 2-form. In a 4-space, the dual of an area is yet another area. The six independent areas in a 4-space can be grouped into three pairs of dual areas:

$$dx^1 \wedge dx^2 \quad \text{and} \quad dx^3 \wedge dx^4$$

$$dx^3 \wedge dx^1 \quad \text{and} \quad dx^2 \wedge dx^4$$

$$dx^2 \wedge dx^3 \quad \text{and} \quad dx^1 \wedge dx^4$$

3-6 Exterior Derivatives

The gradient of a scalar is an example of an exterior derivative. From section 3-1,

$$\bar{d}f = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \qquad (3\text{-}2)$$

The scalar f is a 0-form and the exterior derivative takes the 0-form into a 1-form. The exterior derivative can be extended to the general case by the following rules$^{(17)}$:

(i) $\bar{d}(\omega + \lambda) = \bar{d}\omega + \bar{d}\lambda$

(ii) $\bar{d}(\omega \wedge \mu) = \bar{d}\omega \wedge \mu + (-1)^p \omega \wedge \bar{d}\mu$

(iii) $\bar{d}(dx^k) = 0$ (3-22)

In this ω and λ are p-forms and μ is any q-form.

The exterior derivative is best illustrated by some examples.

3-6-1

In the Cartesian coordinate system, let:

$$\omega = H_1\, dx + H_2\, dy + H_3\, dz$$

using equation 3-22, rule (i),

$$\bar{d}\omega = \bar{d}\,(H_1\ dx) + \bar{d}\,(H_2\ dy) + \bar{d}\,(H_3\ dz) \tag{3-23}$$

Using rule (ii),

$$\bar{d}\,(H_1\ dx) = \bar{d}\,H_1 \wedge dx + (-1)^o\, H_1 \wedge \bar{d}\,(dx)$$

The second term is zero by rule (iii)

$$\therefore\ \bar{d}\,(H_1\ dx) = \bar{d}\,H_1 \wedge dx$$

$$= \frac{\partial H_1}{\partial x}\ dx \wedge dx + \frac{\partial H_1}{\partial y}\ dy \wedge dx + \frac{\partial H_1}{\partial z}\ dz \wedge dx \tag{3-24}$$

In this, $dx \wedge dx = 0$

$$dy \wedge dx = -dx \wedge dy$$

Substitution in equation 3-24 results in:

$$\bar{d}\,(H_1\,dx) = -\frac{\partial H_1}{\partial y}\,dx \wedge dy + \frac{\partial H_1}{\partial z}\,dz \wedge dx$$

Similar substitutions in equation 3-23 yield:

$$\bar{d}\omega = \left(\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y}\right) dx \wedge dy + \left(\frac{\partial H_3}{\partial y} - \frac{\partial H_2}{\partial z}\right) dy \wedge dz + \left(\frac{\partial H_1}{\partial z} - \frac{\partial H_3}{\partial x}\right) dz \wedge dx \qquad (3\text{-}25)$$

The terms in the brackets of equation 3-25 are recognised as those of the components of the curl of a vector $\vec{H}$. The exterior derivative is a generalisation of the concept of curl. The exterior derivative of a scalar (0-form) results in a vector (1-form). The exterior derivative of a 1-form, such as ω, results in a 2-form (e.g. equation 3-25). In general, the exterior derivative takes the p-form into a (p+1) form.

The bracketted terms of equation 3-25 give the projections of the area $\bar{d}\omega$ on the three coordinate planes. Taking the dual of ω,

$$*\bar{d}\omega = \left(\frac{\partial H_3}{\partial y} - \frac{\partial H_2}{\partial z}\right) dx + \left(\frac{\partial H_1}{\partial z} - \frac{\partial H_3}{\partial x}\right) dy + \left(\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y}\right) dz \qquad (3\text{-}26)$$

The resulting 1-form is the conventional curl of a vector $\vec{H}$. Denoting the exterior derivative by the expression "generalised curl", it is seen that, for Euclidean 3-space,

gradient of a scalar $\equiv$ generalised curl of an 0-form

conventional curl of a vector $\equiv$ dual of the generalised curl of a 1-form

3-6-2-

The example considered here is a 2-form, $\tilde{F}$, in a 4-dimensional space.

$$\tilde{F} = F_{12}\, dx^1 \wedge dx^2 + F_{13}\, dx^1 \wedge dx^3 + F_{14}\, dx^1 \wedge dx^4$$

$$+ F_{23}\, dx^2 \wedge dx^3 + F_{24}\, dx^2 \wedge dx^4 + F_{34}\, dx^3 \wedge dx^4$$

In terms of the skewsymmetric matrix F_{ab},

$$\tilde{F} = \sum_{(a,b)} F_{ab}\, dx^a \wedge dx^b$$

The brackets enclosing a, b indicate summation with

$1 \leqslant a < b \leqslant 4$

Considering now the exterior derivative,

$$d\tilde{F} = \sum_{(a,b),c} \frac{\partial F_{ab}}{\partial x^c} dx^c \wedge dx^a \wedge dx^b, \tag{3-27}$$

there are three possible values for c:

$c < a < b$

$a < c < b$

$a < b < c$

When a, b and c are so ordered

$$d\tilde{F} = \sum_{(c,a,b)} \frac{\partial F_{ab}}{\partial x^c} dx^c \wedge dx^a \wedge dx^b + \sum_{(a,c,b)} \frac{\partial F_{ab}}{\partial x^c} dx^c \wedge dx^a \wedge dx^b$$

$$+ \sum_{(a,b,c)} \frac{\partial F_{ab}}{\partial x^c} dx^c \wedge dx^a \wedge dx^b$$

$$= \sum_{(a,b,c)} \left(\frac{\partial F_{bc}}{\partial x^a} - \frac{\partial F_{ac}}{\partial x^b} + \frac{\partial F_{ab}}{\partial x^c} \right) dx^a \wedge dx^b \wedge dx^c$$

$$= \sum_{(a,b,c)} \left(\frac{\partial F_{bc}}{\partial x^a} + \frac{\partial F_{ca}}{\partial x^b} + \frac{\partial F_{ab}}{\partial x^c} \right) dx^a \wedge dx^b \wedge dx^c$$

(3-28)

3-7 Generalised Divergence

The generalised divergence takes a p-form into a (p-1) form. The conventional divergence of a vector gives a scalar. The extension of this concept to a p-form is done in three stages: (i) the dual of ω, $= *\omega$, gives a (n-p) form (ii) the exterior derivative, $\bar{d} * \omega$ takes the (n-p) form into a (n-p+1) form (iii) a dual of this (n-p+1) form, $*\bar{d} * \omega$, gives the required (p-1) form. Wheeler(16) defines the divergence or "codifferential" operation as:

$$\bar{\delta}\omega = (-1)^{np+n+s+1} *\bar{d} * \omega \qquad (3-29)$$

This will be illustrated by some examples.

3-7-1

In Cartesian coordinates, let a 1-form ω be given as:

$$\omega = B_1 \, dx + B_2 \, dy + B_3 \, dz$$

$$* \omega = B_1 \, dy \wedge dz + B_2 \, dz \wedge dx + B_3 \, dx \wedge dy$$

$$\bar{d} * \omega = \left(\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z} \right) dx \wedge dy \wedge dz$$

$$*\bar{d} * \omega = \left(\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z} \right) \qquad (3-30)$$

Equation 3-30 is recognised as the conventional divergence of a vector in a 3-space.

3-7-2

In the space-time coordinate system, consider a 2-form $\tilde{H}$ given as:

$$\tilde{H} = H_1 \; dy \wedge dz + H_2 \; dz \wedge dx + H_3 \; dx \wedge dy$$

$$+ H_4 \; dx \wedge dt + H_5 \; dy \wedge dt + H_6 \; dz \wedge dt$$

The equations,

$$*(dy \wedge dz) = -(dx \wedge dt) \qquad (3\text{-}19)$$

$$*(dx \wedge dt) = dy \wedge dz \qquad (3\text{-}20)$$

derived in Section 3-5-2 will be used here to obtain the dual of $\tilde{H}$.

$$*\tilde{H} = -H_1 \; dx \wedge dt - H_2 \; dy \wedge dt - H_3 \; dz \wedge dt$$

$$+ H_4 \; dy \wedge dz + H_5 \; dz \wedge dx + H_6 \; dx \wedge dy$$

The next operation in the codifferential is the exterior derivative.

$$d*\tilde{H} = \left(\frac{\partial H_4}{\partial x} + \frac{\partial H_5}{\partial y} + \frac{\partial H_6}{\partial z} \right) dx \wedge dy \wedge dz \quad .$$

$$+\left(\frac{\partial H_4}{\partial t} + \frac{\partial H_2}{\partial z} - \frac{\partial H_3}{\partial y}\right) dy \wedge dz \wedge dt$$

$$+ \ldots$$

To obtain the dual of this, the relations

$$*(dx \wedge dy \wedge dz) = -dt$$

$$*(dy \wedge dz \wedge dt) = -dx$$

are used (Appendix 1)

$$*d*\tilde{H} = -\left(\frac{\partial H_4}{\partial t} + \frac{\partial H_2}{\partial z} - \frac{\partial H_3}{\partial y}\right) dx - \ldots - \ldots$$

$$-\left(\frac{\partial H_4}{\partial x} + \frac{\partial H_5}{\partial y} + \frac{\partial H_6}{\partial z}\right) \quad dt \qquad (3\text{-}31)$$

The codifferential is seen here to take the 2-form $\tilde{H}$ into a 1-form.

3-7-3

The Laplacian of a vector, $\vec{A}$, is written in vector notation, using the gradient, curl and divergence operators as:

$$\text{grad}\,(\text{div}\,\vec{A}\,) - \text{curl}\,(\text{curl}\,\vec{A}\,)$$

or, $\vec{\nabla}\,(\vec{\nabla}\cdot\vec{A}\,) - \vec{\nabla}\times(\vec{\nabla}\times\vec{A}\,)$

In the notation used in exterior differential forms, the Laplacian of a p-form ω in any n-dimensional space is:

$$-\bar{d}\ \bar{\delta}\,\omega \quad - \quad \bar{\delta}\ \bar{d}\,\omega$$

The form ω is called "harmonic" if the Laplacian of ω is zero(17).

3-8 Poincare Lemma

If ω is a p-form,

then:

$$\bar{d}\,(\bar{d}\,\omega\ ) = 0 \tag{3-32}$$

This is known as Poincare Lemma and will be examined in this section. Let ω be a p-form expressed as:

$$\omega = \sum_{(H)} A_H\ dx^H$$

where $H = \{ h_1, h_2, \ldots , h_p \}$

The brackets enclosing H denote the ordering:

$$1 \leqslant h_1 < h_2 < \ldots < h_p \leqslant n$$

$$\bar{d}\omega = \sum_{(H),i} \frac{\partial A_H}{\partial x^i} dx^i \wedge dx^H$$

$$\bar{d}(\bar{d}\omega) = \sum_{(H),i,j} \frac{\partial^2 A_H}{\partial x^j \partial x^i} dx^j \wedge dx^i \wedge dx^H \qquad (3\text{-}33)$$

Now,

$$\frac{\partial^2 A_H}{\partial x^j \partial x^i} = \frac{\partial^2 A_H}{\partial x^i \partial x^j}$$

but,

$$dx^j \wedge dx^i = - dx^i \wedge dx^j$$

In equation 3-33, if the summation is written as:

$$\sum_{(H)} \left\{ \sum_{i,j} \left(\frac{\partial^2 A_H}{\partial x^j \partial x^i} dx^j \wedge dx^i \right) \wedge dx^H \right\}$$

the bracketted term is seen to be zero, giving $\bar{d}(\bar{d}\omega) = 0$.

In Euclidean 3-space, the application of Poincare Lemma to a scalar, f, and a vector, $\tilde{A}$, gives:

$$* \bar{d}(\bar{d} f) = 0 \qquad (3\text{-}34)$$

$$* \bar{d} * (* \bar{d} \tilde{A}) = * \bar{d}(\bar{d} \tilde{A}) = 0 \qquad (3\text{-}35)$$

This is just one way of expressing the well known relations

curl (gradient) = 0

div (curl) = 0

3-9 Generalised Stokes' Theorem

The theorem is stated as:

$$\oint_{\partial\Sigma} \omega = \int_{\Sigma} \bar{d}\omega \qquad (3\text{-}36)$$

In this, ω is a p-form and $\bar{d}\omega$, the exterior derivative of ω, a (p+1) form.. The latter is integrated over a (p+1) dimensional region Σ. The boundary of Σ, denoted by $\partial\Sigma$ has p-dimensions. The p-form ω is integrated over this region.

3-9-1

Consider a 1-form $\tilde{H}$ in a 3-space. Let,

$$*\bar{d}\,\tilde{H} = \tilde{J} \qquad (3\text{-}37)$$

$$\text{i.e. } \left(\frac{\partial H_3}{\partial y} - \frac{\partial H_2}{\partial z} \right) dx + (\ldots)\, dy + (\ldots)\, dz$$

$$= J_1\, dx + J_2\, dy + J_3\, dz$$

In conventional vector analysis, equation 3-37 stands for:

$$\text{curl } \vec{H} = \vec{J}$$

In fig. 4b, Σ describes an area whose boundary is given by the closed curve $\partial\Sigma$. The integral, $\oint_{\partial\Sigma} \tilde{H}$, is the line integral of $\tilde{H}$ around the closed loop. In the conventional vector analysis, it is written as:

$$\oint \vec{H} \cdot \vec{dl} \quad (\vec{dl} \text{ is the elemental vector length along the curve})$$

In equation 3-36, the integral

$$\int_{\Sigma} \bar{d}\,\omega = \int_{\Sigma} \bar{d}\,\tilde{H} = \int_{\Sigma} *\tilde{J}$$

In the conventional vector analysis, it is written as:

$$\iint \vec{J} \cdot \vec{n}\, dS \quad (\vec{n} \text{ and } dS \text{ are as shown in fig. 4b})$$

Another application of Stokes' theorem is in the equation relating electric flux and charge. In vector form:

$$\oiint_{\partial\Sigma} \vec{D} \cdot \vec{n}\, dS = \iiint_{\Sigma} \rho\, dV$$

In this relation, Σ is a volume region bounded by a closed surface $\partial\Sigma$. The vector $\vec{D}$ is the electric flux density and ρ is charge density. In the notation of differential forms,

$$\int_{\partial\Sigma} *\tilde{D} = \int_{\Sigma} \bar{d} *\tilde{D} = \int_{\Sigma} * \rho$$

When a region is bounded by subregions (e. g. a triangular area bounded by three straight lines), the incidence matrices described in Section 2-5 are found to be useful. The integrations can be carried out separately over the subregions and put together with the aid of incidence matrices.

3-10 Relation to Tensor Analysis

The exterior product, exterior derivative and star operator (dual) are examined in tensor notation in this section. A 2-form was earlier expressed as a product of a skewsymmetric matrix and base vectors:

$$\mu = \frac{1}{2} \sum_{\substack{a=1 \\ b=1}}^{3} A_{ab}\, dx^a \wedge dx^b \qquad (3\text{-}6)$$

In Einstein summation convention, repeated indices themselves imply summation and the sign $\sum$ is omitted. In this convention,

$$\mu = \tfrac{1}{2} A_{ab} \ dx^a \wedge dx^b \qquad (3\text{-}38)$$

If the coordinates X are transformed into a new system $\bar{X}$, the 1-form μ is given in terms of a new matrix $\bar{A}$ as:

$$\mu = \tfrac{1}{2} \bar{A}_{\alpha\beta} \ d\bar{x}^{\alpha} \wedge d\bar{x}^{\beta}$$

If the unit vectors transform as:

$$dx^a = C^a_{\alpha} \ d\bar{x}^{\alpha},$$

$\bar{A}_{\alpha\beta}$ is seen to transform as:

$$\bar{A}_{\alpha\beta} = C^a_{\alpha} \ C^b_{\beta} \ A_{ab} \qquad (3\text{-}39)$$

A set of physical or geometrical entities, which is represented by the aggregate of a set of components following transformation laws, such as equation 3-39, is known as a tensor(20, 21). A tensor transforming as A_{ab}, involving the transformation tensors, C, is called a covariant tensor. When the transformation involves inverse of C, the tensor is a contravariant tensor - e.g.

$$A^{\alpha} = \gamma^{\alpha}_{a} A^{a}, \quad \text{where} \quad \gamma^{\alpha}_{a} = (C^{a}_{\alpha})^{-1}$$

In equation 3-6, A_{ab} is either zero $(a = b)$, or $(-A_{ba})$, $a \neq b$, and A_{ab} is a skewsymmetric tensor. A differential p-form, in terms of a skewsymmetric tensor of rank p is:

$$\mu = \frac{1}{p!} A_{\alpha\beta\ldots\pi} \, dx^{\alpha} \wedge dx^{\beta} \wedge \ldots \wedge dx^{\pi} \qquad (3\text{-}40)$$

3-10-1 Exterior Products of Tensors

The exterior product of two 1-forms,

$$\omega = A_1 \, dx^1 + A_2 \, dx^2 + A_3 \, dx^3 + \ldots = A_{\alpha} \, dx^{\alpha}$$

and

$$\lambda = B_1 \, dx^1 + B_2 \, dx^2 + B_3 \, dx^3 + \ldots = B_{\beta} \, dx^{\beta}$$

is:

$$\begin{aligned} \omega \wedge \lambda = & \ (A_1 B_2 - A_2 B_1) \, dx^1 \wedge dx^2 \\ & + (A_1 B_3 - A_3 B_1) \, dx^1 \wedge dx^3 + \ldots \\ & + (A_2 B_3 - A_3 B_2) \, dx^2 \wedge dx^3 + \ldots \\ = & \ \tfrac{1}{2} (A_{\alpha} B_{\beta} - A_{\beta} B_{\alpha}) \, dx^{\alpha} \wedge dx^{\beta} \end{aligned}$$

and the skewsymmetric tensor

$$F_{\alpha\beta} = (A_{\alpha} B_{\beta} - A_{\beta} B_{\alpha}) \qquad (3\text{-}41)$$

is thus seen to give the exterior product. Now, if another 1-form μ defined as:

$$\mu = G_{\gamma} \; dx^{\gamma}$$

is multiplied by the above 2-form $\omega \wedge \lambda$ i. e.

$$\omega \wedge \lambda \wedge \mu = (\tfrac{1}{2} F_{\alpha\beta} \, dx^{\alpha} \wedge dx^{\beta}) \wedge (G_{\gamma} \; dx^{\gamma})$$

$$= \frac{1}{3!} H_{\alpha\beta\gamma} \; dx^{\alpha} \wedge dx^{\beta} \wedge dx^{\gamma}$$

then,

$$H_{\alpha\beta\gamma} = (F_{\alpha\beta} G_{\gamma} + F_{\beta\gamma} G_{\alpha} + F_{\gamma\alpha} G_{\beta}) \qquad (3\text{-}42)$$

The exterior product may be extended to higher order forms by the use of the generalised "Kronecker delta". The simplest Kronecker delta is:

$$\delta^{i}_{a} = 0 \quad (i \neq a)$$

$$= 1 \quad (i = a)$$

The more general form of delta is given by first forming a matrix whose elements are the Kronecker deltas of the type δ^i_a and then taking the determinant of this matrix. For example:

$$\delta^{i\,j}_{a\,b} = \text{determinant of} \begin{vmatrix} \delta^i_a & \delta^i_b \\ \delta^j_a & \delta^j_b \end{vmatrix}$$

$$= (\delta^i_a \; \delta^j_b - \delta^i_b \; \delta^j_a)$$

It is seen that in equation 3-41

$$F_{\alpha\beta} = \delta^{\pi\sigma}_{\alpha\beta} \quad A_\pi \quad B_\sigma$$

Similarly equation 3-42, in terms of a Kronecker delta, is:

$$H_{\alpha\beta\gamma} = \frac{1}{2!} \; \delta^{\pi\sigma\mu}_{\alpha\beta\gamma} \quad F_{\pi\sigma} \; G_\mu$$

The Kronecker delta $\delta^{\pi\sigma\mu}_{\alpha\beta\gamma}$ is given by the determinant of the matrix:

$$\begin{vmatrix} \delta^\pi_\alpha & \delta^\pi_\beta & \delta^\pi_\gamma \\ \delta^\sigma_\alpha & \delta^\sigma_\beta & \delta^\sigma_\gamma \\ \delta^\mu_\alpha & \delta^\mu_\beta & \delta^\mu_\gamma \end{vmatrix}$$

In general the exterior product of a p-form F and a q-form G is given by:

$$H_{abc\ldots r} = \left(\frac{1}{p!} \times \frac{1}{q!}\right) \delta^{mn\ldots pq\ldots s}_{abc\ldots r} F_{mn\ldots p} G_{q\ldots s}$$

3-10-2 Inner Product of Tensors

The inner product of base vectors, when the coordinate system is not orthonormal, was given as:

$$((dx^i, \ dx^j)) = g^{ij} \tag{3-12}$$

The tensor formed with the components g^{ij} plays a very important role in tensor analysis. Its inverse, $(g^{ij})^{-1} = g_{ij}$, is known as the metric tensor. The word "metric" means "measure" and the metric tensor is used in geometry to express length and angle between directions and enables the formulation of invariants in general.

The inner product of two 1-forms

$$((\omega, \lambda)) = ((A_\alpha \ dx^\alpha, \ B_\beta \, dx^\beta))$$

$$= A_\alpha B_\beta \, g^{\alpha\beta} = A_\alpha (B_\beta \, g^{\alpha\beta})$$

The latter bracketted term is written as B^α in tensor notation;

it gives the "contravariant" components of the tensor B. The shifting of the index is known as "raising" in tensor language. The inner product in terms of B^{α} is:

$$((\omega, \lambda)) = A_{\alpha} B^{\alpha}$$

The repeated index α sums the product of the corresponding components of A and B. This process is known as "contraction" in tensor language.

If two p-forms are expressed in terms of coefficients $F_{ab\ldots p}$ and $G_{mn\ldots s}$, the inner product of these is:

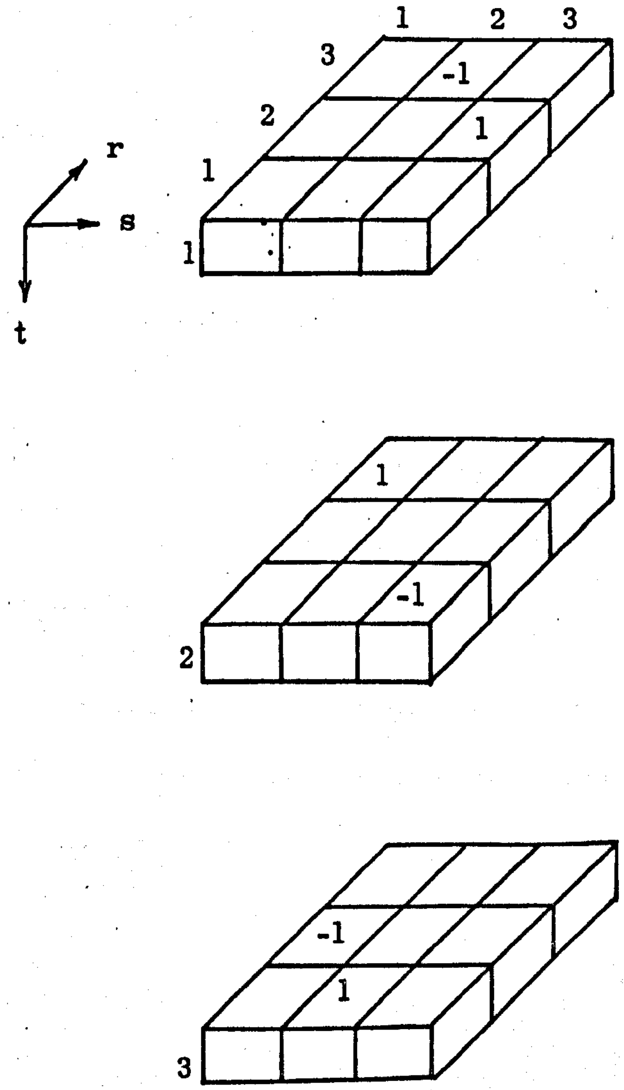
$$\frac{1}{p!} F_{ab\ldots p} G^{ab\ldots p}$$

(The skewsymmetric tensor, G, is expressed in terms of contravariant components). This is a special of contraction in which the contracted system is not merely reduced in rank but in fact results in a scalar.

3-10-3 Dual Tensors

A special skewsymmetric matrix "e" is first introduced, whose terms are: +1, -1 and 0. In 3-dimensions:

$e^{rst} =$



In n-dimensions,

$$e^{\alpha\beta\ldots\pi} = +1,\ -1,\ \text{or},\ 0$$

depending on whether

(i) an even permutation of α, β, ... π will restore~~d~~ the sequence 1, 2, ..., n

(or)

(ii) an odd permutation will restore it

(or)

(iii) any index is repeated

The e just defined is useful in dealing with determinants and dual tensors$^{(20)}$. In terms of e, the determinant of the transformation tensor C^a_α is:

$$|C| = e^{\alpha\beta\ldots\pi} \quad C^1_\alpha\ C^2_\beta\ \ldots\ C^n_\pi$$

The transformation law for "e" is:

$$|C| \quad e^{ab\ldots n} = e^{\alpha\beta\ldots\pi} \quad C^a_\alpha\ C^b_\beta\ \ldots\ C^n_\pi$$

Because of the appearance of the determinant C in the above equation, e is called a relative tensor of weight one. A tensor of weight one is also called a tensor density and is used in the formulation of a network model for Maxwell's electromagnetic field equations. The determinant of the metric tensor, g, is a tensor of weight two. Using the square-root of g, another skewsymmetric tensor is formed:

$$\mathcal{E}^{ab\ldots n} = \frac{1}{\sqrt{g}} e^{ab\ldots n}$$

which transforms as an absolute tensor.

If a tensor "F" of rank p has "f" as its dual tensor, then(20):

$$f^{\pi\mu\ldots\sigma} = \frac{1}{p!} F_{\alpha\beta\ldots\delta} \mathcal{E}^{\alpha\beta\ldots\delta\pi\mu\ldots\sigma}$$

$$= \frac{1}{p!} \frac{1}{\sqrt{g}} F_{\alpha\beta\ldots\delta} e^{\alpha\beta\ldots\delta\pi\mu\ldots\sigma} \qquad (3\text{-}43)$$

(The author has modified slightly the definition given by Brand(20). In equation 3-43, the sequences of F and $\mathcal{E}$ are reversed as compared to reference 20, page 370. This new definition agrees with the star operator, as defined by Flanders(17))..

The significance of equation 3-43 can be illustrated by taking the example in fig. 4a. If $\omega = A_\alpha dx^\alpha$ and $\lambda = B_\beta dx^\alpha$, then from equation 3-41,

$$\omega \wedge \lambda = \tfrac{1}{2} F_{\alpha\beta}\, dx^{\alpha} \wedge dx^{\beta}$$

where, $F_{\alpha\beta} = (A_{\alpha}B_{\beta} - A_{\beta}\, B_{\alpha})$

In the expanded form,

$$\omega \wedge \lambda = F_{12}\, dx^{1} \wedge dx^{2} + F_{23}\, dx^{2} \wedge dx^{3} + F_{31}\, dx^{3} \wedge dx^{1}$$

$$\mu = *(\omega \wedge \lambda) = F_{12}\, dx^{3} + F_{23}\, dx^{1} + F_{31}\, dx^{2}$$

$$= \tfrac{1}{2} F_{\alpha\beta}\, e^{\alpha\beta\gamma}\, dx^{\gamma} = f^{\gamma}\, dx^{\gamma} \qquad (3\text{-}44)$$

If curvilinear coordinates are used, equation 3-44 assumes the more general form:

$$\mu = \tfrac{1}{2} F_{\alpha\beta}\ \mathcal{E}^{\alpha\beta\gamma}\ g_{\gamma\pi}\ dx^{\pi} = f^{\gamma}\, g_{\gamma\pi}\, dx^{\pi}$$

The appearance of the term $\sqrt{g}$ was not discussed in Section 3-5, because orthonormal coordinates were assumed there. Equation 3-14 can now be generalised by redefining the unit n-dimensional volume:

$$d\sigma = \sqrt{g}\ dx^{1} \wedge dx^{2} \wedge \ldots \wedge dx^{n} \qquad (3\text{-}14b)$$

If the metric tensor has a negative determinant, such as in the space-time coordinate system, the modulus is taken to obtain the square root of g (i. e. $\sqrt{|g|}$).

3-10-4 Exterior Derivatives - Tensor Form

The generalised Kronecker delta, used in Section 3-10-1 to formulate exterior product, is employed here to generalise exterior products (generalised curl of tensors). If $T_{bc\ldots g}$ is a skewsymmetric tensor of rank p, the generalised curl is[20]:

$$\frac{1}{p!} \quad \delta^{abc\ldots g}_{mn\ldots q} \quad \frac{\partial T_{bc\ldots g}}{\partial x^a} \qquad (3\text{-}45)$$

For a scalar, f,

$$\bar{d}f = \left(\frac{1}{0!} \ \delta^m_a \ \frac{\partial f}{\partial x^m}\right) dx^a$$

$$= \delta^m_a \left(\frac{\partial f}{\partial x^m}\right) dx^a = \frac{\partial f}{\partial x^m} \ dx^m$$

For a vector, H,

$$\bar{d}H = \tfrac{1}{2}\left(\frac{1}{1!} \ \delta^{ab}_{mn} \ \frac{\partial H_b}{\partial x^a}\right) \quad dx^m \wedge dx^n$$

$$= \tfrac{1}{2}(\delta^a_m \quad \delta^b_n - \delta^a_n \quad \delta^b_m) \frac{\partial H_b}{\partial x^a} dx^m \wedge dx^n$$

$$= \tfrac{1}{2}(\frac{\partial H_n}{\partial x^m} - \frac{\partial H_m}{\partial x^n}) dx^m \wedge dx^n$$

This is seen to agree with equation 3-25.

For a skewsymmetric tensor $\tilde{F}$, of rank two,

$$\bar{d} \tilde{F} = \frac{1}{3!} (\frac{1}{2!} \delta^{abc}_{mnk} \frac{\partial F_{bc}}{\partial x^a}) dx^m \wedge dx^n \wedge dx^k$$

$$= \frac{1}{3!} (\frac{\partial F_{nk}}{\partial x^m} + \frac{\partial F_{km}}{\partial x^n} + \frac{\partial F_{mn}}{\partial x^k}) dx^m \wedge dx^n \wedge dx^k$$

This is seen to agree with equation 3-28.

The divergence is obtained by taking the dual of the curl of the dual of a tensor. For a skewsymmetric tensor $T^{bc\ldots m}$,

$$\text{div } T = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^m} (T^{bc\ldots m} \sqrt{g}) \qquad (3\text{-}46)$$

From section 3-10, it is seen that in tensor analysis the maze of indices often makes it difficult to observe the differences between various types of quantities. It also appears to lack the substantial body of results established once and for all in exterior forms. The exterior calculus is found to be helpful in generalising the concepts of curl, divergence and dual, particularly with orthonormal frames. Exterior derivatives are based on equality of mixed partials. Tensor analysis may have to be used, when second order partial derivatives change their value when the sequence of differentiation is changed.

The exterior calculus developed in the present chapter is applied to electromagnetic field equations in Chapter 4. In terms of these Maxwell's curl and divergence equations assume a more general form. Line integrals of 1-forms, surface integrals of 2-forms and volume integrals of 3-forms are used to construct a network model for the electromagnetic field. Kirchhoff's voltage and current laws for the network interpret Poincare Lemma for the differential forms. Stokes' theorem reduces some of the surface and volume integrals to line and surface integrals. In this form they interpret Faraday's and Ampere's Laws. As the study undertaken in this Chapter is valid for any general curvilinear reference system, the network model derived for the electromagnetic field has an invariant structure. The parameters of the network, however, depend on the coordinate axes chosen.

In particular, a term, $\sqrt{g}$, appears in surface and volume integrals of "tensor densities" (tensors multiplied by $\sqrt{g}$). This term depends on the metrical properties of the coordinates.

The generalisation of exterior derivatives to space-coordinate system gives two tensors - two groups of interacting fields - for the electromagnetic field. As a consequence of the two tensors a third tensor, called "Stress-energy Tensor", arises. Maxwell's electromagnetic stresses and Poynting's energy-flow vector form the components of this tensor. These are examined in Chapter 4.

CHAPTER 4

MAXWELL'S ELECTROMAGNETIC EQUATIONS

Maxwell's equations presented in a general manner in this chapter will be used to construct a network model. The region of electromagnetic field studied is first divided into several small subregions, integrations performed in the subregions being displayed as network quantities. When Maxwell's equations are expressed in exterior differential form, the integrations can be performed in a general coordinate system. This leads to a network model, the structure of which is independent of the choice of coordinates. The network parameters will, however, depend on the system used. In tensor notation, the integration involves, in some cases, tensor densities and scalar densities.

Maxwell's equations expressed in space-time coordinate system uses two field tensors F_{mn} and H^{mn}. A stress energy tensor is seen to result from the product terms of field tensors F_{mn} and H^{mn}, and gives Maxwell's electromagnetic stresses and Poynting's energy flow vector. The forces on small regions can be obtained by the integration of the appropriate stress-tensor densities. This will be illustrated in a later section when evaluating the torque of an electric machine.

Maxwell's equations summarise, mathematically, the macroscopic field theory. The term "macroscopic" excludes quantum electrodynamics and microscopic study of atoms. In "field" theory, force and matter are assumed to be distributed continuously through space (as opposed to discrete points and action-at-a-distance theory).

In M.K.S. system of units, Maxwell's equations, in vector notation, are:

$$\text{(a)} \quad \text{curl}\ \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\text{(b)} \quad \text{curl}\ \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\text{(c)} \quad \text{div}\ \vec{B} = 0$$

$$\text{(d)} \quad \text{div}\ \vec{D} = \rho \qquad (4\text{-}1)$$

The properties of a stationary media give further equations:

$$B = \mu \eta_o H \qquad (\eta_o = 4 \pi \times 10^{-7})$$
$$D = \kappa \varepsilon_o E \qquad (\varepsilon_o = 8.854 \times 10^{-12})$$
$$J = \sigma E \qquad (4-2)$$

For frames of reference in uniform relative motion, the Lorentz transformation leaves Maxwell's equations (4-1) invariant. The macroscopic parameters μ, κ and σ are also subject to transformation, and equations 4-2 are no longer valid.

4-1 Exterior Differential Form of Maxwell's Equations

First a set of 1-forms is defined as follows:

$$E = E_1 \, dx + E_2 \, dy + E_3 \, dz$$
$$H = H_1 \, dx + H_2 \, dy + H_3 \, dz$$
$$B = B_1 \, dx + B_2 \, dy + B_3 \, dz$$
$$D = D_1 \, dx + D_2 \, dy + D_3 \, dz$$
$$J = J_1 \, dx + J_2 \, dy + J_3 \, dz \qquad (4-3)$$

Partial derivatives with respect to time will be denoted by a dot over the letters. Maxwell's equations, in terms of exterior forms, are:

$$(a) \quad * \bar{d}\,(E) = -\dot{B}$$

$$(b) \quad * \bar{d}\,(H) = J + \dot{D}$$

$$(c) \quad * \bar{d} * (B) = 0$$

$$(d) \quad * \bar{d} * (D) = \rho \qquad (4\text{-}4)$$

Equations 4-4 are examined below, in a Cartesian coordinate system, using results from sections 3-6-1 and 3-7-1.

4-1-1

Equation 4-4(a) gives, on expansion,

$$* \bar{d}\,(E) = *\bar{d}\,(E_1\, dx + E_2\, dy + E_3\, dz)$$

$$= *\bar{d}\,(E_1\, dx) + *\bar{d}\,(E_2\, dy) + *\bar{d}\,(E_3\, dz)$$

$$\bar{d}\,(E_1\, dx) = \bar{d}E_1 \wedge dx$$

$$= (\frac{\partial E_1}{\partial y}\, dy \wedge dx + \frac{\partial E_1}{\partial z}\, dz \wedge dx)$$

$$\therefore * \bar{d}\,(E) \;=\; *\left(\frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y}\right) dx \wedge dy \;+\; *\left(\frac{\partial E_3}{\partial y} - \frac{\partial E_2}{\partial z}\right) dy \wedge dz$$

$$+ \; * \left(\frac{\partial E_1}{\partial z} - \frac{\partial E_3}{\partial x}\right) dz \wedge dx$$

For the Cartesian coordinate system,

$$* \,(dx \wedge dy) \;=\; dz \quad \text{and so on}$$

$$\therefore * \bar{d}\,(E) \;=\; \left(\frac{\partial E_3}{\partial y} - \frac{\partial E_2}{\partial z}\right) dx \;+\; \left(\frac{\partial E_1}{\partial z} - \frac{\partial E_3}{\partial x}\right) dy$$

$$+ \left(\frac{\partial E_2}{\partial x} - \frac{\partial E_1}{\partial y}\right) dz$$

$$= \left(- \frac{\partial B_1}{\partial t}\right) dx \;+\; \left(- \frac{\partial B_2}{\partial t}\right) dy \;+\; \left(- \frac{\partial B_3}{\partial t}\right) dz$$

i. e. $\quad * \bar{d}\,(E) \;=\; -\dot{B}$

This is the "exterior" form of the vector equation curl $\vec{E} = -(\partial \vec{B} / \partial t)$. The relation between this curl (to be called "conventional curl") and the exterior derivative $\bar{d}$ (to be called "generalised curl") may be stated: -

conventional curl = dual of generalised curl

Equation 4-4(b) can be expanded likewise.

4-1-2

Equation 4-4(c) is now examined.

$$*(B) = B_1 \, dy \wedge dz + B_2 \, dz \wedge dx + B_3 \, dx \wedge dy$$

$$\bar{d} * (B) = \frac{\partial B_1}{\partial x} dx \wedge dy \wedge dz + \frac{\partial B_2}{\partial y} dy \wedge dz \wedge dx + \frac{\partial B_3}{\partial z} dz \wedge dx \wedge dy$$

$$= \left(\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z} \right) dx \wedge dy \wedge dz$$

$$*\bar{d} * (B) = \left(\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z} \right) = 0$$

Equation 4-4(d) is expanded similarly.

4-1-3

The divergence equations can be seen to follow from Poincare Lemma derived in Section 3-8.

For the 1-form E,

$$\bar{d} \, (\bar{d} \, E) = 0 \qquad \text{(Poincare Lemma)}$$

This can be rearranged as:

$$*\bar{d} * \left[*\bar{d} \, (E) \right] = 0$$

Substituting $*\bar{d} \, (E) = -\dot{B} = -(\partial B / \partial t)$, it is seen that:

$$*\bar{d} * \left(\frac{\partial B}{\partial t} \right) = 0$$

If the components of flux density, B, and its derivatives are continuous, the commutation of operators ($*\bar{d}*$) and ($\partial/\ \partial t$) is permissible(22), i. e.

$$\frac{\partial}{\partial t}\ [* \bar{d}\ * (B)\ ]\ =\ 0$$

It follows that at every point in the field, the divergence of magnetic flux density, B, is unchanging with time. If ever in its past history the field has vanished, this constant must be zero. Since one may reasonably suppose that the initial creation of the field was at a finite time ago, we conclude that the divergence of flux density, B, is zero(22).

Likewise, $\bar{d}\ [\ \bar{d}\ (H)\ ]\ =\ 0$ gives:

$$*\bar{d} * (J)\ +\ \left\{ *\bar{d} * (\frac{\partial D}{\partial t}) \right\}\ =\ 0 \qquad (4\text{-}5)$$

The conservation of charge is expressed as:

$$(\ \partial J_1/\ \partial x) + (\ \partial J_2/\ \partial y) + (\ \partial J_3/\ \partial z) + (\ \partial \rho/\ \partial t)\ =\ 0$$

or,

$$*\bar{d} * (J) + \frac{\partial \rho}{\partial t}\ =\ 0 \qquad (4\text{-}6)$$

Substitution of equation 4-6 in 4-5 yields, on commuting the operators ($*\bar{d}*$) and ($\partial / \partial t$),

$$\frac{\partial}{\partial t}\left\{ -\rho + *\bar{d}*(D) \right\} = 0$$

If, again, we suppose that at some time in the past history the field has vanished, the divergence of electric flux density, D, is seen to give the charge density.

4-2 Integral Form of Maxwell's Equations

In Section 3-9, Stokes' theorem was expressed, in exterior form as:

$$\oint_{\partial\Sigma} \omega = \int_{\Sigma} \bar{d}(\omega) \qquad (3\text{-}36)$$

From equation 3-36 and 4-4(a), it is seen that:

$$\oint_{\partial\Sigma} E = \iint_{\Sigma} \bar{d}(E) = -\iint_{\Sigma} *\dot{B} \qquad (4\text{-}7)$$

In this integration, Σ is an area enclosed by a curve $\partial\Sigma$

In vector notation, equation 4-7 is:

$$\oint_{C} \vec{E} \cdot \vec{dl} = -\iint_{S} \dot{\vec{B}} \cdot \vec{n}\, dS \qquad (4\text{-}8)$$

($\vec{dl}$ is the vector elemental length along the closed curve c and $\vec{n}$ is the normal unit vector of an elemental area dS - fig. 4b). Equation 4-8 is seen to give Faraday's law, on commuting the integral and dot operators; the emf around a closed path is the negative rate of change of flux enclosed by the path.

For the divergence equation $*\bar{d} *D = \rho$, Stokes' theorem gives:

$$\oiint_{\partial\Sigma} *(D) = \iiint_{\Sigma} \bar{d} * (D) = \iiint_{\Sigma} * \rho \qquad (4\text{-}9)$$

In this integration Σ is a volume enclosed by a surface $\partial\Sigma$. In vector notation, equation 4-9 is:

$$\oiint \vec{D} \cdot \vec{n}\ dS = \iiint \rho\ dV \qquad \text{(Gauss' theorem)}$$

The theorem states that the total electric flux integrated over a surface enclosing a volume equals the charge inside it.

For equations 4-4(b) and (c):

$$\oint_{\partial\Sigma} H = \iint_{\Sigma} * (J + \dot{D})$$

$$\oiint_{\partial\Sigma} *B = 0 \qquad (4\text{-}10)$$

In a later section, the integrals of the field quantities over infinitesimal regions will be represented by network quantities in order to establish a network model for Maxwell's equations.

The integration of a p-form ω over a p-dimensional region Σ is carried out in three stages: (i) an elemental subregion $d\Sigma$ in the region Σ is expressed as a p-form. (ii) a scalar is formed by the inner products of the two p-forms ω and $d\Sigma$. (iii) this scalar is integrated using the usual integral methods in calculus. The same result is achieved by replacing the base vectors $\vec{dx}^k$ in the differential form by the ordinary differentials dx^k. For example, if:

$$E = E_1 \vec{dx}^1 + E_2 \vec{dx}^2 + E_3 \vec{dx}^3,$$

then

$$\int_\Sigma E = \int (E_1\, dx^1 + E_2\, dx^2 + E_3\, dx^3)$$

The integration is carried out in the usual way. It must be remembered that, in the preceding sections, the arrows above the base vectors $\vec{dx}^k$ have been omitted for convenience. However, this need not give rise to any confusion provided it is remembered that under the integration sign, dx^k represents ordinary differentials; elsewhere it gives the base vectors.

4-3 Maxwell's Equations in Space-Time Coordinates

In relativistic treatment, time is considered as the fourth coordinate. In Section 3-4, the cotime, ŧ, given by the product ct was introduced. The cotime has the dimension of length and its inner product is:

$$((d\text{ŧ}\ ,\ d\text{ŧ})) \;=\; -1 \qquad \text{(Section 3-4)}$$

In the space-time system, Maxwell's equations are expressed in terms of two 2-forms, $\tilde{F}$ and $\tilde{H}$. The 2-form $\tilde{F}$ contains the space components of magnetic flux density, B, and electric force, E. The 2-form $\tilde{H}$ contains magnetic force, H and electric flux density D. In terms of these, the exterior differential equations are:

$$\bar{d}\,(\tilde{F}) \;=\; 0 \qquad (4\text{-}11)$$

$$\bar{\delta}\,(\tilde{H}) \;=\; *\,\bar{d}\,*\,(\tilde{H}) \;=\; \mathcal{S} \qquad (4\text{-}12)$$

where, $\mathcal{S}$ is a 1-form containing current and charge densities. The 2-form $\tilde{F}$ is obtained from vector and scalar potentials as follows.

A 1-form can be formed, consisting of the vector potential, A, of the magnetic field and the scalar potential, ψ of the electric field:

$$A = A_1\,dx + A_2\,dy + A_3\,dz + A_4\,dt \qquad (4\text{-}13)$$

$$(A_4 = \psi\ /c\ )$$

The exterior derivative of the 1-form A gives a 2-form $\tilde{F}$:

$$\begin{aligned}\tilde{F} = \bar{d}\,(A) = {} & \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) dx \wedge dy + \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) dy \wedge dz \\ & + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) dz \wedge dx + \left(\frac{\partial A_4}{\partial x} - \frac{\partial A_1}{\partial t}\right) dx \wedge dt \\ & + \left(\frac{\partial A_4}{\partial y} - \frac{\partial A_2}{\partial t}\right) dy \wedge dt + \left(\frac{\partial A_4}{\partial z} - \frac{\partial A_3}{\partial t}\right) dz \wedge dt\end{aligned}$$

$$(4\text{-}14)$$

The first three bracketed terms are recognised as the magnetic flux densities B_3, B_1 and B_2 given by the vector equation, $\vec{B} = \text{curl}\,\vec{A}$. The last three bracketed terms of equation 4-12 are E_1/c, E_2/c and E_3/c respectively, given by the vector equation, $\vec{E} = -\,\partial\vec{A}/\ \partial t \quad + (\text{grad}\ \psi\ )$.

In terms of flux density B and electric force E, equation 4-14 is:

$$\tilde{F} = \bar{d}(A) = B_1\, dy \wedge dz + B_2\, dz \wedge dx + B_3\, dx \wedge dy$$

$$+ (E_1/c)\, dx \wedge dt + (E_2/c)\, dy \wedge dt + (E_3/c)\, dz \wedge dt \qquad (4\text{-}15)$$

Poincare Lemma states that:

$$\bar{d}\,(\bar{d}\,A) = \bar{d}\,(\tilde{F}) = 0 \qquad (4\text{-}16)$$

The exterior derivative of the 2-form $\tilde{F}$ is (from Section 3-6-2):

$$\bar{d}\,(\tilde{F}) = \sum_{(a,b,c)} \left(\frac{\partial F_{bc}}{\partial x^a} + \frac{\partial F_{ca}}{\partial x^b} + \frac{\partial F_{ab}}{\partial x^c} \right) dx^a \wedge dx^b \wedge dx^c \qquad (3\text{-}28)$$

In this equation F_{ab} gives the coefficient of the 2-form $\tilde{F}$ corresponding to the term $dx^a \wedge dx^b$.

The bracketed term above must be zero in order to satisfy equation 4-16.

$$\boxed{\left(\frac{\partial F_{bc}}{\partial x^a} + \frac{\partial F_{ca}}{\partial x^b} + \frac{\partial F_{ab}}{\partial x^c} \right) = 0} \qquad (4\text{-}17)$$

When a = 1, b = 2 and c = 3

$$\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z} = 0$$

or,

$$\text{div } \vec{B} = 0$$

The equation, curl $\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$, results when c = 4 and (a, b) = (1, 2) (1, 3) and (2, 3). Thus half of Maxwell's equations appear in the space-time system as:

$$\bar{d}\,(\tilde{F}) = 0 \qquad (4\text{-}11)$$

The other half appears in terms of a second 2-form, $\tilde{H}$, as follows:

4-3-1

The second 2-form, $\tilde{H}$, contains the magnetic field vector, H, and electric field vector, D. Using the relations, $B = \eta_o H$, $D = \mathcal{E}_o E$ and $1/\sqrt{\mathcal{E}_o \eta_o} = c$, in free space, the 2-form $\frac{1}{\eta_o} \tilde{F}$ ($\eta_o = 4\pi \times 10^{-7}$) is seen to expand to:

$$H_1\, dy \wedge dz + H_2\, dz \wedge dx + H_3\, dx \wedge dy$$
$$+ cD_1\, dx \wedge dt + cD_2\, dy \wedge dt + cD_3\, dz \wedge dt$$

(4-18)

Denoting this 2-form by $\tilde{H}$, the divergence of $\tilde{H}$ is (from Appendix 2):

$$\bar{\delta}\,(\tilde{H}) = *\bar{d}*(\tilde{H}) = J_1\,dx + J_2\,dy + J_3\,dz - c\,\rho\,dt \qquad (4\text{-}19)$$

For free space, current and charge densities are zero, and,

$$\bar{\delta}\,(\tilde{H}) = *\bar{d}*(\tilde{H}) = 0 \qquad \text{(for free space)} \qquad (4\text{-}20)$$

For other media, equation 4-18 can still be used to define the second field form $\tilde{H}$. It is no longer related to the first field form $\tilde{F}$ by the constant η_o. The divergence of $\tilde{H}$ is not zero but gives a 1-form

$$\bar{\delta}\,(\tilde{H}) = s \qquad \text{(for other media)} \qquad (4\text{-}12)$$

where,

$$s = J_1\,dx + J_2\,dy + J_3\,dz - c\,\rho\,dt \qquad (4\text{-}21)$$

4-4 Tensor Density (21, 23, 24)

In general, a quantity transforming as a tensor multiplied by $\sqrt{g}$ is called a tensor density (g is the determinant of the metric tensor). This arises in field analysis when the dual of field vectors are considered and the integration of such quantities used to construct the field network model. In general coordinates, the element of volume is(21):

$$dV = \sqrt{g} \quad dx^1 \, dx^2 \, dx^3$$

This is invariant under coordinate transformations.

Writing $d\Upsilon$ for the product $(dx^1 \, dx^2 \, dx^3)$,

$$dV = \sqrt{g} \quad d\Upsilon \qquad (4\text{-}22)$$

If T is a scalar, i. e. an invariant function of position, then $T\sqrt{g} \quad d\Upsilon$ is an invariant (Eddington(24)). The integral $\iiint T\sqrt{g} \quad d\Upsilon$ is also invariant. Each unit cube (whose edges are dx^1, dx^2 and dx^3) contributes the amount $T\sqrt{g}$ to this invariant. Accordingly $T\sqrt{g}$ is called a scalar density. An example of this type is the mass of a volume, given in polar coordinates by $\iiint \rho_0 r \, (dr \, d\theta dz)$. In this integration

ρ_0 is the specific gravity of the medium and $r = \sqrt{g}$.

If T is a vector or a higher order tensor, say $T^{\alpha\beta}$, then

$$\iiint T^{\alpha\beta} \sqrt{g} \quad d\Upsilon$$

is generally not a tensor; although this is a sum of a number of tensors, it implies a summation of tensors not located at the same point. But as the region of integration is made smaller and smaller, the transformation law approaches more and more nearly that of a single tensor(24). Thus the tensor density $T^{\alpha\beta} \sqrt{g}$ can at least be integrated over infinitesimal regions.

4-4-1 Tensor Density Form of Maxwell's Equations

The vector equation, curl $\vec{A} = \vec{B}$, is first examined in terms of tensor densities. In exterior form the vector equation is expressed as:

$$*\bar{d}\,(A) = B \qquad (4\text{-}23)$$

The exterior derivative of the vector, A_b, in tensor notation is (from Section 3-10-4, equation 3-45):

$$\frac{1}{1!} \delta^{ab}_{mn} \frac{\partial A_b}{\partial x^a} = \left(\frac{\partial A_n}{\partial x^m} - \frac{\partial A_m}{\partial x^n} \right) \quad (4\text{-}24)$$

The dual of the tensor given in the bracketed term above is (from Section 3-10-3, equation 3-43):

$$\frac{1}{2!} \frac{1}{\sqrt{g}} \left(\frac{\partial A_n}{\partial x^m} - \frac{\partial A_m}{\partial x^n} \right) e^{mnp} \quad (4\text{-}25)$$

(e^{mnp} is the skewsymmetric unit matrix). Equation, curl $\vec{A} = \vec{B}$, now becomes:

$$\frac{1}{2!} \frac{1}{\sqrt{g}} \left(\frac{\partial A_n}{\partial x^m} - \frac{\partial A_m}{\partial x^n} \right) e^{mnp} = B^p \quad (4\text{-}26)$$

This can be reformulated as:

$$\left(\frac{\partial A_n}{\partial x^m} - \frac{\partial A_m}{\partial x^n} \right) = \left(\sqrt{g}\, B^p \right) e_{pmn} \quad (4\text{-}27)$$

The tensor-density, $\sqrt{g}\, B^p$, is usually written as $B^{p'}$.

Likewise using tensor densities (denoted by primed quanties), Maxwell's equations are:

$$\left(\frac{\partial E_n}{\partial x^m} - \frac{\partial E_m}{\partial x^n} \right) = - \left(\frac{\partial B^{p'}}{\partial t} \right) e_{pmn}$$

$$\left(\frac{\partial H_n}{\partial x^m} - \frac{\partial H_m}{\partial x^n} \right) = \left(J^{p'} + \frac{\partial D^{p'}}{\partial t} \right) e_{pmn}$$

$$\frac{\partial B^{p'}}{\partial x^{p}} = 0$$

$$\frac{\partial D^{p'}}{\partial x^{p}} = \rho' \qquad (\rho' = \rho\sqrt{g}) \qquad (4\text{-}28)$$

4-5 Four Dimensional Field Tensors

The analysis of Maxwell's equations in space-time coordinates given in Section 4-3 can now be brought into tensor notation. The field equations are expressed in terms of two field tensors F_{mn} and H^{mn} as:

$$\frac{\partial F_{nk}}{\partial x^{m}} + \frac{\partial F_{km}}{\partial x^{n}} + \frac{\partial F_{mn}}{\partial x^{k}} = 0$$

$$\frac{1}{\sqrt{g}} \quad \frac{(H^{mn} \sqrt{g})}{x^{n}} = s^{m}$$

The field tensor F_{mn} is derived from a tensor,

$$A_{m} = \begin{array}{|c|c|c|c|}\hline A_1 & A_2 & A_3 & A_4 \\ \hline \end{array} \qquad (4\text{-}29)$$

A_1, A_2 and A_3 are the components of vector potential in 3-space used in Section 4-4-1 and A_4 is given in terms of the scalar electric potential, ψ, as $A_4 = \psi / c$;
x^1, x^2, x^3 are the generalised coordinates in 3-space;
coordinate $x^4 = ct$, gives cotime.

The exterior derivative of the tensor A_m is (from Section 3-10-4, equation 3-45):

$$\frac{1}{1!} \delta^{ab}_{mn} \frac{\partial A_b}{\partial x^a} = \left(\frac{\partial A_n}{\partial x^m} - \frac{\partial A_m}{\partial x^n} \right) = F_{mn} \qquad (4\text{-}30)$$

when m, n = 1, 2, 3, equation 4-30 gives the components of magnetic flux density B (see equation 4-27). When n = 4, F_{m4} gives E_m/c.

In matrix form, the field tensor F_{mn} is:

$$F_{mn} = \begin{array}{c|c|c|c|c|}
{}^{\quad n\rightarrow}_{m\downarrow} & 1 & 2 & 3 & 4 \\ \hline
1 & & B^3\sqrt{g} & -B^2\sqrt{g} & \frac{E_1}{c} \\ \hline
2 & -B^3\sqrt{g} & & B^1\sqrt{g} & \frac{E_2}{c} \\ \hline
3 & B^2\sqrt{g} & -B^1\sqrt{g} & & \frac{E_3}{c} \\ \hline
4 & -\frac{E_1}{c} & -\frac{E_2}{c} & -\frac{E_3}{c} & \\ \hline
\end{array} \qquad (4\text{-}31)$$

The exterior derivative of F_{mn} is (from Section 3-10-4, equation 3-45):

$$\frac{1}{2!}\,\delta^{abc}_{mnk}\,\frac{\partial F_{bc}}{\partial x^a}$$

$$= \left(\frac{\partial F_{nk}}{\partial x^m} + \frac{\partial F_{km}}{\partial x^n} + \frac{\partial F_{mn}}{\partial x^k}\right)$$

$$= 0 \qquad (4\text{-}32)$$

This gives half of Maxwell's equations. In vector notation they are:

$$\text{curl } \vec{E} + \partial\vec{B}/\partial t = 0$$

$$\text{div } \vec{B} = 0$$

4-5-1

It was pointed out in Section 4-3 that the other two Maxwell's equations are given in terms of a second field tensor in the space-time coordinate system.

Proceeding in a similar manner, a second field tensor given in free space by F_{mn}/η_o is examined. Its divergence is found to be zero. The divergence of a tensor of rank two, in tensor notation is (from Section 3-10-4, equation 3-46):

$$\frac{1}{\sqrt{g}} \frac{\partial(H^{mn}\sqrt{g})}{\partial x^n}$$

The second field tensor H_{mn} is brought into the contravariant form by multiplying it with the inverse metric tensor and raising the indices.

$$H^{mn} = H_{ab}\, g^{am}\, g^{bn}$$

In matrix form,

$$H^{mn} = \begin{array}{c|c|c|c|c|}
{}^{\;\;n\,\rightarrow}_{m\,\downarrow} & 1 & 2 & 3 & 4 \\
\hline
1 & & \frac{H_3}{\sqrt{g}} & -\frac{H_2}{\sqrt{g}} & -cD^1 \\
\hline
2 & -\frac{H_3}{\sqrt{g}} & & \frac{H_1}{\sqrt{g}} & -cD^2 \\
\hline
3 & \frac{H_2}{\sqrt{g}} & -\frac{H_1}{\sqrt{g}} & & -cD^3 \\
\hline
4 & cD^1 & cD^2 & cD^3 & \\
\hline
\end{array} \qquad (4\text{-}33)$$

The divergence is zero.

i. e.

$$\frac{1}{\sqrt{g}} \; \frac{\partial (H^{mn} \sqrt{g})}{\partial x^n} = 0 \qquad \text{(for free space)}$$

For other media, the second field tensor H^{mn} can still be used. It is no longer related to the first field tensor by the constant η_o. Its divergence is not zero.

$$\frac{1}{\sqrt{g}} \; \frac{\partial (H^{mn} \sqrt{g})}{\partial x^n} = s^{m} \qquad \text{(for other media)} \qquad (4\text{-}34)$$

where,

$$\mathcal{S}^m = \overset{\rightarrow m}{\begin{array}{|c|c|c|c|}\hline J^1 & J^2 & J^3 & c\rho \\ \hline \end{array}} \qquad (4\text{-}35)$$

Equation 4-34 gives the Maxwell relations:-

$$\text{curl}\ \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J}$$

$$\text{div}\ \vec{D} = \rho$$

The determinant of the metric tensor in space-time coordinate is negative. When the square root is taken, the modulus is used, i. e. $\sqrt{-g}$

4-5-2 Stress-Energy Tensor

Maxwell uses products of the type $B^m H_n$ and $E_p D^q$ to express his magnetic and electric stresses (the space in which the fields are present being conceived to be in a state of tension analogous to that of an elastic medium). Poynting uses products of the type $E_r H_s$ to arrive at his energy flow vector. The forces on small regions can be obtained by the integration of the appropriate stress-tensor densities. This will be illustrated in a later section for an electric machine. To arrive at a general form of the stress-energy tensor, the field tensors F_{mn} and H^{mn} developed earlier are mutliplied and contracted to give $F_{mn} H^{np}$ (i.e. the matrices given by equation 4-31 and 4-33 are multiplied.) However, a slight modification is necessary; a term $\frac{1}{4} F_{rs} H^{sr}$ is subtracted from the diagonal elements of the matrix product $F_{mn} H^{np}$. Rainich$^{(25)}$ shows this firstly for Newtonian fields and then extends the modification for electromagnetic fields.

Using the letter T for the stress-energy tensor,

$$T^p_m = F_{mn} H^{np} - \frac{1}{4} \delta^p_m F_{rs} H^{sr} .$$

In matrix form,

$$T^{p}_{m} = \begin{array}{c|c|c|c|c} m \backslash p & 1 & 2 & 3 & 4 \\ \hline 1 & B^1H_1 + E_1D^1 - W & B^2H_1 + E_1D^2 & B^3H_1 + E_1D^3 & c\sqrt{g}(B^2D^3 - B^3D^2) \\ \hline 2 & B^1H_2 + E_2D^1 & B^2H_2 + E_2D^2 - W & B^3H_2 + E_2D^3 & c\sqrt{g}(B^3D^1 - B^1D^3) \\ \hline 3 & B^1H_3 + E_3D^1 & B^2H_3 + E_3D^2 & B^3H_3 + E_3D^3 - W & c\sqrt{g}(B^1D^2 - B^2D^1) \\ \hline 4 & \frac{1}{c\sqrt{g}}(E_2H_3 - E_3H_2) & \frac{1}{c\sqrt{g}}(E_3H_1 - E_1H_3) & \frac{1}{c\sqrt{g}}(E_1H_2 - E_2H_1) & W \end{array} \tag{4-36}$$

$$W = \text{total stored energy} = \tfrac{1}{2} B^{m} H_{m} + \tfrac{1}{2} E_{n} D^{n}$$

In equation 4-36, when m = p = 1,

$$T^{1}_{1} = B^1H_1 + E_1D^1 - \tfrac{1}{2}B^mH_m - \tfrac{1}{2}E_nD^n$$

$$= \tfrac{1}{2}(B^1H_1 - B^2H_2 - B^3H_3) + \tfrac{1}{2}(E_1D^1 - E_2D^2 - E_3D^3)$$

gives the tensile stress. The terms T^2_2 and T^3_3 give tensile stresses along the other coordinates. The tengential stresses are given by terms of the type T^1_2.

In the matrix given by equation 4-36 the last row (T^p_4) gives the energy flow vector given by Poynting and the total stored energy. The first three terms of the last column (T^4_m) represent a tensor known as electromagnetic momentum tensor.

4-6 Physical Components

A vector can be resolved along the three directions given by the base vectors of a coordinate system. The magnitudes of the three components are known as physical components. They can be related to covariant or contravariant tensor components using the metric tensor. In Cartesian coordinates; there is no distinction between the various components, since the metric tensor is unity. The permeability, conductivity and permittivity of a media are given in terms of physical components by certain constants (for iron, a B-H curve is usually given instead of a constant). As integrals of tensor components are used on the field network model, the relation between the tensor and tensor-density components and the properties of the media are examined here.

4-6-1

In 2-dimensional polar coordinates, for example, the base vectors are rleated to the Cartesian unit vectors as:

$$\vec{dr} = \cos\theta \ \vec{dx} + \sin\theta \ \vec{dy}$$

$$\vec{d\theta} = -(\sin\theta / r) \ \vec{dx} + (\cos\theta / r) \ \vec{dy}$$

This is seen to follow from the transformation:

$$x = r \cos\theta \quad \text{and} \quad y = r \sin\theta$$

The vector dot product $\vec{d\theta} \cdot \vec{d\theta} = 1/r^2$. It follows that the base vector $\vec{d\theta}$ is not of unit length and varies from point to point. If a tensor is given by a single component E_θ, in vector form it will be $E_\theta \ \vec{d\theta}$. The magnitude of this vector, $E_{(\theta)}$, is clearly E_θ / r.

4-6-2

If the metric tensor, in a general orthogonal coordinate system is given as:

$$g_{\alpha\beta} = \begin{array}{c|c|c|c} \alpha \backslash \beta & 1 & 2 & 3 \\ \hline 1 & (h_1)^2 & & \\ \hline 2 & & (h_2)^2 & \\ \hline 3 & & & (h_3)^2 \\ \hline \end{array}$$

then, the relation between the tensor components of electric force, E, and the tensor-density components of electric flux density, D, is:

$$\varepsilon_1 = \frac{\sqrt{g}D^1}{E_1} = \frac{h_2h_3}{h_1} \kappa \varepsilon_0$$

$$\varepsilon_2 = \frac{\sqrt{g}D^2}{E_2} = \frac{h_3h_1}{h_2} \kappa \varepsilon_0$$

$$\varepsilon_3 = \frac{\sqrt{g}D^3}{E_3} = \frac{h_1h_2}{h_3} \kappa \varepsilon_0 \qquad (4\text{-}37)$$

(. κ is the relative permittivity of the media and

$\mathcal{E}_o$ = 8.854 x 10^{-12}; M.K.S. system of units are used)

Similar equations can be written for the magnetic field quantities.

CHAPTER 5

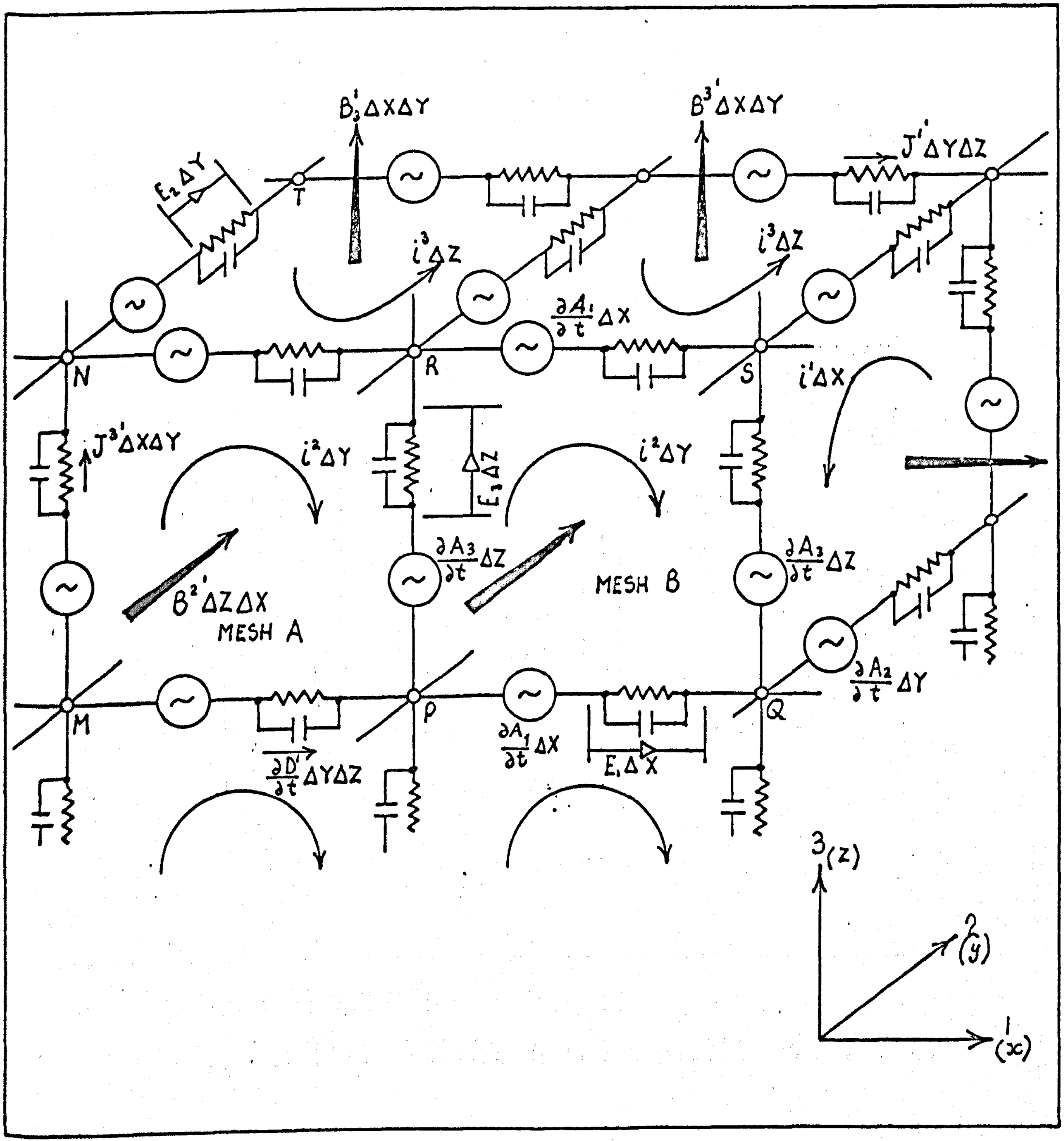
NETWORK MODEL FOR MAXWELL'S EQUATIONS

The network model for the region of space studied is established by first dividing the region into subregions, small enough to give the desired degree of accuracy. The subregions are blocks formed with edges Δx^1, Δx^2 and Δx^3. Any curvilinear reference system may be used to give the coordinates x^1, x^2 and x^3 and the blocks may be of uneven lengths in different directions. Line, surface and volume integration of the magnetic and electric field quantities (expressed as exterior differential forms) is carried out over the blocks. Some of the integrals reduce to simpler forms by the use of Stokes' theorem. The resulting integrals are displayed as network quantities. The model developed by Kron(26) makes use of ideal transformers in addition to inductors, capacitors and resistors. Kirchhoff's algebraic relations for the network model translate the differential relations in field equations, summarised by Maxwell, for electromagnetic field into micro-mesh network quantities. The algebraic diagram developed in Chapter 2 (Section 2-3) is used with the field network to bring out the differential structural relations.

In the algabraic analysis of Kron's network model it is difficult to display clearly on the algebraic diagram, the various equations of Maxwell, and in a manner in which it can be developed for a more general case than in three dimensions. A different type of network model is first introduced. In this, the integrals of the differential forms associated with electric field quantities are lumped into branches connecting adjacent points i. e. the edges forming blocks shown in Fig. 5a. The branches consist of capacitors and resistors. Integrals of the differential forms of the magnetic field quantities are associated with the surfaces of the blocks. Since the electric and magnetic field quantities are interacting, the resulting model is called an interconnected model. The algabraic structural relations are studied for this model. In Kron's network model, the magnetic field quantities are also lumped into branches using inductors and ideal transformers. This model is more suitable for practical applications using network analysers.

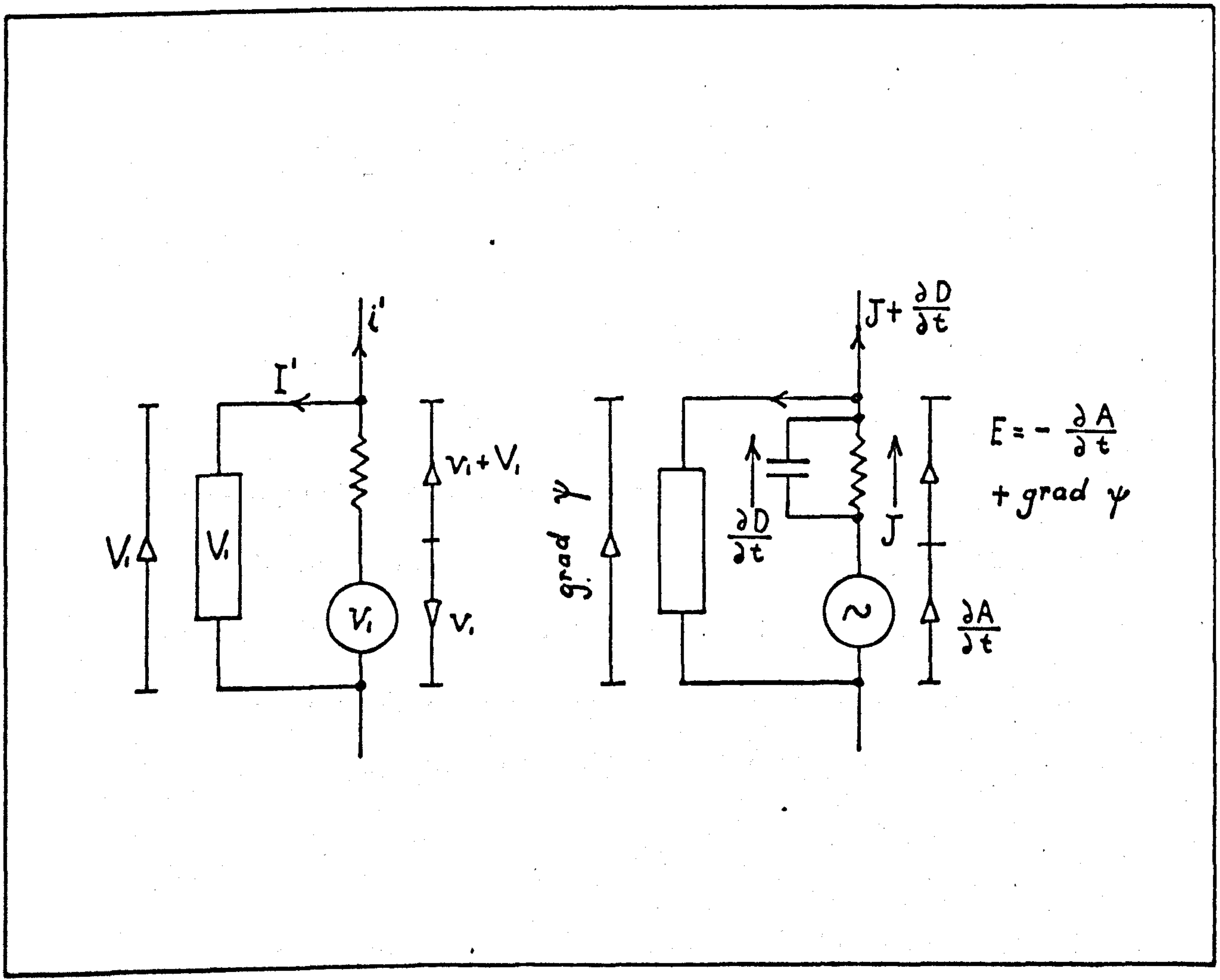
5-1 Interconnected Model

The branches of the network model in Fig. 5a are constructed by considering the line integral of electric force,



PART (a)

FIG. 5. INTERCONNECTED NETWORK MODEL FOR FIELDS



PART (b) PRIMITIVE 1-NETWORK

FIG. 5. INTERCONNECTED NETWORK MODEL FOR FIELDS

$\int E$, and the surface integrals of current density, $\iint * J$, and electric flux density, $\iint * D$, as follows. The exterior 1-form for the electric force,

$$E = E_1 \;\overrightarrow{dx} + E_2 \;\overrightarrow{dy} + E_3 \cdot \overrightarrow{dz} ,$$

is integrated along the edges of the blocks as:

$$\int E = \int E_1 \,dx + E_2 \,dy + E_3 \,dz \qquad (5\text{-}1)$$

Under the integration sign, dx, dy and dz are treated as ordinary differentials (Section 4-2). For example, along the edge PQ (Fig. 5a), dy = dz = 0, and the integral is $E_1 \Delta X$, and is shown by the voltage drop across an impedance. Along branches NT and PR they are $E_2 \Delta Y$ and $E_3 \Delta Z$ respectively.

For current density J, the exterior 2-form is:

$$*J = J^1 \sqrt{g}\, dy \wedge dz + J^2 \sqrt{g}\, dz \wedge dx + J^3 \sqrt{g}\, dx \wedge dy \qquad (5\text{-}2)$$

The star denotes that the 2-form above is a dual of the directional vector $J_1 \;\overrightarrow{dx} + J_2 \;\overrightarrow{dy} + J_3 \;\overrightarrow{dz}$. The dual is integrated over an area normal to the branch. The term $\sqrt{g}$ arises out of the star operation in taking the dual of vector J^m (Section 3-10-3). For the branch PQ, the integration of the 2-form is performed over a small area Σ (Fig. 6a):

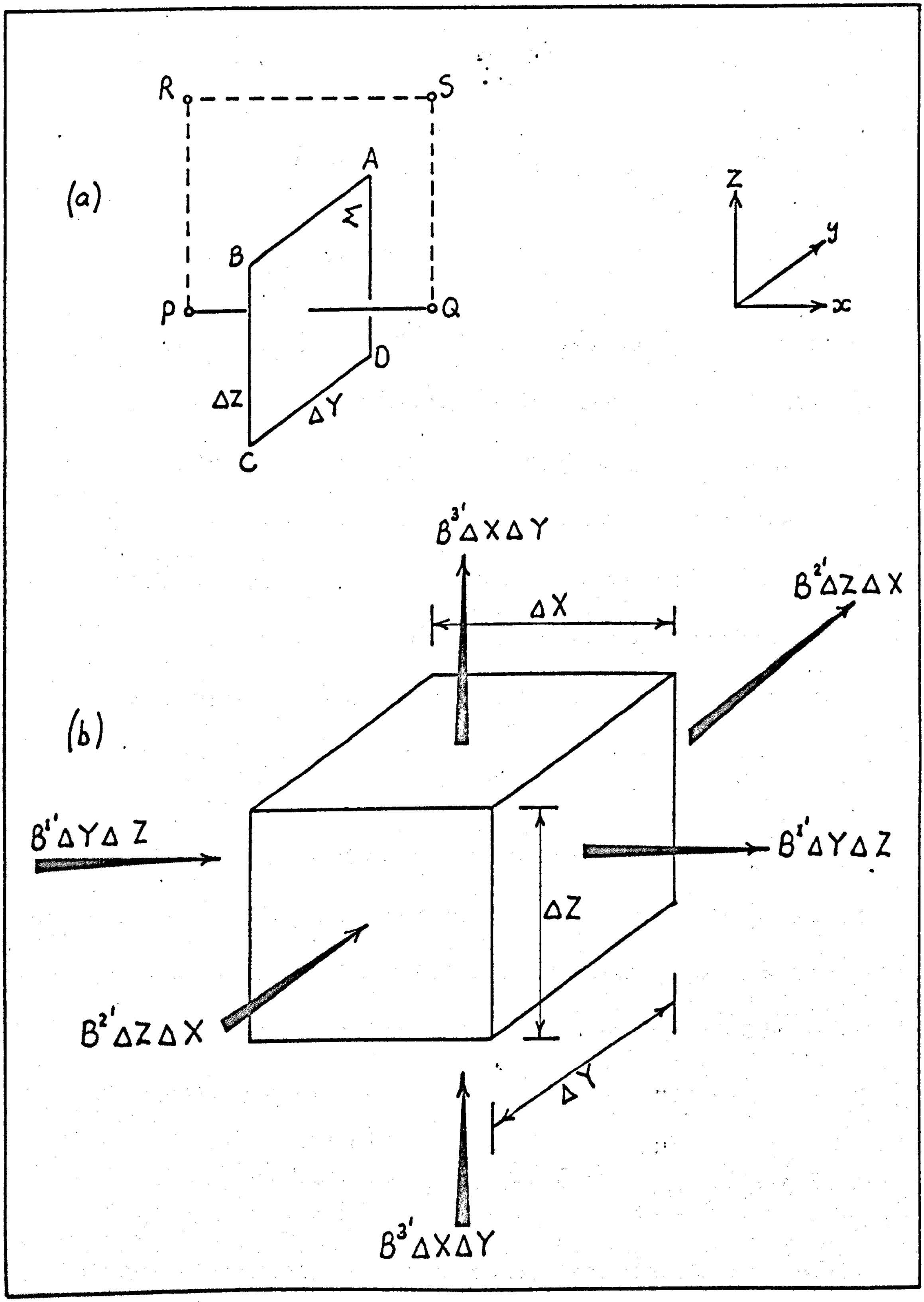


FIG. 6. ELEMENTAL BLOCKS AND REGIONS

$$\iint_{\Sigma} * J = \iint_{\Sigma} J^1 \sqrt{g}\, dy \;\; dz \quad (\text{since } dx = 0 \text{ on } \Sigma\,)$$

$$= J^1 \sqrt{g}\, \Delta Y \, \Delta Z$$

The quantity $J^1\sqrt{g}$, $= J^{1'}$, is a "tensor density" component of the current density (Section 4-4). The integral $J^{1'}$ $\Delta Y \, \Delta Z$ appears in the network as a current through the resistor. This gives the conduction current. The displacement current, $\frac{\partial D^{1'}}{\partial t}$ $\Delta Y \, \Delta Z$ is, likewise, given by a current through the capacitor. The voltage drops and currents in y- and z-directions are expressed in terms of the corresponding vector components.

5-1-1

The values of the resistances and capacitances can be evaluated in terms of the conductivity and permittivity of the media as follows. The relation between the tensor components in terms of κ and σ is given in Section 4-6-2 for orthogonal systems:

$$\mathcal{E}_1 = \frac{\sqrt{g}\; D^1}{E_1} = \frac{h_2 h_3}{h_1} \kappa \, \mathcal{E}_o \quad \text{etc.} \qquad (\text{equation } 4\text{-}37)$$

(In these equations, g is the determinant of the metric tensor:

$$g_{\alpha\beta} = \begin{array}{c|c|c|c} & \beta\rightarrow & & \\ \hline \alpha\downarrow & (h_1)^2 & & \\ \hline & & (h_2)^2 & \\ \hline & & & (h_3)^2 \end{array}$$

κ is the permittivity of the media; $\mathcal{E}_o = 8.854 \times 10^{-12}$).

The capacitance in branch PQ (Fig. 5a) is then:

$$C_1 = \mathcal{E}_1 \frac{\Delta Y \, \Delta Z}{\Delta X} = (\frac{h_2 h_3}{h_1})(\frac{\Delta Y \, \Delta Z}{\Delta X}) \, \kappa \, \mathcal{E}_o \tag{5-3}$$

Similarly, the conductance in branch PQ is:

$$G_1 = (\frac{h_2 h_3}{h_1}) \, (\frac{\Delta Y \, \Delta Z}{\Delta X}) \, \sigma \tag{5-4}$$

The resistance in branch PQ = $R_1 = 1/G_1$. The impedance values in y- and z- directions are also evaluated in terms of κ and σ. If the coordinate system is not orthogonal, the simple relations of equation 4-37 can no longer be used. The resulting model would then have mutual resistance and capacitance terms.

5-1-2

In this section the voltage induced in a mesh of the network model, due to varying flux density is represented by a set of generators. For example, mesh PRSQ (mesh B) in Fig. 5a links a flux

$$= \iint_{\substack{\text{area}\\ \text{PRSQ}}} * B = B^2 \sqrt{g'} \; \Delta Z \, \Delta X = B^{2'} \; \Delta Z \; \Delta X$$

In this integral the star denotes that the 2-form is a dual of directional vector $B_1 \vec{dx} + B_2 \vec{dy} + B_3 \vec{dz}$.

The induced mesh voltage is given, according to Faraday's law, by the negative rate of change of flux i. e. $-(\partial B^{2'} / \partial t) \Delta Z \; \Delta X$. The same result is obtained directly from Maxwell's equations using Stokes' theorem from Section 4-2, equation 4-7,

$$\int_{\partial \Sigma} E = - \iint_{\Sigma} * \dot{B}$$

In this integration, Σ is an area enclosed by a curve $\partial \Sigma$. The line integral of E is given by voltage drops across the impedances. For the mesh PRSQ, they must equal the induced mesh voltage shown by generators (since the total mesh voltage is zero by Kirchhoff's law). The term, $-\iint_{\Sigma} {}^{*}\dot{B}$, must, therefore, represent the induced voltage.

The induced voltage, in a mesh, due to varying flux density is represented by a set of four generators. For example, in mesh B, there are two generators of the type ($\partial A_1/ \partial t) \Delta X$ along branches PQ and RS; two of the type ($\partial A_3/ \partial t) \Delta Z$ along branches PR and QS. Although the generators in branches PQ and RS are both shown as ($\partial A_1/ \partial t) \Delta Z$, they represent numerically different quantities.

$$(\dot{A}_1 \Delta X)_{RS} - (\dot{A}_1 \Delta X)_{PQ} = (\frac{\partial \dot{A}_1}{\partial z} \Delta Z) \Delta X$$

(the dot over the letters denotes time derivatives). Similarly,

$$(\dot{A}_3 \Delta Z)_{QS} - (\dot{A}_3 \Delta Z)_{PR} = (\frac{\partial \dot{A}_3}{\partial x} \Delta X) \Delta Z$$

The total mesh voltage due to generators in this mesh

$$= (\frac{\partial \dot{A}_1}{\partial z} - \frac{\partial \dot{A}_3}{\partial x}) \Delta Z \Delta X = \dot{B}^{2'} \Delta Z \Delta X$$

This corresponds to the equation:

$$\vec{B} = \text{curl } \vec{A}$$

where, $\vec{A}$ is the magnetic vector potential. In Section 2-1, the mesh voltages in electric circuits were expressed in terms of branch voltages as: $v'_c = (C_c)_t \cdot v$ (equation 2-5). For the field network this relation gives:

$$\frac{\partial B}{\partial t} = \text{curl } \frac{\partial A}{\partial t}$$

The total voltage drop across the branch PQ. Fig. 5a, is:

$$V_{PQ} = \frac{\partial A_1}{\partial t} \Delta X + E_1 \Delta X$$

Using the field equation

$$\vec{E} = - \partial\vec{A}/ \partial t + (\vec{\text{grad}} \ \psi) ,$$

$$V_{PQ} = (\text{grad } \psi)_1 \Delta X = \frac{\partial \psi}{\partial x} \Delta X$$

$$= (\Delta \psi)_{PQ} \qquad (5\text{-}6)$$

The integral $B^{2'} \Delta Z \Delta X$ as such is not shown by any network parameter. It is associated with areas such as PRSQ. This quantity can be considered to exist in a circuit consisting of 2-dimensional elements such as $\Delta X\ \Delta Y$, $\Delta Y\ \Delta Z$ and $\Delta Z\ \Delta X$. Such a circuit is a 2-chain and is considered further in Section 5-2-3.

For the interconnected model, it is seen that impedances consist of resistors and capacitors. The conduction and displacement currents are given by the surface integrals of dual quantities, $*J$ and $*\dot{D}$. The resulting terms are of the type $j^1 \sqrt{g}\ \ \Delta Y\ \Delta Z$ and $\frac{\partial D^1}{\partial t} \sqrt{g}\ \ \Delta Y\ \Delta Z$. The voltage drop across the impedances are given by the line integral of the electric force, E(i. e. terms of the type $E_1\ \Delta X$). The "generator" voltages are given in terms of magnetic vector potential by terms of the type $\frac{\partial A_1}{\partial t}\ \Delta X$. When the induced mesh voltage is expressed as a sum of the branch "generator" voltages, i. e. $(C_c)_t \cdot v$, the induced voltage takes the form

$$- \frac{\partial B}{\partial t} = -\text{curl}\ \frac{\partial A}{\partial t}$$

The total branch voltage is given in terms of a scalar potential as $\Delta \psi$.

5-1-3

The mesh currents of the network are examined in this section. Their relation to branch currents, given in electric circuit theory by the equation: $i = C_c \cdot i^{c'}$ is studied. The relation between mesh currents and the magnetic force, H is first analysed. The mesh currents contribute to magnetomotive forces. The mmf's across the blocks are given by the line integrals $\int H$. These are of the type $H_1 \, \Delta X$. The mesh currents are not shown as i's but as $i^1 \, \Delta X$, $i^2 \, \Delta Y$ and $i^3 \, \Delta Z$, so that the magnetic force, H, can be directly related to "i".

The mesh currents $i^1 \, \Delta X$, $i^2 \, \Delta Y$ and $i^3 \, \Delta Z$ in the network of Fig. 5a contribute to a magnetic field according to Ampêre's law. The magnetic field due to sources outside the region under consideration can be evaluated in terms of a scalar magnetic potential Ω [27]. The total magnetic force in terms of these is:

$$H = i + \text{grad}\, \Omega = i + \bar{d}\Omega \qquad (5\text{-}7)$$

The exterior differential (denoted by $\bar{d}$) of equation 5-7 gives:

$$\bar{d}H = \bar{d}i + \bar{d}(\bar{d}\Omega)$$

By Poincare Lemma, since Ω is a scalar field (exterior 0-form), $\bar{d}(\bar{d}\Omega) = 0$, so that

$$\bar{d}H = \bar{d}i \qquad (5\text{-}8)$$

The exterior differential, $\bar{d}i$, will now be shown to relate branch currents to mesh currents. The branch current PR (in Fig. 5a) can be expressed in terms of two mesh currents of the type $i^2 \Delta Y$ for meshes A and B and two other mesh currents of the type $i^1 \Delta X$ lying in y-z plane. The contribution of the mesh currents A and B to the branch current of PR is:

$$(i^2 \Delta Y)_{\text{mesh B}} - (i^2 \Delta Y)_{\text{mesh A}} = \frac{\partial i^2}{\partial x} \Delta X \, \Delta Y$$

Similarly the two mesh currents in y-z plane contribute a term $= - \frac{\partial i^1}{\partial y} \Delta Y \, \Delta X$. The branch current PR,as a sum of the mesh currents is, then:

$$\left(\frac{\partial i^2}{\partial x} - \frac{\partial i^1}{\partial y} \right) \Delta X \; \Delta Y$$

The same branch current, in terms of resistor and capacitor currents is:

$$(J^{3'} + \partial D^{3'} / \partial t) \, \Delta Z \, \Delta Y$$

i. e.

$$J^{3'} + \partial D^{3'} / \partial t = \left(\frac{\partial i^2}{\partial x} - \frac{\partial i^1}{\partial y} \right) \qquad (5\text{-}9)$$

In mesh analysis of electric circuits (Section 2-1), this relation between branch and mesh currents was written in the form $i = (C_c) \cdot i^{c'}$.

Equation 5-9 can be extended to other branches as well to give the general relation

$$*(J + \dot{D}) = \bar{d}\, i \qquad (5\text{-}10)$$

From equations 5-8 and 5-10, it is seen that

$$\bar{d} H = *(J + \dot{D})$$

giving the Maxwell equation: curl $\vec{H} = \vec{J} + \partial\vec{D}/\partial t$.

5-1-4

In the nodal analysis of electric circuits (Section 2-2), Kirchhoff's current law is expressed in the form:

$$(A_o)_t \cdot i = 0$$

This is now examined in terms of field quantities. At the node R (Fig. 5a), the branch currents add up to zero (Kirchhoff's current law). The branch currents RS and NR add to, at node R

$$\left\{(J^{1'} + \dot{D}^{1'})\Delta Y\, \Delta Z\right\}_{RS} - \left\{(J^{1'} + \dot{D}^{1'})\Delta Y\, \Delta Z\right\}_{NR}$$

$$= \frac{\partial}{\partial x} (J^{1'} + \dot{D}^{1'}) \Delta X \, \Delta Y \, \Delta Z$$

The other branch currents likewise add to:

$$\frac{\partial}{\partial y} (J^{2'} + \dot{D}^{2'}) \Delta X \, \Delta Y \, \Delta Z \quad \text{and} \quad \frac{\partial}{\partial z} (J^{3'} + \dot{D}^{3'}) \Delta X \, \Delta Y \, \Delta Z$$

The addition of all currents at node R is then:

$$\left\{ \frac{\partial}{\partial x} (J^{1'} + \dot{D}^{1'}) + \frac{\partial}{\partial y} (J^{2'} + \dot{D}^{2'}) + \frac{\partial}{\partial z} (J^{3'} + \dot{D}^{3'}) \right\} \Delta X \, \Delta Y \, \Delta Z$$

This is zero by Kirchhoff's current law,

$(A_o)_t \cdot i = 0,$ giving

$$\text{div } (J + \dot{D}) = 0 \tag{5-11}$$

The divergence equation, div D $= \rho$, follows as a consequence of equation 5-11 (in Section 4-1-3 this was established assuming conservation of charge). To each node, such as node R in Fig. 5a, is attached six capacitor plates. The charges on the capacitor plates add up to a total of $\rho' \, \Delta X \, \Delta Y \, \Delta Z$.

For the interconnected model, it is seen that branch current-mesh current relation, $i = C_c \cdot i^{c'}$, leads to the equation: curl $\vec{H} = \vec{J} + \partial\vec{D}/\partial t$. It is also seen that Kirchhoff's current law, $(A_o)_t \cdot i = 0$, results in: div $(\vec{J} + \partial\vec{D}/\partial t) = 0$. From this the divergence equation, div $\vec{D} = \rho$, follows.

5-2 Algebraic Diagram

The algebraic structural relations of the network can be summarised in the form of the flow diagram or "algebraic diagram" developed in Section 2-3 for electric circuits. Fig. 7a extends the algebraic diagram developed in Section 2-3 (Fig. 2) to circuits formed by 2-dimensional regions (2-chains) of the type $\Delta X \, \Delta Y$, $\Delta Y \, \Delta Z$ and $\Delta Z \, \Delta X$. These will be called "2-networks". A closed volume bounded by surfaces would represent a generalised 2-mesh, for example, a cube bounded by six squares. It was pointed out in Section 5-1-2 that field quantities of the type $B^{2'} \, \Delta Z \, \Delta X$ do not appear in the field network as such, but are associated with the corresponding area elements. These will be examined in terms of 2-networks. First, the 1-circuit is examined.

5-2-1 1-Circuit Voltages

Fig. 7a recapitulates the algebraic diagrams of Figs. 2 and 3 (Sections 2-3 and 2-5). The transformation C^1_c links closed-mesh currents to branch currents, and A_{o1} links the node-pair (open-mesh) voltages to branch voltages. The voltage, v_1, represents impressed voltage from a source

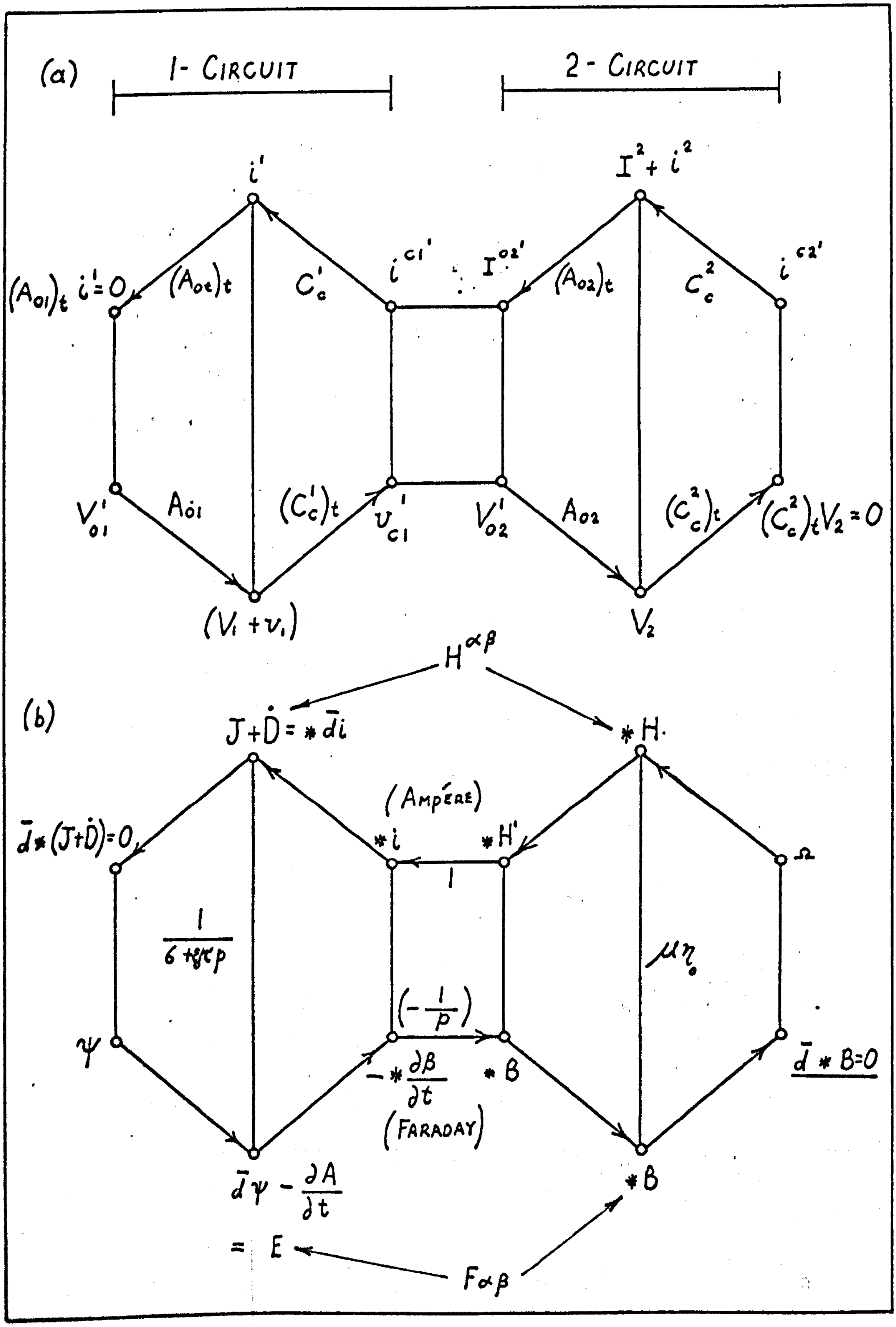


FIG. 7 ALGEBRAIC DIAGRAM FOR FIELD NETWORK

connected in series with the impedance element. The response voltage, V_1, denotes the total voltage drop across the branch. The voltage drop across the impedance is $(V_1 + v_1)$. The current, i^1 is a response due to the source voltage and flows through the closed meshes of a network.

The branch currents of the field network are of the type $(J^{1'} + \dot{D}^{1'})\ \Delta Y\ \Delta Z$, $(J^{2'} + \dot{D}^{2'})\ \Delta Z\ \Delta X$ and $(J^{3'} + \dot{D}^{3'})\ \Delta X\ \Delta Y$. The branch impedances are here seen to consist of resistors in parallel with capacitors. The impedance drops are $E_1\ \Delta X$, $E_2\ \Delta Y$ and $E_3\ \Delta Z$. The impedance is of the type $1/(\sigma + \epsilon_o \kappa \frac{d}{dt})$. These are shown in the algebraic diagram by long vertical lines (Fig. 7b). The "primitive" elements of the field network are shown in Fig. 5b. It is seen that the impedance drop, E, is split into two parts: (i) $\bar{d}\psi$ (or grad ψ) corresponding to the total branch voltage drop, V_1, and (ii) $\partial A/\partial t$ corresponding to the generator voltage, v_1. The electric force, E, is split into its scalar and vector potentials.

In the circuit analysis of Section 2-1, it was pointed out that:

$$(C_c^1)_t\ (V_1 + v_1) = (C_c^1)_t\ V_1 + (C_c^1)_t\ v_1 = 0 + v'_{c1} \qquad (5\text{-}12)$$

The term $(C_c^1)_t\ V_1$ sums up voltage drops of the branches in a closed mesh and is zero by Kirchhoff's laws. In the field network this corresponds to $\bar{d}\,(\bar{d}\,\psi)$ which is zero by Poincare Lemma. This is demonstrated using equation 5-6 in Section 5-1-2.

$$V_{PQ} = \frac{\partial \psi}{\partial x}\ \Delta X$$

For the closed mesh PQSR in Fig. 5a,

$$V_{PQ} + V_{QS} + V_{SR} + V_{RP} = 0$$

i. e.

$$\left(\frac{\partial \psi}{\partial x}\right)_{PQ} \Delta X + \left(\frac{\partial \psi}{\partial z}\right)_{QS} \Delta Z$$

$$-\left(\frac{\partial \psi}{\partial x}\right)_{RS} \Delta X - \left(\frac{\partial \psi}{\partial z}\right)_{PR} \Delta Z = 0$$

i. e.

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial z}\right)\ \Delta Z\ \Delta X - \frac{\partial}{\partial z}\left(\frac{\partial \psi}{\partial x}\right) \Delta X\ \Delta Z = 0$$

Similar analysis for other meshes gives the general relation $\bar{d}\,(\bar{d}\,\psi) = 0.$

In equation 5-12 the term $(C_c^1)_t \cdot v_1$ was analysed for field networks in Section 5-1-2. The mesh voltage in terms of the branch generator voltages were found to be of the form:

$$\text{curl } \frac{\partial A}{\partial t} = \frac{\partial B}{\partial t}$$

Equation 5-12 thus translates into circuit terms the differential equation:

$$\bar{d}\,(E) = \bar{d}\,(\bar{d}\gamma - \frac{\partial A}{\partial t}) = \bar{d}\,(\bar{d}\gamma) - \bar{d}\,(\frac{\partial A}{\partial t}) = 0 - *\,(\frac{\partial B}{\partial t}) \quad (5\text{-}13)$$

The algebraic diagram Fig. 7 shows these results.

5-2-2- 1-Circuit Currents

In electric circuit theory, the branch current - mesh current relation is given in Section 2-1 as: $i^1 = C^1_c \, i^{cl'}$. For the field network, the analysis in Section 5-1-3 gave equation 5-10:

$$\bar{d}\,i = *(J + \dot{D}) \qquad (5\text{-}10)$$

Further, in circuit theory, Kirchhoff's current law for the nodes is expressed in Section 2-2 in terms of transformation matrix A_{ol} as:

$$(A_{ol})_t \, i^1 = 0$$

For the field network, the analysis in Section 5-1-4 gave equation 5-11:

$$\bar{d} * (J + \dot{D}) = 0 \qquad (5\text{-}11)$$

Such constraints on the circuit currents and voltages and the corresponding field quantities are shown in Fig. 7. Also, combining equations 5-10 and 5-11,

$$\bar{d} * (J + \dot{D}) = \bar{d}\,(\bar{d}\, i) = 0 \qquad (5\text{-}15)$$

This is a statement of Poincare Lemma which in network terms leads to:

$$(A_{ol})_t \,.\, i^1 = (A_{ol})_t \,.\, C^1_c \,.\, i^{cl'} = 0$$

5-2-3- 2-Circuit Voltages

In the network of Fig. 5a, the magnetic field quantities are not shown as such by any network parameters. The magnetic flux density, for example, is integrated over regions such as $\Delta X\, \Delta Y$, $\Delta Y\, \Delta Z$ and $\Delta Z\, \Delta X$. To an area PQ SR is attached an integral $B^{2'}\, \Delta Z\, \Delta X$ (Fig. 5a) and so forth. These quantities do not appear on the branches of the network. The areas can be assumed to form a network of 2-dimensional regions. For example, a cube is a closed volume bounded by six square (or rectangular) areas. A closed mesh or loop in 1-circuit theory is a chain of lines enclosing an area. A closed

2-mesh (for a 2-network consisting of area elements) would give a chain of areas enclosing a volume such as a cube.

The surface integrals of magnetic flux density $B^{1'} \Delta Y \Delta Z$, $B^{2'} \Delta Z \Delta X$, $B^{3'} \Delta X \Delta Y$ and three similar quantities appear on the six faces of a cube $\Delta X \Delta Y \Delta Z$ (Fig. 6b). They add up to:

$$\left(\frac{\partial B^{1'}}{\partial x} + \frac{\partial B^{2'}}{\partial y} + \frac{\partial B^{3'}}{\partial z}\right) \Delta X \Delta Y \Delta Z$$

The bracketed term gives the divergence of magnetic flux density. The sum of the integrals on the six faces of the cube is therefore zero. If the surface integrals are treated as 2-circuit voltages, V_2, they add up to zero on a closed 2-mesh. This is shown on the algebraic diagram by the relation $(C_c^2)_t \cdot V_2 = 0$ (Fig. 7) - an extension of the Kirchhoff's voltage law for closed meshes of a 2-circuit.

5-2-4 2-Circuit Currents

In electric circuit theory, the impressed current, I, is applied across the two boundaries of a branch or a chain of branches forming an open path. The boundaries are called node-pairs and the current, I, as an impressed node-pair current. For 2-circuits, the boundary of an area is the mesh surrounding the area. The mesh current, i^1, can be regarded as an impressed 2-circuit current I^2. The algebraic diagram (Fig. 7) can then summarise the 2-circuit relations. This relationship is now examined in terms of field quantities.

Maxwell equation, $\text{curl}\,\vec{H} = \vec{J} + \partial\vec{D} / \partial t$, becomes in exterior notation:

$$\bar{d}H = *(J + \dot{D}) \qquad \text{(equation 4-4b)}$$

The integral of the 2-forms above, over an area $\Delta Y\ \Delta Z$ (Fig. 6a) is:

$$\iint_{\Sigma} \bar{d}H = \iint_{\Sigma} *(J + \dot{D})$$

$$= \oint_{\partial\Sigma} H \qquad \text{(by Stokes theorem)}$$

The symbol $\partial\Sigma$ stands for the boundary of the area, i. e. A - B - C - D - A.

The surface integral $\iint_{\Sigma} *(J + \dot{D})$ has been represented earlier in the network model, by branch currents. In Fig. 6a, the surface integral over the area Σ is shown as branch current PQ. That is, the dual of the directional vector,

$J_1 \vec{dx} + J_2 \vec{dy} + J_3 \vec{dz}$, gives a 2-form, $*J$. This quantity is integrated over areas such as Σ . The resulting surface integrals, $\iint_{\Sigma} *J$, are now associated, not with the areas such as Σ , but with the duals of these areas. In Fig. 6a, branch PQ represents the dual of area Σ . The integrals could be represented in a symbolic form as $* \iint_{\Sigma} * (J + \dot{D})$. The integral of the magnetic field vector, H, along a line can likewise be associated with a surface normal to the line. The line integral $\int_{B}^{A} H$ in Fig. 6a, $= H_2 \Delta Y$, could be associated with the surface PQ SR. Symbolically it can be written as $* \int H$ and is shown on the algebraic diagram as a 2-circuit current (i. e. as $I^2 + i^2$) (Fig. 7).

In Section 5-1-3, the magnetic force, H, was considered to be a sum of the effects of sources outside the region under consideration and those inside. Equation 5-7 gave:

$$H = i + \text{grad}\,\Omega = i + \bar{d}\,\Omega$$

For the area PQ RS in Fig. 6a, this corresponds to

$$H_2 \Delta Y = i^2 \Delta Y + (\Delta \Omega)_{BA} \qquad (5\text{-}15)$$

In equation 5-15, the mesh current $i^2 \Delta Y$ can be regarded as an impressed 2-circuit current I^2. The algebraic diagram (Fig. 7) summarises the equations relating the 2-circuit quantities.

Maxwell's differential equations are thus seen to be summarised by an algebraic diagram. The network model translates the differential structure into an algebraic structure. Kirchhoff's laws interpret the electromagnetic laws. However, a more general form of Kirchhoff's laws to cover 2-circuits is used.

5-2-5

The field tensors F_{mn} (equation 4-31) and H^{mn} (equation 4-33) can be written in terms of partitioned submatrices as:

$$F_{mn} = \left[\begin{array}{c|c} \overset{\sim}{*}B & \frac{E}{c} \\ \hline -E/c & o \end{array}\right] \quad \text{and} \quad H^{mn} = \left[\begin{array}{c|c} \overset{\sim}{*}H & -cD \\ \hline cD & o \end{array}\right]$$

The matrices above have been partitioned into space- and time-varying terms. The TILDE over star denotes the dual of a vector in 3-space and not the dual of the vector in the four space-time coordinate system. The grouping of the electric and magnetic field vectors above is indicated on the algebraic diagram, Fig. 7b. The field tensors F_{mn} and H^{mn} are seen to be expressed by the generalised current and voltage tensors. It is seen that the voltage terms are E (covariant tensor) and *B. Although B has been expressed as a contravariant tensor, B^{α}, the dual of B (= $\varepsilon_{\alpha\beta\gamma} B^{\gamma}$) is doubly covariant. The current terms are $J + \dot{D}$ (contravariant tensor) and *H (doubly contravariant tensor). These are expressed as cohomology and homology sequences in Fig. 8. This is developed further in Section 5-2-6.

The Kirchhoff's voltage laws for closed meshes are seen to give the field relations: $\mathrm{curl}\,\vec{E} + \partial\vec{B}/\partial t = 0$ and $\mathrm{div}\vec{B} = 0$. In terms of field tensor F_{mn}, the two relations are combined into one equation:

$$\left(\frac{\partial F_{nk}}{\partial x^{m}} + \frac{\partial F_{km}}{\partial x^{n}} + \frac{\partial F_{mn}}{\partial x^{k}}\right) = 0 \qquad (4\text{-}32)$$

The current laws of Kirchhoff for open meshes give the field relations: curl $\vec{H} = \vec{J} + \partial\vec{D}/\partial t$ and div $(\vec{J} + \partial\vec{D}/\partial t) = 0$. In terms of field tensor H^{mn}, the two relations are combined into one equation:

$$\frac{1}{\sqrt{g}} \frac{\partial(H^{mn}\sqrt{g})}{\partial x^n} = \mathcal{S}^m \tag{4-34}$$

<u>5-2-6</u>

When the field variation with respect to time is sinusoidal and has a frequency, f, the operator $\partial / \partial t$ can be replaced by $j\omega$ (= $j\,2\pi f$). The algebraic diagram can then be simplified to that of Fig. 8. The algebraic diagram also shows possible extensions (dotted lines) using incidence matrices shown in Fig. 3. (Section 2-5). The incidence matrices connect points to lines, lines to planes, planes to cubes and so forth. They can be expressed by a matrix product of the transformation matrices as shown in Section 2-5:

$$C_c^o \cdot (A_{o1})_t = M_1^o \qquad \text{(equation 2-21)}$$

$$C_c^1 \cdot (A_{o2})_t = M_2^1 \qquad \text{(equation 2-24)}$$

similarly,

$$C_c^2 \cdot (A_{o3})_t = M_3^2$$

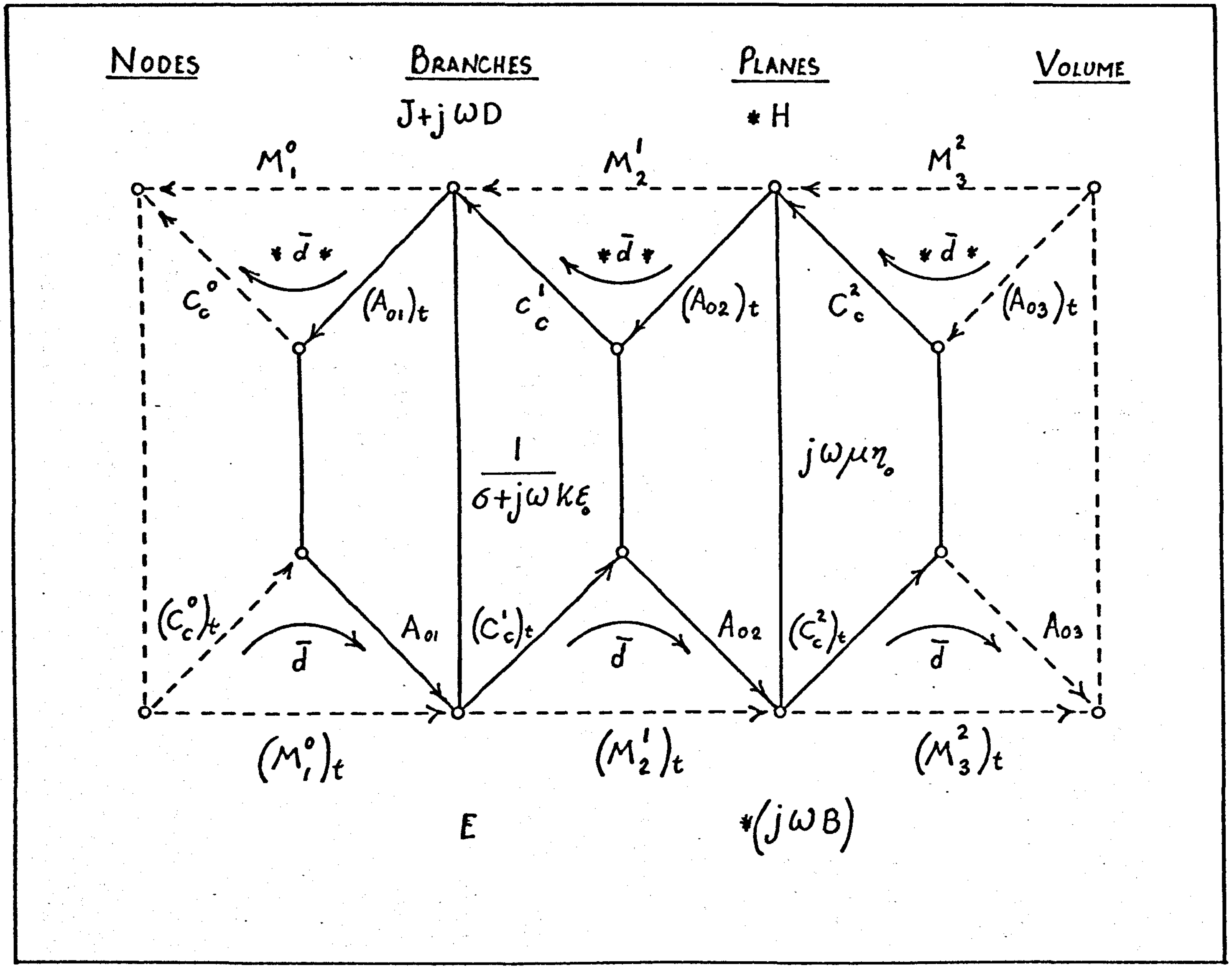


FIG. 8. EXTENDED ALGEBRAIC DIAGRAM FOR FIELD NETWORK

The covariant voltage tensors are said to be in cohomology sequence in topological language. The contravariant current tensors are in homology sequence.

The algabraic diagram of Fig. 8. also shows how the generalised curl and divergence of the field quantities are obtained in network terms. The generalised curl in exterior notation is "$\bar{d}$". The gradient of the scalar potential, ψ , gives a vector $\bar{d}\psi$. The generalised curl of electric force, E, results in a 2-form, $\bar{d}E = -*\dot{B}$. The generalised curl of this 2-form is: $\bar{d}(\bar{d}E) = -\bar{d}(*\dot{B}) = 0$, i.e. the divergence of flux density = 0. The generalised divergence is given the notation $*\bar{d}*$. In this notation, $*\bar{d}*(J + \dot{D}) = 0$. Also the curl equation, curl $\vec{H} = \vec{J} + \partial\vec{D}/\partial t$ can be rearranged as:

$$*\bar{d}*\ (*H) = (J + \dot{D})$$

The generalised divergence takes a 2-form into a 1-form (vector) and a 1-form into a 0-form (scalar).

Fig. 9 is reproduced from Kron's analysis of multidimensional space filters (Reference 6). This is seen to be an extension of the algebraic diagram developed for the inter-

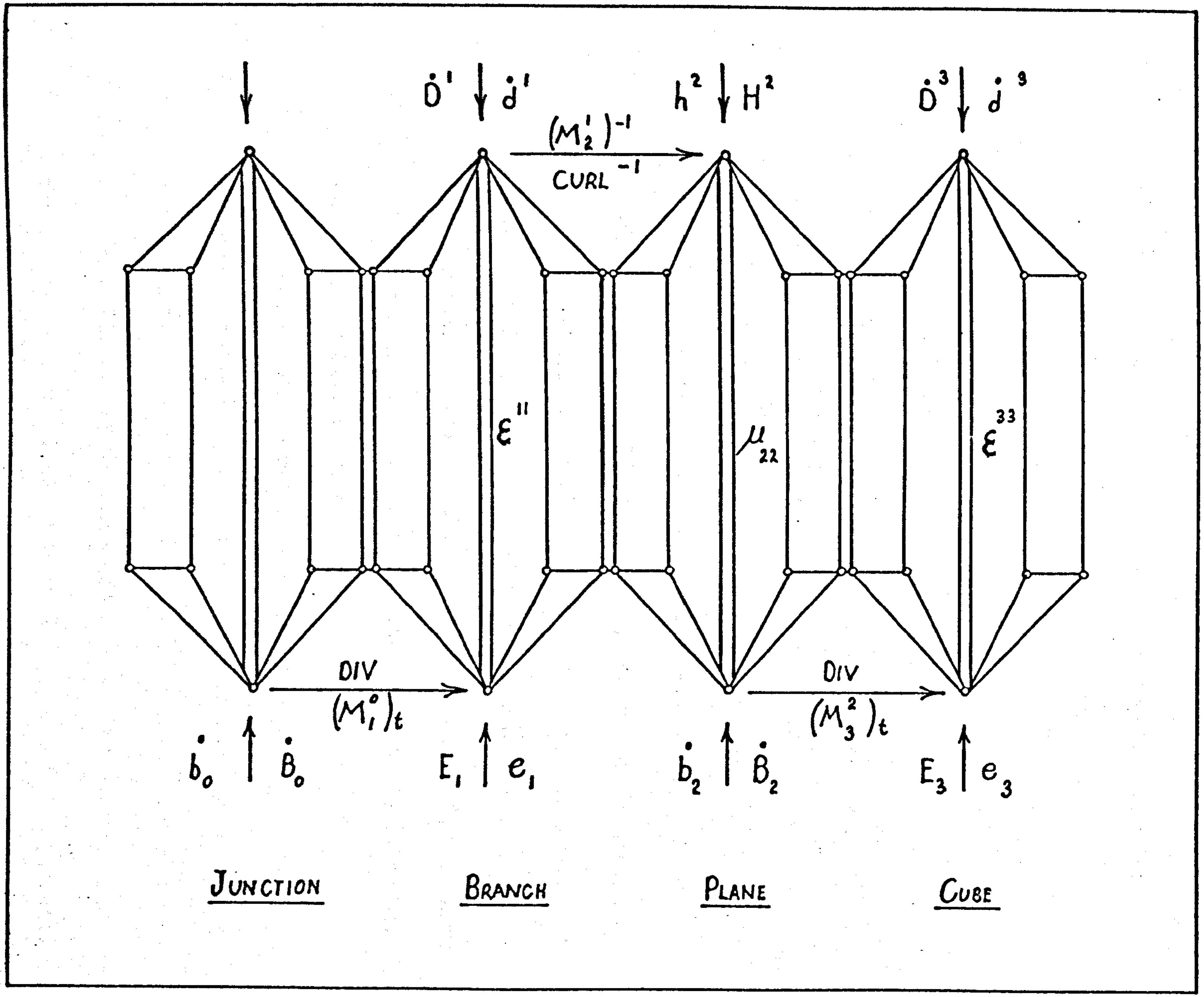


FIG.9. KRON'S ALGEBRAIC DIAGRAM FOR SPACE FILTERS

connected model. Kron's analysis, however, includes abstract quantities like magnetic poles and magnetic conduction currents.

5-3 Kron's Network Model(26)

In the network model developed by Kron (Fig. 10), the magnetic field quantities so far associated with surfaces are lumped into inductors and ideal transformers. Fig. 11 shows the elements of this network in z-x planes. In each mesh such as ABCDEFGH (Fig. 11), there are, in addition to the impedance elements, two ideal transformers (i. e. one-to-one ratio transformers with zero leakage reactances and resistances, zero magnetising currents and therefore infinite self and mutual inductances). Meshes such as ABCDEFGH consisting of inductors, capacitors, resistors and ideal transformers will be referred to as "large" meshes (as opposed to "small" meshes EFKL etc. consisting of inductors and ideal transformers only). Each "large" mesh consists of the two primaries of two ideal transformers and their secondaries, placed diagonally opposite to the primaries in the same mesh. The polarities of the windings are such

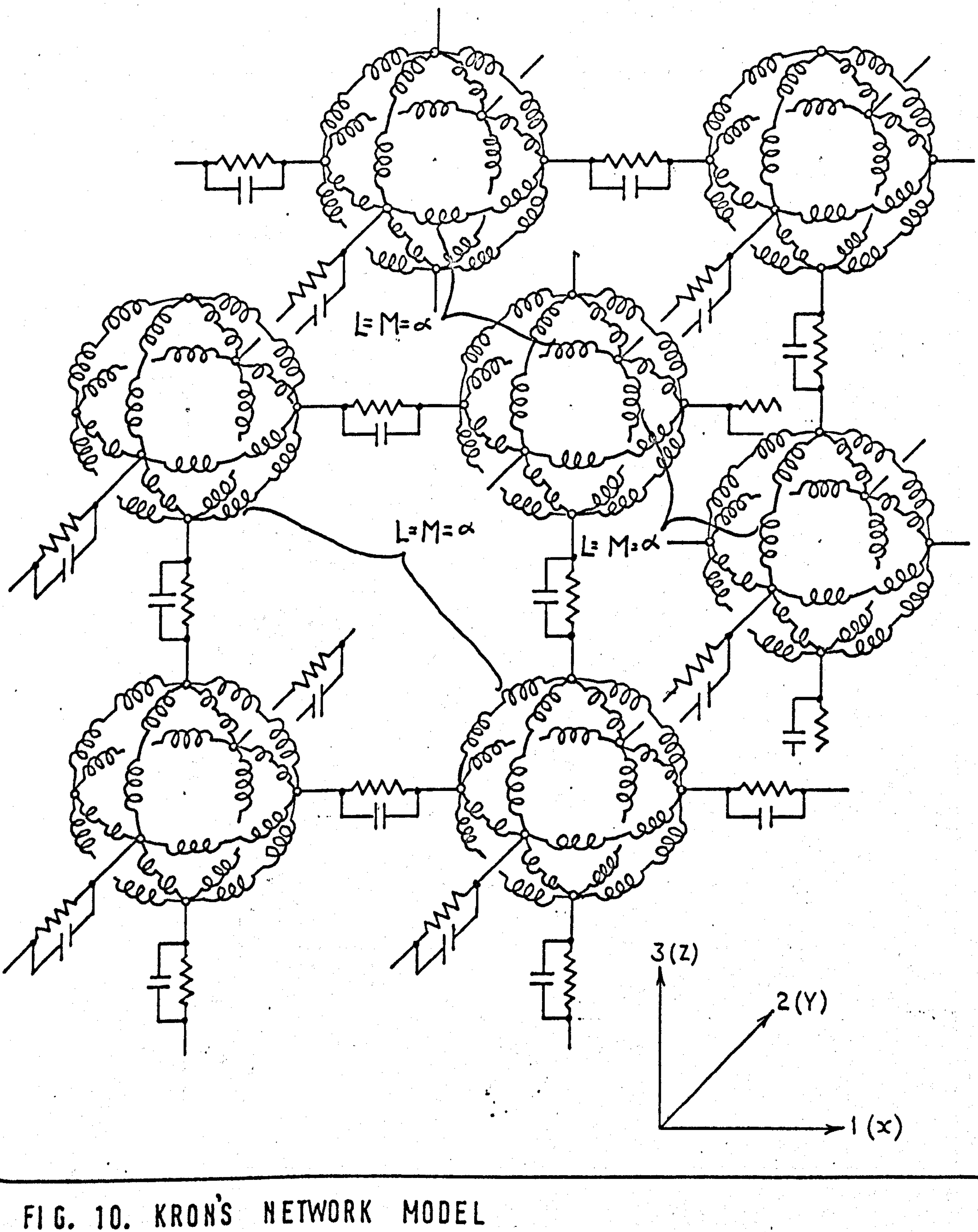


FIG. 10. KRON'S NETWORK MODEL

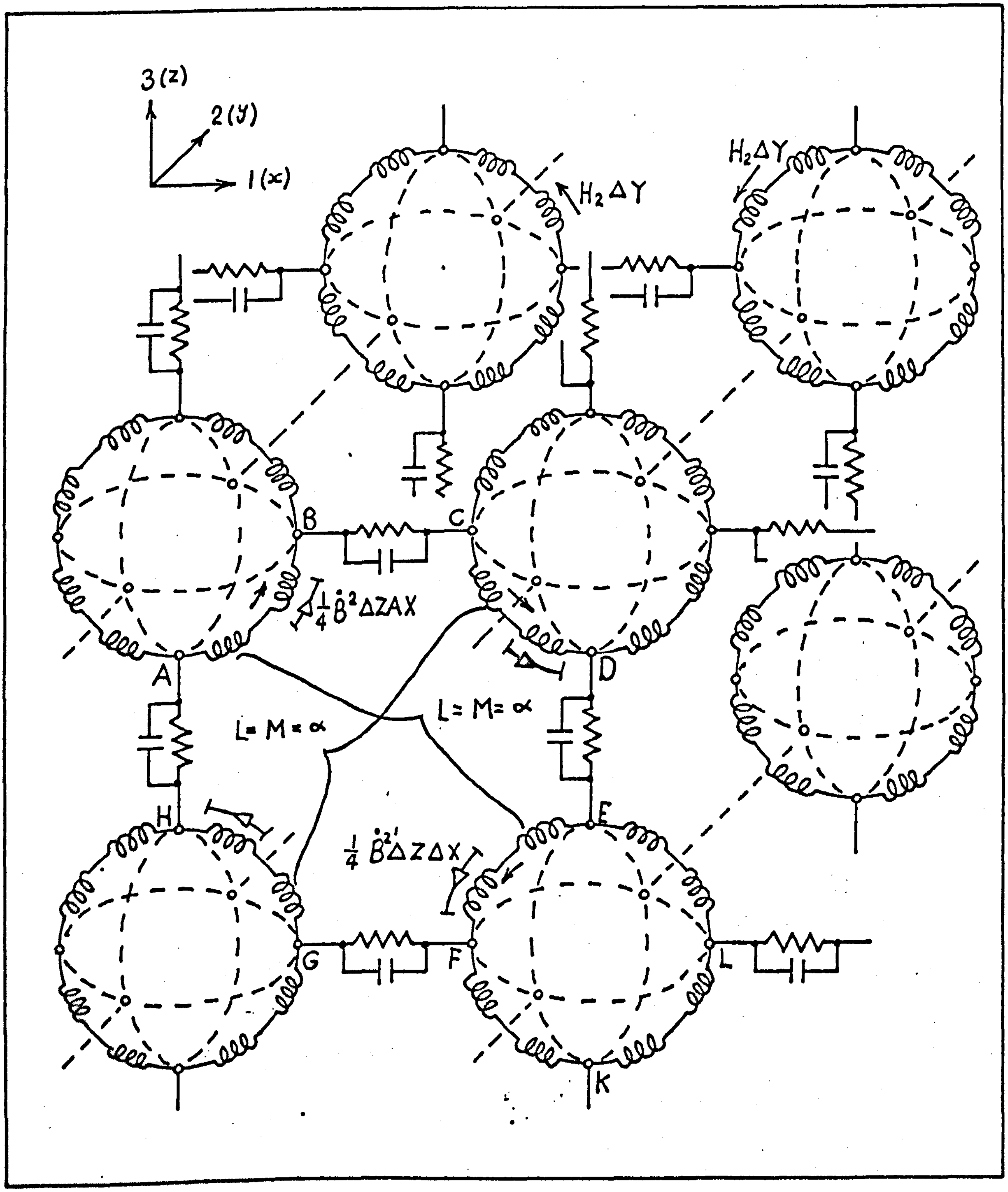


FIG 11 KRON'S NETWORK MODEL IN X-Z PLANES

that, around the "large" mesh each primary voltage cancels the corresponding secondary voltage. The inductors in the meshes of Fig. 11 carry a current of the type H_2 ΔY and their voltage drops are of the type $\frac{1}{4}\,\dot{B}^{2'}$ $\Delta Z\ \Delta X$, where $\dot{B}^{2'} = \partial B^{2'}/\ \partial t$. The resistors, capacitors, their currents and voltage drops are exactly the same as those of the network developed earlier (Fig. 5). The inductance values in z-x planes are given by $\frac{1}{4}\,B^{2'}\ \Delta Z\ \Delta X\ /H_2\ \Delta Y$. Their values in terms of the permeability of the media are obtained by extending equations 5-3 and 5-4 to the inductances:

$$L_2 = \frac{1}{4}\left(\frac{h_3 h_1}{h_2}\right)\left(\frac{\Delta Z\ \Delta X}{\Delta Y}\right)\mu\,\eta_o \qquad (\ \eta_o = 4\,\pi\ \mathrm{x}10^{-7}) \quad (5\text{-}16)$$

The inductances in other planes follow similarly.

5-3-1

The Maxwell equation, $\mathrm{curl}\ \vec{E} = -\ \partial\vec{B}/\ \partial t$ was expressed in Section 5-1-2, in integral form as:

$$\oint_{\partial\Sigma} E = -\iint_{\Sigma} * \dot{B}$$

In particular, for the large mesh (Fig. 11), the impedance drops across the capacitor-resistor branches were associated with the line integral of E. The surface integral of $*\dot{B}$ resulted in ($-\frac{\partial B^{2'}}{\partial t}$ $\Delta Z\ \Delta X$). In the network of Fig. 11, the above surface integral is given by four voltage drops, each a quarter of the integral, across the inductors. The primary and secondary voltages of the ideal transformers cancel out in the large mesh. The voltage drops around the mesh ABCDEFGH add to:

$$(E_3\ \Delta Z)_{HA} + (E_1\ \Delta X)_{BC} - (E_3\ \Delta Z)_{DE}$$

$$-(E_1 \Delta X)_{FG} + \frac{\partial B^{2'}}{\partial t}\ \Delta Z\ \Delta X = 0,$$

satisfying Kirchhoff's voltage law.

5-3-2

The mesh currents in the small meshes such as EFKL (Fig. 11) are zero, since the impedance offered by the ideal transformers is infinite to such mesh currents(28). If currents are allowed to circulate in the small meshes, the resulting network will no longer be homogeneous. It follows that the currents in the large meshes are of the type $H_2\ \Delta Y$.

For the previous network model (Fig. 5), the mesh currents were of the type $i^2 \Delta Y$ related to $H_2 \Delta Y$ by equation 5-7 as:

$$H = i + \bar{d}\Omega$$

The scalar magnetic potential Ω expressed the effects of magnetic sources outside the region under consideration. In Kron's network model, the ideal transformers at the boundaries would be energised from outside the region. The scalar magnetic potentials are thus already taken into account. Moreover, the network propagates electromagnetic waves and not simply currents and voltages.

When the branch currents were expressed as a sum of mesh currents in Fig. 5a (Section 5-1-3), equation 5-10 resulted:

$$*(J + \dot{D}) = \bar{d}i$$

Since the mesh currents in Kron's network (Fig. 11) are given in terms of the magnetic field vector, H, the above equation now takes the form:

$$*(J + \dot{D}) = \bar{d}H$$

(i.e. Maxwell's equation $\operatorname{curl} \vec{H} = \vec{J} + \partial\vec{D}/\partial t$).

The divergence equation, div $\vec{B} = 0$, is derived in Appendix 2 in two stages (i) the flux density, B, is related to the flux linkages of the ideal transformers (ii) the divergence of B is seen to be zero as a consequence of the general equation div (curl) = 0. The divergence equation, div $\vec{D} = \rho$, is derived as shown in Section 5-1-4.

Practical application of Kron's network model for field problems such as waveguides and electric machines is examined in Chapter 6. It is, however, seen that for developing the algebraic relations, the network model developed earlier in the thesis is more convenient. The exterior differential relations are given in terms of algebraic structural relations. Moreover, the interconnected model presents the relations in a form in which it can be extended for more abstract studies such as those of Kron's wave automaton using magnetic poles and magnetic conduction currents in more than three dimensions. Kron's network model for Maxwell's equations, using ideal transformers, is essentially a 1-circuit model. It is, therefore, more convenient in practical solutions of field problems. The interconnected model is also seen to express the 4-dimensional field tensors F_{mn} and H^{mn} as generalised voltages and currents.

CHAPTER 6

APPLICATION OF FIELD NETWORK MODEL

The network model described in Chapter 5 representing Maxwell's field equations will be examined here in its application to waveguides and electric machines ... Solution of field problems by analogue techniques and equivalent electrical circuits is a well known engineering approach. Karplus(29) has summarised a wide variety of field-plotting and analogue simulation methods.

Kron's network model however has been derived in terms of general coordinates. The general network can be simplified wherever, in an application, symmetry, known distribution and other such properties are present. In deriving his network, Kron places the emphasis on physical properties associated with integrals over infinitesimal regions rather than merely satisfying mathematically a set of partial differential equations. This process is physically more satisfying since the whole space of the network can be imagined to be filled with fields represented by these integrals.

6-1 Two-Dimensional Electromagnetic Field Problems

In a wide variety of problems, variation of field quantities along one of the coordinate axes is negligibly small (or at least the coordinates can be transformed to satisfy this condition). Such cases will be called "2-dimensional field problems." It should be emphasised here that, in this thesis, the term 2-dimensional field problems denotes the lack of variation of the field quantities along a particular direction,and does not suggest that all field quantities lie in 2-dimensional planes. If the coordinate, x^2 (= y), be taken in this direction, the partial derivatives of the electromagnetic field quantities with respect to y are:-

$$\frac{\partial B^{2'}}{\partial y} = \frac{\partial E_3}{\partial y} = \frac{\partial E_1}{\partial y} = 0 \quad \text{etc.}$$

also,

$$\frac{\partial D^{2'}}{\partial y} = \frac{\partial H_3}{\partial y} = \frac{\partial H_1}{\partial y} = 0 \quad \text{etc.} \qquad (6\text{-}1)$$

Making use of this property, Maxwell's equations (Section 4-4-1, equations 4-28) split into two independent parts.

(a)

$$\frac{\partial B^{2'}}{\partial y} = 0$$

$$\frac{\partial E_1}{\partial z} - \frac{\partial E_3}{\partial x} = - \frac{\partial B^{2'}}{\partial t}$$

$$- \frac{\partial H_2}{\partial z} = J^{1'} + \frac{\partial D^{1'}}{\partial t}$$

$$- \frac{\partial H_2}{\partial x} = J^{3'} + \frac{\partial D^{3'}}{\partial t}$$

$$\frac{\partial D^{1'}}{\partial x} + \frac{\partial D^{3'}}{\partial z} = \rho'$$

(b)

$$\frac{\partial B^{1'}}{\partial x} + \frac{\partial B^{3'}}{\partial y} = 0$$

$$- \frac{\partial E_2}{\partial z} = - \frac{\partial B^{1'}}{\partial t}$$

$$\frac{\partial E_2}{\partial x} = - \frac{\partial B^{3'}}{\partial t}$$

$$\frac{\partial H_1}{\partial z} - \frac{\partial H_3}{\partial x} = J^{2'} + \frac{\partial D^{2'}}{t}$$

$$\frac{\partial D^{2'}}{\partial y} = 0$$

(6-2)

It is noted that part (a) of equation 6-2 contains the terms $B^{2'}$, E_1, E_3, H_2, $J^{1'}$, $J^{3'}$, $D^{1'}$ and $D^{3'}$ and ρ' only; part (b) contains the terms $B^{1'}$, $B^{3'}$, E_2, H_1, H_3, $J^{2'}$ and $D^{2'}$ only. This suggests that the two sets of equations can be solved independently of each other. Kron's network model (Fig. 10) also splits up into two parts under this condition - the network of Fig. 11 and the network of Fig. 12. The solution of 2-dimensional field problems could be carried out

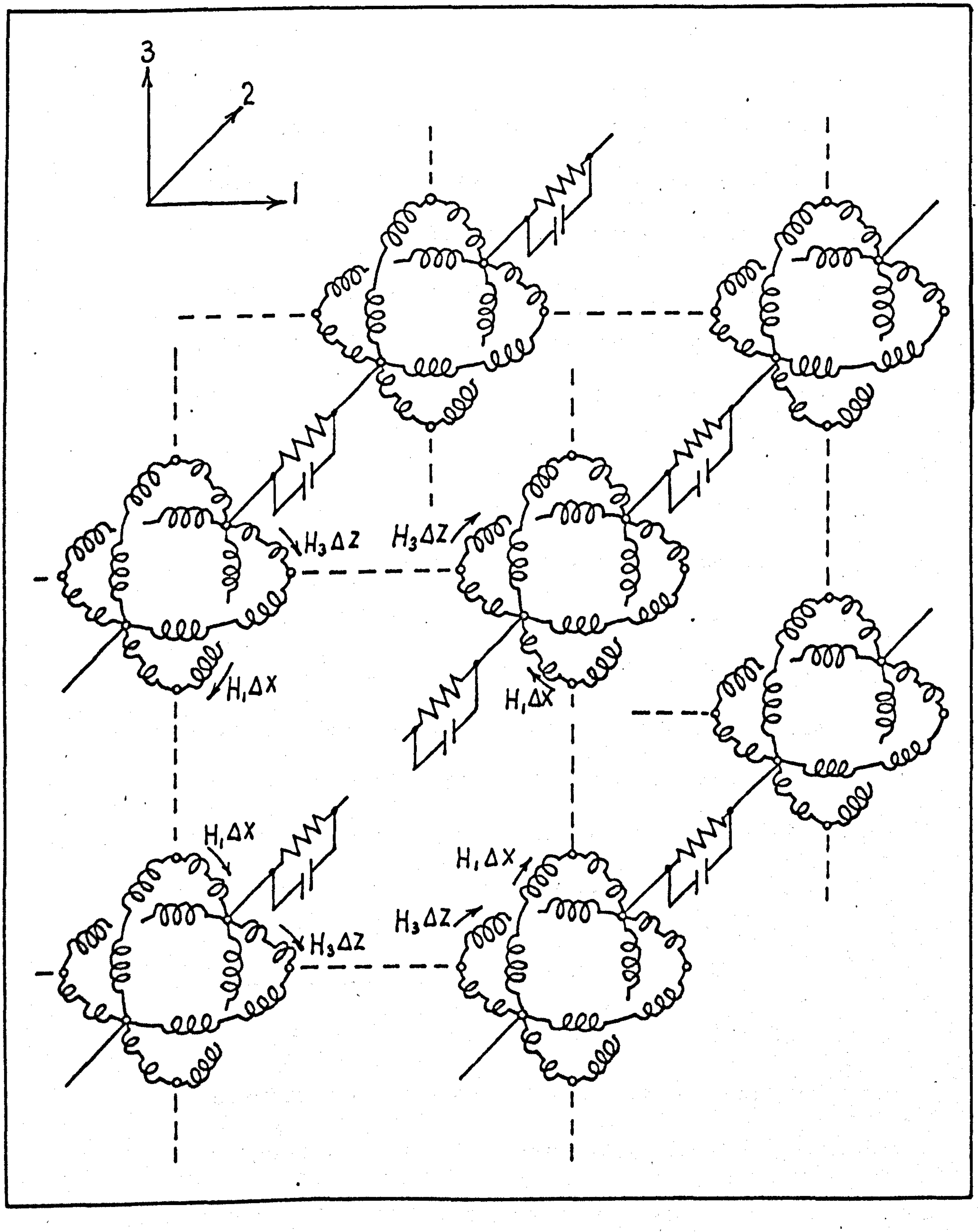


FIG 12 TWO DIMENSIONAL FIELD MODEL

separately on the two networks. It is noted that in the network of Fig. 11, all the elements are in z-x planes. The network of Fig. 12 can be further simplified to the network of Fig. 13. Equations 6-2(b) are seen to be satisfied by the latter network. The ideal transformers are implied in the network model of Fig. 13 although not actually shown. It was pointed out in Section 5-3 that the two windings of an ideal transformer are placed diagonally opposite in a "large" mesh and have equal and opposite voltages across them. For 2-dimensional field problems, in the network of Fig. 12, the two adjacent ideal transformers in a "large" mesh also have equal and opposite voltages across them. For numerical computations involving network of the type in Fig. 12, the simplified version of Fig. 13 can be used, in which it is not necessary to use ideal transformers.

6-2 Rectangular Waveguides

To bring out the order of errors involved in discretisation of field problems, numerical computation of a rectangular waveguide problem is studied. Rectangular and circular waveguides transmitting signal under "TE mode" can be simulated by the field network of Fig. 13. The symbol

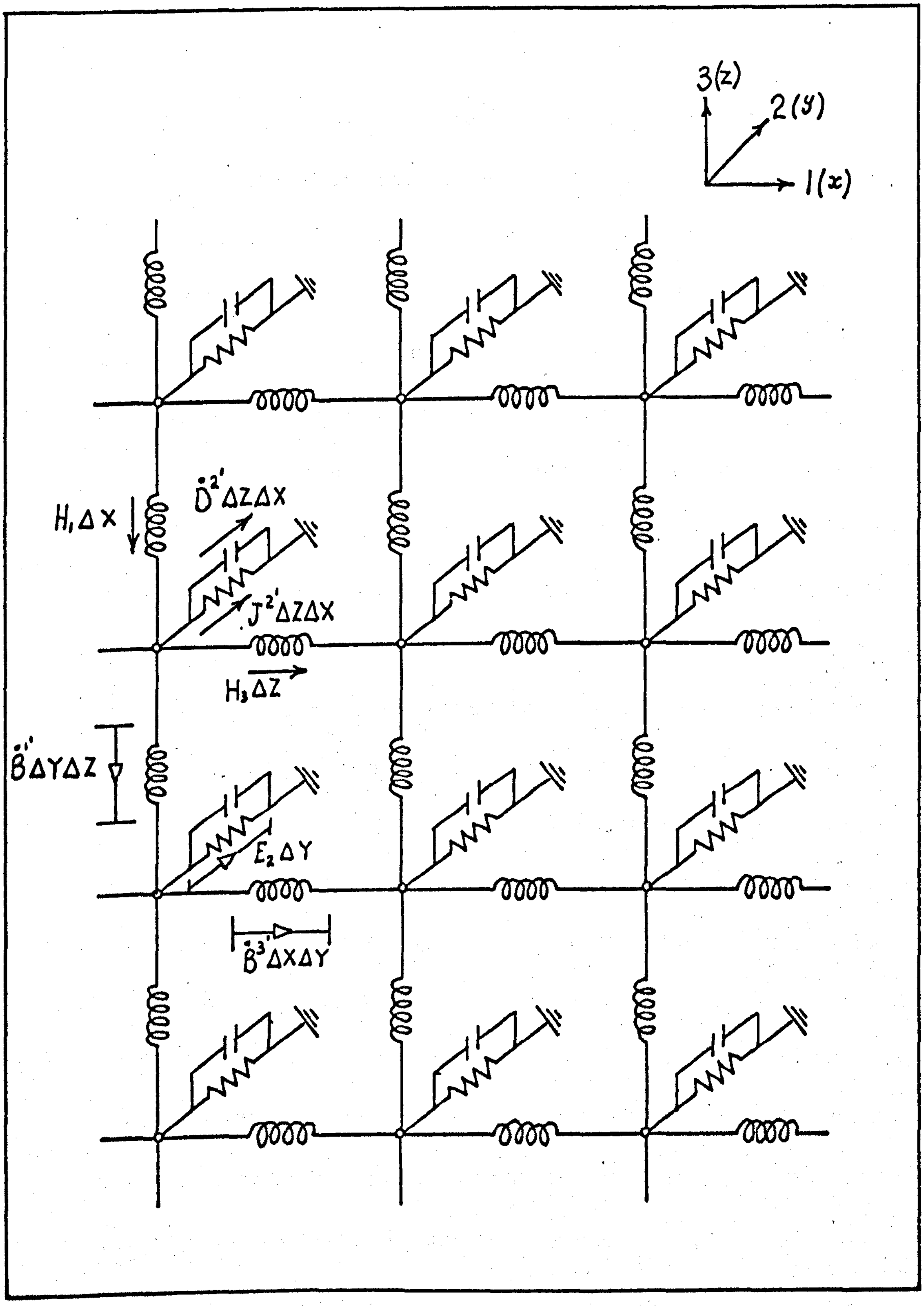


FIG. 13. SIMPLIFIED TWO DIMENSIONAL FIELD MODEL

TE indicates that the electric field is everywhere transverse to the axis of transmission. As shown later, the network of Fig. 13. can also be used to simulate electric machines when end effects are either ignored or when these are calculated separately and then superposed. A brief description of the rectangular waveguides follows.

Waveguides are used in radars, communications etc. when frequencies of the order of 10^{10} cycles per second have to be transmitted. The electromagnetic waves are guided by metallic walls through a hollow tube. The attenuation of the waves, as they are transmitted across, will depend upon the conductivity of the metal and the frequency.

Fig. 14a shows one mode of transmission in which the electric field is transverse to the axis of transmission. The magnetic field is shown by dotted lines and the electric field by solid lines. There can be no tangential component of electric field at a perfect conductive surface. For, if there were, it would exert a force on the charges within the conductor and move them from one point to another until the electric field was reduced to zero. In the rectangular waveguide, the electric field near two of the conducting walls is tangential and

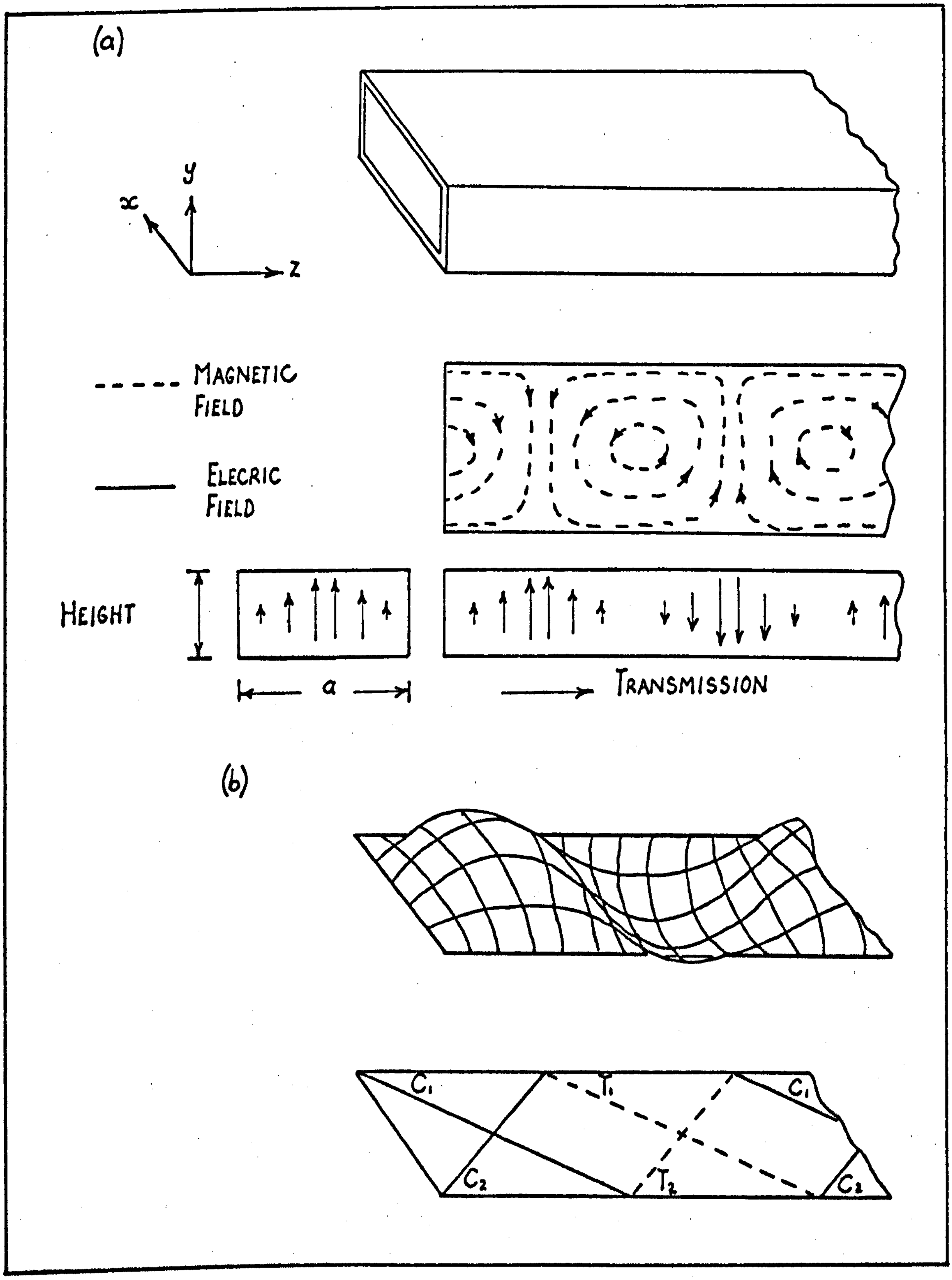


FIG. 14. RECTANGULAR WAVEGUIDE (TE$_{1,0}$ MODE)

has zero intensity and it is normal to the other two walls. It is assumed that the field quantities do not vary along the height of the waveguide. Further, the variations along the width of the waveguide and along the axis of transmission are assumed sinusoidal. This mode of transmission is known as the $TE_{1,o}$ mode(30). The subscript "1" denotes that the number of maxima of the electric field along the width is one. The subscript "o" denotes that there is no variation along the height.

The electric field components in x- and z-directions are zero. The electric field in y-direction is given by(31):

$$E_2 = \hat{E}_2 \times \sin\left(\frac{x}{a}\pi\right) \times \sin(\beta z - \omega t)$$

where the "phase constant"

$$\beta = \sqrt{\left(\frac{\omega}{c}\right)^2 - \left(\frac{\pi}{a}\right)^2} \qquad (6\text{-}3)$$

$$(\omega = 2\pi f; \quad c = \text{velocity of light})$$

The instantaneous waveform is shown in Fig. 14b. The waveform can be regarded as a combination of two plane waveforms(31) with crests C_1 and C_2 and troughs T_1 and T_2 as shown in Fig. 14b. In electric machine analysis a pulsating field is often resolved into two oppositely revolving fields. In particular, in equation 6-3, when

$$\frac{\omega}{c} = \frac{\pi}{a} \qquad \text{i.e.} \quad \beta = 0$$

$$E_2 = - \hat{E}_2 \sin\left(\frac{x}{a}\pi\right) \sin \omega t$$

This expression describes a "standing wave" and the two component waves could be thought of as racing from one wall to another in opposite sense. The frequency under this condition, i.e. the phase constant $\beta = 0$, is known as cut-off frequency. At wavelengths longer than the cut-off wavelength, the wave will be rapidly attenuated.

6-3 Numerical Computation

The matching impedances for a waveguide have been calculated by a theoretical formula and also by the network model of Fig. 13. The error incurred in using the field network is plotted as a function of the subdivisions.

If a transmission line is terminated with the characteristic impedance at the frequency of the applied signal, there will be no reflections from the end of the line; the only signal appearing on the line will be the incident wave. If some other terminating impedances are used, a portion of the incident wave will be reflected and the total signal appearing on the line will be the sum of the incident and reflected waves. The theoretical characteristic impedance is given in terms of wave impedance:

$$Z_w = 377 \frac{\omega}{\beta c} \qquad (6\text{-}4)$$

$$(377 = \sqrt{\eta_o / \varepsilon_o} \;)$$

For the network of Fig. 13, the theoretical matching impedance terminating each section is:

$$Z = Z_w \cdot \frac{\Delta Y}{\Delta X} \qquad (6\text{-}5)$$

The waveguide studied was selected from the American Services Standard List(30). It has a cross-section of 4.3 inch x 2.15 inch. The recommended operating range of frequency for $TE_{1,o}$ mode is 1.70 - 2.60 kmc/sec. For the first part of computation, a frequency of 2×10^9 c/s was selected. For this frequency,

$$\frac{\beta c}{\omega} = 0.731955$$

The width, 4.3 inch = 11 cm, was first subdivided into five parts (ΔX). For varying subdivisions, ΔZ, along the axis of transmission, the inductance and capacitance values of the field network of Fig. 13 were determined. The boundary condition, $E_y = 0$ at the surface of the side walls, i.e. no tangential electric field at conducting surfaces, is represented in the field network by short-circuiting the admittances to the ground along these planes. Further, a lossless transmission line was assumed so that, apart from the terminal impedance, none of the other elements in the network of Fig. 13 include resistors. With $\Delta X = 2.2$ cm (five subdivisions), $\Delta Z = 2.2$ cm and $f = 2 \times 10^9$ cycles/second the parameters are:

$$\beta c/\omega = 0.731955$$

$$X_L = j\,(1.579 \times 10^4)\ \Delta Y \text{ ohm/section}$$

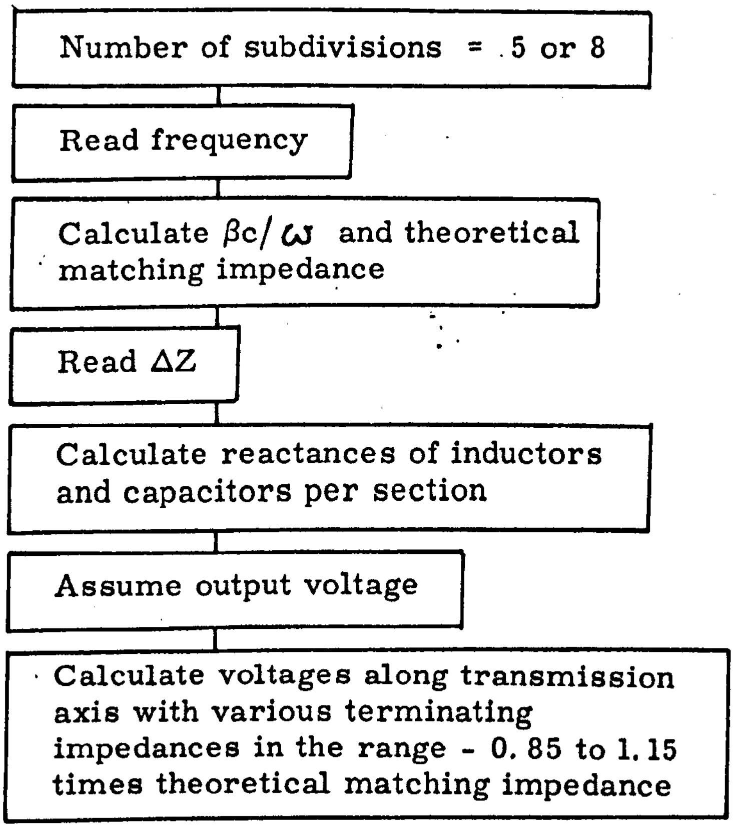
$$X_c = -j\,(1.857 \times 10^4)\ \Delta Y \text{ ohm/section}$$

Theoretical matching impedance

$$= (2.3395 \times 10^4)\ \Delta Y \text{ ohm/section}$$

If $\Delta Z \neq \Delta X$, the inductors along the axis of transmission and those transverse to the axis have different values in the equivalent network.

Calculations were carried out on a digital computer to obtain the matching impedance of the waveguide, as outlined below:



The computed matching impedance is obtained when the voltage along the transmission axis varies in phase angle only and not in amplitude (i. e. no standing waves).

The results are shown plotted in Fig. 15. The error is plotted in a linear scale but the subdivision, ΔZ, along the axis of transmission (as a percentage of the wave length) is

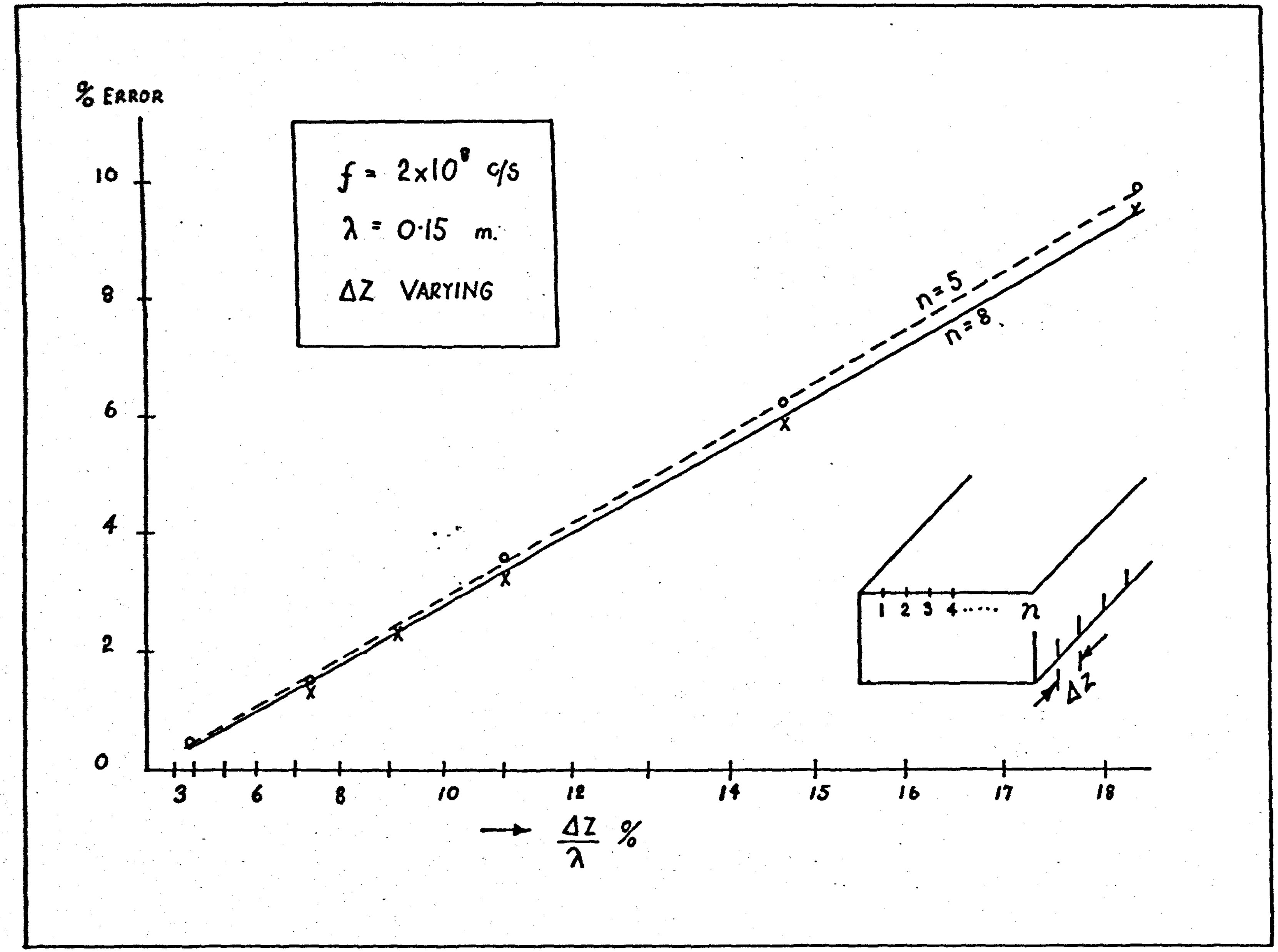


FIG. 15. COMPUTATION OF WAVEGUIDE PROBLEM-FIXED FREQUENCY

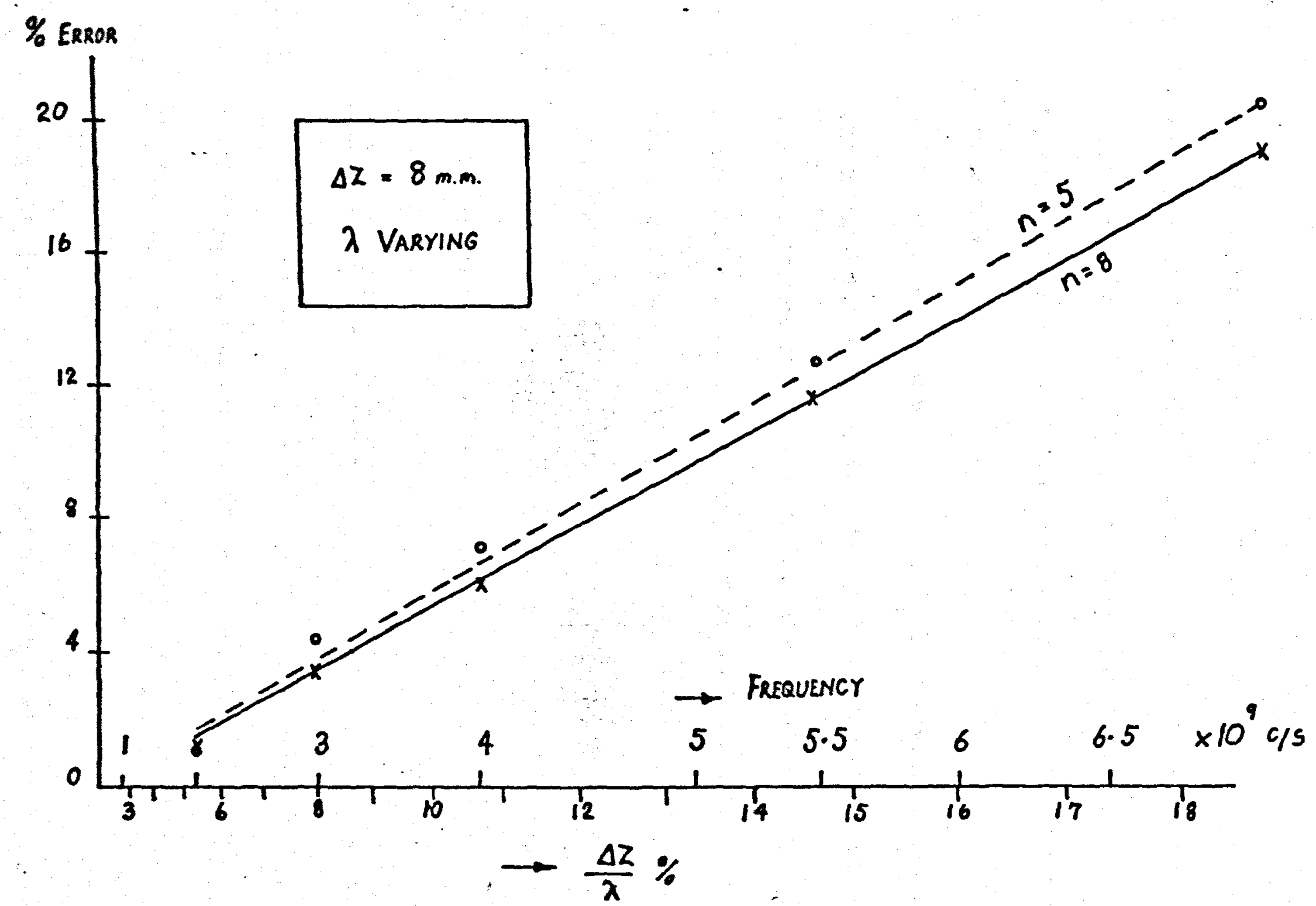


FIG 16 COMPUTATION OF WAVEGUIDE PROBLEM-VARYING FREQUENCY

first squared and then plotted on a linear scale. Calculations were repeated with eight subdivisions of the width of the waveguide instead of five. The resulting curves are seen to be straight lines. The scale of the graph is non-linear along the abscissa.

Calculations were repeated with varying frequencies and a fixed subdivision, ΔZ, along the axis of transmission. The results are shown in Fig. 16. It is observed from Figs. 15 and 16 that a subdivision along the axis of transmission equal to 10% of the wave length gives an error of about 4% whereas a subdivision of 5% of the wave length gives an error of about one per cent. These are seen to agree with the network analyzer studies of Whinnery and Ramo(32).

6-4 Electric Machines

When the variation of the field quantities parallel to the axis of the shaft of an electric machine is ignored, then the 2-dimensional type of network given in Fig. 13 can be used to solve machine problems. Current flow is assumed to be wholly in a direction parallel to the shaft axis (Fig. 22, p. 200) and the magnetic field quantities in this direction put equal to zero. Under sinusoidal distribution, a simpler 1-dimensional "transmission" type of network (Fig. 17b) is shown to result. This agrees with the equivalent network of Barton and Cullen(33) (Fig. 18).

The z-coordinate is taken parallel to the shaft (Fig. 17a). The field quantities do not vary along this axis. The other two coordinates, x^1 and x^2, are the usual polar coordinates r and θ. The network of Fig. 17a gives the parameters in polar coordinates. The permeabilities shown are inhomogeneous (i. e. $\mu_r \neq \mu_\theta$). This arises when, in a theoretical analysis, the effects of slots and teeth are simulated by a single medium having different permeability in different directions(34). The reason for this is that the slots cut in the steel punchings of electric machines offer higher reluctance to fluxes parallel to the air-gap whereas most of the radial flux is carried by the highly permeable teeth. The resistances and permeabilities in directions parallel to and perpendicular to the air-gap are appropriately averaged.

The network of Fig. 17b is obtained as below, from the network of Fig. 17a, when sinusoidal distribution is assumed.

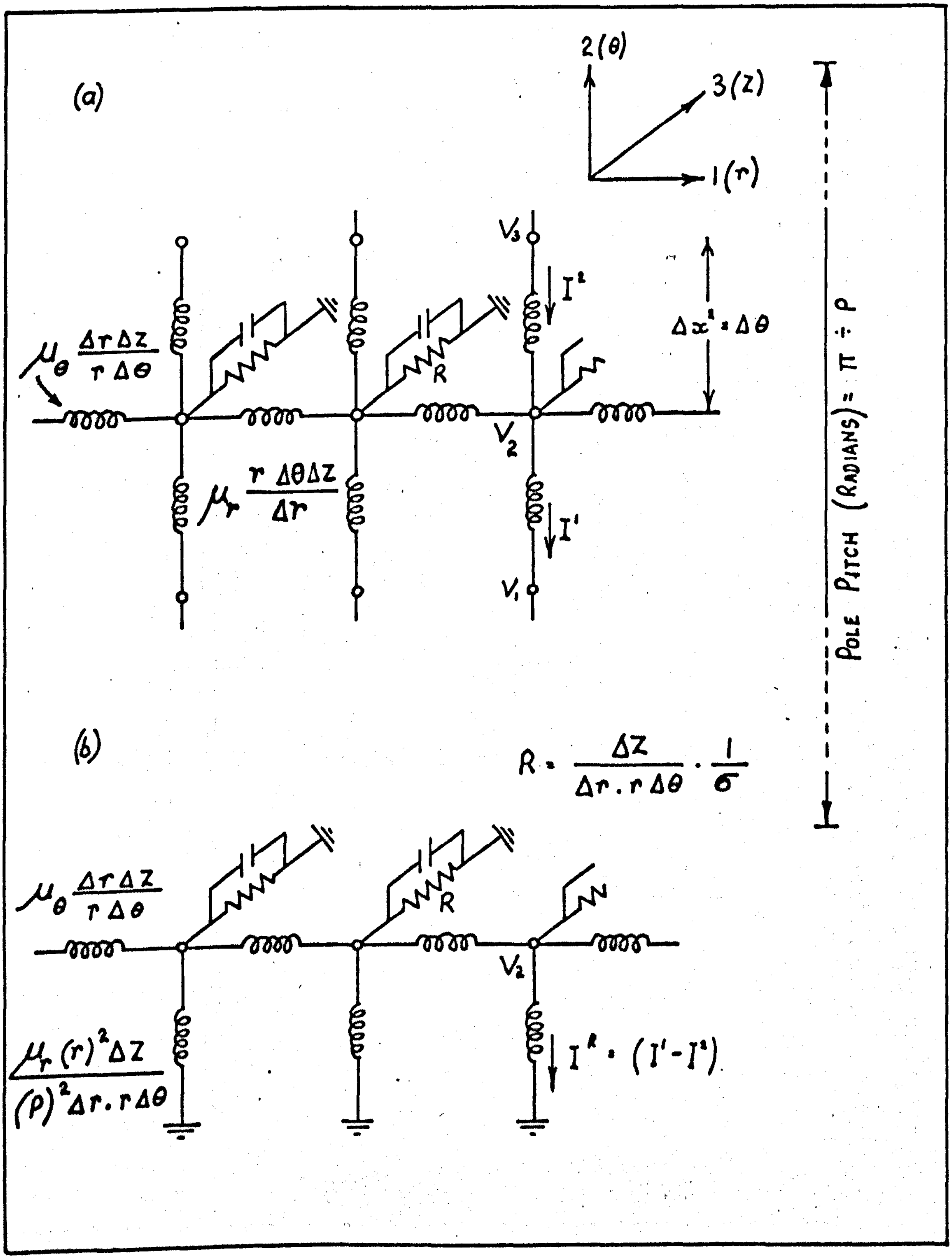


FIG 17 TRANSMISSION LINE TYPE NETWORK FOR ELECTRIC MACHINES

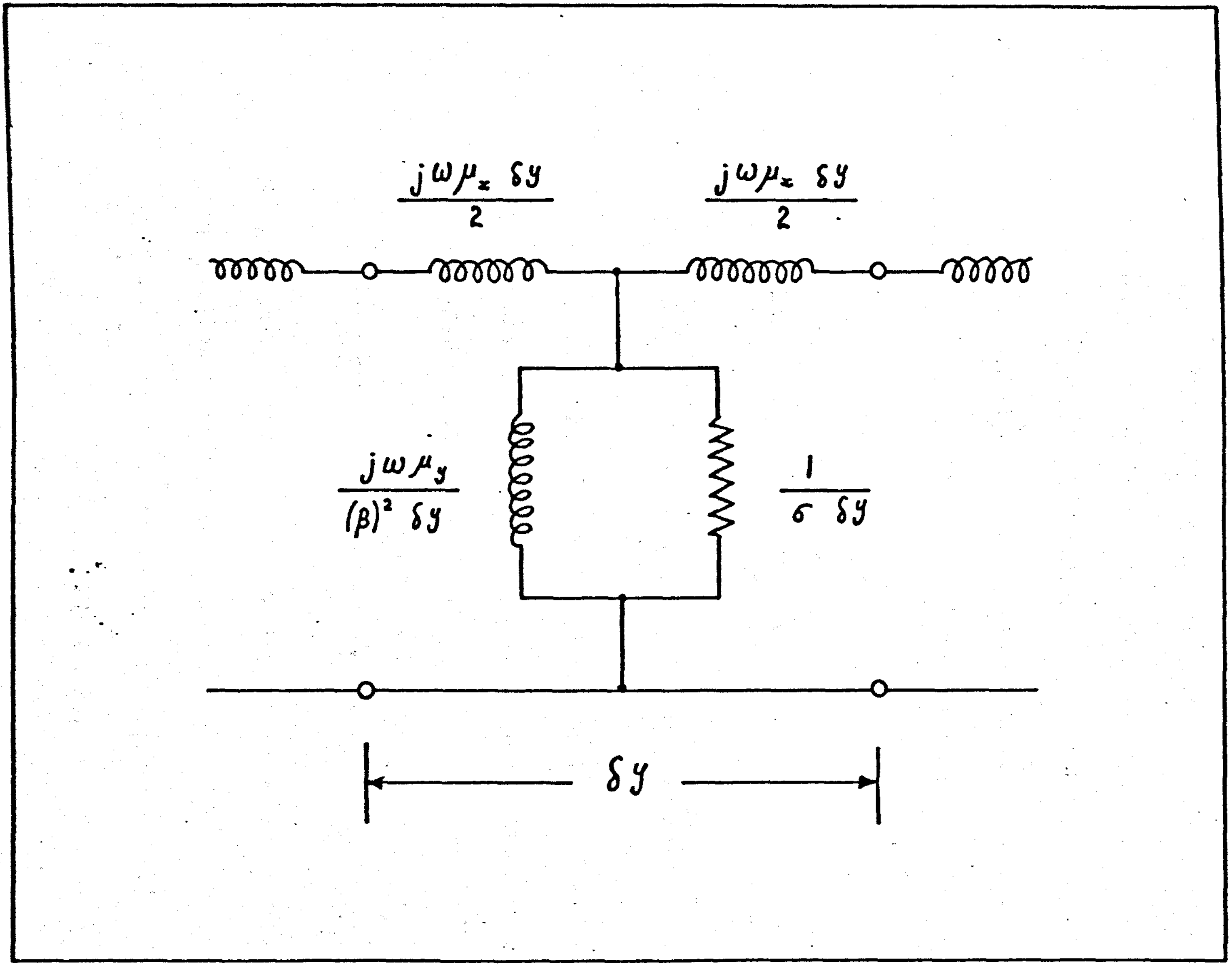


FIG.18. ELEMENT OF EQUIVALENT TRANSMISSION LINE
(REPRODUCED FROM REFERENCE 33)

The voltage V_2 shown in the network can be expressed as $\hat{V}\sin(P\theta)$, since sinusoidal distribution is assumed (P = pairs of poles). The voltages V_1 and V_3 are given by $\hat{V}\sin\left[P(\theta - \Delta\theta)\right]$ and $\hat{V}\sin\left[P(\theta + \Delta\theta)\right]$ respectively. The currents, I^1 and I^2, are given in terms of these voltages as:

$$(\mu_r \frac{r\Delta\theta \,.\, \Delta Z}{\Delta r})\ pI^1 = (V_2 - V_1) \qquad (6\text{-}6)$$

$$(\mu_r \frac{r\Delta\theta \,.\, \Delta Z}{\Delta r})\ pI^2 = (V_3 - V_2) \quad ; \quad p = d/dt \qquad (6\text{-}7)$$

The resultant current leaving the node is: $I^R = I^1 - I^2$. In terms of I^R, a relation is obtained by subtracting equation 6-7 from 6-6, giving:

$$(\mu_r \frac{r\Delta\theta \,.\, \Delta Z}{\Delta r})\ pI^R = (2V_2 - V_1 - V_3) \qquad (6\text{-}8)$$

In equation 6-8, the voltages can be expressed as sine functions of the maximum voltage as:

$$\begin{aligned} 2V_2 - V_1 - V_3 &= 2\hat{V}\sin(P\theta) - \hat{V}\sin\left[P(\theta - \Delta\theta)\right] \\ &\quad - \hat{V}\sin\left[P(\theta + \Delta\theta)\right] \\ &= 2\hat{V}\sin(P\theta)\left\{1 - \cos P(\Delta\theta)\right\} \\ &\simeq V_2\,(P\,.\,\Delta\theta)^2 \end{aligned} \qquad (6\text{-}9)$$

Substituting equation 6-9 in equation 6-8 and rearranging,

$$\left\{ \frac{\mu_r (r)^2 \Delta Z}{(P)^2 \Delta r . r\Delta\theta} \right\} pI^R = V_2 \qquad (6\text{-}10)$$

$$\text{i.e.} \quad L_R \, pI^R = V_2$$

That is, a single inductor grounded at one end and connected to the node V_2 can simulate the two currents I^1 and I^2. The value of the inductance is given by the bracketed term of equation 6-10. This procedure can be carried out at every node in the r-direction, resulting in a network of the type in Fig. 17b. This is analogous to the representation of a transmission line. For electric machines, the value of the capacitance is negligibly small and the network can be further simplified to that of Fig. 18.

Fig. 18 is reproduced from reference 33. This agrees with the network of Fig. 17b with the following substitutions:

$$\mu_r = \mu_y$$

$$\mu_\theta = \mu_x$$

$$\Delta r = \delta y$$

$$\Delta z \div r\Delta\theta = 1$$

$$\frac{r}{P} = \frac{1}{\beta}$$

Capacitance is ignored. Moreover in reference 33 cartesian coordinates are assumed i. e. the air-gap and teeth dimensions are considered small compared to the radius.

This transmission line representation of machine with sinusoidal distribution has been derived by first considering a more general network, then ignoring end effects, assuming sinusoidal distribution and a suitable coordinate system. In reference 33 the same final result is derived by first simplifying Maxwell's equations in a particular coordinate system and then comparing it with a transmission line network. Such network studies in electric machines can be usefully applied to solve problems such as eddy-currents or skin effects in conductors(33) and solid iron effects(43).

Kron's network model is seen to cover a wide variety of applications. In particular the $TE_{1,o}$ mode of transmission is similar to electric machines, as far as the network topology is concerned. The resolution into forward and backward revolving fields in electric machines can be compared to the resolution, into plane waves, of the total wave in a wave guide. When the phase constant "β" is zero for the wave guide, the two component waves travel in opposite directions analogous to the resolution of a pulsating field, in an electric machine, into two equal and oppositely rotating fields (revolving field theory of single-phase machines). When the wave guide is terminated by its characteristic impedance, there will be no reflections from the end of the line. The voltage along the transmission axis varies in phase angle only and not in amplitude. In a polyphase electric machine, under balanced condition, a similar distribution results. The resultant field of the various phases are combined into one revolving field. In the case of a wave guide, if it is not properly matched, there will be a standing wave along the transmission axis. In an electric machine, this condition corresponds to the unbalanced case when the two revolving components are unequal, and the resultant field is revolving as well as pulsating.

Resolving a field into a set of two component fields has been used by Kron in his multidimensional wave analysis. (Kron states that in his wave automaton, for the propagation of waves, the p-dimensional and its dual (n-p) dimensional polyhedra serve as a set of orthogonal reference frames).

The subject of electric machines will be examined in further detail particularly with reference to the tensorial analysis of electric machine fields.

CHAPTER 7

FIELD CONCEPTS IN ELECTRIC MACHINES

The previous chapters dealt with stationary fields. That is, the media and the sources of fields are stationary with respect to the observer.

This chapter deals with the concepts of electromagnetic fields in rotating electric machines. Kron uses "straightforward generalisations of conventional rotating electrical machinery" to analyse the field of magnetohydrodynamic generators in his wave automaton. The analysis in a charge coordinate system for the electric machine is seen to lead to more general exterior differential forms. A simplified version of the conventional electric machine is considered in the analysis., The idealised machine envisaged by Park, Kron and others is described. This model represents the machine by means of relatively moving coils with their associated voltages and currents. The parameters are expressed as inductances and resistances. For the field study, a model is described making use of current sheets and distributed parameters. The four dimensional field tensors, introduced in Chapter 4, are utilised to express the magnetic stresses and Poynting energy flow vector. A comparison is made between the discrete model with coils and the continuous model with current sheets.

Kron used tensor analysis in his treatment of electric machines and demonstrated that such an analysis is universal in its application to most kinds of rotating machines and for multi-machine system studies. The field concepts are analysed in these terms. This approach appears to converge with the relativistic treatment of unified field theory.

In developing the field network model for Maxwell's equations (Chapter 5), integrals of differential forms over subregions were defined. Integrals of field quantities over certain regions of the electric machine provide the link between the two treatments mentioned above. The differential forms, however, assume a more general form, when transformations involving non-integrable relations are used. These transformations were used by Kron to demonstrate the fact that most electrical machines are members of a family or group. Throughout the analysis, hysteresis of iron and changes of saturation are ignored. The electric flux density vector, D, is also ignored.

7-1 Park's Idealised Machine(35)

The idealised machine envisaged by Blondel, Park, Kron(2) and others. has at least one cylindrical iron structure (say, the rotor). The other member can be cylindrical (non-salient), or can have a salient structure (Fig. 19). Each member has a layer of winding. On the stator, the winding is usually represented by a coil on the salient pole. (Kron introduces another coil in the inter-polar region). On the rotor (armature), there are two sets of brushes, (hypothetical sets for synchronous machines and induction motors). The brushes are centred on two orthogonal axes through which currents i^{dr} and i^{qr} flow. The rotor winding is assumed to be symmetrically distributed. The axes are called direct- and quadrature- axes. In the diagramatic representation of windings by coils, the axes of coils denote the axes of magnetisation. While the conductors forming the armature rotate, the resultant coils between the brushes are

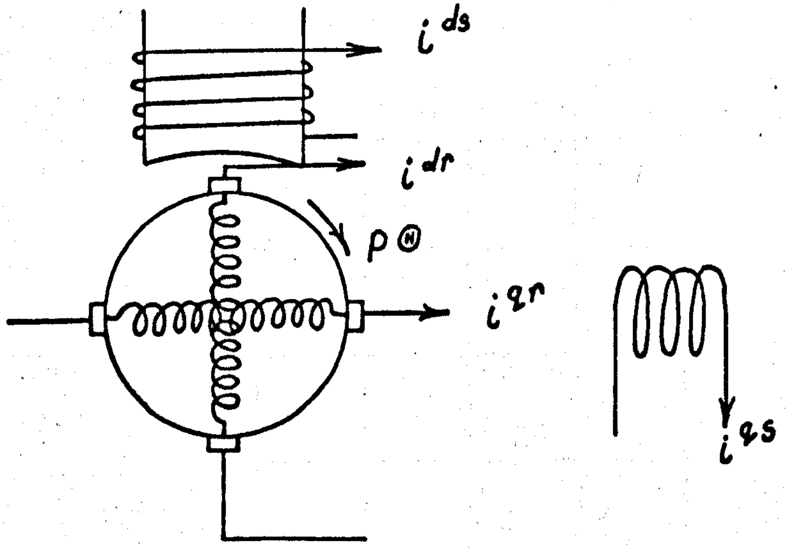


FIG. 19. PARK'S IDEALISED MACHINE

stationary; that is, the coils are composed of different conductors from instant to instant. The dr- and qr- coils (Fig. 19) are therefore known as pseudo-stationary coils, their self and mutual inductances being given in matrix form along with the stator quantities as:

$$L_{ab} = \begin{array}{c|c|c|c|c|} & ds & dr & qr & qs \\ \hline ds & L_{ds} & M_d & & \\ \hline dr & M_d & L_{dr} & & \\ \hline qr & & & L_{qr} & M_q \\ \hline qs & & & M_q & L_{qs} \\ \hline \end{array} \qquad (7\text{-}1)$$

When the armature revolves at a speed of $p\,\Theta$ (capital theta is used here to avoid confusion with polar coordinates), the voltage equation is[(2)]:

$$v_a = R_{ab}\, i^b + L_{ab}\, pi^b + G_{ab}\, i^b\, p\,\Theta \qquad (7\text{-}2)$$

In equation 7-2, v_a is the applied voltage; R_{ab} is the resistance matrix; G_{ab}, the torque matrix, is:

$$G_{ab} = \begin{array}{c|c|c|c|c|} \downarrow a \;\; \xrightarrow{b} & ds & dr & qr & qs \\ \hline ds & & & & \\ \hline dr & & & L'_{qr} & M'_{q} \\ \hline qr & -M'_{d} & -L'_{dr} & & \\ \hline qs & & & & \\ \hline \end{array} \qquad (7\text{-}3)$$

When the flux density wave is sinusoidal the unprimed self and mutual inductances in matrix 7-1 can be used for the primed quantities in matrix 7-3.

The torque applied to the shaft of the machine is:

$$T = Mp^{2}\,\Theta + R_{s}\,p\,\Theta - i_{t} \,.\, G \,.\, i \qquad (7\text{-}4)$$

(M is the rotor inertia; R_s is the frictional resistance; i_t is the transposed current matrix).

7-2 Field Model

The field model makes use of current sheets. The conducting strips carry a finite current per unit width, but as their thickness is reduced to zero, the current per unit area becomes infinite. In such cases the finite current per unit width is used in the analysis. It is called linear current density or line density. It is denoted by the German letter $\mathfrak{J}$ (current density being usually denoted by "J"). The unit for line density is amp per metre. Current sheets are useful simplifications of complicated problems, since they localise current sources and the rest of the region is current-free. Integration over the current sheet region is also easier. Hague(36) has shown that so far as the air-gap field is concerned, windings in slots may be replaced by fictitious current sheets fastened to smooth steel surfaces. Fig. 20 shows two hollow cylindrical tubes (current sheets) which replace the windings shown by the coils in Fig. 19. The line density (linear current density) is assumed to be parallel to the shaft axis at all points. It is also assumed to be sunusoidally distributed over the circle.

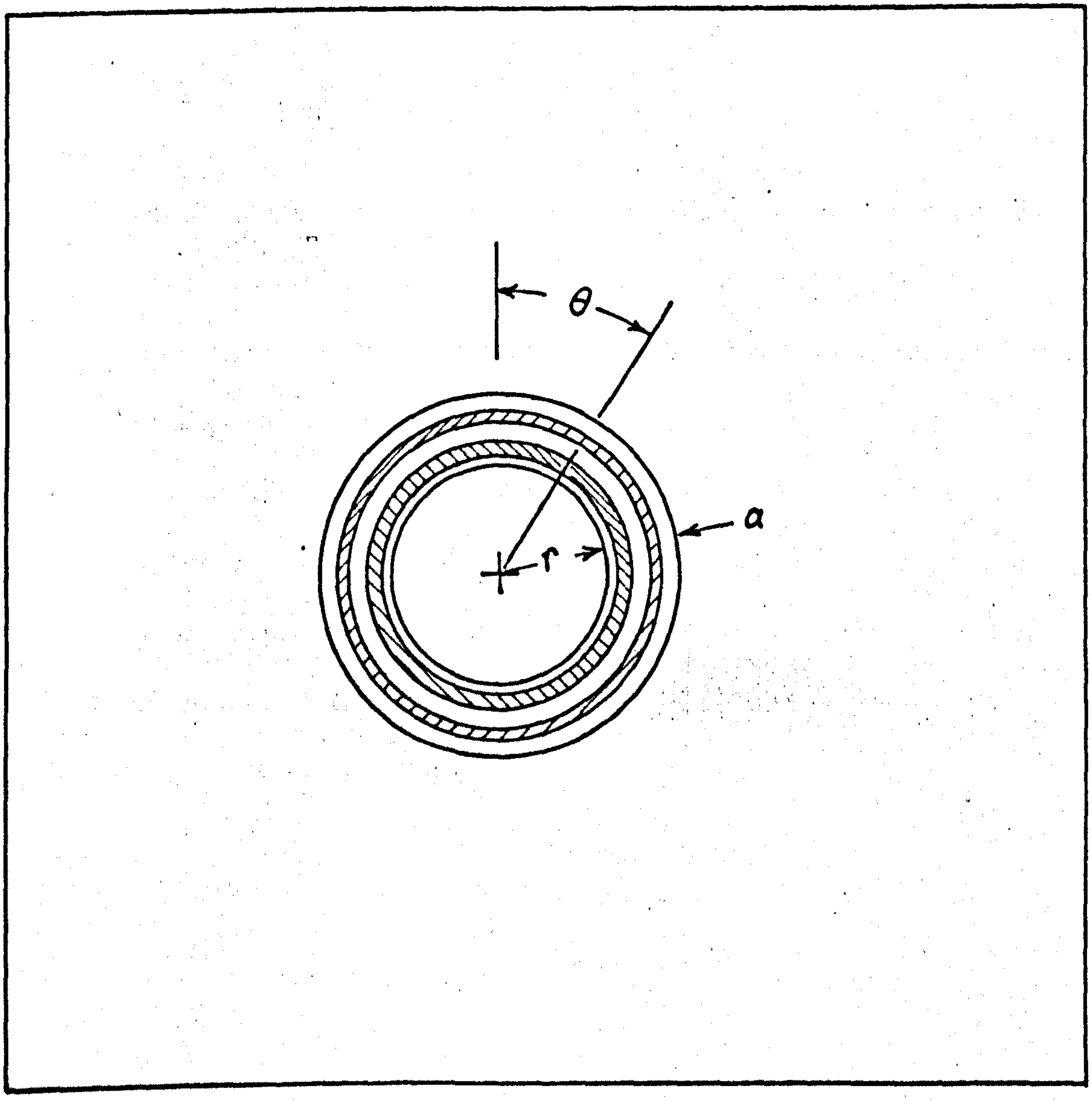


FIG. 20. FIELD MODEL

i. e. for the rotor current sheet,

$$\mathcal{J} = \mathcal{J}^{dr} \cos\theta + \mathcal{J}^{qr} \sin\theta \qquad (7\text{-}5)$$

(superscript dr and qr refer to the direct- and quadrature-axis, described in Section 7-1 for the rotor; angle θ is measured as shown in Fig. 20 from the direct-axis. A 2-pole configuration is assumed).

The following sections establish the relationships between various field quantities. The stress-energy tensor is set up and the electromagnetic torque and power flow are discussed in these terms.

7-2-1 Magnetomotive Force and Magnetic Force

The Maxwell equation, curl $\vec{H} = \vec{J}$, cannot be used as such in the region occupied by the current sheet, since the current density is infinite. However, the integral form using Stokes' theorem (Section 3-9) can be used. In this form

$$\oint_{\partial\Sigma} H = \iint_{\Sigma} \bullet J \qquad (7\text{-}6)$$

where, Σ is an area bounded by a closed curve $\partial\Sigma$

The surface integral of current density, J, gives the total current enclosed by the area Σ. For the current sheet this is finite, and is given by

$$\int_{\theta_1}^{\theta_2} \mathcal{J}\, r d\theta$$

(θ_1 and θ_2 are the angles corresponding to the extremities of the current sheet in the area Σ).

Equation 7-6 is usually known as the "circuited law". It is applied to two small areas ABCD and AB'C'D (Fig. 21), as follows.

First, the small area ABCD (Fig. 21a) is considered. In equation 7-6, the left hand side denotes the line integral of

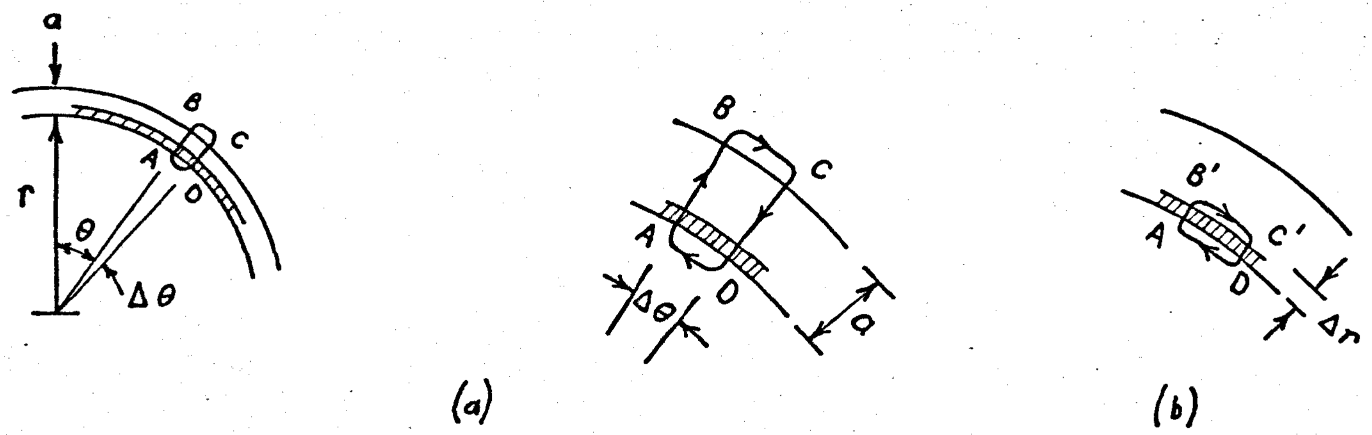


FIG. 21. APPLICATION OF CIRCUITAL LAW

magnetic force, H, around the closed curve $\partial\Sigma$ which, in the present case, is A-B-C-D-A. For paths B-C and D-A, the magnetic force is assumed zero (iron infinitely permeable). For paths A-B and C-D, the variation of the radial magnetic force normal to the air-gap is ignored so that

$$\int_A^B H_r \, dr \simeq (H_r)_{AB} \, a$$

(a is the air-gap length).

Equation 7-6 for the area ABCD is then:

$$(H_r)_{AB} \, a + 0 - (H_r)_{CD} \, a - 0 = \mathcal{J} r \Delta\theta$$

i. e.

$$- \frac{\partial H_r}{\partial \theta} a\Delta\theta = \mathcal{J} r\Delta\theta \qquad (7\text{-}7)$$

Since sinusoidal distribution is assumed, H_r can be expressed as:

$$H_r = H_{dr} \cos\theta + H_{qr} \sin\theta \qquad (7\text{-}8)$$

Substitution of equations 7-8 and 7-5 in equation 7-7 yields:

$$(H_{dr} \sin\theta - H_{qr} \cos\theta) \, a \, \Delta\theta$$

$$= (\mathcal{J}^{dr} \cos\theta + \mathcal{J}^{qr} \sin\theta) \, r \, \Delta\theta$$

i. e.

$$\begin{bmatrix} \mathcal{J}^{dr} \\ \mathcal{J}^{qr} \end{bmatrix} = \frac{a}{r} \begin{bmatrix} -H_{qr} \\ H_{dr} \end{bmatrix} = \frac{a}{r} \begin{bmatrix} & -1 \\ 1 & \end{bmatrix} \begin{bmatrix} H_{dr} \\ H_{qr} \end{bmatrix} \qquad (7\text{-}9)$$

Equation 7-9 gives a relation in terms of dr- and qr-components. The transition from r, θ, z components is achieved by assuming a sinusoidally distributed field. Using a "tilde" over letters to denote dr- and qr-components, equation 7-9 can be expressed in the form:

$$\tilde{\mathcal{J}} = \frac{a}{r} \rho . \tilde{H} \qquad \text{amp/metre} \qquad (7\text{-}10)$$

(ρ is the skew-symmetric rotation matrix).

In equation 7-5 only the rotor current sheet was considered. When the stator is also energised, the stator linear current density is added to the rotor line density in equation 7-9.

Magnetomotive force describes the effectiveness of a coil as a source of magnetic field. As the coil is distributed inside an electric machine, a closed path is used to describe mmf. an example being the path ABCD (Fig. 21a). The line integral, $\oint H$, for this path gives the mmf. Usually a path is chosen with the returning path CD one pole-pitch away from AB. For such a case ${}_C\int^D H_r \; dr = {}_A\int^B H_r \; dr = (H_r)_{AB} \; a$. The total line integral around the closed path, i. e. the mmf, is then $2a(H_r)_{AB}$. Denoting this quantity by the German letter $\mathcal{H}$, equation 7-9 becomes:

$$\begin{bmatrix} \mathcal{J}^{dr} \\ \mathcal{J}^{qr} \end{bmatrix} = \frac{1}{2r} \begin{bmatrix} & -1 \\ 1 & \end{bmatrix} \begin{bmatrix} \mathcal{H}_{dr} \\ \mathcal{H}_{qr} \end{bmatrix} \qquad (7\text{-}11)$$

$$\tilde{\mathcal{J}} = \frac{1}{2r} \rho \, \tilde{\mathcal{H}}$$

7-2-2- Tangential Component of Magnetic Force

Even with the simplified field model, the magnetic field in the air-gap is not purely radial. There is a tangential component of magnetic force in the air-gap. It is necessary, therefore, to establish the order of this component and its value adjacent to the stator and rotor surfaces. For this, a closed path AB'C'D (Fig. 21b) is considered. The integral relation of equation 7-6 now takes the form:

$$(H_r)_{AB'} \ \Delta r + (H_\theta)_{B'C'} \ \Delta\theta - (H_r)_{C'D} \ \Delta r - 0$$

$$= \mathcal{J} \, r \, \Delta\theta \qquad (7\text{-}12)$$

If Δr is made sufficiently small, the magnetic force adjacent to the rotor surface is, from equation 7-12,

$$H_\theta = \mathcal{J} \, r \qquad (7\text{-}13)$$

The physical component, $H_{(\theta)}$, for polar coordinates is equal to H_θ / r (Section 4-6-1). Thus, from equation 7-13,

$$H_{(\theta)} = \mathcal{J} \quad \text{amp/metre} \qquad (7\text{-}14)$$

Comparing equations 7-14 and 7-10, it is seen that the ratio of the two components (radial : tangential) is r : a. Since, in a typical machine, r>>a, the tangential component is of ten ignored. For example, in the integral $\int_A^B H_r \, dr$ (Fig. 21a), the variation of the radial magnetic force, H_r normal to the air-gap was ignored (Section 7-2-1). This would not be so, if the tangential component of magnetic force is a significant quantity compared to the radial force. Care must, however, be taken even if r>>a, when product terms involving the tangential magnetic force arise. This occurs in Maxwell's electromagnetic stresses, Poynting energy-flow vector and in calculating the tangential force on the rotor as shown in Sections 7-3-4 and 7-3-5.

A similar analysis near the stator surface relates the stator line density to the tangential magnetic force adjacent to the stator surface. This need not be equal to the rotor surface H_θ, since the currents carried by stator and rotor coils are not necessarily the same. The tangential force exerted on the stator structure must, of course, equal the force exerted on the rotor structure.

7-2-3 Vector Potential and Flux Density

In general, the direction of the vector potential is the same as that of the element of current by which it is produced(31). Fig. 22 shows the directions of the field quantities B, H, A and also the line density, $\mathcal{J}$, for the field model when a single rotor coil is considered. The arrows are drawn at positions where their maximum values occur. The air-gap is usually very much smaller than the radius so that the field quantities,

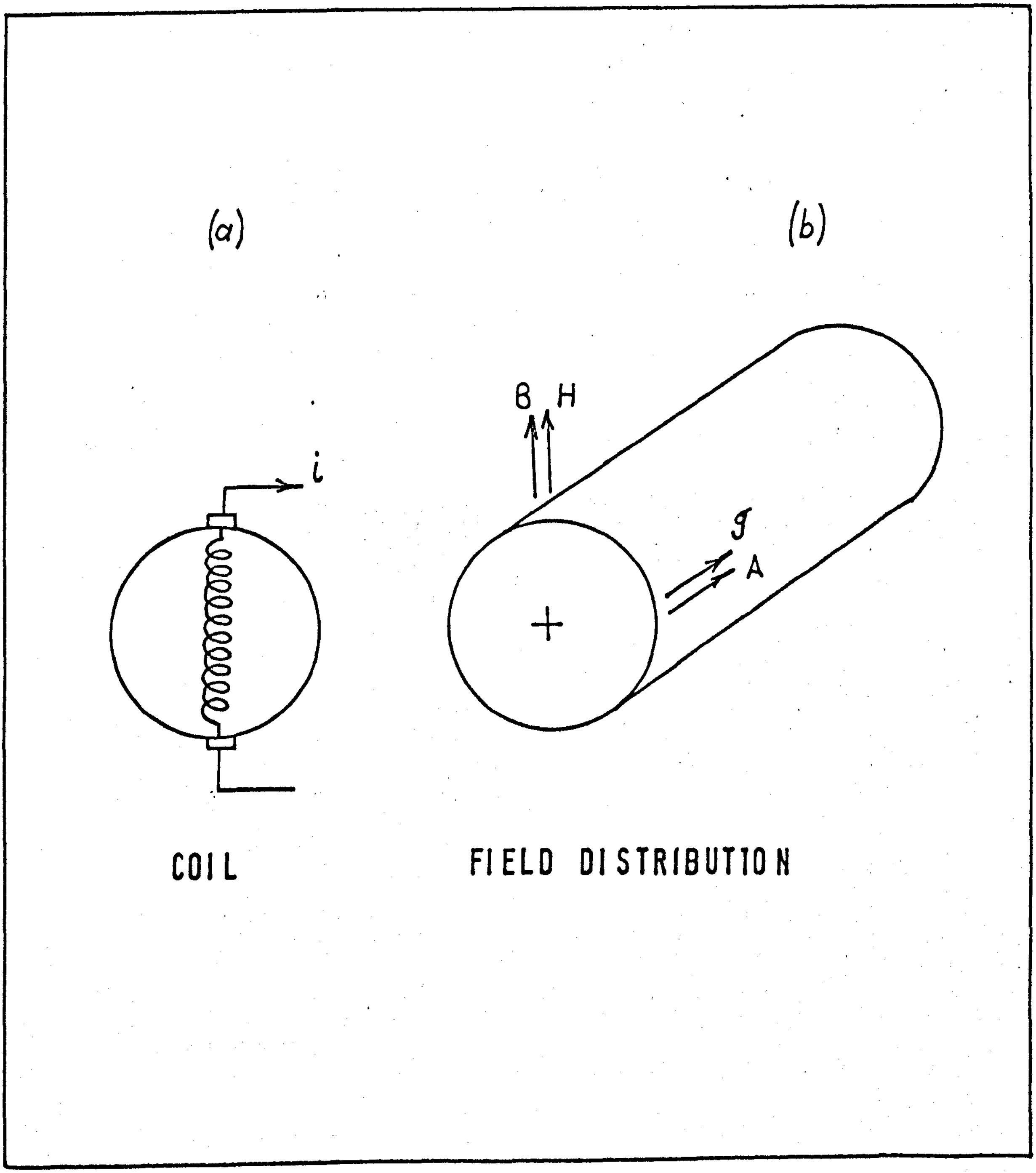


FIG. 22. DIRECTIONS OF FIELD QUANTITES

B and H, are shown nearly radial.

With sinusoidal dis tribution, the vector potential can be expressed as:

$$A_z = A_{dr} \cos\theta + A_{qr} \sin\theta \tag{7-15}$$

The flux density components B^r and B^θ can be expressed in terms of the vector potential as

$$\left(\frac{1}{r} \frac{\partial A_z}{\partial \theta}\right) \quad \text{and} \quad \left(-\frac{1}{r} \frac{\partial A_z}{\partial r}\right)$$

respectively, from the equation, curl $\vec{A} = \vec{B}$. For the radial component, with sinusoidal distribution,

$$B^r = B^{dr} \cos\theta + B^{qr} \sin\theta \tag{7-16}$$

From equation 7-15, $\frac{\partial A_z}{\partial \theta}$ is:

$$-A_{dr} \sin\theta + A_{qr} \cos\theta$$

The equation $B^r = \frac{1}{r} \frac{\partial A_z}{\partial \theta}$, becomes:

$$\begin{bmatrix} B^{dr} \\ B^{qr} \end{bmatrix} = \frac{1}{r} \begin{bmatrix} A_{qr} \\ -A_{dr} \end{bmatrix} = \frac{1}{r} \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \begin{bmatrix} A_{dr} \\ A_{qr} \end{bmatrix} \tag{7-17}$$

Equation 7-17 expresses a relation between dr- and qr-components. With TILDE notation,

$$\tilde{B} = \frac{1}{r} \quad \wp_t \quad \tilde{A} \tag{7-18}$$

($\wp_t$ is the transposed rotation matrix).

A similar analysis can be carried out at the stator surface.

7-2-4 Induced Electric Field

For stationary bodies (i. e. stationary with respect to the observer measuring the field quantities), the induced electric field is:

$$E_z = - \frac{\partial A_z}{\partial t} \tag{7-19}$$

For moving bodies the additional term $\vec{u} \times \vec{B}$ arises, where $\vec{u}$ is the velocity of the moving body with respect to the observer and $\vec{B}$ is the field relative to the observer. For the rotor conductors moving with a velocity $p\ \Theta$, the induced electric field is:

$$E_z = - \frac{\partial A_z}{\partial t} - r\ p\ \Theta\ B^r \tag{7-20}$$

Now with sinusoidal distribution, the induced electric field can be expressed as:

$$E_z = E_{dr} \cos\theta + E_{qr} \sin\theta \tag{7-21}$$

Equations 7-21, 7-15 and 7-16 can be substituted in equation 7-20, giving

$$\begin{array}{|c|}\hline E_{dr} \\ \hline E_{qr} \\ \hline \end{array} = - \begin{array}{|c|}\hline \partial A_{dr} / \partial t \\ \hline \partial A_{qr} / \partial t \\ \hline \end{array} - r \, p\Theta \begin{array}{|c|}\hline B^{dr} \\ \hline B^{qr} \\ \hline \end{array} \tag{7-22}$$

With tilde notation,

$$\tilde{E} = - \frac{\partial \tilde{A}}{\partial t} - r \, p \, \Theta \times \tilde{B} \tag{7-23}$$

Equation 7-18, expressing $\tilde{B}$ in terms of $\tilde{A}$, is substituted in equation 7-23 to give:

$$\tilde{E} = - \frac{\partial \tilde{A}}{\partial t} - p \; \Theta \; \rho_t \tilde{A} \tag{7-24}$$

7-2-5 Four Dimensional Field Tensors

The field tensors in space-time coordinate system developed in Section 4-5 are here applied to electric machine fields. The fourth coordinate, x^4, is taken to denote time in seconds. The first field tensor is from equation 4-31,

$$F_{mn} = \begin{array}{c|c|c|c|c|}
{}^{\rightarrow n}_{m \downarrow} & r & \theta & z & t \\ \hline
r & & & -rB^{\theta} & \\ \hline
\theta & & & rB^{r} & \\ \hline
z & rB^{\theta} & -rB^{r} & & E_z \\ \hline
t & & & -E_z & \\ \hline
\end{array} \tag{7-25}$$

The second field tensor, is from equation 4-33,

$$H^{mn} = \begin{array}{c|c|c|c|c|} {}_{m}^{\;n} & r & \theta & z & t \\ \hline r & & & -\frac{H_\theta}{r} & \\ \hline \theta & & & \frac{H_r}{r} & \\ \hline z & \frac{H_\theta}{r} & -\frac{H_r}{r} & & \\ \hline t & & & & \\ \hline \end{array} \qquad (7\text{-}26)$$

The stress-energy tensor is, from equation 4-36,

$$T^{m}_{n} = \begin{array}{c|c|c|c|c|} {}_{n}^{\;m} & r & \theta & z & t \\ \hline r & \frac{1}{2}B^r\, H_r & B^\theta\, H_r & & \\ \hline \theta & B^r\, H_\theta & -\frac{1}{2}B^r\, H_r & & \\ \hline z & & & -\frac{1}{2}B^r\, H_r & \\ \hline t & -\frac{E_z\, H_\theta}{r} & \frac{E_z\, H_r}{r} & & \frac{1}{2}B^r\, H_r \\ \hline \end{array}$$

$$(7\text{-}27)$$

In equation 7-27, in the diagonal terms, the addition of the quantity $B^{\theta} H_{\theta}$ is ignored when compared with $B^{r} H_{r}$.

The stress-energy tensor gives Maxwell's magnetic stresses and Poynting's energy-flow vector. The tangential stress can be related to the torque on the rotor and the radial Poynting component to power flow. These relationships are developed in Sections 7-3-4 and 7-3-5.

7-3 Relation between Field Model and Lumped Parameter Model

The following sections relate the field quantities to terminal quantities, for the idealised machine. The performance equations are seen to be obtained directly from Maxwell's equations. Field-terminal relations such as current density-coil currents, vector potential-coil flux linkages, induced electric force-induced emf, tangential magnetic stress-torque, and Poynting radial energy flow-power flow are described.

7-3-1 Current Density-Terminal Current Relation

Fig. 23a shows the representation of a distributed winding, by a coil. The coil carries a current i. Fig. 23b shows the equivalent field model, with a current sheet having a line density $\hat{\mathcal{J}}\sin\theta$, (peak value denoted by a wedge over the letter). If this maximum value is directed along an axis at right angles to the axis A-A', then the field produced by the current sheet is in the same direction as the coil of Fig. 23a. Moreover, for the two models to be equivalent, the mmf around the closed path (shown by the dotted line) must be equal to that produced by the coil. For the coil representation, the mmf is Ni (N being the number of turns of the coil). For the field model,

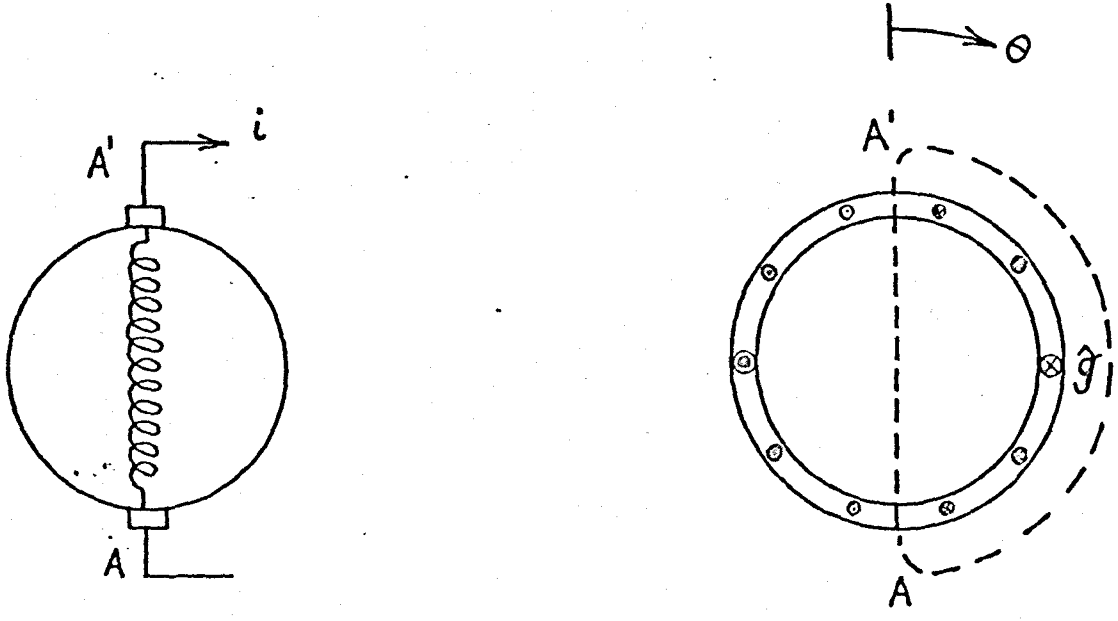
$$\text{mmf} = \int_0^{\pi} (\hat{\mathcal{J}}\sin\theta)\ r\,d\theta = 2\,\hat{\mathcal{J}}\,r$$

i.e. $Ni = 2\,\hat{\mathcal{J}}\,r$

or,

$$\hat{\mathcal{J}} = \frac{Ni}{2r} \qquad (7\text{-}28)$$

Denoting the directions by the superscripts dr and qr, equation 7-28 becomes:



(a)

COIL REPRESENTATION

(b)

CURRENT SHEET

FIG. 23.

$$\mathcal{J}^{qr} = \frac{N\, i^{dr}}{2r} \qquad (7\text{-}29)$$

This space-relation is shown in Fig. 24. A similar analysis for the coil current i^{dr} gives:

$$\mathcal{J}^{dr} = -\frac{N\, i^{qr}}{2r} \qquad (7\text{-}30)$$

Equations 7-29 and 7-30 are combined into one matrix equation:

$$\begin{array}{|c|}\hline \mathcal{J}^{dr} \\ \hline \mathcal{J}^{qr} \\ \hline\end{array} = \frac{N}{2r} \begin{array}{|c|}\hline -i^{qr} \\ \hline i^{dr} \\ \hline\end{array} = \frac{N}{2r} \begin{array}{|c|c|}\hline & -1 \\ \hline 1 & \\ \hline\end{array} \begin{array}{|c|}\hline i^{dr} \\ \hline i^{qr} \\ \hline\end{array}$$

or

$$\tilde{\mathcal{J}} = \frac{N}{2r}\, \rho \;\; \tilde{i} \qquad \text{amp/metre} \qquad (7\text{-}31)$$

From equations 7-31 and 7-11 it follows that

$$\tilde{\mathcal{H}} = N\tilde{i} \qquad (7\text{-}32)$$

That is, the mmf's are given by the ampere turns of the winding and the axes of magnetisation are given by the axes of the coils, as stated earlier.

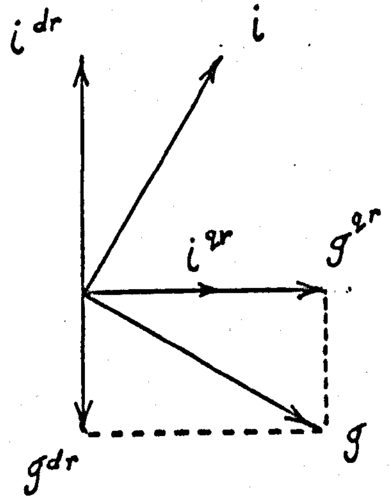


FIG.24. SPACE VECTOR DIAGRAM

The space relations between currents and current densities are illustrated in Fig. 24.

7-3-2 Vector Potential - Flux Linkage Relation

The inductance of a winding can be calculated in terms of vector potential as(27, 37):

$$L = \frac{\iint A . J \, dS}{i^2} \quad x \quad \text{length of conductor} \qquad (7\text{-}33)$$

(dS is an elemental area on the cross-section of the conductor).

The integration is carried out over the cross-section of the conductor. For the current sheets, equation 7-33 reduces to:

$$L = \frac{\int A . \mathcal{J} \, r \, d\theta}{i^2} \quad x \quad \text{length of conductor} \quad (7\text{-}34)$$

For the coil of Fig. 23. with sinusoidal distribution, $A = \hat{A} \sin\theta$ and $\mathcal{J} = \hat{\mathcal{J}} \sin\theta$ (peak values denoted by wedges over letters).

Equation 7-34 becomes:

$$L = \frac{\ell}{i^2} \int_{0}^{2\pi} \hat{A} \sin\theta \quad \hat{\mathcal{J}} \sin\theta \quad r d\theta$$

$$= \frac{\pi \ell r}{i^2} \hat{A} \, \hat{\mathcal{J}} \qquad (7\text{-}35)$$

(ℓ = rotor length)

Substituting $\hat{\mathcal{J}} = Ni/2r$ (equation 7-28).

$$L = \frac{\pi N \ell}{2i} \hat{A}$$

or,

$$\hat{A} = \frac{2}{\pi N \ell} L\, i \qquad \text{weber/metre} \qquad (7\text{-}36)$$

Equation 7-36 gives the peak value of the vector potential due to a current i in the coil shown in Fig. 23, and is seen to be directed along an axis at right angles to the coil representation of the winding. Denoting the directions by subscripts dr and qr, equation 7-36 becomes:

$$A_{qr} = \frac{2}{\pi N \ell} L_{dr}\, i^{dr}$$

Similarly, a coil in the quadrature axis carrying a current i^{qr} produces a vector potential $A_{dr}\cos\theta$, where

$$A_{dr} = -\frac{2}{\pi N \ell} L_{qr}\, i^{qr}$$

i. e.

$$\begin{array}{|c|}\hline A_{dr} \\ \hline A_{qr} \\ \hline \end{array} = -\frac{2}{\pi N l} \begin{array}{|c|}\hline L_{qr}\, i^{qr} \\ \hline -L_{dr}\, i^{dr} \\ \hline \end{array}$$

$$= -\frac{2}{\pi N l}\begin{array}{|c|c|}\hline & 1 \\ \hline -1 & \\ \hline \end{array}\ \begin{array}{|c|c|}\hline L_{dr} & \\ \hline & L_{qr} \\ \hline \end{array}\ \begin{array}{|c|}\hline i^{dr} \\ \hline i^{qr} \\ \hline \end{array} \qquad (7\text{-}37)$$

or,

$$\tilde{A} = -\frac{2}{\pi N \ell}\ \mathcal{J}_t\ L\ \tilde{i} \quad \text{weber/metre} \qquad (7\text{-}38)$$

The contribution of the stator winding to the rotor surface vector potential will be expressed in terms of $M_d\ i^{ds}$ and $M_q\ i^{qs}$. The vector potential of the field model is thus seen to be related to the flux linkages of windings in the coil model.

7-3-3 Induced Electric Force and EMF

To arrive at the induced emf, $\mathcal{E}$, in a winding, the induced electric force, E, is integrated around the winding, appropriately weighted according to the distribution of conductors. The coil shown in Fig. 23a has "N" turns.

If the distribution of these turns around the periphery of the rotor is assumed to be approximately sinusoidal, then the number of rotor conductors in a small sector of the rotor subtending $\Delta\theta$ radians at the centre will be $(\frac{N}{2} \sin\theta \, \Delta\theta)$. The induced emf, $\mathcal{E}$, in such a coil, in terms of the induced electric force, E, is:

$$\mathcal{E} = \left\{ \int_0^{2\pi} (\frac{N}{2} \sin\theta) \; E \, d\theta \right\} \text{ x } \text{(length of conductor)}$$

The induced electric force, is, from equation 7-21:

$$E_z = E_{dr} \cos\theta + E_{qr} \sin\theta$$

With this substitution, the integration shown above reduces to:

$$\mathcal{E} = \frac{\pi N \ell}{2} E_{qr}$$

Since the currents and voltages associated with the coil in Fig. 23a are denoted by a suffix "dr", the emf is written as $\mathcal{E}_{dr}$

i. e.

$$\mathcal{E}_{dr} = \frac{\pi N \ell}{2} E_{qr}$$

A similar analysis for a coil in qr-axis yields:

$$\mathcal{E}_{qr} = - \frac{\pi N \ell}{2} . E_{dr}$$

i. e.

$$\begin{bmatrix} \mathcal{E}_{dr} \\ \mathcal{E}_{qr} \end{bmatrix} = \frac{\pi N \ell}{2} \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \begin{bmatrix} E_{dr} \\ E_{qr} \end{bmatrix}$$

or,

$$\tilde{\mathcal{E}} = \frac{\pi N \ell}{2} \rho_t \tilde{E} \qquad \text{volt} \qquad (7\text{-}39)$$

Substitution of $\tilde{E}$ in terms of $\tilde{A}$ from equation 7-24 in 7-39 gives:

$$\tilde{\mathcal{E}} = - \frac{\pi N \ell}{2} \rho_t \left(\frac{\partial \tilde{A}}{\partial t} + \rho_t \tilde{A} \, p \, \Theta \right) \qquad (7\text{-}40)$$

Substitution of $\tilde{A}$ in terms of L. i from equation 7-38 in 7-40 gives:

$$\begin{aligned} \tilde{\mathcal{E}} &= - \left(\frac{\partial L \tilde{i}}{\partial t} + \rho_t L \, \tilde{i} \, p \, \Theta \right) \\ &= - \left(L \frac{\partial \tilde{i}}{\partial t} + \rho_t L \, \tilde{i} \, p \Theta \right) \qquad (7\text{-}41) \end{aligned}$$

Equation 7-41 can be compared with equation 7-2 in which the induced emf is expressed as:

$$- (L_{ab}\, p\, i^b + G_{ab}\, i^b\, p\, \Theta)$$

It is seen that the torque matrix, G_{ab} gives the term $\rho_t L$ in equation 7-41. Equation 7-3 gives the terms of the torque matrix and the primed quantities are equal to the unprimed quantities under sinusoidal distribution.

7-3-4 Electromagnetic Torque

The torque on the rotor due to the electromagnetic field can be established in terms of the tangential Maxwell stress (the space of the electromagnetic field being conceived to be in a state of tension analogous to an elastic medium).

From the stress-energy tensor (equation 7-27), the tangential stress on the rotor surface is:

$$T^r_\theta = B^r H_\theta$$

(The superscript "r" in the tensor component T^r_θ denotes that the stress acts on an area normal to the radial direction; the subscript "θ" denotes the direction of force).

The torque on a small element of area rdθ dz is:

$$T^{r}_{\theta} \; r \, d\theta \, dz = B^{r} H_{\theta} \; r \, d\theta \, dz$$

In polar coordinates the determinant of the metric tensor is: $g = r^2$. A tensor multiplied by $\sqrt{g}$ is a tensor density. The term (T^{r}_{θ} r) is thus the tensor density component of the stress-energy tensor. In Section 4-4, it was pointed out that tensor densities can be integrated over infinitesimal regions. For larger regions, the integration implies a summation of tensors not located at the same point and generally, this is not permissible. For a purely cylindrical structure, however, the direction of the quantity $T^{r}_{\theta} \; r \, d\theta \, dz$ is always normal to the radius. It can therefore be integrated over the rotor surface to give the rotor torque:

$$\int_{0}^{\ell}\int_{0}^{2\pi} T^{r}_{\theta} \; r \, d\theta \, dz = \int_{0}^{\ell}\int_{0}^{2\pi} B^{r} H_{\theta} \; r \, d\theta \, dz \qquad (7\text{-}42)$$

Substitutions of equations 7-16, 7-13 and 7-5 in equation 7-42 yield:

$$\text{Torque} = \pi r^{2} \ell \; (B^{dr} \; \mathcal{J}^{dr} + B^{qr} \; \mathcal{J}^{qr})$$

$$= \pi r^{2} \ell \; (\tilde{B})_{t} \; \tilde{\mathcal{J}} \; \text{newton-metre} \qquad (7\text{-}43)$$

In equation 7-43, the current-density vector is at right angles to the current vector (Fig. 24). The equation is thus seen to correspond to the well known torque equation:

$$T = \phi_{qr}\, i^{dr} - \phi_{dr}\, i^{qr}$$

Further substitutions of equations 7-18, 7-31 and 7-38 in equation 7-43 yield:

$$\text{Torque} = \tilde{i}_t\, L_t\, \rho\, \tilde{i} = \tilde{i}_t\, \rho_t\, L\, \tilde{i} \qquad (7\text{-}44)$$

The electromagnetic torque derived in equation 7-44 can be compared with equation 7-4 in which the generated torque due to currents is expressed as $i_t \cdot G \cdot i$. The torque matrix, G, can be expressed as $\rho_t L$.

In equation 7-42, the tangential component of magnetic force, H_θ, appears. Even though this component is small compared to the radial magnetic force, nevertheless it cannot be ignored in the calculation of tangential magnetic stresses.

7-3-5 Power Flow

The radial component of Poynting's energy-flow vector, T^r_t, can be integrated over the rotor surface to give the radial power flow. From equation 7-27,

$$T_t^r = - \frac{E_z H_\theta}{r} \qquad (7\text{-}45)$$

It is again seen that the tangential component, H_θ, cannot be ignored in Poynting's energy-flow analysis.

Total radial power flow is:

$$P_r = \int_0^\ell \int_0^{2\pi} T_t^r \ r \, d\theta \ dz = - \int_0^\ell \int_0^{2\pi} E_z \ H_\theta \ \ d\theta \, dz$$

Substitution of equations 7-21, 7-13 and 7-5 in the above integral yield:

$$P_r = - \pi r \ell \ \left(E_{dr} \ \mathcal{J}^{dr} + E_{qr} \ \mathcal{J}^{qr} \right)$$

$$= - \pi r \ell \ (\tilde{E})_t \ \tilde{\mathcal{J}} = - \pi r \ell \ (\tilde{\mathcal{J}})_t \ \tilde{E} \ \text{watts}$$

$$(7\text{-}46)$$

Further substitutions of equations 7-41, 7-39 and 7-31 yield:

$$P_r = \tilde{i}_t \left(L \frac{d\tilde{i}}{dt} + \rho_t L \, \tilde{i} \, p \, \Theta \right) \qquad (7\text{-}47)$$

In equation 7-47, the term $\tilde{i}_t L \frac{d\tilde{i}}{dt}$ represents the power flow required to meet the change in magnet stored energy with varying load and the second term, $\tilde{i}_t \ \rho_t \ L \ \tilde{i} \ p \ \Theta = \tilde{i}_t \ G \ \tilde{i} \ p \ \Theta'$, represents the electro-mechanical power conversion.

In the calculation of Poynting's energy-flow vector, the induced electric field has been considered. This is termed "partial field" by Hammond(38). In a conducting material, the induced electric field causes additional charge distribution in order to produce a resultant electric force ρJ (ρ is the resistivity and J is the current density). To quote Hammond " ... it is rather surprising to find that an exploring test charge would not be able to differentiate between the case when the motor is stationary and when it is running. In either event the observed electric force would be ρJ. Neither the applied potential difference nor the induced e. m. f. can be observed separately. " Again "It will be noticed that the Poynting method is exactly analogous to the back-e. m. f. method used in power devices, because the back-e. m. f. is also essentially a partial field, which it is convenient to keep separate for the purposes of computation. "

In Section 7-2-4, the induced electric force for the moving rotor conductors was established. These results have in fact been used in this section to express the Poynting vector (equation 7-47). In Section 7-3-3, the induced electric force been related to the back emf. The analysis in this section is thus seen to correspond with Hammond's partial-field approach.

7-3-6 Units and Dimensions

In the foregoing analysis, the field quantities developed for the field model have been related to the terminal quantities of Park's idealised machine. Maxwell's equations for the field quantities resulted in the performance equation of the machine. Dimensional balance has been maintained in the field-terminal relations (equations 7-31, 7-38, 7-39, 7-43 and 7-46) and the M. K. S. system of units has been used throughout. The dimensional constants $(N/2r)$, $(\pi N l/2)$, $(\pi r^2 l)$ and $(\pi r l)$ appearing in the equations can all be made unity by considering a machine with unit radius ($r = 1$), unit area per pole ($\pi r l = 1$) and unit turns per pole ($N = 2$ for a 2-pole configuration). Alternatively, a per-unit system for field quantities related to the per-unit system for terminal quantities can be established, to make the constants unity.

In arriving at the field equations and field-terminal relations, hysteresis of iron, changes of saturation and the electric flux density vector, D, are all ignored. Sinusoidal space distribution is assumed. Tangential field components are taken into account.

A 2-pole configuration has been assumed for simplicity, For P pole-pairs, the field terminal relations 7-31 and 7-38 are unaltered if N is now taken as the number of turns of a coil, per pole-pair. The sinusoidal distribution for current density (equation 7-5) then becomes:

$$\mathcal{J} = \mathcal{J}^{dr} \cos (P\theta) + \mathcal{J}^{qr} \sin (P\theta)$$

The generated torque $i_t \, G \, i$ now gives torque per pole-pair. The generated voltage is $G_{ab} \, i^b \, p \, \Theta \; P$.

7-4 Covariant Field Analysis

Kron demonstrated(2, 39) that by using such concepts as covariant derivatives, as developed in Riemannian and non-Riemannian geometry, the performance equations of many kinds of electric machines could be reformulated in a general manner. The equations of the "family" or group of machines could be set up in a way independent of the reference frame adopted. (Each machine is a "reference frame"). The field quantities in a machine could be expressed in terms of covariant derivatives(40, 41). The "field" analysis presented by Kron and analysed in some detail below is not, in fact, the true "point" field description already described in Sections 7-2 and 7-3 in terms of geometrical space-coordinates and time. Now in his generalised theory of electric machines, Kron's reference "coordinates" are charges. He must therefore describe the fields in this abstract "electrical" space. To do this, he defines at least one fundamental field quantity in this space directly. The field equations in the abstract space are then built on this. The quantity which can most readily be defined in terms of charge "coordinates" is the vector potential.

Defined in this manner as in Section 7-4-5, the term which is called "vector potential" is in fact an integral of the "point" vector potential, over local regions of the machine.

The use of "Charge coordinates" enables a unified description of machines to be derived. In addition, the theory as already mentioned, carries over into multi-machine systems, which are coupled mechanically or electrically. Tensor transformations give the family of machine equations. This implies that in such transformations, differentials of field quantities will arise and therefore covariant differentiation of field quantities must be used in the abstract space. In analysing Park's idealised machine, the Park transformations are seen to lead to a skewsymmetric torsion tensor, $S_{\lambda\pi}^{\cdot\cdot\mu}$. Kron pointed out(40) that the analysis of field relations in such terms converges with Novobatzky's proposed relativistic universal field theory(42). In this, five vectors are introduced to bring gravitation and electricity into a single system. The essence of Novobatzky's treatment can be illustrated by two short quotations.

"[The theory] is based on the principle that the connection between adjacent tangential spaces in the four-dimensional world is not affinitive but projective..... the theory of the five-vectors is a projective equivalent to the ordinary four-vectors .... at every world point there is a normal direction N. The four-term space of the tangent vectors is then extended to a five-vector space. Four components of the five-vector unit lie within the tangential space, the fifth has the normal direction."

"Obviously it (the additive introduction of the tensor $S_{\lambda\pi}^{\cdot\cdot\mu}$) relates to a rotation of the rigid vector body after the conclusion of translation, since only a rotation will alter neither the magnitude nor angle. The local axes of rotation can be anywhere according to the values of the S-components. ... It follows from this that $S_{5kl} = - S_{lk5}$ must be skewsymmetrical in the indices k, l:

$$S_{5kl} = F_{kl}, \quad S_{lk5} = -F_{kl} = F_{lk}$$

in which F_{kl} is a skewsymmetric tensor in the four-dimensional world i. e. the electromagnetic field quantity."

Novobatzky attempts to combine gravitational and electromagnetic effects into one unified field theory. A general relativistic theory of this form has still not been successful. However, the electromagnetic field-gravity structure in all the proposed "unified" field theories is surprisingly close to Kron's generalised electro-mechanical presentation of rotating machine theory. The link is in the generalised curl concept. In unified field theory, the tangent space of four-vectors is extended to a five-vector space by considering a normal direction to the tangent 4-space. The generalised curl is expressed in terms of skewsymmetric tensors of the type S_{5kl}. In the machine theory, the current vectors lie in a tangential space in the abstract charge-coordinate system and the rotor angle Θ denotes a direction normal to the "current space." The generalised curl is here expressed in terms of torsion tensors of the type $S_{\beta(S)}^{\cdot\,\cdot\,\gamma}$ (given by equation 7-77; the coordinate $x^S = \Theta$). In both cases the covariant divergence, curl and gradient are used.

In the following analysis, the electromagnetic field equations will be expressed in terms of covariant differentials. The covariant differentials will be derived for Park's reference system and the field quantities examined in these terms. These are then compared with those in the field model described in Section 7-2.

7-4-1 Covariant Differentials

If A_α and A_a are tensor components in two different reference frames with coordinates $\bar{x}^\alpha$ and x^a, they transform as:

$$A_a = C_a^\alpha A_\alpha$$

(C_a^α is the transformation tensor and repeated index α in the above equation implies a summation with $\alpha = 1, 2, \ldots, n$).

It is seen that the partial derivatives do not, in general, transform as tensors:

$$\frac{\partial A_a}{\partial x^b} = \frac{\partial (C_a^\alpha A_\alpha)}{\partial x^b} = \frac{\partial C_a^\alpha}{\partial x^b} A_\alpha + C_a^\alpha \frac{\partial A_\alpha}{\partial x^b}$$

$$= \frac{\partial C_a^\alpha}{\partial x^b} A_\alpha + C_a^\alpha \frac{\partial A_\alpha}{\partial \bar{x}^\beta} \frac{\partial \bar{x}^\beta}{\partial x^\beta}{}^b$$

i. e.

$$\frac{\partial A_a}{\partial x^b} = \frac{\partial C_a^\alpha}{\partial x^b} A_\alpha + C_a^\alpha C_b^\beta \frac{\partial A_\alpha}{\partial \bar{x}^\beta} \qquad (7\text{-}48)$$

The appearance of the first term on the right-hand side of equation 7-48 contravenes the tensor law of transformation.

However, the "covariant derivative" of a vector is a tensor$^{(21)}$:

$$\frac{\delta A_\alpha}{\delta \bar{x}^\beta} = \frac{\partial A_\alpha}{\partial \bar{x}^\beta} - \Gamma_{\alpha\beta}^{\pi} A_\pi \qquad (7\text{-}49)$$

where, the "affine connection" is:

$$\Gamma_{\alpha\beta}^{\pi} = \tfrac{1}{2} g^{\pi\delta} \left(\frac{\partial g_{\delta\alpha}}{\partial \bar{x}^\beta} + \frac{\partial g_{\beta\delta}}{\delta \bar{x}^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial \bar{x}^\delta}\right) \qquad (7\text{-}50)$$

In equation 7-50, $g_{\alpha\beta}$ etc. is the "metric tensor", and $g^{\pi\gamma}$ gives the terms of its inverse. The covariant derivative $\frac{\delta A_a}{\delta x^b}$ is defined similarly. It can be shown that covariant derivatives transform as tensors$^{(21)}$:

$$\frac{\delta A_a}{\delta x^b} = C_a^\alpha \, C_b^\beta \, \frac{\delta A_\alpha}{\delta \bar{x}^\beta} \qquad (7\text{-}51)$$

The law of transformation for the affine connection $\Gamma_{\alpha\beta}^{\pi}$ is:

$$\Gamma_{ab}^{p} = \Gamma_{\alpha\beta}^{\pi} \, C_a^\alpha \, C_b^\beta \, C_\pi^p + C_\sigma^p \, \frac{\partial C_a^\sigma}{\partial x^b} \qquad (7\text{-}52)$$

From equation 7-50, it is observed that the affine connection here is symmetric, because the metric tensor $g_{\alpha\beta}$ is symmetric.

i. e.

$$\Gamma^{\pi}_{\alpha\beta} = \Gamma^{\pi}_{\beta\alpha}$$

In equation 7-50, the bracketed term on the right-hand side is known as the Christoffel symbol and is usually denoted by $[\alpha\beta, \delta]$.

7-4-2 Exterior Differentials - Covariant Form

The exterior derivative of a vector (1-form) is, from Section 3-10-4, equation 3-45:

$$\frac{1}{1!} \delta^{ab}_{mn} \frac{\partial A_b}{\partial x^a} = \left(\frac{\partial A_n}{\partial x^m} - \frac{\partial A_m}{\partial x^n} \right) \tag{7-53}$$

A term, $\Gamma^{p}_{mn} A_p$ can be added to andsubtracted from equation 7-53. Now using the definition of covariant derivative (equation 7-49), it is seen that, the exterior derivative of a vector can be written as:

$$\left(\frac{\delta A_n}{\delta x^m} - \frac{\delta A_m}{\delta x^n} \right) = \frac{1}{1!} \delta^{ab}_{mn} \frac{\delta A_b}{\delta x^a} \tag{7-54}$$

In fact, the exterior derivative of a tensor of rank p in terms of covariant differentials is obtained by replacing the partial derivatives in equation 3-45 by covariant derivatives. i. e.

$$\frac{1}{p!}\,\delta^{abc\ldots g}_{mn\ldots q}\;\frac{\partial T_{bc\ldots g}}{\partial x^{a}}$$

$$=\frac{1}{p!}\,\delta^{abc\ldots g}_{mn\ldots q}\;\frac{\delta T_{bc\ldots g}}{\delta x^{a}} \qquad (7\text{-}55)$$

For the four-dimensional electromagnetic field tensor, F_{mn}, the exterior derivative (equation 4-32) can be written as:

$$\frac{\delta F_{nk}}{\delta x^{m}}+\frac{\delta F_{km}}{\delta x^{n}}+\frac{\delta F_{mn}}{\delta x^{k}} \qquad (7\text{-}56)$$

The divergence of a tensor in terms of covariant differentials is defined as:

$$\frac{\delta T^{bc\ldots m}}{\delta x^{m}} \qquad (7\text{-}57)$$

For a tensor of rank one, the divergence is:

$$\frac{\delta B^{p}}{\delta x^{p}}=\frac{\partial B^{p}}{\partial x^{p}}+\Gamma^{p}_{ap}\,B^{a} \qquad (7\text{-}58)$$

From equation 7-50,

$$\Gamma^{p}_{ap} = \frac{1}{2} g^{pc} \left(\frac{\partial g_{pc}}{\partial x^{a}}\right) \qquad (7\text{-}59)$$

If the determinant of the metric tensor is denoted by g, then by the law of differentiation of determinants,

$$\frac{\partial g}{\partial x^{a}} = g \quad g^{pc} \left(\frac{\partial g_{pc}}{\partial x^{a}} \right) \qquad (7\text{-}60)$$

Substitutions of equations 7-59 and 7-60 in equation 7-58 yield:

$$\frac{\delta B^{p}}{\delta x^{p}} = \frac{\partial B^{p}}{\partial x^{p}} + \frac{1}{2g} \cdot \frac{\partial g}{\partial x^{a}} \; B^{a}$$

$$= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^{p}} \left(\sqrt{g} \quad B^{p} \right) \qquad (7\text{-}61)$$

Similar analysis for a general case shows that the divergence of a tensor of rank p (equation 7-57) is:

$$\frac{1}{\sqrt{g}} \quad \frac{\partial (T^{bc\ldots m} \sqrt{g})}{\partial x^{m}} \qquad (7\text{-}62)$$

Equation 7-62 is seen to be identical with the expression derived earlier (Section 3-10-4, equation 3-46). In the earlier section the divergence was first defined as the dual of the exterior derivative of the dual of a tensor.

It is seen that the exterior derivatives and divergence operations derived earlier, although in terms of partial derivatives, are in fact tensors. To express them in covariant differential form, the partial derivatives are replaced by covariant derivatives.

In the divergence equation, comparing equations 7-57 and 7-62, it is seen that the affine connection terms appearing in the covariant derivatives also include the term $\sqrt{g}$. This term arises in taking the dual of a tensor.

In some transformations, the relations between the coordinates are implicitly given in terms of an equation between the differentials of the coordinates. When such relations appear in a non-integrable form, the transformation is said to be "non-holonomic". An additional geometric object appears in differential expressions, defined as$^{(39)}$:

$$\Omega^{\gamma}_{\alpha\beta} = \tfrac{1}{2} C^{a}_{\alpha} C^{b}_{\beta} \left\{ \frac{\partial C^{\gamma}_{a}}{\partial x^{b}} - \frac{\partial C^{\gamma}_{b}}{\partial x^{a}} \right\} \tag{7-63}$$

The affine connection (equation 7-50) is defined for such a non-holonomic reference frame as:

$$\Gamma^{\pi}_{\alpha\beta} = g^{\pi\delta}\ [\alpha\beta, \delta] + g^{\pi\delta}\ g_{\beta\sigma}\ \Omega^{\sigma}_{\delta\alpha}$$

$$+ g^{\pi\delta}\ g_{\alpha\sigma}\ \Omega^{\sigma}_{\delta\beta} - \Omega^{\pi}_{\alpha\beta} \qquad (7\text{-}64)$$

7-4-3 Non-Holonomic Transformation in an Electric Machine

Park's transformation is seen to be a special case of the non-holonomic transformation. The skewsymmetric rotation matrix arising out of such a transformation plays an important rôle in the tensor field analysis of machines. (Sections 7-4-5 to 7-4-8).

For the case of a revolving armature synchronous machine, Park's analysis is carried out in terms of hypothetical stationary brush currents and voltages. Fig. 25 shows the transformation diagramatically. A two-phase synchronous machine is assumed, for convenience. In the figure, a and b denote the axes of magnetisation of the two phase-coils, and d and q are the hypothetical stationary brushes. The relation

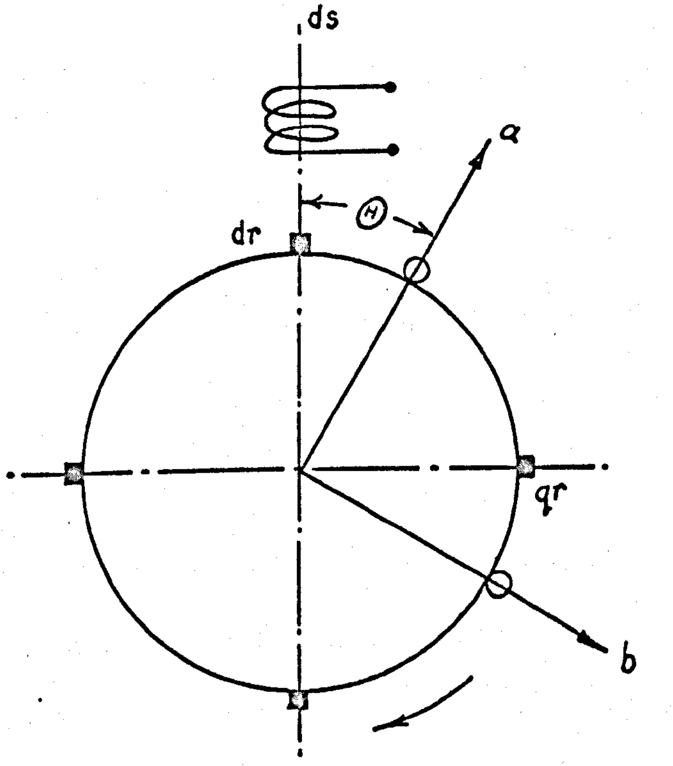


FIG. 25. PARKS AXES TO SLIP RING AXES IN A SYNCHRONOUS MACHINE

between the slip-ring currents and the hypothetical brush currents is:

$$i^{dr} = i^{a} \cos \Theta - i^{b} \sin \Theta$$

$$i^{qr} = i^{a} \sin \Theta + i^{b} \cos \Theta \qquad (7\text{-}65a)$$

Also, the rotor angular displacement is denoted as follows:

$$\bar{x}^{S} = x^{S} \quad (= \Theta) \qquad (7\text{-}65b)$$

The charges flowing across a cross-section of the windings, a and b, are denoted by x^{a} and x^{b} respectively. The currents, in terms of charges, are:

$$i^{a} = \frac{dx^{a}}{dt} ; \qquad i^{b} = \frac{dx^{b}}{dt}$$

Hypothetical charges $\bar{x}^{dr}$ and $\bar{x}^{qr}$ are defined, using equations 7-65 (a) and (b) as:

$$\begin{array}{c} \alpha \\ \hline d\bar{x}^{dr} \\ d\bar{x}^{qr} \\ d\bar{x}^{s} \end{array} = \begin{array}{c|ccc} \alpha \backslash m & a & b & s \\ \hline dr & \cos(x^{s}) & -\sin(x^{s}) & \\ qr & \sin(x^{s}) & \cos(x^{s}) & \\ s & & & 1 \end{array} \quad \begin{array}{c} m \\ \hline dx^{a} \\ dx^{b} \\ dx^{s} \end{array}$$

i. e. $$d\bar{x}^{\alpha} = C^{\alpha}_{m} \; dx^{m} \tag{7-66}$$

It is seen that:

$$\frac{\partial C^{\alpha}_{m}}{\partial x^{n}} \neq \frac{\partial C^{\alpha}_{n}}{\partial x^{m}} \tag{7-67}$$

The left-hand side of equation 7-66 is not then an "exact differential", since C^{α}_{m} cannot be expressed as $\partial \bar{x}^{\alpha} / \partial x^{m}$. Equation 7-66 cannot be integrated to obtain the variables $\bar{x}^{\alpha}$. Such transformations are known as non-holonomic(2, 39). The term "quasi-holonomic" is used when (a) some coordinates, such as x^{S}, transform holonomically (b) the other coordinates transform non-holonomically, and (c) the transformation tensor is a function of the coordinates under heading (a) only.

Using equation 7-66, the non-holonomic object defined by equation 7-63 can be seen to take the form:

$$\Omega^{\gamma}_{\beta(s)} = \frac{1}{2} \begin{array}{c|c|c|c} {}^{\rightarrow\beta}_{\downarrow\gamma} & dr & qr & s \\ \hline dr & & -1 & \\ \hline qr & 1 & & \\ \hline s & & & \end{array} \qquad (7\text{-}68)$$

The matrix in equation 7-68 can be seen to be the skewsymmetric rotation matrix used earlier in field analysis (Sections (7-2) and (7-3)).

7-4-4 Geometric Interpretation of Performance Equation

Kron pointed out the close analogy between the equations of performance of rotating electric machines and the equations of differential geometry of abstract spaces$^{(39)}$. He recognised that the magnetic stored energy of an electric machine remains invariant under such transformations as Park's transformation, as do certain quadratic differential forms in the geometry of abstract spaces. Kron's analysis is briefly reviewed in this section.

In Fig. 25, the magnetic stored energy of the windings, a and b, carrying currents i^a and i^b is:

$$W_m = \tfrac{1}{2} L_{aa} i^a i^a + M_{ab} i^a i^b + \tfrac{1}{2} L_{bb} i^b i^b \qquad (7\text{-}69)$$

(L represents the self, and M the mutual inductance). The stored kinetic energy is $\frac{1}{2} L_{SS}(p \Theta)^2$, where L_{SS} is the rotor inertia. The total stored energy can be expressed as $\frac{1}{2} L_{mn} \frac{dx^m}{dt} \frac{dx^n}{dt}$ (repeated indices implying summation).

In terms of currents i^{dr} and i^{qr} (hypothetical for a synchronous machine) the magnetic stored energy is

$$W_m = \tfrac{1}{2} L_{dr} i^{dr} i^{dr} + \tfrac{1}{2} L_{qr} i^{qr} i^{qr} \qquad (7\text{-}70)$$

Obviously the two expressions for magnetic stored energy (equations 7-69 and 7-70) must be equal. In terms of charges and angular displacement, the total stored energy can be expressed as:

$$W = \frac{1}{2} L_{mn} \frac{dx^m}{dt} \frac{dx^n}{dt}$$

This can be compared to the invariant line element in a geometric space:

$$(ds)^2 = g_{mn} \, dx^m \, dx^n$$

where, g_{mn} is the "metric tensor". For an electric machine, the inductances and inertia serve as the metric tensor. The affine connection, (equation 7-64) in terms of inductances and rotor inertia, is:

$$\Gamma^{\pi}_{\alpha\beta} = \frac{1}{2} L^{\pi\delta} [\alpha\beta, \delta] + L^{\pi\delta} L_{\beta\sigma} \Omega^{\sigma}_{\delta\alpha}$$

$$+ L^{\pi\delta} L_{\alpha\sigma} \Omega^{\sigma}_{\delta\beta} - \Omega^{\pi}_{\alpha\beta} \qquad (7\text{-}71)$$

In Park's reference system (the stationary brush axes), the Christoffel symbol $[\alpha\beta, \gamma]$ is zero since the inductances are constant (the Christoffel symbol term consists of partial derivatives of inductances with respect to charges and angular displacement). Moreover, multiplying equation 7-71 by $L_{\pi\gamma}$, one obtains:

$$\Gamma_{\alpha\beta,\gamma} = \Gamma^{\pi}_{\alpha\beta} L_{\pi\gamma} = L_{\beta\sigma} \Omega^{\sigma}_{\gamma\alpha} + L_{\alpha\sigma} \Omega^{\sigma}_{\gamma\beta} - L_{\pi\gamma} \Omega^{\pi}_{\alpha\beta} \qquad (7\text{-}72)$$

Equation 7-68 gives the Ω terms. Fig. 26 was found by the present author to give $\Gamma_{\alpha\beta,\gamma}$ from equation 7-72. In the figure,

$$G_{\alpha\beta} = 2\,\Omega^{\gamma}_{\alpha(S)} L_{\gamma\beta} = \rho^{\cdot\gamma}_{\alpha} L_{\gamma\beta}$$

and,

$$V_{\alpha\beta} = L_{\alpha\gamma}\, 2\,\Omega^{\gamma}_{\beta(S)} = L_{\alpha\gamma}\, \rho^{\gamma}_{\cdot\beta}$$

i. e.

$$V_{\alpha\beta} = (G_{\beta\alpha})_t$$

Fig. 26 can be explained using the performance equation:

$$v_{\gamma} = R_{\gamma\pi} \frac{dx^{\pi}}{dt} + L_{\gamma\pi} \frac{\delta}{\delta t} \left(\frac{dx^{\pi}}{dt}\right)$$

$$= R_{\gamma\pi} \frac{dx^{\pi}}{dt} + L_{\gamma\pi} \frac{d^2 x^{\pi}}{dt^2} + L_{\gamma\pi} \Gamma^{\pi}_{\alpha\beta} \frac{dx^{\alpha}}{dt} \frac{dx^{\beta}}{dt}$$

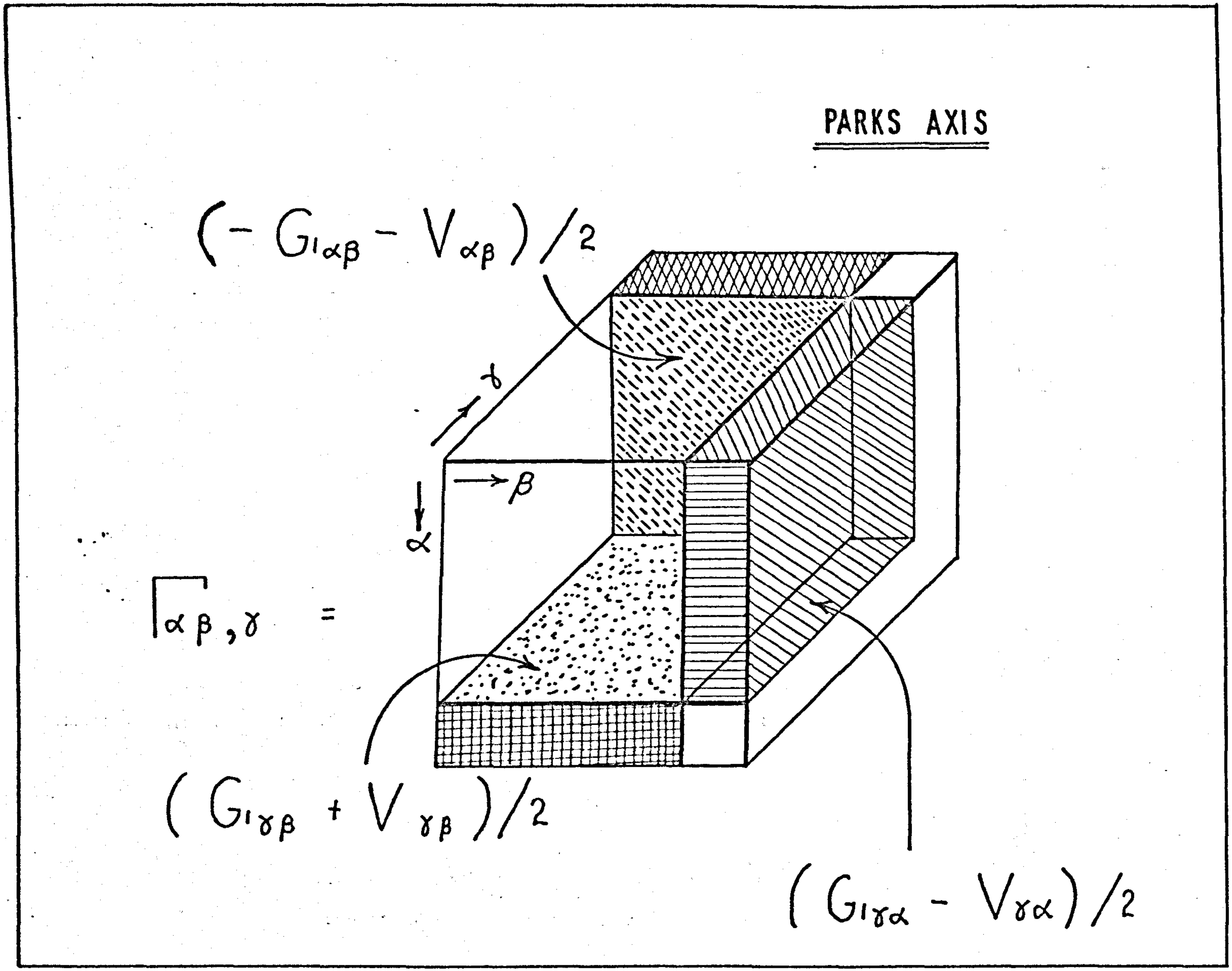


FIG. 26. AFFINE CONNECTION IN PARK'S AXIS

For voltages,

$$v_{\mu} = R_{\mu\pi} i^{\pi} + L_{\mu\pi} \frac{di^{\pi}}{dt} + \Gamma_{\alpha(S),\mu} i^{\alpha} \cdot \frac{dx^{S}}{dt}$$

$$+ \Gamma_{(S)\beta,\mu} \frac{dx^{S}}{dt} i^{\alpha} \qquad (7\text{-}73)$$

The last two terms on the right-hand side of equation (7-73) add to:

$$\tfrac{1}{2} (G_{\mu\alpha} - V_{\mu\alpha}) i^{\alpha} p \Theta + \tfrac{1}{2} (G_{\mu\beta} + V_{\mu\beta}) i^{\beta} p \Theta$$

$$= G_{\mu\alpha} i^{\alpha} p \Theta$$

i. e.

$$v_{\mu} = R_{\mu\pi} i^{\pi} + L_{\mu\pi} \frac{di^{\pi}}{dt} + G_{\mu\pi} i^{\pi} p \Theta \qquad (7\text{-}74)$$

Equation 7-74 can be seen to sum the voltage drops as resistance drop, back e. m. f. due to rate of change of flux linkage and back e. m. f. due to the conductors moving in a magnetic field.

The torque equation is,

$$v_s = R_{s(s)} \frac{dx^S}{dt} + L_{S(S)} \frac{d^2x^S}{dt^2} + \Gamma_{\alpha\beta,\, s}\, i^\alpha\, i^\beta$$

$$= R_{s(s)} \frac{dx^S}{dt} + L_{S(S)} \frac{d^2x^S}{dt^2} - \tfrac{1}{2}\,(G_{\alpha\beta} + V_{\alpha\beta})\, i^\alpha\, i^\beta$$

$$= R_{s(s)} \frac{dx^S}{dt} + L_{S(S)} \frac{d^2x^S}{dt^2} - i^\alpha\, G_{\alpha\beta}\, i^\beta \,.$$

It is seen from both the performance equations and fig. 26 that there is nothing to suggest that the relations have been obtained by first considering a reference system in which the winding currents i^a and i^b are the variables (slip-ring axes) and then transforming them to stationary brush-currents i^{dr} and i^{qr}. The geometric object $\Gamma_{\alpha\beta,\gamma}$ is seen to consist of terms of the type $G_{\alpha\beta}$ and $V_{\alpha\beta}$. These in turn consist of the rotation tensor and the metric tensor. An analysis could therefore be carried out beginning with Park's reference system and defining a tensor

$$P_{\alpha\beta\gamma} \overset{P}{=} \Gamma_{\alpha\beta,\,\gamma} \tag{7-75}$$

(the letter P over the equal sign denoting the validity of the equation in Park's axis only). Kron points out that whereas the earlier analysis dealt with a Riemannian space, the analysis which begins with Park's axis and equation 7-75 deals with a non-Riemannian space. It follows that:

$$P_{[\alpha\beta]\gamma} \overset{P}{=} \Gamma_{[\alpha\beta],\gamma} = -\Omega_{\alpha\beta}^{\pi} L_{\pi\gamma}$$

The square brackets indicate that the asymmetric part with respect to indices α and β is considered here. If the asymmetric part of tensor $P_{\alpha\beta\gamma}$ is denoted by a tensor $S_{\alpha\beta\gamma}$, it follows that:

$$S_{\alpha\beta\gamma} \overset{P}{=} -\Omega_{\alpha\beta}^{\pi} L_{\pi\gamma}$$

Also,

$$S_{\alpha\beta}^{\cdot\cdot\pi} \overset{P}{=} -\Omega_{\alpha\beta}^{\pi} \qquad (7\text{-}76)$$

In the covariant analysis of the electro-magnetic field quantities, the skewsymmetric torsion tensor given by equation 7-76 plays an important role which is investigated in Sections 7-4-5 to 7-4-8. To sum up a holonomic reference frame is the starting point. The transformation to stationary brush axes is quasi-holonomic. That is, the geometric objects

$\Omega_{\alpha\beta}^{\pi}$, $[\alpha\beta,\gamma]$ and $\Gamma_{\alpha\beta,\gamma}$ may be expressed in that frame without the aid of the old reference frame. Consequently there is nothing in the appearance of the transformed geometric objects to characterise them as being in a non-holonomic frame. We now have a choice of regarding the new system either as a Riemannian space referred to a non-holomic reference frame, or else as a non-Riemannian space referred to a holonomic reference frame. With the latter interpretation a torsion tensor $S_{\alpha\beta}^{\cdot\cdot\pi}$ appears which replaces the geometric object Ω. From equations 7-76 and 7-68,

$$2S_{\beta(s)}^{\gamma} = \begin{array}{c|c|c|c|} \gamma\downarrow \; \beta\rightarrow & dr & qr & s \\ \hline dr & & 1 & \\ \hline qr & -1 & & \\ \hline s & & & \\ \hline \end{array} \qquad (7\text{-}77)$$

7-4-5 Vector Potential

The following sections express the electromagnetic field relations in covariant terms. The four-dimensional field tensor, $F_{\alpha\beta}$, developed in Section 4-5, is established here from vector potential considerations. The magnetic flux-density vector and electric-force vector are expressed in terms of this field tensor, $F_{\alpha\beta}$ the magnetic-force vector being expressed in terms of a second four-dimensional field tensor $H^{\alpha\beta}$. The electric flux-density vector is ignored in the analysis. Maxwell's field relations are here examined in terms of the two field tensors $F_{\alpha\beta}$ and $H^{\alpha\beta}$. To bring out the salient features of the analysis, only the rotor winding is considered. It is shown, later, how the analysis can be readily extended for other layers of winding. The results of the following analysis are compared with the field model analysis of Section 7-2.

The four-dimensional field tensor, $F_{\alpha\beta}$, in terms of scalar and vector potentials, ϕ, is obtained from equation 4-30 (Section 4-5). The partial derivatives are replaced by covariant derivatives to give:

$$F_{\alpha\beta} = \left(\frac{\delta \phi_{\beta}}{\delta x^{\alpha}} - \frac{\delta \phi_{\alpha}}{\delta x^{\beta}}\right) \qquad (7\text{-}78)$$

The vector and scalar potentials are here denoted by ϕ rather than by A to avoid confusion with the actual potentials in the machine air-gap. "The reduction of the quantity F_{ik} to the primitive vector ϕ_i (electromagnetic potential) is not based simply on ad-hoc considerations, but is a result of our third requirement that the variational principles must be the only source for the field equations." (Novobatzky$^{(42)}$).

Using the relationships obtained in Section 7-3-2, the flux linkages of the windings can be defined in terms of electromagnetic vector potentials. That is:

$$\phi_\alpha = L_{\alpha\beta}\, i^\beta = \begin{array}{c|c|c|c|} \searrow\!\alpha & dr & qr & s \\ \hline & L_{dr}\ i^{dr} & L_{qr}\ i^{qr} & 0 \\ \hline \end{array} \qquad (7\text{-}79)$$

This relation implies integration carried out over the current sheet surface (Equation 7-35).

The microscopic field quantity (vector potential) associated with points in a non-Riemannian space is thus related to a macroscopic quantity (flux linkage) associated with the physical three dimensional machine space.

The flux density is obtained in terms of vector potential from the equation, curl $\vec{A} = \vec{B}$. In the exterior form of this equation (Section 4-4-1, equation 4-26), the partial derivatives are replaced by covariant derivatives to give:

$$\frac{1}{2!}\,\frac{1}{\sqrt{g}}\left(\frac{\delta\phi_\beta}{\delta x^\alpha} - \frac{\delta\phi_\alpha}{\delta x^\beta}\right) e^{\alpha\beta\gamma} = B^\gamma \qquad (7\text{-}80)$$

Equation 7-80 reduces to:

$$B^{\gamma'} = \begin{array}{l|c|c|c|}
\searrow\gamma & dr & qr & s \\ \cline{2-4}
 & L_{dr}\, i^{dr} & L_{qr}\, i^{qr} & 0 \\ \cline{2-4}
\end{array} \qquad (7\text{-}81)$$

as follows:

$$\begin{aligned}
\left(\frac{\delta\phi_\beta}{\delta x^\alpha} - \frac{\delta\phi_\alpha}{\delta x^\beta}\right) &= \left(\frac{\partial\phi_\beta}{\partial x^\alpha} - \Gamma_{\beta\alpha}^{\;\;\pi}\,\phi_\pi\right) - \left(\frac{\partial\phi_\alpha}{\partial x^\beta} - \Gamma_{\alpha\beta}^{\;\;\pi}\,\phi_\pi\right) \\
&= \left(\frac{\partial\phi_\beta}{\partial x^\alpha} - \frac{\partial\phi_\alpha}{\partial x^\beta}\right) + 2\,\Gamma_{[\alpha\beta]}^{\;\;\;\;\pi}\,\phi_\pi \\
&= \left(\frac{\partial\phi_\beta}{\partial x^\alpha} - \frac{\partial\phi_\alpha}{\partial x^\beta}\right) + 2\,S_{\alpha\beta}^{\;\;\cdot\cdot\pi}\,\phi_\pi \qquad (7\text{-}82)
\end{aligned}$$

The partial derivatives of the flux linkages with respect to the charges and angular displacement of the rotor are zero in Park's axis. Therefore, when equation 7-82 is substituted in equation 7-80, the resulting equation is:

$$\frac{1}{\sqrt{g}} S_{\alpha\beta}^{\cdot\cdot\pi} \quad \phi_{\pi} \quad e^{\alpha\beta\gamma} = B^{\gamma} \tag{7-83}$$

The skewsymmetric torsion tensor $S_{\alpha\beta}^{\cdot\cdot\pi}$ is obtained from equation 7-77. The term $\sqrt{g}\ B^{\gamma}$ is the tensor density component denoted by $B^{\gamma'}$. Equation 7-79, giving ϕ_{π} in terms of flux linkages, can be substituted in equation 7-83. This results in equation 7-81. Equation 7-83 can be expressed as:

$$\boxed{\text{Absolute Curl } \phi = B}$$

The curl operation is extended, in the following section, to a space which includes the time-coordinate as well. In Section 4-5, it was seen that, for the usual space-time system, the curl of a tensor comprising of vector and scalar potentials resulted in a four-dimensional field tensor F_{mn}. Maxwell's equations were then obtained by setting the exterior derivative of F_{mn} equal to zero.

7-4-6 First Field Tensor

The field tensor, $F_{\alpha\beta}$, can be established by extending equation 7-78 to the time-coordinate as well. The scalar potential ϕ_t is taken as zero. The torsion tensor $S_{\alpha\beta}^{\cdot\cdot\pi}$ is extended to the time coordinate by defining(41):

$$S_{\alpha(t)}^{\cdot\cdot\pi} = S_{\alpha(S)}^{\cdot\cdot\pi} \frac{dx^S}{dt} \qquad (7\text{-}83)$$

Maxwell's equations are studied by setting the exterior derivative of $F_{\alpha\beta}$ equal to zero (Section 4-5 equation 4-32).

Equation 7-83 is seen to result from the definition of intrinsic derivative(21):

$$\begin{aligned} \frac{\delta \phi_\alpha}{\delta t} &= \frac{d\phi_\alpha}{dt} - \Gamma^{\pi}_{\alpha\beta} \phi_\pi \frac{dx^\beta}{dt} \\ &= \frac{d\phi_\alpha}{dt} - \Gamma^{\pi}_{\alpha(S)} \phi_\pi \frac{dx^S}{dt} \\ &= \frac{d\phi_\alpha}{dt} - \Gamma^{\pi}_{\alpha(t)} \phi_\pi \end{aligned}$$

where,

$$\Gamma^{\pi}_{\alpha(t)} = \Gamma^{\pi}_{\alpha(S)} \frac{dx^S}{dt}$$

$$S_{\alpha(t)}^{\cdot\cdot\pi} = \Gamma^{\pi}_{[\alpha(t)]} = \Gamma^{\pi}_{[\alpha(S)]} \frac{dx^S}{dt} = S_{\alpha(S)}^{\cdot\cdot\pi} \frac{dx^S}{dt}$$

$$= \begin{array}{c|c|c|c|c} {}_{\pi}\searrow^{\alpha} & dr & qr & s & t \\ \hline dr & & \frac{dx^S}{dt} & & \\ \hline qr & -\frac{dx^S}{dt} & & & \\ \hline s & & & & \\ \hline t & & & & \end{array} = -\rho^{\pi}_{\alpha}\frac{dx^S}{dt} \qquad (7\text{-}84)$$

In equation 7-78, when $\beta = t$,

$$F_{\alpha(t)} = \left(\frac{\delta\phi_t}{\delta x^\alpha} - \frac{\delta\phi_\alpha}{\delta x^t}\right) = \left(\frac{\partial\phi_t}{\partial x^\alpha} - \frac{\partial\phi_\alpha}{\partial x^t}\right) + 2S_{\alpha t}^{\cdot\cdot\pi}\,\phi_\pi$$

$$= \left(0 - \frac{d\phi_\alpha}{dt}\right) - \rho^{\pi}_{\alpha}\frac{dx^S}{dt}\,\phi_\pi$$

$$= -\left(L_{\alpha\beta}\frac{di^\beta}{dt} + \rho^{\pi}_{\alpha}L_{\pi\beta}\,i^\beta\,p\,\Theta\right) = \mathcal{E}_\alpha$$

$$(7\text{-}85)$$

This is seen to be the induced emf given by equation 7-41 (Section 7-3-3). In the earlier analysis, the induced emf was obtained by integrating the induced electric force around the winding.

In terms of flux density and induced emf, the field tensor $F_{\alpha\beta}$ is:

$$F_{\alpha\beta} = \begin{array}{c|c|c|c|c|} \alpha\downarrow \; \rightarrow\beta & dr & qr & s & t \\ \hline dr & & & -B^{qr'} & \mathcal{E}_{dr} \\ \hline qr & & & B^{dr'} & \mathcal{E}_{qr} \\ \hline s & B^{qr'} & -B^{dr'} & & \\ \hline t & -\mathcal{E}_{dr} & -\mathcal{E}_{qr} & & \\ \hline \end{array} \tag{7-86}$$

Maxwell's equations are obtained by setting the exterior derivative of $F_{\alpha\beta}$ equal to zero (Section 4-5). The covariant form of exterior derivative is, from equation 7-56,

$$\frac{\delta F_{\beta\gamma}}{\delta x^{\alpha}} + \frac{\delta F_{\gamma\alpha}}{\delta x^{\beta}} + \frac{\delta F_{\alpha\beta}}{\delta x^{\gamma}} = 0$$

i. e.

$$\frac{\partial F_{\beta\gamma}}{\partial x^{\alpha}} + \frac{\partial F_{\gamma\alpha}}{\partial x^{\beta}} + \frac{\partial F_{\alpha\beta}}{\partial x^{\gamma}}$$

$$+ \quad 2S_{\beta\gamma}^{\cdot\cdot\pi} F_{\pi\alpha} + 2S_{\gamma\alpha}^{\cdot\cdot\pi} F_{\pi\beta} + 2S_{\alpha\beta}^{\cdot\cdot\pi} F_{\pi\gamma} = 0$$

(7-87)

(i) When $\beta = s$, α and γ = dr or qr or s, equation 7-87 reduces to:

$$0 + 0 + 0 + 2S_{(S)\gamma}^{\cdot\cdot\pi} F_{\pi\alpha} + 2S_{\gamma\alpha}^{\cdot\cdot\pi} F_{\pi(s)}$$

$$+ 2S_{\alpha(S)}^{\cdot\cdot\pi} F_{\pi\gamma} = 0$$

The first three terms are zero since the flux linkages in Park's axis do not vary with charges or angular displacement of the rotor. If neither of the indices α and γ is mechanical (i. e. $\neq$ s), the last three terms are also seen to be zero. If $\alpha = s$,

$$2S_{(S)\gamma}^{\cdot\cdot\pi} F_{\pi(s)} + 2S_{\gamma(S)}^{\cdot\cdot\pi} F_{\pi(s)} = 0$$

Thus in all cases the relation:

$$\boxed{\text{absolute divergence } B = 0}$$

is satisfied.

(ii) $\beta = t$, α and γ = dr or qr or s.

Equation 7-87 now reduces to:

$$0 + \frac{\partial F_{\gamma\alpha}}{\partial x^{t}} + 0 + 2S_{(t)\gamma}^{\cdot\,\cdot\,\pi} F_{\pi\alpha} + 2S_{\gamma\alpha}^{\cdot\,\cdot\,\pi} F_{\pi(t)}$$

$$+ 2S_{\alpha(t)}^{\cdot\,\cdot\,\pi} F_{\pi\gamma} = 0$$

If neither of the indices α and γ ~~are~~ is mechanical, all the terms are seen to be zero. If $\alpha = s$,

$$\frac{\partial F_{\gamma(s)}}{\partial x^{t}} + 2S_{(t)\gamma}^{\cdot\,\cdot\,\pi} F_{\pi(s)} + 2S_{\gamma(S)}^{\cdot\,\cdot\,\pi} F_{\pi(t)} = 0$$

From equations 7-77, 7-84 and 7-86, the above relation becomes:

$$-\frac{d}{dt}(\ell\phi) + \ell(-\ell\phi)\frac{dx^{S}}{dt} - \ell(\mathcal{E}) = 0$$

i.e.

$$\mathcal{E} = -\frac{d\phi}{dt} - \ell\phi\, p\Theta$$

This is exactly the same equation as equation 7-85. Thus in all cases the relation:

$$\boxed{\text{absolute curl } {}'\mathcal{E} \;=\; -\text{ absolute } \partial B/\;\partial t}$$

is satisfied.

7-4-7 Second Field Tensor

The second field tensor expresses the magnetic-force vector (equation 4-33, Section 4-5-1). The electric flux density term, D, is ignored so that:

$$H^{\alpha\beta} = \begin{array}{c|c|c|c|c|} {}_{\alpha}\diagdown^{\beta} & dr & qr & s & t \\ \hline dr & & & -\mathcal{H}'_{qr} & \\ \hline qr & & & \mathcal{H}'_{dr} & \\ \hline s & \mathcal{H}'_{qr} & -\mathcal{H}'_{dr} & & \\ \hline t & & & & \\ \hline \end{array} \qquad (7\text{-}88)$$

Maxwell's equations are obtained by setting the divergence of $H^{\alpha\beta}$ equal to the current- and charge-density vector in 4-space (Section 4-5-1). The covariant form of divergence is, from equation 7-57,

$$\frac{\delta H^{\alpha\beta}}{\delta x^{\beta}} = \frac{\partial H^{\alpha\beta}}{\partial x^{\beta}} + \Gamma^{\beta}_{\pi\beta} H^{\alpha\pi} + \Gamma^{\alpha}_{\pi\beta} H^{\pi\beta}$$

In Park's axis, $\Gamma^{\beta}_{\pi\beta} = 0$

Also, since $H^{\pi\beta}$ is skewsymmetric,

$$\Gamma^{\alpha}_{\pi\beta} H^{\pi\beta} = \Gamma^{\alpha}_{[\pi\beta]} H^{\pi\beta} = 2S_{\pi(S)}^{\cdot\cdot\alpha} H^{\pi(S)}$$

Therefore,

$$\frac{\delta H^{\alpha\beta}}{\delta x^{\beta}} = \frac{\partial H^{\alpha\beta}}{\partial x^{\beta}} + 2S_{\pi(S)}^{\cdot\cdot\alpha} H^{\pi(S)} \quad . \tag{7-89}$$

In Park's axis, the magnetic force, $\mathcal{H}$, does not vary with charges or rotor angular displacement. Also $H^{\alpha(t)}$ is zero. The partial derivative term in equation 7-89 is consequently zero. Substituting equation 7-77 in 7-89,

$$\frac{\delta H^{\alpha\beta}}{\delta x^{\beta}} = \begin{array}{c|c|c|c|c|} \alpha & dr & qr & s & t \\ \hline & \mathcal{H}'_{dr} & \mathcal{H}'_{qr} & 0 & 0 \\ \hline \end{array}$$

$$= \begin{array}{c|c|c|c|c|} \alpha & dr & qr & s & t \\ \hline & i^{dr} & i^{qr} & 0 & 0 \\ \hline \end{array} \qquad (7\text{-}90)$$

That is, in the covariant field analysis, the coil currents are defined as the current-density vector. This can be compared with equation 7-31 (Section 7-3-1) in which the current density of the field model is related to the coil currents.

The magnetic stored energy density is given by $\frac{1}{2}(B^{dr}\mathcal{H}_{dr} + B^{qr}\mathcal{H}_{qr})$ which in terms of flux linkage and coil currents is, from equations 7-81 and 7-90:

$$W = \frac{1}{2}(L_{dr}\, i^{dr}\, i^{dr} + L_{qr}\, i^{qr}\, i^{qr}) \qquad (7\text{-}91)$$

That is, in the covariant field analysis, the energy density is, in fact, the integral of the total air-gap magnetic energy.

Equation 7-90 can be compared to equation 7-32 (Section 7-3-1). It is seen that the magnetic force in the covariant analysis does, in fact, represent the mmf in the machine.

Equation 7-90 can be expressed as:

$$\boxed{\text{absolute curl } \mathcal{H} = i}$$

It is also seen that

$$\frac{\delta i^{\alpha}}{\delta x^{\alpha}} = \frac{\partial i^{\alpha}}{\partial x^{\alpha}} + \Gamma^{\alpha}_{\pi\alpha} \, i^{\pi} = 0,$$

(The partial derivatives with respect to charges and angular displacement are zero; when $\alpha = t$, $i^{\alpha} = 0$; moreover, $\Gamma^{\alpha}_{\pi\alpha}$ is zero in Park's axis). The divergence equation can therefore be expressed as:

$$\boxed{\text{absolute div. } i = 0}$$

This equation expresses the conservation of charge (Section 4-1-3, equation 4-6).

7-4-8 Stress-Energy Tensor

The stress-energy tensor is, from equation 4-36,

$$T_{\alpha}^{\beta} = \begin{array}{c|c|c|c|c} \alpha\downarrow\ \to\beta & dr & qr & s & t \\ \hline dr & B^{dr}\mathcal{H}_{dr}-W & B^{qr}\ \mathcal{H}_{dr} & & \\ \hline qr & B^{dr}\ \mathcal{H}_{qr} & B^{qr}\ \mathcal{H}_{qr}-W & & \\ \hline s & & & -W & \\ \hline t & & & \mathcal{E}_{dr}\ \mathcal{H}'_{qr} - \mathcal{E}_{qr}\ \mathcal{H}'_{dr} & W \end{array}$$

(7-92)

The stored energy, W, is given in equation 7-91. In the stress-energy tensor, the term, T_t^s, is seen to give the reactive power.

$$T_t^s = \mathcal{E}_{dr}\ \mathcal{H}'_{qr} - \mathcal{E}_{qr}\ \mathcal{H}'_{dr}.$$

$$= \mathcal{E}_{dr}\ i^{qr} - \mathcal{E}_{qr}\ i^{dr} \qquad (7\text{-}93)$$

From equations 7-85 and ~~8~~ 7-2, it is seen that:

$$v + \mathcal{E} = R\ i$$

That is, applied voltage and back emf add to give the resistance drop. In terms of applied voltage, equation 7-93 is:

$$T_t^s = v_{qr}\ i^{dr} - v_{dr}\ i^{qr}$$

This is, in fact, an expression for the steady state reactive power of a synchronous machine in terms of Park's voltages and currents.

The asymmetric part of the stress tensor is seen to give the electromagnetic torque:

$$T_{dr}^{qr} - T_{qr}^{dr} = B^{qr}\ \mathcal{H}_{dr} - B^{dr}\ \mathcal{H}_{qr}$$

$$= L_{qr}\ i^{qr}\ i^{dr} - L_{dr}\ i^{dr}\ i^{qr} \qquad (7\text{-}94)$$

(from equations 7-81 and 7-90).

Equation 7-94 in terms of torque matrix, G, is:

$$i_t\ G\ i$$

Thus the electromagnetic stress in the covariant analysis is an integral of the tangential stress carried out over the rotor surface of the field model.

The covariant analysis developed by Kron for the performance equation of an electric machine is thus seen to have far deeper field concepts. It connects Legrange's dynamical equations to Maxwell's field equations. The field quantities in such an analysis do not express the machine air-gap field directly, but are given in terms of integrals of machine field vectors over the physical dimensions of the machine.

The analysis can be extended for other layers of winding by considering a rotation type torsion tensor, $S_{\alpha\beta}^{\cdot\cdot\pi}$, for each layer. The rotation tensor is seen in Chapter 8 to relate the absolute changes to apparent changes, when the observer has a varying velocity relative to the winding of an electric machine.

Park's reference system is preferred in analysing Maxwell's equations. The field quantities are independent of the rotor position in this system and partial derivatives with respect to rotor angular displacment are zero. Moreover, in this system, the stator and rotor reference frames are rigidly connected to each other.

In the tensor theory of networks, the closed-path and open-path currents represented, geometrically, orthogonal subspaces. In the tensor theory of electric machines, the electrical and mechanical coordinates were seen to form orthogonal subspaces. In an interconnected multimachine system, there will be both types of subspaces and interconnected subspaces. Kron develop's his theory of wave automaton with such interconnected space-structures, called polyhedra(9). In fact Kron claims that the sources in his multidimensional networks are more general magneto-hydro-dynamic types of generators.(3) The performance equations of such sources would seem to emerge from the field equations of electro-dynamic machines presented in tensor notation in this chapter and fluid-flow equations. The equivalent network for an electromagnetic field, presented earlier, could be coupled with an equivalent network for fluid flow(55). This study is, however, not undertaken in the present thesis.

Kron claims that the oscillatory behaviour of multi-dimensional networks form the backbone of his self-organizing automata. A study of the oscillatory behaviour of the electric machine is undertaken in the next chapter. The tensor concepts used in the present chapter are extended to analyse the synchronising and damping torques during oscillation of an electric machine.

CHAPTER 8

OSCILLATORY BEHAVIOUR OF ELECTRIC MACHINES

The performance of an electric machine under conditions of small oscillations can be studied in Park's axes using the results of Chapter 7. Although the hunting torque components can be evaluated correctly in Park's axes, it is difficult to associate them with the machine constants or the operating conditions. Dreyfus(47) and Nickle and Pierce(48) suggested that the damping torque could be related to the I^2R loss due to the oscillating currents, but no mathematical indication of this appears to have emerged. The nature of damping torque was also examined by Liwschitz(49) who obtained a formula for the positive damping torque from the field winding, but was unable, at that time, to extend this to embrace negative damping from the armature. The reason for the limitations in these cases appears to be that the writers were working with Park's reference frame. In this reference system, during hunting, there is a relative oscillation between the rotating armature windings and the stationary axes of the armature brushes (hypothetical for synchronous machines and induction motors). In the reference frame introduced by Kron(44), the reference axes rotate at a uniform speed with respect to the windings. It has been pointed out in reference 45 that, in Kron's reference frame, it is possible to associate the hunting torque components with an equivalent circuit. A realistic picture of the generation of positive and negative components of synchronising and damping torques during hunting is obtained. There is a simple relation between the reference axes of Park and Kron, and this is examined in Section 8-1.

The tensorial analysis, used in Chapter 7 to study field concepts, can be used to discuss the relationship between the oscillating currents and voltages in the two frames and the association of the various components of hunting torques, with machine constants. These are examined in Sections 8-3 and 8-4.

8-1 Reference Frames of Park and Kron

In Park's reference frame for synchronous machines, the axes for armature and field are both fixed to the field structure. During steady-state operation, with a balanced supply to the armature, the resultant armature m. m. f. rotates synchronously and hence appears stationary relative to the field structure. In Fig. 27, OP represents the steady-state armature m. m. f. (in the figure, the main field winding is on the rotor and the stator contains the armature windings). The components of the armature m. m. f. OP (fig. 27) along Park's direct and quadrature axes will be OA and OB. During hunting, let OP increase to OP' at a given instant, while the rotor angle changes by $\Delta\lambda$. The changes as seen from the d- and q-axes, which move with the rotor, are A_1A_2 and B_1B_2. Obviously,

$$\overrightarrow{A_1A_2} + \overrightarrow{B_1B_2} \neq \overrightarrow{PP'}$$

The changes as seen along Park's axes are not the actual changes induced in the armature currents and voltages. Now, if the axes rotate synchronously (Kron's reference system),

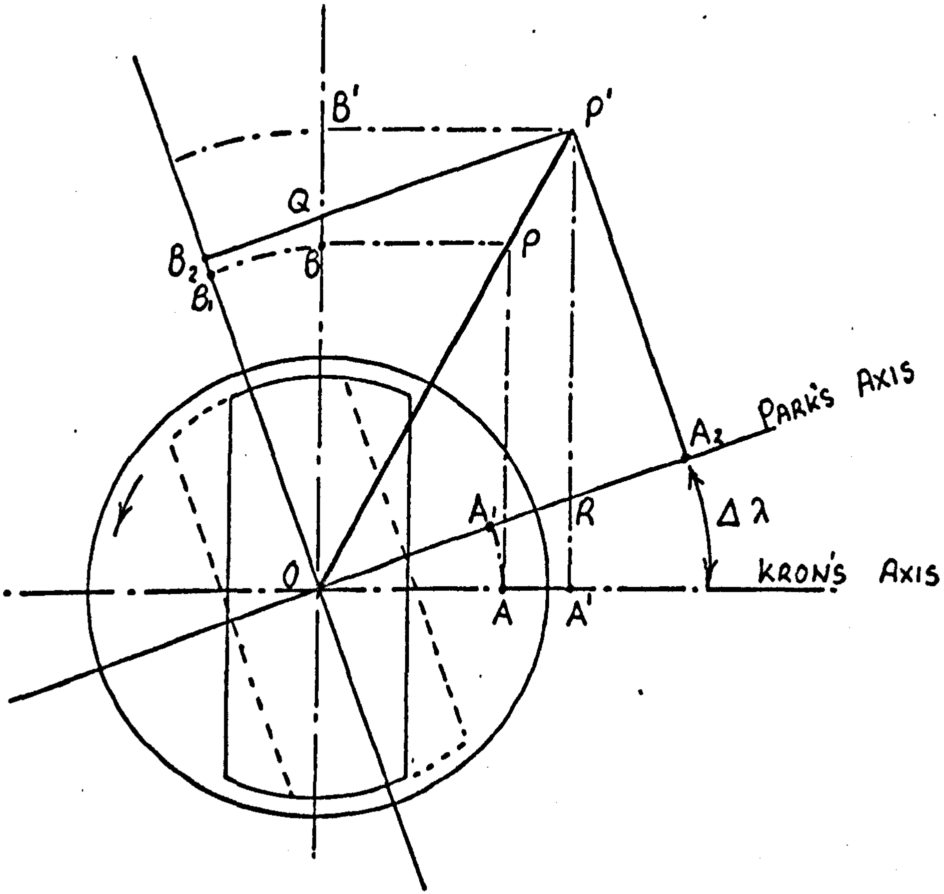


FIG. 27. CHANGES IN ARMATURE CURRENT AND M.M.F. AS SEEN FROM PARK'S AND KRON'S AXES

changes in OP (fig. 27) will be seen as AA' and BB', which add vectorially to give PP'. These are absolute changes seen by Kron's reference frame for the armature. For the field winding, the reference frame is fixed to the field structure (which is the same as Park's axes for the field). That is, Kron uses a hybrid system in which the armature axes rotate at a uniform angular velocity and the field axes hunt along with the field structure. Thus, there is a relative motion between the armature reference axes and the field reference axes during hunting in case of Kron's system. There is, however, no oscillation between the armature and its reference axes, and the armature windings, with its associated transmission lines and transformers, appear as a stationary network. There is a simple mathematical relationship between the quantities shown in fig. 27 along Kron's axes and those along Park's axes. Neglecting second-order effects, from Appendix 4,

$$A_1A_2 \simeq AA' + PA \quad \Delta\lambda$$

$$B_1B_2 \simeq BB' - PB \quad \Delta\lambda \qquad (8\text{-}1)$$

where, $\Delta\lambda = \Delta\Theta$ is the angle representing rotor excursion during oscillation. If OP represents the steady-state armature current, the apparent changes, Δi, seen by Park's axes and the absolute changes, $\tilde{\Delta} i$, seen by Kron's axes can be written, using relations 8-1, as:

$$\begin{bmatrix} \tilde{\Delta} i^{dr} \\ \tilde{\Delta} i^{qr} \end{bmatrix} = \begin{bmatrix} \Delta i^{dr} \\ \Delta i^{qr} \end{bmatrix} - \begin{bmatrix} & -1 \\ 1 & \end{bmatrix} \begin{bmatrix} i^{dr} \\ i^{qr} \end{bmatrix} \Delta\lambda$$

$$\tilde{\Delta} i = \Delta i - \rho \cdot i \quad \Delta\lambda \qquad (8-2)$$

The skewsymmetric rotation matrix, ρ, has been used earlier in Sections 7-2, 7-3 and 7-4 to relate the machine air-gap field quantities to the dr- and qr- components, to relate the field quantities to the terminal quantities for an idealised machine and in the covariant field analysis. The role of the rotation tensor in electric machine analysis is discussed further in Chapter 9.

The changes in armature voltage seen by Park's axes, Δv, and Kron's axes, $\tilde{\Delta} v$, can be related, using a similar analysis, as:

$$\tilde{\Delta} v = \Delta v + \rho_t \cdot v \cdot \Delta \lambda \qquad (8\text{-}3)$$

8-2 Damping and Synchronising Torques

The equation of motion of rotating machine during oscillation can be represented by a second order differential equation as:

$$\Delta T = M p^2 (\Delta \Theta \quad) + T_d \, p \, (\Delta \theta \quad) + T_S \, \Delta \Theta \qquad (8\text{-}4)$$

where ΔT is any externally applied torque, M is the inertia constant of the rotor and T_d and T_S are respectively the damping and synchronising coefficients of torque. The synchronising torque ($T_S \, \Delta \Theta$) is a function of the rotor excursion during small oscillations; the damping torque ($T_d \, p \, (\Delta \Theta$)) is a function of the change in the angular velocity of the rotor during oscillation. These two torque components are, therefore, in phase quadrature with respect to each other. The component ($M p^2 (\Delta \Theta$)) is the inertia torque, due to the angular acceleration of the rotor.

In terms of frictional resistance, R_S, and torque matrix, G, the hunting torque equation is (from Appendix 4):

$$\Delta T = Mp^2 (\Delta\Theta \quad) + R_S\, p\, (\Delta\Theta \quad) - \Delta T_{elect} \qquad (8\text{-}5)$$

where,

$$\Delta T_{elect} = (\tilde{\Delta} i_t \cdot G \cdot i + i_t \cdot G \cdot \tilde{\Delta} i) + i_t \cdot G \cdot \rho \cdot i\, \Delta\lambda - i_t \cdot K \cdot i\, \Delta\lambda \qquad (8\text{-}6)$$

In equation 8-6,

$$K = - \rho_t \cdot G$$

and $\tilde{\Delta} i$ is the oscillating current matrix in Kron's axes.

Comparing equations 8-5 and 8-4, it is seen that R_S gives T_{dm}, the mechanical damping torque; ΔT_{elect} gives T_{de}, the electrical damping torque and T_S, the synchronising torque. Equation 8-6 gives the various components of T_{de} and T_S. This subdivision of the torque equation was originally derived by Kron$^{(44)}$, using tensor calculus and covariant differentiation. This is discussed in Section 8-4.

For sinusoidal oscillations, the substitution, $p = j\,h\,\omega$, is made and equation 8-6 assumes a complex form:

$$-(c \pm j\,d) + (e) \pm (f) \qquad (8\text{-}7)$$

The bracketed groups of terms, in fact, separate out the positive and negative components of the synchronising torque and the damping torque. For example, the last term of equation 8-7, f,is a component of synchronising torque which is always positive for a synchronous motor operating at a lagging power factor. The total quantity ΔT in equation 8-5 is the same whatever reference system is chosen.

For a synchronous motor connected to the mains (considered to be infinite busbars) the components of synchronising torque c, e and f in equation 8-7 where computed and are shown plotted in fig. 28. The computation aspects and machine parameters are discussed in Sections 10-1 and 10-2.

Fig. 28 is reproduced from reference 45 (attached to the thesis). The advantages to be obtained in using the above method of analysis are pointed out there. The various components of synchronising and damping torques could be associated with the design parameters and the state of the machine. The component, c, in fig. 28 is associated with the reactive power of an equivalent circuit (fig. 29). The component,

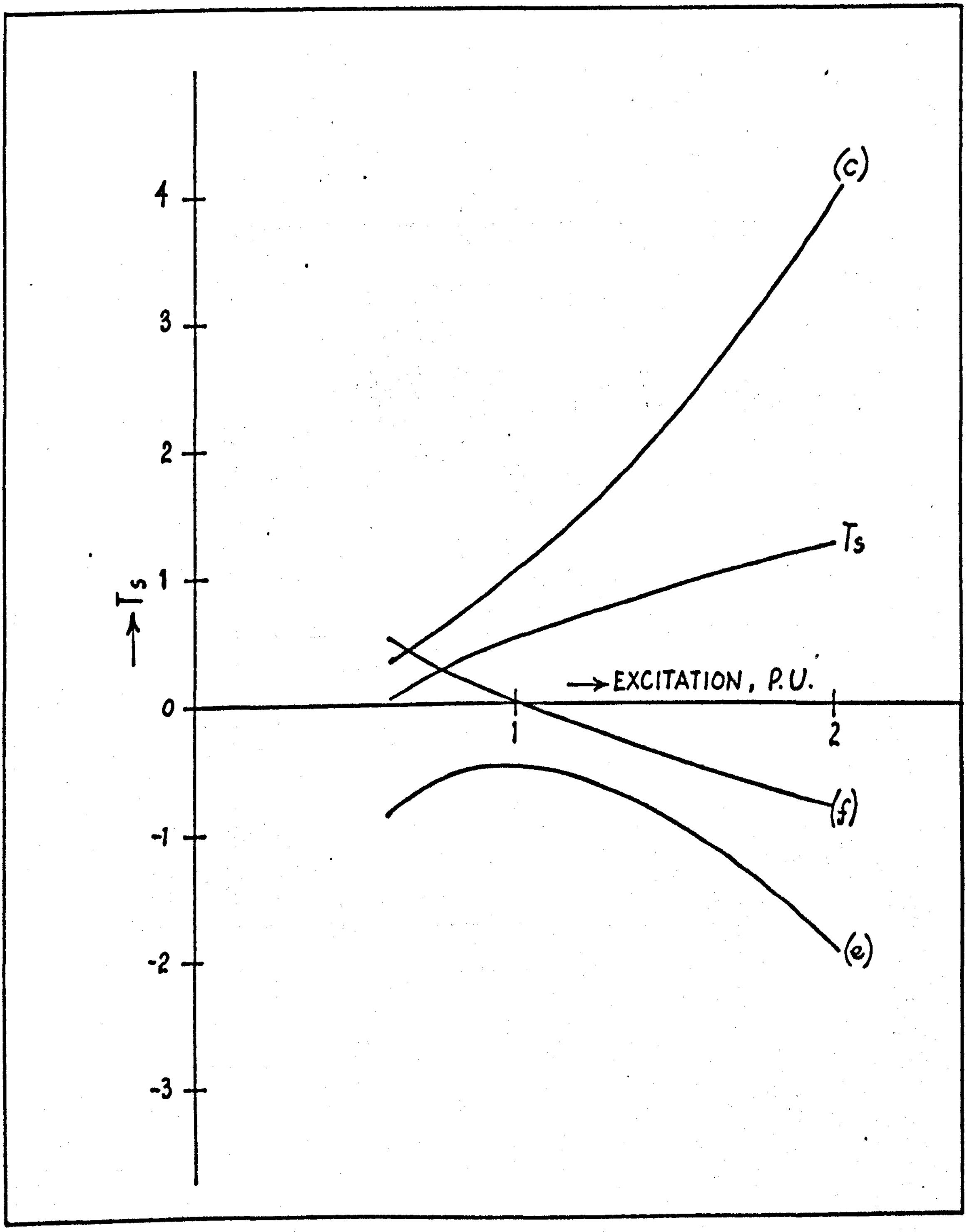
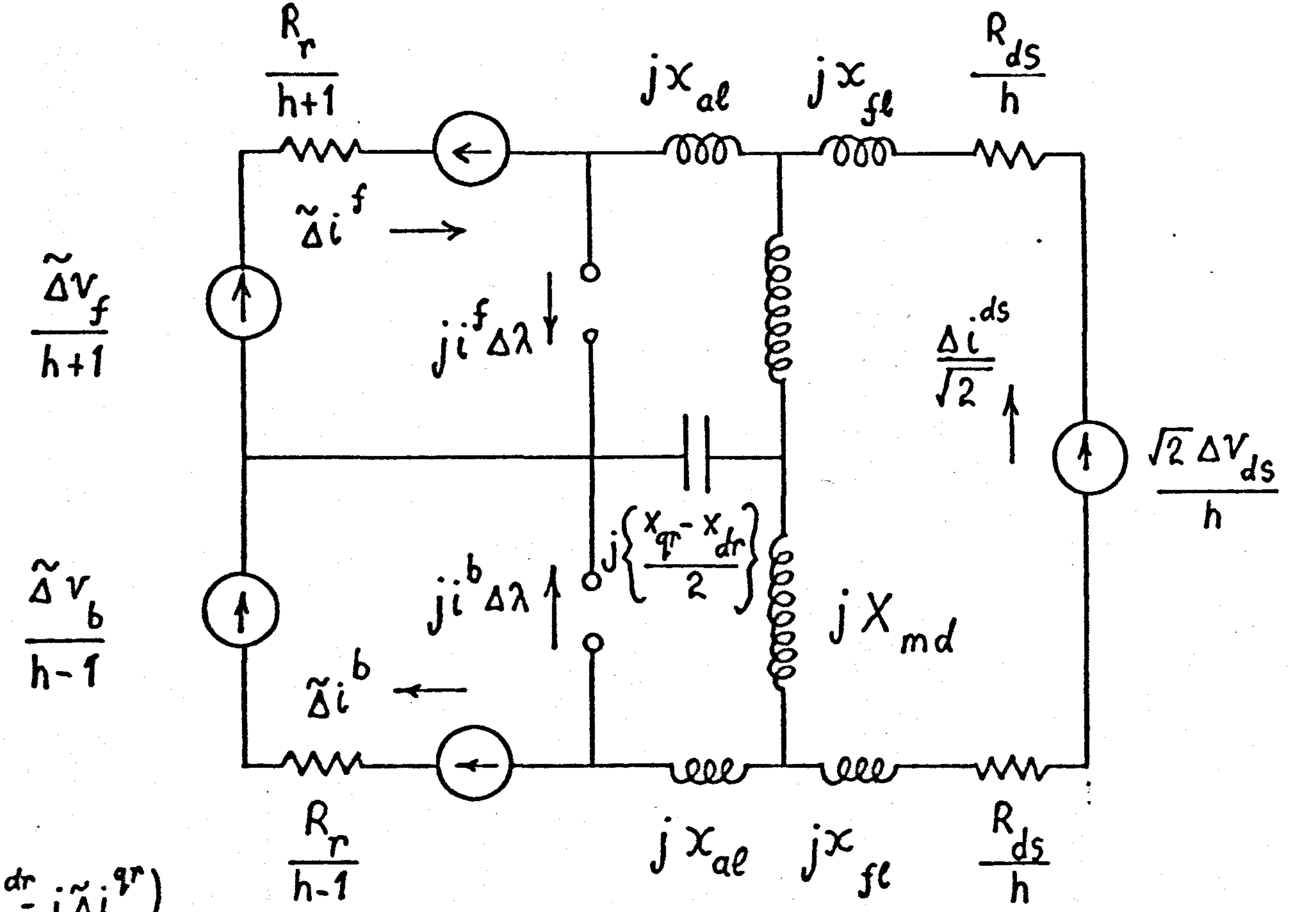


FIG. 28. COMPONENTS OF SYNCHRONISING TORQUE WITH VARYING EXCITATION



$$\tilde{\Delta} i^f = \frac{1}{\sqrt{2}}\left(\tilde{\Delta} i^{dr} - j\tilde{\Delta} i^{qr}\right)$$

$$\tilde{\Delta} i^b = \frac{1}{\sqrt{2}}\left(\tilde{\Delta} i^{dr} + j\tilde{\Delta} i^{qr}\right)$$

FIG. 29. EQUIVALENT NETWORK FOR HUNTING ANALYSIS-KRON'S AXES

f, is the steady-state reactive power in the machine. The component, e, is due to the relative oscillation between the armature and rotating field structure. The components are shown for various values of excitation in fig. 28. The damping torque was also split into two components - positive damping (given also by the ohmic loss in the equivalent circuit (fig. 29) corresponding to the field winding) and negative damping (given by the ohmic loss in the equivalent circuit corresponding to the armature). Positive damping is also provided by any winding (amortisseur) on the field structure. Kron's reference frame is thus seen to indicate the physical nature of the torque components. Earlier authors(47, 48, 49) used Park's reference frame in hunting analysis, and in this system, it is difficult to associate the torque components with the machine constants or the operating conditions.

8-3 Covariant Differentials

In Section 7-4-1, it was pointed out that whereas partial derivatives do not, in general, transform as tensors, covariant derivatives do. It is also seen from equations 8-2 and 8-3 that conventional increments, Δi, $\tilde{\Delta} i$, Δv and $\tilde{\Delta} v$ do not transform as tensors. An extra term appears in these equations, given by $\rho \cdot i \;\; \Delta\lambda$ in one case and $\rho_t \cdot v \;\; \Delta\lambda$ in the other. "Covariant differentials", however, transform as tensors. This is examined here.

The covariant derivative of a tensor, A_α, is defined in Section 7-4-1 by equation 7-49:

$$\frac{\delta A_\alpha}{\delta x^\beta} = \frac{\partial A_\alpha}{\partial x^\beta} - \Gamma^{\pi}_{\alpha\beta} \, A_\pi \qquad (7\text{-}49)$$

The "covariant differentials" follow from equation 7-49:

$$\delta A_\alpha = \Delta A_\alpha - \Gamma^{\pi}_{\alpha\beta} \, A_\pi \quad \Delta x^\beta$$

Similarly,

$$\delta i^\pi = \Delta i^\pi + \Gamma^{\pi}_{\alpha\beta} \, i^\alpha \quad \Delta x^\beta \qquad (8\text{-}8)$$

The covariant differentials transform as tensors:

$$\delta i^\pi = C^\pi_p \cdot \delta i^p \qquad (8\text{-}9)$$

If Greek letters denote Park's axes and Roman letters denote Kron's axes, the transformation tensor, C^π_p, from Park's to Kron's reference system is, from fig. 30,

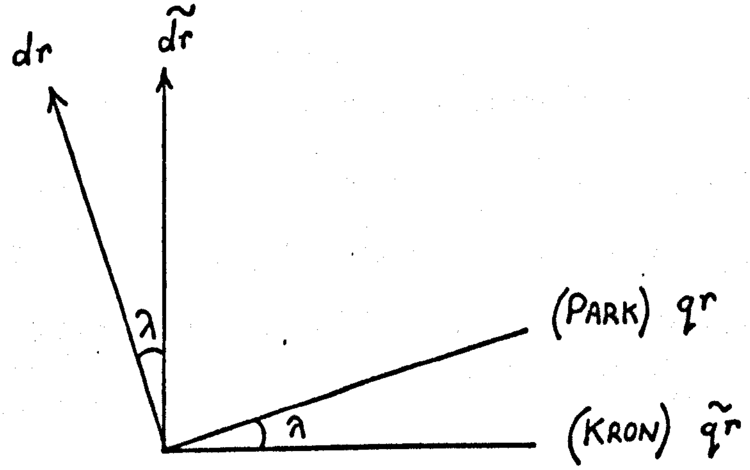


FIG. 30. PARK'S AXIS TO KRON'S AXIS IN A SYNCHROUS MACHINE

$$C_p^{\pi} = \begin{array}{c|c|c|c} \overset{p\,(\text{Kron})}{\pi\,(\text{Park})} & \tilde{dr} & \tilde{qr} & \theta \\ \hline dr & \cos\lambda & -\sin\lambda & \\ \hline qr & \sin\lambda & \cos\lambda & \\ \hline \Theta & & & 1 \end{array} \tag{8-10}$$

In Kron's reference system, used in hunting analysis, the angle λ is zero during steady-state, but during hunting $\Delta\lambda = \Delta\Theta$. The transformation tensor given by equation 8-10 is used to relate the covariant differentials in equation 8-9. In Kron's axis, the covariant differential is:

$$\delta i^p = \tilde{\Delta} i^p + \Gamma^p_{ab} \; i^a \; \Delta x^b \tag{8-11}$$

Substitution of equations 8-11 and 8-8 in equation 8-9 is seen to lead to the relation (Appendix 4):

$$C_p^{\pi} \cdot \tilde{\Delta} i^p = \Delta i^{\pi} - \rho^{\pi}_{\alpha} \cdot i^{\alpha} \; \Delta\lambda \tag{8-12}$$

Moreover, ignoring second order effects

$$C_p^{\pi} \cdot \tilde{\Delta} i^p \simeq \delta_p^{\pi} \cdot \Delta i^p \tag{8-13}$$

(Appendix 4).

When equation 8-13 is substituted in equation 8-12, the resulting equation is the same as equation 8-2 derived earlier using a vector diagram.

Tensor analysis is thus seen to be a useful tool in differentiating between quantities which transform from one reference frame to another as tensors, involving only transformation tensors C or their inverse or both as products, and those which do not. The physical significance, in terms of the machine constants and the operating conditions, of the covariant differentials of currents and voltages in the abstract non-Riemannian space is being examined.

8-4 Tensor Equations of Hunting

Instead of taking the conventional increment, Δ, in the transient equation of a machine, the covariant increment, δ, can be considered to establish the hunting equations. These can then be expanded in terms of conventional increments, Δ, and affine connection, Γ. Such an analysis is seen to lead to the grouping of terms established in Section 8-2 by considering a vector diagram (equation 8-6). The hunting equation is considered here in Kron's freely rotating frame.

In tensor notation, the performance equation of an electric machine is, from Section 7-4-4,

$$v_c = R_{cp} \frac{dx^p}{dt} + L_{cp} \frac{\delta}{\delta t} \left(\frac{dx^p}{dt}\right) \qquad (8\text{-}14)$$

Taking the covariant increment, δ, in the tensor equation above:

$$\delta v_c = \delta\left(R_{cp} \frac{dx^p}{dt}\right) + L_{cp} \delta \left\{\frac{\delta}{\delta t} \left(\frac{dx^p}{dt}\right)\right\} \qquad (8\text{-}15)$$

The covariant differential of the metric tensor, δL_{cp}, is zero(21). In equation 8-15, if the order of covariant operators δ and $\delta / \delta t$ is commuted, an additional tensor appears(44):

$$\delta v_c = \delta\left(R_{cp} \frac{dx^p}{dt}\right) + L_{cp} \frac{\delta}{\delta t} \left\{ \delta \left(\frac{dx^p}{dt}\right) \right\}$$

$$+ K_{mknc} \frac{dx^m}{dt} \frac{dx^n}{dt} \Delta x^k \qquad (8\text{-}16)$$

The Riemann-Christoffel curvature tensor, K_{mknc}, is defined as:(44)

$$K_{mknc} = \frac{\partial \Gamma_{nm,c}}{\partial x^k} - \frac{\partial \Gamma_{nk,c}}{\partial x^m} + \Gamma_{pk,c} \Gamma^p_{nm}$$

$$- \Gamma_{pm,c} \Gamma^p_{nk} - 2 \Gamma_{np,c} \Omega^p_{km}$$

(8-17)

The organisation of the hunting equation in terms of covariant differentials and curvature tensor results in certain terms being added to and subtracted from the conventional equation of hunting. This is examined here.

In equation 8-16, $\Delta x^{k} = \Delta\lambda$, since all the objects are functions of angle λ only. When equation 8-16 is analysed for the hunting torque equation, both indices m and n denote the electrical coordinates. Consequently certain terms become zero and in equation 8-17, the non-zero terms are:

$$\frac{\partial \Gamma_{nm,c}}{\partial x^{(u)}} \ , \ - \Gamma_{pm,c} \ \Gamma^{p}_{n(u)} \quad \text{and} \quad 2 \ \Gamma_{np,c} \ \Omega^{p}_{m(u)}$$

(the bracketed variable u denotes the angle λ). The above terms are expanded in Appendix 4 for voltages and torque. In particular two of the above terms represent the set added to and subtracted from the conventional equation of hunting (Appendix 4).

For the voltage equation the set is:

$$\left\{ \Gamma_{pm,c} \ \Gamma^{p}_{n(u)} - 2 \ \Gamma_{np,c} \ \Omega^{p}_{m(u)} \right\} \frac{dx^{m}}{dt} \ \frac{dx^{n}}{dt} \ \Delta x^{(u)}$$

$$= G \cdot \rho \cdot i \ p\Theta \quad \Delta\lambda \ ,$$

and for the torque equation:

$$\left(\Gamma_{pm,\,(S)} \; \Gamma^{p}_{n(u)} \; -2 \; \Gamma_{np,\,(S)} \; \Omega^{p}_{m(u)} \right) \frac{dx^{m}}{dt} \; \frac{dx^{n}}{dt} \; \Delta x^{(u)}$$

$$= - i_{t} \cdot G \cdot \rho \cdot i \qquad \Delta \lambda \qquad (8\text{-}18)$$

The contribution of the term, $\frac{\partial \Gamma_{nm,\,c}}{\partial x^{(u)}}$, for the voltage equation is:

$$\frac{\partial \Gamma_{nm,\,c}}{\partial x^{(u)}} \; \frac{dx^{m}}{dt} \; \frac{dx^{n}}{dt} \; \Delta x^{(u)} \; = \; \frac{\partial G}{\partial \lambda} \cdot i \quad p\Theta \qquad \Delta \lambda \; ,$$

and for the torque equation:

$$\frac{\partial \Gamma_{nm,\,(S)}}{\partial x^{(u)}} \; \frac{dx^{m}}{dt} \; \frac{dx^{n}}{dt} \; \Delta x^{(u)} \; = - i_{t} \cdot \frac{\partial G}{\partial \lambda} \cdot i \qquad \Delta \lambda$$

$$(8\text{-}19)$$

(Appendix 4)

In torque equation, subtraction of equation 8-18 from 8-19 results in a term:

$$-i_{t} \cdot \frac{\partial G}{\partial \lambda} \cdot i \quad \Delta \lambda \; + \; i_{t} \cdot G \cdot \rho \cdot i \quad \Delta \lambda$$

$$= - i_{t} \cdot \rho_{t} \cdot \frac{\partial L}{\partial \lambda} \cdot i \quad \Delta \lambda \; + \; i_{t} \cdot G \cdot \rho \cdot i \quad \Delta \lambda$$

$$= - i_{t} \cdot \rho_{t} \cdot G \cdot i \quad \Delta \lambda$$

$$= i \, , K \cdot i \quad \Delta \lambda .$$

The torque equation 8-6 derived in Section 8-2 does precisely the same grouping of terms in a simpler way. It is seen that the earlier analysis in fact involves such advanced concepts as curvature tensor and presents the hunting equation in a manner similar to the analysis of disturbed geodesics.

CHAPTER 9

ROTATION TENSOR IN MACHINE DIFFERENTIAL STRUCTURES

In the study of electric machine fields, the rotation tensor, ρ, arose as an orthogonal operator when a sinusoidal distribution of the electromagnetic field quantities was assumed (Section 7-2). When Maxwell's equations were expressed in terms of covariant derivatives in Section 7-4, the transformation of reference frames was seen to lead to the rotation tensor. In covariant divergence and covariant curl operations, an extra term containing the rotation tensor appeared in addition to the partial derivatives, resulting in more general exterior forms. When differentials were considered in small-oscillation study of machines (Chapter 8), the rotation tensor related the apparent and absolute changes.

The rotation tensor is examined now by considering further the covariant derivatives. In Park's reference system for an electric machine, the analysis is shown to lead to a set comprising the rotation tensor, ρ, and another tensor, $\bar{\rho}$. In covariant field analysis, the second tensor, $\bar{\rho}$, does not appear. The presence of $\bar{\rho}$ in hunting analysis complicates the tensor equations unless the problem is linearised and second order effects ignored. These are examined in the following sections.

9-1 Affine Connection $\Gamma_{\alpha\beta}^{\pi}$

Covariant derivatives consist of partial derivatives and in addition certain "affine connection" terms. For example, the covariant derivative of a vector ϕ is, from section 7-4-1,

$$\frac{\delta \phi_{\alpha}}{\delta x^{\beta}} = \frac{\partial \phi_{\alpha}}{\partial x^{\beta}} - \Gamma_{\alpha\beta}^{\pi} \phi_{\pi} \qquad (9\text{-}1)$$

The geometric object, $\Gamma_{\alpha\beta,\gamma}$, is defined as:

$$\Gamma_{\alpha\beta,\gamma} = L_{\gamma\pi} \Gamma_{\alpha\beta}^{\pi}$$

and in Park's reference system for an electric machine this is given by fig. 26.

From the figure, it is seen that the covariant form of affine connection, $\Gamma_{\alpha\beta,\gamma}$, is of the form:

$$\left(\pm \tfrac{1}{2} G \pm \tfrac{1}{2} V \right)$$

where, "G" is the torque matrix given by equation 7-3, (Section 7-1) and "V" is the transpose of "G".

i. e. $$V_{\alpha\beta} = G_{\beta\alpha}$$

It was pointed out in Section 7-3-3 that the torque matrix is related to the inductance matrix by the equation:

$$G = \rho_t \; L$$

i. e. $$G_{\alpha\beta} = \rho_{\alpha}^{\cdot\sigma} L_{\sigma\beta}$$

It follows that:

$$V = L_t \, \rho = L \, \rho$$

i. e. $$V_{\alpha\beta} = L_{\alpha\sigma} \, \rho^{\sigma}_{\cdot\beta}$$

The affine connection, $\Gamma^{\pi}_{\alpha\beta}$, follows from $\Gamma_{\alpha\beta,\gamma}$:

$$\Gamma^{\pi}_{\alpha\beta} = L^{\pi\gamma} \; \Gamma_{\alpha\beta,\gamma} .$$

This is of the form:

$$L^{-1} \; (\pm \tfrac{1}{2} G \pm \tfrac{1}{2} V)$$

It is further seen that:

$$L^{\pi\gamma} \; V_{\gamma\alpha} = L^{\pi\gamma} \; L_{\gamma\sigma} \; \rho^{\sigma}_{\cdot\alpha} = \rho^{\pi}_{\cdot\alpha}$$

$$L^{\pi\gamma} \; G_{\gamma\alpha} = L^{\pi\gamma} \; \rho_{\gamma}^{\cdot\sigma} \; L_{\sigma\alpha} \qquad (9\text{-}2)$$

In equation 9-2, the rotation tensor is pre-multiplied by the inverse metric tensor and post-multiplied by the metric tensor. In Section 3-10-2, it has been pointed out that such operations are represented by "raising" and "lowering" the indices. In equation 9-2 the covariant index of the rotation tensor is raised and the contravariant index lowered. If the resulting tensor is denoted by, $\bar{\rho}$, then:

$$\bar{\rho}^{\pi}_{\cdot\alpha} = L^{\pi\gamma} \rho^{\cdot\sigma}_{\gamma} L_{\sigma\alpha} = L^{\pi\gamma} G_{\gamma\alpha} \qquad (9\text{-}3)$$

The affine connection, $\Gamma^{\pi}_{\alpha\beta}$, in Park's axis is thus seen to be a set comprising the rotation tensor, ρ, and another tensor, $\bar{\rho}$. When the transient and hunting equations are derived, using covariant derivatives and differentials, the two tensors, ρ and $\bar{\rho}$, mentioned above would appear. However, it is seen in the following section that terms containing $\bar{\rho}$ cancel out in the field analysis.

9-2 Field Equations

The exterior derivative of a vector ϕ , in covariant form, is, from Section 7-4-2, equation 7-54,

$$\frac{\delta \phi_\beta}{\delta x^\alpha} - \frac{\delta \phi_\alpha}{\delta x^\beta} = \left(\frac{\partial \phi_\beta}{\partial x^\alpha} - \frac{\partial \phi_\alpha}{\partial x^\beta}\right) + \left(\Gamma^{\pi}_{\alpha\beta} - \Gamma^{\pi}_{\beta\alpha}\right) \phi_\pi$$

$$= \left(\frac{\partial \phi_\beta}{\partial x^\alpha} - \frac{\partial \phi_\alpha}{\partial x^\beta}\right) + L^{\pi\gamma} \left(\Gamma_{\alpha\beta,\gamma} - \Gamma_{\beta\alpha,\gamma}\right) \phi_\pi$$

This represents the curl of the vector in covariant form. In Park's axis, when β = s (mechanical coordinate):

$$\frac{\delta \varphi_{(S)}}{\delta x^\alpha} - \frac{\delta \phi_\alpha}{\delta x^{(S)}} = \left(\frac{\partial \varphi_{(S)}}{\partial x^\alpha} - \frac{\partial \phi_\alpha}{\partial x^{(S)}}\right)$$

$$+ L^{\pi\gamma} \left(\Gamma_{\alpha(S),\gamma} - \Gamma_{(S)\alpha,\gamma}\right) \phi_\pi \qquad (9\text{-}3)$$

The second term on the right-hand side of equation 9-3 reduces to:

$$L^{\pi\gamma} \left(\tfrac{1}{2} G_{\gamma\alpha} - \tfrac{1}{2} V_{\gamma\alpha} - \tfrac{1}{2} G_{\gamma\alpha} - V_{\gamma\alpha}\right)$$

$$= L^{\pi\gamma} (-V_{\gamma\alpha}) = - \rho^{\pi}_{\alpha}$$

In exterior derivatives, the rotation tensor term is added to the partial derivatives. It is also seen from the above equation that the term $L^{\pi\gamma} G_{\gamma\alpha'} = \bar{\rho}^{\pi}_{\alpha'}$ is both added to and subtracted from ρ^{π}_{α} and thus cancels out.

9-2-1 Divergence Equation

The divergence of a two-form, $H^{\pi\beta}$, in covariant form is, from Section 7-4-2, equation 7-57,

$$\frac{\delta H^{\pi\beta}}{\delta x^{\beta}} = \frac{\partial H^{\pi\beta}}{\partial x^{\beta}} + \Gamma^{\pi}_{\alpha\beta} H^{\alpha\beta} + \Gamma^{\beta}_{\alpha\beta} H^{\pi\alpha}$$

$$= \frac{\partial H^{\pi\beta}}{\partial x^{\beta}} + L^{\pi\gamma} \Gamma_{\alpha\beta,\gamma} H^{\alpha\beta} + L^{\beta\alpha} \Gamma_{\alpha\beta,\gamma} H^{\pi\alpha} \qquad (9\text{-}4)$$

In Park's axis, the second term on the right-hand side of equation 9-4 is:

$$L^{\pi\gamma} \Gamma_{\alpha(S),\gamma} H^{\alpha(S)} + L^{\pi\gamma} \Gamma_{(S)\beta,\gamma} H^{(S)\beta}$$

$$= L^{\pi\gamma} \left(\tfrac{1}{2} G_{\gamma\alpha} - \tfrac{1}{2} V_{\gamma\alpha}\right) H^{\alpha(S)} - L^{\pi\gamma} \left(\tfrac{1}{2} G_{\gamma\beta} + \tfrac{1}{2} V_{\gamma\beta}\right) H^{\beta(S)}$$

$$= L^{\pi\gamma}(-V_{\gamma\alpha})\ H^{\alpha(S)} = -\ \rho_{\alpha}^{\ \pi}\ H^{\alpha(S)}$$

The last term of equation 9-4 is:

$$L^{\beta\gamma}\ \Gamma_{(S)\beta,\gamma} = L^{\beta\gamma}(\tfrac{1}{2}\, G_{\gamma\beta} - \tfrac{1}{2}\, V_{\gamma\beta}) = 0$$

Therefore, in Park's axis,

$$\frac{\delta H^{\beta}}{\delta x^{\beta}} = \frac{\partial H^{\beta}}{\partial x^{\beta}} - \rho_{\alpha}^{\ \pi}\ H^{\alpha(S)}$$

One arrives at the same result by considering divergence as the dual of the exterior derivative of the dual of a tensor. It is again seen that the term $L^{\pi\gamma}\ G_{\gamma\alpha'} = \bar{\rho}_{\alpha}^{\ \pi}$, is both added to and subtracted from the divergence equation and thus cancels out.

In the more general form of exterior derivatives used here, in addition to partial derivatives certain "affine connection" terms appear. These are seen in Park's axis to reduce to the rotation tensor. In differential geometry, this would represent the rotation of a "frame" along a curve on a general surface. (Frame denotes a set of mutually orthogonal unit vectors.). In the method of moving frames of E . Gartan, the rate of change of the frame along such a curve is expressed in terms of the frame itself.

i.e.

$$d\vec{e}_r = \Gamma^v_{rs} \; dx^s \; \vec{e}_v \qquad (9\text{-}5)$$

In equation 9-5, the changes in coordinates along the curve are represented by dx^s. The unit vectors are represented by $\vec{e}_v$ and $d\vec{e}_r$ represents the change of the frame. The affine connection Γ^v_{rs} represents the rate of change of the frame. Further development of the subject of moving frames will be found in reference 17.

9-3 Hunting Equations

In the analysis of small oscillation performance Kron(44), Ku(51) and others make use of covariant differentials:

$$\delta i^{\pi} = \Delta i^{\pi} + \Gamma^{\pi}_{\alpha\beta} \; i^{\alpha} \; \Delta x^{\beta}$$

(equation 8-8, Section 8-3).

Computation showed that the covariant differentials lead to complicated equations unless the problem is linearised and second order effects ignored. The analysis can be represented geometrically by a local "current" space tangent to the underlying (non-Riemannian) "charge" space described earlier (Section 7-4-4. It is possible to construct in the tangent space a set of mutually orthogonal unit vectors. Since in an electric machine, the transformations of reference frames are "orthogonal" in nature, the unit vectors will be transformed to another set of mutually orthogonal unit vectors. Such reference frames will have a metric tensor in which the elements along the diagonal are unity and the remaining elements zero. With this metric, the operations carried out in equation 7-64 (Section 7-4-2) results in:

$$\Gamma^{\pi}_{\alpha(S)} = - \rho^{\pi}_{\alpha}$$

i. e.

$$\delta i^{\pi} = \Delta i^{\pi} - \rho^{\pi}_{\alpha} \, i^{\alpha} \, \Delta x^{s} \qquad (9\text{-}6)$$

The above equation can be compared with equation 8-2 (Section 8-1-1):

$$\tilde{\Delta} i = \Delta i - \rho \, i \quad \Delta \lambda \qquad (8\text{-}2)$$

In equation 8-2, the term $\tilde{\Delta} i$ represents absolute changes in currents being given by Kron's axes components.

The rotation tensor arose naturally in Chapters 7 and 8 in the derivation of the machine equations in both field and circuit forms, and in the transient and hunting cases. In the present chapter this tensor has been seen to possess important geometrical significance. The relation between a set of differential equations for electric machines and concepts in differential geometry was first published by Kron and continues to be the basis for all present day work in this area.

CHAPTER 10

COMPUTATION AND EXPERIMENTAL STUDY OF OSCILLATIONS

Computation of small oscillation performance and the experimental work carried out to verify the computation are studied in this chapter.

10-1 Computation of Oscillating Torque Components

In Section 8-2 it was pointed out that the torque components can be computed in terms of rotor excursion, $\Delta\lambda$, and oscillating currents, $\tilde{\Delta} i$, in Kron's axes (equations 8-4, 8-5 and 8-6). The currents, $\tilde{\Delta} i$, can be evaluated from the oscillating voltages, $\tilde{\Delta} v$, by first establishing an equation in terms of Park's axes quantities Δv and Δi, and then using the relations:

$$\tilde{\Delta} v = \Delta v + \rho_t v \Delta\lambda \qquad \text{(equation 8-3)}$$

$$\tilde{\Delta} i = \Delta i - \rho_i \Delta\lambda \qquad \text{(equation 8-2)}$$

The analysis is applied in this section, to a synchronous motor connected to infinite bus-bars. In the analysis, no damper windings are considered, since the machine tested was completely laminated and had no amortisseur winding. The field winding is assumed to be in the direct axis only. The conventional hunting equations for a synchronous machine can be written(2, 46) with the same basic assumptions as made by Park(35) in his original paper on synchronous machines.

$$\begin{bmatrix} \Delta v_{ds} \\ \Delta v_{dr} \\ \Delta v_{qr} \\ \Delta T \end{bmatrix} = \begin{bmatrix} R_{ds}+L_{ds}p & M_d p & & \\ M_d p & R_r + L_{dr}p & L_{qr}p\Theta & L_{qr}i^{qr} \\ -M_d p\Theta & -L_{dr}p\Theta & R_r + L_{qr}p & -M_d i^{ds} - L_{dr}i^{dr} \\ M_d i^{qr} & (L_{dr}-L_{qr})i^{qr} & M_d i^{ds} + (L_{dr}-L_{qr})i^{dr} & R_s + Mp \end{bmatrix} \cdot \begin{bmatrix} \Delta i^{ds} \\ \Delta i^{dr} \\ \Delta i^{qr} \\ \Delta p\Theta \end{bmatrix} \tag{10-1}$$

These equations are obtained by taking conventional increments in the transient equations of a synchronous machine, shown below. The voltage equation 7-2 (Section 7-1) and torque equation 7-4 are combined into one matrix equation:

$$\begin{bmatrix} v_{ds} \\ v_{dr} \\ v_{qr} \\ T \end{bmatrix} = \begin{bmatrix} R_{ds}+L_{ds}p & M_d p & & \\ M_d p & R_r+L_{dr}p & L_{qr}p\Theta & \\ -M_d p\Theta & -L_d p\Theta & R_r+L_{qr}p & \\ M_d i^{qr} & L_{dr} i^{qr} & -L_{qr} i^{dr} & R_s+Mp \end{bmatrix} \bullet \begin{bmatrix} i^{ds} \\ i^{dr} \\ i^{qr} \\ p\Theta \end{bmatrix} \tag{10-2}$$

The substitutions shown by equations 8-2 and 8-3 give:

$$\begin{bmatrix} \Delta v_{ds} \\ \tilde{\Delta} v_{dr} \\ \tilde{\Delta} v_{qr} \\ \Delta T \end{bmatrix} = \begin{bmatrix} R_{ds}+L_{ds}p & M_d p & & -M_d i^{qr} p \\ M_d p & R_r+L_{dr}p & L_{qr}p\Theta & \begin{gathered} -(L_s i^{dr}+M_d i^{ds})p\Theta \\ -L_s i^{qr} p \end{gathered} \\ -M_d p\Theta & -L_{dr} p\Theta & R_r+L_{qr}p & \begin{gathered} -(L_s i^{dr}+M_d i^{ds})p \\ +L_s i^{qr} p\Theta \end{gathered} \\ M_d i^{qr} & L_s i^{qr} & M_d i^{ds}+L_s i^{dr} & \begin{gathered} R_s p + M p^2 \\ M_d i^{ds} i^{dr} + \\ L_s(i^{dr} \cdot i^{dr} - i^{qr} \cdot i^{qr}) \end{gathered} \end{bmatrix} \bullet \begin{bmatrix} \Delta i^{ds} \\ \tilde{\Delta} i^{dr} \\ \tilde{\Delta} i^{qr} \\ \Delta\lambda \end{bmatrix} \tag{10-3}$$

where $L_s = (L_{dr} - L_{qr})$

(The derivation is shown in Appendix 9-1, reference 45).

If the field winding is connected to a constant voltage d.c. supply, $\Delta v_{ds} = 0$. Moreover, the armature windings are connected to infinite bus-bars, so that the absolute changes in voltage, $\tilde{\Delta} v$, in Kron's axes are zero. If the externally applied torque is constant, $\Delta T = 0$. No impressed oscillating voltages and torque are considered here. The hunting impedance matrix given by equation 10-3 is now computed for the synchronous machine, under given operating conditions. The eigen-values are then established from the impedance matrix. A typical set of solutions for the matrix of equation 10-3 in case of the machine studied were:

$e^{(-5.53)314\,t}$ -d.c. component

$e^{(-0.00028 \pm j\,0.012)314\,t}$ - hunting a.c. component

$e^{(-0.15 \pm j\,0.56)314\,t}$ - second a.c. component

The time-constants of the three components are given by the reciprocal of the real parts of the eigen-values. It is seen from the above values that the d.c. component and the second a.c. component decay much more rapidly that the hunting a.c.

component. In the normal operating range of the machine studied, the time constants of the d. c. component and the second a. c. component were less than 8. 5% of the hunting time-period. For the hunting analysis, therefore, their effects can be ignored and the substitution:

$$p = (-u + jh)\,\omega$$

can be made in matrix 10-3 for the operator $p = d/dt$. In the above equation "h" denotes the ratio of hunting frequency to rated frequency and "u" gives the reciprocal time-constant of the oscillations. With this, the impedance matrix of equation 10-3 assumes the form:

$$\begin{array}{c|c|c|} & \text{elect} & \text{mech} \\ \hline \text{elect} & Z_1 & Z_2 \\ \hline \text{mech} & Z_3 & Z_4 \\ \hline \end{array} \qquad (10\text{-}4)$$

where Z_1, Z_2, Z_3 and Z_4 are complex submatrices. For example:

$$Z_4 = R_S(-u + jh)\omega + M\,(-u + jh)^2\,\omega^2$$

$$+ M_d\, i^{ds}\, i^{dr} - (L_{qr} - L_{dr})(i^{dr}\, i^{dr} - i^{qr}\, i^{qr})$$

Eliminating the currents, the matrix equation

$$Z_4 - Z_3 Z_1^{-1} Z_2 = 0 \qquad (10\text{-}5)$$

gives the equation of motion:

$$Mp^2 (\Delta\Theta \;) + T_d p (\Delta\Theta \;) + T_s \Delta\Theta \;= 0$$

(equation 8-4).

With $p = (-u + jh)\omega$, equation 8-4 becomes:

$$M(u^2 - h^2)\omega^2 - u\omega T_d + T_s$$

$$- j\, 2\, u\, h\, \omega^2 M + j\, h\, \omega\, T_d = 0 \qquad (10\text{-}6)$$

Since the real and imaginary parts of equation 10-6 must be separately zero, it follows that:

$$T_s - M h^2 \omega^2 = u\omega\, (T_d - M u \omega \;) \qquad (10\text{-}7)$$

and,

$$T_d = 2\, u\, \omega\, M \qquad (10\text{-}8)$$

In equation 10-7, the right hand side terms are normally very small compared with both T_s and $M h^2 \quad ^2$ so that:

$$T_s - M h^2 \omega^2 \simeq 0$$

In the calculation of torque components, T_s and T_d, it was found that the substitution of $p = j h \omega$ instead of $p = (-u + j h) \omega$ resulted in an error less than 0.13%. This substitution is made in equations 10-3, 10-4 and 10-5. In equation 10-5, the substitution, $p = j h \omega$, results in:

$$Z_4 = j R_S h \omega - M h^2 \omega^2 - i_t \cdot G \cdot \rho \cdot i$$
$$+ i_t \cdot K \cdot i$$

The imaginary part ($R_S h \omega$) in the above equation gives the mechanical damping torque, T_{dm}. The term ($-M h^2 \omega^2$) is the inertia torque. The terms ($i_t \cdot G \cdot \rho \cdot i$) and ($i_t \cdot K \cdot i$) contribute to synchronising torque, T_s, and their significance has already been discussed in Section 8-2. The real part of $Z_3 Z_1^{-1} Z_2$ gives the remaining terms of T_s, and the imaginary part gives T_{de}, the electrical damping torque.

In equations 10-7 and 10-8, the synchronising torque coefficient, T_s, and damping torque coefficient, can be computed theoretically. The quantities ($-M h^2 \omega^2$) and ($2 u \omega M$)

can be established from experimental work as shown in Section 10-4. In these expressions, M is the rotor inertia, $h\omega$ is given by the hunting frequency and $u\omega$ by the time rate of decay of oscillations. These can be measured from hunting tests. Figs. 31 and 32 show the results. Solid lines give theoretical results. Dotted lines give experimental results. The stability limits predicted are found to agree with the laboratory tests. When the synchronising torque becomes negative, the rotor does not oscillate but pulls out of synchronism. When the damping torque becomes negative, the oscillations become larger and larger till the machine eventually loses synchronism. The computer program used is given in Appendix 5.

10-2 Machine Parameters

A universal laboratory machine was used for the hunting tests. The machine has a laminated stator and rotor structure and closely resembles the idealised machine as defined by Park(35). The parameters shown in Table 1 were measured by standard tests.

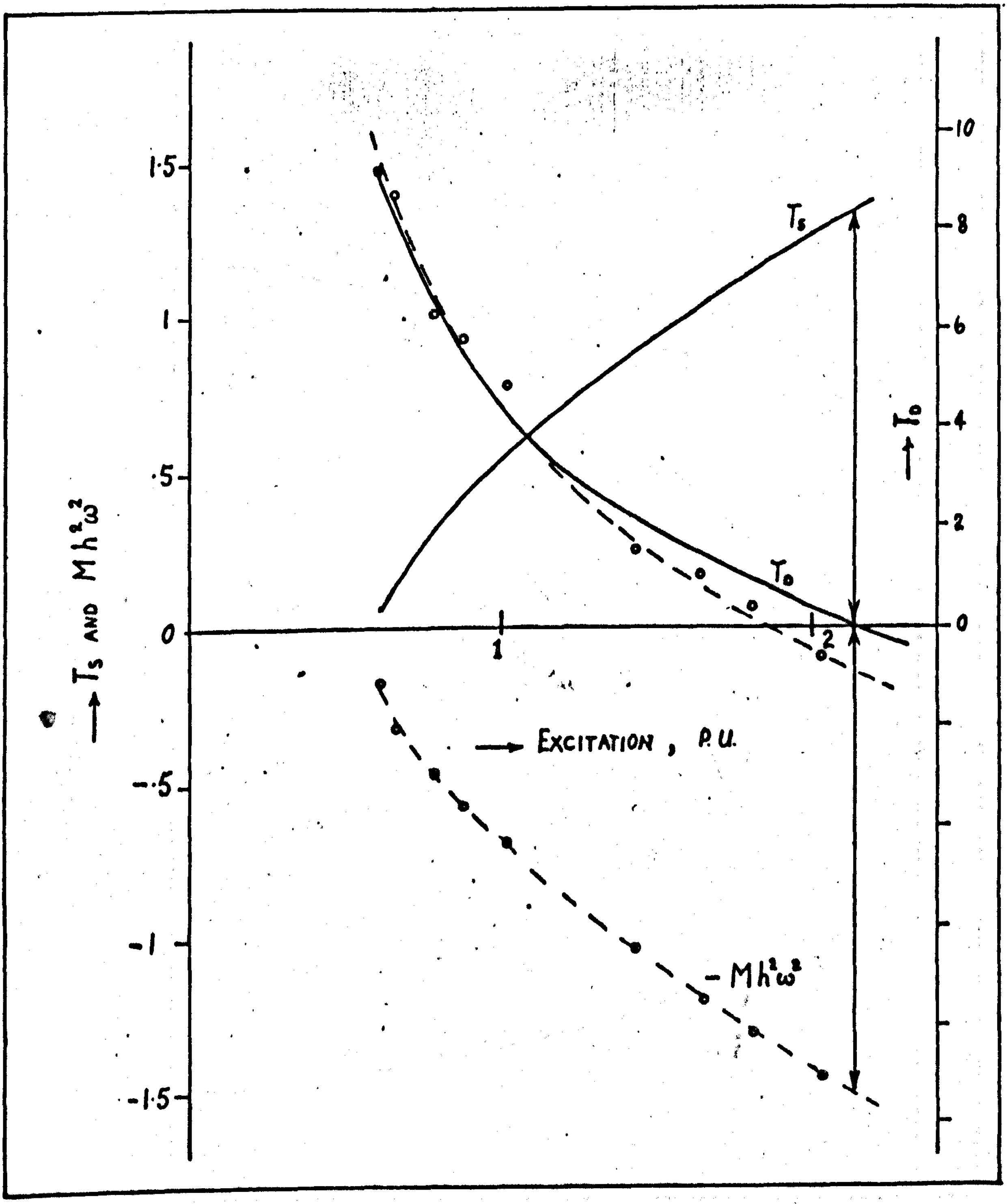


FIG. 31. VARIATION OF T_s AND T_D WITH FIELD EXCITATION

—— COMPUTED

---- EXPERIMENTAL.

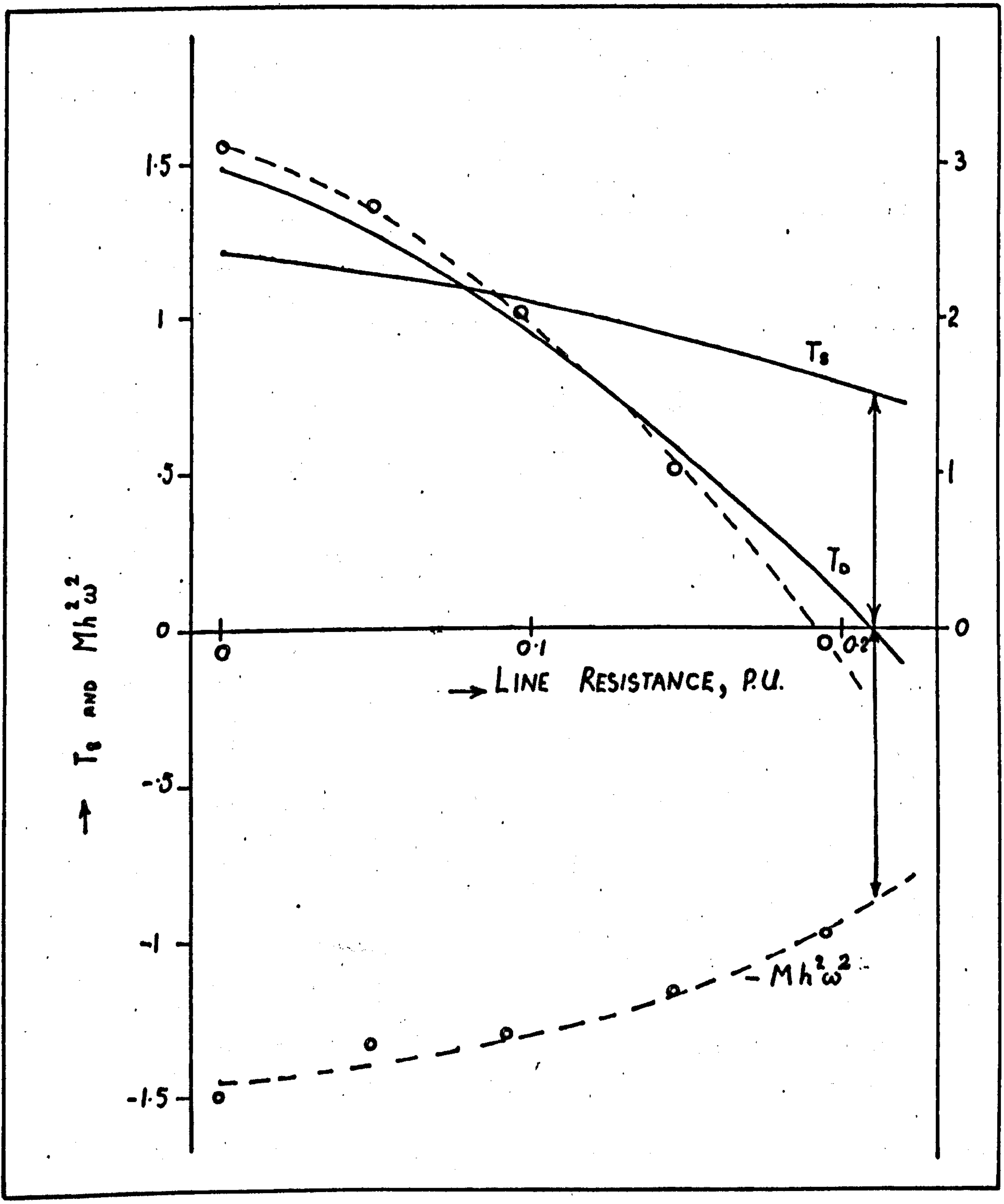


FIG. 32. VARIATION OF T_S AND T_D WITH LINE RESISTANCE

—— COMPUTED

---- EXPERIMENTAL

TABLE 1

Reactances of Machine, Per-Unit Values

Type of machine and rating	Universal machine, 240V, 3-phase, 2. 25A, 2-pole connection
X_{dr} (unsaturated)	0. 99
X_{qr} "	0. 97
X_{ds} "	1. 0
X_{md} "	0. 978
X'_{d} "	0. 0347

The values obtained had previously been found by Sen Gupta to be accurate in the prediction of performance under both steady-state and transient short-circuit conditions(52).

The small air-gap built into the design of this machine for induction motor operation leads to a high ratio of $x_{dr} = x'_{d}$. The armature resistance of the universal machine is also high and was measured by using sub-standard ammeter and voltmeter. The d. c. resistance per phase is 1. 43 ohms (with a delta connection). Skin-effect was ignored because the conductors are small (wire gauge 18 s. w. g.). The a. c. resistance can be reasonably assumed to be equal to the d. c. resistance. In per-unit,

$$R_a = .0078 \text{ p. u.}$$

The rotor winding is tapped for both two-phase and three-phase outputs and connected to slip rings. It is also connected to a commutator for d. c. operation. The brushes on the commutator were lifted for synchronous machine operation. The rotor winding was used as the field, and the d. c. supply connected to two of the slip rings. To include the effect of brushes in the field resistance, the rotor was driven at 3000 r. p. m. and a graph of d. c. current against voltage across input terminals was plotted. The slope of the graph gave an effective resistance of 0. 26 ohm. It is necessary to obtain the field resistance in per-unit values referred to the armature. In Rankin's formula(53):

$$(R_f)_{p.u} = \frac{3}{2} \frac{R_{fo}}{X_{bo}} \left(\frac{I_f}{\frac{3}{2} I_a}\right)^2 \qquad (10\text{-}7)$$

where X_{bo} is the base impedance in ohms, I_f is the field current selected as the base and I_a is the peak rated armature current. The bracketed term in equation 10-7 is called the base-current ratio. To obtain it from a short-circuit test, however, a modification is required to account for the magnetising current needed to overcome the leakage-reactance drop in the short-circuited condition. The p.u. field resistance is now given by:

$$(R_f)_{p.u} = \frac{3}{2}\ \frac{R_{fo}}{X_{bo}}\ \left(\frac{i_f}{\frac{3}{2}\, i_a}\ \ \frac{X_{md}}{X_d}\right)^2 \qquad (10\text{-}8)$$

$$= 0.0165 \ \text{p.u.}$$

In equation 10-8, i_f is the field excitation corresponding to a short circuit of peak value i_a in the armature.

The moment of inertia of the rotor of the universal machine coupled with a d.c. machine is 0.1546 kg - m^2, as specified by the manufacturer. This was checked by a retardation test applying a known power load to the shaft of the machine$^{(54)}$. For this test the frictional power at rated speed was first determined from a no-load test. In per-unit value, M = 0.052 p.u. The inertia constant, $H = M\omega/2 = 8.17$.

10-2-1 Frictional Resistance R_s

The frictional torque for this machine is not directly proportional to the rotor angular speed. The frictional resistance to oscillations was determined by measuring the frictional torque at rated speed from a no-load test and the frictional torques at other speeds by the retardation method, after open-circuiting all the electric circuits. The result is shown plotted in fig. 33. Under this condition,

$$M p^2\Theta + R_S p\Theta = 0$$

The frictional torque, $R_S p \Theta$, is thus equal to the inertia torque. In the retardation test a recording of $p \Theta$, the rotor angular velocity, is obtained. From this the inertia torque can be calculated. First the angular velocity, $p\Theta$, is plotted as a function of time (seconds). The slope of the graph gives the acceleration, $p^2\Theta$. From these results the frictional torque ($= - M p^2\Theta$) is plotted as a function of the rotor angular velocity, $p\Theta$. The graph is shown in fig. 33.

$$R_S = \tan\theta \times \left(\frac{\text{scale of abscissa}}{\text{scale of ordinate}}\right) = .00215$$

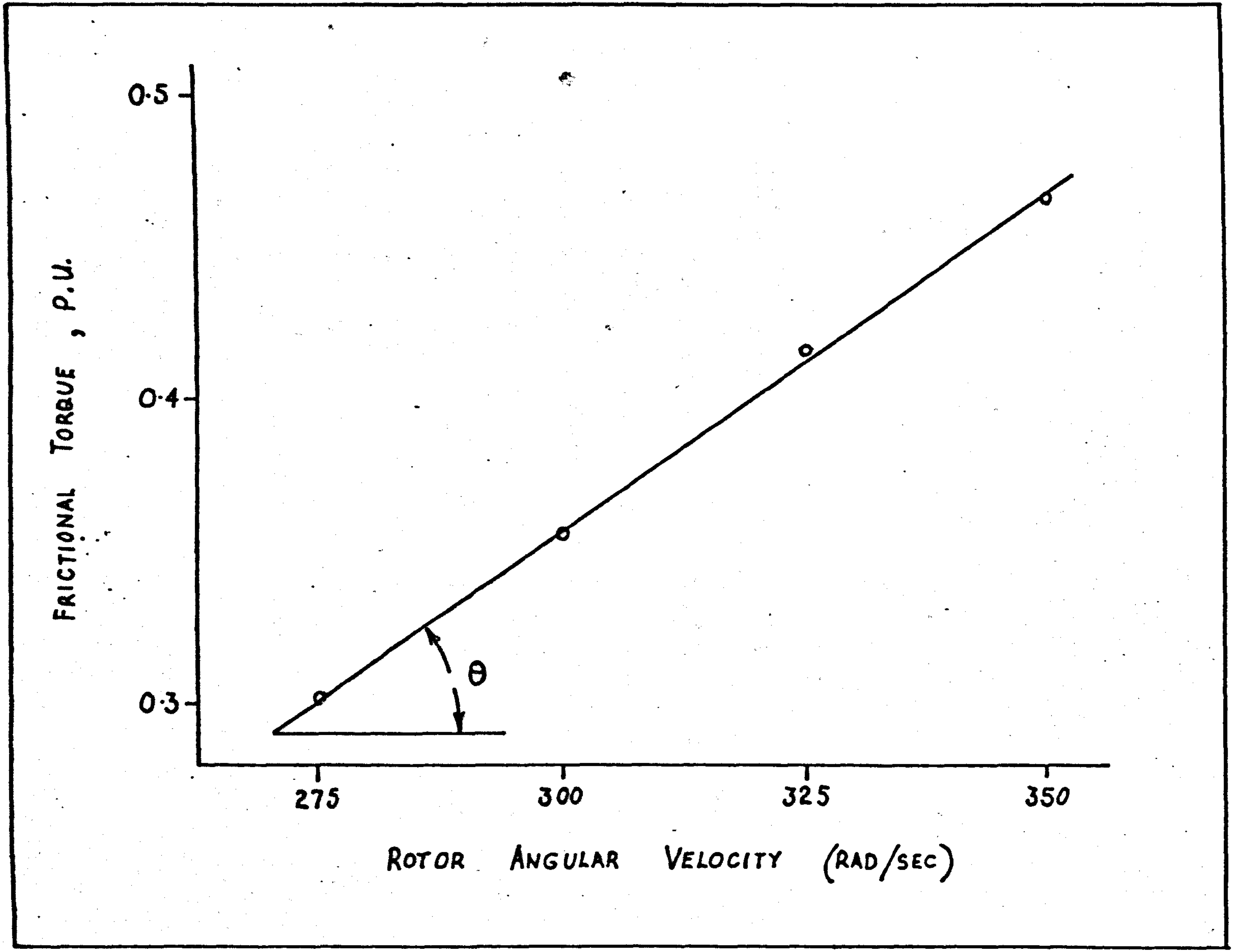


FIG. 33. CALCULATION OF FRICTIONAL RESISTANCE
θ=34°

Therefore $R_S \, p \, \Delta\Theta$ during hunting

$$= R_S \; j \; h \omega \;\; \Delta\Theta = j \, h \, 0.68 \;\; \Delta\Theta$$

The mechanical damping torque coefficient

$$T_{dm} = 0.68 \text{ p.u.}$$

10-3 Experimental Set-up

The circuit diagram for the tests is shown in fig. 34. The field winding was connected to the d. c. supply, as shown through a potentiometer and a series resistance both of which can be varied. This method of control is used to vary the field current, without changing field time-constant. Throughout the tests, the effective field resistance was kept constant at 0.98 ohm (0.0624 p.u.). A 3-phase variable resistance simulates variable line resistance. The universal machine was synchronised to the mains supply (assumed to be an infinite bus-bar), in the usual way. The supply voltage available in the laboratory is 200 volts, 3 phase at 50 c/s.

The machine operated as a 3-phase synchronous motor

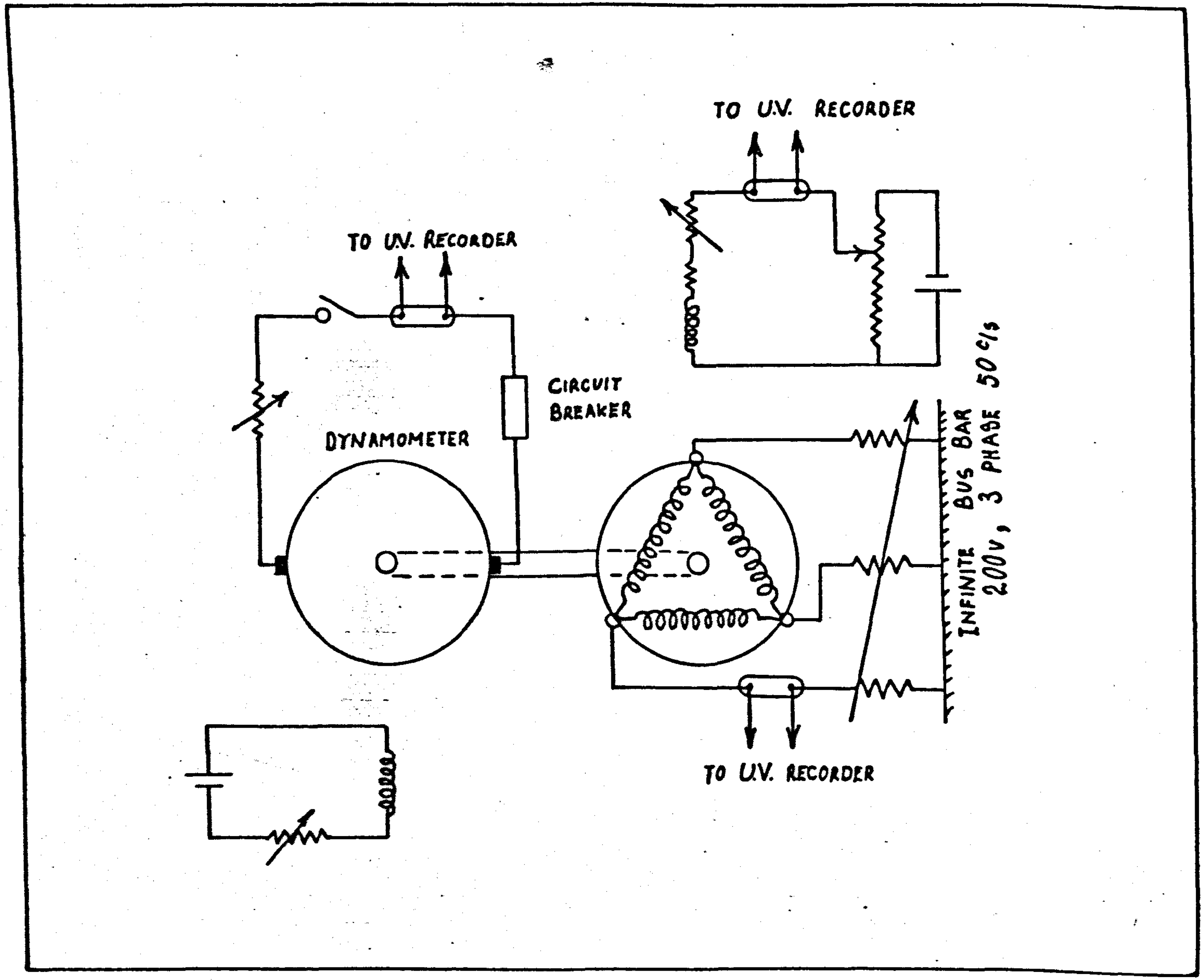


FIG. 34. EXPERIMENTAL SET-UP FOR HUNTING EXPERIMENTS

and was driving a separately excited d. c. machine, with a weak field. To induce oscillations, an impulse load was applied to the d. c. machine, which was otherwise operating with the armature open-circuited. With this arrangement, it was found that the d. c. machine contributed no significant damping to the system. On closing the switch in the d. c. machine circuit, a current flowed in the circuit which was interrupted by the circuit breaker in less than 0.1 second. This produced an impulse load on the synchronous motor which was driving the d. c. generator. The impulse load was adjusted so that the oscillations were small (in accordance with small-oscillation theory), yet large enough to give reasonably accurate results. An oscillation not exceeding ± 5 degree angular excursion was found to be suitable for this purpose. Time is counted after the impulse is over and therefore ΔT, the externally applied torque to the shaft of the synchronous machine, can be taken as zero. The impulse only initiates the oscillations and does not affect the values of T_d and T_s. The d. c. machine rotor inertia and frictional resistance have already been included in the retardation tests and need not be calculated separately.

10-4 Experimental Determination of Inertia and Damping Torques

During hunting tests, the hunting frequency and the rate of decay of motor oscillations were measured, and from these the inertia and damping torques were determined. The hunting frequency, $h\omega$, was measured from the oscillograms of the field and armature currents. The recording instrument was an ultra-violet recorder working on the principle of a duddell oscillograph. A convenient way to determine the rate of decay of the oscillations is to plot the amplitude on a semi-logarithmic graph paper. Now:

$$h = \frac{2\pi}{\omega} \times \text{ oscillation frequency,}$$

and the inertia torque is given by $-M h^2 \omega^2$. The damping torque is given by:

$$T_d = 2 u \omega M = \frac{2 \omega M}{\text{time constant}}$$

(time constant being given by the time in seconds taken by the oscillations to decay to 36.8% of their original values).

This chapter dealt with the computation and experimental aspects of the oscillatory behaviour of electric machines. The stability limits predicted are found to agree within 14% of the laboratory tests. Changes in saturation and effects of space harmonics have been ignored in the analysis.

The computation of the oscillating torque components in Kron's axes was seen in Chapter 8 to lead to a certain grouping of terms, and the components were related to an equivalent circuit. This method of analysis indicates the physical nature of the damping and synchronising torques in an electric machine.

CHAPTER 11

CONCLUSION

As stated in the introduction, this thesis is an attempt to establish the basic steps on which Kron builds the multi-dimensional wave automaton. In the process, the differential relations of Maxwell for electromagnetic fields have been interpreted as algebraic structural relations for an equivalent micro-mesh network. The use of exterior differential calculus has been seen to lead to a network structure independent of the coordinate system. In an orthogonal coordinate system, a model, first proposed by Kron, follows which is physically realisable and both analog and digital solutions are possible. The digital solution for a waveguide problem showed how the errors involved in discretisation of field problems can be made as small as desired.

It is hoped that with the use of the universal approach to differential problems, which exterior forms and exterior calculus provide, the analysis can be extended to multi-dimensional fields and networks. Though analog simulation of such models is not feasible, digital computers are capable of handling "arrays" (generalised matrices) exceeding three dimensions. The systematic organisation of such multidimensional problems can be based on the algebraic or flow diagram presented in the thesis for three dimensional electromagnetic field problems. The more general fields would involve the application of such concepts as generalised Stokes' theorem. Thus much of the ground work has been carried out in the thesis for multidimensional problems. The systematic solution of such problems would also involve a diakoptic treatment of linear networks. Once the structure is established in a general manner, in an actual digital programming schedule, few conceptual points are expected to arise. It is for this reason that exterior forms and exterior calculus, mostly found in advanced mathematical literature, are presented here from an engineering view point. In addition to being universal in its application, the use of such mathematics is seen to lead to equivalent networks physically more satisfying than other types of analog networks.

The algebraic diagram developed for electromagnetic field network model was seen to separate the field quantities into generalised "voltage" and "current" terms. The same division also appears in the four dimensional analysis of field tensors $F_{\alpha\beta}$ and $H^{\alpha\beta}$. These were examined for an electric machine, an electro-dynamic type of network. The tensorial presentation of electric machine fields, first examined by Kron and studied in some detail in the thesis, would seem to be a pointer to the more general magnetohydrodynamic types of generators. The orthogonal electrical and mechanical variables were coupled by means of a torsion tensor, a concept which should help to couple MHD magnetic and fluid-flow networks. The tensor formation of an electric machine has been seen to interpret correctly certain concepts which previously had been ignored (e. g. the tangential magnetic field, physical nature of hunting torques etc.).

The self-organising function of a system such as a wave automaton would be guided by its oscillatory behaviour. The absolute changes, during small oscillations, have been seen to reinterpret correctly the behaviour of an electric machine. Moreover, the tensor hunting equations can be used to study Kron's multidimensional rotating machine (reference 3). Such a machine (purely a theoretical device) could have more than one mechanical axes and be capable of rotating in different directions simultaneously. The freely rotating reference frame used in Chapter 8 could be used to analyse such devices.

In Chapter 5, the connection matrix, C, for the network model expressed the relations between mesh and branch quantities. The same relation satisfied Maxwell's equations (the field relations being expressed in an integral form of exterior differentials). In the electric machine analysis (Chapter 7), the transformation tensor, C, is a function of the mechanical coordinates. The torsion tensor, $S_{\alpha\beta}^{\cdot\cdot\gamma}$, arose out of such transformations and was expressed as a function of C. The more general absolute form of Maxwell's equations resulted, containing terms of the type $S_{\alpha\beta}^{\cdot\cdot\gamma}$. The Lagrangean electrodynamic equations were reinterpreted as Maxwell's equations.

In Chapters 5 and 6 the electromagnetic field equations were viewed as algebraic structural relations for a network model. Many interesting interpretations arose in such a study. The Poincare Lemma for distributed field systems, for example, was interpreted as an algebraic structural relation, $(A_o)_t C_c = 0$, for networks. Conversely, every network can be regarded as being surrounded by an invisible field. Kirchhoff's laws (and Lagrangean equations for dynamical systems) can then be interpreted as Maxwell's field equations. The interaction of network quantities is then studied as an interaction of the electromagnetic waves in the surrounding field. Human memory for example is being studied by some neurologists as a continuum rather than as an on-off process(56). In such a study the waves arriving at many synapto-dendritic junctions are thought to interact and produce patterns similar to interference or Moiré effects. Physicists have already succeeded, using lasers, to capture the diffracted waves from an object both in amplitude and phase using interferometry (a technique called holography) and then project it to "reconstruct the wave-front", the resulting waves being indistinguishable from the original waves. Such holograms are thought possible in neural structures in a human brain. Kron's proposed "artificial brain"(3) and "crystal computers(8)" employ such concepts as propagation of a sequence of electromagnetic and magnetohydrodynamic waves across polyhedral networks and could perhaps lead to a new class of computers.

The study of neural networks may involve determination of a statistically dominant pattern of activity. This would mean that the study of diakoptics of electric circuits has to be extended to cases where the probabilistic patterns of the individual elements are known and the stable overall pattern is required to be assessed. In non-linear systems, the piecewise solution of large-scale systems will depend on the development of a non-linear diakoptics. Another aspect of these networks is the self organising and synchronising nature of behaviour. Wiener, the cyberneticist, calls it the "rhythmic system". Randomly distributed particles in a state of rhythmic oscillation can effect one another through interactions between them. In non-linear systems, the various modes of oscillations of the elements of a group couple together and the mathematics of such oscillating systems is complex. Wiener observed many examples of groups, which by "pooling-effect", produced synchronised rhythms.

As stated earlier, the thesis does not reach as far as Kron's wave automaton. However, it is felt that many of the basic steps have been elucidated. The concepts employed here are general in nature and the possibility of extending these to multidimensional interacting networks, programmed on a computer, does not now seem so remote. The investigation of information processing by such networks was being actively pursued by Dr. Gabriel Kron up to the time of his death. Many aspects of this work have no doubt been clarified by him but have not yet been published.

CHAPTER 12

APPENDICES

12-1 Appendix 1

(i) To show, in the space-time coordinate system,

$$*(dx \wedge dy \wedge dz) = - dt$$

The dual of the 3-form in brackets is normal to all three base vectors dx, dy and dz. The dual is thus a 1-form containing dt term only.

Let $*(dx \wedge dy \wedge dz) = K\, dt$

In equation 3-14 which defines the dual in terms of inner and exterior products, let the exterior form $\lambda = dt$. Then, from equation 3-14,

$$(dx \wedge dy \wedge dz) \wedge dt = ((K\, dt,\ dt))\ d\sigma$$

i. e.

$$d\sigma = ((K\, dt,\ dt))\, d\sigma$$

$$= -K\, d\sigma$$

or,

$$K = -1,$$

and,

$$*(dx \wedge dy \wedge dz) = K\, dt = -dt$$

(ii) To show, in the same system,

$$*(dy \wedge dz \wedge dt) = -dx$$

The dual of the 3-form in brackets is normal to all three base vectors dy, dz and dt. The dual is thus a 1-form containing dx term only. Let λ = dx in equation 3-14 and the dual of the 3-form = K' dx.. Then, from equation 3-14,

$$(dy \wedge dz \wedge dt) \wedge dx = ((K'\, dx\, ,\, dx))\, d\sigma$$

$$= K'\, d\sigma$$

$$dy \wedge dz \wedge dt \wedge dx = -dx \wedge dy \wedge dz \wedge dt$$

$$= -d\sigma$$

i. e.

$$-d\sigma = K'\, d\sigma$$

or, $K' = -1$

$$*(dy \wedge dz \wedge dt) = K'\, dx = -dx$$

12-2 Appendix 2

To show,

$$*\bar{d}*\ (\tilde{H}) = J_1\, dx + J_2\, dy + J_3\, dz - c\ \rho\ dt$$

where,

$$\tilde{H} = H_1\, dy \wedge dz + H_2\ dz \wedge dx + H_3\ dx \wedge dy$$
$$+ cD_1\, dx \wedge dt + cD_2\ dy \wedge dt + cD_3\ dz \wedge dt$$

The divergence of a similar 2-form has been derived in Section 3-7-2. The following substitutions are made in the 2-form of Section 3-7-2:

$$H_4 = cD_1$$

$$H_5 = cD_2$$

$$H_6 = cD_3$$

Equation 3-31 will then take the form:

$$*\bar{d}*(\tilde{H}) = -(c\frac{\partial D_1}{\partial t} + \frac{\partial H_2}{\partial z} - \frac{\partial H_3}{\partial y})\ dx$$
$$-(\ldots)\ dy - (\ldots)\ dz - c(\frac{\partial D_1}{\partial x} + \frac{\partial D_2}{\partial y} + \frac{\partial D_3}{\partial z})\ dt$$

The first three bracketed terms above give the vector expression:

$$c \frac{\partial \vec{D}}{\partial t} - \text{curl } \vec{H} \qquad \text{i. e.} \qquad (-\vec{J})$$

The last bracketed term is:

$$\text{div } \vec{D} \quad \text{i. e.} \quad \rho \text{ , the charge density}$$

It follows that:

$$*\bar{d}* (\tilde{H}) = J_1 \, dx + J_2 \, dy + J_3 \, dz - c \rho \, dt$$

12-3 Appendix 3

To show div. $\vec{B} = 0$, in Kron's network model.

First the flux linkages of the ideal transformer windings are defined in terms of a line integral of a vector $\vec{A}$. The sum of the flux linkages of the "small" meshes in Kron's network is then equated to zero. From this a relation, $\vec{B} = \text{curl } \vec{A}$, is obtained, indicating that the vector $\vec{A}$ is the usual magnetic vector potential. Consequently, div. $\vec{B} = \text{div. } (\text{curl } \vec{A}) = 0$, by Poincare Lemma.

Fig. 35a show a "large" mesh P-R-S-Q in Kron's network by dotted lines. The area PWVS shown by the solid lines is an area formed by joining the diagonals of the large meshes. Fig. 35b shows a "small" mesh at the corner R. The ideal transformer winding marked ④ in the figure will have its secondary winding in the "small" mesh at corner Q. Since the transformer is assumed ideal, the primary and secondary flux linkages should be identical. One may reasonably assume that they will be proportional to the line integral of a vector $\vec{A}$ along the path PS, being the path of symmetry. In fact the flux linkages of the primary (marked ④) at corner R and its secondary at corner Q is defined as:

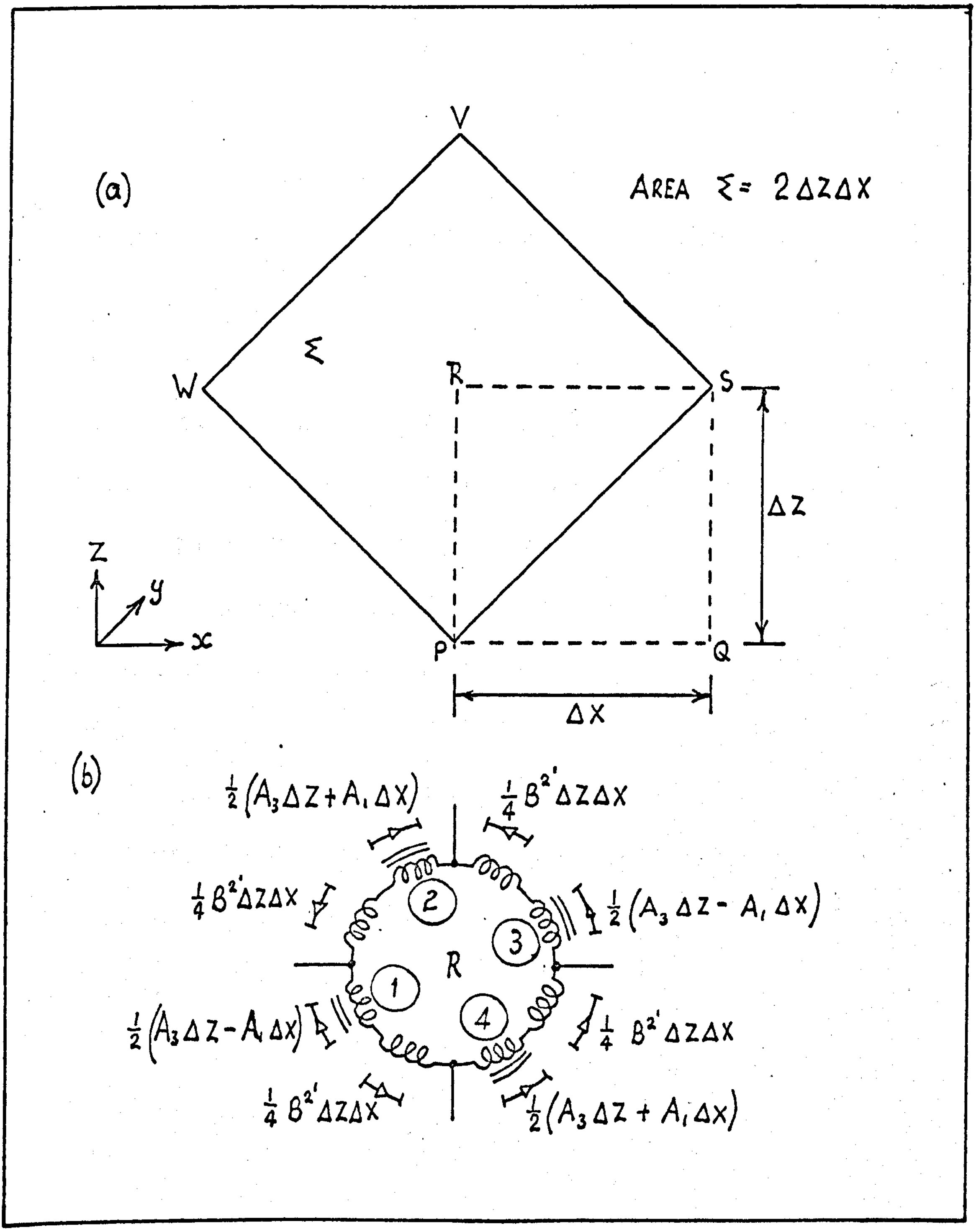


FIG. 35. FLUX LINKAGE OF IDEAL TRANSFORMERS

$$\frac{1}{2}\int_{P}^{S} A = \frac{1}{2}\int_{P}^{S} A_1 \; dx + A_2 \; dy + A_3 \; dz$$

which is equal to $\frac{1}{2}(A_1 \Delta X + A_3 \Delta Z)$. For the ideal transformer windings in the "small" mesh at corner R, marked ①, ②, ③ and ④, the flux linkages are defined in terms of line integrals of vector $\vec{A}$ along paths P-W, W-V, V-S and S-P. The results are given in fig. 35b. The flux linkages of the inductors have already been defined as $\frac{1}{4} B^{2'} \Delta Z \Delta X$ each.

Next, the flux linkage of the "small" mesh at corner R is set equal to zero. For, there are no resistances present in the circuit and constant flux linkage theorem can be applied. If ever in its past history, the field has vanished, this constant flux linkage must be zero. Since one may reasonably assume that the initial creation of the field was at a finite time ago, we conclude that the total flux linkage of the "small" mesh is zero. This gives, for the "small" mesh at corner R,

$$- \frac{1}{4} B^{2'} \Delta Z \Delta X + \frac{1}{2}(A_3 \Delta Z - A_1 \Delta X)_1 - \frac{1}{4} B^{2'} \Delta Z \Delta X$$

$$+ \frac{1}{2}(A_3 \Delta Z + A_1 \Delta X)_2 - \frac{1}{4} B^{2'} \Delta Z \Delta X$$

$$- \frac{1}{2}(A_3 \Delta Z - A_1 \Delta X)_3 - \frac{1}{4} B^{2'} \Delta Z \Delta X$$

$$- \tfrac{1}{2} (A_3 \Delta Z + A_1 \Delta X)_4 = 0$$

i. e.

$$B^{2'} \Delta Z \, \Delta X = \tfrac{1}{2} \Big\{ (A_1)_2 + (A_1)_3 - (A_1)_1$$

$$- (A_1)_4 \Big\} \Delta X + \tfrac{1}{2} \Big\{ (A_3)_1 + (A_3)_2 - (A_3)_3$$

$$- (A_3)_4 \Big\} \quad \Delta Z$$

i. e.

$$B^{2'} \Delta Z \, \Delta X = \cancel{\tfrac{1}{2}} \left(\frac{\partial A_1}{\partial z} \right) \Delta Z \ \Delta X$$

$$- \cancel{\tfrac{1}{2}} \left(\frac{\partial A_3}{\partial x} \right) \Delta X \ \Delta Z$$

This in fact represents the integral relation:

$$\tfrac{1}{2} \iint_{\Sigma} {*B} = \tfrac{1}{2} \iint_{\Sigma} \bar{d} \, A \qquad (12\text{-}1)$$

The area Σ is marked in fig. 35a. By Stokes' theorem,

$$\tfrac{1}{2} \iint_{\Sigma} \bar{d} \, A = \tfrac{1}{2} \int_{\partial \Sigma} A \qquad (12\text{-}2)$$

The boundary, $\partial \Sigma$, for the area Σ is P-S-V-W-P. This was in fact the starting point of the analysis and the procedure is seen to be consistent with Stokes' theorem.

Equation 12-1 is seen to relate the flux density vector, $\vec{B}$, with vector $\vec{A}$ by the equation:

$$\vec{B} = \text{curl}\ \vec{A}$$

The vector $\vec{A}$ used in the analysis is thus the well known magnetic vector potential.

The divergence of B

$$= *\bar{d}* (B) = *\bar{d} \left[\bar{d} (A) \right]$$

$$= 0, \text{ by Poincare Lemma}$$

12-4 Appendix 4

12-4-1 Vector Diagram

To show, in fig. 27, neglecting second order effects,

$$A_1A_2 \simeq AA' + PA \quad \Delta\lambda$$

$$B_1B_2 \simeq BB' - PB \ \Delta\lambda$$

$$A_1A_2 = A_1R + R\,A_2$$

$$A_1R \simeq AA'$$

$$R\,A_2 = P'R \ \sin\ (\Delta\lambda)$$

$$\simeq P'R \ \Delta\lambda$$

$$\simeq PA \ \Delta\lambda$$

$$\therefore A_1A_2 \simeq AA' + PA \ \Delta\lambda$$

$$BB' = BQ + QB'$$

$$BQ \simeq B_1B_2$$

$$QB' = P'B' \ \tan\ (\Delta\lambda)$$

$$\simeq P'B' \quad \Delta\lambda$$

$$\simeq PB \quad \Delta\lambda$$

$$\therefore B_1B_2 \simeq BB' - PB \ \Delta\lambda$$

12-4-2 Hunting Torque Equation

The torque equation in Park's axes is, from equation 7-4, Section 7-1,

$$T = Mp^2\Theta + R_S p\Theta - i_t \cdot G \cdot i \qquad (7\text{-}4)$$

(M is the rotor inertia, R_S is the frictional resistance and G is the torque matrix).

Equation 7-4 can be written as:

$$T = T_{mech} - T_{elect} \qquad (12\text{-}3)$$

where,

$$T_{elect} = i_t \cdot G \cdot i \qquad (12\text{-}4)$$

Taking increment Δ in equation 12-3,

$$\Delta T = \Delta T_{mech} - \Delta T_{elect}$$

i. e.

$$\Delta T = Mp^2 (\Delta\Theta \;) + R_S\, p\, (\Delta\Theta \;) - \Delta T_{elect} \qquad (12\text{-}5)$$

From equation 12-4, it is seen that:

$$\Delta T_{elect} = \Delta\, (i_t \cdot G \cdot i)$$

$$= \Delta i_t \cdot G \cdot i + i_t \cdot \Delta G \cdot i + i_t \cdot G \cdot \Delta i$$

In Park's axes, the inductances are constant during oscillation, so that the term ΔG is zero and,

$$\Delta T_{elect} = \Delta i_t \cdot G \cdot i + i_t \cdot G \cdot \Delta i \qquad (12\text{-}6)$$

In order to obtain an expression in terms of the absolute changes in armature current, $\widetilde{\Delta} i$, seen by Kron's axes, equation 8-2 is substituted in equation 12-6:

$$\begin{aligned} \Delta T_{elect} &= (\Delta \tilde{i}_t \cdot G \cdot i + i_t \cdot G \cdot \widetilde{\Delta} i) + \\ &\quad i_t \cdot \rho_t \cdot G \cdot i \; \Delta\lambda + i_t \cdot G \cdot \rho \cdot i \; \Delta\lambda \\ &= (\widetilde{\Delta} i_t \cdot G \cdot i + i_t \cdot G \cdot \widetilde{\Delta} i) + i_t \cdot G \cdot \rho \cdot i \; \Delta\lambda \\ &\quad - i \cdot K \cdot i \; \Delta\lambda \qquad (12\text{-}7) \end{aligned}$$

where,

$$K = - \rho_t \cdot G = - \rho_t \cdot \rho_t \cdot L \qquad (12\text{-}8)$$

Equations 12-5 and 12-7 correspond to equations 8-5 and 8-6.

12-4-3 Covariant Differentials

To show, that using covariant differentials,

$$c_p^{\pi} \cdot \tilde{\Delta} i^p = \Delta i^{\pi} - \rho_{\alpha}^{\pi} \cdot i^{\alpha} \quad \Delta\lambda \tag{8-12}$$

and,

$$c_p^{\pi} \quad \tilde{\Delta} i^p \simeq \delta_p^{\pi} \cdot \tilde{\Delta} i^p \tag{8-13}$$

From equations 8-8, 8-11 and 8-9, it is seen that,

$$(\Delta i^{\pi} + \Gamma^{\pi}_{\alpha\beta} \, i^{\alpha} \quad \Delta x^{\beta}) = c_p^{\pi} \quad (\tilde{\Delta} i^p + \Gamma^{p}_{ab} \, i^a \quad \Delta x^b) \tag{12-9}$$

Using the law of transformation for the affine connection $\alpha\beta$

(equation 7-52):

$$\Gamma^{p}_{ab} = \Gamma^{\pi}_{\alpha\beta} \; C_a^{\alpha} \; C_b^{\beta} \; C_{\pi}^{p} + C_{\sigma}^{p} \; \frac{\partial C_a^{\sigma}}{\partial x^b}$$

With this substitution, equation 12-9 reduces to:

$$C_p^{\pi} \cdot \tilde{\Delta} i^p = \Delta i^{\pi} - C_p^{\pi} \cdot C_{\sigma}^{p} \cdot \frac{\partial C_a^{\sigma}}{\partial x^b} \cdot i^a \quad \Delta x^b$$

$$= \Delta i^{\pi} - \delta_{\sigma}^{\pi} \cdot \frac{\partial C_a^{\sigma}}{\partial x^b} \cdot i^a \quad \Delta x^b$$

i. e.

$$C^{\pi}_{p} \cdot \tilde{\Delta} i^{p} = \Delta i^{\pi} - \frac{\partial C^{\pi}_{a}}{\partial x^{b}} \cdot C^{a}_{\alpha} \cdot i^{\alpha} \quad \Delta x^{b} \qquad (12\text{-}10)$$

Using the transformation tensor, C, given by equation 8-10, in equation 12-10, it follows that:

$$C^{\pi}_{p} \cdot \tilde{\Delta} i^{p} = \Delta i^{\pi} - \rho^{\pi}_{\alpha} \cdot i^{\alpha} \quad \Delta \lambda \qquad (12\text{-}11)$$

In equation 8-10, the angle λ is zero during steady-state and $\Delta\lambda$ during hunting -

$$\text{i. e. } C^{\pi}_{p} = \begin{array}{c|c} & \xrightarrow{p} \\ \downarrow \pi & \begin{array}{|c|c|c|} \hline \cos\Delta\lambda & -\sin\Delta\lambda & \\ \hline \sin\Delta\lambda & \cos\Delta\lambda & \\ \hline & & 1 \\ \hline \end{array} \end{array}$$

$$\simeq \begin{array}{c|c} & \xrightarrow{p} \\ \downarrow \pi & \begin{array}{|c|c|c|} \hline 1 & -\Delta\lambda & \\ \hline \Delta\lambda & 1 & \\ \hline & & 1 \\ \hline \end{array} \end{array}$$

(neglecting second order effects)

$$\therefore C^{\pi}_{p} \cdot \tilde{\Delta} i^{p} = \begin{array}{c|c|c|c} & p & & \\ \hline \pi & 1 & -\Delta\lambda & \\ \hline & \Delta\lambda & 1 & \\ \hline & & & 1 \\ \hline \end{array} \quad \begin{array}{c|c|} p & \\ \hline & \tilde{\Delta}\, i^{dr} \\ \hline & \tilde{\Delta}\, i^{qr} \\ \hline & \Delta p \Theta \\ \hline \end{array}$$

$$\simeq \begin{array}{c|c|} \pi & \\ \hline & \tilde{\Delta}\, i^{dr} \\ \hline & \tilde{\Delta}\, i^{qr} \\ \hline & \Delta\, p \Theta \\ \hline \end{array} = \delta^{\pi}_{p} \cdot \tilde{\Delta}\, i^{p} \qquad (12\text{-}12)$$

(neglecting second order effects)

Equations 12-11 and 12-12 correspond to equations 8-12 and 8-13.

12-4-4 Tensor Equations of Hunting

Equation 8-14 gives the transient equation of an electric machine in tensor notation:

$$v_c = R_{cp} \frac{dx^p}{dt} + L_{cp} \frac{\delta}{\delta t} \left(\frac{dx^p}{dt}\right)$$

$$= R_{cp} \frac{dx^p}{dt} + L_{cp} \frac{d}{dt} \left(\frac{dx^p}{dt}\right) + \Gamma_{nm,c} \frac{dx^n}{dt} \frac{dx^m}{dt} \qquad (8\text{-}14)$$

Taking increments, Δ, in this equation,

$$\Delta v_c = R_{cp} \Delta\left(\frac{dx^p}{dt}\right) + \frac{\partial L_{cp}}{\partial x^k} \frac{d}{dt} \left(\frac{dx^p}{dt}\right) \Delta x^k$$

$$+ L_{cp} \frac{d}{dt} \left(\Delta \frac{dx^p}{dt}\right) + \frac{\partial \Gamma_{nm,c}}{\partial x^k} \frac{dx^n}{dt} \frac{dx^m}{dt} \Delta x^k$$

$$+ \Gamma_{mn,c} \left(\Delta \frac{dx^n}{dt}\right) \frac{dx^m}{dt} + \Gamma_{nm,c} \frac{dx^n}{dt} \left(\Delta \frac{dx^m}{dt}\right)$$

$$(12\text{-}13)$$

When covariant increments, δ, are considered certain regrouping of terms are involved, and this is examined here in Kron's freely rotating reference frame.

Equation 8-16 gives the tensor hunting equation:

$$\delta v_c = \delta \left(R_{cp} \frac{dx^p}{dt} \right) + L_{cp} \frac{\delta}{\delta t} \left\{ \delta \left(\frac{dx^p}{dt} \right) \right\}$$

$$+ K_{mknc} \frac{dx^m}{dt} \frac{dx^n}{dt} \Delta x^k \qquad (8\text{-}16) .$$

where,

$$K_{mknc} = \frac{\partial \Gamma_{nm,c}}{\partial x^k} - \frac{\partial \Gamma_{nk,c}}{\partial x^m} + \Gamma_{pk,c} \Gamma^p_{nm}$$

$$- \Gamma_{pm,c} \Gamma^p_{nk} - 2 \Gamma_{np,c} \Omega^p_{km} \qquad (8\text{-}17)$$

Kron(50) has shown that the tensor equation 8-16 consists of the conventional equation 12-13 with the following terms added and subtracted:

$$\left(\frac{\partial \Gamma_{nk,c}}{\partial x^m} + \Gamma_{pm,c} \Gamma^p_{nk} + 2 \Gamma_{np,c} \Omega^p_{km} \right) \frac{dx^n}{dt} \frac{dx^m}{dt} \Delta x^k$$

It is seen that the bracketed terms above give the negative components of the curvature tensor (equation 8-17).

(i) Voltage Equation:-

In the voltage equation either index m or index n takes the electrical variable, the other being s, the mechanical variable ($x^S = \Theta$).

The term, $\frac{\partial \Gamma_{nk,c}}{\partial x^m}$, is therefore zero since the objects are functions of the angle λ only.

Also,

$$\Gamma_{pm,c} \quad \Gamma^{p} \quad \frac{dx^n}{dt} \frac{dx^m}{dt} \Delta x^k$$

$$= \Gamma_{p(S),c} \qquad u) \quad p\Theta \quad \Delta\lambda$$

(The index u denotes the variable λ ; i.e. $x^u = \lambda$).

Fig. 26 gives the affine connection in Park's axes. It can be transformed to Kron's axes using equation 7-52:

$$\Gamma^{p}_{ab} = \Gamma^{\pi}_{\alpha\beta} C^{\alpha}_{a} C^{\beta}_{b} C^{p}_{\pi} + C^{p}_{\sigma} \frac{\partial C^{\sigma}_{a}}{\partial x^{b}} \qquad (7\text{-}52)$$

$$\Gamma_{p(S),c} \quad \Gamma^{p}_{n(u)} \quad i^{n} p\Theta \quad \Delta\lambda$$

$$= \Gamma_{\pi(S),\gamma} \; C^{\pi}_{p} \; C^{\gamma}_{c} \; \left\{ C^{p}_{\sigma} \frac{\partial C^{\sigma}_{n}}{\partial \lambda} \right\} \; i^{n} \; p\Theta \quad \Delta\lambda \quad .$$

$$= \tfrac{1}{2} (G_{\gamma\pi} - V_{\gamma\pi}) \; C^{\pi}_{p} \; C^{\gamma}_{c} \; \rho^{p}_{n} \; i^{n} \; p\Theta \; \Delta\lambda$$

$$= \tfrac{1}{2} \; (G_{cp} - V_{cp}) \; \rho^{p}_{n} \; i^{n} \; p\Theta \quad \Delta\lambda \qquad (12\text{-}14)$$

Similarly, the term

$$2 \Gamma_{np,c} \; \Omega^{p}_{km} \; \frac{dx^{n}}{dt} \; \frac{dx^{m}}{dt} \; \Delta x^{k}$$

$$= 2 \; \Gamma_{(S)\; p,\, c} \; \Omega^{p}_{(u)m} \; p\Theta \; i^{m} \; \Delta\lambda$$

$$= \tfrac{1}{2} \; (G_{cp} + V_{cp}) \; \rho^{p}_{m} \; i^{m} \; p\Theta \; \Delta\lambda \qquad (12\text{-}15)$$

The term added and subtracted from the conventional equation of hunting is the sum of the terms given by equations 12-14 and 12-15,

i. e.

$$G_{cp} \; \rho^{p}_{n} \; i^{n} \; p\Theta \quad \Delta\lambda \qquad (12\text{-}16)$$

(ii) Torque Equation:-

In the torque equation both indices m and n denote the electrical variables. The term, $\frac{\partial \Gamma_{nk,(S)}}{\partial x^m}$, is therefore zero. Also,

$$\left(\Gamma_{pm,(S)} \Gamma^{p}_{nk} + 2 \Gamma_{np,(S)} \Omega^{p}_{km} \right) \frac{dx^n}{dt} \frac{dx^m}{dt} \Delta x^k$$

$$= \left(\Gamma_{pm,(S)} \Gamma^{p}_{n(u)} + 2 \Gamma_{np,(S)} \Omega^{p}_{(u)m} \right) i^n i^m \Delta x^{(u)}$$

$$= - i^n G_{np} \rho^{p}_{m} i^m \Delta\lambda \qquad (12\text{-}17)$$

This term is added and subtracted from the conventional equation.

(iii) Curvature Tensor:-

In equation 8-17, the negative terms have already been calculated (12-16 and 12-17). The remaining terms are:

$$\frac{\partial \Gamma_{nm,c}}{\partial x^k} + \Gamma_{pk,c} \Gamma^{p}_{nm} \qquad (12\text{-}18)$$

For the voltage equation,

$$\Gamma_{nm,\,c} \quad \frac{dx^{n}}{dt} \quad \frac{dx^{m}}{dt} = G \; i \; p\Theta$$

and,

$$\frac{\partial \Gamma_{nm,\,c}}{\partial x^{(n)}} \quad \frac{dx^{n}}{dt} \quad \frac{dx^{m}}{dt} \; \Delta x^{(n)} = \frac{\partial G}{\partial \lambda} \; i \; p\Theta \quad \Delta\lambda$$

For the torque equation,

$$\Gamma_{nm,\,(S)} \quad \frac{dx^{n}}{dt} \quad \frac{dx^{m}}{dt} = - i_{t} \; G \; i$$

and,

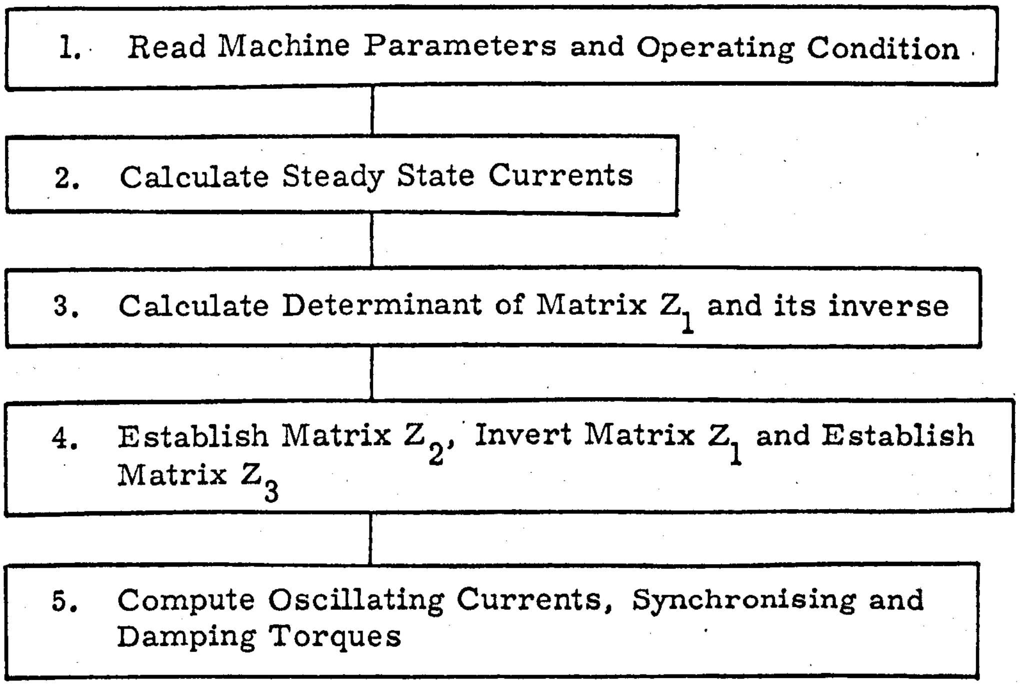
$$\frac{\partial \Gamma_{nm,\,(S)}}{\partial x^{(u)}} \quad \frac{dx^{n}}{dt} \quad \frac{dx^{m}}{dt} \; \Delta x^{(u)} = i_{t} \; \frac{\partial G}{\partial \lambda} \; i \; \Delta\lambda$$

The second term in equation 12-18 is zero for this particular system.

12-5 Appendix 5

Computer Program for Electric Machine Oscillation Study

The program is in Algol language and the computations were carried out on a KDF9 computer. The detailed program follows the flow diagram below:



```
*E43→

begin real delta,emf,line,ra,rf,xd,xq,xf,md,det,id,iq,if,vd,vq,h,sq,
xs,dt,st,p,q,r,s;  integer j,k,m,n;  realarray a,b[1:3,1:3],
c,d,e,f,g[1:3];

xd:=DATA;xq:=DATA;md:=DATA;xf:=DATA;line:=DATA;xs:=xd-xq;
start:  delta:=DATA;emf:=DATA;  ra:=DATA;  rf:=DATA;  h:=DATA;

det:= ra×ra+xd×xq;  det:=1/det;
vd:=-line×sin(delta);  vq:=emf - line×cos(delta);
id:=(ra×vd - xq×vq)×det;  iq:=(xd×vd + ra×vq)×det;
if:=emf/md;
NEWLINE(1);PRINT(if,4,4);PRINT(id,4,4);PRINT(iq,4,4);

p:=ra×ra×rf - h×h×xd×xd×rf - 2×h×h×ra×xd×xf + rf×xd×xd + ra×h×h×md×md
   + ra×h×h×xf×xs + rf×h×h×xd×xs - rf×xd×xs;
q:=h×xf×ra×ra - h×h×h×xf×xd×xd + 2×h×xd×ra×rf - h×md×md×xd
   + h×xf×xd×xd + h×h×h×xd×md×md - rf×ra×h×xs + h×h×h×xd×xf×xs
```

```
   - h×xf×xd×xs - h×h×h×md×md×xs + h×md×md×xs;
sq:=p×p+q×q; sq:=1/sq;  r:=p×sq;  s:=-q×sq;

c[1]:=-h×iq×md×s;  c[2]:=if×md×r + xs×id×r - h×xs×iq×s;
  c[3]:=-h×if×md×s - h×xs×id×s - xs×iq×r;
d[1]:=h×iq×md×r;    d[2]:=if×md×s + h×xs×iq×r + xs×id×s;
  d[3]:=h×if×md×r + h×xs×id×r - xs×iq×s×1;
a[1,1]:=ra×ra+xd×xd-h×h×xd×xd + h×h×xd×xs - xd×xs;
 a[1,2]:=h×h×md×xd - h×h×md×xs;  a[1,3]:=0;
a[2,1]:=-xd×md+h×h×xd×md + md×xs - h×h×md×xs;
  a[2,2]:=ra×rf-h×h×xf×xd + h×h×xf×xs;  a[2,3]:=-rf×xd + rf×xs;
a[3,1]:=ra×md;  a[3,2]:=rf×xd;  a[3,3]:=rf×ra-h×h×xf×xd+h×h×md×md;
b[1,1]:=2×ra×h×xd - h×xs×ra;  b[1,2]:=-h×md×ra;  b[1,3]:=h×md×xd - h×xs×md;
b[2,1]:=-h×md×ra;  b[2,2]:=h×ra×xf+h×rf×xd - rf×h×xs;
  b[2,3]:=-h×xf×xd + h×xf×xs;
b[3,1]:=0;  b[3,2]:=-h×md×md+h×xd×xf;  b[3,3]:=h×xf×ra+h×xd×rf;
 g[1]:=md×iq;   g[2]:=xs×iq;;   g[3]:=md×if + xs×id;
```

```
for m:=1 step 1 until 3 do begin  e[m]:=0; f[m]:=0 end;
for m:=1 step 1 until 3 do begin
for n:=1 step 1 until 3 do begin
e[m]:=e[m]+a[m,n]×c[n]-b[m,n]×d[n];
f[]:=f[m]+a[m,n]×d[n]+b[m,n]×c[n];
end;
NEWLINE(2);PRINT(e[m],4,4);PRINT(f[m],4,4);    end;
st:=md×if×id + xs×id×id - xs×iq×iq;  dt:=0;
for m:=1 step 1 until 3 do begin
st:=st+e[m]×g[m];  dt:=dt+f[m]×g[m];   end;
NEWLINE(3);PRINT(st,4,4);dt:=dt/h;PRINT(dt,4,4);
NEWLINE(6);   goto start ;   end→
```

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