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ELASTIC STABILITY OF FRAMEWORKS

By

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SYNOPSIS

This thesis deals with the elastic instability of frames with complicated sway deformations, subjected to concentrated or distributed external loading and having members of constant or variable flexural rigidity and of trusses having redundant members and supports. Elastic critical loads are predicted by exact and by approximate methods for infinitesimal deformations.

Preface

In this thesis a number of assumptions are made which are normally made in structural analysis. One of them is that the small deflection theory applies. There is an aspect of this assumption which should be considered further by those reading this thesis. Even though the lateral deflections of the individual members is small and hence the individual stiffnesses, carry-over factors etc. are correct, the theory will be in error if the deflections of the joints becomes large as is quite possible near the critical load.

A case in point is Example 1 of Chapter 2 shown in fig. 2.6. The values of the axial forces are determined with the assumption that deflections remain small and these forces yield the critical load given. But if the deflection of the joints are obtained by substitution in equations 2.5a,b,c,d it is seen that they are in fact large. Hence the calculation is in error.

This is not to say that the calculations are therefore of no interest or use. Undoubtedly the axial forces change relatively as the external load is increased, and a calculation ignoring this change is in error. As a step towards the true solution, however, it is possible to calculate the critical load for the particular changes in axial force which would occur if all the deflections did remain small. This is the solution given in Chapter 2.

To obtain the exact solution the very much more difficult mathematics of the elastica and finite joint deflections must be used. This of course is entirely impracticable for real structures. As an alternative, an electronic computer could have been used to obtain a better approximation to the solution.

The intention of this thesis, however, is not to investigate this particular problem more deeply but to concern itself with a wider field. In most structures the translations of the joints is known to be small and in these cases the assumption of small deflections introduces very little error. If application of the external load results in large bending moments appearing in the structure, however, large deflections of the joints may be expected to occur. It is in the case of these structures, like that of fig. 2.6, in which caution must be used in interpreting the results of the calculation.

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Notations

Unless otherwise defined the following notation applies:-

a direction cosine

$A = \alpha/m$

$A' = dA/dP$

$A_i(x) = \frac{1}{\pi}\int_0^\infty \cos(\frac{1}{3}t^3+xt)dt$, Airy integral function

$A'(x) = \frac{d}{dx}(A_i(x))$

b direction cosine

$B = \pi^2\rho/2$

$B_i(x) = \frac{1}{\pi}\int_0^\infty [\sin(\frac{1}{3}t^3+xt) + e^{(-\frac{1}{3}t^3+xt)}]\,dt$, Airy integral function

$B_i'(x) = \frac{d}{dx}(B_i(x))$

c direction cosine in chapters 12 and 13

C_s shear coefficient of prismatic strut under uniform axial load

$C_{s'}$ shear coefficient of prismatic strut under non-uniform axial load

C_b shear coefficient of battened strut

c carry over factor

$c' = dc/dP$

d_1 depth at end 1

E Young's modulus of elasticity

G Modulus of rigidity in chapters 12 and 13

$G = \frac{1}{1 - P/P_c}$, Magnification factor in chapter 8

$G_i(x) = \frac{1}{\pi}\int_0^\infty \sin(\frac{t^3}{3}+xt)dt$, Airy integral function

$G_i'(x) = \frac{d}{dx}(G_i(x))$

H horizontal force component

$\underline{i}$ unit vector

I moment of inertia

$\underline{j}$ unit vector

$J_n(x)$ Bessel function of the first kind of order n

$k = EI/L$, flexural stiffness

$k_G = GI_z/L$, torsional stiffness

$\underline{k}$ unit vector

K stiffness of a complete structure

L, l length

M moment

$(M_F)_1$ fixed end moment at end 1

$m = 1/(1 - \pi^2\rho/2\alpha)$, magnification factor

$\overline{m}$ The power by which the moment of inertia of a non prismatic strut varies along its length

$m_1 = (M_F)_1/WL$

$n = s(1 - m(1+c)/2)$

N number of storeys of a tall building frame

$o = s(-c + m(1+c)/2)$

p distributed axial load per unit length

P_1 axial force at end 1

P_c elastic critical load

$P(x)$ axial force at x

P_e Euler load of members

P_E Euler load of equivalent strut

P_Y Plastic collapse load

qW reaction due to distributed lateral loading W at the end with axial force P_1

Q shear factor due to unit of the ration $\frac{\text{sway}}{\text{length}}$ of the member

R axial force in the rafter

s stiffness factor

$s'=ds/dP$

s_1 modified stiffness of the end with the smallest moment of inertia in chapter 14

modified stiffness of the end with the smallest axial force P_1 in chapter 15

$s''=s(1-c^2)$

sc carry over moment due to unit rotation

$\overline{sc}$ modified carry over moment due to unit rotation

$(sc)'=\frac{d}{dP}(sc)$

T stiffness of a joint in a truss and the "no shear" stiffness of a joint of a portal frame

$u = d_2/d_1$

v shear force in members or vertical force component in frames

w distributed external load per unit length

W external load

x axial coordinate and coordinate axis

y coordinate axis

y(x) deflection at x

$y'(x) = \frac{d}{dx}(y(x))$

$Y_n(x)$ Bessel function of the second kind of order n

z coordinate axis

$\alpha = s(1+c)$

$\alpha' = d\alpha/dP$

$\alpha_1 = s_1 + sc$

θ rotation of a joint

ϕ angle of inclination

Δ determinant

δ sway

$\rho = P/P_e$

∂ increment

γ spring stiffness or slenderness ratio

$\mu = P_1/P_2$

Chapter 1

General introduction

A method has been suggested by Professor W. Merchant[1] that for a particular loading pattern on a structure

$$P_f = f(P_c, P_Y, \eta) \qquad 1.1$$

where

P_f is the failure load of the structure if all the loads are increased slowly and in constant proportion.

P_c is the critical load of the structure i.e the theoretical load at which it would offer no resistance to some applied disturbance if its members remained elastic.

P_Y is the plastic collapse load of the structure i.e the theoretical collapse load if only material properties are taken into account and stability effect ignored.

η is a parameter representing imperfections of manufacture and loading conditions on the structure.

It is the object of this thesis to investigate the elastic stability of structural frameworks under concentrated and distributed external loading. Approximate and exact methods of calculating the elastic critical loads P_c for infinitesimal deformations are established. The influence of

η on the elastic critical load P_c can also be included in the analysis as is demonstrated in chapter 4.

Chapter 2 deals with the stability of symmetrical frames with complicated sways, under symmetrical loading applied at the joints. This has been tackled by establishing two sets of equilibrium equations. The first set of equations gives the axial forces in the members at different stages of loading and the second set of equations gives the stability criterion. The external load which satisfies both sets of equilibrium equations is defined as the critical load.

Chapter 3 is devoted to the approximate estimation of the elastic critical load of multi-bay gabled frames and other more complicated frames. The method described, reduces gabled frames to equivalent rectangular frames with pin joints in the beams. The axial forces in the members were determined approximately by establishing upper and lower bounds to these forces. The method is easy to apply, accurate enough for design purposes and requires little arithmetic to calculate elastic critical loads.

Chapter 4 deals with the elastic instability of unsymmetrical and symmetrical frames under unsymmetrical external loading. This has been tackled by establishing a set of equilibrium equations for determining the forces in the members. Solution of these equations yields a relationship between the external load and the internal axial forces in the members. The maximum external load obtained, is defined as the elastic critical load. This method is also used to investigate the effect of initial imperfection on the elastic critical loads of structures.

Chapter 5 deals with the elastic instability of frames when the influence of pre-buckling deformations is taken into account. The calculations show that an analysis based on this method yields elastic critical loads smaller than those given by the method of chapter 2.

Chapter 6 is devoted to the approximate and exact estimation of elastic critical loads of trusses with redundant members. The approximation has been found to give results within 10% of the exact values. In this chapter it is shown that the elastic critical load of a redundant truss is approximately reached when the forces in the members of the truss are so distributed that as many as possible of the struts meeting at the critical joint of the truss, carry axial forces which are equal proportions of their respective Euler loads.

Chapter 7 deals with the elastic instability of trusses with redundant supports. The approximate method established in chapter 6 is applied to estimate the elastic critical load of trusses with redundant supports. The influence of the redundant reactions at the supports on the critical load is studied.

Chapter 8 deals with the accuracy of Bolton's approximate method. The calculations show that the approximate elastic critical load obtained by Bolton's method is in error by about ± 10%.

Chapter 9 deals with the magnification factor of trusses with redundant members. It is shown that an isolated strut with redundant end restraints has a variable elastic critical load bigger than the initial critical load and lower than the ultimate elastic critical load. Its value depends on the magnitude of initial imperfection in the strut and the external load. In redundant trusses the difference between the first and ultimate critical load is usually small.

Chapter 10 deals with the influence of partial restraint at the supports and the joint connections on the stability of frameworks. The analysis for establishing the relationship between the restraint and the elastic critical load is shown. Calculations show that nearly the full capacity of frames can be obtained by providing moderate restraints at the joint connections and supports.

Chapter 11 deals with the influence of various types of bracing on the stability of portal frameworks. Knee bracing was found to raise the elastic critical of the frames by a large amount. Simple approximate methods for estimating the elastic critical load of knee braced frames are proposed.

Chapter 12 deals with the elastic instability of space frames. In establishing the stability criterion the torsional rigidity of the members was taken into account. The analyses for establishing the stability criteria of several frames with vertical stanchions and one frame with inclined legs are shown.

Chapter 13 deals with the elastic instability of multi-storey portal frames. An approximate method for estimating the critical load of portal frames is proposed. This establishes upper and lower bounds to the elastic critical load. The influence of shear forces on tall portal frames is also investigated. It is shown that an analysis based on orthodox methods, estimates the elastic critical load due to the local failure of the weakest storey.

Chapter 14 deals with the elastic instability of frames having non-uniform members. The moment of inertia of the cross-sections of the members has been taken to vary according to a power $\bar{m}$ of the distance along the members. The standard slope-deflection equation was modified to take into account the varying moment of inertia. This equation indicates new stability functions. A program for the electronic computer was written to tabulate these functions for three values of $\bar{m}$ and four ratios of end depths u. The analyses for determining the elastic critical loads of isolated struts and portal frames is shown. For structure having $\bar{m}$ and u other than those tabulated a graphical technique is suggested.

Chapter 15 deals with the elastic instability of structural frameworks under distribued external loading. The standard slope-deflection equation was again modified, this time to take into account the variation of the axial forces along the members. An expression for the modified stability functions was derived. These functions were tabulated by the computer for different ratios of the end axial forces. The analysis for calculating the elastic critical loads of frames under distributed external loading is shown.

Existing methods of calculating the elastic critical load of portal frames and statically determinate trusses

The methods avaliable for calculating the elastic critical load of structural frameworks are:-

(1) The energy method

(2) The slope-deflection method

(3) The moment distribution method

(4) Modified moment distribution methods:

(a) The successive substitute method

(b) The convergence method

(c) Bolton's method of approximation

(5) Southwell's method

(6) Stiffness method

(1) The energy method

This method is applied to investigate the stability of simple structures and was established by Timoshenko.[2] When a structure carrying a load P less than its critical load P_c, is deformed the bending energy ΔU is greater than the loss in potential energy ΔT. A state of instability is indicated when $\Delta U - \Delta T = 0$.

(2) The slope-deflection method

This method is used mainly in estimating the elastic critical load of frames with complicated sways. An infinitesimal disturbance is applied to the structure which causes rotations at the joints and sways of the members. The moments and forces appearing at the joints are expressed as functions of the deformations and the stability functions of the members, by the modified slope-deflection equation which takes axial forces into account. Equilibrium requires that the total moment and force is zero at the undisturbed joints and infinitesimal at tested joints. This infinitesimal disturbance becomes zero at the critical load. Thus, the determinant Δ formed by the coefficients of the deformations will be the stability criterion. The smallest external load making this determinant vanish will be the critical load.

(3) The moment distribution method

In a moment distribution method, a load factor is chosen, the forces in the members of the structure obtained and hence the value of the stability functions for each member found from tables. An arbitrary moment is applied at any joint and the moments carried over to adjacent joints are obtained. The other joints are balanced in turn as in normal moment distribution, until every joint in the structure except the tested joint is balanced. If the moment carried back to the tested joint is less than the applied moment, the structure is stable.

A higher value of load is now chosen and the process repeated until the moment carried back is equal to the applied disturbing moment. This method is lengthy and requires several complete moment distribution calculations. Also, near the critical load the convergence of calculation becomes very slow and many cycles are required.

(4a) The successive substitute method

This method[3] is used to calculate the elastic critical load of trusses. The fundamental operation is to replace successively each member of a truss by the elastic restraints it provides to the joints it connects. Finally the truss is reduced to one member with elastic restraints at each end and the stability of this member is investigated. This amountsto no more than evaluating the determinant Δ of (2) by systematic physical process. This method can be applied easily to one path trusses but it becomes tedious when the number of joints and paths becomes large.

(4b) Convergence method

Winter and Hoff[4] have described a method of obtaining the value of the elastic critical load using the moment distribution method. In 3 when all the joints except the tested joints are balanced and rebalanced, the end moments will either converge or diverge depending on the external loads. If the process converges to finite values and if the moment remaining at the tested joint does not change sign, the frame is stable. In this case a greater load parameter is chosen and the calculation repeated until the critical load is bracketed. The disadvantage of this method is that a demand for accuracy involves working close to the critical load of the truss. At such a load many cycles of balancing are required before it can be decided whether the moment remaining at the tested joint has become negative or not and whether other joints can be balanced. This method was improved by Dr. A. Bolton.[4] Bolton shown that if disturbing moments corresponding to the joint rotations of the critical mode are applied to the joints of the frame, each joint balanced and carry-over moments calculated, it will be found that exactly the same moments as was originally applied remains at each joint when the frame carries its critical load. Below the critical load every moment is reduced and above the critical load every moment is increased. Since the joint rotations of the critical mode are not avaliable, he proposed as an alternative that an arbitrary pattern of moments should be applied to the loaded frame. Then two or three cycles of balancing and carry-over are carried out. The arithematical total of the balancing moment at every joint is calculated for succeeding cycles. This total will give a satisfactory indication of convergence or divergence of the process.

(4c) Bolton's method of approximation

All the previous methods are exact for calculating the elastic critical load. This is an approximate method. Bolton[5] assumed that only members connected to the least stiff joint of the truss and the joints adjacent to it have a significant effect on the elastic critical load of the truss. In assessing this stiffness, members radiating from joints adjacent to the joint considered are assumed to be fixed at their remote ends. The least stiff joint is located by comparing the stiffness of the joints at the ends of the member which carries the highest fraction of its Euler load. When the rotational stiffness of this joint vanishes, Bolton demonstrates that a good approximation to the elastic critical load of the truss is obtained.

(5) Southwell's method

The southwell plot[6] has been used for the determination of elastic critical loads. The successful application of the Southwell plot depends on the predominant magnification of initial imperfections corresponding to the relevant buckling mode as the critical load is approached. The basic equation of the Southwell plot is

$$\frac{\delta}{\delta_0} = \frac{1}{1 - P/P_c} \qquad 1.2$$

where

δ_0 is initial imperfection at zero load

δ is the imperfection at P load

P_c is the elastic critical load

When δ/δ_o is plotted against $\delta P/\delta_o$, a strainght line is obtained, the inverse gradient of which represent the critical load. The linear relationship holds exactly, provided that the initial imperfection used in deriving the deflected shape under load has the same form as the primary buckling mode. If some other initial form of imperfection is used, the Southwell's plot will not be linear because the exact hyperbolic relationship will not be obtained, although it will be approached with increasing accuracy as the load tends toward the critical load. The ratio δ/δ_o is obtained from the solution of the equilibrium equations given by the slope-deflection equation in which the effects of initial imperfection are included. Two trial loads are required to estimate the critical load.

(6) Stiffness method

This was developed by Dr. D. B. Chandler[1] and is based on the fact that the stiffness of a structure decreases with the increase of the axial forces in its members. At the critical load the stiffness of the structure is zero. Thus it is possible to determine the critical load of the structure by plotting its stiffness $K=Q/\delta$ (where Q is the disturbance and δ the deformation) against the external load. The intercept of the graph on the load axis represents the critical load of the structure.

Chandler has shown that if the disturbance is exciting the primary buckling mode, the stiffness graph will be a straight and if the disturbance is exciting any other buckling mode the stiffness graph will be a curve. The shape of the stiffness curve depends upon the magnitude of the component of the buckling mode present in the disturbance. The larger the component of buckling mode, the smaller is the curvature.

Existing methods of calculating the elastic critical load of redundant trusses

An exact method for calculating the elastic critical load of redundant trusses was developed by Masur.[7] But such calculations are lengthy and complicated for highly redundant trusses. He proposed two theorems which establish a lower and an upper bound to the ultimate elastic critical load of redundant trusses. Usually the gap between the upper bound and lower bound is fairly large and the upper bound is much larger than the ultimate elastic critical load. It is, therefore, advisable to estimate the elastic critical load using the lower bound. He has applied these two theorems to the truss of Figure 1.1. By several trials of arbitrary force distribution and using the convergence process, he obtained $P = 30EI/L^2$ for the lower bound. Applying theorem II, he obtained $P = 75.4EI/L^2$ for the upper bound. This value is much greater than the correct ultimate elastic critical load of $P = 33.7EI/L^2$.

The lower bound theorem will be investigated and it is the object of chapter 6 to improve it to deal with highly redundant trusses with redundant members and redundant supports.

On the analysis of redundant trusses trusses Merchant and Brotton[8] have written a paper on the use of the electronic computer to investigate the force distribution in redundant trusses for different values of the applied external load.

Chapter 2

The elastic instability of frameworks with complicated sways

Introduction

The behaviour of rigidly jointed frameworks is governed by elastic instability, plastic instability and imperfections. This chapter deals with the estimation of the elastic critical load of rigidly jointed, single and multi-bay frameworks with complicated sways and inclined members but confines itself to those cases in which both the geometry and loading are symmetrical. The external loads are assumed to be lumped at the joints.

Owing to the fact that the number of unknowns involved in the analysis increases as the number of joints increases, matrices are used for the determination of the forces in the members and the evaluation of the stability conditions.

The method of analysis presented here is based on the following assumptions:-

(1) A linear relationship exist between the applied actions (moments and forces) and the resulting deformations (rotations and sways). The standard slope-deflection equation which has been used to relate the actions and deformations in the members of the frameworks has been modified to take into account the effect of the axial

forces in the members. For member AB in Figure 2.1 with rotations of both ends and the sway δ, the moments at the ends will be

$$M_{AB} = k\ (s\theta_A + sc\theta_B - \alpha\ \delta/L) \qquad 2.1a$$

and

$$M_{BA} = k\ (sc\theta_A + s\theta_B - \alpha\ \delta/L) \qquad 2.1b$$

The shear force called into play by the moments is

$$v = k/L.\ (\alpha\ \theta_A + \alpha\ \theta_B - 2A\delta/L) \qquad 2.1c$$

where s, c, α, and A are tabulated by Livesley and Chandler (9) in terms of ρ the ratio of the load in the member to its Euler load $\alpha = s(1+c)$ and $k = EI/L$.

The sign convention used regards clockwise rotations of the joints, clockwise moments applied to the ends of the members and sway causing the member as whole to rotate in a clockwise direction as having positive signs. A positive shear force on a member is that which tends to rotate the member in an anti-clockwise direction. This will balance positive bending moments.

(2) Deformations due to the bending moments alone are considered and axial and shear deflections are neglected. For singly connected frames a Williot diagram will be used to relate the sways of the members. For more complicated frameworks where there are three members rigidly connected at some of the joints (even though these are not triangulated) another method is used. This consists of making

vertical and horizontal deflections at each joint, then relating these deflections by considering that the axial deformation of any member is negligible. Usually the Williot diagram when axial shortening is neglected performs the same operation graphically. For member AB in Figure 2.2 subjected to the shown deflections, deformation along AB is

$$(\delta - \delta_o)\cos\phi + (\Delta - \Delta_o)\sin\phi = 0 \qquad 2.2a$$

and the sway of the member is

$$(\delta - \delta_o)\sin\phi - (\Delta - \Delta_o)\cos\phi$$

(3) The deflections of the frameworks are not large enough to change the geometrical configuration of the frame appreciably. For large distortions, the shortening in the member due to bending would also have to be taken into account.

(4) Each member of the framework is assumed to have a constant cross-section.

(5) It is assumed that the material of the framework remains elastic.

Sway freedoms

The number of sway unknowns involved in the analysis depends on the number of joints free to displace, the geometry of the framework (for example if it is symmetrical or unsymmetrical), the pattern of loading (symmetrical or unsymmetrical), and the mode of buckling.

Generally the number of possible deflections of the joints exceeds the number of sway unknowns by the number of the members. For symmetrical frames under symmetrical loads and when considering a symmetrical sway mode there are two cases:-

(a) An odd number of joints

If the possible number of deflections for half the frame, including the central joint is D (the central joint has no horizontal deflection) and the total number of members of half the frame is m, the number of sway components for the whole frame will be 2(D - m).

(b) Even number of joints

The central line will cut one member. If the number of possible deflections for half the frame is D (the two central joints have no horizontal deflection) and the number of members of the half frame including the bisected member is m, the number of sway components for the whole frame will be 2(D - m + 1).

Elastic critical modes

The elastic critical mode is influenced by the geometrical configuration of the framework and the pattern of loading. There are three possible modes.

The anti-symmetrical sway mode is the usual mode of instability in symmetrical frames with symmetrical loading. The symmetrical joints and members have deformations which are equal in value and direction. For unsymmetrical frames the critical mode is usually an unsymmetrical sway mode.

The symmetrical sway mode of instability takes place only in symmetrical frameworks under symmetrical loading. The symmetrical joints and members have deformations which are equal in value and opposite in direction.

In the joint rotation mode the joints do not translate. Instability occurs solely by reason of the rotation of the joints. If the frame is symmetrical and carries symmetrically applied loads, this mode may have either symmetrical or anti-symmetrical rotations of the joints.

Anti-symmetrical sway mode stability criterion

The analysis is based on the fact[10] that the framework at the critical load offers no resistance to any disturbing action (moment or force). The loads on the framework are carried by the components of axial force and bending moment which have been built up by the rotations and sways of the members. An infinitesimal disturbing action is then applied and this will cause new deformations in the framework in such a way as to neutralize, if possible, the effect of the applied disturbance. In this process, the rotations of the joints and the sways of the members will change so that they remain in equilibrium. The size of the new rotations can be obtained from the joint equilibrium equations and the size of the new sways from the sway force equilibrium equations. There is a force corresponding to each sway freedom of the structure. The value of each force can be obtained by statics from the total shear forces in the members involved in each sway mechanism. The sway force multiplied by the displacement of its point of application along its line of action is equal to the sum for all members involved in the mechanism of the shear force in the member multiplied by its sway displacement.

e.g in Figure 2.3

$$F \times \delta = \sum_{i=1}^{3} v_i \delta_i \qquad 2.3$$

where v_i is the shear force in the member due to all rotations and sways and δ_i are the sways for the mechanism considered.

Hence a system of homogeneous linear equations is obtained equal in number to its degree of freedom, and instability is characterized by the vanishing determinant formed by the coefficients of deformations.

Forces in the members

The axial forces in the members must be known so that the values of the stability functions can be obtained from tables. The external loads are supported by the shear forces and the axial forces in the members. When the joints rotate and the members sway, the shear forces change and hence the axial forces in the members also change. For any rotation and sway the modified slope-deflection equation can be used to find the bending moments and hence the shear forces in the members. The joint equilibrium equations and the member equilibrium equations form a system of homogeneous equations containing terms due to the external loads and to the internal force component unknowns. Solution of these equations yields expressions for the values of the force component unknowns at the joints which enable the axial forces in the members to be obtained. These axial forces are thus seen to be functions of the external loads, the physical properties of the members and the stability functions. It is the fact that the axial forces are dependent on the stability functions which makes them difficult to determine. They could be obtained by trial and error.

The member equilibrium equation requires a number of internal reaction components at the joints to balance the axial forces and shears in the members. The expression connecting the value of a typical internal reaction component H with the external load P is

$$H \Delta_1 = P \Delta_2 \qquad 2.4$$

where Δ_1 and Δ_2 are determinants formed by the coefficients of the unknowns and the external loading as is derived in detail later. A value of H/P is assumed, giving the relative forces in the members and then trial values of P are tested to find which value of P satisfies the assumed value of H/P. A new value of H/P is then assumed and the appropriate value of P is obtained. From several such determinations the graph of P against H/P could be drawn. In this graph the maximum value of P obtained is defined as the "symmetrical sway elastic critical load." The anti-symmetrical sway critical load is lower than this value and therefore tests of frame stability must be made at lower values. The graph of P against H/P, however, continues to give the correct axial forces in the members up to the point at which anti-symmetrical sway intervenes and hence the stability tables can be used for the known values of axial force in the members. This procedure is demonstrated in the first numerical example given later. Naturally if there is more than one internal reaction unknowns the process of satisfying several equations would interlock and become very laborious.

The trial and error method of solution also becomes impractically lengthy when the determinants Δ_1 and Δ_2 become large since it involves many cycles of calculation. An alternative numerical technique can be used to find the internal reactions at different stages of loading of the framework. In this process, the following steps are followed:-

(1) The axial forces in the members are assumed to be zero i.e the ρ -values are taken to be zero.

(2) Using these ρ -values the internal reaction unknowns are calculated from equation 2.4.

(3) The axial forces in the members are now determined from statics. One of the members which has an axial force dependent on the internal reaction unknown is chosen as a reference member and it ρ -value is made equal to the value it is intended to test. The ρ -values of all other members are obtained by scaling.

The steps 2 and 3 are repeated until successive values of the internal reaction unknowns remain constant. The number of cycles required will be dependent on the load distribution in the members and whether the value of ρ in the reference member is below or above the critical one. If the reference ρ chosen is below that for the highest possible applied load P, few cycles are required since the calculation converges very rapidly to give the required axial forces in the members, although the number might increases as load approaches the highest value. (This method might fail to give a solution for a reference ρ -value greater than that of the highest load, since the calculation is likely to diverge when one of the determinants approaches zero. Such a test, however, still yield useful information since it indicates an upper bound to the critical load.)

From the value of the internal reaction unknown and the external load a point on the graph can be plotted. To obtain another point on the graph another value of the reference ρ is assumed and the internal reaction unknown is calculated using the relative load parameter of the last test as the initial load distribution. The work involved is very much smaller than before since the relative load parameters will need very little correction. Enough values of the reference ρ are tested until the highest value of the external load is obtained. At this stage of the calculation the axial force in each member for any given external load is known.

If only the elastic critical load given by the anti-symmetrical sway mode is required, the following steps are used:-

(1) The axial forces in the members are assumed to be zero i.e the ρ-values are taken to be zero.

(2) The internal reaction unknowns are calculated from equation 2.4 using these ρ-values.

(3) The relative ρ-values for the members are now calculated from statics.

(4) The critical value of ρ for the refrence member is determined by trial and interpolation using the anti-symmetrical sway mode stability criterion and hence all ρ-values are known. Steps 2, 3 and 4 are repeated and this one repitition will usually give a sufficiently accurate value of the critical load. This process is illustrated by the calculation of the critical load of the two bay gabled frame of

Figure 2.10 given later.

Experimental determination of the elastic critical load

The natural frequency of oscillation of a frame decreases with increasing external load (1). When the vibrational stiffness is plotted against the applied load, an almost straight line relationship is obtained. Curving may occur near the elastic critical load when large deflections begin to have an effect. An estimation of the elastic critical load is made by the extrapolation of the linear part of the graph. In this chapter it is this value of experimental elastic critical load which is compared with the theoretical value.

Theoretical analysis

Frame 1

a) Theoretical analysis of a single bay framework

Since the framework of Figure 2.4 is symmetrical and under symmetrical loading, it will have symmetrical deformations until the anti-symmetrical sway mode interfers. For this frame, the number of sway unknowns is

$$= 2 (3 - 3 + 1)$$

$$= 2$$

These sway unknowns will be taken as the sway of members AB and A'B'. On the application of the external load, the following deformations occur:-

1) Equal and opposite rotation of joints B and B' by θ_B.

2) Equal and opposite rotation of joints C and C' by θ_C.

3) Members AB and A'B' will sway by δ, but in opposite directions.

Members BC and B'C' will sway by $-\delta/\sin\phi$, but in opposite in directions, as is determined from the Williot diagram in Figure 2.4a.

Due to these deformations there will be moments in the members and these are tabulated in Table 2.1.

Joint equilibrium at B and B' requires

$$M_B=\left[(ks)_1+(ks)_2\right]\theta_B+(ksc)_2\theta_C+\left[(k\alpha/L\sin\phi)_2-(k\alpha/L)_1\right]\delta=0 \qquad 2.5a$$

and joint equilibrium at C and C' requires

$$M_C=(ksc)_2\theta_B+\left[(ks)_2+(ks(1-c))_3\right]\theta_C+(k\alpha/L\sin\phi)_2\delta=0 \qquad 2.5b$$

This frame has only one internal reaction force unknown, this is the horizontal force component H, Figure 2.4b. The equilibrium of AB requires that

$$H=(k\alpha/L)_1\theta_B-(2kA/L^2)_1\delta \qquad 2.5c$$

and that of BC requires

$$H\sin\phi-P\cos\phi=(k\alpha/L)_2\theta_B+(k\alpha/L)_2\theta_C+(2kA/L^2\sin\phi)_2\delta \qquad 2.5d$$

Solution of these four equations yields

$$H\,\Delta_1=P\,\Delta_2 \qquad 2.6$$

where

$$\Delta_1=\begin{vmatrix}(ks)_1+(ks)_2 & (ksc)_2 & \left[(k\alpha/L\sin\phi)_2-(k\alpha/L)_1\right]\\ (ksc)_2 & \left[(ks)_2+(ks(1-c))_3\right] & (k\alpha/L\sin\phi)_2\\ \left[(k\alpha/L\sin\phi)_2-(k\alpha/L)_1\right] & (k\alpha/L\sin\phi)_2 & \left[(2kA/L^2)_1+(2kA/L^2\sin^2\phi)_2\right]\end{vmatrix}$$

and

$$\Delta_2=\begin{vmatrix} (ks)_1+(ks)_2 & (ksc)_2 & \left[(k\alpha/L\sin\phi)_2 - (k\alpha/L)_1\right] \\ (ksc)_2 & \left[(ks)_2+(ks(1-c))_3\right] & (k\alpha/L\sin\phi)_2 \\ -(k\alpha/L)_1 & 0 & (2kA/L^2)_1 \end{vmatrix}$$

When the dimensions of the frame and its loading are specified equation 2.6 can be solved numerically to yield a relationship between H and P. The maximum value of P thus obtained, determines the symmetrical sway elastic critical load.

Stability criterion for the anti-symmetrical sway mode

The anti-symmetrical sway will occur at a load lower than the symmetrical sway elastic critical load. In order to establish the condition under which the structure first becomes laterally unstable, it is necessary to consider the equilibrium of the frame in its slightly buckled state. This state can be obtained by superimposing on the symmetrical deformations an infinitesimal anti-symmetrical deformation which corresponds to a set of small variations in the joint rotations and sways of the members. The number of incremental sway unknowns for this mode is

$= 4 \times 2 - 5$

$= 3$

These incremental unknowns will be taken as the incremental sways of AB, BC and A'B'. On the application of infinitesimal equal horizontal forces H_B at joints B and B', the joints and the members deform in the following way:-

1) Equal incremental rotations of joints B and B' by $\partial\theta_B$.

2) Equal incremental rotations of joints C and C' by $\partial\theta_C$.

3) Equal incremental sways of members AB and A'B' by $\partial\delta_1$.

4) Equal incremental sways of members BC and B'C' by $\partial\delta_2$.

Member CC' will sway anti-clockwise by $2\partial\delta_2\cos\phi$ which is obtained from the Williot diagram of Figure 2.5.

Associated with these variation in the deformations are changes in the axial forces but this secondary effect is neglected herein.

Due to these incremental deformations there will be changes in the moments in the members and these are tabulated in Table 2.2. From the consideration of joint equilibrium, the total change of moment at each joint is zero. Hence

$$\partial M_B = \left[(ks)_1 + (ks)_2\right]\partial\theta_B + (ksc)_2\partial\theta_C - (k\alpha/L)_1\partial\delta_1 - (k\alpha/L)_2\partial\delta_2 = 0 \qquad 2.7a$$

$$\partial M_C = (ksc)_2\partial\theta_B + \left[(ks)_2 + (k\alpha)_3\right]\partial\theta_C + \left[(2k\alpha\cos\phi/L)_3 - (k\alpha/L)_2\right]\partial\delta_2 = 0 \qquad 2.7b$$

Due to $\partial\delta_1$ only members AB and A'B' are swayed by $\partial\delta_1$, the shear force equation gives the following relation between H_B and the incremental shear forces in AB and A'B'.

$$2H_B\partial\delta_1 = -\partial\delta_1\left[(k\alpha/L)_1\partial\theta_B - (2kA/L^2)_1\partial\delta_1\right]$$

At the critical load H_B vanihes, hence

$$H_B = -(k\alpha/L)_1\partial\theta_B + (2kA/L^2)_1\partial\delta_1 = 0 \qquad 2.7c$$

Due to $\partial\delta_2$, members BC and B'C' will sway by $\partial\delta_2$ and member

CC' by $2\partial\delta_2\cos\phi$. The shear force sway equation gives

$$2H_C\partial\delta_2\sin\phi = -2\partial\delta_2\left[(k\alpha/L)_2(\partial\theta_B+\partial\theta_C) - (2kA/L^2)_2\partial\delta_2\right] + 2\partial\delta_2\cos\phi\left[(2k\alpha/L)_3\partial\theta_C + (4kA\cos\phi/L^2)_3\partial\delta_2\right]$$

Rearranging this equation and putting $H_C=0$, since the disturbing forces were applied only at B and B', yields

$$H_C\sin\phi = -(k\alpha/L)_2\partial\theta_B + \left[(2k\alpha\cos\phi/L)_3 - (k\alpha/L)_2\right]\partial\theta_C + \left[(2kA/L^2)_2 + (4kA\cos\phi/L^2)_3\right]\partial\delta_2 = 0 \qquad 2.7d$$

These incremental equilibrium equations can be represented in the matrix form as

$$\begin{bmatrix}\partial M_B\\ \partial M_C\\ H_B\\ H_C\sin\phi\end{bmatrix} = \begin{bmatrix}\partial\theta_B\\ \partial\theta_C\\ \partial\delta_1\\ \partial\delta_2\end{bmatrix}\begin{bmatrix}(ks)_1+(ks)_2 & (ksc)_2 & -(\frac{k\alpha}{L})_1 & -(\frac{k\alpha}{L})_2\\ (ksc)_2 & (ks)_2+(k\alpha)_3 & 0 & \left[(\frac{2k\alpha}{L}\cos\phi)_3-(\frac{k\alpha}{L})_2\right]\\ -(\frac{k\alpha}{L})_1 & 0 & (\frac{2kA}{L^2})_1 & 0\\ -(\frac{k\alpha}{L})_2 & \left[(\frac{2k\alpha}{L}\cos\phi)_3-(\frac{k\alpha}{L})_2\right] & 0 & \left[(\frac{2kA}{L^2})_2+(\frac{4kA\cos^2\phi}{L^2})_3\right]\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix} \qquad 2.8$$

The determinant Δ of the square matrix is the anti-symmetrical sway mode stability condition. The load satisfying equation 2.6 and making this determinant vanish is the critical load.

If the rows of the determinant are in the order ($\partial\theta_B$, $\partial\theta_C$, $\partial\delta_1$, $\partial\delta_2$), the columns should be in the order (∂M_B, ∂M_C, H_B, H_C) so that the correct sign of the determinant is obtained. If the assumed load is smaller than the critical one, the determinant will have a positive sign.

Joint rotation mode stability criterion

The criterion for stability in symmetrical joint rotation mode is obtained from Δ_1 by considering only joint rotation and equilibrium and ignoring the sway terms i.e

$$\begin{bmatrix} \partial M_B \\ \partial M_C \end{bmatrix} = \begin{bmatrix} \partial \theta_B \\ \partial \theta_C \end{bmatrix} \begin{bmatrix} (ks)_1 + (ks)_2 & (ksc)_2 \\ (ksc)_2 & (ks)_2 + (ks(1-c))_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad 2.9$$

The criterion for the anti-symmetrical joint rotation mode is similarly obtained from 2.8 by ignoring the coefficients of the sways i.e

$$\begin{bmatrix} \partial M_B \\ \partial M_C \end{bmatrix} = \begin{bmatrix} \partial \theta_B \\ \partial \theta_C \end{bmatrix} \begin{bmatrix} (ks)_1 + (ks)_2 & (ksc)_2 \\ (ksc)_2 & (ks)_2 + (k\alpha)_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad 2.10$$

b) Numerical examples

Example 1

The elastic critical load of the frame in Figure 2.6 is calculated when there are loads only at joints C and C'. The members have equal stiffnesses and Euler loads P_e. In the classical calculation, the following steps are followed:

Step 1

The curve relating the horizontal force component H to the external load is obtained using the numerical technique shown before on page 21. A few values were also calculated by the trial and error process to check the curve given by the numerical technique.

When the lengths and the relative k-values are substituted in equation 2.6, this relationship becomes

$$\frac{H}{P}\cdot\begin{vmatrix} s_1+s_2 & (sc)_2 & 1.414\alpha_2-\alpha_1 \\ (sc)_2 & s_2+(s(1-c))_3 & 1.414\alpha_2 \\ 1.414\alpha_2-\alpha_1 & 1.414\alpha_2 & 2A_1+4A_2 \end{vmatrix} = \begin{vmatrix} s_1+s_2 & (sc)_2 & 1.414\alpha_2-\alpha_1 \\ (sc)_2 & s_2+(s(1-c))_3 & 1.414\alpha_2 \\ -\alpha_1 & 0 & 2A_1 \end{vmatrix} \qquad 2.11$$

The initial value of H/P which corresponds to zero load on the frame is obtained by substituting the values of the stability functions for zero axial load i.e $s=4$, $sc=2$, $\alpha=6$ and $A=6$ in the above equation. This gives

$H/P = 0.485$

With this value of H, the forces in the members and the relative load parameters are calculated. These are tabulated.

Member	AB(1)	BC(2)	CC'(3)
Force	P	1.05P	0.485P
rel. P_e	1	1	1
rel. ρ	1	1.05	0.485

A value of ρ_2 is now assumed and the load parameter ρ of each other member is calculated. Hence when $\rho_2=0.42$, $\rho_1=0.40$ and $\rho_3=0.194$.

From tables(9)

$s_1= 3.4439$ $\alpha_1= 5.5936$ $A_1= 3.6197$

$s_2= 3.4144$ $\alpha_2= 5.5726$ $A_2= 3.5000$ $(sc)_2= 2.1582$

$s_3= 3.7297$ $(sc)_3= 2.0701$

Substituting these values in 2.11 leads to

$H/P = 0.71$

With this new value of H, the forces and the relative load parameters are tabulated

Member	(1)	(2)	(3)
Force	P	1.21P	0.71P
relP_e	1	1	1
rel. ρ	1	1.21	0.71
ρ-values	0.347	0.42	0.246

The same procedure is followed. This leads to

$H/P = 0.734$

This is not far from the value given by the first cycle of calculation and is therefore assumed to be near enough to the exact value. Hence

$$P = \frac{0.42}{1.734 \times 0.707} P_e$$

$$= 0.342\ P_e$$

Following the same procedure, other values of ρ_2 are assumed and the corresponding H-values are calculated. The results obtained are shown in Figure 2.7

Trial and error process

A value of H/P is assumed, say H/P = 1.0. The axial forces and the relative load parameters are calculated. These are tabulated

Member	(1)	(2)	(3)
Force	P	1.414P	P
rel. P_e	1	1	1
rel. ρ	1	1.414	1

The load parameter ρ of the members which result in H/P given by equation 2.11 also being 1.0 is calculated by trial and error process.

First trial

$\rho_1 = 0.34$ $\rho_2 = 1.414 \times 0.34 = 0.48$ $\rho_3 = 0.34$

From tables(9)

$s_{1,3} = 3.5314$ $\alpha_{1,3} = 5.6561$ $A_{1,3} = 3.9782$ $(sc)_{1,3} = 2.1247$

$s_2 = 3.3247$ $\alpha_2 = 5.5093$ $A_2 = 3.1406$ $(sc)_2 = 2.1846$

Substituting these in 2.11, leads to

H/P = 0.93

A higher value of ρ_1 is taken $\rho_1 = 0.36$ and when the calculation is repeated it is found that:

$H/P = 0.99$

By extrapolation the critical ρ_1 is

$$\rho_1 \doteq 0.36 + \frac{0.01 \times 0.02}{0.06}$$
$$\doteq 0.363$$

Likewise the load parameter ρ_1 corresponding to other values of H/P is calculated. These values are also shown on the curve of Figure 2.7

Step 2

The anti-symmetrical elastic critical load is calculated using the stability condition. When the lengths and the relative k-values are substituted in 2.8, the determinant is modified to

$$\begin{vmatrix} s_1+s_2 & (sc)_2 & -\alpha_1 & -\alpha_2 \\ (sc)_2 & s_2+\alpha_3 & 0 & 1.414\alpha_3 - \alpha_2 \\ -\alpha_1 & 0 & 2A_1 & 0 \\ -\alpha_2 & 1.414\alpha_3 - \alpha_2 & 0 & 2A_2+2A_3 \end{vmatrix} = \Delta \qquad 2.12$$

A value of the load parameter ρ_1 which it is hoped will make this determinant vanish is assumed. The value chosen was $\rho_1 = 0.34$. From Figure 2.7, the value of H/P corresponding to this value of ρ_1 is 0.73. The forces in the members and the relative load parameters are therefore

Member	(1)	(2)	(3)
Force	P	1.225P	0.73P
rel. P_e	1	1	1
rel. ρ	1	1.225	0.73
ρ-values	0.34	0.415	0.248

From tables(9)

$s_1 = 3.5314$ $\alpha_1 = 5.6561$ $A_1 = 3.9782$

$s_2 = 3.4144$ $\alpha_2 = 5.5726$ $A_2 = 3.5300$ $(sc)_2 = 2.1582$

$\alpha_3 = 5.7487$ $A_3 = 4.5151$

Substituting these values in equation 2.12 yields

$$\Delta = -100$$

A lower value of $\rho_1 = 0.32$ is therefore tested. Figure 2.7 shows that the value of H/P is 0.63. Thus the load parameters of the other members are $\rho_2 = 0.368$ and $\rho_3 = 0.202$

Substituting the stability functions corresponding to these load parameter in 2.12 gives

$$\Delta = +340$$

Hence the critical load parameter ρ_1 by linear interpolation is

$$\rho_1 \doteq 0.34 - \frac{0.02 \times 100}{440}$$

$$\doteq 0.336$$

Thus the elastic critical load of the framework is

$$2P = 2 \times 0.336\, P_e$$

$$\doteq 0.672 P_e$$

This value corresponds with the value $0.710 P_e$ obtained experimentally from the slope at small deformations on a model of the framework made of bright steel strip members of ½"X 1/6" cross section and 12" length, shown in Figure 2.8. The deflections in this framework became large at an early stage in the test.

Example 2

The elastic critical load of the framework of example 1 is calculated when the external load is applied equally at the four joints of the frame.

Step 1

The curve relating the horizontal force component H to the external load is calculated by trial and error, using equation 2.11 The curve obtained is shown in Figure 2.9. This curve shows that axial force effects have resulted in H being decreased and that the framework has no symmetrical sway mode.

Step 2

The anti-symmetrical elastic load is calculated using the stability condition of equation 2.12. When ρ_1 is assumed to be 0.44, the value of H/P obtained from Figure 2.9 is 0.485. Thus the forces and the relative load parameters are

Member	(1)	(2)	(3)
Force	2P	1.05P	0.485P
rel. P_e	1	1	1
rel. ρ	1	0.525	0.242
ρ -value	0.44	0.231	0.106

From tables(9)

$s_1 = 3.3847$ $\alpha_1 = 5.5516$ $A_1 = 3.3803$

$s_2 = 3.6879$ $\alpha_2 = 5.7692$ $A_2 = 4.6342$ $(sc)_2 = 2.0814$

$\alpha_3 = 5.9006$ $A_3 = 5.3741$ $(sc)_3 = 2.0339$

Substituting these values in 2.12 gives the value of the determinant as

$$\Delta = -66.5$$

A lower value of $\rho_1 = 0.42$ is therefore tested. The value of H/P obtained is again 0.485. The value of the determinant is

$$\Delta = +236$$

By linear interpolation the critical load parameter ρ_1 is

$$\rho_1 \simeq 0.44 - \frac{0.02 \times 66.5}{303.5}$$

$$\simeq 0.436$$

Thus the elastic critical load is

$$4P = 2 \times 0.436\, P_e$$

$$= 0.872 P_e$$

This value corresponds with the value $0.882P_e$ obtained experimentally on a model of the framework, shown in Figure 2.8

Comments on the examples

A careful study of the examples just completed shows that the numerical technique of successive correction, used in obtaining the curve relating H to the external load, converges rapidly to give the exact values of force component after a few cycles of calculation.

In example 1, stability consideration results in an increasing horizontal force component H and a peak value of the external load is attained. This is because the axial force and load parameter ρ of the stanchions are smaller than those of the roof members. The stiffness of the roof members reduces more sharply than those of the stanchions with an increasing external load. Thus the stanchions provide an increasing restraint against lateral displacement, for the roof members. In example 2, the curve relating H and the external load P in Figure 2.9 shows a decreasing horizontal force component with increasing external load and the curve has no peak value. Thus it has no symmetrical sway elastic critical load. This is because the axial force and load parameters of the stanchions are bigger than those of the roof members. With increasing external load, the stiffness of the stanchions is reduced more sharply than that of the roof members and thus less lateral restraint can be provided.

The anti-symmetrical sway elastic critical load of the frames in example 1 and 2, given by the calculation was less than the experimental values by 5.5% and 4.9%. This is because geometrical changes were ignored in establishing the stability criterion and calculating the axial forces in the members.

Frame 2

Two bay gabled frame

The framework of Figure 2.10 will again have symmetrical deformations for the reason given before. The number of sway unknowns is

$$= 2 (5 - 4)$$

$$= 2$$

The sway unknowns will be taken as the sway of AB and A'B'. On the application of the external load, the following deformations occur:

1) Equal and opposite rotations of joints B and B' by θ_B.

2) Equal and opposite rotations of joints C and C' by θ_C.

3) Members AB and A'B' sway by ∂ but in opposite directions.

Members BC and B'C' sway by $- \partial/2\sin\phi$ but in opposite directions.

Members CD and C'D' sway by $+ \partial/2\sin\phi$ but in opposite directions,

as was determined from the Williot diagram of Figure 2.10a.

There is no rotation at joint D since there are equal and opposite deformations in members CD and C'D. Due to the deformations listed above there will be moments in the members and these are tabulated in Table 2.3.

This frame has two internal reaction unknowns. These will be taken as the horizontal force component H and the vertical force component v acting on BC as shown in Figure 2.10b.

Equilibrium of the joints requires that

$$M_B = \left[(ks)_1 + (ks)_2\right]\theta_B + (ksc)_2\theta_C + \left[(k\alpha/2L\sin\phi)_2 - (k\alpha/L)_1\right]\delta = 0 \qquad 2.13a$$

$$M_C = (ksc)_2\theta_B + \left[(ks)_2 + (ks)_3\right]\theta_C + \left[(k\alpha/2L\sin\phi)_2 - (k\alpha/2L\sin\phi)_3\right]\delta = 0 \qquad 2.13b$$

Equilibrium of member AB requires

$$H = (k\alpha/L)_1\theta_B - (2kA/L^2)_1\delta \qquad 2.13c$$

and of BC:

$$H\sin\phi - v\cos\phi = (k\alpha/L)_2(\theta_B + \theta_C) + (kA/L^2\sin\phi)_2\delta \qquad 2.13d$$

and of CD:

$$-H\sin\phi + (P-v)\cos\phi = (k\alpha/L)_3\theta_C - (kA/L^2\sin\phi)_3\delta \qquad 2.13e$$

Solution of these five equations yields

$$H\,\Delta_1 = P\,\Delta_2 \qquad 2.14$$

and

$$v\,\Delta_1 = P\,\Delta_3 \qquad 2.15$$

where

$$\Delta_1 = 2\begin{vmatrix} (ks)_1+(ks)_2 & (ksc)_2 & \left[(k\alpha/2L\sin\phi)_2-(k\alpha/L)_1\right] \\ (ksc)_2 & (ks)_2+(ks)_3 & \left[(k\alpha/2L\sin\phi)_2-(k\alpha/2L\sin\phi)_3\right] \\ \left[(k\alpha/2L\sin\phi)_2-(k\alpha/L)_1\right] & \left[(k\alpha/2L\sin\phi)_2-(k\alpha/2L\sin\phi)_3\right] & \left[(2kA/L^2)_1+(kA/2L\sin\phi)_3 + (kA/2L^2\sin^2\phi)_2\right] \end{vmatrix}$$

$$\Delta_2 = \cot\phi\begin{vmatrix} (ks)_1+(ks)_2 & (ksc)_2 & \left[(k\alpha/2L\sin\phi)_2-(k\alpha/L)_1\right] \\ (ksc)_2 & (ks)_2+(ks)_3 & \left[(k\alpha/2L\sin\phi)_2-(k\alpha/2L\sin\phi)_3\right] \\ -(k\alpha/L)_1 & 0 & (2kA/L^2)_1 \end{vmatrix}$$

and

$$\Delta_3 = \begin{vmatrix} (ks)_1 + (ks)_2 & (ksc)_2 & \left[(k\alpha/2L\sin\phi)_2 - (k\alpha/L)_1\right] \\ (ksc)_2 & (ks)_2 + (ks)_3 & \left[(k\alpha/2L\sin\phi)_2 - (k\alpha/2L\sin\phi)_3\right] \\ \left[(k\alpha/L\sin\phi)_2 - (k\alpha/L)_1\right] & (k\alpha/L\sin\phi)_2 & \left[(2kA/L^2)_1 + (kA/L^2\sin^2\phi)_2\right] \end{vmatrix}$$

When the dimensions of the frame and its loading are specified equations 2.14 and 2.15 can be solved simultaneously to yield relationships between H, v, and P. The maximum value of P thus obtained, determines the symmetrical sway elastic critical load.

Stability criterion for the anti-symmetrical sway mode

The analysis for establishing the anti-symmetrical sway stability criterion will be given briefly. The number of incremental sway unknowns for this mode is

$= 2 \times 5 - 7$

$= 3$

These incremental sway unknowns will be taken as the sway of AB, DE and A'B'. On the application of an infinitesimal disturbing force $2H_D$ at D, the incremental rotation of the joints and the sway of the members will be as follow:-

1) Equal incremental rotations of joints B and B' by $\partial\theta_B$.

2) Equal incremental rotations of joints C and C' by $\partial\theta_C$.

3) Incremental rotation of joint D by $\partial\theta_D$.

4) Equal incremental sways of members AB and A'B' by $\partial\delta_1$.

5) Incremental sway of DE by $\partial\delta_2$.

Equal incremental sways of CB and C'B' by $(\partial\delta_2-\partial\delta_1)/2\sin\phi$.

Equal incremental sways of CD and C'D by $-(\partial\delta_2-\partial\delta_1)/2\sin\phi$. These were determined from the Williot diagram.

Due to these incremental deformations, there will be incremental moments at the ends of the members and these are tabulated in Table 2.4 Following the previous procedure of analysis, equilibrium consideration yields

$$\begin{bmatrix} \partial M_B \\ \partial M_C \\ \partial M_D \\ H_B \\ H_D \end{bmatrix} = \begin{bmatrix} \partial\theta_B \\ \partial\theta_C \\ \partial\theta_D \\ \partial\delta_1 \\ \partial\delta_2 \end{bmatrix} \begin{bmatrix} (ks)_1+(ks)_2 & (ksc)_2 & 0 & \left[\left(\frac{k\alpha}{2L\sin\phi}\right)_2-\left(\frac{k\alpha}{L}\right)_1\right] & -\left(\frac{k\alpha}{2L\sin\phi}\right)_2 \\ (ksc)_2 & (ks)_2+(ks)_3 & (ksc)_3 & \left[\left(\frac{k\alpha}{2L\sin\phi}\right)_2-\left(\frac{k\alpha}{2L\sin\phi}\right)_3\right] & \left[\left(\frac{k\alpha}{2L\sin\phi}\right)_3-\left(\frac{k\alpha}{2L\sin\phi}\right)_2\right] \\ 0 & (ksc)_3 & (ks)_3+\frac{1}{2}(ks)_4 & -\left(\frac{k\alpha}{2L\sin\phi}\right)_3 & \left[\left(\frac{k\alpha}{2L\sin\phi}\right)_3-\left(\frac{k\alpha}{2L}\right)_4\right] \\ \left[\left(\frac{k\alpha}{2L\sin\phi}\right)_2-\left(\frac{k\alpha}{L}\right)_1\right] & \left[\left(\frac{k\alpha}{2L\sin\phi}\right)_2-\left(\frac{k\alpha}{2L\sin\phi}\right)_3\right] & -\left(\frac{k\alpha}{2L\sin\phi}\right)_3 & \left[\left(\frac{2kA}{L^2}\right)_1+\left(\frac{kA}{2L^2\sin^2\phi}\right)_2+\left(\frac{kA}{2L^2\sin^2\phi}\right)_3\right] & -\left[\left(\frac{kA}{2L^2\sin^2\phi}\right)_2+\left(\frac{kA}{2L^2\sin^2\phi}\right)_3\right] \\ -\left(\frac{k\alpha}{2L\sin\phi}\right)_2 & \left[\left(\frac{k\alpha}{2L\sin\phi}\right)_3-\left(\frac{k\alpha}{2L\sin\phi}\right)_2\right] & \left[\left(\frac{k\alpha}{2L\sin\phi}\right)_3-\left(\frac{k\alpha}{2L}\right)_4\right] & -\left[\left(\frac{kA}{2L^2\sin^2\phi}\right)_2+\left(\frac{kA}{2L^2\sin^2\phi}\right)_3\right] & \left[\left(\frac{kA}{2L^2\sin^2\phi}\right)_2+\left(\frac{kA}{2L^2\sin^2\phi}\right)_3+\left(\frac{kA}{L^2}\right)_4\right] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad 2.16$$

The determinant Δ of the square matrix is the anti-symmetrical sway mode stability condition. Any load satisfying equations 2.14 and 2.15 and making this determinant vanish is the critical load.

Numerical example

The elastic critical load of the two bay framework of Figure 2.11 is calculated when all the members have the same length and EI-values and the inclination of the rafters is 45°.

Step 1

When these values are substituted in equations 2.14 and 2.15 the force components expressions become:-

$$\frac{2H}{P}\begin{vmatrix} s_1+s_2 & (sc)_2 & 0.707\alpha_2-\alpha_1 \\ (sc)_2 & s_2+s_3 & 0.707(\alpha_2-\alpha_3) \\ 0.707\alpha_2-\alpha_1 & 0.707(\alpha_2-\alpha_3) & 2A_1+A_3+A_4 \end{vmatrix} = \begin{vmatrix} s_1+s_2 & (sc)_2 & 0.707\alpha_2-\alpha_1 \\ (sc)_2 & s_2+s_3 & 0.707(\alpha_2-\alpha_3) \\ -\alpha_1 & 0 & 2A_1 \end{vmatrix} \qquad 2.17a$$

and

$$\frac{2v}{P}\begin{vmatrix} s_1+s_2 & (sc)_2 & 0.707\alpha_2-\alpha_1 \\ (sc)_2 & s_2+s_3 & 0.707(\alpha_2-\alpha_3) \\ 0.707\alpha_2-\alpha_1 & 0.707(\alpha_2-\alpha_3) & 2A_1+A_3+A_4 \end{vmatrix}$$

$$= \begin{vmatrix} s_1+s_2 & (sc)_2 & 0.707\alpha_2 - \alpha_1 \\ (sc)_2 & s_2+s_3 & 0.707(\alpha_2-\alpha_3) \\ 1.414\alpha_2-\alpha_1 & 1.1.414\alpha_2 & 2A_1+2A_2 \end{vmatrix} \qquad 2.17b$$

The initial value of the force components H and v corresponding to zero external load is calculated by putting s=4, sc=2, α=6 and A=6 in the above equations. This gives

H/P = 0.224

and

v/P = 0.532

Thus the axial forces and the relative load parameters of the members are:

Member	AB(1)	BC(2)	CD(3)	DE(4)
Force	0.532P	0.535P	0.49P	0.936P
rel. P_e	1	1	1	1
rel. ρ	0.57	0.57	0.525	1

Step 2

When the relative lengths, k-values and the angles are substituted in 2.16, the determinant is modified to

$$\begin{vmatrix} s_1+s_2 & (sc)_2 & 0 & \alpha_2-1.414\alpha_1 & -\alpha_2 \\ (sc)_2 & s_2+s_3 & (sc)_3 & \alpha_2-\alpha_3 & \alpha_3-\alpha_2 \\ 0 & (sc)_3 & s_3+\frac{1}{2}s_4 & -\alpha_3 & \alpha_3-0.707\alpha_4 \\ [\alpha_2-1.414\alpha_1] & [\alpha_2-\alpha_3] & -\alpha_3 & 4A_1+2A_2+2A_3 & -2(A_2+A_3) \\ -\alpha_2 & [\alpha_3-\alpha_2] & [\alpha_3-0.707\alpha_4] & -2(A_2+A_3) & 2A_2+2A_3+2A_4 \end{vmatrix} = \Delta \qquad 2.17c$$

The approximate elastic critical load corresponding to the initial relative load parameters is obtained by trial and interpolation.

<u>First trial $\rho_4 = 0.78$</u>

$\rho_{1,2} = 0.445 \qquad \rho_3 = 0.41$

From tables(9)

$s_{1,2} = 3.3847$	$\alpha_{1,2} = 5.5516$	$A_{1,2} = 3.3500$	$(sc)_{1,2} = 2.1669$
$s_3 = 3.4292$	$\alpha_3 = 5.5831$	$A_3 = 3.5598$	$(sc)_3 = 2.1540$
$s_4 = 2.8494$	$\alpha_4 = 5.1838$	$A_4 = 1.3347$	

Substituting these values in 2.17c gives by a lucky chance in this case, the value of the determinant as

$$\Delta = 0$$

Thus the approximate elastic critical load is

$$2P = \frac{0.78 \times 2}{0.936} P_e$$

$$= 1.66\ P_e$$

<u>Step 3</u>

The force components H and v are now recalculated using the approximate critical load parameters obtained in step 2. This immediately gives

$H/P = 0.198$

and

$v/P = 0.505$

The axial forces and the relative load parameters are now:-

Member	(1)	(2)	(3)	(4)
Force	0.505P	0.497P	0.49P	0.99P
rel. P_e	1	1	1	1
rel. ρ	0.51	0.502	0.495	1

Step 4

The elastic critical load parameter ρ corresponding to the revised relative load parameters obtained in step 3 is also obtained by trial and error.

When $\rho_4 = 0.80$ was tested, the value of the determinant was +2830 Another value $\rho_4 = 0.82$ was tested and the value of the determinant was +170. Thus the critical load parameter ρ_4 by linear extrapolation is 0.821 and the elastic critical load is

$$2P = \frac{2 \times 0.821}{0.99} P_e$$

$$= 1.66 P_e$$

This value corresponds with $1.65P_e$ obtained experimentally on a model of the frame.having members lengths of 12", shown in Figure 2.11a

Frame 3

The elastic instability of the frame shown in Figure 2.12 is analysed when the external load is lumped at the joints. This frame will have no deformations until the anti-symmetrical sway mode intervenes. The axial forces in the members can therefore be obtained by statics.

Stability criterion for the anti-symmetrical sway mode

The analysis for establishing the anti-symmetrical sway stability criterion is briefly given. The number of sway unknowns for this mode is

$= 2 \times 4 - 6$

$= 2$

These sway unknowns will be taken as the sway of AB and BC. On the application of an infinitesimal horizontal force $2H_C$ at joint C, the joints and the members will be deformed in the following way:-

1) Joints B and B' rotate clockwise through an angle $\partial\theta_B$.

2) Joints C and C' rotate clockwise through an angle $\partial\theta_C$.

3) Members BC and B'C' will sway by $\partial\delta_1$.

4) Members AB and A'B' will sway by $\partial\delta_2$.

Member BB' will sway by $-2\partial\delta_2\cos\phi$ and member CC' by $-2\partial\delta_1\cos\phi$. Members BC and B'C' will have extra sway of $-\partial\delta_2$. These sways are determined from the Williot diagram shown in Figure 2.13.

Due to these deformations, there will be moments in the members and these are tabulated in Table 2.5.

Equilibrium consideration yields

$$\begin{bmatrix} \partial H_B \\ \partial M_C \\ H_C \\ H_B \end{bmatrix} = \begin{bmatrix} \partial\theta_B \\ \partial\theta_C \\ \partial\delta_1 \\ \partial\delta_2 \end{bmatrix} \begin{bmatrix} [(ks)_1+(k\alpha)_2+(ks)_3] & (ksc)_3 & -(\frac{k\alpha}{L})_3 & [(\frac{k\alpha}{L})_3+(\frac{2k\alpha}{L}\cos\phi)_2-(\frac{k\alpha}{L})_1] \\ (ksc)_3 & (ks)_3+(k\alpha)_4 & [(\frac{2k\alpha}{L}\cos\phi)_4-(\frac{k\alpha}{L})_3] & (\frac{k\alpha}{L})_3 \\ -(\frac{k\alpha}{L})_3 & [(\frac{2k\alpha}{L}\cos\phi)_4-(\frac{k\alpha}{L})_3] & [(\frac{4kA}{L^2}\cos^2\phi)_4+(\frac{2kA}{L^2})_3] & -(\frac{2kA}{L^2})_3 \\ [(\frac{2k\alpha}{L}\cos\phi)_2-(\frac{k\alpha}{L})_1] & (\frac{2k\alpha}{L}\cos\phi)_4 & (\frac{4kA}{L^2}\cos^2\phi)_4 & [(\frac{4kA}{L^2}\cos^2\phi)_2+(\frac{2kA}{L^2})_1] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad 2.18$$

The determinant Δ of the square matrix is the stability criterion. The load making this determinant vanish is the critical load.

<u>Numerical example</u>

The elastic critical load of the structural framework of Figure 2.14 is calculated when equal loads P are applied to the joints. The members have constant EI-values. All the members have equal lengths L except BB' which has a length 2L. Thus member BB' has a stiffness $\frac{1}{2}k$ where k is the stiffness of the other members. The angle of inclination ϕ is 60°. The axial forces and the relative load parameters are:

Member	AB(1)	BB'(2)	BC(3)	CC'(4)
Force	$2P/\sin 60^\circ$	$P\cot 60^\circ$	$P/\sin 60^\circ$	$P\cot 60^\circ$
rel. P_e	1	1	1	1
rel. ρ	1	1	0.5	0.25

When the relative k-values and lengths are substituted in 2.18 the determinant Δ is modified to

$$\begin{vmatrix} s_1+0.5\alpha_2+s_3 & (sc)_3 & -\alpha_3 & \alpha_3+0.25\alpha_2-\alpha_1 \\ (sc)_3 & s_3+\alpha_4 & \alpha_4-\alpha_3 & \alpha_3 \\ -\alpha_3 & \alpha_4-\alpha_3 & A_4+2A_3 & -2A_3 \\ 0.25\alpha_2-\alpha_1 & \alpha_4 & A_4 & A_2/8+2A_1 \end{vmatrix} = \Delta \qquad 2.19$$

The load parameter ρ_1 making Δ vanish is obtained by trial and error.

<u>First trial $\rho_1 = 0.92$</u>

$\rho_2 = 0.92 \quad \rho_3 = 0.46 \quad \rho_4 = 0.23$

From tables(9)

$s_{1,2} = 2.6100 \quad \alpha_{1,2} = 5.0264 \quad A_{1,2} = 0.4864$

$s_3 = 3.3548 \quad \alpha_3 = 5.5305 \quad A_3 = 3.2605 \quad (sc)_3 = 2.1757$

$\alpha_4 = 5.7692 \quad A_4 = 4.6343$

Substituting these values in 2.19 leads to

$$\Delta = -757$$

i.e the frame is unstable and a lower value of ρ_1 is therefore tested.

When $\rho_1 = 0.84 \quad \Delta = -87$

When $\rho_1 = 0.83 \quad \Delta = 0$

Thus the elastic critical load is

$$4P = 2 \times \sin 60 \times 0.83\ P_e$$

$$= 1.435P_e$$

This value corresponds with $4P = (32/22)P_e = 1.45P_e$ obtained experimentally on a model of the frame.

• • •

Frame 4

The elastic instability of the framework in Figure 2.15 is analysed. The framework will have symmetrical deformations until the anti-symmetrical sway mode intervenes. The number of sway unknowns in the symmetrical sway mode is

$$= 2\ (5 - 3)$$

$$= 4$$

These sway unknowns will be taken as the sway of members AB,CB C'B' and A'B'. On the application of the external load, the following deformations occur:-

1) Equal and opposite rotations of joints B and B' by θ_B.
2) Equal and opposite rotations of joints C and C' by θ_C.
3) Equal and opposite sways of AB and A'B' by δ_1.
4) Equal and opposite sways of BC and B'C' by δ_2.

Members CD and C'D sway by $-(\delta_1+\delta_2\sin\phi_1)/\sin\phi_2$ but in opposite directions as is seen from the Williot diagram of Figure 2.15a. Joint D will not rotate because of symmetry. Due to these deformations, there will be moments in the members, these are tabulated in Table 2.6

Joint equilibrium at B and B' requires

$$M_B = \left[(ks)_1 + (ks)_2\right]\theta_B + (ksc)_2\theta_C - (k\alpha/L)_1\delta_1 - (k\alpha/L)_2\delta_2 = 0 \qquad 2.20a$$

and joint equilibrium at C and C' requires

$$M_C = (ksc)_2\theta_B + \left[(ks)_2 + (ks)_3\right]\theta_C + (k\alpha/L)_3(\delta_1 + \delta_2\sin\phi_1)/\sin\phi_2 - (k\alpha/L)_2\delta_2 = 0 \qquad 2.20b$$

The frame has only one internal reaction unknown, this is the horizontal force component H in Figure 2.16. The equilibrium of AB reqires that

$$H = (k\alpha/L)_1(\theta_B - 2\delta_1/m_1L_1) \qquad 2.20c$$

and that of BC requires

$$H\sin\phi_1 - P(1+\mu)\cos\phi_1 = (k\alpha/L)_2(\theta_B + \theta_C - 2\delta_2/m_2L_2) \qquad 2.20d$$

and that of CD requires

$$H\sin\phi_2 - P\cos\phi_2 = (k\alpha/L)_3(\theta_C + 2(\delta_1 + \delta_2\sin\phi_1)/m_3L_3\sin\phi_2) \qquad 2.20e$$

Solution of these five equations yields

$$H\,\Delta_1 = P\,\Delta_2 \qquad 2.21$$

where

$$\Delta_1 = \begin{vmatrix} (ks)_1+(ks)_2 & (ksc)_2 & -\left(\frac{k\alpha}{L}\right)_1 & -\left(\frac{k\alpha}{L}\right)_2 & 0 \\ (ksc)_2 & (ks)_2+(ks)_3 & \left(\frac{k\alpha}{L}\right)_3\frac{1}{\sin\phi_2} & \left[\left(\frac{k\alpha}{L}\right)_3\frac{\sin\phi_1}{\sin\phi_2} - \left(\frac{k\alpha}{L}\right)_2\right] & 0 \\ -\left(\frac{k\alpha}{L}\right)_1 & 0 & \left(\frac{2kA}{L^2}\right)_1 & 0 & 1 \\ -\left(\frac{k\alpha}{L}\right)_2 & -\left(\frac{k\alpha}{L}\right)_2 & 0 & \left(\frac{2kA}{L^2}\right)_2 & \sin\phi_1 \\ 0 & -\left(\frac{k\alpha}{L}\right)_3 & -\left(\frac{2kA}{L\sin\phi_2}\right)_3 & -\left(\frac{2kA}{L^2}\right)_3\frac{\sin\phi_1}{\sin\phi_2} & \sin\phi_2 \end{vmatrix}$$

and

$$\Delta_2 = \begin{vmatrix} (ks)_1+(ks)_2 & (ksc)_2 & -\left(\frac{k\alpha}{L}\right)_1 & -\left(\frac{k\alpha}{L}\right)_2 & 0 \\ (ksc)_2 & (ks)_2+(ks)_3 & \left(\frac{k\alpha}{L}\right)_3 \frac{1}{\sin\phi_2} & \left[\left(\frac{k\alpha}{L}\right)_3 \frac{\sin\phi_1}{\sin\phi_2} - \left(\frac{k\alpha}{L}\right)_2\right] & 0 \\ -\left(\frac{k\alpha}{L}\right)_1 & 0 & \left(\frac{2kA}{L^2}\right)_1 & 0 & 0 \\ -\left(\frac{k\alpha}{L}\right)_2 & -\left(\frac{k\alpha}{L}\right)_2 & 0 & \left(\frac{2kA}{L^2}\right)_2 & (1+\mu)\cos\phi_1 \\ 0 & -\left(\frac{k\alpha}{L}\right)_3 & -\left(\frac{2kA}{L^2}\right)_3 \frac{1}{\sin\phi_2} & -\left(\frac{2kA}{L^2}\right)_3 \frac{\sin\phi_1}{\sin\phi_2} & \cos\phi_2 \end{vmatrix}$$

Equation 2.21 can be solved numerically to yield a relationship between H and P when the dimensions of the frame and its loading are specified.

Stability criterion for the anti-symmetrical sway mode

The analysis for establishing the anti-symmetrical sway stability criterion will be given briefly. The number of incremental sway unknowns for this mode is

$$= 2 \times 5 - 6$$

$$= 4$$

These incremental sway unknowns are taken as the sway of AB, BC C'B' and A'B'. On the application of an infinitesimal horizontal forces H_B at joints B and B', the joints and the members will deform in the following way:-

1) Equal incremental rotations of joints B and B' by $\partial\theta_B$.

2) Equal incremental rotations of joints C and C' by $\partial\theta_C$.

3) Incremental rotation of joint D by $\partial\theta_D$.

4) Equal incremental swaysof members AB and A'B' by $\partial\delta_1$.

5) Equal incremental sways of members BC and B'C' by $\partial\delta_2$.

Equal incremental sways of members CD and C'D by $-\partial\delta_2 \cos\phi_1/\cos\phi_2$ which is determined from the Williot diagram of Figure 2.17.

Due to these incremental deformations, there will be changes in the moments at the ends of the members and these are tabulated in Table 2.7

Equilibrium consideration yields

$$\begin{bmatrix} \partial M_B \\ \partial M_C \\ \partial M_D \\ H_B \\ H_C \end{bmatrix} = \begin{bmatrix} \partial\theta_B \\ \partial\theta_C \\ \partial\theta_D \\ \partial\delta_1 \\ \partial\delta_2 \end{bmatrix} \begin{bmatrix} [(ks)_1+(ks)_2] & (ksc)_2 & 0 & -(k\alpha/L)_1 & -(k\alpha/L)_2 \\ (ksc)_2 & [(ks)_2+(ks)_3] & (ksc)_3 & 0 & (k\alpha/L)_3 \frac{\cos\phi_1}{\cos\phi_2} - (k\alpha/L)_2 \\ 0 & (ksc)_3 & (ks)_3 & 0 & (k\alpha/L)_3 \frac{\cos\phi_1}{\cos\phi_2} \\ -(k\alpha/L)_1 & 0 & 0 & (2kA/L^2)_1 & 0 \\ -(k\alpha/L)_2 & \left[(k\alpha/L)_3 \frac{\cos\phi_1}{\cos\phi_2} - (\frac{k\alpha}{L})_2\right] & \left[(\frac{k\alpha}{L})_3 \frac{\cos\phi_1}{\cos\phi_2}\right] & 0 & (2kA/L^2)_3(\cos\phi_1/\cos\phi_2)^2+(2kA/L^2)_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad 2.22$$

The determinant Δ of the square matrix is the anti-symmetrical sway mode stability criterion. The load satisfying equation 2.21 and making the determinant vanish is the critical load.

Numerical example

The elastic critical load of the framework in Figure 2.18 is estimated when all the members have the same flexural rigidity EI, $\phi_1=60^\circ$, $\phi_2=30^\circ$, $\mu=2$ and $L_3=L_2=L$ and $L_1= 0.459L$. Thus $k_2=k_3=k$ and $k_1=2.18k$. On substituting the relative k-values and the lengths in 2.21 and ignoring the effect of stability i.e putting s=4, sc=2, $\alpha=6$ and A=6, the initial value of H turns out to be

H/P = 1.56

From this value of H and the known vertical force components, the axial forces in the members are found. The axial forces, the relative Euler loads and the relative load parameter of the members are next tabulated.

Member	(1)	(2)	(3)
Force	3P	3.375P	1.85P
rel. P_e	4.75	1	1
rel. ρ	1	5.35	2.94

When the relative k-values and the lengths are substituted in the stability criterion 2.22, the determinant is modified to

$$\begin{vmatrix} 2.18s_1+s_2 & (sc)_2 & 0 & -2.18\alpha_1 & -\alpha_2 \\ (sc)_2 & s_2+s_3 & (sc)_3 & 0 & \alpha_3/1.73-\alpha_2 \\ 0 & (sc)_3 & s_3 & 0 & \alpha_3/1.73 \\ -\alpha_1 & 0 & 0 & 2A_1 & 0 \\ -\alpha_2 & \alpha_3/1.73-\alpha_2 & \alpha_3/1.73 & 0 & 2A_2+\frac{2}{3}A_3 \end{vmatrix} \qquad 2.23$$

The critical load making the determinant vanish is obtained by trial and error.

First trial $\rho_2 = 0.42$

$\rho_1 = 0.0785$ $\quad$ $\rho_3 = 0.231$

$s_1 = 3.8936$ $\quad$ $\alpha_1 = 5.9206$ $\quad$ $A_1 = 5.5258$

$s_2 = 3.4144$ $\quad$ $\alpha_2 = 5.5722$ $\quad$ $A_2 = 3.5000$ $\quad$ $(sc)_2 = 2.1582$

$s_3 = 3.6879$ $\quad$ $\alpha_3 = 5.7692$ $\quad$ $A_3 = 4.6342$ $\quad$ $(sc)_3 = 2.0814$

On substituting these values into 2.23, the value of the determinant is

$\Delta = +64$

i.e the frame is stable. A higher value of ρ_2 is therefore tested.

When $\rho_2 = 0.46$ $\qquad$ $\Delta = -447$

A sufficiently close value of ρ_2 can be obtained by linear interpolation

$$\rho_2 \doteq 0.42 + \frac{0.04 \times 64}{511}$$

$$\doteq 0.425$$

At this stage the approximate value of the axial forces in the members are known and the corresponding stability functions can be read from the stability tables. The same procedure is followed to give:

$H/P = 1.575$

The new value of H is nearly the same as before when the effect stability was neglected and the critical load parameter ρ_2 will be more or less unchanged. The critica l load is therefore

$$6P = \frac{0.425 \times 6}{3.375} P_e$$
$$= 0.755 P_e$$

where P_e is the Euler load of BC.

In an experiment carried out on a model steel frame having L=12" the elastic critical load for this model was found to be $0.819P_e$ giving an error of 7.8%.

• • •

<u>Frame 5</u>

The elastic instability of the frame in Figure 2.19 is analyzed. The framework will have symmetrical deformations until the anti-symmetrical sway mode intervenes. The number of sway unknowns for the symmetrical sway mode is

$$= 2\ (6 - 5)$$
$$= 2$$

These sway unknowns will be taken as the sway of members AB and A'B'. On the application of the external load, the following deformations occur:-

1) Equal and opposite rotations of joints B and B' by θ_B.

2) Equal and opposite rotations of joints E and E' by θ_E.

3) Equal and opposite sways of members AB and A'B' by δ.

Equal and opposite sways of members BC and B'C' by $-\delta\sin\phi_1/\sin\phi_2$.

Equal and opposite sways of members ED and E'D' by $-\delta\sin\phi_1\cot\phi_2$.

Equal and opposite sways of members EB and E'B' by $-\delta\sin\phi_1$, are determined from the Williot diagram of Figure 2.19a.

Due to these deformations there will be moments in the members and these are tabulated in Table 2.8.

There are three internal reaction unknowns. These unknowns will be taken as the shear force v and the axial force H_1 in member ED and the horizontal force component H_2 at joint A. Equilibrium consideration yields

$$\begin{bmatrix} \theta_B \\ \theta_E \\ \delta \\ v \\ H_1 \\ H_2 \end{bmatrix} \begin{bmatrix} [(ks)_1+(ks)_5+(ks)_2] & (ksc)_5 & [(\frac{k\alpha}{L})_2\frac{\sin\phi_1}{\sin\phi_2}+(\frac{k\alpha}{L})_5\sin\phi_1-(\frac{k\alpha}{L})_1] & 0 & 0 & 0 \\ (ksc)_5 & (ks)_4+(ks)_5 & [(\frac{k\alpha}{L})_4\sin\phi_1\cot\phi_2+(\frac{k\alpha}{L})_5\sin\phi_1] & 0 & 0 & 0 \\ (\frac{k\alpha}{L})_5 & (\frac{k\alpha}{L})_5 & [(\frac{2kA}{L^2})_5\sin\phi_1] & 0 & -1 & 0 \\ 0 & (\frac{k\alpha}{L})_4 & [(\frac{2kA}{L^2})_4\sin\phi_1\cot\phi_2] & -1 & 0 & 0 \\ (\frac{k\alpha}{L})_1 & 0 & -(\frac{2kA}{L^2})_1 & 0 & 0 & -\sin\phi_1 \\ (\frac{k\alpha}{L})_2 & 0 & (\frac{2kA}{L^2})_2\frac{\sin\phi_1}{\sin\phi_2} & \cos\phi_2 & \sin\phi_2 & -\sin\phi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -3P\cos\phi_1 \\ -P\cos\phi_2 \end{bmatrix} \qquad 2.24$$

Solution of 2.24 yields the relationship between the external load and the internal reaction unknowns.

Stability criterion for the anti-symmetrical sway mode

The number of incremental sway unknowns for this mode is

= 2 X 6 - 9

= 3

These incremental sway unknowns will be taken as the incremental sways of AB, A'B' and the horizontal deflection at joint E. On the application of an infinitesimal horizontal force $2H_E$ at joint E, the joints rotations and the sways change by the following way:-

1) Equal incremental rotations of joints B and B' by $\partial\theta_B$.

2) Incremental rotation of joint C by $\partial\theta_C$.

3) Incremental rotation of joint D by $\partial\theta_D$.

4) Equal incremental rotations of joints E and E' by $\partial\theta_E$.

5) Equal incremental sways of members AB and A'B' by $\partial\delta_1$.

6) Incrementa l displacement at joint E by $\partial\delta_2$.

There also occur, as obtained from the Williot diagram of Figure 2.19c:

Equal incremental sways of members BC and B'C' by $\partial\delta_2\cos\phi_1/\cos\phi_2$.

Incremental sway of CD by $(\partial\delta_2-\partial\delta_1)(\sin\phi_1-\phi_2)/\cos\phi_2$.

Equal incremental sways of members ED and E'D by $\partial\delta_1\cos\phi_1$.

Equal incremental sways of members EB and E'B' by $\partial\delta_2-\partial\delta_1\sin\phi_1$.

$$
\begin{bmatrix} \partial M_B \\ \partial M_C \\ \partial M_D \\ \partial M_E \\ H_B \\ H_E \end{bmatrix}
=
\left[\begin{matrix}
(ks)_1+(ks)_2+(ks)_5 & (ksc)_2 & 0 & (ksc)_5 \\
2(ksc)_5 & (2ks)_2+(ks)_3 & (ksc)_3 & 0 \\
0 & (ksc)_3 & (ks)_3+(2ks)_4 & 2(ksc)_4 \\
(ksc)_5 & 0 & (ksc)_4 & (ks)_4+(ks)_5 \\
\left[\left(\frac{k\alpha}{L}\right)_2\frac{\cos\phi_1}{\cos\phi_2}-\left(\frac{k\alpha}{L}\right)_1\right] & \left[\left(\frac{k\alpha}{L}\right)_2\frac{\cos\phi_1}{\cos\phi_2}-\left(\frac{k\alpha}{2L}\right)_3\tan\phi_2\cos\phi_1\right] & \left[\left(\frac{k\alpha}{L}\right)_4\cos\phi_1-\left(\frac{k\alpha}{2L}\right)_3\cos\phi_1\tan\phi_2\right] & \left(\frac{k\alpha}{L}\right)_4\cos\phi_1 \\
-\left(\frac{2k\alpha}{L}\right)_5 & -\left(\frac{k\alpha}{L}\right)_3 & -\left(\frac{k\alpha}{L}\right)_3 & -\left(\frac{2k\alpha}{L}\right)_5
\end{matrix}\right.
$$

$$
\left.\begin{matrix}
\left(\frac{k\alpha}{L}\right)_2\frac{\cos\phi_1}{\cos\phi_2}+\left(\frac{k\alpha}{L}\right)_5\sin\phi_1-\left(\frac{k\alpha}{L}\right)_1 & -\left(\frac{2k\alpha}{L}\right)_5 \\
2\left(\frac{k\alpha}{L}\right)_2\frac{\cos\phi_1}{\cos\phi_2}+\left(\frac{k\alpha}{L}\right)_3\frac{\sin(\phi_1-\phi_2)}{\cos\phi_2} & -\left(\frac{k\alpha}{L}\right)_3 \\
2\left(\frac{k\alpha}{L}\right)_4\cos\phi_1+\left(\frac{k\alpha}{L}\right)_3\frac{\sin(\phi_1-\phi_2)}{\cos\phi_2} & -\left(\frac{k\alpha}{L}\right)_3 \\
\left(\frac{k\alpha}{L}\right)_4\cos\phi_1+\left(\frac{k\alpha}{L}\right)_5\sin\phi_1 & -\left(\frac{k\alpha}{L}\right)_5 \\
-\left[\left(\frac{4kA}{L^2}\right)_5\sin\phi_1+\left(\frac{2kA}{L^2}\right)_3\frac{\sin(\phi_1-\phi_2)}{\cos\phi_2}\right. & \left(\frac{kA}{L^2}\right)_3\tan\phi_2\cos\phi_1 \\
\left[\left(\frac{2kA}{L^2}\right)_4\cos^2\phi_1+\left(\frac{2kA}{L^2}\right)_2\left(\frac{\cos\phi_1}{\cos\phi_2}\right)^2-\left(\frac{kA}{L^2}\right)_3\tan\phi_2\frac{\sin(\phi_1-\phi_2)}{\cos\phi_2}\cos\phi_1+\left(\frac{2kA}{L^2}\right)_1\right] & \left[\left(\frac{4kA}{L^2}\right)_5+\left(\frac{2kA}{L^2}\right)_3\right]
\end{matrix}\right]
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
\qquad 2.25
$$

Due to these incremental deformations, there will be changes in the moments at the ends of the members and there are tabulated in Table 2.9.

Equilibrium consideration yields 2.25. The determinant of the square matrix is the anti-symmetrical sway mode stability criterion. Any load satisfying 2.24 and making the determinant vanish is the critical load.

Numerical example

The elastic critica l load of the framework is estimated when $\phi_1=60^\circ$ and $\phi_2=30^\circ$. All the members have the same EI-values. The lengths and the relative k-values are tabulated

Member	AB(1)	BC(2)	CD(3)	DE(4)	EB(5)
length	1.5L	2L	L	1.73L	2L
rel.k	1.33	1	2	1.152	1

Substituting these values in 2.24 and ignoring the effect of the axial forces in the members by making s=4, sc=2, $\alpha=6$ and A=6 gives

$H_1 = v = 0$

and

$H_2/P = 1.73$

The axial forces, relative Euler loads and the relative load parameters of the members are:-

Member	(1)	(2)	(3)	(4)	(5)
Force	3.46P	2P	P	0	P
rel. P_e	1.78	1	4	1.33	1
rel. ρ	1.95	2	0.25	0	1

When the lengths and the relative k-values are substituted in 2.25, the determinant is modified to

$$\begin{vmatrix} 1.33s_1+s_2+s_5 & (sc)_2 & 0 & (sc)_5 & \alpha_2+1.5\alpha_5-3.07\alpha_1 & -\alpha_5 \\ (sc)_2 & s_2+s_3 & (sc)_3 & 0 & \alpha_2+2\alpha_3 & -2\alpha_3 \\ 0 & (sc)_3 & s_3+1.152s_4 & 1.152(sc)_4 & 1.152\alpha_4+2\alpha_3 & -2\alpha_3 \\ (sc)_5 & 0 & 1.152(sc)_4 & 1.152s_4+s_5 & 1.152\alpha_4+1.5\alpha_4 & -\alpha_5 \\ \alpha_2-3.07\alpha_1 & \alpha_2-\alpha_3 & 1.152\alpha_4-\alpha_3 & 1.152\alpha_4 & [2.3A_4+2A_2+4A_3+14.18A_1] & 4A_3 \\ -\alpha_5 & -2\alpha_3 & -1\alpha_3 & -\alpha_5 & -[3A_5+8A_3] & [2A_5+8A_3] \end{vmatrix} = \Delta \qquad 2.26$$

The load making this determinant vanish is obtained by trial and error.

<u>First trial $\rho_2 = 0.88$</u>

$\rho_1=0.866$ $\rho_2=0.88$ $\rho_3=0.11$ $\rho_4=0$ $\rho_5=0.44$

From tables(9)

$s_1= 2.7141$	$\alpha_1=5.0943$	$A_1=0.8504$	
$s_2= 2.6797$	$\alpha_2=5.0718$	$A_2=0.7292$	$(sc)_2= 2.3921$
$s_3= 3.8532$	$\alpha_3=5.8906$	$A_3=5.3478$	$(sc)_3=2.0375$
$s_4= 4.0000$	$\alpha_4=6.0000$	$A_4=6.0000$	$(sc)_4=2.0000$
$s_5=3.3847$	$\alpha_5=5.5516$	$A_5=3.3803$	$(sc)_5=2.1669$

Substituting these values in 2.26 gives

$$\Delta = -790000$$

Second trial $\rho_1 = 0.78$ gives $\Delta = -310000$

Third trial $\rho_1 = 0.74$ gives $\Delta = -545$

Hence the approximate critical load parameter is $\rho_1 = 0.74$. At this stage the approximate values of the axial forces are known and the corresponding stability functions can be read from the stability tables and are substituted in 2.24. This leads to

$H_1 = v = 0$

and

$H_2/P = 1.73$

There is no change in the H_2-value. Thus the elastic critical load parameter is $\rho_1 = 0.74$ and the elastic critical load is

$$6P = \frac{6 \times 0.74}{3.46} P_e$$

$$= 1.28\ P_e$$

where P_e is the Euler load of AB.

This value corresponds with $6P = (54.2/39)P_e = 1.39P_e$ obtained experimentally on the model of the frame having an EI-value of 320 lb.in^2

Chapter 3

An Approximate method of estimating the Elastic Critical Load of Frames with Vertical Stanchions and Inclined Rafters.

1. Introduction

In the past few years many rapid methods for the calculation of the elastic critical loads of rectangular portal frames have been developed. More difficulty arises when dealing with frames having inclined members. This is because redistribution of the axial forces in the members occurs as the loading on the frame is increased, and it is necessary to know these axial forces before the elastic critical load can be calculated. Accordingly a method is needed by which these axial forces can be calculated, exactly or approximately, for any externally applied loads. This additional step must be carried out before a test can be made of the stability of the structure. A further difficulty arising in frames with inclined members, is that it may be necessary to consider several possible sway modes. Experience shows, however, that the anti-symmetrical sway mode is the most important. Accordingly this mode is the only one considered in this chapter.

The approximations described in this chapter reduce gable frames to equivalent rectangular portal frames with pin joints in the beam. Only a little preliminary arithmetic is needed to do this and the determination of the elastic critical load of the equivalent portal is very rapid. Multi-bay gable frames can be dealt with either by reducing them to multi-bay rectangular portals with pin joints in each span, or, with slightly more approximation, to single bay rectangular portals. The calculated results obtained from these approximations agree well with those obtained experimentally on model frames.

2. The forces in the members

It can be shown that the axial force, R, in the rafter of the gabled frame shown in Figure 3.1, is given by the equation

$$R = P\left[\sin\phi + \cos\phi \cdot \cot\phi \cdot \psi\right] \qquad 3.1$$

where

$$\psi = \frac{\left[(ks)_1+(ks)_2\right](2kA/L^2)_1+(k\alpha/L)_1\left[(k\alpha/L\sin\phi)_2 - (k\alpha/L)_1\right]}{\left[(ks)_1+(ks)_2\right]\left[(2kA/L^2)_1+(2kA/L\sin\phi)_2\right]-\left[(k\alpha/L\sin\phi)_2-(k\alpha/L)_1\right]^2}$$

for values of external load less than the critical load. In this equation s, α, and A are stability functions tabulated by Livesley and Chandler. The axial force R cannot be determined immediately since the values of s, α and A cannot be obtained from the tables until the values of the axial forces R and P_1 in the members are known. This can lead to a tedious process of trial and error before accurate values of the axial forces R and P_1 are obtained.

It is possible, however, to obtain an upper and a lower bound to the magnitude of the axial force R in the rafter given by equation (3.1). For the anti-symmetrical sway mode instability must occur at an axial force in the stanchion less than the Euler load P_e of the stanchion since the function n which determines this mode of instability (appendix equation 3.12), becomes equal to $-\infty$ at the Euler load, in the stanchion. For values of axial force in the stanchion less than the Euler load, all values of s, α, A_1 are positive, but A_2 might lie between +6 and -6. Trial of any possible values of $\frac{k_2}{k_1}$ and $\frac{L_2}{L_1}$ shows that ψ has a value between zero and unity. The value zero corresponds to the rafter being much stiffer than the stanchion and the value unity corresponds to the stanchion being very stiff relative to the rafter. Hence when the rafters are very stiff the value of $\frac{R}{P}$ tends to $\sin \phi$ and when the stanchion is very stiff the value of $\frac{R}{P}$ tends to $\frac{1}{\sin \phi}$. For any normal value of relative stiffness, the value of $\frac{R}{P}$ as influenced by stability effects must lie between these two bounds. The lower bound with very flexible stanchions corresponds to the roof being supported on rollers at the eaves. The upper bound with very stiff stanchions corresponds to the roof being fixed against lateral displacement at the eaves. For these extreme cases, the result of calculations gives the critical load values shown in Figure 3.2 for various gable angles ϕ.

For angles of inclination ϕ greater than about 30°, Figure32 shows that the elastic critical load corresponding to these two limits are reasonably close and in this case the middle value of $\frac{R}{P}$ gives a very good approximation to the elastic critical load.

When the inclination ϕ of the rafter becomes small, the two limits give elastic critical loads which are widely spaced. For those frames, however, in which the value of $\frac{P_e}{L}$ of the stanchion is less than twice that of the rafter, the graph of Figure33 obtained from 3.1 shows that $\frac{R}{P}$ lies between unity and $\sin \phi$, which is a much smaller range than that from $\frac{1}{\sin \phi}$ to $\sin \phi$.

To investigate the effect of a small inaccuracy in the value of R the graph shown in Figure 34 was prepared. This shows the critical value of ρ_1 of the stanchion when the load in the rafter is changed, for different ratios of the stiffness of the rafter to that of the stanchion. It is seen that a change in the axial force in the rafter has a much smaller percentage effect on the critical value of ρ_1. Hence it is not to be expected that an error in the relatively small range from $\sin \phi$ to unity will make an appreciable error in the critical load. To obtain a conservative answer and for convenience, the value of $\frac{R}{P}$ is taken as unity for these frames.

For those frames in which the value of $\frac{P_e}{L}$ of the stanchions is bigger than twice and less than four times that of the rafter, the graph of Figure 33 shows that $\frac{R}{P}$ is always bigger than unity but much smaller than $\frac{1}{\sin \phi}$. Hence for these frames $\frac{R}{P}$ is also taken as unity, but the value of the critical load obtained will not now be conservative.

If the value of $\frac{P_e}{L}$ for the stanchion is very much greater than that of the rafter, the value of $\frac{R}{P}$ may be taken as $\frac{1}{\sin \phi}$. Figure 33 shows that this is a reasonable approximation which leads to a conservative estimate of the elastic critical load.

Figure 33 shows the value of $\frac{R}{P}$ when the reduction in stiffness of the members by axial loads is ignored. When this effect is taken into account the value of $\frac{R}{P}$ moves towards the nearer limit $\frac{1}{\sin \phi}$ or $\sin \phi$. Little error therefore is introduced by the approximations given in this section, since the approximation becomes more accurate when the axial force effects are included.

There are a few remaining cases which are not covered by the approximation given above. These are structures in which the inclination of the rafter is less than about 30°, and in which the $\frac{P_e}{L}$ value of the stanchion is more than four times that of the rafter. For these frames it is necessary to calculate the value of $\frac{R}{P}$ from equation 3.1. With this exception, the approximations suggested give reasonable estimates of the axial forces in the members and there is no need to have recourse to successive re-calculation for most frames.

3. Derivation of the stability criterion

Symmetrical single bay gabled frames

It is shown in the appendix that the elastic critical load of a symmetrical gabled frame is the same as that of a rectangular portal with a hinge in the middle of the beam. In the equivalent portal the axial force in the beam is chosen to be the same as the axial force in the rafter of the gabled frame. The length and EI value of each half beam are made the same in both frames. A gabled frame and its equivalent rectangular portal are shown in Figure 3.5a and 3.5b. The stability condition for the anti-symmetrical sway mode of Figure 5a is demonstrated in the appendix to be:

$$K = (km)_1 + (ks'')_2 \qquad 3.2$$

This is also the stability condition for the rectangular portal of Figure 3.5b. In this equation K represents the "no-shear" stiffness of the joints B, and B', k is the usual $\frac{EI}{L}$ value and n and s" are the stability functions tabulated and defined by Livesley and Chandler . To obtain the critical load the least value of the load which makes the "no-shear" stiffness vanish is determined by trial and interpolation. For the load case to be tested the approximations of § 2 will give the ratio of $\rho = \frac{P}{P_e}$ for the rafter and the stanchion. The trial and interpolation process itself is illustrated in later examples.

Unsymmetrical single bay gabled frames

The exact determination of the critical load for these frames involves the evaluation of determinants of the seventh order. To avoid the excessive amount of arithmetic involved, this type of frame is also reduced to an equivalent rectangular portal as shown in Figure 3.6, but this time the equivalence is only approximate. In the equivalent structure again hinged at C, the lengths, the EI values and the axial loads of BC and CD remain the same as in the original structure.

For the equivalent structure the operations table giving the bending moments and forces in the frame is shown in Table 3.1.

Operation	M_{AB}	M_{BA}	M_{BC}	M_{DC}	M_{DE}	M_{ED}	H_E
1) Rot. B	$(ksc)_1\theta_B$	$(ks)_1\theta_B$	$(ks'')_2\theta_B$				$-(k\alpha/L)_1\theta_B$
2) Rot. D				$(ks'')_3\theta_D$	$(ks)_4\theta_D$	$(ksc)_4\theta_D$	$-(k\alpha/L)_4\theta_D$
3) Sway.	$-(k\alpha/L)_1\delta$	$-(k\alpha/L)_1\delta$			$-(k\alpha/L)_4\delta$	$-(k\alpha/L)_4\delta$	$(2kA/L^2)_1\delta$ $+(2kA/L^2)_4\delta$

TABLE 3.1.

α for this table is defined in the appendix and s and sc are stability functions tabulated and defined by Livesley and Chandler.[9]

The equilibrium of joint B requires that:-

$$M_{BA} + M_{BC} = 0$$

hence $$\theta_B = \frac{(k\alpha/L)_1}{(ks)_1 + (ks'')_2} \cdot \delta$$

Joint D is in equilibrium, hence

$$\theta_D = \frac{(k\alpha/L)_4}{(ks)_4 + (ks'')_3} \cdot \delta$$

The total horizontal force at B is therefore:-

$$F = \left[(2kA/L^2)_1 + (2kA/L^2)_4\right]\delta - \left[(k\alpha/L)_1\theta_B + (k\alpha/L)_4\theta_D\right]$$

Substituting for θ_B and θ_D

$$F/\delta = (2kA/L^2)_1 + (2kA/L^2)_4 - \left[\frac{(k\alpha/L)_1^2}{(ks)_1 + (ks'')_2} + \frac{(k\alpha/L)_4^2}{(ks)_4 + (ks'')_3}\right] \qquad 3.3$$

To obtain the critical load from equation 3.3, the least value of the external load which makes the sway stiffness $\frac{F}{\delta}$ vanish is determined by trial and interpolation.

The approximate value of the elastic critical load of the unsymmetrical gabled frame of Figure 3.10A will be determined as an example. In this frame $k_1 = k_1 = k_4$ and $k_3 = 0.577k_1$, $L_1 = L_2 = L_4$ and $L_3 = \sqrt{3}\,L_1$. On substituting the relative values of k and the lengths in equation 3.3, the sway stiffness becomes;

$$\frac{F.L^2}{\delta} = (2A_1 + 2A_4) - \left[\frac{\alpha_1^2}{s_1 + s''_2} + \frac{\alpha_4^2}{s_4 + 0.577s''_3}\right] \qquad 3.4$$

The different axial force in the inclined members obtained by fixing B and D against displacement and making them free to displace respectively, are

Member	Force when B & D are fixed	Force when B & D are free	Mean value
BC	$P \cos 30$	$P \cos^3 30$	$7\sqrt{3}\,P/16$
CD	$P \cos 60$	$P \cos^3 60$	$5P/16$

Following the approximate procedure of section 2, the mean value of the forces in the rafters are used in the calculation which follows. The axial forces, the Euler loads and the relative load parameter are now:-

Member	Force	Euler load	$\frac{P}{P_e}$	relative $\frac{P}{P_e}$
AB	$\frac{7}{4}P$	P_e	$\frac{7}{4}\frac{P}{P_e}$	1
BC	$\frac{7\sqrt{3}}{16}P$	P_e	$\frac{7\sqrt{3}}{16}\frac{P}{P_e}$	0.432
CD	$\frac{5}{16}\cdot P$	$\frac{1}{3}P_e$	$\frac{15}{16}\frac{P}{P_e}$	0.535
DE	$\frac{5}{4}P$	P_e	$\frac{5}{4}\frac{P}{P_e}$	0.715

Knowing the relative load parameters, the critical load parameter $\frac{P}{P_e}$ is obtained by trial and error using the stability conditions 3.4 as follows:-

First trial $\rho_1 = 0.60$

$\rho_1 = 0.60$ $\qquad \rho_2 = 0.26$ $\qquad \rho_3 = 0.322$ $\qquad \rho_4 = 0.43$

From tables (9)

$s_1 = 3.1403$ $\qquad s''_2 = 2.443$ $\qquad s''_3 = 2.3019$ $\qquad s_4 = 3.3996$

$\alpha_1 = 5.3810$ $\qquad \alpha_4 = 5.5621$

$A_1 = 2.4201$ $\qquad A_4 = 3.44015$

$$\frac{F}{\delta} L^2 = (4.84 + 6.88) - \left[\frac{(5.3810)^2}{5.5846} + \frac{(5.5621)^2}{4.7296}\right]$$

$$= 11.72 - (5.19 + 6.55)$$

$$= -0.02$$

This shows that the sway stiffness is negative so a smaller load is tested.

Second trial $\rho_1 = 0.58$

$\rho_1 = 0.58$ $\rho_2 = 0.25$ $\rho_3 = 0.31$ $\rho_4 = 0.415$

$s_1 = 3.1715$ $s''_2 = 2.4673$ $s''_3 = 2.3260$ $s_4 = 3.4200$

$\alpha_1 = 5.4026$ $\alpha_4 = 5.5776$

$A_1 = 2.5404$ $A_4 = 3.5299$

$$\frac{F}{\delta} = (5.08 + 7.06) - \left[\frac{(5.4026)^2}{5.6383} + \frac{(5.5776)^2}{4.762}\right]$$

$$= 12.14 - (5.2 + 6.51)$$

$$= + 0.43$$

i.e. the frame is stable.

The critical value of ρ_1 is now obtained by linear interpolation. To two significant figures its value is 0.60. The total load carried by the gabled frame is

$$3P = \frac{12}{7} \times 0.60 \, P_e$$

$$= 1.03 \, P_e$$

This compared with a value of 0.940 P_e obtained experimentally for a model steel gabled frame.

Multi-bay frames

Symmetrical multi-bay frames can also be reduced to approximately equivalent portal frames in the same way. For example, the two bay gabled frame of Figure3.7a is reduced to the two bay rectangular portal of Figure3.7b. Again the lengths, EI values and axial loads of the half beams in the equivalent portal, remain the same as these of the real frame.

In deriving the stability condition, the joints B, D and B' are displaced horizontally by an amount δ and each joint rotated to balance the sway moment. The sway stiffness $\frac{F}{\delta}$ is given by:-

$$F/\delta = (4kA/L^2)_1 + (2kA/L^2)_3 - \left[\frac{2(k\alpha/L)_1^2}{(ks)_1 + (ks'')_2} + \frac{(k\alpha/L)_3^2}{(ks)_3 + 2(ks'')_2} \right] \quad 3\cdot5$$

This equation is used to illustrate the solution of the frame shown in Figure3.10B. In this frame $k_1 = k_2 = k_3$ and the length of all the members is the same and $\phi = 45^0$. Substituting these values in 3·5, the stability condition becomes:

$$\frac{F}{\delta} L^2 = (4 A_1 + 2 A_3) - \left[\frac{2\alpha_1^2}{S_1 + S_2''} - \frac{\alpha_3^2}{S_3 + 2S_2''} \right] \quad 3\cdot6$$

The forces in the inclined members are

Member	Force when B & C are fixed	Force when B & C are free	Mean value
BC, CD	1.414 P	0.707 P	1.06P

The mean value for the axial force in the rafter is again used in the calculation. The axial forces in the members and the relative load parameters $\frac{P}{P_e}$ are:-

Member	Force	rel. P_e	rel. $\frac{P}{P_e}$
AB	P	1	1.00
BC, CD	1.06 P	1	1.06
DE	2 P	1	2.00

First trial $\rho_1 = 0.44$

$\rho_1 = 0.44$ $\rho_2 = 0.465$ $\rho_3 = 0.88$

From tables (9)

$s_1 = 3.3847$ $s''_2 = 1.9298$ $s_3 = 2.6797$

$\alpha_1 = 5.5516$ $\alpha_3 = 5.0718$

$A_1 = 3.3803$ $A_3 = 0.7292$

$$\frac{F}{\delta} L^2 = (13.52 + 1.46) - \left[\frac{2 \times (5.5516)^2}{5.3145} + \frac{(5.0718)^2}{6.5393}\right]$$

$$= 14.98 - (11.2 + 3.95)$$

$$= -0.17$$

The sway stiffness is thus negative and a lower value of the load is tested.

Second trial $\rho_1 = 0.42$

$\rho_1 = 0.42$ $\rho_2 = 0.445$ $\rho_3 = 0.84$

From tables (9)

$s_1 = 3.4144$ $s''_2 = 1.9830$ $s_3 = 2.7483$

$\alpha_1 = 5.5726$ $\alpha_3 = 5.1168$

$A_1 = 3.5000$ $A_3 = 0.9716$

$$\frac{F}{\delta} L^2 = (14 + 1.94) - \left[\frac{2 \times (5.5726)^2}{5.3974} + \frac{(5.1168)^2}{6.7243}\right]$$

$$= 15.94 - (11.55 + 3.90)$$

$$= +0.49$$

i.e. the frame is stable.

The critical ρ_1 by linear interpolation is $0.44 - 0.02 \times \frac{0.17}{0.56} = 0.435$

and the corresponding elastic critical load is

$$4P = 4 \times 0.435 \, P_e$$

$$= 1.74 \, P_e$$

This value compared with 1.65 P_e obtained experimentally on a model of the gabled frame.

The expression 3·5 given above can be further simplified with very little extra approximation. There is already in existence an approximate method (11) for estimating the elastic critical load of multi-bay rectangular portal frames. In that approximation, the multi-bay portal frame is replaced by a single bay portal frame having each stanchion stiffness equal to half the total stiffness of the stanchions and a beam stiffness equal to the total stiffness of the beams. The axial force in an equivalent stanchion is taken to be half the total of the applied loads, and its Euler load half the sum of the Euler loads of the stanchions.

Following this procedure the original symmetrical multi-bay gabled frame of Fig 3·7 is reduced to a single bay rectangular portal in which:-

(1) Each stanchion stiffness $\bar{k}_1$ is equal to half the sum of the stiffnesses of the stanchions.

(2) Each stanchion load is equal to half the total applied load.

(3) Each stanchion Euler load is equal to half the sum of the Euler loads of the stanchions.

(4) There is a hinge at the middle of the beam, each half beam having a stiffness $\bar{k}_2$ equal to half the sum of the stiffnesses of the inclined members.

(5) The axial forces in the half beams are made equal to half the sum of the axial forces in the inclined members.

The stability condition for estimating the elastic critical load is then simply

$$K = (\bar{k}\,\bar{n})_1 + (\bar{k}\,\bar{s}'')_2 \qquad (3.7)$$

The least value of the load which makes the "no-shear" stiffness K, vanish gives the elastic critical load.

The two bay gabled frame of Figure 3·108 was recalculated in this way. The frame of Figure 3·108 is replaced by a single bay frame with stanchion stiffness $\frac{3}{2}\frac{EI}{L}$ and rafter stiffness $2\frac{EI}{L}$. The force in each stanchion is 2P and that in the rafter is 2 x 1.06 P. Thus the relative stiffness of the rafter is $\bar{k}_2 = \frac{2}{3/2} = 1.33$ and its relative load parameter is $\frac{2 \times 1.06}{2 \times 2 \div 3/2}\ \rho_1 = 0.795\ \rho_1$. Substituting for $\bar{k}_2$ in 3.7, the stability condition becomes:

$$K = \bar{n}_1 + 1.33\ \bar{s}''_2 \qquad (3.8)$$

First trial $\rho_1 = 0.58$

$\rho_1 = 0.58$ $\qquad \rho_2 = 0.46$

From tables (9)

$\bar{n}_1 = -2.5733$ $\qquad \bar{s}''_2 = 1.9438$

$K = -2.5733 + 1.33 \times 1.9438$

$= +0.0167$

i.e. the frame is stable

Second trial $\rho_1 = 0.59$

$\rho_1 = 0.59$ $\qquad \rho_2 = 0.47$

$\bar{n}_1 = -2.7074$ $\qquad \bar{s}''_2 = 1.9115$

$K = -2.7974 + 1.33 \times 1.9115$

$= -0.155$

i.e. the frame is unstable.

The critical ρ_1 by linear interpolation is $0.58 + .01 \times \frac{.0167}{.1717} = 0.581$ and the corresponding elastic critical load is

$$4P = 3 \times 0.581\, P_e$$
$$= 1.74\, P_e$$

In essence this approximation assumed that all ridge joints rotate the same amount, as do all eaves joints. There would in fact be some slight adjustment required since some joints would rotate more or some less than the average. But the effect of these adjustments on the value of the horizontal forces and hence on the sway stiffness is very small.

These two methods can also be used for unsymmetrical multi bay frames. For example, the two bay unsymmetrical gabled frame of Figure3.8a is reduced to the approximately equivalent two bay portal of Figure3.8b. Again the lengths, EI values, and axial forces of the members in the equivalent portal are the same as in the gabled frame. The sway stiffness of the frame of Figure3.8 b is:

$$\frac{F}{\delta} = \left[\left(\frac{2kA}{L^2}\right)_1 + \left(\frac{2kA}{L^2}\right)_4 + \left(\frac{2kA}{L^2}\right)_7\right] - \left[\frac{(k\alpha/L)_1^2}{(ks)_1+(ks'')_2} + \frac{(k\alpha/L)_4^2}{(ks)_4+(ks'')_3+(ks'')_5} + \frac{(k\alpha/L)_7^2}{(ks)_7+(ks'')_6}\right] \quad (3.9)$$

The second approximation can also be used for unsymmetrical multi-bay gabled frames, as for example north light frames. In this case the left hand portion of the equivalent beam is obtained by summing each left hand rafter, the right hand portion of the equivalent beam by summing each right hand rafter and each equivalent stanchion is half the sum of the real stanchions. Several examples, shown in Figure3.10, were calculated using these two approximations and the results obtained are compared with the experimental values in Table3.2.

In multi-bay frames, the axial forces in the rafters near the centre span of the frame will approach $\frac{P}{\sin \phi}$ since they are largely restrained against lateral movement. The outer bays will have progressively smaller loads. If the end rafters are assumed to have an axial force of $P \sin \phi$ and all the others to have an axial force $\frac{P}{\sin \phi}$, a good approximation to the force distribution is obtained.

4. Extension to deal with more complex frames

The exact analysis of the single bay frame of Figure 3.9a will involve dealing with two fourth order determinants. The size of the determinants will grow to the seventh order when another bay is added and to larger sizes when the number of bays is increased.

The approximate method for predicting the elastic critical load of gabled frames can be extended to deal with frames like that of Figure 3.9a in which the stability of the stanchions is the dominant factor in estimating the elastic critical load of the frame.

The frame of Figure 3.9a is replaced by the equivalent gabled frame ABDB'A' of Figure 3.9b to determine the axial forces in the members of the equivalent rectangular portal frame which is obtained as before. The equivalent beam will have an EI value chosen by inspection to be about the average of the members BC and CC' and each half will be of the same length as BD.

Multi-bay frames of this kind, are replaced by equivalent multi-bay portal frames or by equivalent single bay portals. The stability conditions of (3.5) and (3.7) are used in estimating the elastic critical loads of such frames.

When the ridge of a complex frame is only a short distance above the level of the eaves it is better to obtain the horizontal force component H in the equivalent rafter in the following way. The joints B and B' are first assumed to be fixed and the horizontal thrust determined. When B and B' are assumed to be supported on rollers the horizontal thrust is, of course, zero. The value of H used for the equivalent rafter is taken to be the mean of these two values, i.e. one half of the "fixed-end" thrust.

This method can also be applied to other frames in which the linkage of members connecting the two vertical stanchions is stiff enough for failure to occur mainly by sway of the stanchions. A good approximation is likely to be obtained when the Euler load of the equivalent rafter is not less than that of the vertical stanchions. As an example of this method of approximation the critical load of the frame of Figure 3.10E is determined. All the members have the same values of k and L and $\phi = 45^o$. The frame is replaced by a single bay gabled frame having a rafter length of 1.45 L, an EI value equal to that of the original members, and hence a relative k_2 value of 0.688 k_1. The ratio of $\frac{P_e}{L}$ of the stanchion to that of the equivalent rafter is $(1.455)^3 = 3.07$, thus the axial load R, in the rafter is taken to be half the applied load at the apex i.e. R = P. The forces, the relative Euler loads, and the relative load parameter of the equivalent members are:-

Member	Force	rel. P_e	rel. $\frac{P}{P_e}$
AB	2P	2.12	1
BD	P	1	1.06

The stability condition is

$$K = n_1 + 0.688\, s''_2 \qquad (3\cdot10)$$

First trial $\rho_1 = 0.45$

$\rho_2 = 0.477$

From the tables (9)

$n_1 = -1.2537 \qquad s''_2 = 1.8973$

Substituting the value in (10) gives

$K = -1.2537 + 0.688 \times 1.8973$

$= +0.048$

This value is positive and the frame is stable so a larger value of ρ_1 is tested.

Second trial $\rho_1 = 0.46$

$\rho_2 = 0.487$

$n_1 = -1.3357 \qquad s''_2 = 1.8693$

and $K = -1.3357 + 0.688 \times 1.8693$

$= -0.0457$

The critical value of ρ_1 obtained by linear interpolation is 0.455, and the elastic critical load is $2P = 2 \times 0.455\, P_e$

$= 0.91\, P_e$

where P_e is the Euler load of AB.

The value obtained experimentally on a model of the frame having L= 12" and EI = 320 Ib.in^2, was 0.882 P_e.

APPENDIX

The stability criterion for the anti-symmetrical sway mode.

For the frame of Figure 5a we will use the notation:-

$$\alpha = s(1 + c)$$

$$A = \frac{\alpha}{m}$$

where α and A are tabulated and defined by Livesley and Chandler (1). The moments at the ends of the members due to the deformations listed below are:-

Operation	M_{AB}	M_{BA}	M_{BC}	$M_{CB'}$	$H_B = H_{B'}$
1) Rot. B & B' by θ_B	$k_1 s_1 c_1 \theta_B$	$k_1 s_1 \theta_B$	$k_2 s_2 \theta_B$	$k_2 s_2 c_2 \theta_B$	$-\frac{k_1 \alpha_1}{L_1} \theta_B$
2) Rot. C by θ_C			$(ksc)_2 \theta_C$	$(ks)_2 \theta_C$	0
3) Sway B & B' by	$-\frac{k_1 \alpha_1 \delta}{L}$	$-\frac{k_1 \alpha_1 \delta}{L_1}$			$+\frac{2 k_1 A_1 \delta}{L_1^2}$

The equilibrium of joint C requires that:-

$$M_{CB} + M_{CB'} = 0$$

Hence $\quad 2(ks)_2 \, (c_2\theta_B + \theta_c) = 0$

Therefore $\quad \theta_c = - c_2\theta_B$

Joint B is also in equilibrium and hence

$$((ks)_1 + (ks)_2)\,\theta_B + (k\,s\,c)_2\theta_c - \frac{k_1\alpha_1}{L_1} = 0 \qquad (3.11a)$$

Substituting for θ_c and using the definition $s(1 - c^2) = s''$

$$\left\{(ks)_1 + (k\,s'')_2\right\} \theta_B - \frac{k_1 \alpha_1 \delta}{L_1} = 0 \qquad (3.11b)$$

The disturbing force at B to produce these deformations vanishes at the critical load. Hence :-

$$H_B = -\frac{k_1 \alpha_1}{L_1} \theta_B + \frac{2k_1 A_1 \delta}{L_1^2} = 0 \qquad (3.11c)$$

The determinant of the coefficients of the unknowns in 3.11b and 3.11c) is

$$\begin{vmatrix} (ks)_1 + (k\,s'')_2 & -\frac{k_1 \alpha_1}{L_1} \\ -\frac{k_1 \alpha_1}{L_1} & \frac{2k_1 A_1}{L_1^2} \end{vmatrix} \qquad (3.11d)$$

This reduces to

$$\Delta = \frac{2k_1 A_1}{L_1^2} \left((kn)_1 + (k\,s'')_2 \right) \qquad (3.12)$$

The elastic load parameter ρ which makes Δ vanish, is the critical one. The value of the load parameter ρ making A_1 vanish is unity. That making the term $((kn)_1 + (k\,s'')_2)$ vanish is less than unity by an amount depending on the relative stiffnesses of the rafter and stanchion. It follows that $(kn)_1 + (k\,s'')_2) = 0$ is the condition which gives the elastic critical load of the gabled frame since it gives the lowest value of ρ at which the stiffness of the frame becomes zero.

The term $(kn)_1 + (k\,s'')_2)$ is also the "no-shear" stiffness of the joints of the rectangular portal frame shown in Figure35b. In this portal the length of each half beam is the length of the rafter in Figure31a, and the axial force in the beam is equal to that in the rafter.

The critical load of this equivalent portal is obtained when the "no-shear" stiffness vanishes. To estimate the critical load of a gabled frame as in Figure35a, therefore, the structure is reduced to the rectangular portal of Figure35b, and the well known procedure (12) for rectangular portal frames is used.

Chapter 4

Stability of frames with unsymmetry

Introduction

The stability of symmetrical frames subjected to unsymmetrical loading and also unsymmetrical frames with any loading at the joints is examined. The results indicate that the elastic critical loads of the symmetrical frames are appreciably reduced when there is unsymmetry in the loading. The amount of the reduction will be dependent on the degree of unsymmetry in the loading of the frames. The rapid change of the force components was found to be responsible for the major part of the reduction.

In investigating the elastic instability of frameworks, the analysis is ususlly based on the fact[10] that the framework at the critical load offers no resistance to any disturbing action. In the test a disturbing action is applied to the framework which causes fresh deformations in the framework. The action and the deformations in the framework are related by the equilibrium conditions viz:-

$$[\text{action}] = [A]\,[\text{deformation}] \tag{4.1}$$

$[A]$ being a square matrix formed by the coefficients of the deformations and represents the overall stiffness of the framework for a particular mode of elastic instability. Instability occurs when the determinant Δ of the matrix $[A]$ vanishes

i.e $\Delta = 0$ 4.2

Sometimes the axial forces in these frameworks are statically indeterminate and it is well known that the forces in the members must be obtained before a test is made since the determinant contains the stability functions s,c etc. of the members. The internal reaction unknowns H_n can be determined in terms of the applied load P, the geometry of the framework and the stability functions of the members in the form

$$H_n \Delta_1 = P \Delta_n \qquad 4.3$$

where Δ_1 and Δ_n are determinants formed by the coefficients of the unknowns and the external loading as is derived in detail later.

In the case where $\Delta_1 = C\Delta$, where C is an arbitrary constant, the method of analysis for predicting the elastic critical load using equation 4.2 as the criterion for elastic instability is not applicable since this will give rise to very large force components when $\Delta \rightarrow 0$. Equation 4.3 is used instead as the criterion for instability. The number of equations obtained from 4.3 is equal to the number of the internal reaction unknowns. These equations can be solved simultaneously to yield relationships between the internal reaction unknowns and the external loads. The elastic critical load may be defined as the highest load P, thus obtained from the relationship of 4.3 as indicated in Figure 4.5

In the following calculations it is assumed that the modulus of elasticity is constant; the members are straight; of uniform cross-section and obey the simple theory; stresses produced by the loads in the members are within the elastic range; and that deflection due to the bending moment will be the only deformations and that these are small.

Experimental determination of the critical load

The natural frequency of oscillation of a frame decreases with increasing external load (1). When the vibrational stiffness is plotted against the applied load, an almost straight line relationship is obtained. Curving may occur near the elastic critical load when large deflections begin to have an effect. An estimation of the elastic critical load is made by the extrapolation of the linear part of the graph. In this chapter it is this value of experimental elastic critical load which is compared with the theoretical value.

Theoretical solution

The slope-deflection equation is adopted herein to analyse the elastic instability of the frame in Figure 4.1. The loads on the frame are assumed to be lumped at the joints. When the loading is applied the framework will be deformed in the following way:-

1) Joint B rotates clockwise through an angle θ_B.

2) Joint C rotates clockwise through an angle θ_C.

3) Member AB sways clockwise by $\delta \sin\phi_2$.

Member BC sways anti-clockwise by $\delta \sin(\phi_1+\phi_2)$.

Member CD sways clockwise by $\delta \sin\phi_1$. These are obtained from the Williot diagram of Figure 4.1a.

Due to these deformations, there will be moments at the ends of the members which are tabulated in Table 4.1.

Joint equilibrium of B requires that

$M_{BA}+M_{BC}=0$

Substituting for M_{BA} and M_{BD} from Table 4.1 leads to

$$[(ks)_1+(ks)_2]\,\theta_B + (ksc)_2\,\theta_C + \left[\left(\frac{k\alpha}{L}\right)_2 \sin(\phi_1+\phi_2) - \left(\frac{k\alpha}{L}\right)_3 \sin\phi_2\right]\delta = 0 \qquad 4.4a$$

Likewise the equilibrium of joint C requires that

$$(ksc)_2\,\theta_B + [(ks)_2+(ks)_3]\,\theta_C + \left[\left(\frac{k\alpha}{L}\right)_2 \sin(\phi_1+\phi_2) - \left(\frac{k\alpha}{L}\right)_3 \sin\phi_1\right]\delta = 0 \qquad 4.4b$$

There are two force component unknowns for the statically indeterminate frame of Figure 4.1. For convenience, these will be takem as the horizontal force H at one of the supports and the force component v keeping the member BC in equilibrium. Consequently the force components distribution will be as shown in Figure 4.1b. The equilibrium of BC requires that

$$v = \left(\frac{k\alpha}{L}\right)_2(\theta_B+\theta_C) + \left(\frac{2kA}{L^2}\right)_2 \sin(\phi_1+\phi_2)\,\delta \qquad 4.4c$$

and that of AB requires that

$$H - (P-v)\cot\phi_1 = \left(\frac{k\alpha}{L}\right)_1 \frac{\theta_B}{\sin\phi_1} - \left(\frac{2kA}{L^2}\right)_1 \frac{\sin\phi_1}{\sin\phi_2}\,\delta \qquad 4.4d$$

and that of CD requires that

$$-H + (uP - v)\cot\phi_2 = \left(\frac{k\alpha}{L}\right)_3 \frac{\theta_c}{\sin\phi_2} - \left(\frac{2kA}{L^2}\right)_3 \frac{\sin\phi_1}{\sin\phi_2}\,\delta \qquad 4.4e$$

There are five unknowns in five homogenous equations, solution of these equations yields

$$H\,\Delta_1 = P\,\Delta_2 \qquad 4.5$$

and

$$v\,\Delta_1 = P\,\Delta_3 \qquad 4.6$$

where

$$\Delta_1 = \frac{1}{\sin\phi_1\sin\phi_2}\begin{vmatrix} (ks)_1 + (ks)_2 & (ksc)_2 & \left[\left(\frac{k\alpha}{L}\right)_2\sin(\phi_1+\phi_2) - \left(\frac{k\alpha}{L}\right)_1\sin\phi_2\right] \\ (ksc)_2 & (ks)_2 + (ks)_3 & \left[\left(\frac{k\alpha}{L}\right)_2\sin(\phi_1+\phi_2) - \left(\frac{k\alpha}{L}\right)_3\sin\phi_1\right] \\ \left[\left(\frac{k\alpha}{L}\right)_2\sin(\phi_1+\phi_2) - \left(\frac{k\alpha}{L}\right)_1\sin\phi_2\right] & \left[\left(\frac{k\alpha}{L}\right)_2\sin(\phi_1+\phi_2) - \left(\frac{k\alpha}{L}\right)_3\sin\phi_1\right] & \left[\left(\frac{2kA}{L^2}\right)_1\sin^2\phi_2 + \left(\frac{2kA}{L^2}\right)_3\sin^2\phi_1 + \left(\frac{2kA}{L^2}\right)_2\sin^2(\phi_1+\phi_2)\right] \end{vmatrix}$$

$$\Delta_2 = (1+\mu)\cot\phi_1\cot\phi_2\begin{vmatrix} (ks)_1 + (ks)_2 & (ksc)_2 & \left[\left(\frac{k\alpha}{L}\right)_2\sin(\phi_1+\phi_2) - \left(\frac{k\alpha}{L}\right)_1\sin\phi_2\right] \\ (ksc)_2 & (ks)_2 + (ks)_3 & \left[\left(\frac{k\alpha}{L}\right)_2\sin(\phi_1+\phi_2) - \left(\frac{k\alpha}{L}\right)_3\sin\phi_1\right] \\ \left[\left(\frac{k\alpha}{L}\right)_2 - \frac{\mu}{1+\mu}\left(\frac{k\alpha}{L}\right)_1\frac{1}{\cos\phi_1}\right] & \left[\left(\frac{k\alpha}{L}\right)_2 - \frac{1}{1+\mu}\left(\frac{k\alpha}{L}\right)_3\frac{1}{\cos\phi_2}\right] & \left[\left(\frac{2kA}{L^2}\right)_2\sin^2(\phi_1+\phi_2) + \frac{1}{1+\mu}\left\{\left(\frac{2kA}{L^2}\right)_3\frac{\sin\phi_1}{\cos\phi_2} + \mu^2\left(\frac{2kA}{L^2}\right)_1\frac{\sin\phi_2}{\cos\phi_1}\right\}\right] \end{vmatrix}$$

and

$$\Delta_3 = (\cot\phi_1 - \mu\cot\phi_2)\begin{vmatrix} (ks)_1 + (ks)_2 & (ksc)_2 & \left[\left(\frac{k\alpha}{L}\right)_2\sin(\phi_1+\phi_2) - \left(\frac{k\alpha}{L}\right)_1\sin\phi_2\right] \\ (ksc)_2 & (ks)_2 + (ks)_3 & \left[\left(\frac{k\alpha}{L}\right)_2\sin(\phi_1+\phi_2) - \left(\frac{k\alpha}{L}\right)_3\sin\phi_1\right] \\ \left(\frac{k\alpha}{L}\right)_2 & \left(\frac{k\alpha}{L}\right)_2 & \left[\left(\frac{2kA}{L^2}\right)_2\sin(\phi_1+\phi_2)\right] \end{vmatrix}$$

When the dimensions of the framework and its loading conditions are specified equations 4.5 and 4.6 can be solved numerically to yield relationships between H,v and P. The maximum value of P thus obtained, determines the critical load of the frame.

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Numerical examples

Example 1, Unsymmetrical frame

The elastic critical load of the frame of Figure 4.2 is estimated when $\phi_1=60^\circ$, $\phi_2=45^\circ$, $L_2=L_3=L$, $L_1=0.816L$, $\mu=0$ and $(EI)_1=(EI)_2=(EI)_3$. Substituting these values in equations 4.5 and 4.6 gives

$$\Delta_1 = 1.635 \begin{vmatrix} 1.224s_1+s_2 & (sc)_2 & 0.966\alpha_2-1.06\alpha_1 \\ (sc)_2 & s_2+s_3 & 0.966\alpha_2-0.865\alpha_3 \\ 0.966\alpha_2-1.06\alpha_1 & 0.966\alpha_2-0.865\alpha_3 & 1.84A_1+1.87A_2+1.5A_3 \end{vmatrix}$$

$$\Delta_2 = 0.578 \begin{vmatrix} 1.224s_1+s_2 & (sc)_2 & 0.966\alpha_2-1.06\alpha_1 \\ (sc)_2 & s_2+s_3 & 0.966\alpha_2-0.865\alpha_3 \\ \alpha_2 & \alpha_2-1.414\alpha_3 & 1.932A_2+2.448A_3 \end{vmatrix}$$

and

$$\Delta_3 = 0.578 \begin{vmatrix} 1.224s_1+s_2 & (sc)_2 & 0.966\alpha_2-1.06\alpha_1 \\ (sc)_2 & s_2+s_3 & 0.966\alpha_2-0.865\alpha_3 \\ \alpha_2 & \alpha_2 & 1.932A_2 \end{vmatrix}$$

Ignoring the effect of the axial forces in the members i.e taking s=4, sc=2, α=A=6 and substituting in equations 4.5 and 4.6 gives the force component unknowns as

H/P = 0.308

and

v/P = 0.132

The axial forces, the relative Euler load and the relative load parameters of the members are thus

Member	AB(1)	BC(2)	CD(3)
Force	0.904P	0.308P	0.312P
rel. P_e	1.5	1.0	1.0
rel. P/P_e	1.0	0.511	0.518

A value of ρ_1 is assumed and the load parameter ρ of each other member is calculated. Hence when ρ_1=0.60, ρ_2=0.60 X 0.511 = 0.307 and ρ_3 = 0.60 X 0.518 = 0.311

The stability functions corresponding to these load parameters are obtained from the stability tables are substituted into equations 4.5 and 4.6. This yields

H/P = 0.360

and

v/P = 0.152

With these new values of H and v, the forces and the load parameters are recalculated.

Member	(1)	(2)	(3)
Force	0.915P	0.360P	0.362P
rel. P_e	1.5	1.0	1.0
rel. P/P_e	1.0	0.59	0.595
ρ-value	0.60	0.354	0.358

The same procedure is followed. This leads to

H/P = 0.355

and

v/P = 0.147

and the relative load parameters are

$$\rho_1 : \rho_2 : \rho_3 = 1 : 0.583 : 0.59.$$

These new relative load parameters are so close to those of the last cycle that the critical load cannot be far from the value given by the second cycle. The axial force in member AB is 0.915P, thus the external load P is

$$= \frac{1.5 \text{ X } 0.60}{0.915} P_e$$

$$= 0.985 \, P_e$$

P_e being the Euler load of member BC.

Following the same procedure, the external loads calculated for values of ρ are tabulated.

ρ_1	H/P	P/P_e
0	0.308	0
0.60	0.355	0.985
0.80	0.383	1.320
0.90	0.420	1.460
1.00	0.520	1.560

An attempt to calculate the critical load when $\rho_1 = 1.04$ was made. The results for three cycles of calculations are

Trial	v/P	H/P
1	0.201	0.530
2	0.210	0.716
3	0.760	0.0196

The method of calculation thus fails to give any value to P when $\rho_1 = 1.04$

Figure 4.3 shows the numerical results obtained. The elastic critical load P obtained from the graph is $1.56P_e$. This value corresponds with $1.54P_e$ obtained experimentally on a model of the frame having ½"X1/16" steel strip members, shown in Figure 4.4.

Example 2, Symmetrical frame under unsymmetrical loading

The elastic critical load of the frame is estimated when $L_1=L_2=L_3$, $\emptyset_1=\emptyset_2=45^\circ$, $(EI)_1=(EI)_2=(EI)_3$ and $\mu=0$. When these values are substituted in equations 4.5 and 4.6, the determinants become

$$\Delta_1 = \begin{vmatrix} s_1+s_2 & (sc)_2 & 1.414\,\alpha_2-\alpha_1 \\ (sc)_2 & s_2+s_3 & 1.414\,\alpha_2-\alpha_3 \\ 1.414\,\alpha_2-\alpha_1 & 1.414\,\alpha_2-\alpha_3 & 2A_1+4A_2+2A_3 \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} s_1+s_2 & (sc)_2 & 1.414\,\alpha_2-\alpha_1 \\ (sc)_2 & s_2+s_3 & 1.414\,\alpha_2-\alpha_3 \\ 0.707\,\alpha_2 & 0.707\,\alpha_2-\alpha_3 & 2A_2+2A_3 \end{vmatrix}$$

and

$$\Delta_3 = \begin{vmatrix} s_1+s_2 & (sc)_2 & 1.414\,\alpha_2-\alpha_1 \\ (sc)_2 & s_2+s_3 & 1.414\,\alpha_2-\alpha_3 \\ 0.707\,\alpha_2 & 0.707\,\alpha_2 & 2A_2 \end{vmatrix}$$

Following the procedure illustrated above, the numerical results obtained are tabulated as :-

ρ_1	H/P	P/P_e
0	0.50	0
0.60	0.55	0.631
0.90	0.61	0.900
1.00	0.67	0.945
1.06	0.725	0.955
1.10	0.79	0.930

These results are shown in Figure 4.5. The elastic critical load obtained from the graph is $P = 0.955P_e$. This value corresponds with $P = 0.90P_e$, obtained experimentally on a model of the frame.

When the external load P is applied equally at the two joints i.e $\mu = 1.0$, the elastic critical load is $1.60P_e$, bigger than the first load by 67%. For values of μ between zero and unity, the elastic critical load will be between $0.955P_e$ and $1.60P_e$.

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Frames with a small amount of unsymmetry

In frameworks where the unsymmetry in the geometry and loading are in such proportion that the applied loads are carried solely by the axial force in the members and there is no call on the members to resist some of the external load by their flexural regidities, the forces in the members will be constant until instability occurs. The elastic critical load of such frames can be predicted by using 4.2 as a stability criterion.

In other unsymmetrical frames where the portion of the external load carried by bending moments in the members is small or where the variation with the external load of any force component is small, the elastic critical load can be calculated approximately using 4.2. In these frames, the redistribution of the forces in the members becomes rapid when $\Delta_1 = C\,\Delta$ approaches zero. The elastic critical load obtained using 4.2 will be an upper bound to the elastic critical load.

Figure 4.6 shows the curve relating P and the force component v of the framework given in example 1, when μ = 1.0. The highest load obtained from this curve is $1.99P_e$. This value corresponds with $2.12P_e$ obtained by using 4.2 and $2.04P_e$ obtained experimentally.

The difference between the exact elastic critical load and the value obtained from 4.2 will increase as the amount of unsymmetry increases. For example, the approximate elastic critical load of the frame in example 1, obtained from 4.2, using the initial force distribution was $2.41P_e$ giving an error of 55%; the approximate elastic critical load of the frame in example 2, obtained from 4.2 was $1.62P_e$, an error of 68%.

Equation 4.2 was used to calculate the elastic critical load of the unsymmetrical gabled frame of Figure 4.7. The approximate value of the elastic critical load obtained was $1.028P_e$ which corresponds with $0.94P_e$ obtained experimentally.

<u>Initial imperfection in structure</u>

The method of analysis developed in this chapter can also be used to investigate the effect of initial imperfection in frames. The elastic instability of the structure of Figure 4.8 is analyzed when the strut AB has an initial sway δ_o. On the application of external load P, member AB will sway by $\delta - \delta_o$ and joint B will rotate clockwise by θ_B. The moments at the ends of the members are:

Operation	M_{AB}	M_{BA}	M_{BC}
1)Rot. B	$(ksc)_1\theta_B$	$(ks)_1\theta_B$	$3k_2\theta_B$
2)Sway	$-(k\alpha/L)_1(\delta-\delta_o)$	$-(k\alpha/L)_1(\delta-\delta_o)$	

Equilibrium of joint B requires

$$M_B = \left[(ks)_1+(3k)_2\right]\theta_B - (k\alpha/L)_1(\delta - \delta_o) = 0 \qquad 4.7a$$

The structure has only one internal reaction unknown, this is taken as the reaction v at C. The structure has no horizontal force component beaause joint C is on a roller support. Equilibrium of BC gives

$$v = (3k/L)_2\theta_B \qquad 4.7b$$

Equilibrium of member AB requires

$$-(k\alpha/L)_1\theta_B + (2k\alpha/L)_1(\delta/m_1 - \delta_o) = 0 \qquad 4.7c$$

Solution of equations 4.7a and 4.7c yields

$$\theta_B \equiv \frac{(k\alpha)_1(m_1-1)}{(kn)_1+3k_2} \cdot \delta_o/L_1 \qquad 4.7d$$

Substituting equation 4.7d and rearranging, equation 4.7b becomes

$$v \equiv \frac{\alpha_1(m_1-1)}{n_1+3k_2/k_1} \cdot 3/\pi^2 \cdot \delta_o/L_1 \cdot (P_e)_2 \qquad 4.8$$

and the axial force in AB is

$$\rho (P_e)_1 = P + v$$

Equation 4.8 can be solved numerically to yield a relationship between P and v when k_2/k_1 and L_2/L_1 are specified. The maximum value of P thus obtained, gives the elastic critical load of the structure.

Numerical example

The elastic critical load of the structure is calculated when $k_2/k_1 = L_2/L_1 = 1$ and $\delta_o/L = 1/40$. Substituting these values in equation 4.8 becomes

$$v = 0.76X10^{-2} \frac{\alpha_1 (m_1 - 1)}{n_1 + 3} \cdot P_e \qquad 4.9$$

v can be calculated for different values of the load parameter ρ_1.

Case 1 $\rho_1 = 0.5$

From tables (9)

$\alpha_1 = 5.4881$ $m_1 = 1.8168$ $n_1 = - 1.6910$

Substituting these values in 4.9, gives

$$v = \frac{5.4881 \times 0.8168 \times 0.76 \times 10^{-2}}{1.309} \quad P_e$$

$$= 0.026 P_e$$

and the external load is therefore

$$P = (0.5 - 0.026) P_e$$

$$= 0.474 P_e$$

Figure 4.9 shows the numerical results obtained for other values of ρ_1. The elastic critical load obtained from the graph is $P = 0.50P_e$. The elastic critical load of the perfect structure is $P = 0.611P_e$. Figure 4.9 also shows the relationships between P and v obtained when $\delta_o/L = 1/80$ and 1/160 respectively.

Chapter 5

Effect of prebuckling deformations

Introduction

In chapter 2, it was assumed that the frames were free of bending moments before the anti-symmetrical sway elastic critical load is reached. Consequently, there is no bending deformation in any part of the structure until the anti-symmetrical sway mode interfers. Obviously these conditions cannot be satisfied in practical frames. Thus, it is necessary to study the effect of the prebuckling deformations on the stability of the structural frames and to compare the results with those obtained by ignoring these deformations.

E. F. Masur, I. C. Chang, and L. H. Donnell (13) developed a method for analyzing the stability of frames with pin jointed supports, taking into account the effect of initial bending moments. This method is extended to deal with frames having fixed supports and many roof members. The assumptions of chapter 2 still hold and the methods of calculating the forces in the members and the symmetrical sway elastic critical load are the same. Prebuckling deformations are taken into account only when analyzing the anti-symmetrical sway mode of symmetrical frames under symmetrical loading.

In the following, the elastic instability of the gable frame shown in Figure 5.1 is analyzed. Theoretical solutions for both the symmetrical and anti-symmetrical sway modes are obtained and a numerical example is given.

Forces in the members

Since the framework is symmetrical and under symmetrical loading, it will have symmetrical deformations until the anti-symmetrical sway mode interfers. On the application of the external load, the following deformations occur:-

1) Equal and opposite rotation of joints B and B' by θ_B.

2) Equal and opposite sway $-\delta$ of AB and A'B'.

Equal and opposite sway of CB and CB', which is determined from the Williot diagram of Figure 5.1a as $+\delta/\sin\phi$.

Due to these deformations, there will be moments in the members and these are

Operation	M_{AB}	M_{BA}	M_{BC}	M_{CB}	$M_{CB'}$
1) Rot. B & B'	$(ksc)_1\,\theta_B$	$(ks)_1\,\theta_B$	$(ks)_2\,\theta_B$	$(ksc)_2\,\theta_B$	$-(ksc)_2\,\theta_B$
2) Sway δ	$+(\frac{k\alpha}{L})_1\delta$	$+(\frac{k\alpha}{L})_1\delta$	$-(\frac{k\alpha}{L})_2\delta$	$-(\frac{k\alpha}{L})_2\frac{\delta}{\sin\phi}$	$+(\frac{k\alpha}{L})_2\frac{\delta}{\sin\phi}$

Joint equilibrium at B requires

$$M_B=\Big((ks)_1+(ks)_2\Big)\theta_B+\Big((k\alpha/L)_1-(k\alpha/L\sin\phi)_2\Big)\delta = 0 \qquad 5.1a$$

The gabled frame has only one unknown force component, this is the horizontal force component H in Figure 5.1b. The equilibrium of BC requires

$$H = (k\alpha/L)_1\theta_B + (2kA/L^2)_1\delta \qquad 5.1b$$

and the equilibrium of BC requires

$$H\sin\phi - P\cos\phi = (k\alpha/L)_2\theta_B - (2kA/L^2\sin\phi^2)_2\delta \qquad 5.1c$$

Solution of these equilibrium equations yields

$$H\Delta_1 = P\Delta_2 \qquad 5.2$$

$$\theta_B\Delta_1 = P\cot\phi\left((k\alpha/L\sin\phi)_2 - (k\alpha/L)_1\right) \qquad 5.3$$

and

$$\delta\,\Delta_1 = P\cot\phi\left((ks)_1 + (ks)_2\right) \qquad 5.4$$

where

$$\Delta_1 = \begin{vmatrix} (ks)_1 + (ks)_2 & (k\alpha/L)_1 - (k\alpha/L\sin\phi)_2 \\ (k\alpha/L)_1 - (k\alpha/L\sin\phi)_2 & (2kA/L^2)_1 + (2kA/L^2\sin\phi^2)_2 \end{vmatrix}$$

and

$$\Delta_2 = \begin{vmatrix} (ks)_1 + (ks)_2 & (k\alpha/L)_1 - (k\alpha/L\sin\phi)_2 \\ (k\alpha/L)_1 & (2kA/L^2)_1 \end{vmatrix}$$

The axial force in the rafter is

$$R = H\cos\phi + P\sin\phi \qquad 5.5$$

When the dimensions of the frame and its loading are specified equation 5.2 can be solved numerically to yield a relationship between H and P. The maximum value of P thus obtained determines the symmetrical sway elastic critical load.

Stability criterion for the anti-symmetrical sway mode

The anti-symmetrical sway will occur at a load lower than the symmetrical sway elastic critical load. In order to establish the condition under which the structure first becomes laterally unstable, it is necessary to consider the equilibrium of the frame in its slightly buckled state as shown in Figure 5.2. This state can be obtained by superimposing on the symmetrical deformations an infinitesimal antisymmetrical deformation which corresponds to a set of small variation in the joint rotations $\partial\theta$ and sway $\partial\delta$ of the members. These are:

1)Equal incremental rotations of joints B and B' by $\partial\theta_B$.

2) Incremental rotation of joint C by $\partial\theta_C$.

3) Equal incremental sways of AB and A'B' by $-\partial\delta$. The rafters have no changes in their sways since there is equal displacement at B and B'.

Associated with these variations in the deformations are changes ∂P in the axial forces in a typical member which in turn cause changes in the stability functions.

The modified moments at the ends of the members are tabulated in Table 5.1, in which

$$\left.\begin{aligned} \partial s &= \frac{ds}{dP}\partial P \\ \partial(sc) &= \frac{d(sc)}{dP}\partial P \\ \partial\alpha &= \frac{d\alpha}{dP}\partial P \\ \partial A &= \frac{dA}{dP}\partial P \end{aligned}\right\} \qquad 5.6$$

The terms $\frac{ds}{dP}$, $\frac{d(sc)}{dP}$ .....etc, represent respectively the rate of change of the coefficients s,sc,...etc with respect to the axial force P. Their values are given in terms of s,c,α,P and B by the following expressions (13)

$$\left.\begin{aligned} s' &= \frac{ds}{dP} = \frac{s}{2P}(1 - sc^2) \\ c' &= \frac{dc}{dP} = \frac{1+c}{2P}(1 - sc(1-c)) \\ \alpha' &= \frac{d\alpha}{dP} = \frac{\alpha}{2P}(2 - sc) \\ (sc)' &= \frac{d(sc)}{dP} = \frac{s}{2P}(1+2c - sc) \\ A' &= \frac{dA}{dP} = \frac{\alpha}{2P}(2 - sc) - \frac{2B}{P} \end{aligned}\right\} \qquad 5.7$$

The increment in the stability functions of CB are positive and those of CB' are negative. This is because the axial force is increased for CB whilst it is decreased for CB' as in Figure 5.3. The change in the moment ∂M_B at joint B is given by

$$\begin{aligned} \partial M_B = & \left[(ks)_1 + (ks)_2\right]\partial\theta_B + (ksc)_2\,\partial\theta_c + \left(\frac{k\alpha}{L}\right)_1 \partial S \\ & + \left[\left\{(k\partial s)_1 + (k\partial s)_2\right\}\theta_B + \left\{\left(\frac{k\partial\alpha}{L}\right)_1 - \left(\frac{k\partial\alpha}{L\sin\phi}\right)_2\right\} S\right] \end{aligned} \qquad 5.8$$

in which

$$\left.\begin{aligned} \partial s_2 &= \frac{ds_2}{dR}\,\partial R \\ \partial\alpha_2 &= \frac{d\alpha_2}{dR}\,\partial R \\ \partial(sc) &= \frac{d(sc)_2}{dR}\,\partial R \end{aligned}\right\} \qquad 5.9$$

where ∂R is the change in the axial force of the rafter. The horizontal force H is unchanged, thus ∂R is related to ∂P by

$$\partial R = \partial P \sin\phi \qquad 5.10$$

The change in the moment ∂M_C at joint C is given by

$$\partial M_c = 2(ksc)_2\,\partial\theta_B + 2(ks)_2\,\partial\theta_c + 2\left[(k\partial sc)_2\,\theta_B - \left(\frac{k\partial\alpha}{L\sin\phi}\right)_2 \delta\right] \qquad 5.11$$

Joint equilibrium requires that

$$\partial M_B = \partial M_C = 0 \qquad 5.12$$

On substituting 5.6,5.9,5.10,5.12 and rearranging, equation 5.8 becomes

$$\partial M_B = [(ks)_1 + (ks)_2]\,\partial\theta_B + (ksc)_2\,\partial\theta_c + \left(\frac{k\alpha}{L}\right)_1 \partial\delta$$
$$+ \left[\{(ks')_1 + (ks'\sin\phi)_2\}\,\theta_B + \left\{\left(\frac{k\alpha'}{L}\right)_1 - \left(\frac{k\alpha'}{L}\right)_2\right\}\delta\right]\partial P = 0 \qquad 5.13$$

and equation 5.11 becomes

$$\partial M_c = (ksc)_2\,\partial\theta_B + (ksc)_2\,\partial\theta_c$$
$$+ \left[\{k(sc)'\sin\phi\}_2\,\theta_B - \left(\frac{k\alpha'}{L}\right)_2 \delta\right]\partial P = 0 \qquad 5.14$$

The equilibrium of AB requires

$$H = \left\{\left(\frac{k\alpha}{L}\right)_1 \theta_B + \left(\frac{2kA}{L^2}\right)_1 \delta\right\} + \left(\frac{k\alpha}{L}\right)_1 \partial\theta_B + \left(\frac{2kA}{L^2}\right)_1 \partial\delta$$
$$+ \left[\left(\frac{k\partial\alpha}{L}\right)_1 \theta_B + \left(\frac{2k\partial A}{L^2}\right)_1 \delta\right] = 0 \qquad 5.15$$

On substituting 5.1b and 5.6 and rearranging, equation 5.15 becomes

$$\left(\frac{k\alpha}{L}\right)_1 \partial\theta_B + \left(\frac{2kA}{L^2}\right)_1 \partial\delta + \left[\left(\frac{k\alpha'}{L}\right)_1 \theta_B + \left(\frac{2kA'}{L^2}\right)_1 \delta\right]\partial P = 0 \qquad 5.16$$

The equilibrium of BC requires

$$H\sin\phi - (P + \partial P)\cos\phi = \left[\left(\frac{k\alpha}{L}\right)_2 \theta_B - \left(\frac{2kA}{L^2\sin\phi}\right)_2 \delta\right] + \left(\frac{k\alpha}{L}\right)_2 \partial\theta_B$$
$$+ \left(\frac{k\alpha}{L}\right)_2 \partial\theta_c + \left[\left(\frac{k\partial\alpha}{L}\right)_2 \theta_B - \left(\frac{2k\partial A}{L^2\sin\phi}\right)\delta\right] \qquad 5.17$$

On substituting 5.6, 5.9, and 5.10, equation 5.17 becomes

$$\left(\frac{k\alpha}{L}\right)_2 \partial\theta_B + \left(\frac{k\alpha}{L}\right)_2 \partial\theta_c + \left[\left(\frac{k\alpha'}{L}\sin\phi\right)_2 \theta_B - \left(\frac{2kA'}{L^2}\right)_2 \delta + \cos\phi\right]\partial P$$
$$= 0 \qquad 5.18$$

The determinant of the coefficients of the unknowns of equations 5.13, 5.14, 5.16, and 5.18 is

$$\begin{vmatrix} (ks)_1+(ks)_2 & (ksc)_2 & \left(\frac{k\alpha}{L}\right)_1 & [(ks')_1+(ks'\sin\phi)_2]\theta_B+[(\frac{k\alpha'}{L})_1-(\frac{k\alpha'}{L})_2]\delta \\ (ks)_2 & (ks)_2 & 0 & [(k(sc)'\sin\phi)_2]\theta_B-[(\frac{k\alpha'}{L})_2]\delta \\ \left(\frac{k\alpha}{L}\right)_1 & 0 & \left(\frac{2kA}{L^2}\right)_1 & [(\frac{k\alpha'}{L})_1]\theta_B+[(\frac{2kA'}{L^2})_1]\delta \\ \left(\frac{k\alpha}{L}\right)_2 & \left(\frac{k\alpha}{L}\right)_2 & 0 & [(\frac{k\alpha'\sin\phi}{L})_2]\theta_B-(\frac{2kA'}{L^2})_2\delta+\cos\phi] \end{vmatrix}$$

$$= \Delta \qquad 5.19$$

The values of θ_B and δ are obtained from equations 5.3 and 5.4. Equation 5.19 is the anti-symmetrical sway mode stability criterion. Any load satisfying equation 5.2 and making this determinant vanish is the critical load. The coefficients of ∂P in the four equilibrium equations are due to the prebuckling deformations θ_B and δ . Ignoring these coefficients will result in a stability criterion similar to those obtained by the method given in chapter 2.

Numerical example

Forces in the members

The forces in the members and the critical load of the symmetrical sway mode can be obtained from the solution of equation 5.2 which expresses implicitly the horizontal force component H as a function of the applied load. When the dimensions of the gabled frame and its loading conditions are specified equation 5.2 can be solved numerically as in chapter 2. Curve A in Figure 5.4 shows the results obtained for a gabled frame with $L_1=L_2$, $k_1=k_2$, and $\phi=30^\circ$. The maximum value of P given in Figure 5.4 represents the elastic critical load is the symmetrical sway mode.

Anti-symmetrical sway elastic critical load

Anti-symmetrical deformations become possible when the applied loads reache such a magnitude that equations 5.2 and 5.19 are simultaneously satisfied. This implies that both modes are equally possible for the frame under this load.

A numerical solution of equation 5.19 can be obtained by the same procedure as that used previously for solving equation 5.2. By substituting the values of k and L in 5.19, the determinant becomes

$$\begin{vmatrix} s_1+s_2 & (sc)_2 & \alpha_1 & \left[(s_1'+0.5s_2')\theta_B+(\alpha_1'-\alpha_2')\frac{\delta}{L}\right] \\ (sc)_2 & s_2 & 0 & \left[0.5(sc)_2'\,\theta_B-\alpha_2'\frac{\delta}{L}\right] \\ \alpha_1 & 0 & 2A_1 & \left[\alpha_1'\theta_B+2A'\frac{\delta}{L}\right] \\ \alpha_2 & \alpha_2 & 0 & \left[0.5\alpha_2'\,\theta_B-2A_2'\frac{\delta}{L}+0.865\frac{L}{k}\right] \end{vmatrix} = \Delta \qquad 5.20$$

The load parameter ρ_1 of the stanchion, for particular ratio $\frac{\rho_2}{\rho_1}$, makings this determinant vanish is obtained by a trial and error process. θ_B and δ are obtained from equations 5.3 and 5.4.

For $\frac{\rho_2}{\rho_1} = 1.0$

First trial $\rho_1 = 0.50$

From tables (9)

s=3.2945 α=5.4881 c=0.6659 A=3.0207 B=2.4674 sc=2.1936

By using equationss5.7

s'P= -0.757 α'P= -0.532 A'P= -2.9994 (sc)'P= 0.238

By using 5.3 and 5.4

$\theta_B = 5.6 \times 10^{-2} PL/k$

$\delta/L = 6.75 \times 10^{-2} PL/k$

These values are substituted in 5.20 and gives the value of the determinant as

$\Delta = -7.00$

A lower value of ρ_1 is therefore tested.

Second trial

ρ_1=0.48 gives a value to the determinant of $\Delta = -1.74$

By linear extrapolation the load parameter making the determinant vanish is

$$\rho_1 \doteq 0.48 - 0.02 \times \frac{1.74}{5.26} =$$

$$= 0.473$$

and the external load is

$P = 0.473\ P_e$

P_e being the Euler load of the stanchion.

Curve B in Figure 5.4 shows the numerical results obtained for other assumed values of $\frac{\rho_2}{\rho_1}$. The point at which this curve intersects curve A, gives the anti-symmetrical sway elastic critical load which is $P = 0.458\ P_e$.

Curve C in Figure 5.4 is obtained from the solution of the equation

$$(kn)_1 + (ks'')_2 = 0 \qquad 5.21$$

given in chapter 3, which was obtained by ignoring the effect of the prebuckling deformations. The elastic critical load given by this curve is $P = 0.482\ P_e$.

. . .

<u>Numerical procedure</u>

The classical method of calculation is lengthy and tedious, especially when the number of roof members is increased. An alternative numerical procedure is possible. The following steps are followed.

1) Using equation 5.2, H is calculated ignoring the effect of stability on the stiffness of the members i.e s, c, α and A are given the values 4, $\frac{1}{2}$, 6 and 6.

2) The relative values of ρ_1 and ρ_2 are calculated using the H-value obtained in step 1.

3) The critical value of ρ_1 satisfying equation 5.19 is calculated.

4) The H-value for the approximate critical load of step 3 is calculated from equation 5.2.

5) The relative values of ρ_1 and ρ_2 using the H-value obtained in step 4 are calculated.

6) The critical value of ρ_1 which satisfies equation 5.19 is calculated.

Conclusion

The calculation for the frame of Figure 5.1 shows that the elastic critical load is smaller when prebuckling deformations are taken into account than it was when these deformations were ignored as in chapter 2. The two values, however, are very similar. Work carried out on rectangular portal frames with pinned feet by Lu (14) shows a similar reduction in the calculated value of the critical load as was demonstrated in a discussion (15) to Lu's paper.

This indicates that the large amount of extra calculation has little effect on the value of critical load obtained. Moreover, experimental work carried out on models of the frames calculated show that even the values of chapter 2 are lower than the experimental values. For simple frames it can be seen that taking finite deflections into account could explain the increase in the practical values but finite deflections have not been taken into account in the present work.

The calculations of chapter 2, therefore, are perfectly adequate for normal work. The object of this chapter was in fact to show how prebuckling deformations could be taken into account if necessary and also to show that normally prebuckling deformations can be ignored.

Chapter 6

Elastic instability of trusses with redundant members

PART I: Approximate estimation

Introduction

The method of estimating the elastic critical load of rigidly jointed redundant trusses used in this chapter, is based on the fact (7) that the elastic critical load of redundant truss is the largest critical load which can be obtained with any statically possible distribution of forces in its members. Such loads have in the past been determined by first assuming several different but statically possible distributions of forces in the members of the truss. The elastic critical loads were then calculated for each of these distributions and eventually, by a trial and error process, the particular distribution of forces in the members of the truss which gave the highest elastic critical load was obtained. For a highly redundant truss, this procedure can be extremely lengthy, even when critical loads are estimated only approximately as by Bolton(5).

In this chapter it shown that the elastic critical load of a redundant truss is approximately reached when the forces in the members of the truss are so distributed that as many possible of struts meeting at the critical joint of the truss, carry loads P which are equal proportions of their respective Euler load P_e i.e have the same values of P/P_e. The struts chosen for carrying P should be these which have components along the line of action of external load or which indirectly load one of the other members which has such components. The procedure has been found to give results within 10% of those obtained experimentally, and by lengthy repetitive calculations. Bolton's method of estimating the elastic critical load of a truss is used in the present work.

Technique for the redundant truss of Figure 6.1

The development of the technique put forward here for the estimation of the elastic critical load of redundant trusses, can perhaps be most easily followed by dealing with two examples. Let us first consider the cantilever truss of Figure 6.1 which will be assumed to be made from members which have the same cross-sectional properties. This truss, when pin-jointed, has one redundant member. The choice of different values for the load in a member AB, which may be designated the redundant member, gives different statically possible systems of forces in the members of the truss. Some possible systems or patterns are shown in Figure 6.1a to d. For each pattern of load distribution, the elastic critical load can be quickly determined using the Bolton method. The results obtained are those shown in column 7 of Table 6.1, in which P_e is the Euler load of each of the members AB, BC, CD and AD. From column 6 of this table it can be seen that in the critical condition the ratio of the load in the most heavily loaded strut to its Euler load is approximately constant, even though different members become the"most heavily loaded" strut.

Proceeding from Figure 6.1b to 6.1d, the forces in AB and AD decrease continually but although the critical load at first increases it decreases after the member AE becomes the most heavily loaded strut. In fact the maximum value of the critical load occurs in the region where the two struts are equally loaded. This is to be expected because the stiffness of a strut decreases ever more sharply as its axial load is increased. This means that in general if it is possible for an external load to be carried by different patterns of loading in two or more struts the most stiff arrangement will be that in which the two or more struts carry the same fraction of their Euler loads. Any other arrangement which reduces the axial compression in one of the struts at the expense of increasing it in another will give an increase in stiffness of the one strut which is more than offset by the greater reduction in stiffness of the other.

If the P/P_e values of AB and AE are equal, and the compressive force in the member AB is $\emptyset W$, the force in the member AE is $\sqrt{2}(1-\emptyset)W$. Since the Euler load of AE is twice the Euler load of AB, however, the value of P/P_e for AB is $\frac{\emptyset W}{P_e}$ and that of AE $\sqrt{2}(1-\emptyset)\frac{W}{2P_e}$. The fraction of their respective Euler loads which they carry will be equal when:-

$$\phi \frac{W}{P_e} = \sqrt{2}(1 - \phi) \frac{W}{2P_e} \qquad 6.1$$

i.e when $\phi = 0.415$

With this value of ϕ, the members AB,AD and AE all carry equal proportions of their respective Euler loads.

The elastic critical load calculated for this pattern in which ϕ has the value 0.415 is $4.31P_e$ as detailed in Appendix I. - Pg. 128

To check this value, the elastic critical load of the truss was calculated using the lengthy repetitive method already described, in which various values for the load in AB were tried in succession. The result of doing this is shown in Figure 6.2 which gives the elastic critical load of the structure as about $4.42P_e$ i.e $2\frac{1}{2}$% larger than that of our approximate calculation. As a further check a model of the truss 24inches square was made from bright steel strip. The Euler load of the member AB of this truss was 5.5 Ib. the experimental critical load of the truss was 24.5 Ib. The experimental value of the critical load is thus $(24.5/5.5)P_e = 4.44P_e$ i.e 3% higher than the value estimated.

Technique for more highly redundant trusses

The critical joint in the redundant structure of Figure 6.1 which had to be located before the Bolton method could be used, was found with ease. For a more highly redundant truss, the task of locating the least stiff joint is more elaborate. The process involved is nevertheless basically simple and can be generalized in the following procedure:-

(a) The redundant truss is reduced to separate statically determinate trusses by removing a number of members equal to the degree of redundancy of the truss. The forces in the members of these basic trusses, due to the external loads, are then determined by statics. These constitute the basic force patterns from which any pattern can be synthesized.

(b) The patterns of the forces in the trusses are combined two at a time.

(i) to reduce the forces in the most heavily loaded struts

and

(ii) so that the values of the load parameter ρ of the two most heavily loaded struts are equal in the combined patterns.

(c) From the various combination of the basic patterns, the one which gives the highest value of external load for a given value of ρ in its most heavily loaded strut is chosen. For this pattern the weakest joint is found and the forces are further adjusted by making as many as possible of the struts which meet at this joint carry the same fraction of their Euler loads. The struts chosen for adjustment should be those which have components along the line of action of the external load or which indirectly load one of the other members which has such component. During the distribution of the forces meeting at a critical joint, care must be taken that the equilibrium of the forces at any adjacent joint is not upset. The load parameter of any other struts should not exceed the load parameter of the struts which meet at the critical joint. If it turns out that some distant member is required to have a greater ρ -value, the axial load in one or more of the struts meeting at the critical joint is reduced.

(d) The critical load of the truss for this pattern of loading in now estimated.

There is a second way of obtaining the critical distribution of axial forces in the truss. This is based on the fact that the load parameter of any strut does not exceed the critical ρ -value of the truss. A limit is set to the ρ -value of any member of the truss. The axial forces in the struts and tie members meeting at any externally loaded joint are adjusted so that the maximum external load can be carried within the limiting value of ρ . The critical joint is then located and the elastic critical load calculated as illustrated in the next example.

<u>Example</u>

The elastic critical load of the thrice redundant three bay truss of Figure 6.3 is estimated. This has three redundant members. It will be assumed that its members have the same cross sectional properties; also that the truss is subjected to external load at H and the supports only, and that these loads are vertical.

The load W at joint H is carried by the members HC, HI and HJ. The maximum tensile force in HI is $\sqrt{2}P$ since the largest possible compressive force in BC and BA is P where $P = \rho P_e$ and that of HC is $2\sqrt{2}P$ since the Euler load of CI and CJ is twice those of the horizontal and vertical members. We are now in a position to find by statics the reaction at A which is $(1+\sqrt{2})P$. Therefore the applied load at H is $W = 3/2.(1+\sqrt{2})P$. This relation is obtained by taking moments about F for the whole truss. Considering the equilibrium of the forces at joint H, the force in HJ can be determined as a compression of $0.207\sqrt{2}P$. Likewise the equilibrium consideration at joint C yields that the force in CD is a compressive force of P.

The force distribution of the truss is shown in Figure 6.4a but to find the forces in the right hand panel, the magnitude of one force in this panel has to be chosen. If we assume that the force in DK is 2x, the forces in the complete framework are obtained in terms of P and x. Also x can be obtained in terms of P if at the joint D we again assume that the struts DE and DK carry equal proportions of their Euler loads. We then find from Figure 6.4b since DK has twice the Euler load of DE, that

$$\frac{\frac{1}{3}(1+\sqrt{2})P - \sqrt{2}\,x}{P_e} = \frac{2\,x}{2P_e}$$

whence

$$2\,x = P$$

With this value for the force in DK, the forces in every member of the truss and the external load, are obtained in terms of P.

The possible critical joints are B and C. At joint B there are two most heavily loaded struts and one tie member and at joint c, there are four most heavily loaded struts and one tie member, therefore joint B is stiffer than joint C. Hence joint C is the critical joint.

The force distribution of Figure 6.4c could have been obtained using the first method, since most of the struts at the critical joint are "most heavily loaded" struts.

It is now possible to proceed with the Bolton calculation of the critical load by testing the least stiff joint C. As indicated in Appendix II, this gives the critical load as $7.70P_e$.

In order to check this value a model of the truss of Figure 6.3 was made from bright steel strip. The Euler load for member CB was 5.5Ib. and the experimental critical load found was 43.3 Ib. This gives the experimental critical load as $7.85P_e$.

Experimental technique

The stiffness of a structure decreases with the increases of the axial load in its members and is reduced to zero when the truss is under its elastic critical load. The elastic critical load, then, can be estimated by plotting the stiffness of the structure against the load and the intercept of the curve on the load axis being the critical value.

If the disturbing moment applied at a joint is ∂M and the resulting rotation is $\partial \theta$, then

$$\partial M = K \, \partial \theta \qquad 6.2a$$

where K is the rotational stiffness of the joint.

When the truss is not loaded, the rotation caused by the disturbance is $\partial \theta_o$, hence

$$\partial M = K_o \partial \theta_o \qquad 6.2b$$

When the truss is loaded, the rotation caused by the disturbance is $\partial \theta$. hence

$$\partial M = K \partial \theta \qquad 6.2c$$

Eliminating ∂M between 6.2b and 6.2c yields

$$\frac{K}{K_o} = \frac{\partial \theta}{\partial \theta_o} \qquad 6.3$$

During the testing of ~~the~~ a truss for finding the elastic critical load, it is convenient to test the critical joint to obtain the best curve. The elastic critical load of the single bay cantilever truss of Figure 6.1 was estimated by this method. The curve obtained is shown in Figure 6.5.

The ultimate elastic critical load of redundant trusses can also be estimated by using Southwell plot as will be shown in chapter 8 .

In other trusses, it was not possible to test the critical joints because they were in the upper boom. Another technique was used. This was based on the fact that when a disturbance is applied to a structure it will cause a deformation in the structure. The structure, if it is in stable equilibrium, will recover its first state on the removal of the disturbance but will stay in any position if it is in neutral equilibrium. This test was performed om the most heavily loaded strut for determining the elastic critical load of the trusses.

Conclusion

This approach has been applied to several redundant trusses. Exact calculations and experimental tests for such trusses. given in Table 6.4, indicat that this method of estimating the critical load of redundant trusses gives values which are within 10% of the correct values.

PART II

Exact estimation

In redundant trusses, it is shown that the axial forces in the members are so distributed that the highest external load is carried by the redundant structure. This implies that the axial forces are so distributed that the stiffness $K = f_1(k, s, sc, \phi)$ of the structure is the highest possible i.e has a maximum value. For a maximum value of K, the first derivative of K with respect to the force ϕN in the redundant member must be zero.

i.e

$$\frac{dK}{d\phi} = f_2\left\{k, \frac{ds}{d\rho}, \frac{dsc}{d\rho}\right\} = 0 \qquad 6.2$$

This equation contains terms of the first derivative of the stability functions $ds/d\rho$ and $dsc/d\rho$. $ds/d\rho$ and $dsc/d\rho$ are related to the stability functions tabulated by Livesley and Chandler by the equations[13]

$$ds/d\rho = \frac{s}{2\rho}(1 - sc^2) \qquad 6.3i$$

and

$$dsc/d\rho = \frac{s}{2\rho}(1 + 2c - sc) \qquad 6.3ii$$

These functions are tabulated and also plotted in Figure 6.7.

From the maximum characteristic, $\frac{dK}{d\phi}$ in 6.2 will have a positive value for ϕ lower than the critical value ϕ_c and a negative value for ϕ higher than ϕ_c. Hence it is possible to check, for any load distribution whether the load is the elastic critical load or not. Equation 6.2 is used as a guide in estimating the elastic critical load by trials of ϕ.

The approximate method presented in Part I of this chapter will be used as a first trial of ϕ in the calculation as will be demonstrated in the following example.

The elastic critical load of the truss shown in Figure 6.8 will be estimated. The truss when pin jointed has one redundant member. The stiffness of the diagonal members is 0.707k, where k is the stiffness of the horizontal and vertical members.

The axial force in AB is assumed to be ϕW, hence the forces in the members are

$$P_1 = \phi W$$
$$P_2 = \sqrt{2}(1 - \phi)W \qquad 6.4$$
$$P_3 = -(1 - \phi)W$$

The stiffness of the structure for the critical mode is

$$K = k\,(s_1 + s_3 + 0.707\, s_2 - (sc)_3) \qquad 6.5$$

For any particular value of ϕ, the critical load making K vanish is obtained by trial and error. For example when the load parameters of AB and BA' are the same i.e $\phi = 0.74$, the load parameters making K vanish are

$\rho_1 = \rho_2 = 2.54$ and $\rho_3 = -0.896$

and the external load is

$$2W = \frac{2 \times 2.54}{0.74} P_e$$
$$= 6.87\, P_e$$

The value of $dK/d\phi$ is now evaluated to find out whether the calculated 2W is the elastic critical load of the truss or not. Differentiating equation 6.5 with respect to ϕ, gives

$$\frac{dK}{d\phi} = k\left(\frac{ds_1}{d\phi} + 0.707\frac{ds_2}{d\phi} + \frac{ds_3}{d\phi} - \frac{d(sc)_3}{d\phi}\right)$$

$$= k\left(\left(\frac{ds}{d\rho}\right)_1\frac{d\rho_1}{d\phi} + 0.707\left(\frac{ds}{d\rho}\right)_2\frac{d\rho_2}{d\phi} + \left(\frac{ds}{d\rho}\right)_3\frac{d\rho_3}{d\phi} - \left(\frac{dsc}{d\rho}\right)_3\frac{d\rho_3}{d\phi}\right)$$

$$= k\left[\left(\frac{ds}{d\rho}\right)_1\frac{dP_1}{d\phi}\frac{1}{(P_e)_1} + 0.707\left(\frac{ds}{d\rho}\right)_2\frac{dP_2}{d\phi}\frac{1}{(P_e)_2} + \left(\frac{ds}{d\rho}\right)_3\frac{dP_3}{d\phi}\frac{1}{(P_e)_3} - \left(\frac{dsc}{d\rho}\right)_3\frac{dP_3}{d\phi}\frac{1}{(P_e)_3}\right] \qquad 6.6$$

From 6.4

$$dP_1/d\phi = W$$

$$dP_2/d\phi = -\sqrt{2}\,W \qquad 6.7$$

$$dP_3/d\phi = W$$

Substituting these values in 6.6 and taking $(P_e)_1 = (P_e)_3 = (P_e)$ and $(P_e)_2 = \frac{1}{2}P_e$ gives

$$dK/d\phi = \{(ds/d\rho)_1 - 2(ds/d\rho)_2 + (ds/d\rho)_3 - (dsc/d\rho)_3\}\,kW/P_e \qquad 6.8$$

From Figure 6.7, $ds/d\rho$ and $dsc/d\rho$ corresponding to $\rho_{1,2} = 2.54$ and $\rho_3 = -0.896$ are

$$(ds/d\rho)_1 = (ds/d\rho)_2 = -4.9$$

$$(ds/d\rho)_3 = -1.075 \qquad (dsc/d\rho)_3 = +0.195$$

Substituting these values in 6.8 gives

$$dK/d\phi = (-4.9 + 9.8 - 1.075 - 0.195)\,kW/P_e$$

$$= +3.64\,kW/P_e$$

i.e ϕ is lower than the critical value ϕ_c, a higher value of ϕ is tried.

When $\phi = 0.8$, the critical load parameters of the members are

$\rho_1 = 2.77$ $\rho_2 = 1.96$ and $\rho_3 = -0.693$ and the external load is

$$2W = \frac{2 \times 2.77}{0.8} P_e$$

$$= 6.925 P_e$$

$(ds/d\rho)$ and $dsc/d\rho$ corresponding to these load parameters are

$(ds/d\rho)_1 = -6.60$

$(ds/d\rho)_2 = -2.98$

$(ds/d\rho)_3 = -1.11$ $(dsc/d\rho)_3 = +0.22$

Substituting these values in 6.8 gives

$dK/d\phi = -1.97 kW/P_e$

i.e ϕ is higher than the critical value of ϕ_c. Thus ϕ_c lies between 0.74 and 0.8. Any third trial of ϕ must be between these values. The critical load of the truss is $2W = 7.0 P$ obtained by the lengthy method.

Trusses with more than one redundant axial force

Naturally if there is more than one redundant members, the process of satisfying several equations similar to 6.2 would interlock and becomes very laborious. The elastic critical load of the truss shown in Figure 6.9 is estimated. There are four redundant members (with two force unknowns) when the structure is pinjointed. All the members have the same EI and equal length. The axial force in AB and AC are ϕW and ηW respectively. Hence the forces in the members are

$$P_1 = \phi W$$

$$P_2 = \eta W \qquad 6.9$$

$$P_3 = 0.707(2 - \phi - 1.73\eta)\, W$$

The stiffness of the structure is

$$K = k\,(s_1 + 2s_2 + 2s_3) \qquad 6.10$$

The first derivative of 6.10 with respect to ϕ is

$$dK/d\phi = k(ds_1/d\phi + 2\,ds_2/d\phi + 2\,ds_3/d\phi)$$

$$= k\left[\left(\frac{ds}{d\rho}\right)_1 \frac{d\rho_1}{d\phi} + 2\left(\frac{ds}{d\rho}\right)_2 \frac{d\rho_2}{d\phi} + 2\left(\frac{ds}{d\rho}\right)_3 \frac{d\rho_3}{d\phi}\right]$$

$$= \frac{k}{P_e}\left[\left(\frac{ds}{d\rho}\right)_1 \frac{dP_1}{d\phi} + 2\left(\frac{ds}{d\rho}\right)_2 \frac{dP_2}{d\phi} + 2\left(\frac{ds}{d\rho}\right)_3 \frac{dP_3}{d\phi}\right] \qquad 6.11$$

From 6.9 substituting for $dP/d\phi$ gives

$$dK/d\phi = \left[(ds/d\rho)_1 - 1.414\,(ds/d\rho)_3\right] kW/P_e \qquad 6.12$$

Similarly it can be shown that

$$dK/d\eta = \left[2(ds/d\rho)_2 - 2.44\,(ds/d\rho)_3\right] kW/P_e \qquad 6.13$$

When the ρ-values of the members are taken to be equal i.e $\phi = \eta = 0.483$, the ρ-value making K vanish is 2.048 and the external load is

$$2W = \frac{2 \times 2.048}{0.483} P_e$$

$$= 8.45\,P_e$$

$dK/d\phi$ and $dK/d\eta$ are evaluated to find the changes in ϕ and η required to obtain the highest possible value of 2W. $ds/d\rho$ corresponding to $\rho = 2.048$, obtained from Figure 6.7, is -3.17. Substituting this value in 6.12 and 6.13 gives

$$dK/d\phi = -0.414 \times -3.17\ kW/P_e$$

$$= +1.31\,kW/P_e$$

i.e a higher value of ϕ is required.

and

$$dK/d\eta = -0.44 \times -3.17\ kW/P_e$$

$$= +1.395\ kW/P_e$$

i.e a higher value of η is required.

The elastic critical load, $dK/d\phi$ and $dK/d\eta$ for other trials of ϕ and η are shown in the Table below:

ϕ	η	$2W/P_e$	$dK/d\phi$	$dK/d\eta$
0.483	0.483	8.45	+ 1.31	+ 1.395
0.5	0.5	8.51	+ 0.72	+ 0.29
0.52	0.5	8.53	+ 0.32	+ 0.05
0.54	0.49	8.55	- 0.02	+ 0.09

Trusses with more than two joints

When there are more than two joints in the truss, the approximate stiffness K of the truss, given by Bolton[5], is used:-

$$K = T_n - \sum (ksc)^2 / T_f \qquad 6.14$$

where

$T_n = \sum ks$ is the stiffness of the critical joint.

$T_f = \sum ks$ is the stiffness of the adjacent joint.

The first derivative of K with respect to any redundant axial force R is

$$dK/dR = dT_n/dR - \sum \frac{(ksc)^2}{T_f} \left\{ \frac{2(dksc/dR)}{ksc} - (dT_f/dR) / T_f \right\} \qquad 6.15$$

For example the elastic critical load of the truss shown in Figure 6.10 is estimated. The force in AB is taken to be ϕW. Hence the forces in the members are

$P_1 = \phi W$

$P_2 = \sqrt{2}(1 - \phi)W$

$P_3 = -\sqrt{2}\,\phi W$

$P_4 = -(1 - \phi)W$

Joint A is the least stiff joint, hence the approximate stiffness of the truss is

$$K = T_A - 2k(sc)_1^2/T_B - k(sc)_2^2/2T_C \qquad 6.16$$

where

$$T_A = k(2s_1+0.707s_2)$$

$$T_B = k(s_1+0.707s_3+s_4) = T_D$$

$$T_C = k(0.707s_2+2s_4)$$

The first derivative of K with respect to ϕ is

$$\frac{dK}{d\phi} = \frac{dT_A}{d\phi} - 2k^2\frac{(sc)_1^2}{T_B}\left\{2\frac{\frac{d(sc)_1}{d\phi}}{(sc)_1} - \frac{\frac{dT_B}{d\phi}}{T_B}\right\} - \frac{k^2(sc)_2^2}{2T_C}\left\{2\frac{\frac{d(sc)_2}{d\phi}}{(sc)_2} - \frac{\frac{dT_C}{d\phi}}{T_C}\right\} \qquad 6.17$$

where

$$\frac{dT_A}{d\phi} = \left\{2\left(\frac{ds}{d\rho}\right)_1 - 2\left(\frac{ds}{d\rho}\right)_2\right\}\frac{kW}{P_e}$$

$$\frac{dT_B}{d\phi} = \left\{\left(\frac{ds}{d\rho}\right)_1 - 2\left(\frac{ds}{d\rho}\right)_3 + \left(\frac{ds}{d\rho}\right)_4\right\}\frac{kW}{P_e}$$

$$\frac{dT_C}{d\phi} = \left\{-2\left(\frac{ds}{d\rho}\right)_2 + 2\left(\frac{ds}{d\rho}\right)_4\right\}\frac{kW}{P_e}$$

$$\frac{d(sc)_1}{d\phi} = \left(\frac{dsc}{d\rho}\right)_1\frac{kW}{P_e}$$

$$\frac{d(sc)_2}{d\phi} = -2.828\left(\frac{dsc}{d\rho}\right)_2\frac{kW}{P_e}$$

When the load parameters of AB and AC are taken to be equal i.e $\phi=0.74$, the load parameters of the members making K vanish are $\rho_1 = \rho_2 = 1.76$ $\rho_3 = -4.98$ and $\rho_4 = -0.621$ and the external load is

$$W = \frac{1.76}{0.74}P_e$$

$$= 2.379P_e$$

$dK/d\phi$ is evaluated to find the change in ϕ required to obtain the elastic critical load. The stability functions and their derivatives corresponding to the load parameters are

$s_{1,2}=0.8233$ $(sc)^2_{1,2}=10.0587$ $(sc)_{1,2}=3.1715$ $(ds/d\rho)_{1,2}=-2.63$ $\left\{\frac{dsc}{d\rho}\right\}_{1,2}=\pm 1.29$

$s_3= 8.4$ $(ds/d\rho)_3= -0.663$

$s_4= 4.756$ $(ds/d\rho)_4= -1.14$

Substituting these values in 6.17 gives

$$\frac{dK}{d\phi} = \left[0 - \frac{2\times 10.0587}{11.5293}\left\{\frac{2.58}{3.1715} + \frac{2.444}{11.5293}\right\} - \frac{10.0587}{20.186}\left\{-\frac{5.656\times 1.29}{3.1715} - \frac{2.98}{10.093}\right\}\right]\frac{kW}{P_e}$$

$$= \{-1.79 + 1.29\}\ kW/P_e$$

$$= -0.5\ kW/P_e$$

i.e a lowerr value of ϕ is required.

The elastic critical load and $dK/d\phi$ for other trials of ϕ are tabulated below:

ϕ	W/P_e	$(dK/d\phi)/(kW/P_e)$
0.74	2.379	- 0.50
0.72	2.386	+ 0.65
0.70	2.373	+ 1.75

This example was also solved by Bolton* by using a different approach for finding ϕ which gives the elastic critical load of the truss.

* A. Bolton, "The elastic instability of trusses with redundant members" To be published in the Structural Engineer.

Appendix I

Table 6.2 shows a stiffness calculation for the distribution of forces shown in Figure 6.1f. The stiffness calculated is -0.349 units.

The calculation was repeated for a ρ-value of 1.78 in AB, and the stiffness obtained was +0.11. By linear interpolation the value of ρ_1 at the critical load is:-

$$= 1.78 + \frac{0.04 \times 0.11}{0.46}$$

$$= 1.79$$

Therefore

$$W = 1.79P_e/0.415$$

$$= 4.31\ P_e$$

Appendix II

Table 6.3 shows a stiffness calculation for the distribution of forces shown in Figure 6.4c. The stiffness calculated is -5.85 units.

The calculation was repeated for a ρ_1-value of 2.16 and the stiffness obtained was -1.03 units. By linear extrapolation the value of ρ_1 at the critical load is:-

$$= 2.16 - \frac{0.14 \times 1.03}{4.82}$$

$$= 2.13$$

Therefore $P = 2.13\ P_e$

and $W = 2.13 \times \frac{3}{2}(1+\sqrt{2})\ P_e$

$$= 7.7\ P_e$$

Chapter 7

The elastic instability of trusses with redundant supports

In chapter 6 it was shown that an approximation could be used to find with relatively a small amount of arithmetic the critical load of a frame with redundant members. In this chapter the method is used to calculate the critical load of frames which are statically determinate except for the presence of redundant supports and frames which have redundant members as well as redundant supports.

Redundant supports

At small loads the reactions can be obtained by classical methods like strain energy, which assume that the forces in and the extensions of the members are linearly related. At high loads and in particular close to the elastic critical load this assumption no longer applies. Bending of the members either because of magnified initial eccentricity or secondary changes of geometry causes appreciable shortening of the members. This shortening can be introduced into the analysis as an extra term[8] and the effect on the values of the reactions obtained but the analysis then becomes very tedious unless a digital computor is available.

On the other hand merely to find the elastic critical load, at which the frame would buckle even if it remained elastic, is comparatively easy. There is an analogy here with plasticity. To work through the elastic range into the partly plastic range and so to the plastic collapse load requires very tedious calculation. But merely to calculate the plastic collapse load is relatively easy, it is in the intermediate calculation when the frame is partly plastic and partly elastic that the difficulty arises.

In a similar way it is difficult to analyse a frame when buckling is beginning to have a large effect on the members but easy when the critical load is reached and the stiffness of every joint becomes zero. By finding the values of reactions which give the highest critical load it is possible to obtain these values in one computation instead of attempting to find how the original elastic reactions are redistributed in step by step calculations.

It is interesting to note that there are two general types of reaction which must be considered. One is the usual type which induces both compressive and tensile forces in the truss members. The other is the type which induces only tensile forces. For example the bottom tie of a roof truss might act as a catenary supporting loads by direct tension, if a redundant horizontal reaction is introduced at the supports. Obviously this kind of behaviour arises if a member, or a chain of members in a straight line, connects two supports where redundant reactions are provided along the line of the members. For convenience reactions of this type will be referred to as catenary reactions. The importance of catenary reactions is that they allow a very large redistribution of the forces in the members, since tension members are inherently stable and the remaining structure sheds its load onto the stable portion of the structure. With normal reactions this effect does not occur since the ties can only be loaded by the struts with which they must be in statical equilibrium and when the struts have reached the limit of their strength the ties can not be loaded further.

Method of calculation for ordinary redundant reactions

In finding the elastic critical load of a redundant structure there are two stages in the problem. The first is to find the appropriate distribution of forces in the members and the second is to calculate the critical load. This is necessary since the stiffness of the members and hence the critical load depends on the axial loads carried. There is a vicious circle here, because the "appropriate" distribution of forces in the members is that which gives the highest value of the critical load.

It is not necessary to revert to iterative calculations however, since the two stages can be kept separate if the possibility of an error, which seems unlikely to exceed 10% in practice, can be tolerated. It is a common observation amongst engineers calculating critical loads that for any given type of truss the ratio of the axial force in the most heavily loaded strut to its Euler load is very nearly constant. In other words, if it is found that for a certain truss, for example, the biggest axial force at the critical load is 1.9 times the Euler load of the strut concerned, then any other load case, and even if some other strut becomes most heavily loaded, the new critical load will also occur at a pproximately 1.9 times the Euler load of the strut concerned. Hence it is possible in a redundant structure to find the "appropriate" distribution of forces without doing any critical load calculations at all. All that is required is to find the pattern of force distribution in the members which gives the highest value of external load for a given value of P/P_e in any strut of the framework. The P/P_e value has been designated ρ in a very useful table[9] of member stiffnesses which is latter used in the calculation of critical load.

To find the required pattern of forces it is convenient to reduce the redundant structure to a number of statically determinate primary structures by removing a number of reactions equal to the degree of redundancy. These primary statically determinate structures are then combined in the way which gives the highest external load for a given value of ρ . If any strut has a higher value of ρ than the others, the

ρ -value is reduced, if this is possible, by using a greater proportion of the particular primary structure which unloads this member. After a few steps of this kind the critical joint (the weaker of the two joints at the ends of the critical strut) will be determined. The pattern of forces is now adjusted by loading as many as possible of the struts, which meet at the critical joint, to the same value of ρ as the critical strut. The struts chosen for adjustment should be these which have components along the line of action of external load or which indirectly load one of the other members which has such component. This procedure is illustrated in examples 1 and 2.

Example 1

The truss shown in Figure 7.1 has one redundant reaction. All members have been taken as having the same cross-section which means that the Euler loads of each horizontal and vertical member is twice that of the diagonals.

The three primary statically determinate trusses obtained by the elimination of the reactions at H, E and A respectively are shown in Figures 7.2a, 7.2b and 7.2c.

In Figure 7.2a, member CE has the highest ρ -value and C is the critical joint since three struts and one beam meet at C but a tie and two beams meet at E. The external load W for this distribution is easily obtained:-

$$\frac{\sqrt{2}W}{2} \Big/ \frac{P_e}{2} = \rho$$

$$\therefore \quad W = 0.707\rho P_e$$

where P_e is the Euler load of a vertical strut

In Figure 7.2b member CE has again the highest ρ-value and C is again the critical joint. This time:-

$$\sqrt{2}W/3 = \rho P_e/2$$

$$W = 1.06\rho P_e$$

Any combination of the patterns of Figures 7.2a and 7.2b will obviously results in a value of W lying between $0.707\rho P_e$ and $1.06\rho P_e$.

In Figure 7.2c, CE again has the highest ρ-value but this time E is the critical joint. The value of W becomes $P_e/2\sqrt{2}$. Hence this distribution of forces would carry only half the external load of Figure 7.2a for the same limiting value of ρ or one third that of Figure 7.2b. The best distribution of forces for our purpose is thus that of Figure 7.2b which gives the highest external load. Any combination of Figure 7.2b with either 7.2a or 7.2c will give a lower value of external load than the $1.06\ P_e$ of Figure 7.2b.

When an attempt is made to adjust the patterns of load to make as many struts as possible have the same ρ-value, it is realised that conditions of equilibrium result in member FH having a much higher value of ρ for the same external load. For example if member CD is taken to have the same ρ-value as the critical strut CE, which carried load P equilibrium conditions applied in turn at joints D, B, C and F give the axial force in FH as $\sqrt{2}(2-1/\sqrt{2})P$ which is greater than P, the load in the critical strut. Hence FH would have buckled before CE and the external load would have been $0.74P_e$. Example 2 given later shows the procedure used when adjustment of the forces is possible.

The critical load for the pattern of forces in Figure 7.2b was calculated and was found to be $2.62P_e$. As a check on this value a truss was made of bright steel strip of ½"X1/16" cross-section and with panels 2'0" square. The experimental critical load was found to be 15 lbs. and the value of P_e was 5.5 lbs. Hence the experiment gave the critical load as :-

$$W = (15/5.5)P_e$$
$$= 2.73\ P_e$$

It is a matter of interest that the frame supported as in Figure 7.2b has a higher critical load than the same frame supported as in Figure 7.2a although this seems to be the opposite of what might be expected. Experiment, however, confirmed the theory given above since the frame supported as in Figure 7.2a would only carry about 12.5 lbs.

Example 2

The truss shown in Figure 7.3 is similar to that of Figure 7.1 but it has an extra bay and an extra redundant reaction . It is necessary to use this more complicated frame to illustrate the adjustment of the forces in the members meeting at the critical joint when this has been determined as in example 1.

The three primary structures considered are shown in Figures 7.4a, 7.4b and 7.4c (it is obvious from example 1 that there is no point in considering the modes in which the reaction at A is removed).

In Figure 7.4a BC is the most heavily loaded strut and C is the critical joint. The external load W is given by:-

$$W = 4\rho P_e/3$$
$$= 1.33\rho P_e$$

where P_e is the Euler load of BC.

In Figure 7.4b, CE is the most heavily loaded strut and C is again the critical joint. The corresponding external load is:-

$$W = 3\rho P_e/2\sqrt{2}$$

$$= 1.06\rho P_e$$

In Figure 7.4c, CE is again the most heavily loaded strut and C is again the critical joint. The corresponding external load is:-

$$W = \rho P_e/\sqrt{2}$$

$$= 0.707 P_e$$

Since in all three patterns C is the critical joint, any combination of the three will result in C remaining the critical joint. Hence C is assumed to be the critical joint and this takes us over the first stage of the calculation.

The second stage is to adjust the forces in the members meeting at C so that as many struts as possible have the same maximum value.

It is convenient to start with the distribution of Figure 7.4a since W is then a maximum. To check it if is possible to give the struts BC, CD and CE equal ρ values, the distribution of Figure 7.5 was checked. The force in CE is only half that of the forces in the other two struts since its Euler load is only half as big. The forces in the members meeting at F, E, G, I and J cannot be determined due to the indeterminacy of the supports E and I. The patterns of forces for the two cases are shown in Figures 7.6a and 7.6b. (Again it would have been possible to consider the frame supported at A, E and H, but it is immediately apparent that the ρ-value for FH must exceed the critical value).

The external load in Figures 7.6a and 7.6b is the same W =2.707P and either could be used to obtain the approximate critical load. It is apparent that both will give very nearly the same value. In fact when these were calculated both Figure 7.6a and Figure 7.6b gave $W = 2.57P_e$, 1.9% less than the elastic critical load for the mode shown in Figure 7.4b.

Effect of redundant catenary reaction

There are two effects of catenary reactions. The first is the direct action of straight chains of members in carrying the loads and the second is the appearance of large forces in the members which become curved due to joint rotations and which are prevented from shortening.

To illustrate the result of these forces on the elastic critical of structures for four simple cases are shown in Figures 7.7, 7.8,7.9 and 7.10. In the corresponding tables the critical loads for several values of catenary force have been shown. In the structures of Figures 7.9 and 7.10 the ρ-values for all struts have been made equal for the reasons given before. It is seen that the critical load increases as the catenary force increases. The effect of the catenary reaction is large when the struts at the critical joint are connected only to the members carrying this force. The difference between the critical values for R=0 and R=∞ is then about 20%. When the critical joint and its struts are only partially dependent on the redundant reaction as in Figures 7.9 and 7.10, the difference is even smaller.

When there is justification in assuming that the redundant catenary reaction is large, the assumption of fixity of the ends of the struts connected to the members carrying this unknown force, will give values very close to the elastic critical load. This is because the curve relating the elastic critical load to the redundant force R, approaches its limiting value at only moderate values of R. The redundant catenary reaction has a very small effect when the tie member carrying the unknown catenary component is not connected to a joint adjacent to the critical joint.

Example 3

To illustrate these points, the load distribution of the truss of

Figure 7.11 when there is no catenary reaction is shown. The truss members are of constant cross section and the bays are square. Using the approximation[5], the elastic critical load was found to be $2.09P_e$ where P_e is the Euler load of member AB. The most heavily loaded strut in this case was CE, but when a horizontal restraint is imposed, the load distribution changes because the members AF and FE start to deflect as the applied load is increased. Some of the load is carried by the vertical force components of these two members and this results in the rapid increase of the load in members AF and FE, Member CF then starts to carry more load to relieve member CE. The value of the external load is highest when the P/P_e values of members CF and CE are the same. This occurs when the force in CF is twice that in CE, since the Euler load of CF is twice the Euler load of CE. From the consideration of equilibrium at joints C and B, the forces in members BC, BA and BF can now be determined. Joint F is also in equilibrium under the action of the forces in CF and BF and those in AF and FE which have a resultant vertical component of $(0.738W-0.369 \times 0.707W)$. If this is considered with the small deflection of F it shows that the tensile load in AF and FE is very large and is indeterminable since the deflection is indeterminate. The load distribution will therefore be as shown in Figure 7.12 and since R and S are very large, the ends of the members connected to the tie will be considered fixed as shown in Figure 7.13.

The critical load of this frame was also calculated and found to be $3.72P_e$ as compared with the value $2.09P_e$ obtained before. The increase due to providing a horizontal reaction is therefore 78%. To check the value of $3.72P_e$ experimentally, a model of the truss was made of bright mild steel strip. The Euler load of AB was 5.5 lbs. and the elastic critical load 20.5 lbs. Hence the experimental value of the critical load was $3.73P_e$.

Trusses with redundant members as well as redundant supports

It is obvious that the amount of work involved in critical load calculation will increase as the number of redundant members and reactions increases. The number of primary structures to be considered grows and the number of possible combinations of these structures increases even more quickly. In practice it is easy to use the method demonstrated above for up to three or four redundants but for greater number it is better to use the method given in this section.

When there are a large number of redundants it is better to avoid the question of which is the most heavily loaded strut and concentrate immediately on the question of which is the critical joint. It may sometimes be obvious that one particular joint is likely to be critical. In this case all struts meeting at that joint are given the same ρ value and the axial loads in the other members are obtained by equilibrium considerations. If it turns out that no other strut is required to have a greater value of ρ then the one at the joint considered, then this joint is likely to be the critical one. The value of the external load is calculated. Generally, however, there will be some doubt as to which of two or three joints is the critical one. In this case the process is repeated for each of the joints concerned and the one which gives the highest value of external load for the given ρ value is assumed to be the critical one. If it happens that when a joint is tested some distant member is required to have a greater ρ -value, the axial load in one or more of the struts meeting at the tested joint is reduced.

There is a second way of obtaining the critical distribution of axial loads, already given in chapter 6. This is set a limit to the ρ -value for struts anywhere in the frame and then use conditions of equilibrium to adjust the forces in the members so that the maximum external load can be carried. This after a few steps indicates the critical joint. In both cases once the critical joint is located the critical load is determined by the approximation used above. These two approaches have been found to be very quick for even complicated trusses.

For illustration, the elastic critical load of the truss in Figure 7.14 has been estimated. The truss has three redundant members and one redundant support. All the members are taken to have the same cross section and the bays are square. There is one external load W applied at joint L.

The load W at joint L is carried by the tie members LG, LB and LF. The maximum tensile force in LG is $\sqrt{2}P$ and that of LB is $2\sqrt{2}P$ where $P=\rho P_e$ and P_e is the Euler load of LB. For the largest external load W, member LF must provide the largest possible tensile force. Its magnitude is dependent on how much compressive force is possible in BC and CM and the tensile force in CD. The maximum possible compression in BC and CM is P and the tensile force in CD is $\sqrt{2}P$. Considering the equilibrium at joint C, the forces in the members CF and CE can be determined since the forces in the other members are known. The tensile force in CF is $(1+\sqrt{2})P$ and the compressive force in CE is P. The external load W corresponding to this mode of Figure 7.15 is $(2+2\sqrt{2}+1/2)P=5.535P_e$. The possible critical joints are B, H, and E. At joint B there are four most heavily loaded struts and only one tie and at joint H, there are two most heavily loaded struts and one tie member, therefore joint B is weaker than H. At joint E, there are two most heavily loaded struts and two struts having $P/P_e=0.5\rho$ and the four adjacent joints are relatively stiff, therefore it would be expected that joint H is weaker than E.

The elastic critical load parameter ρ was calculated by testing the three joints in turn. The results obtained are tabulated below:-

Joint	B	H	E
$\rho_{Highest}$	2.063	2.076	2.104

This confirms that joint B is the critical joint. Therefore the elastic critical load W corresponding to the mode of Figure 7.15 is

$$W = 5.535 \times 2.063\ P_e$$

$$= 10.4P_e$$

This value was confirmed by an experiment on a model of the truss, which gave a critical load of $(62.7/5.5)P_e = 10.4P_e$.

Chapter 8

Magnification factor of redundant trusses

Statically determinate structures

It is convenient when discussing the magnification factor to consider an isolated pin jointed strut under an axial load. It has been shown, cf. (2), that due to initial imperfection, the deflection of this strut will increase hyperbolically as the load increases and will become very large as the load approaches the critical value. For this strut, the magnification factor G of the initial imperfection is approximately:-

$$G = \frac{1}{1 - P/P_e} \qquad 8.1$$

In the case of statically determinate trusses, the same magnification is used by considering a typical strut in the truss under an axial force ϕW where W is the external load and end restraints mobilized by the other members in the truss. The stiffness of the equivalent restraints are known since there is only one possible distribution of forces in the members of the truss. At the critical load W_c, this strut will have a critical load parameter $\rho_o = \phi W/P_e$. On replacing P in 8.1 by ϕW and P_e by $\rho_o P_e$, the magnification factor becomes

$$G = \frac{1}{1 - \phi W/\rho_o P_e}$$

and replacing $\rho_o P_e$ by ϕW_c yields

$$G = \frac{1}{1 - W/W_c} \qquad 8.2$$

Result 8.2 was suggested by Professor Merchant(16).

Statically indeterminate structures

The effect of indetermancy may be introduced by considering the pin jointed strut to have restraining end springs having a variable stiffness $\bar{K}$ depending on the end rotation θ of the strut such that

$$\bar{K} = S\,\theta \qquad 8.3$$

where S is constant.

In the new strut, the axial force in the strut is known to be P, but the stiffness of the end springs is unknown and is determined once the end rotation is known. Since for every end stiffness there is a corresponding elastic critical load P_c, P_c will be dependent on the end rotation θ.

To establish the elastic stability criterion for determining the elastic critical load P_c, the end rotation at any stage of loading will be assumed to be θ which gives an end stiffness of $S\theta$. Equal and opposite disturbing moments ∂M are then applied at the two ends of the strut which cause infinitesimal end rotations $\partial\theta$. The moment appearing at each end is

$$\partial M = \left[ks(1-c) + S\theta \right] \partial\theta \qquad 8.4$$

Ignoring terms of second order.

At the critical load, the stiffness of this joint $\partial M/\partial \theta$ becomes zero and this equation becomes

$$s(1-c) = - S\theta/k \qquad 8.5$$

In the absence of inital imperfection in the strut, the member will be undeformed as the end load P is increased from zero till it reaches its first buckling load P_e which is obtained by making the righthand side of equation 8.5 zero. As the load P is increased beyond P_e, deformation occurs and consequently the ends will rotate. This mobilizes restraints at the ends. These restraints in turn increase the elastic critical load P_c of the strut since any value of $S\theta/k$ bigger than zero will give P/P_e bigger than unity. The upper limit to the elastic critical load of this strut is $4P_e$ which is obtained when $S\theta/k \rightarrow \infty$. The graph relating the elastic critical load P_c/P_e to the deformation is shown in Figure 8.1.

In the presence of initial imperfection in the strut, the ends will be rotated right from the begining of loading and a restraint is mobilized from the start of loading. The magnitude of rotation will be dependent on the initial imperfections and the axial force in the strut. The relationship between the end rotation and the applied load can be established when the initial imperfections are known. Assume that the above strut has an initial curvature of the form $y_o = a \sin(\pi x/L)$ where 'a' is the deflection at the centre of the member at zero load and the initial rotation of the ends is $\pm a\pi/L$. When an end load P is applied and the ends are free to rotate, the central deflection according to 8.1 will be changed to $\bar{a}$, where

$$\bar{a} = \frac{a}{1 - P/P_e} \qquad 8.6a$$

and the end rotation becomes

$$\begin{aligned} \phi &= \pm \bar{a}\pi/L \\ &= \pm a\pi/L \cdot \frac{1}{1 - P/P_e} \end{aligned} \qquad 8.6b$$

The change in the end rotation is

$$\phi - a\pi/L = \pm a\pi/L \cdot \frac{P/P_e}{1 - P/P_e} \qquad 8.6c$$

If the ends had not been free to rotate during the application of the external load P, the fixed end moments appearing there would have been

$$M_F = \pm ks(1-c). a\pi/L \cdot \frac{P/P_e}{1 - P/P_e} \qquad 8.6d$$

When the strut has end springs, the ends will rotate by $\pm\theta$ and the moments at the ends of the strut are

$$M = \pm \ ks(1-c)\theta - ks(1-c). a\pi/L \cdot \frac{P/P_e}{1 - P/P_e} \qquad 8.6e$$

This moment is equal to the moments provided by the springs which is

$$M = \mp S\theta^2 \qquad 8.6f$$

Substituting 8.6f in 8.6e and rearranging the equation

$$S\theta^2 + ks(1-c)\,\theta - ks(1-c). a\pi/L. \frac{P/P_e}{1 - P/P_e} = 0 \qquad 8.7$$

Solution of equation 8.7 gives the relationship between θ and P for a particular value of a/L. Thus the elastic critical load will be P_e for P = 0 and will be greater than P_e as soon as any load P is applied. The rate of increase in the elastic critical load P_c will be dependent upon the amount of initial imperfection in the strut as shown in Figure 8.2. It follows that the elastic critical load P_c is variable and can be expressed as

$$P_c = \rho P_e$$

where ρ is the critical load parameter obtained from equation 8.5. On replacing P_e in 8.1 by ρP_e ; the magnification factor becomes

$$G = \frac{1}{1 - P/\rho P_e} \qquad 8.8$$

The increased initial imperfection in the strut will push up the elastic critical load P_e nearer to the ultimate value $\rho_c P_e$.

Redundant trusses

Redundant trusses can be treated in the same way. A typical strut having an axial load ϕ'W and with its ends restrained will be considered to investigate the magnification factor of redundnat trusses. The end restraints will have variable stiffnesses depending upon the load distribution in the truss. This distribution changes when the joints are deformed and the change in the stiffness of the restraints will depend upon the amount of rotations. Thus the elastic critical load W_c

will also depend on the deformation in the truss.

In the case of exactly constructed redundant trusses with no eccentricity and ignoring secondary deformations, the force distribution at the working load can be obtained by the classical methods and is a definite one. For this particular force distribution, there is a first elastic critical load W_1. If the applied load W exceeds this value, joint rotations will occur in the truss which causes a redistribution of the forces in the members of the truss such that more external load W can be carried. This redistribution continues until the highest possible load W_c is reached. The truss during this redistribution will have $W_1 < W_{cr} < W_c$ depending on the applied load W.

In redundant trusses, the difference between W_1 and W_c is not so large as that of the isolated strut. Table 8.1 shows numerical values of these loads for different redundant trusses.

Frame	W_1	W_c
Fig. 6.6A	$1.925P_e$	$2.385P_e$
6.1	$4.05\ P_e$	$4.45\ P_e$
6.8	$7.00\ P_e$	$7.00\ P_e$
6.6c	$2.00\ P_e$	$3.42\ P_e$

Table 8.1

In the presence of imperfection in the truss and because of secondary stresses, deformation will grow from the start of loading. This causes a redistribution of the forces in the truss and the elastic critical load W_c being always bigger than W_1 when the external load is applied. Thus a good approximation to the magnification factor can be obtained by taking the ultimate critical load as the elastic critical load.

Southwell plot

Figure 8.1 shows that there is a region near the ultimate elastic critical load P_c where the increase in the end stiffness results in a very small change in the elastic critical load of the strut. This behaviour is also to be expected in redundant trusses near the ultimate elastic critical load. This means that the loading in the truss is largely redistributed at the begining of the loading and the relative redistribution is much smaller near the ultimate distribution. Thus, a good estimation of the ultimate elastic critical load can be obtained by a southwell plot in this region.

An experiment was carried out on a square redundant truss shown in Figure 8.3. A plot of $\frac{\Delta - \Delta_0}{W - W_0}$ against $\Delta - \Delta_0$ is shown in Figure 8.4. Δ_0 being the deflection of a point on AD 0.36L from A at load W_0 and Δ is the deflection of this point at any value of W. Member AD is chosen because its ends do not displace during the deformation of the truss. The curve obtained is smooth running into a straight line for large deformation in the truss. This curve shows that the elastic critical load of the truss is increased as the external load W is increased till it reaches a region where more or less no further redistribution occurs. This region gives a linear relationship between $\frac{\Delta - \Delta_0}{W - W_0}$ and $\Delta - \Delta_0$ and consequently the value of the ultimate elastic critical load is given by the inverse gradient of the straight line. W_c obtained from the plot is $(24.5/5.5)P_e = 4.45P_e$ compared with $4.45P_e$ obtained by the lengthy method of calculation shown in chapter 6.

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Chapter 9

Accuracy of Bolton's method

Introduction

Bolton assumed that only members connected to the least stiff joint of the truss and the joints adjacent to it, have a significant effect on the elastic critical load of the truss. In assessing this stiffness, members radiating from joints adjacent to the joint considered are assumed to be fixed at their remote ends. The least stiff joint is located by comparing the stiffnesses of the two joints at the ends of the member which carries the highest fraction of its Euler load. When the rotational stiffness of this joint vanishes, Bolton demonstrates that for non-redundant trusses a good approximation to the elastic critical load of the truss is obtained.

This approximate method gives a good estimate of the elastic critical load of trusses when there is only one critical joint and there are many members radiating from this joint. But when there are few members connecting the critical joint to adjacent joints or when there is more than one critical joint perhaps because of symmetry, the estimate given by this method may be as much as 10% bigger or smaller than the true elastic critical load, although for statically determinate trusses the error is not likely to be greater than 5%.

A few exact calculations of the elastic critical load of redundant trusses have been made and these values will be compared with the approximate values obtained by Bolton's method.

(i) Few members connecting the critical joint to the adjacent joints

Usually in this case the difference between the exact and the approximate value is small for normal trusses. The stability of the truss of Figure 9.1 will be first investigated using an exact method. There are two possible modes of elastic instability for this truss, the symmetrical and the anti-symmetrical modes.

(a) Symmetrical mode

This mode is only possible in symmetrical trusses under symmetrical loading. Symmetrical joints have equal and opposite joint rotations at the critical load. The stability condition is obtained by applying equal and opposite disturbing moments ∂M_A at joints A and A' which causes equal and opposite rotations $\partial \theta_A$ at joints A and A' as in Figure 9.1a. Joint B will not rotate since there are equal and opposite deformations in the struts connected to this joint. Joints A and A' are in equilibrium, hence

$$\partial M_A = \left[k_1 s_1 + k_2 (s(1-c))_2\right] \partial \theta_A$$

At the critical load, the ratio $\partial M_A / \partial \theta_A$ which measures the stiffness of the joint vanishes, hence

$$s_1 + k\,(s(1-c))_2 = 0 \qquad 9.1$$

where $k = k_2/k_1$

The external load which satisfies 9.1 will be the symmetrical elastic critical load of the truss.

(b) Anti-symmetrical mode

In the case of symmetrical trusses under symmetrical loading, similar joints will have equal rotations. The stability condition is obtained by applying a disturbing moment ∂M_B at joint B which causes a rotation of $\partial\theta_B$ at joint B and $\partial\theta_A$ at joints A and A', shown in Figure 9.1b. Joints A and A' are in equilibrium and the applied disturbance ∂M_B vanishes at the critical load. Hence

$$\partial M_A = \left[(k\,s)_1 + (ks(1-c))_2\right]\partial\theta_A + (ksc)_1\partial\theta_B = 0 \qquad 9.2a$$

$$\partial M_B = 2(ksc)_1\partial\theta_A + 2(ks)_1\partial\theta_B = 0 \qquad 9.2b$$

The determinant Δ of the coefficients of the unknowns is reduced to

$$\Delta = 2k_1^2\left[s_1(s_1 + k(s(1-c)_2) - (sc)_1^2\right] \qquad 9.3$$

The external load which makes Δ vanish is the anti-symmetrical elastic critical load.

According to Bolton's approximate method, the stability condition is

$$\Delta = 2/(s_1 + ks_2)\ .\ \left[s_1(s_1 + ks_2) - (sc)_1^2\right] \qquad 9.4$$

In this condition, the term $\left[s_1(s_1+ks_2) - (sc)_1^2\right]$ has the lowest load parameter ρ and is the stability condition for estimating the elastic critical load. This expression is similar to the exact expression 9.3 except for the presence of $s_1 k(sc)_2$ in 9.3. Therefore the ρ-value satisfying 9.3 must be higher than that satisfying 9.4.

It may be of interest to calculate the values given by 9.1, 9.3 and 9.4 for a particular case. When all the members of the truss are taken to have equal lengths and EI-values and the axial tensile force in AA' is taken to be 2.5 times the force in AB, the load parameter $(P/P_e)_{AB}$ satisfying 9.1 will be bigger than 2.05 since $(s(1-c))_2$ has a positive value. The value of ρ_{AB} satisfying 9.3 is found to be 1.715 and that satisfying 9.4 is 1.671 which is less than the exact value by about 2.5%. These calculations show that the anti-symmetrical mode is the important mode and that Bolton's approximation is 2.5% low. It will be noticed that as the tensile force in AA' is increased, the value of $(sc)_2$ is reduced thus reducing the difference between the exact and the approximate values of ρ_{AB}. The difference will be bigger when the tensile force is reduced. The difference will grow much bigger when the force in AA' becomes compressive. For example, when the compressive forces in all the members are equal, the value of ρ_{AB} satisfying 9.3 is 1.567 and that satisfying 9.4 is 1.295 less than the exact value by 24%.

(ii) Presence of more than one critical joint

Sometimes there is symmetry in loading together with symmetry in the truss which causes more than one joint to become critical at the same time. The elastic critical load of the truss in Figure 9.2 will be estimated by the exact and approximate methods of estimation. There are also two possible modes of elastic instability:-

(a) Symmetrical mode

When equal and opposite disturbing moments ∂M_A are applied at joints A and A', there will be only equal and opposite rotations $\partial\theta_A$ at joints A and A'. The other joints do not rotate since there are equal and opposite deformations in the members connected to joints B and B' shown in Figure 9.2a. The moments ∂M_A at joints A and A' vanish at the critical load so the stability condition is

$$\Delta = 2s_1 + k(s(1-c))_3 = 0 \qquad 9.5$$

where k is the relative stiffness of the diagonal members.

(b) Anti-symmetrical mode

When equal disturbing moments ∂M_A are applied at joints A and A' they will rotate clockwise through an angle $\partial\theta_A$ and joints B and B' through an angle $\partial\theta_B$, shown in Figure 9.2b. Joints B and B' are in equilibrium and the disturbance ∂M_A vanishes at the critical load, hence

$$\partial M_A = \left[(2ks)_1+(k\alpha)_3\right]\partial\theta_A + (2ksc)_1\,\partial\theta_B = 0 \qquad 9.6a$$

$$\partial M_B = (2ksc)_1\partial\theta_A + \left[(2ks)_1+(k\alpha)_2\right]\partial\theta_B = 0 \qquad 9.6b$$

The determinant of the coefficients of the unknowns is reduced to

$$\Delta = (2s_1+k\alpha_3)(2s_1+k\alpha_2) - 4(sc)_1^2 = 0 \qquad 9.7$$

According to Bolton's approximate method, the stability condition is

$$\Delta = (2s_1+ks_3) - \frac{(ksc)_3^2}{2s_1+ks_3} - \frac{2(sc)_1^2}{2s_1+ks_2} = 0 \qquad 9.8$$

If we take k=0.707, the truss to be square, and the load parameter of AA' equal to that of the other struts, the stability condition 9.5 becomes

$$\Delta = 0.707s_1\ (3.828 - c_1)$$

The load parameter ρ_1 for which c_1=3.828 is approximately 1.76. The load parameter $(P/P_e)_1$ satisfying 9.7 and 9.8 are found to be 1.744 and 1.58 respectively. The mode of failure is the anti-symmetrical mode. Therefore, the elastic critical load given by Bolton's methos 9.8 is less than the exact value by about 9.4%.

In this example, Bolton's method underestimates the elastic critical load. Sometimes it overestimates the elastic critical load. For illustration, the elastic critical load of the same truss will be estimated when the diagonal tie member BB' is replaced by two hinged supports at B and B' as shown in Figure 9.3. This truss has similar modes of elastic instability and the stability conditions will be those of the previous truss except for the omission of the stiffness term of the removed tie member. The symmetrical mode condition will be unchanged as it follows that the symmetrical mode load parameter $(P/P_e)_1$ must again be 1.76. The anti-symmetrical mode condition becomes

$$\Delta = s_1(2s_1+k\alpha_2) - 2(sc)_1^2 = 0 \qquad 9.9$$

and the approximation 9.7 becomes

$$\Delta = \left[2s_1+ks_3\right] - \frac{(sc)_1^2}{s_1} - \frac{(ksc)_3^2}{2s_1+ks_3} \qquad 9.10$$

The elastic load parameters ρ_1 satisfying 9.9 and 9.10 are found to be 1.263 a nd 1.342 respectively. When the approximate value of ρ_1 is compared with the critical value of ρ_1, we notice that Bolton's method estimates ρ_1 5.9% higher than the exact value.

When the redundant truss of Figure 9.3 is made statically determinate by removing the supports B and B' we obtain a statically determinate truss with two critical joints. In this truss, the external load is taken be the diagonal strut only. The exact elastic critical load parameter $\rho_{AA'}$ is found to be 2.95 and the approximate value, obtained by Bolton's method is 2.89 with a difference of about 1.7%.

The stability of the truss of Figure 9.2 was also investigated when the truss was loaded as in Figure 9.4. The condition of elastic instability and the critical load parameter ρ are tabulated in Table 9.1

All the members are taken to have the same value of P/P_e. The critical value of P/P_e was also calculated by the exact and approximate methods for the trusses in Figure 9.5. The results obtained are tabulated in Table 9.2.

Mode	Stability condition	P/P_e when k=0.707
(i) Symmetrical mode:		
(a) Diagonals have single curvatures, Figure 4b	$ks(2/k + 1 - c)$	1.76
(b) Diagonals have double curvatures, Figure 4c	$(2-k)s\left(\frac{2+k}{2-k} - c\right)$	1.53
(ii) Anti-symmetrical mode, Figure 4a	$(2+k)s(1+c)$	2.05
Bolton's approximate method.	$(2+k)s - \frac{2(sc)^2}{(2+k)s} - \frac{(ksc)^2}{(2+k)s}$	1.42

Table 9.1

Frame	Approximate P/P_e (1)	Exact P/P_e (2)	(1)/(2)
Fig. 9.5a	2.06	2.11	0.975
Fig. 9.5b	2.09	2.13	0.983

Table 9.2

Chapter 10

Elastically restrained supports and connections

Elastically restrained supports

The elastic critical load of a rigidly jointed portal with pinned feet is considerably lower than that of an identical fixed base rigidly jointed frame. By fixing the stanchions bases of a pinned base frame, the elastic critical load can be increased by a factor of 4 for frameworks having relative beams stiffnesses k bigger than 1.0, and a factor bigger than 4 and up to infinity for k smaller than 1.0. This is shown in the graph in Figure 10.1.

In practice, stanchion bases are neither fully pinned nor fully fixed but they are intermediate between the two extremes i.e there is a rotational restraint. This rotational restraint increases the elastic critical load of the rigidly jointed frame relative to the pin jointed frame. If a reasonable value of the rotational restraint is provided at the bases of the stanchions, the elastic critical load can be increased almost to that of the identical frame with fixed feet. This is shown in the graph in Figure 10.2.

The rotational restraint at the feet of the stanchions is provided either by inserting a restraining beam between the stanchion bases or by using a stiff foundation.[17] In the first case the restraining moment and the rotational deformation are linearly related.

In the second case it is assumed that action (moment) and deformation (rotation) are linearly related. The foundation is represented by a spring having a rotational stiffness γ such that $M = \gamma \theta$ where M is the restraining moment provided by the foundation and θ is the rotation of the foundation which is caused by the deformation of the soil.

In the frames considered, two modes of elastic instability are possible and these are the anti-symmetrical sway and joint rotation modes. Of these, the anti-symmetrical sway mode is the important one since it has the least elastic critical load. Therefore the analysis of the anti-symmetrical sway mode is shown here.

To establish the relationship between the rotational restraint at the bases of the stanchions and the carrying capacity of the frames, the analyses for establishing the stability criterion of a single bay and single storey portal frame and a framework with inclined members are shown.

Single bay portal

On the application of a disturbing moment ∂M; the joints A and A' of the portal in Figure 10.3 will have no shear rotation $\partial\theta_A$ and joints B and B' have no shear rotation $\partial\theta_B$. Due to these deformations, the changes in the moments at the ends of the members are:

Operation	Restraint moment	M_{AB}	M_{BA}	$M_{BB'}$
1)Rot.A&A'	$\gamma\,\partial\theta_A$	$(kn)_1\partial\theta_A$	$-(k\alpha)_1\partial\theta_A$	
2)Rot.B&B'		$-(k\alpha)_1\partial\theta_B$	$(kn)_1\partial\theta_B$	$(k\alpha)_2\partial\theta_B$

Equilibrium consideration yields

$$\partial M_A = (\gamma + (kn)_1)\,\partial\theta_A - (ko)_1\partial\theta_B = 0 \qquad 10.1a$$

$$\partial M_B = -(k\alpha)_1\partial\theta_A + \left[(kn)_1 + (k\alpha)_2\right]\partial\theta_B = 0 \qquad 10.1b$$

The determinant of the coefficients of the unknowns is

$$\begin{vmatrix} \gamma + (kn)_1 & -(k\alpha)_1 \\ -(k\alpha)_1 & (kn)_1 + (k\alpha)_2 \end{vmatrix} \qquad 10.2$$

This is the stability condition, any load making this determinant vanish is the critical load. This equation can be solved numerically when the relative stiffness of the beam and the supports are specified. Some numerical results are tabulated for different values of γ/k_1 when $k_2=k_1$.

γ/k_1	0	1.0	2.0	4.0	10	20	00
$2P/P_e$	0.369	0.649	0.831	1.04	1.268	1.373	1.495

These results are also plotted in Figure 10.2

Framework with inclined members

For the framework of Figure 10.4, only one degree of sway freedom is possible. On the application of a disturbing horizontal force $2H_B$ at joint B, the following deformations occur:

1) Joints A and A' rotate clockwise through an angle $\partial\theta_A$.

2) Joints B and B' rotate clockwise through an angle $\partial\theta_B$.

3) Members AB and A'B' will sway clockwise by $\partial\delta$.

Member BB' sway anti-clockwise by $2\partial\delta\cos\emptyset$, which is obtained from the Williot diagram.

Due to these deformations, there will be moments in the members and these are tabulated

Operation	Restraint moment	M_{AB}	M_{BA}	$M_{BB'}$
1)Rot.A&A'	$\gamma\partial\theta_A$	$(ks\ )_1\partial\theta_A$	$(ksc)_1\partial\theta_A$	
2)Rot.B&B'		$(ksc)_1\partial\theta_B$	$(ks)_1\partial\theta_B$	$(k\alpha)_2\partial\theta_B$
3)Sway		$-(k\alpha/L)_1\partial\delta$	$-(k\alpha/L)_1\partial\delta$	$+(2k\alpha\cos\emptyset/L)_2\partial\delta$

Equilibrium consideration yields

$$\begin{bmatrix}\partial\theta_A\\ \partial\theta_B\\ \partial\delta\end{bmatrix}\begin{bmatrix}(ks)_1+\gamma & (ksc)_1 & -(k\alpha/L)_1\\ (ksc)_1 & (ks)_1+(k\alpha)_2 & (2k\alpha\cos\emptyset/L)_2-(k\alpha/L)_1\\ -(k\alpha/L)_1 & \left[(2k\alpha\cos\emptyset/L)_2-(k\alpha/L)_1\right] & \left[(4kA\cos^2\emptyset/L^2)_2+(2kA/L^2)_1\right]\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \qquad 10.3$$

The determinant of the square matrix is the stability condition. Some numerical results obtained by the solution of the determinant when $k_2=0.707k_1$, $L_1=L_2$ and $\phi=45^\circ$ are obtained for different values of γ/k_1. These are

γ/k_1	0	1.0	2.0	4.0	10	20	∞
$2P/P_e$	0.64	0.807	0.965	1.110	1.265	1.335	1.414

These are also plotted in Figure 10.5.

Elastically restrained connections

The elastic critical load of a rigidly jointed fixed base portal frame is bigger than that of a fixed base portal frames with other joints pinned by a factor of between one and four depending on the relative stiffness of the beam. This is shown in Figure 10.6. For frameworks with inclined members, the difference between the pinned and the rigidly jointed framework is bigger and the factor is greater than four for k_2/k_1 bigger than 0.3, and smaller than 4.0 for values of k_2/k_1 smaller than 0.3, for an inclination $\phi=45^\circ$. The factor will be less for frameworks having $\phi>45^\circ$ and bigger for frameworks having $\phi<45^\circ$. This is shown in Figure 10.7.

Usually, the joint connections are neither fully pinned nor fully rigid but are in an intermediate stage between these extremes when ordinary riveted or bolted connections are used. It is found that some particular relationship exists between the restraining moment at the end of the member and the relative angular rotation between the connection and the member. It is assumed[18] that this relationship is linear of the form $M = \gamma\theta$ where M is the moment at the end of the member, θ is the relative rotation and γ is the rotational stiffness of a spring joining the member to the connection.

The analysis of the anti-symmetrical sway mode will be shown for two frameworks, to establish the relationship between the rotational restraint at the connections and the elastic critical load of these frames. To facilate the analysis, the stability functions will be modified to take into account the elastic restraint of the connections.

Modified stiffness and carry over

For member AB of Figure 10.8 when subjected to moment M_{AB} and moment M_{BA}, the connections will rotate by θ and ϕ at A and B respectively. The ends A and B of the member will rotate through θ_1 and ϕ_1 respectively. The relative rotation between the member and connection at A is therefore $\theta - \theta_1$ and that at B is $\psi - \psi_1$

The end moments are related to the relative rotation by

$M_{AB} = \gamma_1(\theta - \theta_1) = ks\,\theta_1 + ksc\,\psi_1$ 10.4a

$M_{BA} = \gamma_2(\psi - \psi_1) = ksc\theta_1 + ks\,\psi_1$ 10.4b

Putting $\gamma = k\lambda$ in the above equations and rearranging yields

$(s + \lambda_1)\theta_1 + sc\,\psi_1 = \lambda_1\,\theta$ 10.4c

$sc\,\theta_1 + (s + \lambda_2)\psi_1 = \lambda_2\,\psi$ 10.4d

Solution of these two equations gives

$$\theta_1 = \frac{\lambda_1(s+\lambda_2)}{(s+\lambda_1)(s+\lambda_2) - (sc)^2}\,\theta - \frac{\lambda_2\,sc}{(s+\lambda_1)(s+\lambda_2) - (sc)^2}\,\psi \qquad 10.4e$$

and

$$\psi_1 = -\frac{\lambda_1\,sc}{(s+\lambda_1)(s+\lambda_2) - (sc)^2}\,\theta + \frac{\lambda_2(s+\lambda_1}{(s+\lambda_1)(s+\lambda_2) - (sc)^2}\,\psi \qquad 10.4f$$

Substituting for θ_1 and ψ_1 . the end moments become

$M_{AB} = k\,S_{AB}\theta + k\,SC\,\psi$

and 10.5

$M_{BA} = k\,SC\theta + k\,S_{BA}\,\psi$

where

$$S_{AB} = \frac{\lambda_1[s(1-c^2) + \lambda_2]}{(s+\lambda_1)(s+\lambda_2) - (sc)^2}\,s$$

$$SC = \frac{\lambda_1\lambda_2}{(s+\lambda_1)(s+\lambda_2) - (sc)^2}\,sc$$

and

$$S_{BA} = \frac{\lambda_2[s(1-c^2) + \lambda_1]}{(s+\lambda_1)(s+\lambda_2) - (sc)^2}\,s$$

Special cases

(i) $\theta = \phi$ and $\lambda_1 = \lambda_2 = \lambda$

$$M_{AB} = M_{BA} = k(S+SC)\,\theta = k\,\bar{\alpha}\,\theta \qquad 10.5a$$

where

$$\bar{\alpha} = \frac{\lambda}{\lambda+\alpha}\,\alpha$$

(ii) $\phi = 0$ and $\lambda_2 = \infty$

$$S_{AB} = \frac{\lambda}{\lambda + S}\,S$$

$$SC = \frac{\lambda}{\lambda + S}\,SC \qquad 10.5b$$

and

$$S_{BA} = \frac{S(1-C^2) + \lambda}{S + \lambda}\,S$$

Modified sway moments and shear force

When the connections are displaced laterally without rotation the moments $-k\alpha\delta/L$ in the member due to the sway δ will cause end rotation θ_1 and ϕ_1 in the member at A and B respectively. The moments at the connections are

$$M_{AB} = -\gamma_1\theta_1 = ks\theta_1 + ksc\,\phi_1 - k\alpha\delta/L \qquad 10.6a$$

$$M_{BA} = -\gamma_2\theta_1 = ksc\theta_1 + ks\,\phi_1 - k\alpha\delta/L \qquad 10.6b$$

Putting $\gamma = k\lambda$, these equations can be written as

$$(s + \lambda_1)\theta_1 + sc\,\phi_1 = \alpha\delta/L \qquad 10.6c$$

$$sc\,\theta_1 + (s + \lambda_2)\phi_1 = \alpha\delta/L \qquad 10.6d$$

Solution of these two equations gives

$$\theta_1 = \frac{s + \lambda_2 - sc}{(s+\lambda_1)(s+\lambda_2) - (sc)^2} \, \alpha \frac{\delta}{L} \qquad 10.6e$$

and

$$\psi_1 = \frac{s(1-c) + \lambda_1}{(s+\lambda_1)(s+\lambda_2) - (sc)^2} \, \alpha \frac{\delta}{L} \qquad 10.6f$$

Substituting these values to give the modified sway moments

$$M_{AB} = -\frac{\lambda_1 (s(1-c) + \lambda_2)}{(s+\lambda_1)(s+\lambda_2) - (sc)^2} \frac{k\alpha}{L} \delta \qquad 10.6g$$

and

$$M_{BA} = -\frac{\lambda_2 (s(1-c) + \lambda_1)}{(s+\lambda_1)(s+\lambda_2) - (sc)^2} \frac{k\alpha}{L} \delta \qquad 10.6h$$

The shear force balancing the moments is F which is

$$F = (2kA/L^2)\delta - k\alpha/L.\ (\theta_1 + \psi_1) \qquad 10.6i$$

Substituting for θ_1 and ψ_1 and rearranging gives

$$F = \left[\frac{2kA}{L^2} - \frac{k\alpha^2}{L^2} \frac{2s(1-c) + (\lambda_1 + \lambda_2)}{(s+\lambda_1)(s+\lambda_2) - (sc)^2} \right] \delta \qquad 10.6j$$

Special cases

(i) $\lambda_1 = \lambda_2 = \lambda$

$$M_{AB} = M_{BA} = -\frac{\lambda}{\lambda + \alpha} \frac{k\alpha}{L} \delta \qquad 10.6k$$

and

$$F = \left[\frac{2kA}{L^2} - \frac{2k\alpha^2}{L^2} \frac{1}{\lambda + \alpha} \right] \delta \qquad 10.6L$$

(ii) $\lambda_2 = \infty$ and $\lambda_1 = \lambda$

$$M_{AB} = -\frac{\lambda}{\lambda + s} \frac{k\alpha}{L} \delta \qquad 10.6m$$

$$M_{BA} = -\frac{s(1-c)+\lambda}{s+\lambda}\frac{k\alpha}{L}\delta \qquad 10.6n$$

and

$$F = \left[\frac{2kA}{L^2} - \frac{k\alpha^2}{L^2}\frac{1}{1+\lambda}\right]\delta \qquad 10.6o$$

Analysis for establishing the anti-symmetrical sway mode stability criterion

Single bay portal frame

In the frame of Figure 10.9 only one degree of sway freedom is possible and this is the horizontal displacement of joint B. On the application of a horizontal disturbance $2H_B$ at joint B, the following deformations occur:-

1) Joints B and B' rotate clockwise through an angle $\partial\theta_B$.

2) Members AB and A'B' will sway clockwise by $\partial\delta$.

Due to these deformations, there will be moments in the members which are modified to take into account the effect of the elastic connections at joints B and B'. These moments are:

Operation	M_{AB}	M_{BA}	$M_{BB'}$
1)Rot.B&B'	$\frac{\gamma}{\gamma+(ks)_1}(ksc)_1\partial\theta_B$	$\frac{\gamma}{\gamma+(ks)_1}(ks)_1\partial\theta_B$	$\frac{\gamma}{\gamma+(k\alpha)_2}(k\alpha)_2\partial\theta_B$
2)Sway $\partial\delta$	$-\frac{(ks(1-c))_1+\gamma}{(k\alpha)_1+\gamma}(k\alpha/L)_1\partial\delta$	$-\frac{\gamma}{\gamma+(k\alpha)_1}(k\alpha/L)_1\partial\delta$	

Equilibrium consideration yields

$$\begin{bmatrix} \partial\theta_B \\ \partial\delta \end{bmatrix} \begin{bmatrix} \frac{\gamma}{\gamma+(ks)_1}(ks)_1 + \frac{\gamma}{\gamma+(k\alpha)_2}(k\alpha)_2 & -\frac{\gamma}{\gamma+(ks)_1}\left(\frac{k\alpha}{L}\right)_1 \\ -\frac{\gamma}{\gamma+(ks)_1}\left(\frac{k\alpha}{L}\right)_1 & \left(\frac{2kA}{L^2}\right)_1 - \frac{k_1\alpha_1^2}{L_1^2(\gamma+(ks)_1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad 10.7$$

The determinant of the square matrix is the stability condition. This expression can be solved for any value of γ and k_2/k_1 by finding the least load which makes the determinant vanish. The elastic critical load of the portal when $k_2=k_1$ was calculated for different assumed values of γ. The results obtained are tabulated and are also plotted in Figure 10.10.

γ/k_1	0	1.0	2.0	4.0	10	20	∞
$2P/P_e$	0.504	0.652	0.795	0.948	1.182	1.316	1.494

Framework with inclined member

If the supports of the framework of Figure 10.5 are fixed, the stability condition is modified to

$$\begin{vmatrix} (ks)_1+(k\alpha)_2 & (2k\alpha\cos\phi/L)_2 - (k\alpha/L)_1 \\ (2k\alpha\cos\phi/L)_2 - (k\alpha/L)_1 & (4kA\cos^2\phi/L^2)_2 + (2kA/L^2)_1 \end{vmatrix} = 0 \qquad 10.8a$$

When the joints B and B' are assumed to be elastically restrainted, the stability functions in the above stability equation 10.8a will be replaced by the modified values. The stability condition becomes

$$\begin{vmatrix} \left\{\frac{\gamma}{\gamma+(ks)_1}(ks)_1 + \frac{\gamma}{\gamma+(k\alpha)_2}(k\alpha)_2\right\} & \left\{\frac{\gamma}{\gamma+(k\alpha)_2}2\left(\frac{k\alpha}{L}\right)_2\cos\phi - \frac{\gamma}{\gamma+(ks)_1}\left(\frac{k\alpha}{L}\right)_1\right\} \\ \left\{\frac{\gamma}{\gamma+(k\alpha)_2}2\left(\frac{k\alpha}{L}\right)_2\cos\phi - \frac{\gamma}{\gamma+(ks)_1}\left(\frac{k\alpha}{L}\right)_1\right\} & \left\{\left(\frac{2k}{L^2}\right)_2\cos^2\phi\left[2A_2 - \frac{\alpha_2^2}{\gamma+(k\alpha)_2}\right] + \left(\frac{k}{L^2}\right)_1\left(2A_1 - \frac{\alpha_1^2}{\gamma+(ks)_1}\right)\right\} \end{vmatrix} = 0 \qquad 10.8b$$

When k_2/k_1, L_2/L_1 and ϕ are specified this equation can be solved numerically to give the relationship between γ and the the elastic critical load.

The elastic critical load of the framework was estimated for different assumed values of γ when $k_2=0.707k_1$, $L_2=L_1$ and $\phi=45^\circ$ The results obtained are tabulated and are also plotted in the graph of Figure 10.11.

γ/k_1	0	1.0	2.0	4.0	10	20	∞
$2P/P_e$	0.226	0.48	0.66	0.886	1.14	1.261	1.414

Comment on the results

The graphs in Figures 10.4. 10.6. 10.10. 10.11 show that the elastic critical load of frameworks increases as the restraints at the supports and connections are increased. And that the increase is very pronounced for small values of γ. The curves flatten and approach the highest value of $2P(\gamma/k=\infty)$ even at moderate values of γ/k_1. Consequently, almost the full carrying capacity of the framework could be obtained by providing reasonable restraints at the supports and connections.

Chapter 11

Bracing

Introduction

In portal frames anti-symmetrical sway, if it is possible, leads to the lowest value of critical load. If sway is prevented a large increase in the critical load results. In this chapter ways of increasing the anti-symmetrical critical load are discussed.

There are three ways of increasing the critical load for a given weight of structural material. Firstly it is possible to distribute the material between beam and stanchions in the most economic ratio. Secondly, one or more bays of the portal may be stiffened by diagonal bracing. Thirdly one or more bays may be stiffened by knee-braces. For each of these cases an analysis shows the increase in critical load which results. In the case of knee-braced frames an approximation is also proposed for single-bay single-storey portal frames and this is extended to the case of multi-storey and multi-bay frames.

Change of beam and stanchion stiffnesses

Graph Figure 11.1a shows the carrying capacity of a single storey portal framework with the variation of the relative stiffness of the beam to the stanchion. In the process of rasing the elastic critical load of a portal framework, the stiffness of the beam could be increased but the improvement is not significant when the load parameter P/P_e of the stanchion is already high. In the case of higher values of the load parameter P/P_e stiffer stanchions could be used to raise the elastic critical load.

The critical values of P/P_e for a two storey portal frame are shown in graphs Figure 11.1b for different values of beam stiffnesses. Curve A shows the variation of P/P_e for changes of the relative stiffness of the upper beam when the stiffness of the lower beam is kept constant. Curve B shows the variation of P/P_e for changes of the relative stiffness of the lower beam when the stiffness of the upper beam is kept constant. It is seen that increasing the stiffness of the beams begins to have very little effect when the stiffness of the beam is about twice that of the stanchions.

To illustrate the effect of redistribution of the structural weight between beam and stanchions Figure 11.2 has been prepared for the single storey, single bay case. It was assumed that the I-value for a rolled steel joist was related to the weight per unit length by the equation

$$I = 0.615\, w^{2.24} \qquad 11.1$$

The constants in this equation were obtained by plotting logI against logw for commericially available steel sections.

A constant weight requires that

$$2w_1L_1 + w_2L_2 = \text{constant}$$

$$= 2\, w_oL_1 \qquad 11.2$$

where w_1 is the weight per unit lenght of the stanchion.

w_2 is the weight per unit lenght of the beam.

w_o is the weight per unit lenght of the stanchion when all the material is concentrated in the stanchions.

L represents the length.

The relationship between w_1 and w_2 is obtained from 11.1 which gives

$$I_2/I_1 = (w_2/w_1)^{2.24} \qquad 11.3$$

The elastic critical load of any framework is

$$W_c = 2\rho P_e$$

Dividing both sides by $(P_e)_o$ where $(P_e)_o$ is the Euler load of the stanchion when the material is concentrated in the stanchions, yields

$$W_c/(P_e)_o = 2\rho(w_1/w_o)^{2.24} \qquad 11.4$$

since

$$P_e/(P_e)_o = I_1/I_o = (w_1/w_o)^{2.24}$$

$W_c/(P_e)_o$ can be obtained when L_2/L_1 and I_2/I_1 are specified. For the case when $L_2/L_1 = I_2/I_1 = 1.0$ the critical value of the load parameter ρ obtained from tables is

$$\rho = 0.7475$$

Using 11.3 gives

$$w_2 = w_1$$

Substituting these values in 11.2 yields

$$w_1/w_o = \tfrac{2}{3}$$

Substituting these values in 11.4 yields

$W_c/(P_e)_o = 0.61$

In a similar way the values in Table 11.1 were obtained for other ratios of I_2/I_1.

I_2/I_1	0	0.25	0.5	0.75	1.0	2.0	4.0
$W_c/(P_e)_o$	0.5	0.565	0.61	0.616	0.61	0.535	0.31

Table 11.1

If it is assumed that there is no yielding of the material Figure 11.2 shows that, when the lengths of stanchion and beam are equal, the distribution of steel which produces the highest critical load is that which makes the I-value of the beam about three-quarters that of the stanchion. A similar graph can be obtained for any ratio of span to height. Figure 11.2 is sufficient, however, to show that, even apart from any strength requirement, redistribution of material between beam and stanchion is insufficient to produce a significant increase in the critical load. It is shown later that considerable increases can be obtained by diagonal or knee-bracing and these methods are therefore investigated further.

Diagonal bracing

Diagonal bracing might be used to reduce sway in a rigidly jointed portal frame or at the other extreme it might be used to give stiffness to a pin jointed framework which would otherwise be merely a mechanism. Both cases were investigated.

(a) Rigidly jointed frames

If one of the storeys of a multi-storey single bay portal is braced by a diagonal member it acts as a very stiff beam for the storey below. If this lower storey happened to be the critical storey some improvement of the critical load would be obtained as described above but if the ρ- value of the critical stanchions were already high little improvement could be made. If the critical storey itself is braced, however, a considerable increase in the critical load results.

To illustrate this behaviour calculations have been performed for the two storey portal shown in Figure 11.3. When there is no diagonal bracing the elastic critical load is $4P = 1.39P_e$ where P_e is the Euler load of AB. When the upper storey is braced by a member of the same cross-section the critical load rises to $4P = 1.61P_e$. When the critical lower storey is braced the critical load is $4P = 2.58P_e$. Hence bracing the upper storey increased the critical load by 15.6% but bracing the lower storey increased the critical load by 85%.

The analysis of these two cases is set out below.

(i) Bracing the upper storey.

In the framework of Figure 11.3b only one degree of sway freedom is possible. This is the displacement of the whole upper storey which is produced by the sway of the members AB and EF. On the application of a disturbing horizontal force H at joint B, the following deformations occur:-

1) Joint B rotates clockwise through an angle $\partial\theta_B$.

2) Joint C rotates clockwise through an angle $\partial\theta_C$.

3) Joint D rotates clockwise through an angle $\partial\theta_D$.

4) Joint E rotates clockwise through an angle $\partial\theta_E$.

5) Members BA and EF will sway by $\partial\delta$.

Due to these deformations, there will be moments in the members and these are tabulated in Table 11.2.

Equilibrium consideration yields:-

$$\begin{bmatrix} \partial\theta_B \\ \partial\theta_C \\ \partial\theta_D \\ \partial\theta_E \\ \partial\delta \end{bmatrix} \begin{bmatrix} [(ks)_1+(ks)_5+(ks)_7+(ks)_2] & (ksc)_2 & (ksc)_7 & (ksc)_5 & -(\frac{k\alpha}{L})_1 \\ (ksc)_2 & (ks)_2+(ks)_3 & (ksc)_3 & 0 & 0 \\ (ksc)_7 & (ksc)_3 & (ks)_3+(ks)_7+(ks)_4 & (ksc)_4 & 0 \\ (ksc)_5 & 0 & (ksc)_4 & [(ks)_4+(ks)_5+(ks)_6] & -(\frac{k\alpha}{L})_6 \\ -(\frac{k\alpha}{L})_1 & 0 & 0 & -(\frac{k\alpha}{L})_6 & (\frac{2kA}{L^2})_1+(\frac{2kA}{L^2})_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad 11.5$$

The determinant of the square matrix is the stability condition.

The numerical coefficients in the determinant are calculated for the k-value of the diagonal equal to 0.707 and the ρ-value of the lower stanchions twice that of the upper. This yields a determinant:-

$$\begin{vmatrix} s_1+s_2+6.828 & (sc)_2 & 1.414 & 2 & \alpha_1 \\ (sc)_2 & 4+s_2 & 2 & 0 & 0 \\ 1.414 & 2 & s_4+6.828 & (sc)_4 & 0 \\ 2 & 0 & (sc)_4 & s_4+s_6+4 & \alpha_6 \\ \alpha_1 & 0 & 0 & \alpha_6 & 2(A_1+A_6) \end{vmatrix} \quad 11.6$$

The load parameter ρ_1 making this determinant vanish is obtained by trial and interpolation.

First trial $\rho_1 = \rho_6 = 0.84$

$\rho_2 = \rho_4 = 0.42$

From tables[9]

$s_{1,6} = 2.7630 \qquad \alpha_{1,6} = 5.1168 \qquad A_{1,6} = 0.9716$

$s_{2,4} = 3.414$

Substituting these values in 11.6 gives

$\Delta = -2180$

A lower of ρ_1 is therefore tested.

When $\rho_1 = 0.80 \qquad \Delta = +370$

By linear interpolation the critical value of ρ_1 is 0.806 and the critical load of the frame is

$4P = 2 \times 0.806\, P_e$

$= 1.612 P_e$

This value of 4P corresponds with the value $(36.4/22)P_e = 1.65P_e$ obtained experimentally on a model of the braced portal frame.

As the stiffness of the bracing member is increased the carrying capacity of the frame increases but the load is always less than $2P_e$. This is the load most nearly approached when the stiffness of the bracing member is infinitely large.

(ii) Bracing the lower storey.

In the framework of Figure 11.3c one degree of sway freedom is possible. This is the horizontal displacement of joint C which is produced solely by the sway of the stanchions CB and DE. On the application of a disturbing horizontal force H at joint C, the following deformations occur:-

1) Joint B rotates clockwise through an angle $\partial\theta_B$.

2) Joint C rotates clockwise through an angle $\partial\theta_C$.

3) Joint D rotates clockwise through an angle $\partial\theta_D$.

4) Joint E rotates clockwise through an angle $\partial\theta_E$.

5) Members BC and DE will sway by $\partial\delta$.

Due to these deformations there will be moments in the members and these are tabulated in Table 11.3.

Equilibrium consideration yields:-

$$\begin{bmatrix} \partial\theta_B \\ \partial\theta_C \\ \partial\theta_C \\ \partial\theta_D \\ \partial\delta \end{bmatrix} \begin{bmatrix} [(ks)_1+(ks)_7+(ks)_6+(ks)_2] & (ksc)_2 & 0 & (ksc)_6 & (\frac{k\alpha}{L})_2 \\ (ksc)_2 & (ks)_2+(ks)_3 & (ksc)_3 & 0 & (\frac{k\alpha}{L})_2 \\ 0 & (ksc)_3 & (ks)_3+(ks)_4 & (ksc)_4 & (\frac{k\alpha}{L})_4 \\ (ksc)_6 & 0 & (ksc)_4 & (ks)_4+(ks)_6+(ks)_5 & (\frac{k\alpha}{L})_4 \\ +(\frac{k\alpha}{L})_2 & (\frac{k\alpha}{L})_2 & (\frac{k\alpha}{L})_4 & (\frac{k\alpha}{L})_4 & [(\frac{2kA}{L^2})_2+(\frac{2kA}{L^2})_4] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad 11.7$$

The determinant of the square matrix is the stability condition.

When the lengths and the relative k-values are substituted in 11.7, the determinant becomes

$$\begin{vmatrix} s_1+s_2+6.82 & (sc)_2 & 0 & 2 & \alpha_2 \\ (sc)_2 & s_2+4 & 2 & 0 & \alpha_2 \\ 0 & 2 & s_4+4 & (sc)_4 & \alpha_4 \\ 2 & 0 & (sc)_4 & s_5+s_4+4 & \alpha_4 \\ \alpha_2 & \alpha_2 & \alpha_4 & \alpha_4 & 2(A_2+A_4) \end{vmatrix} \qquad 11.8$$

The load parameter making this determinant vanish is obtained as before and turns out to be $\rho_1 = 1.29$. Thus the critical load is

$4P = 2 \times 1.29\ P_e$

$\quad = 2.58P_e$

This value corresponds with the value $(61.2/22)P_e = 2.80P_e$ obtained experimentally on the model of the portal frame.

(b) Pin jointed frames

Sometimes in two storey frames, the upper and the lower beams can be regarded as pin jointed. The diagonal bracing members might also be pin jointed. The stability of such a frame is therefore investigated. With such bracing the elastic critical load is small. For example, if the frame of Figure 11.4a is loaded only at the top, the elastic critical load is $1.05P_e$. When the beams are pin jointed, the elastic critical load is reduced to $0.125P_e$ and when a pin jointed bracing member is added, the elastic critical load is increased to $1.02P_e$.

If the beams of the two storey frame are pin jointed to the stanchions, the stability condition of the portal frame is obtained by considering the stability of a stanchion which is fixed at the foundation and under external loads at the beams positions. The stability condition is obtained by applying a disturbing moment at B which causes no shear rotation $\partial\theta_B$ of joint B. The disturbing moment at the critical load vanishes, thus

$$\partial M_B = \left[(kn)_1 + (kn'')_2\right]\partial\theta_B = 0 \qquad 11.9$$

where

$$n'' = n - o^2/n$$

If only C and C' are loaded by a vertical load 2P, the critical load of the frame will be

$$2P = 2\,(P_e/4)$$

$$= 0.5\,P_e$$

where P_e is the Euler load of AC.

When a pin jointed bracing member is added to the top storey, the stability condition of the frame is also obtained by considering the stability of the stanchion. This time the stability condition is obtained by rotating joints B and B' giving

$$(kn)_1 + (ks'')_2 = 0 \qquad 11.10$$

The elastic critical load of the frame shown in Figure 11.4b is estimated when the stanchions of the frame are of equal lengths and have the same cross section. The stability condition of 11.10 becomes

$$n + s'' = 0$$

By trial and interpolation the critical load parameter ρ is obtained.

First trial $\rho = 0.51$

From tables[9]

$n = 1.8056$ $s'' = 1.7875$

Substituting these values in the stability condition gives

$\Delta = + 0.0181$

A higher value of ρ is therefore tested.

$\rho = 0.52$

$n = 1.7774$ $s'' = 1.8875$

Thus

$\Delta = - 0.1101$

By linear interpolation the critical value of ρ is 0.511 and the elastic critical load is

$$2P = 2 \times 0.511\ P_e$$
$$= 1.022 P_e$$

Knee bracing

This type of bracing may be used for any type of frame. A short diagonal member is used to connect the beam to the stanchions. This method is probably the best, if internal bracing can be allowed, since the elastic critical load of the framework can be raised to several times the original elastic critical load.

For illustration, Figures for the knee braced frame of Figure 11.5

with beam to stanchion stiffness ratio of $\frac{2}{3}$ is given. With no bracing the elastic critical load is $2P = 1.34P_e$ where P_e is the Euler load of AC. When the portal frame is braced as shown in Figure 11.5 by a member having the same moment of inertia as that of the stanchions and having an inclination of $45°$, the elastic critical load is raised to $2P = 5.34P_e$. The ratio of the braced length to the total height of the frame is a half. When the triangulated area BCD is replaced by a stiff area, the elastic critical load is further increased to $2P = 7.68P_e$.

There are three possible modes of instability for the knee braced portal of Figure 11.5:-

(1) Anti-symmetrical sway.

(2) Buckling of the stanchions (Symmetrical failure).

(3) Joint rotation.

The application of disturbance and the resulting deformations will be obtained exactly as shown in chapter 2.

<u>Stability criterion for the anti-symmetrical sway mode</u>

The number of sway unknowns for the frame in Figure 11.5 is

$= 6 \times 2 - 9$

$= 3$

These will be taken as the horizontal deflection of B, B' and C respectively. On the application of a horizontal force H at joint C, the following deformations occur:-

1) Joints B and B' rotate clockwise through an angle $\partial\theta_B$.

2) Joints C and C' rotate clockwise through an angle $\partial\theta_C$.

3) Joints D and D'rotate clockwise through an angle $\partial\theta_D$.

4) Horizontal deflection of C by $\partial\delta$ will give rise to the moments:- $-(k\alpha/L)_1\partial\delta$ at A, B, B', and A'.

5) Horizontal deflection of B and B' by ψ give rise to the moments:-

$$M_{AB} = -k_1[(sc)_1 + \alpha_1 L_2/L_1]\psi$$

$$M_{BA} = -k_1[s_1 + \alpha_1 L_2/L_1]\psi$$

$$M_{DD'} = -k_5(\alpha_5 + 2\alpha_5 L_3/L_5)\psi$$

Due to these deformations, there will be moments in the members and these are tabulated in Table 11.4.

Equilibrium consideration yields:-

$$\begin{bmatrix} \partial\theta_B \\ \partial\theta_C \\ \partial\theta_D \\ \partial\delta \\ \psi \end{bmatrix} \begin{bmatrix} [(ks)_1+(ks)_2+(ks)_4] & (ksc)_2 & (ksc)_4 & -(\frac{k\alpha}{L})_1 & -k_1(S_1+\frac{L_2}{L_1}\alpha_1) \\ (ksc)_2 & (ks)_2+(ks)_3 & (ksc)_3 & 0 & 0 \\ (ksc)_4 & (ksc)_3 & (ks)_4+(ks)_3+(k\alpha)_5 & 0 & -k_5(\alpha_5+\frac{2L_3}{L_5}\alpha_5) \\ [(\frac{k\alpha}{L})_2+(\frac{k\alpha}{L\sin\phi})_4] & [(\frac{k\alpha}{L})_2+(\frac{k\alpha}{L}\cot\phi)_3] & [(\frac{k\alpha}{L\sin\phi})_4+(\frac{k\alpha}{L}\cot\phi)_3-(\frac{2k\alpha}{L}\cot\phi)_5] & 0 & (\frac{2k}{L}\cot\phi)_5(\alpha_5+\frac{2L_3}{L_5}A_5) \\ (\frac{k\alpha}{L})_1 & 0 & 0 & -(\frac{2kA}{L^2})_1 & -[(\frac{k}{L})_1(\alpha_1+\frac{2L_2}{L_1}A_1)] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad 11.11$$

The determinant of the square matrix is the stability condition.

<u>Stability criterion for the buckling of the stanchions mode</u>

In this mode the number of sway unknowns is

$= 2(5-4)$

$= 2$

These will be taken as the horizontal displacement of joints B and B'. In this mode joints C and C' do not displace during the application of the distrubance. On the application of equal and opposite disturbing horizontal forces H at joints B and B', the following deformations occur:-

1) Equal and opposite rotations of joints B and B' through an angle $\partial\theta_B$.
2) Equal and opposite rotations of joints C and C' through an angle $\partial\theta_C$.
3) Equal and opposite rotations of joints D and D' through an angle $\partial\theta_D$.
4) Opposite horizontal deflections of joints B and B' by Δ give rise to the moments

$M_{AB} = -k_1\left[(sc)_1 + \alpha_1 L_2/L_1\right]\Delta$

$M_{BA} = -k_1\left[s_1 + \alpha_1 L_2/L_1\right]\Delta$

$M_{DD'} = -(ks(1-c))_5\,\Delta$

Due to these deformations, there will be moments in the members and these are tabulated in Table 11.5.

Equilibrium consideration yields:-

$$\begin{bmatrix}\partial\theta_B\\ \partial\theta_C\\ \partial\theta_D\\ \Delta\end{bmatrix}\begin{bmatrix}(ks)_1+(ks)_2+(ks)_4 & (ksc)_2 & (ksc)_4 & -k_1\left(s_1+\frac{L_2}{L_1}\alpha_1\right)\\ (ksc)_2 & (ks)_2+(ks)_3 & (ksc)_3 & 0\\ (ksc)_4 & (ksc)_3 & \left[(ks)_3+(ks)_4+(ks(1-c))_5\right] & -(ks(1-c))_5\\ \left[\left(\frac{k\alpha}{L}\right)_2+\left(\frac{k\alpha}{L\sin\phi}\right)_4-\left(\frac{k\alpha}{L}\right)_1\right] & \left[\left(\frac{k\alpha}{L}\right)_2+\left(\frac{k\alpha}{L}\cot\phi\right)_3\right] & \left[\left(\frac{k\alpha}{L\sin\phi}\right)_4+\left(\frac{k\alpha}{L}\cot\phi\right)_3\right] & \left[\left(\frac{k}{L}\right)_1\left(\alpha_1+\frac{2L_2}{L_1}A_1\right)\right]\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix} \quad 11.12$$

The determinant of the square matrix is the stability condition.

Joint rotation mode

In this mode there are no displacement at the joints. The stability conditions for the symmetrical and anti-symmetrical joint rotation modes are obtained from the previous stability conditions by ignoring the coefficients of the lateral deflections.

Of these three modes the anti-symmetrical sway mode is likely to be the lowest and this is calculated for illustration for the knee braced frame of Figure 11.5. The frame has a total height of 2L and beam length 3L and the angle of inclination of the bracing member is 45° and AB/AC = 0.5. The relative k-values and the load parameters P/P_e are

Member	AB(1)	BC(2)	CD(3)	BD(4)	DD'(5)
length	L	L	L	2 L	L
rel.k	1	1	1	0.707	1
rel.P_e	1	1	1	0.5	1
force	P	P	0	0	0
rel.	1	1	0	0	0

When the lengths and the relative k-values are substituted in 11·11, the determinant becomes

$$\begin{vmatrix} 2s+2.828 & sc & 1.414 & -\alpha & -(s+\alpha) \\ sc & s+4 & 2 & 0 & 0 \\ 1.414 & 2 & 12.828 & 0 & -18 \\ \alpha+4.242 & \alpha+6 & -1.758 & 0 & 36 \\ \alpha & 0 & 0 & -2A & -(\alpha+2A) \end{vmatrix} = \Delta \qquad 11.13$$

The critical load parameter ρ making this determinant vanish is obtained by trial and interpolation.

<u>First trial $\rho = 0.67$</u>

From tables[9]

$s = 3.0293 \qquad \alpha = 5.3050 \qquad A = 1.9987 \qquad sc = 2.2758$

Substituting these values in 11.13 leads to

$\Delta = -724$

A lower value of ρ is therefore tested.

$\rho = 0.66$

$s = 3.0453 \qquad \alpha = 5.3159 \qquad A = 2.0589 \qquad sc = 2.2706$

Substituting these values in 11.13 gives

$\Delta = +1950$

By linear interpolation the critical ρ is 0.668 and the critical load is

$2P = 2 \times 0.668 (P_e)_{AB}$

$= 5.34 \, P_e$

where P_e is the Euler load of member AC

This value corresponds with $2P = 5.6 \, P_e$ obtained experimentally on a model of the knee braced portal frame made of bright steel strip.

When the open triangules BCD and B'C'D' are replaced by stiff membrames the critical load is increased to $7.68P_e$. This value was calculated using the stability functions n' and α' defined by Livesley and Chandler.[9] The elastic instability condition becomes

$$(kn')_1 + (k\alpha')_2 = 0 \qquad 11.14$$

The knee bracing member might be regarded as pin jointed to the beam and the stanchion. To work out the extreme case it was assumed that the joint between beam and stanchion was also pinned as in Figure 11.6.

Stability criterion for the anti-symmetrical sway mode (pin jointed)

On the application of a disturbing force 2H at joint C, the following deformations occur:-

1) Joints B and B' rotate clockwise through an angle $\partial\theta_B$.

2) Joints D and D' rotate clockwise through an angle $\partial\theta_D$.

3) Joints C and C' displace horizontally by $\partial\delta$ giving rise to the moment $-(k\alpha/L)\,\partial\delta$ at joints A, B, B' and A'.

4) Horizontal displacement of joints B and B' by Δ which gives rise to the moments

$$M_{AB} = -\,k_1\left[(sc)_1 + \alpha_1 L_2/L_1\right]\Delta$$

$$M_{BA} = -\,k_1\left[s_1 + \alpha_1 L_2/L_1\right]\Delta$$

$$M_{DD'} = -\,k_5\left[\alpha_5 + 2\alpha_5 L_3/L_5\right]\Delta$$

Due to these deformations there will be moments in the members and these are tabulated in Table 11.6.

Equilibrium consideration yields:-

$$\begin{bmatrix} \partial\theta_B \\ \partial\theta_D \\ \partial\delta \\ \mathcal{L} \end{bmatrix} \begin{bmatrix} (ks)_1+(ks'')_2 & 0 & -(\frac{k\alpha}{L})_1 & -k_1(S_1+\frac{L_2}{L_1}\alpha_1) \\ 0 & (ks'')_3+(k\alpha)_5 & 0 & -k_5(\alpha_5+2\frac{L_3}{L_5}\alpha_5) \\ (\frac{ks''}{L})_2 & [(\frac{ks'}{L})_3-(\frac{2k\alpha}{L})_5]\cot\phi & 0 & +(\frac{2k\cot\phi}{L})_5(\alpha_5+2\frac{L_3}{L_5}A_5) \\ (\frac{k\alpha}{L})_1 & 0 & -(\frac{2kA}{L^2})_1 & -(\frac{k}{L})_1(\alpha_1+2\frac{L_2}{L_1}A_1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad 11.15$$

The determinant of the square matrix is the stability condition.

<u>Approximations for knee-braced portals</u>

Critical load calculations have been carried out for a braced portal frame of constant flexural rigidity EI having a total height L and beam length 1.5L. The length of the braced part of the stanchion has been taken to be x.

If the largest load parameter P/P_e of the knee braced portal frame is ρ , the elastic critical load of the frame will be reached

$P = \rho P_e$

where P_e is the Euler load of the part of the stanchion with the highest ρ-value. For x/L less than 0.5, P_e is the Euler load of member AB. Thus

$P = \rho (P_e)_{AB}$

Let the critical load parameter P/P_e of the simple portal frame be ρ_o then its elastic critical load is

$$P_o = \rho_o (P_e)_{AC}$$

and the ratio P/P_o is

$$P/P_o = C\ (P_e)_{AB}/(P_e)_{AC}$$

$$= C/(1-x/L)^2 \qquad 11.16$$

where $C = \dfrac{\rho}{\rho_o}$

For different ratios of x/L and different angles of inclination of the bracing member, the values of C were evaluated by satisfying the stability conditions 11.11, and 11.15 of the anti-symmetrical sway mode. The anti-symmetrical sway mode is the important one for practical values of the ratio x/L but for higher values the buckling of the stanchions and joint rotation modes may become dominant and this is shown in Figure 11.7. The following numerical results were obtained:-

	Rigidly jointed $\phi=30°$	$\phi=45°$	$\phi=60°$	Pin jointed $\phi=45°$
x/L	C	C	C	C
0	1.00	1.00	1.00	1.00
1/6	1.16	1.06	1.00	0.998
1/3	1.20	1.074	0.98	0.945
1/2	----	0.995	0.90	0.745

It is seen from these results that the ratio of the elastic critical loads of rigidly jointed portal frames in which the inclination of the bracing member is $45^\circ - 60^\circ$ differs from the ratio of the Euler loads by less than $\pm$ 10%. The same is true for portals with a pin-jointed knee brace inclined at 45° until the ratio x/L becomes larger than $\frac{1}{3}$.

Therefore instead of using the exact method of analysis, the elastic critical load of a knee braced portal frame could be obtained in practice by multiplying the elastic critical load of the simple portal frame by the ratio of the Euler load of AB to that of AC i.e the bracing factor $1/(1-x/L)^2$.

For inclination Ø smaller than 45°, the elastic critical load of the framework is higher than that given by the approximation. For frameworks with pin jointed knee braces having an inclination of 45° the elastic critical load of the framework is lower than that given by the approximation for large values of x/L. For smaller ratios of x/L the approximate method gives almost exact agreement with the true critical loads. In Figure 11.8 curves of total carrying capacity against the ratio x/L for inclinations 60°, 45° and 30° for rigidly jointed frames and 45° inclination for pin jointed bracing member and the approximate method are shown.

Multi-storey knee braced frames

In the exact estimation of the elastic critical load of braced single storey portal frames, the number of unknowns involved in the analysis was five. For multi-storey frames the number of the unknowns involved will be five times the number of storeys. Therefore the stability condition will be represented by a determinant of size 5n X 5n where n is the number of storeys. Consequently the exact analysis will necessiate the use of an electronic computer and emphasises the necessity of an approximate method.

The approximation based on the bracing factor $1/(1-x/L)^2$ given above can be used to predict the elastic critical load of multi-storey braced frames after certain modifications. The unbraced frame usually has an elastic critical load which is obtained using the classical method. If the lowest storey is assumed to be the weakest and the other storeys are braced by inclined members as in Figure 11.9a the second storey will become much stiffer since the Euler loads of the stanchions of this storey are modified by the bracing factor to

$\bar{P}_e = P_e/(1-x/L)^2$

and the stiffness of the stanchion to

$\bar{k} = k/(1-x/L)$

The stiffness of the beam is also increased. The beam is assumed to be moved down to the level of DD' as in Figure 11.9c. It has been shown in Figure 11.1b that the increase in the stiffness of the upper beams will result in increasing the critical load parameter P/P_e of the lower stanchions and that the difference in the elastic critical load for an infinitely stiff beam and a moderate stiff one is very small. Therefore the new load parameter of the lower stanchions will be estimated using the modified second storey of Figure 11.9c. If the new value of P/P_e is ρ then the elastic critical load of the frame of Figure 11.9c is

$$P = 2\rho P_e$$

If the lower storey is also braced the elastic critical load of the frame of Figure 11.9c will be increased by the bracing factor to

$$P = 2\rho P_e/(1-x/L)^2 \qquad 11.17$$

The approximate elastic critical load of the frame of Figure 11.10 was determined. The members have constant EI value throughout, the two storeys have equal lenght, beams lengths are 1.5 times the stanchion height and $x/L = 0.5$. The critical ρ-value of the lower storey after the modification of the upper storey was found to be 0.70. On substituting these values in equation 11.17, the approximate value of the elastic critical load will be

$$4P = 2 \times 0.7 \, P_e/(\tfrac{1}{2})^2$$
$$= 5.6 \, P_e$$

The experimental value of 4P was $(61.2/9.75)P_e = 6.26P_e$, a difference of about 10.5%. The difference is expected to be smaller for small values of the ratio x/L as shown in the previous approximation.

Multi-bay braced frames

The elastic critical load of multi-bay braced frames can also be estimated approximately using the bracing factor $1/(1-x/L)^2$. The critical load parameter ρ is calculated for the unbraced frame by the exact method of analysis. The elastic critical load of the braced frame is then obtained by multiplying the previous load by the bracing factor $1/(1-x/L)^2$. For example, the elastic critical load of the two bay frame of Figure 11.11a is

$$3P = 2.315\ P_e$$

where P_e is the Euler load of the stanchion.

When bracing members are added to the frame as in Figure 11.11b where $x/L = 0.5$, the approximate elastic critical load will be

$$3P = 2.315\ P_e/(\tfrac{1}{2})^2$$
$$= 9.26\ P_e$$

The experimental value of 3P was $(85/9.75)P_e = 8.71P_e$. The difference is 6.3% and is expected to be smaller for smaller values of x/L.

Effect of beam loading

In the previous frameworks, the applied loads were assumed to be lumped at the top of the stanchions. When the load is distributed equally among the joints of the beams, the elastic critical load will be reduced because axial loads are introduced into the beams and bracing members, and it is well known that the introduction of an axial load in the beam of a simple rectangular portal frame reduces the critical load. This behaviour was noticed experimentally when the loads on the frame of Figure 11.10 were distributed equally at the joints in the beams. The elastic critical load was reduced from 61.2 Ibs to 56 Ibs.

Effect of changing the moment of inertia of the inclined member

To find the effect of the size of moment of inertia of the inclined bracing member, critical load calculations have been carried out for different values for the same single storey portal of Figure 11.5. Using the exact method of solution, the following results were obtained:-

I_{BD}/I_{AB}	0.25	1.00	2.00	∞
ρ_{AB}	0.606	0.668	0.720	0.960

It is seen that the elastic critical load has been increased by about 10% when the moment of inertia was increased from 0.25 to 1.0. The increase is 8% when the moment of inertia is changed from 1.0 to 2.0. The elastic critical load of this framework never exceeds $0.96P_e$ where P_e is the Euler load of AB and this value is attained when the stiffness of the inclined member becomes infinitely large. This shows that most of the increase in the elastic critical load has been obtained when the moment of inertia of the inclined member is about the same as that of the stanchion. The increase in the elastic critical load of the portal frame will be smaller when the ratio x/L becomes less than 0.50. The critical load parameters ρ for an infinite bracing stiffness are tabulated for different values of x/L.

x/L	0	0.1	0.2	0.3	0.4	0.5
ρ_{AB}	0.669	0.727	0.794	0.859	0.915	0.960

Chapter 12

Elastic instability of space frames

Introduction

In this chapter, the elastic stability of space frames is examined. The method of analysis is based on the same assumptions and principles as the plane framework except for the consideration of the torsional rigidity of the framework members. In investigating the elastic instability of frames with inclined members, the standard slope-deflection equation was modified to take into account the effect of the torsional moments as well as axial forces. This modified equation without taking the effect of the axial force was called the slope-deflection-gyration equation.[19]

The solution of a space framework with members having any directions in space has been attacked by establishing the flexural moments and the torsional moments in each member and then projecting these moments along three cartesian axes. Equation 2.3 was used and modified to relate the shear forces and some of the end moments to the sway forces at the joints.

The number of the unknown deformations has been reduced to the minimum by taking advantage of any possible symmetry of the framework and any special geometry. The best possible choice of the cartesian axes has been made for each example demonstrated. In the case of symmetrical framework under symmetrical loading, the symmetrical joints will have either equal and like or equal and unlike deformations depending on the mode of elastic instability tested.

The space frames are assumed to have cylindrical members of circular cross-section and furthermore the frames are loaded so that the axial loads in the members can be determined by statics.

Modes of elastic instability

For symmetrical frameworks under symmetrical loadings there will be the following four modes of instability:-

1) The anti-symmetrical sway mode

In this mode, the rotation of similar joints for a given sway will be equal when the sway is in their plane. This mode includes sways along the coordinate axes and combination of these sways.

2) The sway mode in which the whole structure twists(twisting sway mode)

In this mode, usually the upper horizontal plane of members rotates and this is provided by the sways and torsional rotation of the stanchions, or by the sway of the stanchions and the roof members. The rotations of the similar joints will be equal when the components of displacements at the joints have the same sign and will be equal and unlike when the components of displacements of the joints are in opposite direction.

3) Symmetrical sway mode

In this mode, similar joints will have equal and opposite deformations.

4) Joint rotation mode

This is a non-sway mode, the similar joints will have equal and like rotation if it is an anti-symmetrical joint rotation mode or equal and unlike rotation if it is a symmetrical mode.

Method of analysis

The method of analysis is based on the fact[10] that at the critical load, the stiffness of the framework is reduced to zero. A test disturbance is applied which causes the joints to rotate and displace along the three cartesian coordinate axes x, y, z with the joints remaining in equilibrium i.e the total moment along each axis at each joint is equal to zero. If the disturbing force is along one axis it may not necessarily always cause deformations in the directions of the other axes. The forces at the joints causing the members to sway are related by equation 2.3 to the shear forces in the members. This equation can be modified to take into account cases where one or more joints rotate in the sway mechanism. This modified equation may be written:

$$\text{External Force} \times \begin{matrix}\text{Displacemnet in} \\ \text{direction of this} \\ \text{force}\end{matrix} = \sum_{\text{All members}} \begin{matrix}\text{Shear force X sway} \\ \text{of the member}\end{matrix}$$

$$+ \sum_{\text{All rotated joints}} \begin{matrix}\text{End moment X (joint rotation} \\ \text{due to sway)}\end{matrix} \qquad 12.1$$

This equation is applied to modes in which some of the members are not allowed to sway but geometrical consideration necessitate the ends of other members rotating. This equation gives also as many relationships as the number of sway components.

Actions and deformations are related by either the slope-deflection equation or the modified slope-deflection-gyration equation,taking stability effect and torsion into account. This leads to a system of homogeneous linear equations, instability being characterized by the vanishing determinant formed by the coefficients of the deformations.

Frames with vertical stanchions and beams in two orthogonal planes

The analysis of such frames is relatively simple for simple frameworks. These frames are assumed to have members which are parallel to the cartesian axes. The standard slope-deflection equation is used to relate the deformation and the flexural moments along two axes, and the torsional moment M_z and the axial rotations along the third axis of the member is given by :-

$$M_{zn} = k_G(\theta_{zn} - \theta_{zf}) \qquad 12.2$$

where

$k_G = GI_z/L$

G = modulus of rigidity

and θ_{zn}, θ_{zf} are the axial angular rotations of the near and far ends of the member respectively.

Sign convention

Flexural moments, torsional moments, shear forces and deformations are represented by vectors which have positive signs if they coincide with the positive direction of the three axes.

Theoretical analysis

Frame 1

The stability of the framework shown in Figure 12.1 is investigated. The framework has a height L_1 and a square roof of side length L_2. The torsional stiffness of the stanchions is assumed to be k_{1G} and that of the roof members k_{2G}. The cartesian axes x,y,z are chosen such that the x-axis is along the member BA, y-axis is along the member AD and the z-axis is along the member AA'. Only three modes of elastic instability are possible, the anti-symmetrical sway mode, twisting sway mode and joint rotation mode. The analysis for establishing the various criteria for elastic instability are shown.

Stability criterion for the anti-symmetrical sway mode

There are two degrees of freedom of movement, one along the x-axis and the other is along the y-axis. Other modes of sway can be invoked by the combination of the two sways. Herein only one sway is considered and that is the sway along the y-axis.

If equal disturbing horizontal forces H_y are applied at A and B all the joints will be deformed by $(\theta_x, 0, 0, 0, \delta_y, 0)$ since there are disturbances along the y-axis only and there is no call on the members to resist disturbance along the other axes. Consequently, the problem is reduced to that of considering the stability of the plane frame AA'DD'. It is well known that the "no-shear" stiffness of the joints in the x-plane is the anti-symmetrical sway mode elastic stability criterion.

$$K = (kn)_1 + 6k_2 \qquad 12.3$$

The least load parameter ρ making the "no-shear" stiffness K vanish is the critical ρ-value. This is obtained by a trial and error process.

Stability criterion for the mode in which the whole structure twists

The criterion for elastic instability can be established in two ways which correspond to equation 2.3 and the modified equation 12.1. The first method consists of applying disturbing forces perpendicular to the diagonals of the roof ABCD which rotate the roof bodily in a clockwise direction by ϕ_z where $\phi_z = 2\partial\delta/L_2$. The deformations occur solely in the stanchion members in the form of a sway in which at joint A for example $\partial\delta_x = \partial\delta_y = \partial\delta$ and a torsional rotation of ϕ_z in each stanchion. Due to the symmetry of the framework, all the joints will have equal deformations but the direction of the deformations may be different.

From Figure 12.2b, there is a displacement along the member AD which requires that the forces in the members AA' and DD' should have the same directions. These forces are built up by the sway $\partial\delta$ and the rotation $\partial\theta_x$ in the yz-plane. Thus the rotation $\partial\theta_x$ at A and D are equal. Member AD is rotated by ϕ_z which is due to the sway of members AA' and DD' in opposite direction. This requires that the forces in the stanchions due to the rotations at A and D in the xz-plane should be equal but in opposite directions, thus the rotation $\partial\theta_y$ at A and D are equal and opposite. Likewise the rotations in the xy-plane $\partial\theta_z$ are equal. A further elimination of one of the unknowns can be made by finding the relationship between $\partial\theta_x$ and $\partial\theta_y$. Member AA' sways in the y-direction by $\partial\delta$ which causes a clockwise moment $k\alpha\partial\delta/L_1$ in the x-plane. The member also sways in the x-direction by $\partial\delta$ which causes anti-clockwise moment $k\alpha\partial\delta/L_1$ in the y-plane. Therefore the rotations required to balance the sway moments will be equal in value but opposite in direction i.e. $\partial\theta_y = -\partial\theta_x$. In this case the equilibrium of the moment in either the x or the y-plane should be considered since the relation between $\partial\theta_x$ and $\partial\theta_y$ is already known. The deformations at the joints will be :-

joint A: $(\partial\theta_x, \partial\theta_y(=-\partial\theta_x), \partial\theta_z, \partial\delta_x(=\partial\delta), \partial\delta_y(=\partial\delta), 0)$

joint D: $(\partial\theta_x, -\partial\theta_y(=\partial\theta_x), \partial\theta_z, -\partial\delta_x(=-\partial\delta), \partial\delta_y(=\partial\delta), 0)$

and similarly at joint B: $(\partial\theta_x, \partial\theta_y(=\partial\theta_x), \partial\theta_z, \partial\delta_x(=\partial\delta), -\partial\delta_y(=-\partial\delta), 0)$

Due to these deformations, the moments at the ends of the members meeting at joint A are:-

Member AB	Member AD	Member AA'
$\partial M_x = 2k_{2G}\partial\theta_x$	$\partial M_x = 6k_2\partial\theta_x$	$\partial M_x = k_1 s\,\partial\theta_x + (k\alpha/L)_1\partial\delta$
$\partial M_y = -6k_2\partial\theta_x$	$\partial M_y = -2k_{2G}\partial\theta_x$	$\partial M_y = -k_1 s\partial\theta_x - (k\alpha/L)_1\partial\delta$
$\partial M_z = 6k_2\partial\theta_z$	$\partial M_z = 6k_2\partial\theta_z$	$\partial M_z = k_{1G}\partial\theta_z + k_{1G}2\partial\delta/L_2$

The disturbing forces at the four corners are equal and since the four stanchions are symmetrical each disturbing force is resisted by the shear forces in each stanchion. Equation 12.1 leads to

$$\sqrt{2}\,H\,\partial\delta = (v_x\partial\delta_x + v_y\partial\delta_y + \partial M_z\phi_z)_{AA'} \qquad 12.4a$$

where

$$\partial\delta_x = \partial\delta_y = \partial\delta$$

$$v_x = v_y$$

and $v_y = (k\alpha/L)_1\partial\theta_x + (2kA/L^2)_1\partial\delta$

Substituting these values in 12.4a and knowing that H vanishes at the critical load gives:

$$\sqrt{2}\ H\partial\delta = 2\left[(k\alpha/L)_1\partial\theta_x + (2kA/L^2)_1\partial\delta\right]\partial\delta + k_{1G}(\partial\theta_z + 2\partial\delta/L_2)\ 2\partial\delta/L_2 = 0 \qquad 12.4b$$

Equilibrium at the joints requires that the total moments at joint A along each axis is zero. Hence the equilibrium equations will be:-

$$\begin{bmatrix}\partial M_x = -\partial M_y \\ \partial M_z \\ \sqrt{2}H\end{bmatrix} = \begin{bmatrix}\partial\theta_x \\ \partial\theta_z \\ \partial\delta\end{bmatrix}\begin{bmatrix}2k_{2G}+6k_2+(ks)_1 & 0 & (k\alpha/L)_1 \\ 0 & 12k_2+k_{1G} & 2k_{1G}/L_2 \\ 2(k\alpha/L)_1 & 2k_{1G}/L_2 & (4kA/L^2)_1+4k_{1G}/L_2^2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \qquad 12.4c$$

The determinant of the square matrix can be reduced to

$$\Delta = 2((ks)_1+2k_{2G}+6k_2)(k_{1G}+12k_2)\ \left[(2kA/L^2)_1 - (k\alpha/L)_1^2/((ks)_1+2k_{2G}+6k_2) + 24(k_{1G}k_2/L_2^2)/(k_{1G}+12k_2)\right] \qquad 12.5$$

The lowest load parameter ρ that makes this determinant vanish is the torsional elastic critical ρ-value.

The quantity $\left[(2kA/L^2)_1 - (k\alpha/L)_1^2/((ks)_1+2k_{2G}+6k_2) + 24(k_{1G}k_2/L_2^2)/(k_{1G}+12k_2)\right]$ must vanish at lower value of ρ than $((ks)_1+2k_{2G}+6k_2)$ since the latter occurs in the denominator of a negative term in the former and the term $(k_{1G}+12k_2)$ is a constant. Hence $\left[(2kA/L^2)_1 - (k\alpha/L)_1^2/((ks)_1+2k_{2G}+6k_2) + 24(k_{1G}k_2/L_2^2)/(k_{1G}+12k_2)\right]$ is the expression used to test for instability.

This stability condition can be obtained in another way which corresponds to equation 2.3. In this method no joint rotation is allowed to take place in the stanchions when the displacement is imposed by the disturbance. The roof members will be subjected to sways and the stanchions will sway without rotation. The same joint rotations will take place and the following moments will be in the members at joint A.

Member AA'	Member AB	Member AD
$\partial M_x=(ks)_1\partial\theta_x+(k\sigma/L)_1\partial\delta$	$\partial M_x= 2k_{2G}\partial\theta_x$	$\partial M_x= 6k_2\partial\theta_x$
$\partial M_y=-(ks)_1\partial\theta_x-(k\sigma/L)_1\partial\delta$	$\partial M_y=-6k_2\partial\theta_x$	$\partial M_y= -2k_{2G}\partial\theta_x$
$\partial M_z= k_{1G}\partial\theta_z$	$\partial M_z=6k_2\partial\theta_z-(12k/L)_2\partial\delta$	$\partial M_z=6k_2\partial\theta_z-(12k/L)_2\partial\delta$

Equation 2.3 is used to relate the sway forces at the corners to the shear forces in the members which leads to

$$\sqrt{2}\ H\partial\delta=(v_x\partial\delta_x+v_y\partial\delta_y)_{AA'}+(v_y\partial\delta_y/2)_{AB}+(v_x\partial\delta_x/2)_{AD} \qquad 12.6a$$

since each disturbance stores energy in one stanchion and the halves of two roof members. In this equation

$$(\partial\delta_x=\partial\delta_y)_{AA'}= \partial\delta$$

$$(v_x = v_y)_{AA'}$$

$$(\partial\delta_x=\partial\delta_y)_{AB,AD} = 2\partial\delta$$

$$(v_x)_{AA'}=(k\sigma/L)_1\partial\theta_x+(2kA/L^2)_1\partial\delta$$

and

$$(v_y)_{AB}=(v_x)_{AD}= -\ (12k/L)_2\partial\theta_z-\ (24k/L^2)_2\partial\delta$$

Substituting these values in 12.6a and knowing that H vanishes at the critical load gives

$$\sqrt{2}H =(2k\sigma/L)_1\partial\theta_x-(24k/L)_2\partial\theta_z+\left[(4kA/L^2)_1+(48k/L^3)_2\right]\partial\delta = 0 \qquad 12.6b$$

Equilibrium of the joints requires that the total moment along each axis at each joint is zero. Hence the equilibrium equations represented in matrix form will be:-

$$\begin{bmatrix} \partial M_x = -\partial M_y \\ \partial M_z \\ \sqrt{2}H \end{bmatrix} = \begin{bmatrix} \partial\theta_x & (ks)_1+2k_{2G}+6k_2 & 0 & (k\alpha/L)_1 \\ \partial\theta_z & 0 & k_{1G}+12k_2 & -(24k/L)_2 \\ \partial\delta & (2k\alpha/L)_1 & -(24k/L)_2 & \left[(4kA/L^2)_1+(48k/L^2)_2\right] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad 12.6c$$

The determinant of the square matrix can be reduced to

$$=2((ks)_1+2k_{2G}+6k_2)(k_{1G}+12k_2)\left[(2kA/L^2)_1 - (k\alpha/L)_1^2/((ks)_1+2k_{2G}+6k_2) + 24(k_{1G}k_2/L_2^2)/(k_{1G}+12k_2)\right]$$

Joint rotation mode

The criterion for the elastic instability of the lowest joint rotation mode is obtained when the deformation chosen calls into play no torsional restraint, but only bending. This condition is obtained by considering the symmetrical joint rotation of the plane AA'DD'. The stability condition is simply

$$K = (ks)_1 + 2k_2 \qquad 12.7$$

Numerical example

The elastic critical loads of the framework are calculated when the lengths and the EI-values of all the members are assumed to be constant and the torsional stiffness of each member is 0.77k where k is the flexural stiffness of the members. It follows that the relative flexural stiffness of the members is 1.0 and the "no-shear" stiffness of the joint given in equation 12.3 is :

$K = n + 6$

The load parameter ρ of the stachion when n=-6 is 0.7475 and the anti-symmetrical sway elastic critical load is

$4P = 4 \times 0.7475\ P_e$

$= 2.99\ P_e$

where P_e is the Euler load of the members.

This value corresponds with $(34/11.6)P_e = 2.93P_e$ obtained experimentally on a model of the framework.

The twisting elastic critical load is obtained by using 12.5. when the lengths and the relative k-values are substituted in 12.5, the stability condition becomes

$$\Delta = 2A - \frac{\alpha^2}{s+7.54} + 1.45 \qquad 12.6d$$

The stanchion elastic load parameter ρ making this expression vanish is obtained by a trial and error process.

First trial $\rho = 0.78$

From tables.[9]

$s = 2.8494 \qquad \alpha = 5.1838 \qquad A = 1.3347$

Substituting these values in 12.6d gives

$\Delta = 2.6694 + 1.45 - (5.1838)^2/\ 10.3894$

$= +\ 1.53$

i.e the frame is stable. A higher value of ρ is therefore tested.

When ρ =0.90 Δ= +0.211

When ρ =0.92 Δ= -0.065

The critical load parameter ρ by linear interpolation is 0.916 and the corresponding twisting elastic critical load is

$4P = 4 \times 0.916\ P_e$

$= 3.664\ P_e$

This value corresponds with $(41/11.6)P_e = 3.53\ P_e$ obtained experimentally, shown in Figure 12.3.

The joint rotation elastic critical load is obtained from 12.7 which become

$K = s + 2$

The load parameter ρ whose s= -2 is 2.55 and the corresponding elastic critical load is

$4P = 4 \times 2.55\ P_e$

$= 10.2\ P_e$

<u>Frame 2</u>

The elastic stability of the space frame in Figure 12.4 is considered. The frame has a total height L_1 and beam lengths L_2 and L_3. This frame has two possible modes of elastic instability, the anti-symmetrical sway mode and the joint rotation mode. The anti-symmetrical sway mode only is considered since it has the least elastic critical load.

The cartesian axes x,y,z are chosen such that the x-axis is along the member CA, y-axis is along the member BA and the z-axis is along the member A'A.

Stability criterion for the anti-symmetrical sway mode

There is only one freedom of movement and this is the movement in the x-direction. Due to the symmetry of the frame, when a disturbing force $2H_A$ is applied at joint A similar deformations will take place at A and C of $(0,\partial\theta_y,\partial\theta_z,\partial\delta_x,0,0)$ since the rotations in the y and z planes are called upon to resist the disturbance. Due to these deformations, the following moments will appear in the members meeting at joint A:-

Member AA'	Member AB	Member AC
$\partial M_y=(ks)_1\partial\theta_y-(k\alpha/L)_1\partial\delta$	$\partial M_y=k_{2G}\partial\theta_y$	$\partial M_y=6k_3\partial\theta_y$
$\partial M_z=k_{1G}\partial\theta_z$	$\partial M_z=4k_2\partial\theta_z+(6k/L)_2\partial\delta$	$\partial M_z=6k_3\partial\theta_z$

When the sway at joint A is imposed, the stanchions and the beams BA and DC sway by $\partial\delta$, therefore the shear force sway equation of 2.3 gives:

$$2H_A\partial\delta= 2(v_x\partial\delta_x)_{AA'}+2(v_x\partial\delta_x)_{AB} \qquad 12.7a$$

where

$$\partial\delta_x= \partial\delta$$

$$(v_x)_{AA'}= -\left[(k\alpha/L)_1\partial\theta_y - (2kA/L^2)_1\partial\delta\right]$$

and $(v_x)_{AB} = (6k/L)_2 \partial\theta_z + (12k/L^2)_2 \partial\delta$

Substituting these values in equation 12.7a and knowing that H_A vanishes at the critical load gives:-

$$H_A = -(k\alpha/L)_1 \partial\theta_y + (6k/L)_2 \partial\theta_z + \left[(2kA/L^2)_1 + (12k/L^2)_2\right] \partial\delta = 0 \qquad 12.7b$$

The joints are in equilibrium, hence the total moments along each axis at each joint is zero. Hence equilibrium equations are represented in the matrix form as

$$\begin{bmatrix} \partial M_y \\ \partial M_z \\ H_A \end{bmatrix} = \begin{bmatrix} \partial\theta_y & (ks)_1 + k_{2G} + 6k_3 & 0 & -(k\alpha/L)_1 \\ \partial\theta_z & 0 & k_{1G} + 4k_2 + 6k_3 & (6k/L)_2 \\ \partial\delta & -(k\alpha/L)_1 & (6k/L)_2 & (2kA/L^2)_1 + (12k/L^2)_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad 12.7c$$

The determinant of the square matrix can be reduced to

$$= ((ks)_1 + k_{2G} + 6k_3)(k_{1G} + 4k_2 + 6k_3)\left[(2kA/L^2)_1 + (12k/L^2)_2 - (k\alpha/L)_1^2/((ks)_1 + k_{2G} + 6k_3) - (6k/L)_2^2/k_{1G} + 4k_2 + 6k_3)\right] \qquad 12.8$$

The term $\left[(2kA/L^2)_1 + (12k/L^2)_2 - \frac{(\frac{k\alpha}{L})_1^2}{(ks)_1 + k_{2G} + 6k_3} - \frac{(\frac{6k}{L})_2^2}{k_{1G} + 4k_2 + 6k_3}\right]$ is used to find the lowest load parameter ρ that makes the determinant vanish.

Numerical example

The anti-symmetrical sway elastic critical load of the framework is calculated when all the members have equal lengths and the same cross-section. k_G is taken to be 0.77k where k is the flexural stiffness of the members. When the lengths and the relative k-values are substituted in 12.8, the stability criterion becomes:-

$$\Delta = 2A - \frac{\alpha^2}{s+6.77} + 8.66 \qquad 12.8a$$

The load parameter ρ of the stanchion making this expression vanish is obtained by a trial and error process.

First trial $\rho = 1.54$

From tables.[9]

$s = 1.3653$ $\alpha = 4.2809$ $A = -3.3189$

Substituting these values in 12.8a gives:

$$\Delta = -6.6372 - 2.26 + 8.66$$
$$= -0.2372$$

i.e the frame is unstable, a lower value of ρ is therefore tested.

When $\rho = 1.52$ $\Delta = 0$

Therefore the critical load parameter ρ is 1.52 and the elastic critical load is

$$2P = 2 \times 1.52\ P_e$$
$$= 3.04\ P_e$$

This value corresponds with $(34.2/11.6)P_e = 2.95P_e$ obtained experimentally on a model of the framework.

Frame 3-

The elastic critical load of the two bay framework of Figure 12.5 is estimated. The frame has a total height L_1 and the bays are of width

L_2. The stanchions are assumed to have equal Euler load and axial loads. The cartesian axes x,y,z are chosen as in the previous frame. Only two modes of elastic instability are possible, the joint rotation and the anti-symmetrical sway mode. The analysis for establishing the anti-symmetrical sway mode stability criterion is only shown.

Due to the symmetry there will be a similar deformations at joints A and F of $(0,\partial\theta_y,\partial\theta_z,\partial\delta_1,0,0)$ and at joints B and E $(0,\partial\phi_y,\partial\phi_z,\partial\delta_2,0,0)$ when a disturbance along the x-axis is applied. These deformations will result in the following moments appearing in the members meeting at joint A:-

Member AA'

$$\partial M_y=(ks)_1\partial\theta_y-(k\alpha/L)_1(\partial\delta_1+\partial\delta_2)$$

$$\partial M_z=k_{1G}\partial\theta_z$$

Member AF

$$\partial M_y=6k_3\partial\theta_y$$

$$\partial M_z=6k_3\partial\theta_z$$

Member AB

$$\partial M_y=k_{2G}(\partial\theta_y-\partial\phi_y)$$

$$\partial M_z=4k_2\partial\theta_z+2k_2\partial\phi_z+(6k/L)_2\partial\delta_1$$

and at joint B:-

Member BB'

$$\partial M_y=(ks)_1\partial\phi_y-(k\alpha/L)_1\partial\delta_2$$

$$\partial M_z=k_{1G}\partial\phi_z$$

Member BE

$$\partial M_y=6k_3\partial\phi_y$$

$$\partial M_z=6k_3\partial\phi_z$$

Member BC

$$\partial M_y=k_{2G}\partial\phi_y$$

$$\partial M_z=4k_2\partial\phi_z+(6k/L)_2\partial\delta_2$$

Member BA

$$\partial M_y=k_{2G}(\partial\phi_y-\partial\theta_y)$$

$$\partial M_z=2k_2\partial\theta_z+4k_2\partial\phi_z+(6k/L)_2\partial\delta_1$$

Due to the sway $\partial\delta_1$, members AB, EF, FF' and AA' sway by $\partial\delta_1$

therefore the shear force sway equation 2.3 gives

$$2H_A\partial\delta_1 = 2\left[(v_x\partial\delta_x)_{BA} + (v_x\partial\delta_x)_{AA'}\right] \qquad 12.8b$$

where

$$\partial\delta_x = \partial\delta_1$$

$$(v_x)_{BA} = (6k/L)_2\partial\theta_z + (6k/L)_2\partial\phi_z + (12k/L^2)_2\partial\delta_1$$

and $(v_x)_{AA'} = -\left[(k\alpha/L)_1\partial\phi_y - (2kA/L^2)_1\partial\delta_2\right]$

Substituting these values in 12.8b and knowing H_A vanishes at the critical load gives:-

$$H_A = -(k\alpha/L)_1\partial\theta_y + (6k/L)_2\partial\theta_z + (6k/L)_2\partial\phi_z + \left[(12k/L^2)_2 + (2kA/L^2)_1\right]\partial\delta_1 + (2kA/L^2)_1\partial\delta_2 = 0 \qquad 12.8c$$

When the sway $\partial\delta_2$ is imposed, members CB, BB', FF', DE, EE' and AA' sway by $\partial\delta_2$ therefore the shear force sway equation 2.3 gives:

$$2H_B\partial\delta_2 = 2\left[(v_x\partial\delta_x)_{BB'} + (v_x\partial\delta_x)_{CB} + (v_x\partial\delta_x)_{AA'}\right] \qquad 12.8d$$

where

$$\partial\delta_x = \partial\delta_2$$

$$(v_x)_{BC} = (v_x)_{ED} = (6k/L)_2\partial\phi_z + (12k/L^2)_2\partial\delta_2$$

and $(v_x)_{BB'} = (v_x)_{EE'} = -\left[(k\alpha/L)_1\partial\phi_y - (2kA/L^2)_1\partial\delta_2\right]$

Substituting these values in 12.8d and putting $H_B=0$ since there is no disturbance gives:-

$$H_B = -(k\alpha/L)_1\partial\theta_y - (k\alpha/L)_1\partial\phi_y + (6k/L)_2\partial\phi_z + (2kA/L^2)_1\partial\delta_1 + \left[(12k/L^2)_2 + (4kA/L^2)_1\right]\partial\delta_2 = 0 \qquad 12.8e$$

Equilibrium of the joints and the forces at the joints yields the equilibrium equations represented in matrix form as

$$\begin{bmatrix}\partial M_{yA}\\ \partial M_{zA}\\ \partial M_{yB}\\ \partial M_{zB}\\ H_A\\ H_B\end{bmatrix}=\begin{bmatrix}\partial\theta_y\\ \partial\theta_z\\ \partial\phi_y\\ \partial\phi_z\\ \partial\delta_1\\ \partial\delta_2\end{bmatrix}\begin{bmatrix}(ks)_1+6k_3+k_{2G} & 0 & -k_{2G} & 0 & -\left(\frac{k\alpha}{L}\right)_1 & -\left(\frac{k\alpha}{L}\right)_1\\ 0 & k_{1G}+6k_3+4k_2 & 0 & 2k_2 & \left(\frac{6k}{L}\right)_2 & 0\\ -k_{2G} & 0 & (ks)_1+6k_3+2k_{2G} & 0 & 0 & -\left(\frac{k\alpha}{L}\right)_1\\ 0 & 2k_2 & 0 & k_{1G}+6k_3+8k_2 & \left(\frac{6k}{L}\right)_2 & \left(\frac{6k}{L}\right)_2\\ -\left(\frac{k\alpha}{L}\right)_1 & \left(\frac{6k}{L}\right)_2 & 0 & \left(\frac{6k}{L}\right)_2 & \left[\left(\frac{12k}{L^2}\right)_2+\left(\frac{2kA}{L^2}\right)_1\right] & \left(\frac{2kA}{L^2}\right)_1\\ -\left(\frac{k\alpha}{L}\right)_1 & 0 & -\left(\frac{k\alpha}{L}\right)_1 & \left(\frac{6k}{L}\right)_2 & \left(\frac{2kA}{L^2}\right)_1 & \left[\left(\frac{12k}{L^2}\right)_2+\left(\frac{4kA}{L^2}\right)_1\right]\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\\0\\0\end{bmatrix} \quad 12.9$$

The determinant of the square matrix is the stability condition and the load parameter ρ of the stanchion making the determinant vanish is the critical one.

Numerical example

The elastic critical load of the space framework is calculated when all the members are assumed to have the same cross-section and equal lengths. The torsional stiffness k_G of the members is taken to be 0.77k where k is the flexural stiffness of the members. When the lengths and the relative stiffnesses of the members is substituted in 12.9, the determinant becomes

$$\begin{vmatrix} s+6.77 & 0 & -0.77 & 0 & -\alpha & -\alpha \\ 0 & 10.77 & 0 & 2 & 6 & 0 \\ -0.77 & 0 & s+7.54 & 0 & 0 & -\alpha \\ 0 & 2 & 0 & 14.77 & 6 & 6 \\ -\alpha & 6 & 0 & 6 & 12+2A & 2A \\ -\alpha & 0 & -\alpha & 6 & 2A & 12+4A \end{vmatrix} \qquad 12.9a$$

The load parameter ρ of the stanchion making this determinant vanish is obtained by trial and error. The critical ρ was found to be 0.98 and the elastic critical load is therefore

$$4P = 4 \times 0.98\ P_e$$
$$= 3.92\ P_e$$

This value corresponds with $(44/11.6)P_e = 3.79P_e$ obtained experimentally.

Space frames with inclined members

The analysis for investigating the elastic instability of space frames with inclined members is more complicated. This requires the projection of the flexural and torsional moments of the members along three cartesian axes in considering the equilibrium of the joints. The deflections of the joints in the direction of the axes are also required in determining the sway forces. The way of performing the projection is shown and the derivation of the modified slope-deflection-gyration equation is also given.

Projection of components

A vector can be represented in any coordinate system if its components in another coordinate system are known. Therefore if the two sets of coordinates are x',y',z' and x,y,z with units vectors $\underline{i}',\underline{j}',\underline{k}'$ and $\underline{i},\underline{j},\underline{k}$ along these axes and the components of the vector along these axes are $A_{x'}$, $A_{y'}$, $A_{z'}$, A_x, A_y and A_z respectively, then the vector can be represented by either

$$\underline{A} = A_{x'}\underline{i}' + A_{y'}\underline{j}' + A_{z'}\underline{k}' \qquad 12.10a$$

or

$$\underline{A} = A_x\underline{i} + A_y\underline{j} + A_z\underline{k} \qquad 12.10b$$

The components A_x, A_y, A_z along x,y,z axes could be projected along the other set of axes x',y',z' by the following:-

When the vector $\underline{A}$ represented by 12.10a is scalar multiplied by the unit vector $\underline{i}'$ (scalar product), then $A_{x'}$ is given by

$A_{x'} = \underline{A} \cdot \underline{i}'$

since $\underline{i}' \cdot \underline{i}' = 1$ and $\underline{j}' \cdot \underline{i}' = \underline{k}' \cdot \underline{i}' = 0$

Likewise $A_{y'} = \underline{A} \cdot \underline{j}'$ and $A_{z'} = \underline{A} \cdot \underline{k}'$

If the vector $\underline{A}$ represented by 12.10b is scalar multiplied by the unit vector $\underline{i}'$, then

$$\underline{A} \cdot \underline{i}' = A_x(\underline{i} \cdot \underline{i}') + A_y(\underline{j} \cdot \underline{i}') + A_z(\underline{k} \cdot \underline{i}') \qquad 12.10c$$

where $A_{x'} = \underline{A} \cdot \underline{i}'$ and $(\underline{i} \cdot \underline{i}')$, $(\underline{j} \cdot \underline{i}')$ and $(\underline{k} \cdot \underline{i}')$ are the direction cosines of the x'-axis with respect to x,y,z axes.

If we assume the direction cosines of the x'-axis w.r.t x,y,z to be a_x, a_y, a_z, that of the y'-axis b_x, b_y, b_z and that of the z'-axis to be c_x, c_y, c_z, then equation 12.10c becomes

$$A_{x'} = a_x A_x + a_y A_y + a_z A_z \qquad 12.10d$$

Likewise it can be shown that

$$A_{y'} = b_x A_x + b_y A_y + b_z A_z \qquad 12.10e$$

and

$$A_{z'} = c_x A_x + c_y A_y + c_z A_z \qquad 12.10f$$

Equations 12.10d,e,f can be represented in the matrix form by

$$\begin{bmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{bmatrix} = \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{bmatrix} \qquad 12.11$$

The direction cosines comply with the following conditions:[19]

$$a_x^2 + a_y^2 + a_z^2 = 1$$
$$b_x^2 + b_y^2 + b_z^2 = 1$$
$$c_x^2 + c_y^2 + c_z^2 = 1$$

and

$$\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = 1$$

Following the same procedure it can be shown that

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} A_{x'} \\ A_{y'} \\ A_{z'} \end{bmatrix} \begin{bmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{bmatrix} \qquad 12.12$$

Modified slope-deflection-gyration equation

For any member in space, a set of axes could be defined to represent its torsional axis which is the longitudinal axis of the member and will be taken as the z'-axis. Its pricipal and secondary axes of inertia will be taken as the x'-axis and y'-axis respectively. and these axes are further assumed to be axes of symmetry. The direction of these axes will be defined by their direction cosines as above, with respect to a system of cartesian axes selected x,y,z.

The two joints of any member NF shown in Figure 12.6 in a framework are subjected to two types of deformations such as angular rotation and displacement causing the member to sway. These deformations can be represented by vectors since an angular deformation can be defined as a vector with axis coinciding with the axis of rotation.

At each joint, there will be three angular rotations which will be taken as the rotation in the plane of the selected axes x,y,z and three types of displacement which will be taken as the displacements in the directions of the selected axes x,y,z. At joint N there will be deformations $(\theta_x,\theta_y,\theta_z,\delta_x,\delta_y,\delta_z)_N$ and at joint F there will be deformations $(\theta_x, \theta_y, \theta_z, \delta_x, \delta_y, \delta_z)_F$. These deformations will be projected along the axes of the member by using relationship 12.11

$$\begin{bmatrix} \theta_{x'} \\ \theta_{y'} \\ \theta_{z'} \\ \delta_{x'} \\ \delta_{y'} \end{bmatrix} = \begin{bmatrix} \theta_x \\ \theta_y \\ \theta_z \\ \delta_x \\ \delta_y \\ \delta_z \end{bmatrix} \begin{bmatrix} a_x & a_y & a_z & 0 & 0 & 0 \\ b_x & b_y & b_z & 0 & 0 & 0 \\ c_x & c_y & c_z & 0 & 0 & 0 \\ 0 & 0 & 0 & a_x & a_y & a_z \\ 0 & 0 & 0 & b_x & b_y & b_z \end{bmatrix} \qquad 12.13$$

The deformation along the member has been neglected since it was assumed that the deformations due to the flexural moment are dominant.

Considering the flexural bending moments and torsional moments resulting from these deformations, it will be seen that in the x'-plane, there are two joint rotations $\theta_{Nx'}$ and $\theta_{Fx'}$ and sway of $-(\delta_{Fy'}-\delta_{Ny'})$ which gives rise to

$$M_{Nx'}=EI_{x'}/L.(s\theta_{Nx'}+sc\theta_{Fx'}+(\delta_{Fy'}-\delta_{Ny'})\ \alpha/L) \qquad 12.14a$$

and

$$M_{Fx'}=EI_{x'}/L.(sc\theta_{Nx'}+s\theta_{Fx'}+(\delta_{Fy'}-\delta_{Ny'})\ \alpha/L) \qquad 12.14b$$

In the y'-plane there are two angular rotations $\theta_{Ny'}$ and $\theta_{Fy'}$ and sway of $(\delta_{Fx'}-\delta_{Nx'})$ which gives rise to

$$M_{Ny'}=EI_{y'}/L.(s\theta_{Ny'}+sc\theta_{Fy'}-(\delta_{Fx'}-\delta_{Nx'})\ \alpha/L) \qquad 12.14c$$

and

$$M_{Fy'}=EI_{y'}/L.(sc\theta_{Ny'}+s\theta_{Fy'}-(\delta_{Fx'}-\delta_{Nx'})\ \alpha/L) \qquad 12.14d$$

In the z'-plane, there are only two angular rotations of $\theta_{Nz'}$ and $\theta_{Fz'}$ which gives rise to torsional moments of

$$M_{Nz'}=GI_{z'}/L.(\theta_{Nz'}-\theta_{Fz'}) \qquad 12.14e$$

and

$$M_{Fz'}= - GI_{z'}/L.(\theta_{Nz'}-\theta_{Fz'}) \qquad 12.14f$$

These six equations 12.14a,b,c,d,e,f can be represented in the matrix form as

$$\begin{bmatrix} M_{x'} \\ M_{y'} \\ M_{z'} \end{bmatrix} = \begin{bmatrix} \theta_{x'} \\ \theta_{y'} \\ \theta_{z'} \\ \delta_{x'} \\ \delta_{y'} \end{bmatrix}_N \begin{bmatrix} k_{x'}s & 0 & 0 & 0 & -\alpha/L \\ 0 & k_{y'}s & 0 & \alpha/L & 0 \\ 0 & 0 & k_{z'} & 0 & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} \theta_{x'} \\ \theta_{y'} \\ \theta_{z'} \\ \delta_{x'} \\ \delta_{y'} \end{bmatrix}_F \begin{bmatrix} k_{x'}sc & 0 & 0 & 0 & +\alpha/L \\ 0 & k_{y'}sc & 0 & -\alpha/L & 0 \\ 0 & 0 & -k_{z'} & 0 & 0 \end{bmatrix} \qquad 12.15a$$

and

$$\begin{bmatrix} M_{x'} \\ M_{y'} \\ M_{z'} \end{bmatrix}_F = \begin{bmatrix} \theta_{x'} \\ \theta_{y'} \\ \theta_{z'} \\ \delta_{x'} \\ \delta_{y'} \end{bmatrix}_N \begin{bmatrix} k_{x'}sc & 0 & 0 & 0 & -\alpha/L \\ 0 & k_{y'}sc & 0 & \alpha/L & 0 \\ 0 & 0 & -k_{z'} & 0 & 0 \end{bmatrix}$$

$$\cdot \begin{bmatrix} \theta_{x'} \\ \theta_{y'} \\ \theta_{z'} \\ \delta_{x'} \\ \delta_{y'} \end{bmatrix}_F \begin{bmatrix} k_{x'}s & 0 & 0 & 0 & \alpha/L \\ 0 & k_{y'}s & -0 & -\alpha/L & 0 \\ 0 & 0 & k_{z'} & 0 & 0 \end{bmatrix} \qquad 12.15b$$

These moments are along the axes of the member and these must be projected along the x,y,z axes in considering the equilibrium of the joints and relationship 12.12 is used.

$$\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix} = \begin{bmatrix} M_{x'} \\ M_{y'} \\ M_{z'} \end{bmatrix} \begin{bmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{bmatrix} \qquad 12.16$$

It is seen from 12.16 that (M_x, M_y, M_z) are functions of $(M_{x'}, M_{y'}, M_{z'})$ and from 12.15 that $(M_{x'}, M_{y'}, M_{z'})$ are functions of $(\theta_{x'}, \theta_{y'}, \theta_{z'}, \delta_{x'}, \delta_{y'})$ and from 12.13 that $(\theta_{x'}, \theta_{y'}, \theta_{z'}, \delta_{x'}, \delta_{y'})$ are functions of $(\theta_x, \theta_y, \theta_z, \delta_x, \delta_y, \delta_z)$ so that (M_x, M_y, M_z) can be expressed as functions of $(\theta_x, \theta_y, \theta_z, \delta_x, \delta_y, \delta_z)$ as shown in 12.17

$$\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix}_N = \begin{bmatrix} \theta_x \\ \theta_y \\ \theta_z \\ \delta_x \\ \delta_y \\ \delta_z \end{bmatrix}_N \begin{bmatrix} a_x & a_y & a_z & 0 & 0 & 0 \\ b_x & b_y & b_z & 0 & 0 & 0 \\ c_x & c_y & c_z & 0 & 0 & 0 \\ 0 & 0 & 0 & a_x & a_y & a_z \\ 0 & 0 & 0 & b_x & b_y & b_z \end{bmatrix} \begin{bmatrix} k_{x'}s & 0 & 0 & 0 & -\alpha/L \\ 0 & k_{y'}s & 0 & \alpha/L & 0 \\ 0 & 0 & k_{z'} & 0 & 0 \end{bmatrix} \begin{bmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{bmatrix}$$

$$+ \begin{bmatrix} \theta_x \\ \theta_y \\ \theta_z \\ \delta_x \\ \delta_y \\ \delta_z \end{bmatrix}_F \begin{bmatrix} a_x & a_y & a_z & 0 & 0 & 0 \\ b_x & b_y & b_z & 0 & 0 & 0 \\ c_x & c_y & c_z & 0 & 0 & 0 \\ 0 & 0 & 0 & a_x & a_y & a_z \\ 0 & 0 & 0 & b_x & b_y & b_z \end{bmatrix} \begin{bmatrix} k_{x'}sc & 0 & 0 & 0 & \alpha/L \\ 0 & k_{y'}sc & 0 & -\alpha/L & 0 \\ 0 & 0 & -k_{z'} & 0 & 0 \end{bmatrix} \begin{bmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{bmatrix} \qquad 12.17a$$

or

$$
\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix}_N = \begin{bmatrix} \theta_x \\ \theta_y \\ \theta_z \\ \delta_x \\ \delta_y \\ \delta_z \end{bmatrix}_N
\begin{bmatrix}
[a_x a_x s k_{x'} + b_x b_x s k_{y'} + c_x c_x k_{z'}] & [a_y a_x s k_{x'} + b_y b_x s k_{y'} + c_y c_x k_{z'}] & [a_z a_x s k_{x'} + b_z b_x s k_{y'} + c_z c_x k_{z'}] \\
a_y \quad b_y \quad c_y & a_y \quad b_y \quad c_y & a_y \quad b_y \quad c_y \\
a_z \quad b_z \quad c_z & a_z \quad b_z \quad c_z & a_z \quad b_z \quad c_z \\
[a_x b_x \alpha k_{y'}/L - a_x a_x k_{x'} \alpha/L] & [a_y b_x k_{y'} \alpha/L - b_y a_x k_{x'} \alpha/L] & [a_z b_x k_{y'} \alpha/L - b_z a_x k_{x'} \alpha/L] \\
b_y \quad a_y & b_y \quad a_y & b_y \quad a_y \\
b_z \quad a_z & b_z \quad a_z & b_z \quad a_z
\end{bmatrix}
$$

$$
+ \begin{bmatrix} \theta_x \\ \theta_y \\ \theta_z \\ \delta_x \\ \delta_y \\ \delta_z \end{bmatrix}_F
\begin{bmatrix}
[a_x a_x sc k_{x'} + b_x b_x sc k_{y'} - c_x c_x k_{z'}] & [a_y a_x sc k_{x'} + b_y b_x sc k_{y'} - c_y c_x k_{z'}] & [a_z a_x sc k_{x'} + b_z b_x sc k_{y'} - c_z c_x k'_z] \\
a_y \quad b_y \quad c_y & a_y \quad b_y \quad c_y & a_y \quad b_y \quad c_y \\
a_z \quad b_z \quad c_z & a_z \quad b_z \quad c_z & a_z \quad b_z \quad c_z \\
[-a_x b_x k_{y'} \alpha/L + b_x a_x k_{x'} \alpha/L] & [-a_y b_x k_{y'} \alpha/L + b_y a_x k_{x'} \alpha/L] & [-a_z b_x k_{y'} \alpha/L + b_z a_x k_{x'} \alpha/L] \\
b_y \quad a_y & b_y \quad a_y & b_y \quad a_y \\
b_z \quad a_z & b_z \quad a_z & b_z \quad a_z
\end{bmatrix} \qquad 12.17b
$$

$$
\begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix}_F = \begin{bmatrix} \theta_x \\ \cdot \\ \delta_z \end{bmatrix}_N \Big[\text{same terms but with s and sc interchanged and the sign of } k_{z'} \text{ changed} \Big]
$$

$$
+ \begin{bmatrix} \theta_x \\ \cdot \\ \delta_z \end{bmatrix}_F \Big[\text{same terms but with sc and s interchanged and the sign of } k'_z \text{ changed} \Big]
$$

Shear forces

To determine other equilibrium relationships of actions and deformations it is necessary to find the shear forces along the principal and secondary axes of the members and express them in terms of $(\theta_x, \theta_y, \theta_z, \delta_x, \delta_y, \delta_z)$.

For the member NF the shear force $v_{Nx'}$ is built up by the moments $M_{Ny'}$ and $M_{Fy'}$ in the y-plane and is equal to

$$v_{Nx'} = k_{y'}/L.(\alpha\theta_{Ny'} + \alpha\theta_{Fy'} - (\delta_{Fx'} - \delta_{Nx'})\, 2A/L) \qquad 12.18a$$

and the shear force $v_{Ny'}$ is built up by the moments $M_{Nx'}$ and $M_{Fx'}$ in the x'-plane and is equal to

$$v_{Ny'} = -\, k_{x'}/L.(\alpha\theta_{Nx'} + \alpha\theta_{Fx'} + (\delta_{Fy'} - \delta_{Ny'})\, 2A/L) \qquad 12.18b$$

or in the matrix form as

$$\begin{bmatrix} v_{x'} \\ v_{y'} \end{bmatrix}_N = \begin{bmatrix} \theta_{x'} \\ \theta_{y'} \\ \theta_{z'} \\ \delta_{x'} \\ \delta_{y'} \end{bmatrix}_N \begin{bmatrix} 0 & k_{y'}\alpha/L & 0 & 2k_{y'}A/L^2 & 0 \\ -k_{x'}\alpha/L & 0 & 0 & 0 & 2k_{x'}A/L^2 \end{bmatrix}$$

$$+ \begin{bmatrix} \theta_{x'} \\ \theta_{y'} \\ \theta_{z'} \\ \delta_{x'} \\ \delta_{y'} \end{bmatrix}_F \begin{bmatrix} 0 & k_{y'}\alpha/L & 0 & -2k_{y'}A/L^2 & 0 \\ -k_{x'}\alpha/L & 0 & 0 & 0 & -2k_{x'}A/L^2 \end{bmatrix} \qquad 12.19$$

The corresponding shear forces at the other end of a member will be of the same value but in opposite direction.

i.e

$$\begin{bmatrix} v_{x'} \\ v_{y'} \end{bmatrix}_N = - \begin{bmatrix} v_{x'} \\ v_{y'} \end{bmatrix}_F$$

From 12.19 it is seen that $(v_{x'}, v_{y'})$ are functions of $(\theta_{x'}, \theta_{y'}, \theta_{z'}, \delta_{x'}, \delta_{y'})$ and from 12.13 that $(\theta_{x'}, \theta_{y'}, \theta_{z'}, \delta_{x'}, \delta_{y'})$ are are functions of $(\theta_x, \theta_y, \theta_z, \delta_x, \delta_y, \delta_z)$ therefore $(v_{x'}, v_{y'})$ can be expressed as a function of $(\theta_x, \theta_y, \theta_z, \delta_x, \delta_y, \delta_z)$ as is shown in 12.20.

$$\begin{bmatrix} v_{x'} \\ v_{y'} \end{bmatrix}_N = \begin{bmatrix} \theta_x \\ \theta_y \\ \theta_z \\ \delta_x \\ \delta_y \\ \delta_z \end{bmatrix}_N \begin{bmatrix} a_x & a_y & a_z & 0 & 0 & 0 \\ b_x & b_y & b_z & 0 & 0 & 0 \\ c_x & c_y & c_z & 0 & 0 & 0 \\ 0 & 0 & 0 & a_x & a_y & a_z \\ 0 & 0 & 0 & b_x & b_y & b_z \end{bmatrix} \begin{bmatrix} 0 & k_{y'}\alpha/L & 0 & 2k_{y'}A/L^2 & 0 \\ -k_{x'}\alpha/L & 0 & 0 & 0 & 2k_{x'}A/L^2 \end{bmatrix}$$

$$+ \begin{bmatrix} \theta_x \\ \theta_y \\ \theta_z \\ \delta_x \\ \delta_y \\ \delta_z \end{bmatrix}_F \begin{bmatrix} a_x & a_y & a_z & 0 & 0 & 0 \\ b_x & b_y & b_z & 0 & 0 & 0 \\ c_x & c_y & c_z & 0 & 0 & 0 \\ 0 & 0 & 0 & a_x & a_y & a_z \\ 0 & 0 & 0 & b_x & b_y & b_z \end{bmatrix} \begin{bmatrix} 0 & k_{y'}\alpha/L & 0 & -2k_{y'}A/L^2 & 0 \\ -k_{x'}\alpha/L & 0 & 0 & 0 & -2k_{x'}A/L^2 \end{bmatrix} \qquad 12.20a$$

or

$$\begin{bmatrix} V_{x'} \\ V_{y'} \end{bmatrix}_N = \begin{bmatrix} b_x k_{y'}\alpha/L & b_y k_{y'}\alpha/L & b_z k_{y'}\alpha/L & a_x 2k_{y'}A/L^2 & a_y 2k_{y'}A/L^2 & a_z 2k_{y'}A/L^2 \\ -a_x k_{x'}\alpha/L & -a_y k_{x'}\alpha/L & -a_z k_{x'}\alpha/L & b_x 2k_{x'}A/L^2 & b_y 2k_{x'}A/L^2 & b_z 2k_{x'}A/L^2 \end{bmatrix} \begin{bmatrix} \theta_x \\ \theta_y \\ \theta_z \\ \delta_x \\ \delta_y \\ \delta_z \end{bmatrix}_N$$

$$+ \begin{bmatrix} b_x k_{y'}\alpha/L & b_y k_{y'}\alpha/L & b_z k_{y'}\alpha/L & -a_x 2k_{y'}A/L^2 & -a_y 2k_{y'}A/L^2 & -a_z 2k_{y'}A/L^2 \\ -a_x k_{x'}\alpha/L & -a_y k_{x'}\alpha/L & -a_z k_{x'}\alpha/L & -b_x 2k_{x'}A/L^2 & -b_y 2k_{x'}A/L^2 & -b_z 2k_{x'}A/L^2 \end{bmatrix} \begin{bmatrix} \theta_x \\ \theta_y \\ \theta_z \\ \delta_x \\ \delta_y \\ \delta_z \end{bmatrix}_F \qquad 12.20b$$

The moments and shear forces expressed in terms of the deformations at the joints will have signs which are automatically given by the direction cosines.

<u>Direction cosines of the axes of a member with circular section in space</u>

For any member in space, the axis of torsion can be defined since its projections along the cartesian coordinate axes x,y,z can be easily computed. Some difficulties arise when the direction cosines of the other axes are determined.

Let the unit vector along the torsional axis by $\underline{z}'$ then

$$\underline{z}' = c_x\underline{i} + c_y\underline{j} + c_z\underline{k} \qquad 12.21a$$

where

$c_x = (x - x_o)/L$, $c_y = (y - y_o)/L$ and $c_z = (z - z_o)/L$

If the unit vector along the axis of the member x'-axis has direction cosines a_x, a_y, a_z and this unit vector is taken to be perpendicular to the z-axis then

$$\underline{x}' \cdot \underline{z} = 0 \qquad 12.21b$$

where

$$\underline{x}' = a_x\underline{i} + a_y\underline{j} + a_z\underline{k}$$

and $\underline{z} = z\,\underline{k}$

It follows from 12.21b that

$$a_z\, z = 0$$

but $z \neq 0$ thus $a_z = 0$ 12.21c

$\underline{z}'$ and $\underline{x}'$ are the two axes of the member and these are perpendicular to one another hence

$$\underline{z}' \cdot \underline{x}' = a_x c_x + c_y a_y = 0 \qquad 12.21d$$

From the conditions of the direction cosines

$$a_x^2 + a_y^2 + a_z^2 = 1 \qquad 12.21e$$

If the unit vector $\underline{x}'$ is measured in the positive direction of the x-axis then 12.21c, 12.21d and 12.21e gives

$$a_x = + (c_y/c_x)/\sqrt{1+(c_y/c_x)^2} \qquad 12.21f$$

and

$$a_y = - 1/\sqrt{1+(c_y/c_x)^2} \qquad 12.21g$$

If the unit vector along the third axis is $\underline{y}' = b_x\underline{i} + b_y\underline{j} + b_z\underline{k}$ and this vector is perpendicular to the $\underline{x}'$ and $\underline{z}'$ vectors, then

$$\underline{z}' \cdot \underline{y}' = c_x b_x + c_y b_y + c_z b_z = 0 \qquad 12.21h$$

$$\underline{x}' \cdot \underline{y}' = a_x b_x + a_y b_y = 0 \qquad 12.21i$$

The direction cosines of the third vector satisfies

$$b_x^2 + b_y^2 + b_z^2 = 1 \qquad 12.21j$$

Solution of equations 12.21h,i,j gives

$$b_z = + 1/\sqrt{\left\{(a_y/a_x)^2 + 1\right\}\left\{c_z/(c_y - c_x a_y/a_x)\right\}^2 + 1} \qquad 12.21k$$

$$b_y = - c_z b_z/(c_y - c_x a_y/a_x) \qquad 12.21l$$

and

$$b_x = - a_y b_y/a_x \qquad 12.21m$$

These direction cosines satisfy the condition[19]

$$\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = 1$$

allowing the arithmetic to be checked.

Frame 4

The elastic critical load of the framework of Figure 12.7 with inclined members is determined. The three modes of elastic instability are possible but the analysis of the critical mode is only shown. A model of the framework was made of members having equal lengths and flexural rigidity EI. The members had circular sections. When the framework was loaded it was found that the second mode of instability was the critical mode since any dynamic disturbance applied to the frame causes the dominant torsional vibration only. The cartesian axes x,y,z are chosen as in frame 1. Due to the number of parameters given by 12.17 it is more convenient to deal with a frame with known direction cosines. The inclined members are assumed to make 45° with the z-axis and their projections on the xy-plane make also 45° with the x-axis. Hence equations 12.21f,g,k,l,m gives the direction cosines of member AA' as

$$\begin{bmatrix} -0.707, & -0.707, & 0 \\ +0.5, & -0.5, & 0.707 \\ -0.5, & +0.5 & 0.707 \end{bmatrix}$$

Stability criterion for the twisting sway mode

As in frame 1, if the deformations at joint A are $(\partial\theta_x, \partial\theta_y, \partial\theta_z, \partial\delta_x(=\partial\delta), \partial\delta_y(=\partial\delta), 0)$ then the deformations at joints B and D will be $(-\partial\theta_x, \partial\theta_y, \partial\theta_z, \partial\delta_x(=\partial\delta), -\partial\delta_y(-\partial\delta), 0)$ and $(\partial\theta_x, -\partial\theta_y, \partial\theta_z, -\partial\delta_x(-\partial\delta), \partial\delta_y(\partial\delta), 0)$

respectively. Due to these deformations the following moments will appear at the ends of the members meeting at joint A.

Member AB

$\partial M_x = 2k_{2G}\partial\theta_x$

$\partial M_y = (k\alpha)_2\partial\theta_y$

$\partial M_z = (k\alpha)_2\partial\theta_z - (2k\alpha/L)_2\partial\delta$

Member AD

$\partial M_x = (k\alpha)_2\partial\theta_x$

$\partial M_y = 2k_{2G}\partial\theta_y$

$\partial M_z = (k\alpha)_2\partial\theta_z - (2k\alpha/L)_2\partial\delta$

The moments at the end of member AA' are obtained by substituting the direction cosines in 12.17 and putting $\partial\delta_x = \partial\delta_y = \partial\delta$ and $\partial\delta_z = 0$ which gives

$$\begin{bmatrix}\partial M_x\\ \partial M_y\\ \partial M_z\end{bmatrix}_A = \begin{bmatrix}\partial\theta_x\\ \partial\theta_y\\ \partial\theta_z\\ \partial\delta\end{bmatrix}\begin{bmatrix}0.75(ks)_1+0.25k_{1G} & 0.25(ks)_1-0.25k_{1G} & 0.3535((ks)_1-k_{1G}) & 0.707(k\alpha/L)_1\\ 0.25(ks)_1-0.25k_{1G} & 0.75(ks)_1+0.25k_{1G} & -0.3535((ks)_1-k_{1G}) & -0.707(k\alpha/L)_1\\ 0.3535((ks)_1-k_{1G}) & -0.3535((ks)_1-k_{1G}) & 0.5((ks)_1+k_{1G}) & (k\alpha/L)_1\end{bmatrix}$$

For the reason given in frame 1, each disturbing force at each corner stores energy in one inclined member and the halves of two roof members. Thus the shear force sway equation 2.3 gives

$$2H\partial\delta = (v_{x'}\partial\delta_{x'} + v_{y'}\partial\delta_{y'})_{AA'} + (v_y\partial\delta_y/2)_{AB} + (v_x\partial\delta_x/2)_{AD} \qquad 12.22a$$

where

$$(v_y)_{AB} = -\left[(2k\alpha/L)_2\partial\theta_z - (4kA/L^2)_2\partial\delta\right]$$

$$(v_x)_{AD} = -\left[(2k\alpha/L)_2\partial\theta_z - (4kA/L^2)_2\partial\delta\right]$$

$$(\partial\delta_y)_{AB} = (\partial\delta_x)_{AD} = \partial\delta$$

$(\partial\delta_{x'})_{AA'}$ and $(\partial\delta_{y'})_{AA'}$ are obtained by using 12.11

$$\begin{bmatrix}\partial\delta_{x'}\\ \partial\delta_{y'}\end{bmatrix}_{AA'} = \begin{bmatrix}\partial\delta_x\\ \partial\delta_y\end{bmatrix}\begin{bmatrix}-0.707 & -0.707\\ 0.5 & -0.5\end{bmatrix}$$

But at joint A $\partial\delta_x = \partial\delta_y = \partial\delta$ hence

$$\begin{bmatrix}\partial\delta_{x'}\\ \partial\delta_{y'}\end{bmatrix}_{AA'} = \left[\partial\delta\right]\begin{bmatrix}-1.414\\ 0\end{bmatrix}$$

$(v_{x'})_{AA'}$ is obtained from 12.20 which is

$$\left[v_{x'}\right]_{AA'} = \begin{bmatrix}\partial\theta_x\\ \partial\theta_y\\ \partial\theta_z\\ \partial\delta\end{bmatrix}\left[0.5(k\alpha/L)_1 \quad -0.5(k\alpha/L)_1 \quad 0.707(k\alpha/L)_1 \quad 2.828(kA/L^2)_1\right]$$

Substituting these values in 12.22a gives

$$\left[\sqrt{2}H\right] = \begin{bmatrix}\partial\theta_x\\ \partial\theta_y\\ \partial\theta_z\\ \partial\delta\end{bmatrix}\left[0.707(k\alpha/L)_1 \quad -0.707(k\alpha/L)_1 \quad \{(k\alpha/L)_1-(4k\alpha/L)_2\} \quad \{(8kA/L^2)_2 + (4kA/L^2)_1\}\right] \qquad 12.22b$$

At the critical load H vanishes therefore the above expression is equal to zero. Equilibrium at the joints requires that the total moments at joint A along each axis is zero. Hence the equilibrium equations will be:

$$\begin{bmatrix}\partial M_x\\ \partial M_y\\ \partial M_z\\ \sqrt{2}H\end{bmatrix}=\begin{bmatrix}\partial\theta_x\\ \partial\theta_y\\ \partial\theta_z\\ \partial\delta\end{bmatrix}\begin{bmatrix}(k\alpha)_2+2k_{2G}+0.75(ks)_1+0.25k_{1G} & 0.25((ks)_1-k_{1G}) & 0.3535((ks)_1-k_{1G}) & 0.707(k\alpha/L)_1\\ 0.25((ks)_1-k_{1G}) & (k\alpha)_2+2k_{2G}+0.75(ks)_1+0.25k_{1G} & -0.3535((ks)_1-k_{1G}) & -0.707(k\alpha/L)_1\\ 0.3535((ks)_1-k_{1G}) & -0.3535((ks)_1-k_{1G}) & (2k\alpha)_2+0.5(ks)_1+0.5k_{1G} & (k\alpha/L)_1-(4k\alpha/L)_2\\ 0.707(k\alpha/L)_1 & -0.707(k\alpha/L)_1 & (k\alpha/L)_1-(4k\alpha/L)_2 & (8kA/L^2)_2+(4kA/L^2)_1\end{bmatrix}\begin{bmatrix}0\\0\\0\\0\end{bmatrix} \qquad 12.23$$

The determinant of the square matrix is the stability condition and the load parameter ρ making the determinant vanish is the critical one which is obtained by a trial and error process.

Numerical example

The framework was assumed to have equal lengths and EI-values therefore the relative flexural stiffness of all the members is equal. The torsional stiffness of the members is assumed to be 0.77k where k is the flexural stiffness. When the lengths and the relative k-values are substituted in 12.23, the determinant becomes

$$\begin{vmatrix}\alpha_2+0.75s_1+1.73 & 0.25s_1-0.192 & 0.3535s_1-0.272 & 0.707\alpha_1\\ 0.25s_1-0.192 & \alpha_2+0.75s_1+1.73 & -0.3535s_1+0.272 & -0.707\alpha_1\\ 0.3535s_1-0.272 & -0.3535s_1+0.272 & 2\alpha_2+0.5s_1+0.3535 & \alpha_1-4\alpha_2\\ 0.707\alpha_1 & -0.707\alpha_1 & \alpha_1-4\alpha_2 & 8A_2+4A_1\end{vmatrix} \qquad 12.23a$$

The forces, the relative Euler loads and the relative load parameters of the members are:

Member	AA'(1)	AB,AD(2)
force	1.414P	0.707P
rel.P_e	1	1
rel. ρ	1	0.5

The critical load parameter ρ is obtained by trial and error.

First trial ρ_1 =0.84

ρ_2 =0.42

From tables.[9]

s_1=2.7483 α_1=5.1168 A_1=0.9716

α_2=5.5726 A_2=3.5000

Substituting these values in 12.23a gives

$$\Delta = +\ 4710$$

i.e the frame is stable, a higher value of is therefore tested.

When ρ_1 =0.92 Δ = +59

The elastic critical load of the framework is therefore

$$4P = \frac{4 \times 0.92}{1.414} P_e$$

$$= 2.6\ P_e$$

This value corresponds with $(28/11.7)P_e = 2.393P_e$ obtained experimentally.

Chapter 13

Multi-bay and multi-storey building frames with vertical stanchions

Introduction

In multi-bay and multi-storey building frames, the size of the matrices used in estimating the elastic critical loads becomes very large and this necessiates the employment of the electronic computer. In these frames there are three possible modes of elastic instability in the absence of external influences. These influences may introduce other modes of stability such as partial sway mode. These modes will be investigated to determine the critical mode.

It is well known that the mode of instability where no lateral movement is allowed has an elastic critical load bigger than other modes. To investigate whether the sway or the twisting sway mode is the critical one, the elastic critical loads of the two modes of the frame in Figure 12.1 were estimated for different beam stiffnesses. The results obtained are tabulated below:-

k_2/k_1	sway critical ρ_1	twisting sway critical ρ_1
0	0.25	0.25
1	0.75	0.916
2	0.855	1.009
∞	1.00	1.123

These results shows that the anti-symmetrical sway elastic critical load is the least load for fixed stanchion feet.

The sway deformation in the twisting sway mode is a combination of stanchion sway and either torsional rotation of the stanchions or the sway of the roof members. The extra torsional rotation or the sway of the roof members increases the stiffness of the frame, thus increasing its elastic critical load. When the torsional stiffness k_G of the members tends to zero, the twisting elastic critical load approaches the anti-symmetrical sway elastic critical load since the twisting sway stability condition of 12.5 is then modified to

$$\Delta = 2(kA)_1((kn)_1+6k_2) \qquad 13.1$$

when $k_{1G} = k_{2G} = 0$

This is the stability condition of the anti-symmetrical sway mode, since $\{(kn)_1+6k_2\}$ has the lowest elastic load parameter ρ that makes Δ vanish. Therefore, the sway elastic critical load is a lower bound to the twisting elastic critical load. It can be concluded that the critical sway mode is the particular mode which requires the least number of members to sway whilst satisfying geometrical and equilibrium conditions. This is true for frames with partially restrained stanchion feet and the analysis to be carried out was made easier by the modification of the stability functions.

Modified stability functions

The stability functions are modified to take into account the

the partial fixity of the supports. The supports are replaced by three springs resisting deformations in the three dimensions at the feet of the stanchions. The stiffness of each of these springs is assumed to be K. Derivation of an expressions for the modified functions are obtained by imposing deformation at one end of the member and deforming the other end to balance the moment. From the resultant moment at the first end and the shear force in the member, the modified functions are obtained.

For member AB connected to a spring at B and subjected to an end moment M_{AB} at A, the ends will be rotated through angles θ_A and θ_B respectively. Due to these deformations, the moments at the ends are tabulated:

Operation	M_{spring}	M_{BA}	M_{AB}
1. Rot. A		$ksc\theta_A$	$ks\theta_A$
2. Rot. B	$K\theta_B$	$ks\theta_B$	$ksc\theta_B$

B is in equilibrium, hence

$$ksc\theta_A + (K+ks)\theta_B = 0$$

which gives

$$\theta_B = -\,ksc\theta_A\,/(K+ks)$$

A is also in equilibrium, hence

$$M_{AB} = ks\theta_A + ksc\theta_B$$

$$= ks'\theta_A$$

where

$$s' = s\left(\frac{s'' + K/k}{s + K/k}\right)$$

Using the same procedure, it can be shown that the shear force due to the rotation of A is $k\alpha'\theta_A/L$ and that the sway moment at A due to a sway δ of A is $- k\alpha'\delta/L$ where

$$\alpha' = \alpha\left(1 - \frac{sc}{s + K/k}\right)$$

The sway shear force is $2kA'\delta/L^2$ where

$$A' = A - \frac{\alpha^2}{2(s+K/k)}$$

and the torsional moment appearing at A due to the torsional rotation ϕ_A is $k_G'\phi_A$ where

$$k_G' = k_G\frac{K/k}{k_G/k+K/k}$$

The no-shear bending moment appearing at A due to the rotation θ_A is $kn'\theta_A$ where

$$n' = n - \frac{o^2}{n+K/k}$$

The two elastic critical loads of the frame are then calculated using the modified stability functions obtained above. The results obtained by assuming different beam and spring stiffnesses are tabulated below:

K/k_1	k_2/k_1	sway critical ρ_1	twisting sway critical ρ_1
0	0	0	0
0	1	0.184	0.195
0	∞	0.25	0.25
2	1	0.412	0.545
∞	1	0.75	0.916

The elastic critical load of the twisting sway mode in multi-bay building frames will be much higher than that of the sway mode. This is because rotation of any floor level as a whole will call into play sways in every beam at that level which greatly increases the stiffness of the structure as indicated by frames 2 and 3 in chapter 12.

Plane frames

Approximate methods for estimating the sway elastic critical load.

Multi-bay frames

There is already an approximate method[11] for estimating the elastic critical load in which the multi-bay frame is replaced by a single bay frame having each beam stiffness equal to the sum of the beam stiffnesses at the corresponding level in the multi-bay frame stanchions stiffnesses, Euler loads and axial forces equal to the sum of those of the multi-bay frame. A few calculations using this approximation were made. The exact values and the approximate values which show satisfactory agreement, are tabulated for different numbers of bays. The frames are assumed to have members of equal lengths and stiffnesses and to have inner stanchions which carry double the axial forces of the external stanchions.

Number of bay	(1) Exact $\Sigma P/P_e$	(2) Approx. $\Sigma P/P_e$	(2)/(1)
1	1.495	1.495	1.000
2	2.295	2.388	1.040
3	3.161	3.260	1.031
4	4.014	4.122	1.027
5	4.862	4.982	1.025

Tall building frames

It is well recognised that the sway mode of instability of tall portal frames has the least elastic critical load and instability usually occurs in this mode, in the absence of lateral bracing and horizontal constraints at each floor level. There are two possible modes of sway instability:-

(i) Instability due to local failure of the weakest storey.

Usually this mode of instability is possible in multi-bay and multi-storey or single bay multi-storey portal frames. An approximate method for predicting this elastic critical load is given.

(ii) Instability due to the failure of the whole building as a built-up strut.

This mode of instability is possible in multi-storey tall buildings with a single or a small number of bays. The tall building frame becomes unstable in the same way a battened strut behaves.

Approximate method for local failure of the weakest storey

The approximate method is based on the fact that the deformation in the weakest storey at the critical load of the frame will be large compared with those of the adjacent storeys. These deformations will be sway of the stanchions of the storey concerned with joint rotations to wipe out the shear forces and keep the joints in equilibrium. The storeys near the weakest storey will be affected by the large deformation but the amount of deformation will be less as the stiffnesses of the other storeys become bigger. For convenience, the no shear stability functions will be used in obtaining the stability conditions and therefore the deformation will be reduced to a no shear rotation of the joints only.

Each joint at the critical storey will undergo a rotation $\emptyset$ when a disturbance is applied and as result the adjacent joint will be subjected to a moment carried from the critical storey joints, causing these joints to rotate through an angle θ. The value of θ is smaller than $\emptyset$ because the adjacent storeys joints are stiffer than the joints of the critical storey. When θ is taken to be zero i.e when the adjacent joints are fixed, the elastic critical load given by this mode will be bigger than the exact elastic critical load. And when θ is taken to be equal to that of the critical joints $\emptyset$, the elastic critical load given by this mode will be lower than the exact value, following the argument that $\theta < \emptyset$.

These two cases i.e $\theta = 0$ and $\theta = \emptyset$ will be taken as a method for estimating upper and lower bounds of the elastic critical load. The exact value of the elastic critical load will be nearer to one or other of the bounds depending on the conditions in the adjacent storeys. The mean value of the two bounds will be taken as an approximation to the elastic critical load of tall portal frames. It is believed to give values good enough for practical purposes.

To increase the accuracy of the method, more joints could be taken giving the proper deformations to these joints but leaving the last joint either fixed or rotated through an angle similar to the one before the last. This will be illustrated in obtaining the stability conditions.

The upper bound of the elastic critical load has been proposed by Dr. Bolton[12] as an approximate method for estimating the elastic critical load of tall building frames.

Derivation of the stability conditions

1) First approximation

Let the weakest storey be the rth storey. Rotate the upper and the lower joints of this storey through $\theta_{A'}$ and θ_A respectively. For obtaining the upper bound of the elastic critical load, joints B and B' will be fixed. The moments appearing at joints A and A' vanish at the critical load hence

$$M_{A'} = T_{A'}\theta_{A'} - (ko)_r\,\theta_A = 0$$

$$M_A = -(ko)_r\theta_{A'} + T_A\theta_A = 0$$

where T is the no-shear stiffness of the joint and is equal to $(kn)_r + (kn)_{r\pm1} + 6k_B$.

The determinant of the coefficients of unknowns is reduceable to

$$\Delta = T_A T_{A'}(1 - \eta_r) \qquad 13.2a$$

where

$$\eta_r = (ko)_r^2 / T_A T_{A'}$$

For the lower bound of the elastic critical load, joints B and B' shown in Figure 13.1 will be rotated through $\Theta_{A'}$ and Θ_A respectively. The moments appearing at joints A' and A will become

$$M_{A'}=(T_{A'}-(ko)_{r-1})\Theta_{A'}-(ko)_r\Theta_A=0$$

$$M_A=-(ko)_r\Theta_{A'}+(T_A-(ko)_{r+1})\Theta_A=0$$

The stability condition of 13.2a will be also modified to

$$\Delta = T_A T_{A'}(1-\eta_r') \qquad 13.2b$$

where

$$\eta_r' = (ko)_r^2/(T_{A'}-(ko)_{r-1})(T_A-(ko)_{r+1})$$

Any load parameter ρ making $\eta_r=1$ will be the upper bound of the elastic critical load parameter and that making $\eta_r'=1$ will be the lower bound of the critical load parameter since the terms in the brackets become zero before T_A and $T_{A'}$.

2) Second approximation

The proper deformations of the (r-1)th, rth and (r+1)th storeys are considered. For the upper bound of the elastic critical load, the upper joints of the (r-2)th storey and the lower joints of the (r+1)th storey will be fixed. Accordingly, the moments appearing at the joints are:

$$M_{B'}=T_{B'}\Theta_{B'}-(ko)_{r-1}\Theta_{A'}=0$$

$$M_{A'}=-(ko)_{r-1}\Theta_{B'}+T_{A'}\Theta_{A'}-(ko)_r\Theta_A=0$$

$M_A = -(ko)_r\theta_{A'} + T_A\theta_A - (ko)_{r+1}\theta_B = 0$

$M_B = -(ko)_{r+1}\theta_A + T_B\theta_B = 0$

The determinant of the coefficients of unknowns is reduceable to a simpler form and in this expression the term which gives the lowest load parameter is

$$\Delta = (1 - \eta_{r-1})(1 - \eta_{r+1}) - \eta_r \qquad 13.3a$$

where

$\eta_{r-1} = (ko)^2_{r-1}/T_{A'}T_{B'}$ and $\eta_{r+1} = (ko)^2_{r+1}/T_A T_B$

For obtaining the lower bound of the elastic critical load, the upper joints of the (r-2)th storey and the lower joints of the (r+2)th storey will be rotated through $\theta_{B'}$ and θ_B respectively. It follows that the stability condition of 13.3a will be modified to

$$= (1 - \eta'_{r-1})(1 - \eta'_{r+1}) - \eta_r \qquad 13.3b$$

where

$\eta'_{r-1} = (ko)^2_{r-1}/T_{A'}(T_{B'} - (ko)_{r-2})$ and $\eta'_{r+1} = (ko)^2_{r+1}/T_A(T_B - (ko)_{r+2})$

3) <u>Third approximation</u>

The proper deformations of the (r-2)th, (r-1)th, rth, (r+1)th and (r+2)th storeys are considered. For the upper bound of the elastic critical load, the upper joints of the (r-3)th storey and the lower joints of the (r+3)th will be fixed. Then the stability condition will be

$$\Delta = (1-\eta_{r-2}-\eta_{r-1})(1-\eta_{r+2}- {}_{r+1}) - \eta_{r} \quad (1-\eta_{r-2})(1-\eta_{r+2}) \qquad 13.4a$$

For the lower bound, the joints of the (r-3)th storey will be rotated through the same angle as the (r-2)th and similarly the joints of the (r+3)th storey. The stability condition 13.4a will be modified to

$$\Delta = (1-\eta'_{r-2}-\eta_{r-1})(1-\eta'_{r+2}-\eta_{r+1}) - \eta_{r} \quad (1-\eta'_{r-2})(1-\eta'_{r+2}) \qquad 13.4b$$

The stability conditions 13.4a and 13.4b are reduceable to any of the previous approximations by fixing some of the joints to cause some of the η terms to vanish. In special cases, when the weakest storey is at the bottom, the stability conditions, starting from the bottom storey and the lowest joint A, are modified to

Approximation	Stability condition for the upper bound		Stability condition for the lower bound	
First Approx.	$T_A = 0$	13.5a	$T_A - (ko)_2 = 0$	13.5b
Second Approx.	$1-\eta_2 = 0$	13.6a	$1-\eta'_2 = 0$	13.6b
Third Approx.	$1-(\eta_2+\eta_3)=0$	13.7a	$1-(\eta_2+\eta'_3)=0$	13.7b

Partial fixity at the supports

Sometimes, the supports of a portal frames are not rigidly jointed but only partial fixity is provided by some means such as joining the two supports by a cross beam etc. The bounds of the elastic critical load will be estimated in the same way. The η term of the bottom storey will be unchanged and the η' term due to the supports will be equal to η since there is no storey beyond the supports to modify the η term.

Numerical examples

Example 1

The elastic critical load of the frame in Figure 13.2 will be estimated using the three approximate methods. All the members have the same cross-section and equal lengths. The stanchions of the bottom storey have the highest relative load parameter P/P_e and it follows that the bottom storey is the weakest one. So the conditions 13.5, 13.6 and 13.7 will be used in the estimation. On substituting the relative stiffnesses, the "no-shear" stiffnesses of the joints are:

$T_A = n_1 + n_2 + 6$

$T_B = n_2 + n_3 + 6$ $\qquad T_B - (ko)_3 = n_2 + n_3 + 6 - o_3$

$T_C = n_3 + n_4 + 6$ $\qquad T_C - (ko)_4 = n_3 + n_4 + 6 - o_4$

$\eta_2 = o_2^2 / T_A T_B$

$\eta_3 = o_3^2 / T_B T_C$ $\qquad \eta_3' = o_3^2 / T_B (T_C - o_4)$

Using equations 13.5a and 13.5b, the calculations for predicting the bounds are shown in Tables 13.1a and 13.1b. Using the other two approximations 13.6 and 13.7, the bounds obtained are tabulated, together with the exact value obtained by the classical method of prediction using 7X7 determinant, in Table 13.2

Approximation	Upper bound ρ_1	Lower bound ρ_1	mean value ρ_1	Exact value ρ_1
13.5c	0.652	0.54	0.596	
13.6	0.564	0.52	0.542	0.546
13.7	0.546	0.534	0.540	

Table 13.2

Example 2

The elastic critical load of the frame in Figure 13.2 will be estimated when there is a concentrated loads only at the top floor of the portal frame. In this frame the weakest storey is probably the second one. The second approximation of 13.3 will be used to evaluate the upper and the lower bounds of the elastic critical load. The no-shear stiffness of all the joints will be equal to

$T = 2n+6$ and $T-(ko) = 2n+6-o$

The term $\eta_{r+1}=0$ since the supports are fixed, $\eta_{r-1}=\eta_r=\eta=(o/T)^2$ and $\eta'=o^2/T(T-o)$. Accordingly equations 13.3a and 13.3b become:

$$\Delta = 1 - 2\eta \qquad 13.8a$$

$$\Delta = 1 - (\eta+\eta') \qquad 13.8b$$

The calculations for predicting the two bounds are shown in Tables 13.3a and 13.3b. The mean value of P/P_e obtained is 0.4322.

As the number of storeys in this frame is increased, the weakest storey will move upward. For example when the third storey is taken to be the weakest one, the mean value of P/P_e will be 0.419. When the number of the storeys becomes fairly large, the joints far from the foundations will rotate more or less the same amount, then the stability condition becomes

$$\Delta = T - 2o \qquad 13.8c$$

The load parameter ρ making Δ vanish is about 0.402. This is the lowest possible critical load parameter ρ for the uniform storeys shown in Figure 13.2. The exact values of the elastic critical load parameter ρ are tabulated for different number of storeys.

Number	1	2	3	4	5	6	7	∞
P/P_e	0.748	0.523	0.462	0.436	0.423	0.415	0.410	0.402

Example 3

The two bounds of the elastic critical load of the frame in Figure 13.3 will be estimated. In this frame, the bottom storey is the critical one. The second approximation of 13.3 will be used. The exact value of the elastic critical load was obtained by R.B.L Smith and W. Merchant.[20]

The no-shear stiffnesses of the joints A, B and C are

$T_A = 3\times 4 + 2.22n_1$

$T_B = 2.22n_1 + 3.1n_2 + 6$

$T_C = 3.1n_2 + 2.6n_3 + 6$

$\eta_1 = (2.22o_1)^2/T_AT_B$

$\eta_2 = (3.1o_2)^2/T_BT_C \qquad \eta_2' = (3.1o_2)^2/T_B(T_C - 2.6o_3)$

Equations 13.3a and 13.3b are reduced to

$$\Delta = 1 - \eta_1 - \eta_2 \qquad 13.9a$$

$$\Delta = 1 - \eta_1 - \eta_2' \qquad 13.9b$$

The calculations for estimating the two bounds are shown in Tables 13.4a and 13.4b. The results obtained can be tabulated as follows:

Upper bound ρ_1	Lower bound ρ_1	mean value ρ_1	Exact value ρ_1
0.431	0.398	0.414	0.415

This method has been applied to other frames of Figure 13.4. The second storey is taken to be the weakest storey. The approximations of 13.3 was used. The results obtained are tabulated below:

Number of storeys	1	2	3	4	5	7
Upper bound ρ_1				0.258	0.213	0.164
Lower bound ρ_1				0.250	0.196	0.134
Mean value ρ_1				0.254	0.204	0.149
Exact value ρ_1	0.748	0.478	0.338	0.257	0.206	0.145

Instability of the whole building frame

This kind of instability is possible in a very tall building when each individual storey is well designed and when there are few bays. The curve relating P/P_e to N, the number of storeys, is shown in Figure 13.5. This curve is obtained by the classical method of estimation[21]. This curve shows that after a certain number of storeys, reduction in the elastic critical load is very small as the number N is increased and that there is a least value of P/P_e equal to 0.402 which is attained when N is infinitely large. It could be concluded that after a certain value of N, increase in N has in practice no effect on the elastic critical load. This is not true, however, and the elastic critical load given by the previous calculation is merely that of the local failure mode of the weakest storey and is not the elastic critical load of the whole building frame. Therefore, the classical method of prediction ceases to be applicable in the estimation of the elastic critical load of tall building frames. The method of estimation must be modified to take into account the number of the storeys of the building. In the previous method it was found that the effect of the axial deformations of the stanchions has a very small effect on the elastic critical load of the portal frames when the number of storeys is small and therefore an assumption was made that the axial deformations are negligible. To obtain the correct value of the elastic critical load, these deformations must be considered. Consideration of these deformations will show the reduction of the stiffness of the critical storey joints and the amount of reduction is usually dependent on the number of storeys N. Such analysis and calculation will be very long and tedious especially for very tall building where the size of the matrices of the elastic stability condition rapidly necessiates the employment of an electronic computer. Actual estimation is not shown but the basic equations are given. The following steps are to be followed:-

1) Apply a disturbance which causes a no-shear rotations in the joints of the frame such that all the joints remain in equilibrium. For the rth joint, the moment is

$$\partial M_r = \{(kn)_r + (kn)_{r+1} + (k\alpha)_{Br}\}\partial\theta_r - (ko)_r\partial\theta_{r-1} - (ko)_{r+1}\partial\theta_{r+1} + (k\alpha/L)_{Br}\sum_{x=1}^{r} 2u_x = 0 \qquad 13.10a$$

where

n, o, and α are the stability functions, and

u_x is the axial deformation of each stanchion. In each storey there will be extension in one stanchion and contraction in the other.

2) There will be a shear force v_r on each cross beam of size

$$v_r = (2k\alpha/L)_{Br}\partial\theta_r + (2kA/L^2)_{Br}\sum_{x=1}^{r} 2u_x \qquad 13.10b$$

3) Each stanchion is deformed by u_r where

$$u_r = \frac{1}{(E\bar{A}/L)_r}\sum_{x=r}^{N} v_x \qquad 13.10c$$

wher $\bar{A}$=cross sectional area

From these three conditions, there will be 3N force and deformation unknowns and will be also 3N relationships. A square determinant can be written down which is the stability condition. Any load parameter P/P_e which makes this determinant vanish will be the critical load parameter of the frame.

Special case

When N is very large and the stanchions have equal EI, L and carry the same axial force, the joint rotations, axial deformations and the shear force on the beam can be assumed to be equal to mean values $\overline{\partial\theta}, \bar{v}, \bar{u}$. The three conditions 13.10 can be modified to

$$\partial M = [(2kn - 2ko) + (k\alpha)_B]\overline{\partial\theta} + (k\alpha/L)_B\, 2r\bar{u} = 0 \qquad 13.11a$$

$$\bar{v} = (2k\alpha/L)_B\overline{\partial\theta} - (2kA/L^2)_B\, 2r\bar{u} \qquad 13.11b$$

$$\bar{u} = \frac{1}{(E\bar{A}/L)} \quad (N+1-r)\ \bar{v} \qquad 13.11c$$

The determinant of the coefficients of unknowns is reduceable to

$$\Delta = 2(kn - ko) + (k\alpha)_B \frac{1}{1 + \phi r(N+1-r)} \qquad 13.12$$

where

$\phi = (4kA/L^2)_B/(E\bar{A}/L)$

When $N=\infty$, 13.12 becomes

$= 2k(n - o)$

The lowest load parameter ρ making $(n-o)$ vanish is zero. This shows that the critical load of a frame which is an infinite number of storeys is in fact zero, as is to be expected by analogy with an infinitely long strut of finite cross-section.

The estimation of the elastic critical load of tall frames using the above method 13.10 is not practical due to the number of unknowns involved. There is another way of predicting the approximate value of the elastic critical load of special frames. The tall building frame behaves like a battened strut built up from two or more main members which carry the axial load and which are held at a fixed distance by the beams. Therefore, the elastic critical load of the tall building can be estimated by considering the stability of the equivalent strut.

Equivalent battened strut

The shearing force in a strut reduces the value of the elastic critical load by a factor C called "shear coefficient". For the case of battened struts, the shear coefficient depends upon two factors: battens spacing and batten dimensions. According to Timoshenko[2] the elastic critical load of a solid strut under uniform axial load is

$$P_{crit} = C_s \, P_c \qquad 13.13$$

where

$$C_s = \frac{1}{1 + n_1 P_c / AG}$$

$P_c = \rho P_e$

ρ = load parameter P/P_e depends upon the end conditions of the strut.

P_e = Euler load of the strut.

n_1 a numerical factor depending on the shape of the cross section.

A = cross sectional area

G = modulus of rigidity

For a battened strut

$$P_{crit} = C_b \, P_c \qquad 13.14$$

According to Timoshenko[2]

$$C_b = \frac{1}{1 + P_c \left[\frac{ab}{12EI_b} + \frac{a^2}{24EI_c} \frac{1}{1-\beta} + \frac{n_2 a}{bA_b G} \right]}$$

where

a = length of the stanchion

b = length of the batten

I_c = moment of inertia of the stanchion

I_b = moment of inertia of the batten

$\beta = P_{cri}/2\pi^2 EI_c/a^2$

This is an approximate value of C_b since an assumption is made that points of contraflexure occur at the middle of the stanchions. The coefficient C_b can be expressed in another form as

$$C_b = \frac{1}{1 + \frac{\rho}{N^2}\pi^2 \frac{I}{I_c}\left[\frac{1}{24k} + \frac{1}{24(1-\beta)} + \frac{n_2 E}{G}\frac{1}{\gamma_b^2}\right]}$$

where

$I/I_c = 2+\frac{1}{2}(b/a)^2\gamma_c$

k=relative stiffness of the batten

γ= slenderness ratio

I= moment of inertia of the equivalent solid strut

The elastic critical load of a solid strut under non-uniform axial force is

$$(P_1+P_2)_{cri}/2 = C_s' P_c \qquad 13.15$$

where

$$C_s' = \frac{1}{1 + n_1 C_p P_c/A_s G}$$

P_1, P_2 are the axial loads at the ends of the strut

$P_c = \rho P_e$

ρ is the load parameter $(P_1+P_2)/2P_e$ its value depends upon the ratio P_1/P_2 and the ends conditions.

C_p=factor depends upon P_1/P_2. It has a value of unity when P_1/P_2=1 and about 0.879 when P_1/P_2= 0. The approximate evaluation of

C_p and the derivation of the elastic critical load of a solid strut fixed at one end and free at the other is obtained approximately by the energy method, shown in the appendix.

For battened struts subjected to non-uniform axial load, the shear coefficient C_b' will be similar to C_b except P_c is replaced by C_pP_c in the denomintor. ρ =0.3996 for P_1/P_2=0. For other values of P_1/P_2 the load parameter ρ can be obtained from the stability tables of chapter 15.

The above expressions for the shear coefficients are applied only to building frames having storeys of equal height and the same stanchion properties. For other frames, the problem can be tackled by estimating C_s of a solid strut having variable flexural rigidity EI and in which the change in slope of the deflection curve produced by shearing is variable.

Numerical example

The elastic critical load due to the local failure of the weakest storey of the frame in Figure 13.6 is

$$P_c = 2\rho P_e$$

$$= 2\rho N^2 P_E/(I/I_c) \qquad 13.16$$

where P_E is the Euler load of the solid equivalent strut.

When γ is taken to be 50 equation 13.16 becomes

$$P_c/P_E = \rho N^2/626 \qquad 13.17a$$

It was shown in example 2 that for $N \gg 7$, ρ will be about 0.402, hence equation 13.17a becomes

$$P_c/P_E = 0.641 \times 10^{-3}\, N^2 \qquad 13.17b$$

For $N \leq 7$, the values of ρ can be obtained from the table in example 2. Substituting these results in 13.17a, the curve relating P_c/P_E and N is obtained and is shown in Figure 13.7.

The curve relating P_c/P_E to N for the equivalent battened strut is obtained by using equation 13.14 taking $n_1/\bar{A}G = 3$ and $\rho = 0.25$ and ignoring the effect of stability. The modified equation is

$$P_c/P_E = \frac{0.25}{1 + 390/N^2} \qquad 13.17c$$

The curve obtained from 13.17c is also shown in Figure 13.7. These curves show that the load parameter P_c/P_E for the local storey instability becomes very large for big values of N and tend to infinity when N becomes infinity. Equation 13.17c shows that the load parameter P_c/P_E increases as N is increased but it has a limit of 0.25 when $N=\infty$. This equation also shows that the effect of the shearing is pronounced for small N and the effect decays as the number of storeys is increased.

Appendix

Take the deflection curve of the strut in Figure 13.8 to be approximately[2]

$$y = \delta\,(1 - \cos \pi x/2L) \qquad 13.18$$

The bending moment at any section mn is

$$M = \int_x^L p(\eta - y)\, d\xi \qquad 13.19a$$

Substituting for y in this expression and also observing that

$$\eta = \delta\,(1 - \cos \pi\xi/2L) \qquad 13.19b$$

we obtain

$$M = p\delta \left[(L-x)\cos \pi x/2L - 2L/\pi \,.(1-\sin \pi x/2L) \right] \qquad 13.19c$$

Substituting this in the expression for the strain energy of bending we obtain

$$\Delta U_m = \int_0^L M^2 dx/2EI = \frac{\delta^2 p^2 L^3}{2EI}\left(\frac{1}{6} + \frac{9}{\pi^2} - \frac{32}{\pi^3}\right) \qquad 13.20$$

The shear force at section mn is

$$Q = dM/dx = -\,p\,\frac{dy}{dx}\int_x^L d\xi$$

$$= -\,p(L - x)\,\frac{dy}{dx}$$

$$= -\,p(L - x)\,\frac{\pi\delta}{2L}\,\sin \pi x/2L \qquad 13.21$$

Substituting this in the expression for the strain energy of shear, we obtain

$$\Delta U_s = \int_0^L n_1 Q^2\, dx/2AG = \frac{n_1 p^2}{AG}\,\frac{\delta^2 \pi^2 L}{16}\left(\frac{1}{3} - \frac{2}{\pi^2}\right) \qquad 13.22$$

The total work done by the loads during buckling is

$$\Delta T = \frac{p}{2}\int_0^L (L-x)\left(\frac{dy}{dx}\right) dx$$

$$= \frac{\pi^2 \delta^2 p}{8}\left(\frac{1}{4} - \frac{1}{\pi^2}\right) \qquad 13.23$$

At the critical load, the strain energy of the strut is equal to the work done by the external loads. Hence

$$\Delta U_m + \Delta U_s = \Delta T \qquad 13.24$$

Substituting 13.20, 12.22 and 13.23 in equation 13.24 we obtain an approximate critical load

$$(pL/2)_c = P_c/(1 + C_p P_c n_1/AG) \qquad 13.25$$

where

$$P_c = \rho P_e$$

$$\rho = \frac{1}{8}(1/4 - 1/\pi^2) / (1/6 + 9/\pi^2 - 32/\pi^3)$$

$$= 0.3996114$$

$$C_p = (1/3 - 2/\pi^2) / (1/4 - 1/\pi^2)$$

$$= 0.879015$$

Chapter 14

Elastic instability of frameworks with non-uniform members

Introduction

Members in portal frames sometimes have a varying cross section, the section being increased where an increase in the bending moment is expected to make the members approximately uniform in strength. Such members are quite frequently met with in engineering structures especiably in frameworks where the flexural rigidity of the members plays a role in carrying the external load applied to these frameworks i.e in rigid frameworks where the loads on the framework are taken by the framework components of axial forces and bending moments in the members. In this chapter a practical method of estimating the elastic critical loads of frameworks with uniformly tapered members in which the moment of inertia of the cross section varies according to a power of the distance along the member, is presented using tabulated stability functions. The stability functions are obtained by solving the basic moment equation at any section along the tapered members. The exact solution of this equation is shown and the results obtained are checked by an approximate solution using the finite difference method.

The subject of the stability of isolated struts having particular ends conditions has been investigated by different writers [22,23,24,25]. Good approximate values of the elastic critical loads of struts having either pinned or fixed ends can be obtained, using the approximate methods proposed by some of the writers. But the approximate methods fail when the conditions at the ends are neither pinned nor fixed. Therefore, there are limitations on the previous work when the partial fixity provided by the adjacent members in the framework is considered and when a foundation provides some restraint.

The method of analysis used and the assumptions made are similar to those of chapter 2, except that the stiffness, carry over and the shear factors in the standard slope-deflection equation are different. The stiffnesses of the two ends of the tapered strut are different, the stiffness s_1 of the smaller end being less than the stiffness s_2 of the other end. The moments appearing at each end of the member due to unit rotation of the other end are the same and will be referred to as $\overline{sc}$. The moments appearing at the ends of the member due to the sway of one end are different and will be referred to as α_1 and α_2. The values of α_1 and α_2 are shown in the appendix to be $s_1+\overline{sc}$ and $s_2+\overline{sc}$. The values of these stability functions have been calculated taking into account the effect of the axial load and the change in the moment of inertia from one end to the other.

Therefore, with this notation, the modified slope-deflection equation for the end moments of the member AB of Figure 14.1 due to the rotations θ_1 and θ_2 at the ends and the sway δ are

$$M_1 = k_1(s_1\theta_1 + \overline{sc}\theta_2 - \alpha_1\delta/L) \qquad 14.1a$$

and

$$M_2 = k_1(\overline{sc}\theta_1 + s_2\theta_2 - \alpha_2\delta/L) \qquad 14.1b$$

and the shear force balancing the moments is

$$v = k_1/L.\ (\alpha_1\theta_1 + \alpha_2\theta_2 - Q\ \delta/L) \qquad 14.1c$$

where

$Q = \alpha_1 + \alpha_2 - \pi^2 \rho_1$ (given in the appendix)

The moments and the shear force have been expressed in term of the k-value and the load parameter ρ of the smaller end.

The purpose of the calculation which follows is to evaluate s_1, s_2, $\overline{sc}$, and Q in a form suitable for computer use and to obtain tables of some values which could be used in desk calculation. These tabulated values have been used in the calculation of the elastic critical loads of non-prismatic isolated struts and portal frames with non-uniform stanchions.

Derivation of the fundamental equation

The strut considered has a uniform taper in one plane. The point of intersection of the two sides of the tapered strut will be taken as the origin of coordinates. The smaller end of the strut will be denoted as end 1 and the larger as end 2. The depths of the cross sections at 1 and 2 are d_1 and d_2 respectively and the length of the strut is L. Therefore the depth of cross section d_x may be expressed by

$$d_x = d_1\left(\frac{x}{a}\right) = d_2\left(\frac{x}{b}\right) \qquad 14.2a$$

in which d_x is the depth of any cross section located at distance x from the origin which is itself at distance a from the end 1 and b from the end 2. The moment of inertia of the cross sectional area of the strut about the axis of buckling is expressible in the form

$$I_x = I_1\left(\frac{x}{a}\right)^{\bar{m}} = I_2\left(\frac{x}{b}\right)^{\bar{m}} \qquad 14.2b$$

in which I_x is the moment of inertia at distance x from the origin, I_1 denotes the moment of inertia of the smaller end 1 and I_2 denotes the moment of inertia of the larger end 2. $\bar{m}$ is a shape factor that depends on the cross sectional shape and dimensions of the strut. The shape factor $\bar{m}$ may be evaluated from equations 14.2a and 14.2b which yields the condition

$$m = (\mathrm{LOG}\, I_2/I_1) \,/\, \mathrm{LOG}\, u \qquad 14.2c$$

where

$$u = d_2/d_1$$

Therefore the shape factor can be determined once the dimensions of the end cross sections are known. For struts of rectangular cross section the shape factor m is equal to either 1 or 3 depending upon the axis about which buckling occur. A strut having an open web or an open base section consisting of equal areas at the corners of the cross section has a value of 2 (approximately) and a tapered strut with solid circular cross section has a value of $\overline{m}= 4$. For struts of wide flange shape or closed box section, the shape factor will be between the limits $\overline{m}=2.1$ and $\overline{m}=2.6$ (23).

The differential equation of the deflection curve of a slightly bent ideal strut under the action of a uniform compression, can be obtained by considering the bending moment M_x at any section along the member at a distance x from the origin.

$$M_x = P\,y - (M_1+M_2)\left(\frac{x-a}{b-a}\right) + M_1 \qquad 14.3a$$

By beam theory, the moment and the radius of curvature is related by

$$M_x = \frac{EI_x}{R} \qquad 14.3b$$

and

$$\frac{1}{R} = -\frac{d^2y}{dx^2} \qquad 14.3c$$

hence

$$M_x = -EI_x\frac{d^2y}{dx^2} \qquad 14.3d$$

Subsituting 14.2b and 14.3d and rearranging, equation 14.3a becomes

$$EI_1\left(\frac{x}{a}\right)^{\overline{m}}\frac{d^2y}{dx^2} + Py = M_1\left(\frac{x-a}{b-a} - 1\right) + M_2\left(\frac{x-a}{b-a}\right) \qquad 14.4$$

in which P is the axial load, y represents lateral deflection of the strut and M_1 and M_2 are the applied end moments.

Stability functions

In finding the elastic critical loads of frames with uniform members, the stability functions tabulated by Livesley and Chandler have been found of value. In a similar manner it was decided to tabulate functions for non-prismatic members.

The stability functions for different values of $\overline{m}$ and u were evaluated by solving the fundamental equation 14.4. The expressions for the stiffnesses and shear factors are given later. In the case in which $\overline{m}=4$, the stability functions were found to be simple functions of the stability functions of the strut with constant moment of inertia which are tabulated by Livesley and Chandler[9]. These are

$s_1 = u\,s$

$s_2 = u^3 s$

$\overline{sc} = u^2 sc$

$s_1'' = u\,s''$

$s_2'' = u^3 s''$

$\alpha_1 = u\,s + u^2 sc$

$\alpha_2 = u^3 s + u^2 sc$

$Q = u\,s + u^3 s + 2u^2(sc - B)$

and

$\varrho_1 = u^2 \varrho$

where s, sc, s". B and ϱ are the stability functions of the uniform strut.

Therefore. the stability functions for any value of ρ and u can be obtained and in turn the elastic critical load of any frame.

For $\bar{m}$ other than 4, the stability functions are not simply related to those of the uniform strut. Therefore, the stability functions were tabulated for four values of u.

$\bar{m}$	u
3.0	1.5, 2.0, 2.5, 3.0
2.4	2.0, 2.5, 3.0, 3.5
2.0	2.0, 3.0, 4.0, 5.0

For values of u between these, the elastic critical load can be calculated by graphical and calculation technique.

Elastic instability of struts

The elastic critical loads of struts with the middle portions of uniform cross sections shown in Figure 14.2 will be determined. The struts will be considered to be hinged at the ends. The length of the strut will be taken to be L and the length of the middle portion will be denoted by z. The moment of inertia of the middle portion will be I_2 and that of the ends is I_1.

There are two possible sway modes of elastic instability which are equilivalent to the first and the second buckling modes of the uniformly cross section strut with hinged ends. It is well known that the first mode has the lowest value of the two elastic critical loads. Therefore the analysis of the first mode will be shown only.

If equal disturbing forces H are applied at B and B' the following deformations will occur:-

1) Joints B and B' will rotate in opposite directions through an angle $\partial\theta_B$.

2) The portions AB and A'B' will sway in opposite directions by $\partial\delta$.

Due to these deformations there will be moments at the ends of each of the three portions of the strut. These moments are tabulated.

Operation	M_{AB}	M_{BA}	$M_{BB'}$
1) Rot. B&B'		$(ks''_2)_1\partial\theta_B$	$(ks(1-c))_2\partial\theta_B$
2) Sway $\partial\delta$	$-(k\alpha_1\partial\delta/L)_1$	$-(k\alpha_2\partial\delta/L)_1$	
3) Balance A&A'	$+(k\alpha_1\partial\delta/L)_1$	$+(k\alpha_1\partial\delta/L)_1\left(\frac{\overline{sc}}{s_1}\right)_1$	

Joints B and B' are in equilibrium, hence

$$\partial M_B = \left[(ks(1-c))_2 + (ks''_2)_1\right]\partial\theta_B + \left(k\alpha_1\frac{\overline{sc}}{s_1} - \alpha_2\right)_1\partial\delta/L = 0 \qquad 14.5a$$

The disturbing force at B and B' is

$$H = -(ks''_2)\,\partial\theta_B/L + \left(kQ - k\alpha_1\left(1+\frac{sc}{s_1}\right)\right)_1\partial\delta/L^2 \qquad 14.5b$$

At the critical load this force H vanishes since the stiffness of the strut becomes zero.

The term $(\alpha_1 \frac{\overline{sc}}{s_1} - c_2')_1$ can be reduced to the simpler form $-s''_{21}$ by using the definitions of α_1' and c_2'. Likewise the term $(Q - \alpha_1'(1+ \frac{sc}{s_1}))_1$ is reduceable to $(s''_2 - \pi^2 \rho_1)_1$. It follows that the equilibrium equations become

$$[(ks(1-c))_2+(ks''_2)_1]\partial\theta_B - (ks''_2)_1 \partial\delta/L = 0 \qquad 14.5c$$

$$-(ks''_2)_1 \partial\theta_B/L + k_1(s''_2 - \pi^2\rho_1)_1 \partial\delta/L^2 = 0 \qquad 14.5d$$

The determinant of the coefficients of the unknowns is

$$\begin{vmatrix} (ks(1-c))_2+(ks''_2)_1 & -(ks''_2)_1 \\ -(ks''_2)_1 & k_1(s''_2 - \pi^2\rho_1)_1 \end{vmatrix} = 0 \qquad 14.6$$

This is the condition for the elastic instability, any load parameter ρ_{11} satisfying this condition is the critical one.

The elastic critical load of the strut is expressed as KEI_2/L^2 by some writers (24), where K is a quantity given in tables for different values of $\overline{m}$, u and z/L. K is here expressed in terms of ρ_{11}, u, $\overline{m}$ and z/L, to check the calculations with other writers' results.

Let the critical load parameter of the smaller end be ρ_{11}, hence

$$P = \rho_{11}(P_e)_{11}$$

$$= \frac{\rho_{11}}{u^{\overline{m}}} (P_e)_{21} \qquad \text{from } 14.2$$

$$= \frac{\rho_{11}}{u^{\overline{m}}} \cdot \frac{\pi^2 EI_{21}}{(\frac{L-z}{2})^2}$$

$$= \frac{4\rho_{11}\pi^2}{u^{\overline{m}}(1-z/L)^2} \cdot EI_2/L^2$$

Thus

$$K = \frac{4\pi^2 \rho_{11}}{u^{\overline{m}}(1-z/L)^2} . \qquad 14.7$$

Numerical examples

Example 1. Taper so that $\overline{m}=4$

For this case of $\overline{m}$, the elastic critical load can be calculated for any value of u. The stability functions s''_{21} and ρ_{11} are replaced by the equivalent stability functions of the uniform cross section strut. Thus the stability condition 14.6 is modified to

$$\begin{vmatrix} s''_1 + \frac{1}{u^3}\frac{(ks(1-c))_2}{k_1} & -s''_1 \\ -s''_1 & s''_1 - \frac{\pi^2}{u}\rho_1 \end{vmatrix} = 0 \qquad 14.8$$

The elastic critical loads will be calculated for different values of z/L and u.

(i) $z/L = 0$ and $u=1.78$ $(I_1/I_2 = 0.1)$

In this case the stiffness of the middle portion becomes infinity so that the stability condition 14.8 becomes

$$s''_1 - \frac{\pi^2}{1.78}\rho_1 = 0 = \triangle$$

and equation 14.7 gives

$$K = \frac{4\pi^2}{\sqrt{10}}\rho_1$$

The value of ρ_1 which satisfied the stability condition is obtained by trial and error process.

<u>First trial ρ_1 =0.38</u>

From tables(9)

$s_1'' = 2.1532$

Substituting these value in the stability condition gives

$$\Delta = 2.1532 - 2.11$$

$$= + 0.0432$$

A higher value of ρ_1 is, therefore, tested.

When $\rho_1 = 0.40$ $\quad\quad \Delta = - 0.1179$

By linear interpolation the critical ρ_1 is 0.385, and K = 4.8.

For other values of u, the ρ_1 and K values obtained are tabulated in Table 14.1.

I_1/I_2	u^4	Δ	ρ_1	K
0.1	10	$s'' - 5.55\rho_1$	0.385	4.80
0.2	5	$s'' - 6.64\rho_1$	0.338	5.96
0.4	2.5	$s'' - 7.9\ \rho_1$	0.298	7.45
0.6	1.67	$s'' - 8.73\rho_1$	0.276	8.45
0.8	1.25	$s'' - 9.4\ \rho_1$	0.260	9.16
1.0	1.0	$s'' - \pi^2\rho_1$	0.25	π^2

Table 14.1

(ii) <u>z/L = 0.6 and u = 1.78 (I_1/I_2= 0.1)</u>

The relative k and ρ values are tabulated

Member	AB(1)	BB'(2)
rel. I	1.0	10.0
rel. L	1.0	3.0
rel. k	1.0	10/3
rel. P_e	1.0	10/9
rel. ρ	10/9	1.0

The load parameter of the uniform strut is $\rho_1 = \rho_{11}/u^2$

$$= 10\rho_2/9\sqrt{10}$$

$$= 0.351\,\rho_2$$

When the relative k values and the u value are substituted in the stability criterion 14.8, the determinant is modified to

$$\begin{vmatrix} s_1'' + 0.593\,(s(1-c))_2 & -s_1'' \\ -s_1'' & s_1'' - 5.55\rho_1 \end{vmatrix} = \triangle \qquad 14.8a$$

and equation 14.7 gives

$$K = \frac{4\,\pi^2}{10 \times 0.16}\,\rho_{11}$$

$$= \frac{4\,\pi^2}{9 \times 0.16}\,\rho_2$$

The critical ρ_2 satisfying 14.8a is obtained by trial and error.

<u>First trial $\rho_2 = 0.32$</u>

$\rho_1 = 0.32 \times 0.351 = 0.112$

From tables(9)

$s_1'' = 2.7716$

$s_2 = 3.5602 \qquad (sc)_2 = 2.1166$

Substituting these values in the stability condition 14.8a gives:

$$\Delta = \begin{vmatrix} 3.6266 & -2.7716 \\ -2.7716 & 2.1476 \end{vmatrix}$$

$$= +1.36$$

A higher value of ρ_2 is therefore tested.

When $\rho_2 = 0.34 \qquad \Delta = -1.65$

By linear interpolation, the critical ρ_2 is 0.329 and therefore K = 9.03. This value compares with 9.08 obtained by Dinnik.[24]

Example 2. Taper so that $\bar{m}=3$

For values of $\bar{m}$ other than 4, the elastic critical load can be calculated for certain values of u. For intermediate values of u, a graphical method will be used to determine the elastic critical load. This will give results which are good enough for practical purposes because the curves relating the critical load and u are smooth. This will be shown in determining the elastic critical load of the same struts for $\bar{m}=3$.

(i) $z/L = 0$ and $u = 1.5$

As before, the stiffness of the middle portion is infinite and the stability condition 14.6 becomes:-

$$\Delta = s''_{21} - \pi^2 \rho_{11} \qquad 14.9$$

and equation 14.7 becomes

$$K = \frac{4\pi^2 \rho_{11}}{(1.5)^3} \qquad 14.9a$$

The value of ρ_{11} satisfying 14.9 is obtained by trial and error using the stability table for $\overline{m}=3$ and u=1.5.

First trial $\rho_{11} = 0.57$

$s''_{21} = 5.8724$

Substituting these values in 14.9 gives

$\Delta = 5.8724 - 5.62$

$= + 0.2524$

A higher value of ρ_{11} is therefore tested.

When $\rho_{11} = 0.60$ $\quad \Delta = - 0.1352$

By linear interpolation, the critical ρ_{11} is 0.589 and K=6.90. For other values of u, the critical load parameter ρ_{11} and the K-values are tabulated below.

u	1	1.5	2.0	2.5	3.0
ρ_{11}	0.25	0.589	1.08	1.737	2.556
K	π^2	6.90	5.32	4.40	3.74

These values are plotted as curve A in Figure 14.3. The elastic critical load of struts having intermediate values of u can be read from the graph. On the same curve are the values given by Dinnik.[24]

(ii) z/L = 0.2 and u=1.5

The relative k and ρ values are tabulated below

Member	AB(1)	BB'(2)
rel. I	1.0	3.375
rel. L	2.0	1.0
rel. k	1.0	2 X 3.375
rel. P_e	1.0	4 X 3.375
rel. ρ	1.0	0.074

When the relative k values are substituted in the stability condition 14.6, the determinant becomes

$$\begin{vmatrix} s''_{21} + 6.75\,(s(1-c))_2 & -\,s''_{21} \\ -\,s''_{21} & s''_{21} - \pi^2 \rho_{11} \end{vmatrix} \qquad 14.9b$$

and equation 14.7 becomes

$$K = \frac{4\,\pi^2\,\rho_{11}}{(1.5)^3 \times 0.64}$$

The critical load parameter ρ_{11} satisfying 14.9b is obtained by trial and error.

First trial ρ_{11} = 0.39

ρ_2 = 0.39 X 0.074 = 0.0288

From table $\bar{m}$=3 and u=1.5

s''_{21} = 6.4355

From tables(9)

$s_2 = 3.9683 \quad (sc)_2 = 2.0079$

Substituting these values in 14.9b gives:-

$$\Delta = \begin{vmatrix} 19.6355 & -6.4355 \\ -6.4355 & 2.5955 \end{vmatrix}$$

$$= +9.5$$

A higher value of ρ_{11} is therefore tested.

When $\rho_{11} = 0.42$ $\quad \Delta = +2.85$

and when $\rho_{11} = 0.45$ $\quad \Delta = -4.10$

By linear interpolation ρ_{11} is 0.432 and K= 7.90. The same procedure was followed in calculating K for other values of u using the tabulated stability tables of $\overline{m}=3$. The numerical results obtained are:-

u	1.0	1.5	2.0	2.5	3.0
ρ_{11}	0.25	0.432	0.852	1.448	2.203
K	π^2	7.90	6.57	5.71	5.05

Curve B in Figure 14.3 shows these results. Also on this curve are the values given by Dinnik.[24] Curve C shows the numerical results obtained when z/L = 0.4.

The elastic critical loads of strut having $\overline{m}$ other than those tabulated can also be estimated using the tables. Figure 14.3a shows the relationships, obtained from the tables, between P/P_{e_1} and u for

$\overline{m}$ = 4,3 and 2 for pin jointed strut. From Figure 14.3a, Figure 14.3b is obtained which shows the relationship between P/P_{e_1} and $\overline{m}$ for particular value of u.

Portal frames

It is necessary to be able to investigate the elastic stability of rectangular portal frames with non-prismatic members. In the following analyses the beam is assumed to have uniform cross section and the stanchions to have variable cross sections as shown in Figure 14.4.

There are two possible modes of elastic instability, the joint rotation and the anti-symmetrical sway modes.

1) Joint rotation mode

On the application of equal and opposite disturbing moments ∂M at the two joints, the two joints will rotate in opposite directions through an angle $\partial\theta_B$. Hence

$$\partial M = \left[(ks_2)_1+(2k)_2\right]\partial\theta_B \qquad 14.10$$

The stiffness of the joint $\left[(ks_2)_1+(2k)_2\right]$ is the stability condition, any load making the stiffness of the joint vanish is the critical load and is obtained by trial and error.

2) Anti-symmetrical sway mode

In this mode there is only one freedom of sway movement, that is the horizontal displacement of the joints. When a disturbing horizontal force $2H_B$ is applied, the following deformations occur:

1) Joints B and B' rotate clockwise through an angle $\partial\theta_B$.

2) Members AB and A'B' sway by $\partial\delta$.

Due to these deformations, there will be moments at the ends of the members. These are tabulated as follows:-

Operation	M_{AB}	M_{BA}	$M_{BB'}$
1) Rot. B&B'	$(k\overline{sc})_1 \partial\theta_B$	$(ks_2)_1 \partial\theta_B$	$6k_2 \partial\theta_B$
2) Sway $\partial\delta$	$-(kc'_1)_1 \partial\delta/L$	$-(kc'_2)_1 \partial\delta/L$	

The joints are in equilibrium, hence

$$\partial M_B = \left[(ks_2)_1 + 6k_2\right] \partial\theta_B - (kc'_2)_1 \partial\delta/L = 0 \qquad 14.11a$$

The total horizontal force at B is

$$2H_B = 2\left[-(kc'_2)_1 \partial\theta_B + (kQ)_1 \partial\delta/L^2\right] \qquad 14.11b$$

At the critical load, the stiffness of the frame becomes zero resulting in H_B being zero also. The determinant of the coefficients of the unknowns is

$$\begin{vmatrix} (ks_2)_1 + 6k_2 & -(kc'_2)_1 \\ -(kc'_2)_1 & (kQ)_1 \end{vmatrix} = \Delta = 0 \qquad 14.12$$

This is the anti-symmetrical sway mode elastic stability condition.

Numerical example

The elastic critical load of the joint rotation mode will be calculated when $k_2/k_1 = 1.0$, $\overline{m}=3$ and $u=1.5$. Thus from 14.10 the stiffness of the joint is

$$\Delta = s_{21} + 2$$

The critical load parameter making the stiffness of the joint vanish is obtained by trial and error.

First trial $\rho_{11} = 4.20$

From table $\overline{m}=3$ and $u=1.5$

$$s_{21} = -1.8722$$

Substituting this value in the stability condition gives:-

$$\Delta = -1.8722 + 2$$

$$= +0.1278$$

A higher value of ρ_{11} is therefore tested.

Second trial $\rho_{11} = 4.23$

$$s_{21} = -2.0262$$

and the joint stiffness is

$$\Delta = -2.0262 + 2$$

$$= -0.0262$$

By linear interpolation the critical ρ_{11} is 4.225 and the elastic critical load of the joint rotation mode is

$$2P = 2 \times 4.225 \, P_{e_{11}}$$

$$= 8.45 P_{e_{11}}$$

where $P_{e_{11}}$ is the Euler load of the smaller end of the stanchion.
For other values of u the elastic critical loads are tabulated:-

u	1.0	1.5	2.0	2.5	3.0
$2P/P_{e_{11}}$	5.103	8.45	12.66		23.584

<u>Anti-symmetrical sway elastic critical load</u>

The elastic critical of the portal is calculated when $k_2/k_1=8$ $\overline{m}=3$ and u=2.0. The stability condition 14.12 is then modified to

$$\begin{vmatrix} s_{21}+48 & -\alpha_{21} \\ -\alpha_{21} & Q_1 \end{vmatrix} = \Delta \qquad 14.12a$$

The critical load parameter ρ_{11} satisfying 14.12a is obtained by trial and error.

<u>First trial $\rho_{11}= 2.56$</u>

From table $\overline{m}=3$ and u=2

$s_{21}= 13.0731$ $\alpha_{21}= 19.93$ $Q_1= 6.13$

Substituting these values in 14.12a yields

$\Delta = -24$

A lower value of ρ_{11} is therefore tested.

When $\rho_{11}= 2.52$ $\Delta = +4.5$

By linear interpolation the critical load parameter is 2.526 and the anti-symmetrical elastic critical load is

$$2P = 2 \times 2.526\, P_{e_{11}}$$
$$= 5.052 P_{e_{11}}$$

For other values of u and k_2/k_1, the critical loads are tabulated in Table 14.2. The results are also plotted in Figure 14.5. The curves obtained are smooth. The elastic critical load of portals for the intermediate values of u can be read from the graphs with good accuracy.

k_2/k_1	$2P/P_{e_{11}}$				
	u=1.0	u=1.5	u=2.0	u=2.5	u=3.0
0	0.5	0.713	0.932	1.094	1.320
1.0	1.493	2.051	2.40	2.712	2.877
2.0	1.712	2.632	3.310	3.760	4.128
4.0	1.843	3.100	4.275	5.248	5.964
8.0	1.914	3.408	5.052	6.680	8.136
16	1.960	3.582	5.540	7.716	9.960
∞	2.000	3.770	6.096	8.982	12.452

Table 14.2

Appendix

Methods of solution of the differential equation

Exact method

The right hand side of equation 14.4 can be reduced to zero by replacing y by z where

$$z = y - \frac{M_1}{P}\left(\frac{x-a}{b-a} - 1\right) + \frac{M_2}{P}\left(\frac{x-a}{b-a}\right) \qquad 14.13a$$

Thus the differential equation 14.4 becomes

$$\frac{d^2z}{dx^2} + \varpi^2 x^{-m} z = 0 \qquad 14.13b$$

where

$$\varpi^2 = \frac{Pa^m}{EI_1}$$

This equation is a transformation of a Bessel equation. The Bessel equation of the form(26)

$$\frac{d^2z}{dx^2} - \frac{2\alpha-1}{x}\frac{dz}{dx} + \left[\beta^2\gamma^2 x^{2\gamma-2} + \frac{\alpha^2 - n^2\gamma^2}{x^2}\right] z = 0 \qquad 14.13c$$

has a general solution

$$\left.\begin{array}{l} z = x^{\alpha}\left[A J_n(\beta x^{\gamma}) + B Y_n(\beta x^{\gamma})\right] \\ z = x^{\alpha}\left[A J_n(\beta x^{\gamma}) + B J_{-n}(\beta x^{\gamma})\right] \end{array}\right\} \qquad 14.13d$$

according as n is an integer or not. A and B are the constants of integration. Therefore the solution can, at once, be written down in terms of Bessel functions, by given particular values to the constants α, β, γ and n. Comparing the two equations 14.13b and 14.13c we find that

$$\alpha = \frac{1}{2}\ ; \quad \gamma = \frac{-m+2}{2}\ ; \quad n = \mp\frac{1}{-m+2}\ ; \quad \beta = \mp\frac{2\varpi}{2-m} \qquad 14.13e$$

Hence the general solution of the fundamental equation 14.4 is

$$y = \sqrt{x}\left[AJ_n(\beta x^{\gamma}) + BY_n(\beta x^{\gamma})\right] + \frac{M_1}{P}\left(\frac{x-a}{b-a} - 1\right) + \frac{M_2}{P}\left(\frac{x-a}{b-a}\right)$$

$$y = \sqrt{x}\left[AJ_n(\beta x^{\gamma}) + BJ_{-n}(\beta x^{\gamma})\right] + \frac{M_1}{P}\left(\frac{x-a}{b-a} - 1\right) + \frac{M_2}{P}\left(\frac{x-a}{b-a}\right) \qquad 14.13f$$

depending on whether n is an integer or not.

There are four unknowns A, B, M_1, and M_2 which have to be determined from the following conditions at the ends

$$\text{at } x = a \quad y = 0 \text{ and } \frac{dy}{dx} = \theta_1$$

$$\text{and at } x = b \quad y = 0 \text{ and } \frac{dy}{dx} = \theta_2 \qquad 14.13g$$

<u>Derivation of expressions for the stability functions</u>

<u>Taper so that $\bar{m}=4$</u>

In 14.13e if $\bar{m}$ is replaced by 4.0, the values of the constants become

$$\gamma = -1; \quad n = +\tfrac{1}{2}; \quad \beta = +\varpi = +\sqrt{\frac{Pa^4}{EI_1}}$$

and the general solution is

$$y = \sqrt{x}\left[AJ_{\frac{1}{2}}\left(\frac{\varpi}{x}\right) + BJ_{-\frac{1}{2}}\left(\frac{\varpi}{x}\right)\right] + \frac{M_1}{P}\left(\frac{x-a}{b-a} - 1\right) + \frac{M_2}{P}\left(\frac{x-a}{b-a}\right) \qquad 14.14a$$

The Bessel functions of order $\frac{1}{2}$ and $-\frac{1}{2}$ are expressible in the form

$$AJ_{\frac{1}{2}}\left(\frac{\varpi}{x}\right) = \frac{A}{\sqrt{\frac{\pi\varpi}{2}}}\frac{\sin(\varpi/x)}{\sqrt{1/x}} = A'\sqrt{x}\,\sin\frac{\varpi}{x}$$

$$BJ_{-\frac{1}{2}}\left(\frac{\varpi}{x}\right) = \frac{B}{\sqrt{\frac{\pi\varpi}{2}}}\frac{\cos(\varpi/x)}{\sqrt{1/x}} = B'\sqrt{x}\,\cos\frac{\varpi}{x} \qquad 14.14b$$

Substituting in equation 14.14 and rearranging, the equation becomes

$$y = x\left[A'\sin\frac{\varpi}{x} + B'\cos\frac{\varpi}{x}\right] + \frac{M_1}{P}\left(\frac{x-a}{b-a} - 1\right) + \frac{M_2}{P}\left(\frac{x-a}{b-a}\right) \qquad 14.14c$$

The first derivative of equation 14.14c is

$$\frac{dy}{dx} = A'\left(\sin\frac{\varpi}{x} - \frac{\varpi}{x}\cos\frac{\varpi}{x}\right) + B'\left(\cos\frac{\varpi}{x} + \frac{\varpi}{x}\sin\frac{\varpi}{x}\right) + \frac{M_1+M_2}{P(b-a)} \qquad 14.14d$$

The values of the constants A' and B' are obtained by substituting the two end conditions y =0 at x=a and x=b in equation 14.14c. Which yields

$$A' = \frac{1}{\sin\left(\frac{\varpi}{a}-\frac{\varpi}{b}\right)}\left[\frac{M_1}{a}\cos\frac{\varpi}{b} + \frac{M_2}{b}\cos\frac{\varpi}{a}\right]$$

$$B' = -\frac{1}{\sin\left(\frac{\varpi}{a}-\frac{\varpi}{b}\right)}\left[\frac{M_1}{a}\sin\frac{\varpi}{b} + \frac{M_2}{b}\sin\frac{\varpi}{a}\right] \qquad 14.14e$$

The values of M_1 and M_2 are obtained by substituting the two conditions $\frac{dy}{dx} = \theta_1$ at x=a and $\frac{dy}{dx} = \theta_2$ at x=b in equation 14.14d.

$$M_1 = \frac{EI_1}{L}.u.\left[\frac{\alpha(1-2\alpha\cot 2\alpha)}{\tan\alpha - \alpha}\right]\theta_1 + \frac{EI_1}{L}.u^2.\left[\frac{\alpha(1-2\alpha\cot 2\alpha)}{\tan\alpha - \alpha}\right]\left[\frac{2\alpha - \sin 2\alpha}{\sin 2\alpha - 2\alpha\cos 2\alpha}\right]\theta_2 \qquad 14.14f$$

$$M_2 = \frac{EI_1}{L}.u^2\left[\frac{\alpha(1-2\alpha\cot 2\alpha}{\tan\alpha - \alpha}\right]\left[\frac{2\alpha - \sin 2\alpha}{\sin 2\alpha - 2\alpha\cos 2\alpha}\right]\theta_1 + \frac{EI_1}{L}.u^3.\left[\frac{\alpha(1-2\alpha\cot 2\alpha}{\tan\alpha - \alpha}\right]\theta_2 \qquad 14.14f'$$

where

$$2\alpha = \frac{\pi}{u}\sqrt{\frac{P}{P_{e_1}}}$$

It is observed[9] that

$$\left[\frac{\alpha(1-2\alpha\cot 2\alpha}{\tan\alpha - \alpha}\right] = s$$

and

$$\left[\frac{\alpha(1-2\alpha\cot 2\alpha}{\tan\alpha - \alpha}\right]\left[\frac{2\alpha - \sin 2\alpha}{\sin 2\alpha - 2\alpha\cos 2\alpha}\right] = sc$$

where s and sc are the stability function tabulated by Livesley and Chandler.

The end moments due to the rotations of the ends can be expressed as

$$M_1 = EI_1/L. \, (s_1\theta_1 + sc\,\theta_2) \qquad 14.14g$$

$$M_2 = EI_1/L. \, (sc\,\theta_1 + s_2\theta_2)$$

where s_1 and s_2 are the stiffnesses of the smaller and larger ends respectively and sc is the moment appearing at either end due to unit rotation of the other. Comparsion of 14.14f and 14.14g yields

$$\left. \begin{aligned} s_1 &= u\,s \\ s_2 &= u^3 s \\ \overline{sc} &= u^2 sc \end{aligned} \right\} \qquad 14.14h$$

Modified stiffnesses s_1'' and s_2''.

If $M_2 = 0$ i.e end 2 is pinned, then we define $M_1 = k_1 s_1'' \theta_1$

where

$$s_1'' = \frac{s_1 s_2 - (\overline{sc})^2}{s_2}$$

and if $M_1 = 0$ then $M_2 = k_1 s_2'' \theta_2$

where

$$s_2'' = \frac{s_1 s_2 - (\overline{sc})^2}{s_1}$$

and these are expressible in terms of the stability functions tabulated by Livesley and Chandler as

$$\left. \begin{aligned} s_1'' &= u\,s'' \\ s_2'' &= u^3 s'' \end{aligned} \right\} \qquad 14.14i$$

Sway factors α_1, α_2, and Q

If one end of a pin jointed non-prismatic strut is displaced laterally by δ, the strut will rotate bodily by a clockwise rotation $\phi = \delta/L$ as in Figure 14.6a. If the two ends are rotated such that the two end portions are parallel to the initial position as shown in Figure 14.6b, end moments will appear at the ends. These are

$$M_1 = - k_1(s_1+\overline{sc}) \delta/L$$

$$= - k_1\alpha_1\delta/L \qquad 14.14j$$

where

$$\alpha_1 = s_1+ \overline{sc}$$

and

$$M_2 = - k_1(s_2+ \overline{sc}) \delta/L$$

$$= - k_1\alpha_2\delta/L \qquad 14.14k$$

where

$$\alpha_2 = s_2+ \overline{sc}$$

Thus the modified slope-deflection equations becomes

$$M_1 = k_1(s_1\theta_1+ \overline{sc}\ \theta_2 - \alpha_1\ \delta/L) \qquad 14.15a$$

and

$$M_2 = k_1(\overline{sc}\ \theta_1+ s_2\theta_2 - \alpha_2\ \delta/L) \qquad 14.15b$$

The force required to keep the strut in the position shown in Figure 14.6b is

$$F = -\frac{M_1+ M_2}{L} - P\,\delta/L$$

Substituting for M_1 and M_2 using 14.14j and 14.14k, gives

$$F = + k_1(\alpha_1' + \alpha_2')\,\delta/L^2 - P\,\delta/L$$

$$= + k_1(\alpha_1' + \alpha_2' - \pi^2\rho_1)\,\delta/L^2$$

$$= + k_1 Q\,\delta/L^2 \qquad 14.15c$$

where

$$Q = \alpha_1' + \alpha_2' - \pi^2\rho_1$$

These stability functions are expressible in terms of the stability functions tabulated by Livesley and Chandler as

$$\alpha_1' = u\,s + u^2 sc \qquad 14.16a$$

$$\alpha_2' = u^3 s + u^2 sc \qquad 14.16b$$

and

$$Q = su(1+u^2) + 2u^2(sc - B) \qquad 14.16c$$

where

$$2B = \pi^2\rho$$

Taper so that $\overline{m} = 3$

When $\overline{m} = 3$, the constants in 14.13e are

$$\gamma = -\tfrac{1}{2}\ ;\ n = +1\ ;\quad \beta = 2\varpi = 2\sqrt{\frac{Pa^3}{EI_1}}$$

and the general solution of the basic equation is

$$y = \sqrt{x}\left[A J_1\left(\frac{2\varpi}{\sqrt{x}}\right) + B\,Y_1\left(\frac{2\varpi}{\sqrt{x}}\right)\right] + \frac{M_1}{P}\left(\frac{x-a}{b-a} - 1\right) + \frac{M_2}{P}\left(\frac{x-a}{b-a}\right) \qquad 14.17a$$

The first derivative of equation 14.17a is

$$\frac{dy}{dx} = A\left[\frac{J_1\left(\frac{2\varpi}{\sqrt{x}}\right)}{\sqrt{x}} - \frac{\varpi}{x} J_0\left(\frac{2\varpi}{\sqrt{x}}\right)\right] + B\left[\frac{Y_1\left(\frac{2\varpi}{\sqrt{x}}\right)}{\sqrt{x}} - \frac{\varpi}{x} Y_0\left(\frac{2\varpi}{\sqrt{x}}\right)\right] + \frac{M_1 + M_2}{P(b-a)} \qquad 14.17b$$

This equation can be expressed in terms of Bessel functions of the second order as

$$\frac{dy}{dx} = \frac{\varpi}{x}\left[A J_2\left(\frac{2\varpi}{\sqrt{x}}\right) + B Y_2\left(\frac{2\varpi}{\sqrt{x}}\right)\right] + \frac{M_1+M_2}{P(b-a)} \qquad 14.17c$$

but it is more convenient to use equation 14.17b in tabulating the stability functions. The values of the constants A and B are obtained by substituting the conditions y=0 at x=a and x=b in equation 14.17a.

$$\left.\begin{aligned} A &= \frac{1}{PC_1}\left[\frac{M_1}{\sqrt{a}} Y_1(\bar{b}) + \frac{M_2}{\sqrt{b}} Y_1(\bar{a})\right] \\ B &= -\frac{1}{PC_1}\left[\frac{M_1}{\sqrt{a}} J_1(\bar{b}) + \frac{M_2}{\sqrt{b}} J_1(\bar{a})\right] \end{aligned}\right\} \qquad 14.17d$$

where

$$C_1 = J_1(\bar{a}) Y_1(\bar{b}) - Y_1(\bar{a}) J_1(\bar{b})$$

and

$$\bar{a} = 2\phi\ , \quad \bar{b} = \frac{2\phi}{\sqrt{u}} \quad \text{and} \quad \frac{\pi}{u-1}\sqrt{\frac{P}{P_{e_1}}} = \phi$$

The values of M_1 and M_2 are obtained by substituting the two conditions $\frac{dy}{dx} = \theta_1$ at x=a and $\frac{dy}{dx} = \theta_2$ at x=b in equation 14.17c.

$$M_1 = \frac{EI_1}{L}\cdot\pi^2\frac{P}{P_{e_1}}C_1 \frac{\frac{C_1}{u} + \frac{u-1}{u\sqrt{u}}\phi C_3}{\left[\left\{uC_1+(u-1)\phi C_2\right\}\left\{\frac{C_1}{u}+\frac{u-1}{u\sqrt{u}}\phi C_3\right\} - \left\{\frac{u-1}{\sqrt{u}}\frac{1}{\pi}+C_1\right\}^2\right]}\theta_1$$

$$- \frac{EI_1}{L}\cdot\pi^2\frac{P}{P_{e_1}}C_1 \frac{\frac{u-1}{\sqrt{u}}\frac{1}{\pi}+C_1}{\left[\left\{uC_1+(u-1)\phi C_2\right\}\left\{\frac{C_1}{u}+\frac{u-1}{u\sqrt{u}}\phi C_3\right\} - \left\{\frac{u-1}{\sqrt{u}}\frac{1}{\pi}+C_1\right\}^2\right]}\theta_2 \qquad 14.17e$$

and

$$M_2 = -\frac{EI_1}{L}\pi^2\frac{P}{P_{e_1}}C_1 \frac{\frac{u-1}{\sqrt{u}}\frac{1}{\pi}+C_1}{\left[\left\{uC_1+(u-1)\phi C_2\right\}\left\{\frac{C_1}{u}+\frac{u-1}{u\sqrt{u}}\phi C_3\right\} - \left\{\frac{u-1}{\sqrt{u}}\frac{1}{\pi}+C_1\right\}^2\right]}\theta_1$$

$$+ \frac{EI_1}{L}\pi^2\frac{P}{P_{e_1}}C_1 \frac{uC_1+(u-1)\phi C_2}{\left[\left\{uC_1+(u-1)\phi C_2\right\}\left\{\frac{C_1}{u}+\frac{u-1}{u\sqrt{u}}\phi C_3\right\} - \left\{\frac{u-1}{\sqrt{u}}\frac{1}{\pi}+C_1\right\}^2\right]}\theta_2 \qquad 14.17f$$

where

$$C_2 = J_1(\bar{b})\,Y_0(\bar{a}) - Y_1(\bar{b})\,J_0(\bar{a})$$

and

$$C_3 = Y_0(\bar{b})\,J_1(\bar{a}) - J_0(\bar{b})\,Y_1(\bar{a})$$

From these two expressions for the end moments, it follows that the stiffnesses and $\overline{sc}$ are

$$S_1 = \pi^2 \frac{P}{P_{e_1}} c_1 \frac{\frac{c_1}{u} + \frac{u-1}{u\sqrt{u}}\phi c_3}{\left[\left\{uc_1 + (u-1)\phi c_2\right\}\left\{\frac{c_1}{u} + \frac{u-1}{u\sqrt{u}}\phi c_3\right\} - \left\{\frac{u-1}{\sqrt{u}}\frac{1}{\pi} + c_1\right\}^2\right]}$$

$$S_2 = \pi^2 \frac{P}{P_{e_1}} c_1 \frac{uc_1 + (u-1)\phi c_2}{\left[\left\{uc_1 + (u-1)\phi c_2\right\}\left\{\frac{c_1}{u} + \frac{u-1}{u\sqrt{u}}\phi c_3\right\} - \left\{\frac{u-1}{\sqrt{u}}\frac{1}{\pi} + c_1\right\}^2\right]} \qquad 14.17g$$

$$\overline{sc} = -\pi^2 \frac{P}{P_{e_1}} c_1 \frac{\frac{u-1}{\sqrt{u}}\frac{1}{\pi} + c_1}{\left[\left\{uc_1 + (u-1)\phi c_2\right\}\left\{\frac{c_1}{u} + \frac{u-1}{u\sqrt{u}}\phi c_3\right\} - \left\{\frac{u-1}{\sqrt{u}}\frac{1}{\pi} + c_1\right\}^2\right]}$$

The expressions for the modified stiffnesses and sway factors are

$$s_1'' = \frac{s_1 s_2 - (\overline{sc})^2}{s_2}$$

$$s_2'' = \frac{s_1 s_2 - (\overline{sc})^2}{s_1}$$

$$\alpha_1 = s_1 + \overline{sc} \qquad 14.17h$$

$$\alpha_2 = s_2 + \overline{sc}$$

$$Q = \alpha_1 + \alpha_2 - \pi^2 \rho_1$$

<u>Taper so that $\overline{m}=2.4$</u>

When $\overline{m} = 2.4$, the constants in 14.13e become

$$\gamma = -\frac{1}{5} \;;\quad n = \frac{5}{2} \;;\quad \beta = 5\varpi = 5\sqrt{\frac{Pa^{2.4}}{EI_L}}$$

and the general solution is

$$y = \sqrt{x}\left[A\, J_{\frac{5}{2}}\left(\frac{5\varpi}{x^{1/5}}\right) + B\, J_{-\frac{5}{2}}\left(\frac{5\varpi}{x^{1/5}}\right)\right] + \frac{M_1}{P}\left(\frac{x-a}{b-a} - 1\right) + \frac{M_2}{P}\left(\frac{x-a}{b-a}\right) \qquad 14.18a$$

The first derivative of equation 14.18a is

$$\frac{dy}{dx} = A\, \frac{\varpi}{x^{7/10}}\, J_{7/2}\left(\frac{5\varpi}{x^{1/5}}\right) - B\, J_{-7/10}\left(\frac{5\varpi}{x^{1/5}}\right) + \frac{M_1 + M_2}{P(b-a)} \qquad 14.18b$$

When the end conditions are substituted in equations 14.18a and 14.18b, the values of the unknowns are

$$A = \frac{1}{PC_1}\left[\frac{M_1}{\sqrt{a}}\, J_{-\frac{5}{2}}(\bar{b}) + \frac{M_2}{\sqrt{b}}\, J_{-\frac{5}{2}}(\bar{a})\right]$$

$$14.18c$$

$$B = -\frac{1}{PC_1}\left[\frac{M_1}{\sqrt{a}}\, J_{\frac{5}{2}}(\bar{b}) + \frac{M_2}{\sqrt{b}}\, J_{\frac{5}{2}}(\bar{a})\right]$$

$$M_1 = \frac{EI_1}{L}\cdot \pi^2 \frac{P}{P_{e_1}} C_1 \frac{\frac{\phi(u-1)}{u^{6/5}} C_3 + C_1}{\left[\left\{\phi(u-1)C_2 + C_1\right\}\left\{\frac{\phi(u-1)}{u^{6/5}} C_3 + C_1\right\} - \left\{C_1 - \frac{0.4}{\pi}\frac{u-1}{\sqrt{u}}\right\}^2\right]}\theta_1$$

$$- \frac{EI_1}{L}\cdot \pi^2 \frac{P}{P_{e_1}} C_1 \frac{C_1 - \frac{0.4}{\pi}\frac{u-1}{\sqrt{u}}}{\left[\left\{\phi(u-1)C_2 + C_1\right\}\left\{\frac{\phi(u-1)}{u^{6/5}} C_3 + C_1\right\} - \left\{C_1 - \frac{0.4}{\pi}\frac{u-1}{\sqrt{u}}\right\}^2\right]}\theta_2 \qquad 14.18d$$

$$M_2 = -\frac{EI_1}{L}\cdot \pi^2 \frac{P}{P_{e_1}} C_1 \frac{C_1 - \frac{0.4}{\pi}\frac{u-1}{\sqrt{u}}}{\left[\left\{\phi(u-1)C_2 + C_1\right\}\left\{\frac{\phi(u-1)}{u^{6/5}} C_3 + C_1\right\} - \left\{C_1 - \frac{0.4}{\pi}\frac{u-1}{\sqrt{u}}\right\}^2\right]}\theta_1$$

$$+ \frac{EI_1}{L}\cdot \pi^2 \frac{P}{P_{e_1}} C_1 \frac{\phi(u-1)C_2 + C_1}{\left[\left\{\phi(u-1)C_2 + C_1\right\}\left\{\frac{\phi(u-1)}{u^{6/5}} C_3 + C_1\right\} - \left\{C_1 - \frac{0.4}{\pi}\frac{u-1}{\sqrt{u}}\right\}^2\right]}\theta_2 \qquad 14.18e$$

where

$$C_1 = J_{5/2}(\bar{a}) \cdot J_{-5/2}(\bar{b}) - J_{-5/2}(\bar{a}) \cdot J_{5/2}(\bar{b})$$

$$C_2 = J_{-5/2}(\bar{b}) \cdot J_{7/2}(\bar{a}) - J_{5/2}(\bar{b}) \cdot J_{-7/2}(\bar{a})$$

$$C_3 = J_{-5/2}(\bar{a}) \cdot J_{7/2}(\bar{b}) + J_{5/2}(\bar{a}) \cdot J_{-7/2}(\bar{b})$$

$$\phi = \frac{\pi}{u - 1} \quad \frac{P}{P_{e_1}}$$

$$\bar{a} = 5\,\phi$$

$$\bar{b} = \frac{5\,\phi}{u^{1/5}}$$

The Bessel functions involved in this analysis are expressible in the form[26]

$$J_{5/2}(z) = \sqrt{\frac{2}{\pi z}}\left[\left(\frac{3}{z^2} - 1\right)\sin z - \frac{3}{z}\cos z\right]$$

$$J_{-5/2}(z) = \sqrt{\frac{2}{\pi z}}\left[\frac{3}{z}\sin z + \left(\frac{3}{z^2} - 1\right)\cos z\right]$$

$$J_{7/2}(z) = \sqrt{\frac{2}{\pi z}}\left[\left(\frac{15}{z^3} - \frac{6}{z}\right)\sin z - \left(\frac{15}{z^2} - 1\right)\cos z\right] \qquad 14 \cdot 18f$$

$$J_{-7/2}(z) = -\sqrt{\frac{2}{\pi z}}\left[\left(\frac{15}{z^2} - 1\right)\sin z + \left(\frac{15}{z^3} - \frac{6}{z}\right)\cos z\right]$$

Thus the stiffnesses and $\overline{sc}$ are

$$S_1 = \pi^2 \frac{P}{P_{e_1}} C_1 \frac{\frac{\phi(u-1)}{u^{6/5}} C_3 + C_1}{\left[\left\{\phi(u-1) C_2 + C_1\right\}\left\{\frac{\phi(u-1)}{u^{6/5}} C_3 + C_1\right\} - \left\{C_1 - \frac{0.4}{\pi} \frac{u-1}{\sqrt{u}}\right\}^2\right]}$$

$$S_2 = \pi^2 \frac{P}{P_{e_1}} C_1 \frac{\phi(u-1) C_2 + C_1}{\left[\left\{\phi(u-1) C_2 + C_1\right\}\left\{\frac{\phi(u-1)}{u^{6/5}} C_3 + C_1\right\} - \left\{C_1 - \frac{0.4}{\pi} \frac{u-1}{\sqrt{u}}\right\}^2\right]}$$

$$\overline{sc} = -\pi^2 \frac{P}{P_{e_1}} C_1 \frac{C_1 - \frac{0.4}{\pi} \frac{u-1}{\sqrt{u}}}{\left[\left\{\phi(u-1) C_2 + C_1\right\}\left\{\frac{\phi(u-1)}{u^{6/5}} C_3 + C_1\right\} - \left\{C_1 - \frac{0.4}{\pi} \frac{u-1}{\sqrt{u}}\right\}^2\right]}$$

The expressions for the modified stiffnesses and the sway factors are similar to those given in case $\overline{m}=3$.

<u>Taper so that $\overline{m} = 2$</u>

When $\overline{m}=2$, the reduced fundamental equation is

$$EI_1 \left(\frac{x}{a}\right)^2 \frac{d^2z}{dx^2} + Pz = 0 \qquad 14.19a$$

The integral of this equation can be obtained as (Timo.[2] 127)

$$z = \sqrt{\frac{x}{a}}\left[A \sin\left\{\beta \log\left(\frac{x}{a}\right)\right\} + B \cos\left\{\beta \log\left(\frac{x}{a}\right)\right\}\right] \qquad 14.19b$$

and the general solution as

$$y = \sqrt{\frac{x}{a}}\left[A \sin\left\{\beta \log\left(\frac{x}{a}\right)\right\} + B \cos\left\{\beta \log\left(\frac{x}{a}\right)\right\}\right] + \frac{M_1}{P}\left(\frac{x-a}{b-a} - 1\right) + \frac{M_2}{P}\left(\frac{x-a}{b-a}\right) \qquad 14.19c$$

The first derivative of equation 14.19c is

$$\frac{dy}{dx} = A\left[\frac{1}{2}\frac{1}{\sqrt{ax}} \sin\left\{\beta \log\left(\frac{x}{a}\right)\right\} + \frac{1}{\sqrt{ax}} \beta \cos\left\{\beta \log\left(\frac{x}{a}\right)\right\}\right]$$
$$+ B\left[\frac{1}{2}\frac{1}{\sqrt{ax}} \cos\left\{\beta \log\left(\frac{x}{a}\right)\right\} - \frac{1}{\sqrt{ax}} \beta \sin\left\{\beta \log\left(\frac{x}{a}\right)\right\}\right] + \frac{M_1 + M_2}{P(b-a)} \qquad 14.19d$$

On substituting the usual boundary conditions in equations 14.19c and 14.19d the values of the unknowns are

$$A = -\frac{1}{\sin\psi}\left[\frac{M_1}{P}\cos\psi + \frac{M_2}{P}\sqrt{\frac{a}{b}}\right] \qquad 14.19e$$

$$B = \frac{M_1}{P}$$

$$M_1 = \frac{EI_1}{L}\cdot\pi^2\frac{P}{P_{e_1}}\,\frac{\frac{u+1}{2u}\sin\psi - \frac{u-1}{u}\beta\cos\psi}{\left[\frac{(u-1)^2}{u}(\frac{1}{4}-\beta^2)\sin\psi + \left\{\frac{2(u-1)}{\sqrt{u}} - \frac{(u^2-1)}{u}\cos\psi\right\}\beta\right]}\,\theta_1$$

$$-\frac{EI_1}{L}\cdot\pi^2\frac{P}{P_{e_1}}\,\frac{\sin\psi - \frac{u-1}{\sqrt{u}}\beta\cos\psi}{\left[\frac{(u-1)^2}{u}(\frac{1}{4}-\beta^2)\sin\psi + \left\{\frac{2(u-1)}{\sqrt{u}} - \frac{(u^2-1)}{u}\cos\psi\right\}\beta\right]}\,\theta_2 \qquad 14.19f$$

$$M_2 = -\frac{EI_1}{L}\cdot\pi^2\frac{P}{P_{e_1}}\,\frac{\sin\psi - \frac{u-1}{\sqrt{u}}\beta\cos\psi}{\left[\frac{(u-1)^2}{u}(\frac{1}{4}-\beta^2)\sin\psi + \left\{\frac{2(u-1)}{\sqrt{u}} - \frac{(u^2-1)}{u}\cos\psi\right\}\beta\right]}\,\theta_1 \qquad 14.19g$$

$$+\frac{EI_1}{L}\cdot\pi^2\frac{P}{P_{e_1}}\,\frac{\frac{u+1}{2}\sin\psi - (u-1)\beta\cos\psi}{\left[\frac{(u-1)^2}{u}(\frac{1}{4}-\beta^2)\sin\psi + \left\{\frac{2(u-1)}{\sqrt{u}} - \frac{(u^2-1)}{u}\cos\psi\right\}\beta\right]}\,\theta_2$$

where

$$\psi = \beta\log\left(\frac{a}{b}\right) \quad \text{and} \quad \beta = \sqrt{\frac{Pa^2}{EI_1} - \frac{1}{4}}$$

Thus the stiffnesses and $\overline{sc}$ are

$$S_1 = \pi^2\frac{P}{P_{e_1}}\,\frac{\frac{u+1}{2u}\sin\psi - \frac{(u-1)}{u}\beta\cos\psi}{\left[\frac{(u-1)^2}{u}(\frac{1}{4}-\beta^2)\sin\psi + \left\{\frac{2(u-1)}{\sqrt{u}} - \frac{(u^2-1)}{u}\cos\psi\right\}\beta\right]}$$

$$S_2 = u\,S_1 \qquad 14.19h$$

$$\overline{sc} = -\pi^2\frac{P}{P_{e_1}}\,\frac{\sin\psi - \frac{u-1}{\sqrt{u}}\beta\cos\psi}{\left[\frac{(u-1)^2}{u}(\frac{1}{4}-\beta^2)\sin\psi + \left\{\frac{2(u-1)}{\sqrt{u}} - \frac{(u^2-1)}{u}\cos\psi\right\}\beta\right]}$$

Taper so that $\bar{m} = 1$

When $\bar{m} = 1$, the constants in 14.13e are

$$\gamma = \frac{1}{2} \; ; \; n = +1 \quad , \; \beta = 2\varpi \quad \text{and} \quad \varpi^2 = \frac{Pa}{EI_1} = \frac{\phi^2}{a}$$

and the general solution is

$$y = \sqrt{x}\left[A\,J_1(2\varpi\sqrt{x}) + B\,Y_1(2\varpi\sqrt{x})\right] + \frac{M_1}{P}\left(\frac{x-a}{b-a} - 1\right) + \frac{M_2}{P}\left(\frac{x-a}{b-a}\right) \qquad 14.20a$$

The first derivative of equation 14.20a is

$$\frac{dy}{dx} = \varpi\left[A\,J_0(2\varpi\sqrt{x}) + B\,Y_0(2\varpi\sqrt{x})\right] + \frac{M_1 + M_2}{P(b-a)} \qquad 14.20b$$

On substituting the boundary conditions in equations 14.20a and 14.20b, the values of the unknowns are

$$A = \frac{1}{PC_1}\left[\frac{M_1}{\sqrt{a}}\,Y_1(\bar{b}) + \frac{M_2}{\sqrt{b}}\,Y_1(\bar{a})\right]$$

14.20c

$$B = -\frac{1}{PC_1}\left[\frac{M_1}{\sqrt{a}}\,J_1(\bar{b}) + \frac{M_2}{\sqrt{b}}\,J_1(\bar{a})\right]$$

$$M_1 = \frac{EI_1}{L}\cdot\pi^2\frac{P}{P_{e_1}}C_1\,\frac{C_3\phi\,\frac{(u-1)}{\sqrt{u}} + C_1}{\lambda}\,\theta_1 + \frac{EI_1}{L}\pi^2\frac{P}{P_{e_1}}C_1\,\frac{\frac{u-1}{\sqrt{u}}\frac{1}{\pi} - C_1}{\lambda}\,\theta_2 \qquad 14.20d$$

$$M_2 = \frac{EI_1}{L}\cdot\pi^2\frac{P}{P_{e_1}}C_1\,\frac{\frac{u-1}{\sqrt{u}}\frac{1}{\pi} - C_1}{\lambda}\,\theta_1 + \frac{EI_1}{L}\pi^2\frac{P}{P_{e_1}}C_1\,\frac{\phi(u-1)C_2 + C_1}{\lambda}\,\theta_2 \qquad 14.20e$$

where

$$C_1 = J_1(\bar{a})\,Y_1(\bar{b}) - J_1(\bar{b})\,Y_1(\bar{a})$$

$$C_2 = J_0(\bar{a})\,Y_1(\bar{b}) - J_1(\bar{b})\,Y_0(\bar{a})$$

$$C_3 = Y_1(\bar{a})\,J_0(\bar{b}) - J_1(\bar{a})\,Y_0(\bar{b})$$

$$\lambda = \left[\left\{\phi(u-1)\,C_2 + C_1\right\}\left\{C_3\,\frac{\phi(u-1)}{\sqrt{u}} + C_1\right\} - \left\{C_1 - \frac{u-1}{\sqrt{u}}\,\frac{1}{\pi}\right\}^2\right]$$

$$\bar{a} = 2\phi \quad , \quad \bar{b} = 2\phi\sqrt{u} \quad \text{and} \quad \phi = \frac{\pi}{u-1}\sqrt{\frac{P}{P_{e_1}}}$$

Approximate solution of equation 14.4

It was necessary to check a few values of the basic stability functions s_1, s_2 and $\overline{sc}$ given by the exact method before tabulating the stability functions. The finite difference method of solution[27] was used. This method of solution is based upon the solution of a linear set of finite didderence equations.

The origin of the coordinates will be taken as the end 1. Thus the moment of inertia I_x at any cross section at distance x from the end 1 is

$$I_x = I_1 (1 + (u-1)\, x/L)^{\overline{m}} \qquad 14.21a$$

and the fundamental differential equation is

$$\frac{d^2y}{dx^2} + \frac{P}{EI_x} y = \frac{M_1}{EI_x}(x/L - 1) + \frac{M_2}{EI_x}(x/L) \qquad 14.21b$$

If the strut is now considered to be divided into N equal spaces numbered at 0 at the end 1 and proceeding to N at the end 2, the first and the second derivatives at any point n can be approximated by their difference equivalents (27)

$$\left(\frac{dy}{dx}\right)_n \simeq \frac{y_{n+1} - y_{n-1}}{2h} \qquad 14.21c$$

$$\left(\frac{d^2y}{dx^2}\right)_n \simeq \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2}$$

where h is the interval between the points i.e $h = L/N$ since these are equally spaced. Using these approximations, the differential equation 14.21b becomes

$$y_{n-1} + (h^2P/EI_x - 2)y_n + y_{n+1} + (1 - n/N)\,h^2M_1/EI_x - n/N.h^2M_2/EI_x = 0 \qquad 14.22$$

Since the strut has been divided into N equal spaces there can be a maximum number of (N+1) finite difference equations and corresponding (N+5) unknowns ($y_o, \ldots y_N, y_{-1}, y_{N+1}, M_1, M_2$). Four additional equations can be formulated from the conditions at the ends. It follows that the number of relationships are sufficient to determine the unknowns.

One of the end stiffnesses will be evaluated at each cycle of calculation. Thus one end will be rotated while the other is held fixed. If the end 1 is rotated through an angle θ_1 and the other end 2 is fixed, the boundary conditions are

$$y_o = y_N = 0 \qquad 14.23a$$

$$\left(\frac{dy}{dx}\right)_o = \theta_1 \quad \text{and} \quad \left(\frac{dy}{dx}\right)_N = 0$$

By using the approximate equivalent of the first derivative, the following is obtained

$$y_{N+1} = y_{N-1} \qquad 14.23b$$

$$y_{-1} = y_{+1} - 2h\theta_1$$

The moments M_1 and M_2 which are caused by the rotation θ_1 can always be denoted as

$$M_1 = EI_1/L.\ s_1\theta_1 \qquad 14.23c$$

$$M_2 = EI_1/L.\ \overline{sc}\theta_1$$

where s_1 is the stiffness of the smaller end and $\overline{sc}$ is the moment appearing at end 2 due to unit rotation. If we define a set of quantities

$$K_n = PL^2/N^2EI_x - 2$$
$$\alpha_n = EI_1/EI_x \cdot \frac{N - n}{N^3} \qquad 14.23d$$
$$\beta_n = -EI_1/EI_x \cdot n/N^3$$

these together with the expressions for the moments 14.23c reduce equation 14.22 to

$$y_{n-1} + K_n y_n + y_{n+1} + \alpha_n s_1 L\theta_1 + \beta_n sc L\theta_1 = 0 \qquad 14.24$$

Applying this equation to the points along the strut and using the results of 14.23b in considering the end points, the following set of equation is obtained

$$\begin{array}{lll}
n=0 & \beta_0 \overline{sc} L\theta_1 + 2y_1 & \alpha_0 s_1 L\theta_1 = 2L\theta_1/N \\
n=1 & \beta_1 \overline{sc} L\theta_1 + K_1 y_1 + y_2 & \alpha_1 s_1 L\theta_1 = 0 \\
n=2 & \beta_2 \overline{sc} L\theta_1 + y_1 + K_2 y_2 + y_3 & \alpha_2 s_1 L\theta_1 = 0 \\
n=3 & \beta_3 \overline{sc} L\theta_1 \quad + y_2 + K_3 y_3 + y_4 & \alpha_3 s_1 L\theta_1 = 0 \qquad 14.25a \\
 & & \\
n=N-1 & \beta_{N-1} \overline{sc} L\theta_1 & y_{N-2} + K_{N-1} y_{N-1} + \alpha_{N-1} s_1 L\theta_1 = 0 \\
n=N & \beta_N \overline{sc} L\theta_1 & 2y_{N-1} + \alpha_N s_1 L\theta_1 = 0
\end{array}$$

These equations can be represented in matrix form as

$$\begin{bmatrix} \beta_0 & 2 & & & & & \alpha_0 \\ \beta_1 & K_1 & 1 & & & & \alpha_1 \\ \beta_2 & 1 & K_2 & 1 & & & \alpha_2 \\ \beta_3 & & 1 & K_3 & 1 & & \alpha_3 \\ & & & & & & \\ \beta_{N-1} & & & & 1 & K_{N-1} & \alpha_{N-1} \\ \beta_N & & & & & 2 & \alpha_N \end{bmatrix} \begin{bmatrix} \overline{sc}\,L\theta_1 \\ y_1 \\ y_2 \\ y_3 \\ \\ y_{N-1} \\ s_1 L\theta_1 \end{bmatrix} = \begin{bmatrix} \frac{2}{N} L\theta_1 \\ 0 \\ 0 \\ 0 \\ \\ 0 \\ 0 \end{bmatrix} \qquad 14.25b$$

The quantities of 14.23d can be expressed in more useful form as

$$K_n = C_n\ P/P_{e_1} - 2$$

where

$$C_n = (\pi/N)^2(1/(1+(u-1)n/N)^m)$$

$$\alpha_n = (N - n)/N^3.\ 1/(1+(u-1)n/N)^m$$

$$\beta_n = -\ n/N^3.\ 1/(1+(u-1)n/N)^m$$

If end 2 is now rotated through an angle θ_2 and end 1 is fixed, the new boundary conditions are

$$y_o = y_N = 0 \qquad 14.26a$$

$$\left(\frac{dy}{dx}\right)_o = 0 \quad \text{and} \quad \left(\frac{dy}{dx}\right)_N = \theta_2$$

By using 14.21c, the following is obtained

$$y_{-1} = y_{+1} \qquad 14.26b$$

$$y_{N+1} = 2h\theta + y_{N-1}$$

The new end moments are

$$M_1 = EI_1/L.\overline{sc}\theta_2 \qquad\qquad 14.26c$$
$$M_2 = EI_1/L.\ s_2\theta_2$$

If the new expressions for M_1 and M_2 are subsituted in equation 14.22 together with the defined quantities of 14.23d equation 14.22 becomes

$$y_{n-1} + K_n y_n + y_{n+1} + \alpha_n \overline{sc} L\theta_2 + \beta_n s_2 L\theta_2 = 0 \qquad\qquad 14.27$$

By similar reasoning, the matrix form for calculating s_2 will be

$$\begin{bmatrix} \beta_0 & 2 & & & & & \alpha_0 \\ \beta_1 & K_1 & 1 & & & & \alpha_1 \\ \beta_2 & 1 & K_2 & 1 & & & \alpha_2 \\ \beta_3 & & 1 & K_3 & 1 & & \alpha_3 \\ & & & & & & \\ \beta_{N-1} & & & & 1 & K_{N-1} & \alpha_{N-1} \\ \beta_N & & & & & 2 & \alpha_N \end{bmatrix} \begin{bmatrix} s_2 L\theta_2 \\ y_1 \\ y_2 \\ y_3 \\ \\ y_{N-1} \\ \overline{sc} L\theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \\ 0 \\ -\frac{2}{N} L\theta_2 \end{bmatrix} \qquad 14.28$$

Tabulation of the stability functions

Exact values

An α-code programme to calculate the stability functions has been developed using the English Electronics "DEUCE" Computer. The expressions given in the appendix were used in the tabulation. Three programmes for the cases $\bar{m}$ = 2, 2.4, and 3 were arranged to tabulate the stability functions for any value of u. The stages in the programme are listed below:-

(1) Read in the values of the constants, the starting value of ρ_1, the increment $\partial\rho_1$ and u.

(2) Using the initial value of ρ_1 generate the values of the basic stability functions s_1, s_2, $\overline{sc}$. There was a facility for calculating the Bessel functions of zero and first order. This was used in the case $\bar{m}$=3. For the case $\bar{m}$=2.4, .. subroutines were used to calculate the Bessel functions since they are functions of the trigonometrical functions.

(3) Generate other stability functions using the basic stability functions and store in succession.

(4) Print out the value of ρ_1 followed by the corresponding values of the stability functions.

(5) Change ρ_1 to $\rho_1+\partial\rho_1$ and repeat steps 2, 3 and 4.

Another separate α-code programme was used to tabulate these functions in table form.

Approximate values

To check a few values of the basic stability functions, the basic differential equation was solved by the finite difference method. This requires two programmes, an α-code and GIP programme. The stages of the α-code programme are as follows:-

(1) Read in the values of the constants, the starting value of ρ_1, the increment $\partial\rho_1$ and u.

(2) Using the initial value of ρ_1 generate the values of K_n, α_n, and β_n.

(3) Place the numbers and the values calculated in the right places in the square matrix 14.25b.

(4) Punch out the square matrix on cards.

(5) Change ρ_1 to $\rho_1 + \partial\rho_1$ and repeat steps 2, 3, and 4.

The stages in the GIP programme is listed:-

(1) Read in the matrices.

(2) Invert the square matrix.

(3) Transpose the inverted matrix.

(4) Multiply the transposed matrix and the column matrix.

(5) Print out the results (column matrix).

Chapter 15

Elastic instability of frameworks with distributed loading

Introduction

In the previous chapters it was assumed that the external loads are lumped at the joints and that all members carry constant axial forces with fixed lines of actions passing through the joints at the ends of each member. In practice, the loads on the framework are often distributed along the members of the frame. The total external load diagram of the framework indicates that there is a variation in the vertical force component from one end of the member to the other. It follows that there are variable axial forces in the members when the vertical forces have a component along the members. When the members deform, the distributed loading displaces in accordance with the deformations and the line of action of the components along the member will be dependent on the displacement of each section from the initial position. Thus the problem becomes one of considering a variable axial force with a variable line of action. The basic equation for this case will be different from the one used by Livesley and Chandler for calculating the stability functions. Therefore the stability functions used for finding the elastic critical load of some frames with distributed loads will be different from those tabulated by Livesley and Chandler.

The values of the stability functions presented here were obtained by solving the basic equation of the shear force on any section along the member. The axial force in the strut was assumed to vary linearly from one end to another. The stability functions were found to be dependent on two parameter; ρ the ratio of the mean value of the end axial forces to the Euler load of the strut and the ratio of the end axial forces μ .

The elastic critical load of an isolated strut free at one end and fixed at the other, subjected to its own weight, was investigated by Timoshenko and other writers(2) using Bessel functions in the solution of the differential equation of the buckled strut. The stability of this strut was also considered when in addition to the strut weight there was a concentrated compressive load at the free end. The values of the compressive forces were tabulated for different ratio of the distributed load to the critical load of uniformly loaded strut free at one end. By using the energy method, the stability of a hinged strut submitted to the action of its own weight in addition to compressive force, was also considered. Tyler and Christiano (28) have given an analytic method of solution of a beam column with partial uniformly distributed lateral load with variable axial force whose line of action does not change as the column deflects. They used the Airy integral functions in solving the basic equation. Their analysis has no direct use in this work since the line of action in the problem at hand deflects as the strut deforms but it helps in solving the basic equation of the shear force.

The analyses investigating the stability of some complete structures are also given. The assumptions of chapter 2 still hold. The method of analysis is similar to that of chapter 2 except that a modified slope-deflection equation is used with the new values of the stability functions. The stiffness of the end with the lower axial force is s_1 and that with the higher axial force is s_2 and the moment carried over, due to unit rotation, is $\overline{sc}$. α_1 and α_2 are the shear factors due to the rotations of the joints. α_1 and α_2 and Q are the sway moments and the shear factor due to unit of the ratio sway/length of the member.

For a member 12 rotated at both ends through angles θ_1 and θ_2 and subjected to sway δ, the end moments and the shear force are

$$M_1 = k\,(s_1\theta_1 + \overline{sc}\,\theta_2 - \alpha_1\delta/L) \qquad 15.1a$$

$$M_2 = k\,(\overline{sc}\,\theta_1 + s_2\theta_2 - \alpha_2\delta/L) \qquad 15.1b$$

and

$$v = k/L\,(\alpha_1\theta_1 + \alpha_2\theta_2 - Q\,\delta/L) \qquad 15.1c$$

The method for calculating the forces in the members is similar to that of chapter 2 except that the effect of the fixed end moments is included. The values of the fixed end moments were tabulated for struts with variable axial forces and an expression for their values, in terms of the stability functions tabulated by Livesley and Chandler for struts with constant axial forces was derived.

The purpose of the calculations which follow is to express s_1, s_2, $\overline{sc}$, α_1, α_2 and Q in a form suitable for computer evaluation.

Derivation of the basic equations

The distributed loads along the members of a framework have different effects depending on the inclination of these members.

(1) If the members are horizontal and the external loading is perpendicular to them, there will be end moments at the ends of the members and the axial forces in the members and their lines of actions will be constant. The basic equation for determining the fixed end moments when the external load is uniformly distributed is

$$EI\frac{d^2y}{dx^2} + Py = -M_F - wLx/2 + wx^2/2 \qquad 15.2$$

where y is the deflection of a section at distance x from 1.

M_F is the fixed end moment.

w is the distributed load per unit length.

P is the axial force in the member

The distribution of the external load might vary with the distance along the member. In this case, the basic equation for evaluating the fixed end moments is

$$EI\frac{d^4y}{dx^4} + P\frac{d^2y}{dx^2} = w(x) \qquad 15.3$$

where w(x) denotes the distributed load per unit length at distance x.

The loading may be continuous or with a discontinuity or might be concentrated at one or more points along the member. This requires the solution of a number of separate differential equations depending on the number of the discontinuities in the loading. Two functions are introduced to obtain a single differential equation applying to the entire span of the strut. These functions are the Heaviside unit[29] H(x-a) and the δ-function $\delta(x-a)$.

H(x-a) is defined as function which has a value of zero in the interval x=0 and $x \leq a$ and at $x \geq a$ assumes a value of unity which is maintained for the remaining portion of the strut.

$$H(x-a) \begin{cases} 0 & 0 \leq x < a \\ 1 & a \leq x \leq L \end{cases}$$

The δ function is defined as function which has a value of zero at $x \leq a$ and $x > a$ and assume a value of unity at x=a

$$\delta(x-a) \begin{cases} 0 & a \geq x \geq a \\ 1 & x = a \end{cases}$$

The basic equation for patially distributed loading is

$$EI\frac{d^4y}{dx^4} + P\frac{d^2y}{dx^2} = \left[H(x-a) + H(x-b) + \ldots\right] w \qquad 15.4a$$

and that of the concentrated loading is

$$EI\frac{d^4y}{dx^4} + P\frac{d^2y}{dx^2} = W_1\delta(x-a) + W_2\delta(x-b) + \ldots. \qquad 15.4b$$

The solutions of the last two equations can be obtained by using the Laplace transform where the Laplace transform of H and δ functions are

$$\mathcal{L}H(x-a)\,f(x) = e^{-ap}f(p)$$

$$\mathcal{L}\delta(x-a) = 1$$

Since it is not possible to tabulate the values of the fixed end moments for the various cases of discontinuity due to the variation in the positions of the external loads, the solutions of equations 15.4a and 15.4b is not shown and another technique is used for evaluating the fixed end moments using the already tabulated stability functions (9). For example, in the case of a concentrated external load at the mid-span, the point of loading is assumed to be a joiht undergoing deformation to balance the transverse loading. So that the fixed end moment for the sttut loaded at the mid-span can be obtained by causing the mid-point to deflect by δ. The moment appearing at the ends of the strut is

$$M_F = \mp(k\alpha/l)\,\delta \qquad 15.5a$$

where k and l are the stiffness and length of half the strut. Half the external load W is carried to each end by each half of the strut, hence the equilibrium equation of each half requires

$$W\ell/2 - (2kA\delta/\ell) = 0 \qquad 15.5b$$

whence

$$\delta = \frac{W\,\ell^2}{4kA} \qquad 15.5c$$

Substituting for δ

$$M_F = \mp W\ell/4 \,.\, \alpha/A \qquad 15.5d$$

α/A = m (by previous definition) and L =2 ℓ; hence

$$M_F = \mp \frac{m}{8}\, WL \qquad 15.6$$

In calculating M_F , the load parameter ρ of half the strut is used and not that of the whole strut in reading m from the tables(9).

The same procedure can be used in evaluating the fixed end moments in the presence of more than one point load on the span or in case of partially distributed load, using the expression derived for the fixed end moment and the stability functions(9).

(2) If the member is not horizontal, the distributed loading has a component along the member due to vertical loads. The loads inclined to the axis of the strut can be resolved into two components, one along the axis and one perpendicular to it. The basic equation can be obtained by considering an element of length dx along the strut AB of Figure 15.1. It is in equilibrium, so that the total vertical force on the element is zero. Hence

$$- V + w\,dx + (V + dV) = 0$$

whence

$$\frac{dV}{dx} = - w \qquad 15.7a$$

Taking moments about n and assuming that the angle between the axis of the strut and the centre line of the element is small, we obtain

$$M + w\, dx.dx/2 + (V+dV)dx - (M+dM) + (P(x) + pdx).dy - pdx/2.dy=0$$

Neglect terms of second order, this equation becomes

$$\frac{dM}{dx} - P(x)\frac{dy}{dx} = V \qquad 15.7b$$

The expression for the curvature of the axis of the strut is

$$EI\frac{d^2y}{dx^2} = - M \qquad 15.7c$$

Thus the basic equation is

$$EI\frac{d^3y}{dx^3} + P(x)\frac{dy}{dx} = - V \qquad 15.8$$

The value of V will be dependent on the deformation of the member and the end moments. Its value will be constant along the strut in the absence of the distributed lateral loading. This equation is used for the evaluation of the new stability functions and the fixed end moments.

There might be a discontinuity in the loading along the inclined members, in this case, the evaluation of the stability functions and the fixed end moments will require the solution of a set of basic equations depending on the number of the discontinuities. For example, the basic equations for the strut in Figure 15.1a are

$$EI\frac{d^3y_1}{dx^3} + P_1\frac{dy_1}{dx} = -V \qquad \text{(i)}$$

$$EI\frac{d^3y_2}{dx^3} + P(x)\frac{dy_2}{dx} = -V \qquad \text{(ii)} \qquad 15.9$$

$$EI\frac{d^3y_3}{dx^3} + P_2\frac{dy_3}{dx} = -V \qquad \text{(iii)}$$

In the case of concentrated loads on the inclined members, the set of the basic equations will be similar to equations (i) and (iii). The solutions of these equations will be more involved since it requires the evaluation of many constants.

<u>Stability tables</u>

To facilate the calculation of the elastic critical loads of structural frames taking the effect of distributed loads into account, it was necessary to prepare new tables of stability functions. These functions are dependent on two parameters μ and P/P_e, where μ is defined as P_1/P_2 and $P=(P_1+P_2)/2$. P_1 and P_2 are the smaller and larger axial forces at the ends of the strut and P_e is the Euler load. The value of μ must therefore lie between zero and unity. Livesley and Chandler tables[9] are for the special case when μ is unity.

The stability functions were tabulated for compressive force only and for μ = 0, 0.1, 0.2, 0.3 and 0.4.

For values of μ between these values and between 0.4 and 1.0, a graphical method must be used to obtain the values of the stability functions as will be demonstrated in the later examples. The expressions for the stability functions are those given in the appendix.

In calculating the force components in structural frameworks, it was necessary to replace the distributed loading by equivalent concentrated loads at the ends with fixed end moments. Thus expressions for these ends loads and moments were also obtained when the stability functions hadbbeen determined and these are also tabulated. The fixed end moments occuring with a uniformly distributed loading W are expressed as varying coefficients of WL. These fixed end moments are unequal at the two ends, the coefficients for the end with the smaller axial force will be termed m_1 and the other m_2. The end loads are also unequal. The load at the end with the smaller axial force is qW and that at the other end is (1-q)W is also tabulated.

In cases in which the distributed lateral loading is not perpendicular to the axis of the strut, the fixed end moments will both be the same as before but the length L of the strut in the expression m_1WL is replaced by Lcosϕ where ϕ is the angle between the applied external load and the perpendicular to the axis of the strut.

It is shown in the appendix that the fixed end moments for uniformly distributed loading W acting on struts carrying uniform axial forces can be expressed in terms of the Livesley and Chandler stability functions as

$$M_F = \mp \phi' WL \qquad 15.9$$

where

$$\phi' = \frac{1}{2B}\left(1 - \frac{s(1 - c)}{2}\right)$$

Theoretical analysis for frames having members with constant axial forces

As an illustration, the elastic instability of the frame in Figure 15.2 was analyzed. Since the frame is symmetrical and under symmetrical loading, it will have symmetrical deformations until the anti-symmetrical sway mode intervenes. On the application of the external load 2P on the beam BB', joints B and B' rotate in opposite direction by θ_B. The moments at the ends of the members are;-

Operation	M_{AB}	M_{BA}	$M_{BB'}$
1)Fixed end moment			$-M_F$
2)Rot. B&B'	$(ksc)_1\theta_B$	$(ks)_1\theta_B$	$(ks(1-c))_2\theta_B$

Joint equilibrium at B requires

$$\left[(ks)_1+(ks(1-c))_2\right]\theta_B - M_F = 0 \qquad 15.10a$$

Hence

$$\theta_B = \frac{M_F}{(ks)_1+(ks(1-c))_2} \qquad 15.10b$$

The frame has only one force component unknown, this is the horizontal force component H, shown in Figure 15.2a. The equilibrium of AB requires that

$$H \sin\phi - P \cos\phi = (k\alpha/L)_1\theta_B \qquad 15.10c$$

Substituting for θ_B and replacing M_F by $2\phi_2' PL_2$, equation 15.10c becomes

$$H/P = 1/\sin\phi \left[\cos\phi + L_2/L_1 \cdot 2\alpha_1\phi_2'/\{s_1 + k_2/k_1 \cdot (s(1-c))_2\} \right] \qquad 15.11$$

and the force in AB is

$$R = H\cos\phi + P\sin\phi \qquad 15.12$$

When the dimensions of the frame and its loading are specified equation 15.11 can be solved numerically to yield a relationship between H and P.

Stability criterion for the anti-symmetrical sway mode

To establish the condition for the anti-symmetrical sway mode, an infinitesimal disturbing horizontal force $2H_B$ is applied at B which causes the following incremental deformations:-

1) Equal incremental rotations of joints B and B' by $\partial\theta_B$.

2) Equal incremental sway of AB and A'B' by $\partial\delta$.

Member BB' sway by $-2\partial\delta\cos\phi$, as determined from the Williot diagram.

Due to these incremental deformations, the incremental moments at the ends of the members are:-

Operation	M_{AB}	M_{BA}	$M_{BB'}$
1)Rot.B&B'	$(ksc)_1\partial\theta_B$	$(ks)_1\partial\theta_B$	$(k\alpha)_1\partial\theta_B$
2)Sway	$-(k\alpha/L)_1\partial\delta$	$-(k\alpha/L)_1\partial\delta$	$+(2k\alpha\cos\phi/L)_2\partial\delta$

Eqilibrium consideration yields

$$\begin{bmatrix}\partial M_B \\ H_B\sin\phi\end{bmatrix} = \begin{bmatrix}\partial\theta_B \\ \partial\delta\end{bmatrix}\begin{bmatrix}[(ks)_1+(k\alpha)_2] & [(2k\alpha\cos\phi/L)_2-(k\alpha/L)_1] \\ [(2k\alpha\cos\phi/L)_2-(k\alpha/L)_1] & [(2kA/L^2)_1+(4kA\cos^2\phi/L^2)_2]\end{bmatrix} = \begin{bmatrix}0 \\ 0\end{bmatrix} \quad 15.13$$

The determinant of the square matrix is the anti-symmetrical sway mode stability condition. Any load satisfying equation 15.11 and making the determinant vanish is the critical load.

Numerical examples

Example 1

The forces in the members and the critical load in the symmetrical mode can be obtained from the solution of equation 15.11. When the dimensions of the frame and its loading conditions are specified, equation 15.11 can be solved numerically.

The frame has members of the same EI-value and length, ϕ is 45° and the external load 2P is distributed uniformly along BB'.

P in equation 15.11 is replaced by R and H using equation 15.12. When the k-values and the lengths are substituted in the modified equation obtained from 15.11, the equation becomes

$$1 - (2\frac{R}{H} - 1.414)(0.707 + \frac{2\phi_2'\alpha_1}{s_1 + (s(1-c))_2}) = 0 \qquad 15.14$$

ϕ_2 is given by 15.9

The load parameter ρ_1 of the inclined member, for the particular value of ρ_2, satisfying equation 15.14 is obtained by a trial and extrapolation process.

For $\rho_2 = 0.60$

from tables(9)

$s_2 = 3.1403$ $(sc)_2 = 2.2407$ $B_2 = 2.9609$

Thus $\phi_2' = \frac{1}{5.9218}(1 - \frac{3.1403 - 2.2407}{2})$

$= 0.093$

Equation 15.14 becomes

$$1 - (\frac{2\rho_1}{0.6} - 1.414)(0.707 + \frac{0.093 \times 2\alpha_1}{s_1 + 0.8996}) = 0 \qquad 15.14a$$

<u>First trial $\rho_1 = 0.78$</u>

From tables(9)

$s_1 = 2.8494$ $\alpha_1 = 5.1838$

Substituting these values in 15.14a

$$\Delta = 1 - \left(\frac{1.56}{0.6} - 1.414 \right)\left(0.707 + \frac{10.3676 \times 0.093}{3.7490}\right)$$

$$= 1 - 1.14$$

$$= - 0.14$$

A lower value of ρ_1 is therefore tested.

When $\rho_1 = 0.74$ $\Delta = - 0.013$

ρ_1 by linear extrapolation is

$$\simeq 0.74 - 0.04 \times \frac{0.013}{0.127}$$

$$\simeq 0.736$$

Equation 15.12 gives

$$P = (1.414 \times 0.736 - 0.60)\ P_e$$

$$= 0.443 P_e$$

and $H/P = 0.6/0.443 = 1.352$

Curve A in Figure 15.3, shows the numerical results obtained for several assumed values of ρ_2. The value of $2P = 1.26\ P_e$ given at the top of curve A represents the elastic critical load of the frame in the symmetrical joint rotation mode. This differs from the value $2.54P_e$ obtained when the load 2P is lumped at the joints.

Anti-symmetrical sway elastic critical load

Anti-symmetrical deformations become possible when the applied loads reaches such a magnitude that equations 15.11 and 15.13 are simultaneously satisfied.

A numerical solution of equation 15.13 can be performed. By substituting the values of k, ϕ and the lengths in equation 15.13, the determinant becomes

$$\begin{vmatrix} s_1 + \alpha_2 & 1.414\,\alpha_2 - \alpha_1 \\ 1.414\alpha_2 - \alpha_1 & 2A_1 + 2A_2 \end{vmatrix} \qquad 15.15$$

The load parameter ρ_1 of AB, for the particular value of H/P, making the determinant 15.15 vanish is obtained by a trial and interpolation process. Curve B in Figure 15.3 shows the numerical results obtained. The point at which this curve intersects with the other curve A, gives the anti-symmetrical sway elastic critical load which is $2P=1.60P_e$ when the loads are lumped at the joints and $1.175P_e$ when the external load 2P is distributed along BB'.

Example 2

The elastic instability of the same frame will be investigated when $\phi = 90^\circ$. When the values of k, L and ϕ are substituted in 15.11, the equation becomes

$$\frac{H}{P} + \frac{2\phi_2'\alpha_1}{s_1+(s(1-c))_2} = 0 \qquad 15.16$$

Equation 15.16 can be solved numerically as shown in example1.

Curve A in Figure 15.4. shows the numerical results obtained when the external load 2P is distributed along BB'. The symmetrical elastic critical load corresponding to this load case is $2P=3.27P_e$. When the load 2P is lumped at the joints B and B', the symmetrical elastic critical load is $2P= 5.10P_e$.

Anti-symmetrical sway elastic critical load

The stability criterion for the portal frame in the anti-symmetrical sway mode is

$$(kn)_1 + (k\alpha)_2 = 0 \qquad 15.16a$$

Equation 15.16a can be solved numerically to yield a relationship between H and P. The curve B in Figure 15.4 shows the numerical results obtained. The anti-symmetrical sway elastic critical load of the portal when the load is lumped at the joints is $2P= 1.495P_e$ and is $1.486P_e$ when the load is distributed along BB'.

Comment on the numerical examples

Based on the results presented herein, the following conclusions may be drawn regarding the stability of frames under distributed external loads.

(1) The critical load associated with the symmetrical mode of instability is considerably reduced when the loads are not lumped at

the joints.

Due to lateral loading, there will be lateral deflection of the beams which tends to draw the eaves inward, reducing the compressive force in the beam. This may delay the appearance of symmetrical elastic instability. When the load is approaching the symmetrical elastic critical load, the central deflection of the beam will become large and this renders the analysis given here invalid. This would necessiate the consideration of finite deflections and the effects of change of geometry on the analysis.

(2) The anti-symmetrical sway mode is also influenced by lateral loading. In portal frames, the reduction is very small, of the order of a few precent. In frames where the axial forces in the most heavily loaded strut is dependent on the force component unknown, H, the reduction is considerably larger.

Theoretical analysis for structures having members with variable axial forces

1. Isolated struts

The elastic instability of isolated struts under different end conditions and different ratios of end axial forces, will be investigated.

(i) <u>A strut with one end is fixed and the other free</u>

To establish the stability condition, an infinitesimal horizontal disturbing force H_A is applied at A, shown in Figure 15.5. Joint A will sway by $\partial\delta$ and rotate by $\partial\theta_A$. The end moments are:-

Operation	M_{BA}	M_{AB}
1) Rot. A	$k\ \bar{s}\bar{c}\,\partial\theta_A$	$k\ s_1\partial\theta_A$
2) Sway $\partial\delta$	$-\ k\alpha_2\partial\delta/L$	$-\ k\alpha_1\partial\delta/L$

Joint equilibrium at A requires that:-

$$k\ s_1\partial\theta_A - k\ \alpha_1\partial\delta/L = 0 \qquad 15.17a$$

The horizontal force H_A at A is resisted by the moments in AB and this force vanishes at the critical load, hence

$$H_A = -\ (k\alpha/L)\partial\theta_A + (kQ/L^2)\partial\delta \qquad 15.17b$$

The determinant of the coefficients of the unknowns is

$$= (k/L)^2(\ s_1Q - \alpha_1^2) \qquad 15.18$$

Any load making this determinant vanish is the critical load. Equation 15.18 can be solved to yield a relationship between μ and the external load. A typical calculation is shown for $\mu = 0$.

<u>First trial $\rho = 0.4$</u>

From table $\mu = 0$

$s_1 = 3.7177 \quad \alpha_1 = 5.1699 \quad Q = 7.1666$

Substituting these values in 15.18 gives:-

$\frac{L^2\Delta}{k^2} = 26.6 - 26.8$

$= -0.2$

i.e the strut is unstable and a lower value of ρ is tested.

<u>Second trial ρ =0.38</u>

From table $\mu=0$

$s_1 = 3.7328 \quad \alpha_1 = 5.2135 \qquad Q = 7.4132$

$\frac{L^2\Delta}{k^2} = 27.7 - 27.2$

$= + 0.5$

i.e the strut is stable.

The critical ρ by linear interpolation is 0.394.

By definition

$\mu = P_1/P_2$

$P = (P_1+P_2)/2$

$= \rho P_e$

From these definitions it can be shown that

$$P_2 = \frac{2\rho P_e}{1+\mu} \qquad 15.19$$

Thus the total load carried by the strut is

$P_2 = 2 \times 0.394\, P_e$

$= 0.788 P_e$

For other values of μ, the critical ρ-values and P_2 are tabulated in Table 15.1

μ	ρ	P_2/P_e
0	0.394	0.788
0.1	0.360	0.655
0.2	0.335	0.558
0.3	0.315	0.485
0.4	0.300	0.429
1.0	0.250	0.250

Table 15.1

(ii) A strut with both ends pin-jointed

The stability condition is

$$s_1'' = s_2'' = 0 \qquad 15.20$$

This equation can be used to establish the relationship between u and the external load P_2. The critical ρ values and P_2 obtained are tabulated in Table 15.2 for different values of μ.

μ	ρ	P_2/P_e
0	0.941	1.813
0.1	0.958	1.740
0.2	0.972	1.620
0.3	0.981	1.510
0.4	0.987	1.400
1.0	1.000	1.000

Table 15.2

(iii) A strut with one end pin-jointed and the other fixed

There are two cases, the first is when the end with the bigger axial force is pin- jointed . The stability condition for this case is

$$s_2 = 0 \qquad 15.21a$$

Using the stability tables, the critical ρ-values and P_2 obtained by satisfying 15.21a for different values of μ are tabulated in Table 15.3.

μ	ρ	P_2/P_e
0	1.520	3.040
0.1	1.600	2.910
0.2	1.671	2.790
0.3	1.736	2.670
0.4	1.794	2.540
1.0	2.046	2.046

Table 15.3

The other case is when the end with the smaller axial force is pin-jointed. The stability condition is then

$$s_1 = 0 \qquad 15.21b$$

For various values of μ, the critical ρ-values and P_2, obtained from 15.21b are tabulated in Table 15.4

μ	ρ	P_2/P_e
0	2.660	5.319
0.1		
0.2	2.478	4.120
0.3	2.397	3.680
0.4	2.325	3.320
1.0	2.046	2.046

Table 15.4

The numerical results given in the tables are shown in Figure 15.6. For any value of μ other than those given, the critical load can be obtained from Figure 15.6.

2. <u>Frame containing members with variable axial forces</u>

The elastic instability of the frame in Figure 15.7 is analyzed when the external load 2P is distributed uniformly along the members AB and A'B'. On the application of the external load 2P, the two members will deflect but the joint B will not rotate because of symmetry. The vertical reactions at A and A' are P and at joint B there is only a horizontal force component H. This is resolved into two components along and perpendicular to the members. The perpendicular component $H\sin\phi$ necessary to keep each member in equilibrium can be obtained as:-

$$H \sin\phi = qP \cos\phi$$

whence

$$H/P = q \cot\phi \qquad 15.22$$

When the angle ϕ is specified, equation 15.22 can be solved numerically to yield a relationship between H and P.

In equation 15.22, q is known for certain values of μ. For intermediate values of μ, q can be obtained graphically. Figure 15.8 shows the curves relating q and μ for different values of ρ.

The stability condition for the anti-symmetrical joint rotation mode is

$2k\,s_1 = 0$ 15.23

Any load satisfying equation 15.22 and 15.23 will be the critical load of the structure.

Numerical example

The elastic critical load of the structure will be determined when $\phi = 45^\circ$. Equation 15.22 becomes

$H/P = q$

The axial force in AB at B is	0.707H
The axial force in AB at A is	0.707(H+P)
The mean axial force in AB is	0.707(H +0.5P)

$\rho\,P_e$

Hence the external load is $$2P = \frac{2.828\,\rho}{0.5 + H/P}\,P_e$$

and

$$\mu = \frac{0.707\,H}{0.707(H+P)}$$

Substituting for H/P, u becomes

$$\mu = \frac{q}{1+q}$$

For different values of ρ, the tabulated q-values are plotted against μ. Figure 15.8 shows the curves obtained by taking ρ =1.0, 2.0, 2.5, and 3.0.

Ignoring the effect of stability i.e q = 0.5, gives

$$H/P = 0.5$$

and

$$\mu = \frac{0.5}{1 + 0.5}$$

$$= 0.333$$

From Figure 15.8, the q-value corresponding to ρ =1.0 and μ=0.333 is 0.5156. Thus

$$\mu = \frac{0.5156}{1+0.5156}$$

$$= 0.342$$

q is corrected again. When ρ =1.0 and μ=0.342, q is 0.515. Thus

$$\mu = \frac{0.515}{1+0.515}$$

$$= 0.340$$

Thus the value of H/P corresponding to ρ =1.0 is 0.515, and the external load is

$$2P = \frac{2.828 \times 1.0}{0.5+ 0.515} P_e$$

$$= 2.78 \, P_e$$

Curve A in Figure 15.9 shows the numerical results obtained when ρ =2.0, 2.5, and 3.0.

Equation 15.23 can be also solved to yield a relationship between H and P. But this time it can be solved using the tabulated stability functions directly. For example, when $\mu=0.2$, the load parameter ρ whose s_1 is zero is 2.478. Thus the H/P-value and the external load are

$$H/P = \frac{0.2}{1 - 0.2}$$

$$= 0.25$$

and

$$2P = \frac{2.828 \times 2.478}{0.5 + 0.25} P_e$$

$$= 9.33P_e$$

Curve B in Figure 15.9 shows the numerical results obtained when μ = 0, 0.2, 0.3, and 0.4. This curve intersects curve A at $2P = 6.30P_e$ which is therefore the anti-symmetrical critical load. When the external load is lumped at joint B, the elastic critical load is $2P = 2.9P_e$.

3. Frame with variable axial forces and sway

The elastic instability of the frame in Figure 15.10 will be analyzed when the external load 2P is distributed along the inclined members only. On the application of the external load 2P, the joints

B and B' rotate by equal and opposite amounts θ_B. Thus the moments at the ends of the members are:-

Operation	M_{AB}	M_{BA}	$M_{BB'}$
1)Fixed end moments	$-M_{21}$	$+M_{11}$	
2)Rot. B&B'	$(ksc)_1\theta_B$	$(ks_1)_1\theta_B$	$(ks(1-c))_2\theta_B$

Joint equilibrium at B requires

$$\left[(ks_1)_1+(ks(1-c))_2\right]\theta_B + M_{11} = 0 \qquad 15.25a$$

where $M_{11} = m_{11}PL_1\cos\emptyset$

Hence

$$\theta_B = -\frac{m_{11}}{(ks_1)_1+(ks(1-c))_2}PL_1\cos\emptyset \qquad 15.25b$$

At the supports there are vertical force components P and at the joints B and B' there are horizontal force components H only. Member BB' is in equilibrium since there are equal and opposite rotations at the ends of the member. Member AB is in equilibrium under the action of the external load P, the moments and H, hence

$$H\sin\emptyset - q_1 P\cos\emptyset - (k\alpha_1/L)_1\theta_B = 0 \qquad 15.25c$$

Substituting for θ_B and rearranging the equation

$$\frac{H}{P} = \cot\emptyset\left\{q_1 - \frac{\alpha_{11}\,m_{11}}{s_{11}+k_2/k_1.(s(1-c))_2}\right\} \qquad 15.26$$

When $\emptyset$ and k_2/k_1 are specified, equation 15.26 can be solved to yield a relationship between H and P.

Anti-symmetrical sway mode stability criterion

To establish the condition for the anti-symmetrical sway mode, an infinitesimal disturbing horizontal force $2H_B$ is apllied at B which causes the following incremental deformations:-

1) Equal incremental rotations of joints B and B' by $\partial\theta_B$.

2) Equal incremental sways of AB and A'B' by $\partial\delta$.

Member BB' sways by $-2\partial\delta\cos\emptyset$, as obtained from the Williot diagram.

Due to these incremental deformations, the incremental moments at the ends of the members are:-

Operation	M_{AB}	M_{BA}	$M_{BB'}$
1) Rot. B&B'	$(ksc)_1\partial\theta_B$	$(ks_1)_1\partial\theta_B$	$(k\alpha)_2\partial\theta_B$
2) Sway $\partial\delta$	$-(k\alpha_2/L)_1\partial\delta$	$-(k\alpha_1/L)_1\partial\delta$	$+(2k\alpha\cos\emptyset/L)_2\partial\delta$

Equilibrium consideration yields.

$$\begin{bmatrix}\partial M_B \\ H\sin\emptyset\end{bmatrix} = \begin{bmatrix}\partial\theta_B \\ \partial\delta\end{bmatrix} \begin{bmatrix}(ks_1)_1+(k\alpha)_2 & (2k\alpha\cos\emptyset/L)_2-(k\alpha_1/L)_1 \\ (2k\alpha\cos\emptyset/L)_2-(k\alpha_1/L)_1 & (4kAc\overset{2}{o}s\emptyset/L^2)_2+(kQ/L^2)_1\end{bmatrix}$$

$$= \begin{bmatrix}0 \\ 0\end{bmatrix} \qquad 15.27$$

Any load satisfying 15.26 and making the determinant of the square matrix vanish is the critical load.

Numerical example

The elastic critical load of the frame will be estimated when $k_2=k_1$, $L_2=L_1$ and $\phi=45°$.

The axial force in AB at B is $0.707\ H$

The axial force in AB at A is $0.707(H+P)$

Thus

$$\mu = \frac{H/P}{1 + H/P} \qquad 15.28a$$

Equation 15.28a can be rearranged to give H/P in term μ as

$$H/P = \frac{\mu}{1 - \mu} \qquad 15.28b$$

The mean axial force in AB is $R = 0.707(H/P + 0.5)\ P$

$$= \rho P_e$$

and the external load is

$$2P = \frac{2.828\ \rho_1}{0.5 + H/P} P_e \qquad 15.28c$$

The load parameter of AB is $\rho_1 = R/P_e$

and that of BB' is $\rho_2 = H/P_e$

Hence

$$\frac{\rho_2}{\rho_1} = \frac{1.414\ H/P}{0.5 + H/P} \qquad 15.28d$$

When the values of k and ϕ are substituted in 15.26, the equation becomes

$$\frac{H}{P} = q_1 - \frac{\alpha_{11} \quad m_{11}}{s_{11}+(s(1-c))_2} \qquad 15.28e$$

The ratio H/P when the effects of axial forces in the members is ignored is:-

$$H/P = 0.5 - \frac{6 \times 1/12}{6}$$

$$= 0.4167$$

From equations 15.28a and 15.28d

$$\mu = \frac{0.4167}{1.4167}$$

$$= 0.294$$

and

$$\frac{\rho_2}{\rho_1} = \frac{1.414 \times 0.4167}{0.9167}$$

$$= 0.642$$

A value of ρ_1 is assumed and its tabulated stability functions s_{11}, α_{11} and q_1 are plotted against μ as shown in Figure 15.11. This gives the values of the stability functions at values of μ other than those tabulated in the stability tables.

From Figure 15.11, the stability functions of ρ_1=1.0 corresponds to μ=0.294 are

$s_{11} = 2.87$ $\alpha_{11} = 4.23$ $q_1 = 0.5165$ and $m_{11} = 0.0996$

and from the stability tables (9), the stability functions of ρ_2=0.642 are

$s_2 = 3.0771$ $(sc)_2 = 2.2605$

Substituting these values in 15.28e, gives

$$H/P = 0.5165 - \frac{4.23 \times 0.0996}{3.682}$$

$$= 0.4022$$

and $\mu = 0.286$

The difference in the μ-value is small and the difference in the stability functions will also be small. Thus the external load is

$$2P = \frac{2.828 \times 1.0}{0.5+0.4022} P_e$$

$$= 3.14\ P_e$$

Following the previous procedure, the H/P-value and the elastic critical load is calculated when $\rho_1 = 1.40$. The results are shown in Figure 15.12, curve A.

Equation 15.27 can also be solved to yield a relationship between H and P. This time, the tabulated stability functions are used directly in the solution of equation 15.27. For example, when $\mu = 0.2$, equation 15.28b gives

$$H/P = \frac{0.2}{1 - 0.2}$$

$$= 0.25$$

and equation 15.28d gives

$$\frac{\rho_2}{\rho_1} = \frac{1.414 \times 0.25}{0.5+ 0.25}$$

$$= 0.47$$

When the k-values and the lengths are substituted in 15.27, the determinant becomes

$$\begin{vmatrix} s_{11} + \alpha_2 & 1.414\alpha_2 - \alpha_{11} \\ 1.414\alpha_2 - \alpha_{11} & 2A_2 + Q_1 \end{vmatrix} = 0 \qquad 15.28f$$

The load parameter ρ_1 satisfying 15.28f is obtained by a trial and error process.

<u>First trial $\rho_1 = 1.24$</u>

$\rho_1 = 1.24$ $\qquad \rho_2 = 0.583$

From table $\mu = 0.2$

$s_{11} = 2.6482$ $\qquad \alpha_{11} = 3.5847$ $\qquad Q_1 = -3.3209$

From tables (9)

$\alpha_2 = 5.4026$ $\qquad A_2 = 2.5404$

Substituting these values in 15.28f

$$\Delta = \begin{vmatrix} 8.0508 & 4.0653 \\ 4.0653 & 1.7599 \end{vmatrix}$$

$= -2.4$

A lower value of is therefore tested.

When $\rho_1 = 1.20$ $\qquad \Delta = +3.8$

ρ_1 by linear interpolation is 1.225 and the critical load is

$$2P = \frac{2.828 \times 1.225}{0.5 + 0.25} P_e$$

$= 4.62\ P_e$

Curve B in Figure 15.12, shows the numerical results obtained when $\mu = 0$, 0.2, 0.3 and 0.4. This curve intersects curve A at $2P = 3.60\ P_e$. The carrying capacity of this frame is $1.6P_e$ when the external load is lumped at the joints B and B'.

APPENDIX

Solution of the basic equations

Equation 15.2

$$EI\frac{d^2y}{dx^2}+Py=-M_F-\frac{wLx}{2}+\frac{wx^2}{2}$$

The gerenal solution of this equation is

$$y=A\sin\varpi x+B\cos\varpi x-\left(\frac{M_F}{P}+\frac{wEI}{P^2}\right)-\frac{wLx}{2P}+\frac{wx^2}{2P} \qquad 15.29a$$

where A and B are constants.

The first derivative of equation 15.29a is

$$\frac{dy}{dx}=A\varpi\cos\varpi x-\varpi B\sin\varpi x-\frac{wL}{2P}+\frac{wx}{P} \qquad 15.29b$$

When x=0, y=0, equation 15.29a gives

$$B=\frac{M_F}{P}+\frac{wEI}{P^2} \qquad 15.29c$$

Also when x=L, y=0, equation 15.29a gives

$$0=A\sin\psi+B\cos\psi-\left(\frac{M_F}{P}+\frac{wEI}{P^2}\right) \qquad 15.29d$$

Substituting for B, equation 15.29a gives

$$A=\frac{1}{\sin\psi}(1-\cos\psi)\left(\frac{M_F}{P}+\frac{wEI}{P^2}\right) \qquad 15.29e$$

where

$$\psi=\varpi L \quad \text{and} \quad \varpi^2=\frac{P}{EI}$$

When x=0, $\frac{dy}{dx}=0$. Substituting these values in 15.29b gives

$$0=\varpi A-\frac{wL}{2P} \qquad 15.29f$$

Substituting for A, equation 15.29f becomes

$$\frac{\varpi}{\sin\psi}(1-\cos\psi)\left(\frac{M_F}{P}+\frac{wEI}{P^2}\right)=\frac{wL}{2P} \qquad 15.29g$$

Rearranging equation 15.29g and substituting for and gives

$$M_F=\frac{wL^2}{\pi^2\frac{P}{P_e}}\left(1-\frac{\pi}{2}\sqrt{\frac{P}{P_e}}\cot\pi\sqrt{\frac{P}{P_e}}\right) \qquad 15.29h$$

By definition (9)

$$s = \frac{(1-2\alpha \cot 2\alpha)\alpha}{\tan\alpha - \alpha}$$

and

$$c = \frac{2\alpha - \sin 2\alpha}{\sin 2\alpha - 2\alpha \cos 2\alpha}$$

where

$$\alpha = \frac{\pi}{2}\sqrt{\frac{P}{P_e}}$$

Using these definitions, it can be shown that

$$s(1 - c) = \pi\sqrt{\frac{P}{P_e}} \cot \pi\sqrt{\frac{P}{P_e}} \qquad 15.29i$$

Substituting 15.29i in 15.29h and using the definition $\pi^2 \frac{P}{P_e} = 2B$, gives

$$M_F = \frac{1}{2B}(1 - s(1-c)/2)\, wL^2 \qquad 15.30$$

<u>Equation 15.8</u>

$$EI\frac{d^3y}{dx^3} + P(x)\frac{dy}{dx} = -V$$

The point of zero axial force will be taken as the origin of the coordinates x,y. Let this point to be at distance a from end 1 and b from end 2. Then the axial force at any point along the strut can be expressed as

$P(x) = P_1(x/a) = P_2(x/b)$

where P_1 and P_2 are the axial forces at end 1 and end 2 respectively. Thus equation 15.8 can be written as

$$EI\frac{d^3y}{dx^3} + P_1\left(\frac{x}{a}\right)\frac{dy}{dx} = -V \qquad 15.31a$$

Put $\frac{dy}{dx} = z$ and $\varpi^3 = \frac{P_1}{EIa}$, equation 15.31a becomes

$$\frac{d^2z}{dx^2} + \varpi^3 x z = -\frac{V}{EI} \qquad 15.31b$$

The complementary solutions of this equation can be shown to be function of Bessel functions of order $\frac{1}{3}$ and $-\frac{1}{3}$, but the solution of the standard equation (30,31,32)

$$\frac{d^2y}{dx_1^2} - x_1 y = f(x_1) \qquad 15.31c$$

is known in term of Airy integral functions. Equation 15.31b is transformed to the standard equation 15.31c by putting

$$x_1 = -\varpi x \qquad 15.31d$$

Equation 15.31b becomes

$$\frac{d^2z}{dx_1^2} - x_1 z = D \qquad 15.31e$$

where

$$D = -V/EI\varpi^2$$

The solution of equation 15.31e when D is constant is

$$z = \alpha A_i(x_1) + \beta B_i(x_1) - \pi D G_i(x_1) \qquad 15.31f$$

where α and β are constants and $A_i(x_1)$, $B_i(x_1)$, $G_i(x_1)$ are Airy Integrals

By definition

$$z = \frac{dy}{dx} = \frac{dy}{dx_1}\frac{dx_1}{dx} = -\varpi \frac{dy}{dx_1} \qquad 15.31g$$

Thus equation 15.31f becomes

$$-\varpi \frac{dy}{dx_1} = \alpha A_i(x_1) + \beta B_i(x_1) - \pi D G_i(x_1) \qquad 15.31h$$

The integral of equation 15.31h is

$$-\varpi y = \alpha \int^{x_1} A_i(t)dt + \beta \int^{x_1} B_i(t)dt - \pi D \int^{x_1} G_i(t)dt + \gamma \qquad 15.31i$$

and the first derivative of equation 15.31h is

$$-\varpi \frac{d^2y}{dx_1^2} = \alpha A_i'(x_1) + \beta B_i'(x_1) - \pi D G_i'(x_1) \qquad 15.31j$$

Derivation of s_1, s_2, $\overline{sc}$, q_1, and q_2

When x=a, $x_1 = -\varpi a = \bar{a}$ and $\frac{dy}{dx_1} = \frac{dy}{dx}\frac{dx}{dx_1} = -\frac{1}{\varpi}\theta_1$. Substituting these values in 15.31h gives

$$+\theta_1 = \bar{\alpha} A_i(\bar{a}) + \beta B_i(\bar{a}) - \pi D G_i(\bar{a}) \qquad 15.32a$$

When x=b, $x_1 = -\varpi b = \bar{b}$ and $\frac{dy}{dx_1} = -\frac{1}{\varpi}\theta_2$. Substituting these values in 15.31h gives

$$+\theta_2 = \bar{\alpha} A_i(\bar{b}) + \beta B_i(\bar{b}) - \pi D G_i(\bar{b}) \qquad 15.32b$$

Solution of equations 15.32a and 15.32b yields

$$\bar{\alpha} = \frac{1}{\mathcal{L}_1}\left[B_i(\bar{b})\theta_1 - B_i(\bar{a})\theta_2 + \pi \mathcal{L}_2 D\right] \qquad 15.32c$$

and

$$\beta = \frac{1}{\mathcal{L}_1}\left[-A_i(\bar{b})\theta_1 + A_i(\bar{a})\theta_2 + \pi \mathcal{L}_3 D\right] \qquad 15.32d$$

where

$$\mathcal{L}_1 = \begin{vmatrix} A_i(\bar{a}) & B_i(\bar{a}) \\ A_i(\bar{b}) & B_i(\bar{b}) \end{vmatrix}$$

$$\mathcal{L}_2 = \begin{vmatrix} G_i(\bar{a}) & B_i(\bar{a}) \\ G_i(\bar{b}) & B_i(\bar{b}) \end{vmatrix}$$

and

$$\mathcal{L}_3 = \begin{vmatrix} A_i(\bar{a}) & G_i(\bar{a}) \\ A_i(\bar{b}) & G_i(\bar{b}) \end{vmatrix}$$

When x=a, $x_1 = \bar{a}$ and y = 0. Substituting these values in 15.31i gives

$$0 = \bar{\alpha}\int^{\bar{a}} A_i(t)\,dt + \beta\int^{\bar{a}} B_i(t)\,dt - \pi D\int^{\bar{a}} G_i(t)\,dt + \gamma \qquad 15.32e$$

Also when x=b, $x = \bar{b}$ and y=0. Substituting these values in 15.31i gives

$$0=\bar{\alpha}\int^{\bar{b}} A_i(t)\,dt + \beta\int^{\bar{b}} B_i(t)\,dt - \pi D\int^{\bar{b}} G_i(t)\,dt + \gamma \qquad 15.32f$$

Subtracting 15.32f from 15.32e, to eliminate γ gives the resulting equation.

$$\bar{\alpha}\,\psi_4 + \beta\,\psi_5 - \pi D\,\psi_6 = 0 \qquad 15.32g$$

where

$$\int^{\bar{a}} A_i(t)\,dt - \int^{\bar{b}} A_i(t)\,dt = \psi_4$$
$$\int^{\bar{a}} B_i(t)\,dt - \int^{\bar{b}} B_i(t)\,dt = \psi_5$$
$$\int^{\bar{a}} G_i(t)\,dt - \int^{\bar{b}} G_i(t)\,dt = \psi_6$$

Substituting equation 15.32c and 15.32d in 15.32g gives

$$\pi D = \Phi_1\,\theta_1 + \Phi_2\,\theta_2 \qquad 15.32h$$

where

$$\Phi_1 = \frac{\psi_4 B_i(\bar{b}) - A_i(\bar{b})\,\psi_5}{\psi_1\psi_6 - (\psi_2\psi_4 + \psi_3\psi_5)}$$

and

$$\Phi_2 = \frac{A_i(\bar{a})\,\psi_5 - B_i(\bar{a})\,\psi_4}{\psi_1\psi_6 - (\psi_2\psi_4 + \psi_3\psi_5)}$$

When x=a, x = $\bar{a}$ and $\left(\frac{d^2y}{dx^2}\right)_1 = -\frac{M_1}{EI}$ i.e $\frac{d^2y}{dx_1^2} = -\frac{1}{\varpi^2}\frac{M_1}{EI}$. Substituting these values in equation 15.31j gives

$$\frac{M_1}{EI\varpi} = \bar{\alpha}\,A_i'(\bar{a}) + \beta\,B_i'(\bar{a}) - \pi D\,G_i'(\bar{a}) \qquad 15.32i$$

Substituting equations 15.32c, 15.32d and 15.32h in 15.32i gives

$$\frac{M_1}{EI} = \frac{\varpi}{\psi_j}\left\{(\psi_7 + \psi_8\Phi_1)\,\theta_1 + \left(\frac{1}{\pi} + \psi_8\Phi_8\right)\theta_2\right\} \qquad 15.32j$$

where

$$A_i(\bar{a})\,B_i'(\bar{a}) - A_i'(\bar{a})\,B_i(\bar{a}) = \frac{1}{\pi} \quad \text{Identity}$$

$$\psi_7 = B_i(\bar{b})\,A_i'(\bar{a}) - A_i(\bar{b})\,B_i'(\bar{a})$$

and

$$\psi_8 = \psi_2\,A_i'(\bar{a}) + \psi_3\,B_i'(\bar{a}) - \psi_1\,G_i'(\bar{a})$$

Also when x=b, $x_1=\bar{b}$ and $\left(\frac{d^2y}{dx^2}\right)_2=+\frac{M_2}{EI}$ or $\left(\frac{d^2y}{dx_1^2}\right)_2=+\frac{1}{\varpi^2}\frac{M_2}{EI}$. Substituting these values in equation 15.31j gives

$$-\frac{M_2}{EI}\frac{1}{\varpi}=\bar{\alpha}A_i'(\bar{b})+\beta B_i'(\bar{b})-\pi D G_i'(\bar{b}) \qquad 15.32k$$

Substituting equations 15.32c, 15.32d and 15.32h in 15.32k gives

$$-\frac{M_2}{EI}=\frac{\varpi}{\psi_1}\left\{\left(-\frac{1}{\pi}+\psi_{10}\Phi_1\right)\theta_1+\left(\psi_9+\psi_{10}\Phi_2\right)\theta_2\right\} \qquad 15.32L$$

where

$$B_i(\bar{b})A_i'(\bar{b})-A_i(\bar{b})B_i(\bar{b})=-\frac{1}{\pi} \qquad \text{(identity)}$$

$$\psi_9=A_i(\bar{a})B_i'(\bar{b})-A_i'(\bar{b})B_i(\bar{a})$$

and

$$\psi_{10}=\psi_2 A_i'(\bar{b})+\psi_3 B_i'(\bar{b})-\psi_1 G_i'(\bar{b})$$

If we define

$$M_1=\frac{EI}{L}(s_1\theta_1+\bar{sc}\,\theta_2) \qquad 15.32m$$

$$M_2=\frac{EI}{L}(\bar{sc}\,\theta_1+s_2\theta_2) \qquad 15.32n$$

$$v=\frac{EI}{L^2}(q_1\theta_1+q_1\theta_2) \qquad 15.32o$$

Comparsion of equations 15.32j and 15.32m; 15.32k and 15.32n; and 15.32h and 15.32o yields

$$s_1=\frac{\varpi L}{\psi_1}(\psi_7+\psi_8\Phi_1)$$

$$\bar{sc}=\frac{\varpi L}{\psi_1}\left(\frac{1}{\pi}+\psi_8\Phi_2\right)$$

$$=\frac{\varpi L}{\psi_1}\left(\frac{1}{\pi}-\psi_{10}\Phi_1\right) \qquad 15.33$$

$$s_2=-\frac{\varpi L}{\psi_1}(\psi_9+\psi_{10}\Phi_2)$$

$$q_1=\frac{(\varpi L)^2}{\pi}\Phi_1$$

$$q_1=\frac{(\varpi L)^2}{\pi}\Phi_2$$

Derivation of the sway factors α_1, α_2, and Q

The basic equation, its solution 15.31h, integral 15.31i and its first derivative 1".31j are the same as before but the boundary conditions are different.

When x=a, $x_1=\bar{a}$ and $\frac{dy}{dx_1} = \theta_1 = 0$, also

when x=b, $x = \bar{b}$ and $\frac{dy}{dx} = \theta_2 = 0$.

Th"s putting $\theta_1 = \theta_2 = 0$ in equations 15.32c, and 15.32d, the new values of $\bar{\alpha}$ and β are

$$\bar{\alpha} = \frac{\psi_2}{\psi_1} \pi D \qquad 15.34a$$

and

$$\beta = \frac{\psi_3}{\psi_1} \pi D \qquad 15.34b$$

When x=a, $x_1 = \bar{a}$ and y =0. Substit"ting in 15.31i gives

$$0 = \bar{\alpha} \int^{\bar{a}} A_i(t)dt + \beta \int^{\bar{a}} B_i(t)dt - \pi D \int^{\bar{a}} G_i(t)\,dt + \gamma \qquad 15.34c$$

When x=b, $x_1 = \bar{b}$ and y = δ. Substituting in 15.31i gives

$$-\delta\varpi = \bar{\alpha} \int^{\bar{b}} A_i(t)dt + \beta \int^{\bar{b}} B_i(t)dt - \pi D \int^{\bar{b}} G_i(t)dt + \gamma \qquad 15.34d$$

Subtracting 15.34d from 15.34c to eliminate γ yields

$$\delta\varpi = \bar{\alpha}\,\psi_4 + \beta\,\psi_5 - \pi D\,\psi_6 \qquad 15.34e$$

Substituting equations 15.34a and 15.34b in 15.34e yields

$$\pi D = -\frac{\varpi L\,\psi_1}{\psi_1\psi_6 - (\psi_2\psi_4 + \psi_3\psi_5)} \qquad 15.34f$$

When x=a, x_1=a and $\left(\frac{d^2y}{dx_1^2}\right) = -\frac{1}{\varpi^2}\frac{M_1}{EI}$. Substituting these values in equation 15.31j and using 15.34a, 15.34b and 15.34f, the following relationship is obtained

$$\frac{M_1}{EI} = -\varpi^2 L \frac{\psi_8}{\psi_1\psi_6 - (\psi_2\psi_4 + \psi_3\psi_5)} \frac{\delta}{L} \qquad 15.34g$$

When x=b, $x = \bar{b}$ and $\left(\frac{d^2y}{dx_1^2}\right) = +\frac{1}{\varpi^2}\frac{M_2}{EI}$. Substituting these values in equation 15.31j and using 15.34a, 15.34b and 15.34f, the following relationship is obtained

$$-\frac{M_2}{EI} = -\varpi^2 L \frac{\psi_{10}}{\psi_1\psi_6 - (\psi_2\psi_4 + \psi_3\psi_5)} \frac{\delta}{L} \qquad 15.34h$$

If we define

$$M_1 = -\frac{EI}{L}\alpha_1\frac{\delta}{L} \qquad 15.34i$$

$$M_2 = -\frac{EI}{L}\alpha_2\frac{\delta}{L} \qquad 15.34j$$

$$v = -\frac{EI}{L^2}Q\frac{\delta}{L} \qquad 15.34k$$

Comparasion of 15.34i and 15.34g; 15.34j and 15.34h and 15.34k and 15.34f yields

$$\alpha_1 = +(\varpi L)^2 \frac{\psi_8}{\psi_1\psi_6 - (\psi_2\psi_4 + \psi_3\psi_5)}$$

$$\alpha_2 = -(\varpi L)^2 \frac{\psi_{10}}{\psi_1\psi_6 - (\psi_2\psi_4 + \psi_3\psi_5)} \qquad 15.35$$

$$Q = +\frac{(\varpi L)^3}{\pi} \frac{\psi_1}{\psi_1\psi_6 - (\psi_2\psi_4 + \psi_3\psi_5)}$$

Derivation of m_1, m_2, and q

The basic equation 15.8 is the same but this time the shear force is dependent on x since there is a distributed load w. The shear force at any section is D + (x-a) w where D is the reaction at end 1 due to the distributed load and the deformation in the strut. The basic equation can be written as

$$EI\frac{d^3y}{dx^3}+P_1\left(\frac{x}{a}\right)\frac{dy}{dx}=D+w(x-a) \qquad 15.36a$$

Using the previous definitions equation 15.36a is transformed to

$$\frac{d^2z}{dx_1^2}-x_1z=D'+\frac{wa}{P_1}x_1 \qquad 15.36b$$

where

$$D'=\frac{D-wa}{EI\varpi^2}$$

The solution of equation 15.36b is

$$z=\bar{\alpha}A_i(x_1)+\beta B_i(x_1)-\pi D'G_i(x_1)+\frac{wa}{P_1} \qquad 15.36c$$

The integral of equation 15.36c is

$$-\varpi y=\bar{\alpha}\int^{x_1}A_i(t)dt+\beta\int^{x_1}B_i(t)dt-\pi D'\int^{x_1}G_i(t)dt+\frac{wa}{P_1}x_1+\gamma \qquad 15.36d$$

and the first derivative of equation 15.36c is

$$-\varpi\frac{d^2y}{dx_1^2}=\bar{\alpha}A_i'(x_1)+\beta B_i'(x_1)-\pi D'G_i'(x_1) \qquad 15.36e$$

When x=a, $x_1=\bar{a}$ and $\frac{dy}{dx}$ = 0, i.e z=0. Substituting these values in equation 15.36c yields

$$0=\bar{\alpha}A_i(\bar{a})+\beta B_i(\bar{a})-\pi D'G_i(\bar{a})+\frac{aw}{P_1} \qquad 15.36f$$

Also when x=b, $x_1=\bar{b}$ and z=0. Substituting these values in equaion 15.36c yields

$$0=\bar{\alpha}A_i(\bar{b})+\beta B_i(\bar{b})-\pi D'G_i(\bar{b})+\frac{aw}{P_1} \qquad 15.36g$$

Solution of equations 15.36f and 15.36g yields

$$\alpha=\frac{1}{\Psi_1}\left\{-\frac{aw}{P_1}(B_i(\bar{b})-B_i(\bar{a})+\pi D'\Psi_2)\right\} \qquad 15.36h$$

and

$$\beta=\frac{1}{\Psi_1}\left\{-\frac{aw}{P_1}(A_i(\bar{a})-A_i(\bar{b})+\pi D'\Psi_3)\right\} \qquad 15.36i$$

When x=a, $x_1=\bar{a}$ and y =0. Substituting these values in 15.36d yields

$$0=\bar{\alpha}\int^{\bar{a}}A_i(t)dt+\beta\int^{\bar{a}}B_i(t)dt-\pi D'\int^{\bar{a}}G_i(t)dt+\frac{aw}{P_1}\bar{a}+\gamma \qquad 15.36j$$

Also when x=b, $x_1=\bar{b}$ and y =0. Substituting these values in 15.36d gives

$$0=\alpha\int^{\bar{b}} A_i(t)dt+\beta\int^{\bar{b}} B_i(t)dt-\pi D'\int^{\bar{b}} G_i(t)dt+\frac{aw}{P_1}\bar{b}+\gamma \qquad 15.36k$$

Subtracting 15.36k from 15.36k and substituting for α and β yields

$$\pi D'=\left[\phi_1+\phi_2-\frac{\psi_1(\bar{a}-\bar{b})}{\psi_1\psi_6-(\psi_2\psi_4+\psi_3\psi_5)}\right]\left[-\frac{aw}{P_1}\right] \qquad 15.36L$$

Equation 15.36L can be further reduced using 15.33 and 15.35 to

$$D=\left\{1-\frac{EI}{L^2P_1}(q_1+q_2-Q)\right\}\left\{aw\right\} \qquad 15.36L'$$

When x=a, $x_1=\bar{a}$ and $\left(\frac{d^2y}{dx_1^2}\right)=-\frac{1}{\varpi^2}\frac{M_1}{EI}$. Substituting these values in 15.36e and substituting for α, β and $\pi D'$ yields

$$\frac{M_1}{EI}=-\frac{\varpi}{\psi_1}\left[\psi_7+\frac{1}{\pi}+\psi_8\phi_1+\psi_8\phi_2-\frac{\psi_1\psi_8(\bar{a}-\bar{b})}{\psi_1\psi_6-(\psi_2\psi_4+\psi_3\psi_5)}\right]\frac{aw}{P_1} \qquad 15.36m$$

Equation 15.36m can be further reduced using 15.33 and 15.35 to

$$\frac{M_1}{EI}=-\left[S_1+\overline{SC}-\alpha_1\right]\frac{aw}{P_1} \qquad 15.36m'$$

Also when x=b, $x_1=\bar{b}$ and $\left(\frac{d^2y}{dx_1^2}\right)=+\frac{1}{\varpi^2}\frac{M_2}{EI}$. Substituting these values in 15.36e and using 15.36h, 15.36i, 15.36L, 15.33 and 15.35, gives

$$\frac{M_2}{EI}=\left[S_2+\overline{SC}-\alpha_2\right]\frac{a\omega}{P_1} \qquad 15.36n$$

If we define

$\mu = P_1/P_2 = a/b$

The mean value of axial force is $P = (P_1+P_2)/2$

which gives $P_1 = \frac{2\mu}{1+\mu}P$ 15.36o

The length of the strut is $L = b - a$

Hence $a = \frac{\mu}{1-\mu}L$ 15.36p

$M_1 = m_1wL^2$ 15.36q

$M_2 = m_2 w L^2$ 15.36r

$v = q\ wL$ 15.36s

Comparsion of 15.36m' and 15.36q; 15.36n and 15.36s and 15.36L' and 15.36t using 15.36o and 15.36p yields

$$m_1 = -\frac{1+\mu}{2(1-\mu)} \frac{1}{P/P_e}(s_1 + \overline{sc} - \alpha_1)$$

$$m_2 = -\frac{1+\mu}{2(1-\mu)} \frac{1}{P/P_e}(s_2 + \overline{sc} - \alpha_2)$$

$$q = \left[\frac{\mu}{1-\mu} - \frac{1+\mu}{1-\mu} \frac{1}{2\pi^2 \frac{P}{P_e}}(q_1 + q_2 - Q)\right] \qquad 15.37$$

It is convenient to tabulate the stability functions in term P/P_e. Hence the parameters $\bar{a}$, $\bar{b}$ and ϖL as expressed in terms P/P_e and u are

$$\bar{a} = -\left[\frac{2\pi^2}{(1-\mu)^2(1+\mu)} \frac{P}{P_e}\right]^{\frac{1}{3}} . u$$

$$\bar{b} = \bar{a}/\mu$$

$$\varpi L = \frac{1-\mu}{\mu}\bar{a}$$

Modified stability functions

The factors s_1'', s_2'', q_1'' and q_2'' are those of the strut pinned at one end. These values can be obtained as functions of the stability functions which have already been obtained. When joint 2 is pin-jointed and a moment M_1 is applied at 1, joint 1 will rotate clockwise through an angle θ_1 and joint 2 will rotate clockwise through an angle θ_2 to balance the moment ksc θ_1 appearing at 2. Hence

$$k\ \overline{sc}\ \theta_1 + k\ s_2\theta_2 = 0$$

which gives

$$\theta_2 = - \frac{\overline{sc}}{s_2} \theta_1 \qquad 15.38a$$

The moment at end 1 is

$$M_1 = k \left(s_1\theta_1 + \overline{sc}\ \theta_2\right)$$

Substituting for θ_2 yields

$$M_1 = k \frac{s_1 s_2 - (\overline{sc})^2}{s_2} \theta_1 \qquad 15.38b$$

If we define

$$M_1 = k\ s_1''\theta_1 \text{, then } s_1'' = \frac{s_1 s_2 - (\overline{sc})^2}{s_2} \qquad 15.38c$$

The shear force required to keep the pin-jointed strut in equilibrium is

$$\frac{k}{L}\left[q_1\theta_1 + q_2\theta_2\right] = \frac{k}{L}\left[q_1 - q_2 \frac{(\overline{sc})^2}{s_2}\right]\theta_1 \qquad 15.38d$$

If we define the shear force by

$$\frac{k}{L} q_1'' \theta_1 \qquad 15.38e$$

Comparsion of 15.38d and 15.38e yields

$$q_1'' = \left[q_1 - q_2 \frac{(\overline{sc})^2}{s_2}\right] \qquad 15.38f$$

Following the same procedure, it can be shown that

$$s_2'' = \left[\frac{s_1 s_2 - (\overline{sc})^2}{s_1}\right] \qquad 15.38g$$

and

$$q_2'' = \left[q_2 - q_1 \frac{(\overline{sc})^2}{s_1}\right] \qquad 15.38h$$

Electronic Computer Programming

The stability functions were found to be function of the Airy integral functions, $A_i(t)dt$, $B_i(t)dt$, $G_i(t)dt$, $\int A_i(x)$, $\int B_i(x)$ $\int G_i(x)$, $A_i'(x)$, $B_i'(x)$. $G_i'(x)$. These functions are series and their expressions can be obtained from (31, 33, 34). The number of terms taken in the evaluation of these functions will be dependent on the argument x of the functions. For values of x less than unity, the first few terms in the series will be sufficient to give the values of the functions, but once the x-value exceeds unity, more terms must be taken and for large values of x, the asymptotic expansion of these functions must be used.

On the Liverpool University Electronic Computer, there is a facility for calculating these functions. This necessiated writting a subroutine consisting of a number of first order differential equations equal in number to that of the Airy integral functions. If the values of these functions are given for some starting value x of the argument, then the values of these functions at x +h can be obtained. The h-value was obtained for equal interval of 0.02 of the load parameter P/P_e of the strut.

The functions $A_i(x)$ and $B_i(x)$ are independent solutions of the differential equation

$$y'' = xy$$

Then $A_i(x)$ and its integral and derivative will satisfy

$$A_i''(x) = x\, A_i(x)$$

and likewise for $B_i(x)$ and its integral and derivative satisfy

$$B_i''(x) = x\, B_i(x)$$

The function $G_i(x)$ and its integral and derivative satisfy

$$G_i''(x) - x\, G_i(x) = -\frac{1}{\pi}$$

Then the set of the differential equations will be

$$\frac{d}{dx}\left(\int^x A_i(t)dt \right) = A_i(x)$$

$$\frac{d}{dx}\left(A_i(x) \right) = A_i'(x)$$

$$\frac{d}{dx}\left(A_i'(x) \right) = x\, A_i(x)$$

$$\frac{d}{dx}\left(\int^x B_i(t)dt \right) = B_i(x)$$

$$\frac{d}{dx}\left(B_i(x) \right) = B_i'(x)$$

$$\frac{d}{dx}\left(B_i'(x) \right) = x\, B_i(x)$$

$$\frac{d}{dx}\left(\int^x G_i(t)dt \right) = G_i(x)$$

$$\frac{d}{dx}\left(G_i(x) \right) = G_i'(x)$$

$$\frac{d}{dx}\left(G_i'(x) \right) = x\, G_i(x) - \frac{1}{\pi}$$

Once the subroutine has calculated these functions, the machine will proceed to calculate the stability functions. The expressions for $\overline{sc}$ was given in two forms in 15.33 which have the same numerical value. The programme was made to calculate these values which was taken as a measure of the accuracy of the evaluation. Some of the results obtained are tabulated below and it will be noticed that q_1 and q_2 as calculated on the machine were found to be equal to α_1 and α_2.

	$\mu = 0.2$	
ρ	0.02	0.44
s_1	3.98240602	3.58747105
$\overline{sc}$	2.00662220	2.16775032
$\overline{sc}$	2.00662246	2.16774610
s_2	3.96481301	3.17718846
q_1	5.96703040	5.23746903
q_2	5.99344707	5.87043593
α_1	5.96703053	5.23747479
α_2	5.99344772	5.87043697
Q	11.7630032	6.72105253
m_1	0.0835812285	0.089418234
m_2	0.0836364873	0.0907577145
q	0.500312483	0.507638205

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