Mathematical Platonism and Set-theoretic Indeterminacy

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Abstract

In this work, I will be looking at the issues raised by *set-theoretic indeterminacy* for a Gödelian platonist, who holds that there is a universe of independently existing mathematical objects and that there are objective unique truth-values for any set-theoretic statement. After careful consideration of the philosophical and mathematical issues involved, I claim that Gödelian platonism is untenable. In Chapter 1, I examine different forms of mathematical platonism and I elucidate their features. In particular, I distinguish between a *substantive* form (Gödel's platonism) and an *operational* form (*anti-constructivism*). I also make it clear that I will be concerned with set-theoretic Gödelian platonism. In Chapter 2, I examine the *indeterminacy phenomenon* in set theory through a detailed analysis of the most famous open conjecture, the Continuum Hypothesis (CH). In Chapter 3, I move on to describe the main philosophical orientations with regard to the indeterminacy phenomenon and I show how *model-theoretic relativity* is the main source of trouble for platonism. In Chapter 4, I examine the theoretical ancestry of Gödel's conceptions (which may date back to Cantor's philosophy of the infinite) and Gödel's philosophy of indeterminacy. In Chapter 5 and Chapter 6, I deal with, respectively, Maddy's *set-theoretic naturalism* and *plenitudinous platonism* (in the form presented by Balaguer, FBP), and I raise some objections against these conceptual frameworks. In Chapter 7, I propose abandoning ontological platonism and I defend a mild form of *conceptual realism* resting upon the notion of *non-arbitrary expansions*. Finally, in Chapter 8, I tackle the problem of *insolubility* in contemporary set theory and I advise that operational platonism, *qua* anti-constructivism, as described in Chapter 1, is the only bit of platonism which could be upheld.

Contents

Introduction

This work deals with the ontology and the semantics of contemporary set theory, as interpreted by and seen from the point of view of mathematical platonism. Its main purpose is to prove that mathematical platonism in its most frequently discussed contemporary form, Gödelian Platonism, is affected by thorny ontological issues which make it implausible.

To be more precise, the central issue is the so-called set-theoretic indeterminacy, namely the fact that any description of a universe conceived of as a unique and existent reality satisfying rigid ontological requirements falls short of a satisfactory characterisation. While this does not necessarily represent a threat to the view that set theory is the foundation of mathematics, for a platonist the inability to fix up ontological indeterminacy is not a satisfactory state of affairs.

The present work offers a diagnosis of the problem, an examination of the relations between platonism and set-theoretic ontology and, finally, the tentative proposal to discard (Gödelian) *substantive* platonism while upholding *operational* platonism, insofar as this latter seems to fit the goals of conceptual *set-theoretic objectivism.*

Set-theoretic objectivism aims to address set-theoretic ontology within an interpretive framework which is not based on *objects.* My claim is that while objects and structures play a key role in mathematical intuition, a role widely acknowledged by mathematical platonism, they have to be reconsidered in the light of the study of *concepts.* The work of connecting these three aspects of mathematical intuition represents, in my opinion, the bulk of a satisfactory philosophical description of set-theoretic objectivism.

Whereas emphasis upon the role of objects and structures has been a staple of realism in the philosophy of mathematics (see, in particular, Russell (1903, 1919), Gödel (*1951, 1944, 1947), Hellman (1989), Shapiro (1997), Resnik (1997)), *realist conceptualism* has been paid attention only after reconsideration of Gödel's work, especially in the light of its alleged connections to Husserl's phenomenology. As a matter of fact, there is no *prima facie* need to recover

Husserl's phenomenology to address the role of *conceptualism* within mathematics. Overall, a synthesis operated across different works and contributions can constitute a good historical reference. In particular, the Cantorian synthesis (see Cantor (1883b)), although fragmentary, is a good starting point and represents the historical ancestry of a good deal of later realist and platonist theorisations.

While focus on concepts remains the point of convergence of the different threads of this work, the largest part of it is devoted to the examination of the relations between platonism and set theory, with no attempt forcibly to shove my solution into the philosophical arena.

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When I started working on the problem, I assumed that I should first clarify what mathematical platonism and set-theoretic indeterminacy are. Additionally, I should precisely state why the latter is a problem for the former. Only then, I thought, I would be able to measure the solutions offered against a concrete and firm conceptual background.

Unfortunately, both platonism and indeterminacy turn out to be two very slippery items. Some of the difficulties in presenting the key notions and facts and in handling them, will be apparent in many sections of this work. I found I had to carry out a work of synthesis, recapitulation and simplification in order to orient the discussion. What follows is the outcome of a struggle for clarity and for a comprehensible examination.

It is well-known that preliminary definitions are the hardest to deal with in a piece of philosophical work. In this case the status and the content of such notions as platonism, indeterminacy, universe of sets, to mention only a few, is particularly controversial.

Platonism is a perfect case study. Historically, 'platonism' has been used to refer to many things, including Plato's philosophy, the platonic tradition in different ages and platonic *revival* throughout history. Although its *core* *theory* seems to be well-defined, there is a lot of theoretical variance among its different historical appearances, a variance which is compounded by later and further interpretations.

The same applies to mathematical platonism, although the historical range is necessarily narrower. The controversy is made even sharper by the fact that mathematicians or set-theorists assume platonism to be something which does not coincide necessarily with what philosophers think: Fraenkel-Bar-Hillel-Levy (1973), Kunen (1980) or Kanamori (1996) are interesting examples of an attempt briefly to deal with the platonistic attitude within a set-theoretic working environment, with a strong effort to refine previous formulations. Their view is essentially *operational* rather than *substantive.* The difference between these two views will be discussed in Chapter 1.

The classical Bernays (1935) is not necessarily the point of view of contemporary platonists and Shapiro's distinction between *realism in ontology* and *realism in truth-value* (2000), which recasts Dummett's distinction between platonism and *truth-value realism* (Dummett (1978)), later echoed by many textbooks on the philosophy of mathematics might not be the last word. Feferman (1987) describes platonism essentially as *anti-constructivism* (see Chapter 1), while the Quinean and the post-Quinean approach is more keen on the intra-scientific and intra-mathematical indispensability and the dialogue between physical sciences and mathematical methodologies (see Quine (1963), Maddy (1990, 1997), Leng (2010)).

Tait (2009) is an example of the bitter controversy raging over platonism. Commenting on Hersh's characterisation (Hersh (1999)), which turns out to be fairly standard, he says:

> I hope that it is clear that much of what Hersh takes to be characteristic of Platonism, when it is not simply silly, can be understood to express nothing more than grammatical facts about the language of mathematics, so long as we take this language literally and do not attempt to translate it into another language, about other things. [...] Those who speak Platonism in the same vein as Hersh undoubtedly *feel* that they are saying

> something different from or at least more than the rather deflationary assertions that I am drawing from Hersh's words. ((2009), p. 8)

What Tait seems to adumbrate here is that the controversy over what is legitimately thought to be platonism also brings to surface a more general controversy over what attitude one should take with regard to the semantics of language and logic. This latter topic, although fascinating, cannot be even briefly addressed in this work.

To sum up, platonism is something which everyone seems to understand but which remains controversial, to the point that one sometimes doubts that there is such a view as platonism. This leads to the consideration that, whenever mathematical platonism is addressed, a significant part of the work should also be devoted to explaining what it *ought to be* rather than to what it *is*, giving up the idea that one is approaching a well-defined notion or conception. Accordingly, in the present work, alongside the examination of the problem, it has been instrumental to develop an autonomous and, inevitably biased, view of what mathematical platonism is or could be like. Eventually, *set-theoretic platonism*, in its Gödelian form, turned out to be the most suitable template.

When it comes to formulating the problem of set-theoretic indeterminacy, things get even murkier. Indeterminacy inevitably implies the point of view of someone who seeks or assumes a form of determinacy in ontology alongside the assumption that there *must* be something like a set-theoretic ontology. The expression, even linguistically, seems to be already suffering from a philosophical bias and, therefore, what one philosopher or mathematician would feel correct to describe as indeterminacy, some other would not even recognise as a palpable phenomenon. Set-theorists will agree on the fact that there is an underdeterminacy of the universe of sets according to the point of view of ZFC, but presumably only some of them will attach philosophical significance to this fact. The existence of undecidable statements in ZFC does not necessarily point to a general ontological indeterminacy of what ZFC is supposed to talk about, the cumulative hierarchy, interpreted as the *platonic* universe of sets.

If there is a universe, in platonistic terms, then it makes sense to speak of its indeterminacy as resulting from our theories. If there is no universe (in platonistic terms), indeterminacy is, philosophically, just a *façon de parler*. If one is a *set-theoretic foundationalist*, then it seems that some degree of ontological commitment to the existence of a universe is necessary; otherwise, one can just restrict the range of mathematical ontology to the finite or see all talk about ontology as *nonsensical*. While many set-theorists are always ready to deflate all ontological commitments (see Shelah (1991), Kanamori (2003)), others see the question as crucial and strictly linked to the exploration of the mathematics of sets (for instance, see Woodin (2011a)).

So much for methodology. In the introductory picture which follows, I will now proceed to explain what the problem under consideration is and my point of view in very general terms.

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A mathematical platonist believes in the existence of a world of mathematical entities, whose features are detected within a process of *discovery* by the mathematician. It is assumed (or, it ought to be assumed) that this world is *existent* in a genuine sense, although, at first glance, it is not certainly easy to specify by virtue of what form of existence it exists.

A common interpretation, focusing on semantics (following Frege (1950), see also Linnebo (2009)), has it that platonism just claims that our mathematical discourse and formal statements, in order to be meaningful, have to refer to objects. Reference must not be vacuous, that is it must be satisfied by an existing universe of objects. The true statements of mathematics are true by virtue of their reference to real properties of these objects.[1]

The ontological Gödelian framework assumes that sets are independent existents, in a (presumably) stratified existing universe. Given its current foundational status, set theory has proved to be particularly suitable to provide, at least in principle, a specimen of what the mathematical universe could be like.

[1]This interpretation of semantics does not automatically imply the necessity of a full-fledged platonistic ontology, see Chapter 1 for further details.

In search of a model-theoretic characterisation of the world of sets, several considerations led Zermelo to the unified picture of such world, V, which is defined recursively as:

$$V_0 = \emptyset$$

$$V_{\alpha+1} = \mathcal{P}(V_\alpha)$$

$$V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$$

where λ is a limit ordinal.

V is the cumulative hierarchy of pure sets, which starts with $\emptyset$, and goes through all finite and transfinite stages (for the infinite part, the Axiom of Infinity and the Axiom of Replacement are required), using the relatively simple and well-understood operations of Power-set and Union (see Boolos (1971)).

Set-reductionism has become a sort of *default* position for mathematics (accordingly, platonism has ever since been characterised as *set-theoretic realism* (Maddy (1989, 1996)). Through appropriate coding, however unnatural and contrived this might appear, it is possible to reduce all mathematical entities to an orderly set-theoretic hierarchy.[2]

V is an everlasting conquest within mathematics, given its unifying character and its internal elegance. Although simplistic in its very nature, it is also a very practical means to depict a full-fledged mathematical ontology. Alongside V, Zermelo established the natural closure of the axioms at V_κ, the rank of the first *inaccessible cardinal* (Zermelo (1930)). Inaccessibility, launched by Hausdorff in 1908, represented the first achievement in the expansion of the transfinite after Cantor.[3] Given the second-order character of his axiomatisation and its consequent *quasi-categoricity*, Zermelo was confident enough that he had secured Cantorian set theory and all later set-theoretic development within a well-defined, refined, elegant and maximally unifying mathematical ontology.

[2] *Set-reductionism* could also be seen as the proposition which says: for some ordinal α, every mathematical object is a set $x \in V_\alpha$.

[3] κ is (weakly) inaccessible if it is a *limit* and a *regular* (if it is not the sum of $< \kappa$ cardinals) cardinal.

Moreover, inclined as he was to platonistic stances, he could not doubt that this would represent, on the one hand, the firm ground for the mathematics of the future and, on the other, the full expression of the belief in the ultimate universe of mathematics.

How do a platonist's views accommodate to V? Very well, it would seem at first. I want to express this fact through the formulation of the following recapitulating principle:

Principle 1 (Set-theoretic Platonism) *There is a (unique) world of sets (V) and all mathematical statements about sets are true or false in that world.*

Principle 1 instantiates the platonic universe in an adequately describable structure. Unfortunately, the picture of this universe, as resulting from the axioms, is filled with relevant gaps and holes. Consider the following set-theoretic statements:

Continuum Hypothesis (CH). Given an infinite $X \subseteq \mathbb{R}$, if $|X| \neq \aleph_0$, then $|X| = |\mathbb{R}|$.

Generalised Continuum Hypothesis (GCH). $(\forall\alpha)\ 2^{\aleph_\alpha} = \aleph_{\alpha+1}$.

Suslin's Hypothesis (SH). There is no Suslin's line.[4]

Lebesgue measurability of Σ^1_2-reals. All Σ^1_2-reals are Lebesgue-measurable.[5]

[4]Suslin's line arises from a weakening of the definition of the real line. The real line is *dense*, *complete* and *separable* (has a countable dense subset). Suslin's line is dense, complete but not separable. Instead of separability, it has a weaker property, the *countable chain condition* (*c.c.c.*, every family of pairwise disjoint subsets is at most countable). The problem arises whether Suslin's line exists. First, Jensen (1972) proved that, in L, SH is false and, afterwards, it was proved that SH follows from the combinatorial principle known as Martin's Axiom (MA, see Chapter 2, Section 2.2). Since MA is independent of ZFC, so must be SH.

[5]Σ^1_2 sets of reals are particular sets which arise in *descriptive set theory*. Sets in the Borel hierarchy (a hierarchy of subsets of $\mathbb{R}$) are all measurable and so are *analytic sets* (continuous images of Borel sets) and *co-analytic sets*, complements of analytic sets. Σ^1_2 reals are continuous images of co-analytic sets and it has been proved that the problem whether

This is just a very short list of problems which do not have a solution in ZFC (Zermelo's axiomatisation plus the Axiom of Replacement and the Axiom of Choice). This means that the V which ZFC is talking about is strongly underdetermined.

However, ZFC^2 (second-order ZFC), in virtue of the *quasi-categoricity theorem* (Zermelo (1930)), requires that any model of the theory is isomorphic to the transitive structure $< V_\kappa, \in >$. Therefore, in all (full) models of ZFC^2 the CH has a determinate (unique) truth-value and, therefore, before first-order logic became the predominant formal framework for mathematical logic, V seemed to be immune from indeterminacy.[6]

Unfortunately, second-order logic is at least as controversial as platonism and there are hardly any mathematicians who would see ZFC^2 as the proper environment for their mathematical investigations. Although it is not direct evidence against it, the fact that all most prestigious and well-known set-theoretic textbooks, (see Jech (2003) and the most comprehensive Kanamori - Foreman (2010)) assign no role to second-order logic and arguments is, in a sense, revealing. Second-order logic remains a side dish within mathematical discussions and is mainly the subject of philosophical speculation. Besides, there are philosophers who utterly dislike second-order logic for, presumably, no less convincing reasons than those for which other philosophers keep it in such a high esteem.

However, there will be much to say about second-order logic and I will turn back to this later. In general, it is understood that my primary concern is about first-order axiomatic set theory. As a consequence of the wide manipulative machinery connected to first-order ZFC, it can be shown that all the previously mentioned statements are undecidable in ZFC and that the world the theory talks about is essentially underdetermined by that theory. How did this happen?

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they are *Lebesgue measurable* is undecidable in ZFC. I refer the reader to Bagaria (2009) for a good introductory presentation of all these results. See also Chapter 2, Section 2.1.

[6]The point of the choice of logic is crucial, as Zermelo perfectly knew, and he embarked upon a controversy with Skolem over the relativity of first-order set-theoretic notions. Skolemian relativity is absent in (full) second-order logic. See Shapiro (1991) and also Chapter 3, Section 2.

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The understanding of the contemporary situation with the semantics of axiomatic set theory and of the basic indeterminacy of the universe of discourse of the ZFC axioms is tied to the understanding of the intricate model-theoretic machinery through which it is possible to talk about models of (fragments of) ZFC or any of its extensions.

Philosophically, the development of model theory and of its methodologies within a strict first-order framework paved the way to the recognition of a shift in focus from intuitions over *objects* to intuitions over *structures*.

Rather than trying to settle the CH solely relying on knowledge concerning well-defined infinitary objects, mathematicians turned to considering structures which satisfy the ZFC axioms and to proving that not all structures which model ZFC also model CH. A model-theoretic structure of ZFC is a domain of objects along with a set of relations. $< M, \in>$ is said to be a standard structure for ZFC, if the $\in$-relation is interpreted as the usual 'member of' relation on sets.

There are essentially two methodologies to produce standard models of ZFC and its extensions: through *inner models*, which made their debut in Gödel's seminal work on the *constructible hierarchy* (Gödel (1938)) and through *forcing*, which was devised by Paul Cohen in 1963 in order to prove the *consistency* of $\neg CH$ with ZFC.

Recall that a statement ϕ is independent from a theory T if and only if T does not prove or refute ϕ. If T is consistent, then it has a model. If $M \models T$ and $M \models \phi$, then it must follow that $T \nvdash \neg\phi$. The same applies to the case where $M \models \neg\phi$, it must follow that $T \nvdash \phi$. If there are two models of T, M and M', in which ϕ and $\neg\phi$ are respectively true, then ϕ must be independent from T.

Now, first, Gödel showed that $L \models ZFC$, where L is recursively defined as:

$$L_0 = \emptyset$$

$$L_{\alpha+1} = def(L_\alpha)$$

$$L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$$

where λ is a limit ordinal. In L, at each stage, one only considers subsets of the previous stages which can be *defined*, that is sets for which there is a definable first-order property. The nature and the extent of definability can be specified in very rigorous terms (see Jech (2003), Devlin (1984)).

Taking advantage of the properties of L, Gödel was able to show that $L \models GCH$ and, in particular, that CH holds in L.

On the other hand, in very rough terms, models obtained through *forcing* can be very aptly described through a characterisation of V as the Boolean-valued universe $V^{\mathbb{B}}$, where at each stage, rather than sets, one first takes into account the characteristic (2-valued) functions of sets of V and, then, $\mathbb{B}$-valued functions of sets, thus getting[7]

$$V^{\mathbb{B}} = \bigcup_{\alpha} V_\alpha^{\mathbb{B}}$$

Usually the goal of forcing is to provide *independence proofs*. Each *forcing* extension (or *generic* extension) can be precisely defined through the choice of a particular $\mathbb{B}$, each one representing a model of ZFC where the value of the continuum, for instance, can be manipulated in such a way that any value, apart from those explicitly ruled out by König's theorem[8], is admissible. In particular, there are extensions where the CH is false.

This very sketchy summary will be supplemented by further details in the next chapters. For the time being, this helps me to state the following principle:

Principle 2 (Model-theoretic multiverse indeterminacy) *There is such a richness of possibilities arising from model-theoretic manipulations, viz. inner*

[7]It should be noticed that $V^{\mathbb{B}}$ is not an extension, rather, a re-formulation of V, in which a set of *forcing conditions* (equivalent to a particular *complete Boolean algebra* $\mathbb{B}$) can be adequately defined. The interpretation of Cohen's work through Boolean-valued models emerged shortly after Cohen's proof (1963) through the joint work of Solovay and Scott. A full description of this construction cannot even be hinted at here. See Bell (2005).

[8]See Chapter 2, Section 2.1.

models and generic extensions, as to make it plausible that there are different ontological frameworks (universes) where mathematical statements (may) have mutually contradictory truth-values.

This principle contradicts the previously stated Principle 1 and this situation gives rise to the problem which I am considering in this work.

This clash between the platonist's beliefs and the harsh model-theoretic reality can give rise to different reactions. Some of the possible counter-arguments are examined in the following chapters.

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Platonism has been a driving force for developing set theory. However thinly committed to strong ontological assumptions, at least operationally platonism carries with itself the idea that the process of creation of new entities will allow us to discover the ultimate universe of sets. If only because of this, acceptance of the transfinite, of the *finitised*[9] actual infinite and of a good deal of new mathematical notions was made possible. Interest in set theory was triggered by interest in the unique possibilities offered by the subject to achieve conceptual, ontological and foundational generality for mathematics. Because of the indeterminacy phenomenon, not only platonism, but also the possibility to carry out this project seems to be at risk. Whatever philosophy a mathematician might be inclined to accept, he will always have a penchant for the idea that there are truths which come, to use Erdős's words, from the Book. But what if the Book is *underdetermined* or theoretically neutral or *multiversist*?

Enquiries into large cardinals and new axioms were similarly prompted by this uncomfortable situation. But, as we shall see in due course, the pain provoked by the friction between the two aforementioned Principle 1 and 2 is not soothed by any ontological proliferation, so to speak.

If the *generic multiverse* generated by all generic extensions (that is, each of the previously mentioned $V^{\mathbb{B}}$) and inner models seem to be the main focus

[9]See Chapter 8, Section 2.

of the recent research, considerable attention has been elicited by an attempt to counter the *multiversist* position laying emphasis on *absoluteness properties.* Absoluteness refers to the feature of a specific property of being preserved in all different models of a theory. E.g., the property of *ordered pair* ($< a, b >$) is absolute across the multiverse, whereas that of *cardinal number*, as amply predictable, is not. Within the *generic multiverse*, there are certainly some absolute properties. In the presence of some particular axioms (*forcing axioms*), more absolute properties pop out.

The logic of *generic absoluteness* has been extensively investigated by Woodin within the framework of the so-called Ω-logic. Some of its results are no doubt staggering (see Woodin (2001) and Bagaria (2006)) and a summary of some technical details is offered in Chapter 2. In general, there is a wide controversy over whether the results obtained are, in any sense, natural and, hence, acceptable in the light of the need to extend ZFC. What is certain is that their interpretation does not lead unequivocally towards the re-casting of an *ultimate-universe view*, along the lines of a platonistic conception.

The multiversist view seems to be in better shape, even from a platonistic point of view, if one, for instance, is inclined to adopt something like Hamkins's recently proposed *set-theoretic geology* (Hamkins (2009)), where the full import of model-theoretic indeterminacy as connected to forcing is given a more proper conceptual interpretation. However, to appreciate the full strength of multiversism for the realist camp, one has to inevitably turn to some form of plentiful platonism, like Balaguer's full-blooded platonism, which is not necessarily more easily defendable than Gödelian platonism.

There is a more obvious move that a platonist might be tempted to make. Faced up with the present situation, she might simply say that the ultimate universe of mathematical objects is *unknowable* or, rephrased in the jargon of set-theoretic platonism, whatever might be the ultimate universe of sets, whether V or not, this universe is not knowable or, at least, fully understandable. If V is such a universe, then V is not knowable.

This *platonistic retreat* does not come without problems. I label it *extreme* *platonism* and it will be examined in Chapter 3. For the time being, a *prima facie* and most obvious issue is represented by the fact that such a generic belief in an existent world of mathematical entities, without any further specification and strong substantiation, takes its toll on the defensibility of platonism. Not knowing what the universe is like does not seem to offer good chances of a full philosophical defence of its existence.

Secondly, it seems to me that claiming that V exists (in the platonistic sense) but is unknowable cannot be strongly differentiated from supporting the *multiverse interpretation*. For, on the multiversist interpretation of set-theoretic ontology, the claim (A): 'V *is unknowable*' is a true principle, even if one does not really want to refer to what (A) purports to refer to, namely the unique *existing* universe of sets.

Put in other terms, a person who believes that the ultimate universe of mathematical objects is not knowable having in mind a platonistic characterisation of that universe does not deviate from the person who says that there is a multiverse because there is no unique (ultimate) V. Thus, the platonistic retreat may collapse into *model-theoretic multiversism*.

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One further remark. While writing this work, I realised that many side topics were becoming prominent. The gradual evolution of the ideas showed that the deep interconnections between set-theoretic concepts and notions, indeterminacy and undecidability were given a new and intriguing look if examined in the light of platonism. This encouraged further elucidation of the concepts and, first of all, further elucidation of what platonism actually meant and what set theory, considered as a whole, is really searching for in its investigations.

Therefore, it turned out that, to a certain extent, the chapters could be read as independent and self-contained parts. As I said above, the work aims to reconstruct the relation between platonism and set-theoretic investigations and its internal structure reflects this foremost goal. Nonetheless, a diagnosis and a possible solution are clearly enunciated on the way to the end.

The structure is as follows.

In Chapter 1 and Chapter 2, platonism and indeterminacy are introduced and more technical details are provided. In particular, Gödelian platonism is established as the form in which we are interested.

In Chapter 2 more set-theoretic background is revealed. Along with the examination of some technical details, some recent developments which aim to found *ontological uniqueness* on the intricacies of *generic absoluteness* are taken into account.

If indeterminacy affects negatively the platonistic view of mathematics, there is also much debate on the anti-realist camp, on how the issue should be dealt with. In general, there is a wide debate on indeterminacy and undecidable statements *per se*, spanning from Field-style deflationism to Hauser's methodological and conceptualistic anti-constructivism. Feferman's Weylian programme of foundations trims set theory of some of its most relevant parts, but it is philosophically fertile. Fans and supporters of second-order logic will also be paid the due attention. However, there will be no attempt to delve into such questions as whether first-order is preferable to second-order logic. Only, some of the features of second-order logic will be taken into consideration, given their strong relevance for our purposes (Chapter 3).

The main strain of contemporary platonism, Gödelian platonism, only briefly mentioned in Chapter 1, is then subject to careful examination. Chapter 4 consists of two sections. One, historical, aims to trace Gödel's philosophical background to Cantorian philosophy. In the wider picture, it seems that elements of one have ebbed away into the other, only to confirm that the roots of much contemporary platonism lie in Cantor's philosophy of mathematics. The second, more philosophical, section focuses on Gödel's philosophy of indeterminacy, as described in the Cantor paper (Gödel (1947/64)).

Full-blooded platonism emerges as the most eminent philosophical attempt of backing *multiversism*. The problems which it raises may be no less demanding than those which it solves, as will be shown. Moreover, this doctrine does not seem to overcome the drawbacks of Gödelian platonism. Maddy's naturalism stands out as a deliberately minimalist form of platonistic objectivism. However, many of the principles it involves can be found within a standard platonistic conception. The main asset of naturalism lies in its purported ontological neutrality. However, the fact remains that, if one adopts, even in the light of a self-explanatory naturalistic necessity, maximisation principles, it is difficult to eliminate all residues of ontology. Chapters 5 and 6 give an overview of these two contemporary forms of realism.

Finally, Chapter 7 and 8 will be my contribution to the description of a form of *set-theoretic objectivism* based on *concept expansion* and *inevitable* properties. Historically there are many instances of this attitude and the framework does not have to be laid out from scratch. No historical systematic doctrine, however, is held here or given particular preference. Phenomenology may be a persuasive reference but a full examination of it is besides the goal of the work. Rather than attempting to found set-theoretic objectivism upon an external philosophical bunch of doctrines, one should rather pay attention to the internal web of concepts and methodologies, starting with very general notions such as set, actual infinite, ordinal, cardinal, order, universe. Only through the clarification of these concepts, in the spirit of Cantor, Gödel and Zermelo, but not at variance with that of all the pioneers who have contributed to the development of set theory, König, Hausdorff, Fraenkel, Cohen (many more should be mentioned), one is able to see how set-theoretic intuitions are dialectically intertwined to produce what is likely to be an *objective* view of the mathematical infinite. Hence, given my lack of interest in *ontological rigidity*, indeterminacy is finally reduced to the conceptual inexhaustibility of set-theoretic conceptualisations and to the existence of conceptual incompatibilities.

Chapter 1

Platonism

1.1 Operational Platonism: Bernays

The first and main character of the drama we are presenting is platonism. Given the difficulties briefly evoked in the Introduction, the meaning attached to the term has not always been consistent. In the literature one can find at least two different presentations of platonism, one more *operational* or *methodological* in character and the other more *substantive* and philosophically articulated. In many cases, these two kinds of presentation cannot be clearly distinguished.

A general and classical overview of the features of platonism with a strong emphasis on its *operational* characters is given by Bernays in his classical paper (Bernays (1935)). While talking about definitions of mathematical (geometric) entities in Hilbert's axiomatisation of Euclid's *Elements*, Bernays underscores the fact that, whereas Euclid talks about *constructed* entities, Hilbert assumes that these entities pre-exist any construction. Accordingly, statements referring to these entities mean that the mathematician refers to classes of objects which are generally assumed to be *existent* in their own right: points, straight lines, shapes. Completed domains of these objects also pre-exist any of their members and, therefore, a single object is not singled out as the result of a *construction*, but rather by showing that it belongs to a certain domain (class) of objects.

All this leads Bernays to assert that:

> This example shows already that the tendency of which we are speaking consists in viewing the objects as cut off from all links with the reflecting subject. Since this tendency asserted itself especially in the philosophy of Plato, allow me to call it 'platonism'. The value of platonistically inspired mathematical conceptions is that they furnish models of abstract imagination. These stand out by their simplicity and logical strength. They form representations which extrapolate from certain regions of experience and intuition. ((1935), in [11], p. 259)

This picture is supplemented by further details which emerge later in Bernays's examination. Here follows a brief summary of the characters which he ascribes to platonism (in square brackets, my glosses):

1. Existential statements, ranging over domains of objects, as opposed to constructed objects [e.g., $(\exists x)\ P(x)$... as opposed to 'let us construct the object x such that x is P...']

2. Excluded middle [In particular: given any property P and a class of objects A, either $(\forall x \in A)\ P(x)$ or $(\exists x \in A)\ \neg P(x)$.]

3. Objects are acausal, mind-independent *abstracta* cut off from sensory experience.

4. The infinite is treated in a way which eventually makes it analogous to the finite. [Also: existential statements range over infinite domains in the same way as they range over finite domains].

5. Definitions can be *non-constructive*, i.e. reference to a pre-existing totality (domain) of objects, to which one object belongs is admitted (*impredicativity*).

6. Domains are *complete*. Therefore, the meaning of the universal quantifier ($\forall$) is the range of *all* values which it can take in a given domain. Completed domains are *existent* mathematical objects.

This list is very nice, as it represents a sort of *minimal closure*, so to speak, of all features of platonism. Let us now examine more closely the operational features of platonism in set theory as described by Feferman.

1.2 Operational Platonism: Anti-Constructivism

In Feferman ((1987), p. 165), an alternative operational list is given:

1. Sets as independent existents.
2. Actual infinity [All infinities can be thought of as *simple* objects, namely transfinite sets].
3. Arbitrary subsets [Subsets of a given (infinite) set, even if not arising from any construction, are admitted. Unrestricted subset formation].
4. Power-sets [see above].
5. The Axiom of Choice [equivalent to the *well-ordering theorem* (Zermelo (1904)): any set can be well-ordered. It implies that any (infinite) set is a *transfinite cardinal*].
6. Relations and functions as sets [Set-reductionism].
7. Objectivity of truth and classical logic.

As one can see, some items of this list fit into or are equivalent to Bernays's ones. What strikes one most is the fact that the principles which Feferman defines to be typical of a platonistic attitude are, in a way or the other, encapsulated in the axioms of set theory. *Prima facie*, this would suggest that Feferman is simply wrong and that set theory does not represent an entirely platonistic foundation of mathematics. The only feature which appears in Feferman's list and which is undoubtedly ascribable to platonism is the belief in the independent existence of sets. Apart from this latter, the other principles enumerated above belong to what I would, more moderately, define *methodological anti-constructivism* (featuring in the bullet point 5 of Bernays's enumeration), which simply consists in the recognition that constructivism is unsuitable to develop set theory. In other terms, a Brouwerian or Weylian constructivist would not accept *impredicative* definitions of sets and, consequently, could never accept the majority of the principles enumerated above.

In Feferman's view, however, at least as far as the article mentioned is concerned, there is a strong connection between the *ontological* claim that sets really exist and the other principles of the platonistic operational attitude. Feferman's conclusion seems to be that Cantorian set theory is a clear manifestation of both *substantive* and *operational* platonism in mathematics and that, insofar as the axioms of set theory are based on it, this latter also necessarily is.

In this work I will take it that Feferman's enumeration, except for principle 1, defines operational platonism as being *anti-constructivism* in mathematics. Later on, I will support the idea that anti-constructivism is the only bit of platonism which may be necessary in set theory.

The fact remains that, whether or not anti-constructivism can be separated from ontological platonism (as represented by Principle 1 and bullet point 1 in Feferman's list) is controversial. Hauser and Maddy (see, respectively, Chapter 3 and 5) think so. Feferman, as said, seems clearly to argue against this view.[10] At any rate, this interesting issue will be briefly re-examined in Chapter 8.

1.3 Gödelian Platonism

I will now give my presentation of the *substantive* form of philosophical platonism, that is Gödel's platonism, with which I shall be concerned in this work.

When talking very informally about platonism and its nature, mathematicians usually refer to something stronger than the *meaning* of the logical syntax, of the logical rules and of the axioms. First and foremost, platonism seems to be concerned with the existence of a reality which is not physical and is clearly beyond logic and human minds but which, nonetheless, exists as legitimately as, to paraphrase Russell's words (Russell (1919)), animals exist for zoology. Analogously, what mathematicians find to be theorems within their informal and formal reasoning ought not to be ascribed to the processes of their minds

[10] For further details concerning Feferman's conception, see Chapter 3.

solely, but viewed as the outcome of a rigorous process of *discovery*.

This view is very informally presented in the oft-quoted passage of G. H. Hardy's *A Mathematician's Apology*:

> I believe that mathematical reality lies outside us, that our function is to discover or *observe* it, and that the theorems which we prove, and which we describe grandiloquently as 'our creations', are simply our notes of our observation. ((1940), p. 35)

Along the same lines, but in a different context, when Gödel talks about Russell's *vicious circle principle*, he argues that the principle applies to a non-platonistic conception, as only in a non-platonistic environment the domain of discourse is supposed to be created from within the logical or mathematical system which one is using. For a platonist, this would be at variance with the idea that there are pre-existing mathematical entities which are the real subject of any mathematical discourse. These objects usually form *completed* domains, which, in turn, can be seen as objects in their own right, maybe layered within a super-structure like V or Ω.[11] All paradoxes involving self-reference and impredicativity (such as Russell's paradox) do not affect such orderly (pre-existing) structures. Therefore, Gödel concludes:

> It seems to me that the assumption of such objects is quite as legitimate as the assumption of physical bodies and there is quite as much reason to believe in their existence. They are in the same sense necessary to obtain a satisfactory system of mathematics as physical bodies are necessary for a satisfactory theory of our sense perceptions and in both cases it is impossible to interpret the propositions one wants to assert about these entities as propositions about the 'data', i.e., in the latter case the actually occurring sense perceptions. ((1944), in [11], pp. 456-7)

The adherence of Gödel to platonism is further testimonied, among many, by his expression of faith in the objectivity of something like the truth-value of the CH as resting upon a pre-existing ontology:

[11]The class of all ordinals.

> [...] If the meanings of the primitive terms of set theory as explained on pages [474-75] and in footnote [11][12] are accepted as sound, it follows that the set-theoretical concepts and theorems describe *some well-determined reality* [my italics], in which Cantor's conjecture is either true or false ((1947), in [11], p. 476).[13]

This well-determined reality *guides*, in Gödel's opinion, our mathematical theories in such a way as to make them internally consistent and able to give a satisfactory solution to all most relevant problems. It is only because we believe in the *pre-existence* of that reality that we believe to be able to develop objective mathematical theories and also find objective solutions for most problems.

> For someone who considers mathematical objects to exist independently of our constructions and of our having an intuition of them individually, and who requires only that the general mathematical concepts must be sufficiently clear for us to be able to recognize their soundness and the truth of the axioms concerning them, there exists, I believe, a satisfactory foundation of Cantor's set theory in its whole original extent [...] ((1947), in [11], p. 475).[14]

Gödelian platonism is an *epistemically strong* and *optimistic* view of mathematics (and set theory): we have access to the *determinate* reality of sets and we are (will be) also able to discover the objective truth-values of all set-theoretic statements. Inasmuch as it links mathematical objectivity in truth-values to an externally existing mathematical ontology, Gödelian platonism also provides a strong justification for continuing our investigations in the presence of statements undecidable from very strong axioms (ZFC). Whether the present situation in set theory really warrants this optimistic line of thought and whether

[12]In Benacerraf-Putnam's reprint of the article (see [11], pp. 470-85).

[13]For further details on Gödel's conception and his views on set-theoretic indeterminacy, see Chapter 4.

[14]Gödel's platonism I will also sometimes call *standard platonism* or, simply, *platonism*. The reason is, Gödelian platonism seems to be the most accurate approximation of what mathematical platonism is thought to be in common talk (although many commentators have viewed Frege's platonism as more standard and historically more influential than Gödel's platonism).

or not it justifies the belief in an independently existing universe of sets will be examined later on.

To sum up, following, but, at the same time, partly deviating from a recent survey of the subject (Linnebo (2009)), one can see Gödelian platonism as the conjunction of the following statements:

EXISTENCE Mathematical objects exist *per se*, necessarily and eternally. The union of all mathematical objects represents a self-sufficient reality, the *universe* of mathematics.

TRUTH All mathematical statements, which are adequately and conveniently expressed in a formal context which is able to describe, at least, a relevant fragment of the mathematical reality, have a truth-value. Namely, all formally meaningful statements about properties of that reality are either true or false.

INDEPENDENCE The mathematical universe is remote from physical reality and sense experience. Accordingly, it is physically acausal. Furthermore, a mathematical object is not equivalent to any mental procedure, linguistic construct or convention.

In addition to this, a Gödelian is also committed to this strong epistemic view:

STRONG ACCESS We have *access* to and have a *determinate* conception of the universe of sets.

Appendix: Alternative Forms of Platonism

I have just said that Gödelian platonism is the conjunction of the following four properties: EXISTENCE, TRUTH, INDEPENDENCE and STRONG ACCESS. This characterisation seems to be rather faithful to a genuine and intuitive understanding of what platonism is, along the *naive* lines which one could find in Hardy's quote, but other options would be, in principle, possible. For instance, one could assert TRUTH without committing to the other two. Viceversa, one could assert INDEPENDENCE and EXISTENCE and feel free not to subscribe to TRUTH.[15]

Part of the discussion concerning contemporary platonism is represented by the different *combinatorial* options offered by this picture.

Therefore, it seems to be reasonable to introduce one further classification, which is meant to highlight these different attitudes rather than specify any internal, so to speak, features of platonism. Shapiro (2000, 2005) distinguishes between two forms of realism (platonism):

> Define *realism in ontology* to be the view that at least some mathematical objects exist objectively. According to ontological realism, mathematical objects are prima facie abstract, acausal, indestructible, eternal, and not part of space and time. Since mathematical objects share these properties with Platonic Forms, realism in ontology is sometimes called 'Platonism'. [...]
>
> Define *realism in truth-value* to be the view that mathematical statements have objective truth-values independent of minds, languages, conventions, and such of mathematicians. ((2005), p. 6)

Following this classification, a realist in ontology would be a person who holds INDEPENDENCE and EXISTENCE, whereas a realist in truth-value would just hold TRUTH. This further distinction is *prima facie* required by the distinction between (mathematical) ontology and (logical) semantics. A person may hold the belief that the CH is true, because there is an (objectively true) axiom which implies that the statement

[15]Asserting EXISTENCE alone, according to Linnebo (2009), would correspond to *anti-nominalism*, the semantic view which asserts that reference to existent objects is non-vacuous. It should be noted that STRONG ACCESS is completely *independent* from the other properties: a person who only believes in the objectivity of mathematical reality or in the objectivity of truth may or may not believe that ontology and truth are fully accessible and determinate.

is true. This sort of realist does not care about the ontological status of the entities which are involved in the formulation of the CH.

Linnebo (2009) claims that realism in truth-value is not necessarily platonistic. While a realist in ontology has good reasons for being also a realist in truth-value, a realist in truth-value might even reject the idea that there are mathematical objects and, thus, be a *nominalist.*[16] Linnebo's article lays emphasis on the fact that *real* platonism implies EXISTENCE and INDEPENDENCE and that TRUTH ought not to be viewed as one of its features. But we can surmise that a realist in truth-values *might* also believe that our ability to find a determinate (objective) semantics for mathematics is the result of the existence of objective, mind-independent truths and, thus, uphold (a reformulated version of) INDEPENDENCE within his platonistic account.

On the other hand, one might just want to defend the existence of a mathematical ontology, while denying that there is any pre-existing truth concerning this ontology. This person might have a very relaxed philosophical attitude concerning the truth-value of mathematical statements, leaving the task of saying something about it to the working mathematician. Finally, he might assert that *semantic indeterminacy* does not rule out a rigid ontology and that, consequently, the problem under consideration here is not really relevant. Later in this work, I will have something to say about whether an *objects-platonist* should be concerned about indeterminacy.

There are at least two more philosophical conceptions belonging to or directly tied to platonistic conceptions and this work will examine both in two separate chapters: *naturalism* in the Maddyan form and Balaguer's *plenitudinous platonism.* In the wide debate about the objectivity of mathematics and of its foundations, these two conceptions stand out as attempts to reform the old (standard or non-standard) views. However related to platonism, they follow peculiar lines of thought which make them fairly distinct from the characterisation presented above. If Gödelian platonism is assumed to be the standard view and *conservatism* over Gödelian platonism implies that its most important features are preserved, then these two forms are *non-conservative.*

Maddy's naturalism disavows all the three principles, for the essential reason that, following our reasoning on Gödelian platonism, it disavows any founding ontology (Maddy (1996, 1997)) and, along with and by virtue of this choice, rejects the principle

[16]For example, Hellman (1989) would be an instance of *realism in truth-values.* His *eliminativist* account of structures should be able to delete any reference to structures as *existent* entities.

that the objectivity of mathematical truths would flow from that ontology. Objectivity is based on intra-mathematical maxims and extrinsic criteria.

Balaguer's point of view as presented in Balaguer (1995, 1998) is non-conservative in that it posits the existence of many universes of objects and, thus, substantially rejects TRUTH.

These two conceptions are not necessarily deliberately meant to solve the old problems connected to Gödelian platonism, including the problem considered here. However, it is plausible to conjecture that they arise as responses to some of those problems and that they may want to be seen as reforms of the standard view. I label them non-conservative reforms, as, although inspired by it, they (sometimes deliberately) are interpreted as trying to overcome Gödelian platonism under many aspects. The extent and strength of these reforms for the problem of the indeterminacy will be examined in Chapter 5 and Chapter 6.

Chapter 2

Indeterminacy

2.1 Set-theoretic Indeterminacy

The second and last character I have to present is set-theoretic indeterminacy. First, let us try to fix some terminology. The ZFC theory of sets is not indeterminate under any explicit sense of the word. In ZFC, some statements pertinent to some set-theoretic properties are undecidable. This means that such statements cannot be proved or disproved. An incomplete (and *essentially* incompletable) list has been provided in the Introduction.

Gödel (1938), using the inner model L and Cohen (1963), using *forcing* over a countable *ground* model of ZFC, proved that the CH is independent of ZFC. In particular, using forcing, it is possible to show that $2^{\aleph_0} = \aleph_{13}$ or $2^{\aleph_0} = \aleph_{112}$ is as consistent with ZFC as $2^{\aleph_0} = \aleph_1$. The CH is, therefore, not decided by the axioms of ZFC.

The same applies to Suslin's hypothesis (SH): *there are no Suslin lines.* As already said, L and forcing models have provided us with a proof that it is independent of ZFC.

However unwelcome these results may appear, the set-theorist must feel some sort of relief as a consequence of independence proofs: any attempt to prove or disprove these statements within ZFC must necessarily be a waste of time and, consequently, he is free to turn to other (possibly solvable) set-theoretic problems.

As a general remark, if a statement ϕ is not provable from a theory T, this does not mean that one should automatically lose interest in attempting to prove or disprove

ϕ from theories stronger than T. In particular, the Gödelian platonist's construal of the undecidability of ϕ, as suggested by his understanding of Gödel's incompleteness theorem, implies that ϕ has a truth-value even if ϕ is not provable from a particular theory T. The Gödelian is thus especially interested in verifying that, by providing T with a suitable axiomatic strengthening, certain statements are decidable. A person who does not hold any belief in the objectivity of truth may nonetheless want to know what theories prove what and, accordingly, feel some pressure to extend T.

However, in the cases under consideration, the undecidability results engendered a clear feeling that a relevant subset of set-theoretic problems could not be settled and that, as a consequence, our overall conception of sets was being put into question. Some of the reasons for such a quandary are listed below.

1. The axioms of ZFC already seem to offer the most convenient framework in which mathematics can be easily modelled and *reproduced*. In fact, the first few transfinite levels (up to and including $V_{\omega+\omega}$ (far below the closure point of the axioms which is the *first inaccessible cardinal*)) are sufficient to embed the rest of mathematics successfully. Indeed, ZFC is too wide a theory for most mathematical purposes; subsystems of it are equally efficient (see Simpson (1999)). Therefore, extensions of ZFC seem to be hardly justifiable in view of our *non-set-theoretic* mathematical purposes. In particular, some mathematicians hold the extreme view that a mathematical problem is either decided by ZFC or is simply undecidable.

2. The solution of problems such as the CH is crucial for our understanding of the world of sets. The CH, in its extended form (the Generalised Continuum Hypothesis, GCH), if true, would model the fine structure of the *cardinal hierarchy* and the behaviour of cardinal exponentiation in a highly regular and precise way. In the absence of substantive results about the CH and the GCH, one cannot say much about cardinal exponentiation in ZFC.[17]

[17] At least with regard to regular cardinals. With regard to singular cardinals of uncountable cofinality, we have the beautiful Silver's theorem:

Theorem 1 (Silver's Theorem) *Given any singular cardinal $\aleph_\alpha$, such that $cf(\aleph_\alpha) > \omega$, if for all $\lambda < \alpha$, $2^{\aleph_\lambda} = \aleph_{\lambda+1}$, then $2^{\aleph_\alpha} = \aleph_{\alpha+1}$.*

Another famous statement which can only be shown to be consistent with ZFC is the Singular Cardinal Hypothesis (SCH), which extends Silver's results to singular cardinals of *countable* cofinality.

3. There are also historical reasons for seeing these hypotheses as thoroughly relevant to the understanding of the world of sets. Set theory was initiated by generalisations of Cantor's results concerning the *transfinite* cardinality of particular *point sets* (Cantor (1879-1884)). The push to pursue and investigate a theory of transfinite sets was essentially given by the necessity to understand the structure of the continuum. The failure to identify the power of the continuum is staggering, if measured against the expectations of early set theory. The proof of this is: as a consequence of Cohen's result, many mathematicians lost interest in set theory as a *foundational* framework.

4. Partly as a consequence of 1., 2. and 3., and partly because of other philosophical considerations, undecidability in ZFC seems to affect our understanding of sets more than, for instance, undecidability in number theory questions our understanding of natural numbers. Therefore, undecidability seems inevitably to lead to the question whether we actually have a definite *conception* of sets and to what degree of accuracy this conception is legitimately represented by V.

Now, since it is straightforward to show that $V \models ZFC$,[18] if only because we deliberately wanted V to be the intuitive well-founded super-structure of sets, and since V seems to correspond to our general understanding of the universe of sets, then one is left with no alternative but to assert that V is underdetermined by ZFC. Talk of *set-theoretic indeterminacy* is, therefore, a legitimate way of recapitulating this uncomfortable state of affairs.

In other terms, the *indeterminacy problem* can be defined to be arising from the contrast between the fact that the axioms of ZFC are held by many to be the foundation of mathematics and the fact that, in spite of this, they do not settle all the properties of their *intended* super-structure satisfactorily.

At the operational level, the situation is different. The search for independence proofs has given a strongly positive contribution to set theory. For a start, the elucidation of what a model of ZFC should or could be has been triggered by the prediction of the undecidability of some problems.

As mentioned, historically, Gödel commenced an enterprise which marks the shift

[18]As a consequence of Gödel's second incompleteness theorem, ZFC cannot prove its own consistency and, thus, that it has a model. Of course, there are stronger theories which prove that ZFC has a model.

from the primitive Cantorian initiatives to a more mature model-theoretic setting, made possible by Skolem's work on non-standard models. The Löwenheim-Skolem theorems and the existence of *compactness* properties for first-order logic were instrumental. Contrary to Zermelo's second-order claims as encapsulated in his *quasi-categoricity theorem*, undecidability gave rise to a whole host of results on how to find model extensions and manipulations. As a consequence, paradoxically, Skolemian relativity was to be greeted with satisfaction rather than discontent, and what seemed to refute Cantor's naive realism and point to a failure of the early conceptions turned into a deeper understanding of the world of sets.

Indeed, the power of first-order logic lies exactly in this flexibility. Granted, this situation may be uncomfortable for a Gödelian platonist, as ZFC-undecidability cannot be easily fixed and, if one believes in the form of platonism which has been described in Chapter 1, then there is no easy fix for indeterminacy (see Chapter 3).

On the opposite side, Cohen's line of thought (Cohen (1966, 2011)) has since established itself as a very austere, skeptical and cautious interpretation of our efforts to understand the full set-theoretic ontology. The natural consequence of this view is that to say something about the size of the continuum is *impossible* and one should simply accept this fact. The semantic approach does not fix this situation. On the contrary, it compounds things, because it shows that there is no fixed reference of the notion of *universe of sets.*

Whatever the matter may be, the undecidability of the CH in ZFC has not implied the death of the problem. Other moves are possible. In section 2 a brief account of these developments is given.

But first, a remark about what would seem to be an obvious move to make.

The CH could be added as a new axiom to ZFC, as happened with the Axiom of Choice. As a matter of fact, this is not a convenient way to circumvent the difficulties, since there are many reasons to think that it is definitely *not* an axiom.[19] To begin with, an axiom is either evident or unanimously acceptable (or accepted), features which do not seem to be ascribable to any of the currently undecidable statements (see Maddy (1988, 1997)).

Secondly, as has become clear after Cohen's proof, the size of the continuum is too severely underdetermined to enable set-theorists to assert that one of the many

[19]To use Shelah's jargon, the CH or, rather, the GCH could be seen as *semi-axioms*, only acceptable in view of their (limited) operational success. See Chapter 8, Section 1.

possible sizes should be taken as an axiom.

2.2 From Cantor to Woodin. A Philosophical History of the *CH*

At the beginning of his recent and extremely detailed survey of the subject, Kanamori states:

> The whole transfinite landscape can be viewed as having been articulated by Cantor in significant part to solve the Continuum Problem. Zermelo's axioms can be construed as clarifying the set existence commitments of a single proof, of his Well-Ordering Theorem. ((2007), p. 1)

If this assertion is true, it would no doubt be noteworthy that two of the main features of our present conception of sets, *viz.* the extension of numbers into the transfinite and the well-ordering principle, originated from the attempt of solving just one problem: the Continuum Problem.

If, historically, some mathematical areas have been developed in the wake of single significant problems,[20] this is certainly the case of set theory, to the point that the history of its development can be studied alongside that of the Continuum Problem. Therefore, Cohen's monograph in the sixties seemed to say something definitive on the present and future state of this branch of mathematics.

In many set-theorists' opinion, the situation today is rather different and Kanamori's interesting diagnosis is that, when Cantor-Gödel's (and, I would add, Zermelo's) realist line of thought ceased to seem relevant, set theory became what makes it a fairly distinguished theory among all mathematical ones, insofar as it deals with '*analyzing mathematical propositions and gauging their consistency strength*' ((2007), 1).

Whatever the matter may be, it is certain that any account of indeterminacy is tied to the history of the most well-known undecided statement of *ZFC*, that

[20]In this list, presumably, one could also include the Prime Number Distribution Problem for number theory.

is the CH:

> Any subset of the continuum is either countable or has the cardinality of the continuum. (Cantor, 1878)

When Cantor delivered his famous hypothesis, he thought that the solution would be found very soon, if not within weeks, certainly within some years. He showed some sort of faith in the truth of his claim and announced that he would be able to present a proof shortly. As we know, all attempts were unsuccessful. As a recognition of its status, the problem was put by Hilbert at the top of his famous list of, at the time, twenty-three most important unsolved problems (1900).

Is it possible to sketch a philosophical history of the CH in a few paragraphs? What follows is an attempt in this direction. However, most technical details of proofs and additional results will have to be sought directly by the reader in other sources, maybe those cited in the bibliography. This account does not aim to be comprehensive or mathematically detailed. Its purpose is to show, through the use of empirical evidence, the most salient facts of the history of the Continuum Problem.

As we shall see, in its more recent stage, this history features the attempt to find a *consistent set of principles* for the theory of sets, which would imply, as an axiom or as a consequence of an axiom, the GCH, the most important *regularity properties* of definable sets of reals (e.g., the *perfect set property*) along with large cardinal assumptions and the *full determinacy* of sets of reals (or a milder form of determinacy, such as PD). *Prima facie*, it would seem that the task is impossible: whenever one of the above mentioned principles enters the arena, some other ceases to be valid, in a sort of endless *dialectical* splitting of possibilities.

However, the case for the feasibility of this programme has received strong support from Woodin's results, which show that the programme is successful at the level of second-order reals and that it may also be so for more complex structures. His contribution will figure in the last subsection and the consequences of this dramatic alternative for a platonist will be thoroughly examined later in this work.

2.2.1 Beginnings

Despite its straightforward nature, the Continuum Problem foreshadowed the use of a very controversial principle, on which the whole of set theory is founded, the *well-ordering principle (WOP)*. In declaring the *WOP* a natural law of thought (Cantor (1883)), Cantor seemed to prevent all criticisms about the architecture of the transfinite and, at the same time, shift the focus from the intractability of this principle to the intractability of the Continuum Problem.

Eventually, the early Zermelian axiomatisation (Zermelo (1908)) which strongly relied upon a proof of the *WOP* seemed to hinder further skepticism, but the problematic nature of the *well-ordering theorem*, later recast by Zermelo as the Axiom of Choice, was only overshadowed by that of the Continuum Problem, as the latter presupposes the former.[21]

However, at the time, Zermelo's optimism was connected to the *quasi-categoricity* proof of ZFC^2. And, after all, the historical paradoxes had found a fix within Zermelo's final axiomatisation (Zermelo (1930)) (which also featured the Axiom of Replacement and the Axiom of Foundation, this latter establishing the well-foundedness of the universe of sets).

Now, taking for granted, as a methodological premise, that the Axiom of Choice should be considered a true principle of set theory and mathematics and that, therefore, the Continuum Problem in the form: is $2^{\aleph_0} = \aleph_1$? is a fully meaningful problem, in this first subsection we turn our attention to Cantor's early efforts to find a proof, which had concentrated on a property of definable sets of reals, the *perfect set property*.

A point set of the real line $A \subseteq \mathbb{R}$ is closed if and only if its complement is an *open set*. A closed interval, for instance, is a closed set A of points of the real line.

[21] The meaningfulness of the Continuum Problem is preserved, even if one does not accept the Axiom of Choice, as it would still make sense to ask whether any uncountable subset of the continuum has the same power as the continuum. However, in that case, the power of the continuum could not be any $\aleph_\alpha$, for any ordinal α.

An accumulation point of A is any $a \in A$ in each of whose neighbourhoods (any interval containing the point) there are infinite members of A. The example of the closed interval will do. $[0, 1]$, construed as a set of points has an infinite number of accumulation points, more precisely, all of its points are accumulation points, as one can easily check. A closed set can be more precisely characterised as a set which contains all of its accumulation points. If $a \in A$ is not an accumulation point, then a is an *isolated point.* Consider the set of all $\{1/n : n \in \mathbb{N}\} \cup \{0\}$. This set contains its only one accumulation point, 0 and is, thus, closed.

Sets which contain all of their accumulation points and no isolated points are called *perfect sets.* The above mentioned closed interval $[0, 1]$ is a perfect set.

Cantor was able to show that all perfect sets have the cardinality of the continuum. As a culmination of his research on the topic, he was also able to produce a very refined result, which proves that any *closed set* has a perfect subset:

Theorem 2 (Cantor-Bendixson) *If P is an uncountable closed set, then $P = A \cup B$, where A is perfect and B is at most countable. Hence, $|P| = 2^{\aleph_0}$.*

Cantor-Bendixson is the deepest result about the size of the continuum which Cantor was to achieve. Along with the corollary of König's theorem about cardinal sums and products, which rules out certain values of the continuum,[22]

Theorem 3 (König's theorem's corollary) $cf(2^{\aleph_\alpha}) > \aleph_\alpha$.

Cantor-Bendixson has been the most one could hope to prove from ZFC about the size of the continuum for a long time.

[22]For the sake of completeness, I will state König's theorem which says that, given two increasing sequences of cardinals, κ_i and λ_i, if $(\forall i \in I)\ \kappa_i < \lambda_i$, then $\sum_{i \in I} \kappa_i < \prod_{i \in I} \kappa_i$. The theorem cited above is an easy corollary. A consequence of both is that $\mathfrak{c} \neq \aleph_\omega$ (where $\mathfrak{c}$ is the power of the continuum), as $cf(\aleph_\omega) = \omega$ and if $2^{\aleph_0} = \mathfrak{c} = \aleph_\omega$, then, by the theorem cited in the text, $cf(\aleph_\omega) > \aleph_\omega$, which is impossible.

Now, uncountable open sets had been shown to be of the same cardinality as the continuum, so all the most relevant cases seemed to be settled. The Cantor-Bendixson theorem led Cantor to conjecture that every subset of the real line might contain a perfect subset:

Perfect Set Hypothesis (*PSH*) *Every uncountable subset of the real line has a perfect subset.*

Evidently $PSH \to CH$. Unfortunately, the work on the perfect set hypothesis was ultimately unsuccessful and in 1908, Felix Bernstein gave an example of an uncountable subset of the continuum without the *perfect set property.*

However, the work on definable sets of reals and Cantor's initiatives were developed by the French analysts Borel, Baire and Lebesgue at the turn of the century. The subject was then taken up by the Russian school of Lužin, Aleksandrov and Suslin years later and all the combined efforts of these pioneers led to the rising of *descriptive set theory*, the theory of definable sets of reals.

If Cantor had mainly worked on the concept of the *derivative* A' of a set A, which contains all the points of accumulation of A and, thence, had proceeded to iterate the operation of *derivation* into the transfinite in order to achieve a classification of the cardinalities of point sets, new generalisations were soon devised by Borel and Luzin, which would pave new ways of looking at the continuum.

The focus was, once again, on trying to prove that regularity properties such as the *perfect set property* held along these hierarchies. Meanwhile, Lebesgue had presented his dissertation on measure and integration (1902), which offered a precise definition of measurability of point sets. Thence came the notion of *Lebesgue measurability*, which soon became another important regularity property.

The hierarchy of Borel sets, $\mathcal{B}$, arises very naturally from the previous conceptualisations. Let

Σ_1^0 = the collection of all open sets

Π_1^0 = the collection of all closed sets

One can then define recursively a hierarchy of sets through countable unions and intersections in the following way[23]:

$$\Sigma_\alpha^0 = \{A : A = \bigcup_n^\infty A_n\}$$

where each A_n belongs to some Π_β for some $\beta < \alpha$.

$$\Pi_\alpha^0 = \{A : A = \bigcap_n^\infty A_n\}$$

where each A_n belongs to some Σ_β for some $\beta < \alpha$.

This guarantees that at each stage of the hierarchy new Borel sets are defined. However, it can be shown that this process is exhausted by ω_1 steps. Hence, it is immediate that

$$\mathcal{B} = \bigcup_{\alpha<\omega_1} B_\alpha$$

where each B_α belongs to some stage of the previously defined hierarchy. Put in other terms, Borel sets are all closed and open sets of the real line which are closed under countable intersection and union.

But then one can go further, defining the hierarchy of *projective sets*. First, one takes the collection of all *analytic sets*, those sets which are continuous images of Borel sets in the Baire space[24] and then goes further iterating complementation and projection.

[23] See Jech (2003), p. 140.

[24] Baire space ($\mathcal{N} = \omega^\omega$) is the set of all sequences of natural numbers, which is an equivalent characterisation of the concept of the continuum.

The projection of a set $A \subset X \times Y$ is the operation of taking all the x-coordinates for which there is a y-coordinate such that (x, y) belongs to A. Put in simpler terms, it is the operation of projecting a two-dimensional point set into a one-dimensional point-set.[25] Through an appropriate mapping, *analytic* sets are defined as projections of Borel sets. The projective hierarchy can be extended into the transfinite in the same way as Borel hierarchy, using iteratively *complementation* and *projection*. The projective hierarchy is a generalisation of the Borel hierarchy, as Borel sets are precisely the Δ^1_1 sets of the projective hierarchy.[26]

The Borel hierarchy is particularly well-behaved, as every Borel set has a perfect subset and, then, *PSH* and, in particular, *CH* hold in it. Moreover, Borel sets are all Lebesgue measurable. Suslin (1917) also showed that uncountable analytic sets must have a perfect subset and, therefore, be well-behaved in the sense specified.

Further generalisations lead to the construction of new hierarchies of sets, if one takes continuous images of analytic sets (coanalytic) (Σ^2_α sets of reals, as Σ^1_α sets of reals are sets in the projective hierarchy) and, after that, again continuous images of coanalytic sets (Σ^3_α) and so on. The more one proceeds with generalisation the more difficult it is to ascribe regularity properties to sets of reals.

Undecidability with regard to regularity properties already arises for sets of reals at the Σ^1_2 level. For instance, that Σ^1_2 sets are Lebesgue measurable is undecidable in *ZFC*, whereas one can show that all analytic sets are Lebesgue measurable.

The search for regularity properties and the study of the structure of definable sets of reals has enormously enlarged and enriched the set-theoretic landscape, providing new startling results. Connections to older and more recent areas have cropped up. As shown, undecidability already steps in quite soon in the hierarchy of definable sets, which shows, once again, how harsh can indeterminacy be, at the level of *ZFC*.

[25] Obviously, projection can be analogously extended to any subset of $\mathbb{R}^n$, with $n \in \mathbb{N}$.
[26] $\Delta^1_n = \Sigma^1_n \cap \Pi^1_n$.

Now, what if one wants that all sets in the projective hierarchy (Σ^1_α) are well-behaved, namely that they are Lebesgue measurable and have the perfect subset property? There is a weakening of the *Axiom of Determinacy* that is *Projective Determinacy*, which does the job.

Mycielski and Steinhaus's Axiom of Determinacy (AD) arises in the context of determinacy of games where two players pick up 0 and 1, in turn, constructing finite binary sequences. If any of the two sequences belongs to a previously defined subset of the real line, then one of the players wins. That it is always the case that, given any game, either of the two players wins (that is, has a *winning strategy*) is asserted by the Axiom of Determinacy. Although intuitive, AD is inconsistent with ZFC and, therefore, was soon discarded as an option.

However, Projective Determinacy (PD) emerged as a plausible alternative, as it is not incompatible with the Axiom of Choice.

It is one of the most beautiful conquests of contemporary set theory to have proved that the acceptance of PD is equivalent to the acceptance of a *large cardinal axiom* about a class of particular large cardinals (Woodin's cardinals).

Theorem 4 (Martin-Steel, 1985) *If there are infinitely many Woodin cardinals, then every projective set is determined, that is PD holds.*

Martin-Steel's theorem, complemented by the following:

Theorem 5 (Woodin-Shelah, 1984) *If there are infinitely many Woodin cardinals, then every projective set is Lebesgue measurable and also PSH holds.*

gives us the result we were searching for. Under PD, all projective sets are Lebesgue measurable and are well-behaved, that is, no counterexample to the CH can come from the projective hierarchy.

While many questions about more sophisticated hierarchies of reals are still undecided, at the level of the projective hierarchy it seems that PD gives us a very satisfactory and *regular* picture. The mentioned results are probably the highest point reached by investigation into the CH from the point of view of the definable continuum since Cantor's time and have, thus, crowned and done justice to his earlier hopes and efforts.

The theorems just mentioned are eminently beautiful, interesting and important also because they connect two seemingly mutually unrelated areas of set-theoretic investigations, descriptive set theory and large cardinals, thus providing, at least, extrinsic reasons for accepting an axiom like PD. This remarkable feature of showing the connectedness of different approaches is another success of the work on the CH.

2.2.2 Models and Absoluteness

A leap forward and a decisive turn in the history of the Continuum Problem was achieved after it was established that the axiomatic apparatus of ZFC was not able to settle the problem.

What was only a suspicion transformed into a matter of fact after the joint work of Gödel (1938) and Cohen (1963). Their work launched a full-fledged model-theoretic approach, along the lines of the work of pioneers like Skolem. Focus on V as the underlying ontology of the world of sets has since become the primary concern of set-theorists and the methodologies that Gödel and Cohen developed provide the main tools for carrying out the task through model-theoretic manipulations.

In Gödel's work, V was thinned out to an *inner model*, where arbitrary subsets were replaced by first-order definable subsets, that is sets for which there existed a defining property. In L, the hierarchy of definable sets, as presented in the Introduction, using Skolem's approach and Mostowski's transitive collapse, it can be shown that all subsets of natural numbers are definable in fewer than ω_1 stages and, as a consequence, that the CH (and, by a similar line of reasoning, also the GCH) hold in it.

L has since become the archetypal *inner model* of ZFC, that is, a transitive model which contains all ordinals. After Gödel, Jensen's work helped to shape the fine structure of L with an increasing precision and identify the combinatorial principles which are connected to it, like the $\Diamond$ principle.

This latter is a very important and intensely studied combinatorial principle which gives us interesting information about the structure of $\aleph_1$.[27]

The principle is mathematically interesting *per se*, regardless of its connection to set theory and, therefore, it is all the more striking that it is implied by the Axiom of Constructibility ($V = L$).

Moreover, $L \models \neg SH$ (Jensen, 1972), that is the existence of Suslin's lines is consistent with the axioms of ZFC, a result analogous to that concerning CH discovered by Gödel. That SH would soon be shown to be independent seemed to be plausible after Jensen's proof, in exactly the same way as was the independence of the CH from Gödel's proof of its consistency with ZFC. The final proof was reached after the crucial identification of a combinatorial principle, Martin's Axiom (MA) from which SH would naturally follow.[28]

All this showed the power of the semantic approach: combinatorial principles, like $\diamondsuit$ and MA, are each strongly connected to alternative model-theoretic manipulations, thus establishing a remarkable and profound connection between combinatorics and universes.

Now we come to *forcing*. Its development has helped set-theorists to produce a lot of independence proofs, and, given the predominance of metamathematical methods in contemporary set theory, it and its refinements have become

[27]For the sake of completeness, I will give the definition of $\diamondsuit$. First we have to define *closed unbounded* and *stationary sets* in ω_1:

Definition 1 (Closed Unbounded Set) *A set P is closed unbounded in ω_1 if $sup(P) \in P$ and $sup(P) = \omega_1$.*

Definition 2 (Stationary Set) *A set is stationary if its intersection with all closed unbounded sets in ω_1 is non-empty.*

And then we get $\diamondsuit$:

Definition 3 ($\diamondsuit$) *There exists an at most countable sequence W_α in ω_1 such that the set $A = \{\alpha < \omega_1 | X \cap \alpha \in W_\alpha\}$, where X is any subset of ω_1, is stationary.*

[28]Martin's Axiom is a generalisation of a series of analogous principles holding for a generic cardinal κ (MA_κ), and, in particular, it implies that MA_κ is true for all $\kappa < 2^{\aleph_0}$. Its standard formulation is beyond the scope of this brief account. Again, I refer the reader to Jech (2003) for a comprehensive survey of this and other strong combinatorial statements.

a fundamental area of research. Unfortunately, the richness of its complicated machinery cannot be expounded here. I will just explain its general features in connection to the CH.

The basic idea is that we want to add enough reals to a countable model of ZFC to show that the CH must necessarily fail. However, it is not easy to carry out this construction, because the reals to be added have to be chosen very carefully, in order to prevent the model from collapsing into a constructible structure. Therefore, reals must have a certain degree of *genericity*. Generic reals are coded by finite binary sequences, whose structure is a partial ordering $\mathbb{P}$ and the properties which make them generic are *forced* by the choice of suitable segments of $\mathbb{P}$. After one generic (Cohen) real is successfully added to the initial ground model M to generate the desired extension of the model, $M[G]$, then one can reiterate the process, for instance, $\aleph_2$-many times, to get the failure of the CH.

After the advent of L and of forcing, plenty of set-theoretic statements were shown to be formally unprovable from ZFC. Forcing was strengthened, refined and recast in its iterated form. Also, it was noticed that many forcing notions were associated with important combinatorial principles, as the previously mentioned Martin's Axiom. One peculiarity of these notions was that, through them, *generic absoluteness* obtains, namely their consequences are absolute across all *set generic extensions* induced by forcing (see, in particular, Bagaria (2000) for a comprehensive survey).

This led mathematicians to identify and study more carefully generic absoluteness and the forcing notions associated with this. The topic is relevant for our discourse, especially in relation to the most recent developments. Some details are provided in what follows.

Further generalisations following Cohen's discoveries showed that a forcing notion can be made equivalent to a complete *Boolean algebra* $\mathbb{B}$. The universe of sets deriving from forcing over the universe itself is the Boolean-valued universe $V^{\mathbb{B}}$ which was presented in the Introduction:

$$V^{\mathbb{B}} = \bigcup_{\alpha} V_{\alpha}^{\mathbb{B}}$$

This universe is just a convenient fiction, as no forcing extension of the universe itself is really possible. Its utility lies in the fact that it helps to define the notion of *generic absoluteness.*

A trivial absoluteness is that of Δ_0 formulas, that is formulas which contain only bounded quantifiers[29]. A more interesting example is the absoluteness of L in all inner models of ZFC, which implies that CH is absolute in all inner models. This absoluteness for inner models is reached through the Axiom of Constructibility, which is absolute in all inner models.

The same rationale motivates the so-called *forcing axioms*, that is to find a principle which holds across the Boolean-valued universe and whose consequences, analogously, cannot be destructed by forcing. The analogy works very well, as the Axiom of Constructibility is equivalent to the $\Diamond$ principle and forcing axioms are, likewise, associated to other combinatorial principles like MA.

All forcing axioms examined thus far converge to indicate that $\mathfrak{c} = \aleph_2$. However, as for the Axiom of Constructibility, the problem is whether and in what sense any of these axioms is intuitively plausible and acceptable. The question is, obviously, of great moment.

A perfect example of reasoning associated to forcing axioms is that offered by Woodin's results. As we will see, in that case, however, a generic absoluteness argument is tied to a very sophisticated analysis of a particular structure, with a strong attention to its *completeness.* This does not necessarily apply to all the other cases where a forcing axiom is taken into account.

2.2.3 Very Large Infinite

Large cardinals were briefly mentioned in subsection 2.1.1 in connection to important results of descriptive set theory. Historically, they can be seen as a generalisation of the reasons behind Cantor's extension of numbers into the

[29]Such as $(\exists v \in w)\ \phi$ or $(\forall v \in w)\ \phi$.

transfinite. However, many threads now connect them to the bulk of contemporary set theory and they represent an autonomous thriving area of research.

They play an essential role in our history, as shown by the previously mentioned theorems but their introduction, as we shall see, has not eased the situation.

The study of large cardinals was launched by Hausdorff in a seminal paper on *inaccessibility* (see Introduction). Thereafter, new and increasingly bigger cardinals were defined, up to an inconsistent extension of concepts which was revealed by Kunen in 1971.

Although they can be seen as arising from generalisations of purely infinitary concepts and properties (Kanamori (2003)), at least from some large cardinals onwards a convenient expository framework is provided by model theory: in general, one takes elementary embeddings of the universe and defines their critical points (Drake (1974), Kanamori (2003)).

For the definition of a measurable cardinal, one takes an elementary embedding[30] of V into a subuniverse (inner model) M

$$j : V \to M$$

such that, for some α, $j(\alpha)$ ceases to be the identity (that is, $j(\alpha) \neq \alpha$). If such an embedding exists, then the critical point of j is a *measurable* cardinal.

We can generate other large cardinals through making M as close as possible to V. Using the Axiom of Choice, Kunen's theorem (Kunen, 1971) establishes that there is no (non-trivial) elementary embedding of V with itself, a condition which was required by the definition of *Reinhardt cardinals*. Since then, Reinhardt cardinals are, therefore, seen as the upper bound of all large cardinal extensions.

As said, large cardinals are generalisations of infinitary concepts and inaccessibility is the ancestor of all large cardinals. The concept seemed to generalise quite well the properties of ω and project them into an entirely new stack of big cardinals.

[30]For a definition of elementary embedding, see Chapter 3, Section 2.

ω is the only known cardinal beneath the first inaccessible which is (strong) limit and regular. Recall that a cardinal κ is regular if and only it is not the sum of fewer than κ cardinals. ω is obviously regular. But ω is also strong limit, as $(\forall n < \omega)\ 2^n < \omega$. Is this property of being inaccessible ascribable to any other initial ordinal in V? Well, there are reasons to think that V is too big to have just one inaccessible cardinal and that, therefore, there must be an α such that V_α is inaccessible. This line of reasoning is connected to an intuitive *reflection principle* which asserts:

Principle 3 (Principle) *Given any first-order property P, if P is ascribable to V, then there is a least $\alpha \in On$[31] such that $V_\alpha \models P$.*

Informally, a similar reflection principle lies behind many large cardinal extensions. The rationale underlying the acceptance of this view is that V is unknowable and that, whatever is ascribed to V, must belong to lower ranks within V.

Here, the unknowability of V is, thus, a purely *operational* principle. Is there any connection between this operational *unknowability principle* and the weak platonist's claim that the real universe is not knowable? In the next chapter, we will examine whether this can actually be the case.

Now, more and more large cardinals have been produced over the years and their bearing on our knowledge of set theory is immense.

Large cardinals have produced staggering results, like the following beautiful theorem by Scott:

Theorem 6 (Scott) *If there is a measurable cardinal, then $V \neq L$.*

which marvellously expresses the transcendence of V over L through the set existence assumption of a single large cardinal.

However, quite disappointingly and rather unpredictably, further results by Solovay and Lévy (1967) have shown that, even in the presence of the strongest currently known large cardinal assumptions, the CH is not decided.

[31] On is another label for the class of all ordinals.

This amounts to the fact that adding new bits to the cumulative hierarchy does not have relevant consequences on the size of the continuum, thus shattering Gödel's hopes that axioms of infinity might be decisive. This does not prove that large cardinals do not have a direct bearing on the *structure* of the continuum. The above mentioned theorems clearly prove this. But, sadly, they are ineffective in connection to the CH. Therefore, reliance upon ontological considerations, such as those expressed by Gödel, motivating the belief in truth-value objectivism, proved to be, at least as far as these results are concerned, equally ineffective. This hint will be developed philosophically in the later chapters.

Apart from their relation to the CH, the intrinsic utility of large cardinals for set theory is huge. For instance, it has been noticed that all large cardinals form, more or less, a linearly ordered hierarchy induced by consistency strength.

As is well-known, all consistency proofs must be *relative*, as Gödel's second incompleteness theorem rules out direct proofs (i.e., no theory powerful enough to capture *primitive recursive arithmetic* can prove its own consistency). For instance, the consistency of the CH with ZFC is the statement: $Con(ZFC) \leftrightarrow Con(ZFC + CH)$, which means that, *if* ZFC is consistent, then it remains so if one adds CH. This is rephrased as CH having the same consistency strength of ZFC. However, a theory T' is consistency stronger than a theory T if $Con(T') \rightarrow Con(T)$ (and not *vice versa*). For instance, $Con(ZFC + IC) \rightarrow Con(ZFC)$, where IC is the statement: 'there is an inaccessible cardinal'.

As has been said, consistency strength induces a linear order on large cardinals. As a consequence, using large cardinals, one can easily compare the strength of two set-theoretic statements, since their equiconsistency with two different large cardinals automatically results in an *absolute consistency strength measure*.

Only this suffices to prove that, although the CH is not decided by any currently known large cardinals, their theoretical strength and notions associated have completely re-shaped the way we think of the universe of sets.

2.2.4 Ω-logic, Ultimate L and the CH

Now let's turn to the last and, maybe, most exciting part of this brief history. This section is almost entirely drawn from Bagaria (2000, 2005, 2008) and is only meant to provide a very rough picture of Woodin's breakthroughs.

The readers will have by now realised that whether the CH has or has not been solved depends on their (arguable) choice. What is certain is that Woodin's results, once again, show how the CH is a driving force for the development of set theory along the lines of more internal cogency and conceptual unification.

If generic absoluteness is fostered as a relevant criterion to choose new axioms, the exact nature of its logic should be provided. This has recently been done by Woodin (Woodin (1999, 2001)) through the introduction of the so-called Ω-logic. Ω-logic is a natural extension of standard logic, more precisely it is a strengthening of the usual relations of provability and satisfiability of standard logic.

In order to set the scene conveniently, first one has to define what the intended semantic environment of Ω-logic is. As we saw in the last sections, through an appropriate coding, every forcing notion derived from a partial ordering $\mathbb{P}$ can be extended to a complete Boolean algebra $\mathbb{B}$. The result of this extension is the *construction* of the Boolean-valued universe of sets, $V^{\mathbb{B}} = \bigcup_\alpha V_\alpha^{\mathbb{B}}$.

This latter construction is the naturally resulting picture of a universe of forcing extensions, and it shares all the features of V. Absoluteness, therefore, qualifies more precisely as *absoluteness* across the Boolean-valued universe. It follows that a statement ϕ is *generically* absolute if and only if for all $\mathbb{B}$, ϕ is true in $V^{\mathbb{B}}$. Generic absoluteness obviously implies that whatever is true in the Boolean-valued universe, for all Boolean algebras, is true in V.

Ω-logic attempts to capture this notion more precisely. The satisfiability relation is reformulated in the following way:

Given a theory T and a formula ϕ,

$$V_\alpha^{\mathbb{B}} \models T \rightarrow (T \models_\Omega \phi \leftrightarrow (\forall \mathbb{B})\ V_\alpha^{\mathbb{B}} \models \phi)$$

$V_\alpha^{\mathbb{B}}$ is a standard model of ZFC and, therefore, $V_\alpha^{\mathbb{B}} \models Con(ZFC)$. This shows how strong an extension is Ω-logic as compared to standard first-order logic. $Con(ZFC)$ is an undecidable statement in first-order ZFC, but is decided in Ω-logic.[32] This suggests the following generalisation:

$$T \models \phi \rightarrow T \models_\Omega \phi$$

The converse, as seen, does not hold.

There is a corresponding provability relation ($\vdash_\Omega$) which can be defined in Ω-logic but its formulation is too complicated even to be mentioned here. It involves the existence of *universally Baire sets of reals* and deviates substantially from the natural understanding of provability in standard logic.

It should be remarked that the definition of validity in Ω-logic is itself invariant under forcing, that is, for any forcing extension $\mathbb{P}$, it is possible to prove that the proposition 'ϕ is valid in T according to Ω-logic' must be an Ω-valid statement.

As for standard logic, one might ask whether Ω-logic is sound and complete. Soundness is an easy result to prove, that is

$$T \vdash_\Omega \phi \rightarrow T \models_\Omega \phi$$

holds.

Completeness is what is known as Woodin's Ω-conjecture:

Claim 1 (Ω-conjecture) *If there is a proper class of Woodin cardinals, then Ω-logic is complete.*

The conjecture has not been proved yet, but there are some hints that it might be correct (see Steel (2004)). Woodin's argument concerning the CH depends on this conjecture and so there is a genuine interest in establishing whether it is correct or wrong.

What is interesting is that, in turn, the Ω-conjecture depends on the existence of a proper class of Woodin cardinals, thus marking a deep connection, once again, between ranks of the higher infinite and lower-level structures.

[32] However, this does not violate Gödel's Incompleteness Theorems, as it is always possible to find sentences which are undecidable even from the point of view of Ω-logic.

Now that the scene setting has been made, I will turn to Woodin's argument.

Let us consider the hierarchy of hereditary sets $H(\kappa)$, that is sets who transitive closure has a cardinality less than κ. $H(\omega)$ is simply the structure of all finite sets and is equivalent to $\mathbb{N}$, as it contains all the hereditarily finite sets, that is sets whose transitive closure is finite.

$H(\omega_1)$ is the structure of subsets of $\mathbb{N}$ but it is not equivalent to the powerset of $\mathbb{N}$. Actually, this would hold only in the presence of the CH. $H(\omega_1)$ is also the structure of all projective sets and we have seen that projective sets are all well-behaved in the presence of PD.

This is a crucial point, as Woodin stresses that PD is the best possible candidate because it *settles* the theory of $H(\omega_1)$. A further reason is that its consistency strength can be exactly measured using Woodin cardinals and, thus, reasons for believing in large cardinals gives us automatically strong reasons for accepting it.

Woodin asks whether there is something analogous for the next larger structure of sets of reals, that is $H(\omega_2)$. Therefore, his result should be seen in connection with this *desideratum.*

Theorem 7 (Woodin) *From the point of view of Ω-logic (and if the Ω-conjecture is correct), that is from the point of view of* generic absoluteness, *each axiom which will work for $H(\omega_2)$ in the same way as PD for $H(\omega_1)$, that is an axiom which will settle the structure of $H(\omega_2)$, must imply the failure of the CH.*

Moreover, there is an axiom, which Woodin calls $(*)$, which does the job quite well and, in particular, as other forcing axioms, implies that $\mathfrak{c} = \aleph_2$.

Woodin's result is very strong. Note that the informal formulation of the property of settling the structure of $H(\omega_2)$ can be made very precise. In particular, it can be reformulated as a *completeness* result.[33]

There is, therefore, no inherent vagueness in the above mentioned theorem.

[33] In the following way:

$$(ZFC + (*) \vdash_\Omega \ulcorner H(\omega_2) \models \phi \urcorner) \leftrightarrow H(\omega_2) \models \phi$$

However, as we have seen, its acceptance is hanging upon the correctness of the Ω-conjecture, and, in turn, the consistency of Woodin cardinals.

There has been a lot of debate surrounding Woodin's results (see Bellotti (2005)).

One obvious qualm, which generalises to all forcing axioms, is about how genuine is an axiom which attains generic absoluteness for a certain structure. All such axioms, if accepted on the grounds of their inducing generic absoluteness, would seem to be too much a contrived and *ad hoc* choice. As already said, forcing axioms have certainly a combinatorial character and, therefore, carry some information about the *world of sets*. However, in this case their use is made in view of establishing generic absoluteness and not for their nature as encoding some *true* properties of the world of sets.

The second remark is about the meaning to be attached to the requirement that the axiom satisfactorily solves all problems of sets of reals at the level of $H(\omega_2)$. While one may want PD to work in this way, because there may be other strong intuitive reasons for accepting PD (connection to large cardinals), it is not so obvious that this completeness might be significant for structures higher than $H(\omega_1)$ and that, accordingly, this would be a motivating criterion for picking axioms like $(*)$.

Woodin himself has recently suggested a change of direction (Woodin (2011b)). His research now seems to be oriented towards finding a generalisation of L through the construction of a hierarchy of inner models. It is known that inner models provide a very natural semantic environment for large cardinals, meaning that many large cardinal assumptions hold in L or one of its generalisations. However, as Scott's theorem states, L is ruled out by measurable cardinals. However, suitable generalisations which do not imply the Axiom of Constructibility can be provided (see Jensen (1995)).

One of the staggering results proved by Woodin and Shelah is that *generic*

where $\ulcorner H(\omega_2) \models \phi \urcorner$ codes the proposition which says that ϕ is *true* in $H(\omega_2)$ and, analogously,

$$(ZFC + (*) \vdash_\Omega \ulcorner H(\omega_2) \models \neg\phi \urcorner) \leftrightarrow H(\omega_2) \models \neg\phi$$

where $\ulcorner H(\omega_2) \models \neg\phi \urcorner$ codes the proposition which says that ϕ is *false* in $H(\omega_2)$.

absoluteness applies to some inner models like $L(\mathbb{R})$, the inner model which contains all ordinals and all real numbers. This absoluteness is equiconsistent with the existence of a very large large cardinal, a supercompact cardinal:

Theorem 8 (Shelah-Woodin) *If there is a supercompact cardinal, then there is an embedding*

$$L(\mathbb{R}) \to L(\mathbb{R})^{V^{\mathbb{P}}}$$

which is the identity, for all *forcing notions* $\mathbb{P}$.

In other terms, in the presence of a *supercompact* cardinal, every set of reals constructed in $L(\mathbb{R})$ must preserve its properties in any forcing extension. This is supplemented by other amazing results of Woodin and Shelah, which showed that in $L(\mathbb{R})$ all sets of reals are determined, that is AD holds in them. Therefore, reals in this particular structure must be extremely well-behaved, in particular, have all *regularity properties.*

Elaborating upon these results, Woodin has recently delivered the idea that one should search for an *ultimate* L, a suitably re-defined constructible hierarchy which serves as the natural structure for all large cardinal assumptions. Of course, in this structure the CH would hold.

Whichever of the two programmes one would feel like subscribing to, it is but certain that Woodin's attempts are tied to an *ultimate-universe* view. Arguing against the CH, Woodin has, at the same time, tried to show that the Continuum Problem is not vague, as, in his proof, the truth or falsity of the CH clearly draw the boundary between different characterisations of $H(\omega_2)$. Therefore, Woodin's underlying philosophical motivations would seem to square with those of Gödelian platonism.[34]

However, as we shall see, his *ultimate-universe* view, although grounded on such platonistic leanings, may not conform to the requirement that a unanimously accepted solution of this kind of problems should come from the acceptance of *true* axioms. On this interpretation, his results may substantially disconfirm Gödel's original hopes.

[34]See also Chapter 8 of this work for more details.

Chapter 3

The Philosophy of Indeterminacy

3.1 Perspectives

Now that the scene setting has been made, we can turn our attention to the twofold problem which is examined in this work: what is the relation between platonism and set-theoretic indeterminacy? Is platonism refuted by set-theoretic indeterminacy?

The fact that ZFC does not decide statements like the CH has some bearings on our conception of sets regardless of one's philosophical orientation. Accordingly, this section also aims to provide a general overview of different philosophical coping strategies. What follows is a description of the debate on set-theoretic indeterminacy, which, I think, will especially help put into focus the difficulties related to platonism.

In particular, the picture which has been provided in the first section needs extending. A clarification of the relation between platonism and its founding ontology has to be indicated, as part of the problem consists in determining whether the belief in a platonic ontology as founding the objectivity of set-theoretic statements is questioned by set-theoretic indeterminacy.

Overall, the philosophical strategies adopted can be grouped into two main categories: the first comprises those views which meet the need to spell out why it is reasonable to pursue further the search for a solution to the unsolved problems, with no, or hardly, any effort to tie these reasons to a pre-existing philosophical conception. Put in other terms, in the presence of undecidability, one is required to indicate whether a particular problem might have a solution, whatever it may be, and by what means the solution should be attained. The second category consists of philosophical conceptions which explain what indeterminacy is and how and whether it should be fixed.

As the history of the CH shows, in many cases it is hardly possible to separate the mathematical from the philosophical.

For instance, Cohen's point of view, as expressed in Cohen (1966) is firmly rooted in mathematical evidence, but also offers interesting philosophical insights. While not disavowing the possibility of finding a solution, he says that:

> In essence, one can feel that set theory is merely a highly successful shell which has nothing to do with "real sets" but at best describes some type of mental process used in describing the real objects such as integers. The great defect with this view is that it leaves unexplained why this fiction is successful and how a presumably incorrect intuition has led us to such a remarkable system. ((1966), p. 151)

There, he also casts his prophecy, which seems to fit into the contemporary developments, that the CH might eventually come to be viewed as false. The reasons provided contain traces of philosophical elaborations upon the *intuitive* notion of the continuum, which, in his opinion, cannot be reached through a process of *piecemeal* construction from below using the Axiom of Replacement.

> Thus $\mathfrak{c}$ is greater than $\aleph_n$, $\aleph_\omega$, $\aleph_\alpha$ where $\alpha = \aleph_\omega$ etc. This point of view regards $\mathfrak{c}$ as an incredibly rich set given to us by a one bold new axiom, which can never be approached by any piecemeal process of construction. ((1966), p. 151)

Leaving aside Cohen's own point of view, what was clear after the publication of his outstanding contributions was that the problem itself would not be solved easily, particularly because, at the time, enquiry into large cardinals and consistency results was at its beginning. Soon it was rightly and dramatically pointed out that the hope to single out a fairly invariant universe of sets had been shattered once and for all. Although resurrected by the *ultimate-universe* conception induced by *generic absoluteness* or *ultimate L*, the hope to restore the unique heaven of sets has since been seriously and, on some set-theorists' opinion, definitively, undermined (see, for example, Shelah (2003) and Hamkins (2009) in Chapter 8).

But well before Cohenian *multiversism* was revealed, Skolem's work had pointed to the intrinsic relativity of set-theoretic notions. As said, Skolem's conception had been strongly countered by Zermelo and the relativism implied by Skolem's theorems had been all too quickly discarded as a minor shortcoming of the language of first-order logic.

In his thorough-going analysis of the topic, Putnam (1980) addresses model-theoretic relativity vindicating Skolem's vocabulary and achievements. Model-theoretic relativism as arising from Skolemian and Cohenian, so to speak, relativity is the strongest threat to Gödelian platonism. In my opinion, Putnam correctly identifies its target as being what he calls *moderate platonism*, in which he would presumably subsume Gödelian platonism. On the contrary, as we shall see, those forms of platonism which do not claim STRONG ACCESS (which he generically labels *extreme platonism*) along with *verificationism* should, on his view, be resilient to the threat posed. However, I will argue that *extreme platonism*, which Tait calls *super-realism* is too theological to result philosophically convincing.

As a result of Putnam's unsurpassed analysis, Gödelian platonism has but two equally painful alternatives: either falling into extreme platonism, dropping STRONG ACCESS and becoming epistemically *weak*, or accepting that mathematical objectivity might be accounted for in a different way, first of all by giving up the pretension that we have a definite conception of the *universe of sets*. As we shall see, either of the two alternatives is fraught with problems, although the latter a lot less.[35]

Kai Hauser's re-interpretation of Gödel's initiatives clearly conforms to the purpose of promoting the second alternative, that of accounting for objectivity along different lines of reasoning. However, Hauser's position in the debate is relevant under many aspects. In his lucid article (Hauser (2002)), he emphasises the fact that platonism is used as an excuse to deprive the CH of its meaningfulness. Feferman is the clearest example of this attitude. Feferman's attempt to demonstrate the vacuousness and (partial) unintelligibility of the problem is, in Hauser's opinion, strongly dependent upon the presumption that a problem such as the Continuum Problem is meaningful only for platonists. Hauser's arguments aim to bring forward evidence that this is not the case.

Unfortunately, as said in Chapter 1, in Feferman's view, *operational* platonism (*anti-constructivism*) is as much imbued with a full-fledged platonistic conception as its substantive and philosophical counterpart. Set theory thrives on *operational platonism* and, therefore, Hauser's subtle distinctions do not hold. This, ultimately, leads to a stalemate in the debate between the objectivist camp (represented by Hauser) and the supporter of a *predicativist* conception of sets like Feferman.

Hartry Field's analysis proceeds along Feferman's lines, insofar as it gives up set-theoretic objectivism, but it leaves room for a moderate *finitary objectivism*, motivated by the claim that the notion of finitude is not problematic.

Given the ontological adequacy and lack of indeterminateness of second-order axioms, these latter could be viewed as being naturally subservient to the claims of mathematical realism. However, preference for a second-order account does not come without a price. Above all, as informally pointed out by Putnam and Feferman (and by Quine, in the first place) the lack of indeterminateness of its vocabulary and ontology is eminently due to the conspicuous set-theoretic setting in which it is cast. Even if this did not really seem to be an issue, the fact remains that one has the persistent feeling that second-order determinacy is

[35]Indeed, the re-structuring of the notion of objectivity had already been provided by Gödel himself, through the coinage of *conceptual realism* (see Chapter 7). Gödel's general view on the issue of indeterminacy will be examined in Chapter 4.

somewhat artificial, especially if measured against practical set-theoretic reasoning. Such determinacy seems to be built in from the start and this would reveal, in Feferman's words (Feferman (2011)), that any attempt to see determinacy as a bonus of second-order logic must imply a vicious circle.

As seen, at the very core of the issue of indeterminacy lies the problem of whether we have a proper understanding of the notion of *universe of sets*. Different orientations and views develop the theme in different directions and it is also clear that also those mathematicians or philosophers whose contributions belong to the first category cannot fully escape the need to provide some philosophical details and reflections. Some of these contributions to the debate are examined in what follows.

3.2 Model-theoretic Relativism

3.2.1 Skolemian Relativity

As is known, the notion of *model* is a key one in logic. Roughly speaking, a model A for a theory T is a structure in which all $t \in \Sigma$, where Σ is the set of all the closures of sentences of T, are true. Models spell out the semantic content of a particular theory and, given the results of *completeness* of the deductive calculus within first-order theories, they show that the semantic approach is equivalent to the syntactic one.

We have already seen a particular subset of models, ZFC-transitive models, like $< M, \in >$, that is models which contain the elements of all their elements. Using Mostowski's theorem, it is possible to prove that any model of ZFC which is a set and in which Extensionality holds is uniquely equivalent to its *transitive collapse*. This formidable result turns out to be essential in many model-theoretic constructions for set theory, like Gödel's definition of the constructible hierarchy. Mostowski's theorem also shows that models of ZFC are essentially and standardly transitive models and one need not add any further conditions. V is, obviously, such a model, as can be shown very easily.

A general template for a model is an ordered pair $< A, R >$ where A is a domain of objects and R is the set of all relations defined on the domain. For instance, $\mathcal{M} = \{\mathbb{N}, <, +, \times, s, =\}$ is the standard (intended) structure of Peano's axioms for arithmetic, where the relations of $+$, $\times$ and successor (s) are defined in the usual way. This shows that the notion of model is, at the outset, naturally tied to the intuitive *ontology* of a particular theory. It is but natural to think that the underlying ontology of number theory consists of natural numbers with the basic relations on them as defined by Peano's axioms. Analogously, models for ZFC are transitive classes, that is fragments of V or of its subuniverses with the $\in$-relation defined in the usual way. So far so good.

Unfortunately, the tie between the intuitive ontology of a theory and its model-theoretic structure is not fixed in first-order logic. As noted before, the great advantage or disadvantage (depending on the point of view) of first-order logic lies in its flexibility. Structures can be easily *extended* to larger non-isomorphic structures or elementarily embedded onto submodels of a certain structure[36] and this is exactly what is shown by the Löwenheim-Skolem theorems:

Theorem 9 (Downwards Löwenheim-Skolem) *If a theory T has an infinite model, then it has a countable model.*

whose upwards counterpart is

Theorem 10 (Upwards Löwenheim-Skolem) *If a theory T has an infinite model, then it has model of any cardinality α.*

Some of the counterintuitive consequences of these theorems are well-known. With regard to the upwards direction, the theorem implies that first-order arithmetic has *non-standard uncountable* models. What these models are like can be shown very easily through a standard addition of new constants to the domain. With regard to the downwards direction, it must necessarily follow that ZFC

[36]Two structures $\mathfrak{A}$ and $\mathfrak{B}$ are said to be isomorphic if there is a mapping of the elements of their domains which is truth-preserving and, if D and D' are their respective domains, $|D| = |D'|$. The elementary embedding of a structure into another is a truth-preserving injection $f : A \rightarrow B$ of the domain A of one onto a subset of the domain B of the other.

has a countable model. By the elementarity of the embedding which can be easily generated, therefore, all that is true in an uncountable model $\mathcal{M}'$ of ZFC must, therefore, be true in a countable model $\mathcal{M}$. This remarkable consequence of model-theoretic results seemed to be paradoxical and, thus, elicit the idea that there was something wrong with the formalism of first-order set theory, as expressed by:

Skolem's Paradox *If ZFC is consistent, then it has a countable model ($\mathcal{M}$). Therefore $\mathcal{M} \models$'there is an uncountable set' (Θ).*

The paradoxicality involved has been subject to an extensive investigation and debate. However, *prima facie* it is clear that it does not lead to any antinomy or contradiction within ZFC. It must be necessarily so, as all talk about models of ZFC must informally take place outside ZFC. Also, externally, we see that there is a clash between the fact that Θ is a theorem of ZFC which holds in a countable structure, but internally, from the point of view of $\mathcal{M}$, the situation is different. The function which maps uncountable sets to the set of the real numbers, for instance, *does not have to lie* inside $\mathcal{M}$. Inside $\mathcal{M}$, all talk about uncountable sets can be *relativised* to the available sets, that is countable sets, with no loss of information about the external reality, so to speak.

Therefore, what Skolem's paradox actually shows is that first-order set-theoretic notions are *relative to* structures and that the reference of the language of first-order set theory is not fixed by the axioms.

As seen, although Skolem's paradox is not really an issue mathematically, his theorems must necessarily have drastic consequences for our conception of the relations between ontology and semantics. As a first and foremost consequence, one should, at least, admit that the natural and intuitive tie between ontology and semantics is severed and there is no hope to fix this situation. The expressivity of first-order logic is too weak to model ontology accurately.

The ideology, initiated by Skolem, which proclaims that any ontological consideration regarding set theory is ruled out by set-theoretic relativity can be conveniently labelled *model-theoretic relativism.* Actively pursued by Skolem and, especially, by anti-objectivist philosophers, this ideology represents a major threat to the Gödelian platonist's conceptions.

In the previous section, we have met another form of model-theoretic relativity, that arising from forcing (see Principle 1). However, whereas Skolemian relativity seems to be more general insofar as it is rooted in our language, forcing reveals the indeterminacy of a particular structure, V. However, the combination of these forms of model-theoretic relativism leads to the same disheartening conclusion: using first-order logic, there is no hope to model our ontological presuppositions with a very high degree of precision.

3.2.2 Putnam's Argument

That model-theoretic relativity obviously has consequences for a platonist is clearly and convincingly argued by Hilary Putnam in Putnam (1980). I find Putnam's demolition of the Gödelian's convictions definitive and, therefore, I will examine the argument in detail. Putnam's argument aims to show the following three facts:

1. The relativity of set-theoretic notions can be extended to the relativity of the truth-value and, thus, the *determinacy* of many set-theoretic problems.

2. As a consequence of (1), the problem invests not only a Gödelian platonist, who believes that the objectivity of the truth-value of all set-theoretic statements is determined by an underlying independently existing ontology, but also a truth-value realist, who just cares about getting determinate truth-values.

3. Finally, set-theoretic skolemisation extends to the semantic determinacy of all our scientific language and beliefs, inasmuch as all our theories are formulated within a convenient first-order logical framework.

Here, I will just be concerned with (1) and (2) and I will not deal with the most radical consequences of Putnam's argument.

(1) is shown using the example of the Axiom of Constructibility.[37] In accordance with our mathematical practice, both intrinsic and extrinsic criteria may be scrutinised in order to reach a verdict about whether this axiom is true or false.

For instance, one reason for believing that $V = L$ is false might be that, as a consequence of Scott's theorem mentioned in Chapter 2, the axiom would rule out the existence of some large cardinals. Now, in the case of large cardinals, the reason for accepting them seems to be mainly intrinsic, as they derive from generalisations of Cantorian generating principles. Therefore, in the presence of sufficiently strong large cardinals assumptions, $V = L$ should be false on purely intrinsic grounds.

Now, Putnam argues that in the light of model-theoretic relativity, all such debate about the acceptance of new axioms is (partly) meaningless. For, he says, let's suppose that one reasons exactly in the same way as shown above and says that $V = L$ is false.

He is able to construct a Π_2-sentence of set theory which says, roughly, that, given any countable subset of reals S, there is a constructible ω-model[38] M where S is represented. Since countable subsets of reals can be coded by reals through a standard procedure, then the sentence says that, given any real s, there is a constructible ω-model where s is represented. Therefore, our Π_2-sentence will say that there is a constructible model for any real.

The remarkable fact is that Π_2-sentences are absolute, by Shoenfield's theorem (1967), and, therefore, our sentence must also be true in all models, in particular, it must be true in V. Hence, Putnam's conclusion:

Theorem 11 (Putnam, 1980) *Given any real s, there is a countable ω-model M, such that $M \models ZF + V = L$ and s is represented in M.*

[37] Which says that the cumulative hierarchy (V) is equal to Gödel's *constructible hierarchy* (L).

[38] An ω-model is a countable model which behaves in the same way as the structure of the natural numbers.

Now, we are claiming that $V = L$ is false *in reality*. It must follow that there are non-constructible reals. However, by Theorem 11, we can always find a constructible model which contains s.

Therefore, how could a Gödelian platonist who thinks that $V \neq L$, because he is convinced that this is the case in the *real* world of mathematics, ever justify his view on other criteria than the existence of *occult qualities*, to use Putnam's words?[39] The point is expressed in the following passage:

> The claim that Gödel makes is that "$V = L$" is false in "reality". But what on earth can this mean? It must mean, at the very least, that in the case just envisaged, the model we have described in which "$V = L$" holds would not be the *intended model*. But why not? [...] Perhaps someone will say that "$V \neq L$" (or something which implies that V does not equal L) should be added to the axioms of ZF as an additional "theoretical" constraint. [...] But while this may be acceptable from a non-realist standpoint, it can hardly be acceptable from a realist standpoint. [...] A realist like Gödel holds that we have access to an "intended interpretation" of ZF, where the access is not simply linguistic. ((1980), in [11], p. 425).

Putnam's argument, therefore, aims to establish the general point that the notion of *external*, independent reality, as conceived of by Gödel, is made useless and, even unintelligible, by our model-theoretic manipulation of sets.

The second point which Putnam tries to establish is that this affects also a semantic realist's conception. This does not need any appeal to Skolemian relativity, since, as shown by Principle 1, it is Cohenian, so to speak, model-theoretic relativity which leads to this conclusion.

[39]Of course, if model-theoretic considerations are seen as legitimate *ontological* considerations, the argument could simply be used to assert that $V = L$ is actually true, as one could argue that models of ZFC do not ultimately distinguish between constructible and non-constructible reals. But this would not be sufficient for a Gödelian platonist, as he may only want to judge of the truth of the axiom presumably on the basis of his alleged *pre-theoretic* access to the universe of sets. However, *generic absoluteness* arguments involve this sort of model-theoretic reasoning, as we have seen. But then, again: to what extent is this reasoning natural and genuine for a platonist?

Putnam makes no mention of forcing in that work. He just says that there seems to be a limit to our mathematical observations and, therefore, there will always be some indeterminacy. Putnam sees this as a natural extension of his argument, but, to my surprise, does not push the point any further to show how this indeterminacy could be assumed to be a radical form of indeterminacy in the present context of contemporary set theory.

3.2.3 The model-theoretic challenge to Gödelian Platonism and Extreme Platonism

It is my opinion that, as much as universe indeterminacy, the referential (in the Skolemian sense) indeterminacy of first-order logic is one of the main issues affecting Gödelian platonism.

Now, as shown by the history of the Continuum Problem, focus on objects is not sufficient to provide solutions to this situation and some sort of reasoning on models as an attempt to settle set-theoretic questions of the level of complexity of the CH has become indispensable. However, even so, it is not clear how model-theoretic reasoning, propelled by generic absoluteness, is really able to provide viable solutions. Moreover, Skolemian relativity seems to stand out as an overarching and unsurmountable objection to any attempt to restore a link between a Gödelian's purported *strong access* to the universe of sets and first-order semantics.

One alternative for Gödelian platonists would be that of modifying his views so as to retreat to extreme platonism. An extreme platonist holds INDEPENDENCE, EXISTENCE and TRUTH. However, contrary to Gödelians, this platonist does not hold STRONG ACCESS, rather a weaker version of it:

WEAK ACCESS The universe of sets is only partly knowable and accessible.

It is argued in what follows that this solution is a very weak response to the problem and that, therefore, it makes sense to describe it as a retreat to *extreme platonism.* In essence, it is the retreat to the belief in the existence of Putnam's *occult qualities.*

To many commentators, what I call extreme platonism is platonism in its pure essence (again, see Linnebo (2009)). Referential indeterminacy is not a problem, it is said, for this conception, as it does not claim that we are able to find all the objective truth-values of all mathematical and, in particular, set-theoretic statements.

To me, this view has always appeared troublesome and hardly defendable. I shall try to explain why.

It is clear that, by this conception, there is no link between the mathematician's access to an independently existing ontology and the mathematical theories which establish the truth-value of a mathematical statement. An extreme platonist holds that all set-theoretic statements have a truth-value, even if we are not able to find them. However, presumably a platonist mathematician of this sort will have to deal with mathematical theories as any other mathematician. He will then find that some areas of mathematics are more *determinate* than others, which, in his interpretation, means that he has a stronger access to some areas of mathematics than others. Then I would ask: what reasons would this person provide to account for this phenomenon?

A possible solution seems to be hinted at by Dummett in Dummett (1963). While addressing the notion of *intuitive provability* of a statement, he says:

> This need not mean, of course, that a platonist is committed to holding that *every* mathematical statement possesses a definite truth-value. His only claim is that, once we have definitely assigned the range of our variables and the application of our primitive predicates, all statements formed from these predicates by means of the sentential operators and of quantification over this range acquire a definite truth-value, *true* or *false*. This presumably holds for number theory or analysis; but it is still open to him to allow that for some theories - set theory, for instance - we have not yet assigned the range of the variables in a completely determinate manner, and hence that in such a theory there may be statements for which we have not yet determined a definite truth-value. ((1963), p. 164)

On my reading of Dummett's quote, number theory and analysis will not be subject to indeterminacy, whereas set theory will. But if Dummett is right, then my question would translate into: what criteria would this person provide to draw clear boundaries between number theory and set theory? I argue that this might be a very problematic task.

For the *set-reductionist*, set theory is the foundation of mathematics and, therefore, all mathematical concepts are re-translatable into set-theoretic notions. Unless one thinks that this translation is ontologically non-conservative, that is, that sets only vaguely resemble numbers, spaces, algebraic structures, but are essentially different from all such items, one has to admit that set-theoretic indeterminacy *is* simply mathematical indeterminacy.

Even disregarding set-reductionism, we know that problems of set theory have been shown to have bearings even on our understanding of finite sets and finite combinatorics (see Friedman (1986, 1998) and Hauser (2002)). Although the point is controversial, it seems that it is reasonable to assume that Gödel's programme of enlarging the transfinite structure of sets will have more of such consequences on finite sets in the future. This does not necessarily mean that such results make number theory less determinate, only that it seems problematic clearly to separate the realm of number theory from that of set theory.

On the other hand, one can say that it is a matter of fact that set theory and the sort of problems which are connected to it are sometimes regarded very uncomfortably by other mathematicians, because there is too much generality involved in their formulation. The point is stressed by Harvey Friedman:

> There was a growing realization [in the mathematical community, *my note*] that the cause of these difficulties was excessive generality in the formulation of the problems which allowed for pathological cases which were radically different in character from normal mathematical examples. ((2000), p. 436)

However, in what sense is set theory dealing with pathological cases? Unless one has strong reasons to believe that set theory is somehow pathological, one should not venture into the claim that it can be successfully severed from the rest of mathematics. It could be done practically (that is, one could avoid any reference to sets in mathematics), but, in view of its foundational status, I do not see how this could ever be possible *ontologically*. Even assuming that natural numbers are clearly different from sets, it would remain to be explained why they seem to be better understood by appealing to transfinite ordinals.

If Dummett's statement only prefigures a sort of provisional position, which will cease to be valid when we will fully make sense of the notion of universe of sets and will have found a clear reference for this notion, I do not see how this position would substantially differ from that of a Gödelian platonist.

WEAK ACCESS can be unpacked as involving an argument from *unknowability* or *inaccessibility*, which clearly has a theological ancestry.

Unknowability Argument *Mathematical objects and truths exist independently of our minds and theorisations. We are not able to perceive all the mathematical relations which obtain among mathematical objects, that is we do not have access to the whole of its internal structure.*

To me, the reasons for embracing such an argument have always been far from being clear, and, consequently, I view as all the more surprising that much platonistic discourse is based on it, with no attempt to clarify its assumptions.

It has been, in my opinion, correctly pointed out that this argument leads directly to the belief in the existence of a *speculative* mathematics. In the following quote, Tait provides an interesting definition of speculative mathematics in terms of *super-realism*:

> An important consequence of super-realism and, as I believe, a telling objection to it, is that it implies an alienation of truth in mathematics from what we actually do: mathematics becomes *speculative* in the sense that even the most elementary computations, deductions and propositions must answer to a reality which we, at best, can only partially comprehend and about which we could be wrong. If there are grounds for truth and existence and they are not the axioms, then the axioms could be false. ((2009), p. 3)

The characterisation of extreme platonism as *super-realism* is quite intriguing. If one embraces this interpretation, one has to conjecture the existence of a deeper, possibly incomprehensible level of mathematical reality, which our presumably finite minds are not able to spell out completely.

The reader will judge autonomously whether this argument has any plausibility, apart from its identification with Tait's *super-realism.*

The Unknowability Argument is sometimes recast as an argument resting upon Cantor's Absolute Infinite. In this section, there is no time to deal with Cantor's conception (see below, Chapters 4 and 7), but some brief hints can be given.

Cantor (1883, 1887, 1888) came to see the infinite as essentially coming in two forms: as the *Transfinitum* (Transfinite), which is the proper actual infinite and as the *Absolute Infinite*, which is God. To be precise, the Absolute Infinite also plays another role in set theory, as an internal limitation principle, but for now I will just be concerned with its metaphysical identification with God.

Since the Absolute is God, then, by any sufficiently reasonable theology, the Absolute is unknowable, or not fully knowable. Therefore, this would justify the view that our access to the infinite is partial and only partially mathematically comprehensible. We have seen a technical related use of this principle (Chapter 2), within the Reflection Principle, as providing the philosophical motivating reasons for expanding the realm of the transfinite. Therefore, the Unknowability Argument, tied to some mathematical re-translation of Cantor's Absolute Infinite, may become a legitimate *operational* principle of set theory. However, how it would be possible to extend its operational use to an epistemological position about the incomprehensibility of the mathematical reality is far from clear.

The Unknowability Argument might also be construed as an argument which says that our minds are *finite* and that, therefore, there is no chance to fully understand the infinite. However, even if we had some sort of neuroscientific support in favour of this thesis, I guess that this would not help much, as, in that case, one should explain how it is neurophysiologically possible to conceive of something like the Absolute Infinite (although in an obscure and limited way) and, at the same time, not be able to capture its features.

To be clear, I am not claiming that there is not at the moment and there will never be any conceivable principle which would make the Unknowability Argument plausible. What I am just saying is that it is hard to see how this strong metaphysical presupposition would help the interpretation of set-theoretic indeterminacy in view of all our currently known mathematical practice. Of course, this will always depend on how robust one's metaphysical assumptions are. However, insofar as the knowledge of mathematics has always been strongly linked to practising mathematics, robust metaphysical assumptions seem to generate more that sense of alienation evoked by Tait rather than a sense of reassurance.

To summarise, it seems to me that extreme platonism is very weak. It might be true that an extreme platonist might be better placed to face up with the indeterminacy phenomenon, but this brings in a very high price, in my opinion, that of embracing very strong and hardly justifiable assumptions.[40]

Alternatively, a Gödelian platonist who does not want to commit to extreme platonism does not have any other alternative but try to reformulate mathematical objectivity on different grounds. The topic is partly explored within the Hauser section and resumed, again, in Chapter 4 and Chapter 7.

[40] On an incidental note, the argument from the *finiteness* of our minds turns out to be quite the opposite of the fairly standard platonistic view that our minds exceed a computer's ability to *understand* mathematical facts. In particular, our minds would be more powerful than a Turing machine. This would be shown, for instance, by our ability to capture the incompleteness of formal systems like Peano's arithmetic. Hints of this conception can be traced to Gödel (*1951), although the interpretation of this work is still controversial (see Boolos's introduction, in Gödel (1995)). An interpretation of Gödel's incompleteness theorems along the lines indicated has been provided by authors who have explicitly endorsed a platonistic conception of the mathematical reality. In particular, this interpretation has been strongly advocated and defended by Lucas (1961) and Penrose (1989, 1994, 2005). Therefore, historically it would seem that any argument from unknowability and the finiteness of our minds would be at odds with a Gödelian platonist's own claim that our minds are essentially non-mechanistic and, therefore, able to understand infinitary concepts.

As noted, the mismatch between our understanding of the reality and the reality itself seems to be a staple of much platonism (or is very frequently ascribed to platonism). Maybe these platonists, even unconsciously, would just be paying their homage to Plato's emphasis on the struggle between *doxa* and *episteme* in our knowledge acquisition.

3.3 Second-Order Set Theory

First-order model-theoretic relativism stands out as an incredibly strong objection to mathematical platonism. Although we have seen that some measures ought to and can be taken, their efficacy is dubious.

Now, we have mentioned second-order logic a certain number of times thus far. It turns out that one of the natural coping strategies which a Gödelian platonist might want to adopt is that of appealing to the features of second-order logic to relieve himself of the burden of indeterminacy. As already noted, after all, the early axiomatisations of natural numbers, real numbers and sets had been cast in the language of second-order logic. And it was only after Skolem's and Gödel's extensive use of a first-order framework that second-order was temporarily sidelined, only to pop up again in the most recent debate on the foundations of mathematics.

Developments in the study of subsystems of arithmetic also feature a second-order account (again, see Simpson (1999)). The case for second-order logic has been exhaustively presented by Stewart Shapiro in Shapiro (1991).

Generally speaking, the central thesis of Shapiro and all second-order supporters is that second-order logic is in a better position to define something like the universe of sets, given the categoricity of second-order formal theories. Therefore, the switch to second-order languages would be necessary in the light of strong concerns for ontological issues.

Moreover, thriving on the features of second-order logic, some of its supporters are naturally inclined to accept a structuralist account of mathematical ontology. Obviously, this is not implicit in the choice of second-order logic. One can regard second-order logic as the *true* logic and, at the same time, refrain from any particular ontological account of mathematics (in particular, from structuralism). However, in recent times, some form of structuralism has proved to be the standard choice for many supporters of the preferability of second-order.

In this section I will argue that, whatever one might think of second-order set theory in philosophical terms, it is dubious whether the issue of indeterminacy is addressed by it efficaciously. While I think that second-order logic fully reveals the importance of thinking in structural terms in mathematics, I am not convinced that this move allows us to deliver hopes that, by it, one gets a more definite picture of the universe of sets.

3.3.1 Categoricity

In second-order logic, the quantifiers bind individual variables and predicate letters. The language of second-order logic, L_2, is the same as the language of first-order logic, except that quantification over relations is admitted.

One of the main assets of second-order logic is its expressive strength. By expressive strength, I mean the ability to capture some notions to an extremely high degree of precision.

In second-order logic I can define a sentence ϕ_{UN} which *univocally* says that there is an uncountable set. Analogously, I can *univocally* express that there is an infinite set using one sentence ϕ_{IN}. Many other mathematical properties and relations can be analogously *univocally* defined. For a comprehensive list, see Shapiro (1991). Now, what does *univocally* mean exactly? It means that the sentence in question is true if and only if it is true in a model which contains an *uncountable* set, in the case of ϕ_{UN}, or if and only if the model is *infinite*, in the case of ϕ_{IN}.

In first-order logic, as we saw, the domain of objects, even when specified, can be easily *collapsed*, that is embedded into a substructure, or extended to a larger structure quite easily, as a result of the Löwenheim-Skolem theorems. Therefore, first-order analogues of second-order sentences must fail to be *only* true in one structure with some univocally definable properties.

It is not difficult, however, to show that model-theoretic relativity does not apply to second-order logic. The reason is that, in second-order logic, we can talk about subsets of the domain directly and, therefore, all second-order properties of mathematical objects (or, more simply, second-order mathematical objects) can be directly singled out.

However, this expressive strength comes with a price.

Second-order deductive calculus DED_2, as opposed to the deductive calculus of first-order logic, is incomplete. This means that there are second-order valid formulas which are not provable. On the contrary, it was shown by Gödel that first-order deductive calculus is complete.

This fundamental incompleteness, therefore, represents one major shortcoming of second-order logic.

However, if one reformulates arithmetic, real analysis and set theory in a second-order framework, then one gets a bonus which really marks a strong difference with first-order.

Theorem 12 (Dedekind, 1888) *The second-order theory of arithmetic, PA^2 is* categorical, *that is: any two models of it must be isomorphic.*

Theorem 13 *The second-order theory of real analysis, that is the formal theory of real numbers, is categorical.*

Theorem 14 (Zermelo, 1930) *1) $V_\kappa \models ZFC^2$, where κ is the least inaccessible cardinal. 2) Any two models of ZFC^2 are either isomorphic or one is isomorphic to an initial segment of the other, that is, one is an* end extension *of the other (quasi-categoricity).*

Categoricity probably represents one of the most appealing features of second-order logic. It compensates the supporter of second-order set theory for the loss represented by the incompleteness of the second-order deductive apparatus.

But, what if one wants to achieve deductive completeness for second-order logic? Well, this is possible, if one switches to a reformulated second-order semantics, consisting of Henkin models.[41]

Recall that a model $\mathcal{M}$ is a structure $< M, R >$ endowed with a domain (M) and a set of relations (R). In first-order logic, the domain is represented by objects over which first-order variables range. In second-order, the variables range also over relations, and we have seen that relations are subsets of M^n, the *generalised product* of the domain.

[41] Other interpretive frameworks are also possible, see Shapiro (2005) for a comprehensive overview.

Now, a Henkin model is a structure $< M^D, R >$ where the variables range over a smaller subset of the original domain and, thus, not all relations are admissible. Only relations definable in M^{D^n} are. If one adopts Henkin semantics for second-order logic, then one achieves a complete deductive calculus. Unfortunately enough, all the shortcomings (or advantages, depending on the point of view) of first-order logic, such as model-theoretic relativity, Löwenheim-Skolem theorems and *compactness* will hold in the Henkin interpretation of second-order logic.

So, the situation is the following: either one adopts *full models* and gives up completeness, but gains categoricity results or one gives up categoricity and achieves deductive completeness. The choice is not easy. The are philosophical reasons for preferring one semantic framework over the other. In general, strong concerns over ontology will lead to the acceptance of full models, which express, as shown, the most distinctive character of second-order as opposed to first-order logic.

3.3.2 Indeterminacy Fixed?

It is important to understand what it means for second-order set theory to be categorical. In ZFC^2 I can define a sentence ϕ_{CH} which asserts the CH. I can analogously define the sentence which expresses the negation of the CH, $\phi_{\neg CH}$. Now, I can show that one of these two sentences is a logical truth in ZFC^2. The reason why ϕ_{CH} or $\phi_{\neg CH}$ must be a logical truth of second-order logic derives from the expressive strength of second-order logic: ϕ_{CH} is defined in such a way that it is true only in a structure where the CH is true.

This fact, along with the categoricity result established by Zermelo, would lend support to the following claim:

Claim 2 (Second-Order Supporter) *In second-order logic the structure of the universe of sets is fully determined, as shown by the strict correspondence between the content of a set-theoretic statement and its instantiation in the set-theoretic hierarchy* $< V_\kappa, \in >$.[42]

[42]Of course, any extensions of V_κ as a result of the addition of large cardinal axioms can

Now, is Claim 2 correct? The question is not of little moment: if Claim 1 is correct, then the problem of indeterminacy would simply reduce to the problem of choosing the best logic. A second-order supporter will, then, say that, since there is no convincing argument against embracing a second-order account of mathematics, then the problem is solved.

In particular, a Gödelian platonist will see this as a confirmation of her idea that there is an objective truth-value to statements like the CH because there is a *well-determined* independently existing ontology which is fully captured by a second-order account of set theory. Finally, everybody, regardless of their particular philosophical orientations, will admit to the fact that we have an understanding of the world of sets and that such understanding is captured by second-order set theory.

But is it all so simple?

What follows is an attempt to show that the optimism implied by Claim 1 is unjustified. I will just present some of the main criticisms of this conception, including mine, but I am warning the reader that what follows is not going very deeply into the question, as this would require a separate and self-contained work.

First of all, Putnam (1980) has convincingly argued that the semantics of second-order logic is as indefinite as its calculus. In particular, as we have seen, there can be different frameworks for second-order logic, while, for first-order logic, everybody will more easily agree on the main semantic facts.

As we have just seen, the choice of the semantics has primacy, as from this choice follows a series of results which have strong bearings on our conception of second-order logic. But, if the semantics is not intuitively understood, how can one ever make a reasonable and justified choice? In particular, if one is especially concerned about ontology, then one should choose a characterisation based on *full models*. To me, this seems the least arguable choice, as if there is any reason to choose second-order logic, then categoricity is the reason. However, if one

be shown to be quasi-categorical in second-order logic in the same sense as V_κ, where κ is the first inaccessible cardinal. See Väänänen (2001).

chooses ontology and categoricity, one will have to live with the fundamental incompleteness of the second-order deductive calculus.

This latter issue is not less important. There are philosophers who think that this inherent incompleteness implies that second-order logic is not a sufficiently strong logic. Since the main task of logic is to provide the strongest possible deductive apparatus, which is what first-order logic succeeds in accomplishing through its *completeness*, then second-order logic is clearly an inadequate account of logical reasoning. Hence comes the predicament of a second-order logic supporter who chooses full rather than Henkin models.

On the other hand, if he chooses Henkin models, then all the advantages of second-order will vanish instantaneously, making the choice of second-order less justifiable from the ontological point of view.

I have a much stronger concern about the validity of Claim 1. When I say that ϕ_{CH} seems to reflect a fully determinate set-theoretic ontology because of the way it is formulated, do I also necessarily mean that, in virtue of this, the Continuum Problem becomes a more determinate problem? In simpler terms, what contribution, in terms of solutions to the Continuum Problem, does something like second-order purported determinacy bring? Granted, I would certainly be happy to say that there is a fully determined universe of sets if this sort of information were *actually* conveyed by ϕ_{CH}. But the truth is, ϕ_{CH} does not mean that the universe of sets is *ontologically* more determined in a second-order framework, because from ZFC^2 comes no more hint at whether the CH is true or false than comes from ZFC.

But then one could reply: but your problem was with the set-theoretic reality (universe), not with the insolubility of some particular set-theoretic statements. Consequently, if your main concern is about indeterminacy, then the problem of how the Continuum Problem will be solved should be sidelined. But then, I think I would answer back that, in the absence of more substantive information, the sort of determinacy induced by ϕ_{CH} is unsatisfactory.

Moreover, I would add that first-order model-theoretic relativity is a better way to represent that sort of *pre-theoretical* intuitive indeterminacy of the set-theoretic universe (as also carried forward by my brief history of the CH) which is only deceitfully by-passed by second-order logic. First-order logic lets this pre-theoretical indeterminacy emerge very naturally, while second-order set theory just disguises this situation giving an artificial picture of the world of sets. On this view, second-order determinacy is a clear disadvantage and, in some way, a fraud. Ignasi Jané says:

> The equivalence of the validity of particular pure second-order sentences with set-theoretical assertions suggests a skeptical view of the *determinacy* [my italics] of canonical second-order consequence. Namely, it suggests that whenever we speak of canonical second-order consequence, we are alluding only to a highly underdetermined relation, and that difficult set-theoretical decisions have to be made in order to *turn* it into a determinate one. And many of such decisions are open issues in set theory. ((2005), p. 798)

Jané reverses the classical assessment of the alleged advantages of second-order determinacy. This determinacy is suspicious, as it depends on statements which are so much debated to be, as the CH is, still open issues in set theory. It is not first-order set theory which is indeterminate, on this interpretation, rather it is second-order set theory which belies our currently well-known mathematical indeterminacy and, thus, betrays the purpose of representing mathematical facts adequately.[43]

This is also, for example, Jouko Väänänen's point of view. In Väänänen (2001), he actively argues against the claim that second-order logic is a better candidate for an examination of mathematical ontology, as he thinks that, as far as the mathematician is concerned, full or Henkin models do not really make any difference. This is because, Väänänen argues, the mathematician uses a sort of informal pre-theoretical logic which she, only afterwards, tries to accommodate to a formal framework.

[43]Quine's (Quine (1941)) and others' opinion notwithstanding which wants that second-order logic is 'set theory in disguise'. But this is hardly paradoxical: since second-order logic is already set theory, it seems to come with a built-in, but artificial, set-theoretic determinacy.

Like Jané, Väänänen also stresses the point that a second-order supporter is in a worse position than a first-order supporter, because the former cannot provide adequate reasons for the indeterminacy of problems like the Continuum Problem, in spite of the *full* determinacy of the world he is talking about.

> ...if we try to analyse why we are not able to decide, e.g., the Continuum Hypothesis, on the basis of ZFC_2, it seems very plausible to develop a theory about what the second-order quantifiers range over. The first-order set theory ZFC is exactly such a theory, and it is indeed the strongest currently available tool for investigating formalizations of second-order logic. But this means we are back in the Henkin semantics of second-order logic that full second-order logic was supposed to avoid. ((2001), p. 519)

As anticipated, the opinions reported above do not represent in the least a full case against second-order logic. But, at least, it seems to me that they bring in some evidence to contest that Claim 1 is sound and to assert that a Gödelian platonist's adoption of a second-order grounding of set-theoretic intuitions and determinacy may not be decisive for the issue under consideration.

3.4 Field's nominalistic Deflationism

In the second tranche of this chapter, we will mostly deal with conceptions which are explicitly in contrast with platonism or see it as irrelevant in relation to the problem of set-theoretic indeterminacy. Some of these conceptions are strongly deflationary, in the sense that they suggest that either problems like the CH (and all other unsolved problems, for that matter) are meaningless or that (and, sometimes, only by virtue of that), they will never be determined.

Hartry Field's view, as presented in Field (2001), is the instance of an influential line of thought which maintains that, painful though this may appear, the undecidable statements of set theory are radically indeterminate.

> [...] I don't see any pretheoretic reason why such statements should be assumed to have determinate truth value. Their lack of determinate truth

> value seems to me to be fully compatible with accepted methodology in mathematics. I have already pointed out that the recognition of indeterminacy in no way forces us to give up classical reasoning. Also, we can still advance aesthetic criteria preferring certain values of the continuum over others; we must now view these not as *evidence that* the continuum has a certain value, but rather as *reasons for refining our concepts so as to give* the continuum that value, but I do not see this as in violation of any uncontroversial methodological demand. ((2001), p. 342)

We have seen the challenge that model-theoretic relativity and the existence of the generic multiverse posit to Gödelian platonism. Putnam's argument showed that our language, at least the first-order mathematical language, does not fix univocally its interpretation and that, therefore, any attempt to capture ontology, within that conceptual framework, is doomed to fail. A realist's vocabulary and epistemological purposes are thoroughly pressurised by Putnam's argumentation. However, whether this should lead to an extreme anti-objectivism is not clear from Putnam's analysis.

On one hand, Field rejects extreme anti-objectivism, but, on the other, he declares that there is no point in trying to fix indeterminacy, as there is no point in making any choice among equally referentially indeterminate universes of set theory. The reason why he rejects anti-objectivism, however, is that he wants to actively argue in favour of the referential clarity of the notion of *finitude*. Therefore, Field's analysis goes a step further than Putnam's, in that referential indeterminacy is diagnosed as being a characterising feature of the infinite. At the same time, providing an account of finitude as immune from (set-theoretic) indeterminacy, Field wants to secure empirical science from model-theoretic indeterminacy, an enterprise which goes along with his previously undertaken attempt to ban all non-finitistic vocabulary and ontology from science (see Field (1980)).

The examination of Field's position is, therefore, interesting, in that it clearly exposes the main objectives of nominalism in ontology: infinitary language and concepts are the source of antinomies or incongruences, starting with Skolemian relativity to end up with the *generic multiverse* and should, therefore, be regarded as meaningless. It is no surprise, then, that given the meaninglessness of such questions as the Continuum Problem one can decide of its truth or falsity, for instance, only on *aesthetic* grounds. There is no fact of the matter to whether the CH is true or false, essentially because there is no universe of sets which is instantiated by our infinitary language and infinitary mathematical practice.

As already mentioned, Field thrives on Putnam's argumentations, but, at the same time, wants to keep the determinacy of, at least, the finitary portion of our mathematical (and scientific) language and ontology.

He argues that, if one disavows even the determinacy of the finitary part of mathematics, then it is not very easy to account for our inferential practices overall, as our notion of logical system, proof and validity clearly rest upon an intuitive grasp of the notion of *finitude.*

He says:

> I think that anti-objectivism has considerable plausibility for the typical undecidable sentences of set theory. It has much less plausibility for the undecidable sentences of elementary number theory: these strike almost everyone pretheoretically as having determinate truth-value, though we may not know what it is. ((2001), p. 337)

In a sense, Field is certainly right. *Prima facie*, the belief that statements concerning positive integers should not have a determinate truth-value would sound as odd. This is because we seem to clearly know what structure we are talking about, when we utter statements referring to natural numbers. This structure is that of basic arithmetic. However, as already pointed out in the previous sections, the question whether it is actually possible to draw a line between number theory, for instance, and set theory is highly controversial. While it is certainly intuitively clear that the notion of *finite* is a lot less controversial than that of infinite (or transfinite, for that matter), it is less clear how to fix this situation in an epistemologically uncontroversial argument.

In particular, there are statements of number theory, with a clear combinatorial character, which have been shown to be decidable only assuming the existence of *large infinities* (see Friedman (1986)).

But one does not have to turn to large cardinals to see that the finite is hardly separable from the infinite.

As known, statements such as *Con(PA)* (see Gentzen (1936)) and *Goodstein's theorem* (see Paris-Kirby (1982)) are not decidable in PA but are proved through the assumption of the existence of ordinals up to ϵ_0. This, at least, shows that it is always problematic to draw a line between finitary and infinitary intuitions. After all, this is crudely shown by Gödel's incompleteness theorems, as was explicitly emphasised by Gödel (see Gödel (1931)), that the addition of ordinals might be the key to the decidability of particular statements. Finally, this is also the rationale underlying strong Axioms of Infinity.

But let us turn back to Field's argument. He is not claiming that arithmetic is complete by any sense of the word. He is just assuming that the notion of finitude which is dealt with in arithmetic is immune from referential indeterminacy. However, he is not suggesting that extreme anti-objectivism is completely wrong either. He just affirms that it seems to be implausible.

Given the importance of the role played by the existence of non-standard models in Putnam's argumentation, Field's move is that of preventing the collapse of structures of arithmetic into non-standard structures.

This he accomplishes, reformulating his theory of sets S to contain a *finiteness* predicate, which is implied by what he calls *cosmological* assumptions:

1. Time is infinite

2. Time is Archimedean

Given any set of events (Z) conceived of as a *discrete* structure in the standard sense (that is, Z (1) has an earliest and latest member and (2) given any two members a and b of Z, the distance between a and b is at least one second), the cosmological assumptions imply that Z is finite.

Finally, if we take set theory with the cosmological assumptions and define a finiteness predicate ($\mathcal{F}$), we are guaranteed that the extension of $\mathcal{F}$ comprises only *standardly* finite sets. This would account for our belief in the determinacy of the notion of *finitude.*

The procedure, *per se*, is not really relevant on the mathematical side, as the finiteness predicate and the cosmological assumptions are already contrived to prevent non-standard objects from satisfying the finiteness predicate. Field's point is that our fullest theory of sets must, at least, be limited by these theoretical constraints, if it wants to achieve the foundation of finite mathematics. This he shows using a physical vocabulary, because he wants to secure the determinacy of the notion of finitude as arising from our empirical inferential practices and ontology.

The implicit corollary of this argument is that, since there is no *natural cosmology* of set theory, then there is no way to found set-theoretic determinacy. Hence comes the radical indeterminacy of set theory.

The shortcomings of this conception are clear to Field himself: one has to use physical assumptions to secure the definiteness of mathematical concepts and this is certainly not an uncontroversially correct procedure.

Field, like Putnam, has a point when he seems to suggest that an intuitive cosmology for the infinite would not fix its interpretation in terms of ontology contrary to what his cosmological assumptions do for the finite. Incidentally, thence also comes the very strong suggestion that one should be very open-minded on how to settle the CH. Balaguer's full-blooded platonism (see Chapter 6) is a direct descendant of this line of thought and fully elaborates upon it. However, in view of my previous remark about the inseparability of positive integers from higher ordinals, Field's strategy seems to be rather weak.

That there is an intuitive, although not necessarily *physical*,[44] *incomplete*, pre-theoretic grasp of the transfinite [45] seems to be shown, on the contrary, by decades of set-theoretic practice, which clearly demonstrate that the extensions

[44]However, the existence of a physical continuum whose structure should turn out to be isomorphic to that of $\mathbb{R}$, may provide cosmological, in the *physical* sense, reasons for extending the determinacy of the number domain, at least, to the determinacy of $\mathbb{R}$.

[45]One could say: a set of principles of the level of complexity of Cantor's generating principles (see Hallett (1984) for a thorough discussion of these principles), strongly resembling and reflecting those for the natural numbers. It can be easily shown that large cardinals, for instance, build on top of these quite successfully.

of very basic principles (Feferman's motto: *a little bit goes a long way*) has given what seems to be an internally consistent picture of the infinite. And if this is indeed true, then why would one retreat to a more modest ontology? The reason would be that handling finite notions is well-established in our mathematical practice. But then again: how about the natural extendability of these notions?

For instance, the well-ordering theorem or the well-foundedness of the hierarchy of sets, aren't they a natural extension of features relating to an ω-sequence?

In conclusion, while Field's efforts to found our understanding of the finite (and, therefore, of the natural numbers) upon intuitive cosmological assumptions seems to be perfectly sound, unfortunately they cannot rule out the existence of natural extensions of his assumptions or the formulation of new ones in line with the need to explain the set-theoretic procedures.

Consequently, it seems to me that the role of cosmological assumptions will be but merely confirmative and not explanatory of our presupposition, that we understand the finite a lot better than we understand the infinite.

3.5 Feferman and Hauser on the Vagueness of the CH

The examination of Feferman's and Hauser's contributions to the debate is motivated by one main reason.

As we saw in the Introduction, apart from any other consideration of its intrinsic philosophical value, if anything, platonism has been a driving force in the development of set theory. The reasons are partly historical and partly practical: for one thing, Cantor and many other set-theorists who developed further his early intuitions all had realist views about sets. The intertwined relationship between, although unorthodox, realist views and the conceptual building of the theory of sets has been clarified by a lot of scholarly literature (see, especially, Dauben (1979), Hallett (1984), Lavine (1998), Jané (2005)).

Secondly, the majority of operationally valid principles of set theory fit better into a platonistic conception of sets than anything else. Even very informally, then, this provides an explanation of why platonism (in both its *ontological* and *operational* constituents) has been denounced by authors such as Feferman as the underlying philosophy of sets, which, at the same time, represents its metaphysics and operative mode of working.

Not only does Feferman say this, but he also adds that the reason why one would want to pursue the search for new axioms and new set-theoretic principles, after Cohen's proof of the independence of CH from ZFC, is essentially a belief in platonism.

Overall, Feferman's attack is, therefore, directed *at the whole bulk of contemporary set theory* as being irremediably imbued with platonism. Feferman's line of thought recasts the Brouwerian and Weylian attempts to deliver mathematics from metaphysics and extend foundational work to demonstrate that what is needed for expressing all of essential mathematics is a much smaller subset of what is used in contemporary set theory.

Starting with a negative assessment of all set-theoretic mathematics (Feferman (1987)), insofar as this latter would be dependent upon Cantor's ideology, Feferman's philosophical contribution has proceeded to proclaim the *inherent vagueness* of the Continuum Problem and declare its meaninglessness, along with the rejection of the new axioms as a conceptually legitimate and mathematically fruitful trend of research (Feferman (1999, 2000)). It has since focused on the various conceptions of the *continuum* which are implied by mathematics (Feferman (2008)) as justifying its inherent vagueness and finally settled on a new philosophical account of mathematics as based on *conceptual structuralism* (Feferman (2011)), by which the objectivity of mathematics is reached through an inter-subjective agreement on motivating concepts and methodologies.

On the other hand, Hauser's article on the Continuum Problem (Hauser (2002)) has since been the most comprehensive response to Feferman's challenge of the basic tenets of the contemporary philosophy of set theory.[46]

Hauser acknowledges the problems related to platonism, but argues in favour

[46] Other examples of detailed counterarguments are Maddy (2000) and Steel (2000), that is, the arguments presented by the respective authors at the same panel meeting where Feferman presented his theses.

of the existence of an objective solution to problems such as the CH independently of any philosophical and ontological underpinning. As a matter of fact, he attempts a reformulation of platonism along the lines of the conceptual reform which had been very shyly initiated by Gödel in the 40s. Thus, he gives full emphasis and fuel to the previously mentioned Gödelian platonist's need to switch to another definition of objectivity. We will be briefly looking at this conception in Chapter 7.

In spite of this, he defends the view that contemporary set theory is meaningful independently of any prior philosophical presumption or justifying underlying ontology. However, the principles which he defends are, in Feferman's view, due to a platonistic pre-conception, as Feferman conflates substantive and operational platonism, as we saw in Chapter 1. Therefore, the two views cannot, ultimately, be reconciled.

I hope that this preview has convincingly shown that it is worth examining the position of these two authors. The quarrel between the two, in my opinion, voices very accurately two irreconcilable positions on the foundations of mathematics. Among other things, Feferman shows how one can get rid of the problem very easily, while Hauser shows how platonism is invested by set-theoretic indeterminacy and in what sense it should be reformulated.

But, above all, despite their irreconcilable views, they show that the problem of platonism is the key to most of the philosophical issues of set theory.

3.5.1 Axioms

First, Feferman states that the new axioms which are supposed to settle the CH hardly have any consequences for contemporary mathematical work. He makes the distinction between *structural* and *foundational* axioms. *Structural* axioms can be found in most mathematical areas, as being specifically related to each area's motivating purposes and methodologies. They

> [...] act as axioms in the sense that they provide a framework in which certain kinds of operations and lines of reasoning are appropriate whereas others are not. [...] Without trying to argue this further, I take it that

> the value of these kinds of structural axioms for the organization of mathematics is now indisputable. ((1999), p. 100)

On the contrary, *foundational* axioms are but those self-evident statements which represent the foundation of mathematical knowledge. But, whereas structural axioms really contribute to the study of a particular mathematical theory and are constantly used by mathematicians with the purpose of studying specific mathematical objects, foundational axioms hardly necessitate any mentioning in the mathematician's routine work. Such are, for instance, Peano-Dedekind axioms for arithmetic or Zermelo-Fraenkel axioms for set theory.

However, on Feferman's view, there stands a dramatic difference between these two categories of foundational axioms. Axioms for arithmetic are basic, rooted in a well-established experience and have gained the widest consensus within the mathematical community. Axioms for set theory have failed to gain this status. As seen, the situation is compounded by the their referential indeterminacy.

In particular, Feferman argues that, in set theory, indeterminacy is triggered by the Power Set Axiom. Far from being a natural principle the power-set operation gives rise both to highly impredicative reasoning and also to the notion of *arbitrary subset*. He says:

> It is argued by set-theorists nowadays that the axioms of ZFC are evident for the universe V of sets consisting of all objects in some V_α. But the intuition for that is a far cry from what leads one to accept Dedekind-Peano axioms. Among other things, what this takes for granted is that there is an objective notion of arbitrary subset of a given set. ((1999), p. 102).

While Putnam and Field essentially blame model-theoretic relativity as the cause of referential indeterminacy, Feferman focuses on the notion of *arbitrary subset* as being obscure. Philosophically, this obscurity is tied to impredicative definitions. Impredicativity violates the following principle:

Principle 4 (Predicativity) *Definitions of entities which refer to complete totalities to which these entities belong are not admitted.*

While our mathematical practice teaches us to deal with definite subsets of ℕ, no mathematical practice, according to Feferman, teaches us what to do with an object like the *impredicatively defined* arbitrary subset, which exists only by virtue of the existence of a full power set principle.

Now, following Weyl (1918) and Brouwer's intuitionistic constructivism, Feferman rejects *impredicative* language and definitions. Thriving on Poincaré's and Russell's works on the *vicious circle principle*[47], he also argues that all contemporary set theory is interwoven with impredicativity and, thus, the entire architecture of it is philosophically dubious.

Is the notion of an arbitrary subset and the relative impredicativity conception really an issue for set-theorists? In Hauser's view, the answer is definitely no.

To Hauser, the roots of such hostility towards impredicativity are to be found in philosophical pre-conceptions which are also hostile to the widespread use of the actual infinite in mathematics. Hauser's purpose is to show that these pre-conceptions are no more serious and invalidating for the development of set theory than any early metaphysical objection to its development. In particular, the notion of 'set' and 'set of', as clearly founding the cumulative hierarchy, have not resulted in any inconsistency and the resulting structure has been, so far, tremendously successful to the point that all relevant mathematics can be embedded in it. The known antinomies do no harm to it, as long as one turns the intuitive Cantorian approach into the precise Zermelian axiomatic one.

As to the consistency of the *extended* hierarchy including large cardinals, Hauser makes it clear that

> while the possibility of an inconsistency always has to be acknowledged, the progress made in the inner-model program together with the elaborate web of equiconsistency results built around the large-cardinal hierarchy provide compelling evidence that at least a non-trivial segment of the large-cardinal axioms (and *a fortiori* ZFC) is free from contradiction. ((2002), p. 262)

[47]Which, according to these authors, is implied by the *violation* of Principle 4. See Russell (1908).

As already pointed out, Hauser is an active supporter of the contemporary developments of set theory and, in full accordance with Woodin's point of view as presented in Woodin (2001, 2011a), he thinks that axioms like $(*)$ within Ω-logic, that is, *generic absoluteness properties* and the work on the fine structure of L and its generalisations into the *ultimate* L which give a natural interpretive super-structure of large cardinals, will give a relevant contribution to the settling of the CH. There are other reasons accounting for the acceptance of new axioms into the body of mathematics. As mentioned, there is a direct link between finite and infinite mathematics, as emerging from independence results in arithmetic: take, for example, the mentioned Paris-Kirby proof of the unprovability of Goodstein's theorem (see Potter (2004) for a detailed survey) or Paris-Harrington's result on Ramsey's theorem and Friedman's on a variant of Kruskal's theorem. In particular, this latter result also provides a consistency strength measure using the linear hierarchy of large cardinals (in particular, the equiconsistency with Mahlo cardinals).

The work of Paris, Harrington and Friedman represents a corner-stone of modern unprovability theory in this respect and Feferman fully recognizes the importance of these achievements. However, in relation to the independent number-theoretic statements whose proof requires the assumption of some *large cardinals*, Feferman's opinion is that the results achieved are not as relevant as the ones which do not imply any more than standard formalised arithmetic. In Feferman's words, that we need large cardinals in order to *prove* a statement of number theory begs the question of the consistency of these large cardinal assumptions. Feferman says:

> The conclusion to be drawn (concerning these results, *my note*) is not nearly as clear as for the earlier work, since the truth of ϕ is now *not* a result of ordinary mathematical reasoning, but depends essentially on acceptance of $1 - Con(S)$. It is begging the question to claim this shows we need axioms of large cardinals in order to settle the truth of such ϕ, since our *only* reason for accepting that truth lies in our belief of the consistency of those axioms. ((1999), p. 108)

Therefore, the contention that large cardinals should be accepted on extrinsic grounds should cease to be valid as the proof of their consistency is obviously not derivable from their consequences. On the other hand, their intrinsic plausibility will always be a far cry, if the intrinsic plausibility of more basic axioms like the Power Set Axiom is far from being granted (because of its impredicativity).

But, above all, what strikes Feferman as decisively against the acceptance of large cardinals is the fact that they do not help settle problems of the complexity of the CH.

> But the striking thing, despite all progress, is that contrary to Gödel's hopes, the Continuum Hypothesis is *still* undecided by these further axioms, since it has been shown to be independent of all remotely plausible axioms of infinity, including MC[48] [...]. That may lead one to raise doubts not only about Gödel's program but also about its very presumptions. Is CH a definite problem as Gödel and many current set-theorists believe? Is the continuum itself a mathematical entity? If it has only *Platonic* existence, how can we access its properties? ((1999), p. 107)

Hence his prediction:

> [...] the Continuum Hypothesis is an inherently vague problem that *no* new axiom will settle in a convincingly definite way. Moreover, I think the Platonistic philosophy of mathematics that is currently claimed to justify set theory and mathematics more generally is thoroughly unsatisfactory and that some other philosophy grounded in inter-subjective *human* conceptions will have to be sought in order to explain the apparent objectivity of mathematics. ((1999), pp. 109-10)

Feferman's statements remain a bit ambiguous, as Hauser shows. On the one hand, Feferman says that the Continuum Problem is inherently vague, because it relies upon the impredicative and obscure notion of power set (arbitrary subset). On the other hand, he seems to assume that one of the reasons why the CH is vague is that no new axiom will settle it. It does not seem to be very clear for which of the two reasons, then, one should regard the CH as vague.

[48] The axiom which says that there exists a *measurable cardinal*.

Hauser rejects the thesis of the vagueness and also adds that:

> [...] past experience has shown that there is essentially only one way of extending the theory provided in ZFC which remains faithful to our primary intuitions of the concept of set, namely by postulating the existence of large cardinals. ((2002), p. 269)

The results achieved so far and the new procedures developed

> underscore the role of large-cardinal hierarchy as the natural superstructure for ZFC. ((2002), p. 269)

3.5.2 Weylian vs Cantorian Programme

The main reason why Feferman sees set theory as lying in such a predicament is that he is a strong supporter of Weyl's reform of the conceptual foundation of analysis, as provided in Weyl (1918). As already noted, his distaste for the notion of an arbitrary subset is, therefore, determined by a strong philosophical pre-conception, which, starting with Brouwer, has strongly criticised the use of impredicative definitions and related concepts.

Weyl's proposal has been instrumental in the development of Feferman's views (see, especially, Feferman (1987)). Weyl takes a middle position between radical Brouwerian intuitionism and Cantor's full-fledged realism, so to speak. In *Das Kontinuum*, his main purpose is that of delivering mathematics from all impredicative concepts and trying to secure its foundations on arithmetically defined sets of objects.[49]

Since Feferman's concern is essentially predicativity along Weyl's suggestions, then it must follow that set theory is a highly implausible candidate to be a secure foundation of mathematics. In his Feferman (1987) and Feferman (2005), he also gains further support from alternative reformulations of

[49]For instance, Weyl would not accept the *least upper bound principle* in analysis: *any bounded set of reals has a least upper bound*, as this is clearly impredicative. However, he admits of a reformulated least upper bound principle when dealing with sequences of reals indexed by natural numbers.

axiomatic set theory, especially those based upon suitable restrictions of comprehension axioms, to the claim that predicative reformulations are as effective as ZFC (for a comprehensive overview, see Simpson (1999)).

Therefore, his focus on predicativity ties with a higher goal: that of reformulating mathematics within the safe boundaries of either finitary notions or notions which do not imply an ordinal complexity higher than that of ω or strictly larger countable ordinals.

Of course, his appeal to strict finitism or something of this sort would automatically sound odd to any contemporary set-theorist, for reasons which are obvious. As Hauser points out, to reject impredicativity essentially amounts to rejecting the actual infinite, namely, the whole transfinite landscape. Why would a set-theorist be attracted by this?

But Feferman's aim is not to convince set-theorists, but mathematicians: he just wants to show that there might be alternative formulations of set theory which dispense with all the Cantorian machinery, but which are equally foundationally powerful. On Feferman's view, this would show that going into the obscure depths of impredicative mathematics is not required of the standard mathematician.

On this point, Hauser answers him back that, even if one accepted the intuitionists' or constructivists' ideology, the results of a reformulation of mathematics along those lines would be modest and, moreover, in some cases these results are not sufficient to carry out the mathematics we do. Therefore, they would not meet the goal for which they were pursued.

The focus on predicativity is also motivated by epistemological reasons.

Foundationally, as already noted, the *predicativity programme* provides the push to sharpen our mathematical notions and matches up with a view of philosophy of mathematics as the search for the boundaries of intelligible mathematics, as much as, in the wake of Kant's philosophy, epistemology deals with the search for the boundaries of intelligible knowledge.

In Feferman's words:

> ...one answer to the question "What is predicativity" is that it is a con-

cept applicable to different foundational stances given by the rejection of the actual infinite for various domains, coupled with its possible limited acceptance for others. Then the logical problem in each case is to characterize exactly the limits of that particular stance. [...] And for philosophy, its value is to provide sharp, informative explications in terms of which arguments can more pointedly be mounted in favor of, or opposed to, one or another foundational stance. ((2005), p. 621)

The following quote from Hauser works very well as a response to Feferman's position and ties in with a philosophy of mathematics construed as a recapitulation and description of mathematical practice, even when it goes beyond the safe territory of predicativity:

> Mathematical development occurs only when the human mind becomes conscious of its independence and evolves through idealization, abstraction, and generalization. In this way the problems about prime numbers, but just as well the questions of transfinite set theory, arose. Their investigation is of *intrinsic* interest regardless of irrelevant considerations about their role in the description of physical phenomena. ((2002), p. 268)

3.5.3 Platonism and Anti-platonism

As is clear from the preceding subsection, the Feferman-Hauser debate on the meaningfulness of problems such as the CH is oriented by the main question of whether platonism[50] is philosophically and mathematically legitimate and to what extent it is involved in the development of mathematics and, in particular, set theory.

Feferman's answer is that platonism is strongly involved, while Hauser thinks that platonism is involved, but mathematical questions such as the CH are legitimate and answerable regardless of any prior philosophical presumption. In particular, the CH is a valid problem even for an anti-platonist. Moreover, platonism can be re-structured in a way which makes it suitable for an adequate account of mathematical knowledge.

[50] As ever, the kind of platonism I am referring to here is Gödelian platonism.

Therefore, Hauser's position, although not originally conceived as an apology of platonism, turns into an argument in favour of the plausibility of a suitably reformulated platonism. His restructured platonistic conception is shown in Hauser (2001) and briefly resumed in Hauser (2002):

> At the outset, mathematical propositions are treated as having determinate truth values, but no attempt is made to describe their truth by relying on a specific picture of mathematical objects. Instead one seeks to 'exhibit' the truth or falsity of mathematical propositions by rational and reliable methods. [...] This reflects the widespread agreement among philosophers (and mathematicians) about what counts as evidence for the truth of a proposition [...]. In this sense the determinateness of CH can be maintained without getting caught up in the metaphysical issue of platonism. ((2002), p. 266)

To summarise, Hauser's position is very similar to that of a Maddyan naturalist (see Chapter 5 and 7 for further details), who believes in the objectivity of truth, but not in the existence of a platonic universe.

What should we make of this distinction at this point? Hauser's position seems to be no better than that of the Gödelian platonist. The only difference is that, while a Gödelian platonist has a problem in trying to explain how she has access to the universe of sets and to the truth or falsity of certain set-theoretic statements, Hauser's realist should just try to show how an objective truth-value could be determined within our formal theories, presumably according to certain *naturalistic* criteria.

On the other hand, as mentioned, Feferman's definition of platonism conflates *operational* and *substantive* platonism and, therefore, on Feferman's point of view, Hauser's attempt to disentangle the objectivity of truth-values from a rigid ontology does not do any better. That the first and foremost problem lurking inside set-theoretic indeterminacy is platonism, already in its operational from (anti-constructivism), is clear from Feferman's many statements to this effect. On the contrary, Hauser thinks that platonism is largely irrelevant. However, in the meantime, he tries to formulate a different conception of objectivity, which may share many tracts with Gödelian platonism.

And now for a couple of concluding remarks.

From the examination of the Hauser-Feferman philosophical exchange, there seems to be enough evidence to assert that platonism is a crucial issue in relation to the objectivity of questions such as the CH. At the same time, a platonist has a double problem with set-theoretic indeterminacy, as, for one thing, he has to provide an account of how there can be objective truth-values and, secondly, how these objective truth-values happen to be indeterminate.

As Hauser's position shows, there is some room for arguing that mathematical objectivity does not commit us to endorse mathematical platonism. Unfortunately, it seems that if one lays emphasis on operational aspects of platonism, then one is forced to admit that the most part of set theory has been shaped by platonistic pre-conceptions. Feferman's challenge is, therefore, to the bulk of mathematics as conceived in set-theoretic terms rather than to some particular portions or some conceptions of the continuum.[51]

Hauser has a point when he says that a set-theorist need not be a platonist. And his attempt to dismantle Feferman's perspective aims to bring evidence drawn from mathematical practice. However, Feferman has a point too when he says that there have been and there are alternative views of mathematical knowledge as arising from different philosophical methodologies. It is of no interest, in Feferman's opinion, whether set theory, in its contemporary form, has established itself as the most common foundational framework. This is a necessarily extrinsic argument, which does not settle the intrinsic necessity of set-theoretic mathematics. Of course, arguments from real practice always have a strong plausibility, but one should always remember that the acceptance of the Axiom of Choice among mathematicians does not make this axiom mathematically more plausible.

In conclusion, whatever the matter may be, what I hope my readers will have by now been convinced of, after reading this section, is that platonism is not

[51] Although, his interest in the conceptual disentanglement of the notion of continuum is shown by Feferman (2008).

only philosophically, but also mathematically crucial for a large set of questions which concern the foundations and the development of mathematics.

Chapter 4

Gödelian Platonism

4.1 Overview of the chapter

As already said, Gödelian platonism is the standard form of platonism with which we are concerned in this work. Some of its general features have been already described.

In accordance with the general template specified in Chapter 1, the standard platonist is seen as holding EXISTENCE, INDEPENDENCE, TRUTH and STRONG ACCESS. In particular, TRUTH has a special relationship with STRONG ACCESS, as the Gödelian holds that we have a *definite* and *determinate* conception of the universe of sets, whereby we are able to find the truth-values of all set-theoretic statements.

So far, we have not scrutinised Gödel's views on set-theoretic indeterminacy. Its features were assumed to be implicit in the preceding sections. The present chapter aims to fill the gap. In the last subsection, the key features of Gödel's philosophy of indeterminacy are examined and his argument in favour of the objectivity of the CH is presented.

Gödel's conceptions, constantly evoked and discussed in this work, have attracted a lot of attention and, as Boolos says in Gödel (1995), the resurgence of mathematicians' and philosophers' interest in platonism has probably been fuelled by its being advocated by Gödel himself. Furthermore, Gödel's late adherence to that form of platonism which he calls *conceptual realism* has been reconsidered, along with its implications, by pointing out its connections to various significant philosophical positions. I will briefly deal with conceptual realism in Chapter 7, while examining the problem of mathematical objectivity.

The examination of the ancestry of Gödel's conceptions has been taken into account in many works (see, especially, Wang (1974, 1997)). In this chapter, I will propose my somewhat unorthodox interpretive route, which aims to establish a connection between Gödel's thought and Cantor's philosophy.

4.2 Part I: The Ancestry. Gödel's Cantorianism

In this section I shall argue that Gödel's platonism is significantly indebted to Cantor's philosophy of mathematics.

The platonistic character of Cantor's philosophy has been extensively exposed in the literature and we can surmise that *standard platonism* would certainly be a philosophical system with which Cantor would feel at ease. However, it is still rather unclear whether Cantor's conceptions can be plainly labelled platonistic and in what sense. There are some strains of Cantor's philosophy which are not necessarily reducible to any, however unorthodox, form of platonism and which, on the contrary, would rather be regarded as being non-platonistic (or only marginally platonistic).[52] However, it is certainly reasonable to allocate Cantor's conceptions to the extreme realist camp.

[52]It is likely that Cantor's overall philosophy of mathematics would not fit into any of Linnebo's or Shapiro's proposed classifications of realism (see Chapter 1). The examination of his philosophy reveals a wider and more sophisticated picture, which clearly exhibits elements of Hegelian and Spinozian *monism*. In Cantor (1885), there are traces of Pythagoreanism and this philosophy has been acknowledged as playing a prominent role in his work by scholars such as Michael Hallett (see Hallett (1984)). As an aside, a direct reference to Plato can be found in several *loci* of Cantor's works, e.g. Cantor (1883b), and he explicitly traces his own philosophical system back to Plato's and Spinoza's systems in the endnote 6 of that work. In a letter to Eulenberg dating to 1886, Cantor affirms that his conception is similar to Augustine's, thus recasting his platonism as a philosophy continuous with Christian rational theology. At any rate, henceforth, I may sometimes use the somewhat glib label 'Cantorian platonism' to refer to his philosophy.

Until the end of his life, he kept struggling with the Continuum Problem and the search for a proof of the *CH*, whose truth he was never to doubt. His unwavering faith in the correctness of his conception of the transfinite was hardly shaken by the paradoxes. Moreover, in his last years, his philosophical conceptions evolved towards a theologically complex system. The philosophical (Scholastic) subtleties cropping up in his late philosophical contributions bewildered both mathematicians and philosophers. The ultimate reason for turning to theology was probably that he felt some pressure to provide evidence that the transfinite ought to be seen as a legitimately existing domain of our reality, insofar as this latter was the work of God.

But before he embarked upon such speculations, in *Grundlagen* he had already laid out the structure of the theory of transfinite ordinals and powers (cardinals), along with the bulk of his philosophical doctrines. Further details supplementing the conceptual framework expressed in Cantor (1883b), including those appearing in the philosophical epistles sent to mathematicians, theologians and philosophers, may sometimes help to clarify his intentions,[53] but they do not alter the general features of the picture. Therefore, my examination will essentially draw upon Cantor (1883b).

Coming to Gödel, his philosophy of mathematics has given rise to several interpretations and there is a lack of general consensus on how to better construe it. In recent times, after the publication of Gödel's *Nachlass* (1986) and the appearance of Wang's books based on the conversations which the author had with Gödel himself (see Wang (1974, 1997)), the scholarly literature has essentially identified Leibniz and Husserl as being the main philosophical sources of Gödel's thought and has, thus, reconsidered Gödel's views in the light of *monadology* and *phenomenology* (see, for instance, Hauser (2006), Tieszen (2002, 2005, 2011)). However, all interpretations substantially agree about the standard platonistic character of Gödel's two main published philosophical essays, Gödel (1944) and Gödel (1947).[54] In Gödel (*1951), Gödel declares quite neatly

[53] See, in particular, Cantor (1885) and Cantor (1887-88).

[54] As late as 1968, Russell affirmed that 'Gödel turned out to be an unadulterated Platonist', contrary to many's expectation, Russell seems to suggest.

his adherence to platonism. As a corollary of the arguments advanced in that famous lecture, he says that:

> So the alternative [between absolute undecidability or infinite transcendence of our mind over machines, *my note*] seems to imply that mathematical objects and facts (or at least *something* in them) exist objectively and independently of our mental acts and decisions, that is to say, ||it seems to imply|| some form or other of Platonism or "realism" as to the mathematical objects. ((*1951), in [64], pp. 311-2)

Earlier, in Gödel (1944), he had already pointed out that the existence of mathematical objects had to be seen as legitimate as the assumption of the existence of physical objects (see the passage quoted in Chapter 1). In Gödel (1947), his platonistic belief clearly orients his verdict about the full meaningfulness of the Continuum Problem.

In the literature, one does not find any hint at the fact that Gödel's philosophy may share some traits with Cantor's conceptions. In what follows, I will not deal with the question whether Gödel was actually an adherent, so to speak, of Cantor's philosophy, nor am I interested in the question whether there is any textual basis to claim that Gödel drew upon Cantor's thought.

My claim is that, however things might be historically, one can claim that there is some evidence that Gödelian platonism is connected to Cantor's philosophy. As a matter of fact, my claim is slightly stronger. I assert that Gödel's conception is, in its main features, to a substantial extent, reducible to Cantor's conception.

But first, I want to dispel one possible misinterpretation of my purposes. Here, I am not trivially arguing that, insofar as set theory has been created by Cantor, then it must necessarily follow that the most eminent set-theorists and logicians like Gödel have been influenced, in one way or another, by Cantor's philosophy. Rather, my purpose is to show that Gödel's philosophy might follow from a partial re-elaboration of Cantor's philosophy and that, if one adopts this view, some interpretive problems related to Gödel's conceptions may find plausible interpretive solutions.

In the following subsections, I will try and show how some of the main facts concerning Gödel's platonism can be successfully related to Cantor's positions. The presence of Cantor's thought is more neatly detectable in Gödel's Cantor paper. Also some of Gödel's most puzzling statements about the epistemology and the ontology of sets can be found in that work. Accordingly, I will mostly compare Cantor's *Grundlagen*'s conception to that of Gödel (1947). In particular, I will focus on the notions of existence, objectivity and fruitfulness, as being the three main areas of philosophical interest for both authors.

4.2.1 Existence

In *Grundlagen*, Cantor mentions two forms of existence of mathematical objects, which, in his opinion, are interrelated. Gödel in his Cantor paper also seems to adumbrate this distinction, although, as we will see, his language and theoretical context differ from Cantor's.

Cantor says that:

> We can speak of the actuality of the integers, finite as well as infinite, in two *senses*; but strictly speaking they are the same relationships in which in general the reality of any concepts and ideas can be considered. First, we may regard the integers as actual insofar as, on the basis of definitions, they occupy an entirely determinate place in our understanding, are well distinguished from all other parts of our thought, and stand to them in determinate relationships, and thus modify the substance of our mind in a determinate way; let us call this kind of reality of our numbers their *intrasubjective* or *immanent reality*. But then, reality can also be ascribed to numbers to the extent that they must be taken as an expression or copy of the events and relationships in the external world which confronts the intellect, or to the extent that, for instance, the various number-classes (I), (II), (III), etc. are representatives of powers which occur in physical and mental nature. I call this second kind of reality the *transubjective* or the *transient reality* of the integers. ((1883b), pp. 895-6)[55]

[55] Page numbers refer to [33], in which Cantor (1883b) has been reprinted in English translation.

Let's examine this passage in detail.

The *immanent* reality of mathematical objects is a form of purely *conceptual* reality. Mathematical objects exist insofar as they are *concepts*, whose meaning and properties are expressed by an intricate web of logical relationships. This does not mean that Cantor thinks that mathematics is entirely reducible to logic, at least, not in the same way as posited by Frege's *logicism*. Nor does it seem that he is committed to saying that, inasmuch as they are concepts, they should be regarded as mental constructs, along the lines of *psychologism*. Refraining himself from any further explanation about the nature of concepts, Cantor just surmises that these latter occupy a definite and determinate 'position' in a seamless web of pure mathematical relationships. On this view, the study of the relationships among concepts is the task of pure mathematics.

Methodologically, Cantor seems to suggest, mathematics develops by showing the internal logical relationships among concepts. On this view, for instance, the proof of a theorem is reduced to showing that a certain concept A (corresponding to a mathematical object) has a certain property Φ. Φ, presumably, is attributed to A, by showing that A shares the same properties of other objects which fall under Φ.

One might legitimately ask whether mathematical concepts can be distinguished from non-mathematical concepts. In other terms, is there any specific mathematical intuition, as opposed to a general form of intuition, which provides the means to detect the properties of a mathematical object? It does not seem so. In the passage mentioned above, Cantor states that 'immanence' refers to both mathematical and non-mathematical concepts. As a consequence, the way we connect our mathematical ideas must follow from and closely resemble the logical processes underlying the connections among ideas of all sorts.

Cantor is eager to clarify that the conceptual existence of mathematical objects is not reducible to 'existence' in Kantian terms.[56]

[56]Cantor's aversion for Kantian philosophy is clearly expressed in *Grundlagen*. Gödel's strong interest for Kant is witnessed, among other works, by Gödel (1949), in which, however, he explicitly disavows Kantian philosophy. Wang (1974, 1997) mentions Kant among the main sources of Gödel's thought, along with Leibniz and Husserl. Traces of Kantism, probably

I will try to show where Cantor's conception of immanence is conspicuously at work. More details on this conception will be provided in Chapter 7. As known, Cantor claims that the *well-ordering principle* is a law of thought (again, see Cantor (1883b)). This does not mean that the well-ordering principle implies the existence of a corresponding, innate mental construct. Essentially, one derives this notion from the study of the logical relationships among such notions as, for instance, *order, set, actual infinite.* Accordingly, it should be possible to obtain the well-ordering principle, solely upon reflection on the concepts of *order, set, actual infinite.*

Another example: when Cantor affirms that he could not go any further in his mathematical investigations without introducing the concept of *transfinite* number, he hints at the fact that, upon reflection on the meaning of some iterative processes in analysis, he reached, inevitably and necessarily, the notion of actual infinite number.

Put in simpler terms, the 'immanence' view is that mathematical ideas develop as the result of a generative process of interconnection of previously established ideas or, to use Cantor's words, through the identification of the 'position' that a mathematical idea occupies in the abstract space of (mathematical) thought. We shall return to this in Chapter 7.

Now let's turn to the *trans-subjective* notion of existence. However surprising this may appear, Cantor affirms that mathematical ideas are also *instantiated* in the physical reality. He speaks of transfinite numbers as being physical 'powers'.[57]

Trans-subjective existence is ascribed to integers, real numbers, all equally *indispensable* for our scientific discourse, but also to the hierarchy of cardinal

mediated by Husserlian phenomenology, are detectable in his work. Parsons (1995), p. 71, calls Gödel's conceptual realism *transcendental realism*, laying emphasis on its Kantian ancestry. At the same time, he finds that there is an unresolved tension between Gödel's avowal of some form of conceptual trascendentalism and his expressions of unmitigated objects-platonism, such as one can find in Gödel (1944) and Gödel (1947).

[57]The term is most ambiguous. Most probably, it is borrowed from the physics or biology of Cantor's time and, possibly, inspired to the jargon of non-mechanistic and qualitative physics, whose full advent Cantor warmly encouraged (see, especially, Ferreirós (2004)).

powers ($\aleph_1$, $\aleph_2$, ..., $\aleph_\omega$, ...). Confirmation of the *trans-subjective* existence of mathematical entities is not necessarily found at the very outset. It might take some time to see it confirmed in a well-established physical model or theory.[58]

Both in the published and unpublished works, Gödel seems to be essentially concerned, to use Cantorian terminology, with the immanent existence of mathematical concepts. This accounts for the view, expressed in Gödel (*1951) and Gödel (1947), that the problem of the existence of mathematical objects is a sort of replica of the problem of the existence of *universals.*

Moreover, it is plausible to conjecture that the ideas entertained by Gödel are very similar to those expounded by Cantor. In other passages (see Gödel (1944), for instance), the necessary existence of mathematical objects is justified by the need to have a satisfactory theory of mathematics, in particular, of sets. In his famous and oft-quoted passage from Gödel (1964), he talks of objects which are clearly perceived:

> But, despite their remoteness from sense experience, we do have something like a perception also of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true. ((1964), in [11], pp. 483-484[59])

In a way that is reminiscent of Cantor's formulations, Gödel gives his rendition of the view that immanent mathematical concepts are existent and objective:

> Evidently the 'given' underlying mathematics is closely related to the abstract elements contained in our empirical ideas. It by no means follows, however, that the data of this second kind, because they cannot be associated with actions of certain things upon our sense organs, are something purely subjective, as Kant asserted. ((1964), *ibidem.*)

[58]It should be noted that Cantor did not just claim that mathematics is indispensable for physics in the same way as naturalism or naturalised platonism do. His claim is, metaphysically, much stronger. He affirms that mathematical entities are immanent to the physical structure of the universe, something more similar to the claims of the natural philosophy of Plato's *Timaeus* or Aristotle's *Metaphysics*. However, Spinoza's *Ethics* seems to be his main source.

[59]Henceforward, all page references from Gödel (1947/64) refer to [11].

Mathematical concepts, Gödel seems to suggest, are grounded on objective procedures of refinement of sensory data, which, in the end, make concepts force upon us their truth. Subsequent refinements of these procedures is also the key to producing always new mathematical ideas, along the lines of Cantor's methodology. This process, in particular, might help settle the unsolved problems of set theory, as explicitly declared in the following quote:

> What, however, perhaps more than anything else, justifies the acceptance of this criterion of truth in set theory [that is, truth arising from conceptual refinements, *my note*] is the fact that continued appeals to mathematical intuition are necessary not only for obtaining unambiguous answers to the questions of transfinite set theory, but also for the solution of the problems of finitary number theory, where the meaningfulness and the unambiguity of the concepts entering into them can hardly be doubted. ((1964), p. 485)

Moreover, the generation of new concepts is important for clarifying or extending, so to speak, the range of the old concepts.

Gödel's and Cantor's conceptions also look alike in envisaging the process of mathematical 'discovery' as a constant augmentation of our knowledge of the relationships among concepts. To Gödel, the proof of this was given by the discovery of the fundamental inexhaustibility of mathematics (set theory), as implied by its fundamental *incompleteness.* On his view, the process of extending ZFC through new axioms is, therefore, the natural outcome of the attempt to extend our maps of mathematical concepts and, *ipso facto*, the reach of the (relative) completeness of our theories.

Gödel does not mention physical instantiations of mathematical objects. However, the following passage might conceal a reference to Cantor's notion of *trans-subjective existence*:

> They (*scil.* the data of kind specified) [the data of mathematical experience, *my note*] may represent an aspect of objective reality, but, as opposed, to sensations, their presence in us may be due to another kind of relationship between ourselves and reality. ((1964), p. 484)

My reconstruction, due to the lack of further details, is, obviously, only an educated guess.

However, surely Gödel's allusion to the connection between mathematical objects and ourselves can be successfully interpreted in the light of Cantor's statements. The point is developed in the following subsection.

4.2.2 Objectivity

On Cantor's view, there is a connection between the immanent and trans-subjective existence of mathematical objects. The connection is given by the existence, ontologically, of just one level of reality, exactly as conjectured by philosophers like Spinoza. Therefore, the distinction between concepts (mental) and objects (physical) will cease in a holistic theory of the world.

This does not diminish the importance of 'immanence' as an epistemological criterion, nor does it mean that *trans-subjective* existence is prior to the immanent one. What we can and ought to know about mathematics is already revealed by the immanent structure of mathematical objects.

However, the aforementioned connection between ourselves and the physical world accounts for the adequacy of our mathematical concepts to represent an externally and independently existing world of mathematical objects. In other terms, epistemologically, Cantor's *holism* provides an account of why our mathematical concepts are *adequate*, in a way which, at the ontological level, is reflected by his *metaphysical monism.* This is how Cantor expresses his conception:

> Because of the thoroughly realistic but, at the same time, no less idealistic foundation of my point of view, I have no doubt that these two sorts of reality always occur together in the sense that a concept designated in the first respect as existent always also possesses in certain, even infinitely many, ways a transient reality. [...] This linking of both realities has its true foundation in the *unity* of the all to which ourselves belong. - The mention of this linking has here only one purpose: that of enabling one to derive from it a result which seems to me of very great importance for

> mathematics, namely, that mathematics, in the development of its ideas has *only* to take into account the *immanent* reality of its concepts and has *absolutely no* obligation to examine their *transient* reality. ((1883b), p. 896)

Although what Cantor means by the expression *'unity of the world to which ourselves belong'* is fairly clear, we need to be more accurate. What follows is an attempt to dispel all obscurities.

The realist assumes that mathematical entities are existent, mind-independent objects, as expressed by EXISTENCE and INDEPENDENCE. The idealist, on the other hand, thinks that mathematical objects, quite as much as any other object of our thought, are essentially a product of our intellect. The unity at which Cantor hints at is, then, the doctrine that any object of our thought has a *non-ideal* (we could say, non-mental, but the term would not necessarily fit into Cantor's purposes) equivalent in the external reality.

Accordingly, Cantor thinks that all mathematical objects have *real* correlates, which, however, mathematicians do not have to take into account, as, methodologically, the mere consideration of immanent (ideal) concepts is sufficient to develop mathematics.

Following that, Cantor also claims that the objectivity of mathematics is accounted for by *metaphysical monism.*

Claim 3 (Cantor's Monism) *Our thoughts (concepts) have physical correlates in the physical reality and physical objects have mental correlates in our thoughts.*

If monism is true, the question whether and in what sense mathematics is objective is reduced, by Cantor's philosophical reduction, to that of the sense in which our thoughts and perceptions about the physical world are objective. This is accomplished by invoking *holism* as an underlying epistemology.

With regard to the historical ancestry of his ideas, Cantor says, in one of *Grundlagen*'s endnotes, that both Plato and Spinoza adhered to this form of monism. He is also eager to clarify that the Kantian point of view, identifying the foundation of our knowledge only with the subjective (transcendental) forms of intuition, is wrong, as it separates the forms of knowledge from the sources of knowledge, which reside in the physical world and, ultimately, in the unity between the physical and the mental.

Now, is *monism* a doctrine also held by Gödel?

My conjecture is that, when he claims that the 'immediately given' of mathematical intuition may be due to 'another kind of relationship between ourselves and reality' he may be hinting at the ultimate unity of the idealist and realist elements along the lines of Cantor's characterisation. Of course, we cannot have any proof of this. The passage is quite murky and, what is worse, Gödel makes no attempt to clarify further his view. But the mere fact that he mentions the possibility of a hitherto unknown or concealed connection between ourselves and reality provides, at least, some evidence that he may have agreed with Cantor's conception.

In my opinion, this is also made more plausible by the fact that Gödel shares Cantor's view that Kant's definition of objectivity as entirely relying upon the transcendental structure of subjectivity is too narrow to include an adequate account of the objectivity of mathematical knowledge (see quote above). However, unlike Cantor, he may have agreed with Kant about the fact that the immediately given is derived through a process of refinement of empirical data.

I do not see any reason for denying that Gödel may also have thought that the objectivity of this process (through which the fundamental structure of mathematical knowledge is brought to light and 'immanently' clarified) is guaranteed by a stronger metaphysical connection between our minds and the physical reality.[60]

[60]One reason to deny this fact is that many commentators would take it as implying some sort of *mysticism*, which they reject as a plausible epistemology. I reproduce Potter's (2001) quote from Chihara (1990) as a proof of this widespread distaste: 'Gödel's appeal to mathematical perceptions to justify his belief in sets is strikingly similar to the appeal to mystical experiences that some philosophers have made to justify their belief in God. Mathematics begins to look like a kind of theology' (p. 21). Tait (2009) also exemplifies this attitude. Interestingly, in the paper mentioned, Potter claims that Gödel's appeal to intuition, far from implying mysticism, is very close to Dummett's conceptions of *indefinitely extensible concepts*.

In any case, like Cantor, he was less interested in the trans-subjective existence of objects than in the development of mathematics through immanent existence. Incidentally, it should be noted that, on Cantor's view, immanence just requires of the mathematician to check the *consistency* of her notions. Therefore, the immanence view also accounts for the extreme freedom of the mathematician to explore new ideas and create new objects. As Cantor puts it, *'mathematics, in its development, is entirely free'*. This freedom has to be interpreted as the epistemological trademark of mathematics, but also as a methodological manifesto for its development. As we will see, future naturalism will thrive on this philosophical stance.

4.2.3 Fruitfulness

In the light of monism, one would just conjecture that the pursuit of the *immanent* conceptual development would represent the only methodological criterion to justify the procedures of mathematics and, in particular, set theory. As a matter of fact, Cantor also envisages some correctives, so to speak, to the general methodological procedures indicated, along the lines of naturalistic concerns.

These concerns are reiterated by Gödel within his epistemological sketch in Gödel (1947) and, then, fully exploited and extended by Maddy (see, in particular, Maddy (1996, 1997)). Further details on Maddy's elaborations on Cantor's and Gödel's *ur-naturalism* will be given in Chapter 5.

Cantor says:

> It is not necessary, I believe, to fear, as many do, that these principles [the freedom in introducing new concepts, *my note*] present any danger to science. For in the first place the designated conditions, under which alone the freedom to form numbers can be practised, are of such a kind as to allow only the narrowest scope for discretion. Moreover, every mathematical concept carries within itself the necessary corrective: if it is fruitless or

I am not very keen on mystical arguments either, as shown in Chapter 3 while discussing *extreme platonism*. However, I do not take this to be a strong reason to deny that, historically, Gödel may have embraced a theological epistemology, like that proposed by Spinoza and Cantor.

> unsuited for the purpose, then that appears very soon through its usefulness, and it will be abandoned for lack of success. But every superfluous constraint on the urge to mathematical investigation seems to me to bring with a much greater danger, all the more serious because in fact absolutely no justification for such constraints can be advanced from the essence of the science. ((1883b), p. 896)

Let us examine Cantor's words in detail. Procedurally, mathematicians take care only of the immanent development of their conceptualisations. So long as these latter are consistent, the only harm that science (mathematics) may suffer from them is that they are not useful for its development.

This reveals Cantor's attention for what one would call *extrinsic* criteria for the acceptance of mathematical principles and axioms. On Cantor's view, if some new concept is unsuited to our mathematical purposes, it will be abandoned for 'lack of success'. However, this stance does not seem at odds with Cantor's professed realism: it just requires that our existence claims must follow the practice.

It seems that Cantor would not be keen on an inflationary ontology. If, on the one hand, on the immanence view, it can be said that all possible objects exist, thus prefiguring the existence of a *plenitude* of them (see Chapter 6), it is also true that only those objects which prove to be instrumental for the development of mathematics are regarded as legitimate items of our mathematical ontology.

Moreover, closer inspection reveals that naturalism (or *naturalised platonism*, so to speak) is a corollary of the freedom of mathematicians. This freedom should also entail responsibility, and therefore, the only court of appeal for mathematical ontology cannot but be mathematics itself.

It is also interesting to note that Cantor is the ancestor of the view that philosophy conceived of as a *first philosophy* is harmful to the development of mathematics.[61] For instance, he says that the theory of functions, the *core*

[61] However, Cantor does not seem to uphold this principle always. For instance, he uses 'first philosophy' (metaphysics and theology) in many circumstances in order to defend set theory from external criticism. At any rate, his contemporaries and old critics of set theory, in the same way as contemporary Feferman-like critics do, regarded sets as already imbued

business of modern analysis, developed throughout the 19th century, would have been hindered, in its development, by extra-mathematical constraints.

To summarise Cantor's thought: working without constraints in mathematics means that the mathematician develops new concepts (objects) and studies the properties of these concepts putting aside all philosophical worries. Philosophical work is complementary, but not integral to mathematical work. The development of mathematics follows the route of its internal *immanence*, that is, of the ramified structure of mathematical concepts.

Finally, we come to Gödel's hints at naturalised platonism. As shown, among others, by Maddy, he is also especially concerned with defining criteria of acceptance of mathematical axioms which are justified on extrinsic grounds. In particular, in his Cantor paper, this aspect has strong bearings on his attempt to show that the CH is a fully meaningful statement, even if it should turn out to be independent of ZFC (which was shown some years later, as known).

In the following passage, Gödel departs, for a moment, from all philosophical concerns about mathematical objects and settles on a more loose methodological platform, based on the external success of motivating concepts and axioms.

> Besides mathematical intuition, there exists another (though only probable) criterion of the truth of mathematical axioms, namely their fruitfulness in mathematics and, one may add, possibly also in physics. This criterion, however, though it may become decisive in the future, cannot yet be applied to the specifically set-theoretic axioms (such as those referring to great cardinal numbers), because very little is known about their consequences in other fields. ((1964), p. 485)

In particular, Gödel is looking at *large cardinal axioms* as an example of (extrinsic) success, which he, at that time, could only conjecture. The following quote also strikes me as resembling Cantor's above mentioned passage:

> Even disregarding the intrinsic necessity of some new axiom, and even in case it has no intrinsic necessity at all, a probable decision about its truth is possible in another way, namely, inductively by studying its 'success'.

with too much metaphysics.

> Success here means fruitfulness in consequences, in particular in 'verifiable' consequences [...]. There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole field, and yielding such powerful methods for solving problems [...] that, no matter whether or not they are intrinsically necessary, they would have to be accepted in the same sense as any well-established physical theory. ((1964), p. 477)

Gödel's expressions of naturalism want that we should grant ourselves as much mathematical ontology for our mathematical purposes as needed by our standard mathematical procedures. Large cardinals might even turn out to be partly irrelevant for the purpose of settling problems like the CH, but they might have consequences even for the most seemingly remote areas of mathematics.

On the one hand, this ties perfectly with Gödel's conception that sets are necessary to obtain a satisfactory theory of mathematics, exactly in the same way as physical bodies are necessary for a satisfactory theory of the universe. In particular, large cardinals might prove to be necessary to sharpen our concept of set.

On the other hand, this also matches up with his methodological objectivism: the extension of our web of concepts, as following from our practice, allows us to attain the highest possible degree of objectivity in our theories (see, also, Gödel (1944)). At the outset, there may not be any pre-formed ontology of mathematics, but the naturalistic purpose of obtaining satisfactory theories of mathematics may also help produce a clearer ontological picture.

4.2.4 Concluding Remarks

Here follow some final remarks.

As we have seen, when Gödel talks about a possible connection between ourselves and reality, he may have had in mind Cantor's belief in the ultimate unity of ideal and real. In accordance with this stance, Gödel may also have inclined for some sort of epistemological spinozism.

That a full-fledged spinozist conception is nowhere to be found in his writings may be due to the fact that he did not want to pursue this point further, as he realised, exactly as Cantor had done before him, that this is not really the key methodological point. Or, more simply, he did not want to be entangled in the hardships of metaphysics. As far as the development of mathematics is concerned, consideration of the *trans-subjective* reality of mathematics is not needed. What I called the 'immanence view' is sufficient.

In other terms, that the objectivity of mathematics may be ultimately rooted in a 'special' relationship between our minds and the external reality is a metaphysical stance, which, although probable, cannot be further investigated. On the other hand, the 'immanent' structure of mathematical knowledge can be studied and described, possibly using some sorts of Husserlian procedures.

Historically, if my interpretation is correct, one could say that Gödel's approach, thriving on a Cantorian philosophical setting, turns to making sense of Cantor's intuition that mathematics is a science of pure concepts. In the meantime, this latter idea had been successfully bolstered by Husserl (see, in particular, Husserl (1913, 1931)), whose phenomenology, according to later testimonies, Gödel endorsed at some point (see Wang (1997)). Thus doing, he still paid his tribute to Cantor, as he expanded on Cantor's philosophical conceptions, using a different (but closely related to Cantor's) method of analysis, such as the one provided by Husserlian phenomenology.

Unfortunately we cannot but speculate on what beliefs Gödel truly held, as even the unpublished essays do not enlighten us on the crucial points of his philosophy. And, again, he was also very reluctant to talk about metaphysics.

In the light of the picture presented in this chapter, it seems now clearer why Gödel had so many difficulties in spelling out a coherent theory of mathematical intuition. If the theory of the existence of such intuition is really related to Cantor's philosophy, then he may have clearly understood its implicit difficulties. In the end, he resolved not to tackle the problem of intuition from the point of view of its original Cantorian formulation. In any case, from Cantor's philosophy he could not draw any further clarification about this thorny issue.

4.3 Part II: Philosophy of Indeterminacy

Gödel's views on set-theoretic indeterminacy are exposed in Gödel (1947) and again in Gödel (1964). In both versions, Gödel aims to establish, essentially, the following facts:

1. On the grounds of the platonistic belief in the existence of sets, it is perfectly reasonable to claim that there is a solution to the Continuum Problem. This belief is epitomised by Gödel's claim that Cantorian set theory describes, as already mentioned several times, a 'well-determined reality'. There exists a form of mathematical intuition which accounts for the belief in the existence of sets in the same way as sensory intuition accounts for the belief in the existence of physical bodies.

2. Even if the problem of the existence of sets is put aside as irrelevant, it is still possible to acknowledge the meaningfulness of the CH within a research programme aimed to extend ZFC. Hence, it is probable that an objective solution to the CH might be found, as a consequence of new axioms, in particular Axioms of Strong Infinity, whose acceptance, although not directly motivated by intrinsic reasons, may derive from their verifiable consequences.

Gödel is essentially concerned about fostering further development of set theory as a general strategy to settle the unsolved problems. In the light of these developments, even the issue of realism (platonism) becomes irrelevant. However, as we have seen, many commentators find that point 2. is essentially motivated by point 1. On this interpretation, the search for new axioms should be essentially regarded as integral to the search for the ultimate universe of sets along the lines of platonism.

However, as Kanamori has pointed out (see Kanamori (1996, 2003)), Gödel-Cantor's realism was certainly instrumental in pushing forward the reach of the transfinite landscape at the outset, but then the field gained full autonomy from the original motivating philosophy. In particular, as we have seen in depth in Chapter 2, earlier Gödelian hopes were rebutted by the discovery that variance throughout forcing extensions of the truth-value of statements like the CH could not be stopped by the acceptance of strong axioms.

However, Gödel's philosophy of indeterminacy, in the face of its practical failure, still motivates attempts to settle the CH along the lines of the programme expounded by Gödel himself. Moreover, the ultimate-universe view, if anything, still draws on Gödel's belief in the objective existence of sets.

In what follows some more details about Gödel's conceptions are described.

4.3.1 Referential Determinacy

In accordance with the requirements of his realism, Gödel rejects any attempt to show that set theory should be viewed in the same way as geometry, namely as a multiverse theory, where the CH plays the same role as that of Euclid's *fifth postulate*. According to this view, there is no single set theory. The simile between the two cases is made sharper by the fact that non-Euclidean geometry is shown to be consistent within Euclidean geometry exactly in the same way as $ZFC + \neg CH$ is shown to be consistent within ZFC.[62]

Arguing against a splitting of set theory analogous to the splitting of geometry, Gödel maintains that, although, in principle, there is no geometry which is *formally* preferable over others, certainly some geometry may be preferable for the purposes of an adequate description of the physical world. In other terms, we cannot be neutral when it comes to connecting mathematical theories to scientific descriptions. Hence, if we are required to make a choice about which geometry fits best into the description of our physical universe, then the problem of whether the fifth postulate is true or false becomes relevant. The ascertainment of its truth or falsity will, in particular, depend on our theory of the world.

He writes:

> In geometry, e.g., the question as to whether Euclid's fifth postulate is true retains its meaning if the primitive terms are taken in a definite

[62] In particular, it can be formally proved (see Hilbert (1899)) that there is a model of Euclidean geometry where the fifth postulate is false.

> sense, i.e., as referring to the behavior of rigid bodies, rays of light, etc. The situation in set theory is similar, the difference is only that, in geometry, the meaning usually adopted today refers to physics rather than to mathematical intuition and that, therefore, a decision falls outside the range of mathematics. ((1964), p. 483)

In accordance with this view, referential determinacy should automatically obtain when our theories refer to a well-established ontology. If the underlying interpretation of Euclid's *Elements* is the physical world, then it may well be the case that Euclid's geometry is not suitable for this task. In this case, one would be forced to assume that the fifth postulate is false. Analogously, Gödel claims, if sets exist, as claimed by a realist, then the question of whether the CH is true or false is perfectly meaningful.

4.3.2 Intuition of Sets

Referential determinacy, that is, the acknowledgement of the (unique) reality of sets, is mediated by the existence of a form of mathematical intuition which elaborates upon sensory data (the *given* of experience). This intuition allows us to perceive sets objectively. The data provided by this form of intuition are as objective as empirical data, though not relying upon any known connection between the mind and the physical reality. This is briefly hinted at in the passage mentioned in the preceding sections:

> Evidently the 'given' underlying mathematics is closely related to the abstract elements contained in our empirical ideas. It by no means follows, however, that the data of this second kind, because they cannot be associated with action of certain things upon our sense organs, are something purely subjective, as Kant asserted. ((1964), p. 484)

The claim that we have an intuition of sets (which forces upon us the truth of the axioms) has posed various interpretive problems. In Part I, we have seen how some difficulties may be overcome by comparing Gödel's conception to Cantor's one. In particular, I claimed that Gödel's aversion for what he calls Kant's subjectivism may be related to Cantor's *monism*.

The argument advanced in Gödel (1947/64) should be seen as complementary to that in Gödel (*1951). In essence, Gödel was convinced that through the incompleteness theorems he had shown that our minds infinitely surpass the powers of any mechanical device. In particular, statements such as $Con(PA)$ cannot be captured by any system of primitive recursive arithmetic, whereas human minds can easily accomplish this task. This would prove that human minds have developed an intuition which machines do not possess and will never possess. In particular, our intuition must refer to the external and independently existing reality of sets.

However plausible this argument may seem, it does not provide us with any hint about what this intuition is like in scientific terms. In the light of my Cantorian interpretation, this is hardly surprising.

Moreover, among other things, Gödel also assumed that our minds are not reducible to our brains. Therefore, he may have conjectured that there is more to our mathematical intuition than the properties and the products of a finite number of neural connections.

We have also seen that, in stating this, Gödel may have, instead, clearly referred to Husserl's phenomenology. However, even in the light of a phenomenological interpretation, the problem remains of whence this intuition comes and of how it works in a mathematical context.

4.3.3 Asymmetry

Finally, even disregarding the problem of the existence of sets, one can still assert the meaningfulness of statements such as the CH, as, for instance, Hauser does (see Chapter 3), suggesting to extend ZFC.

In order to do this, one has to be aware of what extending ZFC means. Gödel provides an example which shows that, contrary to what asserted by the doctrine of axiomatic plenitude invoked by Errera (see Errera (1952)), extensions require some understanding of the universe of sets.

As already explained, let IC be the statement: 'there is an inaccessible cardinal'.

Now, $ZFC + IC$ is an extension of the number domain of ZFC, whereas

$ZFC + \neg IC$ denies that there is anything after all ordinals produced through the axioms of ZFC. Hence, $ZFC + \neg IC$ is just trivially equivalent to ZFC.[63] Now, $ZFC + IC$ and $ZFC + \neg IC$ are *asymmetric.*

Their consequences are also asymmetric. The former is an extension which strongly bears on how we conceive of the world of sets.

The consequences of the choice of $ZFC + IC$ are enormous, since to accept that there are cardinals which go well beyond the previously defined levels dramatically modifies our view of sets. It has been suggested that large cardinal assumptions require a leap of faith comparable to the one required for accepting the Axiom of Infinity, that is the existence of ω.[64]

Therefore, even if any ontological worry is left aside, this proves that we cannot be completely neutral on the choice of the axioms. In particular, Gödel thinks that a choice should be oriented by ontological criteria, so to speak. However, as we have seen, the acceptance of some new axiom may only be justified, extrinsically, by its verifiable consequences:

> ...besides mathematical intuition, there is another (though only probable) criterion of the truth of mathematical axioms, namely their fruitfulness in mathematics and, one may add, also in physics. [...] The simplest case of an application of the criterion under discussion arises when some set-theoretical axiom has number-theoretical consequences verifiable by computation up to any integer. ((1964), p. 484)

Eventually, Gödel suggests, one should come to recognise that the features of our set-theoretic intuition clearly account for the meaningfulness of the CH and for the hope that it will be settled. In his words:

[63] In the sense that the ordinals producible in ZFC are exactly the same as those producible in $ZFC + \neg IC$. In other terms, ZFC and $ZFC + \neg IC$ are *equiconsistent*, whereas $ZFC + IC$ is *consistency stronger* than $ZFC + \neg IC$.

[64] Incidentally, it should be noted that this asymmetry does not analogously apply to alternative theories like Euclidean and non-Euclidean geometry, at least not in the sense expressed by IC: non-Euclidean geometry does not bring in more 'abstract space', so to speak, than Euclidean geometry, whereas IC brings in more numbers. This fact may give further support to Gödel's view that Errera's invoked plenitude for set theory is unacceptable.

The mere psychological fact of the existence of an intuition which is sufficiently clear to produce the axioms of set theory and an open series of extensions of them suffices to give meaning to the question of the truth or falsity of propositions like Cantor's continuum hypothesis. ((1964), p. 485)

Chapter 5

Naturalism

5.1 Set-theoretic Naturalism

Quine has famously characterised naturalism as being

> ...the recognition that it is within science itself, and not in some prior philosophy, that reality is to be identified and described. ((1981), p. 21)

Accordingly, a naturalistic philosopher does not attach any significance to extra-scientific philosophy, as far as this latter aims to provide the ontology of science. All talk about ontology is admitted only as long as it is continuous with the scientific discourse.

Now, in this work, I am not interested in naturalism *per se*, as a self-contained and independent philosophy of science and mathematics. I just want to present *set-theoretic naturalism*, that is naturalism applied to set theory, as has been exposed and promoted by Penelope Maddy in recent years.[65] The main reason of interest is the fact that some features of set-theoretic naturalism have strong bearings on the present discussion.

Besides, since set-theoretic naturalism (which I will sometimes address simply as *naturalism* henceforward for the sake of linguistic economy) seems suc-

[65]See, especially, Maddy (1996, 1997, 2011) for a comprehensive overview of set-theoretic naturalism. An alternative form of naturalism is, for instance, that presented by Burgess, which is also briefly discussed and contrasted with her own by Maddy in Maddy (2005).

cessfully to dismiss all concerns about set-theoretic indeterminacy, I think that it could be seen by a realist as an alternative coping strategy. One obvious reason is that Maddy's philosophical ancestry (Quine and Gödel) is platonistic.

A second one is the following. Naturalism implies that the objectivity of the overall mathematical enterprise has to be sought within mathematics itself and not outside the boundaries of the procedures and techniques used by the mathematician. As a result of this, it also implies the demise of all extra-mathematical ontological concerns. Therefore, as can be inferred from Maddy's intimations, I would presume that a naturalist would not find as philosophically problematic that some important set-theoretic statements are not decided by ZFC. He simply does not envisage any bearings coming from this fact for his very thin ontological presumptions.

However, in claiming a purely internal (methodological) objectivity, Maddy's *metaphilosophical* view, as described in Maddy (1996), seems to be relapsing, so to speak, into the hardships of *platonism*. Maddy suggests that ontological considerations ought not to be relevant for the purposes of our mathematical discourse. But she also actively supports the work on extensions of ZFC (see, in particular, Maddy (1988)). It is certainly suggested by the current practice that the philosophically neutral discussion within the set-theoretic community about what axioms or principles should be held to be true is generally sufficient, in most cases, for our mathematical purposes.

At the same time, it can be convincingly shown that, in many cases, especially those involving *believing the axioms*, the discussion is already oriented by the prior endorsement of some form of philosophy. And indeed, the metaphilosophical maxims which Maddy presents, although not bowing to the agenda of a *first philosophy*, can be easily re-construed as metaphilosophical reformulations of platonistically oriented operational maxims. As always, a prior issue is whether operational platonism can be epistemologically merged with substantive platonism. Although, as seen, there is no conclusion to be drawn thus far, from the debate emerges some evidence that operational platonism (anti-constructivism) has been and is still crucial within set-theorists' work.

But, rather than viewing it as an intrinsic weakness, I claim that this is actually an interesting feature of Maddy's naturalism, insofar as it goes along with my arguments in favour of anti-constructivism (see Chapter 8). Therefore, although I am no adherent of naturalism in this work, I will claim that my very thin objectivist hypothesis is also somewhat indebted to Maddy's naturalism.

Obviously, my (possibly unintended) interpretation does not imply that Maddy's view arises as a response to set-theoretic indeterminacy. But I will assume that, on this interpretation, set-theoretic naturalism can be successfully framed as a reform of platonism along the line of a deflation of all ontological concerns. On another, maybe more faithful, interpretation, set-theoretic naturalism is a form of mathematical naturalism which deflates all extra-mathematical concerns about ontology and objectivity, at least when these latter notions are construed in the classical philosophical form.

More of this in the last subsection.

5.2 Introitus. Three forms of Realism

Admittedly, Maddy's naturalism is akin to and a descendant of Quine's naturalism, from which it also differs. Quine's idea is that there is a conspicuous continuity between scientific and mathematical knowledge and no boundary can be traced between them. This is essentially due to Quine's *holism*, the view that both mathematics and science equally belong to a *seamless web of knowledge*, described by a unique, all-embracing theory of knowledge (T), which provides an explanation of both empirical and mathematical 'phenomena'. The main, *residual*, ontological fact of T is that, since mathematical entities are treated on a par with any other entities within the scientific discourse, they have to be conceived of as existing for the same reasons why one would conceive of molecules, atoms, plants, bacteria and stars as existents.[66]

All extra-scientific ontological demands and pre-conceived ontological tem-

[66]That existence claims should be essentially empirical has been challenged by Quine's denunciation of the flawed nature of classical empiricism, which can especially be found in Quine (1951).

plates, if incompatible with the scientific discourse, ought to be dismissed. Another natural outcome of Quine's holism is the fact that, since reference to mathematical entities is unavoidable and *indispensable* for our scientific purposes, the assumption of the existence of these entities is integral to the progress of science.

Theoretical indispensability would, thus, justify EXISTENCE, and it is only on this condition that Quinean realism can be conceptually subsumed by mathematical platonism. But its resemblance to traditional mathematical platonism is only remote. The main reason is that Quinean realism is only justified by the scientific discourse. In accordance with the ideology of its inventor, it is not even a philosophical conception, as, in Quine's own words:

> ...naturalism is the abandonment of the goal of a first philosophy. It sees natural science as an inquiry into reality, fallible and corrigible but not answerable to any supra-scientific tribunal, and not in need of any justification beyond observation and the hypothetico-deductive method. ((1981), in [101], p. 498)

Given Quine's rather deflationary view of truth as related to ontology, it follows that what we need to account for is just what theories would suit best a particular context of investigation. These theories shall have peculiar features which Quine calls *theoretical virtues.*

Quine's realism is the first strain of realism which Maddy addresses in declaring her ancestry, only to depart from it substantially. In Maddy's opinion, Quine's view has major shortcomings.

First of all, this form of naturalism does not really account for the existence of very simple and well-understood mathematical structures in the way we would expect from a plausible theory of mathematical objects. For instance, the natural-number structure seems to be too well-understood and well-intuited a notion to be acceptable only in view of its scientific applicability.

Another obvious problem for a set-theoretic naturalist like Maddy is that all mathematical entities which are not indispensable for our scientific purposes, on Quine's view, have to be dismissed as meaningless. Some such entities would be transfinite sets (or, presumably, in view of scientific purposes, infinite sets not beyond $2^{\mathbb{N}}$ or $\mathbb{R}^{\mathbb{R}}$).[67] This too quick dismissal of relevant portions of mathematics has especially bewildered more moderate naturalists like Maddy and Burgess. The consequence is: this form of realism is inadequate for Maddy's purposes.

The second form of realism, which is historically relevant for naturalists, is Gödelian platonism. While this is maximalist with regard to the mathematical ontology, its reliance upon EXISTENCE and INDEPENDENCE, insofar as they violate the spatio-temporal referentiality of most scientific discourse, makes it untenable and unsuitable for the naturalist. Similarly, its reference to the powers of a mysterious intuition, which would justify the belief in STRONG ACCESS, contradicts the need for strong scientific evidence.

Finally, the last form of realism from which Maddy departs is her own, as presented in a famous book (Maddy (1990)). She labelled her view, at the time, Compromise Platonism but, ultimately, she felt dissatisfied by it. Compromise Platonism is an attempt of reconciling Quinean indispensability and Gödelian epistemological views. The project failed partly because there seemed to be no sufficient grounds to assert direct perception of sets and partly because Maddy ceased to believe in Quinean *indispensability*.

After departing from all these forms of realism, Maddy turned to naturalism, in its *set-theoretic* form, as essentially required of the wide foundational goals of set theory within mathematics.

5.3 Naturalism at a Glance

5.3.1 Principles

In some sense, Maddy sees set theory to be to mathematics as physics is to science. Much like physics, whose basic entities and domain of discourse are *irreducible* to simpler ones, is a sort of general ontological framework for science,

[67] Although, as Maddy points out, Quine came to view them as, at least, continuous with our most generally accepted *mathematical grammar* and, therefore, not devoid of meaning (see Maddy (2005), p. 445).

analogously, sets are irreducible to simpler objects and, thus, would represent the most elementary ground of our mathematical discourse. The naturalist's preference for sets is the analogue, in mathematics, of physicalism in science.

This powerful simile is also meant to convey the idea that mathematics can be shown to be successfully *naturalised* within set theory.

Maddy's claim is that set theory is

> the ultimate court of ontological appeal; it provides the ontological framework in which the practice of mathematics is carried out. ((1996), p. 512)

Therefore, set theory, with its talk about transfinite ordinals and cardinals, indiscernibles, well-founded sets, transitive models, is indispensable, within mathematics, as much as physics, with all its talk of particles, wave-lengths, fields, etc. is within science. Maddy is not very keen on scientific indispensability arguments *per se* to justify mathematical ontology, but she obviously recognises that the efficacy of set theory for mathematics is tied to what Quine would call its theoretical virtues: simplicity, fecundity, scope, among others.

However, while in physics empirical observation contributes substantially to our existence claims, in set theory it is not very easy to find something analogous. Therefore, it is the main burden of the set-theoretic naturalist to explain the principles and the methodology which should work as the equivalent of observation within set theory. Put in other terms, the goal of set-theoretic naturalism would be, pushing further the simile with physics, that of spelling out the *observational constraints* of set theory.

The following list mentions its most salient features.

1. **Demise of robust realism.** Gödelian realism is flawed, also because, as noted already, we do not have an explanation of what is Gödel's mathematical intuition, in a way which conforms to scientifically acceptable standards. This obscurity leaves key epistemological issues and, in particular Benacerrafian concerns (see Benacerraf (1973)), unanswered. However, some Gödelian philosophy can be incorporated in naturalism, particularly when Gödel seems to be defining some sort of protocols, that

is *extrinsic* criteria to assess the validity of an axiom (see, in particular, Gödel (1947)). Maddy does not exclude, however, the endorsement of some sort of *thin realism* (see also Maddy (2011)). In general, however, robust realism endangers the scientific enterprise.

2. **Pursuit of intra-mathematical philosophy.** Set-theoretic naturalism requires of the ontology to be in accordance with mathematical practice. Therefore, any prior philosophical assumptions have to be put aside. The highest tribunal for mathematical ontology has to be mathematics and mathematical procedures. Maddy is no apostate, so to speak. She is aware that philosophy is necessary. But the relation between philosophy and mathematics (set theory) has to be re-structured. Admittedly, the boundary is not easy to trace. This quote should provide some clarification:

   > What I am suggesting is this: if your interest is in set theoretic methodology (as mine is), and in particular, if you want to know how those methods can be justified and extended (as I do), then you should attend to the details of practice. As a corollary, the demise of set theoretic realism should not inspire a search for a new extra-mathematical metaphysics. But this is not to say that philosophy is useless. [...] Philosophical views can be extremely helpful, even essential, in a heuristic sense, and, in fact, though I have no example at hand to cite, I suppose an erstwhile philosophical view might migrate and eventually become so thoroughly entangled with actual scientific or mathematical practice as to become a legitimate methodological maxim. ((1996), p. 503)

3. **Extra-mathematical philosophy must yield to metaphilosophy.** Naturalism claims that philosophy has only an inspirational, not a justificatory role in mathematics. Our philosophy, as continuous with mathematical practice, should help reconstruct mathematical practice rather than give prescriptions on the grounds of extra-mathematical conceptions. In this capacity, naturalism just states some meta-principles (maxims)

which have to guide our understanding, reconstruction and maybe further expansion of portions of mathematical lore. Since the focus is on sets, our maxims will refer to set-theoretic principles and methodologies, in a way which is in accordance with set-theoretic work.

However, this metaphilosophical undertaking is not necessarily in contrast, as may be presumed, with a robust platonistic ancestry, such as Gödel's emphasis on protocols like extrinsic criteria. Maddy justifies the intrusion of Gödelianism in the following way:

> ...though one strain in Gödel's writings involves a truly philosophical Realism, another actually consists of a series of realistic-sounding methodological principles: allow infinitary methods in metamathematics, don't require existence proofs to provide constructions or definitions, allow impredicative definitions, regard axiomatic set theories as extendible. All these principles can be seen as arising out of practice and as guiding successful subsequent practice, in the manner of a true methodological maxim like Mechanism. ((1996), p. 502)

Maddy stresses the point that these maxims and their practical application have an obvious philosophical ancestry. But the philosophy at work here does not imply the belief in some sort of prior metaphysical construction.

Nonetheless, as already pointed out, Maddy does not understate the inspirational role also of more robust philosophical conceptions in mathematics (or science). She concedes that, even if not fully self-consciously, scientists and mathematicians can be actively influenced by philosophical conceptions and, then, fully exploit their philosophical presumptions to carry out their scientific work. But they shall never substitute their own philosophy with the concrete practice of science or mathematics. The underlying philosophy of some of the creators of modern science or of mathematics is integral to the creation history of their theories, but not to the strength and the importance of their results.

5.3.2 Methodology

Now we come to the bulk of Maddy's version of naturalism, as represented by her focus on a naturalist methodology. The set-theoretic naturalist will evaluate set-theoretic results in view of particular methods and procedures. These latter are gauged on his anti-philosophical (metaphilosophical) presumptions in such a way as to prevent the intrusion of any philosophical bias. The general features have been indicated, but something more detailed needs being addressed.

The overall purpose of defining observational procedures in set theory is that of confining it within the solid walls of intra-mathematical practice, whatever this may mean. Obviously, a definition of mathematical practice is *prima facie* problematic, as to give an account of what the mathematician's work exactly consists in, in terms of observation is, presumably, no less problematic than to give an account of what mathematics is. However, it is certainly true that mathematical practice is, at least, a well-understood notion within the mathematical community. At least, we have some sort of *sociological* evidence that so are things.

However, if one zooms in on the picture, what contemporary set theory shows is that, even within the boundaries of strict intra-mathematical discussion, there is strong disagreement among mathematicians about relevant issues. If there are portions of mathematics which are open to philosophical interpretation, set theory is no doubt one of them. Consequently, even the mathematician most reluctant to philosophising needs to make decisions which hinge upon his philosophical presumptions.

Therefore, defining protocols is not a trivial undertaking. In general, one assumes that mathematicians, while possibly disagreeing over their philosophical presumptions, will be easily reconciled by one central argument, that from the *utility of their methods.* Often, it is only the expedience of a particular procedure or principle, and nothing else, which is decisive for its acceptance. This is how Zermelo synthetically surveys the case for the Axiom of Choice:

> ...so long as the relatively simple problems mentioned here remain inaccessible to Peano's [choice-free] expedients, and so long as, on the other

> hand, the principle of choice cannot be definitely refuted, no one has the right to prevent the representatives of productive science from continuing to use this 'hypothesis' - as one may call it for all I care - and developing its consequences to the greatest extent, especially since any possible contradiction inherent in a given point of view can be discovered only in that way ((1908), in [99], p. 1127)

Expedience, we said. One could also say very grossly: anything goes if no inconsistency arises. Unfortunately, unlike the Axiom of Choice, there are set-theoretic principles whose evaluation, using both *intrinsic* and *extrinsic* criteria, is not so easy and their utility is, of course, a matter of debate, to say the least. Moreover, their consistency is only conjectured. Utility essentially ties in with *success*. But, even when extrinsic success is clearly witnessed, other legitimate qualms arise. At least, Gödelian platonism cuts through these difficulties, in that it requires that there must be strong intrinsic evidence, in particular, truth in the universe of sets, as described, presumably, by our best theory of sets. Unfortunately, this does not seem a very easily definable protocol of observation.

5.3.3 Maxims

In order to relieve this quandary, Maddy proposes an idealised methodological procedure, which features the set-theoretic community making supposedly unbiased decisions based on the standards of mathematical practice.

Each time a controversy arises, according to the procedure described in Maddy (1996), the set-theoretic community, disguised as an impartial judge, should decide which of the two involved parties has presented the case most successfully and, therefore, proposes the correct solution.

Let's see how all this works.

Maddy develops quite extensively the naturalist's case against $V = L$ in both Maddy (1996) and Maddy (1997). The point of departure is that $V = L$ is generally rejected by set-theorists. A naturalist can thrive on the technical details (in accordance with Maddy's motto: 'attend to the details of practice') of this case and show practically her *modus operandi*.

Maddy's examination is essentially aimed to establish that $V = L$ violates one essential methodological principle within set-theoretic practice, that of maximality (labelled as MAXIMISE).

MAXIMISE is a structural principle, which allows one to expand the universe of sets as remotely as the known methodologies allow. MAXIMISE is, therefore, the underlying metaphilosophical rationale behind the most part of set theory. More practically, suppose that we show, as is feasible, that we only need two levels of infinity, natural and real numbers (at most three, if one also wants to admit subsets of reals). This would imply that the we do not need more infinity in mathematics than what goes beyond $\beth_2$.[68] MAXIMISE implies that there is no intra-mathematical reason for not iterating the $\beth$-function well beyond that limit. The only reason against the generalisation of this procedure would be a strong philosophical argument, that is a philosophical pre-conception, against the usefulness (and, thus, meaningfulness) of numbers $> \beth_2$.

Now, on Quine's strict orthodoxy, maybe one could find these arguments sound. But since Maddy's naturalist departs from Quine's strictly orthodox naturalistic presumptions, he will say that the unbounded iteration of the $\beth$-function is a legitimate move. This echoes the way early criticisms of Cantor's theory of the transfinite, as based on iterations of particular procedures (point set derivation beyond the ω-th level) had been dismissed by Cantor himself on the grounds of what seems to be an instance of naturalised epistemology.

Consider again, for example, Cantor's famous statement about existence claims:

[68]The $\beth$-function computes the iteration of the power-set operation producing the following hierarchy of cardinalities:

$$\beth_0 = \aleph_0$$

$$\beth_1 = 2^{\aleph_0}$$

and, in general,

$$\beth_{\alpha+1} = 2^{\beth_\alpha}$$

Finally, if α is a limit ordinal, then for all $\beta < \alpha$,

$$\beth_\alpha = sup(\beth_\beta)$$

Hence, $\beth_2$ is the cardinality of set of all real functions ($\mathbb{R}^{\mathbb{R}}$).

> In particular, in the introduction of new numbers, it is only obligated to give definitions of them which will bestow such a determinacy and, in certain circumstances, such a relationship to the older numbers that they can in any given instance be precisely distinguished. As soon as a number satisfies all these conditions it can and must be regarded in mathematics as existent and real. ((1883b), p. 896)

Here, Cantor stresses that new objects only be required to be *conceptually* continuous with older objects though a process of growing generalisation. For the existence of transfinite ordinals, therefore, it would not be required more than their distinguishability (and, at the same time, their conceptual kinship to) other classes of numbers. The processes of iteration which motivate the production of the natural numbers are the same which motivate the extension of the number system into Ω through Cantorian generating principles. Analogous *maximality* arguments would apply to the $\beth$-function.

A set-theoretic naturalist will accept the evidence brought before her, which clearly establishes that maximality is the central *desideratum* of most set-theoretic constructions, and, accordingly consent to the following:

Principle 5 (Maximise) *Our set-theoretic procedures should go in the direction of maximality.*

The application of this principle can be found everywhere in set theory, embracing transfinite ordinals, large cardinals, forcing extensions and the *generic multiverse*, that is the universe arising from all *generic extensions* (the $V^{\mathbb{B}}$ described in the Introduction).

Let's go back to $V = L$. This latter is *restrictive* with regard to the width of the transfinite hierarchy and, thus, it blatantly violates MAXIMISE.[69]

There is another principle which is generally held by set-theorists and mathematicians to be pre-eminent in their practice.

[69]Notwithstanding the $(V = L)$-lobbyist's efforts (which Maddy declares as ill-founded) aimed to show that $ZFC + V = L$ is maximising over $ZFC + \exists 0^\sharp$. $0^\sharp$ is a set of *indiscernibles*, whose existence arises from the embedding of L, the constructible universe, in itself. The existence of $0^\sharp$ and of its transfinite generalisation mark an essential internal transcendence of V over L. If $0^\sharp$ does not exist, then V must be very similar to L.

Principle 6 (Unify) *Our set-theoretic procedures aim to give as much as possible a unified picture of the universe of sets.*

In some respect, UNIFY acts as the natural corrective of MAXIMISE.[70] There are so many compelling examples of UNIFY in set theory that it is hardly possible to give a comprehensive list.

I presume that, naturalistically, V could be seen as the result of the strongest possible unification of set-theoretic procedures and results. Through V, the notion of well-foundedness, well-order, ordinal, power-set, transitive set and, in practice, the most part of all set-theoretic notions are successfully shown to be part of a unique incremental, infinite framework. Therefore, on the naturalist's view, rather than being the result of a chimerical jump onto the arbitrary and the indefinite, V should be viewed as the result of a naturalising attitude, in full accordance with set-theoretic practice.

For the naturalism-minded, UNIFY is obviously also the guiding principle with regard to internal ontology. In naturalistic terms, this would justify the search for the unification represented by the *ultimate-universe* view. Maddy says:

> If one of the goals of the practice of set theory is to provide such an ontological framework, it can only do so only providing a single, unified theory. It has been suggested that the mathematician's low opinion is partly due to the overwhelming turn toward independence results; ...he wants set theory to deliver a single batch of fundamental assumptions strong enough for all his purposes. The goal of providing such assumptions provides the set theorist with a motivation to adopt UNIFY. ((1996), p. 512)

However, Maddy makes it clear that maxims such as the above mentioned principles can also be in tension. When this happens, the set-theoretic community might consider shifting focus from one to another and, consequently, accept that

[70] Incidentally, Cantor's *generating principle* CLASS (see Chapter 7, Section 3.2), whose aim is to gather all ordinals of the same cardinality into well-defined totalities (the $\aleph$s) is, analogously, a sort of limitation principle in the Absolutely Infinite class of all ordinals.

$V = L$ could be of better use if one prefers UNIFY over MAXIMISE. On this ongoing work of calibration of metaphilosophical priorities, then the CH could be decided in one or the other way. This leads to the following remark:

> This line of thought suggests a naturalistic interpretation of the familiar remark that 'CH might not have a determinate truth-value' or 'CH may be inherently ambiguous'. It might happen that the constraints provided by the various current and future maxims of set theoretic practice will not settle the size of the continuum. ((1996), p. 512)

Here is the naturalistic bottom line to indeterminacy: more work on the axioms will show us whether there is a *naturalistically* acceptable fact of the matter to the unsolved problems. It might turn out that the internal tension between mutually inconsistent requirements does not allow us to uniquely settle, say, the CH. But this will be the natural outcome of an internal, so to speak, procedure, that of pondering why and in which way we want a particular solution to the CH. As anticipated, this does not have any bearings on our undertakings, as the Gödelian platonist's pre-theoretical and intuitive *ontological determinacy* is ruled out.

5.4 Critical Discussion

Set-theoretic naturalism offers a lot of material on which to pause and reflect.

In general, it seems that the reasons why one would not accept it are the same as those for which one would reject it *tout court* (see Weir (2005) for an overview). The debate is very wide and articulated and touches upon several internal issues. However, for my purposes, I will just be concerned with the relevance of naturalism for issues like set-theoretic indeterminacy or the existence of a universe of sets, as envisaged by a platonist. The following remarks are only aimed to orient the discussion.

First of all, I have a very general worry about Maddy's conception, which is worth expounding. Does naturalism efficaciously respond to Feferman's concerns about the fact that set theory is essentially platonistic in its essence? If anything, naturalism provides reasons for believing in the objectivity of mathematical questions without hanging upon ontological matters. A naturalist does not see the extension into large cardinals or the adoption of $V = L$ as required by the truth of large cardinal axioms or the truth of L, but by their success in instantiating and thriving on the two, among others, metaphilosophical maxims which motivate set theory as a whole. Analogously, this person would see a solution of the CH as something related to further internal developments of our theories, not as the consequence of a belief in the universe of sets.

Now, Maddy and her fellow naturalists say that ontology is not even touched upon in this discussion of axioms. But, as we have seen, one of the radical reasons for indeterminacy is the instability of our model-theoretic elaborations of V. This indeterminacy is not easily fixed by any structural axiom, except for $V = L$ or *forcing axioms*. What this shows is that we may need principles oriented by *ontological* considerations in order to settle set-theoretic problems and that the way our set-theoretic ontology is designed enters into the solution of important set-theoretic problems.

Now, in contemporary set theory model-theoretic reasoning is so deeply nested in our reasoning that there is no hope to talk about sets without talking about universes. It is true that, maybe, model-theoretic reasoning does not live up to the standards of truth required by a Gödelian and is a mere formalistic shortcut. However, at present, I do not see any other way to talk about existing universes than through model-theoretic frameworks, unless one is willing to take up with *extreme platonism*.

In other terms, there might be other reasons for accepting such maxims as MAXIMISE, but it is also likely that our maxims will be oriented, in one way or the other, at making sense of our talk of *universes*. The consequence is that any set-theoretic maxim, as derived from observational constraints, has to deal with models of sets, which imply, at least, some sort of prior assumptions on how sets are made. For example, the iterative hierarchy is such a model. I do not see how one can preach the use of maxims which encourage the unification of the world of sets and, at the same time, refrain from ontological presuppositions.

In a sense, this is obvious. The maxims of set theory are made in such a way as to allow us to build the ontology of set theory, that is the whole transfinite landscape. Analogously, the maxims of physics have to be made in such a way as to allow us to talk about the basic entities of physics, of which we already have a sort of intuition. Therefore, our maxims must conform to a pre-theoretical grasp of the notions we want to inject into the body of set-theoretic practice. But, then, any such pre-conceptions must always be prior to our metaphilosophy and orient it, rather than being oriented by it.

The naturalist may answer back that this is true, but that this does not testify to the presence of any ontological presuppositions within our maxims. Our maxims have developed in a way which conforms to practice. If practice sanctions a strong interest for ontology, then we should attend to it neutrally. But this, in my opinion, confirms Feferman's concerns on platonism as being the operative mode of set theory, rather than dissolving them. After all, this is exactly what Feferman claims, namely that an interest for a maximal unbridled ontology, call it *Gödelian platonism* (even if it is merely construed as anti-constructivism) or *thin realism*, is the root of the pathologies of set theory.

As to the problem of realism, Maddy thinks that the acceptance of the procedures of set theory may commit us to nothing more than what she labels Thin Realism (see Maddy (2011)), that is the belief that there is a world of sets which is tied to our set-theoretic experience. We could say, a very thin form of EXISTENCE. However, given the increasing complexity of the picture of the world of sets, its unification in the iterative conception, its model-theoretic instability and its extensions into large cardinals, to cite a few among many results, it seems that a person who accepts the existence, in naturalistic terms, of a world of sets should also accept a bit more than Thin Realism. In particular, all the bulk of procedures, techniques and tools which reveal a strong interest for maximality. On this assumption, Thin Realism cannot adequately explain away Feferman's concerns, as operational platonism already happens to be built in it, and our maxims only happen to be confirming this fact.

Finally, we come to indeterminacy. That the CH might never be settled because of an inextinguishable tension between our maxims is an interesting view on the problem, but, if my reasoning is correct and our maxims contain ontological descriptions, I take it that this would just mean that the failure to indicate an ultimate universe will always result in the failure to settle problems such as the CH. If this is true, what stronger evidence for the meaningfulness of the CH than a platonist's could a naturalist bring forward? In fact, in what sense the naturalist's and the platonist's positions diverge with respect to this issue seems to me unclear.

In the face of these difficulties, like Hauser, Maddy tenaciously claims that the CH is meaningful because it is not dependent on any platonistic assumption. She suggests that:

> ...we needn't even argue that the answer to the CH is somehow predetermined, that there is a pre-existing answer out there for us to discover. Instead, we need to assess the prospects of finding a new axiom that is well-suited to the goals of set theory and also settle CH. ((2000), p. 2)

But I ask: what does it mean that an axiom which settles the CH has to suit to the goals of set theory, if not that it should be in accordance with our maxims which, as we have seen, might already be oriented by a platonistic pre-theoretical grasp of ontology? On the naturalist's view presented here, set theory should develop in accordance with the purpose of founding a maximal and (probably) unified mathematical ontology of sets, which is exactly what platonism claims in *operational* and, at the same time, also *substantive* terms.

In conclusion, my view is that set-theoretic naturalism and its allegedly neutral maxims also contain an ontological description of sets which does not seem to depart substantially from that of platonism. I argue, however, that this might not necessarily be a weakness of naturalism, as it provides some evidence that, in order to reach some sort of objectivity within set theory, one may need to embrace platonism, at least in its *anti-constructivist* form. In Chapters 7 and 8 this topic will be partly resumed.

Chapter 6

Full-blooded Platonism

6.1 Plenitude

In the Introduction, it has been mentioned that the *multiversist* maintains that our understanding of the universe of sets necessarily leads to the recognition of the existence of multiple universes of sets. A multiversist platonist would, then, in particular, hold that each universe of sets is an independently existing universe of sets. He would, therefore, keep INDEPENDENCE and EXISTENCE, significantly relaxing, at the same time, the grip on TRUTH, so as to allow the existence of different and mutually incompatible truth-assignments to the indeterminate set-theoretic statements.

Obviously, if there are many universes of sets, it is hard to see how there is an objective truth-value to set-theoretic statements like the CH. However, as we shall see, such a platonist can still claim that, given one universe of sets, each set-theoretic statement has a unique truth-value in it.

In general, all forms of plenitudinous platonism imply the acceptance of a *plenitude principle* like the following:

Principle 7 (Universe plenitude) *Each logically consistent theory, in particular each logically consistent theory of sets, instantiates an existing universe of mathematical objects (sets).*

At the level of objects, plenitude requires the existence of all objects which represent the semantic counterpart of a consistent theory. Therefore, analogously, one can define a principle of plenitude which guarantees the plenitudinous existence of as many objects as posited by logically consistent theories:

Principle 8 (Objects plenitude) *All objects which can exist, that is, all objects that can be conceived of as existing in a logically consistent way, do exist.*

The relation between the first and the second principle of plenitude is clear: the consistency of a theory is ontologically equivalent to the existence of all the universes (models) of that theory and, therefore, to the existence of all the objects posited by that universe.

In other terms, by Principle 7 and 8, the only requirement for the existence of objects is that they are objects of the universes of logically consistent theories. The latter point matches up with the first-order *completeness* requirement: any logically consistent theory has a model (universe).

Plenitudinous platonism is very well suited to respond to the present situation in set theory as eminently characterised by model-theoretic relativity. In particular, it is well suited to represent the ontological picture arising from the generic multiverse induced by *forcing*. Consequently, its main achievement is to show that the challenge to Gödelian platonism posed by Principle 2 in the Introduction is deflated through the acceptance of a plenitudinous ontology.

However, there are several issues related to this conception: do plenitudinous accounts do justice to the notion of objectivity inherent in mathematical platonism? Do they reflect adequately the platonistic notion of existence? Is the acceptance of a plenitudinous ontology really required by the current developments of set theory? Is the claim that we are able to produce consistent theories and objects applicable to all mathematical objects and theories?

Some of these issues will be examined in the last section of this chapter.

Plenitudinous platonism has been advocated by Balaguer (see especially Balaguer (1995, 1998)) and has also been proposed in a different form by Linsky and Zalta (see Zalta (1983), Linsky - Zalta (1995)). In this chapter I will be mostly concerned with Balaguer's plenitudinous platonism, which has been labelled *full-blooded platonism* (*FBP*). As with Maddy's naturalism, my goal is not to refute or subscribe to this conception. Essentially, I will be describing its main features, which are no doubt relevant for the examination of the problem under discussion in this work, and, in the last section, I will try and bring forward some evidence that adopting *FBP* is not less problematic than adopting standard platonism.

6.2 Balaguer's Response to Benacerraf's Argument

First of all, it is important to bear in mind that Balaguer's proposal is an attempt to respond to Benacerraf's epistemological argument (see Benacerraf (1973)). Accordingly, the setting of its ontological framework is essentially subservient to this purpose.

Benacerraf's argument challenges the platonist's claim that he can have knowledge of independently existing mathematical entities. Its premise is that the only reliable account of the knowledge of objects is causal: one knows P because one has had a causal interaction with the objects and facts addressed by P. For physical objects and facts, we can indicate sensory perception as the medium of causal interaction. Unfortunately, there is no analogous medium to be indicated with respect to mathematical objects, as the platonic universe is, by definition, *acausal* and *non-spatio-temporal*. As seen, Gödel claimed that we have a mathematical intuition which would work in the same way as sensory perception but his claim is a far cry from a well-established fact. Maddy (1990) attempted to fill the gap but, admittedly, the results were unsatisfactory.

Given this situation, Benacerraf argues that a platonist cannot give a proper epistemological account of her access to the world of mathematical entities. The conclusion cannot but follow: mathematical platonism is epistemologically bankrupt.

Balaguer's response consists in providing an explanation of what it actually means for a platonist to have access to the realm of mathematical entities.

Here follows the structure of his argument.

1. Mathematicians are able to yield logically consistent theories. At least, they are able to check whether their theories are consistent or inconsistent.

2. Mathematical theories are logically consistent descriptions of mathematical objects, as all consistent theories are satisfied by some objects.

3. To have access to mathematical objects amounts to producing logically consistent theories.

4. All accessible, in the sense of *FBP*, objects exist. [*FBP*'s principle of plenitude: *if there are any mathematical objects, then all possible mathematical objects exist.*]

5. *Conclusion*: mathematicians have access to all existing mathematical objects.

The ontological commitment of *FBP*'s principle of plenitude is clarified by Balaguer in the following way:

> *FBP* can be expressed very intuitively (but also very sloppily) as the view that all possible mathematical objects exist. To give a more precise formulation of the view, we need to get rid of the *de re* modality; thus we might say that *FBP* is the view that all mathematical objects which possibly *could* exist actually *do* exist, or perhaps that there exist mathematical objects of all kinds. ((1995), p. 304)

The first upshot of plenitude is clear. As we have seen, Gödelian platonists are credited with claiming not only that they have an intuition of the mathematical universe, but also that this intuition is replete with a lot of ontological information, which allows them to see mathematical truths.[71] Therefore, they are also required to give an account of how they have access to all of these truths. Plenitude principles corroborate the Gödelian platonist's epistemic views, by

[71]This is precisely the epistemological conception expressed by STRONG ACCESS.

positing that all mathematical objects and universes are on a par, so long as the criterion of consistency is satisfied. Therefore, the access to the truth about mathematical objects requires no more than checking the consistency of some theories.

Plenitude is usually countered insofar as it would imply that one cannot any longer refer to the whole world of sets, but only to *some* sets.

> ...[FBP] seems to forbid us - for no good reason - from speaking of *all* sets. That is, even if the mathematical realm is as robust as FBP suggests, we ought to be able to develop a theory of *all* sets and say whether CH is true in this theory. My response to this is that we *can* develop such a theory. Tell me what you mean by 'set', and I will give you a theory of *all* objects falling under that concept. ((1995), p. 316)

Plenitude, thus, also requires that we no longer claim to have only one definition of *set*. Therefore, the *FBP*-ist's position is also very well suited to respond to model-theoretic relativity, as presented by Putnam. As we will see in the last section, however, as a result of the view that the notion of set is not unique, *FBP* might be likely to fall into a form of model-theoretic relativity which is even stronger than all those considered thus far. Balaguer says:

> According to FBP, both ZFC and ZFC + not-C truly describe parts of the mathematical realm: but there is nothing wrong with this, because they describe different parts of that realm. This might be expressed by saying that ZFC describes the universe of $sets_1$, while ZFC + not-C describes the universe of $sets_2$, where $sets_1$ and $sets_2$ are different kinds of things. ((1995), p. 315)

For the time being, I shall only be concerned with Balaguer's epistemological project. The argument sketched above sets out to prove that the platonist can claim to have access to mathematical objects without appealing to mysterious forms of intuition.

This is because, on Balaguer's view, access to mathematical objects is formally equivalent to the ability to produce consistent theories. Therefore, the interaction of the mathematician with objects reduces to his grasp of consistent mathematical theorising, so to speak.

Although one can always slip into inconsistent theorisations, nobody will seriously doubt that, in general, mathematicians are able to forestall moves which lead to inconsistencies and concentrate on consistent theorisations.

It is a bit bewildering, *prima facie*, that logical consistency is equalled to existence, as it would seem that this is not what platonists mean when they talk about the 'existence of mathematical objects'. However, in view of what we have said in the previous chapters, it seems fairly reasonable to assume that the model-theoretic understanding of existence has primacy over any other form of understanding.

On the other hand, whether Balaguer's argument is really able to account for the claim that platonists have a genuine knowledge of mathematical entities along the lines of Benacerraf's requirements is a different story. A pre-requisite to believe in Balaguer's position is to endorse FBP. But Balaguer tries to show that endorsing FBP as a background epistemological theory is *operationally implicit* in our mathematical practice in the same way as endorsing the existence of an external physical world is operationally implicit in our way of describing the physical world.[72]

This Balaguer accomplishes by introducing the epistemological distinction between an *internalist* and an *externalist* account of one's beliefs. This latter only requires us to show that our procedures of knowledge acquisition are reliable, whereas the former also demands an explanation of why one thinks these procedures to be reliable.

In Balaguer's opinion, Benacerraf's problem only requires of the platonist to give an *externalist* account of his procedures of knowledge acquisition. The main reason is that all accounts of the procedures of knowledge acquisition with regard to physical entities are also externalist. In accordance with the require-

[72]Balaguer says that an FBP-ist is, thus, the equivalent in mathematics of an EWA-ist (EWA = External World Assumption) in physics, that is, the equivalent of a person who assumes the existence of an external world. In Balaguer (1995), Balaguer tries to dispel (in my opinion, successfully) the waves of criticism directed at this position.

ments of the causal theory of knowledge, we can account for the knowledge of physical bodies through sensory interaction. Thus we can, analogously, provide an externalist account for mathematical knowledge through FBP.

As a result of the full adequacy of our externalist account, Balaguer seems to suggest that the adoption of FBP does not require prior knowledge of the features of FBP, exactly in the same way as the adoption of a causal theory of knowledge does not imply prior knowledge of causal interactions.

6.3 Multiversism and Objectivity

We have said that any logically consistent theory must describe a portion of the mathematical realm, as envisaged by FBP's principle of plenitude.

But we should say more about what this FBP's multiverse is like. In particular, since we can have different axiomatisations of the notion of set, one, for instance, where the Axiom of Choice is true and one where it is not, one where the Axiom of Determinacy is true and one where it is not and so on, all of these theories will necessarily describe different universes.

But there is more. There is an extension of ZFC where the CH is true and one where it is false, and, by the principle of plenitude, both theories describe two mutually incompatible but equally existing universes.

The multiverse of FBP, therefore, comprises not only all possible models of one single theory, but also all possible models of all logically consistent theories. Each theory automatically induces one description of the mathematical realm.[73] In particular, since there are different descriptions of sets, then there will be different universes of sets. All these different universes of sets are equally and legitimately existing universes of mathematical objects.

The FBP-ist's interpretation of set-theoretic indeterminacy must, therefore, be that each theory does not instantiate only one universe, insofar as, given the fundamental incompleteness of each theory, each of them is equiconsistent with many other theories. Since, for instance, $Con(ZFC) \leftrightarrow Con(ZFC + CH)$ and

[73] Actually, more than one, as ZFC has infinitely many models where different set-theoretic facts hold or do not hold.

$Con(ZFC) \leftrightarrow Con(ZFC + \neg CH)$, then we have at least two possible different universes.

However, one might wonder whether this gives rise to a contradiction. In particular, since the CH and its negation can be both true at the same time, it would follow that $CH \wedge \neg CH$ is also true. The trick is that, obviously, $CH \wedge \neg CH$ cannot be true in *any single* logically consistent theory, whereas in each of them, either of the statements may be true. From the point of view of each single theory, therefore, there is no contradiction.

There is a *prima facie* worry about multiversism. CH is neither true nor false from the axioms of ZFC. Therefore, at first glance, one does not feel at ease with the idea that the CH is both true and false in different set-theoretic universes. This is because we feel that such a choice would depend upon objective facts as, presumably, implied by the objective relations between the continuum and other set-theoretic concepts.

However, in the light of such a plenitudinous universe, the problem of set-theoretic indeterminacy, in the form: 'what axioms would one accept which decide the CH?' will inevitably be deflated. Since it is possible to show that there are consistent ZFC-theories where the CH is true and others where it is false, then we just have to live with this plurality of options. But if we want to salvage, so to speak, a form of objectivity, then, Balaguer argues, one might also try and define criteria which would justify the preference of a universe over another one. He says:

> There are at least two ways in which FBP-ists can salvage the objective bite of mathematical disputes. The first has to do with the notion of *inclusiveness*, or *broadness*: the dispute over CH, for instance, might be construed as a dispute about whether ZF+CH or ZF+not-CH characterizes a broader notion of 'set'. And a second way in which FBP-ists can salvage the objective bite is by pointing out that certain mathematical disputes are disputes about whether some sentence is true in a *standard model*. ((1995), p. 317)

FBP's characterisation of the platonist's notion of objectivity and truth is a bit uncomfortable. In the next section, we shall briefly see why. In general, as said, Balaguer is more interested in giving a viable account of a platonist epistemology than in dealing with the question of objectivity. He is aware of the difficulties related to set-theoretic indeterminacy for a standard platonist. However, given the preferability of *FBP*'s epistemology over non-full-blooded epistemologies, he seems to conclude that there is no alternative way to fix the problem. He says:

> The most important advantage that FBP has over non-full-blooded versions of platonism (i.e., versions of platonism in which there is just *one* kind of set and in which CH is either true or false) is that all of the latter fall prey to Benacerraf's epistemological argument. If non-full-blooded platonism were correct, it would be a mystery how we could ever know whether CH was true or false; or, in the lingo of FBP, it would be a mystery how we could know whether *the* universe of sets was a universe of $sets_1$, or $sets_2$, or... ((1995), p. 317)

This latter quote also features one further point in favour of *FBP*: not only does this conception allow one to explain how it is possible to be a platonist and have knowledge of independently existing mathematical entities, but it also leads to the demise of TRUTH as implied by the existence of a uniquely determined ontology.[74]

In the next section I will present some of the main problems with which I think that the *FBP*-ist's conceptions may be fraught.

6.4 Critical Discussion

In this section I will just assume that a platonist might want to use *FBP* essentially as a response to the problem of set-theoretic indeterminacy. Balaguer

[74]Among other advantages of *FBP* is '..that it reconciles the objectivity of mathematics with the extreme *freedom* that mathematicians have.' ((1995), p. 318). The resemblance between this claim and Cantor's analogous statements contained in Cantor (1883b) is no doubt striking. In general, Balaguer's conception seems to be strongly indebted to Cantor's very sketchy epistemology of mathematics, as presented in *Grundlagen*.

has tried to argue that the main reason to accept FBP is that it allows one to give a coherent picture of a platonist epistemology, whereas my main interest is for its ontological claims.

The remarks which follow aim to show that, if FBP is essentially preferred for its ontology, its acceptance implies some woeful issues. I will take a neutral stance on the problem of whether FBP is successful in delivering a coherent conception of epistemology.

Obviously, criticisms directed at the FBP-ist's ontology might have bearings on its overall plausibility. However, I guess that Balaguer's project could still be advocated even if one has strong concerns only on the plausibility of its ontological claims.

The platonist multiversist I will be describing is not necessarily a thorough-going FBP-ist, but his way of thinking is very close to that of an FBP-ist or, in any case, to someone who holds principles of plenitude.

In general, the most serious strain of criticism invests the relation between axioms and models, which FBP reverses: model-theoretic understanding is prior to concerns over the correct axioms. In particular, this also implies a quietist stance about the fact that, in set theory, there are infinitely many universes of sets which are produced by the same axioms.

Now, this interpretation of set theory may be shown to be substantially deviating from mathematical practice and also from the very expectations of a platonist. The point is fully developed in the next subsections.

6.4.1 Relativity and Ontological Plenitude

To begin with, we should ask whether Cohenian relativity can be construed as saying that there are different universes of sets, as envisaged by FBP's principle of plenitude.

As seen, this is obvious in FBP's intentions and, to some degree, also in the *standard* interpretation of contemporary set theory (see, for example, Martin (2001)).

But one could say that what Cohen's method proves is only that V is indeterminate, not that there are different, existing in the platonist's sense, descriptions of V. Let ZFC_α, where $\alpha \in On$, the theory derived from the addition of $2^{\aleph_0} = \aleph_\alpha$ to ZFC. All of these theories, apart from those where α is inconsistent with König's theorem, are all equally consistent theories.

Balaguer claims that for all such α, all these theories describe portions of the mathematical realm. A first and foremost concern about this conception is whether it could ever legitimately count as a platonist construal of the universe of sets. The difficulties arising in connection with this point are acutely presented by Potter:

> ...for a view to count as realist [platonist]... it must hold the truth of sentences in question to be metaphysically constrained by their subject matter more substantially than Balaguer can allow. A realist [platonist] conception of a domain is something we win through to when we have gained an understanding of the nature of the objects the domain contains and the relations that hold between them. ((2004), p. 11)

Moreover, for a mathematician whose main concern is about the plausibility of the axioms, the CH or its negation are not and cannot possibly be axioms. Therefore, the CH or the statement, for instance, $2^{\aleph_0} = \aleph_{\omega_2}$, do not both represent a legitimate way of extending ZFC. In view of this, the fact that the axioms have infinitely many different model-theoretic instantiations is largely immaterial.

Strong and fruitful extensions of ZFC are those in which some axioms would *imply* a certain value of the power of the continuum. But the process of extension, here, would not be tied to model-theoretic considerations, as the addition of an axiom will be justified *a priori* and not in the light of the model-theoretic constructions which satisfy it.[75]

Suppose we come to accept a new axiom A, which, in particular, implies the CH. The supporter of the priority of axioms over models would say that only such cases would provide interesting extensions of ZFC and a way of modelling

[75] Although, as said many times, several axiom candidates are strongly oriented by model-theoretic considerations (in particular, $V = L$ and forcing axioms).

the ontology in a meaningful and substantive way. In particular, once accepted A, then the CH becomes indestructible, that is, in any model of $ZFC + A$, the CH would be true. Maybe, there would still be generic extensions where the value of other undecided properties of ZFC could be changed. But, at least, by A, the value of the CH would be invariant across the multiverse and, therefore, in the light of our looking for an objective value of the CH, we would just want to prefer that picture of the multiverse to others for very good reasons.

Indeed, this is what Gödel suggested as being the primary aim of the extensions of ZFC, a point about which, I guess, all set-theorists would agree: only strong axioms can extend ZFC meaningfully.

But there is another issue investing the preference for model-theoretic notions over investigations about axioms. The issue is that Balaguer's conception may imply a sort of radical indeterminacy of our set-theoretic concepts. Colyvan and Zalta have exposed this inherent weakness of FBP very clearly:

> As soon as a platonist postulates a plenitude of mathematical objects, it becomes a question as to how the singular terms of our most fundamental mathematical theories can have denotations. If all possible sets exist, there will be an ω that exists in virtue of the truth of ZF + CH, an ω which exists in virtue of the truth of ZF + AC, etc. So which of these sets does the singular term 'ω' in ZF denote? [...] A traditional non-plenitudinous platonist assumes that there is exactly one true set theory, and so may suppose that 'ω' in ZF has a unique denotation. But this is not an option for Balaguer's full-blooded platonism. ((1999), p. 345)

To summarise Colyvan and Zalta's objection: according to Balaguer, we do not have a fixed reference for sets. But this may imply a radical indeterminacy of many set-theoretic notions which are *uniquely* interpreted. If we have different universes, which represent different kinds of sets, we will also have different concepts, for instance, of ω. However, *prima facie*, there seems to be no need to postulate that set-theorists denote different objects when they are talking about very simple objects such as ω. Hence, *prima facie*, FBP leads to a very implausible interpretation of mathematical practice.

This argument shows that, at least potentially, a plenitudinous interpretation of the multiverse can put at risk even the most basic assumptions about sets, inflating model-theoretic indeterminacy to an intolerable point.[76]

6.4.2 Does the existence of a Multiverse really imply a Plenitudinous Ontology?

To use a metaphor, FBP could be seen as a sort of *inflation theory* of the mathematical universe. At some point, in the history of the mathematical universe, an event occurred which produced an inflation of the universe and its collapse into many different universes. Multiversism seems to require that this collapse occurs at the level of our infinitary concepts, as forcing does not affect the finitary part of mathematics. On the contrary, if Colyvan and Zalta's criticism is sound, an FBP-ist may be forced to view multiversism as affecting all the objects (universes) of mathematics, including natural numbers (arithmetic).

While I think that Balaguer is right in construing our ontological considerations as deriving from model-theoretic constructions, I am not convinced that the only possible solution is that of postulating such an inflation of our ontology. One may simply assert, for instance, that there is no chance to have a single

[76]In Linsky - Zalta (1995), the authors follow a different approach in order to prevent radical denotational indeterminacy within a plenitudinous framework. They define a form of plenitudinous platonism (which they label *platonized naturalism*), in which *abstract* objects are conceived of as encoding, rather than exemplifying, properties attributed to them by their respective theories. For instance, the objects of PA encode precisely those properties which are attributed to them by the axioms of PA and the objects of ZFC those properties which are attributed to them by the axioms of ZFC. The rationale for the introduction of the notion of encoding is that encoding is incomplete with respect to some properties. In particular, let Φ_{CH} be the property that '2^ω is equal to ω_1' and $\Phi_{\neg CH}$ the property that '$2^\omega \neq \omega_1$'. On Linsky and Zalta's account, the object 2^ω is *encoding incomplete* with respect to Φ, as neither of these two properties can be attributed by the axioms of ZFC to it. Thus, $ZFC + CH$ and $ZFC + \neg CH$ should be viewed as extending the encoding of ZFC with regard to 2^ω, while preserving a unique reference to it. In other terms, 2^ω is the same object in ZFC, $ZFC+CH$ and $ZFC + \neg CH$, but in the first one it is *encoding incomplete* (with respect to Φ_{CH} and its negation), in the second and in the third it is subject to different *encodings*. Unlike FBP, this conception gives priority to axioms over models.

determinate universe, while disavowing the '*increasing of our ontology to the limit*' (Colyvan - Zalta (1999), p. 338).

Secondly, the point of view of the supporters of generic absoluteness described in Chapter 2 agrees with Balaguer's conception about the fact that model-theoretic considerations have primacy. But then, one might use the same model-theoretic considerations to reverse Balaguer's theoretical purpose and restore stability and invariance across the multiverse, by introducing a single (or more) *forcing axiom*. I guess that, epistemologically, the ultimate-universe view, as arising from these endeavours, would be as easily justifiable as FBP's implicit multiversism.

The purpose of all these constructions is to show that, even in the presence of multiple universes, amalgamations are always possible. But even prior to these attempts, Zermelo's quasi-categoricity theorem pursued the same line of reasoning. A similar argument can be found in Martin (2001).

Martin argues that, if there are strong reasons to believe that the standard structure of arithmetic is well-understood and is unique up to isomorphism (as seen, even nominalists like Field supports this claim (see Field (1989, 2001)), then the same reasons may be used to argue in favour of the contention that there is a unique isomorphism between any two V-structures which can be defined along all transfinite stages.[77] Martin's construction is very similar to that used by Zermelo to prove his quasi-categoricity theorem.

However, by using a line of reasoning essentially based on second-order logic, contrary to some second-order logic supporters, Martin does not want to show that the set-theoretic realm is not indeterminate. He just wants to prove that a plenitudinous universe-ontology may be adequately amalgamated into a single set-theoretic universe, in which, presumably, only one truth-assignment is pos-

[77]That is, in mathematical terms, if $V^{\mathfrak{M}_1}$ and $V^{\mathfrak{M}_2}$ are two structures satisfying the axioms of ZFC, then there is a unique isomorphism $\pi : V^{\mathfrak{M}_1} \to V^{\mathfrak{M}_2}$ or, otherwise, that an *amalgamation* of these two structures is always possible. As a matter of fact, in order to accomplish this, Martin is forced to assume Cantor's Absolute Infinite Principle and the Well-Ordering Theorem. However, in his construction, these two principles amount to no more than two well-established procedural principles.

sible. Martin's conclusion is that the use of a plenitudinous ontology in order to prove that there cannot (or ought not to) be any unique truth-assignment to statements such as the CH is, at best, misguided.

6.4.3 Asymmetry and Consistency Strength

If one is more interested in axioms as being prior to model-theoretic considerations, then one cannot but aver that any two logically consistent axiomatisations encode different information concerning sets and that their differences are substantial. While this may not be a problem for a plenitudinous platonist, the issue of the relevance of our theories is certainly crucial for mathematicians. It is my opinion that FBP simply does not address this issue adequately: Balaguer claims that we may want to bring forward some criteria to prefer one theory over another but he doesn't say which.

For a start, as shown in Chapter 4 through the discussion of Gödel's stance, there are extensions of ZFC which are accepted because they are intrinsically plausible, namely, as arising from generalisations of well-established and well-describable procedures.

Again, let ZFC_{IC} be ZFC plus the axiom which postulates the existence of an inaccessible cardinal. ZFC_{IC} and ZFC are clearly asymmetric, so to speak, in their ontological claims, although there is some evidence that they might be both consistent. This would be sufficient for an FBP-ist to claim that these two theories describe equally legitimate universes of set theory and, thus, that they are equally *acceptable*.

But consistency, here, is a bit vacuous and unproductive for the acceptance of ZFC_{IC}. The fact that it is possible to formulate a logically consistent theory of inaccessible cardinals is very scant evidence that inaccessible cardinals should find a place within our mathematical procedures. One might want stronger reasons for accepting a stronger axiomatic theory and these reasons, generally, go along with the reasons for accepting previously defined procedures.

Granted, this asymmetry works essentially at the level of a procedural justification of mathematics. It justifies the acceptance of ZFC on the grounds that it is essential for our mathematical purposes, while it makes the acceptance of ZFC_{IC} more dubious insofar as this latter is remotely connected to our mathematical practice.[78] Moreover, ontologically this asymmetry is largely irrelevant: larger models are treated in the same way as smaller ones.

Yet, procedurally, it is very dubious that the consistency of a theory says something relevant, in the mathematical sense, about its *acceptability.*

There is a final objection, this time directed at FBP's epistemology, which is worth mentioning here. FBP wants that all logically consistent (possible) objects exist. The consistency of these objects is essential here, as it is exactly on the assumption that we can have an adequate conception of consistency that FBP rests.

However, as we have seen, the consistency strengths of our theories largely differ and there seems to be a linear ordering of their strength on a scale of cardinals which goes as far as large cardinals. As can be proved easily, $Con(ZFC_{IC}) \rightarrow Con(ZFC)$, whereas the converse does not hold. In this case, we say that the consistency strength of ZFC_{IC} is greater than that of ZFC.

This clearly shows that ZFC_{IC} is a stronger theory than ZFC, but, at the same time, that the reasons for conjecturing its consistency must be definitely stronger than those which would support the consistency of ZFC. In general, the farther one goes in climbing the scale of large cardinals, the more difficult it seems to receive a confirmation of their consistency.

Therefore, what the existence of a scale of consistency strengths may show is that our talk of consistency becomes the more obscure the farther theories depart from those bits of mathematics which one has very natural reasons to believe to be consistent (e.g., arithmetic). Moreover, there may also be an analogous scale in our ability to evaluate the consistency of theories and this ability may not be necessarily deployed homogeneously across the whole range of consistency strengths.

To summarise: if, on an internalistic account, one has reasons to believe that

[78] Even so, as shown in the previous chapters, this latter claim is probably misguided, as large cardinals may have strong bearings even on our conception of finite sets.

FBP is true because of our ability to evaluate consistency, then this view might be threatened by the existence of a scale of different degrees of consistencies.

Chapter 7

Set-theoretic Objectivism

7.1 Overview of the Chapter

As shown in Chapter 3, Gödelian platonism is pressurised by set-theoretic indeterminacy. In particular, failure to gain confirmation of its ontological claims from contemporary set theory may represent such a serious drawback for this conception of mathematics as to make it overall untenable.

Does this imply that we had better no longer care about the objectivity of mathematics? Is it still possible to account for the objectivity of sets without being entangled in the controversial ontological claims of platonism? If the answer is yes, would there still be a role for platonism to play? What should we think of indeterminacy from the set-theoretic objectivist's perspective?

In this and the next chapter, I will attempt to answer these questions, providing, at the same time, my personal view of what set-theoretic objectivism should be. In particular, I shall argue that the belief in the objectivity of our set-theoretic procedures does not imply the belief in the truth-value uniqueness of statements like the CH.

Objectivism has been held and expounded by several authors and, as has been shown in the previous chapters, is rooted in various theoretical perspectives. First of all, it was already a part of Cantorian philosophy (see Cantor (1883b, 1885, 1887-88)) to view sets as integral to the objective metaphysical design of the world. Notwithstanding its theological framing, Cantor's philosophy essentially identifies objectivity of sets with the immanent (conceptual) content of set-theoretic propositions.

Cantor's *immanence view* is the ancestor of set-theoretic objectivism. Husserl and his philosophical method, possibly inspired by Cantor's philosophical doctrines (see Ortiz (1997), Tieszen (2002)), take up the challenge to show that the realm of objectivity is tied to that of *introspection.*

Gödel's characterisation of objectivity (Gödel (1944, 1947, *1951)) equally shares Cantor's and Husserl's motivating conceptions. However, it still oscillates between standard platonism and full-fledged *conceptual realism.* In the end, his emphasis on objectivity rather than objects seems to reflect the abandoning of a thorough-going objects-platonism.

Plenitudinous platonism follows a different route, insofar as it conjectures that there is a *plenum* of objects which satisfy all possible determinations. Balaguer (see Balaguer (1995, 1998)) wants objectivity to be resting upon the saturation of ontology, whereas other forms of plenitudinous platonism (such as Zalta and Linsky's *naturalised platonism*) require the description of a theory of *abstracta* as non-spatiotemporal objects. Standard naturalists like Quine, Putnam and Maddy baulk at any non-mathematical ontological claim and encourage the acceptance of methodological holism as a way to (conditional) objectivity.

Forms of objectivism have also been voiced by scholars coming from other philosophical backgrounds. For instance, recently Feferman (Feferman (2011)), in the wake of Searle's suggestions (see Searle (1995)), has spoken of *intersubjective* criteria of objectivity while keeping ontological commitment to a minimum.

Kai Hauser (Hauser (2001)) recasts objectivity in the spirit of Maddy's naturalism as a study of *theory formation,* substantially drawing upon Wang's (Wang (1997)) and Husserl's intimations. In particular, we will be concerned with Hauser's perspective in the second section of this chapter.

There is a wide variety of positions and perspectives on set-theoretic objectivity. The common trait of all the aforementioned conceptions is that most of our ontological concerns should be dropped. A second shared feature is their emphasis on our *conceptual practices* rather than on our *intuition of objects* (and, hence, on our ontological presumptions).

Therefore, in this chapter I will have to say something about the rise of the importance of concepts in the question of objectivity and on how, in particular, this has intersected with platonic realism.

It is not clear to me whether a platonist who lays emphasis on conceptual objectivity should automatically be considered a semantic realist, that is a realist who drops INDEPENDENCE and EXISTENCE and keeps only TRUTH. On the one hand, Gödel would admittedly be such a platonist, in the light of his late adherence to *conceptual realism.* However, as we have seen, he never completely dropped his faith in the truth of the axioms as reflecting truth in an independently existing world.

In any case, if the world of concepts is conceived of as a *single*, existent world of *concepts,* I do not see the point of abandoning a flawed single-universe objects-ontology to embark on a single-universe concepts-ontology.

The project of some objectivists is that of showing that set-theoretic indeterminacy is a transient phenomenon. On this perspective, the ultimate theory of sets will have to settle practically all most relevant set-theoretic problems. In my opinion, this route aims to circumvent the difficulties arising from Gödelian platonism, but ends up requiring the same determinacy which is conjectured by Gödelian platonists: the objectivity which is implicit in their conceptions may not be significantly less demanding than that required by Gödelian platonists.

Other objectivists (among them, presumably, naturalists) will just require that a sort of agreement on what solutions ought to be found for the unsolved problems may be reached at some point. Solutions may be negotiated by mathematicians in view of its consequences for their work. As happened with the Axiom of Choice, the acceptance of new axioms would, then, follow the route of extrinsic considerations. Objectivity of sets would, then, more or less amount to a sociological notion, hanging on the balanced decisions of the mathematical community.

In the last section of this chapter, I will expound my take on objectivity, which is thinner than that proposed by other set-theoretic objectivists, but also diverging from that of naturalists.

As I said in the Introduction, I entertain the idea that a set theorist has intuitions over *objects* (structures) and over *concepts* (properties of objects and structures). The study of the interrelations between these two kinds of ontology represents the foundation of objectivism. Contrary to other objectivists, I maintain that a genuine ontological intuition is an essential part of our conception of sets. This is because objects are also part of a process of conceptualisation and they do not need, for their existence, the postulation of platonic universes or INDEPENDENCE. However, I do not exclude the possibility that concepts such as sets may have an alternative form of existence. The matter is beyond the scope of this work.

Upon an accurate reconsideration of what sets are like conceptually, I also argue that we should allow for indeterminacy as being unavoidable. Central to this reconstruction is, finally, the role of *operational platonism* qua anti-constructivism: this latter is instrumental for the *immanent* development of set theory.

I will conclude this section with one quote. The goal of present and future set-theoretic objectivism, in all its forms, should be that of making sense of the following words of Dehornoy, who has studied very exotic objects, braids, only to find that they are close relatives of very large sets. From this fact he infers that:

> It seems to us that the role of set theory in such cases [as that of braids, *my note*] is quite similar to the role of physics when the latter gives heuristic evidence for some statements that mathematicians are to prove subsequently... we think that the braid order is an application of set theory, and, more precisely, of this most speculative part of set theory that investigates large cardinals. ((2000), p. 600)

7.2 Ontology dropped

We should now pause a moment to recollect all the multiple threads of our discussion. As said, Gödelian platonism is in trouble with its ambition of viewing set theory as an objective description of a universe of objects (sets). The reason for the trouble was clear from the beginning, but resuming it will help us tidy up the picture once and for all.

A Gödelian platonist thinks that the objectivity of set-theoretic propositions is related to the objectivity of a world which has certain describable properties.[79] For reasons which have become increasingly compelling along the development of set theory, the platonic universe has assumed the traits of V, that is, of the *iterative hierarchy* of sets.

A pre-theoretical reason for seeing the platonic universe of sets as coinciding with V is that V encodes the ontological information coming from the axioms.[80] Philosophically, V also meets with the requirements of scientific simplicity, regularity and cleanliness. Its primacy can also be advocated indirectly, appealing to our practical purposes in mathematics. On this view, the dispensability of the Axiom of Foundation, for instance, although justifiable, would be seen as impractical, in the light of our purposes.

Unfortunately, as has been widely shown, V is strongly underdetermined by the axioms. But even in the presence of substantive characterisations of V via extensions of ZFC, one can always find models of our theories which are mutually incompatible. This is essentially due to the strong model-theoretic relativity of first-order logic. Second-order logic proposes a fix, which is somewhat uncomfortable in many aspects (see Chapter 3).

Now, it is said that first-order models work as approximations of V. But this obviously implies platonism, insofar as it implies that there exists a fully characterised V (which we may find at some point). Therefore, it seems more correct

[79] Extreme platonists hold that not all the properties of this world are describable (which amounts to Weak Access), but, as we saw in Chapter 3, they are not able to provide strong justifications for this claim.

[80] Obviously, the fact that V is the platonic universe of mathematics is, in turn, a consequence of platonism's being based, in its Gödelian form, on *set-reductionism*.

to say that these models retain all the general features of V, giving each its own version of what V is like. However, the ultimate-universe supporter will reply that the generic multiverse induced by forcing can be made invariant through suitable axioms. We leave it open whether axioms of generic absoluteness are genuine axioms and, above all, whether they will ever succeed in reinforcing our belief in a single universe.

In the meantime, I will claim that, on any reasonable view of the strength of our logical procedures, the *single universe view* is untenable.

To summarise:

Claim 4 *Gödelian Platonism is committed to the view that 1) there is a single universe of sets and that 2) we have access to it. Different reasons lead to viewing this universe as V. Unfortunately, model-theoretic relativity shows that our theories invariably talk about many different and incompatible universes. Therefore, Gödelian platonism is flawed.*

It has been suggested that a feasible alternative would be that of endorsing a multiversist view of sets. Sets do not form just a single class of entities (universe). On the contrary, platonistic multiversism requires that different universes are different existents, allowing us to come to terms with Claim 1. Is multiversism any better in relation to set-theoretic indeterminacy?

In Chapter 6, I mentioned some of the most salient objections to this view. In particular, it was shown that any characterisation of a universe as existent by the construction of a corresponding model falls short of relevance. For instance, the fact that there is a compatibility between a certain set of axioms and different values of the cardinality of the continuum is a largely immaterial fact for a mathematician. But there is another complication. As pointed out by several authors, multiversism implies a strong referential variance. The purpose of multiversism is to account for referential indeterminacy in some areas (the continuum) but may end up fostering referential indeterminacy in all areas of set theory.

Ontologically, naturalism would be seen as more favourable in this respect. However, I have brought forward some potential objections, to the effect that our naturalistic goals (as encoded by our maxims) are hiddenly oriented by ontological considerations. Or maybe I am wrong about this and naturalism successfully deontologises set theory.

At any rate, what, in my opinion, all this suggests is that the real reason of trouble for a platonist is the pretension to be able to deliver a reasonable conception of *platonic existence*. The natural conclusion is that *mathematical objects should not be viewed as existing* in platonistic terms, that is, they should not be viewed as existing independently of our descriptions.

Of course, for a Gödelian platonist EXISTENCE and INDEPENDENCE go together. Therefore, one might also think that I am encouraging the denial of the existence of mathematical objects *tout court*, which is a slightly different claim. But I think that it is clear enough from the context what form of existence I am not seeing as legitimate.[81]

7.3 Concepts

7.3.1 Sets as Concepts

The theory that sets are concepts emerged quite early in the historical development of set theory. As seen, it had been already formulated by Cantor in *Grundlagen* and is one of the main features of his philosophy of mathematics.

As known, Frege's treatment of natural numbers is conceptual (see, in particular, Frege (1884, 1893/1903)), although Frege's idea that all truths of arithmetic are *analytic* is actively countered by Gödel (see Gödel (1944)) and also substantially deviates from Cantor's conceptions (see also Maddy (1997) and Potter (2000, 2004) for further clarifications on this point).

The way I want to frame the discussion of sets as concepts is essentially related to Cantor-Gödel realistic line of thought. In particular, my account wants to focus on the notion of *conceptual expansion*, which is inherent in both

[81] In other terms, I am not denying EXISTENCE. On Linnebo's classification (Linnebo (2009)), this would, at least, amount to anti-nominalism, that is, the claim that mathematical objects are epistemologically irreducible to empirical objects.

Cantor's and Gödel's conception, but is not very much favoured within Frege's account of logic (see below).

The emergence of *conceptual realism* in Gödel's thought may have been sped by his adherence to Husserl's phenomenology.[82] Whatever the matter may be historically, it is no doubt true that many of Gödel's philosophical claims can be correctly interpreted in the light of his adherence to Husserl's system. As a consequence, some references to phenomenological notions, methodologies and terminology will be made on the go.

Focus on concepts causes a clear shift of interest from models (objects) to axioms (theories). As we have already seen, the question whether axioms are prior to semantics was crucial in view of an assessment of Balaguer's efforts and of the various principles of plenitude. On that view, all theories are equally acceptable, so long as they are consistent. In particular, each semantic framework represents a universe of that theory, so long as that theory is consistent (as it obviously is, if it has a model).

Focus on concepts represents the reversal of this perspective. We should no longer care about descriptions of universes of objects as required by the ontological requirements implicit in Gödelian platonism but, rather, focus on the objectivity of our theories as the outcome of the search for *objective axioms*.

But now a question naturally arises: how do we find objective axioms? Well, in accordance with the methodological premises of conceptualism, we are supposed to be able to elaborate upon the notion of *set* and *set of*, list the most relevant properties which can be attached to these notions and view our refined conceptualisations as legitimately representing objective axioms. The burden of specifying the correct procedure is usually passed onto the hands of a theory which will explain how we acquire knowledge, like those described by naturalists.

A conceptualistic form of set-theoretic naturalism has been recently cham-

[82] Of which we are, as usual, informed by Wang (see Wang (1997), p. 164: 'Available evidence indicates that from 1959 on Gödel studied Husserl's work carefully for a number of years. His library includes all of Husserl's major writings, many marked with underlinings and marginal comments, and accompanied by inserted pages mostly in Gabelsberger shorthand.')

pioned by Hauser (see Hauser (2001, 2002)). I want briefly to examine his point of view here, as I wish to contrast his conception with mine.

Hauser has spoken of inversion of priorities as the main feature of his project. Now, as known (see Chapter 3), he thinks that the recent work in set theory, partly expounded in Chapter 2, clearly demonstrates this tendency in the set-theoretic operational context. In particular, he seems to think that it would be possible to find an objective truth-value to the CH, in the face of the collapse of the *single-universe theory.*

His perspective is, therefore, that of the *truth-value realist* who does not believe in an independently existing truth. Rather, objective truth-values are found within the context of objective theories. The objectivity of these latter, in turn, is the final product of a process of conceptual selection, refinement and generalisation.

He says:

> ...the primary epistemological concept is evidence rather than truth. Much of my account of set-theoretic practice can be construed in terms of the epistemological primacy of *objectivity over objects*, and parts of it will re-inforce structural similarities between prevailing maxims of theory formation in set theory and the natural sciences. ((2001), p. 246)

Hauser's goal, as he clearly sets forth, is not very different from that of naturalistically-minded philosophers like Maddy. The only difference is that, in Hauser's view, the tension between maxims and incompatible solutions will be resolved, eventually, by showing that some *theories* are preferable to others. All this will occur during the formation stage of our theories, which involves a selection of suitable logical properties of sets, and thence comes his idea of *objectivity in theory formation.* The final process rests upon procedures of *verification, confirmation* and *falsification* analogous to those used in the scientific practice.

Commenting on analogous remarks made by Martin (see Martin (1998)) about the plausibility of Axioms of Determinacy, Hauser says that:

> Martin describes two important consequences of determinacy [...] that have had a profound influence on the development of both recursion the-

> ory and set theory. As emphasized by Martin the most remarkable (from the methodological view point) aspect of these lemmata is that they provide a striking example of (theoretical) *prediction and confirmation* in the natural sciences. ((2001), pp. 254-5)

Hauser lists some criteria which will have to be checked within our testing procedures, like *relevant asymmetries* induced by different axioms.

For instance, the fact that any axiom which gives a complete characterisation of the theory of $< H(\omega_2), \in >$ must imply, from the point of view of Ω-logic, the falsity of the CH, points to a remarkable asymmetry between CH and $\neg CH$, measured against a particular class of axioms.

The most interesting point of Hauser's formation theory is that one does not start with a pre-formed ontology of sets. Platonism provides an obvious justification for finding objective truth-values to statements like the CH, because it is rooted in an ontological belief. Hauser's objectivity in theory formation, while denying the need of any ontological presumption, does not exclude the possibility that some ontological beliefs may be derived from *direct verification.*

Hence his claim that, in the light of the existence of a precise characterisation of large cardinals via inner models and their suitability to work as a sort of *yardstick* for the strength of set-theoretic statements, one should conclude that:

> The level of understanding attained in this way is what provides the main motivation for accepting the large cardinal hierarchy and its calibration in terms of relative consistency strength as the natural super-structure for ZFC. ((2001), p. 260)

I see one main difficulty within this conception: what counts as *evidence for confirmation* in mathematical testing? Naturalists like Quine and Putnam would respond that evidence is intertwined to the usefulness of axioms within our fullest scientific theories. This does not seem to be Hauser's point of view. But if this is true, how is it possible to account for the fact that an axiom has *received confirmation* from its testing within a set-theoretic environment?

Moreover, I also suspect that Hauser's concealed objective is that of supporting an *ultimate-universe view* indirectly, by showing that the assumption of certain axioms would restore uniqueness of solutions. At least, his wholehearted support of Woodin's programme goes in that direction.

In particular, as said for Maddy's naturalism, forcing axioms have bearings on some ontological properties, in particular *invariance under forcing*. Therefore, the choice of any of these axioms must imply *ontological non-neutrality*.

If my interpretation is correct, the hidden goal of this form of set-theoretic objectivism would, then, be an old one: that of gathering indirect evidence that the universe of sets is so and so. As we have seen, this reiterates the issues connected to Gödelian platonism and ought not to be accepted without full awareness of its shortcomings.

7.3.2 Conceptual Realism

In *Grundlagen*, Cantor sketches a procedure to elaborate the immanent structure of mathematical concepts, so as to produce stronger and stronger refinements:

> One posits a thing with properties that at the outset is nothing other than a name or a sign A, and then in an orderly fashion gives it different, or even infinitely many, intelligible predicates whose meaning is known on the basis of ideas that are already at hand, and which may not contradict one another. In this way one determines the connection of A to the concepts already at hand, in particular to related concepts. If one has reached the end of this process, then one has met all the preconditions for awakening the concept A which slumbered inside us, and it comes into being accompanied by the intrasubjective reality which is all that can be demanded of a concept [...]. ((1883b), pp. 918-9)

There are at least two things worth noting in this passage.

First of all, the methodology envisaged by Cantor. At the outset, there is a simple, raw concept. Afterwards, one tries to establish connections with other already available concepts. This amounts to specifying all the possible relations which the concept has with other concepts. In particular, this may require that one takes into account an infinite list of properties. Cantor does not provide any further clarifications on how the procedure should be carried out.[83] Jané explains the procedure in the following way:

> This is like characterising the concept by means of a consistent set of axioms, but not as an isolated theory, because, in order to ensure that the new concept be suitably embedded into the body of mathematics, Cantor demands that the meaning of the predicates in question (thus the meaning of the terms occurring in the axioms) be known 'on the basis of already available ideas'. ((2010), p. 195)

An obvious case study is the theory of the transfinite in the way Cantor himself developed it. Cantor's generating principles can be very neutrally construed as encoding the most salient properties of the notion of *number* within a framework of conceptual interrelationships. An alternative way to put things using the set-theoretic jargon is that the generating principles produce all *definite* finite and infinite collections of objects. It is no doubt striking that all this is made possible by the sole analysis of a very simple concept like that of (whole) number.[84]

[83]The idea may relate to Leibniz's description of an analogous procedure in his *Monadology*, §34 and §35 (see Leibniz (1989)) and, in particular, to Leibniz's search for a *characteristica universalis*, a derivation system which would show how complex concepts can be reduced to simpler ones and complex statements broken into simpler self-evident analytic truths of the same complexity of *identities*. Axiomatic metaphysics along the lines of Leibniz's intimations was also one of Gödel's philosophical objectives (his hope that we may one day succeed in this undertaking is delivered in the last lines of Gödel (1944)). In general, the influence of Leibniz's thought upon the evolution of conceptual realism and, in particular, Gödel, is widely acknowledged by scholars. However, as said, Gödel's belief in the objectivity of truth rules out the reduction of mathematical truth to analyticity, if the latter is conceived of as *tautologicity* in the truth-functional sense. In particular, against Carnap, Gödel denies that truth is merely a syntactical notion (see Gödel (*1953)). Again, his argument rests upon his Incompleteness Theorems.

[84]It should be noted that at the time when Cantor devised the generating principles, he had not yet come out with a clear notion of *Menge* (set). In *Grundlagen*, he saw transfinite ordinals as new (whole) numbers rather than as sets of numbers. Accordingly, transfinite arithmetic was seen as a continuation of finite arithmetic. Set-reductionism clearly emerged only after he abandoned the generating approach and defined ordinals as isomorphism types of well-ordered sets (for this aspect of the evolution of set theory, see Dauben (1979), Hallett (1984), Ferreirós (1999) and Jané (2010)).

The generating principles can be rephrased in the following *operational* way:

1. [SUCCESSOR] Given any definite number α, there exists the successor of α, $\alpha + 1$.

2. [LEAST UPPER BOUND] Given any infinite increasing sequence of finite numbers $\{\alpha_n\}$, where $n \in \mathbb{N}$, there exists a least β which is not the successor of any of the numbers of the sequence.

3. [CLASS] Numbers defined through the iteration of 1. and 2. can be grouped into classes of different, ascending cardinality.[85]

Now, what Cantor claims is that these three principles arise by reflecting on the relationships that the idea of number has with, for instance, the concepts of *addition*, *sequence*, *order*, *smallest number*, already defined, presumably, at previous stages of the process of conceptual analysis.

In principle, it should be possible to detect all the other concepts which can be linked to the original concept, so as to produce, as Jané says, what we might call, in this case, the *axioms of numbers*.[86]

The generating principles are only one of the many examples of how Cantor's procedure finds a concrete application in set theory and, more generally, in mathematics. On Cantor's perspective, it is clear that no rule would be definable *a priori* as mathematics, entirely free in its development, can, at a certain point of its development, always establish new connections in unpredictable ways.

The second point on which I want to lay emphasis is Cantor's idea that a concept may be *slumbering* inside us and is awakened with all its manifold connections by the procedure indicated. The words that Cantor uses seem to

[85]The first application of LEAST UPPER BOUND yields ω, but from that point on, it has to be reformulated so as to generate the least upper bounds of countable sequences of ordinals, that is, limit ordinals after ω. CLASS and a suitably extended form of LEAST UPPER BOUND (resting upon the *cofinality principle*) allow the formation of initial ordinals: ω_0, ω_1, ω_2,... CLASS is not sufficient to generate the first singular initial ordinal ω_ω and, therefore, has to be reformulated to allow the formation of singular initial ordinals (see Jané (2010)).

[86]In particular, Wang (1974) and Jané (2005b) show how Zermelo's axiomatisations (1908, 1930) can be derived from Cantor's generating principles.

point to the existence of internal psychological processes of *reminiscence*, of which Husserl must certainly have been aware when he initiated his work on phenomenology (see, again, Ortiz (1997)). In particular, they would prove that the human mind has the innate ability to awaken latent knowledge and make it available to our conscious processes. This process seems to coincide with Husserl's definition of *introspection*.

According to Husserl (see, in particular, Husserl (1913)), the process of knowledge acquisition is based upon two main reductions: the *eidetic* reduction which allows one to apprehend objects in their *givenness*, that is, in their *conceptual essentiality* and the *transcendental* reduction, which allows one to apprehend objects as the result of their reduction to *conscious acts* within our minds. In particular, the latter plays the most significant role within Husserl's phenomenology. The transcendental reduction leads to discovering the *noetic constitution* of an object, that is the way our consciousness becomes aware of a certain object. It should be noted that 'object', in the Husserlian sense, need not be a material object. Any sufficiently delimited content of our consciousness, so long as it is taken into account by our intentionality (the faculty of consciousness itself), is an *object*. Objects, in turn, can be grouped into different conceptual regions and different ontologies of objects can, accordingly, be provided. In particular, a *regional* ontology of mathematical objects.

What is most important, the comprehension of an *eidos*, that is, the perception of a pure object in the Husserlian sense is not mediated by any sensory faculty. However, it is analogous to sensory perception, in that it is not simply the outcome of a passive reception of stimuli.

Accordingly, conceptual essences are aspects of an (ideal) object in the same way as physical properties are aspects of a material object: both are the outcome of a process of *internal* objective elaboration of the conscious mind.

Cantor thought that the ultimate root of the objectivity of mathematics was the fundamental unity between its immanent and trans-subjective aspects, but immanence is sufficient to yield objective axioms.

Gödel analogously claims, as we saw in Chapter 4, that the objectivity of our concepts may depend on a relationship between ourselves and the external reality. I interpreted this claim as echoing Cantor's metaphysical monism. However, phenomenologically, this relationship might be conceived of as the ability of a conscious mind to refine the sensory material through its transcendental faculties to the point that objects are perceived in their full phenomenological objectivity (*givenness*). Conceptual objectivity is seen by both as something which can be attained through a process of gradual expansion, whose features resemble Husserl's description of the process of constitution (*Fundierung*).

7.3.3 Uniqueness Properties

However, it is not clear whether this conceptual objectivity is related, and in what sense, to *uniqueness* (in the forms, respectively, of unique theories, axioms or universes). For, no doubt, Gödel thought that there was a unique objective response to the Continuum Problem and a unique objective truth-value of the CH. Conceptual realism does not necessary encourage this view.

Hauser represents this uncomfortable state of affairs in the following way:

> [On Gödel's view, *my addition*], the meaning of the continuum problem is tied on the unfolding of concepts through successive refinements of mathematical intuition. One difficulty with this is why it should lead to a unique resolution of CH, for our intuitions could conceivably evolve into different directions inducing us to formulate axioms with opposite outcomes of CH. ((2006), p. 540)

On the one hand, it is true that the process of conceptualisation is, in principle, inexhaustible and that one might stumble upon more general principles which provide a stronger and deeper unification.

For instance, this happened when Cantor shifted his focus from numbers to sets, that is, from numbers to arbitrary finite or infinite collections of objects. The same rationale is behind Cantor's ultimate adoption of the dichotomy between *consistent multiplicities*, which can be measured by a finite or transfinite number, and inconsistent multiplicities, like Ω or V which transcend any finite or transfinite measure.

As an aside, inconsistent multiplicities emanate from the concept of *absolute infinity*, the form of infinite which cannot be further increased (nor determined,[87] for that matter, see Chapter 2 and 3). The notion has proved to be one of the most fertile intuitions within set theory and provides an internal unification of motivating principles for the set-theoretic hierarchy. As we saw in Chapter 2, the *reflection principle*:

Principle 9 (Reflection) *If V can be ascribed a certain property, then there must exist some α, such that V_α already has that property.*

descends from it. This is how Gödel viewed the principle:

> 8.7.9 All the principles for setting up the axioms of set theory should be reducible to a form of Ackermann's principle: the Absolute is unknowable. The strength of this principle increases as we get stronger and stronger systems of set theory. The other principles are only heuristic principles. Hence, the central principle is the reflection principle, which presumably will be understood better as our experience increases. Meanwhile, it helps to separate out more specific principles which either give some additional information or are not yet seen clearly to be derivable from the reflection principle as we understand it now.[88] (in [145], p. 283)

On the other hand, it seems that this inexhaustibility may just leave us with an open-ended and fundamentally incompletable conceptual framework. In the light of this, with respect to Gödel, it is likely that he may have favoured the idea that some properties would reflect regularly across the universe of sets. In other

[87]The Absolute Infinite doctrine is stated in theological terms in Cantor ((1883b), p. 891): '...the true Infinite, which is in God, permits no determination whatsoever. As to the latter point I fully agree, and cannot do otherwise; the proposition '*omnis determinatio est negatio*' is for me entirely beyond question' and resumed in a different fashion in Cantor (1899) as the central pillar of Cantor's *limitation of size* doctrine: there are *inconsistent* infinite multiplicities.

[88]In particular, the unknowability of V rests upon the objective (provable) fact that, were it knowable, then it could be characterised, thus violating its being an inconsistent multiplicity.

terms, we should not expect to find always new properties while exploring the conceptual domain of relationships about sets. This might account for Gödel's belief that sets are highly regular objects, as shown by the following remark:

> 8.7.5 *Uniformity of the universe of sets* (analogous to the uniformity of nature). The universe of sets does not change its character substantially as one goes from smaller to larger sets or cardinals. In some cases, it may be difficult to see what the analogous situations or properties are. But in case of simple and, in some sense, "meaningful" properties it is pretty clear that there is no analogue except the property itself. (in [145], p. 281)

Surely if there is a fixed universe of properties (concepts), then it would be more likely that we will be able to find some of them which adequately and uniquely provide the truth-value of the *CH*. In other terms, in this case, there must be axioms which, as Gödel often says, solve practically all problems of set theory. On this interpretation, uniqueness would, then, be connected to implicit *principles of regularity* holding for set-concepts as much as for set-objects. But then, once again, we would go back to the *single-universe* template, which has been shown to be highly problematic.

To sum up, it seems to me that, exactly in the same way as in Cantor's, in Gödel's conceptual realism there is an unresolved tension between idealist and realist elements (see, again, Jané (1995, 2010) and Hallett (1984)). The idealist part would be content with establishing the immanent objectivity of mathematics through the phenomenological transcendental reduction, whereas the realist part, although embedded within a full idealistic perspective, requires the persistence of the single-universe hypothesis as an underlying philosophy and, hence, the belief in *uniqueness properties.*

7.4 Set-theoretic Objectivism

7.4.1 General Features

The problem of what form of objectivism is most suitable for our purposes seems to me essentially connected to the question of how important we think the requirement of uniqueness to be. In the version of set-theoretic objectivism I am briefly proposing below, uniqueness is ruled out, while retaining the full objectivity of sets.

But, first of all, let me review the forms of objectivism which have been, in one way or another, propounded in this work, mine being the last one:

- [***Gödelian platonism***] Set theory is about an independently existing and immutable universe of sets, whose features are encoded by all the objective set-theoretic truths. Set-theoretic truths are independent of our minds.

- [***Semantic Realism***] Set theory is about set-theoretic truths, that is, true statements on sets. Each set-theoretic statement has an objective truth-value which may be independent of our minds.

- [***Hauser's conception/Naturalism***] Set theory is about different theories of sets, whose objectivity can be measured using reliable scientific methods. In particular, preference for one theory over others is the outcome of the process of investigating which one is most suitable for our purposes. Objectivity attained in this way will provide objective justifications for assigning a *unique* truth-value to set-theoretic statements.

- [***My conception***] Set theory is the most adequate theoretical environment for mathematics, insofar as it provides a general ontological framework for it and an adequate representation of our foundational intuitions, as based, respectively, on objects (structures) and concepts. Its objectivity lies in the internal inevitability of its *immanent* development. In order to guarantee that its internal development is maximal, operational platonism (anti-constructivism) is needed. Anti-constructivism is, therefore, needed not only for justifying the belief in the uniqueness of solutions of set-theoretic problems, but also, more essentially, for guaranteeing the conceptual development of set theory.

The point of platonism will be developed in the next chapter. For now, I want to contrast my conception with the third one. The goal of set-theoretic objectivism as I intend it is not that of supporting intra-mathematical developments, disregarding all external philosophical constraints. Mathematical concepts come in different forms and there is no need to distinguish between philosophical and mathematical concepts. In other terms, on my interpretation, naturalism is a philosophy which justifies set-theoretic procedures, insofar as they are rooted in set-theoretic and, more generally, mathematical practice. On the contrary, set-theoretic objectivism is just the idea that set theory gives the most adequate representation of our mathematical theorisations, in particular, providing an adequate ontological framework, but does not put any stress on the decisions made by the mathematical community about what theory is preferable. This is a particularly sensitive point, as, in some naturalists' and conceptual objectivists' view, our decisions are made in view of *preferred* theories and, consequently, *preferred* truth-values. These alternative goals account for the different uses of the word 'objective' in the two theoretical frameworks.

7.4.2 Inevitability and Non-Uniqueness

Let me argue this point in more detail. As set-theoretic objectivists, we should view set theory as an objective part of our knowledge. But the main reason for viewing it as objective is that it rests upon an *internally* objective development of some mathematical concepts. Therefore, I do not need to presuppose that there is a pre-formed ontology of sets or a pre-existing class of set-theoretic truths which account for its objectivity. I claim that its procedures, its internal methodology and, in particular, its immanent conceptual development should already be viewed as an instance of objective knowledge.

I do not rely on any particular background theory for defending this view and I do not believe that there is something like an intuition, as Gödel puts it, which allows one to perceive *sets*. And I am also not sure that we have an introspection sharp enough which is able to produce sharp set-theoretic concepts and intuitions.

Although I am not sympathetic with the phenomenological view that there is something like a class of *founding acts* of our thought which produces objective refinements of our mathematical concepts, I think that phenomenology, especially if interpreted as a pre-theoretical explanation of scientifically detectable processes, might have a strong point to make about how we gain objective knowledge. But, as a general methodological attitude, in this work I want to keep a neutral position on the background epistemology. Accordingly, my aim is not that of investigating *how* an objective set theory might be achieved.

Again, the reason is that any account of the objectivity of set theory strongly depends on an account of the overall objectivity of our knowledge. So the question must be re-formulated as: how can one account for objective knowledge? Unfortunately, I have no answer to this question and, in particular, I have no *new* theory to propose.

My claim is that set-theoretic objectivism is concerned with the objectivity of some set-theoretic conceptual procedures. However, objectivity is a more general problem, which does not require a special treatment within set theory. At the same time, objectivity is clearly detectable from how we conceptualise sets.

Therefore, I also argue that the existence of objectivity is a simple fact deriving from *plain observation*: set-theorists, like all other mathematicians in their respective areas of specialisation, clearly see that there is a *conceptual inevitability* in their theorisations, that is, that some concepts have to be developed in some obliged directions on the grounds of specifiable internal interrelationships.

What we need to do in order to describe the objectivity of sets is elaborating upon notions like that of *inevitability*.

Consider, for instance, the sequence of transfinite ordinals, which is treated, at the outset, in the same way as the sequence of finite numbers. Therefore, one may conjecture that the commutativity of finite ordinals is preserved in the transfinite. However, closer inspection of the notion of transfinite ordinal, insofar as a transfinite ordinal represents a unique type of well-order, reveals that commutativity does not hold.[89] Whatever his background philosophy of mathematics, a mathematician will have to concede that, on the grounds of the simple concepts involved in the more complex concept of transfinite ordinal, it is an *objective fact* that transfinite ordinals are not commutative. One could say that, under the only plausible interpretation of transfinite ordinals as well-ordered sets, the non-commutativity of transfinite ordinals is an inevitable fact.

In a sense, this fact seems to substantiate Hauser's idea that we have to seek *confirmation* of our intuitions. Non-commutativity becomes, therefore, after we have tested (conceptually) our original presumptions, an inevitable property of transfinite ordinals, thus revealing that there are *pre-formed paths* in the way we expand our concepts.

The search for set-theoretic objectivity could, then, be seen as the search for the class of all procedures which show the *inevitability* of the properties of sets after removing all the obvious, immediately detectable, properties which are implicit in our primitive notions. Hence, set-theoretic objectivism, as I see it, is just the claim that *there exist inevitable properties of sets.* What distances this from the Gödelian or the plenitudinous platonist's conception of objectivity is the fact that I do not assume that the objectivity of sets rests upon one or many pre-existing universes of sets. The objective properties of sets are viewed as deriving from the *process* of *expanding* the initial properties rather than acquiring knowledge on an objectively pre-existing domain of objects. Therefore, there is not even any need to posit the existence of objects to describe their inevitable properties.

My elaborations on the notion of *inevitability* make it resemble to the notion of *non-arbitrary expansion,* which is proposed and investigated in Buzaglo (2002). Buzaglo aims to show that there is a huge number of non-trivial conceptual expansions in mathematics which have made successful contributions

[89] $\omega+1$ and $1+\omega$ are not *order-isomorphic* and, therefore, they are not the same. Cantorian set theory cannot make sense of transfinite ordinals without a robust conception of *order.* This shows that they cannot be treated in the same way as finite ordinals, for which the concept of *ordinality* is equivalent to that of *cardinality.*

to the discipline. Leaving aside the most remarkable of them, Cantor's transfinite, the notion of real and complex number, of logarithm and others have been introduced expanding the conceptual range of previously established notions.

This aspect of mathematical practice and the logic which governs it has always been acknowledged a minor role in all reconstructions of mathematical enterprise, essentially in virtue of, Buzaglo argues, Frege's opposition to them. Buzaglo's aims are expressed in the following passage:

> While Frege claims that the idea of expansions detracts from the principles of reference and sense, and that therefore there cannot be a logic that includes this process, I claim that there can be a logic that includes non-arbitrary expansions, and that there are convincing reasons to believe that a certain type of expansion expresses human rationality. [...] These will eventuate in a different conception of logic that is not confined to a general study of the space of reference and truth *after* they have already been consolidated, but also includes an analysis of how this space is established. ((2002), p. 3)

A logic of concept expansion is also manifest in phenomenology and, in general, Cantor-Gödel's conceptualism seems to thrive on it. As I said, there is no need to embrace any particular conception and we have to be aware of all the different possible interpretive reconstructions of how we can detect forms of objectivity (non-arbitrariness) in our conceptualisations.[90]

I now want to argue that the objectivity of set theory, as I have described it, does not imply that we may not have unique solutions to some set-theoretic problems.

Consider, again, the two following claims:

1. There is a single universe of sets.

2. There is a single theory of sets which settles all most relevant problems of set theory.

[90]Although clearly pertaining to a wider domain of problems and theoretical perspectives, interesting insights on the 'construction' and expansion of concepts can also be found in Hacking (1999).

Given the interpretation of objectivity along the lines indicated above, I do not see how the fact that there are objective set-theoretic procedures must necessarily imply (1) and (2). Rather, I argue that such an inference would be misguided.

Here follow some considerations to the point. First of all, if we allow for a complex structure of concepts with different ramifications and, possibly, dialectical splittings, it must go without saying that there might be some mutually incompatible directions within our conceptualisations. In other words, immanent developments of set-theoretic concepts must work as *paths of a tree*, and, hence, there must be incompatible *paths*.

In set theory we find relevant conceptual bifurcations pretty much everywhere. Each bifurcation must correspond to different choices. The unity that one achieves with the complex axiomatic machinery of ZFC is not sufficient, in my opinion, to reduce the complexity of this tree-like whole of conceptualisations. Unity could be seen as desirable only in view of the *extrinsic* preferability of a unified axiomatic theory over scattered and, possibly, non-homogeneous intuitions about sets.

Whether the Axiom of Choice should be or should not be included among our axioms is essentially decided by showing that it is useful to obtain a satisfactory theory of sets. I do not deny this fact, but one should always bear in mind, however, that this is far from proving that the Axiom of Choice is conceptually inevitable *across all paths of the tree*, if we stick to this metaphor. At the same time, I agree about the fact that it might be inevitable along one or more paths, thus revealing its deep connections to other set-theoretic concepts.

In other terms, *inevitability across* the tree-like and manifold structure of concepts is not a constituent of my view of objectivity.[91]

If this latter were considered a legitimate principle, it would be analogous to the belief in a single-universe theory, which is exactly what we want to avoid

[91] However, I acknowledge the fact that some set-theoretic principles, like Extensionality, may be considered inevitable across all conceptualisations, because their role is vital for obtaining, to use Gödel's words, a satisfactory (well-developed) theory of sets. In general, the more complex principles are, the more likely is the occurrence of multiple conceptualisations.

here. To me, the existence of incompatibility fully marks the difference between *set-objects* and *set-concepts* and represents the demise of ontological concerns about objects in the platonistic sense.

Allow me to use the Continuum Problem once again to exemplify this situation. Its intractability may be due to several factors, but one may also conjecture the existence of a form of conceptual incompatibility between our views of the continuum. The point will also be further developed in Chapter 8.

One conception of the continuum comes from geometry and it wants the continuum to resemble the structure of our spatial intuitions. The other comes from arithmetic and views the continuum as a class of sets of points, as a superstructure of all collections of points in the space. These two conceptions may not be reconcilable.

I have always particularly liked the way Jensen describes this fact. He compares the tension and the opposition between L and V to the tension and opposition between two opposite tendencies: that towards arithmetisation (which he calls Pythagoreanism) and that towards geometrisation (which he calls Newtonianism). The former wants to reduce all mathematical concepts to the clarity of the notion of whole finite number and, accordingly, within set theory, it aims to reduce the complexity of the *iterative structure* to the simplicity of the *constructible hierarchy*. The latter wants to keep the primordial force of the unfettered geometrical intuition of space within set theory allowing the existence of all possible subsets of a given set. In his words:

> I would like to propose a - necessarily somewhat speculative - hypothesis. Could it be that the duality in modern set theory is nothing but a new manifestation of an ancient conflict between two points of view - I almost want to say two emotional states - which have always existed in mathematics? I call them the arithmetical and geometrical points of view. I also call the first one the Pythagorean point of view, for Pythagoras expressed it in its purest form: Everything consists of numbers. In other words, every mathematical structure can be interpreted in the natural structure of the positive integers. This idea is naturally very attractive; it gives to all of mathematics the intuitive clarity of the natural numbers. I conjecture

> that, if it could really be carried out, it would still be the dominant point of view today. In reality, however, the geometric point of view has been dominant since the rise of analysis. Thus I also call it the Newtonian point of view. The Newtonian directs his gaze to the real rather than the natural numbers. He is less impressed by their clarity than by their boundless multiplicity. The real numbers constitute a gigantic, unfathomable sea. For every principle that generates real numbers, there must be a number not attainable by that principle. This excludes the possibility of an interpretation of the real numbers within the natural numbers. ((1995), p. 401)

If the arithmetical and the geometric conceptualisations of the continuum are incompatible, as I surmise, then I would expect that one should, at least, be aware of this fact in view of the need to find a solution to the Continuum Problem. In particular, I argue that this fact has strong bearings on the *solvability* of problems of the same complexity of the Continuum Problem.[92]

Although my characterisation might seem vague, it can be made precise and uncontroversial by showing how the axioms of set theory may make the combinatorial content of the notion of the continuum more or less explicit, by pointing to incompatible ways which go in one or another of the two directions mentioned by Jensen (see Chapter 2).

Moreover, rather than seeing incompatibilities as cracks in the homogeneous surface of mathematics, I argue that they show that our concepts are objective,[93] as there is no way to force upon them the pre-formed requirement that they *give rise to unique expansions.*

Finally, the account which I have given may also give good reasons for not accepting axioms only on the basis of their ontological desirability, so to speak.

[92]I want to stress the fact this does not imply the view that the concept of the continuum is *vague*, possibly only that it is *multiple*, nor does this point to an intrinsic vagueness of the concept of *set*, unless one interprets 'vague' as meaning 'subject to further determinations'. On this view, all concepts must be vague.

[93]The argument had already been advanced, in a different form, by Gödel. However, Gödel (see Gödel (*1951, *1953)) drew from this fact the conclusion that mathematical objects are not created by us, as, otherwise, we would have *complete* knowledge of them.

If one takes the demise of uniqueness seriously, then one does not have to worry about the underlying objects-framework (universe-framework) more than about other objective properties of sets.

We certainly have a genuine intuition of sets as objects, as shown by our model-theoretic constructions. But they do not necessarily prevail over other intuitions. Therefore, in my version of set-theoretic objectivism, ontological properties are on a par with other properties of sets. There is nothing special about the fact that some axioms may lead to forms of ontological uniqueness: this must be viewed as an internal development of our ability to single out *interesting* properties of sets.

Chapter 8

Indeterminacy Reconsidered

8.1 Insolubility and Genuine Indeterminacy

8.1.1 Multiple Solutions

In this last chapter, I will be concerned with the following questions: What is the current status of the open problems mentioned in the Introduction, among which the Continuum Problem is probably the most significant and well-known? In what sense, if any, could one say that the Continuum Problem has been solved?

One of the most remarkable outcomes of set-theoretic indeterminacy is the fact that it has forced us to re-think the notion of *solution of a mathematical problem*. In standard mathematical terms, a problem is solved if we have established some mathematical results that are strong enough to fill the relevant gaps in our knowledge. Sometimes, a problem comes with one or more conjectures, which will turn out to be true or false. In mathematics there is only one well-established criterion to decide whether a conjecture is true or false: we have to find a *proof* of its truth or of its falsity.

Do these simple methodological principles also apply to undecidable set-theoretic statements like the CH? As shown in Chapter 2, after the development of the axiomatisation, it quickly became apparent that the Continuum Problem was intractable, and that the best one could hope to find was a proof that a particular class of axioms was or was not able to settle it. In other terms, it was realised that its mathematical treatment would have been essentially metamathematical.

Metamathematical results also provide a diagnosis of why the Continuum Problem is so hard to tackle using standard mathematical methods: within ZFC, which encodes the most part of our mathematical intuitions, no solution can be found. Moreover, this situation remains unchanged even in the presence of stronger Axioms of Infinity.

Therefore, the reasoning goes, since all of our intuitions about mathematics are encoded by ZFC or its suitable extensions, there must be hardly any hope to find a solution to the Continuum Problem using and expanding on our current intuitions.

This argument is maybe too extreme and also slightly besides the point. Metamathematical proofs do not rule out that standard mathematical proofs of a statement can be found at some point, unless the *absolute undecidability* of a particular set-theoretic statement has been established, something which is clearly beyond our reach.[94] For instance, if one had come up with a proof that Fermat's last theorem is unprovable in PA before Wiles's work, this proof would not have prevented Wiles from establishing his result. The situation with the Continuum Problem is certainly different, as we know that very strong systems of mathematics are not able to settle it. However, as said, this fact alone cannot account for the belief in its *absolute undecidability.*

At any rate, it is out of question that present mathematical work on the Continuum Problem essentially amounts to metamathematical work on the provability of complex set-theoretic statements.[95]

[94]Obviously, this will depend upon how far one thinks standard mathematics to go. If one deems standard mathematics to coincide with ZFC, then a proof of the unprovability of a statement from ZFC would be equivalent to a proof of its unprovability in standard mathematics (and, thus, presumably, the proof would be taken to be a proof of its *absolute undecidability*.)

[95]An exception is maybe represented by some results in cardinal arithmetic, within ZFC, which may have some bearings on the Continuum Problem. For instance, Shelah's *pcf*-theory shows that in ZFC there is a bound to the value of $2^{\aleph_\omega}$: if the GCH holds for all cardinals

This fact marks a first operational difference between what 'being solved' means for a standard mathematical conjecture and what it means for the Continuum Problem.

A second remarkable difference, clearly connected to the first, is that attempts to find a unique solution have ushered in *multiple* solutions. Therefore, in a sense, one could say that the Continuum Problem has *too many* solutions and that all of them are equally acceptable. Independence results show that ZFC and its extensions also provide multiple solutions to many important set-theoretic statements. This fact already implies the demise of the *single-solution* template.

This may be hard to digest for standard mathematicians. Some of them may even want metamathematics to play no relevant role in mathematical enterprise.

In fact, mathematics and metamathematics are *de facto* complementary in current research, in particular, set-theoretic research. The main reason is that some mathematical problems may happen to be too complex to be adequately expressed and solved using only one particular set of axioms. The only option we would be left with, therefore, is simply to check metamathematically what kind of solution (solutions) are possible.

There is still a third difference between the Continuum Hypothesis and other undecidable statements. Whereas we know that some of them (e.g., Gödel's sentence, Goodstein's theorem, etc.) are true but unprovable from certain axioms, we have no clue about whether the CH is true or false. This fact lays even more emphasis on the crucial role played by platonism with respect to insolubility, insofar as it provides informal reasons for continuing our investigations even in such desperate cases.

In what follows, I have tried to give an overview of some of the most influential positions on insolubility. In the next subsection, I shall briefly present my view.

As a conclusion of his work which shows that, through the acceptance of Woodin cardinals, the axiom (*) and Ω-logic as a background logical framework,

below $\aleph_\omega$, then $2^{\aleph_\omega} < \aleph_{\omega_4}$.

the CH is false, Woodin says:

> So, is the Continuum Hypothesis solvable? Perhaps I am not completely confident the "solution" I have sketched is the solution, but it is for me convincing evidence that there *is* a solution. Thus, I now believe the Continuum Hypothesis is solvable, which is a fundamental change in my view of set theory. While most would agree that a clear resolution of the Continuum Hypothesis would be a remarkable event, it seems relatively few believe that such a resolution will ever happen. ((2001), p. 690)

As we saw in Chapter 1, Woodin's solution would imply that the CH, under suitable assumptions, is false. At the same time, it would seem that Woodin would be essentially content with showing that, under suitable methodologies, the CH can find a solution (and, hence, that it is a meaningful question). One such methodology is that which arises from the study of *generic absoluteness* properties induced by particular axioms. But on my interpretation, it seems implicit in what he says that this is not necessarily the only valid methodology.

It is therefore less surprising that Woodin himself (as said in Chapter 2), has recently turned his attention to an alternative strategy, the search for a generalisation of L (ultimate L) which would represent the natural model-theoretic structure for all large cardinals. Since large cardinals are thought to be the natural *super-structure* of set theory, it would follow that ultimate L would be the ultimate universe of sets. Hence, it would also follow that the CH is true (see Woodin (2011a, 2011b)).

We do not presently know whether this endeavour of Woodin's will ever be successful. What is sure is that his recent work is strongly oriented by ontological considerations related to *absoluteness properties.* As we have seen, these properties are particularly desirable in view of the strong regularity that they imply. In particular, they also imply that there is a unique universe of sets whose features we will succeed in describing. This belief is integral to the platonistic belief in an independently existing and fully intelligible universe of sets. Platonism, therefore, provides strong motivations for Woodin's investigations and views on insolubility.

In an interesting survey of the subject in 2003, Foreman (Foreman (2003)) has advanced the argument that such ontological considerations are irrelevant for the solution of the Continuum Problem. In particular, too strong concerns over absoluteness properties would be misguided. Foreman says that we already have a clear example of absoluteness which does not work:

> The theory of L is completely forcing absolute; in fact it is absolute between models of set theory with the same ordinals, a much stronger property. ((2003), p. 22)

However, the rejection of $V = L$ is widespread among set-theorists, essentially because it seems to put an arbitrary restriction on the range of the power-set axiom.[96] Thus, the moral to draw is:

> The considerations of completeness and absoluteness are secondary when considering axioms. The main criterion is what the axioms SAY. ((2003), p. 23)

Foreman proposes a more standard route, by surmising that the solution to the Continuum Problem may come from the adoption of a generalisation of *large cardinals*, that is *generalised large cardinals*.[97] The acceptance of Foreman's proposal along with generalised large cardinals would yield the result that the CH is true. At any rate, not unlike Woodin, Foreman thinks that our aim is that of describing the *true universe of sets*. Hence, again, it seems that the main philosophical motivation for the claim that the CH will be solved is the underlying ultimate-universe conception.

In a recent conference (2011), Hamkins has advanced a cluster of, in my opinion, compelling arguments that the Continuum Problem should not be thought to be solvable using new axioms.

He says that mathematicians' *dream* would be that of finding very simple axioms (principles) which imply CH or $\neg CH$. He proposes two examples where

[96] It should be noticed that Woodin, like the majority of set-theorists, is not a supporter of $V = L$. However, the L he is working on would be naturally seen as an extension ('a trascendent version, say, L^{Ω}' , he says ((2011b), p. 469), of the earlier Gödelian L.

[97] Unlike standard large cardinals, generalised large cardinals are defined through embeddings of the generic multiverse ($V^{\mathbf{B}}$) into suitable portions (subuniverses) of it.

it is clear that the *dream template*, as he calls it, is flawed. The first concerns Freiling's Axiom of Symmetry, which was proposed by Freiling in 1986 in order to prove that the CH is intuitively false.

The axiom says:

Axiom 1 (Axiom of Symmetry (AS)) $(\forall f)\, f : \mathbb{R} \to A\ (\exists x)(\exists y)\ x \notin f(y) \wedge y \notin f(x)$, *where* A *is a* countable *subset of* $\mathbb{R}$.

The axiom was made up by Freiling with the declared purpose of showing that the CH is intuitively false. It is important to notice that its status as an axiom depends on its *extreme intuitiveness*. Now, it can be easily proved that:

Theorem 15 (Freiling, 1986) $AS \to \neg CH$

Proof. The axiom requires that, if we pick up two functions which map the reals (for the sake of simplicity, the reals in the unit interval $(0,1)$) onto countable subsets of reals (countable subsets of the unit interval), we can always find two *random reals* which are not one in the image of the other.

Now suppose that the CH is true. This means that $\mathfrak{c} = \aleph_1$. Now, using the Axiom of Choice, define a well-ordering of the continuum $(\leq)$. Let $x = r_\alpha$ and $y = r_\beta$, with $\alpha \neq \beta$. It must follow that $\alpha, \beta < \omega_1$ by the well-ordering just defined. Now, define $f(x) = \{r_\gamma | \gamma \leq \alpha\}$ and $f(y) = \{r_\gamma | \gamma \leq \beta\}$. Three cases are possible: either $\alpha = \beta$ or $\alpha < \beta$ and $\alpha > \beta$. In all these three cases, AS must be false, as by our construction, either $x \in f(y)$ or $y \in f(x)$ or both hold.

AS seems to be intuitively true. In particular, it encodes our presupposition that, given any two random[98] points of the real line (of the unit interval, for that matter), there are countable mappings which do not contain both. In other terms, if I pick up a random real, it is very intuitive that any countable mapping does not have, among its values, another random real and vice versa (this accounts for the use of the word 'symmetry'). However, the *prima facie* intuitiveness of AS is not sufficient to make this axiom acceptable. The reason

[98]The randomness of the choice is illustrated by Freiling using the procedure of *throwing two darts* at the interval $(0, 1)$.

is, some of the consequences of AS are highly implausible.[99] The consequence of this is that the exclusive reliance upon intuitiveness as the mere criterion for the acceptance of an axiom might turn out to be an impossible task, as we may not have genuine and fully uncontroversial criteria to define *intuitiveness*.

The second example is even more trivial and concerns the following axiom:

Axiom 2 (Power-set Size Axiom (PSA)) $(\forall x)(\forall y)(|x| < |y| \to |\mathcal{P}(x)| < |\mathcal{P}(y)|)$

PSA would seem very natural, even obvious, to many mathematicians. The reason is that it seems pretty intuitive that, if a set A is bigger than a set B, then we should be able to form more subsets from A than from B. PSA obviously holds for the finite. Unfortunately, the cardinal arithmetic of ZFC does not rule out that it is false. Moreover, there are important axioms, like Martin's Maximum,[100] which violate it.

As a consequence, we do not have any clue whether such a simple statement would hold (in our ultimate theory of sets, presumably). We know that PSA is true in the finite, but, as shown, this is not sufficient evidence that it will hold in the infinite.

What Hamkins's reasoning shows is that when we say that a set-theoretic axiom is true *because* it is intuitively true, we may be wide of the mark. But on what other criteria could one possibly rely, if not on intuitiveness? The conclusion must follow, that our search for this kind of axioms, however interesting and far-reaching, will never lead to a spread consensus about their acceptance.

Hamkins's refutation of the (standard) solvability of the Continuum Hypothesis is tied to his belief that multiversism is the correct view about sets. In particular, he has presented a cluster of results known as *set-theoretic geology* (Hamkins (2009)), which interpret the generic multiverse as being a sort of *modal* framework, that is, a super-structure in which each forcing extension is a

[99] AS might be used to reject the Axiom of Choice which, in many mathematicians' opinion, is at least as intuitive as AS.

[100] Martin's Maximum is the strongest possible generalisation of Martin's Axiom, which has been presented in Chapter 2.

possible world of sets. However, it would seem that Hamkins refrains from any platonistic interpretation of this multiverse-framework.[101]

While Hamkins reveals the troubles with the notion of *intuitiveness* of an axiom, Shelah identifies analogous troubles with the notion of *purpose-adequate* axiom. In a lucid article dating to some years ago (Shelah (2003)), Shelah claims that we should live with the existence of multiple theories of sets. He says:

> I do not agree with the pure Platonic view that the interesting problems in set theory can be decided, that we just have to discover the additional axiom. My mental picture is that we have many possible set theories, all conforming to ZFC. ((2003), p. 211)

Shelah, then, moves on to reject the claim that finding an axiom which solves some problems makes that axiom automatically preferable to other axioms. He thinks that axioms which are preferred on extrinsic grounds can, at best, be described as *semi-axioms* and that there are many plausible semi-axioms (including GCH, which is sometimes necessary to simplify our work in cardinal arithmetic). In particular, PD (or $AD^{L(\mathbb{R})}$), which are equiconsistent with large cardinals (see Chapter 2) are good semi-axioms, especially in view of Woodin's requirement of *completeness* of a theory, but not the best semi-axioms.

> Generally I do not think that the fact that a statement solves everything really nicely, even deeply, even being the best semi-axiom (if there is such a thing, which I doubt) is a sufficient reason to say it is a "true" axiom. [...] The judgments of certain semi-axioms as best is based on the groups of problems you are interested in. ((2003), p. 212)

It would seem that Shelah would also share Foreman's concerns that preference for absoluteness properties are essentially rooted in largely immaterial ontological prejudices.

Given his declared anti-platonism, Shelah does not seem inclined to any form of multiversism, but, as explained, he certainly admits that there may

[101] It should be noted that, in accordance with the assumptions of *multiversism*, Hamkins thinks that we should say that the CH has been settled.

be different, even incompatible theories of sets and that the truth of particular statements should be seen as fixed by the particular axiomatic framework which one chooses to pick up.

Shelah's line of reasoning is essentially rooted in moderate formalism, although I think that he may be keen to adhere to some sort of set-theoretic objectivism.[102] In a sense, *formalism* is the default position of most mathematicians or logicians when they face up with such difficulties as the insolubility of problems like those of set theory.

Shelah says that there may be multiple theories of sets, according to what semi-axioms we would judge the best for our mathematical purposes. By granting the GCH the status of a semi-axiom, Shelah just re-states Errera's (see Errera (1952) and Chapter 4) attitude to the Continuum Problem, which wants its epistemological status to be analogous to that of Euclid's *fifth postulate.*

On this view, the solution of open set-theoretic problems is no more relevant than any alleged solution of the problem whether there is only one parallel passing by a point external to a given line.

As Lavine puts it:

> No one worries about whether the parallel postulate is true or false. It is true in some geometries and false in others, and that is that. A crude analogy suggests (as do some other, more refined, arguments) that the situation is analogous for the Continuum Hypothesis: Set theorists should study set theories both with and without the Continuum Hypothesis, and that's all there is to it. There is no further question of absolute truth or falsehood. That attitude is in fact taken by some, and no one would deny that, given our present state of ignorance, it is worthwhile to study set theory both with the Continuum Hypothesis and with its negation. ((1994), p. 243)

However, Lavine also argues that there is a fundamental disanalogy between geometry and set theory: for this latter we have a quasi-categorical axiomatisation (ZFC^2), which implies that the CH has a truth-value. Whether and in what

[102]He says: '..I reject also the extreme formalistic attitude which says that we just scribble symbols on paper or all consistent theories are equal' ((2003), p. 212).

sense this fact should be acknowledged as relevant has already been discussed in Chapter 3.[103]

8.1.2 Genuine Indeterminacy

In the previous chapter, I tried to show that a set-theoretic objectivist does not have to care about uniqueness properties. This is because there may be some conceptual incompatibilities which account for the existence of multiple solutions to just one problem.

Although my work does not contain any proof that platonism is the only reason for believing in TRUTH, I think that I have accumulated a lot of evidence that the supporters of the ultimate theory of sets all share, to some degree, some platonistic beliefs. Even naturalised conceptualists like Hauser, in spite of their disavowal of any platonistic conception, seem to be in line with the view that such an ultimate theory or ultimate universe will be, at some point, found.

Analogously, given the present situation in set theory, it seems plausible to assert that the only reason to believe in the solvability of problems like the Continuum Problem is, at least, a belief in the objectivity of sets as implying uniqueness properties. In my account of set-theoretic objectivism, uniqueness properties are ruled out, hence there is no need to keep solvability in the sense prescribed by the single-solution template.

As a consequence, I am keen on Shelah's hypothesis that there might be more theories of sets. This seems to imply some form of *practical formalism.* However, I argue that this position can also be advocated from an objectivist point of view.

Set-theoretic objectivism as I intend it does not deny that there are objective developments within mathematics. It does not deny that there are objective

[103] Further details about why this position is very uncomfortable are provided by Potter. Potter sets out to make it clear that even if we find a purely set-theoretic argument or principle settling the CH, it would be hard to trace it back to the decidability of the CH in ZFC^2, as 'this [the argument or principle, *my note*] would not automatically make the continuum hypothesis itself logical, since it would follow from the second-order logical truth in question only via the axioms of second-order set theory' ((2004), p. 271).

facts about sets either. However, its grounds for claiming the objectivity of sets are essentially *operational* rather than *ontological*: our mathematical practice reveals *inevitable* properties of sets. Defining the extent and the nature of this inevitability is the task of this form of objectivism.

I would now like to frame genuine indeterminacy once again in the practical context of the study of the continuum.

In a recent paper, Burgess (see Burgess (forthcoming)) traces back the duality arithmetical/geometric continuum to Gödel's conceptions (in particular, Gödel (1947)). I wish to discuss Burgess's line of reasoning, as it helps to reveal some more details about the genuineness of indeterminacy.

Like Jensen, Gödel would acknowledge the existence of two forms of intuition: *geometric* and *set-theoretic*. These forms of intuitions resemble and are in accordance with the two main forms of intuition humans have: *sensory* and *rational*.

Burgess argues that the main difficulty with this conception is that Gödel claims that rational intuition is of a special kind and is not reducible to any known mental faculty. Burgess then goes on to argue that there is hardly any need to posit a new form of intuition and that, for instance, the way we operate within and make inferences about the set-theoretic realm could be *linguistic*.

What is most interesting is that, if this form of intuition is essentially linguistic, it must be necessarily *incomplete*. In particular, Burgess's thesis ties with the syntactic (formalist) view about logic and leads to the conclusion that:

> ..it is a consequence of Gödel's first incompleteness theorem that deduction by first-order logical rules from a fixed finite basis of first-order non-logical axioms will leave some mathematical questions unanswered, whatever the finite fixed basis may be.... in view of Gödel's result, the linguistic picture *tends to suggest* that there must be absolutely undecidable mathematical questions, while the rationalist picture *tends to suggest* that there need not be. (forthcoming, p. 8)

Although my account differs from Burgess's one, I am on his side when he says that indeterminacy should be viewed as a genuine phenomenon, on his view, related to the limits of our linguistic practices, on my own view, related to the limits (internal incompatibilities) of our conceptual practices.

While he puts more emphasis on the limits of *syntax*, I would put more emphasis on the limits of conceptual *semantics*. At any rate, like him, I do not see any reason to claim that our rational intuition, in Gödelian terms, be it phenomenological or linguistic or whatever, is complete.

In conclusion, what would my set-theoretic objectivist say about the current status of open problems in set theory, then? Well, to begin with, he would accept that further mathematical work will provide more insights into the matter and, possibly, more unifying views of the world of sets. However, given the essential incompleteness of our conceptualisations, he doubts that there is any preferable framework. As Shelah puts it, there are good semi-axioms, but there is no best semi-axiom.

Furthermore, as Hamkins says, even if there were one new powerful, all-embracing and far-reaching axiom, how would one describe the psychological experience achieved in other axiomatic frameworks? Shall we say that our perception of *multiple theories* was illusory and that we have now received full enlightenment from one single axiom? How would such an account accommodate to more than a century of speculations on different frameworks and axioms and to all our bulk of independence results?

I think Woodin and Hauser have a point when they say that it can be shown that the CH is not meaningless and that suitable methodologies can allow us to achieve some progress in its understanding. However, given all the considerations made above, the successfulness of any solution will always depend on how strong and persuasive we perceive our own metamathematical arguments to be.

Upon consideration of objectivity as a property of sets not necessarily leading to uniqueness properties, I propose, therefore, interpreting set-theoretic indeterminacy as *integral to our conceptual practices.*

At the same time, I believe that mathematics and metamathematics will have to co-operate more and more closely in the task of measuring the strength of these conceptual practices. The history of set theory has clearly shown that many mathematical problems cannot be dealt with in standard mathematical terms. In particular, it is only through the strengthening of metamathematical analysis that all this has become apparent. The outcome of this is nothing but a beautiful, deep and enormously fruitful achievement.

Shelah puts it this way:

> [Leo, *my note*] Harrington asked me a few years ago: What good does it to you to know all those independence results? My answer was: To sort out possible theorems - after throwing away all relations which do not hold, you no longer have a heap of questions which clearly are all independent, the trash is thrown away and in what remains you find some grains of gold. This is in general a good justification for independence results; a good place where this has worked is cardinal arithmetic - before Cohen and Easton, who would have looked at 2^{ω_1}? ((1991), p. 7)

In the same spirit as this last quote is his subsequent 'rubble removal thesis'. In particular, he lays emphasis on the fact that the proposed interpretation of our procedures will allow us to stick to ZFC as the right operational framework:

> Methods for proving independence, in addition to their intrinsic value, work for us like a sieve: when we have a myriad of problems in some directions and we have tried to prove independence (and many times this results in discarding most of them), we are left with strong candidates for theorems of ZFC. ((2003), p. 219)

8.2 Operational Platonism Reconsidered

In the previous chapters, we have come across platonism in three different contexts: historical, philosophical and operational.

The historical strain is connected to Cantor's work. Although Cantor initially refrained from any philosophical reconstruction of his results about the extension of the number domain, after a while he thought it necessary to provide some philosophical commentary.

At that time, in the face of Cantor's early methodological opposition, the theory of the transfinite was viewed as the province of metaphysics, in that it was based upon the actual infinite. As Gauss had famously put it, the actual infinite was just a *façon de parler* and could not possibly be the subject of mathematical investigation.

In *Grundlagen* (1883b), Cantor made his first attempt to present systematically a new philosophy of the infinite. Some of the most characterising features of this picture have been reviewed in Chapter 4. We have seen that there is no strong reference to platonism in that work and also that an automatic ascription of Cantor's philosophy to standard platonism would be problematic. However, his thorough-going realism is manifest.

Cantor says that his notion of 'set' is indebted to Plato's *Philebus*'s theory of *miktón*. He says:

> I believe that I am defining something akin to the platonic εἶδος or ἰδέα as well as to that which Plato called μικτόν in his dialogue 'Philebus or the Supreme Good'. He contrasts this to the ἄπειρον (i.e. the unbounded, undetermined, which I call the improper infinite) as well as to the πέρας, i.e., the boundary; and he explains it as an ordered 'jumble' of both. Plato himself explains that these concepts are of Pythagorean origin; [...]. ((1883b), p. 916)

However, it is not clear to me whether Cantor attached real significance to the conceptual kinship established. Maybe he just thought that references to Plato's conceptions would prevent further objections.[104] His philosophical goals are further developed in other works, in which he mentions the conceptions of a sizable number of illustrious philosophers.

This first form of platonism should, therefore, be seen as integral to the goal of providing a defence of sets rather than discussing platonism as a philosophy of mathematics *per se*.

[104] On a philological note, Plato does not seem to present in *Philebus* or any other of his works a well-developed theory of the *actual infinite* nor do his conceptions resemble in the least the conceptual machinery relating the theory of the transfinite.

As soon as Cantor's formulation was superseded by Zermelo's breakthroughs in the axiomatisation, sets started being treated as any other mathematical object, which is not in need of any particular metaphysical justification. For instance, the French analysts (Borel, Lebesgue, Baire), although very keen on set-theoretic procedures and concepts, did not show, from the very outset, any interest in the philosophical status of sets.

Once this early, historical form of platonism was progressively phased out, philosophical platonism came to the fore. As I have tried to show throughout, contemporary platonism owes much to Gödel's contributions, but much earlier it had already been endorsed by Frege (whose conceptions I have purposefully abstained from examining) and was also probably implicit in the work of many other mathematicians in the first half of the twentieth century. It certainly became prominent after Gödel declared explicitly his adherence to its motivating ideas.

In this work, I have tried to show that Gödelian platonism is untenable. Strong model-theoretic relativity and the existence of multiple axiomatic frameworks, all equally legitimate, some of which mutually incompatible, is hardly reconcilable with its ontological presumptions about sets. Other forms of platonism, more mystical than philosophical in their essence, raise thorny epistemological issues. As to plenitudinous platonism, while there is at least one form (Zalta and Linsky's *naturalised platonism*) with which I would not be completely unsympathetic,[105] Balaguer's account seems very unnatural.

In Chapter 1, we have seen a presentation of platonism which was strongly focussed on certain *operational* principles, rather than on a thick philosophy as Gödelian platonism is. We saw that, except for point 3. in Bernays's and 1. in Feferman's list about the existence of sets,[106] all the other features which were mentioned were, beyond question, integral to contemporary set theory as consisting of, at least, the axioms of ZFC. However, those principles may also be successfully employed as the underlying rationale for the addition of new

[105]Although their account of *abstracta* is in line with a form of ontological platonism, that is on EXISTENCE and INDEPENDENCE no less than Gödelian platonism.

[106]These *existence claims* I will henceforth refer to as *ontological platonism*.

axioms to ZFC.[107]

As we have seen in Chapter 3, in accordance with his Weylian philosophical leanings, Feferman identifies the crucial problems of set theory as arising in connection with its *impredicativity*. Inasmuch as impredicativity is essentially related to the platonistic conception, the reasoning goes, then set theory must be essentially platonistic. Hauser's view, as we have seen, was that Feferman's argument was flawed and that the issue of platonism is competely irrelevant with respect to the problem of set-theoretic indeterminacy.

I think that there is no need to take sides on the matter, although it seems to me that Maddy's analogous defence of the ontological neutrality of her naturalistic conception, coinciding with Hauser's point of view to a substantial extent, did not fully dispel Feferman's concerns.

However, as anticipated in Chapter 1, a compromise can be found: on the one hand, *ontological platonism* does not have to be necessarily seen as in connection in the least with other operational principles. For instance, a person who believes in the legitimacy of the Axiom of Choice and who thinks that its utility in mathematics is indisputable does not have to be a platonist. On the other hand, as we have seen, it seems reasonable to assert that a platonist has more compelling and convincing reasons to argue the case for the Axiom of Choice than an anti-platonist has. Therefore, I argue that one can assert the separability of Feferman's *operational* principles from *ontological* platonism, although one may also want to specify that those principles are more compatible with, when not directly arising from, a platonistic *ontological* conception. If one wants to separate ontological platonism from the other principles, then one can safely assert that Feferman's operational principles only amount to *anti-*

[107] How exactly Feferman's principles can lead to a justification of the axioms of ZFC and other axioms and contemporary set theory entirely reconstructed from these principles is, unfortunately, beyond the scope of this work. A thorough-going account of the emergence of ZFC from Cantorian set theory can be found in Hallett (1984) and, partly, in Moore (1982). Potter (2004) contains an historically accurate and philosophically robust examination of the philosophy of the axioms and their historical and conceptual justifications. Finally, a standard reference is the somewhat old-fashioned but theoretically and philosophically informed Fraenkel-Bar-Hillel (1962), subsequently revised by Lévy and van Dalen (1973).

constructivism.

On this reading, the amount of platonism we might want to keep in set theory will just imply that set theory cannot be fully developed in a *constructivist* environment. When I say that it cannot be developed, I do not mean that there will inevitably be only one set theory (although, there are, in my opinion, *inevitable* properties of sets that all different theories should attempt to capture). In fact, I am a bit Maddyan here: the contemporary set-theoretic enterprise has fully vindicated the principles of Cantorian and Zermelian set theory (as presented by Feferman) and has fully shown their indispensability to reach an adequate conception of sets.

Therefore, I think that operational platonism *qua* anti-constructivism is the only form of platonism which one would see as befitting the current axioms of sets. Whether or not ontological platonism is inseparable from anti-constructivism should, in my opinion, remain an open problem.[108]

There is one more reason to assert that anti-constructivism is essential in set-theoretic practice. In Chapter 7, I have tried to relate the notion of inevitability to that of non-arbitrary concept expansion. It seems now reasonable to say that concept expansions may thrive on anti-constructivism. It is no wonder that Buzaglo has claimed that classical logic may be seen as an expansion of *constructivistic* logic for the finite: when dealing with *finite* objects, both logics coincide. It is in the infinite that they part company. If our primary concern is over concept expansion in set theory, constructivist logic is clearly inadequate: only through anti-constructivist principles can one hold that the notion of transfinite cardinal number successfully expands on that of *finite* cardinal number.

[108]It is reasonable to assert that probably many set-theorists are *ontological* platonists. We are informed of this again by Shelah (2003). Platonistic leanings seem more prominent in what he calls the California School set theory. In any case, Shelah himself is an example of an acclaimed set-theorist who does not believe in platonism.

8.3 Concluding Remarks

It is the time to draw some conclusions. In my opinion, Gödelian platonism, that is, the form of platonism which has become popular in set theory is untenable. Measured against the present situation of set theory, it seems that its pretensions are too strong. While, on the one hand, I acknowledge its role as a motivating philosophy for doing set theory, on the other hand it seems that its assumptions do not follow from set-theoretic practice.

On the contrary, the form of platonism which I have assumed to be tantamount to *anti-constructivism*, construed as an accurate methodology, has proved to be successful since the earliest times of set theory. In particular, it is doubtful that further progress in the field would have been possible without such a strong methodology. If set theory wants to be the foundation of mathematics, it is hardly conceivable that it can dispense with Cantorian set theory, framed as an operational methodology.

An obvious objection to my claim is that the word 'platonism' in this particular frame of ideas no longer plays any role, in particular the role which the philosophical and mathematical community have historically attributed to it. While I agree with this objection at the surface level, closer inspection suggests that, within the context of operational mathematics, platonism works more as a motivating philosophy than as a forceful system of beliefs.

In the former capacity, platonism (construed as anti-constructivism) seems to fit into our purposes marvellously, while in the latter it seems to be wide of the mark. Set theorists' investigations on the status of the CH do not depend upon the assumption that sets describe a *well-determined reality*. The CH, the GCH, SH, Lebesgue measurability, the search for *regularity* and *absoluteness* properties, *constructibility* and so on will be items on the agenda, regardless of whether it has been established that they describe any perceptible independent reality.

Insofar as operational principles elucidate our efforts, they also provide us with the feeling that there are objective developments of our theories of sets. The task of reaching such a level of clarity and an all-embracing network of concepts and methodologies is extremely rewarding for the mathematician, even in the presence of forms of indeterminacy.

A moral to draw from set-theoretic indeterminacy is that we can no longer afford the innocent view that there is a unique, standard mathematics, which will provide us with concrete results and, on the other side, there is metamathematics which gives us another kind of results, which mathematicians may complacently ignore. Both disciplines are the result of the same effort, that of establishing results on what we *can* know and what we *ought to* investigate. Hopefully, their interrelationship will become increasingly more tight in the near future and many at present contentious points will be clarified. In any case, already available to our interpretation is the insight coming from metamathematics that the notion of solution to a mathematical problem may be subject to revision. This may not concern some mathematicians, as they may see metamathematics remote from their practical goals or because they think their insights about objects are strong enough to yield the desired solutions.

Set-theoretic indeterminacy, if seen in the correct perspective, is an exciting phenomenon, as it pushes our mathematical notions to the limit and encourages awareness of mathematics as a study of the boundaries of our knowledge. In particular, the demise of some strains of ontological concerns and an accurate study of contemporary work on sets has recast set theory, in Kanamori's words, as:

> ...an intriguing mathematical subject where formalized versions of truth and consistency became matters for combinatorial manipulation as in algebra. From Skolem relativism to Cohen relativism the role of set theory for mathematics became even more evidently one of an open-ended framework rather than an elucidating foundation. From this point of view, that the ZFC axioms do not determine the cardinality $2^{\aleph_0}$ of the set of the reals seems an entirely satisfactory state of affairs. ((2003), pp. XVIII-XIX)

We do not have to share all the views expressed by Kanamori in this passage. In particular, as I tried to show, a set-theoretic objectivist can still think that set theory is an elucidating foundation for mathematics, provided he gives up the belief in some uniqueness properties. Consider Cantor's Absolute Infinite principle. Initially regarded by many as a contrived strategy to by-pass set-theoretic inconsistency, it has now become one of the strongest arguments for the extension and the generalisation of Cantor's generating principles. This shows that even the most abstruse philosophical (theological) principles seem to play a role in our mathematical theorising.

There are different forms of set-theoretic objectivism which can be advocated. I have presented mine but I have left to the reader to judge whether alternative forms are plausible. In general, most of them seem to be variations on the theme of Gödel's objectivity. Sometimes, I have only tried to show that the most obvious objections directed at these conceptual frameworks already raise thorny issues. At the same time, extreme anti-objectivism seems to me as implausible as Gödelian platonism. I do not claim that it cannot be successfully advocated, only that its diagnosis of set-theoretic indeterminacy seems to me too extreme to be plausible.

As to the issue whether sets are also objective in other respects, maybe physically objective, I have no answer to provide. I can only say that, in my opinion, this will essentially depend on the degree of co-existence between mental and physical which our scientific theories will allow for in their future developments. On a scientific eliminativist view, sets are objective insofar as they are neurophysiological processes. Possibly our theories about embodied cognition and the role of neural circuitry will clarify completely how a finite number of neural connections can prove objective facts on infinite sets. The key, once again, may be Cantor's finitism.

Finally, one last remark. Platonism is an interesting case study in the philosophy of mathematics. I hope that the examination of the topic will have at least contributed to the view that philosophical conceptions are strongly tied to concrete mathematical problems and that one should take philosophy very seriously in mathematics. In particular, platonism allows one to study and elucidate at least the following notions and problems: 'what is an object?', 'what does its existence mean?', 'is there a universe of objects?'. In my opinion, a correct methodology will always have to take into account the metamathematical embedding of these problems, insofar as this latter contains all the required philosophical thickness.

I also hope to have contributed to demonstrate that there *should* be nothing mystical about platonism. At least, nothing which falls under Wittgenstein's description of mysticism:

> 6.522 There are, indeed, things that cannot be put into words. They *make themselves* manifest. ((2003), p. 89)

(Gödelian) platonism can be put into words. It is only through words that we can and we will always want to assess its validity.

Bibliography

[1] Baaz, M. - Papadimitriou, C.H. - Scott, D.S. - Putnam, H.W. - Harper, C. Jr. (eds.) *Horizons of truth. Kurt Gödel and the Foundations of Mathematics*, Cambridge University Press, Cambridge, 2011

[2] Bagaria, J. - Castells, N. - Larson, P. "An Ω-logic Primer", 2006, in [3], pp. 1-28

[3] Bagaria, J. - Todorcevic, S. *Set Theory. Centre de ricerca matemàtica Barcelona, 2003-2004*, Birkhäuser, Basel, 2006

[4] Bagaria, J. "Bounded forcing axioms as principles of generic absoluteness", *Archive For Mathematical Logic*, Vol. 39, 393-401, 2000

[5] - *Models of Set Theory*, lecture notes of the Master Course held at the University of Barcelona during the Academic Year 2009-2010

[6] Balaguer, M. "A Platonist Epistemology", *Synthèse*, 103, pp. 303-25, 1995

[7] - *Platonism and Anti-platonism in Mathematics*, Oxford University Press, Oxford, 1998

[8] Bell, J. *Set Theory. Boolean-valued Models and Independence Proofs*, Clarendon Press, Oxford, 2005

[9] Bellotti, L. "Woodin on the continuum problem: an overview and some objections", *Logic and philosophy of science*, 3, 2005

[10] Benacerraf, P. "Mathematical Truth", *Journal of Philosophy*, 70, pp. 661-80, 1973 (reprinted in [11], pp. 403-20)

[11] Benacerraf, P. - Putnam, H. (eds.) *Philosophy of mathematics: selected readings*, Cambridge University Press, Cambridge, 1983

[12] Bernays, P. "Sur le platonisme", 1935, (reprinted in [11], pp. 258-71)

[13] Boolos, G. "The iterative conception of set", *Journal of Philosophy*, 68, 8, pp. 215-31, 1971 (reprinted in [11], pp. 486-502)

[14] Burgess, J.P. "Intuitions of three kinds in Gödel's views on the Continuum", forthcoming, in Kennedy, J. *Interpreting Gödel*, Cambridge University Press `http://www.princeton.edu/ jburgess/Goedel.pdf`

[15] Buzaglo, M. *The Logic of Concept Expansion*, Cambridge University Press, Cambridge, 2002

[16] Cantor, G. "Über unendliche lineare Punktmannigfaltigkeiten I-VI", *Mathematische Annalen* , I 1879, vol. 15, pp. 1-7; II, 1880, vol. 17, pp. 357-55; III, 1882, vol. 20, pp. 113-21; IV, 1883a, vol. 21, pp. 51-58; V, 1883b, ["Grundlagen einer allgemeinen Mannigfaltigkeitslehre" (English translation in [33], vol. 2, pp. 878-919)], vol. 21, pp. 545-586; VI, 1884, vol. 23, pp. 453-488

[17] "Über die verschiedenen Standpunkte in Bezug auf das aktuelle Unendliche", *Zeitschrift für Philosophie und philosophische Kritik*, 88, 1885, pp. 224-33

[18] - "Mitteilungen zur Lehre vom Transfiniten I-II", *Zeitschrift für Philosophie und philosophische Kritik*, I, 1887, vol. 91, pp. 81-125; II, 1888, 92, pp. 240-265

[19] - Letter to Dedekind, 3 August 1899, in [33], pp. 931-5

[20] - *Gesammelte Abhandlungen mathematischen und philosophischen Inhalts*, [Zermelo, E. (ed.)], Springer, Berlin, 1932

[21] Chihara, C. *Constructibility and Mathematical Experience*, Oxford University Press, Oxford, 1990

[22] Cohen, P.J. *Set theory and the Continuum Hypothesis*, Benjamin, New York, 1966

[23] - "My Interaction with Kurt Gödel", 2011, (in [1], pp. 435-48)

[24] Colyvan, M. - Zalta, E.N. Review of [7], *Philosophia Mathematica*, 7, pp. 336-49, 1999

[25] Dales, H. G. - Oliveri, G. (eds.) *Truth in Mathematics*, Clarendon Press, Oxford, 1998

[26] Dauben, J.W. *Georg Cantor: his Mathematics and Philosophy of Infinite*, Harvard University Press, Boston, 1979

[27] Dehornoy, P. *Braids and Self-Distributivity*, Birkhäuser, Basel, 2000

[28] Devlin, K. *Constructibility*, Springer, Berlin, 1984

[29] - *The Joy of sets. Fundamentals of contemporary set theory*, Springer, New York, 1993

[30] Drake, F.R. *Set Theory. An Introduction to Large Cardinals*, North-Holland, Berlin, 1974

[31] Dummett, M. *Truth and Other Enigmas*, Harvard University Press, Cambridge, 1978

[32] Errera, A. "Le problème du continu", *Atti dell'Accademia Ligure di Scienze e Lettere*, 9, 1953 (1952)

[33] Ewald, W. B. (ed.) *From Kant to Hilbert. A Source Book in the foundations of mathematics*, 2 vols., Oxford, Oxford University Press, 1996

[34] Feferman, S. - Friedman H. - Maddy P. - Steel, J. "Does Mathematics need new axioms?", *Bulletin of Symbolic Logic*, 6, 4, pp. 401-46, 2000

[35] Feferman, S. "Infinity in mathematics: is Cantor necessary?", 1987, in [142], pp. 151-209

[36] - "Does Mathematics need new axioms?", *American Mathematical Monthly*, vol. 106, pp. 99-111, 1999

[37] - "Why the programs for new axioms need to be questioned", 2000, in [34], pp. 401-13

[38] - "Conceptions of the Continuum", unpublished conference paper, 2008

[39] - "Predicativity", 2005, in [133], pp. 590-624

[40] - "Is the Continuum Hypothesis a definite mathematical problem?", unpublished conference paper, 2011

[41] Ferreirós, J. *Labyrinth of thought. A history of set theory and its role in modern mathematics*, Birkhäuser, Basel, 1999

[42] - "The Motives Behind Cantor's Set Theory: Physical, Biological and Philosophical Questions", *Science in Context*, 17 (1/2), pp. 1-35, 2004

[43] Field, H. *Science Without Numbers*, Pirnceton University Press, Princeton, 1980

[44] - *Realism, Mathematics and Modality*, Blackwell, Oxford, 1989

[45] - *Truth and Absence of Fact*, Oxford University Press, Oxford, 2001

[46] - Foreman, M. "Has the Continuum Hypothesis been settled?", *Logic Colloquium 03*, talk slides, 2003
`www.math.helsinki.fi/logic/LC2003/presentations/foreman.pdf`

[47] Fraenkel, A. *Abstract set theory*, North-Holland, Amsterdam, 1961

[48] Fraenkel, A. - Bar-Hillel, Y. - Levy, A. - Van Dalen, D. *Foundations of set theory*, North-Holland, Amsterdam, 1973

[49] Frege, G. *Die Grundlagen der Arithmetik. Eine logisch mathematische Untersuchung über den Begriff der Zahl*, Köbner, Breslau, 1884 [English translation by J.L. Austin, *The Foundations of Arithmetic. A logico-mathematical enquiry into the concept of number*, Basil Blackwell, Oxford, 1950]

[50] - *Grudgesetze der Arithmetik*, Hermann Pohle, Jena, Band I (1893) - Band II (1903) [English translation (excerpts) in Geach, P. - Black, M. (eds.), *Translation from the Philosophical Writings of Gottlob Frege*, Basil Blackwell Oxford, 1970]

[51] Friedman, H. "Necessary Uses of Abstract Theory in Finite Mathematics", *Advances in Mathematics*, 60, 1, pp. 92-122, 1986

[52] - "Finite Functions and the Necessary Use of Large Cardinals", *Annals of Mathematics*, 148, 3, 1998, pp. 803-893, 1998

[53] "Normal Mathematics will need new axioms", 2000, in [34], pp. 434-46

[54] Gentzen, G. "Die Widerspruchsfreiheit der reinen Zahlentheorie", *Mathematische Annalen*, 112, pp. 493-565, 1936

[55] Gödel, K. "Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme, I", *Monatshefte für Mathematik und Physik*, 38, 173-198, 1931 (reprinted in [62], pp. 144-195)

[56] - "The Consistency of the Continuum Hypothesis", *Annals of Mathematics Studies*, Vol. 3, Princeton, Princeton University Press, 1938 (reprinted in [63], pp. 33 - 101).

[57] - "Russell's mathematical Logic", 1944, in [127], pp. 125-153 (reprinted in [11], pp. 447-469)

[58] - "What is Cantor's Continuum Problem?", 1947, *American Mathematical Monthly*, vol. 54, pp. 515-525 (1964 version reprinted in [11], pp. 470-485)

[59] - "A remark on the relationship between relativity theory and idealistic philosophy", 1949, in [128], pp. 555-562 and [63], pp. 202-207

[60] - "Some basic theorems on the foundations of mathematics and its implications", lecture manuscript (address to the American Mathematical Society), known as Gibbs Lecture, *1951, reprinted in [64], pp. 304-23

[61] - "Is mathematics syntax of language?", *1953, in [64], pp. 334-63

[62] - *Collected Works. I: Publications 1929 - 1936*, [S. Feferman, S. Kleene, G. Moore, R. Solovay, and J. van Heijenoort (eds.)], Oxford, Oxford University Press, 1986

[63] - *Collected Works. II: Publications 1938 - 1974*, [S. Feferman, J. Dawson, S. Kleene, G. Moore, R. Solovay, and J. van Heijenoort (eds.)], Oxford, Oxford University Press, 1990

[64] - *Collected Works. III: Unpublished essays and lectures*, [S. Feferman, J. Dawson, S. Kleene, G. Moore, R. Solovay, and J. van Heijenoort (eds.)], Oxford, Oxford University Press, 1995

[65] - *Collected Works. IV: Correspondence A-G*, [S. Feferman, J. Dawson, S. Kleene, G. Moore, R. Solovay, and J. van Heijenoort (eds.)], Oxford, Oxford University Press, 2003

[66] - *Collected Works. V: Correspondence H-Z*, [S. Feferman, J. Dawson, S. Kleene, G. Moore, R. Solovay, and J. van Heijenoort (eds.)], Oxford, Oxford University Press, 2003

[67] Hacking, I. *The Social Construction of What?*, Harvard University Press, Harvard, 1999

[68] Hájek, P. - Valdés-Villanueva, L. - Westerståhl (eds.) *Logic, Methodology, Philosophy of Science. Proceedings of the Twelfth INternation Congress*, King's College Publications, London, 2005

[69] Hallett, M. *Cantorian set theory and limitation of size*, Clarendon Press, Oxford, 1984

[70] Hamkins, J. "Some Second Order Set Theory", 2009, in [121], pp. 36-50

[71] - "A Tutorial in Set-theoretic Geology", Birkbeck Summer School, lecture slides, 2011

[72] Hardy, G. *A Mathematician's Apology*, Cambridge University Press, 1940 (1992)

[73] Hauser, K. "Objectivity over Objects: A Case Study in Theory Formation", *Synthèse*, 128, 3, pp. 245-85, 2001

[74] - "Is the Continuum Problem Inherently Vague", *Philosophia Mathematica*, vol. 10, pp. 257-285, 2002

[75] - "Gödel's program revisited. Part I: The turn to Phenomenology", *Bulletin of Symbolic Logic*, 12, 4, pp. 529-590, 2006

[76] Heijenoort, J. van (ed.) *From Frege to Gödel: a source book in mathematical logic, 1879-1931*, Harvard University Press, Cambridge (MA), 1967

[77] Hellman, G. *Mathematics Without Numbers*, Oxford, Clarendon Press, 1989

[78] Hersh, R. *What is Mathematics Really?*, Oxford University Press, Oxford, 1999

[79] Hilbert, D. *Grundlagen der Geometrie*, Teubner, Leipzig, 1903 (1899) [English translation: *The Foundations of Geometry*, Open Court, Chicago, 1980]

[80] Husserl, E. *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie. Erstes Buch: Allgemein Einführung in die Reine Phänomenologie*, Niemeyer, Halle, 1913 [English translation: *Ideas Pertaining to a Pure Phenomenology and to a Phenomenological Philosophy-First Book: General Introduction to a Pure Phenomenology* (translated by F. Kersten), Nijhoff, The Hague, 1982]

[81] - *Cartesianische Meditationen und Pariser Vorträge* [1931], (hrsg. von S. Strasser), Nijhoff, 1950, Den Haag [English translation: *Cartesian Meditations*, (translated by D. Cairns), Kluwer, Dordrecht, 1988]

[82] Jané, I. "The role of the absolute infinite in Cantor's conception of set", 1995, *Erkenntnis*, vol. 42, pp. 375-402, 1995

[83] - "Higher-Order Logic Reconsidered", 2005a, in [133], pp. 781-810, 2005a

[84] - "The iterative conception of sets from a Cantorian perspective", 2005b in [68], pp. 373-93

[85] - "Idealist and Realist Elements in Cantor's Thought", *Philosophia Mathematica*, 18, 2, pp. 193-226, 2010

[86] Jech, T. *Set theory*, Academic Press, New York, 2003

[87] Jensen, R. "Inner Models and Large Cardinals", *The Bulletin of Symbolic Logic*, 1, 4, pp. 393-407, 1995

[88] Kanamori, A. "The mathematical development of set theory from Cantor to Cohen" *Bulletin of Symbolic Logic*, 2, 1-71, 1996

[89] - *The Higher Infinite*, Springer, Berlin, 2003

[90] Kanamori, A. - Foreman, M. *Handbook of Set Theory*, 3 vols., Springer, Berlin, 2010

[91] Kunen, K. *Set Theory. An Introduction to Independence Proofs*, North Holland, Amsterdam, 1980

[92] Lavine, S. *Understanding the Infinite*, Harvard University Press, Cambridge (MA), 1994

[93] Leibniz, G.W. *Philosophical Essays* (translated by R. Ariew and D. Garber), Hackett, Indianapolis, 1989

[94] Leng, M. *Mathematics and Reality*, Oxford University Press, Oxford, 2010

[95] Linnebo, Ø. "Platonism in the Philosophy of Mathematics", *Stanford Encyclopedia of Philosophy*, 2009 `http://plato.stanford.edu/entries/platonism-mathematics/`

[96] Linsky, B. - Zalta, E.N. "Naturalized Platonism versus Platonized Naturalism", 1995, *Journal of Philosophy*, 92 (10), pp. 525-55

[97] Lucas, J. "Minds, Machines and Gödel", *Philosophy*, 36, pp. 112-27, 1961

[98] Maddy, P. "Believing the Axioms I-II", *Journal of Symbolic Logic*, I, 53, 2, pp. 481-511, 1988; II, 53, 3, pp. 736-64, 1988

[99] - "The roots of contemporary platonism", *Journal of Symbolic Logic*, 54 (4), pp. 1121-44, 1989

[100] - *Realism in mathematics*, Clarendon Press, Oxford, 1990

[101] - "Set-theoretic Naturalism", *Bulletin of Symbolic Logic*, 61, 2, pp. 490-514, 1996

[102] - *Naturalism in mathematics*, Oxford University Press, Oxford, 1997

[103] - "Does Mathematics need new axioms?", 2000, in [34], pp. 413-22

[104] - "Three forms of Naturalism", 2005, in [133], pp. 437-59

[105] - *Defending the Axioms*, Oxford University Press, Oxford, 2011

[106] Martin, D.A. "Mathematical Evidence", 1998, in [25], pp. 215-32

[107] - "Multiple Universes of Sets and Indeterminate Truth Values", *Topoi*, 20, pp. 5-16, 2001

[108] Ortiz, C. "Did Georg Cantor influence Edmund Husserl?", *Synthèse*, 113, 1, pp. 145-70, 1997

[109] Paris, J. - Kirby, L. "Accessible independence results for Peano arithmetic", *Bulletin of London Mathematical Society*, 14, pp. 285-93, 1982

[110] Parsons, C. "Platonism and Mathematical Intuition in Kurt Gödel's Thought", *Bulletin of Symbolic Logic*, 1 (1), pp. 44-74, 1995

[111] Penrose, R. *The Emperor's New Mind*, Oxford University Press, Oxford, 1989

[112] - *Shadows of the mind*, Oxford University Press, Oxford, 1994

[113] - *The Road to Reality*, Knopf, New York, 2005

[114] Potter, M. *Reason's nearest kin. Philosophies of Arithmetic from Kant to Carnap*, Oxford University Press, Oxford, 2000

[115] - "Was Gödel a Gödelian Platonist?", *Philosophia Mathematica*, 9 (3), pp. 331-46, 2001

[116] - *Set Theory and its Philosophy*, Clarendon Press, Oxford, 2004

[117] Putnam, H. "Models and Reality", *Journal of Symbolic Logic*, 45, 3, pp. 464-82, 1980 (reprinted in [11], pp. 421-46)

[118] Quine, W. van O. "Whitehead and the rise of modern logic", 1941, in [126], pp. 127-63

[119] - "Two Dogmas of Empiricism", *The Philosophical Review*, 60, pp. 20-43, 1951

[120] - *Theories and Things*, Harvard University Press, Cambridge(MA), 1981

[121] Ramanujam, R. - Sarukkai, S (eds.) *Logic and Its Applications*, Springer, Berlin, 2009

[122] Resnik, M. *Mathematics as a Science of Patterns*, Oxford, Oxford University Press, 1997

[123] Russell, B. *The Principles of Mathematics*, W.W. Norton & Company, London - New York, 1903 (1996)

[124] "Mathematical Logic as Based on the Theory of Types", *American Journal of Mathematics*, 30, 222-262, 1908

[125] - *Introduction to Mathematical Philosophy*, New York, MacMillan, 1919

[126] Schilpp, P.A. (ed.) *The philosophy of Alfred North Whitehead*, New York, Tudor, 1941

[127] - (ed.) *The philosophy of Bertrand Russell*, Northwestern University Press, Evanston and Chicago, 1944

[128] - (ed.) *Albert Einstein: Philosopher-Scientist*, Open Court, La Salle, 1949

[129] Searle, J. *The construction of social Reality*, Free Press, New York, 1995

[130] Shapiro, S. *Foundations without Foundationalism. A case for Second-Order Logic*, Oxford University Press, Oxford, 1991

[131] - *Philosophy of Mathematics: Structure and Ontology*, Oxford, Oxford University Press, 1997

[132] - *Thinking about Mathematics*, Oxford University Press, Oxford, 2000

[133] - (ed.) *The Oxford Handbook of Philosophy of Mathematics*, Oxford University Press, Oxford, 2005

[134] Shelah, S. "The future of Set Theory", *Israel Mathematical Conference Proceedings vol. 6*, 1991, `http://shelah.logic.at/files/E16.pdf`

[135] - "Logical Dreams", *Bulletin of the American Mathematical Society*, 40, 2, pp. 203-28, 2003

[136] Simpson, S. *Subsystems of Second-Order Arithmetic*, Cambridge University Press, Cambridge, 1999

[137] Steel, J. "Mathematics needs new axioms", 2000, in [34], pp. 422-33

[138] Tait, W.B. "Beyond the Axioms: the question of Objectivity in Mathematics", *Philosophia Mathematica*, 9, 21-36, 2009

[139] Tieszen, R. "Gödel and the Intuition of Concepts", *Synthèse*, 133 (3), pp. 363-91, 2002

[140] - *Phenomenology, Logic and the Philosophy of Mathematics*, Cambridge University Press, Cambridge, 2005

[141] - *After Gödel: Platonism and Rationalism in Mathematics and Logic*, Oxford University Press, Oxford, 2011

[142] Toraldo di Francia, G. (ed.) *L'Infinito nella Scienza*, Istituto della Enciclopedia Italiana, Roma, 1987

[143] Väänänen, J. "Second-Order Logic and Foundations of Mathematics", *The Bulletin of Symbolic Logic*, 17, 4, pp. 504-20, 2001

[144] Wang, H. *From mathematics to Philosophy*, Routledge and Kegan Paul, London, 1974

[145] - *A logical journey. From Gödel to Philosophy*, MIT Press, Cambridge (MA), 1997

[146] Weir, A. "Naturalism Reconsidered", 2005, in [133], pp. 460-82

[147] Weyl, H. *Das Kontinuum*, von Veit und Comp., Leipzig, 1918 [English translation: *The Continuum: A critical examination of the foundations of Analysis*, Dover, New York, 1994]

[148] Wittgenstein, L. *Logisch-Philosophisch Abhandlung*, *Annalen der Naturphilosophie*, XIV, 3/4, 1921 [English translation: *Tractatus Logico-Philosophicus*, translated by D.F. Pears and B. McGuinness, Routledge, London-New York, 2003]

[149] Woodin, W.H. - Heller, M. (eds.) *Infinity. New Research Frontiers*, Cambridge University Press, Cambridge, 2011

[150] Woodin, W.H. *The Axiom of Determinacy, Forcing Axioms and the Nonstationary Ideal*, De Gruyter, Berlin-New York, 1999

[151] - "The Continuum Hypothesis I-II", *Notices of the American Mathematical Society*, 48, 6, pp. 567-76; 48, 7, 681-90, 2001

[152] - "The Realm of the Infinite", 2011a, in [149], pp. 89-118

[153] - "The Transfinite Universe", 2011b, in [1], pp. 449-474

[154] Zalta, E.N. *Abstract Objects. An Introduction to Axiomatic Metaphysics*, Reidel, Dordrecht, 1983

[155] Zermelo, E. "Beweis daß jede Menge wohlgeordnet werden kann", *Mathematische Annalen*, vol. 59, 1904 (English translation in [76], pp. 183-98)

[156] - "Untersuchungen über die Grundlagen der Mengenlehre", *Mathematische Annalen*, 65, 261-81, 1908 (English translation in [76], pp. 199-215)

[157] Zermelo, E. "Über Grenzzahlen und Mengenbereiche: Neue Untersuchungen über die Grundlagen der Mengenlehre", *Fundamenta Mathematicae*, 16, pp. 29-47, 1930

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