Statistical Exploration of Heterotic Pati-Salam String Vacua

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Αφιερώνεται στους Γονείς μου Παναγιώτη και Ρένα

Dedicated to my parents Panayiotis and Rena

Abstract

To this date, experimental searches for exotics (fractionally charged states) have yielded negative results, thus imposing strong limitations on their existence at low energies. In this work we argue that a possible scenario that could explain the experimental data is that such states are simply not present at the low energy limit. This thesis presents the phenomenology of the first string model which does not have any exotic states at the massless level, and where only the top quark Yukawa coupling exists at the tri-linear level superpotential. More specifically we present its spectrum, cubic superpotential and a viable semi-realistic phenomenological scenario which is supported by a specific set of F- and D-flat solutions. The discovery of this model is the result of the statististical exploration of a class of heterotic Pati-Salam vacua, out of which we managed to extract three generation exophobic models with the required representations needed to induce spontaneous breaking to the Standard Model. We also exhibit the derivation of the analytic formulae that permitted the exact identification of several properties of a string vacuum, and thus allowed its distinction among supersymmetric vacua that share the same gauge group.

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I would like to take this opportunity to express my gratitude to various people who have contributed towards the completion of this thesis. Firstly I would like to thank my supervisor Alon E. Faraggi, for opening the doors of theoretical physics research to me, and for assigning me interesting problems. I consider myself extremely fortunate to have had the opportunity to collaborate with genuinely interesting and charismatic people. I have had many insightful and often entertaining discussions in and out of physics with Costas Kounnas, the advice of whom helped with my progression and is deeply appreciated. I have benefited from various discussions and e-mail exchanges with John Rizos the guidance of whom was crucial for the completion of this thesis. I would also like to thank Ben Assel with whom I spent a large amount of time discussing our subject and cross-checking calculations.

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Declaration

The work described in this thesis was carried out in the Department of Mathematical Sciences at the University of Liverpool from October 2007 to June 2011. It is the original work of the author, unless otherwise acknowledged within the text. It has not been submitted previously for a degree at this or any other University.

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Chapter 1

Introduction: The Standard Model and Beyond

THE UNIFIED description of the laws of physics under a single theoretical construct has been a long-pursued goal among the theoretical physics community. Newton unified celestial with terrestrial physics, Maxwell unified electricity with magnetism, Einstein space with time, De Broglie waves with particles. These are some of the most important corner stones towards a unified picture for describing nature. The conceptual and calculational foundations of contemporary physics are based on the fact that a physical system's properties can be understood by studying its underlying symmetries [1]. Consequently our efforts towards unification concentrate on extending the space-time and internal symmetries, which dictate our currently most successful theory for understanding subatomic physics: The Standard Model (SM). Our biggest effort at the moment is the creation of a consistent quantum theory of gravity which effectively describes the observed data. In this chapter we briefly describe the Standard Model and some of the most explored theories of unification which are relevant to the purpose of this thesis.

1.1 The Standard Model: An Effective Field Theory

Our experimentally-verified, and thus low-energetic, knowledge for the electroweak and strong interactions of elementary particles is described by the Standard Model, which is a four dimensional relativistic gauge theory based on the internal symmetry group $SU(3)_C \times SU(2)_L \times U(1)_Y$. All the elementary particles are considered to be point-like, and forces emerge as the result of the Lagrangian invariance under the pre-stated gauge group. According to the Standard Model the force carriers for the electroweak and strong interactions are gauge bosons found in the adjoint representation of $SU(3)_C \times SU(2)_L \times U(1)_Y$.

Force Carrier	Q_{em}	Spin
g	0	1
γ	0	1
$W^{\pm}$	± 1	1
Z^0	0	1

Table 1.1: *The gauge bosons of the strong and electroweak interactions*

The three families of matter are described by fermionic fields in the fundamental representation of the SM gauge group.

Matter Field	$SU(3)_C$	$SU(2)_L$	$U(1)_Y$
Q_L	3	2	1/3
u_L^c	$\bar{3}$	1	−4/3
d_L^c	$\bar{3}$	1	2/3
L	1	2	−1
e_L^c	1	1	2

Table 1.2: *Matter content and representations of the lightest family*

The Higgs scalar is responsible for the breaking of the electroweak symmetry $SU(2)_L \times U(1)_Y$ to the observed $U(1)_{em}$, by accumulating a non-zero vacuum expectation value (VEV) $\langle \phi \rangle$. This mechanism assigns mass to the weak bosons Z and $W^{\pm}$, while it leaves the photon massless. Chiral fermions acquire mass through their Yukawa-like interactions with the Higgs multiplet: $Q_L \phi u_L^c$, $Q_L \bar{\phi} d_L^c$ and $L \phi e_L^c$. The Higgs VEV is determined by the Higgs potential:

$$V(\phi) = -m^2\phi^*\phi + \lambda(\phi^*\phi)^2, \tag{1.1}$$

which in turn fixes the scale of the electro-weak symmetry breaking:

$$M_W \simeq \langle\phi\rangle \simeq \frac{m}{\sqrt{\lambda}} \simeq 10^2 GeV. \tag{1.2}$$

Despite all of its successes the Standard Model is highly unlikely to be the complete theory for describing nature due to its phenomenological origins, which introduce too many free parameters and leave too many fundamental questions unresolved. Some of the main open questions concerning the SM are: why is $SU(3)_C \times SU(2)_L \times U(1)_Y$ the correct quantum field theory to describe particle interactions and why are there three families of quarks and leptons? Furthermore what is the origin for the quark and lepton masses? Moreover one finds it difficult to accept the fact that it has 19 arbitrary parameters whose values are carefully chosen in order to fit the data. Finally, the most obvious weakness of the standard model is its inability to incorporate gravity, a theory of the geometry of space-time, within its construction.

Our current strategy in dealing with the Standard Model's weaknesses is to consider it as an approximation, a low energy effective theory of a more basic theory which makes sense at higher energies.

The SM faces "infinities" issues coming from the integrand $\int d^4k \rightarrow \infty$. This problem is dealt by introducing the regulator $\int_0^\infty d|k| \rightarrow \int_0^\Lambda d|k|$. Our former understanding of a renormalisable theory considered Λ to be a mathematical parameter that yields cut-off independent results in the limit $\Lambda \rightarrow \infty$, by redefining the finite number of parameters appearing in the Lagrangian. From an effective theory's point of view, this cut-off is considered to be a parameter that corresponds to the energy scale at which the effects of the new physics beyond the effective field theory become important. Summarising, we can say that the Standard Model, although tremendously successful, is valid up to a certain energy scale and it constitutes the low energy manifestation of a more fundamental theory.

1.2 Supersymmetry and the MSSM

Supersymmetry (SUSY) is an abstract extension of the Poincare group, which commutes with the gauge symmetries. It relates bosons and fermions when a supercharge operator Q acts on either: $Q|F/B\rangle = |B/F\rangle$. Bosons and fermions which are related through supersymmetric transformations are called superpartners, they differ by spin 1/2, and they inhabit the same supermultiplet. The squared mass operator commutes with the supercharge and with all the spacetime rotation and translation operators, so all the particles within the same supermultiplet must have equal momentum eigenvalues and therefore equal masses.

Partner	Spin	Superpartner	Spin
Photon	1	Photino	1/2
Quark	1/2	Squark	0
Electron	1/2	Selectron	0
Gluon	1	Gluino	1/2

Table 1.3: *Supersymmetric partners' spin, differs by 1/2.*

Within the minimal supersymmetric extension of the Standard Model [2, 3], gauge bosons and their supersymmetric partners, the gauginos, are found in vector multiplets, whereas quarks and leptons and their superpartners (squarks and sleptons) are found in chiral multiplets. Although exact invariance under supersymmetry predicts that superpartners should have the same mass and quantum numbers, the superpartners have never been observed. Therefore if such a symmetry exists, it should be valid at higher energy scales than the currently accessible ones and it is broken at lower enegy in a way that allows superpartners to acquire masses heavier than their experimentally observed counterparts.

1.3 Grand Unification

The idea of Grand Unification[4] is based on embedding the SM gauge group G, in a larger simple group H. The low energy gauge structure then emerges as follows: $H \longrightarrow G : SU(3) \times SU(2) \times U(1) \rightarrow SU(3) \times U(1)_{em}$. The combination of SUSY and Grand Unified Theories (GUTs)[5] gives birth to SUSY-GUTs. For a recent review see e.g [6, 7]. SUSY-GUTs belong to some of the most explored scenarios beyond the Standard Model, because they can provide appealing insights to numerous problems.

1.4 Extra Dimensions

At a first glance the task of unifying forces seems to be an impossible procedure, in the sense that gravity is described by space-time symmetries while all the other forces are described by internal symmetries. Kaluza and Klein in their attempt to unify general relativity with electromagnetism, overcame this conceptual obstacle by promoting the internal $U(1)_{em}$ symmetry into a space-time symmetry. More specifically the Einstein-Hilbert action was generalised so that it admits the $M_4 \times S^1$ space as a solution:

$$S = -\frac{1}{16\pi G_5} \int dx^5 \int d^4x \sqrt{-g} R. \tag{1.3}$$

If we impose periodicity in the fifth dimension for all the fields:

$$x^5 : x^5 \rightarrow x^5 + 2\pi R, \tag{1.4}$$

then under the 5D coordinate transformation

$$x^A : x^A \rightarrow x^A + \epsilon^A(x), \tag{1.5}$$

the metric transforms as: $h'_{AB} = h_{AB} - \partial_A \epsilon_B - \partial_B \epsilon_A$. Assuming that no metric components or any fields depend on x^5 we get that $h'_{\mu 5} = h_{\mu 5} - \partial_\mu \epsilon_5$. After assigning new names to $h_{\mu 5}$ and ϵ_5 we get the usual electromagnetic gauge transformation:

$$A'_\mu = A_\mu - \partial_\mu \lambda. \tag{1.6}$$

Therefore a theory of both gravity and electromagnetism can emerge, if one adds one extra compact dimension in a purely gravitational theory. From this example an important conceptual idea related to the present work emerges:

extra dimensions appear as internal symmetries to a lower dimensional observer. It is also important to note that a massless field ϕ which depends on the four ordinary and on one extra dimension will be described by an equation for massive particles after periodicity on the extra dimension is imposed. After Fourier expanding the field ϕ in respect to the periodic coordinate, we get that each mode Φ_m corresponds to a mode of mass: $M^2 \sim \frac{m}{R^2}$ where m is the mode number. For $m \neq 0$, and when the compactification radius is sufficiently small, i.e $R \to 0$, all the Kaluza-Klein modes become infinitely massive and thus escape detection.

1.5 String Theory and String Phenomenology

String theory [8, 9, 10, 11] is a consistent theory of quantum gravity which at low energies gives rise to gauge theories, scalar fields and chiral fermions. As a unifying theory it incorporates, reproduces, and provides new perspectives on, several concepts and extensions of the Standard Model of particle physics. The above-mentioned achievements occur by generalising the concept of the point particle to the case of an extended one-dimensional object propagating in a multidimensional space-time. Anomaly cancellation allows the existence of five different superstring theories in ten dimensions.

Type	Gauge Group	Supersymmetry
Type I (open)	$SO(32)$	chiral $N = 1$
Type IIA	–	non-chiral $N = 2$
Type IIB	–	chiral $N = 2$
Heterotic	$E_8 \times E_8$	chiral $N = 1$
Heterotic	$SO(32)$	chiral $N = 1$

Table 1.4: *The five consistent string theories in* $D = 10$

String phenomenology [12] provides a bridge between string theory and the observed low energy Standard Model data. It also provides new insights that would be difficult or impossible to obtain using standard field theoretical techniques [13]. The route that we follow in order to identify signatures of string theory in the currently observed data is the construction and analysis of four dimensional string models. The construction of such models demands a four-dimensional formulation of string theory. It is believed that the extra dimensions are compactified on an internal manifold, whose size is sufficiently small to have escaped detection. Although a string theory Lagrangian contains only one free parameter, known as the string tension, the compactification of the additional six dimensions introduces a much larger number of free parameters, the values of which must be fixed in a way that provides phenomenologically viable scenarios.

The work described in this thesis explores whether there can exist semi-realistic string vacua, which do not contain any fractionally charged representations at the effective field theory limit. Such vacua would back up contemporary experiments which have searched for such exotic particles without yielding any evidence for their existence. An example of such an experiment is the case of an improved and highly automated Millikan oil drop technique. Drops of silicon oil are ejected through a silicon orifice and fall through the air under the influence of gravity and a vertical alternating electric field. The positions of the drops are imaged by a digital camera, which is interfaced to a computer. The electric charge for a drop is then calculated by solving the following system of equations: $mg + E_{\downarrow}Q = 6\pi\eta r u_{\downarrow}$ and $mg - E_{\uparrow}Q = 6\pi\eta r u_{\uparrow}$.

1.6 List of Publications

Parts of this thesis have been published in scientific journals.

- B. Assel, K. Christodoulides, A. E. Faraggi, C. Kounnas, and J. Rizos,
  Exophobic Quasi-Realistic Heterotic String Vacua,
  Phys.Lett. B683 (2010) 306313

- B. Assel, K. Christodoulides, A. E. Faraggi, C. Kounnas, and J. Rizos,
  Classification of Heterotic Pati-Salam Models,
  Nucl. Phys B844 (2010), 365

- K. Christodoulides, A. E. Faraggi, and J. Rizos,
  Top Quark Mass in Exophobic Pati-Salam Heterotic String Model,
  Phys.Lett. B702 (2011) 81-89

1.7 Organising the chapters

The chapters of this thesis are organised as follows.

- Chapter 2: A general introduction to the bosonic string and superstring is presented. We discuss canonical and light-cone quantisation, aspects of the bosonic and fermionic partition functions, and the occurrence of modular invariance. We also introduce the concept of compactification on a circle.

- Chapter 3: We begin the chapter with some generalities on the heterotic string in its bosonic formulation, followed by the description of the toroidal compactification. We then proceed by providing a very generic background on $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold compactifications, in order to introduce the concepts of twist vectors, shift vectors and Wilson lines. After showing the equivalence between fermions and bosons in two dimensions we proceed to presenting the heterotic string in its four dimensional fermionic formulation.

- Chapter 4: We present the consistency constraints and model building rules of the four-dimensional free fermionic models and we explain the general derivation of the spectrum. In the second part of the chapter, we present a specific free fermionic model, the main characteristic of which is the absence of exotics in the massless spectrum. The standard analysis of flat directions is also exhibited.

- Chapter 5: In this chapter we exhibit the classification techniques that we developed in order to perform the statistical exploration for a class of Pati-Salam string vacua. The methodology described provides supersymmetric, three family models with electroweak breaking and which are free of exotic representations at the massless level.

- Chapter 6: In this chapter we give a short summary of the issues discussed and results obtained. We also provide some potentially interesting outlooks and suggestions for future research.

Chapter 2

Elements of String Theory

THIS CHAPTER introduces the fundamental notions of the bosonic string and the superstring by discussing their equations of motion, symmetries, conserved currents, and canonical and light-cone quantisation. The constraints imposed by modular invariance on the one loop bosonic partition function and compactification on a circle are also explored.

2.1 The Classical Relativistic Bosonic String

We consider the string to be embedded in a D-dimensional space-time M. A point of the string $X = X^{\mu}(\sigma, \tau)$, $\mu = 0,\ 1.....,\ D-1$, is then parametrised by the proper time τ and the spatial co-ordinate along the string σ. The part of space swept by the string is called the world sheet.

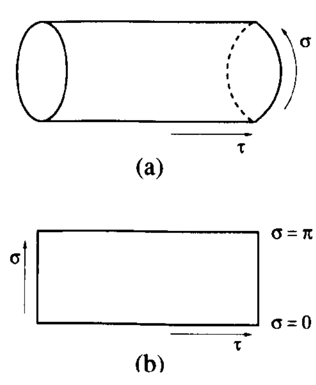

Figure 2.1: *a) Closed string worldsheet* *b)Open string worldsheet*

The kinematic and dynamical properties of the string are derived from the minimisation of the Nambu-Goto (NG) action which defines the area swept by the bosonic string:

$$S_{NG} = -T \int_{\Sigma} d^2\sigma |det\, G_{\alpha\beta}|^{1/2}, \tag{2.1}$$

where T is the tension (energy per length) of the string and $G_{\alpha\beta}$ is the induced metric on the world-sheet:

$$G_{\alpha\beta} = \frac{\partial X^{\mu}}{\partial \sigma^{\alpha}} \frac{\partial X^{\nu}}{\partial \sigma^{\beta}} G_{\mu\nu}, \tag{2.2}$$

where $\alpha, \beta = 0, 1$ are the world-sheet indices corresponding to the worldsheet coordinates τ and σ, and $G_{\mu\nu}$ is the background metric. The square root in the NG action makes the quantisation procedure inconvenient and for this reason we introduce an equivalent action consisting of more fields. The Polyakov action is given by

$$S = -\frac{T}{2} \int_{\Sigma} d^2\sigma \sqrt{-h} h^{\alpha\beta} \partial_{\alpha} X^{\mu} \partial_{\beta} X^{\nu} G_{\mu\nu} = -\frac{T}{2} \int_{\Sigma} d^2\sigma \sqrt{-h} h^{\alpha\beta} G_{\alpha\beta}, \tag{2.3}$$

where $h^{\alpha\beta}$ is an auxiliary field which corresponds to the intrinsic metric of the worldsheet. The additional field $h^{\alpha\beta}$ does not propagate and doesn't affect the kinematics of the string. Both actions yield the same equations of motion for X^{μ}, and we can derive the NG action from the Polyakov action. This can be verified by varying the Polyakov action with respect to the intrinsic metric. The equation of motion, which is the energy-momentum tensor, will then contain expressions which include the induced metric and the intrinsic metric. After some basic algebra we get that $\sqrt{detG_{\alpha\beta}} = \frac{1}{2}\sqrt{-h} h^{\alpha\beta} G_{\alpha\beta}$ and therefore the equivalence between the NG and the Polyakov action becomes obvious. The Polyakov action is invariant under specific transformations:

- Poincare transformations $\delta X^{\mu} = \Lambda^{\mu}_{\nu} X^{\nu} + b^{\mu}$ which transform the world-sheet metric as $\delta h^{\alpha\beta} = 0$.

- Worldsheet diffeomorphisms or else reparametrisations of the world-sheet coordinates $(\sigma, \tau) \rightarrow (\sigma', \tau')$. Under reparametrisations the metric is transformed as $h_{\alpha\beta} = \frac{\partial \sigma'^{\gamma}}{\partial \sigma^{\alpha}} \frac{\partial \sigma'^{\delta}}{\partial \sigma^{\beta}} h'_{\gamma\delta}(\sigma', \tau')$.

- Weyl transformations of the world sheet metric $h_{\alpha\beta} \rightarrow e^{\phi(\sigma,\tau)} h_{\alpha\beta}$.

After setting $h_{\alpha\beta} = \eta_{\alpha\beta}$ and using light-cone coordinates $\sigma^{\pm} = \tau \pm \sigma$, the Polyakov action takes the form:

$$S_P = T \int d^2\sigma(\partial_+ X^\mu \partial_- X^\nu G_{\mu\nu}). \tag{2.4}$$

Varying the action given by (2.4) yields the wave equation for the relativistic string:

$$\partial_+ \partial_- X^\mu = 0. \tag{2.5}$$

Whereas the variation of the action (2.3) with respect to the intrinsic metric yields a vanishing energy momentum tensor, this condition does not follow from the variation of the action (2.4). In this case, we need to impose $T_{\alpha\beta} = 0$ as an independent constraint. The general solution of the wave equation (2.5) can be written in terms of left $X_L^\mu(\tau + \sigma)$ and right $X_R^\mu(\tau - \sigma)$ moving terms in the following way: $X^\mu(\tau, \sigma) = X_L^\mu(\tau + \sigma) + X_R^\mu(\tau - \sigma)$. Such a solution for the closed string can be written as an expansion of Fourier modes

$$X_L^\mu(\tau + \sigma) = \frac{\chi^\mu}{2} + \frac{l_s^2}{2} p^\mu(\tau + \sigma) + i\frac{l_s}{2} \sum_{k \neq 0} \frac{\alpha_k^\mu}{k} e^{-ik(\tau+\sigma)}, \tag{2.6}$$

$$X_R^\mu(\tau - \sigma) = \frac{\chi^\mu}{2} + \frac{l_s^2}{2} \bar{p}^\mu(\tau - \sigma) + i\frac{l_s}{2} \sum_{k \neq 0} \frac{\tilde{\alpha}_k^\mu}{k} e^{-ik(\tau-\sigma)}. \tag{2.7}$$

This solution satisfies the closed string's periodicity condition $X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + 2\pi)$ and hence the condition $p^\mu = \bar{p}^\mu$. The former expansions have introduced the Regge slope parameter α' which is connected to the characteristic length of the string $\frac{l_s^2}{2} = \alpha'$ and the string tension $T = \frac{1}{2\pi\alpha'}$. The previous relations also indicate the fact that the degrees of freedom describing the classical relativistic string are

- The center of mass coordinate χ^μ.
- The total momentum of the string p^μ.
- The vibrational modes of the string α_k^m.

By imposing boundary conditions which are consistent with the equations of motion, other solutions which describe open strings with free endpoints and open strings with fixed endpoints can be found. Our study only utilises closed strings and we will therefore concentrate on this case.

2.1.1 Closed Bosonic String Quantisation and Spectrum

We will now demonstrate the covariant quantisation of the bosonic string. In order to achieve covariant quantisation we impose equal-time commutation relations on the position X^μ and conjugate momenta $\pi_\mu(\sigma,\tau) = \frac{\partial L}{\partial(\partial_\tau X^\mu)}$,

$$\begin{aligned} [X^\mu(\sigma,\tau), \pi^\nu(\sigma',\tau)] &= i\eta^{\mu\nu}\delta(\sigma-\sigma') \\ [X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)] &= 0 \\ [\pi^\mu(\sigma,\tau), \pi^\nu(\sigma',\tau)] &= 0. \end{aligned} \tag{2.8}$$

Utilising the mode expansions of the bosonic string we get the following expressions for the 1st quantised closed bosonic string:

$$\begin{aligned} [x^\mu, p^\nu] &= i\eta^{\mu\nu} \\ [\alpha^\mu_m, \alpha^\nu_n] &= m\,\eta^{\mu\nu}\delta_{m+n,0} \\ [\tilde\alpha^\mu_m, \tilde\alpha^\nu_n] &= m\,\eta^{\mu\nu}\delta_{m+n,0} \\ [\tilde\alpha^\mu_m, \alpha^\nu_n] &= 0. \end{aligned} \tag{2.9}$$

After the covariant quantisation procedure, the parameters x^μ and p^μ, α^μ_n and $\tilde\alpha^\mu_n$, hence X^μ, are promoted into operators which obey analogous commutation relations with the harmonic oscillator. We now have to find the states on which the pre-stated operators act. In analogy with the harmonic oscillator we define a number operator $N_m = \alpha_{-m}{\cdot}\alpha_m, \quad m \geq 1$, whose eigenstates satisfy the relation $N_m|i_m\rangle = i_m|i_m\rangle$. A closed string state will be given by $|S\rangle = |L\rangle \otimes |R\rangle$ since the right and left movers act independently. Hence, in order to get the closed string spectrum, we have to define two different number operators which are defined by infinite sums over the modes:

$$N_L = \alpha_{-m}{\cdot}\alpha_m, \qquad N_R = \tilde\alpha_{-m}{\cdot}\tilde\alpha_m. \tag{2.10}$$

Following a procedure where we use the number operators $N_L = \alpha_{-m}.\alpha_m$ and $N_R = \tilde{\alpha}_{-m}.\tilde{\alpha}_m$ along with the commutation relations $[\alpha^\mu_m, \alpha^\nu_n] = m\,\eta^{\mu\nu}\delta_{m+n,0}$, $[\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] = m\,\eta^{\mu\nu}\delta_{m+n,0}$, we can define the ground state $i_m = \tilde{i}_n = 0$ as a state that is annihilated by the operators α^μ_m and $\tilde{\alpha}^\mu_n$, $\forall\ m, n > 0$. For $m, n > 0$ the operators α^μ_{-m} and $\tilde{\alpha}^\mu_{-n}$ raise the eigenvalues of the number operator N_L and N_R by m and n respectively. It then follows that the states of the theory are obtained by a successive application of creation operators on the ground state.

Yet there exists a "problem" in the theory's spectrum: the presence of the metric $\eta_{\mu\nu}$ makes it possible for negative norm states to exist via the relation $[\alpha^\mu_m, \alpha^\nu_{-m}] = \eta^{\mu\nu} m\delta_{m-m,0}$. These unphysical states are truncated by imposing constraints on the theory's spectrum. The constraints arise from the fact that the variation of the Polyakov action with respect to the intrinsic metric demands that the energy momentum tensor vanishes: $T_{\alpha\beta} = 0$. When changing to complex coordinates the holomorphic and anti-holomorphic part of the energy-momentum tensor are expanded into Laurent series where the Laurent coefficients are the Virasoro operators:

$$T_{zz}(z) = \sum \frac{L_m}{z^{m+2}} \tag{2.11}$$

$$T_{\bar{z}\bar{z}}(\bar{z}) = \sum \frac{\bar{L}_m}{\bar{z}^{m+2}}. \tag{2.12}$$

The Virasoro operators describe the modes of the energy-momentum tensor and after quantisation they are given by the normal-ordered relations

$$L_m = \frac{1}{2}\sum :\alpha_{m-n}.\alpha_n: \quad , \quad \bar{L}_m = \frac{1}{2}\sum :\bar{\alpha}_{m-n}.\bar{\alpha}_n: \quad . \tag{2.13}$$

These operators satisfy commutation relations called the Virasoro Algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{12}(m^3 - m)\delta_{m+n,0} , \tag{2.14}$$

$$[\bar{L}_m, \bar{L}_n] = (m-n)\bar{L}_{m+n} + \frac{D}{12}(m^3 - m)\delta_{m+n,0} , \tag{2.15}$$

where the last term of the right-hand side is the central extension and it vanishes for $m = 0, \pm 1$. It should be noted that the former relations without the central extension constitute the Witt algebra of conformal transformations. The Virasoro operators can be used to eliminate the unphysical states of the theory by requiring their expectation value to vanish for a physical state $|\Psi\rangle$:

$$\langle\Psi|L_m - \alpha\delta_{m,0}|\Psi\rangle = 0 \tag{2.16}$$

$$\langle\Psi|\bar{L}_m - \alpha\delta_{m,0}|\Psi\rangle = 0. \tag{2.17}$$

The former relations take into account that the operators $L_m, \bar{L}_m$ with $m \neq 0$ do not have a normal ordering ambiguity. Therefore the term $\alpha\delta_{m,0}$ ensures that we only get a contribution α due to normal ordering ambiguity in the case of L_0. A physical state $|\Psi\rangle$ must then satisfy:

$$(L_m - \alpha\delta_{m,0})|\Psi\rangle = 0 \quad , \quad (\bar{L}_m - \alpha\delta_{m,0})|\Psi\rangle = 0. \tag{2.18}$$

For $m > 0$ the Virasoro operators annihilate the physical state: $L_m|\Psi\rangle = 0$, $\bar{L}_m|\Psi\rangle = 0$. For $m = 0$ we have the "mass shell" condition: $(L_0 - \alpha)|\Psi\rangle = 0$, $(\bar{L}_0 - \alpha)|\Psi\rangle = 0$.

The Virasoro operators L_0 and $\bar{L}_0$ can be written in terms of the number operators:

$$L_0 = \frac{1}{8\pi T}p^\mu p_\mu + N_L \quad , \quad \bar{L}_0 = \frac{1}{8\pi T}p^\mu p_\mu + N_R. \tag{2.19}$$

Both the sum and difference of L_0 and $\bar{L}_0$ annihilate the physical states:

$$(L_0 + \bar{L}_0 - 2\alpha)|\Psi\rangle = 0 \quad , \quad (L_0 - \bar{L}_0)|\Psi\rangle = 0. \tag{2.20}$$

The condition $(L_0 - \bar{L}_0)|\Psi\rangle = 0$ is called the level matching condition. In order to understand the physical interpretation of our theory we must obtain the mass spectrum. The mass operator M^2 comes by using the relation $M^2 = -p^\mu p_\mu$ when acting on states with the Virasoro operators. Taking to consideration that the absence of negative norm states requires that $\alpha = 1$, then the mass operator is given by

$$M^2 = -p^\mu p_\mu = \frac{2}{\alpha'}(N_L + N_R - 2). \tag{2.21}$$

The ground state of the closed bosonic string $|0,k\rangle$ is found when $N_L = N_R = 0$:

$$M^2 = -\frac{4}{\alpha'}. \tag{2.22}$$

As we can see this is a tachyon, meaning that the bosonic string admits an unstable vacuum and thus cannot be a physical theory. The first excited state, though, consists of the graviton, implying that a modification of the theory could lead to a consistent theory of quantum gravity.

Another way of string quantisation is the light-cone quantisation. As the name implies this method utilises the light-cone coordinates for space-time:

$$X^{\pm} = \frac{1}{\sqrt{2}}(X^0 \pm X^{D-1}). \tag{2.23}$$

The former definition seperates the D space-time coordinates X^μ into the null coordinates $X^\pm$ and the $D-2$ transverse coordinates X^i. Since the two sets of coordinates are treated differently Lorentz invariance is no longer manifest. In the light-cone gauge we choose

$$X^+ = \chi^+ + l_s^2 p^+ \tau, \tag{2.24}$$

which corresponds to setting $\alpha_n^+ = 0$ for $n \neq 0$.

The mass formula is given in terms of the transverse oscillators:

$$M^2 = 2p^+p^- - p^ip_i = \frac{2}{\alpha'}(N + \tilde{N} - 2\alpha), \tag{2.25}$$

where N and $\tilde{N}$ are defined by

$$N = \sum_{n=1}^{+\infty} \alpha_{-n}^i \alpha_{n\,i} \quad , \tag{2.26}$$

$$\tilde{N} = \sum_{n=1}^{+\infty} \tilde{\alpha}_{-n}^i \tilde{\alpha}_{n\,i} \quad , \tag{2.27}$$

and α is a normal ordering constant satisfying the relation:

$$\alpha = \frac{D-2}{24}. \tag{2.28}$$

The former relation is obtained by using the so called ζ-function regularisation. The dimension of the theory, $D = 26$, and the value of the normal ordering constant, $\alpha = 1$, follow from the closure of the Lorentz algebra in the light-cone gauge.

The lower excited states are the massless states:

$$\sum_{1\leq i,j\leq D-2} R_{i,j}\, a^i_{-1}\tilde{a}^j_{-1}|k\rangle, \tag{2.29}$$

These states can be split into three sets:

- $$\sum_{1\leq i,j\leq D-2} S_{i,j}\, a^i_{-1}\tilde{a}^j_{-1}|k\rangle, \tag{2.30}$$

  where the matrix $S_{i,j}$ is symmetric and traceless. These states correspond to the one-particle graviton states.

- $$\sum_{1\leq i,j\leq D-2} A_{i,j}\, a^i_{-1}\tilde{a}^j_{-1}|k\rangle, \tag{2.31}$$

  where the matrix $A_{i,j}$ is antisymmetric. These states correspond to the one-particle states of a Kalb-Ramond field, which is an antisymmetric tensor field $B_{\mu\nu}$.

- $$S\, a^i_{-1}\tilde{a}^i_{-1}|k\rangle, \quad (i \text{ summed from } 1 \text{ to } D-2), \tag{2.32}$$

  where S is a constant coming from the decomposition of $R_{i,j}$ into a symmetric traceless part, an antisymmetric part and a part proportional to the identity matrix. This state corresponds to the dilaton.

2.1.2 Compactification

In the case where we compactify one dimension of the theory on a circle [14, 15, 16, 17, 18] we separate the fields into uncompactified fields X^μ_L, X^μ_R, $\mu = 0, ..., D-2$ which "live" on a $M^{1,D-1}$ space-time and the compactified fields X^I_L, X^I_R, $I = D-1$ which "live" on an internal space K.

The equations of motion of the compactified fields have an analogous form with the uncompactified ones. The compactified fields obey the condition $X^I = X^I + 2\pi R$, where R is considered to be the compactification radius. This condition, along with the fact that momentum eigenstates are single valued, yields the following relation for the internal momentum quantum number m : $p^I + \bar{p}^I = \frac{m}{R}$. The compact coordinates obey the following relation:

$$X^I(\tau, \sigma + 2\pi) = X^I(\tau, \sigma) + 2\pi R n, \tag{2.33}$$

where n is the winding number (integer) which tells us how many times the closed string winds around the compact dimension of radius R. The quantity $(p^I - \bar{p}^I)$ is called the winding contribution. After compactifying the 26th dimension in the Kaluza-Klein fashion i.e on a circle, the mass formula becomes: $\alpha' m^2 = (\frac{nR}{\alpha'})^2 + (\frac{K}{R})^2 + 2(N_R + N_L) - 4$ and the level matching condition becomes $N_R - N_L = nK$.

2.1.3 One-Loop Bosonic Amplitude and Modular Invariance

All the possible closed string states can be found by studying loop diagrams. The one-loop string amplitude corresponds to a world-sheet with the topology of the torus, so we need to integrate over all the metrics

$$g_{\alpha\beta} = \frac{\alpha}{\tau_2} \begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix},$$

where the Teichmuller parameter $\tau \in \mathbb{C}$, $Im\tau > 0$ parametrises a family of conformally inequivalent tori, and $\alpha > 0$ is a continuous real parameter. Since the torus satisfies the identifications

$$z \equiv z + 2\pi \quad \text{and} \quad z \equiv z + 2\pi\tau \tag{2.34}$$

it can be expressed as a parallelogram. Tori parametrised by τ are considered to be conformally equivalent if they are connected through the so called modular

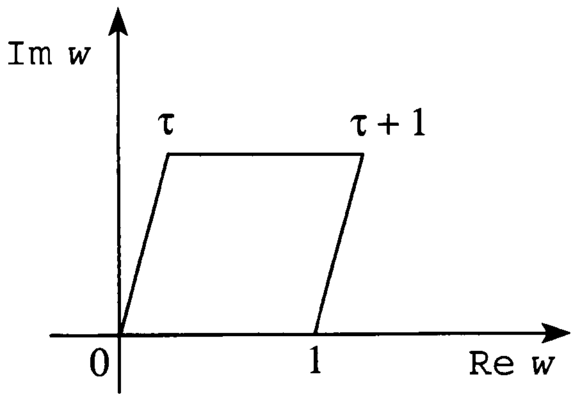

Figure 2.2: *The torus as a parallelogram*

transformations

$$T : \tau \longrightarrow \tau + 1 \quad \text{reparametrises the torus,} \tag{2.35}$$

$$S : \tau \longrightarrow -1/\tau \quad \text{reorients the torus, i.e swaps } \sigma^1 \text{ with } \sigma^2. \tag{2.36}$$

The modular group can be re-written more compactly as:

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d}, \quad a, b, c, d \in \mathbb{Z},\ ab - cd = 1. \tag{2.37}$$

The moduli space M is therefore given by $M \cong \mathbb{H}^+/PSL(2; Z)$, where $\mathbb{H}^+$ is the upper half of the complex τ plane. The fundamental domain

$$|\tau| \geq 1 \quad , \quad Re\tau \in [-\frac{1}{2}, +\frac{1}{2}) \quad , \tag{2.38}$$

determines the values of τ for which we get inequivalent tori.

The total one-loop partition function requires the integration over all the independent values of τ, i.e over the fundamental domain D:

$$\int_D \frac{d\tau d\bar{\tau}}{Im(\tau)^2} \tag{2.39}$$

where $\frac{d\tau d\bar{\tau}}{Im(\tau)^2}$ is a modular invariant quantity. In analogy with Quantum Mechanics, where a given amplitude is given by $\sum_n \langle n|e^{-iHt}|n\rangle = Tr(e^{-iHt})$, the

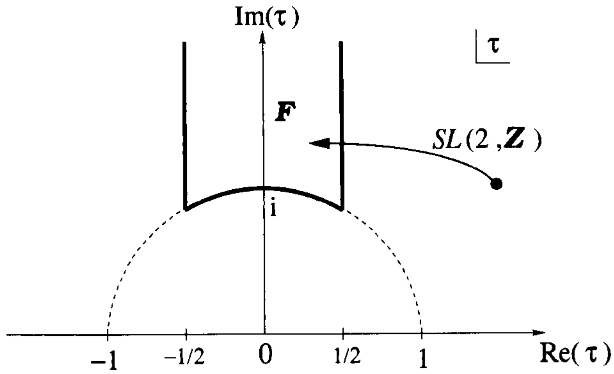

Figure 2.3: *The fundamental domain*

vacuum to vacuum amplitude for a rectangular torus is

$$Z[\tau] = Tr(e^{-2\pi(Im\tau)H}). \tag{2.40}$$

In the generic case for which Re $\tau \neq 0$, and after defining $q = e^{2\pi\tau}$ and $\bar{q} = e^{-2\pi i\bar{\tau}}$, the partition function becomes

$$Z[\tau] = Trq^{L_0-c/24}\bar{q}^{\bar{L}_0-\bar{c}/24}. \tag{2.41}$$

Taking in consideration the Dedekind Eta function $\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ and its modular transformations

$$\eta(\tau+1) = e^{i\pi/12}\ \eta(\tau) \quad , \qquad \eta(-\frac{1}{\tau}) = \sqrt{-i\tau}\ \eta(\tau) \quad , \tag{2.42}$$

we get that the one-loop, modular invariant, vacuum to vacuum amplitude for a free boson is

$$Z[\tau] = \frac{1}{\sqrt{\tau_2}}\frac{1}{|\eta(\tau)|^2}. \tag{2.43}$$

Compactification on a circle changes the partition function into

$$Z_{circle}[\tau] = \frac{1}{|\eta(\tau)|^2}\sum_{m,n} q^{\frac{1}{2}(\frac{m}{R}+\frac{Rn}{2})^2}\bar{q}^{\frac{1}{2}(\frac{m}{R}-\frac{Rn}{2})^2}, \tag{2.44}$$

where the contibution from $\frac{1}{|\eta(\tau)|^2}$ is due to bosonic oscillators acting on the vacuum and the term $q^{\frac{1}{2}(\frac{m}{R}+\frac{Rn}{2})^2}$ is due to the Kaluza-Klein and winding modes.

2.2 Classical Superstrings

The bosonic string contains 26 bosonic fields and no fermions, therefore only forces and no matter. Also, the lowest energy state is a tachyon, a particle mode with negative mass squared, which means that the vacuum state of the theory is unstable. The former problems are remedied by extending the bosonic string to the superstring. After including D free fermionic terms Ψ^μ to the Ramond-Neveu-Schwarz (RNS) action it takes the form

$$S = -\frac{T}{2}\int_\Sigma d^2\sigma\,(\partial_\alpha X^\mu \partial^\alpha X_\mu + i\,\bar{\Psi}^\mu \rho^\alpha \partial_\alpha \Psi_\mu) \tag{2.45}$$

where X^μ are the bosonic fields, Ψ^μ are the Majorana spinors $\Psi = \binom{\Psi_-}{\Psi_+}$ and ρ^α are the two dimensional Dirac matrices on the world-sheet

$$\rho^0 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \rho^1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{2.46}$$

This action is invariant under specific global supersymmetric transformations

$$\delta X^\mu = \bar{\epsilon}\Psi^\mu \tag{2.47}$$

$$\delta \Psi^\mu = -i\rho^\alpha \partial_\alpha X^\mu \epsilon\,, \tag{2.48}$$

where we have introduced the SUSY transformation parameter $\epsilon = \binom{\epsilon_-}{\epsilon_+}$ which is a Majorana spinor. The components of ϵ are Grassman numbers. Varying the RNS action yields the equations of motion. In terms of light-cone coordinates $\sigma^\pm = \tau \pm \sigma$, X^μ and Ψ^μ obey the following relations:

$$\partial_+\partial_- X^\mu = 0 \tag{2.49}$$

$$\partial_+\Psi^\mu_- = \partial_-\Psi^\mu_+ = 0. \tag{2.50}$$

As in the purely bosonic case, the general solution for the former equations is a decomposition into left and right movers. Equations of motion also imply the decoupling between the left and right-moving world-sheet fields Ψ^μ: $\Psi_+ = \Psi_+(\sigma^+)$ and $\Psi_- = \Psi_-(\sigma^-)$. The conserved currents associated with the RNS action are are the result of translational invariance and supersymmetry transformations. Translational invariance yields momentum

$$P^\mu_\alpha = T\partial_\alpha X^\mu. \quad (2.51)$$

Invariance under supersymmetric transformations yields the supercurrent

$$J^\mu_\alpha = \frac{1}{2}\rho^\beta\rho_\alpha\Psi^\mu\partial_\beta X_\mu. \quad (2.52)$$

Variation of the RNS action with respect to the world-sheet metric yields the energy-momentum tensor

$$T_{\alpha\beta} = \partial_\alpha X^\mu\partial_\beta X_\mu + \frac{i}{4}\bar{\Psi}^\mu\rho_\alpha\partial_\beta\Psi_\mu + \frac{i}{4}\bar{\Psi}^\mu\rho_\beta\partial_\alpha\Psi_\mu. \quad (2.53)$$

The energy-momentum tensor and the supercurrent can be written more compactly using worldsheet light-cone coordinates:

$$T_{++} = \partial_+X_\mu\partial_+X^\mu + \frac{i}{2}\Psi^\mu_+\partial_+\Psi_{+\mu} \quad (2.54)$$

$$T_{--} = \partial_-X_\mu\partial_-X^\mu + \frac{i}{2}\Psi^\mu_-\partial_-\Psi_{-\mu} \quad (2.55)$$

$$J_+ = \Psi^\mu_+\partial_+X_\mu \quad (2.56)$$

$$J_- = \Psi^\mu_-\partial_-X_\mu. \quad (2.57)$$

2.2.1 Mode Expansions and Boundary Conditions

The vanishing of boundary terms when varying the RNS action with respect to Ψ^μ imposes that fermionic coordinates can take either periodic or antiperiodic boundary conditions.

- The Ramond (or R) boundary conditions, which are periodic:

$$\psi^\mu_\pm(\sigma^0, \sigma^1 = \pi) = \psi^\mu_\pm(\sigma^0, \sigma^1 = 0), \quad (2.58)$$

- The Neveu-Schwarz (or NS) boundary conditions, which are antiperiodic:

$$\psi^{\mu}_{\pm}(\sigma^0, \sigma^1 = \pi) = -\psi^{\mu}_{\pm}(\sigma^0, \sigma^1 = 0). \tag{2.59}$$

Then the equations of motion lead to the following mode expansions :

$$\psi^{\mu}_{+} = \sum_{n \in \mathbb{Z}} d^{\mu}_{n} e^{-2in\sigma^{+}}, \quad \text{for R boundary conditions,} \tag{2.60}$$

$$\psi^{\mu}_{+} = \sum_{r \in \mathbb{Z}+\frac{1}{2}} b^{\mu}_{r} e^{-2ir\sigma^{+}}, \quad \text{for NS boundary conditions,} \tag{2.61}$$

and similar expressions for right-moving fermion fields, providing another set of mode-oscillators $\tilde{d}^{\mu}_{n}, \tilde{b}^{\mu}_{r}$.

2.2.2 Canonical Quantisation and Spectrum

The next step is to quantise the theory, which is done by assuming the canonical bosonic commutation and fermionic anti-commutation relations. The bosonic coordinates are quantised using the same procedure described in (2.8), which in turn yields the relations (2.9). In order to quantise the world-sheet fermion fields we impose the equal time anti-commutation relations

$$\{\Psi^{\mu}_{\alpha}(\sigma, \tau), \Psi^{\nu}_{\beta}(\sigma', \tau)\} = \pi \eta^{\mu\nu} \delta_{\alpha\beta} \delta(\sigma - \sigma'), \tag{2.62}$$

which lead to

$$\{d^{\mu}_{m}, d^{\nu}_{n}\} = \eta^{\mu\nu} \delta_{m+n,0}, \quad \text{in the R-sector,} \tag{2.63}$$

$$\{b^{\mu}_{r}, b^{\nu}_{s}\} = \eta^{\mu\nu} \delta_{r+s,0}, \quad \text{in the NS-sector.} \tag{2.64}$$

Once again this theory includes negative-norm states which will have to be removed using the Super-Virasoro generators. The Super-Virasoro operators are generalisations of the Virasoro operators which are extended by the inclusion of a Virasoro operator which is associated with fermionic oscillators:

$$L_m \longrightarrow L_m^{(B)} + L_m^{(F)} \, . \tag{2.65}$$

In addition to the former generator we get generators arising from the supercurrent:

$$G_r = \sum \alpha_m . b_{r+m} \quad \text{in the NS sector} \tag{2.66}$$

$$F_m = \sum \alpha_{-n} . d_{m+n} \quad \text{in the R sector.} \tag{2.67}$$

In order to obtain the open superstring spectrum we have the freedom to act with periodic or anti-periodic fermionic oscillators on the vacum, yielding the so called Ramond and Neveu-Schwarz sectors. The NS sector has a unique vacuum state and all excited states are obtained by applying creation operators $\alpha^{\mu}_{-n}, b^{\mu}_{-r}$ with positive half-integer frequencies, and they describe bosons. The R-states are obtained by applying creation operators $\alpha^{\mu}_{-n}, d^{\mu}_{-m}$ with integer frequencies on the degenerate ground state. The R-ground states are obtained by the zero mode oscillators $\alpha^{\mu}_0, d^{\mu}_0$ acting on the vacuum. The zero modes d^{μ}_0 commute with the Hamiltonian and therefore do not change the energy of the vacuum state. Furthermore they satisfy the commutation relations of gamma matrices and thus realise a spinor representation of the Lorentz group. Therefore they constitute a degeneracy which is interpreted as space-time spin and hence the vacuum and excited R-states are fermions.

Since in the case of the closed superstring the boundary conditions for the fermions are chosen independently for left and right movers we get bosonic or fermionic states according to the following cases:

NS–NS	Boson	$\otimes$	Boson	$=$	Boson,
NS–R	Boson	$\otimes$	Fermion	$=$	Fermion,
R–NS	Fermion	$\otimes$	Boson	$=$	Fermion,
R–R	Fermion	$\otimes$	Fermion	$=$	Boson.

2.2.3 The GSO Projection

The spectrum admits an imaginary mass (tachyon) indicating that the vacuum is unstable. The "unphysical" states of the theory are truncated using the Gliozzi-Scherk-Olive (GSO) projection. In the NS sector we keep states with an odd number of fermion excitations and reject states with an even number of fermion excitations. This is done by defining a fermion number operator

$$F = \sum b_{-r}.b_r \ . \tag{2.68}$$

We then define a parity operator which determines the states that we can have in the theory and removes the tachyon:

$$P_{NS} = \frac{1}{2}[1 - (-1)^F] \ . \tag{2.69}$$

In the R sector we define the Klein operator which is given by

$$(-1)^F = \pm\Gamma^{11} \ , \tag{2.70}$$

for massless states. The operator $\Gamma^{11} = \Gamma^0\Gamma^1......\Gamma^9$ is a 10D chirality operator which acts on spinors, defining whether they have positive or negative chirality:

$$\Gamma^{11}\psi = \pm\psi \ . \tag{2.71}$$

For the GSO projections of the R sector, we use the following projection operator:

$$P_{\pm} = \frac{1}{2}[1 \mp \Gamma_{11}(-1)^F] \ . \tag{2.72}$$

It should be noted that when one realises the spectrum from a partition function, the GSO projection arises as a requirement of modular invariance.

Chapter 3

Heterotic Strings in Ten and Four Dimensions

THIS CHAPTER briefly reviews the conceptual foundations and spectrum of the heterotic string, and then exhibits aspects of the orbifold and fermionic compactifications. Our discussion aims to introduce the direct formalism of the heterotic string in four dimensions that in turn will lead to four dimensional model building.

3.1 The $E_8 \times E_8$ Heterotic String

Up until now we have studied string theory where world-sheet supersymmetry was realised for both the left and right movers. The heterotic string includes only left fermionic worldsheet fields, thus realising supersymmetry only among the left movers, and it therefore combines characteristics from both the bosonic string and the superstring. The heterotic string in ten dimensions [19] requires that the 16 mismatched dimensions, also known as gauge degrees of freedom, $I = 1, .., 16$, are compactified on a 16-torus which is defined by an even and self-dual lattice Λ. The lattice Λ is spanned by a set of basis vectors $e^I_\alpha, \alpha = 1, ..., 16$ and where the coordinates obey the following relation:

$$X^I \sim X^I + 2\pi \mathcal{L}^I, \quad \mathcal{L} \in \Lambda, \tag{3.1}$$

where $\mathcal{L}$ denotes the winding.

The states of the theory are obtained by acting on the vacuum with the tensor product of the left quantised movers with the right quantised movers. The left massless states in the lightcone gauge are given by the NS fermionic oscillators $b^i_{-1/2}$ or the R fermionic zero modes d^i_0 acting on the vacuum. The right massless states include $|p^I\rangle_R$ which carry momentum but have no oscillators acting on the vacuum, and states where the oscillators $\tilde{\alpha}^i_{-1}$ or $\tilde{\alpha}^I_{-1}$ act on the vacuum $|0\rangle_R$. Physical states are constrained by the mass-shell constraint

$$M_L^2 = M_R^2 \,, \tag{3.2}$$

which also leads to the absence of tachyonic states. That is because the only left mover with negative M_L^2 is the NS ground state with $M_L^2 = -2$ whereas the only right-mover with negative M_R^2 is the bosonic string state with $M_R^2 = -4$. Combining the right and left quantised massless movers, the following states arise:

- The supergravity $D = 10$, $N = 1$ multiplet

$$b^i_{-1/2}|0\rangle_L \otimes \ \tilde{\alpha}^j_{-1}|0\rangle_R \ \ , \tag{3.3}$$

  and the corresponding supersymmetric partners:

$$|q_s\rangle_L \otimes \ \tilde{\alpha}^j_{-1}|0\rangle_R \ . \tag{3.4}$$

- Gauge bosons

$$b^i_{-1/2}|0\rangle_L \otimes \ \tilde{\alpha}^I_{-1}|0\rangle_R \,, \quad b^i_{-1/2}|0\rangle_L \otimes \ |p^I\rangle_R \,, \quad p^2 = 2 \tag{3.5}$$

  and their corresponding gauginos

$$|q_s\rangle_L \otimes \ \tilde{\alpha}^I_{-1}|0\rangle_R \,, \quad |q_s\rangle_L \otimes \ |p^I\rangle_R \,, \quad p^2 = 2 \,. \tag{3.6}$$

Here $|q_s\rangle_L$ denotes the spinor representation 8_s of $SO(8)$ and is given by $q_s = (\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$ with an even number of plus signs. Also $i, j = 1, ..., 8$ and $I = 1, ..., 16$ for all of the previous states.

3.2 Compactification on a Six-Torus

The connection of string theory with the real world is widely believed to be achieved by choosing the ten dimensional space as $M_{10} = M_{3,1} \times M_6$. In other words the six extra dimensions are compactified onto a six-dimensional internal space, the dimensions of which are small enough to have escaped detection. At the moment the geometry and dimensions of the internal manifold are constrained by generic phenomenological guidelines and mathematical consistency rather than a strict principle. One of the earliest compactification schemes considers the background manifold to be a flat torus [20].

Due to its periodicities the six dimensional torus is defined as the quotient space $T^6 \equiv \mathbb{R}^6/\Gamma$ where $\Gamma = \{n_\alpha e_\alpha, n_\alpha \in \mathbb{Z}, \alpha = 1, .., 6\}$ is the lattice spanned by the basis vectors e_α. Although simple and exactly solvable, this compactification scenario is not realistic because the obtained four dimensional spectrum yields $N = 4$ supersymmetry. In order for the toroidal compactification scenario to become more realistic we impose finite symmetries on the toroidal lattice, thus defining an orbifold [21, 22].

3.3 $\mathbb{Z}_2 \times \mathbb{Z}_2$ Orbifold

We take the case where the six-dimensional torus lattice is the product of three two-tori: $T^6 = T^2_{(1)} \times T^2_{(3)} \times T^2_{(5)}$. Each two dimensional torus $T^2_{(\alpha)} = \mathbb{R}^2/\Gamma_{(\alpha)}$ is spanned by $\Gamma_{(\alpha)} = \{n_i e_i | i = \alpha, \alpha+1\}$ where $\alpha = 1, 3, 5$. We consider each complexified pair $Z^\alpha = [X^{(2\alpha-1)} + iX^{(2\alpha)}]$ of the the six extra dimensions to be compactified on the analogus periodic complex plane, denoted by $\mathbb{C}_\alpha$.

By defining $\mathbb{Z}_2 \times \mathbb{Z}_2$ as the point group P, we are actually introducing rotations to pairs of the three complex planes (Z^1, Z^2), (Z^2, Z^3), (Z^1, Z^3) which correspond to the compactified directions. The complex plane Z^0, which corresponds to the uncompactified transverse coordinates, remains unchanged under the orbifold group action. Each element (twist) of the point group generates rotations to a different pair of complex planes.

The twists [23] of the point group are denoted as $\{\mathbb{I},\ \theta_1,\ \theta_2,\ \theta_1 \circ \theta_2\}$.

The two generators of the group $\theta_1 = diag(e^{i2\pi u_1^1}, e^{i2\pi u_1^2}, e^{i2\pi u_1^3})$ and $\theta_2 = diag(e^{i2\pi u_2^1}, e^{i2\pi u_2^2}, e^{i2\pi u_2^3})$, are associated with the twist vectors $u_1 = (0, \frac{1}{2}, -\frac{1}{2}, 0)$ and $u_2 = (0, 0, \frac{1}{2}, -\frac{1}{2})$. The twists can then be considered as the results of linear combinations of the twist vectors

$$\begin{aligned} \mathbb{I} &\Leftrightarrow 0u_1 + 0u_2 \\ \theta_1 &\Leftrightarrow 1u_1 + 0u_2 \\ \theta_2 &\Leftrightarrow 0u_1 + 1u_2 \\ \theta_1 \circ \theta_2 &\Leftrightarrow 1u_1 + 1u_2. \end{aligned} \tag{3.7}$$

Each point of the internal space is subjected to symmetries due to the toroidal periodicities $n_\alpha e_\alpha$ and the twists $ku_1 + lu_2$ $(k, l = 0, 1)$ imposed by orbifolding the internal manifold. We define the space group $S = \{ku_1 + lu_2,\ n_\alpha e_\alpha \mid k,\ l = 0,\ 1 \ \ n_\alpha \in \mathbb{Z}\}$ as the combination of the point group and the translations associated with Γ. The elements of the space group for which $k = l = 0$, i.e just lattice translations, constitute the so called untwisted sector while for any other values of k and l we have the so called twisted sectors. The strings of the untwisted sector are closed before twisting, whereas for the twisted sectors the closed string boundary conditions are only satisfied upon the action of a point group element.

The action of the space group S is extended to an action on the 16 gauge degrees of freedom. Elements of the space group S are then mapped into elements of the gauge twisting group G as follows:

$$\mathbb{Z}_2 \times \mathbb{Z}_2 : (\theta_1^k \circ \theta_2^l, n_\alpha e_\alpha) \mapsto (kV_1 + lV_2, n_\alpha A_\alpha) \ , \tag{3.8}$$

where V_i are the shift vectors, and A_α represent Wilson lines [24]. Wilson lines are crucial for phenomenology since they project out gauge bosons and gauginos. An element of the gauge twisting group acts on the 16 gauge degrees of freedom as follows:

$$\mathbb{Z}_2 \times \mathbb{Z}_2 : X^I \to X^I + kV_1^I + lV_2^I + n_\alpha A_\alpha^I, \quad I = 1, ..., 16. \tag{3.9}$$

The $\mathbb{Z}_2 \times \mathbb{Z}_2$ toroidal orbifold is then the quotient space $\mathbb{O}_{\mathbb{Z}_2\times\mathbb{Z}_2} \equiv \mathbb{R}^6/S \times T^{16}/G$, and the elements of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold are given by

$$\mathbb{Z}_2 \times \mathbb{Z}_2 : (\theta_1^k \circ \theta_2^l, n_\alpha e_\alpha; kV_1 + lV_2, n_\alpha A_\alpha) \in S \times G. \tag{3.10}$$

The surviving states of the orbifold theory are $S \otimes G$ invariant, i.e they remain invariant under the action of the orbifold elements.

Although orbifolding techniques are a powerful tool for the exploration of string vacua, we will not give any more details because these techniques were not used for the purpose of this thesis. Nevertheless, the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold is considered to be directly related to free fermionic compactifications [25, 26] and it can provide useful intuition concerning these algebraic constructions.

3.4 The Free Fermionic Formulation of the Heterotic String

The bosonic formulation of the heterotic string requires that the extra dimensions and gauge degrees of freedom are compactified on some manifold. This section introduces an alternative formulation of the heterotic string, where all the extra dimensions and gauge degrees of freedom are represented as two-dimensional free fermions propagating on the string world-sheet. This formulation is a result of the equivalence between worldsheet bosons and fermions.

The Operator Product Expansions (OPEs) between bosonic operators $e^{\pm iX(z)}$ are

$$\begin{aligned} e^{iX(z)}e^{-iX(0)} &= \frac{1}{z} + O(z), \\ e^{iX(z)}e^{iX(0)} &= O(z), \\ e^{-iX(z)}e^{-iX(0)} &= O(z), \end{aligned} \tag{3.11}$$

while the OPEs among two complexified Majorana-Weyl fermions

$\psi = \frac{1}{\sqrt{2}}(\psi^1 + i\psi^2)$ and $\bar{\psi} = \frac{1}{\sqrt{2}}(\psi^1 - i\psi^2)$ are

$$\begin{aligned} \psi(z)\bar{\psi}(0) &= \frac{1}{z} + O(z), \\ \psi(z)\psi(0) &= O(z), \\ \bar{\psi}(z)\bar{\psi}(0) &= O(z). \end{aligned} \tag{3.12}$$

Comparing the two equations the equivalence between bosons and fermions in a two dimensional conformal field theory becomes obvious:

$$\psi(z) \cong e^{iX(z)}, \qquad \bar{\psi}(z) \cong e^{-iX(z)}. \tag{3.13}$$

The heterotic string action can be written in a form that utilises the equivalence between bosons and fermions. In this formulation the 16 extra gauge degrees of freedom, can be reinterpreted as fermionic fields. The heterotic string only considers the following left-moving space-time fields:

$$X_+^\mu(z), \quad \psi_+^\mu(z), \quad \mu = 0, ..., 9 \ , \tag{3.14}$$

and the following right-moving fields:

$$X_-^\mu(\bar{z}), \quad \lambda_-^\alpha(\bar{z}), \quad \mu = 0, ..., 9, \ \alpha = 1, ..., 32. \tag{3.15}$$

The ten dimensional heterotic action then realises supersymmetry only among the left movers:

$$S = -\frac{T}{2}\int d^2\sigma[\sum_{\mu}^{9}(\partial_\alpha X_\mu \partial^\alpha X^\mu - 2i\psi_+^\mu \partial_- \psi_+^\mu) - 2i\sum_{\alpha=1}^{32} \lambda_-^\alpha \partial_+ \lambda_-^\alpha]. \tag{3.16}$$

Provided that the boundary conditions for λ_-^α are all the same, meaning that all of the internal fermions are either periodic or anti-periodic, we obtain the $SO(32)$ heterotic string. A different choice for the boundary conditions of the internal fermions, where 16 of them are chosen to be periodic and 16 of them are chosen to be anti-periodic, leads to having two additional sectors in our theory and yields the $E_8 \times E_8$ heterotic string. At this point it is worth noting that the normal-ordered product between two internal fermions constitutes a world-sheet current J. The OPE between two such currents is given by

$$J^A(z_1)J^B(z_2) = \frac{k\delta^{AB}}{(z_1 - z_2)^2} + i\frac{f^{ABC}J^C}{(z_1 - z_2)} , \tag{3.17}$$

where f^{ABC} are the structure constants of the $SO(32)$ or the $E_8 \times E_8$ gauge group, and k is the level of the algebra.

Since we are interested in a direct 4D formulation of the heterotic string we restrict the space-time index to run only from 0 to 3, and we consider the rest of the dimensions to be extra internal degrees of freedom [27]. Each of the left and right moving free fermions contribute to the conformal anomaly by $\frac{1}{2}$, and therefore the conformal anomaly in four dimensions ($D_L = D_R = 4$) vanishes:

$$C_L = -26 + 11 + D_L + \frac{D_L}{2} + \frac{18}{2} \quad , \qquad C_R = -26 + D_R + \frac{44}{2}. \tag{3.18}$$

A vanishing conformal anomaly is equivalent to eliminating the negative norm states in the covariant quantisation, and closing the Lorentz algebra in light-cone gauge.

At this point we are left with 74 fields in our theory:

$$\begin{aligned} X^\mu(z,\bar{z}), &\quad \mu = 0, ..., 3 \\ \bar{X}^\mu(z,\bar{z}), &\quad \mu = 0, ..., 3 \\ \psi^\mu(z), &\quad \mu = 0, ..., 3 \\ \lambda^i(z), &\quad i = 1, ..., 18 \\ \bar{\lambda}^j(\bar{z}), &\quad j = 1, ..., 44, \end{aligned} \tag{3.19}$$

where $\lambda(z), \lambda(\bar{z})$ denote real world-sheet fermions. After imposing the light-cone gauge we are left with a total of 68 fields, since only the transverse coordinates of X^μ, $\bar{X}^\mu$ and Ψ^μ are kept.

Chapter 4

Four-Dimensional String Models

IN ORDER to provide insights on certain phenomenological aspects, we follow an approach where we consider that the theory which effectively describes our low energy universe is a SUSY GUT which also incorporates gravity. For this purpose we study $k = 1$ heterotic string models [28] where the SO(10) GUT gauge group is further broken by Wilson lines. This chapter introduces the construction constraints and analysis techniques of semi-realistic Free Fermionic Models [29, 30], and finally exhibits how a specific model can inspire a phenomenologically viable scenario.

4.1 Rules of String Model Building

Models in the free fermionic formulation [31, 32, 33] are constructed by specifying a finite set of spin structures [34], or else boundary condition vectors (b.c vectors):

$$\vec{b}_i = \{\alpha(\psi^1), \alpha(\psi^2), \alpha(\lambda^1), ..., \alpha(\lambda^{18}) | \alpha(\bar{\lambda}^1), ..., \alpha(\bar{\lambda}^{44})\} \ , \quad i \in \mathbb{N}_0 \backslash \{0\} \ , \quad (4.1)$$

and their corresponding generalised GSO (GGSO) coefficients $C\binom{\vec{\mathbf{b}}_i}{\vec{\mathbf{b}}_j}$. Each boundary condition vector is a spin structure containing the phases $\alpha(f)$ picked up by world-sheet fermions when parallel transported along the torus' non-contractible loops:

$$f \longrightarrow -e^{i\pi\alpha(f)} f \ , \quad (4.2)$$

where f is a fermionic field and the minus sign is conventional. When two fermion fields λ^i, λ^j have the same set of boundary conditions in every spin structure they are combined into a complex fermion:

$$\lambda_{ij} = \frac{1}{\sqrt{2}}(\lambda^i + i\lambda^j)\ , \quad \text{or to its conjugate} \quad \lambda^*_{ij} = \frac{1}{\sqrt{2}}(\lambda^i - i\lambda^j)\ . \tag{4.3}$$

The low-energy properties of each model depend on the values of the GGSO coeffiecients and the phases picked up by the following fermionic fields:

$\psi^\mu_{1,2}$	Correspond to the superpartners of the transverse uncompactified coordinates.
$\bar{\Psi}^{1...5}$	Correspond to fermionised gauge degrees of freedom. Define the underlying Observable $SO(10)$ symmetry.
$\bar{\Phi}^{1...8}$	Correspond to fermionised gauge degrees of freedom. Define the Hidden symmetry.
$\bar{\eta}^{1...3}$	Correspond to fermionised gauge degrees of freedom. Define the local abelian symmetries related to a family.
$\{\chi^I, y^I, \omega^I\}, I = 1,..,6$	The fermionised left moving compactified dimensions and their superpartners.
$\{\bar{y}^I, \bar{\omega}^I\}, I = 1,...,6$	The fermionised right moving compactified dimensions.

Linear combinations of basis vectors' spin structures give rise to sectors which contain the physical states of the theory. The basis vectors then form a finite additive group Ξ through the operation

$$\vec{\alpha} + \vec{\beta} = \{\vec{\alpha}(f_1) + \vec{\beta}(f_1), ..., \vec{\alpha}(f_{64}) + \vec{\beta}(f_{64})\}\ , \tag{4.4}$$

where $\vec{\alpha}(f_i) + \vec{\beta}(f_i), \forall\ i = 1, ..., 64$ denotes the addition between the ith component of the spin structures $\vec{\alpha}, \vec{\beta}$, and the underlying algebra is the result of (4.2) :

$$0 + 0 = 0, \quad 0 + 1 = 1 + 0 = 1, \quad 1 + 1 = 0. \tag{4.5}$$

Any two given spin structures $\vec{\alpha}$, $\vec{\beta}$ contribute to a modular invariant partition function

$$Z(\tau,\bar\tau) = \sum_{\text{spin structures } \vec\alpha,\vec\beta} C\binom{\vec\alpha}{\vec\beta} Z\binom{\vec\alpha}{\vec\beta}(\tau,\bar\tau), \tag{4.6}$$

where the GGSO coefficients $C\binom{\vec\alpha}{\vec\beta}$ are introduced in order to guarantee modular invariance. The choice of the GGSO coefficients admits some arbitrariness and plays a crucial phenomenological role. The total partition function $Z\binom{\vec\alpha}{\vec\beta}$ is the product of the partition functions associated to each fermion field:

$$Z\begin{bmatrix}0\\0\end{bmatrix} = \sqrt{\frac{\theta_3}{\eta}}, \qquad Z\begin{bmatrix}0\\1\end{bmatrix} = \sqrt{\frac{\theta_4}{\eta}}, \tag{4.7a}$$

$$Z\begin{bmatrix}1\\0\end{bmatrix} = \sqrt{\frac{\theta_2}{\eta}}, \qquad Z\begin{bmatrix}1\\1\end{bmatrix} = \sqrt{\frac{\theta_1}{\eta}}, \tag{4.7b}$$

where θ_i are the Jacobi theta functions

$$\theta[^a_b] = \sum_{n\in\mathbb{Z}} q^{\frac{(n-\frac{a}{2})}{2}} e^{2\pi i(-\frac{b}{2})(n-\frac{a}{2})}, \tag{4.8}$$

and where

$$\theta[^1_1] = \theta_1, \qquad \theta[^1_0] = \theta_2, \qquad \theta[^0_0] = \theta_3, \qquad \theta[^0_1] = \theta_4. \tag{4.9}$$

These formulae should be complex conjugated for the right-movers.

The consistency of each model is guaranteed by the so called Antoniadis-Bachas-Kounnas (ABK) rules which arise from the modular invariance of the partition function. The partition function for a four dimensional heterotic string is given by the expression(we now drop the arrow over the basis vectors):

$$Z = \int \left[\frac{d\tau d\bar\tau}{\tau_2^2}\right] Z_B^2 \sum_{\text{spin str.}} c\begin{bmatrix}\alpha\\\beta\end{bmatrix} \prod_{f=1}^{64} Z_F\begin{bmatrix}\alpha_f\\\beta_f\end{bmatrix}, \tag{4.10}$$

where the integration measure is modular invariant, Z_B^2 is the contribution of the two uncompactified transverse bosonic coordinates expressed in terms of the Dedekind eta function, and Z_F is the fermionic contribution which is expressed in terms of Jacobi theta functions. Under the modular transformation $\tau \longrightarrow \tau+1$, the Dedekind eta function and the Jacobi theta functions transform as shown below:

$$\eta \longrightarrow e^{\frac{i\pi}{12}}\eta \ ; \ \theta_1 \longrightarrow e^{\frac{i\pi}{4}}\theta_1 \ ; \ \theta_2 \longrightarrow e^{\frac{i\pi}{4}}\theta_2 \ ; \ \theta_3 \longleftrightarrow \theta_4, \tag{4.11}$$

while under $\tau \longrightarrow -\frac{1}{\tau}$

$$\eta \longrightarrow (-i\tau)^{1/2}\eta \ ; \ \frac{\theta_1}{\eta} \longrightarrow e^{-\frac{i\pi}{2}}\frac{\theta_1}{\eta} \ ; \ \frac{\theta_2}{\eta} \longleftrightarrow \frac{\theta_4}{\eta} \ ; \ \frac{\theta_3}{\eta} \longleftrightarrow \frac{\theta_3}{\eta}. \tag{4.12}$$

Modular invariance of the partition function leads to the following conditions:

$$C\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = e^{\frac{i\pi(\alpha.\alpha+1.1)}{4}}C\begin{pmatrix}\alpha\\ \beta-\alpha+1\end{pmatrix} \tag{4.13}$$

$$C\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = e^{\frac{i\pi(\alpha.\beta)}{2}}C\begin{pmatrix}\beta\\ -\alpha\end{pmatrix} \tag{4.14}$$

$$C\begin{pmatrix}\alpha\\ \beta\end{pmatrix}C\begin{pmatrix}\alpha'\\ \beta'\end{pmatrix} = \delta_\alpha\delta_{\alpha'}e^{\frac{-i\pi(\alpha.\alpha')}{2}}C\begin{pmatrix}\alpha\\ \beta+\alpha'\end{pmatrix}C\begin{pmatrix}\alpha'\\ \beta'+\alpha\end{pmatrix}, \tag{4.15}$$

where $\delta_\alpha = e^{i\pi\alpha(\psi^\mu_{1,2})}$ and the inner product between two vectors α, β is defined by

$$\alpha.\beta = \left(\frac{1}{2}\sum_{\text{left real fermions}} - \frac{1}{2}\sum_{\text{right real fermions}}\right)\alpha(f)\beta(f). \tag{4.16}$$

If we set $\alpha' = \gamma$ and $\beta' = -\alpha$ in (4.15), combine the result with (4.14) and normalise $C\binom{0}{0} = 1$ we get that $C\binom{\alpha}{0} = \delta_\alpha$. Next, we define a set of vectors Ξ

$$\Xi = \left\{\alpha \ \middle| \ c\begin{bmatrix}\alpha\\ 0\end{bmatrix} = \delta_\alpha\right\}. \tag{4.17}$$

It can be shown that this set is an abelian additive group, and that a vector which has all of its components periodic belongs to that group: $1 \in \Xi$. A finite abelian group is isomorphic to a direct sum of $\mathbb{Z}_N$ factors: $\Xi = \bigoplus_{i=1}^{k}\mathbb{Z}_{N_i}$. This means that there exists a basis $\{b_1, ..., b_k\}$ generating Ξ such that

$$\sum_{i=1}^{k} m_i b_i = 0 \iff m_i = 0 \mod N_i \ \ \forall i \ , \tag{4.18}$$

where N_i is the smallest positive integer for which $N_i b_i = 0$. By setting $\alpha = b_i$ and $\beta = b_j$ and raising both sides to N_{ij} (least common multiple of N_i and N_j) in (4.14), we obtain the relations $N_{ij} b_i b_j = 0 \bmod 4$ and $N_i b_i b_i = 0 \bmod 4$. In the case that N_i is even we get the condition $N_i b_i b_i = 0 \bmod 8$.

According to the ABK rules the GGSO phases between spin structures which only take periodic or anti-periodic boundary conditions are constrained to take the values 1 or -1. The rest of the ABK rules regarding the the GGSO phases for this specific case, are the following:

$$\begin{aligned} C\binom{\alpha}{\alpha} &= -e^{\frac{i\pi\alpha.\alpha}{4}} C\binom{\alpha}{1} \\ C\binom{\alpha}{\beta} &= e^{\frac{i\pi\alpha.\beta}{2}} C\binom{\beta}{\alpha} \\ C\binom{\alpha}{\beta+\delta} &= e^{i\pi\alpha(\psi^\mu)} C\binom{\alpha}{\beta} C\binom{\alpha}{\delta} \ . \end{aligned} \tag{4.19}$$

The ABK rules regarding b.c vectors where the entries only take the values 0 or 1 are the following:

$$\begin{aligned} \sum_i m_i b_i = 0 \quad & \text{if} \quad \forall i,\ m_i = 0 \, mod \, 2 \\ b_i.b_j \quad &= \quad 0 \, mod \, 2 \ \ \forall i,j \\ b_i.b_i \quad &= \quad 0 \, mod \, 4 \ \ \forall i \\ b_1 = \vec{1} \ \ & (\vec{1} \in \Xi) \ . \end{aligned} \tag{4.20}$$

4.2 The Spectrum

After the Fourier modes of each of the real or complex fermions are promoted into operators they can act on the vacuum to form the states of each model. Each sector $\alpha \in \Xi$ corresponds to a Hilbert space of states which is obtained by acting with raising operators on the vacuum. The state $\Psi^{\mu}_{-1/2}\bar{\Psi}^{1}_{-1/2}\bar{\Psi}^{3*}_{-1/2}|0\rangle_{NS}$, for example denotes a gauge boson which is obtained by acting with the NS raising operators of Ψ^{μ}, $\bar{\Psi}^{1}$ and $\bar{\Psi}^{3*}$ on the vacuum of the NS sector. When acting on the vacuum with zero modes we have to take into consideration that the anti-commutation relation for the fermionic modes ψ_n is given by $\{\psi_n, \psi_m\} = \delta_{n+m,0}$ which in turn gives $\{\psi_n, \psi_0\} = 0$. This means that the zero mode ψ_0 takes the ground state to another ground state. We label the "up" state of the degenerate vacuum, where no oscillator acts on the vacuum, as $|+\rangle$. In the case that a zero mode acts on the vacuum the "down" state of the degenerate vacuum is denoted as $|-\rangle$. The degenerate vacuum is denoted as $|\pm\rangle$ where $F_\alpha(|+\rangle) = 0$ and $F_\alpha(|-\rangle) = -1$ [35]. $F_\alpha(f)$ is the fermion number operator. In the case of a complex fermionic oscillator acting on the vacuum $F_\alpha(f) = 1$, and for a complex conjugate fermion $F_\alpha(f^*) = -1$. The initial spectrum of each model is truncated by a set of constraints on the physical states. The GGSO projection selects the surviving states $|s\rangle_\alpha$ of the sector α by satisfying the relation:

$$e^{i\pi b_i F_\alpha}|s\rangle_\alpha = \delta_\alpha C\binom{\alpha}{b_i}^{*}|s\rangle_\alpha \quad \forall i, \text{where} \ \ \delta_\alpha = e^{i\pi\vec{\alpha}(\psi^{\mu})} \ , \qquad (4.21)$$

and

$$b_i F_\alpha = \left(\sum_{\text{real + complex left}} - \sum_{\text{real + complex right}} \right) b_i(f) F_\alpha(f).$$

In order to obtain the massless states of each model we have to utilise the constraints

$$\begin{aligned} M_L^2 &= -\frac{1}{2} + \frac{\alpha_L.\alpha_L}{8} + \sum_{\text{left fermions}} N(f) \ , \\ M_R^2 &= -1 + \frac{|\alpha_R.\alpha_R|}{8} + \sum_{\text{right fermions}} \tilde{N}(f) \ , \\ M_L^2 &= M_R^2 = 0 \ , \end{aligned} \qquad (4.22)$$

where α_L and α_R are the parts of the vector α corresponding only to the left and right complex fermions respectively.

The simplest set of basis vectors we can choose is to have only the vector $\vec{1}$ in the basis. Then we have $\Xi = \{\vec{1}, \vec{0}\}$ and the physical states are found in only two sectors. The states in the $\vec{1}$ sector are massive, whereas in the NS-sector the ground state satisfies $M_L^2 = M_R^2 = -\frac{1}{2}$, yielding tachyonic states. These tachyonic states $\bar{\lambda}^j|0>_{NS}$ are obtained by acting on the vacuum $|0>_{NS}$ with one right fermionic creation operator. Then we have massless states obtained by acting on the vacuum in several ways :

- $\psi^\mu \partial \bar{X}^\nu |0\rangle_{NS}$,
  where $\partial \bar{X}^\nu$ is a bosonic creation operator and ψ^μ is a fermionic creation operator. These states correspond to the four-dimensional graviton, dilaton and antisymmetric tensor.

- $\psi^\mu \bar{\lambda}^i \bar{\lambda}^j |0\rangle_{NS}$, $i, j = 1, ..., 44$.
  The gauge bosons of the gauge group SO(44).

- $\{\chi^I, y^I, \omega^I\} \partial \bar{X}^\nu |0\rangle_{NS}$, $I = 1, ..., 6$.
  Each fermion triplet $\{X^I, y^I, \omega^I\} = \lambda^k$, $k = 1, .., 18$ yields the three generators of the Ith $SU(2)$ and therefore these states are identified with the gauge bosons of $SU(2)^6$.

- $\lambda^k \bar{\lambda}^i \bar{\lambda}^j |0\rangle_{NS}$.
  These are scalar fields.

The GGSO projections in the NS-sector are performed as follows :

-

$$
\begin{aligned}
e^{i\pi \vec{1} F_{\vec{0}}} \psi^\mu \partial \bar{X}^\nu |0\rangle_{NS} &= e^{i\pi(1+0)} \psi^\mu \partial \bar{X}^\nu |0\rangle_{NS} \\
&= -\psi^\mu \partial \bar{X}^\nu |0\rangle_{NS} \\
&= 1 \times (-1) \times \psi^\mu \partial \bar{X}^\nu |0\rangle_{NS} \\
&= \delta_{\vec{0}} C\binom{\vec{0}}{\vec{1}}^* \psi^\mu \partial \bar{X}^\nu |0\rangle_{NS} .
\end{aligned}
$$

So the states $\psi^\mu \partial \bar{X}^\nu |0\rangle_{NS}$ verify the GGSO condition and thus remain in the spectrum (In this calculation we used $C\binom{\vec{0}}{\vec{1}} = -1$).

- $$\begin{aligned} e^{i\pi\vec{1}F_{\vec{0}}}\psi^{\mu}\bar{\lambda}^{i}\bar{\lambda}^{j}|0\rangle_{NS} &= e^{i\pi(+1-1-1)}\psi^{\mu}\bar{\lambda}^{i}\bar{\lambda}^{j}|0\rangle_{NS} \\ &= -\psi^{\mu}\bar{\lambda}^{i}\bar{\lambda}^{j}|0\rangle_{NS} \\ &= \delta_{\vec{0}}C\begin{pmatrix}\vec{0}\\ \vec{1}\end{pmatrix}^{*}\psi^{\mu}\bar{\lambda}^{i}\bar{\lambda}^{j}|0\rangle_{NS}\ . \end{aligned}$$

- $$\begin{aligned} e^{i\pi\vec{1}F_{\vec{0}}}\lambda^{k}\partial\bar{X}^{\nu}|0\rangle_{NS} &= e^{i\pi(+1+0)}\lambda^{k}\partial\bar{X}^{J}|0\rangle_{NS} \\ &= -\lambda^{k}\partial\bar{X}^{\nu}|0\rangle_{NS} \\ &= \delta_{\vec{0}}C\begin{pmatrix}\vec{0}\\ \vec{1}\end{pmatrix}^{*}\lambda^{k}\partial\bar{X}^{\nu}|0\rangle_{NS}\ . \end{aligned}$$

- $$\begin{aligned} e^{i\pi\vec{1}F_{\vec{0}}}\lambda^{k}\bar{\lambda}^{i}\bar{\lambda}^{j}|0\rangle_{NS} &= e^{i\pi(+1-1-1)}\lambda^{k}\bar{\lambda}^{i}\bar{\lambda}^{j}|0\rangle_{NS} \\ &= -\lambda^{k}\bar{\lambda}^{i}\bar{\lambda}^{j}|0\rangle_{NS} \\ &= \delta_{\vec{0}}C\begin{pmatrix}\vec{0}\\ \vec{1}\end{pmatrix}^{*}\lambda^{k}\bar{\lambda}^{i}\bar{\lambda}^{j}|0\rangle_{NS}\ . \end{aligned}$$

So, in this model we have the graviton, the dilaton and the antisymmetric tensor (which are present in every model), gauge bosons which constitute the unrealistic gauge group $SU(2)^6 \times SO(44)$, and scalars. This model, although consistent, is far from being promising since it has a large gauge group, absent fermions, no supersymmetry and contains tachyonic states. The various properties of a model can drastically change by adding more b.c vectors and their corresponding GGSO phases.

4.2.1 Semi-Realistic Models

Adding supersymmetry resolves some of the above-mentioned problems. In the supersymmetric sector, world-sheet SUSY is realised non-linearly and the world-sheet supercurrent [36], is given by

$$T_F = \psi^\mu \partial \chi_\mu + f^{IJK} \chi^I \chi^J \chi^K \ , \tag{4.23}$$

where f^{IJK} are the structure constants of a semi-simple Lie group of dimension 18. In the case of realistic free fermionic models the $\chi^I (I = 1, ..., 18)$ transform in the adjoint representation of $SU(2)^6$ and are denoted by $(\chi^I\ ,\ y^I\ ,\ \omega^I)$ and $I = 1, ..., 6$.

The simplest supersymmetric extension of the pre-mentioned model is obtained by including one additional b.c vector $S = \{\psi^{1,2}, \chi^{1,..,6}\}$ and choosing values for the GGSO phases in a modular-invariant fashion. Since the spectrum for the NS sector was found before, and the sector $1 + S$ is massive, we will concentrate on the spectrum of sector S which contains the superpartners of the untwisted sector. The spectrum is obtained by two right-moving fermionic oscillators acting on the degenerate vacuum which is formed by the zero modes of the periodic complexified fermions $\psi^{1,2}, \chi^{1,2}, \chi^{3,4}$ and $\chi^{5,6}$. The massless states are $\overbrace{|\pm\rangle}^{\psi^{1,2}}\overbrace{|\pm\rangle}^{\chi^{1,2}}\overbrace{|\pm\rangle}^{\chi^{3,4}}\overbrace{|\pm\rangle}^{\chi^{5,6}}\partial\bar{X}^\nu$ and $\overbrace{|\pm\rangle}^{\psi^{1,2}}\overbrace{|\pm\rangle}^{\chi^{1,2}}\overbrace{|\pm\rangle}^{\chi^{3,4}}\overbrace{|\pm\rangle}^{\chi^{5,6}}\phi^\alpha\phi^\beta$. After performing the GGSO projections we are left with four gravitini and the superpartners of the particles coming from the NS sector. In order to achieve $N = 1$ supersymmetry one has to add two additional b.c vectors.

The structure of each additional b.c vector is constrained by the requirement of a well-defined supercurrent. The eighteen left-moving fermions are devided into six triplets $\{\chi_i, y_i, \omega_i\}$ in the adjoint representation of $SU(2)^6$. The allowed boundary condition for each triplet depends on the boundary condition of the world-sheet fermion ψ^μ_{12}.

CASE 1		CASE 2	
$b(\psi^\mu_{12}) = 1$	(1, 0, 0)	$b(\psi^\mu_{12}) = 0$	$(1, 1, 0)$
	(0, 1, 0)		$(1, 0, 1)$
	(0, 0, 1)		$(0, 1, 1)$
	(1, 1, 1)		$(0, 0, 0)$.

After adding more b.c vectors we reach a point where the gauge group becomes $SO(10) \times$ OTHER, where OTHER denotes the horizontal $U(1)$ symmetries along with the hidden gauge group. Further breaking of the $SO(10)$ symmetry into one of its maximal subgroups is achieved by adding Wilsonian sectors [35]. The resulting observable subgroup depends on the boundary conditions of the worldsheet fermions $\{\bar{\Psi}^{1,\cdots,5}\}$ of the Wilsonian sector. We can distinguish the following cases:

- $b\{\bar{\Psi}^{1,\cdots,5}\} = \{\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\} \longrightarrow SU(5) \times U(1)$
- $b\{\bar{\Psi}^{1,\cdots,5}\} = \{11100\}$
  OR $\longrightarrow SO(6) \times SO(4)$
  $b\{\bar{\Psi}^{1,\cdots,5}\} = \{00011\}$
- The direct breaking of the initial gauge group into $SU(3)\times SU(2)\times U(1)_c\times U(1)_L$ at the string scale is achieved by combining both the previous two steps.

4.3 A Model without Exotics

In the next section we will exhibit the phenomenology of a specific semi-realistic free fermionic model [37] which was discovered by utilising the methodology described in the next chapter. We construct an $N = 1$ supersymmetric $SU(4) \times SU(2)_L \times SU(2)_R$ model supplemented by an $SO(4) \times SO(4) \times SO(8)$ hidden gauge symmetry and three abelian factors according to the guidelines of Ref. [38, 39, 40]. This is an exemplary model, whose purpose is to show that there exist Pati-Salam models which are free of fractionally charged states [41, 42] at the massless level, and where only the top quark trilinear Yukawa coupling can be obtained. The F- and D-flat solutions are constrained by the requirements of GUT and electroweak symmetry breaking, and the acquirement of heavy mass by coloured gauge fields.

4.4 The Model

The basis is formed by the following thirteen spin structures:

$$
\begin{aligned}
v_1 = 1 &= \{\psi^\mu,\ \chi^{1,\dots,6}, y^{1,\dots,6}, \omega^{1,\dots,6} | \\
&\qquad \bar{y}^{1,\dots,6}, \bar{\omega}^{1,\dots,6}, \bar{\eta}^{1,2,3}, \bar{\psi}^{1,\dots,5}, \bar{\phi}^{1,\dots,8}\}, \\
v_2 = S &= \{\psi^\mu, \chi^{1,\dots,6}\}, \\
v_{2+i} = e_i &= \{y^i, \omega^i | \bar{y}^i, \bar{\omega}^i\},\ i = 1, \dots, 6, \\
v_9 = b_1 &= \{\chi^{34}, \chi^{56}, y^{34}, y^{56} | \bar{y}^{34}, \bar{y}^{56}, \bar{\eta}^1, \bar{\psi}^{1,\dots,5}\}, \\
v_{10} = b_2 &= \{\chi^{12}, \chi^{56}, y^{12}, y^{56} | \bar{y}^{12}, \bar{y}^{56}, \bar{\eta}^2, \bar{\psi}^{1,\dots,5}\}, \\
v_{11} = z_1 &= \{\bar{\phi}^{1,\dots,4}\}, \\
v_{12} = z_2 &= \{\bar{\phi}^{5,\dots,8}\}, \\
v_{13} = \alpha &= \{\bar{\psi}^{4,5}, \bar{\phi}^{1,2}\}\ ,
\end{aligned}
\tag{4.24}
$$

where we have denoted only the fermions with periodic boundary conditions. The first two basis vectors generate a model with $N = 4$ space–time supersymmetry and an $SO(44)$ gauge group in four dimensions. The next six basis vectors $e_i, i = 1, ..., 6$ reduce the gauge symmetry to $SO(32) \times U(1)^6$. The basis vectors b_1, b_2 reduce the gauge group to $SO(10) \times U(1)^2 \times SO(18)$ and reduce $N = 4$ to $N = 1$. The basis vectors z_1 and z_2 reduce the hidden gauge symmetry arising from the Neveu–Schwarz (NS) sector to $SO(10) \times U(1)^3 \times SO(8) \times SO(8)$. The vector α corresponds to a Wilson line, and it breaks the observable symmetry to $SO(6) \times SO(4) \times U(1)^3$, and the hidden gauge group to $SO(4) \times SO(4) \times SO(8)$.

The GGSO coefficients were chosen to be:

$$
\begin{array}{c} \\ 1 \\ S \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ b_1 \\ b_2 \\ z_1 \\ z_2 \\ \alpha \end{array}
\begin{array}{c}
\begin{array}{ccccccccccccc} 1 & S & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & b_1 & b_2 & z_1 & z_2 & \alpha \end{array} \\
\left(\begin{array}{rrrrrrrrrrrrr}
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 \\
-1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 & 1 & -1 & -1 & 1 \\
-1 & -1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & 1 & -1 & 1 & 1 \\
-1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 & 1 & -1 \\
-1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \\
-1 & -1 & 1 & -1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & 1 & 1 \\
-1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 \\
-1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 & 1 \\
-1 & -1 & 1 & -1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 \\
-1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1
\end{array}\right)
\end{array} . \tag{4.25}
$$

Apart from singlets, the observable massless states contain three chiral generations, one pair of heavy Higgs states to break the Pati-Salam symmetry [43] and light Higgs bi-doublets needed for generating fermion masses and electroweak symmetry breaking. The twisted and untwisted spectrum, and the gauge invariant terms which participate in the superpotential [44, 45], can be found in Appendix A, and they were derived by a computer program written in MATLAB. We will now roughly sketch how this program was written.

First we create "functions" where the underlying algebra, the addition and inner product between two vectors is defined. The b.c vectors and the independent GGSO coefficients (above the diagonal) are inserted manually by the user. The program then ensures the consistency of the model by verifying the ABK rules for the b.c vectors, and generates the remaining GGSO phases. The user must initially specify which are the real, complex and conjugate fermionic oscillators of the model, and input their corresponding fermion numbers. In order to find all the possible $2^{13} = 8192$ linear combinations of b.c vectors which constitute the sectors of the model a routine finds all the possible "numbers" of the binary system that can be expressed with 13 digits. In terms of our purpose each binary number corresponds to a different sector, the components of which are found when the entries of the binary number are multiplied with the corresponding b.c vectors and then added. Based on the fact that we are only interested in the massless sectors, the program then finds the quantities $\alpha_L\alpha_L$, $\alpha_R\alpha_R$, N_L and N_R for each sector. The sectors for which both N_L and N_R are equal or greater than zero such that they satisfy (4.22), are considered to be massless. They are separated from the massive sectors and are grouped into: observable spinorial sectors, hidden spinorial sectors, exotic spinorial sectors and vectorial sectors. At this stage we are interested in generating all the possible states of a sector and then truncating them with GGSO projections. Each state corresponds to a "Fermion Number Vector"(FNV) which contains the fermion numbers of each state. In the case of the NS sector all the possible FNVs are generated by combining the fermion number of one left oscillator with the fermion numbers of two right oscillators acting on the vacuum. For spinorial sectors, the program checks the non-zero entries of a sector and, in order to find the appropriate FNVs, it assigns all the possible combinations of 0 and -1 to the corresponding entry of the FNV. In order to generate the FNVs of the vectorial sectors the techniques of the NS and spinorial sectors are combined. In order to be able to perform the GGSO projections it was essential to write routines which calculate the δ_α and $C\binom{\alpha}{b_i}$ for each sector α and b.c vector b_i. Then another routine would perform the GGSO projections for each state (which corresponds to a specific FNV), and keep only the FNVs that survived the 13 GGSO projections. After this procedure is finished for all the states, we are left with FNVs and the sector they belong to. Then, all the local and global charges of each state were found. The states are then grouped into representations based on their charges. In order to find the cubic superpotential, we must take in to consideration that each trilinear term should be SUSY invariant, gauge invariant ,and in general, it should not consist of vanishing correlators. It is important to note that a sector $S+\alpha$ contains the superpartners of the sector α and the states of both sectors belong to the same supermultiplet.

We should note that at the cubic level superpotential only the top quark acquires mass through the interaction term $\bar{F}_{2R}F_{3L}h_1$. The fermions of the other two families are considered to acquire mass through non-renormalisable [46] interaction terms. The interaction terms found in the superpotential provide general guidelines on how to interpret the several fields found in the spectrum. The general way representations are combined in order to yield the three families and GUT Higgs are as follows:

$$3 \times (4, 2, 1) + 3 \times (\bar{4}, 1, 2) \rightarrow \text{ Three families,}$$
$$1 \times (4, 1, 2) + 1 \times (\bar{4}, 1, 2) \rightarrow \text{ GUT Higgs.}$$

The heaviest family is formed by the representations $\bar{F}_{2R}$ and F_{3L} and it gets its mass from trilinear interactions. The rest of the fermionic chiral fields are considered to acquire their mass through higher-order interactions. We consider that the Higgs pair that breaks the Pati-Salam symmetry to the Standard Model is $H = F_{1R}$ and $\bar{H} = \bar{F}_{1R}$. That is because, at the level of the cubic superpotential, both these representations remain massless and they can provide masses to potentially hazardous triplets through the missing partner mechanism. According to this mechanism [47], triplets that could mediate proton decay, can acquire heavy masses through the vacuum expectation value of the singlet representations embedded in additional spinorial and anti-spinorial representations of $SO(10)$. In our case these singlets are the superpartners of the right-handed neutrino and right-handed anti-neutrino found in the pair of heavy Higgs representations, and therefore the missing partner mechanism is realised as follows: $HDH \; + \; \bar{H}D\bar{H} \rightarrow d^c D_3 \langle \nu^c \rangle \; + \; \bar{d}^c \bar{D}_3 \langle \bar{\nu}^c \rangle$.

In each string model there typically exists a moduli space of solutions to the F and D flatness constraints which are supersymmetric [48, 49, 50, 51]. Although much of the study concerning string vacua involves the analysis and classification of these flat directions, here we will merely exhibit that such a solution can be found for this particular model. For this reason we constrain our study to cases where VEVs are acquired only by singlets and non-abelian singlets which are charged exclusively under the observable gauge group.

The set of D-flat constraints is given by

$$\langle D_A \rangle = \langle D_\alpha \rangle = 0 \ ,$$

$$D_A = \left[\sum Q_A^k |\chi_k|^2 + \xi\right] \ , \tag{4.26}$$

$$D_\alpha = \left[\sum Q_\alpha^k |\chi_k|^2\right] \ , \ \alpha \neq A \ , \tag{4.27}$$

$$\xi = \frac{g^2(\mathrm{Tr}Q_A)}{192\pi^2} M_{\mathrm{Pl}}^2 \ , \tag{4.28}$$

where χ_k are the singlet fields which acquire VEVs of order $\sqrt{\xi}$, and ξ is the Fayet-Iliopoulos term corresponding to the anomalous abelian symmetry and is generated due to the Green-Schwarz mechanism. The Q_A^k and Q_α^k denote the anomalous and non–anomalous charges, and $M_{\mathrm{Pl}} \approx 2 \times 10^{18}$ GeV denotes the reduced Planck mass. The solution of such a system of equations is of course not unique. From the previous set of equations it becomes obvious that we must uncover the combination of the abelian symmetries that constitutes the anomalous $U(1)$. Such an anomalous abelian symmetry is a generic characteristic of string models. The string vacuum initially contains three anomalous abelian $U(1)$ symmetries.

$$\mathrm{Tr}U(1)_1 = -12 \ ; \ \mathrm{Tr}U(1)_2 = -24 \ ; \ \mathrm{Tr}U(1)_3 = -12 \ . \tag{4.29}$$

The anomalies can be rotated into two anomaly-free

$$U(1)_1' = U(1)_1 - U(1)_3 \tag{4.30}$$

$$U(1)_2' = U(1)_1 - U(1)_2 + U(1)_3 \ , \tag{4.31}$$

and one anomalous combination as expected

$$U(1)_A' = U(1)_1 + 2\,U(1)_2 + U(1)_3 \ , \ \mathrm{Tr}U(1)_A = -72 \ . \tag{4.32}$$

The anomalous $U(1)_A$ is broken by the Green–Schwarz mechanism [52, 53, 54] in which a potentially large Fayet–Iliopoulos D–term, ξ, is generated by the VEV of the dilaton field. Such a D–term would in general break supersymmetry, unless there is a direction which will acquire a VEV cancelling the Fayet–Iliopoulos ξ–term, and thus restore supersymmetry. After imposing $\bar{F}_{2R} = \bar{F}_{3R} = \bar{F}_{4R} = 0$, the D–flatness constraints for the singlets and non-abelian singlets of the model are given by

$$\begin{aligned}
U(1)'_1 \; : \; & \left(|\Phi_{12}|^2 - |\bar{\Phi}_{12}|^2\right) + \left(|\Phi_{\bar{1}2}|^2 - |\bar{\Phi}_{\bar{1}2}|^2\right) + 2\left(|\Phi_{\bar{1}3}|^2 - |\bar{\Phi}_{\bar{1}3}|^2\right) \\
& - \left(|\Phi_{23}|^2 - |\bar{\Phi}_{23}|^2\right) + \left(|\Phi_{\bar{2}3}|^2 - |\bar{\Phi}_{\bar{2}3}|^2\right) + \frac{1}{2}\sum_{i=1,2}\left(|\zeta_i|^2 - |\bar{\zeta}_i|^2\right) \\
& -\frac{1}{2}\sum_{i=5,6}\left(|\zeta_i|^2 - |\bar{\zeta}_i|^2\right) + \sum_{i=3,4,7}\left(|\zeta_i|^2 - |\bar{\zeta}_i|^2\right) - \frac{1}{2}|F_{1R}|^2 = 0 \quad (4.33) \\
U(1)'_2 \; : \; & 2\left(|\Phi_{\bar{1}2}|^2 - |\bar{\Phi}_{\bar{1}2}|^2\right) + 2\left(|\Phi_{13}|^2 - |\bar{\Phi}_{13}|^2\right) - 2\left(|\Phi_{\bar{2}3}|^2 - |\bar{\Phi}_{\bar{2}3}|^2\right) \\
& + \left(|\zeta_1|^2 - |\bar{\zeta}_1|^2\right) + 2|\xi_-|^2 + \frac{1}{2}\left(|\bar{F}_{1R}|^2 - |F_{1R}|^2\right) = 0 \quad (4.34) \\
U(1)'_A \; : \; & 3\left(|\Phi_{12}|^2 - |\bar{\Phi}_{12}|^2\right) - \left(|\Phi_{\bar{1}2}|^2 - |\bar{\Phi}_{\bar{1}2}|^2\right) + 2\left(|\Phi_{13}|^2 - |\bar{\Phi}_{13}|^2\right) \\
& +3\left(|\Phi_{23}|^2 - |\bar{\Phi}_{23}|^2\right) + \left(|\Phi_{\bar{2}3}|^2 - |\bar{\Phi}_{\bar{2}3}|^2\right) - \frac{1}{2}\left(|\zeta_1|^2 - |\bar{\zeta}_1|^2\right) \\
& +\frac{3}{2}\sum_{i=2,5,6}\left(|\zeta_i|^2 - |\bar{\zeta}_i|^2\right) + 3|\xi_+|^2 - |\xi_-|^2 \\
& -\frac{1}{2}|F_{1R}|^2 - |\bar{F}_{1R}|^2 = +\frac{3\, g^2 M^2}{16\pi^2} \equiv \xi \; , \quad (4.35)
\end{aligned}$$

where g is the gauge coupling in the effective field theory and M is the reduced Planck mass $M \equiv M_{Planck}/\sqrt{8\pi}$.

Along with D-flatness we must also ensure F-flatness. The set of F–flatness constraints are obtained by requiring that the superpotential's derivatives vanish:

$$\langle F_i^\dagger \equiv -\frac{\partial W}{\partial \eta_i} \rangle = 0 \; , \quad (4.36)$$

where η_i are all the singlet fields that appear in the model. The singlet superpotential is given by

$$
\begin{aligned}
&\frac{W_\Phi}{g\sqrt{2}} = \\
&\bar{\Phi}_{13}\xi_-\xi_+ + \Phi_{23}\bar{\Phi}_{12}\Phi_{\overline{13}} + \Phi_{13}\bar{\Phi}_{12}\Phi_{\overline{23}} + \Phi_{23}\bar{\Phi}_{13}\Phi_{\overline{12}} + \Phi_{\overline{12}}\Phi_{\overline{23}}\bar{\Phi}_{\overline{13}} \\
&+\Phi_{13}\bar{\Phi}_{23}\bar{\Phi}_{\overline{12}} + \Phi_{12}\bar{\Phi}_{23}\bar{\Phi}_{\overline{13}} + \Phi_{\overline{13}}\bar{\Phi}_{\overline{12}}\bar{\Phi}_{\overline{23}} + \Phi_{12}\bar{\Phi}_{13}\bar{\Phi}_{\overline{23}} \\
&+{\zeta_1}^2\bar{\Phi}_{\overline{12}} + {\bar{\zeta}_1}^2\Phi_{\overline{12}} + \left({\zeta_3}^2 + {\zeta_4}^2 + {\zeta_7}^2\right)\bar{\Phi}_{\overline{13}} + \left({\bar{\zeta}_3}^2 + {\bar{\zeta}_4}^2 + {\bar{\zeta}_7}^2\right)\Phi_{\overline{13}} \\
&+\frac{1}{2}\bar{\zeta}_2\bar{\zeta}_5\xi_+ + {\zeta_2}^2\bar{\Phi}_{12} + \left({\zeta_5}^2 + {\zeta_6}^2\right)\bar{\Phi}_{23} + \Phi_{12}{\bar{\zeta}_2}^2 + \Phi_5\left(\zeta_1\bar{\zeta}_1 + \zeta_2\bar{\zeta}_2\right) \\
&+\Phi_2\left(\zeta_5\bar{\zeta}_5 + \zeta_6\bar{\zeta}_6\right) + \Phi_{23}\left({\bar{\zeta}_5}^2 + {\bar{\zeta}_6}^2\right) + \Phi_4\zeta_7\bar{\zeta}_7 + \frac{\zeta_4\zeta_5\bar{\zeta}_2}{\sqrt{2}} + \frac{\zeta_2\bar{\zeta}_3\bar{\zeta}_5}{\sqrt{2}}
\end{aligned}
\tag{4.37}
$$

The electroweak Higgs doublets come in pairs and are accommodated in the Pati–Salam bi-doublets h_1, h_2, h_3. Their mass matrix is

$$
M_h \sim \begin{array}{c} \\ h_1 \\ h_2 \\ h_3 \end{array}
\begin{array}{c}
\begin{array}{ccc} h_1 & h_2 & h_3 \end{array} \\
\begin{pmatrix}
\Phi_{13} & \frac{\zeta_1}{\sqrt{2}} & 0 \\
\frac{\zeta_1}{\sqrt{2}} & \bar{\Phi}_{23} & 0 \\
0 & 0 & \Phi_{23}
\end{pmatrix}
\end{array} . \tag{4.38}
$$

In order to keep h_1 massless we need to impose the condition

$$
\Phi_{13}\,\bar{\Phi}_{23} - \frac{\zeta_1^2}{2} = 0. \tag{4.39}
$$

Next we discuss the colour–triplet mass matrix in our string derived Pati–Salam model. Three pairs of colour–triplets arise in the model from the untwisted Neveu–Schwarz sector, and are accommodated in the sextet of the Pati–Salam $SU(4)$. We denote these by $D_i = d_i(3,1,-\frac{1}{3}) + d_i^c(\bar{3},1,\frac{1}{3})$ and $\bar{D}_i = \bar{d}_i(3,1,-\frac{1}{3})) + \bar{d}_i^c(\bar{3},1,\frac{1}{3})$. An additional sextet arises in the model from a twisted sector. A further pair of colour triplets is obtained from the heavy Higgs states, $\bar{F}_{1R}$ and F_{1R}, that are used to break the Pati–Salam symmetry, and must get a VEV of the order of the GUT scale. We denote the colour triplet of F_{1R} by d_{1R} and the colour triplet of $\bar{F}_{1R}$ by $\bar{d}_{1R}^c$. At the cubic level the colour triplet mass matrix then takes the form

$$M_D = \begin{array}{c} \\ d_1^c \\ d_2^c \\ d_3^c \\ \bar{d}_1^c \\ \bar{d}_2^c \\ \bar{d}_3^c \\ d_4^c \\ \bar{d}_{1R}^c \end{array} \begin{array}{c} \begin{array}{cccccccc} d_1 & d_2 & d_3 & \bar{d}_1 & \bar{d}_2 & \bar{d}_3 & d_4 & d_{1R} \end{array} \\ \left(\begin{array}{cccccccc} 0 & \bar{\Phi}_{12} & \bar{\Phi}_{13} & 0 & \bar{\Phi}_{\overline{12}} & \bar{\Phi}_{\overline{13}} & 0 & F_{1R} \\ \bar{\Phi}_{12} & 0 & \bar{\Phi}_{23} & \Phi_{\overline{12}} & 0 & \bar{\Phi}_{\overline{23}} & \xi_- & 0 \\ \bar{\Phi}_{13} & \bar{\Phi}_{23} & 0 & \Phi_{13} & \Phi_{\overline{23}} & 0 & 0 & 0 \\ 0 & \Phi_{\overline{12}} & \Phi_{13} & 0 & \Phi_{12} & \Phi_{13} & 0 & 0 \\ \Phi_{\overline{12}} & 0 & \Phi_{\overline{23}} & \Phi_{12} & 0 & \Phi_{23} & \xi_+ & 0 \\ \bar{\Phi}_{\overline{13}} & \bar{\Phi}_{\overline{23}} & 0 & \Phi_{13} & \Phi_{23} & 0 & 0 & 0 \\ 0 & \xi_- & 0 & 0 & \xi_+ & 0 & \Phi_{13} & 0 \\ 0 & \bar{F}_{1R} & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right) \end{array} . \tag{4.40}$$

We have $\det(M_D) \sim \Phi_{13}^3$ so in order to keep triplets heavy and h_1 light we need $\{\Phi_{13}, \zeta_1, \bar{\Phi}_{23}\} \neq 0$.

Next, we examine the pattern of symmetry breaking. The following 9 parameter solution

$$\left\{\Phi_3, \Phi_4, \Phi_6, \bar{\Phi}_{23}, \Phi_{\overline{23}}, \bar{\Phi}_{\overline{23}}, \Phi_{\overline{13}}, \bar{\Phi}_{\overline{13}}, \bar{\Phi}_{12}\right\} , \tag{4.41}$$

satisfies all F-flatness equations while ensuring electroweak breaking, and that all triplets become massive. The F-flatness equations (4.42)-(4.52) are obtained after imposing $\Phi_1 = \Phi_2 = \xi_+ = \xi_- = \zeta_i = \bar{z}_i = 0$, $\forall i = 3, \ldots, 7$.

$$\Phi_5 = -\frac{2i}{\sqrt{3}}\frac{\bar{\Phi}_{12}}{\bar{\Phi}_{23}}\sqrt{\frac{\Phi_{\bar{13}}\Phi_{\bar{23}}\bar{\Phi}_{\bar{23}}}{\bar{\Phi}_{\bar{13}}}} \tag{4.42}$$

$$\Phi_{23} = \frac{\Phi_{\bar{23}}\bar{\Phi}_{\bar{23}}}{\bar{\Phi}_{23}} \tag{4.43}$$

$$\Phi_{13} = -\frac{\Phi_{\bar{13}}\bar{\Phi}_{\bar{23}}}{3\bar{\Phi}_{23}} \tag{4.44}$$

$$\bar{\Phi}_{13} = -\frac{3\bar{\Phi}_{23}\bar{\Phi}_{\bar{13}}}{\bar{\Phi}_{\bar{23}}} \tag{4.45}$$

$$\Phi_{12} = -\frac{\bar{\Phi}_{12}\Phi_{\bar{13}}\Phi_{\bar{23}}\bar{\Phi}_{\bar{23}}}{3\bar{\Phi}_{23}{}^2\bar{\Phi}_{\bar{13}}} \tag{4.46}$$

$$\Phi_{\bar{12}} = \frac{\bar{\Phi}_{12}\Phi_{\bar{13}}\bar{\Phi}_{\bar{23}}}{3\bar{\Phi}_{23}\bar{\Phi}_{\bar{13}}} \tag{4.47}$$

$$\bar{\Phi}_{\bar{12}} = -\frac{\bar{\Phi}_{12}\Phi_{\bar{23}}}{\bar{\Phi}_{23}} \tag{4.48}$$

$$\zeta_1 = i\sqrt{\frac{2}{3}}\sqrt{\Phi_{\bar{13}}\bar{\Phi}_{\bar{23}}} \tag{4.49}$$

$$\bar{\zeta}_1 = -\sqrt{2\Phi_{\bar{23}}\bar{\Phi}_{\bar{13}}} \tag{4.50}$$

$$\zeta_2 = i\sqrt{\frac{2}{3}}\sqrt{\frac{\Phi_{\bar{23}}}{\bar{\Phi}_{23}}}\sqrt{\Phi_{\bar{13}}\bar{\Phi}_{\bar{23}}} \tag{4.51}$$

$$\bar{\zeta}_2 = \sqrt{2\bar{\Phi}_{23}\bar{\Phi}_{\bar{13}}}\,. \tag{4.52}$$

The triplet mass matrix determinant is

$$\det M_D = -\frac{64}{27}\frac{F_{1R}\bar{F}_{1R}\bar{\Phi}_{12}\Phi_{\bar{13}}{}^3\Phi_{\bar{23}}{}^2\bar{\Phi}_{\bar{23}}{}^3}{\bar{\Phi}_{23}{}^3}\,, \tag{4.53}$$

and thus all triplets are massive.

For this F–flatness solution, the three D–flatness equations (4.33–4.35) depend on seven parameters, $|\bar{\Phi}_{23}|$, $|\Phi_{\bar{23}}|$, $|\bar{\Phi}_{\bar{23}}|$, $|\Phi_{\bar{13}}|$, $|\bar{\Phi}_{\bar{13}}|$, $|\bar{\Phi}_{12}|$, and $|F_{1R}| = |\bar{F}_{1R}|$. Setting $|F_{1R}| = |\bar{F}_{1R}| = M_{\rm G} = 0.02\sqrt{\xi}$ the D–flatness equations can be solved numerically in terms of three parameters. Choosing, for example, $|\bar{\Phi}_{23}| = |\bar{\Phi}_{\bar{13}}| = \frac{1}{2}|\bar{\Phi}_{\bar{23}}| = \chi$ we can solve numerically for $|\Phi_{\bar{13}}|$, $|\bar{\Phi}_{23}|$ and $|\bar{\Phi}_{12}|$. The numerical solution is shown in figure 4.1 and was found by John Rizos.

In figure 4.2 we plot the mass of the two lightest colour triplets for the one parameter solution displayed in figure 4.1. From the figure we note that for

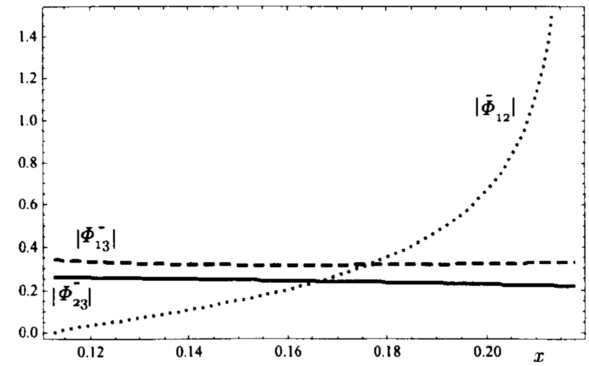

Figure 4.1: *Solution of of the D-flatness equations for* $|\Phi_{\bar{1}3}|$, $|\bar{\Phi}_{23}|$ *and* $|\bar{\Phi}_{12}|$ *as a function of* $\chi = |\bar{\Phi}_{23}| = |\bar{\Phi}_{\bar{1}3}| = \frac{1}{2}|\bar{\Phi}_{\bar{2}3}|$ *(all VEVs are in units of* $\sqrt{\xi}$*).*

singlet VEVs of the order of $0.1\sqrt{\xi}$ the lightest triplet mass is of the order of $0.4M_{\mathrm{GUT}}$. Thus the additional colour triplets are heavy enough to protect the proton from decaying through dangerous triplet-mediated dimension-5 operators [55].

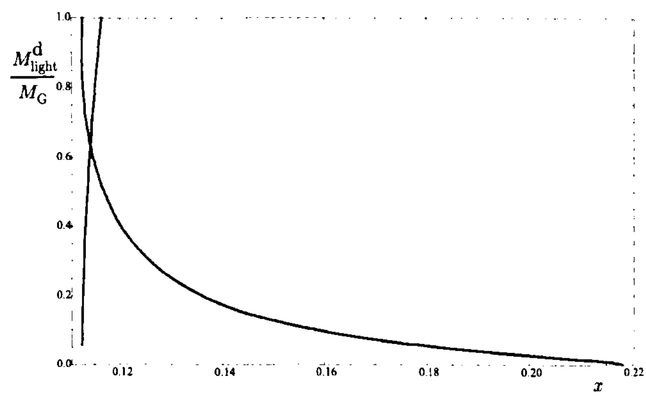

Figure 4.2: *The ratio of the two lightest colour triplet mass over* M_{GUT} *as a function of* $\chi = |\bar{\Phi}_{23}| = |\Phi_{\bar{1}3}| = \frac{1}{2}|\bar{\Phi}_{\bar{2}3}|$ *(in units of* $\sqrt{\xi}$*).*

Chapter 5

Classification of Heterotic String Models

STRING PHENOMENOLOGY's initial direction of research was the construction and analysis of individual models [56, 57, 58, 59, 60, 61] like the one exhibited in the previous chapter. Individual models have managed to show that there can indeed exist a relation between string theory and reality, but the insights they provide do not suffice towards the realisation of a vacua selection mechanism. Recently, another direction of research is being explored: the statistical exploration of large classes of vacua [62, 63, 64, 65, 66, 67, 68].

In this chapter we will illustrate the conceptual and calculational tools that were developed in order to achieve the classification of a class of Pati-Salam string vacua [69, 70, 71]. In particular, we have developed a combination of analytical and computational techniques that allow us to identify models from their low energy properties. The models under examination preserve the canonical $SO(10)$-GUT embedding of the weak hypercharge while the $SO(10)$ breaking is achieved by the addition of Wilsonian sectors. These models necessarily contain fractionally charged states [72], however the absence of such states from the massless spectrum is supported by experimental searches [73]. Apart from the basic phenomenological requirements:

- Four uncompactified dimensions
- $N = 1$ space-time supersymmetry

- Three generations of chiral matter
- SO(10) embedding of the hypercharge

we have searched for models that could provide an absence of exotic representations at the massless level and mechanisms for the breaking of the gauge group down to the Standard Model.

5.1 Generating the Class of Vacua

The class of Pati-Salam vacua is generated by defining a fixed basis (4.24) and varying the generalised GSO phases. $N = 1$ SUSY is achieved by choosing the following GGSO phases:

$$c\begin{bmatrix}1\\1\end{bmatrix} = c\begin{bmatrix}S\\1\end{bmatrix} = c\begin{bmatrix}S\\e_i\end{bmatrix} = c\begin{bmatrix}S\\b_m\end{bmatrix} = c\begin{bmatrix}S\\z_n\end{bmatrix} = c\begin{bmatrix}S\\\alpha\end{bmatrix} = -1, \quad (5.1)$$
$$i = 1, ..., 6,\ m = 1, 2,\ n = 1, 2,$$

leaving 66 independent coefficients,

$$c\begin{bmatrix}e_i\\e_j\end{bmatrix}, i \geq j, \quad c\begin{bmatrix}b_1\\b_2\end{bmatrix}, \quad c\begin{bmatrix}z_1\\z_2\end{bmatrix}, \quad c\begin{bmatrix}1\\b_A\end{bmatrix}, \quad c\begin{bmatrix}1\\z_A\end{bmatrix}$$
$$c\begin{bmatrix}e_i\\z_n\end{bmatrix}, c\begin{bmatrix}e_i\\b_A\end{bmatrix}, c\begin{bmatrix}b_A\\z_n\end{bmatrix}, \quad c\begin{bmatrix}1\\\alpha\end{bmatrix}, c\begin{bmatrix}e_i\\\alpha\end{bmatrix}, c\begin{bmatrix}b_A\\\alpha\end{bmatrix}, c\begin{bmatrix}z_A\\\alpha\end{bmatrix}$$
$$i, j = 1, \ldots 6\ ,\ A, B, m, n = 1, 2.$$

By Varying the GGSO phases we can generate $2^{(13*12)/2-12} \approx 7.3 * 10^{19}$ distinct vacua, each of which manifests different low energy properties. Although these vacua are in principle different to each other it is expected that there exists some degeneracy at the effective field theory limit.

5.2 Counting the Twisted Matter Spectrum

In the previous chapter we explained how each sector is a linear combination of b.c vectors. Since we are interested in performing a classification it is practical to be able to express sectors with common features (i.e *they provide observable* spinorial states) in a single expression. For this reason we express a set of 16 sectors as B^i_{pqrs}, where p, q, r, s are coefficients of some of the b.c vectors constituting a sector which take values 0 or 1, and i counts the number of these expressions. B^i_{pqrs} yield states which manifest as spinorial and vectorial representations charged under the observable, hidden, or both gauge groups. The counting of spinorial and vectorial representations is realised by utilising the so-called projectors. Each sector B^i_{pqrs} corresponds to a projector $P^i_{pqrs} = 0, 1$ which is an entity expressed in terms of GGSO coefficients and determines the survival or not of a sector. The projectors are nothing more than an alternative way of expressing the GGSO projection under specific conditions. The relation

$$e^{i\pi b_i F_\alpha}|s\rangle_\alpha = \delta_\alpha c\binom{\alpha}{b_i}^* |s\rangle_\alpha \ ,$$

makes it clear that the survival of a specific state characterised by F_α, with respect to a sector b_i, depends on the product $b_i F_\alpha$ and the GGSO coefficient $C\binom{\alpha}{b_i}^*$. If we want to perform, for example, GGSO projections on the states of a sector α for which $\delta_\alpha = -1$, then the previous relation becomes

$$e^{i\pi b_i F_\alpha}|s\rangle_\alpha = -c\binom{\alpha}{b_i}^* |s\rangle_\alpha \ .$$

In the case that the sectors α and b_i do not have any common elements, i.e $\{\alpha\} \cap \{b_i\} = \emptyset => b_i F_\alpha = 0$, then the survival condition becomes

$$-c\binom{\alpha}{b_i}^* = 1 \ .$$

The former relation naturally leads to the so-called projector, the value of which determines whether a state will be truncated or not from the spectrum:

$$P = \frac{1}{2}[1 - c\binom{\alpha}{b_i}^*] \ .$$

Using the appropriate formalism the projectors can be expressed as a system of linear equations with p,q,r and s as unknowns, which makes their computational manipulation more feasible. The solutions of a specific system of equations yield the different combinations of p,q,r,s for which sectors survive the GGSO projections. This formalism is more suitable and much more flexible for a computer-oriented analysis. In order to achieve the transition to this formalism, the following notation is introduced:

$$c\begin{bmatrix} a_i \\ a_j \end{bmatrix} = e^{i\pi(a_i|a_j)} \ , \ (a_i\,|a_j) = 0,1 \ . \tag{5.2}$$

The new expression implies properties which can be easily derived after performing standard algebraic methods involving the GGSO coefficients.

$$(a_i\,|a_j + a_k) = (a_i\,|a_j) + (a_i\,|a_k) \ , \forall\ a_i : \{\psi^\mu\} \cap a_i = \emptyset \tag{5.3}$$

$$(a_i\,|a_j) = (a_j\,|a_i) \ , \forall\ a_i, a_j : a_i \cdot a_j = 0 \text{ mod } 4 \ . \tag{5.4}$$

5.2.1 Observable Spinorial States, Representations and Projectors

The chiral spinorial representations of the observable $SO(6)\times SO(4)$ arise from the sectors

$$\begin{aligned} B^{(1)}_{pqrs} &= S + b_1 + pe_3 + qe_4 + re_5 + se_6 \\ &= \{\psi^\mu, \chi^{12}, (1-p)y^3\bar{y}^3, p\omega^3\bar{\omega}^3, (1-q)y^4\bar{y}^4, q\omega^4\bar{\omega}^4, \\ &\qquad (1-r)y^5\bar{y}^5, r\omega^5\bar{\omega}^5, (1-s)y^6\bar{y}^6, s\omega^6\bar{\omega}^6, \bar{\eta}^1, \bar{\psi}^{1..5}\} \qquad (5.5) \\ B^{(2)}_{pqrs} &= S + b_2 + pe_1 + qe_2 + re_5 + se_6 \\ B^{(3)}_{pqrs} &= S + b_3 + pe_1 + qe_2 + re_3 + se_4 \ , \end{aligned}$$

where $b_3 = b_1 + b_2 + x = 1 + S + b_1 + b_2 + \sum_{i=1}^{6} e_i + \sum_{n=1}^{2} z_n$. Provided that we do not have any enhancements, these states fall into representations of $SO(6) \times SO(4) \times U(1)^i$ or equivalently into representations of $SU(4) \times SU(2)_L \times SU(2)_R \times U(1)^i$: $(\mathbf{4}, \mathbf{2}, \mathbf{1})$, $(\mathbf{4}, \mathbf{1}, \mathbf{2})$, $(\bar{\mathbf{4}}, \mathbf{2}, \mathbf{1})$, $(\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2})$. In order to determine the particle content of each different representation we utilised the following normalisations for the hyperchage and the electromagnetic charge:

$$Y = \frac{1}{3}(Q_1 + Q_2 + Q_3) + \frac{1}{2}(Q_4 + Q_5) \tag{5.6}$$

$$Q_{em} = Y + \frac{1}{2}(Q_4 - Q_5) \ , \tag{5.7}$$

where the Q_i charges of a state arise due to Ψ^i for $i = 1, ..., 5$.
The following table summarises the eigenvalues of the electroweak $SU(2)\times U(1)$ Cartan generators, with respect to states which fall into the chiral observable Pati-Salam representations:

Representation	$\bar{\psi}^{1,2,3}$	$\bar{\psi}^{4,5}$	Y	Q_{em}
$(\bar{\mathbf{4}},\mathbf{1},\mathbf{2})$	$(+,+,+)$	$(+,+)$	1	1
	$(+,+,+)$	$(-,-)$	0	0
	$(+,-,-)$	$(+,+)$	1/3	1/3
	$(+,-,-)$	$(-,-)$	-2/3	-2/3
$(\mathbf{4},\mathbf{1},\mathbf{2})$	$(-,-,-)$	$(-,-)$	-1	-1
	$(-,-,-)$	$(+,+)$	0	0
	$(+,+,-)$	$(-,-)$	-1/3	-1/3
	$(+,+,-)$	$(+,+)$	2/3	2/3
$(\bar{\mathbf{4}},\mathbf{2},\mathbf{1})$	$(+,+,+)$	$(+,-)$	1/2	1,0
	$(+,-,-)$	$(+,-)$	-1/6	1/3,-2/3
$(\mathbf{4},\mathbf{2},\mathbf{1})$	$(-,-,-)$	$(+,-)$	-1/2	-1,0
	$(+,+,-)$	$(+,-)$	1/6	-1/3,2/3

Families and anti-families in the context of these models can be formed only if we combine the surviving states of two different sectors:

$$\begin{aligned} 16 &= (4,2,1) + (\bar{4},1,2) = N_{4L} + N_{\bar{4}R} \\ \bar{16} &= (4,1,2) + (\bar{4},2,1) = N_{4R} + N_{\bar{4}L} \ . \end{aligned} \tag{5.8}$$

A phenomenologically viable model must, of course, consist of only 3 families:

$$N_{4L} - N_{\bar{4}L} = N_{\bar{4}R} - N_{4R} = 3 \ . \tag{5.9}$$

In order to be able to distinguish between $N_{4L}, N_{\bar{4}L}, N_{\bar{4}R}$ and N_{4R}, one has to define Representation Operators that will determine the representations in which the states of each observable sector will fall. The operators $X^{iSU(4)}_{pqrs} = \pm 1$ that define the $SU(4)$ chirality ($\mathbf{4}$ or $\bar{\mathbf{4}}$) for B^1_{pqrs} , B^2_{pqrs} and B^3_{pqrs} respectively are:

$$
\begin{aligned}
X^{1_{SU(4)}}_{pqrs} &= C\begin{pmatrix} B^1_{pqrs} \\ S+b_2+\alpha+(1-r)e_5+(1-s)e_6 \end{pmatrix} \\
X^{2_{SU(4)}}_{pqrs} &= C\begin{pmatrix} B^2_{pqrs} \\ S+b_1+\alpha+(1-r)e_5+(1-s)e_6 \end{pmatrix} \\
X^{3_{SU(4)}}_{pqrs} &= C\begin{pmatrix} B^3_{pqrs} \\ S+b_2+\alpha+(1-p)e_1+(1-q)e_2 \end{pmatrix} .
\end{aligned} \tag{5.10}
$$

The representation operators $X^{i_{SU(2)_{L/R}}}_{pqrs} = \pm 1$ determine the $SU(2)_{L/R}$ representations ((**1**,**2**) or (**2**,**1**)) for B^1_{pqrs} , B^2_{pqrs} and B^3_{pqrs} respectively. In the following expressions $V_i = S + b_i + \alpha + \chi$.

$$
\begin{aligned}
X^{1_{SU(2)_{L/R}}}_{pqrs} &= C\begin{pmatrix} B^1_{pqrs} \\ V_1+(1-p)e_3+(1-q)e_4+(1-r)e_5+(1-s)e_6 \end{pmatrix} \\
X^{2_{SU(2)_{L/R}}}_{pqrs} &= C\begin{pmatrix} B^2_{pqrs} \\ V_2+(1-p)e_1+(1-q)e_2+(1-r)e_5+(1-s)e_6 \end{pmatrix} \\
X^{3_{SU(2)_{L/R}}}_{pqrs} &= C\begin{pmatrix} B^3_{pqrs} \\ V_3+(1-p)e_1+(1-q)e_2+(1-r)e_3+(1-s)e_4 \end{pmatrix} .
\end{aligned} \tag{5.11}
$$

The explicit expressions for the 48 projectors related to the observable chiral matter are:

$$
\begin{aligned}
P^{(1)}_{pqrs} &= \frac{1}{4}\left(1-c\begin{pmatrix} e_1 \\ B^{(1)}_{pqrs} \end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} e_2 \\ B^{(1)}_{pqrs} \end{pmatrix}\right) \\
&\cdot\frac{1}{4}\left(1-c\begin{pmatrix} z_1 \\ B^{(1)}_{pqrs} \end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} z_2 \\ B^{(1)}_{pqrs} \end{pmatrix}\right) \\
P^{(2)}_{pqrs} &= \frac{1}{4}\left(1-c\begin{pmatrix} e_3 \\ B^{(2)}_{pqrs} \end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} e_4 \\ B^{(2)}_{pqrs} \end{pmatrix}\right) \\
&\cdot\frac{1}{4}\left(1-c\begin{pmatrix} z_1 \\ B^{(2)}_{pqrs} \end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} z_2 \\ B^{(2)}_{pqrs} \end{pmatrix}\right) \\
P^{(3)}_{pqrs} &= \frac{1}{4}\left(1-c\begin{pmatrix} e_5 \\ B^{(3)}_{pqrs} \end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} e_6 \\ B^{(3)}_{pqrs} \end{pmatrix}\right) \\
&\cdot\frac{1}{4}\left(1-c\begin{pmatrix} z_1 \\ B^{(3)}_{pqrs} \end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} z_2 \\ B^{(3)}_{pqrs} \end{pmatrix}\right) .
\end{aligned} \tag{5.12}
$$

The analytic expressions for each different projector $P^{1,2,3}_{pqrs}$ respectively, are given in a matrix form $\Delta^i W^i = Y^i$:

$$\begin{pmatrix} (e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\ (z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6) \\ (z_2|e_3) & (z_2|e_4) & (z_2|e_5) & (z_2|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_1) \\ (e_2|b_1) \\ (z_1|b_1) \\ (z_2|b_1) \end{pmatrix}$$

$$\begin{pmatrix} (e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6) \\ (z_2|e_1) & (z_2|e_2) & (z_2|e_5) & (z_2|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_3|b_2) \\ (e_4|b_2) \\ (z_1|b_2) \\ (z_2|b_2) \end{pmatrix} \quad (5.13)$$

$$\begin{pmatrix} (e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\ (z_1|e_1) & (z_1|e_2) & (z_1|e_3) & (z_1|e_4) \\ (z_2|e_1) & (z_2|e_2) & (z_2|e_3) & (z_2|e_4) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_5|b_3) \\ (e_6|b_3) \\ (z_1|b_3) \\ (z_2|b_3) \end{pmatrix} .$$

5.2.2 Hidden Matter, Representations and Projectors

Although hidden matter representations can be used in order to further constrain the space of vacua we did not consider them in this study. Nevertheless, utilising the hidden spinorial sectors and projectors can be used in further studies, in order to distinguish between models which seem to be equivalent at the effective field theory limit.

Hidden matter states come from 96 sectors, 48 of which yield **((2,1),(2,1))** representations of $SO(4)_1 \times SO(4)_2 = SU(2)_1 \times SU(2)_2 \times SU(2)_3 \times SU(2)_4$:

$$\begin{aligned} B^{(4)}_{pqrs} &= B^{(1)}_{pqrs} + x + z_1 = S + b_1 + pe_3 + qe_4 + re_5 + se_6 + x + z_1 \\ &= \{\psi^\mu, \chi^{12}, (1-p)y^3\bar{y}^3, p\omega^3\bar{\omega}^3, (1-q)y^4\bar{y}^4, q\omega^4\bar{\omega}^4, \\ &\qquad (1-r)y^5\bar{y}^5, r\omega^5\bar{\omega}^5, (1-s)y^6\bar{y}^6, s\omega^6\bar{\omega}^6, \bar{\eta}^{23}, \bar{\phi}^{1..4}\} \qquad (5.14) \\ B^{(5)}_{pqrs} &= B^{(2)}_{pqrs} + x + z_1 = S + b_2 + pe_1 + qe_2 + re_5 + se_6 + x + z_1 \\ B^{(6)}_{pqrs} &= B^{(3)}_{pqrs} + x + z_1 = S + b_3 + pe_1 + qe_2 + re_3 + se_4 + x + z_1 \ , \end{aligned}$$

where $x = 1 + S + \sum_{i=1}^{6} e_i + z_1 + z_2 = \{\bar{\eta}^{123}, \bar{\psi}^{12345}\}$. The expressions for the projectors corresponding to $B^{(4,5,6)}_{pqrs}$ are given below.

$$\begin{aligned}
P^{(4)}_{pqrs} &= \frac{1}{8}\left(1 - c\binom{e_1}{B^{(4)}_{pqrs}}\right) \cdot \left(1 - c\binom{e_2}{B^{(4)}_{pqrs}}\right) \\
&\quad \cdot \left(1 - c\binom{z_2}{B^{(4)}_{pqrs}}\right) \\
P^{(5)}_{pqrs} &= \frac{1}{8}\left(1 - c\binom{e_3}{B^{(5)}_{pqrs}}\right) \cdot \left(1 - c\binom{e_4}{B^{(5)}_{pqrs}}\right) \\
&\quad \cdot \left(1 - c\binom{z_2}{B^{(5)}_{pqrs}}\right) \\
P^{(6)}_{pqrs} &= \frac{1}{8}\left(1 - c\binom{e_5}{B^{(6)}_{pqrs}}\right) \cdot \left(1 - c\binom{e_6}{B^{(6)}_{pqrs}}\right) \\
&\quad \cdot \left(1 - c\binom{z_2}{B^{(6)}_{pqrs}}\right) .
\end{aligned} \tag{5.15}$$

Their corresponding analytic expressions are:

$$\begin{pmatrix} (e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\ (z_2|e_3) & (z_2|e_4) & (z_2|e_5) & (z_2|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_1 + x + z_1) \\ (e_2|b_1 + x + z_1) \\ (z_2|b_1 + x + z_1) \end{pmatrix}$$

$$\begin{pmatrix} (e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\ (z_2|e_1) & (z_2|e_2) & (z_2|e_5) & (z_2|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_2 + x + z_1) \\ (e_2|b_2 + x + z_1) \\ (z_2|b_2 + x + z_1) \end{pmatrix} \tag{5.16}$$

$$\begin{pmatrix} (e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\ (z_2|e_1) & (z_2|e_2) & (z_2|e_3) & (z_2|e_4) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_3 + x + z_1) \\ (e_2|b_3 + x + z_1) \\ (z_2|b_3 + x + z_1) \end{pmatrix} .$$

The remaining 48 sectors are given by

$$\begin{aligned}
B^{(7)}_{pqrs} &= B^{(1)}_{pqrs} + \chi + z_2 \;=\; S + b_1 + pe_3 + qe_4 + re_5 + se_6 + \chi + z_2 \\
&= \{\psi^{\mu}, \chi^{12}, (1-p)y^3\bar{y}^3, p\omega^3\bar{\omega}^3, (1-q)y^4\bar{y}^4, q\omega^4\bar{\omega}^4, \\
&\quad (1-r)y^5\bar{y}^5, r\omega^5\bar{\omega}^5, (1-s)y^6\bar{y}^6, s\omega^6\bar{\omega}^6, \bar{\eta}^{23}, \bar{\phi}^{5..8}\} \qquad (5.17) \\
B^{(8)}_{pqrs} &= B^{(2)}_{pqrs} + \chi + z_2 \;=\; S + b_2 + pe_1 + qe_2 + re_5 + se_6 + \chi + z_2 \\
B^{(9)}_{pqrs} &= B^{(3)}_{pqrs} + \chi + z_2 \;=\; S + b_3 + pe_1 + qe_2 + re_3 + se_4 + \chi + z_2 \,.
\end{aligned}$$

The states of $B^{(7,8,9)}_{pqrs}$ fall into the spinorial $\mathbf{8}$ and antispinorial $\bar{\mathbf{8}}$ representations of $SO(8)$. The 48 projectors $P^{7,8,9}_{p,q,r,s}$ corresponding to $B^{7,8,9}_{p,q,r,s}$ are the following:

$$\begin{aligned}
P^{(7)}_{pqrs} &= \frac{1}{4}\left(1 - c\begin{pmatrix} e_1 \\ B^{(7)}_{pqrs}\end{pmatrix}\right)\cdot\left(1 - c\begin{pmatrix} e_2 \\ B^{(7)}_{pqrs}\end{pmatrix}\right) \\
&\cdot\frac{1}{4}\left(1 - c\begin{pmatrix} z_1 \\ B^{(7)}_{pqrs}\end{pmatrix}\right)\cdot\left(1 - c\begin{pmatrix} \alpha \\ B^{(7)}_{pqrs}\end{pmatrix}\right) \\
P^{(8)}_{pqrs} &= \frac{1}{4}\left(1 - c\begin{pmatrix} e_3 \\ B^{(8)}_{pqrs}\end{pmatrix}\right)\cdot\left(1 - c\begin{pmatrix} e_4 \\ B^{(8)}_{pqrs}\end{pmatrix}\right) \\
&\cdot\frac{1}{4}\left(1 - c\begin{pmatrix} z_1 \\ B^{(8)}_{pqrs}\end{pmatrix}\right)\cdot\left(1 - c\begin{pmatrix} \alpha \\ B^{(8)}_{pqrs}\end{pmatrix}\right) \qquad (5.18) \\
P^{(9)}_{pqrs} &= \frac{1}{4}\left(1 - c\begin{pmatrix} e_5 \\ B^{(9)}_{pqrs}\end{pmatrix}\right)\cdot\left(1 - c\begin{pmatrix} e_6 \\ B^{(9)}_{pqrs}\end{pmatrix}\right) \\
&\cdot\frac{1}{4}\left(1 - c\begin{pmatrix} z_1 \\ B^{(9)}_{pqrs}\end{pmatrix}\right)\cdot\left(1 - c\begin{pmatrix} \alpha \\ B^{(9)}_{pqrs}\end{pmatrix}\right) \,.
\end{aligned}$$

The analytic expressions for $P^{7,8,9}_{p,q,r,s}$ are given below:

$$\begin{pmatrix}
(e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\
(e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\
(z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6) \\
(\alpha|e_3) & (\alpha|e_4) & (\alpha|e_5) & (\alpha|e_6)
\end{pmatrix}
\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}
=
\begin{pmatrix}
(e_1|b_1 + x + z_2) \\
(e_2|b_1 + x + z_2) \\
(z_1|b_1 + x + z_2) \\
(\alpha|b_1 + x + z_2)
\end{pmatrix}$$

$$\begin{pmatrix}
(e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\
(e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\
(z_1|e_1) & (z_1|e_2) & (z_1|e_5) & (z_1|e_6) \\
(\alpha|e_1) & (\alpha|e_2) & (\alpha|e_5) & (\alpha|e_6)
\end{pmatrix}
\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix}
=
\begin{pmatrix}
(e_1|b_2 + x + z_2) \\
(e_2|b_2 + x + z_2) \\
(z_1|b_2 + x + z_2) \\
(\alpha|b_2 + x + z_2)
\end{pmatrix} \qquad (5.19)$$

$$\begin{pmatrix} (e_5\,|e_1) & (e_5\,|e_2) & (e_5\,|e_3) & (e_5\,|e_4) \\ (e_6\,|e_1) & (e_6\,|e_2) & (e_6\,|e_3) & (e_6\,|e_4) \\ (z_1\,|e_1) & (z_1\,|e_2) & (z_1\,|e_3) & (z_1\,|e_4) \\ (\alpha\,|e_1) & (\alpha\,|e_2) & (\alpha\,|e_3) & (\alpha\,|e_4) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1\,|b_3+x+z_2) \\ (e_2\,|b_3+x+z_2) \\ (z_1\,|b_3+x+z_2) \\ (\alpha\,|b_3+x+z_2) \end{pmatrix} .$$

5.2.3 Exotic States, Representations and Projectors

Spinorial exotics are obtained from 192 sectors.

$$\begin{aligned} B^{(10)}_{pqrs} &= B^{(1)}_{pqrs}+\alpha \;=\; S+b_1+pe_3+qe_4+re_5+se_6+\alpha \\ &= \{\psi^{\mu},\chi^{12},(1-p)y^3\bar{y}^3,p\omega^3\bar{\omega}^3,(1-q)y^4\bar{y}^4,q\omega^4\bar{\omega}^4, \\ &\qquad (1-r)y^5\bar{y}^5,r\omega^5\bar{\omega}^5,(1-s)y^6\bar{y}^6,s\omega^6\bar{\omega}^6,\bar{\eta}^1,\bar{\psi}^{1..3},\bar{\phi}^{1..2}\} \qquad (5.20) \\ B^{(11)}_{pqrs} &= B^{(2)}_{pqrs}+\alpha \;=\; S+b_2+pe_1+qe_2+re_5+se_6+\alpha \\ B^{(12)}_{pqrs} &= B^{(3)}_{pqrs}+\alpha \;=\; S+b_3+pe_1+qe_2+re_3+se_4+\alpha\ . \end{aligned}$$

The expressions for $B^{13,14,15}_{p,q,r,s}$ are given by: $B^{10,11,12}_{p,q,r,s}+z_1$. The states of $B^{10,11,12}_{p,q,r,s}$ fall into the representations of $SU(4)_{obs}\times SU(2)_1\times SU(2)_2$ while the states of $B^{13,14,15}_{p,q,r,s}$ fall into the representations of $SU(4)_{obs}\times SU(2)_3\times SU(2)_4$. The representations and observable charges of these states are given below :

Representation	$\bar{\psi}^{1,2,3}$	$\bar{\phi}^{1,2}$ or $\bar{\phi}^{3,4}$	Y	Q_{em}
	$(+,+,+)$	$(+,+)$	1/2	1/2
$(\bar{\mathbf{4}},\mathbf{1},\mathbf{2})$	$(+,+,+)$	$(-,-)$	1/2	1/2
	$(+,-,-)$	$(+,+)$	-1/6	-1/6
	$(+,-,-)$	$(-,-)$	-1/6	-1/6
	$(-,-,-)$	$(-,-)$	-1/2	-1/2
$(\mathbf{4},\mathbf{1},\mathbf{2})$	$(-,-,-)$	$(+,+)$	-1/2	-1/2
	$(+,+,-)$	$(-,-)$	1/6	1/6
	$(+,+,-)$	$(+,+)$	1/6	1/6
$(\bar{\mathbf{4}},\mathbf{2},\mathbf{1})$	$(+,+,+)$	$(+,-)$	1/2	1/2
	$(+,-,-)$	$(+,-)$	-1/6	-1/6
$(\mathbf{4},\mathbf{2},\mathbf{1})$	$(-,-,-)$	$(+,-)$	-1/2	-1/2
	$(+,+,-)$	$(+,-)$	1/6	1/6

We can therefore summarise all the previous results by saying that sectors coming from $B^{10,11,12,13,14,15}_{p,q,r,s}$, give rise to $(\mathbf{4},1,1)$ and $(\bar{\mathbf{4}},1,1)$ representations under the Pati-Salam gauge group, with fractional electric charges $\pm\frac{1}{2}$ and $\pm\frac{1}{6}$.

The projectors corresponding to $B^{10,11,12}_{p,q,r,s}$ are:

$$\begin{aligned}
P^{(10)}_{pqrs} &= \frac{1}{4}\left(1-c\begin{pmatrix} e_1 \\ B^{(10)}_{pqrs}\end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} e_2 \\ B^{(10)}_{pqrs}\end{pmatrix}\right) \\
&\quad \cdot\frac{1}{4}\left(1-c\begin{pmatrix} z_2 \\ B^{(10)}_{pqrs}\end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} \alpha+z_1 \\ B^{(10)}_{pqrs}\end{pmatrix}\right) \\
P^{(11)}_{pqrs} &= \frac{1}{4}\left(1-c\begin{pmatrix} e_3 \\ B^{(11)}_{pqrs}\end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} e_4 \\ B^{(11)}_{pqrs}\end{pmatrix}\right) \\
&\quad \cdot\frac{1}{4}\left(1-c\begin{pmatrix} z_2 \\ B^{(11)}_{pqrs}\end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} \alpha+z_1 \\ B^{(11)}_{pqrs}\end{pmatrix}\right) \\
P^{(12)}_{pqrs} &= \frac{1}{4}\left(1-c\begin{pmatrix} e_5 \\ B^{(12)}_{pqrs}\end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} e_6 \\ B^{(12)}_{pqrs}\end{pmatrix}\right) \\
&\quad \cdot\frac{1}{4}\left(1-c\begin{pmatrix} z_2 \\ B^{(12)}_{pqrs}\end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} \alpha+z_1 \\ B^{(12)}_{pqrs}\end{pmatrix}\right) .
\end{aligned} \tag{5.21}$$

We can get the expressions for $P^{13,14,15}$ if we substitute $B^{10,11,12}\longrightarrow B^{11,12,13}$ and $\alpha+z_1\rightarrow\alpha$. The previous projectors can be expressed in matrix form:

$$\begin{pmatrix} (e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\ (z_2|e_3) & (z_2|e_4) & (z_2|e_5) & (z_2|e_6) \\ (\alpha+z_1|e_3) & (\alpha+z_1|e_4) & (\alpha+z_1|e_5) & (\alpha+z_1|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_1+\alpha) \\ (e_2|b_1+\alpha) \\ (z_1|b_1+\alpha) \\ (z_2|b_1+\alpha) \end{pmatrix}$$

$$\begin{pmatrix} (e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\ (z_2|e_1) & (z_2|e_2) & (z_2|e_5) & (z_2|e_6) \\ (\alpha+z_1|e_1) & (\alpha+z_1|e_2) & (\alpha+z_1|e_5) & (\alpha+z_1|e_6) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_2+\alpha) \\ (e_2|b_2+\alpha) \\ (z_1|b_2+\alpha) \\ (z_2|b_2+\alpha) \end{pmatrix} \quad (5.22)$$

$$\begin{pmatrix} (e_5|e_1) & (e_5|e_2) & (e_5|e_3) & (e_5|e_4) \\ (e_6|e_1) & (e_6|e_2) & (e_6|e_3) & (e_6|e_4) \\ (z_2|e_1) & (z_2|e_2) & (z_2|e_3) & (z_2|e_4) \\ (\alpha+z_1|e_1) & (\alpha+z_1|e_2) & (\alpha+z_1|e_3) & (\alpha+z_1|e_4) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_3+\alpha) \\ (e_2|b_3+\alpha) \\ (z_1|b_3+\alpha) \\ (z_2|b_3+\alpha) \end{pmatrix} .$$

We can get the analytical expressions for $P^{13,14,15}$ if we substitute $\alpha + z_1 \rightarrow \alpha$. Then we have an additional 96 sectors producing exotics in the spinorial representations.

$$\begin{aligned} B^{(16)}_{pqrs} &= B^{(1)}_{pqrs} + \alpha + \chi = S + b_1 + pe_3 + qe_4 + re_5 + se_6 + \alpha + \chi \\ &= \{\psi^\mu, \chi^{12}, (1-p)y^3\bar{y}^3, p\omega^3\bar{\omega}^3, (1-q)y^4\bar{y}^4, q\omega^4\bar{\omega}^4, \\ &\qquad (1-r)y^5\bar{y}^5, r\omega^5\bar{\omega}^5, (1-s)y^6\bar{y}^6, s\omega^6\bar{\omega}^6, \bar{\eta}^2, \bar{\eta}^3, \bar{\psi}^{4..5}, \bar{\Phi}^{1..2}\} \quad (5.23) \\ B^{(17)}_{pqrs} &= B^{(2)}_{pqrs} + \alpha + \chi = S + b_2 + pe_1 + qe_2 + re_5 + se_6 + \alpha + \chi \\ B^{(18)}_{pqrs} &= B^{(3)}_{pqrs} + \alpha + \chi = S + b_3 + pe_1 + qe_2 + re_3 + se_4 + \alpha + \chi . \end{aligned}$$

The expressions for $B^{19,20,21}_{p,q,r,s}$ are given by $B^{16,17,18}_{p,q,r,s} + z_1$. The states of $B^{16,17,18}_{p,q,r,s}$ fall into the representations of $SU(2)_L \times SU(2)_R \times SU(2)_1 \times SU(2)_2$, while the states of the sectors coming from $B^{16,17,18}_{p,q,r,s} + z_1$ fall into representations of $SU(2)_L \times SU(2)_R \times SU(2)_3 \times SU(2)_4$. The representations and observable charges of these states are given below:

Representation	$\bar{\psi}^{4,5}$	$\bar{\phi}^{1,2}$ or $\bar{\phi}^{3,4}$	Y	Q_{em}
	$(+,+)$	$(+,+)$	1/2	1/2
$((\mathbf{1},\mathbf{2}),(\mathbf{1},\mathbf{2}))$	$(+,+)$	$(-,-)$	1/2	1/2
	$(-,-)$	$(+,+)$	-1/2	-1/2
	$(-,-)$	$(-,-)$	-1/2	-1/2
$((\mathbf{1},\mathbf{2}),(\mathbf{2},\mathbf{1}))$	$(+,+)$	$(+,-)$	1/2	1/2
	$(-,-)$	$(+,-)$	-1/2	-1/2
$((\mathbf{2},\mathbf{1}),(\mathbf{1},\mathbf{2}))$	$(+,-)$	$(+,+)$	0	-1/2,1/2
	$(+,-)$	$(-,-)$	0	-1/2,1/2
$((\mathbf{2},\mathbf{1}),(\mathbf{2},\mathbf{1}))$	$(+,-)$	$(+,-)$	0	-1/2,1/2

The "mixed" states from $B^{16,17,18,19,20,21}_{p,q,r,s}$ give rise to $(\mathbf{1},\mathbf{2},\mathbf{1})$ and $(\mathbf{1},\mathbf{1},\mathbf{2})$ representations under the Pati-Salam gauge group with fractional electric charges $\pm\frac{1}{2}$. The projectors corresponding to $B^{16,17,18}_{p,q,r,s}$ are

$$\begin{aligned}
P^{(16)}_{pqrs} &= \frac{1}{8}\left(1-c\begin{pmatrix} e_1 \\ B^{(16)}_{pqrs}\end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} e_2 \\ B^{(16)}_{pqrs}\end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} z_2 \\ B^{(16)}_{pqrs}\end{pmatrix}\right)\\
P^{(17)}_{pqrs} &= \frac{1}{8}\left(1-c\begin{pmatrix} e_3 \\ B^{(17)}_{pqrs}\end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} e_4 \\ B^{(17)}_{pqrs}\end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} z_2 \\ B^{(17)}_{pqrs}\end{pmatrix}\right)\quad(5.24)\\
P^{(18)}_{pqrs} &= \frac{1}{8}\left(1-c\begin{pmatrix} e_5 \\ B^{(18)}_{pqrs}\end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} e_6 \\ B^{(18)}_{pqrs}\end{pmatrix}\right)\cdot\left(1-c\begin{pmatrix} z_2 \\ B^{(18)}_{pqrs}\end{pmatrix}\right).
\end{aligned}$$

In order to get the expressions for $P^{19,20,21}_{p,q,r,s}$ we have to substitute $B^{16,17,18}_{p,q,r,s} \rightarrow B^{19,20,21}_{p,q,r,s}$.

$$\begin{pmatrix} (e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\ (z_2|e_3) & (z_2|e_4) & (z_2|e_5) & (z_2|e_6) \end{pmatrix}\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_1+\alpha+\chi) \\ (e_2|b_1+\alpha+\chi) \\ (z_2|b_1+\alpha+\chi) \end{pmatrix}$$

$$\begin{pmatrix} (e_3|e_1) & (e_3|e_2) & (e_3|e_5) & (e_3|e_6) \\ (e_4|e_1) & (e_4|e_2) & (e_4|e_5) & (e_4|e_6) \\ (z_2|e_1) & (z_2|e_2) & (z_2|e_5) & (z_2|e_6) \end{pmatrix}\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1|b_2+\alpha+\chi) \\ (e_2|b_2+\alpha+\chi) \\ (z_2|b_2+\alpha+\chi) \end{pmatrix} \quad (5.25)$$

$$\begin{pmatrix} (e_5\,|e_1) & (e_5\,|e_2) & (e_5\,|e_3) & (e_5\,|e_4) \\ (e_6\,|e_1) & (e_6\,|e_2) & (e_6\,|e_3) & (e_6\,|e_4) \\ (z_2\,|e_1) & (z_2\,|e_2) & (z_2\,|e_3) & (z_2\,|e_4) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} (e_1\,|b_3+\alpha+\chi) \\ (e_2\,|b_3+\alpha+\chi) \\ (z_2\,|b_3+\alpha+\chi) \end{pmatrix} .$$

We can get the analytical expressions for $P^{19,20,21}$ if we substitute $\alpha+\chi \to \alpha+\chi+z_1$ in the previous expressions.

5.2.4 Vectorial States, Representations and Projectors

The 48 sectors that provide vectorial states are

$$\begin{aligned} B^{(1)}_{pqrs}+\chi &= S+b_1+pe_3+qe_4+re_5+se_6+\chi \\ &= \{\psi^\mu,\chi^{12},(1-p)y^3\bar{y}^3,p\omega^3\bar{\omega}^3,(1-q)y^4\bar{y}^4,q\omega^4\bar{\omega}^4, \\ &\quad (1-r)y^5\bar{y}^5,r\omega^5\bar{\omega}^5,(1-s)y^6\bar{y}^6,s\omega^6\bar{\omega}^6,\bar{\eta}^2,\bar{\eta}^3 \qquad (5.26) \\ B^{(2)}_{pqrs}+\chi &= S+b_2+pe_1+qe_2+re_5+se_6+\chi \\ B^{(3)}_{pqrs}+\chi &= S+b_3+pe_1+qe_2+re_3+se_4+\chi\ . \end{aligned}$$

Each surviving sector can potentially give rise to states which belong to observable or hidden $SO(2N)$ vectorial representations:

- $\{\bar{\psi}^{123}\}|R\rangle^{(i)}_{pqrs}$, $i=1,2,3\to$ Vectorial representation of $SO(6)$,
- $\{\bar{\psi}^{45}\}|R\rangle^{(i)}_{pqrs}$, $i=1,2,3\to$ Vectorial representation of $SO(4)$,
- $\{\bar{\phi}^{12}\}|R\rangle^{(i)}_{pqrs}$, $i=1,2,3\to$ Vectorial representation of $SO(4)$,
- $\{\bar{\phi}^{34}\}|R\rangle^{(i)}_{pqrs}$, $i=1,2,3\to$ Vectorial representation of $SO(4)$,
- $\{\bar{\phi}^{5..8}\}|R\rangle^{(i)}_{pqrs}$, $i=1,2,3\to$ Vectorial representation of $SO(8)$,

where $|R\rangle^{(i)}_{pqrs}$ is the degenerate Ramond vacuum of the $B^{(i)}_{pqrs} + \chi$ sector, and $i = 1, 2, 3$.

The corresponding projectors are:

$$\begin{aligned} P^{(i)(\bar\psi_{123})}_{pqrs} &= \frac{1}{4}\left(1 - c\binom{e_{2i-1}}{B^{(i)}_{pqrs}+x}\right)\cdot\left(1 - c\binom{e_{2i}}{B^{(i)}_{pqrs}+x}\right) \\ &\cdot\frac{1}{4}\left(1 - c\binom{z_1}{B^{(i)}_{pqrs}+x}\right)\cdot\left(1 - c\binom{z_2}{B^{(i)}_{pqrs}+x}\right) \\ &\cdot\frac{1}{2}\left(1 - c\binom{\alpha}{B^{(i)}_{pqrs}+x}\right) \end{aligned}$$

$$\begin{aligned} P^{(i)(\bar\psi_{45})}_{pqrs} &= \frac{1}{4}\left(1 - c\binom{e_{2i-1}}{B^{(i)}_{pqrs}+x}\right)\cdot\left(1 - c\binom{e_{2i}}{B^{(i)}_{pqrs}+x}\right) \\ &\cdot\frac{1}{4}\left(1 - c\binom{z_1}{B^{(i)}_{pqrs}+x}\right)\cdot\left(1 - c\binom{z_2}{B^{(i)}_{pqrs}+x}\right) \\ &\cdot\frac{1}{2}\left(1 + c\binom{\alpha}{B^{(i)}_{pqrs}+x}\right) \end{aligned}$$

$$\begin{aligned} P^{(i)(\bar\Phi_{12})}_{pqrs} &= \frac{1}{4}\left(1 - c\binom{e_{2i-1}}{B^{(i)}_{pqrs}+x}\right)\cdot\left(1 - c\binom{e_{2i}}{B^{(i)}_{pqrs}+x}\right) \\ &\cdot\frac{1}{4}\left(1 + c\binom{z_1}{B^{(i)}_{pqrs}+x}\right)\cdot\left(1 - c\binom{z_2}{B^{(i)}_{pqrs}+x}\right) \\ &\cdot\frac{1}{2}\left(1 + c\binom{\alpha}{B^{(i)}_{pqrs}+x}\right) \end{aligned} \tag{5.27}$$

$$\begin{aligned} P^{(i)(\bar\Phi_{34})}_{pqrs} &= \frac{1}{4}\left(1 - c\binom{e_{2i-1}}{B^{(i)}_{pqrs}+x}\right)\cdot\left(1 - c\binom{e_{2i}}{B^{(i)}_{pqrs}+x}\right) \\ &\cdot\frac{1}{4}\left(1 + c\binom{z_1}{B^{(i)}_{pqrs}+x}\right)\cdot\left(1 - c\binom{z_2}{B^{(i)}_{pqrs}+x}\right) \\ &\cdot\frac{1}{2}\left(1 - c\binom{\alpha}{B^{(i)}_{pqrs}+x}\right) \end{aligned}$$

$$\begin{aligned} P^{(i)(\bar\Phi_{5678})}_{pqrs} &= \frac{1}{4}\left(1 - c\binom{e_{2i-1}}{B^{(i)}_{pqrs}+x}\right)\cdot\left(1 - c\binom{e_{2i}}{B^{(i)}_{pqrs}+x}\right) \\ &\cdot\frac{1}{4}\left(1 - c\binom{z_1}{B^{(i)}_{pqrs}+x}\right)\cdot\left(1 + c\binom{z_2}{B^{(i)}_{pqrs}+x}\right) \\ &\cdot\frac{1}{2}\left(1 - c\binom{\alpha}{B^{(i)}_{pqrs}+x}\right) . \end{aligned}$$

The explicit analytic expressions corresponding to $B^{(i)}_{pqrs} + \chi$, $i = 1, 2, 3$ are:

$$\Delta^{(1)}_v = \begin{pmatrix} (e_1|e_3) & (e_1|e_4) & (e_1|e_5) & (e_1|e_6) \\ (e_2|e_3) & (e_2|e_4) & (e_2|e_5) & (e_2|e_6) \\ (z_1|e_3) & (z_1|e_4) & (z_1|e_5) & (z_1|e_6) \\ (z_2|e_3) & (z_2|e_4) & (z_2|e_5) & (z_2|e_6) \\ (\alpha|e_3) & (\alpha|e_4) & (\alpha|e_5) & (\alpha|e_6) \end{pmatrix}$$

$$Y^{(1)}_{\bar\psi^{123}} = \begin{pmatrix} (e_1 \mid b_1 + x) \\ (e_2 \mid b_1 + x) \\ (z_1 \mid b_1 + x) \\ (z_2 \mid b_1 + x) \\ (\alpha \mid b_1 + x) \end{pmatrix} \qquad Y^{(1)}_{\bar\psi^{45}} = \begin{pmatrix} (e_1|b_1 + x) \\ (e_2|b_1 + x) \\ (z_1|b_1 + x) \\ (z_2|b_1 + x) \\ 1 + (\alpha|b_1 + x) \end{pmatrix}$$

$$Y^{(1)}_{\bar\phi^{12}} = \begin{pmatrix} (e_1|b_1 + x) \\ (e_2|b_1 + x) \\ 1 + (z_1|b_1 + x) \\ (z_2|b_1 + x) \\ 1 + (\alpha|b_1 + x) \end{pmatrix} \qquad Y^{(1)}_{\bar\phi^{34}} = \begin{pmatrix} (e_1|b_1 + x) \\ (e_2|b_1 + x) \\ 1 + (z_1|b_1 + x) \\ (z_2|b_1 + x) \\ (\alpha|b_1 + x) \end{pmatrix}$$

$$Y^{(1)}_{\bar\phi^{5..8}} = \begin{pmatrix} (e_1|b_1 + x) \\ (e_2|b_1 + x) \\ (z_1|b_1 + x) \\ 1 + (z_2|b_1 + x) \\ (\alpha|b_1 + x) \end{pmatrix} \tag{5.28}$$

$$\Delta_v^{(2)} = \begin{pmatrix} (e_3\,|e_1) & (e_3\,|e_2) & (e_3\,|e_5) & (e_3\,|e_6) \\ (e_4\,|e_1) & (e_4\,|e_2) & (e_4\,|e_5) & (e_4\,|e_6) \\ (z_1\,|e_1) & (z_1\,|e_2) & (z_1\,|e_5) & (z_1\,|e_6) \\ (z_2\,|e_1) & (z_2\,|e_2) & (z_2\,|e_5) & (z_2\,|e_6) \\ (\alpha\,|e_1) & (\alpha\,|e_4) & (\alpha\,|e_5) & (\alpha\,|e_6) \end{pmatrix}$$

$$Y^{(2)}_{\psi^{,123}} = \begin{pmatrix} (\ e_3 \mid b_2 + x\ ) \\ (\ e_4 \mid b_2 + x\ ) \\ (\ z_1 \mid b_2 + x\ ) \\ (\ z_2 \mid b_2 + x\ ) \\ (\ \alpha \mid b_2 + x\ ) \end{pmatrix} \qquad Y^{(2)}_{\bar{\psi}^{45}} = \begin{pmatrix} (e_3\,|b_2 + x) \\ (e_4\,|b_2 + x) \\ (z_1\,|b_2 + x) \\ (z_2\,|b_2 + x) \\ 1 + (\alpha\,|b_2 + x) \end{pmatrix}$$

$$Y'^{(2)}_{\bar{\phi}^{12}} = \begin{pmatrix} (e_3\,|b_2 + x) \\ (e_4\,|b_2 + x) \\ 1 + (z_1\,|b_2 + x) \\ (z_2\,|b_2 + x) \\ 1 + (\alpha\,|b_2 + x) \end{pmatrix} \qquad Y^{(2)}_{\bar{\phi}^{34}} = \begin{pmatrix} (e_3\,|b_2 + x) \\ (e_4\,|b_2 + x) \\ 1 + (z_1\,|b_2 + x) \\ (z_2\,|b_2 + x) \\ (\alpha\,|b_2 + x) \end{pmatrix}$$

$$Y^{(2)}_{\bar{\phi}^{5..8}} = \begin{pmatrix} (e_3\,|b_2 + x) \\ (e_4\,|b_2 + x) \\ (z_1\,|b_2 + x) \\ 1 + (z_2\,|b_2 + x) \\ (\alpha\,|b_2 1 + x) \end{pmatrix} \tag{5.29}$$

$$\Delta_v^{(2)} = \begin{pmatrix} (e_5\,|e_1) & (e_5\,|e_2) & (e_5\,|e_3) & (e_5\,|e_4) \\ (e_6\,|e_1) & (e_6\,|e_2) & (e_6\,|e_3) & (e_6\,|e_4) \\ (z_1\,|e_1) & (z_1\,|e_2) & (z_1\,|e_3) & (z_1\,|e_4) \\ (z_2\,|e_1) & (z_2\,|e_2) & (z_2\,|e_3) & (z_2\,|e_4) \\ (\alpha\,|e_1) & (\alpha\,|e_4) & (\alpha\,|e_5) & (\alpha\,|e_6) \end{pmatrix}$$

$$Y^{(3)}_{\bar\psi^{123}} = \begin{pmatrix} (\,e_5\ |\ b_3+x\,) \\ (\,e_6\ |\ b_3+x\,) \\ (\,z_1\ |\ b_3+x\,) \\ (\,z_2\ |\ b_3+x\,) \\ (\,\alpha\ |\ b_3+x\,) \end{pmatrix} \qquad Y^{(3)}_{\bar\psi^{45}} = \begin{pmatrix} (e_5\,|b_3+x) \\ (e_6\,|b_3+x) \\ (z_1\,|b_3+x) \\ (z_2\,|b_3+x) \\ 1+(\alpha\,|b_3+x) \end{pmatrix}$$

$$Y^{(3)}_{\bar\phi^{12}} = \begin{pmatrix} (e_5\,|b_3+x) \\ (e_6\,|b_3+x) \\ 1+(z_1\,|b_3+x) \\ (z_2\,|b_3+x) \\ 1+(\alpha\,|b_3+x) \end{pmatrix} \qquad Y^{(3)}_{\bar\phi^{34}} = \begin{pmatrix} (e_5\,|b_3+x) \\ (e_6\,|b_3+x) \\ 1+(z_1\,|b_3+x) \\ (z_2\,|b_3+x) \\ (\alpha\,|b_3+x) \end{pmatrix}$$

$$Y^{(3)}_{\bar\phi^{5..8}} = \begin{pmatrix} (e_5\,|b_3+x) \\ (e_6\,|b_3+x) \\ (z_1\,|b_3+x) \\ 1+(z_2\,|b_3+x) \\ (\alpha\,|b_3+x) \end{pmatrix}$$

5.3 The Four-Dimensional Gauge Group

The untwisted spectrum is common in all the Pati–Salam vacua that we classify, and the models differ by the states that arise in the twisted sectors. The NS sector gives rise to the gauge group

$$SO(6) \times SO(4) \times U(1)^3 \times SO(4)_1 \times SO(4)_2 \times SO(8), \tag{5.30}$$

which can equivalently be written as :

$$\begin{aligned} \text{Observable} &: \; SO(6)_{obs} \times SU(2)_L \times SU(2)_R \times U(1)_1 \times U(1)_2 \times U(1)_3 \\ \text{Hidden} &: \; SU(2)_1 \times SU(2)_2 \times SU(2)_3 \times SU(2)_4 \times SO(8)_{hid}. \end{aligned} \tag{5.31}$$

In addition to NS sector, gauge bosons may arise from other sectors:

$$\mathbf{G} = \left\{ \begin{array}{cccccc} z_1, & z_2, & \alpha, & \alpha + z_1, & & \\ x, & z_1 + z_2, & \alpha + z_2, & \alpha + z_1 + z_2, & \alpha + x, & \alpha + x + z_1 \end{array} \right\}. \tag{5.32}$$

When the gauge bosons of a sector transform under a subgroup of the NS gauge group, the specific gauge group will undergo enhancement. In Appendix B we present the types of enhancements which can potentially occur from different sectors, assuming that only one set of conditions is satisfied in each distinct case. We have cases where the observable, hidden, or both gauge groups are enlarged. An enlargement of the observable gauge group, for example, implies that chiral matter would be obtained from the **6** and $\bar{\mathbf{6}}$ of $SU(6)$. This would result in losing the $SO(10)$ embedding of the hypercharge, and would make it impossible to form a complete $SO(10)$ family from a net number of representations. "Mixed enhancements" would be the worst case scenario since they would unify observable and hidden states under the same representations, resulting in phenomenologically unacceptable consequences which contradict all the current low energy data. Due to simplicity, computational power limitations, and acceptable phenomenology, we restricted the class of vacua to the cases where we do not have any enhancements.

5.4 Results

Using the algebraic expressions presented in the previous sections we can analyse the entire massless spectrum for a given choice of GGSO projection coefficients that completely specify a specific string model. These formulae are inserted into a computer program which is used to scan the space of string vacua produced by random generation of the one–loop GGSO projection coefficients. The number of possible configurations is $2^{51} \sim 10^{15}$, which is too large for a complete classification. For this reason a random generation algorithm is utilised, and the characteristics of the model for each set of random GGSO projection coefficients are extracted. This procedure was followed and produced a three generation Pati–Salam string model that does not contain any exotic massless states as described in the previous chapter. We use this methodology to classify the Pati–Salam free fermionic string models with respect to some phenomenological criteria. The observable sector of a heterotic string Pati–Salam model is characterized by 9 integers $(n_g, k_L, k_R, n_6, n_h, n_4, n_{\bar{4}}, n_{2L}, n_{2R})$, where

$$
\begin{aligned}
& n_{4L} - n_{\bar{4}L} = n_{\bar{4}R} - n_{4R} = n_g = \#\text{ of generations} \\
& n_{\bar{4}L} = k_L = \#\text{ of non chiral left pairs} \\
& n_{4R} = k_R = \#\text{ of non chiral right pairs} \\
& n_6 = \#\text{ of } (\mathbf{6},\mathbf{1},\mathbf{1}) \\
& n_h = \#\text{ of } (\mathbf{1},\mathbf{2},\mathbf{2}) \\
& n_4 = \#\text{ of } (\mathbf{4},\mathbf{1},\mathbf{1})\text{ (exotic)} \\
& n_{\bar{4}} = \#\text{ of } (\bar{\mathbf{4}},\mathbf{1},\mathbf{1})\text{ (exotic)} \\
& n_{2L} = \#\text{ of } (\mathbf{1},\mathbf{2},\mathbf{1})\text{ (exotic)} \\
& n_{2R} = \#\text{ of } (\mathbf{1},\mathbf{1},\mathbf{2})\text{ (exotic)} \ .
\end{aligned}
$$

Using the methodology outlined in Section 5.2 we obtain analytic formulae for all these quantities. The spectrum of a viable Pati–Salam heterotic string model should have:

$n_g = 3$ three chiral generations,

$k_L \geq 0$ heavy mass can be generated for non chiral pairs,

$k_R \geq 1$ at least one Higgs pair to break the PS symmetry,

$n_6 \geq 1$ at least one required for missing partner mechanism ,

$n_h \geq 1$ at least one light Higgs bi–doublet,

$n_4 = n_{\bar{4}} \geq 0$ heavy mass can be generated for vector–like exotics,

$n_{2L} = 0 \bmod 2$ heavy mass can be generated for vector–like exotics,

$n_{2R} = 0 \bmod 2$ heavy mass can be generated for vector–like exotics.

We next explore the space of Pati–Salam free fermionic heterotic string vacua. We perform a statistical sampling in a space of 10^{11} models out of the total of 2^{51}. Using a computer FORTRAN95 program running on a single node of the Theoretical Physics Division of University of Ioannina, HPC cluster, we were able to obtain the relative data within a period of one week. This corresponds to examining approximately 1/20000 models in this class. Increasing the sample by one order of magnitude is within the cluster capabilities, however, as already checked by using a 10^9 and a 10^{10} random sample. The results obtained are similar to the ones presented below. Some of the results are presented in Figures 5.1-5.6 and Table 1.

In Figure 5.1 the number of models versus the number of generations is displayed. Of note in Figure 5.1 is the absence of any models with 7, 9, 11, 13, 14 and 15 generations. This may indicate that these cases are completely forbidden or are extremely unlikely cases in the space of all possibilities.

In Figure 5.2 we display a three-dimensional plot, the number of models versus the number of generations and the total number of exotic fractionally charged states. As seen from the figure, the distribution exhibits a peak for models with zero chiral generations and a non-vanishing number of exotic multiplets, and decreases with increasing and decreasing number of exotics. Moreover, we find no correlation between the absence of fractionally charged exotic states and the number of generations. We can have exophobic models

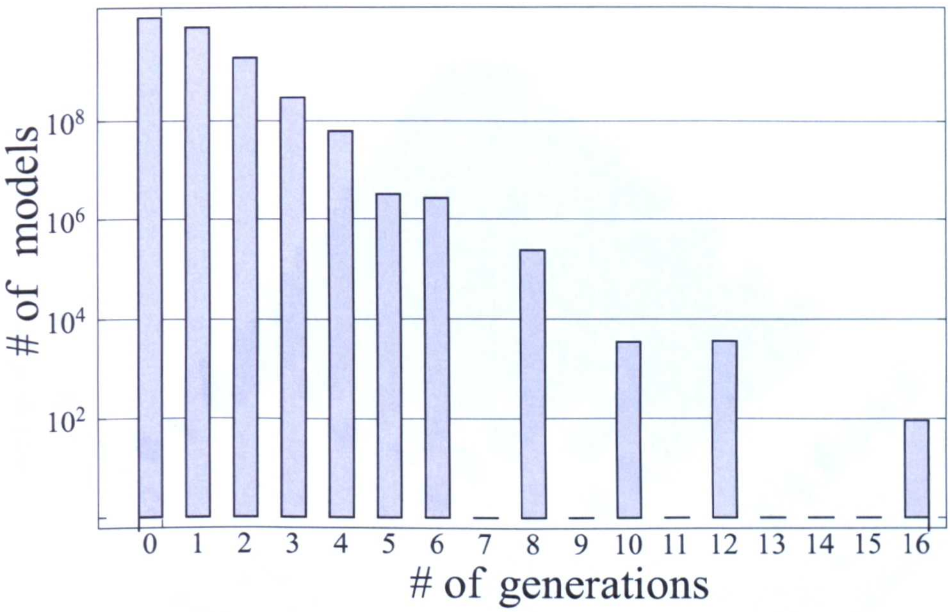

Figure 5.1: *Number of models versus number of generations (n_g) in a random sample of* 10^{11} *GGSO configurations.*

for all values of n_g.

However, in the case of models without any exotic multiplets, we observe the following relation between the number of chiral generations (n_g), the number of Higgs bi–doublets (n_h) and sextets (n_6):

$$n_g \pmod 2 = n_h \pmod 2 = n_6 \pmod 2 \tag{5.33}$$

As noted from Table C.1, found in Appendix C, the number of Higgs bi–doublets and sextets is indeed odd or even depending on the number of generations. Another important phenomenological point to note from table C.1 is the existence of exophobic models with a varying number of Higgs bi–doublet representations.

In Figure 5.3 we display the multiplicities of models versus the number of generations in the case of exotic-free models. As seen from the figure the number of models decreases with increasing number of generations. The same exclusion of models with some number of generations noted in Figure 5.1 is also seen in Figure 5.2 for the same cases.

Figure 5.4 displays the total number of three generation models versus the number of exotic fractionally charged states in a given three generation model.

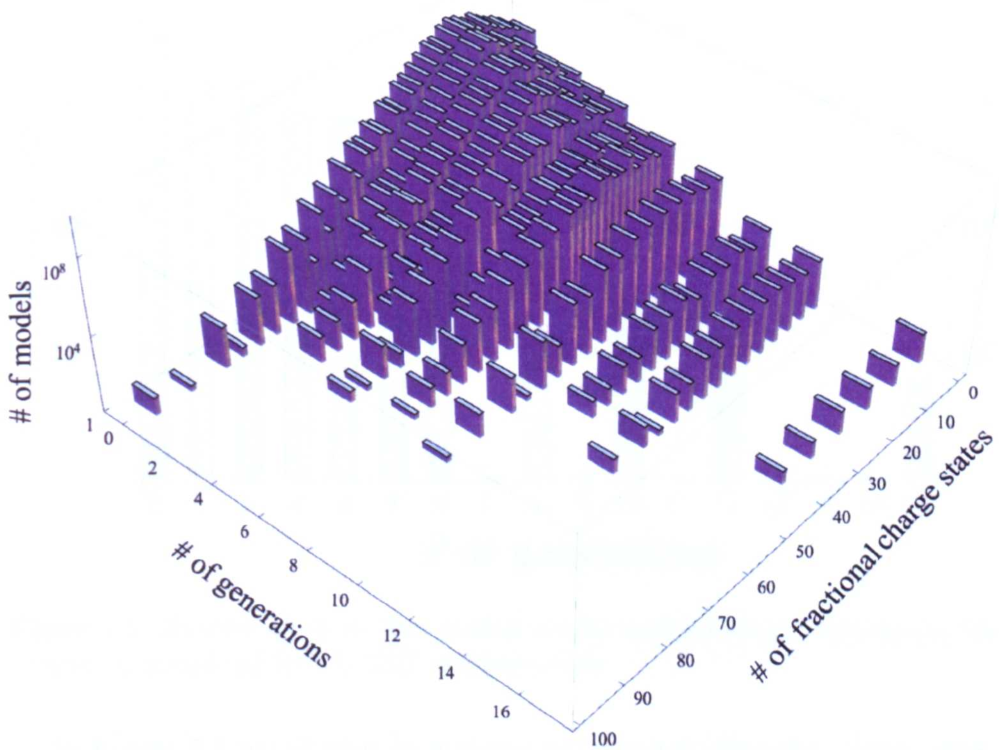

Figure 5.2: *Number of models versus number of generations (n_g) and total number of exotic multiplets in a random sample of* 10^{11} *GGSO configurations.*

As seen from the figure the total number of exophobic three generation models is slightly less than 10^6, which is roughly $1/10^5$ of the entire sample.

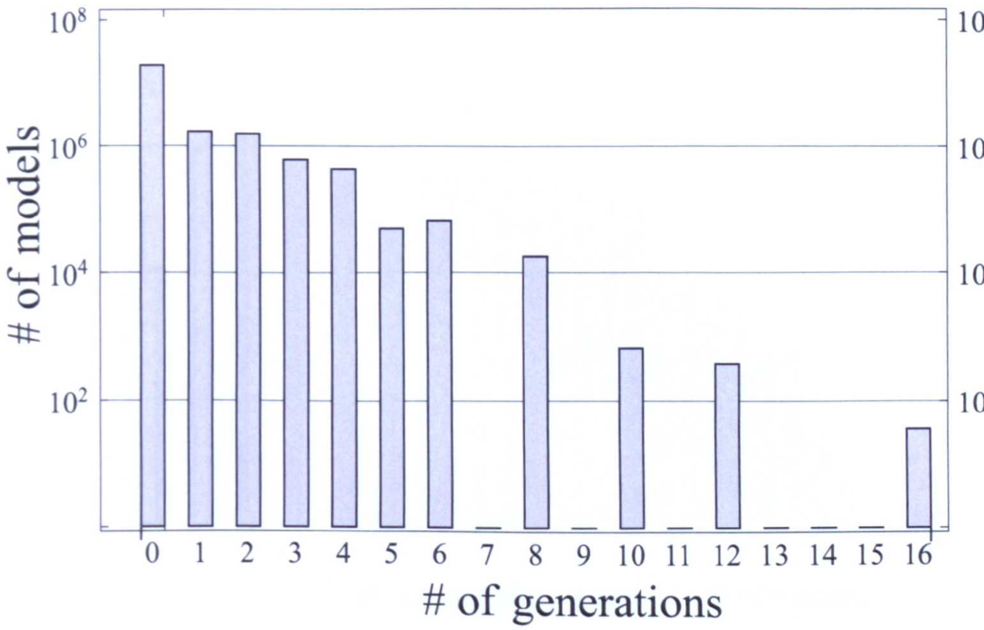

Figure 5.3: *Number of exotic free models versus number of generations (n_g) in a random sample of* 10^{11} *GGSO configurations.*

In Figure 5.5 we display in a three dimensional plot, the total number of three generation models versus the number of exotic $SU(4)$ **4**-plets and number of exotic $SO(4)$ $\mathbf{2_L}$ and $\mathbf{2_R}$ doublets. In Figure 5.6 we display in a three dimensional plot the number of three generation models versus the number of additional non-chiral representations in the $(\bar{\mathbf{4}}, \mathbf{1}, \mathbf{2_R}) \oplus (\mathbf{4}, \mathbf{1}, \mathbf{2_R})$, $(\mathbf{4}, \mathbf{2_L}, \mathbf{1}) \oplus (\bar{\mathbf{4}}, \mathbf{2_L}, \mathbf{1})$, and additional $(\mathbf{6}, \mathbf{1}, \mathbf{1})$ multiplets of $SU(4) \times SU(2)_L \times SU(2)_R$. Finally, in Table 5.1 we tabulate the number of models with sequential imposition of phenomenological constraints. The total number of models in the sample is 10^{11}. We first impose that there is no enhancement of the four dimensional gauge symmetry. Roughly 80% percent of the models satisfy this criteria. Next, we impose that the generations form complete families. In other words, there is no chiral representation of the Pati–Salam gauge group that is not accompanied by the representation that completes it to a representation of $SO(10)$. So the entire chiral spectrum is contained in complete representations of $SO(10)$ decomposed under the Pati–Salam subgroup. Roughly 1/5 of the previous set satisfy this criterion. The restriction to three chiral generations further reduces the number of models by two orders of magnitude. Imposing the existence of heavy string

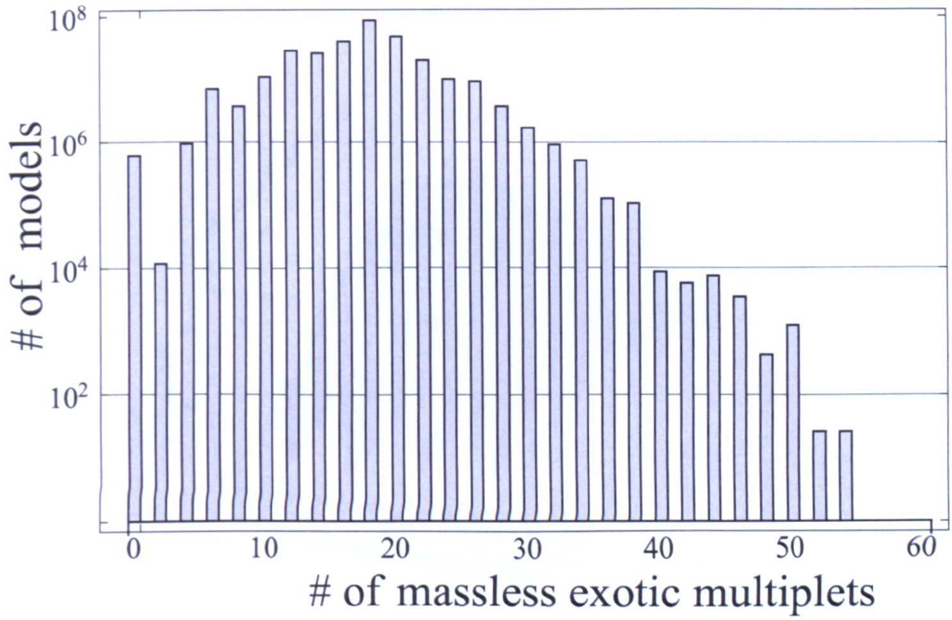

Figure 5.4: *Number of 3-generation models versus total number of exotic multiplets in a random sample of* 10^{11} *GGSO configurations.*

states to break the Pati–Salam gauge symmetry to the Standard Model gauge group leads to a reduction by another order of magnitude. The requirement of Standard Model Higgs doublets does not lead to a further reduction because as noted above in (5.33), the total number of Higgs bi–doublets is equal to the number of chiral generations modulo 2. Therefore, the existence of three chiral generations necessarily implies a non–zero number of Higgs bi–doublets in the spectrum. Finally, imposing the absence of massless exotics reduces the number of models by further two orders of magnitude. Therefore, the reduction from the initial sample is by roughly six orders of magnitude, *i.e* one in every 10^6 models satisfy all of these constraints. Given that the total number of vacua in the space of models scanned is of the order of 10^{15}, we expect that 10^9 of the models satisfy these criteria, which leaves a substantial number to accommodate further phenomenological constraints. For example, requiring minimal number of PS breaking Higgs ($k_L = 0, k_R = 1$) truncates further by a factor of 4 the number of models as seen in line (g). Furthermore, approximately 1/4 of these models have also a minimal Standard Model Higgs sector with ($n_h = 1$).

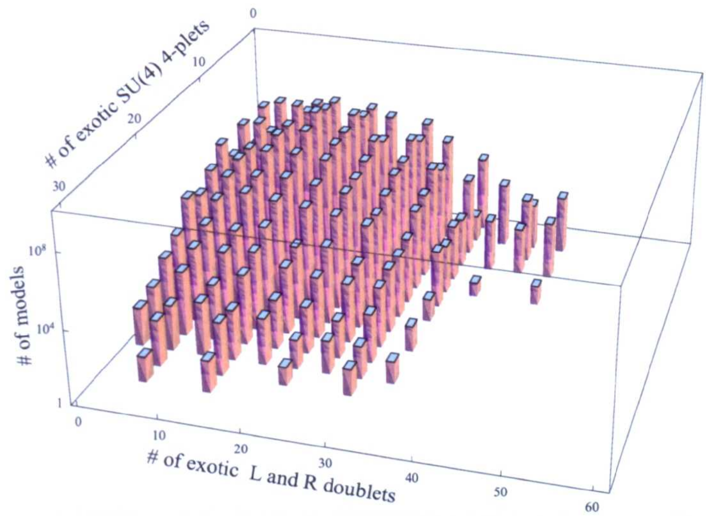

Figure 5.5: *Number of 3-generation models versus number of exotic $SU(4)$ multiplets and total number of L plus R exotic $SU(2)$ doublets in a random sample of 10^{11} GGSO configurations. We note that the exophobic cases correspond to the upper left column.*

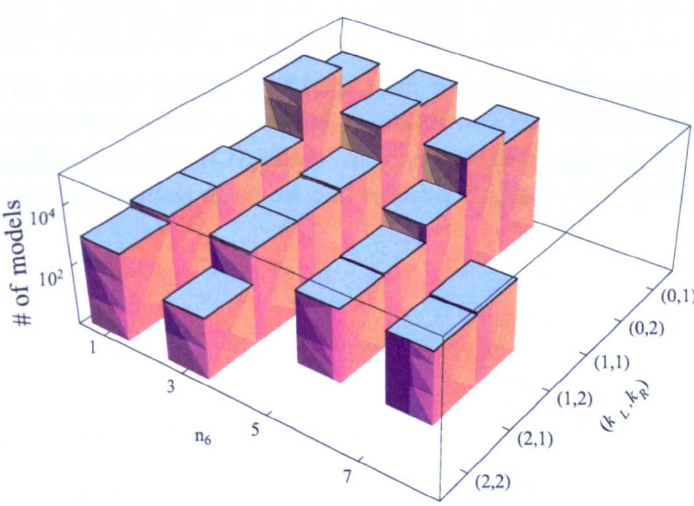

Figure 5.6: *Number of 3-generation models versus number of additional non-chiral left and right pairs (k_L, k_R) and additional $(6,1,1)$ $SU(4)$ reps (n_6) in a random sample of 10^{11} GGSO configurations. We note that accommodating the heavy Higgs states necessitates $k_R = 1$. By (5.33) the minimal case in realistic models also requires $n_6 = 1$.*

	Constraint	# of models in sample	Probability	Estimated # of models in class
	None	100000000000	1	2.25×10^{15}
(a)	+ No gauge group enhancements	78977078333	7.90×10^{-1}	1.78×10^{15}
(b)	+ Complete families	22497003372	2.25×10^{-1}	5.07×10^{14}
(c)	+ 3 generations	298140621	2.98×10^{-3}	6.71×10^{12}
(d)	+ PS breaking Higgs	23694017	2.37×10^{-4}	5.34×10^{11}
(e)	+ SM breaking Higgs	19191088	1.92×10^{-4}	4.32×10^{11}
(f)	+ No massless exotics	121669	1.22×10^{-6}	2.74×10^{9}
(g)	+ Minimal PS Higgs	31804	3.18×10^{-7}	7.16×10^{8}

Table 5.1: *Pati–Salam models statistics with respect to phenomenological constraints imposed on massless spectrum. Constraints in second column act additionally. Omitting constraint (e) does not change the results of (f), (g) since all massless exotic free models have an odd number of pairs of SM Higgs doublets.*

Chapter 6

Conclusions

IN THIS chapter we will give a short summary of the issues discussed in this thesis, and finally give some suggestions for further research.

We started by giving an overview of the Standard Model and theories of unification such as supersymmetry, extra dimensions and grand unification in Chapter 1. In the same chapter we briefly described how string theory attempts to explain physical reality under a single theoretical framework. An introduction to the basics of the bosonic string and superstring, including quantisation and compactification, were given in Chapter 2. In Chapter 3 the $E_8 \times E_8$ heterotic string and orbifold compactifications were introduced.

String theory enables the construction of phenomenological models that provide a self-consistent framework for the exploration of gauge interactions. The free fermionic compactifications of the $E_8 \times E_8$ heterotic string, which were described in Chapter 4, have provided some of the most realistic models to date, providing insights on gauge coupling unification, proton stability, top quark mass etc. The work conducted for the purpose of this thesis was concentrated in providing insights as to why we have not yet been able to observe any particles in exotic representations. By exploring a large class of heterotic Pati-Salam vacua in the way described in Chapter 5 we were able to show that such fractionally charged states might as well not exist at low energies. Furthermore, we have shown that there possibly exists a plethora of heterotic Pati-Salam vacua which provide both plausible phenomenology and an absence of massless exotic representations at the same time. This was achieved by utilising analytical and computational tools that distinguished models according to their chiral content. The distinction between different vacua was done by counting the observable and exotic chiral representations. Although these vacua are in principle different to each other, it is expected that there exists some degeneracy at the effective field theory limit. This degeneracy can be reduced if we also take into consideration the hidden representations of each model. It would also be interesting to see whether such "exophobic" vacua exist in Pati-Salam models with a different configuration of boundary condition vectors.

One such "exotic-free" model is exhibited in Chapter 4. In the same chapter we gave the ABK consistency rules, its spectrum and superpotential. The specific model combines a number of virtues, such as GUT and electroweak symmetry breaking, acquisition of heavy mass by coloured gauge fields which may mediate proton decay, an absence of massless exotic representations, and the existence of a trilinear top quark Yukawa coupling. Its phenomenology is studied and constrained by the analysis of supersymmetric flat directions. The numerical solution of the F- and D-flat constraints was affected by the existence of an anomalous $U(1)$, which is a generic feature of free fermionic models. Possible future directions regarding the phenomenology of isolated "exotic-free" models would be to find higher-order terms in the superpotential and a systematic classification of the F- and D-flat solutions.

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Appendix A

Spectrum and Superpotential

Untwisted sextet fields:

$$D_1 = (6,1,1)_{1,\ 0,\ 0}\ ,\ \chi^{12}$$
$$D_2 = (6,1,1)_{0,\ 1,\ 0}\ ,\ \chi^{34}$$
$$D_3 = (6,1,1)_{1,\ 0,\ 0}\ ,\ \chi^{56}$$
$$\bar{D}_1 = (6,1,1)_{-1,\ 0,\ 0}\ ,\ \chi^{12}$$
$$\bar{D}_2 = (6,1,1)_{0,\ -1,\ 0}\ ,\ \chi^{34}$$
$$\bar{D}_3 = (6,1,1)_{0,\ 0,\ -1}\ ,\ \chi^{56}\ .$$

Untwisted Singlets:

$$\Phi_{23} = (1,\ 1,\ 1)_{0,\ 1,\ 1}$$
$$\Phi_{\bar{23}} = (1,\ 1,\ 1)_{0,\ 1,\ -1}$$
$$\bar{\Phi}_{23} = (1,\ 1,\ 1)_{0,\ -1,\ -1}$$
$$\bar{\Phi}_{\bar{23}} = (1,\ 1,\ 1)_{0,\ -1,\ 1}$$
$$\Phi_{13} = (1,\ 1,\ 1)_{1,\ 0,\ 1}$$
$$\Phi_{\bar{13}} = (1,\ 1,\ 1)_{1,\ 0,\ -1}$$
$$\bar{\Phi}_{13} = (1,\ 1,\ 1)_{-1,\ 0,\ -1}$$
$$\bar{\Phi}_{\bar{13}} = (1,\ 1,\ 1)_{-1,\ 0,\ 1}$$
$$\Phi_{12} = (1,\ 1,\ 1)_{1,\ 1,\ 0}$$
$$\Phi_{\bar{12}} = (1,\ 1,\ 1)_{1,\ -1,\ 0}$$
$$\bar{\Phi}_{12} = (1,\ 1,\ 1)_{-1,\ -1,\ 0}$$
$$\bar{\Phi}_{\bar{12}} = (1,\ 1,\ 1)_{-1,\ 1,\ 0}\ .$$

Untwisted-Ungauged Singlets:

$$\Phi_1 = (1,1,1)_{0,0,0}\ ,\ \chi^{12},\ \bar{y}_1, \bar{\omega}_1$$
$$\Phi_2 = (1,1,1)_{0,0,0}\ ,\ \chi^{12},\ \bar{y}_2, \bar{\omega}_2$$
$$\Phi_3 = (1,1,1)_{0,0,0}\ ,\ \chi^{34},\ \bar{y}_3, \bar{\omega}_3$$
$$\Phi_4 = (1,1,1)_{0,0,0}\ ,\ \chi^{34},\ \bar{y}_4, \bar{\omega}_4$$
$$\Phi_5 = (1,1,1)_{0,0,0}\ ,\ \chi^{56},\ \bar{y}_5, \bar{\omega}_5$$
$$\Phi_6 = (1,1,1)_{0,0,0}\ ,\ \chi^{56},\ \bar{y}_6, \bar{\omega}_6\ .$$

Observable Spinorial Representations:

Sector	Field	$SU(4)\times SU(2)_L\times SU(2)_R$	$U(1)_1$	$U(1)_2$	$U(1)_3$
$S+b_2+e_6$	$\bar{F}_{1R}$	$(\bar{4},1,2)$	0	$-1/2$	0
$S+b_1+e_4+e_5$	$\bar{F}_{2R}$	$(\bar{4},1,2)$	$1/2$	0	0
$S+b_1+e_3+e_4+e_5+e_6$	F_{1R}	$(4,1,2)$	$-1/2$	0	0
$S+b_2+e_1+e_6$	F_{1L}	$(4,2,1)$	0	$-1/2$	0
$1+b_1+b_2+z_1+z_2+e_4+e_5+e_6$	F_{2L}	$(4,2,1)$	0	0	$-1/2$
$1+b_1+b_2+z_1+z_2+e_3+e_5+e_6$	$\bar{F}_{3R}$	$(\bar{4},1,2)$	0	0	$1/2$
$1+b_1+b_2+z_1+z_2+e_1+e_4+e_5+e_6$	$\bar{F}_{4R}$	$(\bar{4},1,2)$	0	0	$-1/2$
$1+b_1+b_2+z_1+z_2+e_1+e_3+e_5+e_6$	F_{3L}	$(4,2,1)$	0	0	$1/2$

Observable Vectorial Representations:

Sector	Field	$SU(4)\times SU(2)_L\times SU(2)_R$	$U(1)_1$	$U(1)_2$	$U(1)_3$
$S+b_1+b_2+e_2$	ζ_1	$(1,1,1)$	$1/2$	$-1/2$	0
	$\bar{\zeta}_1$	$(1,1,1)$	$-1/2$	$1/2$	0
$S+b_1+b_2+e_1+e_2+e_3+e_4$	ζ_2	$(1,1,1)$	$1/2$	$1/2$	0
	$\bar{\zeta}_2$	$(1,1,1)$	$-1/2$	$-1/2$	0
$1+b_2+z_1+z_2+e_3+e_4+e_6$	ζ_3	$(1,1,1)$	$1/2$	0	$-1/2$
	ζ_4	$(1,1,1)$	$1/2$	0	$-1/2$
	$\bar{\zeta}_3$	$(1,1,1)$	$-1/2$	0	$1/2$
	$\bar{\zeta}_4$	$(1,1,1)$	$-1/2$	0	$1/2$
	D_4	$(6,1,1)$	$-1/2$	0	$-1/2$
	ξ^+	$(1,1,1)$	$1/2$	1	$1/2$
	ξ^-	$(1,1,1)$	$1/2$	-1	$1/2$
$1+b_2+z_1+z_2+e_1+e_3+e_4+e_6$	h_1	$(1,2,2)$	$-1/2$	0	$-1/2$
$1+b_1+z_1+z_2+e_1+e_2+e_4$	h_3	$(1,2,2)$	0	$-1/2$	$-1/2$
$1+b_1+z_1+z_2+e_1+e_2+e_3+e_4+e_6$	h_2	$(1,2,2)$	0	$1/2$	$1/2$
$1+b_1+z_1+z_2+e_1+e_2+e_6$	ζ_5	$(1,1,1)$	0	$1/2$	$1/2$
	$\bar{\zeta}_5$	$(1,1,1)$	0	$-1/2$	$-1/2$
$1+b_1+z_1+z_2+e_1+e_2+e_3$	ζ_6	$(1,1,1)$	0	$1/2$	$1/2$
	$\bar{\zeta}_6$	$(1,1,1)$	0	$-1/2$	$-1/2$
$1+b_2+z_1+z_2+e_1+e_2+e_3+e_4+e_5$ $+e_6$	ζ_7	$(1,1,1)$	$1/2$	0	$-1/2$
	$\bar{\zeta}_7$	$(1,1,1)$	$-1/2$	0	$1/2$

Hidden Representations:

Sector	Field	Hidden Gauge Group	$U(1)_1$	$U(1)_2$	$U(1)_3$
$S+b_1+b_2+e_2+e_4$	H^1_{12}	$(2,2,1,1,1)$	$-1/2$	$-1/2$	0
$S+b_1+b_2+e_1+e_2+e_3$	H^2_{12}	$(2,2,1,1,1)$	$1/2$	$-1/2$	0
$1+b_2+z_1+z_2+e_1+e_3+e_4$	H^3_{12}	$(2,2,1,1,1)$	$1/2$	0	$-1/2$
$S+b_1+b_2+e_2+e_3$	H^1_{34}	$(1,1,2,2,1)$	$1/2$	$-1/2$	0
$S+b_1+b_2+e_1+e_2+e_4$	H^2_{34}	$(1,1,2,2,1)$	$-1/2$	$-1/2$	0
$1+b_2+z_1+z_2+e_3+e_4$	H^3_{34}	$(1,1,2,2,1)$	$1/2$	0	$-1/2$
$1+b_1+z_1+z_2+e_1+e_2$	H^4_{34}	$(1,1,2,2,1)$	0	$1/2$	$1/2$
$1+b_1+z_1+z_2+e_1+e_2+e_3+e_6$	H^5_{34}	$(1,1,2,2,1)$	0	$-1/2$	$-1/2$
$S+b_1+b_2+z_1$	H^1_{13}	$(2,1,2,1,1)$	$-1/2$	$-1/2$	0
$S+b_1+b_2+z_1+e_1+e_3+e_4$	H^2_{13}	$(2,1,2,1,1)$	$-1/2$	$1/2$	0
$1+b_2+z_2+e_1+e_3+e_4+e_5+e_6$	H^3_{13}	$(2,1,2,1,1)$	$-1/2$	0	$-1/2$
$1+b_2+z_2+e_1+e_3+e_4+e_5$	H^1_{14}	$(2,1,1,2,1)$	$1/2$	0	$1/2$
$1+b_1+z_2+e_1+e_2+e_4+e_5+e_6$	H^2_{14}	$(2,1,1,2,1)$	0	$1/2$	$1/2$
$1+b_1+z_2+e_1+e_2+e_3+e_4+e_5$	H^3_{14}	$(2,1,1,2,1)$	0	$-1/2$	$-1/2$
$S+b_1+z_2+z_1+e_3+e_4$	H^1_{24}	$(1,2,1,2,1)$	$-1/2$	$1/2$	0
$S+b_1+z_2+z_1+e_1$	H^2_{24}	$(1,2,1,2,1)$	$-1/2$	$-1/2$	0
$1+b_2+z_2+e_3+e_4+e_5+e_6$	H^3_{24}	$(1,2,1,2,1)$	$-1/2$	0	$-1/2$
$1+b_1+z_2+e_1+e_2+e_5+e_6$	H^4_{24}	$(1,2,1,2,1)$	0	$-1/2$	$1/2$
$1+b_1+z_2+e_1+e_2+e_3+e_5$	H^5_{24}	$(1,2,1,2,1)$	0	$1/2$	$-1/2$
$1+b_2+z_1+e_3+e_4+e_5$	H^1_{23}	$(1,2,2,1,1)$	$1/2$	0	$1/2$
$1+b_2+z_1+e_1+e_3+e_4$	Z_1	$(1,1,1,1,8_c)$	$-1/2$	0	$1/2$
$1+b_1+z_1+e_1+e_2+e_5+e6$	Z_2	$(1,1,1,1,8_s)$	0	$-1/2$	$-1/2$
$1+b_1+z_1+e_1+e_2+e_4+e6$	Z_3	$(1,1,1,1,8_c)$	0	$-1/2$	$1/2$
$1+b_1+z_1+e_1+e_2+e_3+e5$	Z_4	$(1,1,1,1,8_s)$	0	$-1/2$	$-1/2$
$1+b_1+z_1+e_1+e_2+e_3+e4$	Z_5	$(1,1,1,1,8_c)$	0	$1/2$	$-1/2$

where the Hidden Gauge Group $\equiv SU(2)_1\times SU(2)_2\times SU(2)_3\times SU(2)_4\times SO(8)$.

Yukawa Couplings	$\bar{F}_{2R}F_{3L}h_1$
Bi-doublet Masses	$h_1h_1\Phi_{13} + h_2h_2\bar{\Phi}_{23} + h_3h_3\Phi_{23} + h_1h_2\zeta_1$
Sextet Terms	$D_1D_2\bar{\Phi}_{12} + \bar{D}_1D_2\Phi_{\overline{12}} + D_1\bar{D}_2\bar{\Phi}_{\overline{12}} + \bar{D}_1\bar{D}_2\Phi_{12}$ $D_1D_3\bar{\Phi}_{13} + \bar{D}_1D_3\Phi_{\overline{13}} + D_1\bar{D}_3\bar{\Phi}_{\overline{13}} + \bar{D}_1\bar{D}_3\Phi_{13}$ $D_2D_3\bar{\Phi}_{23} + \bar{D}_2D_3\Phi_{\overline{23}} + D_2\bar{D}_3\bar{\Phi}_{\overline{23}} + \bar{D}_2\bar{D}_3\Phi_{23}$ $D_1F_{1R}F_{1R} + \bar{D}_1\bar{F}_{2R}\bar{F}_{2R}$ $D_2(\bar{F}_{1R}\bar{F}_{1R} + F_{1L}F_{1L})$ $D_3(\bar{F}_{4R}\bar{F}_{4R} + F_{2L}F_{2L}) + \bar{D}_3(\bar{F}_{3R}\bar{F}_{3R} + F_{3L}F_{3L})$ $D_4(\bar{F}_{2R}\bar{F}_{3R} + D_2\xi^- + \bar{D}_2\xi^+ + D_4\Phi_{13})$
Hidden Terms	$\Phi_{23}(H^3_{14}H^3_{14} + H^5_{34}H^5_{34} + Z_2Z_2 + Z_4Z_4)$ $\Phi_{\overline{23}}(H^4_{24}H^4_{24} + Z_3Z_3)$ $\bar{\Phi}_{23}(H^2_{14}H^2_{14} + H^4_{34}H^4_{34})$ $\bar{\Phi}_{\overline{23}}(H^5_{24}H^5_{24} + Z_5Z_5)$ $\Phi_{13}(H^3_{24}H^3_{24} + H^3_{13}H^3_{13})$ $\Phi_{\overline{13}}Z_1Z_1$ $\bar{\Phi}_{13}(H^1_{23}H^1_{23} + H^1_{14}H^1_{14})$ $\bar{\Phi}_{\overline{13}}(H^3_{34}H^3_{34} + H^3_{12}H^3_{12})$ $\Phi_{12}(H^1_{13}H^1_{13} + H^2_{24}H^2_{24} + H^1_{12}H^1_{12} + H^2_{34}H^2_{34})$ $\Phi_{\overline{12}}(H^1_{24}H^1_{24} + H^2_{13}H^2_{13}$ $\bar{\Phi}_{\overline{12}}(H^1_{34}H^1_{34} + H^2_{12}H^2_{12})$ $\zeta_1Z_1Z_5$ $\bar{\zeta}_1H^1_{14}H^3_{14}$ $\zeta_2H^3_{24}H^4_{24}$ $\bar{\zeta}_2H^3_{34}H^4_{34}$ $\xi^+H^2_{34}H^5_{34}$ $\zeta_6(H^2_{34}H^3_{34} + H^1_{12}H^3_{12})$ $\zeta_7H^1_{24}H^4_{24}$ $H^1_{23}H^5_{24}H^2_{34} + H^3_{24}H^2_{14}H^2_{12}$ $H^1_{14}H^5_{24}H^1_{12} + H^3_{13}H^2_{14}H^1_{34}$
Singlet Terms	$\Phi_2(\zeta_5\bar{\zeta}_5 + \zeta_6\bar{\zeta}_6)$ $\Phi_5(\zeta_1\bar{\zeta}_1 + \zeta_2\bar{\zeta}_2)$ $\Phi_{23}(\Phi_{\overline{13}}\bar{\Phi}_{12} + \bar{\Phi}_{13}\Phi_{\overline{12}} + \bar{\zeta}_5\bar{\zeta}_5 + \bar{\zeta}_6\bar{\zeta}_6)$ $\Phi_{\overline{23}}(\Phi_{13}\bar{\Phi}_{12} + \bar{\Phi}_{\overline{13}}\Phi_{\overline{12}})$ $\bar{\Phi}_{23}(\Phi_{13}\bar{\Phi}_{\overline{12}} + \bar{\Phi}_{\overline{13}}\Phi_{12} + \zeta_5\zeta_5 + \zeta_6\zeta_6)$ $\bar{\Phi}_{\overline{23}}(\Phi_{\overline{13}}\bar{\Phi}_{\overline{12}} + \bar{\Phi}_{13}\Phi_{12})$ $\Phi_{\overline{13}}(\bar{\zeta}_3\bar{\zeta}_3 + \bar{\zeta}_4\bar{\zeta}_4 + \bar{\zeta}_7\bar{\zeta}_7)$ $\bar{\Phi}_{13}\xi^+\bar{\xi}^-$ $\bar{\Phi}_{\overline{13}}(\zeta_3\zeta_3 + \zeta_4\zeta_4 + \zeta_7\zeta_7)$ $\Phi_{12}\bar{\zeta}_2\bar{\zeta}_2$ $\Phi_{\overline{12}}\bar{\zeta}_1\bar{\zeta}_1$ $\bar{\Phi}_{12}\zeta_2\zeta_2$ $\bar{\Phi}_{\overline{12}}\zeta_1\zeta_1$ $\zeta_2\bar{\zeta}_3\bar{\zeta}_5$ $\bar{\zeta}_2(\zeta_4\zeta_5 + \xi^+\bar{\zeta}_5)$

Appendix B

Enhancements

Enhancements of the Observable gauge group

- $\chi = \{\bar{\eta}^{123}, \bar{\psi}^{12345}\}$ is the only sector which can possibly enlarge the observable gauge group. Enhancement takes place when the following conditions are satisfied

Enhancement conditions	Resulting Enhancement
$(\chi\|e_i) = (\chi\|z_n) = 0$	$SU(4)_{obs} \times SU(2)_{L/R} \times U(1)' \rightarrow SU(6)$

The pre-stated conditions hold $\forall\, i = 1, ..., 6$, $n = 1, 2$ and $U(1)'$ is a linear combination of the $U(1)_i$.

Enhancements of the Hidden gauge group

- $z_1 + z_2 = \{\bar{\phi}^{12345678}\}$ is the only sector which can enlarge the hidden gauge group when all of the following conditions are met

Enhancement conditions	Resulting Enhancement
$(e_i\|z_1 + z_2) = (b_k\|z_1 + z_2) = 0$ $\forall\, i = 1, ..., 6,\ k = 1, 2$	$SU(2)_{1/2} \times SU(2)_{3/4} \times SO(8)_{hid} \rightarrow SO(12)$

Mixed gauge group enhancements

Parts of the observable and hidden gauge group can be enhanced simultaneously in multiple cases.

- $\alpha + \chi = \{\bar{\eta}^{123}, \bar{\Psi}^{123}, \bar{\Phi}^{12}\}$

Enhancement Conditions	Resulting Enhancement
$(e_i\|\alpha + \chi) = (z_2\|\alpha + \chi) = 0, \quad \forall i = 1, ..., 6$ $(z_1\|\alpha + \chi) = (\alpha\|\alpha + \chi)$	$SU(4)_{obs} \times SU(2)_{1/2} \times U(1)' \rightarrow SU(6)$

- $\alpha + z_2 = \{\bar{\Psi}^{45}, \bar{\Phi}^{12}, \bar{\Phi}^{5678}\}$

Enhancement Conditions	Resulting Enhancement
$(e_i\|\alpha + z_2) = 0, \quad \forall i = 1, ..., 6$ $(b_1\|\alpha + z_2) = (b_2\|\alpha + z_2)$ $(b_k\|\alpha + z_2) + (z_1\|\alpha + z_2) = (\alpha\|\alpha + z_2)$	$SU(2)_{L/R} \times SU(2)_{1/2} \times SO(8)_{hid} \rightarrow SO(12)$

- $z_1 = \{\bar{\phi}^{1234}\}$ gives rise to different types of enhancements which are summarised in the following table.

Survival Conditions	Resulting Enhancement
$(e_i\|z_1) = (z_2\|z_1) = 0$ $(b_k\|z_1) = 1$	$SU(4)_{obs} \times SU(2)_{1/2} \times SU(2)_{3/4} \rightarrow SO(10)$
$(e_i\|z_1) = (z_2\|z_1) = 0$ $(b_k\|z_1) = 1$	$SU(2)_L \times SU(2)_R \times SU(2)_{2/1} \times SU(2)_{4/3} \rightarrow SO(8)$
$(e_i\|z_1) = (z_2\|z_1) = (b_2\|z_1) = 0$ $(b_1\|z_1) = 1$	$SU(2)_{1/2} \times SU(2)_{3/4} \times U(1) \rightarrow SO(6)$
$(e_i\|z_1) = (z_2\|z_1) = (b_1\|z_1) = 0$ $(b_2\|z_1) = 1$	$SU(2)_{1/2} \times SU(2)_{3/4} \times U(1) \rightarrow SO(6)$
$(e_i\|z_1) = (z_2\|z_1) = (b_k\|z_1) = 0$	$SU(2)_{1/2} \times SU(2)_{3/4} \times U(1) \rightarrow SO(6)$
$(e_j\|z_1) = (z_2\|z_1) = 0$ $(e_i\|z_1) = 1$ AND $(b_1\|z_1) = 0, (b_2\|z_1) = 1, i = 1, 2$ or $(b_1\|z_1) = 1, (b_2\|z_1) = 0, i = 3, 4$ or $(b_1\|z_1) = 1, (b_2\|z_1) = 1, i = 5, 6$	$SU(2)_{1/2} \times SU(2)_{3/4} \rightarrow SO(5)$
$(e_j\|z_1) = (z_2\|z_1) = 0$ $(e_i\|z_1) = 1$ $(b_k\|z_1) = 0$	$SU(2)_{1/2} \times SU(2)_{3/4} \rightarrow SO(5)$
$(e_i\|z_1) = (b_k\|z_1) = 0$ $(z_2\|z_1) = 1$	$SU(2)_{1/2} \times SU(2)_{3/4} \times SO(8)_{hid} \rightarrow SO(12)$

The relations above hold $\forall\ i, j = 1, ...6; i \neq j$ and $k = 1, 2$.

- $\alpha + z_1 + z_2 = \{\bar{\Psi}^{45}, \bar{\Phi}^{34}, \bar{\Phi}^{5678}\}$

Enhancement Conditions	Resulting Enhancement
$(e_i\|\alpha + z_1 + z_2) = 0$ $(b_1\|\alpha + z_1 + z_2) = (b_2\|\alpha + z_1 + z_2) = (\alpha\|\alpha + z_1 + z_2)$	$SU(2)_{L/R} \times SU(2)_{3/4} \times SO(8)_{hid}$ $\rightarrow SO(12)$

The conditions of the previous table hold $\forall i = 1, ..., 6$.

- $z_2 = \{\bar{\phi}^{5678}\}$ can generate enhancements in the following cases

Survival Conditions	Resulting Enhancement
$(e_i\|z_2) = (z_1\|z_2) = (\alpha\|z_2) = 0$ $(b_k\|z_2) = 1$	$SU(4)_{obs} \times SO(8)_{hid} \rightarrow SO(14)$
$(e_i\|z_2) = (z_1\|z_2) = 0$ $(b_k\|z_2) = (\alpha\|z_2) = 1$	$SU(2)_L \times SU(2)_R \times SO(8)_{hid} \rightarrow SO(12)$
$(e_i\|z_2) = (z_1\|z_2) = (b_2\|z_2) = (\alpha\|z_2) = 0$ $(b_1\|z_2) = 1$	$U(1) \times SO(8)_{hid} \rightarrow SO(10)$
$(e_i\|z_2) = (z_1\|z_2) = (b_1\|z_2) = (\alpha\|z_2) = 0$ $(b_2\|z_2) = 1$	$U(1) \times SO(8)_{hid} \rightarrow SO(10)$
$(e_i\|z_2) = (z_1\|z_2) = (b_k\|z_2) = (\alpha\|z_2) = 0$	$U(1) \times SO(8)_{hid} \rightarrow SO(10)$
$(e_j\|z_2) = (z_1\|z_2) = (\alpha\|z_2) = 0$ $(e_i\|z_2) = 1$ AND $(b_1\|z_2) = 0, (b_2\|z_2) = 1, i = 1,2$ or $(b_1\|z_2) = 1, (b_2\|z_2) = 0, i = 3,4$ or $(b_1\|z_2) = 1, (b_2\|z_2) = 1, i = 5,6$	$SO(8)_{hid} \rightarrow SO(9)$
$(e_j\|z_2) = (z_1\|z_2) = (b_k\|z_2) = (\alpha\|z_2) = 0$ $(e_i\|z_2) = 1$	$SO(8)_{hid} \rightarrow SO(9)$
$(e_i\|z_2) = (b_k\|z_2) = 0$ $(\alpha\|z_2) = (z_1\|z_2) = 1$	$SO(4)_1 \times SO(8)_{hid} \rightarrow SO(12)$
$(e_i\|z_2) = (b_k\|z_2) = (\alpha\|z_2) = 0$ $(z_1\|z_2) = 1$	$SO(4)_2 \times SO(8)_{hid} \rightarrow SO(12)$

The relations above hold $\forall\ i,j = 1,....6; i \neq j$ and $k = 1,2$.

- $\alpha + \chi + z_1 = \{\bar{\eta}^{123}, \bar{\Psi}^{123}, \bar{\Phi}^{34}\}$

Enhancement Conditions	Resulting Enhancement
$(e_i\|\alpha + \chi + z_1) = (z_2\|\alpha + \chi + z_1) = 0$	$SU(4)_{obs} \times SU(2)_{3/4} \times U(1)' \rightarrow SU(6)$

The conditions above hold $\forall i = 1, ..., 6$.

- $\alpha = \{\bar{\Psi}^{45}\bar{\phi}^{12}\}$ can also present numerous potential enhancements.

Survival Conditions	Resulting Enhancement
$(e_i\|\alpha) = (z_2\|\alpha) = 0$ $(b_1\|\alpha) = (b_2\|\alpha)$ $(1\|\alpha) = 1 + (b_k\|\alpha) + (z_1\|\alpha)$	$SU(4)_{obs} \times SU(2)_{L/R} \times SU(2)_{1/2} \rightarrow SO(10)$ AND $SU(2)_{L/R} \times SU(2)_{1/2} \times SU(2)_3 \times SU(2)_4 \rightarrow SO(8)$
$(e_i\|\alpha) = (z_2\|\alpha) = 0$ $(b_1\|\alpha) = 1 + (b_2\|\alpha)$ $(1\|\alpha) = (b_1\|\alpha) + (z_1\|\alpha)$	$U(1) \times SU(2)_{L/R} \times SU(2)_{1/2} \rightarrow SO(6)$
$(e_i\|\alpha) = (z_2\|\alpha) = 0$ $(b_2\|\alpha) = 1 + (b_1\|\alpha)$ $(1\|\alpha) = (b_2\|\alpha) + (z_1\|\alpha)$	$U(1) \times SU(2)_{L/R} \times SU(2)_{1/2} \rightarrow SO(6)$
$(e_i\|\alpha) = (z_2\|\alpha) = 0$ $(b_1\|\alpha) = (b_2\|\alpha)$ $(1\|\alpha) = (b_2\|\alpha) + (z_1\|\alpha)$	$U(1) \times SU(2)_{L/R} \times SU(2)_{1/2} \rightarrow SO(6)$
$(e_j\|\alpha) = (z_2\|\alpha) = 0$ $(e_i\|\alpha) = 1$ AND $(b_1\|\alpha) = 1 + (b_2\|\alpha)$ and $(1\|\alpha) = (b_1\|\alpha) + (z_1\|\alpha), \quad i = 1,2$ or $(b_1\|\alpha) = 1 + (b_2\|\alpha)$ and $(1\|\alpha) = (b_2\|\alpha) + (z_1\|\alpha), \quad i = 3,4$ or $(b_1\|\alpha) = (b_2\|\alpha)$ and $(1\|\alpha) = 1 + (b_k\|\alpha) + (z_1\|\alpha), i = 5,6$	$SU(2)_{L/R} \times SU(2)_{1/2} \rightarrow SO(5)$
$(e_j\|\alpha) = (z_2\|\alpha) = 0$ $(e_i\|\alpha) = 1$ $(b_1\|\alpha) = (b_2\|\alpha)$ $(1\|\alpha) = (b_k\|\alpha) + (z_1\|\alpha)$	$SU(2)_{L/R} \times SU(2)_{1/2} \rightarrow SO(5)$
$(e_i\|\alpha) = 0$ $(z_2\|\alpha) = 1$ $(b_1\|\alpha) = (b_2\|\alpha)$ $(1\|\alpha) = (b_k\|\alpha) + (z_1\|\alpha)$	$SU(2)_{L/R} \times SU(2)_{1/2} \times SO(8)_{hid} \rightarrow SO(12)$

The relations above hold $\forall\ i,j = 1,....6; i \neq j$ and $k = 1,2$.

- $\alpha + z_1 = \{\bar{\Psi}^{45}\bar{\phi}^{34}\}$ gives rise to enhancements in the following occasions:

Survival Conditions	Resulting Enhancement
$(e_i\|\alpha+z_1) = (z_2\|\alpha+z_1) = 0$ $(b_1\|\alpha+z_1) = (b_2\|\alpha+z_1)$ $(\alpha\|\alpha+z_1) = 1 + (b_k\|\alpha+z_1)$	$SU(4)_{obs} \times SU(2)_{L/R} \times SU(2)_{3/4}$ $\rightarrow SO(10)$ AND $SU(2)_{L/R} \times SO(4)_1 \times SU(2)_{3/4}$ $\rightarrow SO(8)$
$(e_i\|\alpha+z_1) = (z_2\|\alpha+z_1) = 0$ $1 + (b_1\|\alpha+z_1) = (b_2\|\alpha+z_1) = (\alpha\|\alpha+z_1)$	$U(1) \times SU(2)_{L/R} \times SU(2)_{3/4}$ $\rightarrow SO(6)$
$(e_i\|\alpha+z_1) = (z_2\|\alpha+z_1) = 0$ $1 + (b_2\|\alpha+z_1) = (b_1\|\alpha+z_1) = (\alpha\|\alpha+z_1)$	$U(1) \times SU(2)_{L/R} \times SU(2)_{3/4}$ $\rightarrow SO(6)$
$(e_i\|\alpha+z_1) = (z_2\|\alpha+z_1) = 0$ $(b_1\|\alpha+z_1) = (b_2\|\alpha) = (\alpha\|\alpha+z_1)$	$U(1) \times SU(2)_{L/R} \times SU(2)_{3/4}$ $\rightarrow SO(6)$
$(e_j\|\alpha+z_1) = (z_2\|\alpha+z_1) = 0$ $(e_i\|\alpha+z_1) = 1$ AND $(b_1\|\alpha+z_1) = 1 + (b_2\|\alpha+z_1) = (\alpha\|\alpha+z_1), i = 1,2$ or $(b_1\|\alpha+z_1) = 1 + (b_2\|\alpha+z_1) = 1 + (\alpha\|\alpha+z_1), i = 3,4$ or $(b_1\|\alpha+z_1) = (b_2\|\alpha+z_1) = 1 + (\alpha\|\alpha+z_1), i = 5,6$	$SU(2)_{L/R} \times SU(2)_{3/4} \rightarrow SO(5)$
$(e_j\|\alpha+z_1) = (z_2\|\alpha+z_1) = 0$ $(e_i\|\alpha+z_1) = 1$ $(b_1\|\alpha+z_1) = (b_2\|\alpha+z_1)$ $(1\|\alpha+z_1) = (b_k\|\alpha+z_1) + (z_1\|\alpha+z_1)$	$SU(2)_{L/R} \times SU(2)_{3/4} \rightarrow SO(5)$
$(e_i\|\alpha+z_1) = 0$ $(z_2\|\alpha+z_1) = 1$ $(b_1\|\alpha+z_1) = (b_2\|\alpha+z_1) = (\alpha\|\alpha+z_1)$	$SU(2)_{L/R} \times SU(2)_{3/4} \times SO(8)$ $\rightarrow SO(12)$

Appendix C

Results

n_g	n_h	n_6	# of models	n_g	n_h	n_6	# of models
0	0	0	7389484	0	10	0	948
0	0	2	1645466	0	10	2	3951
0	0	4	1000290	0	10	6	1650
0	0	6	7964	0	10	8	716
0	0	8	35156	0	10	10	2681
0	0	12	125	0	10	14	7
0	0	16	48	0	12	0	1657
0	2	0	1772537	0	12	4	2207
0	2	2	3370245	0	12	8	322
0	2	4	282693	0	12	12	2458
0	2	6	101806	0	14	2	14
0	2	8	240	0	14	10	4
0	2	10	1425	0	16	0	336
0	4	0	1281766	0	16	4	37
0	4	2	314402	0	16	8	98
0	4	4	1272994	0	16	16	121
0	4	6	41240	0	18	2	3
0	4	8	26600	0	20	0	2
0	4	12	695	0	20	4	1
0	4	16	3	0	20	12	2
0	6	0	32801	0	24	0	2
0	6	2	162980	0	24	8	1
0	6	4	42929	0	24	24	1
0	6	6	197305	1	1	1	690074
0	6	10	1077	1	1	3	50495
0	8	0	83905	1	3	1	54719
0	8	2	891	1	3	3	701850
0	8	4	44391	1	3	5	47239
0	8	8	53896	1	5	3	51664
0	8	10	667	1	5	5	91419
0	8	12	198	1	5	7	2408
0	8	16	38	1	7	5	2636

Table C.1: *Multiplicities of massless fractional charge free models in a random sample of* 10^{11} *PS models.*

n_g	n_h	n_6	# of models	n_g	n_h	n_6	# of models
1	7	7	2283	4	0	12	3
2	0	0	159209	4	2	2	145699
2	0	4	2935	4	2	6	2159
2	2	2	1060873	4	2	10	14
2	2	6	15898	4	4	0	4757
2	2	10	243	4	4	4	118796
2	4	0	4435	4	4	8	1546
2	4	4	220673	4	4	12	42
2	4	8	1180	4	6	2	2660
2	6	2	25966	4	6	6	27834
2	6	6	53586	4	6	10	84
2	6	10	52	4	8	0	556
2	8	0	526	4	8	4	2484
2	8	4	1631	4	8	8	7942
2	8	8	5419	4	10	2	24
2	10	2	824	4	10	6	81
2	10	6	61	4	10	10	22
2	10	10	629	4	12	0	37
3	1	1	240224	4	12	4	124
3	1	3	19086	4	12	12	234
3	3	1	20709	4	16	0	1
3	3	3	238714	5	1	1	5743
3	3	5	14007	5	3	3	24930
3	5	3	14932	5	5	5	16949
3	5	5	56886	5	7	7	656
3	5	7	539	6	0	0	9339
3	7	5	591	6	0	4	162
3	7	7	3135	6	2	2	34884
4	0	0	105365	6	2	6	55
4	0	4	3234	6	4	0	184
4	0	8	114	6	4	4	10612

Table C.1 continued.

n_g	n_h	n_6	# of models
6	4	8	26
6	6	2	62
6	6	6	7539
6	6	10	10
6	8	4	34
6	8	8	781
6	10	6	20
6	10	10	187
8	0	0	2543
8	0	8	35
8	2	2	2529
8	4	4	7055
8	4	12	3
8	6	6	1742
8	8	0	19
8	8	8	3328
8	8	16	1
8	10	10	134
8	12	4	4
8	12	12	100
8	16	8	3
8	16	16	4
10	0	0	124
10	2	2	219
10	4	4	112
10	6	6	187
10	8	8	23
12	0	0	47
12	2	2	22
12	4	4	122
12	8	8	145
12	10	10	3
12	12	12	43
16	0	0	7
16	4	4	17
16	8	8	7
16	12	12	4

Table C.1 continued.

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