Families of Skew-symmetric Matrices

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Abstract

Denoting the space of $n \times n$ skew-symmetric matrices, with entries in $\mathbb{C}$, by $Sk(n, \mathbb{C})$ there is a natural notion of equivalence on this space given by an action of the group $Gl(n, \mathbb{C})$. We refer to this as *skew-equivalence*.

In this thesis we consider the classification of families of $n \times n$ skew-symmetric matrices, given by smooth germs $\mathbb{C}^r, 0 \to Sk(n, \mathbb{C})$, up to a related equivalence, $\mathcal{G}$-equivalence, introduced by Bruce, Tari to classify families of symmetric matrices. We use the complete transversal theory of Bruce, du Plessis, Kirk and determinacy results of Damon, Wall, Bruce, du Plessis, to obtain lists of simple $\mathcal{G}$-finitely determined germs. We are assisted in this task by the `Maple` computer package `Transversal`, developed by Kirk.

Our classifications are initiated by considering the action of the appropriate jet-group, $J^1\mathcal{G}$, on the 1-jets of such germs. This amounts to an action of $Gl(n, \mathbb{C})$ on r-dimensional subspaces of $Sk(n, \mathbb{C})$. In particular we consider this action on 2-dimensional subspaces of $Sk(n, \mathbb{C})$ or *pencils* of skew-symmetric matrices. This requires adapting classical techniques, for the reduction of both non-singular and singular pencils of matrices, to the skew-symmetric case.

Non-singular pencils are dealt with by considering representative pairs of skew-symmetric matrices of which at least one is non-singular. Representing these pairs by λ-matrices and considering the associated λ-equivalence we find a complete set of normal forms. Together with a canonical reduction of singular skew-symmetric pencils, derived from the classical approach of Kronecker, we obtain a set of normal forms for pencils of skew-symmetric matrices. In particular we find an explicit list of normal forms for 1-jets of germs $\mathbb{C}^2, 0 \to Sk(4, \mathbb{C})$.

The main result is a complete list of all simple $\mathcal{G}$-finitely determined germs $\mathbb{C}^2, 0 \to Sk(4, \mathbb{C})$. We identify a relationship between aspects of this classification and a classification of families of general 2×2 matrices, $\mathbb{C}^r \to M(2, \mathbb{C})$, carried out by Bruce, Tari. We also consider special families of skew-symmetric matrices whose classification is either fairly straightforward or is obtained from existing classifications of map-germs under $\mathcal{K}$-equivalence.

In addition we perform a selective classification of 3-parameter families of 4×4 skew-symmetric matrices. This is initiated by using our list of normal forms for pencils as a foundation for finding skew-equivalent normal forms for 3-dimensional subspaces of $Sk(4, \mathbb{C})$.

Corresponding to the classifcation of 1-jets of rank r of families of skew-symmetric matrices we identify a dual classification of the 1-jets of corank r of mappings into the dual space $Sk(n, \mathbb{C})^*$.

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Contents

Chapter 1

Introduction

We start with some background notation and results from linear algebra, for which we refer to [Lang].

If V, W are vector spaces over a field K then $Hom(V, W)$ denotes the set of linear maps from V to W. The set of isomorphisms $V \to V$ is denoted by $Gl(V)$. When $W = K$ then $Hom(V, W) = Hom(V, K)$ is the dual of V and denoted by V^*. There is a canonical monomorphism $V \to (V^*)^*$, which is an isomorphism when V is finite dimensional.

The tensor product of V with itself is denoted by $V \otimes V$, and there is a natural bi-linear map $V \times V \to V \otimes V$, $(v, w) \mapsto v \otimes w$. The set $\bigwedge^2 V$ is obtained as the quotient of the tensor product $V \otimes V$ by the submodule generated by elements of the form $v \otimes v$. We define the wedge product of two vectors $v_1, v_2 \in V$ $v_1 \wedge v_2$ to be the image of $v_1 \otimes v_2$ in the quotient $\bigwedge^2 V$. It is linear in both slots since $\otimes$ is. It follows that $0 = (v_1 + v_2) \wedge (v_1 + v_2)$, and since $v_1 \wedge v_1 = v_2 \wedge v_2 = 0$ it is clear that $v_1 \wedge v_2 + v_2 \wedge v_1 = 0$. Clearly any element of $\bigwedge^2 V$ is a linear combination of elements of the form $v_1 \wedge v_2$.

If $\{e_1, \ldots, e_n\}$ is a basis for V then one easily checks that $e_i \wedge e_j$, $1 \leq i < j \leq n$ is a basis for $\bigwedge^2 V$.

We now give a general introduction to skew-symmetric or alternating bilinear forms, and see how skew-symmetric matrices naturally emerge.

Definition 1.0.1 *(a) Let V be vector space defined over a field K. A bilinear form, ϕ, on V is said to be* **alternating** *or* **skew-symmetric** *if $\phi(v, v) = 0$, for all $v \in V$.*

(b) A skew form ϕ is **non-singular** *if $\phi(v, -) : V \to K$ is non-zero for all $0 \neq v \in V$. Otherwise ϕ is* **singular**.

(c) We denote the set of all skew-symmetric bilinear forms on V by $Alt(V)$, where

$$Alt(V) = \{\phi : V \times V \longrightarrow K : \phi(v, v) = 0, \quad \textit{for all } v \in V\}.$$

If $\phi : V \times V \longrightarrow K$ is a skew-symmetric bilinear form, by expanding $\phi(v_1 + v_2, v_1 + v_2)$, v_1, $v_2 \in V$, then

$$\begin{aligned}\phi(v_2, v_1) + \phi(v_1, v_2) &= \phi(v_1 + v_2, v_1 + v_2) - \phi(v_1, v_1) - \phi(v_2, v_2) \\ &= 0.\end{aligned}$$

Hence any skew-symmetric bilinear form is also *antisymmetric*, i.e.

$$\phi(v_2, v_1) = -\phi(v_1, v_2). \tag{1.1}$$

Conversely, if ϕ is an antisymmetric form, then setting $v_2 = v_1$ in (1.1) gives $2\phi(v_1, v_1) = 0$. So if K is not of characteristic 2 then ϕ is skew-symmetric.

Remark 1.0.2 By the above, if K is not of characteristic 2 there is no need to distinguish between skew-symmetric and antisymmetric forms. However, in characterstic 2 this is not the case. For this reason it is preferable to consider skew-symmetric rather than antisymmetric forms.

Relative to a basis $\{e_1, \ldots, e_n\}$ of V a skew-symmetric form, ϕ is specified by an $n \times n$ matrix $A = (a_{ij})$, where $a_{ij} = \phi(e_i, e_j)$. This leads to the following definition.

Definition 1.0.3 *An $n \times n$ matrix $A = (a_{ij})$ with entries over a field K is* **skew-symmetric** *if $a_{ii} = 0$, $a_{ji} = -a_{ij}$, that is A is of the form*

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ -a_{12} & 0 & a_{23} & \cdots & a_{2n} \\ -a_{13} & -a_{23} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & a_{n-1n} \\ -a_{1n} & -a_{2n} & \cdots & -a_{n-1n} & 0 \end{bmatrix}.$$

Lemma 1.0.4 *If K is not of characteristic 2 then a matrix is skew-symmetric if and only if*

$$A^T = -A. \tag{1.2}$$

Proof If $a_{ii} = 0$, $a_{ji} = -a_{ij}$ then clearly $A^T = -A$. However if $A^T = -A$ then $a_{ji} = -a_{ij}$ and $a_{ii} = -a_{ii}$, the latter implying that $2a_{ii} = 0$. So if K is not of characteristic 2 then $a_{ii} = 0$ and A is skew-symmetric. □

The relevance of the matrix constructed above from a basis and a skew form is explained in the following obvious result.

Lemma 1.0.5 *(a) Let V be a vector space, with basis $\{e_1, \ldots, e_n\}$, and ϕ a skew form on V. Then if $v = \sum v_i e_i, w = \sum w_i e_i \in V$ we have $\phi(v, w) = \bar{v}^T A \bar{w}$ where $\bar{v}, \bar{w}$ are the column vectors corresponding to the v_i, w_j, and A is as above.*

(b) A skew form is singular if and only if its matrix representation with respect to any (and hence all bases) is singular.

Lemma 1.0.6 *Any skew-symmetric matrix, A, of odd order n and defined over a field K, not of characteristic 2, has zero determinant.*

Proof Since A is skew-symmetric

$$A^T = -A.$$

Taking determinants

$$\det(A^T) = \det(-A)$$

and hence

$$\det A = (-1)^n \det A$$

which, since n is odd, implies that

$$2 \det A = 0.$$

So as long as K is not of characteristic 2 then

$$\det A = 0.$$

□

We sometimes refer to skew-symmetric matrices of odd or even orders as *odd* or *even* skew-symmetric matrices, accordingly. Note, from now on, we assume the field K is infinite and not of characteristic 2 and hence all matrices, A, defined over K which satisfy (1.2) will be skew-symmetric. The following result will be of use later on.

Lemma 1.0.7 *Given any skew-symmetric matrix of the form*

$$A = \begin{bmatrix} 0 & a_1 & & & & & \\ -a_1 & 0 & a_2 & & & & \\ & -a_2 & 0 & & & & \\ & & & \ddots & & & \\ & & & & 0 & a_s \\ & & & & -a_s & 0 \end{bmatrix},$$

then

$$\det A = \begin{cases} 0 & \textit{if } s \textit{ is even} \\ a_1^2 a_3^2 \cdots a_s^2 & \textit{if } s \textit{ is odd} \end{cases}.$$

Proof If s is even then A is an odd skew-symmetric matrix and so its determinant is zero. If s is odd we have an even skew-symmetric matrix and it can be verified that its determinant is that given above. □

1.1 The Space of Skew-symmetric Matrices

We denote the space of all $n \times n$ skew-symmetric matrices over K by $Sk(n, K)$. Each skew-symmetric matrix is determined by its upper triangular entries and hence we can think of $Sk(n, K)$ as a vector space with

$$\begin{aligned} \dim Sk(n,K) &= (n-1) + (n-2) + \cdots + 1 && (1.3) \\ &= \frac{1}{2}n(n-1). && (1.4) \end{aligned}$$

The standard basis vectors for $Sk(n, K)$ are the set of matrices

$$\{E^{ij} : 1 \leq i < j \leq n\},$$

whose (i, j)th element is 1 and whose (j, i)th element is -1 with all other entries zero. (Alternatively, the (r, s)th entry of E^{ij}, E^{ij}_{rs}, is given by

$$E^{ij}_{rs} = \delta_{ir}\delta_{js} - \delta_{is}\delta_{jr}. \ )$$

Let $N = n(n-1)/2$. Then any $A \in Sk(n, K)$ can be written

$$A = \sum_{1 \le i < j \le n} a_{ij} E^{ij},$$

and can also be represented by the N-tuple $[a_{12}, \ldots, a_{n-1n}]$. (We refer to this as upper triangular representation for A.)

The following result, taken from [Cohn], concerns the reduction of a skew-symmetric bilinear form to a standard skew-symmetric matrix representative.

Theorem 1.1.1 *Let*

$$E = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Given a skew-symmetric bilinear form, ϕ, on a space V (over any field K), the form has even rank $2r$, say, and in a suitably chosen coordinate system its matrix is

$$\begin{aligned} E_r &= \oplus_r E + 0 \\ &= \begin{bmatrix} 0 & 1 & & & & \\ -1 & 0 & & & & \\ & & \ddots & & & \\ & & & 0 & 1 & \\ & & & -1 & 0 & \\ & & & & & 0 \end{bmatrix}, \end{aligned} \tag{1.5}$$

where 0 *is an* $(n-2r) \times (n-2r)$ *null matrix.*

Proof This is proved by showing that V has a basis $u_1, v_1, \ldots, u_r, v_r, w_1, \ldots w_s$, where $\dim V = 2r + s$, rank $\phi = 2r$ and

$$\phi(u_i, v_i) = -\phi(v_i, u_i) = 1,$$

$1 \le i \le r$ with ϕ returning the value 0 for all other choices of pairs of basis elements.

The proof is by induction on $\dim V$. If $\phi = 0$ then $r = 0$ and we have the result. Otherwise, choosing $x, y \in V$ such that $\phi(x, y) \neq 0$, then on dividing x or y by $\phi(x, y)$ we obtain vectors u_1, v_1 for which $\phi(u_1, v_1) = 1$. Furthermore u_1 and v_1 are linearly independent, for if $v_1 = \lambda u_1$ say, then $\phi(u_1, v_1) = \lambda\phi(u_1, u_1) = 0$, which is a contradiction.

Let V_1 be the subspace spanned by u_1 and v_1, and U the subspace given by

$$U = \{z \in V : \phi(z, u_1) = \phi(z, v_1) = 0\}.$$

Clearly, any $x \in V$ can be written as

$$x = \phi(x, v_1)u_1 + \phi(u_1, x)v_1 + x', \tag{1.6}$$

for a unique vector $x' \in V$. In fact it is easily verified that $x' \in U$ and moreover since $V_1 \cap U = \{0\}$ it follows that $V = V_1 \oplus U$. Thus $\dim U = \dim V - 2$ and the result follows by induction on $\dim V$. □

Before discussing skew-symmetric matrices further we consider a related set of canonical objects. Above we defined the exterior product (alternating product), $\bigwedge^2 V$ as

$$\bigwedge^2 V = V \otimes V \,/ Sp\{v \otimes v : v \in V\} \ .$$

If $\dim V = n$ and we choose some basis of V to be $\{e_1, \ldots, e_n\}$ the corresponding basis for $\bigwedge^2 V$ is given by

$$\{e_i \wedge e_j : 1 \leq i < j \leq n\}. \tag{1.7}$$

For convenience we represent a general element of $\bigwedge^2 V$ by the notation, $\hat{v}$, where

$$\hat{v} = \sum_{1 \leq i < j \leq n} \lambda_{ij} e_i \wedge e_j,$$

with $\lambda_{ij} \in K$. Using the property of the wedge product that $e_j \wedge e_i = -e_i \wedge e_j$ we can also write $\hat{v}$ in the form

$$\hat{v} = \sum_{1 \leq r,s \leq n} \mu_{rs} e_r \wedge e_s, \tag{1.8}$$

where $\mu_{sr} = -\mu_{rs} = \lambda_{rs}/2$.

Lemma 1.1.2 *(a) Each skew bilinear form ϕ on V corresponds to a linear map $\bigwedge^2 V \to K$, that is element of $(\bigwedge^2 V)^*$. Conversely any such element corresponds to a skew form $V \times V \to K$.*

(b) Any bilinear form ϕ on a vector space V yields a linear map $\phi^* : V \to V^*$ defined by $\phi^*(v) \mapsto (w \mapsto \phi(v, w))$. In particular, this gives a natural map $Alt(V) \to Hom(V, V^*)$.

Proof (a) Any bilinear form yields a linear map $V \otimes V \to K$; the skew-symmetry shows that this yields a well defined linear map $\bigwedge^2 V \to K$. On the other hand given a linear map $\alpha : \bigwedge^2 V \to K$ define $\phi(v,w) = \alpha(v \wedge w)$. Clearly ϕ is bilinear and $\phi(v,v) = \alpha(0) = 0$.

(b) Is obvious. □

Note that if $\{e_1, \dots, e_n\}$ is a basis for V, then $\{e_i \wedge e_j, 1 \le i < j \le n\}$ is a basis for $\bigwedge^2 V$ and $\{(e_i \wedge e_j)^*, 1 \le i < j \le n\}$ a basis for its dual. Note that $(e_i \wedge e_j)^* = -(e_j \wedge e_i)^*$.

There is a natural notion of equivalence on the space $Alt(V)$ originating with the group of linear automorphisms of V, which we denoted $Gl(V)$.

Definition 1.1.3 *Two elements ϕ_1, ϕ_2 of $Alt(V)$ are* **equivalent** *if for some element $\alpha \in Gl(V)$ we have $\phi_1(v,w) = \phi_2(\alpha(v), \alpha(w))$ for all $v, w \in V$.*

It is easily checked that this gives rise to an equivalence relation on $Alt(V)$. Indeed this follows because there is a corresponding 'action'of the group $Gl(V)$, on the space $Alt(V)$ (or equivalently $(\bigwedge^2 V)^*$). So given $\phi \in Alt(V)$ and $\alpha \in Gl(V)$ then we set $(\alpha.\phi)(v,w) = \phi(\alpha(v), \alpha(w))$. This is a group action in the following sense.

Definition 1.1.4 *By an* **action** *of a group G on a set M we mean a mapping $\Phi : G \times M \to M$, such that for all $x \in M$ and g_1, $g_2 \in G$*

(i) $\Phi(e,x) = x$

(ii) $\Phi(g_1 g_2, x) = \Phi(g_1, \Phi(g_2, x))$,

where e denotes the identity of G. We usually write $g.x$ for $\Phi(g,x)$ and the above then become

(i) $e.x = x$

(ii) $(g_1 g_2).x = g_1.(g_2.x)$.

Given such an action we define an equivalence relation $\sim$ on M by agreeing that $x \sim y$ when there exists an element $g \in G$ for which $y = g.x$. The equivalence classes are called the **orbits** *under the action. Given $x \in M$ the* **orbit through** *x is by definition the equivalence class which contains x, i.e. the set*

$$G.x = \{g.x : g \in G\}.$$

If G is a group which is also a smooth manifold, then G is a **Lie group** *if both the maps $G \times G \to G$,*

$$(g_1, g_2) \mapsto g_1 g_2,$$

and $G \to G$

$$g \mapsto g^{-1}$$

are smooth. If M is a smooth manifold an action of the Lie group G is a **smooth** *map $\Phi : G \times M \to M$ satisfying both properties, (i) and (ii) above, of a group action. The* **Lie algebra**, *LG, of a Lie group, G, is the tangent space to the Lie group at the identity.*

Remark 1.1.5 Classification, up to such an equivalence, amounts to listing orbits, representing each by a suitable *normal form*.

An *invariant* of a group action is some property or function which is constant on orbits.

Naturally when doing calculations one has to choose bases and work with the space of skew-symmetric matrices rather than skew-symmetric forms, $Gl(n, K)$ rather than $Gl(V)$.

Lemma 1.1.6 *The action of $Gl(V)$ on $Alt(V)$ corresponds to the action of $Gl(n, K)$ on $Sk(n, K)$ given by*

$$A \mapsto X^T AX,$$

where $X \in Gl(n, K)$.

It follows (and is easily verifed) that this action preserves skew-symmetry. This leads us to the following definition.

Definition 1.1.7 *Two skew-symmetric matrices A, B are said to be* **skew-equivalent** *if*

$$B = X^T A X,$$

for some matrix $X \in Gl(n, K)$.

In fact, it can be shown that given any skew-symmetric matrix, A, by performing a series of elementary column operations on it, provided we also apply the same elementary operations to its rows, we obtain a skew-equivalent skew-symmetric matrix.

This is dealt with in detail in the next section for the more general case of skew-symmetric matrices defined over a ring R and we refer the reader to Lemma 2.1.9 and Definition 2.1.10, there, for a description of elementary row and column operations on such matrices. So, although the above can be deduced by standard linear algebra, we defer the proof until then, where it is a special case ($R = K$) of Theorem 2.1.14.

Since the reduction under this equivalence corresponds to that given in Theorem 1.1.1 for a skew-symmetric bilinear form, the standard normal forms for skew-symmetric matrices are :

$$N = \oplus_r E \oplus 0, \tag{1.9}$$

with block

$$E = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and 0 an $(n-2r) \times (n-2r)$ null block. These normal forms are representatives of the orbits of $Sk(n, K)$ under $Gl(n, K)$, determined by rank, $2r$, which must be even. Hence $Sk(n, K)$ has finitely many orbits. These orbits are smooth manifolds, and their union is the *stratification* of $Sk(n, K)$.

The proof of the following lemma is taken from [Cohn].

Lemma 1.1.8 *The determinant of a non-singular skew-symmetric matrix is the square of a homogeneous polynomial in its entries.*

Proof By Lemma 1.0.6 any non-singular $n \times n$ skew-symmetric matrix must have n even and we let $n = 2r$. Now let A be the matrix with entries $a_{ij} = t_{ij}$

$(i < j)$, where the t_{ij} are the $n(n-1)/2$ independent upper triangular entries, and $a_{ii} = 0$, $a_{ji} = -a_{ij}$. Hence A is of the form

$$A = \begin{bmatrix} 0 & t_{12} & \cdots & t_{1n} \\ -t_{12} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & t_{n-1n} \\ -t_{1n} & \cdots & -t_{n-1n} & 0 \end{bmatrix}.$$

Let $\mathbb{Q}[t] = \mathbb{Q}[t_{12}, \ldots, t_{n-1n}]$ be the ring of polynomials in the t_{ij}'s with rational coefficients. We can then consider the field given by

$$\begin{aligned} \mathbb{Q}(t) &= \mathbb{Q}(t_{12}, \ldots, t_{n-1n}) \\ &= \left\{ \frac{f(t)}{g(t)} : f, g \in \mathbb{Q}[t] \text{ and } g \neq 0 \right\}. \end{aligned}$$

Clearly, A is defined over this field. We claim that A is non-singular. If not $\det A$ would be $0 \in \mathbb{Q}(t)$. But by choosing certain values, $\bar{t}_{ij}$, for the t_{ij} we can reduce A to

$$\begin{aligned} E_r &= \oplus_r E \\ &= \underbrace{\begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{bmatrix}}_{r blocks}. \end{aligned}$$

and then $\det(A(\bar{t})) = 1$ (from Lemma 1.0.7).

Since the above reduction to normal forms, (1.9), is valid over any field it follows that

$$A = X^T E_r X,$$

where X is a non-singular matrix over $\mathbb{Q}(t)$.

Taking determinants

$$\begin{aligned} \det A &= (\det X)^2 \det(E_r) \\ &= \left(\frac{f}{g}\right)^2, \end{aligned}$$

for some $f, g \in \mathbb{Q}[t]$, where $f, g \neq 0$. Therefore

$$f^2 = g^2 \det A, \tag{1.10}$$

where $\det A$ is a homogeneous polynomial, of degree $2r$, in $\mathbb{Q}[t]$. Since $\mathbb{Q}[t]$ is a unique factorization domain (UFD) it follows that $g|f$ and writing $f/g = F$ we deduce from (1.10) that $\det A = F^2$ is the square of a polynomial in the t_{ij}'s, and hence in the entries of A. Furthermore as noted above this polynomial is homogeneous and of degree r. □

Remark 1.1.9 This polynomial is determined up to sign, which can be fixed so that F reduces to 1 when $A = E_r$. The polynomial determined in this way is called the *Pfaffian* of order $n = 2r$ and is denoted by $Pf(A)$.

We also need the following facts (see [Artin], page 142).

Proposition 1.1.10 *(a) For $A \in Sk(n,K)$, $X \in Gl(n,K)$ we have $Pf(X^T AX) = \det(X).Pf(A)$.*

(b) We write A^{rs} for the matrix obtained from A by deleting rows and columns r and s. Clearly this is itself skew-symmetric, and we write C_{rs} for $(-1)^{r+s-1}Pf(A^{rs})$. Then

$$Pf(A) = \sum_{j=1}^{n} a_{ij}C_{ij}.$$

(This is the analogue of the usual expansion of determinants by rows/columns.)

Proof (a) For the first case we adjoin the variables x_{ij} (entries of a generic matrix X) to the a_{ij}. Clearly $Pf(X^T AX) = \pm\det(X)Pf(A)$; specialising to the case $X = I$ we see that the sign is $+1$.

(b) We may suppose that $n = 2m$ is even. Clearly $Pf(A)$ is linear in the row variables (multiply the i^{th} row and column by t and taking determinant yields $t^2 \det A$). We write D_{rs} for the coefficient (homogeneous of degree $m-1$) of a_{rs} in $Pf(A)$. Consider first the case $r = 1, s = 2$. Now $\det A = Pf(A)^2 = (a_{12}D_{12} + D)^2$ where D does not involve a_{12}. Expanding the determinant of

A by the first row, and the $(1,2)$ minor by its first row it is easy to see that the coefficient of a_{12}^2 in $\det A$ is $\det(A^{12})$, so $D_{12} = \pm C_{12}$. Switching rows and columns we deduce that $D_{rs} = \pm C_{rs}$ for any r, s. So we can deduce that $Pf(A) = \sum_{j=1}^{n} \pm a_{ij} C_{ij}$. It remains to determine the signs. This can be done as follows. We need to choose a particular matrix A which is non-singular, whose (r,s) entry is non-zero, which has no other terms in the r^{th} row. For this we choose the form ϕ with $\phi(e_r, e_s) = 1, \phi(e_i, e_j) = 0$ for $i = r, s, j \neq r, s$. Now write the remaining vectors in order and pair them off consecutively e_a, e_b with $\phi(e_a, e_b) = 1$. Changing basis to bring this to normal form, and some straightforward calculations determines the sign. □

With a view to considering the geometry of stratification of $Sk(n,K)$ we introduce the following space, $\Gamma \subset Sk(n,K) \times K^n$, given by

$$\Gamma = \{(A,v) \in Sk(n,K) \times K^n : Av = 0\}.$$

We then consider orbits of Γ under $Gl(n,K)$. If $X \in Gl(n,K)$ it acts on $(A,v) \in \Gamma$ by

$$X.(A,v) = (X^T A X, X^{-1} v).$$

It can be seen from (1.9) that kernel vectors of N are of the form

$$\omega = (0, \ldots, 0, w_{2r+1}, \ldots, w_n).$$

Given such kernel vectors, ω, we consider an action which gives us a normal form for the set (N, ω). If $Y \in Gl(n,K)$ then let Y^{-1} represent a series of row operations on the last $n - 2r$ rows of ω resulting in the vector

$$e_{2r+1} = (\underbrace{0, \ldots, 0}_{2r}, 1, 0, \ldots, 0).$$

Writing

$$Y = \left[\begin{array}{c|c} I_{2r} & 0 \\ \hline 0 & B \end{array}\right],$$

with B an invertible $(n-2r) \times (n-2r)$ matrix, then it can be seen that these row operations correspond to the premultiplication of the last $(n-2r)$ rows of ω by B^{-1}. The action of this Y on (N, ω) is given by

$$(Y^T N Y, Y^{-1}\omega) = (Y^T N Y, e_{2r+1}),$$

where

$$Y^T N Y = \left[\begin{array}{c|c} E_{2r} & 0 \\ \hline 0 & 0 \end{array}\right],$$

and so N is unaffected by this action irrespective of the submatrix, B. This is as expected since any row operations on the last $n-2r$ rows of a kernel vector ω are accompanied by a series of simultaneous row and column operations on the last $n-2r$ rows and columns of N which are all zero. There are therefore finitely many $Gl(n,K)$ orbits of Γ represented by the normal forms

$$(N = \oplus_r E \oplus 0, e_{2r+1}).$$

Proposition 1.1.11 *The tangent space to the $Gl(n,K)$ orbit through $A \in Sk(n,K)$ is spanned by $\{Y^T A + AY : Y \in M(n,K)\}$.*

Proof Consider the path in $Gl(n,K)$ given by

$$X = I + tY,$$

where Y is any $n \times n$ matrix. Hence the tangent vector to the $Gl(n,K)$ orbit of A (at A) corresponding to this path is

$$\lim_{t \to 0} \left\{ \frac{(I+tY)^T A (I+tY) - A}{t} \right\} \tag{1.11}$$

$$= \quad Y^T A + AY, \tag{1.12}$$

as required. □

Corollary 1.1.12 *The tangent space to the $Gl(n,K)$ orbit through $(A,v) \in \Gamma$ is spanned by $\{(Y^T A + AY, -Yv) : Y \in M(n,K)\}$.*

Proof Clearly, the first components of tangent vectors to the $Gl(n,K)$ orbit of (A,v) are those found in the previous proposition. It remains to consider the second component of these tangent vectors corresponding to $v \in K^n$. Taking the same path, in $Gl(n,K)$, considered in Proposition 1.1.11 it follows that

$$X^{-1} = I - tY + O(2),$$

where $O(2)$ represents terms involving quadratic and higher powers of t. So the tangent vector to the $Gl(n, K)$ orbit of v (at v) corresponding to this path is

$$\lim_{t \to 0} \left\{ \frac{(I - tY)v - v}{t} \right\} = -Yv.$$

Note here that we can neglect the higher powers of t occuring in X^{-1} since they vanish in the limit. Consequently, the tangent vectors to $(A, v) \in \Gamma$ are given by $(Y^T A + AY, -Yv)$, for any $n \times n$ matrix Y. □

By choosing Y to be each E_{ij} for $1 \leq i, j \leq n$, where E_{ij} denotes the matrix with a 1 in the (i, j)th entry and zeros elsewhere (basis vectors of $M(n, K)$), we can use these results to find sets of vectors which span these spaces. In each case, taking all the linearly independent vectors gives us a basis for the tangent space and hence its dimension.

Lemma 1.1.13 *The tangent space to the orbit of*

$$N = \oplus_r E \oplus 0 \in Sk(n, K)$$

has the set of basis vectors given by

$$\left\{ \begin{array}{ll} E^{2k-1j} & 2k \leq j \leq n \\ -E^{2kj} & 2k+1 \leq j \leq n \end{array} \right\},$$

for each $1 \leq k \leq r$.

Hence the dimension of this tangent space (and the dimension of the orbit) is

$$\dim Sk(n, K) - \dim Sk(n - 2r, K) = r(2n - 2r - 1).$$

Proof Note, this proof deals with sparse matrices and so for convenience we leave null blocks of such matrices blank. We first use (1.12) to consider the form these tangent vectors take, for a general skew-symmetric matrix A. Let

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}.$$

If $Y = E_{ij}$ then AY is a matrix whose jth column is the ith column of A, with zeros elsewhere. Furthermore $Y^T = E_{ji}$ and $Y^T A$ is a matrix whose jth row is the ith row of A, with all other entries zero. Hence the tangent vector to the orbit of A corresponding to E_{ij} is given by

$$E_{ij}^T A + AE_{ij} = \begin{pmatrix} & & a_{1i} & & \\ & & \vdots & & \\ a_{i1} & \cdots & a_{ij} + a_{ji} & \cdots & a_{in} \\ & & \vdots & & \\ & & a_{ni} & & \end{pmatrix}.$$

As $A \in Sk(n, K)$ it follows that for $1 \le k \le n$, $a_{ik} = -a_{ki}$ and hence $a_{ij} + a_{ji} = 0$.

Now we consider the tangent vectors to the normal form, N. So for each E_{ij} the corresponding tangent vector is the matrix whose jth row and jth column are obtained, respectively, by superimposing the ith row and ith column of N, with zeros elsewhere. Since the last $n - 2r$ rows (and columns) of the normal form are zero it follows that all tangent vectors corresponding to E_{ij} where $2r + 1 \le i \le n$, are the zero vector. We only need to consider the tangent vectors resulting from directions E_{ij} where $1 \le i \le 2r$.

Consider,

$$N = \left[\begin{array}{cc|cc|c|cc|ccc}
0 & 1 & & & & & & & & \\
-1 & 0 & & & & & & & & \\ \hline
 & & 0 & 1 & & & & & & \\
 & & -1 & 0 & & & & & & \\ \hline
 & & & & \ddots & & & & & \\ \hline
 & & & & & 0 & 1 & & & \\
 & & & & & -1 & 0 & & & \\ \hline
 & & & & & & & 0 & & \\
 & & & & & & & & \ddots & \\
 & & & & & & & & & 0
\end{array}\right]. \tag{1.13}$$

From (1.13) we see that we can obtain the tangent vectors E^{1j} and $-E^{2j}$ by using the first two rows and columns of N, i.e. the first (leading) block. For example we get all tangent vectors of the form $-E^{2j}$ from the directions E_{1j} $(3 \le j \le n)$ in $M(n, K)$. Similarly we get the tangent vectors E^{1j} $(2 \le j \le n)$ from the directions E_{2j}. Note here there are two directions which result in the tangent vector E^{12} namely E_{11} and E_{22} but we choose the later to avoid a range of j which would include the direction E_{12} resulting in the zero vector. In a similar way we use the second block of N to obtain tangent vectors E^{3j} and $-E^{4j}$ given by directions E_{4j}, $4 \leq j \leq n$ and E_{3j}, $5 \leq j \leq n$. Note here that the directions E_{4j} and E_{3j} for $j < 3$ give no new tangent vectors.

It follows that given the kth block of N we obtain the following tangent vectors to N. The tangent vectors corresponding to using the $2k$th column ($2k$th row) of N to fill entries in row $2k-1$ to the right (and column $2k-1$ below) are

$$E^{2k-1j},$$

where $2k \leq j \leq n$, and are obtained from the directions E_{2kj} in $M(n,K)$. The tangent vectors corresponding to filling entries in row $2k$ to the right (and column $2k$ below) are

$$-E^{2kj},$$

where $2k+1 \leq j \leq n$, and are obtained from the directions E_{2k-1j} in $M(n,K)$. Note here that by considering all blocks $1 \leq k \leq r$ we consider all directions, E_{ij}, for $1 \leq i \leq 2r$ which give independent tangent vectors. Clearly since all these (non-zero) vectors are basis elements of $Sk(n,K)$ they are independent.

So the dimension of the tangent space to N is given by

$$\sum_{k=1}^{r}[1+2(n-2k)] = r(2n-2r-1).$$

Notice that this dimension is also given by

$$\dim Sk(n,K) - \dim Sk(n-2r,K) = \frac{1}{2}n(n-1) - \frac{1}{2}(n-2r)(n-2r-1).$$

□

Corollary 1.1.14 *The tangent space to the orbit of*

$$(N = \oplus_r E \oplus 0, e_{2r+1}) \in \Gamma$$

has the set of basis vectors given by

$$\left\{ \begin{array}{c} \left. \begin{array}{ll} (E^{2k-1j},0) & 2k \leq j \leq n, j \neq 2r+1 \\ (E^{2k-1\,2r+1}, -e_{2k}) & \\ (-E^{2kj},0) & 2k+1 \leq j \leq n, j \neq 2r+1 \\ (-E^{2k\,2k+1}, -e_{2k-1}) & \end{array} \right\} 1 \leq k \leq r \\ (0,-e_i) \quad 2r+1 \leq i \leq n \end{array} \right\}$$

and has dimension $r(2n-2r-1)+(n-2r)$.

Proof The first component of any tangent vector to (N, e_{2r+1}) is one of those found in the previous lemma. To find the second component of this vector we need, in addition, to consider the effect of the tangent vector $E_{ij} \in T_I Gl(n,K)$ on $e_{2r+1} \in K^n$ In general if $(A,v) \in \Gamma$ and $v=(v_1,\cdots,v_n)$ then if $Y = E_{ij}$ then the component of the tangent vector in K^n is given by

$$-E_{ij}v = (0,\cdots,0,\underbrace{-v_j}_{ith\,entry},0,\cdots,0).$$

That is, the ith entry of the tangent vector is the negative of the jth entry of v.

If $v = e_{2r+1}$ it follows that $v_{2r+1} = 1$ and $v_1 = \cdots v_{2r} = v_{2r+2} = \cdots v_n = 0$ and so for all directions E_{ij} for which $j \neq 2r+1$ the corresponding component of the tangent vector in K^n is zero. So using Lemma 1.1.13 and considering directions E_{ij}, for which $1 \leq i \leq 2r$, then provided $j \neq 2r+1$ we have tangent vectors to (N, e_{2r+1}) consisting of ordered pairs of the form

$$\left\{ \begin{array}{ll} (E^{2k-1j},0) & 2k \leq j \leq n \\ (-E^{2kj},0) & 2k-1 \leq j \leq n \end{array} \right\},$$

for $1 \leq k \leq r$. However, those tangent vectors obtained by considering the directions E_{i2r+1} will have component $-e_i \in K^n$. So (with $1 \leq i \leq 2r$) we find these tangent vectors are

$$\left\{ \begin{array}{c} (E^{2k-1\,2r+1}, -e_{2k}) \\ (-E^{2k\,2r+1}, -e_{2k-1}) \end{array} \right\},$$

for each $1 \leq k \leq r$.

Since the first components of all these vectors are the independent tangent vectors to N in $Sk(n,K)$, they are independent in $Sk(n,K) \times K^n$, irrespective of their component in K^n. We obtain additional vectors by considering those directions, E_{ij}, which give the zero vector in the first component but whose effect on e_{2r+1} is some non-zero vector in K^n. Recall from Proposition 1.1.13 that such directions are those E_{ij} for which $2r+1 \leq i \leq n$. Clearly by the above argument we only get non-zero components in K^n when we consider the directions E_{i2r+1}. The resulting vectors are

$$(0,-e_{2r+1}),\ldots(0,-e_n).$$

For these to be independent to the vectors, with non-zero components in $Sk(n, K)$, their components in K^n must be independent, which they clearly are since they are basis vectors of K^n. It follows that the dimension of the tangent space to $(\oplus_r E \oplus 0, e_{2r+1}) \in \Gamma$ is

$$\dim Sk(n, K) - \dim Sk(n - 2r, K) + \dim K^{n-2r} = r(2n - 2r - 1) + (n - 2r).$$

□

1.2 Pairs of Skew-symmetric Matrices

Definition 1.2.1 *Let (A_1, A_2), (B_1, B_2) be ordered pairs of skew-symmetric matrices, over K. We say that they are* **skew-equivalent** *if for some invertible matrix X over K we have*

$$B_j = X^T A_j X, \quad j = 1, 2.$$

Definition 1.2.2 *Given a pair (A_1, A_2) then we call the roots $(\mu : \lambda)$, on the projective line, PK, which satisfy*

$$\det(\mu A_1 + \lambda A_2) = 0$$

the eigenvalues of this pair. Associated with each (distinct) eigenvalue $(\mu_0 : \lambda_0)$ there is an eigenvector $\mathbf{u}_0 \neq \mathbf{0}$ satisfying

$$(\mu_0 A_1 + \lambda_0 A_2)\mathbf{u}_0 = \mathbf{0}.$$

Consider any pair $(X^T A_1 X, X^T A_2 X)$, where $X \in Gl(n, K)$, which is skew-equivalent to (A_1, A_2). This pair has eigenvalues, $(\mu : \lambda)$, given by the roots of

$$\begin{aligned} \det(\mu X^T A_1 X + \lambda X^T A_2 X) &= \det[X^T(\mu A_1 + \lambda A_2)X] \\ &= (\det X)^2 \det(\mu A_1 + \lambda A_2), \end{aligned}$$

which since $\det X \neq 0$, are the eigenvalues of (A_1, A_2). Hence eigenvalues are an invariant of skew-equivalent pairs.

1.3 Pencils of Skew-symmetric Matrices

Consider a pair of skew-symmetric matrices (A_1, A_2) neither of which is null or a scalar multiple of the other. This pair form a *basis* for a 2-dimensional family of skew-symmetric matrices obtained by the linear combination

$$\alpha_1 A_1 + \alpha_2 A_2,$$

where α_i are scalars in K.

We call this family a *pencil* of skew-symmetric matrices. The skew-symmetric matrices obtained by fixing the scalar parameters are known as *members* of the pencil. Since A_1, A_2 are skew-symmetric it can be easily verified that for all $(\alpha_1, \alpha_2) \in K^2$ $\alpha_1 A_1 + \alpha_2 A_2$ is also skew-symmetric. We refer to pencils obtained from odd or even pairs as odd and even pencils respectively.

Clearly a pencil is determined by any pair of independent matrices in it. In what follows we aim to use these independent pairs to describe pencils. For this reason, in $Sk(n, K)$, null matrices are of no interest, and we also need to identify a matrix with all its scalar multiples. To do this we consider the projective space of $Sk(n, K)$ and denote it by $P(n, K)$. As a result any two distinct points of $P(n, K)$ represent a pair of independent skew-symmetric matrices. With this in mind we formally define pencils of skew-symmetric matrices as follows.

Definition 1.3.1 *A* **pencil** *of skew-symmetric matrices is a line in the projective space, $P(n, K)$, of such matrices. It is, of course, determined by any pair of distinct points on that line. Hence given such a pair (A_1, A_2) of skew-symmetric matrices we can represent the pencil, A, they determine by*

$$A = \mu A_1 + \lambda A_2$$

where the ratio $(\mu : \lambda) \in PK$ are the coordinates of points of the pencil.

The action of the general linear group, $Gl(n, K)$, on $Sk(n, K)$, mentioned above, produces an action on the projective space $P(n, K)$. This leads us to the following definition.

Definition 1.3.2 *Two pencils are said to be skew-equivalent if one line is taken to the other by an element of $Gl(n, K)$.*

Alternatively since any pencil is determined by two distinct points on it, then if we can find a pair of distinct points on each pencil which are skew-equivalent, according to Definition 1.2.1, the pencils they determine are said to be skew-equivalent. This leads to the following lemma.

Lemma 1.3.3 *Let the pencils A, B be defined by (distinct) pairs (A_1, A_2), (B_1, B_2) respectively. Then the pencils are skew-equivalent if and only if for some*

$$\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \in Gl(2, K)$$

we have $(\alpha A_1 + \beta A_2, \gamma A_1 + \delta A_2)$ and (B_1, B_2) are skew-equivalent as pairs.

Proof As stated above the pencil A is determined by the independent pair (A_1, A_2) and can be expressed by

$$A = \mu A_1 + \lambda A_2. \tag{1.14}$$

We can find a pair of members of A by choosing two ratios $(\alpha : \beta)$, $(\gamma : \delta)$. This pair, $(\alpha A_1 + \beta A_2, \gamma A_1 + \delta A_2)$, is distinct if these two ratios are distinct, that is,

$$\frac{\alpha}{\beta} \neq \frac{\gamma}{\delta}$$

or $\alpha\delta - \beta\gamma \neq 0$. If this is the case, then we can also express the pencil in terms of this new pair or basis by

$$\begin{aligned} A &= \rho(\alpha A_1 + \beta A_2) + \sigma(\gamma A_1 + \delta A_2) \\ &= (\rho\alpha + \sigma\gamma)A_1 + (\rho\beta + \sigma\delta)A_2, \end{aligned}$$

where $(\rho : \sigma) \in PK$. Comparing with (1.14) we see that the coordinates $(\mu : \lambda)$ of points of A, relative to the old basis, can be expressed in terms of the coordinates $(\rho : \sigma)$ of points, relative to the new basis, by the linear map

$$\begin{pmatrix} \mu \\ \lambda \end{pmatrix} = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \begin{pmatrix} \rho \\ \sigma \end{pmatrix}.$$

Hence the condition above for the pair $(\alpha A_1 + \beta A_2, \gamma A_1 + \delta A_2)$ to be distinct, and hence determine the pencil A, is just the requirement that the matrix

$$\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \in Gl(2, K).$$

(We refer to this matrix as the *change of basis* matrix.) If this pair is skew-equivalent to the pair (B_1, B_2), then it follows from Definitions 1.2.1, 1.3.1 and 1.3.2 that the two pencils are skew-equivalent. □

In this context, the following well known result is useful later.

Lemma 1.3.4 *(The Three Point Lemma) Given three distinct points $\mathbf{x}_1 = (x_1 : y_1)$, $\mathbf{x}_2 = (x_2 : y_2)$, $\mathbf{x}_3 = (x_3 : y_3)$ on the projective line PK^1, there is an element $g \in Gl(2, K)$ (or a unique element $g \in PGl(2, K)$) such that*

$$g\mathbf{x}_1 = (1 : 0), \qquad g\mathbf{x}_2 = (0 : 1), \qquad g\mathbf{x}_3 = (1 : 1);$$

□

1.4 Singular and Non-singular Pencils

Given an $n \times n$ pencil,

$$A = \mu A_1 + \lambda A_2,$$

it has determinant

$$\begin{aligned} \Delta &= \det(\mu A_1 + \lambda A_2) \\ &= q_0\mu^n + q_1\mu^{n-1}\lambda + \cdots + q_n\lambda^n, \end{aligned}$$

a homogeneous polynomial in $K[\mu, \lambda]$. Then (the polynomial function) $\Delta : K^2 \longrightarrow K$ vanishes at those points $(\mu : \lambda)$ of the pencil for which $\mu A_1 + \lambda A_2$ is a singular matrix.

Definition 1.4.1 *A pencil is said to be* **singular** *if all its members are singular, that is for all $(\mu : \lambda) \in PK$, $\Delta = 0$. As K is infinite this is true if and only if Δ is identically zero, i.e. $q_0 = \cdots = q_n = 0$.*

Note if a skew-symmetric pencil, $A = \mu A_1 + \lambda A_2$, is odd then it follows from Lemma 1.0.6 that, for each $(\mu : \lambda) \in PK$, $\det A$ is zero. Hence all odd skew-symmetric pencils are singular.

A *non-singular* pencil is one for which Δ is a non-zero polynomial, i.e at least one of the $q_i \neq 0$. (Note that $q_0 = \det A_1$, $q_n = \det A_2$.) Over $\mathbb{C}$ this determinant can be resolved into linear factors

$$\Delta(\mu, \lambda) = (\alpha_1\mu - \beta_1\lambda)\cdots(\alpha_n\mu - \beta_n\lambda)$$

and the singular members of the pencil are given by $\beta_i A_1 + \alpha_i A_2$. There are thus n or less distinct singular members, one for each different root $(\beta_i : \alpha_i)$ of Δ. All other members are non-singular. Recall from Definition 1.2.2 that these distinct roots of Δ are the eigenvalues of the pair (A_1, A_2).

In particular for a non-singular skew-symmetric pencil, A, its determinant Δ must be a non-zero polynomial of even degree, $2r$ say. In fact by considering A as a matrix whose entries are linear in (μ, λ) it follows from Lemma 1.1.8 that we can write

$$\Delta(\mu, \lambda) = (\underbrace{p_0\mu^r + p_1\mu^{r-1}\lambda + \cdots + p_r\lambda^r}_{Pf(A)})^2,$$

and we refer to the polynomial $Pf(A)$ as the *Pfaffian* of the skew-symmetric pencil, A. Hence for a skew-symmetric pencil, $A = \mu A_1 + \lambda A_2$, the eigenvalues of (A_1, A_2) (and the corresponding singular points of A) are given by the roots of $Pf(A)$.

Lemma 1.4.2 *Let be A a non-singular skew-symmetric pencil, defined by the distinct pair (A_1, A_2). The eigenvalues of any distinct pair (or basis) of A, $(\alpha A_1 + \beta A_2, \gamma A_1 + \delta A_2)$, are obtained, from the eigenvalues of (A_1, A_2), by the action of an invertible linear map. Furthermore the corresponding singular members and eigenvectors are preserved.*

Proof Recall from the Proof of Lemma 1.3.3 that any distinct pair $(\alpha A_1 + \beta A_2, \gamma A_1 + \lambda A_2)$ has the associated change of basis matrix

$$\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \in Gl(2, K).$$

If $(\mu_0 : \lambda_0)$ is an eigenvalue of (A_1, A_2) (and $\mathbf{u}_0$ the corresponding eigenvector) then $\mu_0 A_1 + \lambda_0 A_2$ is a singular member of A. From the Proof of Lemma 1.3.3 it follows that the corresponding eigenvalue, $(\rho_0 : \sigma_0)$ is given by

$$\begin{pmatrix} \rho_0 \\ \sigma_0 \end{pmatrix} = \frac{1}{\alpha\delta - \gamma\beta} \begin{pmatrix} \delta & -\gamma \\ -\beta & \alpha \end{pmatrix} \begin{pmatrix} \mu_0 \\ \lambda_0 \end{pmatrix}.$$

Since we only require the ratio $(\rho_0 : \sigma_0)$ this is given by $[\delta\mu_0 - \gamma\lambda_0 : -\beta\mu_0 + \alpha\lambda_0]$. Hence the eigenvalues of the pair $(\alpha A_1 + \beta A_2, \gamma A_1 + \lambda A_2)$ are obtained from the eigenvalues of (A_1, A_2) by the action of the inverse of the change of basis matrix. It can be verified that the corresponding singular element of A, in terms of the pair $(\alpha A_1 + \beta A_2, \gamma A_1 + \lambda A_2)$, is still $\mu_0 A_1 + \lambda_0 A_2$ (up to some scalar multiple) and hence the corresponding eigenvector is preserved. □

Lemma 1.4.3 *If two skew-symmetric pencils A and B, defined by distinct pairs (A_1, A_2) and (B_1, B_2) respectively, are skew-equivalent then they both have the same number of singular elements. Furthermore, there is a map, of the type described in Lemma 1.4.2, between each eigenvalue of (A_1, A_2) and (B_1, B_2) where the corresponding singular elements of A and B are skew-equivalent as skew-symmetric matrices.*

Proof If A and B are skew-equivalent then, by Lemma 1.3.3, for some

$$\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \in Gl(2, K),$$

the pairs $(A'_1, A'_2) = (\alpha A_1 + \beta A_2, \gamma A_1 + \delta A_2)$ and (B_1, B_2) are skew-equivalent. If $(\mu_0 : \lambda_0)$ is an eigenvalue of (A_1, A_2) then $\mu_0 A_1 + \lambda_0 A_2$ is the corresponding singular element of A. By Lemma 1.4.2 $(\rho_0 : \sigma_0)$ is the corresponding eigenvalue of the pair (A'_1, A'_2) and

$$\mu_0 A_1 + \lambda_0 A_2 = \rho_0 A'_1 + \sigma_0 A'_2.$$

Since (A'_1, A'_2) and (B_1, B_2) are skew-equivalent

$$(B_1, B_2) = X^T (A'_1, A'_2) X$$

and

$$\begin{aligned} \mu_0 A_1 + \lambda_0 A_2 &= \rho_0 (X^{-1})^T B_1 X^{-1} + \sigma_0 (X^{-1})^T B_2 X^{-1} \\ &= (X^{-1})^T (\rho_0 B_1 + \sigma_0 B_2) X^{-1}, \end{aligned}$$

where $\rho_0 B_1 + \sigma_0 B_2$ is the singular element of B corresponding to the common eigenvalue, $(\rho_0 : \sigma_0)$, of pairs (A'_1, A'_2) and (B_1, B_2). Hence

$$\rho_0 B_1 + \sigma_0 B_2 = X^T (\mu_0 A_1 + \lambda_0 A_2) X,$$

as required. □

Chapter 2

Non-singular Pencils

This chapter is concerned chiefly with non-singular pencils. We start by generalising the notion of skew-equivalence of skew-symmetric matrices, described in Definition 1.1.7, by considering these matrices to be defined over a ring R. Subsequently we derive results about their decompositions and invariants under this equivalence. By adapting these results to the particular case, of λ-equivalence of λ-matrices, we obtain results very useful for the classification of pairs of skew-symmetric matrices (as described in Definition 1.2.1).

Most of the results in the first two sections of this chapter are either taken directly from or are based on results given in Chapter 7 (Pgs. 104–122) of [HrtHwk].

2.1 Classifying skew-symmetric matrices over ID's

In this section we study skew-symmetric matrices defined over integral domains (ID's). We first need a few definitions.

Definition 2.1.1 *An integral domain (ID), R, is a commutative ring with a* 1 *which has no zero divisors.*

From here on assume the ring R is an ID unless we specify otherwise.

Definition 2.1.2 *An element $u \in R$ is a* **unit** *of R if*

$$vu = 1,$$

for some $v \in R$.

Definition 2.1.3 *We say that two elements a, $b \in R$ are* **associates** *if*

$$a = ub,$$

for some unit $u \in R$; this is an equivalence relation and we write $a \sim b$.

Definition 2.1.4 *An $s \times s$ matrix, X, over R is invertible if there exists an $s \times s$ matrix Y satisfying*

$$XY = YX = I_s.$$

Lemma 2.1.5 *A matrix X is invertible if and only if $\det X$ is a unit in R.*

Proof If X is invertible then by Definition 2.1.4 there exists an $s \times s$ matrix Y over R such that

$$XY = I_s.$$

So, taking determinants

$$\det(XY) = (\det X)(\det Y) = 1,$$

which by Definition 2.1.2 implies that $\det X$ is a unit of R.
Conversely, if $\det X$ is a unit of R by Definition 2.1.2 there exists an element $(\det X)^{-1} \in R$ such that

$$(\det X)(\det X)^{-1} = 1.$$

For any matrix over R,

$$X.\mathrm{adj}\, X = \mathrm{adj}\, X.X = (\det X)I_s,$$

so multiplying through by $(\det X)^{-1}$ gives

$$X.(\det X)^{-1}\mathrm{adj}\, X = (\det X)^{-1}\mathrm{adj}\, X.X = I_s,$$

and by Definition 2.1.4, with $Y = (\det X)^{-1}\mathrm{adj}\, X$, X is invertible. □

Definition 2.1.6 *Let A and B be two matrices over R of the same size. Then B is said to be* **equivalent** *to A (over R) if there exist invertible matrices X and Y over R such that*

$$B = YAX.$$

Definition 2.1.7 *We define a list of special square matrices, with entries in R, as follows:*

(i) F_{ij} is the matrix obtained from the identity matrix by interchanging rows i and j;

(ii) $G_i(u)$ is the **diagonal** *matrix with a unit u of R in the ith row and 1's elsewhere on the diagonal;*

(iii) $H_{ij}(r)$, for any $r \in R$ and $j \neq i$, is the matrix obtained from the identity matrix by adding r times row j to row i. Thus $H_{ij}(r)$ has 1's on the diagonal, r in the (i, j) place and zeros elsewhere.

(iv) $\bar{H}_{ij}(r)$ is defined in the same way as $H_{ij}(r)$ with the word 'row' replaced by 'column'. So $\bar{H}_{ij}(r)$ has 1's on the diagonal, r in the (j, i) place and zeros elsewhere.

Note that $\bar{H}_{ij}(r) = H_{ji}(r)$ but it is useful to have both definitions.

Lemma 2.1.8 *The above matrices F_{ij}, $G_i(u)$, $H_{ij}(r)$ and $\bar{H}_{ij}(r)$ are all invertible.*

Proof From Definition 2.1.7 we find that $\det F_{ij} = -1$, $\det G_i(u) = u$ and $\det H_{ij}(r) = \bar{H}_{ij}(r) = 1$. So the determinants of these matrices are units of R and therefore by Lemma 2.1.5 the matrices are all invertible. □

Lemma 2.1.9 *The effect of postmultiplying a given matrix of the appropriate size*

(1) by F_{ij} is to interchange columns i and j,

(2) by $G_i(u)$ is to multiply column i by u,

(3) by $\bar{H}_{ij}(r)$ is to add r times column j to column i.

The effect of premultiplying a given matrix of the appropriate size

(4) by F_{ij} is to interchange rows i and j,

(5) by $G_i(u)$ is to multiply row i by u,

(6) by $H_{ij}(r)$ is to add r times row j to row i.

Proof These results follow by standard matrix multiplication. □

Definition 2.1.10 *The operations, 1–3, described in Lemma 2.1.9 are known as the* **elementary column operations** *on a matrix, and those of 4–6 as the* **elementary row operations**.

We have the following theorem.

Theorem 2.1.11 *Two matrices A and B are equivalent if it is possible to pass from one to the other by a sequence of elementary row/column operations.*

Proof To carry out an elementary row (column) operation on a matrix, A, by Lemma 2.1.9 one only has to perform that operation on the identity matrix and premultiply (postmultiply) A by the result. Since by Lemma 2.1.8 the matrices performing these tasks are invertible, any of these elementary operations will transform the matrix, A, into an equivalent one. In particular the sequence of row/column operations needed to pass from A to B is represented by a sequence of matrices of the type listed in Definition 2.1.7 given by

$$X_1, \dots, X_r, Y_1, \dots, Y_s, \tag{2.1}$$

where the matrices X_i represent the elementary column operations and the Y_j represent the elementary row operations. It then follows by Lemma 2.1.9 that

$$B = Y_s \cdots Y_1 A X_1 \cdots X_r. \tag{2.2}$$

Given such a list of row/column operations, where for each of the two types the order in which they are performed is provided, as in (2.1), then the associativity of matrix multiplication allows us considerable flexibilty when performing these operations on a matrix. For example, to obtain the matrix B from A in (2.2), we could start by performing the first p column operations, represented by the sequence $X_1, \ldots, X_p$, $(p < r)$, and follow these by the first q row operations represented by $Y_1, \ldots, Y_q$, $(q < s)$, then we could perform another sequence of column operations starting with X_{p+1} and so on. (We could for example perform alternate single row and column operations from each sequence.)

As mentioned above by Lemma 2.1.8 each of $X_1, \ldots, X_r$ and $Y_1, \ldots, Y_s$ is invertible and so rewriting (2.2) as

$$B = YAX$$

where $X = X_1 \cdots X_r$ and $Y = Y_s \cdots Y_1$ it follows that X and Y are invertible. Therefore by Definition 2.1.6 A and B are equivalent. □

When dealing with skew-symmetric matrices we need to refine the equivalence given in Definition 2.1.6 so that it preserves the skew-symmetry of a matrix. This leads us to the following definition.

Definition 2.1.12 *Let A and B be two* **skew-symmetric** *matrices over R of the same size. Then B is* **skew-equivalent** *to A (over R) if there exists an invertible matrix, X, over R such that*

$$B = X^T AX.$$

So, skew-equivalence is the special case of equivalence where, $Y = X^T$. Furthermore if the ring over which the matrices are defined is a field, K, we have the skew-equivalence of matrices over K refered to, in Definition 1.1.7, in Chapter 1. As mentioned there, it is easily verified that skew-equivalence preserves skew-symmetry.

Definition 2.1.13 *Given an elementary row (or column) operation on a matrix we refer to the same elementary operation applied to its columns (rows) as its* **counterpart column (row) operation**. *Furthermore the action of this row (or column) operation followed by its counterpart column (row) operation is called a* **simultaneous row and column operation**.

Lemma 2.1.14 *Two skew-symmetric matrices A and B are skew-equivalent if it is possible to pass from one to the other by series of (elementary) simultaneous row and column operations.*

Proof Given the matrix, A, there are three possible elementary simultaneous row and column operations we can perform on it:

(i) interchanging columns i and j followed by interchanging rows i and j. It can been seen from Lemma 2.1.9 that this action is represented by the matrix

$$F_{ij}AF_{ij}$$

and since it can be verified that $F_{ij} = F_{ij}^T$, this matrix is skew-equivalent to A.

(ii) multiplying column i by a unit $u \in R$ followed by multiplying row i by u. This action results in the matrix

$$G_i(u)AG_i(u)$$

where, clearly, since $G_i(u)$ is a diagonal matrix $G_i(u) = G_i(u)^T$ and we have a matrix skew-equivalent to A.

(iii) adding $r \in R$ times column j to column i followed by adding r times row j to row i. This corresponds to the matrix

$$H_{ij}(r)A\bar{H}_{ij}(r)$$

and since $H_{ij}(r) = \bar{H}_{ji}(r) = \bar{H}_{ij}(r)^T$ it follows that this matrix is skew-equivalent to A.

So any elementary row and column operation results in a skew-equivalent matrix. Note that given a series of elementary column operations $X_1, \ldots, X_r$ by the above their counterpart row operations can be written $X_1^T, \ldots, X_r^T$ and if we can pass from A to B by the corresponding series of simultaneous row and columns operations then

$$\begin{aligned} B &= (X_r^T \cdots X_1^T)A(X_1 \cdots X_r) \\ &= (X_1 \cdots X_r)^T A(X_1 \cdots X_r) \\ &= X^T AX, \end{aligned}$$

where $X = X_1 \cdots X_r$. □

Note here that if $R = K$, i.e. A, B are skew-symmetric matrices defined over the field K, then, as remarked in Chapter 1, we deduce from this lemma that if B can be obtained from A by a sequence of elementary simultaneous row and column operations then it is skew-equivalent to A. Here the elementary row and column operations, described in Lemma 2.1.9, are just the standard elementary operations on matrices defined over a field.

Before going any further we add the following remark, which is useful when performing simultaneous row and column operations on a skew-symmetric matrix.

Remark 2.1.15 Given a skew-symmetric matrix $A = (a_{ij})$, the effect of simultaneous row and column operations of types (ii) and (iii) on this matrix are fairly easy to interprete. However the result of simultaneous row and column *interchanges* on a skew-symmetric matrix is less obvious.

Suppose, starting with

$$A = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1i} & \cdots & a_{1j} & \cdots & a_{1n} \\ -a_{12} & 0 & \cdots & a_{2i} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & & \vdots & & \vdots \\ -a_{1i} & -a_{2i} & \cdots & 0 & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & \ddots & \vdots & & \vdots \\ -a_{1j} & -a_{2j} & \cdots & -a_{ij} & \cdots & 0 & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots & & \vdots & \ddots & \vdots \\ -a_{1n} & -a_{2n} & \cdots & -a_{in} & \cdots & -a_{jn} & \cdots & 0 \end{bmatrix}, \tag{2.3}$$

we wish to interchange columns C_i and C_j, followed by interchanging rows R_i and R_j. Call the resulting skew-equivalent matrix A'.

Clearly, any elements which do not lie in either C_i, C_j, R_i or R_j are unchanged in A'. In other words by drawing a vertical line through both columns C_i and C_j of A and a horizontal line through both rows R_i and R_j the entries remaining are preserved in these positions in A'.

Secondly, the entries of the two columns C_i and C_j which *don't* lie in either R_i or R_j are interchanged in the same manner as for standard column interchanges on a matrix. In other words, looking down both columns C_i and C_j of A, excluding the entries at the intersection of a veritcal and horizontal line, pairs of entries in these columns, occuring in the same row, are interchanged in A'.

Thirdly, in A', the entries a_{ii} and a_{ij} are interchanged with a_{jj} and a_{ji}, respectively, i.e. the iith and jjth entries of A' are a_{jj} and a_{ii} respectively and the ijth and jith entries of A' are a_{ji} and a_{ij} respectively. In particular, since we are considering skew-symmetric matrices (i.e. $a_{ji} = -a_{ij}$), interpreting the resulting entries of A' in terms of the entries of A, this amounts to switching two zeros on the main diagonal and changing the sign of the ijth and jith entries of A.

We represent these three effects, on the matrix A in (2.3), by the following *incomplete* matrix where any missing entries (in rows R_i and R_j) are denoted by $\oslash$,

$$\vec{A} = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1j} & \cdots & a_{1i} & \cdots & a_{1n} \\ -a_{12} & 0 & \cdots & a_{2j} & \cdots & a_{2i} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & & \vdots & & \vdots \\ \oslash & \oslash & \cdots & 0 & \cdots & -a_{ij} & \cdots & \oslash \\ \vdots & \vdots & & \vdots & \ddots & \vdots & & \vdots \\ \oslash & \oslash & \cdots & a_{ij} & \cdots & 0 & \cdots & \oslash \\ \vdots & \vdots & & \vdots & & \vdots & \ddots & \vdots \\ -a_{1n} & -a_{2n} & \cdots & -a_{jn} & \cdots & -a_{in} & \cdots & 0 \end{bmatrix}.$$

Finally, the remaining entries of A' are given by interchanging entries between rows R_i and R_j in the usual way, these entries being those which lie in neither C_i or C_j of A. This ensures that A' is skew-symmetric and the remaining entries are filled according to the definition of a skew-symmetric matrix (i.e. $a'_{ji} = -a'_{ij}$).

The main result of this section concerns skew-symmetric matrices defined over Principal Ideal Domains (PIDs). Before this, we provide a standard result for PIDs.

Lemma 2.1.16 *Given any PID R and $r_1, \dots, r_n \in R$, then the ideal in R generated by $r_1, \dots, r_n$, denoted by $\langle r_1, \dots, r_n \rangle = \langle r \rangle$, where $r = \gcd\{r_1, \dots, r_n\}$.*

Proof If R is a PID then $\langle r_1, \dots r_n \rangle = \langle r \rangle$, for some $r \in R$. So

(i) $r|r_1, \ldots, r|r_n$ (since $r_1, \ldots r_n \in \langle r \rangle$);

(ii) $r = \alpha_1 r_1 + \cdots \alpha_n r_n$, $\alpha_i \in R$ (since $r \in \langle r_1, \ldots, r_n \rangle$).

Firstly, (i) tells us that r is a common divisor of $r_1, \ldots, r_n$. Furthermore if d is any common divisor of $r_1, \ldots, r_n$ then (ii) implies that $d|r$ and therefore $r = \gcd\{r_1, \ldots r_n\}$. □

We also introduce some convenient notation.

Definition 2.1.17 *An $n \times n$ skew-symmetric matrix of the form*

$$\left[\begin{array}{cccccccc|ccc}
0 & a_1 & & & & & & & & & \\
-a_1 & 0 & & & & & & & & & \\
& & 0 & a_2 & & & & & & & \\
& & -a_2 & 0 & & & & & & & \\
& & & & \ddots & & & & & & \\
& & & & & & 0 & a_r & & & \\
& & & & & & -a_r & 0 & & & \\
\hline
& & & & & & & & 0 & & \\
& & & & & & & & & \ddots & \\
& & & & & & & & & & 0
\end{array}\right],$$

is referred to as skewdiag$(a_1, a_2, \ldots, a_r)$.

Theorem 2.1.18 *Any $n \times n$ skew-symmetric matrix A with entries in a PID, R, is skew-equivalent to a matrix, skewdiag$(d_1, d_2, \ldots, d_r)$, of the form*

$$\left[\begin{array}{cccccccc|ccc}
0 & d_1 & & & & & & & & & \\
-d_1 & 0 & & & & & & & & & \\
& & 0 & d_2 & & & & & & & \\
& & -d_2 & 0 & & & & & & & \\
& & & & \ddots & & & & & & \\
& & & & & & 0 & d_r & & & \\
& & & & & & -d_r & 0 & & & \\
\hline
& & & & & & & & 0 & & \\
& & & & & & & & & \ddots & \\
& & & & & & & & & & 0
\end{array}\right],$$

where $d_1|d_2|\cdots|d_r$, for some $r \geq 0$.

Proof The proof is an adaption of the proof of Theorem 7.10 in [HrtHwk] for the skew-symmetric case.

It is instructive to prove this theorem first for the special case of a Euclidean domain since in this case the argument is clearer and it is this situation which we shall be concerned with later when we study λ-matrices. We then modify the argument for the general PID case.

<u>Case when R is a Euclidean domain</u>

So given an $n \times n$ skew-symmetric matrix, A over a Euclidean domain R (equipped with a Euclidean function ϕ) we show how to reduce A by means of simultaneous row and column operations to a skew symmetric matrix of the form $\text{skewdiag}(d_1, \ldots, d_r)$ where $d_1 | \cdots | d_r$.

<u>First Stage</u>

We aim to reduce A to the skew-equivalent $n \times n$ skew-symmetric matrix C of the form:

$$C = \left[\begin{array}{cc|ccc} 0 & d_1 & 0 & \cdots & 0 \\ -d_1 & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & C^* & \\ 0 & 0 & & & \end{array}\right], \tag{2.4}$$

where d_1 divides each entry of the skew-symmetric sub-matrix, C^*. We refer to the 2×2 skew-symmetric block in the top left-hand corner of a skew-symmetric matrix (or submatrix) as the leading diagonal block and the $(1, 2)$ entry as the entry in the leading diagonal block.

We describe below a finite sequence of simultaneous row and column operations which when performed on $A = (a_{ij})$ yield either a skew-symmetric matrix of the form (2.4) or a skew-symmetric matrix $B = (b_{ij})$ satisfying the condition

$$\phi(b_{12}) < \phi(a_{12}). \tag{2.5}$$

In the latter case we carry out a further sequence of simultaneous row and column operations. Then we reach either (2.4), in which case we stop, or a matrix whose entry in its leading diagonal block has ϕ-value reduced further and we continue. Eventually after a finite number of steps we must reach (2.4) since otherwise the ϕ-values of the entries in the leading diagonal blocks of the matrices obtained form a strictly decreasing infinite sequence of non-negative integers, which is not possible.

The sequence of operations is as follows. If A is the zero matrix we are already at (2.4). Otherwise A has at least two non-zero entries which by suitable simultaneous row and column interchanges can be moved into the leading diagonal block. We therefore assume $a_{12} \neq 0$ and A is of the form:

$$A = \left[\begin{array}{cc|ccc} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ -a_{12} & 0 & a_{23} & \cdots & a_{2n} \\ \hline -a_{13} & -a_{23} & & & \\ \vdots & \vdots & & A^* & \\ -a_{1n} & -a_{2n} & & & \end{array}\right],$$

where A^* is a skew-symmetric sub-matrix.

There are three possibilities.

Case 1 There is an entry a_{1j} ($j \geq 3$) in the first row such that a_{12} doesn't divide a_{1j}.

As R is a Euclidean domain we can write

$$a_{1j} = a_{12}q + r,$$

where either $r = 0$ or $\phi(r) < \phi(a_{12})$. Since a_{12} doesn't divide a_{1j} then $r \neq 0$ and so $\phi(r) < \phi(a_{12})$. By subtracting q times the second column from the jth column and then interchanging the second and jth columns we replace a_{12} by r. To preserve the skew-symmetry we must accompany each of the above by their counterpart row operations and by doing so we replace $-a_{12}$ by $-r$. Consequently we achieve (2.5) (Note that $q = 0$, i.e $\phi(a_{1j}) = \phi(r) < \phi(a_{12})$ is a special case of the above and to get (2.5) we just interchange the second and jth columns.)

Case 2 There is an entry $-a_{2j}$ in the second column such that a_{12} doesn't divide $-a_{2j}$. In this case

$$-a_{2j} = a_{12}q + r,$$

where $\phi(r) < \phi(a_{12})$, and we can proceed as in Case 1 first operating with rows instead of columns, and then applying the corresponding counterpart column operations required to preserve skew-symmetry (to reach (2.5)).

Case 3 a_{12} divides every entry in the first row and second column.

In this case by simultaneous row and column operations, involving subtracting suitable multiples of the second column from the other columns, we can replace all the entries of the first row and column, other than a_{12} and $-a_{12}$ respectively, by zeros. Similarly by simultaneous row and column operations, involving adding multiples of the first row to the other rows, we can kill off all elements other than a_{12} in the second column and all elements other than $-a_{12}$ in the second row. The resulting skew-symmetric matrix is of the form:

$$D = \left[\begin{array}{cc|ccc} 0 & a_{12} & 0 & \cdots & 0 \\ -a_{12} & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & & & \\ \vdots & \vdots & & D^* & \\ 0 & 0 & & & \end{array}\right],$$

where D^* is a skew-symmetric sub-matrix. If a_{12} divides every element of D^* we have (2.4). Otherwise there is an entry d_{ij} $(j > i \geq 3, j \geq 4)$ in D^* such that a_{12} doesn't divide d_{ij}. In that case we add the ith row to the first row which along with the counterpart column operation gives us Case 1. Consequently, by the same argument used there, we obtain a skew-equivalent matrix with an entry, b_{12}, in the leading diagonal block for which $\phi(b_{12}) < \phi(a_{12})$. This gives us condition (2.5).

So the outcome in each of the three cases is a skew-symmetric matrix skew-equivalent to A which either has the form (2.4) or satisfies (2.5). As previously mentioned repeated application brings us to (2.4) after a finite number of steps thereby completing the first stage of the reduction.

Conclusion of the reduction

Having reached (2.4) we have effectively reduced the size of the matrix with which we are dealing. The above process can then be applied to the submatrix C^* reducing its size further. We note that any simultaneous elementary operations on C^* correspond to simultaneous elementary operations on C which don't affect the first two rows and columns. Furthermore any simultaneous elementary operations on C^* give a new skew-symmetric matrix whose entries will be linear (over R) combinations of the old ones and therefore these new entries will still all be divisible by d_1. The result will be a matrix of the form

$$\left[\begin{array}{cc|cc|ccc} 0 & d_1 & 0 & 0 & 0 & \cdots & 0 \\ -d_1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & -d_2 & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & 0 & 0 & & & \\ \vdots & \vdots & \vdots & \vdots & & C^{**} & \\ 0 & 0 & 0 & 0 & & & \end{array}\right],$$

where $d_1 | d_2$ and d_2 divides every entry in the skew-symmetric sub-matrix C^{**}. We carry on this procedure further reducing the effective size of the matrix and leaving a trail of (leading) diagonal blocks as we go. In due course we reach a matrix of the form:

$$\left[\begin{array}{ccccccc} 0 & d_1 & & & & & \\ -d_1 & 0 & & & & & \\ & & 0 & d_2 & & & \\ & & -d_2 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & d_r \\ & & & & & -d_r & 0 \\ & & & & & & & 0 \end{array}\right].$$

General case

The argument here is not very different from that for a Euclidean domain, the main difference being that simultaneous row and column operations are no longer sufficient alone but need to be supplemented by another kind of 'secondary operation' to effect the reduction.

We wish to mimic the above method but to do so we first need to find a replacement for the Euclidean function ϕ. We do this by introducing a length function on R^* (the non-zero elements of R). If $r \in R^*$ then r can be written in the form

$$r = up_1 \dots p_m,$$

where u is a unit, the p_i are primes in R and $m \geq 0$. The integer m in this expression is unique and we define $\ell(r) = m$ and call this the *length* of r. Clearly

$$\ell(rr') = \ell(r) + \ell(r'), \tag{2.6}$$

where $r, r' \in R^*$.

We show how to reduce a $n \times n$ skew-symmetric matrix A over R to the required skew diagonal matrix by a sequence of operations which correspond to the post and pre multiplication of an invertible matrix and its transpose.

First Stage

As for the Euclidean case this consists of successive applications of a certain sequence of (simultaneous) operations chosen so that each application either leads to (2.4) or to the modified case of (2.5) where ϕ is replaced by ℓ.

Modification to the sequence of operations is only needed in cases 1 and 2 and it is sufficient to explain what happens in case 1.

Here we have $a_{12} \neq 0$ (if not we can make this so by suitable simultaneous row and column interchanges) and there is some a_{1j} such that a_{12} doesn't divide a_{1j} for some $2 < j \leq n$. For notational convenience we can, by a simultaneous row and column interchange, suppose $j = 3$.

Since R is a PID we know from Lemma 2.1.16 that the non-empty set $\{a_{12}, a_{13}\}$ has an hcf d and that the ideals $R\langle a_{12}, a_{13}\rangle$ and $R\langle d\rangle$ are equal. From the former we have

$$a_{12} = dy_2, \qquad a_{13} = dy_3, \tag{2.7}$$

where, since a_{12} doesn't divide a_{13}, y_2 is not a unit. Therefore $\ell(y_2) \geq 1$ and by (2.6)

$$\ell(d) < \ell(a_{12}).$$

Using $R\langle a_{12}, a_{13}\rangle = R\langle d\rangle$ we can write

$$d = x_2 a_{12} + x_3 a_{13},$$

for some x_2, $x_3 \in R$. Then from (2.7) we have

$$d = d(x_2 y_2 + x_3 y_3),$$

hence $x_2 y_2 + x_3 y_3 = 1$. Therefore the determinant of the $n \times n$ matrix

$$S = \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & x_2 & -y_3 & 0 \\ 0 & x_3 & y_2 & \\ \hline & 0 & & I_{n-3} \end{array}\right]$$

is 1 and so this matrix is invertible. It follows that the skew-symmetric matrix S^TAS is skew-equivalent to A and the entry in its leading diagonal block is

$$x_2a_{12} + x_3a_{13} = d,$$

where $\ell(d) < \ell(a_{12})$. So the matrix S^TAS is a matrix satisfying the modified version of (2.5) where the function ϕ is replaced by ℓ.

The Conclusion of the Reduction

This follows exactly as in the Euclidean case. □

We can use the previous proof to establish a result stronger than Lemma 2.1.14 for skew-symmetric matrices defined over a Euclidean domain.

Theorem 2.1.19 *Two skew-symmetric matrices A, B defined over a Euclidean domain, R, are skew-equivalent if and only if one can pass from one to the other by a series of simultaneous row and column operations.*

Proof We have already established, in Lemma 2.1.14, that if we can pass from A to B by a series of simultaneous row and column operations then B is skew-equivalent to A. It remains to prove the converse. Suppose B is skew-equivalent to A that is by Definition 2.1.12

$$B = X^TAX, \tag{2.8}$$

for some invertible matrix X. Consider this matrix X,

$$X = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}.$$

Using techniques described above in the proof of the Euclidean case of Theorem 2.1.18 we can, by column operations only and in a finite number of steps, reduce X to

$$X' = \left(\begin{array}{c|c} x'_{11} & 0 \ \cdots \ 0 \\ \hline x'_{21} & \\ \vdots & X_1 \\ x'_{n1} & \end{array}\right). \tag{2.9}$$

Before proceeding it is worth remarking that any matrix obtained from an invertible matrix by applying elementary row/column operations is also invertible. This follows from Lemma 2.1.8 and the fact that the subset of units of R is closed. Consequently $\det X' = u$ is a unit and so from (2.9)

$$\det X' = x'_{11} \det X_1 = u.$$

So, for some $v \in R$, $uv = 1$ and

$$(x'_{11} \det X_1)v = x'_{11}(\det X_1 \cdot v) = 1$$

from which we find that x'_{11} is a unit. By a column operation of type 2 given in Lemma 2.1.9 we can assume the leading diagonal entry of X' is 1 in which case $\det X' = \det X_1$ which by the previous remark is a unit. Hence we have an $(n-1) \times (n-1)$ invertible submatrix X_1 and by proceeding as above, confining any column operations to the last $n-1$ columns of X', we can reduce X' to the form

$$X'' = \left(\begin{array}{c|cccc} 1 & 0 & 0 & \cdots & 0 \\ \hline x''_{21} & 1 & 0 & \cdots & 0 \\ x''_{31} & x''_{32} & & & \\ \vdots & \vdots & & X_2 & \\ x''_{n1} & x''_{n2} & & & \end{array}\right),$$

where X_2 is an invertible $(n-2) \times (n-2)$ submatrix. By induction we can therefore reduce X by elementary column operations to

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ x_{21} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ x_{n1} & \cdots & x_{nn-1} & 1 \end{pmatrix}$$

and hence it can be seen by further column operations we reach I_n. It follows by standard linear algebra that by performing the same column operations used above to pass from X to I_n on the identity matrix we obtain the inverse X^{-1} of X. We can write

$$X^{-1} = X_1 \cdots X_r,$$

where $X_1, \ldots, X_r$ are, in order, the matrices of types (1)–(3) representing the elementary column operations used. So

$$\begin{aligned} X &= (X_1 \cdots X_r)^{-1} \\ &= X_r^{-1} \cdots X_1^{-1}. \end{aligned}$$

Since the reciprocal of any elementary column operation is also an elementary column operation (of the same type) it follows that postmultiplying the matrix A by X is the same as carrying out a sequence of elementary column operations on A and so (2.8) implies that one can obtain B by a sequence of simultaneous row and column operations, as required. □

Corollary 2.1.20 *Two matrices A, B, defined over a Euclidean domain, R, are equivalent if and only if one can pass from one to the other by a sequence of elementary row and column operations.*

Proof If we can pass from A to B by a series of row and column operations then by Theorem 2.1.11 A and B are equivalent.

Conversely, if A, B are equivalent then

$$B = YAX, \tag{2.10}$$

for some invertible matrices X, Y.

In the proof of Theorem 2.1.19, above, we showed that by a sequence of elementary column operations, represented by matrices $X_1, \ldots, X_r$ of types (1)–(3) in Lemma 2.1.9, we can reduce X to I_n and by applying these same column operations to I_n we obtain the inverse

$$X^{-1} = X_1 \cdots X_r$$

of X. Hence by postmultiplying A by $X = X_r^{-1} \cdots X_1^{-1}$ we are applying a sequence of elementary column operations on A.

Similarly, by applying row operations to the invertible matrix, Y, it can be reduced to I_n by a sequence of elementary row operations, represented by matrices, $Y_1, \ldots, Y_s$, of types (4)–(6) in Lemma 2.1.9. By applying these to I_n we can express the inverse, Y^{-1}, as

$$Y^{-1} = Y_s \cdots Y_1.$$

Hence

$$\begin{aligned} Y &= (Y_s \cdots Y_1)^{-1} \\ &= Y_1^{-1} \cdots Y_s^{-1}, \end{aligned}$$

where $Y_1^{-1}, \dots, Y_s^{-1}$ also represent elementary row operations since the reciprocal of an elementary row operation is also an elementary row operation (of the same type). So premultiplying A by Y has the effect of carrying out this sequence of row operations on A.

Hence it follows from (2.10) that B can be obtained from A by a sequence of elementary row and column operations. □

2.2 Invariants

Preliminary

We define a k-minor of any matrix A over R to be the determinant of a $k \times k$ sub-matrix obtained from A by deleting a suitable number of rows and columns (leaving the order of the remaining rows and columns unchanged). Thus a k-minor is an element of R.

Definition 2.2.1 *Let A be any $s \times t$ matrix over R and $1 \leq k \leq \min\{s, t\}$. We define $I_k(A)$ to be the ideal of R generated by all the k-minors of A.*

In the following lemma we obtain a result, concerning the ideals generated by the $k \times k$ minors of a special type of diagonal matrix, which will be useful later on.

Lemma 2.2.2 *Consider the matrix*

$$B = \begin{bmatrix} b_1 & 0 & 0 & & & & & \\ 0 & b_2 & 0 & & & & & \\ 0 & 0 & b_3 & & & & & \\ & & & \ddots & & & & \\ & & & & b_s & & & \\ & & & & & 0 & & \\ & & & & & & \ddots & \\ & & & & & & & 0 \end{bmatrix},$$

with $b_1 | b_2 | b_3 | \cdots | b_s$. Then the ideal of all its $k \times k$ minors is principal and is equal to

$$I_k(B) = \begin{cases} \langle b_1 b_2 \cdots b_k \rangle & \textit{if } k \leq s \\ \langle 0 \rangle & \textit{otherwise} \end{cases}.$$

Proof If $k > s$ then $I_k(B) = \langle 0 \rangle$ since all $k \times k$ minors are zero. If $k \leq s$ any non-zero member of $I_k(B)$ is clearly a sum of elements of the form $\alpha b_{i_1} \cdots b_{i_k}$, where $\alpha \in R$, and we may suppose $i_1 < \cdots < i_k$. Since $b_j | b_{i_j}$ for $1 \leq j \leq k$ we deduce that $b_1 \cdots b_k$ divides every member of $I_k(B)$ and since clearly $b_1 \cdots b_k \in I_k(B)$ we have the required result. □

Lemma 2.2.3 *Let A, B be $s \times t$ matrices over R and suppose that they are equivalent over R. Then*

$$I_k(A) = I_k(B)$$

for $1 \leq k \leq \min\{s, t\}$.

Proof This result is Lemma 7.14 in [HrtHwk] and we closely follow their proof. We first need the following observation. Let D be any $m \times m$ matrix over R, and write $D = (\mathbf{d_1}, \ldots, \mathbf{d_m})$, where $\mathbf{d_1}, \ldots, \mathbf{d_m}$ denote the columns of D. Furthermore let $\mathbf{d_1'}$, $\mathbf{d_1''}$ be two other column vectors of length m with entries in R. Then

$$\det(\mathbf{d_1'} + \mathbf{d_1''}, \mathbf{d_2}, \ldots, \mathbf{d_m}) = \det(\mathbf{d_1'}, \mathbf{d_2}, \ldots, \mathbf{d_m}) + \det(\mathbf{d_1''}, \mathbf{d_2}, \ldots, \mathbf{d_m})$$

and

$$\det(r\mathbf{d_1}, \mathbf{d_2}, \ldots, \mathbf{d_m}) = r \det(\mathbf{d_1}, \mathbf{d_2}, \ldots, \mathbf{d_m}) \qquad \text{if } r \in R.$$

These facts can be proved by expanding the left hand determinants by the first column. Clearly similar remarks apply to the other columns. (Note these are a generalisation of standard results for determinants of matrices over a field.) Thus, if we have some $m \times m$ matrix D and replace its ith column by some R-linear combination of columns $\mathbf{e_1}, \ldots, \mathbf{e_k}$, then the determinant of the resulting matrix is a R-linear combination of the determinants of the matrices obtained from D by replacing its ith column by $\mathbf{e_1}, \ldots, \mathbf{e_k}$ in turn. We can use this principle to obtain, from the following, a result neccessary for the proof.

Suppose we are given some collection $\mathbf{c_1}, \ldots, \mathbf{c_l}$ of column vectors of length m with entries in R. Let $\mathcal{S}$ be the set of all $m \times m$ matrices which can be formed from these columns (with repeats allowed). Let C be any $m \times m$ matrix whose columns are ***R*-linear combinations** of $\mathbf{c_1}, \ldots, \mathbf{c_l}$. Then applying the principle, described in the previous paragraph, to each column in succession it follows that $\det C$ is a R-linear combination of elements $\det W$ for $W \in \mathcal{S}$. Therefore $\det C$ belongs to the ideal of R generated by the elements $\det W$ as W runs over $\mathcal{S}$.

Having established this we are ready to prove the result. Let $A = (\mathbf{a_1}, \ldots, \mathbf{a_t})$ be any $s \times t$ matrix over R, and let X be any $t \times t$ matrix over R. Consider AX. The ith column of this matrix is given by the column vector

$$\sum_{j=1}^{t} x_{ji}\mathbf{a_j}. \tag{2.11}$$

We examine a typical $i \times i$ submatrix E of AX, letting $J = \{j_1, \ldots, j_i\}$ be the collection of its rows written down in natural order. Then from (2.11) the columns of E are R-linear combinations of 'partial columns' of A, that is, of column $\mathbf{a_k^J}$ got by selecting the $j_1, \ldots, j_i$-th entries of $\mathbf{a_k}$. Therefore, by the result in the previous paragraph, the corresponding i-minor $\det E$ is an R-linear combination of elements

$$\det(\mathbf{a_{k_1}^J}, \ldots, \mathbf{a_{k_i}^J}), \tag{2.12}$$

determinants of $i \times i$ submatrices formed from selections of the columns $\mathbf{a_1}^J, \ldots, \mathbf{a_t}^J$. The determinant (2.12) is zero unless all of $k_1, \ldots k_i$ are distinct and in this case we can bring its columns into the same order as they occur in A, by a series of column interchanges, with the resulting determinant differing only by sign. Consequently, determinants of the form in (2.12) are, up to sign, i-minors of A. So it follows that, if E is any $i \times i$ submatrix of AX, then $\det E$ is a R-linear combination of i-minors of A and so belongs to the ideal $I_i(A)$ generated by these minors. Therefore

$$I_i(AX) \subseteq I_i(A). \tag{2.13}$$

By similar arguments with rows we find that, for any $s \times s$ matrix Y,

$$I_i(YA) \subseteq I_i(A). \tag{2.14}$$

So, replacing the matrix A by YA, on applying (2.13) followed by (2.14) we deduce that

$$I_i(YAX) \subseteq I_i(A). \tag{2.15}$$

If Y and X are invertible and $B = YAX$, then $A = Y^{-1}BX^{-1}$. Immediately, from (2.15)

$$I_i(B) \subseteq I_i(A),$$

and furthermore, applying (2.15) to $Y^{-1}BX^{-1} = A$ we also have

$$I_i(A) \subseteq I_i(B).$$

Hence $I_i(A) = I_i(B)$, as required. □

We are ready to prove the following theorem.

Theorem 2.2.4 *A $n \times n$ skew-symmetric matrix, A, with entries in a PID R is skew-equivalent to a matrix of the form*

$$D = \left[\begin{array}{cccccccc|ccc} 0 & d_1 & & & & & & & & & \\ -d_1 & 0 & & & & & & & & & \\ & & 0 & d_2 & & & & & & & \\ & & -d_2 & 0 & & & & & & & \\ & & & & \ddots & & & & & & \\ & & & & & & 0 & d_r & & & \\ & & & & & & -d_r & 0 & & & \\ \hline & & & & & & & & 0 & & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & 0 \end{array}\right],$$

where $d_1|\cdots|d_r$ and the d_i's are unique up to units. Furthermore, the ideals $I_k(A)$ generated by the $k \times k$ minors of A are given by

$$I_k(A) = \left\{ \begin{array}{cc} 0 & \text{if } k > 2r \\ \langle d_1^2 \cdots d_{s-1}^2 d_s \rangle & \text{if } k = 2s-1 \text{ is odd} \\ \langle d_1^2 \cdots d_s^2 \rangle & \text{if } k \text{ is even i.e. } k = 2s \end{array} \right\} \quad 1 \leq k \leq 2r \quad ,$$

where $1 \leq s \leq r$.

Conversely, given a skew-symmetric matrix A for which the even ideals $I_{2s}(A) = \langle g_s^2 \rangle$, $1 \leq s \leq r$ with g_s^2 the principal generators, then (setting $g_0 = 1$) A is skew-equivalent to a matrix as above with

$$d_s = \frac{g_s}{g_{s-1}}. \tag{2.16}$$

Proof By Theorem 2.1.18 A is skew-equivalent to a matrix

$$D = \left[\begin{array}{ccccccc|ccc}
0 & d_1 & & & & & & & & \\
-d_1 & 0 & & & & & & & & \\
& & 0 & d_2 & & & & & & \\
& & -d_2 & 0 & & & & & & \\
& & & & \ddots & & & & & \\
& & & & & 0 & d_r & & & \\
& & & & & -d_r & 0 & & & \\
\hline
& & & & & & & 0 & & \\
& & & & & & & & \ddots & \\
& & & & & & & & & 0
\end{array}\right],$$

for some $d_1 | \cdots | d_r$. By applying Lemma 2.2.3 to the special case of skew-equivalence it follows that

$$I_k(A) = I_k(D).$$

Furthermore by interchanging columns of D we obtain the equivalent diagonal matrix

$$\tilde{D} = \begin{bmatrix}
d_1 & & & & & & & & & \\
& -d_1 & & & & & & & & \\
& & d_2 & & & & & & & \\
& & & -d_2 & & & & & & \\
& & & & \ddots & & & & & \\
& & & & & d_r & & & & \\
& & & & & & -d_r & & & \\
& & & & & & & 0 & & \\
& & & & & & & & \ddots & \\
& & & & & & & & & 0
\end{bmatrix}$$

and by Lemma 2.2.3 it follows that

$$I_k(A) = I_k(\tilde{D}).$$

If $k > 2r$ then by Lemma 2.2.2 $I_k(A) = \langle 0 \rangle$. However for $1 \leq k \leq 2r$ we need to consider odd and even ideals separately. So, again by Lemma 2.2.2: if k is odd, i.e. $k = 2s - 1$, we have

$$I_{2s-1}(A) = \langle d_1^2 \cdots d_{s-1}^2 d_s \rangle$$

and if k is even, i.e. $k = 2s$ we have

$$I_{2s}(A) = \langle d_1^2 \cdots d_s^2 \rangle,$$

where $1 \leq s \leq r$. It remains to check how unique these d_i are. Assume A is also skew-equivalent to a matrix

$$D' = \left[\begin{array}{cccccccc|ccc} 0 & d'_1 & & & & & & & & & \\ -d'_1 & 0 & & & & & & & & & \\ & & 0 & d'_2 & & & & & & & \\ & & -d'_2 & 0 & & & & & & & \\ & & & & \ddots & & & & & & \\ & & & & & & 0 & d'_r & & & \\ & & & & & & -d'_r & 0 & & & \\ \hline & & & & & & & & 0 & & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & 0 \end{array}\right],$$

where $d'_1|\cdots|d'_r$. Then by applying the above, for $1 \leq s \leq r$, we have

$$I_{2s-1}(A) = \langle d_1^2 \cdots d_{s-1}^2 d_s \rangle = \langle d_1'^2 \cdots d_{s-1}'^2 d'_s \rangle$$

when $k = 2s - 1$ is odd and

$$I_{2s}(A) = \langle d_1^2 \cdots d_s^2 \rangle = \langle d_1'^2 \cdots d_s'^2 \rangle$$

when $k = 2s$ is even. This implies that

$$d_1^2 \cdots d_{s-1}^2 d_s \sim d_1'^2 \cdots d_{s-1}'^2 d'_s \tag{2.17}$$

and

$$d_1^2 \cdots d_s^2 \sim d_1'^2 \cdots d_s'^2 \tag{2.18}$$

respectively.

Set $e_s = d_1^2 \cdots d_{s-1}^2 d_s$, $e'_s = d_1'^2 \cdots d_{s-1}'^2 d'_s$ and $f_s = d_1^2 \cdots d_s^2$, $f'_s = d_1'^2 \cdots d_s'^2$. Then by (2.17) we have $e_s = u_s e'_s$ for some unit $u_s \in R$. Therefore

$$f_s = d_s e_s = d_s u_s e'_s$$

and also, by (2.18), for some unit $v_s \in R$,

$$f_s = v_s f'_s = v_s d'_s e'_s$$

whence $d_s u_s = v_s d'_s$ and $d_s = u_s^{-1} v_s d'_s$. Hence $d_s \sim d'_s$ as required.

Finally given the even ideals $I_{2s}(A) = \langle g_s^2 \rangle$ where, for each $1 \leq s \leq r$, g_s^2 is a principal generator of $I_{2s}(A)$ it follows that, up to constant multiples,

$$g_s = d_1 \cdots d_s.$$

Hence setting $g_0 = 1$, for each $1 \leq s \leq r$, d_s is given by the expression

$$d_s = \frac{g_s}{g_{s-1}}.$$

□

The following definition follows from this result.

Definition 2.2.5 *We call the sequence $d_1, \ldots, d_r$ a sequence of invariant factors of A over the PID, R, and skewdiag($d_1, \ldots, d_r$) an invariant factor matrix for A. In particular, we say the* **rank** *of A is $2r$.*

Theorem 2.2.6 *Two $n \times n$ skew-symmetric matrices A and B over a PID R are skew-equivalent if and only if they have (to within associates) the same sequence of invariant factors over R.*

Proof $\Longrightarrow$ By Theorem 2.2.4 A is skew-equivalent to a matrix of the form skewdiag$(d_1, \ldots, d_r)$ with, up to units, invariant factors $d_1, \ldots, d_r$. So since B is skew-equivalent to A it follows that B is also skew-equivalent to skewdiag$(d_1, \ldots, d_r)$ and hence, by Theorem 2.2.4 again, has the same sequence of invariant factors.

$\Longleftarrow$ If A and B have, up to units, the same sequence of invariant factors then A is skew-equivalent to $D =$ skewdiag$(d_1, \ldots d_r)$ and B is skew-equivalent to $D' =$ skewdiag$(d'_1, \ldots, d'_r)$, where $d_k \sim d'_k$ for $1 \leq k \leq r$. By applying a sequence of elementary (simultaneous) row and column operations, involving multiplication of a row and column by a unit, it follows that D and D' are skew-equivalent. Hence by the commutativity of (skew-)equivalence A and B are skew-equivalent. □

This theorem is the most significant consequence of the above work. Our aim is to apply this and some of the above to the special case of λ-matrices.

2.3 λ-Matrices

The following section is inspired by Chapter III of [Ferrar].

Definition 2.3.1 *A λ-matrix, A, is a matrix over the Euclidean domain $K[\lambda]$, where $K[\lambda]$ is a ring of polynomials in a variable λ with coefficients in the field K, and the Euclidean function, ϕ, is taken to be the degree of the polynomials.*

The determinant, $\det A$, of a square λ-matrix is, in general, a polynomial in λ. However, it may also be a constant, independent of λ. In particular if this constant is zero, i.e $\det A = 0$ for all values of λ, we say $\det A$ is identically zero.

Definition 2.3.2 *(i) The square λ-matrix, A, is said to be* **singular** *when $\det A$ is identically zero and* **non-singular** *otherwise.*

(ii) The λ-matrix A has **rank** *r (≥ 1) if r is the largest integer for which not all minors of order r are identically zero.*

(Note : by Theorem 2.2.4, (ii) corresponds, precisely, to the description of rank given in Definition 2.2.5.)

Corollary 2.3.3 *A square λ-matrix, X, is invertible if and only if $\det X$ is a non-zero constant.*

Proof This is just Lemma 2.1.5 restated for $R = K[\lambda]$, the units of which are nonzero constants. □

The equivalence of λ-matrices, corresponding to Definition 2.1.6, is referred to as λ-equivalence, so that two λ-matrices, A, B are said to be λ-equivalent if there exist invertible λ-matrices, X, Y, such that

$$B = YAX.$$

We refer to elementary row/column operations on a λ-matrix as *elementary λ-transformations.* They are the row/column operations described in Lemma 2.1.9 applied to the ring $K[\lambda]$. For completeness we restate them.

Definition 2.3.4 *The elementary λ-transformations of a matrix are*

(i) the interchange of rows (columns) i and j;

(ii) the multiplication of a row (column) i by a non-zero constant;

(iii) the addition of a polynomial multiple of a row (column) j to a row (column) i.

Here we are mainly concerned with skew-symmetric λ-matrices and, accordingly, we define an equivalence corresponding to that given in Definition 2.1.12.

Definition 2.3.5 *Two skew-symmetric matrices, A and B are skew λ-equivalent if there exists an invertible λ-matrix X such that*

$$B = X^T AX.$$

As we have seen in Theorem 2.1.19 this notion of skew-equivalence corresponds to applying a series of simultaneous elementary row and column operations to A. We refer to such operations on a λ-matrix as simultaneous λ-transformations.

By our definition of λ-matrices it is clear that Theorems 2.1.18, 2.2.4 and 2.2.6 are applicable for skew-symmetric λ-matrices.

We use skew λ-equivalence to prove a result about non-singular skew-symmetric pencils. First we need a few preliminaries concerning general λ-matrices. Note that throughout the following we denote constant matrices, over K, by small letters and λ-matrices by capital letters.

Definition 2.3.6 *Let A be a (square) λ-matrix. It is said to be of* degree k *when λ^k is the highest power of λ occuring among its entries. Such a matrix may be written in the form*

$$A = a_k\lambda^k + a_{k-1}\lambda^{k-1} + \cdots + a_0,$$

where a_k, $a_{k-1}, \cdots, a_0$ are matrices over K and a_k is not the null matrix; i.e it can be thought of as a polynomial in λ with matrix coefficients.

Lemma 2.3.7 *Consider the λ-matrices A, B given by*

$$A = a_k\lambda^k + a_{k-1}\lambda^{k-1} + \cdots + a_0,$$

$$B = b_l\lambda^l + b_{l-1}\lambda^{l-1} + \cdots + b_0,$$

where $a_k \neq 0$ and $\det b_l \neq 0$. Then the products AB and BA are λ-matrices of degree $k + l$.

Proof Given the two (constant) matrices a_k and b_l, where b_l is non-singular, then, from Chapter I, §10 of [Ferrar], the matrices $a_k b_l$ and $b_l a_k$ have the same rank as a_k. In particular if a_k is not the null matrix then neither is $a_k b_l$ or $b_l a_k$ and the result follows. □

The next lemma is an analogue (for polynomials with matrix coefficients) of the standard property of the Euclidean domain $K[\lambda]$.

Lemma 2.3.8 *Consider two λ-matrices A, B written in the form given in Lemma 2.3.7, where b_l is non-singular. Then there is a unique pair of matrices Q_1 and R_1 for which*

$$A = Q_1 B + R_1,$$

and either $R_1 = 0$ or R_1 is a λ-matrix of degree less than l (possibly a constant). There is also a unique pair of matrices Q_2 and R_2 for which

$$A = BQ_2 + R_2$$

and either $R_2 = 0$ or R_2 is a λ-matrix of degree less than l (possibly a constant).

Proof We start by proving the following result.

For k, $l \geq 0$, given any two λ-matrices

$$A = a_k\lambda^k + a_{k-1}\lambda^{k-1} + \cdots + a_0,$$

$$B = b_l\lambda^l + b_{l-1}\lambda^{l-1} + \cdots + b_0,$$

where $a_k \neq 0$ and b_l is non-singular, then there exists a pair of matrices Q, R, where R has degree less than l, such that

$$A = QB + R.$$

If $l > k$, then we choose $Q = 0$ and $R = A$. So assume $l \leq k$. If $k = 0$ then $l = 0$ and $A = a_0$, $B = b_0$ are constant matrices with b_0 non-singular. Consequently we can write

$$A = \underbrace{(AB^{-1})}_{Q} B + \underbrace{0}_{R}.$$

For some general $k > 0$, by choosing $Q_1 = a_k b_l^{-1}\lambda^{k-l}$ we can write

$$A = Q_1 B + R_1, \tag{2.19}$$

where $R_1 = A - Q_1 B$ has degree less than k. Assuming the above hypothesis is true for all λ-matrices A of degree $r < k$ then we can write

$$R_1 = Q_2 B + R_2,$$

for some R_2 of degree $< l$. So from (2.19)

$$\begin{aligned} A &= Q_1 B + Q_2 B + R_2 \\ &= (Q_1 + Q_2) B + R_2 \end{aligned}$$

and setting $Q = Q_1 + Q_2$ and $R = R_2$ the hypothesis holds for k. Since we have already shown that it holds for $k = 0$ it follows by induction that the hypothesis holds for all $k \geq 0$. It remains to show that the pair Q, R are unique. If

$$A = Q_1 B + R_1 = Q_2 B + R_2$$

then

$$(Q_1 - Q_2) B = R_2 - R_1. \tag{2.20}$$

Assuming $Q_1 - Q_2$ is non-zero then since b_l is non-singular it follows by Lemma 2.3.7 that the LHS of this equation has degree $\geq l$ but by the above hypothesis the RHS of this equation has degree $< l$ and we have a contradiction. Consequently our assumption is incorrect and

$$Q_1 - Q_2 = 0$$

which implies that

$$Q_1 = Q_2 \qquad \text{and} \qquad R_1 = R_2$$

as required. The proof of the second statement is similar with the quotient Q a post-multiplier of B, instead of a premultiplier. □

The following theorem, which is of considerable use for the study of non-singular pencils, is Theorem 9 in Chapter III, §9.2 of [Ferrar].

Theorem 2.3.9 *Let a_1, a_2, b_1, b_2 be matrices over K and let a_2 and b_2 be non-singular. Then if the (λ)-matrices*

$$A = a_1 + \lambda a_2, \qquad B = b_1 + \lambda b_2$$

are λ-equivalent there are non-singular matrices p, q over K for which

$$B = qAp.$$

In particular $b_1 = q a_1 p$, $b_2 = q a_2 p$.

Proof If A and B are λ-equivalent there are invertible λ-matrices P, Q for which

$$B = QAP. \tag{2.21}$$

Consider the two λ-matrices Q and B. As b_2 is non-singular, by Lemma 2.3.8, there exist a unique pair of matrices Q_1, q for which

$$Q = BQ_1 + q, \tag{2.22}$$

where, since B is linear in λ, q is a constant matrix (possibly zero). Similarly for the two λ-matrices P^{-1} and A there exist a unique pair of matrices S_1, s such that

$$P^{-1} = S_1 A + s, \tag{2.23}$$

where s is a constant matrix, possibly zero. Then from (2.21) we have

$$B(S_1 A + s) = (BQ_1 + q)A,$$

which on rearranging gives

$$B(S_1 - Q_1)A = qA - Bs. \tag{2.24}$$

We consider both sides of this equation in turn. Since A and B are linear in λ and q and s are constant matrices the RHS of this equation is at most linear in λ.

If we assume $S_1 - Q_1$ is not the null matrix but a λ-matrix of degree $t \geq 0$, the LHS is

$$(b_1 + \lambda b_2)(S_1 - Q_1)(a_1 + \lambda a_2). \tag{2.25}$$

Since b_2 and a_2 are both non-singular, by Lemma 2.3.7, (2.25) is of degree $t+2$ in λ and cannot be identical to the RHS of (2.24). We conclude our assumption to be incorrect and that $S_1 - Q_1 = 0$. So from (2.24)

$$qA = Bs. \tag{2.26}$$

It remains to show q and s to be non-singular. Using Lemma 2.3.8, let

$$P = P_1 B + p.$$

Then since $I = PP^{-1}$

$$I = (P_1 B + p)(S_1 A + s)$$

and

$$I - ps = (P_1 B + p)S_1 A + P_1 Bs.$$

But from (2.26) it follows that

$$I - ps = (P_1BS_1 + pS_1 + P_1q)A. \tag{2.27}$$

We consider both sides of this equation. If $(P_1BS_1 + pS_1 + P_1q)$ is not the null matrix then since $A = a_1 + \lambda a_2$ and a_2 is non-singular, by Lemma 2.3.7, the RHS is of at least degree 1 in λ. However the LHS is a constant matrix and we have a contradiction. So $(P_1BS_1 + pS_1 + P_1q)$ is the null matrix and from (2.27) $I = ps$, i.e. s is non-singular with inverse p. From (2.26) we have

$$qAp = B,$$

and by writing $Q^{-1} = AT_1 + t$ and considering $QQ^{-1} = I$ by a similar argument we can show that q is also non-singular and the result follows. □

In the following section we will consider pairs, (a_1, a_2), of skew-symmetric matrices of which at least one, a_2, is non-singular. Associated to this pair we have a linear λ-matrix $a_1 + \lambda a_2$. So if the pairs (a_1, a_2) and (b_1, b_2) are skew-equivalent this is the same as saying that, for some invertible matrix p over K,

$$b_1 + \lambda b_2 = p^T(a_1 + \lambda a_2)p. \tag{2.28}$$

With this in mind we state the following theorem, which is, principally, an adaption of the above Theorem 2.3.9 for the skew-symmetric case.

Lemma 2.3.10 *Let (a_1, a_2), (b_1, b_2) be skew-symmetric pairs with a_2 non-singular. Then they are skew-equivalent if and only if the the skew-symmetric λ-matrices*

$$A = a_1 + \lambda a_2, \qquad B = b_1 + \lambda b_2$$

are skew λ-equivalent.

Proof Clearly, if (a_1, a_2) and (b_1, b_2) are skew-equivalent then for some invertible matrix p over K

$$b_1 + \lambda b_2 = p^T(a_1 + \lambda a_2)p$$

and $b_1 + \lambda b_2$ is also skew λ-equivalent to $a_1 + \lambda a_2$.

If $A = a_1 + \lambda a_2$ and $B = b_1 + \lambda b_2$ are skew λ-equivalent there is an invertible λ-matrix P for which

$$B = P^TAP. \tag{2.29}$$

Furthermore, by Theorem 2.2.4, $\det(a_1 + \lambda a_2)$ and $\det(b_1 + \lambda b_2)$ differ by a non-zero constant factor. By considering the polynomial $\det(a_1 + \lambda a_2)$ we see that any terms of degree n are given by

$$\det(\lambda a_2) = \lambda^n \det a_2,$$

(similarly the terms of degree n in the polynomial $\det(b_1 + \lambda b_2)$ are $\lambda^n \det b_2$) and since a_2 is non-singular this polynomial has degree n. Hence the degree of $\det(b_1 + \lambda b_2)$ is n and b_2 is also non-singular. The remainder of the proof is an adaption of the Proof of Theorem 2.3.9.

Given the two λ-matrices P, B then as b_2 is non-singular by Lemma 2.3.8 there exist a unique pair of matrices P_1, p for which

$$P = P_1 B + p, \tag{2.30}$$

where, since B is linear in λ, p is a constant matrix (possibly zero). Similarly for the two matrices P^{-1}, A there exist a unique pair of matrices S_1, s such that

$$P^{-1} = S_1 A + s \tag{2.31}$$

where s is a constant, possibly zero. Since B is skew-symmetric

$$\begin{aligned} P^T &= B^T P_1^T + p^T &(2.32)\\ &= -BP_1^T + p^T &(2.33)\end{aligned}$$

and from (2.29) we have

$$B(S_1 A + s) = (-BP_1^T + p^T)A.$$

Consequently,

$$B(S_1 + P_1^T)A = p^T A - Bs. \tag{2.34}$$

We now consider both sides of this equation in turn. Since A and B are linear in λ and p and s constant matrices the RHS of this equation is at most linear in λ.

If we assume $S_1 + P_1^T$ is not the null matrix but a λ-matrix of degree t, where $t \geq 0$, the LHS becomes

$$(b_1 + \lambda b_2)(S_1 + P_1^T)(a_1 + \lambda a_2).$$

Furthermore, since both a_2 and b_2 are non-singular, by Lemma 2.3.7 it has degree $t+2$ and so the LHS cannot be identical to the RHS. We deduce that our assumption is incorrect and $S_1 + P_1^T = 0$. So from (2.34)

$$Bs = p^T A. \tag{2.35}$$

It remains to show that $s^{-1} = p$. Since $I = PP^{-1}$ then

$$I = (P_1 B + p)(S_1 A + s)$$

and

$$I - ps = (P_1 B + p) S_1 A + P_1 B s.$$

But, by the above, $Bs = p^T A$ and so

$$I - ps = (P_1 B S_1 + p S_1 + P_1 p^T) A.$$

We consider both sides of this equation. If $(P_1 B S_1 + p S_1 + P_1 p^T)$ is not the null matrix then since $A = a_1 + \lambda a_2$ and a_2 is non-singular by Lemma 2.3.7 the RHS is of at least degree 1 in λ. The LHS, however, is a constant matrix and we have a contradiction. So $(P_1 B S_1 + p S_1 + P_1 p^T)$ is the null matrix and $I = ps$, that is s is non-singular with inverse p. It therefore follows from (2.35) that

$$B = p^T A p,$$

and (a_1, a_2) and (b_1, b_2) are skew-equivalent pairs. □

2.4 Non-singular Skew-symmetric Pairs

In the following all skew-symmetric matrices have entries in $\mathbb{C}$.

We consider the set, U, of pairs of even $n \times n$ skew-symmetric matrices (A_1, A_2), with A_2 non-singular. Let $n = 2r$. The set U is an open subset of $Sk(n, \mathbb{C}) \times Sk(n, \mathbb{C})$ and, as described in Definition 1.2.1 of Section 1.2, skew-equivalent pairs are those lying in the same orbit of the action of $Gl(n, \mathbb{C})$ on this set. Expressing such pairs by a linear skew-symmetric λ-matrix $A_1 + \lambda A_2$, by Lemma 2.3.10 and Theorem 2.2.6, two such pairs are skew-equivalent if and only if they have the same sequence of invariant factors. (A sequence of invariant factors corresponds to a single orbit of the set of such pairs.)

So given any such pair, expressed as a linear skew-symmetric λ-matrix, with a sequence $d_1, \ldots, d_r$ of invariant factors our aim is to find the simplest linear λ-matrix with the same sequence of invariant factors. This will then be a normal form for any pair with this sequence of invariant factors. In fact if we can find a set of such normal forms yielding all possible invariant factors then any pair will be equivalent to one in our set.

We start by finding the invariant factors for such a pair, (A_1, A_2), represented by the $n \times n$ linear skew-symmetric λ-matrix

$$A = A_1 + \lambda A_2,$$

where A_2 is non-singular. Then, by Lemma 1.1.8 of Chapter 1,

$$\det A = \det(A_1 + \lambda A_2) = (f(\lambda))^2,$$

for some $f(\lambda)$. It follows from the proof of Lemma 2.3.10 that $f(\lambda)$ has degree r and so over $\mathbb{C}$ we can factorize it as

$$f(\lambda) = \prod_{i=1}^{r}(b_i\lambda + c_i),$$

where $b_i \neq 0$ for all i. Furthermore by Theorem 2.2.4 $\det A = (d_1 \cdots d_r)^2$ where $d_1, \ldots, d_r$ are the invariant factors of A. It therefore follows that up to constants

$$d_1 \cdots d_r = \lambda^{v_r} \prod_{i=1}^{q}(a_i\lambda + 1)^{s_{ir}},$$

where q is the number of distinct non-zero roots of $f(\lambda)$. (Here v_r is the number of factors of f for which $c_j = 0$, the a_i correspond to some b_j/c_j, $c_j \neq 0$.) The integer s_{ir} then denotes the multiplicity of the linear factor $(a_i\lambda + 1)$ and $v_r \geq 0$ the multiplicity of the zero root. Note if $v_r = 0$ then $f(\lambda)$ has no zero root which corresponds to A_1 also being non-singular.

Using (2.16) from Theorem 2.2.4 we can find the invariants of any λ-matrix from the principal generators of its even ideals. Since these principal generators are square the following notation is useful.

Definition 2.4.1 *If $I_{2k} = \langle g \rangle$ we write $\sqrt{I_{2k}}$ for $\langle \sqrt{g} \rangle$. (This is not to be confused with the radical of I_{2k}.)*

Hence we write down the generators $g_k = d_1 \cdots d_k$ for the ideals $\sqrt{I_{2k}(A)}$ as follows

$$\begin{array}{rcccl} g_1 & = & d_1 & = & \lambda^{v_1} \prod_{i=1}^{q} (a_i\lambda + 1)^{s_{i1}} \\ g_2 & = & d_1 d_2 & = & \lambda^{v_2} \prod_{i=1}^{q} (a_i\lambda + 1)^{s_{i2}} \\ & & \vdots & & \vdots \\ g_r & = & d_1 d_2 \cdots d_r & = & \lambda^{v_r} \prod_{i=1}^{q} (a_i\lambda + 1)^{s_{ir}}, \end{array} \tag{2.36}$$

where for each $1 \leq i \leq q$, $s_{i1} \leq s_{i2} \leq \cdots \leq s_{ir}$, and $v_1 \leq v_2 \leq \cdots \leq v_r$. So on finding the principal generators, g_j^2, of A using (2.16) its invariant factors, $d_1, \ldots, d_r$, $k = 1, \ldots r$, are given by

$$\begin{aligned} d_k &= \frac{g_k}{g_{k-1}} \\ &= \lambda^{v_k - v_{k-1}} \prod_{i=1}^{q} (a_i\lambda + 1)^{s_{ik} - s_{ik-1}}, \end{aligned} \tag{2.37}$$

where $d_1 | \cdots | d_r$.

Given any such sequence of invariant factors, the following section proposes a normal form.

2.4.1 Normal Forms

Consider the block of the form

$$\underbrace{\left[\begin{array}{cc|cc|c|cc} 0 & a_i\lambda+1 & & & & & \\ -a_i\lambda-1 & 0 & 1 & & & & \\ \hline & -1 & 0 & a_i\lambda+1 & & & \\ & & -a_i\lambda-1 & 0 & 1 & & \\ \hline & & & -1 & & & \\ & & & & \ddots & & \\ & & & & & 1 & \\ \hline & & & & -1 & 0 & a_i\lambda+1 \\ & & & & & -a_i\lambda-1 & 0 \end{array}\right]}_{2r_{ij}}, \tag{2.38}$$

where $a_i \neq 0$ and r_{ij} is the number of 2×2 blocks

$$\left[\begin{array}{cc} 0 & a_i\lambda+1 \\ -a_i\lambda-1 & 0 \end{array}\right].$$

It is more convenient to refer to the size of such a block by the number r_{ij} of its constituent 2×2 blocks as opposed to its actual size $2r_{ij}$.

For each distinct a_i we construct a direct sum of $n(i)$ blocks of the above form of sizes $r_{i1}, r_{i2}, \ldots, r_{in(i)}$ arranged so that $r_{i1} \leq r_{i2} \leq \cdots \leq r_{in(i)}$. These sizes depend on the multiplicities, s_{ik}, of the factor $(a_i\lambda + 1)$, in the principal generator g_k of each of the even ideals. Supposing the determinant of a pair has q distinct non-zero roots $a_1, \ldots, a_q$ its proposed normal form, N, consists of a direct sum of these objects. Corresponding to any zero eigenvalue of a pair its normal form has in addition a direct sum of $n(0)$ blocks of the form

$$\underbrace{\left[\begin{array}{cc|cc|c|cc}
0 & \lambda & & & & & \\
-\lambda & 0 & 1 & & & & \\ \hline
& -1 & 0 & \lambda & & & \\
& & -\lambda & 0 & 1 & & \\ \hline
& & & -1 & & & \\
& & & & \ddots & & \\
& & & & & 1 & \\ \hline
& & & & -1 & 0 & \lambda \\
& & & & & -\lambda & 0
\end{array}\right]}_{2r_{0j}}, \qquad (2.39)$$

of sizes $r_{01} \leq \cdots \leq r_{0n(0)}$, determined by the multiplicity of λ in the principal generators of the even ideals of $A_1 + \lambda A_2$. We can therefore express such a normal form, N by the following shorthand:

$$\text{distinct(non-zero)} \left\{ \begin{array}{ccccc}
a_1 & r_{11} & r_{12} & \cdots & r_{1n(1)} \\
\vdots & & & & \\
\vdots & & & & \\
\underline{a_q} & \underline{r_{q1}} & \underline{r_{q2}} & \cdots & \underline{r_{qn(q)}} \\
0 & r_{01} & r_{02} & \cdots & r_{0n(0)}
\end{array}\right. , \qquad (2.40)$$

where for each $0 \leq i \leq q$, $r_{i1} \leq r_{i2} \leq \cdots \leq r_{in(i)}$.

Our aim is to show that any sequence of invariant factors can be realised by a normal form of this type.

2.4.2 Invariants of the Normal Form

Given such a normal form we first need to find its invariants. We start by using (2.16) to calculate the invariants of blocks of the type in (2.38).

Lemma 2.4.2 *The skew-symmetric λ-matrix*

$$\Omega = \underbrace{\left[\begin{array}{cc|cc|c|cc} 0 & a_i\lambda+1 & & & & & \\ -a_i\lambda-1 & 0 & 1 & & & & \\ \hline & -1 & 0 & a_i\lambda+1 & & & \\ & & -a_i\lambda-1 & 0 & 1 & & \\ \hline & & & -1 & \ddots & & \\ & & & & & 1 & \\ \hline & & & & -1 & 0 & a_i\lambda+1 \\ & & & & & -a_i\lambda-1 & 0 \end{array}\right]}_{2r_{ij}},$$

has invariant factors $d_1 = d_2 = \cdots = d_{r_{ij}-1} = 1$, $d_{r_{ij}} = (a_i\lambda+1)^{r_{ij}}$, and is therefore skew λ-equivalent to

$$E(a_i, r_{ij}) = \left[\begin{array}{cc|cc|cc|cc} 0 & 1 & & & & & & \\ -1 & 0 & & & & & & \\ \hline & & 0 & 1 & & & & \\ & & -1 & 0 & & & & \\ \hline & & & & \ddots & & & \\ & & & & 0 & 1 & & \\ & & & & -1 & 0 & & \\ \hline & & & & & & 0 & (a_i\lambda+1)^{r_{ij}} \\ & & & & & & -(a_i\lambda+1)^{r_{ij}} & 0 \end{array}\right].$$

Proof The block Ω contains $r_{ij} - 1$ blocks

$$E = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

So with the exception of $I_{2r_{ij}}$ each even ideal I_{2k} $(1 \leq k \leq r_{ij} - 1)$ contains a $2k \times 2k$ minor of the form

$$\left[\begin{array}{cc|cc|c|c|cc}
0 & 1 & & & & & & \\
-1 & 0 & a_i\lambda+1 & & & & & \\ \hline
 & -a_i\lambda-1 & 0 & 1 & & & & \\
 & & -1 & 0 & & & & \\ \hline
 & & & & \ddots & & & \\
 & & & & & & a_i\lambda+1 & \\ \hline
 & & & & & -a_i\lambda-1 & 0 & 1 \\
 & & & & & & -1 & 0
\end{array}\right].$$

By Lemma 1.0.7 this matrix has determinant 1, so $1 \in I_{2k}$ which implies that

$$I_{2k} = \langle 1 \rangle,$$

for $1 \leq k \leq r_{ij} - 1$.
Furthermore from the same lemma we see that $\det \Omega = (a_i\lambda + 1)^{2r_{ij}}$ from which we deduce that

$$I_{2r_{ij}} = \langle (a_i\lambda + 1)^{2r_{ij}} \rangle.$$

It follows from (2.16) that

$$d_1 = d_2 = \cdots = d_{r_{ij}-1} = 1 \quad \text{and} \quad d_{r_{ij}} = (a_i\lambda + 1)^{r_{ij}}.$$

So from Theorem 2.2.4 Ω is skew λ-equivalent to

$$E(a_i, r_{ij}) = \left[\begin{array}{cc|cc|cc|cc}
0 & 1 & & & & & & \\
-1 & 0 & & & & & & \\ \hline
 & & 0 & 1 & & & & \\
 & & -1 & 0 & & & & \\ \hline
 & & & & \ddots & & & \\
 & & & & 0 & 1 & & \\
 & & & & -1 & 0 & & \\ \hline
 & & & & & & 0 & (a_i\lambda+1)^{r_{ij}} \\
 & & & & & & -(a_i\lambda+1)^{r_{ij}} & 0
\end{array}\right],$$

as required. □

By a similar argument we can deduce the following, about blocks of the type in (2.39).

Corollary 2.4.3 *The skew-symmetric λ-matrix*

$$\underbrace{\left[\begin{array}{cc|cc|cc|cc}
0 & \lambda & & & & & & \\
-\lambda & 0 & 1 & & & & & \\
\hline
& -1 & 0 & \lambda & & & & \\
& & -\lambda & 0 & 1 & & & \\
\hline
& & & -1 & & & & \\
& & & & \ddots & & & \\
& & & & & & 1 & \\
\hline
& & & & & -1 & 0 & \lambda \\
& & & & & & -\lambda & 0
\end{array}\right]}_{2r_{0j}},$$

has invariant factors $d_1 = d_2 = \cdots = d_{r_{0j}-1} = 1$, $d_{r_{0j}} = \lambda^{r_{0j}}$, and is therefore skew λ-equivalent to

$$E(0, r_{0j}) = \left[\begin{array}{cc|cc|cc|cc}
0 & 1 & & & & & & \\
-1 & 0 & & & & & & \\
\hline
& & 0 & 1 & & & & \\
& & -1 & 0 & & & & \\
\hline
& & & & \ddots & & & \\
& & & & 0 & 1 & & \\
& & & & -1 & 0 & & \\
\hline
& & & & & & 0 & \lambda^{r_{0j}} \\
& & & & & & -\lambda^{r_{0j}} & 0
\end{array}\right].$$

Proof The argument is the same as that for Lemma 2.4.2 with the linear factor $(a_i\lambda + 1)$ replaced by λ. □

Remark 2.4.4 So given any constituent block, Ω, of the normal form represented by (2.40), by Theorem 2.1.19 (applied over the Euclidean domain $K[\lambda]$) we can obtain the block $E(a_i, r_{ij})$ by a series of simultaneous λ-transformations. By virtue of the normal form being a direct sum, the effects of such λ-transformations are confined to the block concerned. By applying this argument to each of its blocks, it follows that the normal form (represented by (2.40)) is skew λ-equivalent to a direct sum of the form

$$M = \bigoplus_{i=1}^{q} \bigoplus_{j=1}^{n(i)} E(a_i, r_{ij}) \bigoplus_{j=1}^{n(0)} E(0, r_{0j}).$$

By calculating the invariant factors of this sum, M, we find the invariant factors for the skew λ-equivalent normal form, N. Before doing this we state a couple of properties of M which will be refered to later.

Lemma 2.4.5 *We denote the number of constituent blocks of the λ-matrix*

$$M = \bigoplus_{i=1}^{q}\bigoplus_{j=1}^{n(i)} E(a_i, r_{ij}) \bigoplus_{j=1}^{n(0)} E(0, r_{0j}),$$

by p', so

$$p' = \underbrace{\sum_{i=1}^{q} n(i)}_{p} + n(0).$$

The number of blocks

$$E = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

contained in M, is denoted by m' and is given by

$$m' = \underbrace{\sum_{i=1}^{q}\sum_{j=1}^{n(i)}(r_{ij} - 1)}_{m} + \sum_{j=1}^{n(0)}(r_{0j} - 1).$$

Proof The expression for p' is self-explanatory. The expression for m' follows from a consideration of the form taken by the blocks $E(a_i, r_{ij})$ and $E(0, r_{0j})$. □

The integer m' is an important invariant when considering the ideals generated by even minors of M. In fact we can use it to deduce the following.

Lemma 2.4.6 *If $I_{2k}(M)$ is the ideal of all $2k \times 2k$ minors of the sum M then if $k \leq m'$ we have*

$$I_{2k}(M) = \langle 1 \rangle.$$

Proof If $k \leq m'$ one of the $2k \times 2k$ minors is $\det(\oplus_k E) = 1$. So $I_{2k}(M) = \langle 1 \rangle$ as required. □

By Lemma 2.1.16 an ideal $I = \langle f_1, \ldots, f_n \rangle$, where $f_1, \ldots, f_n \in K[\lambda]$, has the principal generator $f = \gcd\{f_1, \ldots, f_n\}$. With this in mind, we are ready to consider the invariants of M.

Lemma 2.4.7 *Given a sum*

$$M = \bigoplus_{i=1}^{q} \bigoplus_{j=1}^{n(i)} E(a_i, r_{ij}) \bigoplus_{j=1}^{n(0)} E(0, r_{0j})$$

with p' and m' the quantities defined and expressed in Lemma 2.4.5, consider the ideal $I_{2k}(M)$ generated by all $2k \times 2k$ minors of M. If $k - m' \leq 0$ then $I_{2k}(M) = \langle 1 \rangle$. But if $k - m' > 0$ there are two possibilities:

(a) if, for all $0 \leq i \leq q$, $p' - n(i) \geq k - m'$ then $\sqrt{I_{2k}(M)} = \langle 1 \rangle$;

(b) if $k - m' + n(i) - p' = t_{ik} > 0$ for some $0 \leq i \leq q$ then

$$\sqrt{I_{2k}(M)} = \langle \lambda^{R_{0k}} \prod_{i=1}^{q} (a_i\lambda + 1)^{R_{ik}} \rangle,$$

where $R_{ik} = \sum_{j=1}^{t_{ik}} r_{ij}$ and $R_{ik} = 0$ if $t_{ik} \leq 0$. [1]

If M has the sequence of invariant factors $\delta_1, \ldots, \delta_r$ these are given by

$$\delta_k = \lambda^{R_{0k} - R_{0k-1}} \prod_{i=1}^{q} (a_i\lambda + 1)^{R_{ik} - R_{ik-1}}.$$

$1 \leq k \leq r$, where for each $0 \leq i \leq q$, $R_{i0} = 0$.

Proof The case $k - m' \leq 0$ is covered by Lemma 2.4.6. We need to consider the possibility $k > m'$. Let $s = k - m' > 0$. By suitable λ-transformations (interchanging rows and columns, multiplying rows or columns by -1) we can find a diagonal matrix M', λ-equivalent to M, of the form

$$M' = I_{2m'} \oplus D \oplus D_0$$

where

$$D = \oplus_{i=1}^{q} D_i$$

[1]The subscript k in t_{ik} and R_{ik} refers to the ideal, I_{2k}, we are dealing with and hence for the main purposes of this Lemma is fixed. This notation however will be useful later on.

and $D_i = \operatorname{diag}(d_{i1}, d_{i2}, \ldots, d_{i2n(i)})$ are $2n(i) \times 2n(i)$ diagonal blocks given by

$$D_i = \begin{bmatrix} (a_i\lambda+1)^{r_{i1}} & & & & \\ & (a_i\lambda+1)^{r_{i1}} & & & \\ & & \ddots & & \\ & & & (a_i\lambda+1)^{r_{in(i)}} & \\ & & & & (a_i\lambda+1)^{r_{in(i)}} \end{bmatrix}$$

and $D_0 = \operatorname{diag}(d_{01}, d_{02}, \ldots, d_{02n(0)})$ is the $2n(0) \times 2n(0)$ diagonal block given by

$$D_0 = \begin{bmatrix} \lambda^{r_{01}} & & & & \\ & \lambda^{r_{01}} & & & \\ & & \ddots & & \\ & & & \lambda^{r_{0n(0)}} & \\ & & & & \lambda^{r_{0n(0)}} \end{bmatrix}.$$

It follows from the ordering of the r_{ij}'s (including the r_{0j}'s) in our original normal form, described in Section 2.4.1, that each D_i $(0 \le i \le q)$ is a diagonal block, of the type described in Lemma 2.2.2. From Lemma 2.2.3 $I_{2k}(M) = I_{2k}(M')$. We therefore want to find the generators of $I_{2k}(M')$. In fact we only need to consider those minors of M' containing the $2m' \times 2m'$ identity $I_{2m'}$ since any other $2k \times 2k$ minor will be a multiple of one of these (the multiple being a product of terms of $(D \oplus D_0)$ replacing the unused 1's of $I_{2m'}$). To complete these minors we need to choose a further $2s = 2k - 2m'$ elements from $(D \oplus D_0)$ and it is not too difficult to see that

$$I_{2k}(M') = I_{2s}(D \oplus D_0).$$

The generators of $I_{2s}(D \oplus D_0)$ are products of non-negative numbers, $s(i)$, of elements chosen from each block D_i respectively where

$$\sum_{i=1}^{q} s(i) + s(0) = 2s.$$

Furthermore by Lemma 2.2.2 it is sufficient to choose the first $s(i)$ elements in each block D_i. So

$$I_{2k}(M') = I_{2s}(D \oplus D_0)$$

has generators of the form:

$$d_{11} \cdots d_{1s(1)} d_{21} \cdots d_{2s(2)} \cdots d_{q1} \cdots d_{qs(q)} d_{01} \cdots d_{0s(0)},$$

where $\sum_{i=1}^{q} s(i) + s(0) = 2s$ and $0 \leq s(i) \leq 2n(i)$ for each $0 \leq i \leq q$. Here $s(i) = 0$ means there are no elements chosen from D_i. Note here that for $1 \leq j \leq n(i)$

$$d_{i2j-1} = d_{i2j} = (a_i\lambda + 1)^{r_{ij}}$$

and

$$d_{02j-1} = d_{02j} = \lambda^{r_{0j}}.$$

When finding a generator the minimum number of diagonal elements we can choose involving some a_i is

$$2s - (2p' - 2n(i)) = 2t_{ik}.$$

If $2t_{ik} \leq 0$ it follows that $(a_i\lambda + 1)$ is not a divisor of all the generators of $I_{2s}(D \oplus D_0)$. However if $2t_{ik} > 0$ every minor of $I_{2s}(D \oplus D_0)$ has a factor consisting of at least $2t_{ik}$ powers of $(a_i\lambda + 1)$. So the gcd of the minors has a factor consisting of a product of the lowest $2t_{ik}$ powers of $(a_i\lambda + 1)$. By the ordering of our original normal form and since, for $1 \leq j \leq n(i)$,

$$d_{i2j-1} = d_{i2j} = (a_i\lambda + 1)^{r_{ij}}$$

it follows that this product is the product of the first t_{ik} pairs of elements in D_i i.e.

$$(a_i\lambda + 1)^{2r_{i1}} \cdots (a_i\lambda + 1)^{2r_{it_{ik}}}.$$

Consequently, by considering each $1 \leq i \leq q$, a factor of the gcd is given by

$$\prod_{i=1}^{q} (a_i\lambda + 1)^{2R_{ik}},$$

where $R_{ik} = \sum_{j=1}^{t_{ik}} r_{ij}$ and $R_{ik} = 0$ if $t_{ik} \leq 0$. Furthermore if $t_{ik} \leq 0$ for all $1 \leq i \leq q$ then none of the elements $(a_i\lambda + 1)$ is a common divisor of all the generators of $I_{2s}(D \oplus D_0)$.

Similarly the minimal number of elements of D_0 appearing in a generator is given by

$$2s - (2p' - 2n(0)) = 2t_{0k}.$$

If $2t_{0k} \leq 0$ then λ is not a divisor of all the generators of $I_{2s}(D \oplus D_0)$. However if $2t_{0k} > 0$ by a similar argument as above the gcd has a factor

$$\lambda^{2R_{0k}},$$

where $R_{0k} = \sum_{j=1}^{t_{0k}} r_{0k}$ is a sum of the first t_{0k} pairs of elements of D_0.

It follows that if for all $0 \leq i \leq q$ we have $t_{ik} \leq 0$ then

$$I_{2s}(D) = \langle 1 \rangle.$$

Otherwise the gcd is given by

$$\lambda^{2R_{0k}} \prod_{i=1}^{q} (a_i \lambda + 1)^{2R_{ik}},$$

and using Lemma 2.1.16 and the notation in Definition 2.4.1 the result follows.

The invariant factors, $\delta_1, \ldots, \delta_r$, are then found using (2.16). □

So given a normal form, N, of the type described in Section 2.4.1, it follows from Remark 2.4.4 that Lemma 2.4.7 gives an algorithm for finding its sequence of invariant factors.

The following theorem is the principal objective of this chapter. To establish it we need to prove two things.

(i) First we show that any sequence of invariant factors (represented by (2.37)) can be realised by a normal form, N, of the above type.

(ii) Distinct normal forms N yield distinct sets of invariant factors. This is done by showing that there is a ***unique*** normal form corresponding to a choice of invariant factors.

Theorem 2.4.8 *Every pair of skew-symmetric matrices $A_1 + \lambda A_2$ with A_2 non-singular is skew-equivalent to a unique normal form $N = N_1 + \lambda N_2$ of the type described in Section 2.4.1 and represented by the shorthand*

$$\textit{distinct(non-zero)} \left\{ \begin{array}{ccccc} a_1 & r_{11} & r_{12} & \cdots & r_{1n(1)} \\ \vdots & & & & \\ \vdots & & & & \\ a_q & r_{q1} & r_{q2} & \cdots & r_{qn(q)} \\ \hline 0 & r_{01} & r_{02} & \cdots & r_{0n(0)} \end{array} \right. , \tag{2.41}$$

with $r_{i1} \leq r_{i2} \leq \cdots \leq r_{in(i)}$ for $0 \leq i \leq q$.

Proof Denoting the invariant factors of N by $\delta_1, \ldots, \delta_r$, from Lemma 2.4.7, for each $1 \leq k \leq r$,

$$\delta_k = \lambda^{R_{0k}-R_{0k-1}} \prod_{i=1}^{q} (a_i\lambda + 1)^{R_{ik}-R_{ik-1}}, \tag{2.42}$$

where $R_{ik} = \sum_{j=1}^{t_{ik}} r_{ij}$ with $t_{ik} = k - m' + n(i) - p'$ and $R_{ik} = 0$ if $t_{ik} \leq 0$. (Note, if $k = 1$ then for each $0 \leq i \leq q$, $R_{i0} = 0$.)

From (2.37), for $1 \leq k \leq r$, the invariant factor d_k of any pair $A = A_1 + \lambda A_2$ (with A_2 non-singular) is of the form

$$d_k = \lambda^{v_k - v_{k-1}} \prod_{i=1}^{q} (e_i\lambda + 1)^{s_{ik}-s_{ik-1}},$$

where for each $1 \leq i \leq q$, $s_{i1} \leq s_{i2} \leq \cdots \leq s_{ir}$, and $v_1 \leq v_2 \leq \cdots \leq v_r$ and $d_1 | \cdots | d_r$.

We need to show that we can find a set of a_i's, r_{ij}'s as in (2.41) so that $d_k = \delta_k$ for each k. Obviously, we choose each $a_i = e_i$ leaving us to find the r_{ij}'s.

We need to solve the equations

$$s_{ik} - s_{ik-1} = R_{ik} - R_{ik-1}, \tag{2.43}$$

for each $1 \leq i \leq q$, and

$$v_k - v_{k-1} = R_{0k} - R_{0k-1}. \tag{2.44}$$

We deduce from (2.43) that

$$s_{ik} - s_{ik-1} = \begin{cases} r_{ik-m'+n(i)-p'} & \text{if } k - m' + n(i) - p' \geq 1 \\ 0 & \text{otherwise} \end{cases}. \tag{2.45}$$

In this way (by considering $1 \leq k \leq r$) for each $1 \leq i \leq q$ we obtain the values for the r_{ij} in terms of the s_{ik}'s. Similarly it can be shown that if (2.44) holds then

$$v_k - v_{k-1} = \begin{cases} r_{0k-m'+n(0)-p'} & \text{if } k - m' + n(0) - p' \geq 1 \\ 0 & \text{otherwise} \end{cases}. \tag{2.46}$$

Hence we can get the values for the r_{0j}'s in terms of the v_k's. Furthermore (for each sequence of invariant factors) these values of r_{ij} are unique thus showing that no two N's with distinct sets of a_i's, r_{ij}'s are skew-equivalent. Finally, recall that these normal forms are constructed so that for each $0 \leq i \leq q$ the r_{ij} are non-decreasing through $0 \leq j \leq n(i)$, a fact used in the proof of Lemma 2.4.7 and therefore a requirement for our argument. It remains to check that this is true for the r_{ij}'s found from the s_{ik}'s and v_k's. We first want to show that for each $1 \leq i \leq q$ then for $1 \leq k \leq r$

$$r_{ik-1-m'+n(i)-p'} \leq r_{ik-m'+n(i)-p'}.$$

It follows from (2.45) that

$$r_{ik-1-m'+n(i)-p'} = s_{ik-1} - s_{ik-2}, \tag{2.47}$$

$$r_{ik-m'+n(i)-p'} = s_{ik} - s_{ik-1}. \tag{2.48}$$

If we denote the power of the factor $(a_i\lambda + 1)$ appearing in the invariant d_k by $\deg d_k^{a_i}$ it follows from (2.37) that

$$\deg d_{k-1}^{a_i} = s_{ik-1} - s_{ik-2} \tag{2.49}$$

$$\deg d_k^{a_i} = s_{ik} - s_{ik-1}. \tag{2.50}$$

Furthermore since $d_{k-1}|d_k$ then

$$\deg d_{k-1}^{a_i} \leq \deg d_k^{a_i}$$

and so from (2.49) and (2.50) it follows that

$$s_{ik-1} - s_{ik-2} \leq s_{ik} - s_{ik-1}.$$

The required result then follows from (2.47) and (2.48). It remains to show that

$$r_{0k-1-m'+n(0)-p'} \leq r_{0k-m'+n(0)-p'},$$

for $1 \leq k \leq r$. By (2.46) this amounts to showing that

$$v_{k-1} - v_{k-2} \leq v_k - v_{k-1}.$$

Denoting the power of the factor λ appearing in the invariant, d_k, by $\deg d_k^{\lambda}$ it follows from (2.37) that

$$\deg d_{k-1}^{\lambda} = v_{k-1} - v_{k-2} \tag{2.51}$$

$$\deg d_k^{\lambda} = v_k - v_{k-1}. \tag{2.52}$$

Again since $d_{k-1}|d_k$

$$\deg d_{k-1}^{\lambda} \leq \deg d_k^{\lambda}$$

and (2.51), (2.52) give the required result.

Consequently given the invariant factors of any linear skew-symmetric λ-matrix A we can find a unique skew-equivalent normal form of the type described in Section 2.4.1. □

Recall from Section 1.3 of Chapter 1 that the pair (A_1, A_2) determine the pencil

$$A = \mu A_1 + \lambda A_2$$

for points $(\mu : \lambda)$ on the projective line, PK. We can also represent this pencil by the affine chart

$$A_1 + \lambda' A_2,$$

where $\lambda' = \lambda/\mu$, along with the single matrix A_2 corresponding to the point at infinity, $(0:1)$. Furthermore recall that the eigenvalues of the pair (A_1, A_2) are those points $(\mu : \lambda)$ satisfying $\det(\mu A_1 + \lambda A_2) = 0$ and correspond to singular members of the pencil, A. As A_2 is non-singular it follows that the point at infinity is not an eigenvalue and hence the eigenvalues, $(1 : \lambda')$, of the pair (A_1, A_2) are given by the roots of

$$\det(A_1 + \lambda' A_2) = 0.$$

We deduce that the pair (A_1, A_2) has eigenvalues given by the roots of

$$\det(A_1 + \lambda' A_2) = \lambda'^{v_r} \prod_{i=1}^{q} (a_i \lambda' + 1)^{s_{ir}} = 0.$$

Hence these eigenvalues are

$$\begin{array}{lll} (1:0) & \text{of multiplicity } v_r & \\ (a_i : -1) & \text{of multiplicity } s_{ir} & 1 \leq i \leq q \end{array}.$$

If the normal form of the pair (A_1, A_2) is (N_1, N_2) then, since eigenvalues are an invariant of skew-equivalent pairs, these are also the eigenvalues (including multiplicities) of the pair (N_1, N_2).

It follows from Lemma 1.3.3 that the pencil, A, determined by the pair (A_1, A_2) is skew-equivalent to a pencil determined by the pair (N_1, N_2), this being the special case where the change of basis matrix is the identity. So $N_1 + \lambda N_2$ is also a normal form for a pencil. Note here that, by Lemma 1.4.3, for each common eigenvalue of (A_1, A_2) and (N_1, N_2) the corresponding singular elements of each pencil are skew-equivalent.

2.4.3 Classifying Pencils

If the pencil is singular see Chapter 3.

If the pencil is non-singular by a linear change of basis we can always choose a representative pair of matrices, which are both non-singular, and then use Theorem 2.4.8 to find a skew-equivalent normal form. This normal form can be simplified by further changes of basis.

2.4.4 Classifying Pairs

When classifying pairs we do not have the luxury of changes of basis. Any pair for which at least one is non-singular has a normal form as found above using Theorem 2.4.8.

Alternatively, given a pair of singular matrices (A_1, A_2) then they either determine a singular pencil or not. In the latter case there are a pair of a finite number, $Q \leq n$, of points where the (non-singular) pencil they determine meets the set of singular matrices. By choosing a pair of non-singular matrices on this pencil we can, again by Theorem 2.4.8, find a normal form pair (B_1, B_2). The pencil determined by this pair also meets the singular set in Q points. If we denote the set of pairs of these points by $\{(S_i, S_j), 1 \leq i < j \leq Q\}$ then by Lemma 1.4.3 the original pair (A_1, A_2) is skew-equivalent to one of $\binom{Q}{2}$ possible normal forms, (S_i, S_j).

Chapter 3

Singular Skew-symmetric Pencils

In this chapter we consider the reduction of singular skew-symmetric pencils. The method we adopt for this is inspired by work in Chapter IX of [TurnAit].

In Section 3.2 we describe an initial reduction which breaks a singular skew-symmetric matrix pencil down into a series of canonical singular submatrices, possibly with an additional non-singular sub-pencil. Then in Section 3.3, using the results of the previous chapter for dealing with the non-singular part, we refine this reduction further to obtain normal forms for singular pencils.

An overview of the initial reduction is useful. We start this by establishing that any singular (skew-symmetric) pencil, $\Lambda = \lambda A + \mu B$, has a non-trivial kernel vector (with homogeneous polynomial entries in λ, μ) of a *minimal* degree, k_1. This minimal degree is a (skew-equivalent) invariant of the pencil. By the action of $Gl(n, K)$ on this kernel vector we obtain a canonical vector and from the associated skew-equivalent action of $Gl(n, K)$ on the pencil Λ we obtain a corresponding canonical singular (skew-symmetric) sub-matrix, while reducing the remainder of the pencil to a smaller skew-symmetric sub-pencil, Γ. The size of this singular block is determined by the minimal degree k_1. In addition there are various by-products of the reduction which need to be cleared away and we consider this in Section 3.3.

If the submatrix, Γ, is singular it can be reduced in a similar way by finding a canonical kernel vector of minimal degree $k_2 \geq k_1$, thereby separating a further canonical singular (skew-symmetric) sub-block from the remainder of the pencil and so on. By definition, the minimal degrees of successive subpencils, Γ, form a non-decreasing sequence. In due course the process stops, in particular when a *non-singular* sub-pencil, Γ is produced.

First we need a few preliminaries concerning pencils which we keep as general as possible. However, when necessary, we specialise some of these for the skew-symmetric case. We start by generalising some of the definitions introduced in Sections 1.3 and 1.4 of Chapter 1.

Definition 3.0.9 *Let A, B be $n' \times n$ matrices defined over K. The* **pencil** *determined by A and B is the family of matrices $\lambda A + \mu B$, where $(\lambda : \mu)$ are points of the projective line PK^1.*

In the following such pencils will be denoted by uppercase greek letters (i.e. Λ, Ω).

Definition 3.0.10 *(i) Two pencils Λ, Ω are said to be* **strictly equivalent** *if for some invertible constant matrices P and Q, of appropriate sizes, we have*

$$\Omega = Q\Lambda P.$$

(ii) If $\Lambda = \lambda A + \mu B$, $\Omega = \rho C + \sigma D$ then Λ and Ω are called **equivalent** *if $\rho(\alpha A + \gamma B) + \sigma(\beta A + \delta B)$ and $\rho C + \sigma D$ are strictly equivalent, where the associated linear change of coordinates is given by*

$$\begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} \rho \\ \sigma \end{pmatrix},$$

with the **change of basis matrix**

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in Gl(2, K).$$

Remark 3.0.11 We often omit the prefix 'strict', in (i), when it is clear which equivalences we are using.

Definition 3.0.12 *The* **rank** *of any $n' \times n$ pencil $\Lambda = \lambda A_1 + \mu A_2$ is defined to be the maximal rank of $\lambda A_1 + \mu A_2$ as $(\lambda : \mu)$ varies over PK^1.*

This rank is determined by the minors of Λ.

Lemma 3.0.13 *Let I_j denote the ideal in $K[\lambda, \mu]$ generated by the $j \times j$ minors of Λ. Then rank Λ is the largest k for which $I_k \neq 0$.*

Proof If k is the largest integer for which $I_k \neq 0$, then some $k \times k$ minor, Δ, of Λ is a non-zero polynomial. As K has characteristic zero, $\Delta(\lambda, \mu) \neq 0$ for some $(\lambda : \mu)$ so the maximal rank $\Lambda \geq k$. But since for $j > k$ all $j \times j$ minors vanish identically $\Lambda(\lambda, \mu)$ has rank $\leq k$ for all $(\lambda : \mu)$. It follows, therefore, that Λ has maximal rank k. □

Definition 3.0.14 *An $n' \times n$ pencil Λ is* **singular** *if rank $\Lambda < \min\{n', n\}$.*

Remark 3.0.15 It follows from this definition and the previous lemma that the condition, given in Section 1.4, for an $n \times n$ pencil Λ to be singular (i.e. that $\det \Lambda \equiv 0$) is equivalent to rank $\Lambda < n$.

The following definition introduces the notion of *kernel vectors* of a pencil.

Definition 3.0.16 **Polynomial kernel vectors,** $\mathbf{v}$, *of the pencil Λ are $n \times 1$ vectors, whose entries are polynomials in λ and μ, for which $\Lambda\mathbf{v} = \mathbf{0}$.*

Similarly, we define **polynomial nullifying vectors** *to be $1 \times n'$ vectors, $\mathbf{u}$, whose entries are polynomials in λ and μ, for which $\mathbf{u}\Lambda = \mathbf{0}$.*

Note, since the nullifying polynomial vectors defined here are not needed for the main objectives of this chapter the following results deal specifically with the kernel vectors. Similar results will also hold for the nullifying vectors.

In fact we need only consider homogeneous polynomial (kernel) vectors of our pencils as shown by the following lemma.

Lemma 3.0.17 *Let $\mathbf{v} = \mathbf{v}_0 + \cdots + \mathbf{v}_s$ where $\mathbf{v}_i$, $0 \leq i \leq s$, is a vector with homogeneous entries of degree i. Then if $\Lambda\mathbf{v} = \mathbf{0}$ then $\Lambda\mathbf{v}_i = \mathbf{0}$.*

Proof Since the entries of Λ are linear in λ and μ this is obvious. □

Definition 3.0.18 *If a pencil Λ has a homogeneous polynomial kernel vector,* $\mathbf{v}$, *with entries of degree k we say that the columns of Λ have* **dependence of order** k.

If Λ has a homogeneous polynomial kernel vector $\mathbf{w}$ *of degree m, and no relation of column dependence holds for a homogeneous polynomial kernel vector of degree less than m, then m is the* **minimal order** *of column dependence of Λ and* $\mathbf{w}$ *has* **minimal degree**.

Similarly, our pencil Λ is said to be *row dependent* of order k' if there exists a homogeneous nullifying vector $\mathbf{u}$, with entries of degree k', which satisfies $\mathbf{u}\Lambda = \mathbf{0}$. The smallest degree, m', of such a vector is the minimal order of *row dependence* of Λ. The following result applies to skew-symmetric pencils.

Proposition 3.0.19 *The minimal orders of row and column dependence of a skew-symmetric pencil Λ, have the same value. We call this value the minimal order of dependence of Λ.*

Proof Let Λ have column dependence of minimal order k and row dependence of minimal order k'. So

$$\Lambda\mathbf{v} = \mathbf{0},$$

where $\mathbf{v}$ is a minimal homogeneous kernel vector of degree k. Therefore

$$(\Lambda\mathbf{v})^T = \mathbf{0}$$

which implies that

$$\mathbf{v}^T\Lambda^T = \mathbf{0}$$

and since Λ is skew-symmetric

$$-\mathbf{v}^T\Lambda = \mathbf{0}.$$

But $\mathbf{v}^T$ has degree k, so $k \geq k'$ since otherwise the minimality of k' is contradicted.

We also know that

$$\mathbf{u}\Lambda = \mathbf{0},$$

where $\mathbf{u}$ is a homogeneous vector of minimal degree k'. Therefore

$$(\mathbf{u}\Lambda)^T = \mathbf{0}$$

which implies

$$\Lambda^T \mathbf{u}^T = \mathbf{0}$$

and, since Λ is skew-symmetric, that

$$-\Lambda \mathbf{u}^T = \mathbf{0}.$$

Since $\mathbf{u}^T$ has degree k' then for the minimality of k to be preserved $k' \geq k$. So $k' = k$ as required. □

Lemma 3.0.20 *The minimal order of column (row) dependence of a pencil, Λ, is invariant under*

(i) equivalent transformation of Λ,

(ii) homogeneous non-singular transformation of λ, μ to ρ, σ; i.e change of basis.

Proof We consider column dependence. Letting $\mathbf{v}$ be a homogeneous kernel vector of minimal degree k then

$$\Lambda \mathbf{v} = \mathbf{0}.$$

(i) Consider an equivalent matrix $Q\Lambda P$, where $\det P \neq 0$, $\det Q \neq 0$, and suppose that $Q\Lambda P\mathbf{v}_1 = \mathbf{0}$ where $\mathbf{v}_1$ is a vector of minimal order, less than k, in the kernel of $Q\Lambda P$. Since Q is invertible then

$$\Lambda P \mathbf{v}_1 = \mathbf{0}$$

where, since the elements of P are constants, the vector $P\mathbf{v}_1$ in the kernel of Λ must be, like $\mathbf{v}_1$, of lower order than k. But this contradicts the assumption that k is the minimal order of column dependence of Λ. Hence $\mathbf{v}_1$ cannot be of lower order than $\mathbf{v}$. Equally $\mathbf{v}$ cannot be of lower order than $\mathbf{v}_1$ since if it were then

$$\begin{aligned} Q\Lambda P(P^{-1}\mathbf{v}) &= Q\Lambda\mathbf{v} \\ &= \mathbf{0}, \end{aligned}$$

would contradict the minimality of $\mathbf{v}_1$. So the minimal order of dependence is invariant under equivalent transformation.

(ii) If we consider a linear change from λ, μ to ρ, σ then $\mathbf{v}$ is transformed into a homogeneous polynomial vector $\mathbf{v}'$ in the variables ρ, σ. Such a transformation cannot raise the degree in $\mathbf{v}'$, though it might lead to the lowering of it through the cancelling of some common factor in the transformed elements. In such a case however, since this transformation is non-singular, we can transform back from ρ, σ to λ, μ resulting in a vector of lower degree than $\mathbf{v}$ which would contradict its minimality. Hence the minimal order k is also invariant under change of basis as required.

The result for row dependence follows similarly. □

The following corollary to this result applies to skew-symmetric pencils.

Corollary 3.0.21 *The minimal order of dependence of a skew-symmetric pencil, Λ, is invariant under*

(i) skew-equivalent transformation of Λ,

(ii) homogeneous non-singular transformation of λ, μ to ρ, σ; i.e change of basis.

Proof Since, by Proposition 3.0.19, the minimal orders of row and column dependence of a skew-symmetric matrix are the same we only need to prove the results for column dependence. To show that column dependence is invariant under skew-equivalence we simply replace Q by P^T in the proof of Lemma 3.0.20 part (i). Part (ii) follows exactly as before. □

In the following section we establish some results which are useful for the reduction of singular pencils.

3.1 Key Lemmas

Lemma 3.1.1 *Let Λ be an $n' \times n$ pencil (where $n' \geq n$) and suppose it has k columns yielding an $n' \times k$ pencil of rank $(k-1)$. Then Λ has a homogeneous kernel vector of degree $(k-1)$.*

Proof Let Λ_1 be the $n' \times k$ pencil formed by these k columns. Since Λ_1 has rank $(k-1)$ then by Lemma 3.0.13 it has a non-zero $(k-1) \times (k-1)$ minor and by choosing k rows of Λ_1 we can form a $k \times k$ matrix Ω which contains this minor. It also follows from Lemma 3.0.13 that Ω has rank $(k-1)$ and $\det \Omega = 0$. Consequently

$$\Omega \mathrm{adj} \Omega = \mathbf{0}.$$

But since Ω contains the above non-zero $(k-1) \times (k-1)$ minor, adjΩ is non-zero and furthermore each of its non-zero entries is homogeneous and of degree $(k-1)$. We choose a non-zero column, $\mathbf{w}$, of adjΩ which is therefore annihilated by the k rows of Ω. For generic (λ, μ) the ranks of $\Omega(\lambda, \mu)$ and $\Lambda_1(\lambda, \mu)$ are both $(k-1)$ and so by standard linear algebra the rowspace of $\Lambda_1(\lambda, \mu)$ is spanned by the rows of $\Omega(\lambda, \mu)$. So, in particular, the rows of $\Lambda_1(\lambda, \mu)$ are linear combinations of the rows of $\Omega(\lambda, \mu)$ and since Ω annihilates the vector, $\mathbf{w}$, it follows that, for almost all (λ, μ), $\Lambda_1(\lambda, \mu)$ kills $\mathbf{w}(\lambda, \mu)$. Furthermore since K has characteristic zero we deduce that $\Lambda_1 \mathbf{w} = \mathbf{0}$ for all values of (λ, μ). Consequently we have found a $k \times 1$ homogeneous vector, $\mathbf{w}$, of degree $(k-1)$ which is killed by Λ_1. If we assign the suffices $i_1, \ldots, i_k$ to the k columns of Λ which form Λ_1 and write

$$\mathbf{w} = \begin{pmatrix} w_{i_1} \\ \vdots \\ w_{i_k} \end{pmatrix},$$

then it follows that

$$\Lambda \omega = \mathbf{0},$$

where ω is the $n \times 1$ vector with entries w_{i_j}, $(1 \leq j \leq k)$, in rows i_j and zeros elsewhere. □

Corollary 3.1.2 *If Λ is an $n' \times n$ pencil (where $n' \geq n$) of rank $r < n$ then it has a homogeneous polynomial kernel vector of degree r.*

Proof Since Λ has rank r, by Lemma 3.0.13, it has a non-zero $r \times r$ minor and all higher order minors are zero. Since $r < n$ we can find $r + 1$ columns of Λ which yield an $n' \times (r + 1)$ pencil of rank r by choosing them so that they contain this $r \times r$ minor. The rest then follows from Lemma 3.1.1. □

Lemma 3.1.3 *Suppose an $n' \times n$ pencil, Λ, has a homogeneous polynomial kernel vector, $\mathbf{v}$, of degree $k \leq n - 1$. Then Λ is equivalent to a pencil with* *kernel vector either*

$$\mathbf{v}(k) = \begin{pmatrix} \lambda^k \\ \lambda^{k-1}\mu \\ \vdots \\ \mu^k \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad \textit{or} \qquad \mathbf{v}' = \begin{pmatrix} v_1 \\ \vdots \\ v_k \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

for some v_i, of degree k, not all zero.

Proof If Λ has a homogeneous polynomial kernel vector

$$\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

of degree k then $\Lambda\mathbf{v} = \mathbf{0}$. This vector, $\mathbf{v}$, can be represented by an $n \times (k+1)$ matrix, C, whose j rows yield the coefficients of the monomials for each of the v_j. Premultiplication of $\mathbf{v}$ by an invertible matrix, P^{-1}, corresponds to row operations on $\mathbf{v}$. Furthermore the resulting vector $P^{-1}\mathbf{v}$ is a kernel vector of the pencil $Q\Lambda P$ which is equivalent to Λ. If C has rank $(k+1)$ then by row operations it can be reduced to the form

$$\begin{bmatrix} I_{k+1} \\ 0 \end{bmatrix}$$

in which case the resulting kernel vector $P^{-1}\mathbf{v}$ is $\mathbf{v}(k)$. Otherwise if the rank of C is not maximal then by row operations it can be reduced to a form with k or fewer non-zero rows and the resulting kernel vector is $\mathbf{v}'$. □

Remark 3.1.4 The same result holds for a skew-symmetric pencil subject to skew-equivalence.

3.2 Initial Reduction of Singular Skew-symmetric Pencils

Note that in this section, when we apply the methods and terminology of linear algebra to our pencils, we are in fact applying it to the matrices obtained for generic (λ, μ) and use the fact that K has characteristic zero to show that the results will then hold for all (λ, μ). By Lemma 3.0.20 the minimal order of (column) dependence of the pencil Λ is invariant under equivalence (and change of basis). In the following proofs we will frequently use the fact that this minimality must be preserved to rule out, by contradiction, certain results.

We start with the following theorem for *square* singular pencils.

Theorem 3.2.1 *Let Λ be an $n \times n$ singular pencil. Then it has a homogeneous polynomial kernel vector* $\mathbf{v}$ *of minimal degree $k < n$. Furthermore Λ is equivalent to a pencil with kernel vector* $\mathbf{v}(k)$.

Proof Since Λ is singular, rank$\Lambda = r < n$. So by Corollary 3.1.2, it has a homogeneous polynomial kernel vector of degree $r < n$. Therefore there exists a polynomial kernel vector $\mathbf{v}$ of minimal degree $k \leq r$. So $k < n$. By Lemma 3.1.3 Λ is equivalent to a pencil, $Q\Lambda P$, with kernel vector either $\mathbf{v}(k)$ or a non-zero vector $\mathbf{v}'$ which has its last $n - k$ entries zero. In the latter case the first k columns of $Q\Lambda P$ are dependent and therefore have rank $\leq (k-1)$. By Lemma 3.1.1 we deduce that $Q\Lambda P$ has a kernel vector of degree $\leq (k-1)$ thereby contradicting the minimality of k. Consequently we can rule out this case. Recall from Lemma 3.1.3 that

$$\mathbf{v}(k) = \begin{pmatrix} \lambda^k \\ \lambda^{k-1}\mu \\ \vdots \\ \mu^k \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

$\square$.

Previously, we haven't been particularly concerned with the type of singular pencils we are dealing with. This is because, when finding polynomial kernel vectors of a pencil, its type (i.e. general, skew-symmetric etc) is not significant since these vectors depend on column operations on the pencil and it is immaterial whether or not we are also performing the same row operations (on the pencil). However, clearly, when it comes to considering the actual reduction of a pencil, Λ, we must be more precise.

As a consequence from here on we restrict our attention to skew-symmetric pencils subject to skew-equivalence. We start with the following result, the first part of which is just a revamped version of Theorem 3.2.1. Note it will sometimes be more convenient to denote a pencil skew-equivalent to Λ also by Λ.

Lemma 3.2.2 *Consider an $n \times n$ singular skew-symmetric pencil Λ. Then, for some $k < n$ it is skew-equivalent to a pencil of the form*

$$\lambda \left[\begin{array}{ccc|ccc} 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & -D^T & \\ 0 & \cdots & 0 & & & \\ \hline 0 & & & & & \\ \vdots & D & & & A & \\ 0 & & & & & \end{array}\right] + \mu \left[\begin{array}{ccc|ccc} 0 & \cdots & 0 & & & \\ \vdots & \ddots & \vdots & & -D^T & \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \hline & & 0 & & & \\ & D & \vdots & & B & \\ & & 0 & & & \end{array}\right], \tag{3.1}$$

where A and B are $(n-k-1) \times (n-k-1)$ skew-symmetric matrices and D is an $(n-k-1) \times k$ matrix. Furthermore this skew-symmetric pencil has a homogeneous polynomial kernel vector of the form

$$\mathbf{w}(k) = \begin{pmatrix} \lambda^k \\ -\lambda^{k-1}\mu \\ \vdots \\ (-1)^k \mu^k \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Proof As just stated, the first part of this proof follows a similar argument to that used, in the proof of Theorem 3.2.1, for a general square pencil.

Since Λ is singular rank$\Lambda = r < n$. So, by Corollary 3.1.2, it has a homogeneous polynomial kernel vector of degree $r < n$. Therefore it has a polynomial kernel vector $\mathbf{v}$ of *minimal* degree $k < n$. By a similar argument to that used to prove Lemma 3.1.3 Λ is skew-equivalent to a pencil $P^T \Lambda P$ with either a (non-zero) kernel vector $\mathbf{v}'$, which has its last $n-k$ entries zero, or a vector $\mathbf{w}(k)$ of the form

$$\mathbf{w}(k) = \begin{pmatrix} \lambda^k \\ -\lambda^{k-1}\mu \\ \vdots \\ (-1)^k\mu^k \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Then by a similar argument to that used in the proof of Theorem 3.2.1 we can rule out the first possiblity. Note we obtain $\mathbf{w}(k)$, rather than the vector $\mathbf{v}(k)$ given in the proof of Lemma 3.1.3, by slightly altering the row operations on the matrix of coefficients C thereby reducing it to the form

$$\begin{bmatrix} J_{k+1} \\ 0 \end{bmatrix},$$

where the $(k+1) \times (k+1)$ matrix, J_{k+1}, is given by

$$J_{k+1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & (-1)^k \end{pmatrix}.$$

We have therefore reduced Λ to

$$\Lambda = \lambda A + \mu B,$$

with $A = (a_{ij})$, $B = (b_{ij})$ $n \times n$ skew-symmetric matrices, such that

$$\Lambda\mathbf{w}(k) = (\lambda A + \mu B)\mathbf{w}(k) = \mathbf{0}. \tag{3.2}$$

From (3.2) we get the following identities:

$$\sum_{j=1}^{k+1}(-1)^{j-1}(a_{ij}\lambda + b_{ij}\mu)\lambda^{k+1-j}\mu^{j-1} = 0, \quad 1 \le i \le n.$$

For each $1 \le i \le n$ we have a homogeneous polynomial of degree $k+1$ which must be identically zero. Consequently the coefficients of each of the monomials must vanish and we get the following conditions on the entries of A and B:

$$\begin{array}{rcl} a_{i1} & = & 0 \\ a_{i2} & = & b_{i1} \\ & \vdots & \\ a_{ik+1} & = & b_{ik} \\ b_{ik+1} & = & 0. \end{array}$$

We have therefore established that if (3.2) holds, matrices A, B take the form:

$$A = [0DA_1] \qquad \text{and} \qquad B = [D0B_1],$$

where D is an $n \times k$ matrix, A_1 and B_1 two $n \times (n-k-1)$ matrices and 0 is an $n \times 1$ column of zeros.

Consider the first $k+1$ equations of (3.2) which are equivalent to $(\lambda[0D]_{k+1} + \mu[D0]_{k+1})\mathbf{w}(k) = \mathbf{0}$ i.e to the vanishing of

$$\left(\lambda \begin{bmatrix} 0 & d_{11} & \cdots & d_{1k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & d_{k+11} & \cdots & d_{k+1k} \end{bmatrix} + \mu \begin{bmatrix} d_{11} & \cdots & d_{1k} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ d_{k+11} & \cdots & d_{k+1k} & 0 \end{bmatrix}\right) \begin{bmatrix} \lambda^k \\ \vdots \\ (-1)^k \mu^k \end{bmatrix}. \tag{3.3}$$

Since Λ is skew-symmetric it follows that the $(k+1) \times (k+1)$ block $(\lambda[0D]_{k+1} + \mu[D0]_{k+1})$ is skew-symmetric which implies that both $[0D]_{k+1}$ and $[D0]_{k+1}$ are skew-symmetric. As the first column of $[0D]_{k+1}$ is zero by skew-symmetry its first row must also be zero i.e $d_{11} = \cdots = d_{1k} = 0$. Consequently the first row of $[D0]_{k+1}$ is zero which in turn by skew-symmetry implies that its first column is zero i.e $d_{21} = \cdots d_{k+11} = 0$. So the first row and column of $(\lambda[0D]_{k+1} + \mu[D0]_{k+1})$ are zero and by induction it follows that $[0D]_{k+1}$ and $[D0]_{k+1}$ are null matrices. Using this we have the skew-equivalent pencil in (3.1). □

Lemma 3.2.3 *The $(n-k-1) \times k$ matrix, D, in the reduction of Λ given by (3.1) has maximal rank, k.*

Proof The proof is by contradiction. If rank $D < k$ then by row operations on D we obtain a matrix D' with less than k non-zero rows. By carrying out the corresponding simultaneous row and column operations on the pencil given by

(3.1) we obtain the skew-equivalent pencil

$$\lambda \left[\begin{array}{ccc|ccc} 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & -D'^T & \\ 0 & \cdots & 0 & & & \\ \hline 0 & & & & & \\ \vdots & D' & & & A^* & \\ 0 & & & & & \end{array}\right] + \mu \left[\begin{array}{ccc|ccc} 0 & \cdots & 0 & & & \\ \vdots & \ddots & \vdots & & -D'^T & \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \hline & & 0 & & & \\ & D' & \vdots & & B^* & \\ & & 0 & & & \end{array}\right], \tag{3.4}$$

where A^* and B^* are $(n-k-1) \times (n-k-1)$ skew-symmetric matrices. The first $(k+1)$ columns form the $n \times (k+1)$ pencil

$$\lambda \left[\begin{array}{ccc} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ \hline 0 & & \\ \vdots & D' & \\ 0 & & \end{array}\right] + \mu \left[\begin{array}{ccc} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ \hline & & 0 \\ & D' & \vdots \\ & & 0 \end{array}\right]$$

which has less than k non-zero rows. It follows that all its $k \times k$ and $(k+1) \times (k+1)$ minors are identically zero and it has rank $\leq k-1$. Consequently Lemma 3.1.1 implies that the skew-equivalent pencil (3.4) has a homogeneous polynomial kernel vector of degree $\leq k-1$ which contradicts the minimality of k. $\square$

Since D has maximal rank, k, by a series of row operations we can reduce it to the form

$$\begin{bmatrix} -I_k \\ 0 \end{bmatrix}.$$

By carrying out these row operations accompanied by their counterpart column operations we can reduce Λ to the skew-equivalent pencil

$$\Lambda = \left[\begin{array}{c|c|c} 0 & Q & 0 \\ \hline -Q^T & F & G \\ \hline 0 & -G^T & \Gamma \end{array}\right], \tag{3.5}$$

where 0 denotes, as usual, null matrices and Q is the $(k+1) \times k$ pencil given by

$$Q = \begin{bmatrix} \mu & 0 & \cdots & 0 \\ \lambda & \mu & \ddots & \vdots \\ 0 & \lambda & \ddots & 0 \\ \vdots & \ddots & \ddots & \mu \\ 0 & \cdots & 0 & \lambda \end{bmatrix},$$

G is a $k \times (n-2k-1)$ pencil and F, Γ are skew-symmetric pencils of order k and $(n-2k-1)$ respectively.

Remark 3.2.4 We make a point which proves useful later on. Given a pencil Λ' of the form in (3.1) i.e. which satisfies

$$\Lambda' \mathbf{w}(k) = \mathbf{0},$$

then the skew-equivalent pencil $\Lambda = P^T \Lambda' P$, in (3.5), which is obtained from it by simultaneous row and column operations on its last $n-(k+1)$ rows and columns satisfies

$$(P^T \Lambda' P) P^{-1} \mathbf{w}(k) = \mathbf{0}.$$

Hence Λ has the polynomial kernel vector $P^{-1}\mathbf{w}(k) = \mathbf{w}(k)$ since the action of P^{-1} on the vector $\mathbf{w}(k)$ corresponds to applying row operations to its last $(n-k-1)$ rows which are all zeros.

So simultaneous row and column operations on the last $(n-k-1)$ rows and columns of Λ preserve its polynomial kernel vector $\mathbf{w}(k)$.

By further skew-equivalent transformation we aim to kill off all the entries in the sub-pencil, F, of (3.5). Considering the action, on this pencil, by the invertible matrix

$$\left[\begin{array}{c|c|c} I_{k+1} & U & 0 \\ \hline 0 & I_k & 0 \\ \hline 0 & 0 & I_{n-2k-1} \end{array}\right],$$

where U is some (unknown) $(k+1) \times k$ matrix, we get the skew-equivalent pencil

$$\left[\begin{array}{c|c|c} 0 & Q & 0 \\ \hline -Q^T & -Q^T U + U^T Q + F & G \\ \hline 0 & -G^T & \Gamma \end{array}\right].$$

Hence we need to show that we can find a matrix U such that

$$-Q^T U + U^T Q + F = 0.$$

However before proving this we need the following.

Theorem 3.2.5 *Consider the space of $(r+1) \times r$ matrices, U, of the form*

$$U = \begin{bmatrix} u_1 & \cdots & u_r \\ & U_0 & \end{bmatrix} = \begin{bmatrix} & V_0 & \\ v_1 & \cdots & v_r \end{bmatrix},$$

where U_0 and V_0 are arbitrary $r \times r$ symmetric matrices, then this space has dimension $2r$.

Proof In fact one can show that U is determined uniquely by the choices of $u_1, \dots, u_r$ and $v_1, \dots, v_r$. (Defining the ***skew diagonals*** of the matrix to be the diagonals which cross its main diagonal, then having fixed $u_1, \dots, u_r$, $v_1, \dots, v_r$ each element of the matrix U lies on a skew diagonal meeting some u_i or v_j, and the symmetry conditions force it to take this value.)

The formal proof proceeds by induction on r. The cases $r = 1, 2$ are clear.

Suppose that the result is established for r, and consider the situation of a matrix U which is $(r+2) \times (r+1)$. If we delete the first row and column of this matrix we have exactly the same situation and the space of possible solutions is of dimension $2r$.

However all of the entries in the larger matrix are determined by the symmetry conditions, except from those in places $(1,1)$, $(1,2)$ and $(2,1)$, the latter pair being equal. Consequently the space is of dimension $2r + 2 = 2(r+1)$ as required. □

Lemma 3.2.6 *Let Q be the $(r+1) \times r$ matrix*

$$Q = \begin{bmatrix} \mu & 0 & \cdots & 0 \\ \lambda & \mu & \ddots & \vdots \\ 0 & \lambda & \ddots & 0 \\ \vdots & \ddots & \ddots & \mu \\ 0 & \cdots & 0 & \lambda \end{bmatrix}$$

and Φ any $r \times r$ skew-symmetric pencil then we can find a constant $(r+1) \times r$ matrix U such that

$$-Q^T U + U^T Q = -\Phi. \tag{3.6}$$

Proof If we consider this equation, we see that both sides are $r \times r$ skew-symmetric pencils. The RHS is a pencil of the form $\lambda A_1 + \mu B_1$ where A_1 and B_1 are known $r \times r$ skew-symmetric matrices. The LHS is a pencil of the form $\lambda Q_1 + \mu Q_2$ where Q_1 and Q_2 are $r \times r$ skew-symmetric matrices whose entries are linear combinations of the $r(r+1)$ unknown entries of U. We can therefore think of (3.6) as two sets of linear equations in the $r(r+1)$ entries of U: one set corresponding to the matrices multiplied by λ, i.e $Q_1 = A_1$, and the other, $Q_2 = B_2$ to the matrices multiplied by μ.

Recall that the space of $r \times r$ skew-symmetric matrices has dimension $r(r-1)/2$. It follows therefore that (3.6) results in $r(r-1)$ linear equations in $r(r+1)$ unknowns. If we think of $-Q^T U + U^T Q$ as a linear map

$$\phi : K^{r(r+1)} \longrightarrow K^{r(r-1)},$$

then given any element $-\Phi \in K^{r(r-1)}$ we need to show that there exists an $U \in K^{r(r+1)}$ such that $\phi(U) = -\Phi$, that is ϕ is surjective. For ϕ to be surjective

$$\dim(\mathrm{im}\phi) = r(r-1)$$

and therefore, by the rank-nullity theorem for linear maps,

$$\begin{aligned} \dim(\ker\phi) &= r(r+1) - r(r-1) && (3.7)\\ &= 2r. && (3.8) \end{aligned}$$

So we get the required result if we can show that the space of matrices, U, satisfying $-Q^T U + U^T Q = \mathbf{0}$ has dimension $2r$.

Write $-Q^T U + U^T Q$ as $\lambda Q_1 + \mu Q_2$ (with the entries of the skew-symmetric matrices Q_1 and Q_2 linear combinations of the entries of U). So when considering $-Q^T U + U^T Q = \mathbf{0}$, we can separate the equations for λ and μ, i.e $Q_1 = 0$ and $Q_2 = 0$ respectively.

We write

$$Q = \mu \begin{bmatrix} & I_r & \\ 0 & \cdots & 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 & \cdots & 0 \\ & I_r & \end{bmatrix}$$

and therefore

$$Q^T = \mu \begin{bmatrix} & 0 \\ I_r & \vdots \\ & 0 \end{bmatrix} + \lambda \begin{bmatrix} 0 & \\ \vdots & I_r \\ 0 & \end{bmatrix}.$$

Then $Q_1 = 0$ implies that

$$-\begin{bmatrix} 0 & \\ \vdots & I_r \\ 0 & \end{bmatrix} U + U^T \begin{bmatrix} 0 & \cdots & 0 \\ & I_r & \end{bmatrix} = \mathbf{0}$$

and writing

$$U = \begin{bmatrix} u_1 & \cdots & u_r \\ & U_0 & \\ & & \end{bmatrix}$$

where U_0 is an $r \times r$ matrix, we deduce that $-U_0 + U_0^T = 0$. Furthermore $Q_2 = 0$ implies that

$$-\begin{bmatrix} & 0 \\ I_r & \vdots \\ & 0 \end{bmatrix} U + U^T \begin{bmatrix} & I_r & \\ 0 & \cdots & 0 \end{bmatrix} = \mathbf{0}$$

and if we write

$$U = \begin{bmatrix} & V_0 & \\ & & \\ v_1 & \cdots & v_r \end{bmatrix},$$

where V_0 is an $r \times r$ matrix (whose first row is $u_1 \cdots u_r$) then we find that $-V_0 + V_0^T = 0$.

So if

$$U = \begin{bmatrix} u_1 & \cdots & u_r \\ & U_0 & \\ & & \end{bmatrix} = \begin{bmatrix} & V_0 & \\ & & \\ v_1 & \cdots & v_r \end{bmatrix}$$

satisfies $-Q^T U + U^T Q = 0$ then the two $r \times r$ matrices U_0 and V_0 are both symmetric. Conversely, any matrix U of this form, with U_0 and V_0 both symmetric, is in the kernel of $-Q^T U + U^T Q$. By Lemma 3.2.5 the space of such matrices U has dimension $2r$ as required. □

Lemma 3.2.7 *Consider a skew-symmetric pencil, Λ, of the form*

$$\Lambda = \left[\begin{array}{c|c|c} 0 & Q & 0 \\ \hline -Q^T & F & G \\ \hline 0 & -G^T & \Gamma \end{array}\right], \tag{3.9}$$

where 0 *are null matrices, Q is the $(k+1) \times k$ pencil given by*

$$Q = \begin{bmatrix} \mu & 0 & \cdots & 0 \\ \lambda & \mu & \ddots & \vdots \\ 0 & \lambda & \ddots & 0 \\ \vdots & \ddots & \ddots & \mu \\ 0 & \cdots & 0 & \lambda \end{bmatrix},$$

G is a $k \times (n-2k-1)$ pencil and F, Γ are skew-symmetric pencils of order k and $(n-2k-1)$ respectively. Then by skew-equivalent reduction we obtain an equivalent pencil of the form

$$\left[\begin{array}{c|c|c} 0 & Q & 0 \\ \hline -Q^T & 0 & \lambda R \\ \hline 0 & -\lambda R^T & \Gamma \end{array}\right], \tag{3.10}$$

where R is a constant $k \times (n-2k-1)$ matrix. Furthermore both pencils have the same homogeneous kernel vector $\mathbf{w}(k)$.

Proof The first part follows from the above. By the action on Λ of an invertible matrix of the form

$$\left[\begin{array}{c|c|c} I_{k+1} & U & 0 \\ \hline 0 & I_k & 0 \\ \hline 0 & 0 & I_{n-2k-1} \end{array}\right], \tag{3.11}$$

for some $(k+1) \times k$ matrix, U, we get the skew-equivalent pencil

$$\left[\begin{array}{c|c|c} 0 & Q & 0 \\ \hline -Q^T & -Q^TU + U^TQ + F & G \\ \hline 0 & -G^T & \Gamma \end{array}\right].$$

Then by Lemma 3.2.6 we can choose U such that $-Q^TU + U^TQ + F = 0$. Since the action of (3.11) corresponds to adding rows from the $(k+2)$ to $(2k+1)$ zero rows of $\mathbf{w}(k)$ to its first $(k+1)$ rows, $\mathbf{w}(k)$ is preserved. Consider the reduced pencil,

$$\left[\begin{array}{c|c|c} 0 & Q & 0 \\ \hline -Q^T & 0 & G \\ \hline 0 & -G^T & \Gamma \end{array}\right],$$

where the block G is the $k \times (n-2k-1)$ pencil $G = \lambda g_{ij} + \mu h_{ij}$ where $k+2 \leq i \leq 2k+1$ and $2k+2 \leq j \leq n$. So we can kill off the μh_{ij} in the ijth entry of G by adding a multiple of the column of (3.10), for which there is a $-\mu$ in the ith row of $-Q^T$, to the jth column of (3.10), (i.e by the column operation $C_j + h_{ij}C_{i-k-1}$). In this way we can kill off all the μ terms in G using column operations. Furthermore the counterpart row operations required to preserve skew-symmetry kill off the corresponding μ terms in $-G^T$ and we can reduce (3.10) to the skew-equivalent pencil

$$\left[\begin{array}{c|c|c} 0 & Q & 0 \\ \hline -Q^T & 0 & \lambda R \\ \hline 0 & -\lambda R^T & \Gamma \end{array}\right], \tag{3.12}$$

where R is a constant $k \times (n-2k-1)$ matrix. Since the simultaneous row and column operations used affect only the last $(n-2k-1)$ rows and columns of (3.12) as before the kernel vector of this pencil is unchanged. □

The pencil given in (3.10) is the starting point for the reduction to a normal form. Before a formal description of this reduction we make the following remark.

Remark 3.2.8 If the minimal degree, k_1, of a pencil, Λ, is zero then it has a constant kernel vector, $\mathbf{v}_c$. By row operations we can reduce this vector to

$$\mathbf{v}(0) = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

These row operations correspond to premultiplying $\mathbf{v}_c$ by the invertible matrix P^{-1} and the skew-equivalent pencil $P^T \Lambda P$ has kernel vector $\mathbf{v}(0)$. From this we deduce that the first column of $P^T \Lambda P$ are all zeros and hence by skew-symmetry so is its first row.

We have therefore reduced our pencil to one whose first row and column are null and we are left with an $(n-1) \times (n-1)$ sub-pencil. It could be that this sub-pencil also has minimal degree 0 in which case we introduce another null row and column. Consequently our normal form may start with several null rows and columns. Although we keep this in mind it is more convenient to start the reduction to a normal form by considering the (sub-)pencil with non-zero minimal degree.

Theorem 3.2.9 *Given a singular $n \times n$ skew-symmetric pencil, Λ of minimal degree $k_1 > 0$ we can reduce it to a skew-equivalent pencil of the form*

$$\left[\begin{array}{cc|cc|c|cc|c}
0 & L_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-L_1^T & 0 & \lambda R_{11} & \lambda R_{12} & \cdots & \lambda R_{1s-1} & \lambda R_{1s} & \lambda R_{1s+1} \\
\hline
0 & -\lambda R_{11}^T & 0 & L_2 & \cdots & 0 & 0 & 0 \\
0 & -\lambda R_{12}^T & -L_2^T & 0 & \cdots & \lambda R_{2s-1} & \lambda R_{2s} & \lambda R_{2s+1} \\
\cline{3-4}
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\cline{6-7}
0 & -\lambda R_{1s-1}^T & 0 & -\lambda R_{2s-1}^T & \cdots & 0 & L_s & 0 \\
0 & -\lambda R_{1s}^T & 0 & -\lambda R_{2s}^T & \cdots & -L_s^T & 0 & \lambda R_{ss+1} \\
\cline{6-8}
0 & -\lambda R_{1s+1} & 0 & -\lambda R_{2s+1}^T & \cdots & 0 & -\lambda R_{ss+1}^T & \Psi
\end{array}\right], \tag{3.13}$$

where Ψ is a possibly existent column independent sub-pencil, the blocks L_i are of the form

$$L_i = \overbrace{\begin{bmatrix} \mu & 0 & \cdots & 0 \\ \lambda & \mu & \ddots & \vdots \\ 0 & \lambda & \ddots & 0 \\ \vdots & \ddots & \ddots & \mu \\ 0 & \cdots & 0 & \lambda \end{bmatrix}}^{k_i}$$

and $k_s \geq \cdots \geq k_2 \geq k_1$.

Proof The proof is by induction on n. The initial step has been done above and for completeness we summarise it thus.

Given a singular skew-symmetric pencil with homogeneous kernel vector, $\mathbf{v}$, of minimal degree k_1 then (using Lemmas 3.2.2 and 3.2.3) it is skew-equivalent to a pencil of the form

$$P^T \Lambda P = \left[\begin{array}{c|c} 0 & L_1 \ 0 \\ \hline \begin{array}{c} -L_1^T \\ 0 \end{array} & \Pi \end{array}\right],$$

with

$$L_1 = \begin{bmatrix} \mu & 0 & \cdots & 0 \\ \lambda & \mu & \ddots & \vdots \\ 0 & \lambda & \ddots & 0 \\ \vdots & \ddots & \ddots & \mu \\ 0 & \cdots & 0 & \lambda \end{bmatrix}$$

and Π some skew-symmetric pencil of order $(n - k_1 - 1)$. Furthermore $P^T \Lambda P$ has kernel vector

$$\mathbf{v}(k_1) = P^{-1}\mathbf{v} = \begin{pmatrix} \lambda^{k_1} \\ -\lambda^{k_1-1}\mu \\ \vdots \\ (-1)^{k_1}\mu^{k_1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

For some invertible matrix, Q, we can further reduce Λ to the the skew-equivalent pencil

$$\Lambda_1 = Q^T(P^T\Lambda P)Q = \left[\begin{array}{c|c|c} 0 & L_1 & 0 \\ \hline -L_1^T & 0 & \lambda R_1 \\ \hline 0 & -\lambda R_1^T & \Gamma \end{array}\right], \tag{3.14}$$

where R_1 is a constant $k_1 \times (n-2k_1-1)$ matrix and Γ a skew-symmetric pencil of order $n-2k_1-1$. If $n = 2k_1+1$ we are finished.

This pencil, Λ_1, has a homogeneous polynomial kernel vector $Q^{-1}\mathbf{v}(k_1)$ with *minimal* degree k_1. In fact since the row operations corresponding to premultiplying $\mathbf{v}(k_1)$ by Q^{-1} involve adding rows from the $(n-k_1-1)$ zero rows of $\mathbf{v}(k_1)$, to its first $(k+1)$ rows it follows that $Q^{-1}\mathbf{v}(k_1) = \mathbf{v}(k_1)$.

In general, Λ_1 has a homogeneous polynomial kernel vector

$$\mathbf{v}_1 = \left[\begin{array}{c} X_1 \\ \hline Y_1 \\ \hline Z_1 \end{array}\right],$$

of degree $m \geq k_1$. We therefore have

$$\left[\begin{array}{c|c|c} 0 & L_1 & 0 \\ \hline -L_1^T & 0 & \lambda R_1 \\ \hline 0 & -\lambda R_1^T & \Gamma \end{array}\right] \left[\begin{array}{c} X_1 \\ \hline Y_1 \\ \hline Z_1 \end{array}\right] = \left[\begin{array}{c} 0 \\ \hline 0 \\ \hline 0 \end{array}\right].$$

So

$$L_1 Y_1 = 0, \tag{3.15}$$

$$-L_1^T X_1 + \lambda R_1 Z_1 = 0, \tag{3.16}$$

$$-\lambda R_1^T Y_1 + \Gamma Z_1 = 0. \tag{3.17}$$

From (3.15) we have

$$\begin{bmatrix} \mu & 0 & \cdots & 0 \\ \lambda & \mu & \ddots & \vdots \\ 0 & \lambda & \ddots & 0 \\ \vdots & \ddots & \ddots & \mu \\ 0 & \cdots & 0 & \lambda \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ \vdots \\ y_{k_1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix},$$

where y_i are polynomials in λ, μ of degree m. So

$$\mu y_1 = 0 \qquad \Longrightarrow y_1 \equiv 0$$

$$\lambda y_1 + \mu y_2 = 0 \Longrightarrow y_2 \equiv 0$$

and so on. Consequently the only solution to $L_1 Y_1 = 0$ is $Y_1 = 0$ and it follows that L_1 is column independent. From (3.17)

$$\Gamma Z_1 = 0.$$

If $Z_1 \equiv 0$ is the only solution then Γ is column independent and we are done. Furthermore a corresponding kernel vector is

$$\left[\begin{array}{c} X_1 \\ \hline 0 \\ \hline 0 \end{array}\right],$$

where X_1 is a homogeneous polynomial vector of degree $\geq k_1$ satisfying $-L_1^T X_1 = 0$. One such vector is the minimal kernel vector, $\mathbf{v}(k_1)$, already found.

Otherwise Γ has a non-trivial kernel vector of degree $\geq k_1$ and in particular a minimal polynomial kernel vector Z_1 of degree $k_2 \geq k_1$. (In this case we have the homogeneous polynomial kernel vector, of degree k_2,

$$\mathbf{v}_2 = \left[\begin{array}{c} X_1 \\ \hline 0 \\ \hline Z_1 \end{array}\right],$$

where X_1 satisfies $L_1^T X_1 = \lambda R_1 Z_2$ from (3.16).)

Inductive Step

By induction, we find that Γ is skew-equivalent to

$$\left[\begin{array}{cc|c|cc|c}
0 & L_2 & \cdots & 0 & 0 & 0 \\
-L_2^T & 0 & \cdots & \lambda R_{2s-1} & \lambda R_{2s} & \lambda R_{2s+1} \\
\cline{1-2}
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\cline{4-5}
0 & -\lambda R_{2s-1}^T & \cdots & 0 & L_s & 0 \\
0 & -\lambda R_{2s}^T & \cdots & -L_s^T & 0 & \lambda R_{ss+1} \\
\cline{4-6}
0 & -\lambda R_{2s+1}^T & \cdots & 0 & -\lambda R_{ss+1}^T & \Psi
\end{array}\right],$$

where $k_s \geq \cdots \geq k_2$. The simultaneous row and column operations, required for this reduction, when performed on Λ_1 have the further effect of altering the λR_1 (and $-\lambda R_1^T$) blocks to $\lambda R_1'$ (and $-\lambda R_1'^T$). The result then follows. □

In fact (if it exists) the sub-pencil, Ψ, of this reduced pencil, (3.13), is non-singular as shown by the following result.

Lemma 3.2.10 *Let Ω be a skew-symmetric pencil which is column independent. Then it is also row independent and therefore non-singular.*

Proof If Ω is column independent then

$$\Omega\mathbf{v} = 0$$

has only the trivial solution $\mathbf{v} = \mathbf{0}$. If Ω is row-dependent then for some non-trivial homogeneous polynomial vector $\mathbf{u}$

$$\mathbf{u}\Omega = \mathbf{0}.$$

So $(\mathbf{u}\Omega)^T = \mathbf{0}$ and

$$\Omega^T\mathbf{u}^T = 0.$$

Therefore since Ω is skew-symmetric

$$-\Omega\mathbf{u}^T = 0$$

with $\mathbf{u}^T$ non-trivial. This contradicts the column independence of Ω. So Ω must also be row independent and hence is non-singular. □

3.3 Normal Forms for Skew-symmetric Pencils

We have reduced our pencil to the form

$$\left[\begin{array}{cc|cc|cc|c|cc|c}
0 & L_1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-L_1^T & 0 & \dagger & \dagger & \dagger & \dagger & \cdots & \dagger & \dagger & \dagger \\
\cline{1-4}
0 & \star & 0 & L_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & \star & -L_2^T & 0 & \dagger & \dagger & \cdots & \dagger & \dagger & \dagger \\
\cline{3-6}
0 & \star & 0 & \star & 0 & L_3 & \cdots & 0 & 0 & 0 \\
0 & \star & 0 & \star & -L_3^T & 0 & \cdots & \dagger & \dagger & \dagger \\
\cline{5-6}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\cline{8-9}
0 & \star & 0 & \star & 0 & \star & \cdots & 0 & L_r & 0 \\
0 & \star & 0 & \star & 0 & \star & \cdots & -L_r^T & 0 & \dagger \\
\cline{8-10}
0 & \star & 0 & \star & 0 & \star & \cdots & 0 & \star & M
\end{array}\right],$$

where M is a possibly existent non-singular skew-symmetric pencil, the non-zero entries of blocks denoted by $\star$ are constant multiples of λ and for each such block its diagonally opposite block, denoted by $\dagger$, satisfies $\dagger = -\star^T$.

Starting with the above initial reduction, in this section we describe in concrete terms how to construct normal forms for singular pencils. The principal objective is to clear away the non-zero elements in the submatrices indicated by stars and daggers. To do this we adapt the general method given in [TurnAit] to our skew-symmetric case.

We consider the submatrices, denoted by stars, below the L_i blocks of the matrix. These submatrices are aligned with (that is lie in the same row or column block as) either

(i) two L_i's,

(ii) a L_i and a $-L_j^T$ or

(iii) a L_i and M.

Clearly, if the initial reduction produces no non-singular subpencil only cases (i) and (ii) apply. However if M exists we also need to consider case (iii). This involves incorporating the work in Chapter 2 for simplifying the non-singular sub-block M.

We consider the three cases in turn.

3.3.1 Case (i)

We consider the case where the submatrix is aligned with L_1 and L_2. This submatrix therefore has $k_2 + 1$ rows and k_1 columns aligned with L_1 above and L_2 to the right. For illustrative purposes we represent this submatrix by the matrix of stars given below.

$$\begin{array}{ccc|ccc}
\mu & 0 & 0 & & & \\
\lambda & \mu & 0 & & & \\
0 & \lambda & \mu & & & \\
0 & 0 & \lambda & & & \\
\hline
\star & \star & \star & \mu & 0 & 0 \\
\star & \star & \star & \lambda & \mu & 0 \\
\star & \star & \star & 0 & \lambda & \mu \\
\star & \star & \star & 0 & 0 & \lambda
\end{array}$$

Imagine a diagonal barrier drawn just below the leading diagonal of this submatrix of stars. Since $k_2 \geq k_1$, the top row of stars lies entirely above, and the bottom row entirely below, this barrier.

Each non-zero ijth star represents a term $\alpha\lambda$. If such an entry is *above* the barrier, but not in the final column of stars, it may be moved one step diagonally downwards to the right. This is done by utilizing the λ of L_1 in its own jth column and the μ of L_2 in its own ith row. For example the pair of operations

$$R_5 - \alpha R_2, \qquad C_2 + \alpha C_4,$$

moves the term $\alpha\lambda$ from position (5,1) to (6,2) without changing anything else. If such an entry lies in the final column it can be deleted at once using the λ of L_1 in its own column, e.g the row operation $R_7 - \alpha R_4$ deletes $\alpha\lambda$ from (7,3).

Similarly, for an entry *below* the barrier we can utilize the λ of L_2 in its own row and the μ of L_1 in its own column to move this entry one step diagonally upwards to the left and then repeat this process until it passes off the figure after reaching the first column. So working from left to right above the barrier and from right to left below, by a sequence of row and column operations we can delete every star.

For blocks L_i, $1 \le i \le r$, for every block, $\star$, below it aligned with an L_j, $j > i$, we use this method to kill off each of its entries remembering that after *each* individual (row/column) operation used we need to perform the counterpart (column/row) operation to preserve skew-symmetry. (This will result in the clearing of the diagonally opposite block, $\dagger$, above $-L_j^T$ and to the right of $-L_i^T$.)

In this way, for each L_i, we proceed downwards systematically killing off each block $\star$ aligned with an L_j, $j = i+1, \ldots, r$. We make the following observations:

1) As long as we start with column block L_1 $(i = 1)$ and work from left to right, then any row operations using L_i to clear a block below it only affect this block. (Similarly for the counterpart column operations.)

2) The column operations using L_j to clear the block introduce terms involving λ into the blocks below it and into the blocks above aligned with $-L_k^T$ $(k < j)$. Note, however, the zero blocks we have created above, aligned with L_k $(k < j)$, are preserved. (We have an analogous situation for the counterpart row operations.)

3) These column operations (along with the counterpart row operations)

create a skew-symmetric pencil in the block aligned with L_i and $-L_i^T$ which involves multiples of λ.

Working column block by column block from left to right, systematically killing off each block, $\star$, aligned with a L_i and L_j ($-L_i^T$ and $-L_j^T$), that is by a series of simultaneous row and column operations, we obtain the following skew-equivalent pencil.

$$\left[\begin{array}{cc|cc|cc|c|cc|c}
0 & L_1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-L_1^T & \diamond & 0 & \dagger & 0 & \dagger & \cdots & 0 & \dagger & \dagger \\
\hline
0 & 0 & 0 & L_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & \star & -L_2^T & \diamond & 0 & \dagger & \cdots & 0 & \dagger & \dagger \\
\hline
0 & 0 & 0 & 0 & 0 & L_3 & \cdots & 0 & 0 & 0 \\
0 & \star & 0 & \star & -L_3^T & \diamond & \cdots & 0 & \dagger & \dagger \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & L_r & 0 \\
0 & \star & 0 & \star & 0 & \star & \cdots & -L_r^T & 0 & \dagger \\
\hline
0 & \star & 0 & \star & 0 & \star & \cdots & 0 & \star & M
\end{array}\right],$$

where although some of the blocks $*$ and $\dagger$ have been altered their entries are still constant multiples of λ, and blocks denoted by $\diamond$ are skew-symmetric.

3.3.2 Case (ii)

We represent the submatrix aligned with L_i and $-L_j^T$ by the following figure

$$\begin{array}{cc|cccc}
\mu & 0 & & & & \\
\lambda & \mu & & & & \\
0 & \lambda & & & & \\
\hline
\star & \star & -\mu & -\lambda & 0 & 0 \\
\star & \star & 0 & -\mu & -\lambda & 0 \\
\star & \star & 0 & 0 & -\mu & -\lambda
\end{array}.$$

Again each non-zero star represents a term $\alpha\lambda$. For each such entry we use the λ above and the μ to its right to move it one step diagonally upwards to the right until it is deleted. For example the row operation $R_5 - \alpha R_2$ followed by the column operation $C_2 - \alpha C_4$ moves the element $\alpha\lambda$ from position (5,1) to (4,2). All entries in the final column are deleted using the λ above in the final column of L_i. Working down each column, starting from the LHS and proceeding from left to right, we can delete every star.

For blocks L_i, $1 \leq i \leq r$, for every block, $\star$, below it aligned with an $-L_j^T$, $j > i$, this method can be used to kill off each of its entries provided we remember, after *each* individual (row/column) operation is used, to perform the corresponding the counterpart (column/row) operation required to preserve skew-symmetry. (This will result in the clearing of the diagonally opposite block, $\dagger$, above L_j and to the right of $-L_i^T$)

In this way, for each L_i, we proceed downwards systematically killing off each block $\star$ aligned with an $-L_j^T$, $j = i+1, \ldots, r$. Observe here, that, having previously killed off all the blocks aligned with L_i and L_j, now both the row operations, using L_i, and the column operations, using $-L_j^T$, required to kill off such a block leave the remainder of the matrix unaffected.

Working column block by column block from left to right, we can therefore, by a series of simultaneous row and column operations, kill off every block, $\star$, aligned with a L_i and a $-L_j^T$ (accordingly blocks $\dagger$ aligned with $-L_i^T$ and L_j), to obtain the following skew-equivalent pencil.

$$\left[\begin{array}{cc|cc|cc|c|cc|c}
0 & L_1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-L_1^T & \diamond & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \dagger \\
\cline{1-2}
0 & 0 & 0 & L_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & -L_2^T & \diamond & 0 & 0 & \cdots & 0 & 0 & \dagger \\
\cline{3-4}
0 & 0 & 0 & 0 & 0 & L_3 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -L_3^T & \diamond & \cdots & 0 & 0 & \dagger \\
\cline{5-6}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\cline{8-9}
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & L_r & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & -L_r^T & 0 & \dagger \\
\cline{8-10}
0 & \star & 0 & \star & 0 & \star & \cdots & 0 & \star & M
\end{array}\right].$$

Before considering case (iii) we need to kill off the skew-symmetric pencils, $\diamond$, aligned with blocks L_i and $-L_i^T$. We represent this situation by the following figure.

$$
\begin{array}{ccccc|cccc}
 & & & & & \mu & 0 & 0 & 0 \\
 & & & & & \lambda & \mu & 0 & 0 \\
 & & & & & 0 & \lambda & \mu & 0 \\
 & & & & & 0 & 0 & \lambda & \mu \\
 & & & & & 0 & 0 & 0 & \lambda \\
\hline
-\mu & -\lambda & 0 & 0 & 0 & 0 & \alpha_{12}\lambda & \alpha_{13}\lambda & \alpha_{14}\lambda \\
0 & -\mu & -\lambda & 0 & 0 & -\alpha_{12}\lambda & 0 & \alpha_{23}\lambda & \alpha_{24}\lambda \\
0 & 0 & -\mu & -\lambda & 0 & -\alpha_{13}\lambda & -\alpha_{23}\lambda & 0 & \alpha_{34}\lambda \\
0 & 0 & 0 & -\mu & -\lambda & -\alpha_{14}\lambda & -\alpha_{24}\lambda & -\alpha_{34}\lambda & 0
\end{array}
$$

We can kill off each (j,k)th entry in $\diamond$, by a row operation using the λ term above it in L_i. (Note, as usual, the necessary counterpart column operation kills off the (k,j)th entry.) This introduces a μ term into the $(j,k+1)$th entry. By a column operation, we can use the μ term in $-L_i^T$ to kill this off which, in general, introduces a further λ term diagonally to the right.

For example, in the above figure we can kill off the entries in positions $(7,8)$ and $(8,7)$ by the simultaneous row and column operations:

$$R_7 - \alpha_{23}R_4 \qquad \text{and} \qquad C_7 - \alpha_{23}C_4$$

followed by

$$C_9 - \alpha_{23}C_2 \qquad \text{and} \qquad R_9 - \alpha_{23}R_2,$$

introducing a $\pm\alpha_{23}\lambda$ term to positions $(6,9)$ and $(9,6)$ respectively.

We can kill off all terms in the final column (and row) using R_5 (and C_5). So working in this way from left to right, above the diagonal, we can delete each entry of $\diamond$. Note this example indicates a practical method for obtaining a special case of the result of Lemma 3.2.6.

Procceeding systematically from left to right, using the method just described to kill off each block $\diamond$ aligned with an L_i and a $-L_i^T$, we obtain the skew-equivalent matrix

$$\left[\begin{array}{cc|cc|cc|c|cc|c}
0 & L_1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-L_1^T & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \dagger \\ \hline
0 & 0 & 0 & L_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & -L_2^T & 0 & 0 & 0 & \cdots & 0 & 0 & \dagger \\ \hline
0 & 0 & 0 & 0 & 0 & L_3 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -L_3^T & 0 & \cdots & 0 & 0 & \dagger \\ \hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & L_r & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & -L_r^T & 0 & \dagger \\ \hline
0 & \star & 0 & \star & 0 & \star & \cdots & 0 & \star & M
\end{array}\right]. \tag{3.18}$$

3.3.3 Case (iii)

This requires a lot more work. First we need to consider, M, in more detail.

Firstly, M is a non-singular pencil of even order, $2u$ say. It is given by

$$M = \lambda M_1 + \mu M_2,$$

where, in general, M_1, M_2 are both non-singular skew-symmetric matrices. However, it is also possible that one or both of the basis matrices, M_1, M_2 are singular in which case before proceeding further it is neccessary to change this basis to a pair, (D_1, D_2), of non-singular matrices.

We briefly consider this possibility. Since M has a finite number of singular members we can easily find distinct ratios $(\lambda_1 : \mu_1)$, $(\lambda_2 : \mu_2)$ such that

$$D_1 = \lambda_1 M_1 + \mu_1 M_2, \qquad D_2 = \lambda_2 M_1 + \mu_2 M_2$$

are both non-singular and form a new basis for M i.e

$$M = \bar{\lambda} D_1 + \bar{\mu} D_2.$$

As discussed in Lemma 1.3.3 of Chapter 1, the relationship between old and new coordinates is given via the change of basis matrix, i.e

$$\begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \mu_1 & \mu_2 \end{pmatrix} \begin{pmatrix} \bar{\lambda} \\ \bar{\mu} \end{pmatrix}. \tag{3.19}$$

This necessary change of coordinates changes the entries in pencil (3.18), a particular nuisance being the changes to the singular blocks. We therefore have the skew-equivalent pencil.

$$
\left[\begin{array}{cc|cc|cc|c|cc|c}
0 & L_1' & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-L_1'^T & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \dagger \\
\cline{1-4}
0 & 0 & 0 & L_2' & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & -L_2'^T & 0 & 0 & 0 & \cdots & 0 & 0 & \dagger \\
\cline{3-6}
0 & 0 & 0 & 0 & 0 & L_3' & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -L_3'^T & 0 & \cdots & 0 & 0 & \dagger \\
\cline{5-6}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\cline{8-9}
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & L_r' & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & -L_r'^T & 0 & \dagger \\
\cline{8-10}
0 & \star & 0 & \star & 0 & \star & \cdots & 0 & \star & M
\end{array}\right], \tag{3.20}
$$

where $L_i'(\bar{\lambda}, \bar{\mu})$ is the block obtained from $L_i(\lambda, \mu)$ by the linear change of coordinates given in (3.19) and $M = \bar{\lambda} D_1 + \bar{\mu} D_2$. Note here also that with this substitution the terms in blocks $\star$ (and $\dagger$) involve both $\bar{\lambda}$ and $\bar{\mu}$.

Whether or not such an initial change of basis is necessary the following argument applies in both cases. It is natural to continue with the general case (i.e. when M_1, M_2 are both non-singular) although, despite maintaining the notation $(\lambda : \mu)$ for their coordinates, we prefer to denote the non-singular matrices by D_1, D_2 (thereby incorporating the other possiblity, where any enforced changes to the singular part are temporarily ignored).

We consider the non-singular pencil

$$M = \lambda D_1 + \mu D_2.$$

Representing this pencil by the *pair* (D_1, D_2) of non-singular skew-symmetric matrices, we aim, by the methods described in Chapter 2, to reduce it to one of our normal forms.

As discussed in Section 2.4 we represent this pair by the skew-symmetric μ-matrix (it is more convenient to use μ here rather than the usual λ)

$$M(\mu) = D_1 + \mu D_2.$$

Since M is skew-symmetric

$$
\begin{aligned}
\det M &= \det(D_1 + \mu D_2) \\
&= (f(\mu))^2,
\end{aligned}
$$

for some function $f(\mu)$, which over $\mathbb{C}$ can be factorised as

$$f(\mu) = \prod_{i=1}^{q} (a_i\mu + 1)^{s_{iu}},$$

where $a_i \neq 0$ for all $1 \leq i \leq q$.

The invariant factors of M, d_t are

$$d_t = \prod_{i=1}^{q} (a_i\mu + 1)^{s_{it} - s_{it-1}},$$

for $1 \leq t \leq u$. Recall, these s_{ij}'s are obtained from a representation of the ideals $\sqrt{I_{2k}(M)}$, similar to that of (2.36) in Section 2.4.

Fixing i, then for $1 \leq t \leq u$ we find the non-zero powers $s_{it} - s_{it-1}$ of $(a_i\mu + 1)$ in each invariant factor d_t. If there are $n(i)$ in total we denote these

$$r_{i1} \leq \cdots \leq r_{in(i)}.$$

(Note we ignore any zero powers.) Doing this for each $1 \leq i \leq q$, by Theorem 2.4.8, we can construct a normal form of the type described in Section 2.4.1.

Representing this normal form by

$$N = N_1 + \mu N_2,$$

the pair (N_1, N_2) is skew-equivalent to (D_1, D_2). That is

$$(N_1, N_2) = X^T(D_1, D_2)X$$

and

$$\lambda N_1 + \mu N_2 = X^T(\lambda D_1 + \mu D_2)X.$$

Hence by a series of simultaneous row and column operations on $\lambda D_1 + \mu D_2$ we obtain the skew-equivalent pencil $\lambda N_1 + \mu N_2$.

Returning to our original pencil, (3.18), these simultaneous row and column operations correspond to a series of simultaneous row and column operations on the last $2u$ rows and columns.

These give the skew-equivalent pencil

$$
\left[\begin{array}{cc|cc|cc|c|cc|c}
0 & L_1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-L_1^T & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \dagger \\
\hline
0 & 0 & 0 & L_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & -L_2^T & 0 & 0 & 0 & \cdots & 0 & 0 & \dagger \\
\hline
0 & 0 & 0 & 0 & 0 & L_3 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -L_3^T & 0 & \cdots & 0 & 0 & \dagger \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & L_r & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & -L_r^T & 0 & \dagger \\
\hline
0 & \star & 0 & \star & 0 & \star & \cdots & 0 & \star & N
\end{array}\right], \tag{3.21}
$$

where $N = \lambda N_1 + \mu N_2$, and the blocks $\star$ (and $\dagger$), involving terms in λ have been altered.

Consider this non-singular part :

$$N = \lambda N_1 + \mu N_2.$$

It is the direct sum of blocks :

$$
\underbrace{\left[\begin{array}{cc|cc|c|cc}
0 & a_i\mu + \lambda & & & & & \\
-a_i\mu - \lambda & 0 & \lambda & & & & \\
\hline
 & -\lambda & 0 & a_i\mu + \lambda & & & \\
 & & -a_i\mu - \lambda & 0 & \lambda & & \\
\hline
 & & & -\lambda & \ddots & & \\
 & & & & & \lambda & \\
\hline
 & & & & -\lambda & 0 & a_i\mu + \lambda \\
 & & & & & -a_i\mu - \lambda & 0
\end{array}\right]}_{2r_{ij}}. \tag{3.22}
$$

Furthermore, by an argument similar to that given at the end of Section 2.4.1, the q eigenvalues of (N_1, N_2), (the values of $(\lambda : \mu)$ on the projective line for which $\det N = 0$, see Section 1.2), are $(a_i : -1)$ with multiplicities s_{iu}.

By the *Three Point Lemma* (Lemma 1.3.4) on the complex projective line, we can use an element $p \in Gl(2, \mathbb{C})$ to fix up to three of these distinct eigenvalues to $(0:1)$, $(1:0)$ and $(1:1)$. This $p \in Gl(2, \mathbb{C})$ corresponds to a change of basis of the pencil from (N_1, N_2) to (N_1', N_2') where

$$\lambda N_1 + \mu N_2 = \rho N_1' + \sigma N_2',$$

and the relationship between the new coordinates, (ρ, σ), and the old coordinates (λ, μ) are via the associated change of basis matrix,

$$\begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix} \begin{pmatrix} \rho \\ \sigma \end{pmatrix}. \tag{3.23}$$

(See Lemma 1.3.3 in Chapter 1.)

Since

$$\det(\lambda N_1 + \mu N_2) = (\det X)^2 \prod_{i=1}^{q} (a_i \mu + \lambda)^{s_{iu}},$$

it follows, by the linear coordinate change given by the change of basis matrix, that

$$\det(\rho N_1' + \sigma N_2') = (\det X)^2 \prod_{i=1}^{q} (b_i \sigma + c_i \rho)^{s_{iu}},$$

for some b_i, $c_i \in \mathbb{C}$. So the q distinct eigenvalues of the pair (N_1', N_2') are $(b_i : -c_i)$. So for the first three eigenvalues of this pair to be $(1:0)$, $(0:1)$ and $(1:1)$ the linear map $p \in Gl(2, \mathbb{C})$ sends

$$\begin{array}{llcl} (i=1) & a_1\mu + \lambda & \longmapsto & k_1 \sigma \\ (i=2) & a_2\mu + \lambda & \longmapsto & k_2 \rho \\ (i=3) & a_3\mu + \lambda & \longmapsto & k_3(\rho - \sigma) \\ (i \geq 4) & a_i\mu + \lambda & \longmapsto & b_i \sigma + c_i \rho \end{array},$$

where k_1, k_2, k_3, b_i, $c_i \neq 0$. It can be verfied that the entries of the change of basis matrix required for this are, for any choice of $t \neq 0$, given by $\beta_2 = t$,

$$\alpha_1 = \frac{-a_1(a_2 - a_3)t}{(a_3 - a_1)}, \tag{3.24}$$

$$\alpha_2 = \frac{(a_2 - a_3)t}{(a_3 - a_1)}, \tag{3.25}$$

$$\beta_1 = -a_2 t. \tag{3.26}$$

For each of these four cases we need to consider the effect of this basis change on the blocks (3.22) of $N = \lambda N_1 + \mu N_2$. Before doing this, we introduce the following useful result.

Lemma 3.3.1 *Given a skew-symmetric block of the form*

$$\left[\begin{array}{cc|cc|c|cc}
0 & k_1\sigma & & & & & \\
-k_1\sigma & 0 & l_1(\rho,\sigma) & & & & \\ \hline
 & -l_1(\rho,\sigma) & 0 & k_2\sigma & & & \\
 & & -k_2\sigma & 0 & & & \\ \hline
 & & & & \ddots & & \\
 & & & & & l_r(\rho,\sigma) & \\ \hline
 & & & & -l_r(\rho,\sigma) & 0 & k_r\sigma \\
 & & & & & -k_r\sigma & 0
\end{array}\right], \tag{3.27}$$

where $k_i \neq 0$, $l_i(\rho,\sigma) = \alpha_i\rho + \beta_i\sigma$ ($\alpha_i \neq 0$), then it is skew-equivalent to the block

$$\underbrace{\left[\begin{array}{cc|cc|c|cc}
0 & \sigma & & & & & \\
-\sigma & 0 & \rho & & & & \\ \hline
 & -\rho & 0 & \sigma & & & \\
 & & -\sigma & 0 & & & \\ \hline
 & & & & \ddots & & \\
 & & & & & \rho & \\ \hline
 & & & & -\rho & 0 & \sigma \\
 & & & & & -\sigma & 0
\end{array}\right]}_{2r}. \tag{3.28}$$

Proof The proof is by induction on r

Initial step

Consider the first two blocks of (3.27) :

$$\left[\begin{array}{cc|cc|c}
0 & k_1\sigma & & & \\
-k_1\sigma & 0 & \alpha_1\rho + \beta_1\sigma & & \\ \hline
 & -\alpha_1\rho - \beta_1\sigma & 0 & k_2\sigma & \\
 & & -k_2\sigma & 0 & \alpha_2\rho + \beta_2\sigma \\ \hline
 & & & -\alpha_2\rho - \beta_2\sigma &
\end{array}\right].$$

We perform the following simultaneous row and column operations to eliminate the constants. Note, for each operation described below, for brevity, we omit the proceeding counterpart operation necessary to preserve skew-symmetry.

(1) Scale the second column (i.e. $(1/k_1)C_2$) so that the coefficient of σ in the leading diagonal block is 1.

(2) Add some multiple of the first row to the third (i.e. $R_3 + (\beta_1/k_1)R_1$) to kill off the σ term in this row.

(3) Scale the third row (i.e. $(k_1/\alpha_1)R_3$) so that the coefficient of ρ in this row is -1.

By these simultaneous row and column operations the first two blocks are reduced as shown below

$$
\left.\begin{array}{cc|cc|c}
0 & \sigma & & & \\
-\sigma & 0 & \rho & & \\
\hline
 & -\rho & 0 & \frac{k_1 k_2}{\alpha_1}\sigma & \\
 & & -\frac{k_1 k_2}{\alpha_1}\sigma & 0 & \alpha_2\rho + \beta_2\sigma \\
\hline
 & & & -\alpha_2\rho - \beta_2\sigma &
\end{array}\right. .
$$

Inductive step

Assume the result holds for the first $(s-1)$ blocks, $s \geq 2$, and consider, in particular, the $s-1$, s blocks :

$$
\begin{array}{cc|cc|cc|cc}
0 & \sigma & & & & & & \\
-\sigma & 0 & \rho & & & & & \\
\hline
 & -\rho & 0 & \sigma & & & & \\
 & & -\sigma & 0 & \rho & & & \\
\hline
 & & & -\rho & 0 & k_s\sigma & & \\
 & & & & -k_s\sigma & 0 & \alpha_s\rho + \beta_s\sigma & \\
\hline
 & & & & & -\alpha_s\rho - \beta_s\sigma & 0 & k_{s+1}\sigma \\
 & & & & & & -k_{s+1}\sigma & 0
\end{array}
\tag{3.29}
$$

By the simultaneous row and column operations :

(1)

$$\frac{1}{k_s}C_{2s},$$

(2)

$$R_{2s+1} + \frac{\beta_s}{k_s}R_{2s-1};$$

(this operation introduces a $-(\beta_s/k_s)\rho$ term into position $(2s+1, 2s-2)$),

(3)

$$\frac{k_s}{\alpha_s}R_{2s+1},$$

(4)

$$C_{2s-2} - \frac{\beta_s}{\alpha_s} C_{2s};$$

(this introduces a $-(\beta_s/\alpha_s)\sigma$ term into position $(2s-1, 2s-2)$),

(5)

$$R_{2s-1} - \frac{\beta_s}{\alpha_s} R_{2s-3};$$

(this introduces a $-(\beta_s/\alpha_s)\rho$ term into position $(2s-1, 2s-4)$),

(6)

$$C_{2s-4} - \frac{\beta_s}{\alpha_s} C_{2s-2},$$

these blocks are reduced to the form

$$\begin{array}{cc|cc|cc|cc}
0 & \sigma & & & & & & \\
-\sigma & 0 & \rho + \frac{\beta_s}{\alpha_s}\sigma & & & & & \\
\hline
 & -\rho - \frac{\beta_s}{\alpha_s}\sigma & 0 & \sigma & & & & \\
 & & -\sigma & 0 & \rho & & & \\
\hline
 & & & -\rho & 0 & \sigma & & \\
 & & & & -\sigma & 0 & \rho & \\
\hline
 & & & & & -\rho & 0 & \frac{k_s k_{s+1}}{\alpha_s}\sigma \\
 & & & & & & -\frac{k_s k_{s+1}}{\alpha_s}\sigma & 0.
\end{array}$$

The first $s-1$ blocks are of the form (3.27), with their entries satisfying the associated conditions, so the result follows by induction. □

This result can be used to reduce the types of blocks obtained by the above change of basis. We consider each in turn.

1. The blocks of (N_1', N_2'), corresponding to eigenvalue $(a_1 : -1)$ of (N_1, N_2), are of the form :

$$\underbrace{\left[\begin{array}{cc|cc|c|cc} 0 & k_1\sigma & & & & & \\ -k_1\sigma & 0 & l(\rho,\sigma) & & & & \\ \hline & -l(\rho,\sigma) & 0 & k_1\sigma & & & \\ & & -k_1\sigma & 0 & & & \\ \hline & & & & \ddots & & \\ & & & & & l(\rho,\sigma) & \\ \hline & & & & -l(\rho,\sigma) & 0 & k_1\sigma \\ & & & & & -k_1\sigma & 0 \end{array}\right]}_{2r_{1j}}, \tag{3.30}$$

where $l(\rho,\sigma) = \alpha_1\rho + \beta_1\sigma = \lambda$ is given by (3.23). Clearly from (3.24) $\alpha_1 \neq 0$ and we can apply Lemma 3.3.1 to reduce (3.30) to the skew-equivalent block

$$\underbrace{\left[\begin{array}{cc|cc|c|cc} 0 & \sigma & & & & & \\ -\sigma & 0 & \rho & & & & \\ \hline & -\rho & 0 & \sigma & & & \\ & & -\sigma & 0 & & & \\ \hline & & & & \ddots & & \\ & & & & & \rho & \\ \hline & & & & -\rho & 0 & \sigma \\ & & & & & -\sigma & 0 \end{array}\right]}_{2r_{1j}}. \tag{3.31}$$

For convenience we refer to a direct sum of $n(1)$ blocks of this type as A_1.

2. The blocks of (N_1', N_2'), corresponding to eigenvalue $(a_2 : -1)$ of (N_1, N_2), are of the form :

$$\underbrace{\left[\begin{array}{cc|cc|c|cc} 0 & k_2\rho & & & & & \\ -k_2\rho & 0 & l(\rho,\sigma) & & & & \\ \hline & -l(\rho,\sigma) & 0 & k_2\rho & & & \\ & & -k_2\rho & 0 & & & \\ \hline & & & & \ddots & & \\ & & & & & l(\rho,\sigma) & \\ \hline & & & & -l(\rho,\sigma) & 0 & k_2\rho \\ & & & & & -k_2\rho & 0 \end{array}\right]}_{2r_{2j}}, \tag{3.32}$$

where $l(\rho,\sigma) = \alpha_1\rho + \beta_1\sigma$, as before. We look to apply the result of Lemma 3.3.1 to (3.32). If we switch the ρ and σ in (3.32) then since, from (3.26), it is clear that $\beta_1 \neq 0$ we have a block of the form, (3.27), given in the statement of

Lemma 3.3.1. Applying this lemma and then switching the ρ and σ back we can reduce (3.32) to the skew-equivalent block

$$\underbrace{\left[\begin{array}{cc|cc|c|cc}
0 & \rho & & & & & \\
-\rho & 0 & \sigma & & & & \\ \hline
& -\sigma & 0 & \rho & & & \\
& & -\rho & 0 & & & \\ \hline
& & & & \ddots & & \\
& & & & & \sigma & \\ \hline
& & & & -\sigma & 0 & \rho \\
& & & & & -\rho & 0
\end{array}\right]}_{2r_{2j}} . \qquad (3.33)$$

For convenience we denote a direct sum of $n(2)$ blocks of this type by A_2.

3. The blocks of (N_1', N_2'), corresponding to eigenvalue $(a_3 : -1)$ of (N_1, N_2), are of the form :

$$\underbrace{\left[\begin{array}{cc|cc|c|cc}
0 & k_3(\rho-\sigma) & & & & & \\
-k_3(\rho-\sigma) & 0 & l(\rho,\sigma) & & & & \\ \hline
& -l(\rho,\sigma) & 0 & k_3(\rho-\sigma) & & & \\
& & -k_3(\rho-\sigma) & 0 & & & \\ \hline
& & & & \ddots & & \\
& & & & & l(\rho,\sigma) & \\ \hline
& & & & -l(\rho,\sigma) & 0 & k_3(\rho-\sigma) \\
& & & & & -k_3(\rho-\sigma) & 0
\end{array}\right]}_{2r_{3j}} ,$$

(3.34)

with $l(\rho, \sigma) = \alpha_1\rho + \beta_1\sigma$. Again we look to apply the result of Lemma 3.3.1. Consider the skew-symmetric block obtained from (3.34) by setting $\delta = \rho - \sigma$. Then it follows that

$$l(\rho, \sigma) = (\alpha_1 + \beta_1)\rho - \beta_1\delta,$$

and it can be verified from (3.24) and (3.26) that for distinct non-zero a_1, a_2, a_3, $\alpha_1 + \beta_1 \neq 0$. Therefore we have a block of the form (3.27) given in Lemma 3.3.1. Applying this lemma and afterwards replacing δ by $\rho - \sigma$ we obtain from (3.34) the skew-equivalent block

$$\underbrace{\left[\begin{array}{cc|cc|c|cc}
0 & (\rho-\sigma) & & & & & \\
-(\rho-\sigma) & 0 & \rho & & & & \\ \hline
 & -\rho & 0 & (\rho-\sigma) & & & \\
 & & -(\rho-\sigma) & 0 & & & \\ \hline
 & & & & \ddots & & \\
 & & & & & \rho & \\ \hline
 & & & & -\rho & 0 & (\rho-\sigma) \\
 & & & & & -(\rho-\sigma) & 0
\end{array}\right]}_{2r_{3j}} . \tag{3.35}$$

For convenience we denote a direct sum of $n(3)$ blocks of this type by A_3.

4. The blocks of (N_1', N_2'), corresponding to eigenvalues $(a_i : -1)$, $4 \leq i \leq q$ of (N_1, N_2), are of the form :

$$\underbrace{\left[\begin{array}{cc|cc|c|cc}
0 & b_i\sigma + c_i\rho & & & & & \\
-b_i\sigma - c_i\rho & 0 & l(\rho,\sigma) & & & & \\ \hline
 & -l(\rho,\sigma) & 0 & b_i\sigma + c_i\rho & & & \\
 & & -b_i\sigma - c_i\rho & 0 & & & \\ \hline
 & & & & \ddots & & \\
 & & & & & l(\rho,\sigma) & \\ \hline
 & & & & -l(\rho,\sigma) & 0 & b_i\sigma + c_i\rho \\
 & & & & & -b_i\sigma - c_i\rho & 0
\end{array}\right]}_{2r_{ij}} , \tag{3.36}$$

where as usual $l(\rho, \sigma) = \alpha_1\rho + \beta_1\sigma$. For these cases we cannot eliminate all the constants, so instead the best we can do is to replicate blocks of the type which made up our original normal forms, i.e. of the form (3.22). Again we look to obtain from (3.36) a block of the type given in Lemma 3.3.1. We start by multiplying all the even columns by $1/c_i$ and denoting $e_i = b_i/c_i$. Hence the eigenvalues for these blocks are $(e_i : -1)$. Consider setting $\delta = e_i\sigma + \rho$. Then

$$\frac{1}{c_i} l(\rho, \sigma) = \frac{1}{b_i}\left((\alpha_1 e_i - \beta_1)\rho + \beta_1\delta\right).$$

Considering the respective eigenvalues $(a_i : -1)$, $(e_i : 1)$ of the pairs (N_1, N_2) and (N_1', N_2') then by Lemma 1.4.2 it follows from (3.23) that $a_i = \alpha_1 e_i - \beta_1$. Since $a_i \neq 0$, $\alpha_1 e_i - \beta_1 \neq 0$ and we have a skew-symmetric block of the form given in Lemma 3.3.1. Applying this lemma and then replacing δ by $e_i\sigma + \rho$ in the result we can reduce the block (3.36) to the skew-equivalent block

$$\underbrace{\left[\begin{array}{cc|cc|c|cc}
0 & e_i\sigma+\rho & & & & & \\
-e_i\sigma-\rho & 0 & \rho & & & & \\ \hline
 & -\rho & 0 & e_i\sigma+\rho & & & \\
 & & -e_i\sigma-\rho & 0 & & & \\ \hline
 & & & & \ddots & & \\
 & & & & & \rho & \\ \hline
 & & & & -\rho & 0 & e_i\sigma+\rho \\
 & & & & & -e_i\sigma-\rho & 0
\end{array}\right]}_{2r_{ij}}, \tag{3.37}$$

as desired. For convenience we denote a direct sum of $n(i)$ blocks of this type by B_i where $4 \leq i \leq q$.

Having just described a way of fixing three of the distinct constants of the normal form, N, in (3.21) we need to discuss how this is implemented on the whole pencil.

Essentially this corresponds to a change of basis followed by a series of simultaneous row and column operations on the last $2u$ rows and columns of (3.21). So the nonsingular part is therefore of the form $\bar{N} = \rho\bar{N}_1 + \sigma\bar{N}_2$, which is a sum of q blocks, A_1, A_2, A_3, B_i $4 \leq i \leq q$, consisting, respectively, of a direct sum of the blocks (3.31), (3.33), (3.35) and (3.37) found above.

The effect on the singular blocks, L_i, is that of a change of coordinates. Note if initially we had needed to change the basis (M_1, M_2) of M to the pair of non-singular matrices (D_1, D_2) the further change of coordinates required to fix the eigenvalues can be composed with this initial change to give a *single* linear change of coordinates

$$(\lambda, \mu) \longmapsto (\rho, \sigma),$$

on the original blocks L_i in (3.18). All we can say about the remaining entries in the last $2u$ rows and columns is that they consist of linear terms in ρ and σ.

Before proceeding any further we make the following remark.

Remark 3.3.2 Recall from Section 1.2 that the eigenvalues of a *pair* of skew-symmetric matrices is a skew-equivalent invariant. Furthermore by changes of basis we can only fix up to *three* distinct eigenvalues of a *pencil*. It follows that if a skew-symmetric pencil has four or more distinct eigenvalues then it has *moduli* up to skew-equivalence.

3.3.4 Cleaning up the Singular Blocks

In summary by the work of the previous section we have established that any pencil, (3.18), is skew-equivalent to one of the form :

$$\left[\begin{array}{cc|cc|cc|c|cc|c}
0 & L_1' & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-L_1'^T & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \dagger \\
\hline
0 & 0 & 0 & L_2' & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & -L_2'^T & 0 & 0 & 0 & \cdots & 0 & 0 & \dagger \\
0 & 0 & 0 & 0 & 0 & L_3' & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -L_3'^T & 0 & \cdots & 0 & 0 & \dagger \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & L_r' & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & -L_r'^T & 0 & \dagger \\
0 & \star & 0 & \star & 0 & \star & \cdots & 0 & \star & N
\end{array}\right], \tag{3.38}$$

where $L_i'(\rho,\sigma) = L_i(\lambda,\mu)$ is the block obtained from L_i by a linear change of coordinates

$$\begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \rho \\ \sigma \end{pmatrix}, \tag{3.39}$$

with

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Gl(2,\mathbb{C}).$$

Note the blocks $\star$ and $\dagger$ involve linear terms in ρ and σ.

For the following argument it is convenient to denote such a block $L_i'(\rho,\sigma)$ by $A_{(\rho,\sigma)}$, where

$$A = \left.\left[\underbrace{\begin{matrix} \mu & 0 & \cdots & 0 \\ \lambda & \mu & \ddots & \vdots \\ 0 & \lambda & \ddots & 0 \\ \vdots & \ddots & \ddots & \mu \\ 0 & \cdots & 0 & \lambda \end{matrix}}_{k_i}\right]\right\} k_i + 1,$$

corresponds to $L_i(\lambda,\mu)$.

It is required that such a block $A_{(\rho,\sigma)}$ be reduced to a block of the form

$$B = \left.\left[\underbrace{\begin{matrix} \sigma & 0 & \cdots & 0 \\ \rho & \sigma & \ddots & \vdots \\ 0 & \rho & \ddots & 0 \\ \vdots & \ddots & \ddots & \sigma \\ 0 & \cdots & 0 & \rho \end{matrix}}_{k_i}\right]\right\} k_i + 1.$$

To do this we need to demonstrate the existence of invertible constant matrices P, Q such that

$$QA_{(\rho,\sigma)}P = B. \tag{3.40}$$

The approach adopted here makes direct use of some of the results in Chapter IX of [TurnAit] concerning the reduction of general (singular) pencils. In particular, for brevity we state the key result without proof. (Note : in [TurnAit] the coordinates λ, μ are the other way round.)

Before the change of basis in (3.39), it can be easily verified that the vector

$$v = [\lambda^{k_i}, -\lambda^{k_i-1}\mu, \ldots, (-1)^{k_i}\mu^{k_i}]$$

kills A, i.e

$$vA = 0,$$

where k_i is the minimal order of A. So after the change of basis (3.39) it follows that

$$v_{(\rho,\sigma)}A_{(\rho,\sigma)} = 0, \tag{3.41}$$

where $v_{(\rho,\sigma)}$ is the vector obtained from v by substituting for λ, μ using (3.39).

Let w be the nullifying vector

$$w = [\rho^{k_i}, -\rho^{k_i-1}\sigma, \ldots, (-1)^{k_i}\sigma^{k_i}],$$

of B.

Note that $g \in Gl(2, \mathbb{C})$, in (3.39), gives rise to an invertible linear map

$$g : V_{k_i+1} \longrightarrow V_{k_i+1},$$

where V_{k_i+1} is the space of binary forms of degree k_i, the action of which is given by

$$g.f = f(g^{-1}\mathbf{x}),$$

with $f \in V_{k_i+1}$ and $\mathbf{x} \in \mathbb{C}^2$. Clearly,

$$\{\lambda^{k_i}, -\lambda^{k_i-1}\mu, \ldots, (-1)^{k_i}\mu^{k_i}\}, \qquad \{\rho^{k_i}, -\rho^{k_i-1}\sigma, \ldots, (-1)^{k_i}\sigma^{k_i}\}$$

are both bases for V_{k_i+1} and therefore this linear map is determined by the relation

$$v_{(\rho,\sigma)} = wR^{-1}, \tag{3.42}$$

where R is an invertible $(k_i + 1) \times (k_i + 1)$ constant matrix.

So from (3.41) we can write

$$(v_{(\rho,\sigma)}R)R^{-1}A_{(\rho,\sigma)} = 0,$$

and by (3.42)

$$w(R^{-1}A_{(\rho,\sigma)}) = 0, \tag{3.43}$$

where $R^{-1}A_{(\rho,\sigma)}$ is some pencil in ρ, σ.

Notice, by Lemma 3.0.20, that the minimal order (of row dependence) of A is preserved through both the change of basis, (3.41) and the equivalent transformation of $A_{(\rho,\sigma)}$ to $R^{-1}A_{(\rho,\sigma)}$.

So $R^{-1}A_{(\rho,\sigma)}$ has minimal order k_i and (3.43) is a *minimal relation.*

We quote the following lemma, albeit with notation modified for the present case.

Lemma 3.3.3 *Let Λ be an $n' \times n$ pencil with minimal order of row dependence of degree $k_i < \min\{n', n\}$. There exist invertible constant matrices P_0 and Q_0, over K, which reduce a minimal relation $u\Lambda = 0$ to the form*

$$uQ_0^{-1}Q_0\Lambda P_0 = \omega_{k_i} \begin{bmatrix} B & 0 \\ \rho S & \Lambda_0 \end{bmatrix} = 0,$$

where $\omega_{k_i} = [w, \underbrace{0, \ldots, 0}_{n'-k_i-1}]$, Λ_0 is a pencil with $k_i + 1$ fewer rows and k_i fewer columns than Λ, and S is a submatrix with constant elements. If $k_i = 0$ the B and ρS are non-existent.

Proof See Lemma I of Chapter IX in [TurnAit]. □

Remark 3.3.4 From Lemma 3.0.20 the equivalent pencil $Q_0\Lambda P_0$ also has minimal degree k_i. Also note that the nullifying vector uQ_0^{-1} is obtained from u by successive column operations.

Applying this lemma to (3.43) it follows that

$$wQ_0^{-1}Q_0(R^{-1}A_{(\rho,\sigma)})P_0 = wB = 0, \tag{3.44}$$

where B is as defined above. This is the special case when $\omega_{k_i} = w$.

From (3.44) and Remark 3.3.4 we deduce that $wQ_0^{-1} = w$ and

$$B = \underbrace{(Q_0R^{-1})}_{Q}A_{(\rho,\sigma)}\underbrace{P_0}_{P},$$

as required in (3.40).

So it follows that for each block $L_i'(\rho,\sigma)$ we can perform a series of row and column operations which reduce it to something of the form

$$L_i(\rho,\sigma) = \left[\begin{array}{c} \underbrace{\begin{matrix} \sigma & 0 & \cdots & 0 \\ \rho & \sigma & \ddots & \vdots \\ 0 & \rho & \ddots & 0 \\ \vdots & \ddots & \ddots & \sigma \\ 0 & \cdots & 0 & \rho \end{matrix}}_{k_i} \end{array}\right] \left.\vphantom{\begin{matrix} \sigma \\ \rho \\ 0 \\ \vdots \\ 0 \\ x \end{matrix}}\right\} k_i + 1.$$

Returning to the pencil (3.38), for each block

$$\begin{bmatrix} 0 & L_i' \\ -L_i'^T & 0 \end{bmatrix}, \tag{3.45}$$

we perform the sequence of ***simultaneous*** row and column operations involving the successive row/column operations required to reduce $L_i'(\rho,\sigma)$ to $L_i(\rho,\sigma)$ (and thereby also reducing $-L_i'^T(\rho,\sigma)$ to $-L_i^T(\rho,\sigma)$). In practice after each row/column operation required to reduce the block $L_i'(\rho,\sigma)$ we preceed it, immediately, by its counterpart column/row operation, which reduces $-L_i'^T(\rho,\sigma)$.

Notice, from (3.38), these operations (to be more specific the column operations on blocks $L_i'(\rho,\sigma)$ and counterpart row operations on blocks $-L_i'^T(\rho,\sigma)$ only alter, as regards the rest of the pencil, the non-zero blocks, involving linear terms in ρ and σ, in its last $2u$ rows and columns.

If we denote the non-zero block $\dagger$ aligned with $-L_i^T(\rho,\sigma)$ by G_i and its negative transpose $\star$, aligned below $L_i(\rho,\sigma)$, by $-G_i'^T$ then, using a similar method to that discussed in the proof of Lemma 3.2.7 we can use the σ terms in the blocks $L_i(\rho,\sigma)$ and $-L_i'^T(\rho,\sigma)$ to kill off all the σ terms in G_i and $-G_i^T$.

Briefly, for any block $-L_i^T(\rho,\sigma)$, for each σ term in the k_i rows of $-L_i^T(\rho,\sigma)$ we can, by means of a sequence of column operations, kill off all σ terms in the corresponding row of G_i. These will also introduce constant multiples of ρ into these rows. Furthermore by applying the sequence of counterpart row operations we can use the σ terms in the k_i columns of $L_i^T(\rho,\sigma)$ to kill off all σ terms in the corresponding columns of $-G_i^T$. These simultaneous row and column operations don't change the non-singular pencil $\bar{N}$ since the last $2u$ entries of the columns and rows involved are all zeros.

Doing this for each block (3.45) we obtain the following skew-equivalent pencil.

$$\left[\begin{array}{cc|cc|cc|c|cc|c}
0 & L_1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-L_1^T & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \dagger \\
\hline
0 & 0 & 0 & L_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & -L_2^T & 0 & 0 & 0 & \cdots & 0 & 0 & \dagger \\
\hline
0 & 0 & 0 & 0 & 0 & L_3 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -L_3^T & 0 & \cdots & 0 & 0 & \dagger \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & L_r & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & -L_r^T & 0 & \dagger \\
\hline
0 & \star & 0 & \star & 0 & \star & \cdots & 0 & \star & \bar{N}
\end{array}\right], \tag{3.46}$$

where all parameters are in ρ, σ and each non-zero entry in $\star$ and $\dagger$ are constant multiples of ρ.

As the reader can see we have reduced the pencil, (3.18) of case (iii), to a form where we can employ a similar method, to those used in cases (i) and (ii), to kill off all the non-zero blocks, $\star$ and $\dagger$. We discuss this method presently.

3.3.5 Conclusion of the Reduction

We consider clearing all the entries of blocks, $\star$. Such blocks have $2u$ rows and k_i columns and are aligned with L_i above and $\bar{N}$ to the right. Recall from above that, with one exception, $\bar{N}$ is a direct sum of blocks of the type

$$\underbrace{\left[\begin{array}{cc|cc|c|cc}
0 & e_i\sigma+\rho & & & & & \\
-e_i\sigma-\rho & 0 & \rho & & & & \\ \hline
 & -\rho & 0 & e_i\sigma+\rho & & & \\
 & & -e_i\sigma-\rho & 0 & & & \\ \hline
 & & & & \ddots & & \\
 & & & & & \rho & \\ \hline
 & & & & -\rho & 0 & e_i\sigma+\rho \\
 & & & & & -e_i\sigma-\rho & 0
\end{array}\right]}_{2r_{ij}}, \tag{3.47}$$

where $e_i \neq 0$.

Each block $\star$ is divided into subblocks aligned with an elementary block of $\bar{N}$. Then using each of these elementary blocks, along with the block $L_i(\rho,\sigma)$ above, we can systematically kill off the entries in $\star$.

Reduction type 1

For example consider the case of the block $\star$ aligned with a block $L_1(\rho,\sigma)$ and an elementary block of type (3.47) of size r_{ij}. For illustrative purposes we represent this by the matrix below

$$\begin{array}{ccc|cc|cc|cc}
\sigma & & & & & & & & \\
\rho & \sigma & & & & & & & \\
 & \rho & \sigma & & & & & & \\
 & & \rho & & & & & & \\ \hline
\star & \star & \star & 0 & e_i\sigma+\rho & & & & \\
\star & \star & \star & -e_i\sigma-\rho & 0 & \rho & & & \\ \cline{4-9}
\star & \star & \star & & -\rho & 0 & e_i\sigma+\rho & & \\
\star & \star & \star & & & -e_i\sigma-\rho & 0 & \rho & \\ \cline{4-9}
\star & \star & \star & & & & -\rho & 0 & e_i\sigma+\rho \\
\star & \star & \star & & & & & -e_i\sigma-\rho & 0
\end{array},$$

where $e_i \neq 0$ and each non-zero entry $\star$ represents a term $\alpha\rho$.

Here we apply a similar rule to that used in case (ii) above, provided that each element $\star$ of the first column is treated before those of the second, and each of the second before those of the third, and so on.

Starting with the first column, we proceed downwards killing off each entry, $\star$, by means of a row operation which uses the ρ term above it in $L_1(\rho,\sigma)$. This will introduce a σ term into the adjacent column to the right. By a suitable column operation we can use the $\pm(e_i\sigma+\rho)$ term in the row in which this introduced σ term occurs to kill it off, thereby introducing further ρ terms into this column, hence the need for working column by column from left to right.

By such a pair of row and column operations we can kill off each and every element in the first column. For example, by the pair of operations $R_7-\alpha R_2$ followed by $C_2-\frac{\alpha}{e_i}C_7$ we kill off the term $\alpha\rho$ in position $(7,1)$ of the above matrix, however we also introduce terms $\pm(\alpha/e_i)\rho$ into positions $(7,2)$ and $(9,2)$ respectively.

Having killed off each entry in the first column, by the same method we kill off each entry in the second column, then proceed to the next column and so on. Eventually, we can delete all entries in the final column, in the above case the third column, by a series of row operations using the ρ in the final column of $L_1(\rho,\sigma)$.

Notice, that for this method to work we require $e_i\neq 0$ which as mentioned above is the case, in general. However there is one exceptional type of constituent block of $\bar{N}$ for which this doesn't hold namely the blocks, (3.33), corresponding to eigenvalues $(0:1)$ of the pair $(\bar{N}_1,\bar{N}_2)$. We must therefore treat these blocks differently.

Reduction type 2

Represent such a situation by a matrix of the form :

$$
\begin{array}{ccc|cc|cc|cc}
\sigma & & & & & & & & \\
\rho & \sigma & & & & & & & \\
 & \rho & \sigma & & & & & & \\
 & & \rho & & & & & & \\
\hline
\star & \star & \star & 0 & \rho & & & & \\
\star & \star & \star & -\rho & 0 & \sigma & & & \\
\star & \star & \star & & -\sigma & 0 & \rho & & \\
\star & \star & \star & & & -\rho & 0 & \sigma & \\
\star & \star & \star & & & & -\sigma & 0 & \rho \\
\star & \star & \star & & & & & -\rho & 0,
\end{array}
$$

where, as before, each non-zero element $\star$ is a constant multiple of ρ. Again let $L_1(\rho, \sigma)$ be the block above the $\star$ block.

Here, in contrast to the previous method, to kill off the entries $\star$ we need to work, column by column, *from right to left.*

So starting with the final column, in our case the third column, by a series of row operations, we can first kill off all the entries in this column using the ρ term in the final column of $L_1(\rho, \sigma)$.

We then consider the adjacent column to the left. Proceeding downwards, we can kill off each term $\star$ in this column, by means of a suitable column operation using the ρ term, of the non-singular block, occurring in the same row as this entry. This, in general, introduces a σ term into this column which can be killed off by a row operation using the σ in the same column of $L_1(\rho, \sigma)$. Note that this row operation will also introduce a ρ term into the adjacent column to the left, hence the need for working from right to left.

In this way we can kill off all the entries in the second column. For example, by the pair of operations $C_2 - \alpha C_5$ followed by $R_7 + \alpha R_2$ we can kill off the term $\alpha\rho$ in position $(5, 2)$ of the above matrix. However we also introduce an $\alpha\rho$ term into position $(7, 1)$.

Having killed off all the entries in a column we carry out the same method on the adjacent column to the left and so on until we reach the first column.

In the first column, after deleting an entry by a column operation of the type just described, any subsequently introduced σ terms can simply be killed off by row operations using the σ in the first row of $L_1(\rho, \sigma)$. Hence by proceeding downwards we can delete every entry in the first column.

In summary we have just shown that given any non-zero block $\star$ aligned with a block $L_i(\rho, \sigma)$ above it and any elementary constituent block of $\bar{N}$ to its right then by a series of row and column operations we can kill off all the entries in $\star$. We are ready to conclude the reduction with the following result.

Lemma 3.3.5 *Consider a skew-symmetric pencil of the form given in (3.46)*

$$\left[\begin{array}{cc|cc|cc|c|cc|c}
0 & L_1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-L_1^T & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \dagger \\
\hline
0 & 0 & 0 & L_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & -L_2^T & 0 & 0 & 0 & \cdots & 0 & 0 & \dagger \\
0 & 0 & 0 & 0 & 0 & L_3 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -L_3^T & 0 & \cdots & 0 & 0 & \dagger \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & L_r & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & -L_r^T & 0 & \dagger \\
0 & \star & 0 & \star & 0 & \star & \cdots & 0 & \star & N
\end{array}\right],$$

(where all parameters are in ρ, σ, each non-zero entry in $\star$ and $\dagger$ are constant multiples of ρ and $\bar{N}$ is the direct sum of the q blocks A_1, A_2, A_3, $(B_i : 4 \le i \le q)$ consisting of blocks (3.31), (3.33), (3.35) and (3.37) respectively).

Then this is skew-equivalent to the skew-symmetric pencil

$$\left[\begin{array}{cc|cc|cc|c|cc|c}
0 & L_1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-L_1^T & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & L_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & -L_2^T & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & L_3 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -L_3^T & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & L_r & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & -L_r^T & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & N
\end{array}\right]. \tag{3.48}$$

Proof The proof is by induction.

Consider a general block containing $L_k(\rho, \sigma)$, $1 \le k \le r$. Assume that we can kill off all the preceding blocks $\star$ (and $\dagger$) in the last $2u$ rows (columns), and that any subsequent non-zero entries introduced can also be killed off. Then we would have a matrix of the form:

$$
\left[\begin{array}{cc|c|cc|c|cc|ccc}
0 & L_1 & & & & 0 & & & 0 & \cdots & 0 \\
-L_1^T & 0 & & & & & & & 0 & \cdots & 0 \\
\hline
 & & \ddots & & & & & & & \vdots & \\
\hline
 & & & 0 & L_k & & & & 0 & \cdots & 0 \\
 & & & -L_k^T & 0 & & & & \ddagger & \cdots & \ddagger \\
\hline
 & 0 & & & & \ddots & & & & \vdots & \\
\hline
 & & & & & & 0 & L_r & 0 & \cdots & 0 \\
 & & & & & & -L_r^T & 0 & \dagger & \cdots & \dagger \\
\hline
0 & 0 & & 0 & \bullet & & 0 & \star & S_{11} & & \\
\vdots & \vdots & & \vdots & \vdots & \cdots & \vdots & \vdots & & \ddots & \\
0 & 0 & & 0 & \bullet & & 0 & \star & & & S_{qn(q)}
\end{array}\right], \quad (3.49)
$$

where the blocks whose entries we are describing how to clear are denoted $\bullet$ and $\ddagger$.

As described above we have subdivided the block $\bullet$ and the corresponding block $\ddagger$ to match up with the elementary blocks of $\bar{N}$. We can use the above methods (Reduction types 1 or 2) to kill off each of these subblocks, remembering that after each operation we must perform the corresponding counterpart operation immediately afterwards to preserve skew-symmetry. These counterpart operations contribute to clearing the corresponding subblock of $\ddagger$.

So progressing down the divided block $\bullet$ we can in turn kill off each of these subblocks. We make a few remarks on how this affects the rest of the pencil.

Firstly, any row operations, using L_k, (and counterpart column operations using $-L_k^T$) do not affect the rest of the matrix, only the subblock we are trying to delete.

Alternatively, any column operations, using an elementary block of $\bar{N}$, (and counterpart row operations using the same elementary block) introduce ρ terms, from the remaining non-zero blocks, $\dagger$ (and $\star$) into appropriate blocks in the region directly below L_k (and to the right of $-L_k^T$). Furthermore such operations also introduce a skew-symmetric pencil, of ρ terms only, into the (formerly zero) block aligned with L_k and $-L_k^T$. It is also worth pointing out that these column operations (and counterpart row operations) leave the blocks L_k and $-L_k^T$, themselves, unchanged.

It can also be seen that as we progress down the block, •, using in turn each of the elementary blocks to kill one of its subblocks , that we are continually changing the ρ terms introduced in these regions. So having killed off all the entries in the blocks • and ‡, by simultaneous row and column operations, it remains to kill off these introduced ρ terms. Our matrix (3.49) is therefore skew-equivalent to

$$\left[\begin{array}{cc|c|cc|cc|c|cc|c}
0 & L_1 & & & & & & & & & \\
-L_1^T & 0 & & & & & & & & & 0 \\
 & & \ddots & & & & & & & & \\
 & & & 0 & L_k & & & & & & 0 \\
 & & & -L_k^T & \bar{R}_0 & 0 & \bar{R}_1 & \cdots & 0 & \bar{R}_{r-k} & 0 \\
 & & & & 0 & 0 & L_{k+1} & & 0 & 0 & 0 \\
 & & & & -\bar{R}_1^T & -L_{k+1}^T & 0 & & 0 & 0 & R_{k+1} \\
 & & & & \vdots & & & \ddots & & & \vdots \\
 & & & & 0 & 0 & 0 & \cdots & 0 & L_r & 0 \\
 & & & & -\bar{R}_{r-k}^T & 0 & 0 & \cdots & -L_r^T & 0 & R_r \\
 & 0 & & 0 & 0 & 0 & -R_{k+1}^T & \cdots & 0 & -R_r^T & \bar{N}
\end{array}\right], \tag{3.50}$$

where for $0 \leq i \leq r-k$ the matrices R_i, $\bar{R}_i$ consist of contant multiples of ρ and $\bar{R}_0$ is skew-symmetric.

The reader may recognise, that to kill off blocks $\bar{R}_i$ (and $-\bar{R}_i^T$) where $i \neq 0$ we can just apply the method described above for case (ii) (Section 3.3.2), with λ and μ replaced by ρ and σ respectively. Furthermore, here we also discussed a method for killing off a skew-symmetric, matrix of one parameter, aligned with blocks L_i and $-L_i^T$ and we can apply this method, replacing λ by ρ, to kill off the block $\bar{R}_0$ aligned with L_k and $-L_k^T$. Notice that for both of these methods the simultaneous row and column operations used do not further affect the rest of the matrix as the remainder of the row (column) blocks containing L_i ($-L_i^T$) consist entirely of zero blocks.

So it has been shown that given the kth block, L_k, and assuming that, apart from the L_i blocks ($1 \leq i \leq k-1$), all previous blocks are zero then we can also kill off the non-zero blocks, aligned with $\bar{N}$, below L_k and to the right of $-L_k^T$.

Consider the pencil given in (3.46). It follows that this method can be initially used, to delete from (3.46) all the non-zero elements in its last $2u$ rows and columns, aligned with blocks L_1 and $-L_1^T$. Proceeding in this way from left to right the result follows by induction. □

Matrices of the type given in (3.48) where $\bar{N}$ is possibly non-existent are *normal forms* for a singular skew-symmetric pencil, under skew-equivalence.

Note at this final stage we could, for simplicity, replace the parameters ρ, σ by λ, μ, although obviously they are different to those used above. We conclude this chapter, and the first half of this thesis, with a demonstration of how such normal forms of skew-symmetric matrices are found in practice.

3.4 Deriving Normal Forms for Skew-symmetric Pencils

As we have just demonstrated the normal form for a pair of skew-symmetric matrices under the action of $Gl(2,\mathbb{C}) \times Gl(n,\mathbb{C})$ is

$$\left[\begin{array}{cc|cc|cc|c|cc|c}
0 & L_1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
-L_1^T & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & L_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & -L_2^T & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & L_3 & \cdots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -L_3^T & 0 & \cdots & 0 & 0 & 0 \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & L_s & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & -L_s^T & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & N
\end{array}\right], \tag{3.51}$$

where, for $1 \leq i \leq s$,

$$L_i(\rho,\sigma) = \left.\left[\underbrace{\begin{array}{cccc}
\sigma & 0 & \cdots & 0 \\
\rho & \sigma & \ddots & \vdots \\
0 & \rho & \ddots & 0 \\
\vdots & \ddots & \ddots & \sigma \\
0 & \cdots & 0 & \rho
\end{array}}_{k_i}\right]\right\} k_i + 1$$

and

$$\bar{N} = \rho\bar{N}_1 + \sigma\bar{N}_2 = \begin{bmatrix} A_1 & & & & & 0 \\ & A_2 & & & & \\ & & A_3 & & & \\ & & & B_4 & & \\ & & & & \ddots & \\ 0 & & & & & B_q \end{bmatrix}.$$

Notes :

(i) Here the notation of the number of singular blocks is changed from r to s, so as not to confuse this number with the r_{ij}'s.

(ii) The $A_j|B_j$ notation used for the constituent blocks of the non-singular part $\bar{N}$ is that previously used in Section 3.3.3.

Given some value of n, we describe a general algorithm for listing all distinct normal forms of a pair of $n \times n$ skew-symmetric matrices, over $\mathbb{C}$, under this action.

Consider, first, the singular part of the above normal forms. Each block

$$\begin{bmatrix} 0 & L_i \\ -L_i^T & 0 \end{bmatrix}$$

has size $m_i = 2k_i + 1$ i.e is *odd.* Let

$$\sum_{i=1}^{s} m_i = L.$$

So to determine the sizes of the singular blocks in a normal form we need a partition of L into odd integers, m_i.

The non-singular part has q *distinct* eigenvalues (of which we have fixed the first three). To each eigenvalue, $1 \leq j \leq q$, we associate a positive integer

$$R_j = \sum_{k=1}^{n(j)} r_{jk}.$$

Hence $2R_j$ is the size of the block $A_j|B_j$ with eigenvalue j.

Since there are q eigenvalues, letting

$$U = \sum_{j=1}^{q} R_j,$$

it follows that the size of the non-singular part of a normal form is $2U$. Therefore writing

$$n = L + 2U,$$

we consider all possible non-negative integer pairs (L, U).

The normal forms arising from each pair, (L, U), are determined by the sizes of their constituent singular and non-singular blocks which are found by suitable partitions of L and U respectively. As mentioned above, the sizes of singular blocks in each normal form corresponding to a pair (L, U) are given by one of the partitions of L into odd integers.

For each of these possibilities we also have to determine the sizes of the non-singular blocks. To do this we doubly partition U. In each case the parts resulting from the first partition give the positive integers denoted by R_j above, their number being the number of distinct eigenvalues, q, of the normal form and their size determining that of the blocks corresponding to each of these eigenvalues. Then to determine how each distinct eigenvalue block (denoted by $A_j|B_j$ above) is constructed we partition each of the parts R_j into $n(j)$ parts:

$$R_j = \sum_{k=1}^{n(j)} r_{jk},$$

where $n(j)$ is the number of elementary subblocks of $A_j|B_j$ and r_{jk} $(1 \leq k \leq n(j))$ their sizes.

This algorithm is best demonstrated by an example, which we use as the starting point for calculations in Chapter 6

Example 3.4.1 Find the list of all normal forms of pairs of 4×4 skew-symmetric matrices, over $\mathbb{C}$, under the group action $Gl(2, \mathbb{C}) \times Gl(4, \mathbb{C})$.

This is the case $n = 4$. So we write

$$4 = L + 2U,$$

and starting with $U = 0$ we list all non-negative integer pairs (L, U):

$$(4,0), \qquad (2,1), \qquad (0,2).$$

We consider each of these in turn

(i) Starting with $(4,0)$ we first partition $L = 4$ into odd integers i.e.

$$4 = 3 + 1 = 1 + 1 + 1 + 1.$$

Since $U = 0$ these normal forms have no non-singular part and we can represent the two types arising from $(4,0)$ by

$$(3,1;-), \qquad (1,1,1,1;-). \tag{3.52}$$

(ii) For the second pair, $(2,1)$, $L = 2$ has only one partition into odd integers:

$$2 = 1 + 1.$$

Furthermore, doubly partitioning $U = 1$ just gives the trivial partition and consequently there is a single type

$$(1,1;(1))$$

arising from the pair $(2,1)$. Note we distinguish the parts of the first partition, of U, by enclosing them in parentheses, each one representing a distinct eigenvalue of the non-singular part of the corresponding normal form.

(iii) The third pair, $(0,2)$, has no singular part. However $U = 2$ has two initial partitions:

$$2 \quad \text{and} \quad 1 + 1.$$

We take each partition in turn and partition its parts. So, for example, for the initial trivial partition, 2, we have two possibilities:

$$(2) \quad \text{and} \quad (11).$$

Furthermore for the second partition, $1 + 1$, the only partition of each part is the trivial partition and we have the further possibility:

$$(1),(1).$$

In summary the third pair, $(0,2)$, gives the three types:

$$(-;(2)), \qquad (-;(11)), \qquad (-;(1),(1)).$$

It is clearer from this example how, for a given type, each pair of parentheses enclose the structure of a distinct eigenvalue block.

Notice the last three normal forms are those of non-singular pairs of skew-symmetric matrices. There are, in total, six possible types of normal forms of pairs of 4×4 skew-symmetric matrices and we conclude this example by converting these into matrix form.

(1) Normal form (3,1;-). This has two singular blocks of sizes

$$m_1 = 2k_1 + 1 = 3,$$

$$m_2 = 2k_2 + 1 = 1,$$

respectively. It follows that $k_1 = 1$ and, using the above notation,

$$L_1 = \begin{bmatrix} x \\ y \end{bmatrix}.$$

We have replaced the labels σ and ρ by x and y, respectively, in anticipation of the work in the following chapters.

Since $k_2 = 0$ there is no block L_2 and the second singular block is a single zero, i.e we have a final row and column of zeros. So the corresponding normal form for $(3, 1; -)$ is

$$\left[\begin{array}{ccc|c} 0 & 0 & x & 0 \\ 0 & 0 & y & 0 \\ -x & -y & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]. \tag{3.53}$$

(2) It follows that the normal form, $(1, 1, 1, 1; -)$, represents a matrix whose four singular blocks are all single zeros. In other words we have the null matrix :

$$\left[\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]. \tag{3.54}$$

(3) Normal form $(1, 1; (1))$. The singular part gives two rows and columns of zeros. The non-singular part has a single eigenvalue, $(1 : 0)$, and consists of the block, of size 1 :

$$\begin{bmatrix} 0 & x \\ -x & 0 \end{bmatrix}.$$

The corresponding normal form is therefore

$$\left[\begin{array}{cc|c|c} 0 & x & 0 & 0 \\ -x & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right]. \tag{3.55}$$

The remaining normal forms are non-singular.

(4) Normal form $(-;(2))$. This has one eigenvalue, $(1:0)$, and consists of a single block of size 2, i.e is of the form :

$$\begin{bmatrix} 0 & x & 0 & 0 \\ -x & 0 & y & 0 \\ 0 & -y & 0 & x \\ 0 & 0 & -x & 0 \end{bmatrix}. \tag{3.56}$$

(5) Normal form $(-;(11))$. This also has the single eigenvalue $(1:0)$ but consists of two blocks of size 1 :

$$\left[\begin{array}{cc|cc} 0 & x & 0 & 0 \\ -x & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & x \\ 0 & 0 & -x & 0 \end{array}\right]. \tag{3.57}$$

(6) Finally the normal form $(-:(1),(1))$ has two distinct eigenvalues and consists of the two blocks of size 1 representing each of these eigenvalues, $(1:0)$ and $(0:1)$:

$$\left[\begin{array}{cc|cc} 0 & x & 0 & 0 \\ -x & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & y \\ 0 & 0 & -y & 0 \end{array}\right]. \tag{3.58}$$

3.4.1 Geometrical Interpretation

Any pair of 4×4 skew-symmetric matrices is therefore skew-equivalent to one of the six normal forms derived in Example 3.4.1. We can interpret each such pair as a linear map $A:\mathbb{C}^2,0\to Sk(4,\mathbb{C})$. With this in mind we state the following definition.

Definition 3.4.2 *Given a linear map $A:\mathbb{C}^r,0\to Sk(n,\mathbb{C})$ determined by an r-tuple of skew-symmetric matrices $(A_1,\ldots,A_r)$ the* **jetrank** *of A is defined to* *be the dimension of its image in $Sk(n,\mathbb{C})$ (i.e. the usual rank of this map). The jetrank of $A = x_1A_1 + \cdots + x_rA_r$ is therefore determined by the number of independent matrices in the set $\{A_1, \ldots, A_r\}$.*

We can thus interpret the action of $Gl(r,\mathbb{C}) \times Gl(n,\mathbb{C})$ on r-tuples of skew-symmetric matrices by the standard action of $Gl(n,\mathbb{C})$ on their images in $Sk(n,\mathbb{C})$. This is discussed further in Chapters 6 and 7.

Let P be a hypersurface of $Sk(n,\mathbb{C})$ given by the vanishing of the Pfaffian. Investigating how the images of these linear systems meet P is of some geometrical interest.

Returning to *pairs* of skew-symmetric matrices it follows that they have jetrank ≤ 2 and we can divide the six normal forms of Example 3.4.1 according to their jetrank.

The three normal forms in (3.54), (3.55) and (3.57) represent pairs with jetrank ≤ 1. Clearly, these could simply have been derived from the action of $Gl(4,\mathbb{C})$ on $Sk(4,\mathbb{C})$.

The remaining three normal forms have maximal jetrank, 2, and are normal forms for *pencils* of skew-symmetric matrices. Given a pencil

$$A = xA_1 + yA_2,$$

its Pfaffian is a quadratic

$$Pf(A) = ax^2 + bxy + cy^2.$$

Recall from Chapter 1 that the 'eigenvalues' of a pencil A are given by the roots of its Pfaffian $Pf(A)$. The three normal forms for pencils, A, are distinguished by the nature of their eigenvalues, which give points of the source over which the image of A meets P.

In particular the three possibilities are :

(i) two distinct eigenvalues ; the image of the pencil A meets P over a pair of distinct lines in the source. (Normal form (3.58).)

(ii) a repeated eigenvalue ; the image of A meets P over a repeated line in the source. (Normal form (3.56).)

(iii) $Pf(A)$ vanishes identically on the source and (3.53) gives the normal form for a singular pencil.

We sometimes refer to pencils of type (i) as *non-degenerate* pencils. (Non-degenerate skew pencils are skew pencils with $n/2$ distinct eigenvalues.)

Clearly, this is a relatively simple classification and the corresponding stratifications of pencils of $n \times n$ skew-symmetric matrices will be richer for higher values of n.

Chapter 4

Classification

4.1 Introduction

In this chapter we introduce theory required for the classification of smooth families of skew-symmetric matrices. This classification is motivated by that carried out, in [BrTarSy], on families of *symmetric* matrices. The group action introduced there is also suitable for classifying families of skew-symmetric matrices and since this space is smaller we expect a richer classification. The starting point for any classification is establishing a set of normal forms for 1-jets of these families, up to this equivalence. In the previous two chapters we have dealt with this for the two parameter case.

We start with a few general results about smooth maps taken from [BrGibl] and [Gibson].

Definition 4.1.1 *(i) Given smooth manifolds $X^n \subset \mathbb{C}^{n+r}$, $Y^p \subset \mathbb{C}^{p+s}$ a map $f : X \to Y$ is* **smooth** *if for every $x \in X$ there is an open neighbourhood U of x in $\mathbb{C}^{n+r}$ and a smooth map $F : U \to \mathbb{C}^{p+s}$ with $F|X \cap U = f$.*

(ii) Given f as above, and denoting the tangent spaces to X at x and Y at $f(x)$ by X_x and $Y_{f(x)}$ respectively, the **tangent map** *$Tf_x : X_x \to Y_{f(x)}$ is defined by*

$$Tf_x(u_x) = dF_x(u)_{f(x)},$$

where dF_x is the derivative of F at x.

We state a couple of results concerning transversality.

Definition 4.1.2 *If, for some smooth map $f : X^n \to Y^p$ and some smooth sub-manifold $Q^q \subset Y^p$, the condition*

$$im(Tf_x) + Q_{f(x)} = Y_{f(x)}, \tag{4.1}$$

holds for all $x \in f^{-1}(Q)$ we say that f is **transverse** *to Q.*

Proposition 4.1.3 *Let $f : X^n \to Y^p$ be a smooth mapping and Q^q a smooth submanifold of Y with f transverse to Q. Then $M = f^{-1}(Q)$ is a smooth submanifold of X which either has the same codimension as Q or is empty.*

Proof For a proof we refer to that given for result (1.2) in Chapter II of [Gibson]. □

4.2 Classifying Families of Skew-symmetric Matrices

Some notation.

In what follows $Gl(n, \mathbb{C})$ denotes the group of $n \times n$ invertible matrices over $\mathbb{C}$ and $Sk(n, \mathbb{C})$ the space of $n \times n$ skew-symmetric matrices over $\mathbb{C}$. We shall also write $\mathcal{O}_r$ for the ring of smooth function germs $\mathbb{C}^r, 0 \longrightarrow \mathbb{C}$, and $\mathcal{M}_r$ for its maximal ideal (consisting of functions vanishing at the origin).

We are classifying the set, Sk, of smooth germs

$$A : \mathbb{C}^r, 0 \longrightarrow Sk(n, \mathbb{C}).$$

Such germs can be thought of as r-parameter families of skew-symmetric matrices.

We wish to define the relevant group acting on the space Sk. (For the description of a group action see Definition 1.1.4.) We allow an $\mathcal{R}$ change of coordinates in the parameter space, where $\mathcal{R}$ is the group of diffeomorphisms $\mathbb{C}^r, 0 \longrightarrow \mathbb{C}^r, 0$, the group operation being map composition. In other words germs A, $B : \mathbb{C}^r, 0 \to Sk(n, \mathbb{C})$ will firstly be deemed equivalent if for some diffeomorphism $\phi : \mathbb{C}^r, 0 \to \mathbb{C}^r, 0$, $B = A \circ \phi$.

Furthermore, recall that the action of the group, $Gl(n, \mathbb{C})$, on elements $A \in Sk(n, \mathbb{C})$ given by

$$X.A = X^T AX,$$

$X \in Gl(n, \mathbb{C})$, gives a classification for $Sk(n, \mathbb{C})$.

So to classify *families* of such matrices we employ a parametrised version of this action. Let $\mathcal{H}$ denote the set of germs of smooth mappings $\mathbb{C}^r, 0 \longrightarrow Gl(n, \mathbb{C})$. This set can be given a group structure using the operation of matrix multiplication in the target. There is an obvious action of $\mathcal{H}$ on Sk by $X.A = X^T AX$.

Note that a germ $A : \mathbb{C}^r, 0 \longrightarrow Sk(n, \mathbb{C})$ is an $n \times n$ matrix defined over the integral domain (ID), $\mathcal{O}_r$. We can therefore use some of the definitions and results introduced at the beginning of Chapter 2. Clearly the *units* of $\mathcal{O}_r$ are the germs which do not vanish at the origin. Moreover by Lemma 2.1.5 it follows that $\mathcal{H}$ consists of the *invertible* $n \times n$ matrices over $\mathcal{O}_r$.

Using the descriptions in Lemma 2.1.9 of elementary row and column operations and Definition 2.1.13, regarding simultaneous row and column operations, we can apply the result of Lemma 2.1.14 to the action of $\mathcal{H}$ on Sk and deduce the following result.

Corollary 4.2.1 *Two germs A, $B \in Sk$ are $\mathcal{H}$-equivalent if it is possible to pass from one to the other by a series of elementary simultaneous row and column operations.*

Proof The proof follows from Lemma 2.1.14 and its preamble, with $R = \mathcal{O}_r$. □

In the following to avoid confusion we will refer to such operations as parametrised simultaneous row and column operations.

We can also use earlier work in Section 2.2 to establish invariants of this group action. Recall, from Definition 2.2.1, $I_k(A)$ denotes the ideal of $\mathcal{O}_r$ generated by all the $k \times k$ minors of A.

Corollary 4.2.2 *If two germs A, $B \in \mathcal{S}k$ are $\mathcal{H}$-equivalent then*

$$I_k(A) = I_k(B),$$

for each $1 \leq k \leq n$.

Proof Setting $R = \mathcal{O}_r$, $\mathcal{H}$-equivalence is described in Definition 2.1.12 and is a special case of the equivalence given by Definition 2.1.6. The result then follows directly from Lemma 2.2.3. □

Consider the action of the (direct) product *set*, $\mathcal{G} = \mathcal{R} \times \mathcal{H}$, on $\mathcal{S}k$ which, for elements $g = (\phi, X) \in \mathcal{G}$ and $A \in \mathcal{S}k$, is defined by

$$g.A = (X^{-1})^T (A \circ \phi^{-1}) X^{-1}.$$

Composing the action of two elements, $g_1 = (\phi, X)$, $g_2 = (\psi, Y)$, of this set on $A \in \mathcal{S}k$ we have

$$\begin{aligned} g_1.(g_2.A) &= g_1.\left((Y^{-1})^T (A \circ \psi^{-1}) Y^{-1}\right) \\ &= (X^{-1})^T \left(\left((Y^{-1})^T (A \circ \psi^{-1}) Y^{-1}\right) \circ \phi^{-1}\right) X^{-1} \\ &= (X^{-1})^T \left((Y^{-1} \circ \phi^{-1})^T (A \circ \psi^{-1} \circ \phi^{-1}) (Y^{-1} \circ \phi^{-1}) \right) X^{-1} \\ &= \left((Y^{-1} \circ \phi^{-1}) X^{-1} \right)^T \left(A \circ (\phi \circ \psi)^{-1} \right) (Y^{-1} \circ \phi^{-1}) X^{-1}. \end{aligned}$$

Since $Y^{-1} \circ \phi^{-1} = \left(Y \circ \phi^{-1}\right)^{-1}$ it follows that

$$g_1.(g_2.A) = \left(X\left(Y \circ \phi^{-1}\right)\right)^{-1T} \left(A \circ (\phi \circ \psi)^{-1}\right) \left(X\left(Y \circ \phi^{-1}\right)\right)^{-1},$$

and $\mathcal{G}$ is a *group action*, as defined in Definition 1.1.4, provided we define the corresponding group operation by

$$g_1 g_2 = (\phi, X)(\psi, Y) = \left(\phi \circ \psi, X(Y \circ \phi^{-1})\right).$$

With this operation $\mathcal{G} = \mathcal{R} \times \mathcal{H}$ is a group with identity, (id, I_n), where id is the germ of the identity mapping, and each element (ϕ, X) has the inverse $(\phi^{-1}, X^{-1} \circ \phi)$. Clearly $\mathcal{G}$ is *not* a direct product of the *groups* $\mathcal{R}$ and $\mathcal{H}$. Instead we refer to $\mathcal{G} = \mathcal{R} \times \mathcal{H}$ as the *semi-direct* product of groups $\mathcal{R}$ and $\mathcal{H}$. Using the above action of this group on $\mathcal{S}k$ we have the following equivalence.

Definition 4.2.3 *If $A, B : \mathbb{C}^r, 0 \longrightarrow Sk(n, \mathbb{C})$ are smooth map germs we say that they are $\mathcal{G}$-equivalent if and only if for some $(\phi, X) \in \mathcal{G} = \mathcal{R} \times \mathcal{H}$ we have*

$$B = X^T(A \circ \phi)X. \tag{4.2}$$

Note that the $\mathcal{G}$-orbit of A is

$$\{X^T(A \circ \phi)X : (\phi, X) \in \mathcal{G}\}$$

and we can ignore the inverses necessary for (4.2) to conform to the definition of a group action. (See Definition 1.1.4.)

We can use the result of Corollary 4.2.2 to obtain an invariant of this action of $\mathcal{G}$.

Corollary 4.2.4 *If germs A, $B \in Sk$ are $\mathcal{G}$-equivalent then, for some ring isomorphism $\phi^* : \mathcal{O}_r \to \mathcal{O}_r$,*

$$\phi^*(I_k(A)) = I_k(B),$$

for each $1 \leq k \leq n$.

Proof If A, B are $\mathcal{G}$-equivalent then for some germ of a diffeomorphism, $\phi : \mathbb{C}^r, 0 \to \mathbb{C}^r, 0$, B and $A \circ \phi$ are $\mathcal{H}$-equivalent. So, if $\phi^* : \mathcal{O}_r \to \mathcal{O}_r$ is the corresponding ring isomorphism for ϕ, then by Corollary 4.2.2 the result follows. □

Since the dimension of $Sk(n, \mathbb{C})$ is $n(n-1)/2$, we can think of an element of Sk as a map $\mathbb{C}^r, 0 \longrightarrow \mathbb{C}^N$, where $N = n(n-1)/2$. With this observation, we give the following result, taken from [BrTarSy].

Lemma 4.2.5 *The group $\mathcal{G} = \mathcal{R} \times \mathcal{H}$ acts on the space of mappings $\mathbb{C}^r, 0 \longrightarrow \mathbb{C}^N$ as a subgroup of the corresponding contact group $\mathcal{K}$.*

Proof The action of the group $\mathcal{R}$ in both cases clearly coincides, and $\mathcal{C}$ is the group of mappings

$$\mathbb{C}^r, 0 \longrightarrow Gl(N, \mathbb{C}).$$

The action of $X \in Gl(n, \mathbb{C})$ on $A \in Sk(n, \mathbb{C})$:

$$X.A = X^T A X$$

is linear and invertible. Hence there is a natural group homomorphism

$$\alpha : Gl(n,\mathbb{C}) \longrightarrow Gl(N,\mathbb{C})$$

its image yielding an action on $Sk(n,\mathbb{C}) = \mathbb{C}^N$, which is a subgroup of $Gl(N,\mathbb{C})$.

Since $\mathcal{H}$ is the set of germs $\mathbb{C}^r, 0 \longrightarrow Gl(n,\mathbb{C})$ it follows that α gives a group homomorphism from $\mathcal{H}$ to a subgroup of $\mathcal{C}$. The result then follows. □

Given such an invertible matrix $X = (x_{ij}) \in \mathcal{H}$, where $x_{ij} : \mathbb{C}^r, 0 \to \mathbb{C}$ then the matrix $\tilde{X} \in \mathcal{C}$ is just the matrix representing the (parametrised) linear map $X^T A X$ with respect to the standard basis, $\{E^{ij}\}$, of $Sk(n,\mathbb{C})$.

This is obtained by finding $X^T E^{ij} X$ with respect to $\{E^{ij}\}$ for each $1 \le i < j \le n$. It can be shown , by considering the (lm)th element of $X^T E^{ij} X$, that

$$X^T E^{ij} X = \sum_{1 \le l < m \le n} (x_{il}x_{jm} - x_{jl}x_{im}) E^{lm}$$

and hence

$$\tilde{X}_{(lm)(ij)} = (x_{il}x_{jm} - x_{jl}x_{im}), \quad 1 \le i < j \le n, \ 1 \le l < m \le n,$$

where $(ij) = (i-1)n + j - i(i+1)/2$ (and $(lm) = (l-1)n + m - l(l+1)/2$).

In fact $\mathcal{G}$ is one of Damon's geometric subgroups of $\mathcal{K}$ (for the justification of this see Appendix B) and as a consequence of results of Damon we can use all of the standard techniques of singularity theory (for example those concerning determinacy) to investigate these singularities.

This result also yields a useful invariant of $\mathcal{G}$-equivalent germs.

Lemma 4.2.6 *If two germs A, $B \in Sk$ are $\mathcal{G}$-equivalent they have the same $\mathcal{K}$- and $\mathcal{K}_e$-codimension. (See [Gibson], Pg. 152)*

Proof If A, B are $\mathcal{G}$-equivalent, by Lemma 4.2.5, they are also $\mathcal{K}$-equivalent. Hence since $\mathcal{K}_e$-codimension is an invariant of $\mathcal{K}$-equivalence the result follows. □

To facilitate calculation of the $\mathcal{K}_e$-codimension of a germ we can use the action of the full $\mathcal{K}$ group to reduce it to something more managable. The following lemma provides a technique for doing this.

Lemma 4.2.7 *Two germs A, $B : \mathbb{C}^r, 0 \to \mathbb{C}^N, 0$ represented by N-tuples $\mathbf{a}^T = (a_1, \ldots, a_N)$, $\mathbf{b}^T = (b_1, \ldots, b_N)$, where a_i, $b_i \in \mathcal{M}_r$ $(i = 1, \ldots, N)$, are $\mathcal{K}$-equivalent if, by a series of elementary row operations, we can pass from column* $\mathbf{a}$ *to column* $\mathbf{b}$ *(up to a non-vanishing element of $\mathcal{O}_r$).*

Proof If by a series of elementary row operations we can pass from column $\mathbf{a}$ to column $\mathbf{b}$ then by Theorem 2.1.11, with $R = \mathcal{O}_r$, $Y = \tilde{X}$ and X some 1×1 matrix consisting of a unit $\alpha \in \mathcal{O}_r$, it follows that

$$\begin{aligned} \mathbf{b} &= \tilde{X}\mathbf{a}(\alpha) \\ &= \underbrace{(\alpha\tilde{X})}_{\det\neq 0}\mathbf{a}. \end{aligned}$$

Hence a, $b : \mathbb{C}^r, 0 \to \mathbb{C}^N, 0$ are $\mathcal{K}$-equivalent. □

In the following we discuss some further structure preserved by $\mathcal{G}$-equivalence.

Definition 4.2.8 *The* **discriminant** *of an element $A \in Sk$ is the set $\mathcal{D}(A) = \{x \in \mathbb{C}^r : \det A(x) = 0\}$.*

This invariant is only of use when considering families of skew-symmetric matrices of *even* order. By a paramerised version of Lemma 1.0.6 in Chapter 1 the determinants of families of odd order skew-symmetric matrices are identically zero and hence their discriminants are the whole of $\mathbb{C}^r$.

The following corollary to Definition 4.2.8 is applied to families of skew-symmetric matrices of even order.

Corollary 4.2.9 *Given a smooth germ $A : \mathbb{C}^r, 0 \to Sk(n, \mathbb{C})$, with $n = 2s$, then its discriminant is the zero set of its Pfaffian (the square root of its determinant).*

Proof Note that for $n = 2s$, by Lemma 1.1.8 in Chapter 1,

$$\det A = P(x)^2,$$

where $P(x) = Pf(A)$. Clearly, the discriminant, $\mathcal{D}(A)$, of A is just the zero set of $P(x)$. □

Proposition 4.2.10 *If A, $B : \mathbb{C}^r, 0 \to Sk(n, \mathbb{C})$ (with $n = 2s$) are $\mathcal{G}$-equivalent then their Pfaffians are $\mathcal{K}$-equivalent. Geometrically, this means there is a germ of a diffeomorphism preserving their discriminants, i.e. taking $\mathcal{D}(A)$ to $\mathcal{D}(B)$.*

Proof Since $n = 2s$ we can write

$$\det A = P_1(x)^2, \qquad \det B = P_2(x)^2. \tag{4.3}$$

If A, B are $\mathcal{G}$-equivalent then for some $X \in \mathcal{H}$, $\phi \in \mathcal{R}$

$$B = X^T(A \circ \phi)X,$$

and

$$\begin{aligned} \det B &= (\det X)^2 \det(A \circ \phi) \\ &= (\det X)^2 (\det A) \circ \phi. \end{aligned}$$

From (4.3) it follows that

$$\begin{aligned} P_2(x)^2 &= (\det X)^2 (P_1(x)^2) \circ \phi \\ &= (\det X)^2 (P_1 \circ \phi)^2, \end{aligned}$$

and taking square roots

$$P_2(x) = \pm \det X (P_1 \circ \phi). \tag{4.4}$$

Since $\det X$ is a non-zero function it follows that $P_1(x)$ and $P_2(x)$ are $\mathcal{K}$-equivalent.

From Corollary 4.2.9

$$\mathcal{D}(A) = \{x \in \mathbb{C}^r : P_1(x) = 0\}.$$

From (4.4) $\mathcal{D}(B)$ is the set of points for which $\det X(P_1 \circ \phi)$ vanishes. Furthermore as $\phi \in \mathcal{R}$ there is a smooth change of coordinates in the source taking $\mathcal{D}(A)$ to $\mathcal{D}(B)$ where

$$\mathcal{D}(B) = \{\phi^{-1}(x) : x \in \mathcal{D}(A)\}.$$

□

4.3 Tangent Spaces

For our classification we shall consider only the *simple* singularities, that is those germs, in $\mathcal{S}k$, with neighbourhoods containing finitely many $\mathcal{G}$-orbits.

A key task is to determine the tangent space for the action of $\mathcal{G} = \mathcal{R} \times \mathcal{H}$.

The tangent space to the orbit $\mathcal{G}.A$, through the germ $A \in \mathcal{S}k$, is the image of the "differential", at the identity, of the natural mapping of the group onto this orbit. As usual we have the problem that our groups and spaces are infinite dimensional. Given the germ of a smooth curve $\gamma : \mathbb{C}, 0 \to \mathcal{G}, e$ we can consider

$$\frac{d}{dt}\left(\gamma(t).A\right)\big|_{t=0} = \lim_{t\to 0} \frac{\gamma(t).A - A}{t}. \tag{4.5}$$

One can check that this only depends on

$$\gamma'(0) = \lim_{t\to 0} \frac{\gamma(t) - \gamma(0)}{t},$$

and we define the tangent space $T\mathcal{G}.A$ to be the space spanned by such vectors, (4.5). Since our group is a product, this tangent space is the sum of the tangent space to the $\mathcal{R}$ and $\mathcal{H}$ orbits. So we consider these separately. But the tangent space to the $\mathcal{R}$-orbit of $A \in \mathcal{S}k$ is just the standard $\mathcal{R}$-tangent space to any smooth germ $\mathbb{C}^r, 0 \longrightarrow \mathbb{C}^N$. The following proposition deals with the $\mathcal{H}$-tangent space.

Proposition 4.3.1 *Let $\mathcal{N}$ be the set of smooth germs $\mathbb{C}^r, 0 \longrightarrow M(n, \mathbb{C})$, where $M(n, \mathbb{C})$ is the space of all $n \times n$ matrices over $\mathbb{C}$. The tangent space to the $\mathcal{H}$-orbit through $A \in \mathcal{S}k$ is given by*

$$\{Y^T A + AY : Y \in \mathcal{N}\}.$$

Proof Consider the action on $A \in \mathcal{S}k$, of the path

$$X(t) = I_n + tY \in \mathcal{H},$$

where I_n is the $n \times n$ identity matrix and $Y \in \mathcal{N}$ an arbitrary tangent vector to the group, at I_n. The tangent vector of the resulting path in $\mathcal{S}k$, at $t = 0$, is therefore given by

$$\lim_{t\to 0} \left\{ \frac{(I_n + tY)^T A(I_n + tY) - A}{t} \right\}$$

$$= Y^T A + AY,$$

as required. □

Note : this is just the parametrised version of the calculation used to find the tangent space to elements of $Sk(n, \mathbb{C})$ under the action of $Gl(n, \mathbb{C})$, as given in Proposition 1.1.11.

We are ready to describe the $\mathcal{G}$-tangent space to the orbit of an element $A \in \mathcal{S}k$. Before doing so we need a little notation. In general we shall represent an $n \times n$ skew-symmetric matrix in upper triangular notation. For example the 4×4 matrix

$$A = \begin{bmatrix} 0 & a_1 & a_2 & a_3 \\ -a_1 & 0 & a_4 & a_5 \\ -a_2 & -a_4 & 0 & a_6 \\ -a_3 & -a_5 & -a_6 & 0 \end{bmatrix},$$

is represented by the 6-tuple $(a_1, a_2, a_3, a_4, a_5, a_6)$. We replace the standard basis for $Sk(n, \mathbb{C})$, $\{E^{ij}\}$, with the standard basis for $\mathbb{C}^6$ i.e. $\{e_k\}$, where $k = (ij) = 4(i-1) + j - i(i+1)/2$. For convenience we also refer to the position of the entry a_k as *slot* e_k. We write $A_{x(i)}$ for the matrices

$$\frac{\partial A}{\partial x_i}.$$

So for the above example the corresponding matrix, $A_{x(i)}$, will have upper triangular entries

$$\left(\frac{\partial a_1}{\partial x_i}, \frac{\partial a_2}{\partial x_i}, \frac{\partial a_3}{\partial x_i}, \frac{\partial a_4}{\partial x_i}, \frac{\partial a_5}{\partial x_i}, \frac{\partial a_6}{\partial x_i}\right).$$

The set $\mathcal{S}k$ can be identified with $\mathcal{O}_r^N$, which is an $\mathcal{O}_r$-module. So the tangent space is an $\mathcal{O}_r$-submodule of $\mathcal{O}_r^N$.

Proposition 4.3.2 *(i) The $\mathcal{R}$-tangent space to the orbit of the element $A \in \mathcal{S}k$ is the $\mathcal{O}_r$-module spanned by the $x_i A_{x(j)}$, where $1 \le i, j \le r$.*

(ii) Let $C_{ij}(A)$ (respectively $R_{ij}(A)$) denote the matrix whose jth column (respectively row) is the ith column (respectively row) of A, with zeros elsewhere. Then the tangent space to the orbit of A under the subgroup $\mathcal{H}$ of $\mathcal{C}$ is the $\mathcal{O}_r$-module spanned by the set of skew-symmetric matrices of the form $A_{ij} = C_{ij}(A) + R_{ij}(A)$, $1 \le i, j \le n$.

So the tangent space to the $\mathcal{G}$-orbit of A is

$$T\mathcal{G}.A = \mathcal{M}_r\{A_{x(i)}\} + \mathcal{O}_r\{C_{ij}(A) + R_{ij}(A)\}.$$

We refer to the set

$$\{(A_{x(i)} : 1 \leq i \leq r), (C_{ij}(A) + R_{ij}(A) : 1 \leq i, j \leq n)\}$$

as the set of generators of this tangent space.

Proof The vectors emerging from the action of the $\mathcal{R}$ group are obtained in the usual way. For the $\mathcal{H}$ group, we use the result of Proposition 4.3.1 that tangent vectors to A under this action are given by

$$Y^T A + AY, \tag{4.6}$$

where $Y \in \mathcal{N}$. Let E_{ij} denote the matrix with a 1 in the (i,j)th entry and zeros elsewhere and the set $\{E_{ij}, 1 \leq i, j \leq n\}$ the standard basis of $M(n, \mathbb{C})$. We can represent any element of $\mathcal{N}$ by an $\mathcal{O}_r$-linear combination of these basis vectors. So the tangent space to A is obtained by considering elements $Y = aE_{ij}$ where $a \in \mathcal{O}_r$. Substituting this into (4.6) we find the corresponding tangent vector to be

$$a(E_{ji}A + AE_{ij}) = a(R_{ij}(A) + C_{ij}(A)),$$

as required. □

Again the last part of this proof is just a parametrised version of the argument used to find generators of the tangent space to the action of $Gl(n, \mathbb{C})$ on $Sk(n, \mathbb{C})$. (See the proof of Lemma 1.1.13 in Chapter 1.) As mentioned there, for each pair (i, j), the corresponding generator of the $\mathcal{H}$-tangent space to $A \in Sk$ is the skew-symmetric matrix whose jth row and column is obtained by superimposing the ith row and column of A and leaving the remaining entries zero. For example, taking the matrix A above where $n = 4$ (and $N = 6$) the $\mathcal{H}$-tangent vector corresponding to $(i, j) = (1, 3)$ is the skew-symmetric matrix whose entries are all zero except those in the third (jth) row and column which are, respectively, the first (ith) row and column of A. That is the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -a_1 & 0 \\ 0 & a_1 & 0 & a_3 \\ 0 & 0 & -a_3 & 0 \end{bmatrix},$$

which is $[0,0,0,-a_1,0,a_3]$, in upper-triangular notation. Notice here since A is skew-symmetric superimposing the rows and columns results in cancellation on the leading diagonal.

We also introduce a subgroup of $\mathcal{G}$ which is used in the following section.

Definition 4.3.3 *The subgroup $\mathcal{G}_1 \subset \mathcal{G}$ is defined to be the semi-direct product $\mathcal{G}_1 = \mathcal{R}_1 \times \mathcal{H}_0$, where $\mathcal{R}_1$ is the subgroup of $\mathcal{R}$ consisting of diffeomorphisms, $\mathbb{C}^r, 0 \longrightarrow \mathbb{C}^r, 0$, with 1-jet the identity and $\mathcal{H}_0$ is the subgroup of $\mathcal{H}$ consisting of germs $\mathbb{C}^r, 0 \longrightarrow Gl(n, \mathbb{C})$ with constant part the identity matrix, I_n.*

The following corollary gives the tangent space to the $\mathcal{G}_1$-orbit of a germ $A \in Sk$.

Corollary 4.3.4 *The tangent space to the $\mathcal{G}_1$-orbit of $A \in Sk$ is*

$$T\mathcal{G}_1.A = \mathcal{M}_r^2\{A_{x(i)} : 1 \leq i \leq r\} + \mathcal{M}_r\{C_{ij}(A) + R_{ij}(A) : 1 \leq i,j \leq n\},$$

where the generators $\{(A_{x(i)} : 1 \leq i \leq r), (C_{ij}(A) + R_{ij}(A) : 1 \leq i,j \leq n)\}$ are as described in Proposition 4.3.2.

Proof Since $\mathcal{G}_1 = \mathcal{R}_1 \times \mathcal{H}_0$, the required tangent space is given by the sum

$$T\mathcal{G}_1.A = T\mathcal{R}_1.A + T\mathcal{H}_0.A.$$

As before, we consider the two components of this sum separately. Firstly $T\mathcal{R}_1.A$ is the standard tangent space to the $\mathcal{R}_1$-orbit of a germ $A : \mathbb{C}^r, 0 \longrightarrow \mathbb{C}^N$ i.e.

$$T\mathcal{R}_1.A = \mathcal{M}_r^2\{A_{x(i)} : 1 \leq i \leq r\}.$$

Secondly, to find $T\mathcal{H}_0.A$ we consider the path, $\gamma \in \mathcal{H}_0$, through the identity matrix I_n

$$\gamma(t) = I_n + t\alpha Y,$$

where $Y : \mathbb{C}^r, 0 \longrightarrow M(n, \mathbb{C})$ and $\alpha \in \mathcal{M}_r$. Following a similar argument to that used in Propositions 4.3.1 and 4.3.2 the tangent space to the $\mathcal{H}_0$-orbit of A is given by

$$T\mathcal{H}_0.A = \mathcal{M}_r\{C_{ij}(A) + R_{ij}(A) : 1 \leq i,j \leq n\},$$

where again although the elements $C_{ij}(A) + R_{ij}(A)$ are evaluated as $n \times n$ skew-symmetric matrices they are represented by N-tuples. The result then follows. □

The first key initial simplification shows that we may suppose our germs $A \in Sk$ vanish at the origin.

Proposition 4.3.5 *Given any $A : \mathbb{C}^r, 0 \longrightarrow Sk(n, \mathbb{C})$ with rank $2s$ at the origin then A is $\mathcal{G}$-equivalent to a germ of the form*

$$\bigoplus_s E \bigoplus B,$$

where

$$E = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and $B : \mathbb{C}^r, 0 \longrightarrow Sk(n - 2s, \mathbb{C})$ has $B(0) = 0$.

Proof This proof uses the parametrised simultaneous row and column operations described in Corollary 4.2.1.

If $A(0) \neq 0$ then it has at least two entries, $\pm a$ say, which don't vanish at the origin. By suitable simultaneous row and column interchanges we can move these entries into the leading diagonal block. We can scale the entry, a, in the leading block to unity by multiplying the first row and column by its inverse, $1/a$. By simultaneous row and column operations, involving subtracting appropriate $\mathcal{O}_r$ multiples of the second column from the remaining columns followed by subtracting appropriate $\mathcal{O}_r$ multiples of the first row from the other rows, we can kill off all remaining non-zero entries in the first row and second column (and accordingly the first column and second row). The resulting matrix is of the form:

$$\left[\begin{array}{cc|c} 0 & 1 & 0 \cdots 0 \\ -1 & 0 & 0 \cdots 0 \\ \hline 0 & 0 & \\ \vdots & \vdots & A^* \\ 0 & 0 & \end{array}\right]$$

and we have effectively reduced A to the germ $A^* : \mathbb{C}^r, 0 \longrightarrow Sk(n-2, \mathbb{C})$.

The same process is repeated on this submatrix and continued until we obtain a submatrix $B : \mathbb{C}^r, 0 \longrightarrow Sk(n - 2s, \mathbb{C})$ for which $B(0) = 0$, thereby reaching the required form. $\square$

As a consequence, we can use this reduction to obtain, from any germ $A : \mathbb{C}^r, 0 \longrightarrow Sk(n, \mathbb{C})$, a germ $B : \mathbb{C}^r, 0 \longrightarrow Sk(n-2s, \mathbb{C})$ for which $B(0) = 0$. So we may as well assume, to begin with, that all our germs $A \in \mathcal{S}k$ vanish at $0 \in \mathbb{C}^r$, which we do from now on.

4.4 Classification Theory: Complete Transversals and Determinacy

In the following we seek to list orbits of finitely determined germs, $A \in \mathcal{M}_r\mathcal{O}_r^N$, under the action of the group $\mathcal{G} = \mathcal{R} \times \mathcal{H}$, choosing suitable normal forms as representatives. Classification is done inductively at the jet-level, classifying in turn all $(k+1)$-jets with a given k-jet until determined jets result (or we detect moduli). To do this we employ the method of *complete transversals.*

4.4.1 Complete Transversals

Before discussing specifics we start with a few results concerning *Lie group* actions (see Definition 1.1.4). We first state the following lemma, usually referred to as Mather's Lemma.

Lemma 4.4.1 *(Mather's Lemma) Let G be a Lie group acting smoothly on a finite dimensional manifold V. Let X be a connected submanifold of V. Then X is contained in a single orbit of G if and only if*

(i) for each $x \in X$, the tangent space $T_x(G.x) \supset T_xX$, and

(ii) $\dim T_x(G.x)$ is constant for all $x \in X$.

Proof For the proof see Lemma 3.1 of [MathIV]. □

Lemma 4.4.2 *Let G be a Lie group, H a Lie subgroup of G and $\Phi : G \times X \to X$ a smooth group action. If G, H are connected and $T_x(G.x) = T_x(H.x)$ for all $x \in X$ then $G.x = H.x$ for all $x \in X$.*

Proof We need to show that $G.x$ is contained in a single H orbit, so we apply Mather's Lemma. Here if $y \in G.x$ then $\dim T_y(H.y) = \dim T_y(G.y) = \dim T_x(G.x) = \dim T_x(H.x)$, while $T_y(G.x) = T_y(G.y) = T_y(H.y)$ and we obtain the result. □

We can associate to each $l \in LG$ and $x \in X$ a tangent vector $l.x \in T_x(G.x)$ defined as $d\Phi_{(e,x)}(l,0)$, and write $LG.x$ for $\{l.x : l \in LG\}$. This clearly coincides with $T_x(G.x)$. The following result is Proposition 1.3 of [BrKduP].

Proposition 4.4.3 *Let G be a Lie group acting smoothly on an affine space A, and let W be a vector subspace of V_A (where V_A is identified with the tangent space to A at $x \in A$) with*

$$LG.(x + w) = LG.x \tag{4.7}$$

for all $x \in A$ and $w \in W$. Then

(i) for any $x \in A$ we have

$$x + \{LG.x \cap W\} \subset G.x \cap \{x + W\}.$$

(ii) If $x_0 \in A$ and T is a vector subspace of W satisfying

$$W \subset T + LG.x_0$$

then for any $w \in W$ there exists $g \in G$, $t \in T$ such that $g.(x_0+w) = x_0+t$.

Proof See [BrKduP]. Part (i) follows from the hypothesis in (4.7) and applying Mather's Lemma. Part (ii) is then a consequence of part (i). □

Remark 4.4.4 Proposition 4.4.3 part (ii) is the significant result. It says that the transversal T to the G-orbit of x_0 meets each G-orbit passing through the affine subspace $x_0 + W$ of A. Consequently T is referred to as a *complete transversal.*

The crucial condition required to obtain this result is the hypothesis given in (4.7) which says that for any $x \in A$ the tangent space to the G-orbits of all points in the affine subspace $x + W$ is the same and, what is more, is equal to the tangent space to the $G.x$ orbit at x. In practice this condition is usually replaced by the much sharper condition

$$l.(x+w) = l.x, \tag{4.8}$$

for all $x \in A$, $w \in W$ and $l \in LG$.

Our aim is to use (part (ii) of) Proposition 4.4.3 to classify germs $A : \mathbb{C}^r, 0 \longrightarrow \mathbb{C}^N, 0$, under the action of $\mathcal{G} = \mathcal{R} \times \mathcal{H}$ (or more accurately, the subgroup $\mathcal{G}_1$) and to do this we must first construct some finite dimensional approximations to the space of germs and the group $\mathcal{G}$ (i.e. Lie groups, smooth manifolds).

Finite dimensional approximations to the space of germs are the so called *jet-spaces*, defined as follows.

Definition 4.4.5 *A (k-)jet-space, $J^k(r,N)$, is the space of k-jets of germs $\mathbb{C}^r, 0 \longrightarrow \mathbb{C}^N, 0$. Elements of such a jet-space are N-tuples whose components are polynomials of degree $\leq k$.*

Remark 4.4.6 Each jet-space is a finite dimensional vector space and hence a smooth manifold. We choose a basis for this space, $J^k(r,N)$, to be given by the set of N-tuples or *monomial vectors*

$$x^I e_i = x_1^{i_1} \cdots x_r^{i_r} e_i, \qquad 1 \leq |I| \leq k, \quad 1 \leq i \leq N,$$

where $|I| = i_1 + \cdots + i_r$ and e_i are the standard basis vectors in $\mathbb{C}^N$.

We next consider a finite dimensional approximation to $\mathcal{G}$, which acts on these jetspaces.

Definition 4.4.7 *The finite dimensional approximations to the group $\mathcal{G} = \mathcal{R} \times \mathcal{H}$ are given by the semi-direct product of Lie groups, $J^k\mathcal{G} = J^k\mathcal{R} \times J^k\mathcal{H}$, described as follows.*

(i) The Lie group, $J^k\mathcal{R}$, is the set of k-jets of invertible mappings $\mathbb{C}^r, 0 \longrightarrow \mathbb{C}^r, 0$, with group structure

$$\phi_1 * \phi_2 = j^k(\phi_1 \circ \phi_2), \qquad \phi_1, \phi_2 \in J^k\mathcal{R}.$$

(ii) The Lie group, $J^k\mathcal{H}$, is the set of k-jets of mappings $\mathbb{C}^r,0 \longrightarrow Gl(N,\mathbb{C})$ with group structure

$$\tilde{X}_1 * \tilde{X}_2 = j^k(\tilde{X}_1\tilde{X}_2), \qquad \tilde{X}_1, \tilde{X}_2 \in J^k\mathcal{H}.$$

Note here, if $\mathcal{R}_k \subset \mathcal{R}$ is the normal subgroup of $\mathcal{R}$ consisting of germs whose k-jet is that of the identity, it follows that any element $\phi_k \in \mathcal{R}_k$ preserves the k-jet of an element $\phi \in \mathcal{R}$. We can therefore also represent the Lie group $J^k\mathcal{R}$ by the quotient group $\mathcal{R}/\mathcal{R}_k$.

A similar argument, concerning the normal subgroup $\mathcal{H}_k \subset \mathcal{H}$, consisting of germs whose k-jet is the identity matrix I_N, shows that $J^k\mathcal{H} = \mathcal{H}/\mathcal{H}_k$.

The following defines the corresponding finite dimensional approximation to the group $\mathcal{G}_1$.

Definition 4.4.8 *The Lie group $J^k\mathcal{G}_1$ is the semi-direct product of Lie groups $J^k\mathcal{R}_1 \times J^k\mathcal{H}_0$, where $J^k\mathcal{R}_1 = \mathcal{R}_1/\mathcal{R}_k$ consists of all k-jets of diffeomorphisms, $\mathbb{C}^r,0 \to \mathbb{C}^r,0$, with 1-jet the identity and $J^k\mathcal{H}_0 = \mathcal{H}_0/\mathcal{H}_k$ consists of all k-jets of germs, $\mathbb{C}^r,0 \to Gl(N,\mathbb{C})$ with constant part the identity matrix I_N.*

Having defined the Lie groups, $J^k\mathcal{G}$, $J^k\mathcal{G}_1$ and the smooth manifold, $J^k(r,N)$, on which they act we define the group action as follows.

Definition 4.4.9 *The action of $(\phi,\tilde{X}) \in J^k\mathcal{R} \times J^k\mathcal{H}$ ($J^k\mathcal{R}_1 \times J^k\mathcal{H}_0$) on an element $A \in J^k(r,N)$ is given by*

$$(\phi,\tilde{X}).A = j^k\left[\tilde{X}\left(j^k(A\circ\phi)\right)\right].$$

Note $(\phi,\tilde{X}).A = j^k\left(\tilde{X}(A\circ\phi)\right)$ also.

Lemma 4.4.10 *The $J^k\mathcal{G}$- ($J^k\mathcal{G}_1$-) orbits of $J^k(r,N)$ are all smooth submanifolds of $J^k(r,N)$ and constructible.*

Proof To show they are constructible we note that $J^k\mathcal{G}$ ($J^k\mathcal{G}_1$) is an algebraic group, that is an affine constructible set (indeed the complement in the affine space of an algebraic set) with composition and inverse mapping regular rational. Given $A \in J^k(r,N)$ the map $J^k\mathcal{G} \to J^k(r,N)$,

$$j^k g \mapsto j^k(g.A), \qquad g \in \mathcal{G},$$

is clearly polynomial. So by *Chevalley's Theorem*, see [Mumfrd] Pg. 37, the image is a constructible set. On the other hand the map above has constant rank, so by the Rank Theorem (see [BrocLnd], Pg. 2) the image is locally a smooth submanifold of $J^k(r,N)$. Indeed given any two points on an orbit there is a diffeomorphism $J^k(r,N) \to J^k(r,N)$ preserving the orbit and mapping one point to the other. So every point on an orbit is non-singular. But a non-singular constructible set is a manifold. □

The Lie algebra of $J^k\mathcal{G}$, $L(J^k\mathcal{G})$, generates an $\mathcal{O}_r$-module of vector fields on $J^k(r,N)$ given by

$$j^k\left[\mathcal{O}_r\left\{\alpha_i\frac{\partial}{\partial x_i} : 1 \le i \le r, \sum_{k=1}^{n}\left(a_{ik}\frac{\partial}{\partial a_{jk}} + a_{ki}\frac{\partial}{\partial a_{kj}}\right) : 1 \le i,j \le n\right\}\right], \quad (4.9)$$

where a_{ik} represents the ikth entry of A and $\alpha_i \in \mathcal{M}_r$. The natural effect of $\alpha_i\partial/\partial x_i \in L(J^k\mathcal{G})$ on $A \in J^k(r,N)$ is to give (k-jets of) $\mathcal{M}_r$-multiples of the derivative of the matrix, A, with respect to the source variable x_i. The effect of elements $(a_{ik}\partial/\partial a_{jk} + a_{ki}\partial/\partial a_{kj}) \in L(J^k\mathcal{G})$ on $A \in J^k(r,N)$ is to give (k-jets of $\mathcal{O}_r$-multiples of) the matrix whose jth row and column are obtained by superimposing the ith row and column of A.

Similarly, the Lie algebra of $J^k\mathcal{G}_1$, $L(J^k\mathcal{G}_1)$, generates an $\mathcal{O}_r$-module of vector fields on $J^k(r,N)$,

$$j^k\left[\mathcal{O}_r\left\{\alpha_i\frac{\partial}{\partial x_i} : 1 \le i \le r, \sum_{k=1}^{n}\beta_{ij}\left(a_{ik}\frac{\partial}{\partial a_{jk}} + a_{ki}\frac{\partial}{\partial a_{kj}}\right) : 1 \le i,j \le n\right\}\right], \quad (4.10)$$

where $\alpha_i \in \mathcal{M}_r^2$ and $\beta_{ij} \in \mathcal{M}_r$. The actions of $\alpha_i\partial/\partial x_i$, $(a_{ik}\partial/\partial a_{jk} + a_{ki}\partial/\partial a_{kj}) \in L(J^k\mathcal{G})$ on $A \in J^k(r,N)$ follow from those described for $L(J^k\mathcal{G})$.

Remark 4.4.11 It follows from (4.9) and (4.10) that the tangent space to the $J^k\mathcal{G}$ ($J^k\mathcal{G}_1$)-orbit of a jet $A \in J^k(r,N)$, $TJ^k\mathcal{G}.A$ ($TJ^k\mathcal{G}_1.A$), is spanned by the k-jets of the elements of $T\mathcal{G}.A$ ($T\mathcal{G}_1.A$) given in Proposition 4.3.2 (Corollary 4.3.4).

Each of these Lie groups has a different use in the method of classification. The Lie group $J^k\mathcal{G}$ is used for simplifications discussed later on. Whereas since $J^k\mathcal{G}_1$ has a Lie algebra satisfying Condition (4.8) it is used for the basic classification result. Before stating this result we need one further definition

Definition 4.4.12 *The subspace $\mathcal{M}_r^k\mathcal{O}_r^N/\mathcal{M}_r^{k+1}\mathcal{O}_r^N$ of $J^k(r,N)$, consisting of all N-tuples whose entries are* **homogeneous** *polynomials of degree k, is denoted by $H^k(r,N)$.*

Theorem 4.4.13 *Consider the action of the Lie subgroup $J^{k+1}\mathcal{G}_1$ on a k-jet $A \in J^{k+1}(r,N)$. Given a vector subspace, $T \subset H^{k+1}(r,N)$, satisfying the inclusion*

$$H^{k+1}(r,N) \subset TJ^{k+1}\mathcal{G}_1.A + T \tag{4.11}$$

where $TJ^{k+1}\mathcal{G}_1.A$ is the tangent space to the $J^{k+1}\mathcal{G}_1$-orbit of A at A, then any $(k+1)$-jet of the form

$$A + h, \qquad h \in H^{k+1}(r,N),$$

is $J^{k+1}\mathcal{G}_1$-equivalent to a $(k+1)$-jet of the form $A+t$, for some $t \in T$.

Proof This result is a corollary to Proposition 4.4.3 and is proved in more generality in [BrKduP]. For our situation, taking $A = J^{k+1}(r,N)$, $W = H^{k+1}(r,N)$ and $G = J^{k+1}\mathcal{G}_1$, it follows, from (4.10), that for all $A \in J^{k+1}(r,N)$, $h \in H^{k+1}(r,N)$ and $l \in L(J^{k+1}\mathcal{G}_1)$

$$l.(A+h) = l.A,$$

therefore Condition (4.8) holds and applying Proposition 4.4.3 gives the result. The key here being that the action of each $l \in L(J^{k+1}\mathcal{G}_1)$ on elements $h \in H^{k+1}(r,N)$ always results in jets whose polynomial entries all have degree $k+2$ or higher and therefore drop out in $J^{k+1}(r,N)$.

For example consider the action on $h \in H^{k+1}(r,N)$ of the $\mathcal{R}$-components, $\alpha_i\partial/\partial x_i$, of the Lie algebra $L(J^{k+1}\mathcal{G}_1)$ suggested by (4.10). When differentiating a homogeneous polynomial h_j of degree $k+1$ then multiplying by a function $\alpha_i \in \mathcal{M}_r^2$ guarantees the resulting polynomial has degree $k+2$ or higher. However for those $\mathcal{R}$-components of $L(J^{k+1}\mathcal{G})$, suggested by (4.9), this is not the case when multiplying by functions $\alpha_i \in \mathcal{M}_r$ which have non-zero linear part.

Similarly, the effect of the $\mathcal{H}$-components of $L(J^{k+1}\mathcal{G}_1)$, $L(J^{k+1}\mathcal{G})$ on elements $h \in H^{k+1}(r, N)$ is to multiply certain homogeneous polynomials of degree $k+1$ by function germs $\alpha_i \in \mathcal{O}_r$. To ensure this results in polynomials of degree $k+2$ or higher we require $\alpha_i(0) = 0$, which is true for all $l \in L(J^{k+1}\mathcal{G}_1)$ but not for all $l \in L(J^{k+1}\mathcal{G})$.

So this result only works provided we use the group $\mathcal{G}_1$ (and the corresponding Lie group $J^{k+1}\mathcal{G}_1$) rather than the full group $\mathcal{G}$. $\square$

We refer to this result as The Complete Transversal Theorem and to the affine space, $A + T$, (or even a basis for T) as a *complete transversal* (or a complete $(k+1)$-transversal). This corresponds to the terminology introduced in Remark 4.4.4. We sometimes abbreviate complete transversal to CT.

Given a k-jet A, it is clear that any $(k+1)$-jet with this k-jet is $J^{k+1}\mathcal{G}$-equivalent to some $(k+1)$-jet in the affine space

$$A + H^{k+1}(r, N).$$

However, using Theorem 4.4.13 this space of $(k+1)$-jets can be reduced to a family of representatives, $\{A + t : t \in T\}$, up to $J^{k+1}\mathcal{G}_1$-equivalence. Having done this we refine this family, where possible, by simplifications using elements of the *full* group $\mathcal{G}$, notably scale changes, into a finite list of $(k+1)$-jets. This will be discussed in more detail in Section 4.5. To each of these $(k+1)$-jets we then apply Theorem 4.4.13 over the $(k+2)$-jetspace and so on. We refer to this method of classification as The Complete Transversal Method. As mentioned at the beginning of this section the aim of this method is to eventually obtain a list of determined jets, which represent $\mathcal{G}$-orbits of finitely determined germs of $\mathcal{S}k$. Consequently we need to consider the *determinacy* of a germ/jet.

4.4.2 Determinacy

Throughout the following sections $\mathcal{G} = \mathcal{R} \times \mathcal{H}$ and $\mathcal{G}_1 = \mathcal{R}_1 \times \mathcal{H}_0$ are the subgroups of $\mathcal{K}$ described above.

We first define the notion of determinacy (of a germ).

Definition 4.4.14 *A germ $A : \mathbb{C}^r, 0 \to \mathbb{C}^N, 0$ is said to be k-$\mathcal{G}$-determined when it is $\mathcal{G}$-equivalent to any other germ $B : \mathbb{C}^r, 0 \to \mathbb{C}^N, 0$ with the same k-jet. It is finitely $\mathcal{G}$-determined if it is k-$\mathcal{G}$-determined for some k.*

The same definitions apply replacing $\mathcal{G}$ by $\mathcal{G}_1$.

Consider the following corollary to Theorem 4.4.13.

Corollary 4.4.15 *Let $A : \mathbb{C}^r, 0 \to \mathbb{C}^N, 0$ be a smooth germ and T a subspace of $H^{k+1}(r, N)$ with the property*

$$\mathcal{M}_r^{k+1}\mathcal{O}_r^N \subset T\mathcal{G}_1.A + T + \mathcal{M}_r^{k+2}\mathcal{O}_r^N.$$

Then any germ $B : \mathbb{C}^r, 0 \to \mathbb{C}^N, 0$ with the same k-jet as A is $\mathcal{G}_1$-equivalent to a germ of the form

$$A + t + \Phi,$$

with $t \in T$ and $\Phi \in \mathcal{M}_r^{k+2}\mathcal{O}_r^N$.

Proof This follows immediately from Theorem 4.4.13. □

We also have the following Theorem of Damon.

Theorem 4.4.16 *A smooth germ $A : \mathbb{C}^r, 0 \to \mathbb{C}^N, 0$ is finitely $\mathcal{G}_1$-determined (respectively finitely $\mathcal{G}$-determined) if and only if $\mathcal{M}_r^{k+1}\mathcal{O}_r^N \subset T\mathcal{G}_1.A$ (respectively $T\mathcal{G}.A$) for some k.*

Proof See [Damon]. □

Since the tangent spaces to $\mathcal{G}_1$-orbits of germs $A \in \mathcal{S}k$ are $\mathcal{O}_r$-submodules of $\mathcal{O}_r^N$, we can make use of the following lemma, due to Nakayama.

Lemma 4.4.17 *(Nakayama's Lemma) Let R be a commutative ring with 1, and M an ideal in R such that every element of $1 + M$ (every element of the* *form $1+m$, $m \in M$) is invertible in R. If A is a finitely generated R-module, and B and C are R-modules with A, $B \subset C$ then*

$$A \subset B + M \cdot A \quad \textit{implies that} \quad A \subset B.$$

Proof For a proof we refer to 11.16 and 11.17 of [BrGibl]. □

Bearing in mind the notation employed for germs in $\mathcal{S}k$, the notation used, universally, for the R-modules of this lemma (and in the referenced results) is unfortunate. However to be consistent with the notation used for these results and since the abstract R-modules (A, B, C) will shortly be replaced with $\mathcal{O}_r$-modules, specific to our situation, we keep this notation.

These results can be used to deduce the following determinacy theorem.

Theorem 4.4.18 *A smooth germ $A : \mathbb{C}^r, 0 \to \mathbb{C}^N, 0$ is k-$\mathcal{G}_1$-determined if and only if*

$$\mathcal{M}_r^{k+1}\mathcal{O}_r^N \subset T\mathcal{G}_1.A. \tag{4.12}$$

Proof If A is k-$\mathcal{G}_1$-determined then

$$j^{k+1}A + H^{k+1}(r,N) \subset J^{k+1}\mathcal{G}_1.j^{k+1}A,$$

and taking tangent spaces

$$H^{k+1}(r,N) \subset T(J^{k+1}\mathcal{G}_1.j^{k+1}A).$$

It follows from this that

$$\mathcal{M}_r^{k+1}\mathcal{O}_r^N \subset T\mathcal{G}_1.A + \mathcal{M}_r^{k+2}\mathcal{O}_r^N,$$

and (4.12) is a direct consequence of Nakayama's Lemma. In particular, choosing R to be the ring of function germs $\mathcal{O}_r$, and M its maximal ideal, $\mathcal{M}_r$, it is clear that every element of $1 + \mathcal{M}_r$ is invertible. Let the *module* C be the $\mathcal{O}_r$-module $\mathcal{O}_r^N$ and *module* A be the finitely generated $\mathcal{O}_r$-module $\mathcal{M}_r^{k+1}\mathcal{O}_r^N$. Then choosing *module* B to be the tangent space to the $\mathcal{G}_1$-orbit of germ A, i.e. the $\mathcal{O}_r$-module

$$\mathcal{M}_r^2\{A_{x(i)}\} + \mathcal{M}_r\{C_{ij}(A) + R_{ij}(A)\},$$

by applying Nakayama's Lemma (Lemma 4.4.17) we deduce

$$\mathcal{M}_r^{k+1}\mathcal{O}_r^N \subset T\mathcal{G}_1.A.$$

Conversely since $T\mathcal{G}_1.A$ is an $\mathcal{O}_r$-module the inclusion (4.12) implies that

$$\mathcal{M}_r^{k+s}\mathcal{O}_r^N \subset T\mathcal{G}_1.A,$$

for any $s \geq 1$. We know from Damon's Theorem that A is m-$\mathcal{G}_1$-determined for some m. However we also know, using Corollary 4.4.15, that for any $s \geq 1$ the complete transversal for the $(k+s)$-jet A is empty. This proves the result. □

The following Corollary to Theorem 4.4.18 uses Nakayama's Lemma to give a simpler criterion for the k-$\mathcal{G}_1$-determinacy of a germ.

Corollary 4.4.19 *A smooth germ $A : \mathbb{C}^r, 0 \to \mathbb{C}^N, 0$ is k-$\mathcal{G}_1$-determined if and only if*

$$\mathcal{M}_r^{k+1}\mathcal{O}_r^N \subset T\mathcal{G}_1.A + \mathcal{M}_r^{k+2}\mathcal{O}_r^N \tag{4.13}$$

if and only if (when considered as a k-jet) it has an empty $(k+1)$-transversal.

Proof The first part follows by Nakayama's Lemma (see the first part of the proof of Theorem 4.4.18).

On the otherhand if A has an empty $(k+1)$-transversal then, by Theorem 4.4.13,

$$H^{k+1}(r, N) \subset TJ^{k+1}\mathcal{G}_1.A$$

and again the inclusion

$$\mathcal{M}_r^{k+1}\mathcal{O}_r^N \subset T\mathcal{G}_1.A + \mathcal{M}_r^{k+2}\mathcal{O}_r^N$$

also holds. This argument is reversible. □

We have discussed how to find the degree of $\mathcal{G}_1$-determinacy of a germ. However, we are really interested in the degree of $\mathcal{G}$-determinacy of such a germ.

Consider a k-$\mathcal{G}$-determined germ A. By Definition 4.4.14,

$$A + \mathcal{M}_r^{k+1}\mathcal{O}_r^N \subset \mathcal{G}.A,$$

from which we also have the inclusion

$$A + \mathcal{M}_r^{k+1}\mathcal{O}_r^N \subset \mathcal{G}.A + \mathcal{M}_r^{k+2}\mathcal{O}_r^N.$$

It follows from this that

$$\mathcal{M}_r^{k+1}\mathcal{O}_r^N \subset T\mathcal{G}.A + \mathcal{M}_r^{k+2}\mathcal{O}_r^N,$$

which implies

$$\mathcal{M}_r^{k+2}\mathcal{O}_r^N \subset T\mathcal{G}_1.A + \mathcal{M}_r^{k+3}\mathcal{O}_r^N. \tag{4.14}$$

By applying Nakayama's Lemma to (4.14) then

$$\mathcal{M}_r^{k+2}\mathcal{O}_r^N \subset T\mathcal{G}_1.A,$$

which by Theorem 4.4.18 means that A is $(k+1)$-$\mathcal{G}_1$-determined.

Remark 4.4.20 We have just shown that if A is k-$\mathcal{G}$-determined this implies that A is $(k+1)$-$\mathcal{G}_1$-determined. So, we can use the group $\mathcal{G}_1$ to find a good estimate for the degree of $\mathcal{G}$-determinacy of a germ — this estimate can only ever improve by 1, i.e. it is only ever "out" by at most 1.

We conclude this section with a brief outline of the method of classification we are adopting including some motivation for Section 4.5.

Method of Classification

The classification is done inductively at the k-jet level and starts with a consideration of the case $k = 1$. (For our situation this amounts to the classification of linear combinations of skew-symmetric matrices, under the obvious action of $Gl(r, \mathbb{C}) \times Gl(n, \mathbb{C})$, covered for the case $r = 2$ in Chapters 2 and 3.)

The inductive step is then as follows. Given a k-jet A we find, in $(k+1)$-jet space, a complete transversal $A+T$. Then we need to consider all the constituent $(k+1)$-jets, $A+t$, $t \in T$ in this family. By various simplifications where possible, i.e. by scaling or using Mather's Lemma, which are discussed in Section 4.5, we reduce this family to a finite list of $(k+1)$-jets. Next we consider complete transversals of each $(k+1)$-jet in this list and so on. The process stops for a jet when it has an empty complete transversal (i.e. it is k-$\mathcal{G}$-determined, so all higher jets with this k-jet are $\mathcal{G}$-equivalent to it) or *moduli* are detected; again more about this in Section 4.5.

4.4.3 Results useful for the Unfolding of Germs

In this section we define and prove a few results which are useful when considering the *unfoldings* of finitely $\mathcal{G}$-determined germs. However, since our main interest in the present work is to list these finitely determined germs, we are not particularly concerned with their unfoldings and include the following results both for completeness and possible further study, at a later date. Consequently, any unfolding theory covered here is merely of a superficial nature.

We start with a couple of technical results to be used later on.

Lemma 4.4.21 *Let $\mathcal{O}_r^p$ be an $\mathcal{O}_r$-module. If L, M are finitely generated $\mathcal{O}_r$-submodules of $\mathcal{O}_r^p$ where $L \subset M$ and*

$$\dim_{\mathbb{C}}(\mathcal{O}_r^p/M) = \dim_{\mathbb{C}}(\mathcal{O}_r^p/L) < \infty, \tag{4.15}$$

then $M = L$.

Proof Let $\{g_1, \ldots, g_s\}$, for some integer s, be a basis for $\mathcal{O}_r^p/L$. (Such a basis exists, since from (4.15) $\dim_{\mathbb{C}}(\mathcal{O}_r^p/L)$ is finite.) So any element of $\mathcal{O}_r^p/L$ can be written as a $\mathbb{C}$-linear combination of the g_i.

Then, $\{g_1, \ldots, g_s\}$ span $\mathcal{O}_r^p/M$. For, given any $f \in \mathcal{O}_r^p$, we can write

$$f = \sum_{i=1}^{s} \mu_i g_i + l_1, \tag{4.16}$$

for some $l_1 \in L$ and $\mu_i \in \mathbb{C} : i = 1, \ldots, s$. But since $L \subset M$, l_1 is also in M and we can deduce from (4.16) that any element of $\mathcal{O}_r^p/M$ can be written as $\sum_i \mu_i g_i$, $\mu_i \in \mathbb{C}$. Using (4.15) it follows that $\{g_1, \ldots, g_s\}$ is also a basis for $\mathcal{O}_r^p/M$.

Let $m \in M$ be any element of M. Since $M \subset \mathcal{O}_r^p$ we can write

$$m = \sum_{i=1}^{s} \lambda_i g_i + l, \tag{4.17}$$

for some $l \in L$ and $\lambda_i \in \mathbb{C} : i = 1, \ldots, s$. So

$$\sum_{i=1}^{s} \lambda_i g_i = m - l \in M,$$

(uses $L \subset M$). This implies that in $\mathcal{O}_r^p/M$

$$\sum_{i=1}^{s} \lambda_i g_i = 0. \tag{4.18}$$

So, since we have already shown $\{g_1, \ldots, g_s\}$ to be a basis for $\mathcal{O}_r^p/M$, the only solution to (4.18) is $\lambda_1 = \cdots = \lambda_s = 0$ and from (4.17) we deduce that $M \subset L$ as required. □

Lemma 4.4.22 *Let $V_{n+1} \subset V_n \subset \cdots \subset V_2 \subset V_1$ be a series of inclusions of (complex) vector spaces (or finitely generated $\mathcal{O}_r$-submodules of $\mathcal{O}_r^p$). Suppose*

$$E_j = \{e_1^j, \ldots, e_{m_j}^j\} \subset V_j \quad (\subset V_1),$$

is a basis for V_j/V_{j+1}, $j = 1, \ldots, n$.

Then $E = E_1 \cup \cdots \cup E_{n-1} \cup E_n$ is a basis for V_1/V_{n+1}. In particular,

$$\dim (V_1/V_{n+1}) = \sum_{j=1}^{n} \dim (V_j/V_{j+1}).$$

Proof The proof is by induction on n, where the base case, $n = 1$, is trivial.

Assume the result holds for n, i.e. if

$$E_j = \{e_1^j, \ldots, e_{m_j}^j\} \subset V_j,$$

is a basis for V_j/V_{j+1}, $j = 1, \ldots, n-1$, then

$$E_1 \cup \cdots \cup E_{n-1}$$

is a basis for V_1/V_n. The inductive step is in two parts.

1. We first need to show that E spans V_1/V_{n+1}. Choosing $v_1 \in V_1$, by the inductive hypothesis, we can write

$$v_1 = v' + v_n, \quad v' \in \mathrm{Sp}\{E_1 \cup \cdots \cup E_{n-1}\}, \quad v_n \in V_n. \tag{4.19}$$

But, since $E_n = \{e_1^n, \ldots, e_{m_n}^n\}$ is a basis for V_n/V_{n+1},

$$v_n = v'' + v_{n+1}, \quad v'' \in \mathrm{Sp}\{E_n\}, \quad v_{n+1} \in V_{n+1}. \tag{4.20}$$

So, given any $v_1 \in V_1$, from (4.19) and (4.20), we have

$$v_1 \in \mathrm{Sp}\{E_1 \cup \cdots \cup E_{n-1} \cup E_n\} + V_{n+1},$$

that is E spans V_1/V_{n+1}.

2. We also have to show that $E = E_1 \cup \cdots \cup E_{n-1} \cup E_n$ is an independent set in V_1/V_{n+1}. If

$$\sum_{j=1}^{n}\left(\sum_{i=1}^{m_j} \lambda_{ij} e_i^j\right) \in V_{n+1}, \qquad \lambda_{ij} \in \mathbb{C}, \tag{4.21}$$

(which vanishes in V_1/V_{n+1}) then, it also follows that,

$$\sum_{i=1}^{m_n} \lambda_{in} e_i^n + \sum_{j=1}^{n-1}\left(\sum_{i=1}^{m_j} \lambda_{ij} e_i^j\right) \in V_n,$$

and, since $E_n \subset V_n$,

$$\sum_{j=1}^{n-1}\left(\sum_{i=1}^{m_j} \lambda_{ij} e_i^j\right) \in V_n.$$

By the inductive hypothesis

$$\lambda_{ij} = 0, \qquad \text{for} \quad 1 \leq j \leq n-1, \qquad 1 \leq i \leq m_j, \tag{4.22}$$

and it follows, from (4.21), that $\sum_{i=1}^{m_n} \lambda_{in} e_i^n \in V_{n+1}$ and vanishes in V_n/V_{n+1}.

However, $E_n = \{e_1^n, \ldots, e_{m_n}^n\}$ is a basis for V_n/V_{n+1} so, in addition to (4.22), $\lambda_{in} = 0$, for $1 \leq i \leq m_n$. Hence from (4.21) we have established that $E = E_1 \cup \cdots \cup E_n$ is an independent set in V_1/V_{n+1}.

The result follows by induction. □

We state the following useful lemma.

Lemma 4.4.23 *Given a finitely generated $\mathcal{O}_r$-submodule, $M \subset \mathcal{O}_r^p$ then*

$$\dim_{\mathbb{C}} (\mathcal{O}_r^p/M) < \infty$$

if and only if $\mathcal{M}_r^N \cdot \mathcal{O}_r^p \subset M$ for some integer N.

Proof If, for some N, $\mathcal{M}_r^N \cdot \mathcal{O}_r^p \subset M$ then

$$\dim_{\mathbb{C}} (\mathcal{O}_r^p/M) \leq \dim_{\mathbb{C}} \left(\mathcal{O}_r^p/\mathcal{M}_r^N \cdot \mathcal{O}_r^p\right) < \infty.$$

The converse implication is not so easy. Suppose for some N,

$$\dim_{\mathbb{C}} (\mathcal{O}_r^p/M) \leq N. \tag{4.23}$$

Consider the series of inclusions :

$$M \subset M + \mathcal{M}_r^N \cdot \mathcal{O}_r^p \subset \cdots \subset M + \mathcal{M}_r \cdot \mathcal{O}_r^p \subset M + \mathcal{O}_r^p.$$

It follows directly from these inclusions and (4.23) that

$$N \geq \underbrace{\dim (\mathcal{O}_r^p/M) \geq \dim \left(\mathcal{O}_r^p/(M + \mathcal{M}_r^N \cdot \mathcal{O}_r^p)\right) \geq \cdots \geq \dim (\mathcal{O}_r^p/(M + \mathcal{O}_r^p))}_{N+2 \text{ terms}} = 0.$$

This sequence of inequalities consists of $(N+2)$ integers with *at most* $(N+1)$ different values — between N and 0. So at least one of the inequalities must be an *equality*. There are two possibilities. Either, for some $s \geq 0$

$$\dim (\mathcal{O}_r^p/(M + \mathcal{M}_r^s \cdot \mathcal{O}_r^p)) = \dim \left(\mathcal{O}_r^p/(M + \mathcal{M}_r^{s+1} \cdot \mathcal{O}_r^p)\right) \tag{4.24}$$

or

$$\dim (\mathcal{O}_r^p/M) = \dim \left(\mathcal{O}_r^p/(M + \mathcal{M}_r^N \cdot \mathcal{O}_r^p)\right). \tag{4.25}$$

Using Lemma 4.4.21 we deduce, from (4.24), that

$$M + \mathcal{M}_r^s \cdot \mathcal{O}_r^p = M + \mathcal{M}_r^{s+1} \cdot \mathcal{O}_r^p,$$

which implies

$$\mathcal{M}_r^s \cdot \mathcal{O}_r^p \subset M + \mathcal{M}_r \cdot \mathcal{M}_r^s \cdot \mathcal{O}_r^p,$$

and by Nakayama's Lemma (Lemma 4.4.17)

$$\mathcal{M}_r^s \cdot \mathcal{O}_r^p \subset M.$$

The alternative to this is (4.25), from which, again using Lemma 4.4.21, we deduce that

$$M = M + \mathcal{M}_r^N \cdot \mathcal{O}_r^p,$$

and hence $\mathcal{M}_r^N \cdot \mathcal{O}_r^p \subset M$. □

Corollary 4.4.24 *Given an ideal $I \subset \mathcal{O}_r$ then*

$$\dim_{\mathbb{C}}(\mathcal{O}_r/I) < \infty$$

if and only if, for some N, $\mathcal{M}_r^N \subset I$.

Proof This is just the special case of Lemma 4.4.23 where $p = 1$. □

We give a few definitions adapted from those given in Section 1.3.4 of [Kirk].

Definition 4.4.25 *Consider the germ of a mapping $A_0 : \mathbb{C}^r, 0 \to \mathbb{C}^N, 0$ ($A_0 : \mathbb{C}^r, 0 \to Sk(n, \mathbb{C})$, where $A_0(0) = 0$). An s-parameter* **unfolding** *of A_0 is a germ of a mapping*

$$F : (\mathbb{C}^r \times \mathbb{C}^s, 0) \to (\mathbb{C}^N \times \mathbb{C}^s, 0),$$

that is

$$F : (\mathbb{C}^r \times \mathbb{C}^s, 0) \to (Sk(n, \mathbb{C}) \times \mathbb{C}^s, 0),$$

$$(x, u) \mapsto (A(x, u), u),$$

such that $A_0(x) = A(x, 0)$. The notation $A_u(x) = A(x, u)$ is often used; A_u can be thought of as a deformation of A_0, parametrised smoothly by $u \in \mathbb{C}^s$.

Definition 4.4.26 *Consider two unfoldings $F, G : (\mathbb{C}^r \times \mathbb{C}^s, 0) \to (\mathbb{C}^N \times \mathbb{C}^s, 0)$ of A_0, written $F(x, u) = (A(x, u), u)$, $G(x, u) = (B(x, u), u)$ respectively. Then F, G are* **isomorphic** *if there exist*

$$\phi : \mathbb{C}^r \times \mathbb{C}^s, 0 \to \mathbb{C}^r, 0$$

$$\psi : \mathbb{C}^s, 0 \to \mathbb{C}^s, 0$$

$$\tilde{X} : \mathbb{C}^r \times \mathbb{C}^s \to Gl(N, \mathbb{C}), \qquad (X : \mathbb{C}^r \times \mathbb{C}^s \to Gl(n, \mathbb{C}))$$

with, for small u, ϕ_u, $x \mapsto \phi(x, u)$ (where $\phi_0 : \mathbb{C}^r, 0 \to \mathbb{C}^r, 0$ is a germ of the identity diffeomorphism) and ψ germs of diffeomorphisms and $\tilde{X}(x, 0)$ the identity matrix I_N, such that

$$B(x, u) = \tilde{X}(x, u) A(\phi(x, u), \psi(u)),$$

that is

$$B(x, u) = X^T(x, u) A(\phi(x, u), \psi(u)) X(x, u).$$

In other words B_u is $\mathcal{G}$-equivalent to A_u via an action parametrised smoothly by $u \in \mathbb{C}^s$ (for small u).

For $u \neq 0$, the germs A_u, B_u cannot be considered as germs at the origin with target the null matrix and the germ ϕ_u cannot be considered as a germ of a diffeomorphism which fixes the origin.

We continue with a couple of further definitions.

Definition 4.4.27 *Given an unfolding $F : (\mathbb{C}^r \times \mathbb{C}^s, 0) \to (\mathbb{C}^N \times \mathbb{C}^s, 0)$ and a smooth map $h : \mathbb{C}^t, 0 \to \mathbb{C}^s, 0$ we define an unfolding*

$$h^*F : (\mathbb{C}^r \times \mathbb{C}^t, 0) \to (\mathbb{C}^N \times \mathbb{C}^t, 0)$$

by

$$(x, v) \mapsto (F(x, h(v)), v)\,;$$

h^*F is said to be **induced** *from F by h.*

Definition 4.4.28 *An unfolding F of A_0 is* **versal** *if any unfolding of A_0 is isomorphic to one induced from F.*

At this stage it is convenient to introduce a 'tangent space' similar to those already encountered (see for example the $T\mathcal{G}A$ tangent space defined in Proposition 4.3.2).

Definition 4.4.29 *Consider the mapping $A : \mathbb{C}^r, 0 \to Sk(n, \mathbb{C})$, which can also be thought of as an element $A \in \mathcal{O}_r^N$. The 'extended tangent space' or $\mathcal{G}_e$-tangent space of A, denoted $T\mathcal{G}_e.A$, is defined to be*

$$T\mathcal{G}_e.A = \mathcal{O}_r\{A_{x(i)} : 1 \leq i \leq r,\ C_{ij}(A) + R_{ij}(A) : 1 \leq i, j \leq n\}.$$

The $\mathcal{G}_e$-codimension of A is the codimension of $T\mathcal{G}_e.A$ in $\mathcal{O}_r^N$, that is

$$\mathcal{G}_e\text{-}codimA = \dim_{\mathbb{C}} \left(\mathcal{O}_r^N / T\mathcal{G}_e.A\right).$$

If the $\mathcal{G}_e$-codimension of A is finite we say A is of finite codimension, otherwise A is of infinite codimension.

We state a fundamental result from unfolding theory due to Damon. (See [Damon], [Mart], [Wall].)

Theorem 4.4.30 *An unfolding* $F : (\mathbb{C}^r \times \mathbb{C}^s, 0) \to (\mathbb{C}^N \times \mathbb{C}^s, 0)$ *of* $A_0 \in \mathcal{M}_r \mathcal{O}_r^N$ *is* $\mathcal{G}$*-versal if and only if*

$$T\mathcal{G}_e.A_0 + span_{\mathbb{C}}\{\dot{F}_1, \dots, \dot{F}_s\} = \mathcal{O}_r^N,$$

where the initial speeds $\dot{F}_i \in \mathcal{O}_r^N$ *of* F *are defined by*

$$\dot{F}_i = \frac{\partial A}{\partial u_i}(x, 0), \qquad \text{for } i = 1, \dots, s.$$

From this theorem we have the following corollary.

Corollary 4.4.31 *If* $A_1, \dots, A_s \in \mathcal{O}_r^N$ *form a* $\mathbb{C}$*-spanning set for the complementary space to* $T\mathcal{G}_e.A_0$ *in* $\mathcal{O}_r^N$ *then*

$$F(x, u) = (A_0(x) + \sum_{i=1}^{s} u_i A_i(x), u),$$

is a versal unfolding of A_0*, where* $u = (u_1, \dots, u_s)$.

So, to calculate a versal unfolding of A_0 we need to find a set of such A_i which is a problem at the germ level. However if A_0 is k-$\mathcal{G}_1$-determined then by Theorem 4.4.18 we have $\mathcal{M}_r^{k+1}\mathcal{O}_r^N \subset T\mathcal{G}_1.A_0$ and since $T\mathcal{G}_1.A_0 \subset T\mathcal{G}_e.A_0$ it suffices to calculate the complementary space to $T\mathcal{G}_e.A_0$ in $J^k(r, N)$.

We provide some results which will prove useful for such calculations, starting with the following result obtained using Lemma 4.4.23 above.

Corollary 4.4.32 *Consider a germ* $A : \mathbb{C}^r, 0 \to Sk(n, \mathbb{C})$*, where* $A(0) = 0$*. Then* A *is finitely* $\mathcal{G}_1$*-determined if and only if it has finite* $\mathcal{G}_e$*-codimension.*

Proof This follows immediately from Theorem 4.4.16 and Lemma 4.4.23. □

Having found a finitely determined germ we next look to calculate its $\mathcal{G}_e$-codimension. For this purpose we find the following notation, taken from [Gibson] Pg 156, useful.

Definition 4.4.33 *We define*

$$cod_s A = \dim \frac{T\mathcal{G}_e.A + \mathcal{M}_r^s.\mathcal{O}_r^N}{T\mathcal{G}_e.A + \mathcal{M}_r^{s+1}.\mathcal{O}_r^N},$$

for $s \geq 0$.

Corollary 4.4.34 *Given a mapping $A : \mathbb{C}^r, 0 \to Sk(n, \mathbb{C})$, which is finitely $\mathcal{G}_1$-determined, then for some $k \geq 1$ its $\mathcal{G}_e$-codimension is given by*

$$\begin{aligned}\mathcal{G}_e\text{-}codim A &= \sum_{s=0}^{k} cod_s A \\ &= \sum_{s=0}^{k} \dim \left(\frac{T\mathcal{G}_e.A + \mathcal{M}_r^s.\mathcal{O}_r^N}{T\mathcal{G}_e.A + \mathcal{M}_r^{s+1}.\mathcal{O}_r^N} \right).\end{aligned}$$

Moreover, if $E_s = \{e_i^s : 1 \leq i \leq m_s\}$ is a basis for $T\mathcal{G}_e.A + \mathcal{M}_r^s.\mathcal{O}_r^N / T\mathcal{G}_e.A + \mathcal{M}_r^{s+1}.\mathcal{O}_r^N$ then

$$\bigcup_{s=0}^{k} E_s = \{e_i^s : 1 \leq i \leq m_s, \ 0 \leq s \leq k\},$$

is a basis for $\mathcal{O}_r^N / T\mathcal{G}_e.A$.

Proof Assume A is k-$\mathcal{G}_1$-determined for some $k \geq 1$. Then $\mathcal{M}_r^{k+1}\mathcal{O}_r^N \subset T\mathcal{G}_e.A$ and we have the following series of inclusions :

$$T\mathcal{G}_e.A = T\mathcal{G}_e.A + \mathcal{M}_r^{k+1} \cdot \mathcal{O}_r^N \subset \cdots \subset T\mathcal{G}_e.A + \mathcal{M}_r \cdot \mathcal{O}_r^N \subset \mathcal{O}_r^N.$$

Both results in the statement then follow by applying Lemma 4.4.22. □

Remark 4.4.35 So, for a k-$\mathcal{G}$-determined germ A we can calculate its $\mathcal{G}_e$-codimension (and a basis for the complement of $T\mathcal{G}_e.A$ in $\mathcal{O}_r^N$) by calculating the complement to $T\mathcal{G}_e.A$ in $J^k(r, N)$.

We outline the method employed for these calculations given in [Gibson] (Pgs. 156–159). This concerns how to find

$$\text{cod}_s A = \dim \left(\frac{T\mathcal{G}_e.A + \mathcal{M}_r^s\mathcal{O}_r^N}{T\mathcal{G}_e.A + \mathcal{M}_r^{s+1}\mathcal{O}_r^N} \right),$$

for each $s \geq 0$. The module $\mathcal{M}_r^s \mathcal{O}_r^N$ is generated by the set of basis vectors (of $J^k(r, N)$) :

$$\{x^s e_i, x^{s-1} y e_i, \ldots, y^s e_i : \quad 1 \leq i \leq N\}.$$

We then check which of these basis vectors lie in $T\mathcal{G}_e.A + \mathcal{M}_r^{s+1}\mathcal{O}_r^N$. In practice this involves determining how many of these basis vectors can be obtained from $\mathcal{O}_r$-linear combinations of the generators of $T\mathcal{G}_e.A$ modulo $\mathcal{M}_r^{s+1}\mathcal{O}_r^N$. Then, from the basis vectors not present in $T\mathcal{G}_e.A + \mathcal{M}_r^{s+1}\mathcal{O}_r^N$, we select (a basis for) a supplement for it in $T\mathcal{G}_e.A + \mathcal{M}_r^s\mathcal{O}_r^N$. The number of basis vectors in this supplement is the number $\text{cod}_s A$. Applying Corollary 4.4.34, we use these calculations to obtain a basis for the complement of $T\mathcal{G}_e.A$ in $\mathcal{O}_r^N$ and hence the $\mathcal{G}_e$-codimension of A.

An example of a calculation of this type is provided in the proof of Lemma 6.1.4 in Chapter 6.

A similar $\mathcal{G}$-invariant (to the $\mathcal{G}_e$-codimension) is the $\mathcal{G}$-codimension of a germ (vanishing at the origin).

Definition 4.4.36 *Consider the germ of a mapping $A : \mathbb{C}^r, 0 \to \mathbb{C}^N, 0$. The $\mathcal{G}$-codimension of A is the codimension of $T\mathcal{G}.A$ in $\mathcal{M}_r\mathcal{O}_r^N$, that is*

$$\mathcal{G}\text{-}codim A = \dim_{\mathbb{C}} \left(\mathcal{M}_r\mathcal{O}_r^N / T\mathcal{G}.A\right).$$

If the $\mathcal{G}$-codimension of A is finite we say A is of finite $\mathcal{G}$-codimension, otherwise A is of infinite $\mathcal{G}$-codimension.

Corollary 4.4.37 *Given a mapping $A : \mathbb{C}^r, 0 \to Sk(n, \mathbb{C})$ (where $A(0) = 0$), which is finitely $\mathcal{G}_1$-determined, then for some $k \geq 1$ its $\mathcal{G}$-codimension is given by*

$$\mathcal{G}\text{-}codim A = \sum_{s=1}^{k} \dim \left(\frac{T\mathcal{G}.A + \mathcal{M}_r^s.\mathcal{O}_r^N}{T\mathcal{G}.A + \mathcal{M}_r^{s+1}.\mathcal{O}_r^N} \right).$$

Moreover, if $E_s = \{e_i^s : 1 \leq i \leq m_s\}$ is a basis for $T\mathcal{G}.A + \mathcal{M}_r^s.\mathcal{O}_r^N / T\mathcal{G}.A + \mathcal{M}_r^{s+1}.\mathcal{O}_r^N$ then

$$\bigcup_{s=1}^{k} E_s = \{e_i^s : 1 \leq i \leq m_s, \ 1 \leq s \leq k\},$$

is a basis for $\mathcal{M}_r\mathcal{O}_r^N/T\mathcal{G}.A$.

Proof The result follows by a similar argument to that given for Corollary 4.4.34. $\square$

These results lend themselves naturally to the following finite dimensional interpretation.

Definition 4.4.38 *Consider a jet* $A \in J^k(r,N)$*. Let* $TJ^k\mathcal{G}_e.A$ *and* $TJ^k\mathcal{G}.A$ *be the subspaces of* $J^k(r,N)$ *spanned by the* k*-jets of the elements of* $T\mathcal{G}_e.A$ *and* $T\mathcal{G}.A$*, respectively. Then we define the* $J^k\mathcal{G}_e$*- and* $J^k\mathcal{G}$*-codimensions of* A *to be given by*

$$J^k\mathcal{G}_e\text{-}codimA = \sum_{s=0}^{k} \dim \left(\frac{TJ^k\mathcal{G}_e.A + \mathcal{M}_r^s.\mathcal{O}_r^N}{TJ^k\mathcal{G}_e.A + \mathcal{M}_r^{s+1}.\mathcal{O}_r^N} \right)$$

and

$$J^k\mathcal{G}\text{-}codimA = \sum_{s=1}^{k} \dim \left(\frac{TJ^k\mathcal{G}.A + \mathcal{M}_r^s.\mathcal{O}_r^N}{TJ^k\mathcal{G}.A + \mathcal{M}_r^{s+1}.\mathcal{O}_r^N} \right),$$

respectively.

Remark 4.4.39 It follows that the $\mathcal{G}_e$-($\mathcal{G}$-)codimension of a k-$\mathcal{G}$-determined jet, A, is equal to its $J^k\mathcal{G}_e$-($J^k\mathcal{G}$-)codimension. For specific (usually low) values of k finding the $J^k\mathcal{G}_e$-($J^k\mathcal{G}$-)codimensions of such k-determined jets is easily achieved using the `Transversal` package. Furthermore the calculation process also provides a basis for the complement of $J^k\mathcal{G}_e.A$ ($J^k\mathcal{G}.A$) in $J^k(r,N)$ the former yielding a versal unfolding for A. This is discussed further in Chapter 5.

4.5 Mather's Lemma and Moduli detection

Given some $(k-1)$-jet A, and having established a complete k-transversal $A+T$ in $J^k(r,N)$, we discuss methods for reducing, where possible, the family of k-jets, which lie in this affine space, to a finite number of k-jets.

This is done at the k-jet level by considering the action of the Lie group, $J^k\mathcal{G}$, on the jet-space $J^k(r,N)$. The details of the jet-group $J^k\mathcal{G}$ and its action on $J^k(r,N)$ are provided in Section 4.4.2.

We adopt two basic techniques for determining the orbit structure of a complete transversal. The first involves using the action of explicit members of $J^k\mathcal{G}$ to reduce a set of k-jets to a single representative. The second technique uses Mather's Lemma to identify submanifolds of the complete transversal which are contained in single $J^k\mathcal{G}$-orbits of the jet-space, in each case selecting a suitable representative for the k-jets in this space. This 'refining' of the complete transversal is often effective due to the availability of elements of $J^k\mathcal{G}$ which are not present in the subgroup, $J^k\mathcal{G}_1$, needed for the application of the Complete Transversal Method.

We start by considering the second of these techniques and to do so we restate Mather's Lemma (Lemma 4.4.1) in a form specific to the present situation.

Lemma 4.5.1 *Consider the above action of the Lie group $J^k\mathcal{G}$ on $J^k(r,N)$. Let $X \subset J^k(r,N)$ be a connected manifold. Then X is contained in a single orbit of $J^k\mathcal{G}$ if and only if*

(i) for each jet $x \in X$, the tangent space $T_x(J^k\mathcal{G}.x) \supset T_xX$, and

(ii) $\dim T_x(J^k\mathcal{G}.x)$ is constant for all $x \in X$.

In Section 4.3 we restricted our classification to $\mathcal{G}$-simple germs — roughly speaking germs A for which there is a neighbourhood of A in Sk which meets only finitely many $\mathcal{G}$-orbits. We formalise this notion, in the context of the jet-groups $J^k\mathcal{G}$, with the following definition due to Arnold.

Definition 4.5.2 *Let $\mathcal{G}$ be the group action on Sk defined above. A $\mathcal{G}$-finitely determined map A is $\mathcal{G}$-simple if*

(a) for all $k \geq 1$ the jet $j^kA \in J^k(r,N)$ has a neighbourhood which meets only finitely many $J^k\mathcal{G}$-orbits, say p_k;

(b) p_k remains bounded above as $k \to \infty$.

In fact since A is $\mathcal{G}$-finitely determined (say K-$\mathcal{G}$-determined) we need only show that $J^KA \in J^K(r,N)$ has a neighbourhood meeting only finitely many $J^K\mathcal{G}$-orbits. (The reason is that any orbit near a jet j^kB in $J^k(r,N)$ would meet a transversal to the $J^k\mathcal{G}$-orbit of j^kB. For $k \geq K$, consider $j^KA \in J^k(r,N)$. Since A is K-$\mathcal{G}$-determined $T\mathcal{G}.(j^KA) \supset \mathcal{M}_r^{K+1}\mathcal{O}_r^N$ so the transversal can be chosen to be in $J^K(r,N)$.)

From this we have the following criterion for simplicity, taken from [BrSim].

Lemma 4.5.3 *Let $A \in Sk$ be $\mathcal{G}$-simple. Then for all k and all smooth constructible subsets $Z \subset J^k(r,N)$ through j^kA there is a neighbourhood U of j^kA and a Zariski open subset $V \subset Z$ such that for $y \in V \cap U$*

$$T_y(J^k\mathcal{G}.y) \supset T_yZ.$$

Proof If A is $\mathcal{G}$-simple then, by Definition 4.5.2, the jet j^kA has a neighbourhood, U, meeting only finitely many $J^k\mathcal{G}$-orbits of $J^k(r,N)$ all of which are smooth and constructible (see Lemma 4.4.10). Thus these orbits give a finite constructible partition of $Z \cap U$. Let $X_1, \ldots, X_s$ be the orbits X for which $\dim(X \cap Z) < \dim Z$. Now set $\bar{X} = \mathrm{cl}\,((X_1 \cup \cdots \cup X_s) \cap Z)$. This is Zariski closed by definition and $\dim \bar{X} < \dim Z$. We set $V = Z \backslash \bar{X}$. This is Zariski open and a finite union of orbits Y with $\dim(Y \cap Z) = \dim Z$, so near any $y \in Y \cap Z$, $Y \supset Z$. The result follows. □

We can use the result of this lemma to identify k-jets of germs which are *not* simple.

Corollary 4.5.4 *Consider any smooth constructible set $Z \subset J^k(r,N)$ with the property that*

$$\{y \in Z : T_y(J^k\mathcal{G}.y) \supset T_yZ\}$$

is in some Zariski closed proper subset of Z. Then no germ A with $j^kA \in Z$ is simple. Indeed any neighbourhood of j^kA contains uncountably many orbits.

Proof This is a contradiction of the criterion for simplicity given in Lemma 4.5.3. Indeed on a Zariski open U with j^kA in the closure of U we have $T_y(J^k\mathcal{G}.y) \not\supset T_yZ$ so $\dim(J^k\mathcal{G}.y \cap Z) < \dim Z$, and we must have uncountably many orbits. □

Typically, we attempt to simplify a complete k-transversal, of some $(k-1)$-jet

A_0, consisting of the t-parameter family of k-jets given by

$$Z_a = \{A_a \in J^k(r,N): \ a \in \mathbb{C}^t\}.$$

Clearly such a family is a smooth constructible subset of $J^k(r,N)$ passing through each of its constituent k-jets.

If for *almost all* k-jets, A_a, of this set we find that the inclusion

$$T_{A_a}(J^k\mathcal{G}.A_a) \supset T_{A_a}Z_a,$$

does not hold (by *almost all* we mean, with the exception of the union of a finite number of proper sub-varieties of Z_a) — i.e. Z_a is a set of the type described in Corollary 4.5.4 — then the criterion for simplicity (in Lemma 4.5.3) is not met and it follows that no germ with a k-jet in Z_a is simple. It follows that the family Z_a consists of uncountably many distinct $J^k\mathcal{G}$-types and we have ***moduli***.

Definition 4.5.5 *Let $Z \subset J^k(r,N)$ be a manifold of jets, with $J^k\mathcal{G}$ a Lie group acting on Z. Then elements of Z have $J^k\mathcal{G}$-***moduli*** *if every neighbourhood of every $y \in Z$ meets uncountably many $J^k\mathcal{G}$-orbits.*

In particular Corollary 4.5.4 shows that any family Z of the type described there has moduli.

For our classification detecting the presence of moduli is done computationally using the Maple package `Transversal` and is discussed in the following chapter. Once a complete transversal (or any other family of k-jets) is found to have moduli we can reason, using the above results, that no germ with a k-jet in this family can be simple. Since we are only concerned with the classification of $\mathcal{G}$-simple germs we need not consider these k-jets any further.

If moduli are not present in a family of k-jets we use Mather's Lemma to simplify this family into a union of a finite number of $J^k\mathcal{G}$-orbits, choosing suitable representatives for each. This involves checking both conditions of Lemma 4.5.1 for each of a (finite) number of constructible partitions of Z_a. Again this is done using the package `Transversal` and is discussed in some detail in the following chapter which also provides some illustrative examples of the technique used. Note that checking condition (ii) of Mather's Lemma is equivalent to verifying that the $J^k\mathcal{G}$-codimension of all k-jets in a submanifold is the same.

Having found a family to be a union of a finite number of $J^k\mathcal{G}$-orbits, we then use the $J^k\mathcal{G}$-codimension (of all k-jets in an orbit) to distinguish these orbits. By taking a representative from each distinct orbit we obtain a finite list of k-jets.

In the event that a family of k-jets, Z_a, is contained in a *single* $J^k\mathcal{G}$-orbit — i.e. Mather's Lemma is satisfied for every k-jet in this family — we say the family is $J^k\mathcal{G}$-trivial. In particular if a complete k-transversal of the $(k-1)$-jet A_0 is found to be a $J^k\mathcal{G}$-trivial family we can choose the k-jet A_0 as a representative for the $J^k\mathcal{G}$-orbit containing it.

As will be seen in some of the following calculations, where it is necessary to show a family of k-jets is $J^k\mathcal{G}$-trivial for general values of k (e.g. see the proof of Lemma 6.1.23), checking that both conditions of Mather's Lemma are satisfied over the whole family is no easy task. However, for specific (typically low) values of k, this can be achieved by a fairly simple calculation using `Transversal`, described in the following chapter.

We conclude this section with a brief discussion of the second technique available to us for simplifying families of k-jets, namely 'scale' changes by hand.

Consider a family of k-jets/germs $A_a : \mathbb{C}^r, 0 \to Sk(n, \mathbb{C})$, where $A_a(0) = 0$ for all values of $a \in \mathbb{C}^t$, of the form

$$A_a = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n-1} \\ -a_{12} & 0 & & \vdots \\ \vdots & & \ddots & a_{n-1n} \\ -a_{1n-1} & \cdots & -a_{n-1n} & 0 \end{bmatrix}, \tag{4.26}$$

with $a_{ij} : \mathbb{C}^r, 0 \times \mathbb{C}^t \to \mathbb{C}, 0$. As usual we may regard A_a as a mapping $A_a : \mathbb{C}^r, 0 \to \mathbb{C}^N, 0$ determined by the N-tuple $[a_{12}, \dots, a_{n-1n}]$. So, taking local coordinates, $(x_1, \dots, x_r)$, for $\mathbb{C}^r, 0$ we attempt to simplify the family A_a by the *scale changes*, $\phi_s : \mathbb{C}^r, 0 \to \mathbb{C}^r, 0$ given by

$$(x_1, \dots, x_r) \mapsto (\lambda_1 x_1, \dots, \lambda_r x_r),$$

$\lambda_i \neq 0$, $1 \leq i \leq r$. Because we are dealing with the group $\mathcal{H}$ not all scale changes in the target

$$(a_{12},\ldots,a_{n-1n})\mapsto(\eta_{12}a_{12},\ldots,\eta_{n-1n}a_{n-1n}),$$

($\eta_{ij}\neq 0$, $1\leq i<j\leq n$), are available to us. However if we consider a constant element of $\mathcal{H}$, $\mathbb{C}^r,0\to Gl(n,\mathbb{C})$, given by the diagonal matrix

$$X_c=\begin{bmatrix}\alpha_1 & 0 & \cdots & 0\\ 0 & \alpha_2 & & \vdots\\ \vdots & & \ddots & 0\\ 0 & \cdots & 0 & \alpha_n\end{bmatrix},$$

where $\alpha_1\alpha_2\cdots\alpha_n\neq 0$, then the action $(\phi_s,X_c).A_a=X_c^T(A\circ\phi_s)X_c$ has the effect on the corresponding N-tuple, $A_a=[a_{12},\ldots,a_{n-1n}]$, given by

$$[\eta_{12}(a_{12}\circ\phi_s),\ldots,\eta_{n-1n}(a_{n-1n}\circ\phi_s)],$$

with $\eta_{ij}=\alpha_i\alpha_j$. This method is best illustrated by examples and we refer the reader to the proof of Lemma 6.1.2 in Chapter 6.

4.6 Initial Classification

We are mainly concerned with classifying families of 4×4 skew-symmetric matrices using the techniques of the previous two sections. However beforehand we consider the classification of a few special types of families.

It is useful, for the following proposition, to note that we can define the rank of $A\in\mathcal{S}k$ in a similar way to that described in Lemma 3.0.13, for a pencil. In other words if I_j is the ideal of $\mathcal{O}_r$ generated by the $j\times j$ minors of A then rank A is the largest k for which $I_k\neq 0$. Recall that

$$E=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.$$

Proposition 4.6.1 *When $r=1$ then any germ $A:\mathbb{C},0\longrightarrow Sk(n,\mathbb{C})$ is $\mathcal{G}$-equivalent to something of the form*

$$x^{k_1}E_{s_1}\bigoplus x^{k_2}E_{s_2}\bigoplus\cdots\bigoplus x^{k_t}E_{s_t}\bigoplus 0,$$

where the final zero block is possibly non-existent, $E_s=\oplus_s E$, and the s_i are integers ≥ 1 and $k_1<k_2<\cdots<k_t$.

Proof The proof is by induction on n.

Initial Step

If $A \equiv 0$ we are finished.

Otherwise, since we assume $A(0) = 0$, let $k_1 \geq 1$ be the lowest degree term of all the entries of A. Then we can write

$$A = x^{k_1} A_0, \tag{4.27}$$

with $A_0(0) \neq 0$.

If we denote the rank of A_0 at the origin by $\mathrm{rank}_0 A_0$ then

$$\mathrm{rank}_0 A_0 = 2s_1,$$

for some $s_1 \geq 1$. So by Proposition 4.3.5 we find

$$X^T A_0 X = \left[\begin{array}{ccc|c} E & & & 0 \\ & \ddots & & \vdots \\ & & E & 0 \\ \hline 0 & \cdots & 0 & A_1 \end{array}\right], \tag{4.28}$$

where $X : \mathbb{C}, 0 \longrightarrow Gl(n, \mathbb{C})$ and $A_1(0) = 0$. If $n = 2s_1$ we are finished.

Inductive Step

By induction we find that A_1 is equivalent to

$$x^{l_2} E_{s_2} \oplus \cdots \oplus x^{l_t} E_{s_t} \oplus 0, \tag{4.29}$$

where $l_2 < \cdots < l_t$, and the result follows, taking $k_j = l_j + k_1$ for $j \geq 2$. □

Lemma 4.6.2 *Consider the classification of germs $A : \mathbb{C}^r, 0 \longrightarrow Sk(n, \mathbb{C})$ under the action of the $\mathcal{G} = \mathcal{R} \times \mathcal{H}$ subgroup of the corresponding $\mathcal{K}$-group.*

(i) If $n = 2$ the germs

$$A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$$

are $\mathcal{G}$-equivalent if and only if a and b are $\mathcal{K}$-equivalent. In particular the $\mathcal{G}$-simples are given by classifying the $\mathcal{R}$-simple functions $\mathbb{C}^r, 0 \longrightarrow \mathbb{C}, 0$.

(ii) If $n = 3$ a germ $A : \mathbb{C}^r, 0 \to Sk(3, \mathbb{C})$ is $\mathcal{G}$-finitely determined if and only if the corresponding germ $\mathbb{C}^r, 0 \to \mathbb{C}^3, 0$ is $\mathcal{K}$-finitely determined and two $\mathcal{G}$-finite germs A, B are $\mathcal{G}$-equivalent if and only if the corresponding germs $\mathbb{C}^r, 0 \to \mathbb{C}^3, 0$ are $\mathcal{K}$-equivalent. In particular the $\mathcal{G}$-simples are given by the $\mathcal{K}$-classification of simple germs $\mathbb{C}^r, 0 \longrightarrow \mathbb{C}^3, 0$.

Proof

(i) The first case is easy. If A and B are $\mathcal{G}$-equivalent then

$$\begin{aligned} \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} &= \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} \begin{pmatrix} 0 & b \circ \phi \\ -b \circ \phi & 0 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \\ &= (\alpha\delta - \beta\gamma) \begin{pmatrix} 0 & b \circ \phi \\ -b \circ \phi & 0 \end{pmatrix}, \end{aligned}$$

for some

$$\left(\phi, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \right) \in \mathcal{G},$$

i.e. $a = (\alpha\delta - \beta\gamma)(b \circ \phi)$. So a and b are $\mathcal{K}$-equivalent (which we already know). However if a and b are $\mathcal{K}$-equivalent then $a = \Delta(b \circ \phi)$ for some Δ with $\Delta(0) \neq 0$ and diffeomorphism ϕ. So consider

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \Delta & 0 \\ 0 & 1 \end{pmatrix}$$

above and A and B are $\mathcal{G}$-equivalent. Since the list of $\mathcal{K}$-simple function germs is the same as that for $\mathcal{R}$-simple function germs the result follows.

(ii) We associate a germ $A : \mathbb{C}^r, 0 \longrightarrow Sk(3, \mathbb{C})$ (where $A(0) = 0$),

$$A = \begin{pmatrix} 0 & a_1 & a_2 \\ -a_1 & 0 & a_3 \\ -a_2 & -a_3 & 0 \end{pmatrix},$$

with the germ $\mathbb{C}^r, 0 \to \mathbb{C}^3, 0$, (a_1, a_2, a_3). By Lemma 4.2.5 $\mathcal{G}$ is a connected subgroup of $\mathcal{K}$ and therefore

$$T\mathcal{G}.A \subset T\mathcal{K}.A,$$

for all A. We use Proposition 4.3.2 to find the $\mathcal{G}$-tangent space to this germ. As usual the $\mathcal{R}$-tangent space to this germ is the standard $\mathcal{R}$-tangent space. The $\mathcal{H}$-tangent space is given by

$$\mathcal{O}_r\left\langle \begin{array}{ccc} (a_1,a_2,0), & (0,0,a_2), & (0,0,a_1), \\ (0,a_3,0), & (a_1,0,a_3), & (0,a_1,0), \\ (a_3,0,0), & (a_2,0,0), & (0,a_2,a_3) \end{array} \right\rangle.$$

Furthermore, by a little manipulation of its generators we can show this submodule coincides with the submodule

$$\mathcal{O}_r\left\langle \begin{array}{ccc} (a_1,0,0), & (a_2,0,0), & (a_3,0,0), \\ (0,a_1,0), & (0,a_2,0), & (0,a_3,0), \\ (0,0,a_1), & (0,0,a_2), & (0,0,a_3) \end{array} \right\rangle,$$

which is the $\mathcal{C}$-tangent space to the germ (a_1,a_2,a_3). Therefore the $\mathcal{H}$ and $\mathcal{C}$ tangent spaces to A coincide that is

$$T\mathcal{G}.A = T\mathcal{K}.A,$$

for all A. In particular, at the k-jet level

$$TJ^k\mathcal{G}.A = TJ^k\mathcal{K}.A.$$

So all calculations at the jet level for the $\mathcal{G}$-action (on A) are the same as those for the $\mathcal{K}$-action on A. Furthermore given a k-jet A for which

$$\mathcal{M}_r^{k+1}\mathcal{O}_r^N \subset T\mathcal{G}.A + \mathcal{M}_r^{k+2}\mathcal{O}_r^N,$$

i.e. A is k- or $(k+1)$-$\mathcal{G}$-determined, then it follows that

$$\mathcal{M}_r^{k+1}\mathcal{O}_r^N \subset T\mathcal{K}.A + \mathcal{M}_r^{k+2}\mathcal{O}_r^N,$$

and A is also k- or $(k+1)$-$\mathcal{K}$-determined.

If two $\mathcal{G}$-finite germs A and B are $\mathcal{G}$-equivalent , they are clearly $\mathcal{K}$-equivalent. Conversely if they are $\mathcal{K}$-equivalent, then choose $k \geq 1$ so that both germs are k-$\mathcal{G}$ and k-$\mathcal{K}$-determined. We need only show that the $J^k\mathcal{G}$ and $J^k\mathcal{K}$ orbits of the k-jet of A coincide. This is proved using Lemma 4.4.2, taking $G = J^k\mathcal{K}$ and $H = J^k\mathcal{G}$.

□

4.7 Miscellaneous Results

In the following we provide some background material required later.

4.7.1 Analytic Varieties

We give some basic results for germs of analytic varieties, which are well known results for affine varieties (see [CoxLiOs]).

Definition 4.7.1 *Let $f_1, \ldots, f_s$ be analytic function germs in $\mathcal{O}_r$. Then the set*

$$\mathbf{V}(f_1, \ldots, f_s) = \{x \in \mathbb{C}^r : f_i(x) = 0 \text{ for } 1 \leq i \leq s\},$$

is called the **germ of an analytic variety** *of $\mathbb{C}^r, 0$ defined by $f_1, \ldots, f_s$, or an* **analytic variety germ**.

Definition 4.7.2 *Given an ideal $I \subset \mathcal{O}_r$ we define the analytic variety germ*

$$\mathbf{V}(I) = \{x \in \mathbb{C}^r : f(x) = 0 \text{ for all } f \in I\}. \tag{4.30}$$

Conversely, given an analytic variety germ, $V \subset \mathbb{C}^r, 0$, we have a set $\mathbf{I}(V) \subset \mathcal{O}_r$ given by

$$\mathbf{I}(V) = \{f \in \mathcal{O}_r : f(x) = 0 \text{ for all } x \in V\}. \tag{4.31}$$

Lemma 4.7.3 *The set $\mathbf{I}(V) \subset \mathcal{O}_r$, given by (4.31) in Definition 4.7.2, is an ideal which we refer to as the* **ideal of** V.

Proof Clearly, $0 \in \mathbf{I}(V)$ since the zero germ vanishes on all of $\mathbb{C}^r$, and in particular vanishes on V. Suppose $f, g \in \mathbf{I}(V)$ and $h \in \mathcal{O}_r$. Let x be a point of V sufficiently close to 0. Then

$$f(x) + g(x) = 0 + 0 = 0,$$

implying that $f + g \in \mathbf{I}(V)$, and

$$h(x)f(x) = h(x) \cdot 0 = 0,$$

implying that $hf \in \mathbf{I}(V)$. The result then follows. □

Definition 4.7.4 *Let $I \subset \mathcal{O}_r$ be an ideal. The* **radical** *of I, denoted $\sqrt{I}$, is the set*

$$\{f \in \mathcal{O}_r : f^m \in I \text{ for some integer } m \geq 1\}.$$

Lemma 4.7.5 *If $I \subset \mathcal{O}_r$ is an ideal, then $\sqrt{I}$ is an ideal in $\mathcal{O}_r$ containing I.*

Proof Clearly $I \subset \sqrt{I}$, since $f \in I$ implies that $f^1 \in I$ and hence, by Definition 4.7.4, $f \in \sqrt{I}$.

To show $\sqrt{I}$ is an ideal, suppose $f, g \in \sqrt{I}$. By definition there are positive integers m, l such that $f^m, g^l \in I$. Consider the expansion $(f+g)^{m+l-1}$. Every term in this expansion has a factor $f^i g^j$ with $i + j = m + l - 1$. Since either $i \geq m$ or $j \geq l$, either f^i or g^j is in I, whence $f^i g^j \in I$ and every term in the expansion is in I. Hence $(f + g)^{m+l-1} \in I$ and by definition $f + g \in \sqrt{I}$.

Finally, suppose $f \in \sqrt{I}$ and $h \in \mathcal{O}_r$. Then $f^m \in I$ for some $m \geq 1$. Since I is an ideal, then $(hf)^m = h^m f^m \in I$ and hence $hf \in \sqrt{I}$. It follows that $\sqrt{I}$ is an ideal. □

The following result is taken from [Gunn].

Theorem 4.7.6 *(Hilbert's Nullstellensatz) If $I \subset \mathcal{O}_r$ is an ideal then*

$$\mathbf{I}(\mathbf{V}(I)) = \sqrt{I}.$$

Proof We refer the reader to the proof of Theorem 2 in Chpt III of [Gunn]. □

The main objective of this section is the following result.

Lemma 4.7.7 *Consider the germ of a smooth mapping, $f : \mathbb{C}^r, 0 \to \mathbb{C}^p, 0$ defined by $f = (f_1, \ldots, f_p)$, where each $f_i \in \mathcal{M}_r$. The following three properties are equivalent.*

(i)

$$\dim_{\mathbb{C}} (\mathcal{O}_r / \mathcal{O}_r \langle f_1, \ldots, f_p \rangle) < \infty.$$

(ii) For some N, $\mathcal{M}_r^N \subset \mathcal{O}_r\langle f_1, \ldots, f_p\rangle$.

(iii) (Using the notation introduced in Definition 4.7.1)

$$\mathbf{V}(f_1, \ldots, f_p) = \{0\}.$$

Proof We prove this in two steps. The first shows condition (i) and condition (ii) are equivalent and the second that condition (ii) is equivalent to condition (iii).

Step1

Letting $I = \mathcal{O}_r\langle f_1, \ldots, f_p\rangle$ we just apply Corollary 4.4.24 of Section 4.4.3.

Step2

Firstly, if for some N, $\mathcal{M}_r^N \subset \mathcal{O}_r\langle f_1, \ldots, f_p\rangle$ then for each $1 \leq k \leq r$

$$x_k^N \in \mathcal{O}_r\langle f_1, \ldots, f_p\rangle.$$

So,

$$\mathbf{V}(f_1, \ldots, f_p) \subset \mathbf{V}(x_1^N, \ldots, x_r^N) = \{0\}.$$

The converse inclusion uses the Nullstellensatz for analytic varieties (Lemma 4.7.6). Suppose

$$\mathbf{V}(f_1, \ldots, f_p) = \{0\}.$$

Using notation introduced by Definition 4.7.2, this implies that

$$\mathbf{I}\left(\mathbf{V}(\mathcal{O}_r\langle f_1, \ldots, f_p\rangle)\right) = \mathbf{I}(0) = \mathcal{M}_r.$$

Therefore by Nullstellensatz (Theorem 4.7.6)

$$\mathcal{M}_r = \sqrt{\mathcal{O}_r\langle f_1, \ldots, f_p\rangle},$$

where, by Definition 4.7.4,

$$\sqrt{\mathcal{O}_r\langle f_1, \ldots, f_p\rangle} = \{h \in \mathcal{O}_r : h^m \in \mathcal{O}_r\langle f_1, \ldots, f_p\rangle \text{ for some } m\}.$$

Considering each $x_k \in \mathcal{M}_r$ $(1 \leq k \leq r)$, this implies, for some $M_k \geq 1$, that

$$x_k^{M_k} \in \mathcal{O}_r\langle f_1, \ldots, f_p\rangle.$$

Setting $M = \sum_{k=1}^{r} M_k$, we deduce that

$$\mathcal{M}_r^M \subset \mathcal{O}_r\langle f_1, \dots, f_p \rangle,$$

as required. □

4.7.2 Binary Quartics

There follows a general, but brief, discussion on the space of binary quartics which will be used later on. In particular, to introduce neccessary notation, we quote a couple of results (without proofs), and for further details we refer the reader to Pgs 94-98 of [Newstd].

Definition 4.7.8 *The space, V_{n+1}, of* **binary forms** *consists of polynomials of the form*

$$f(x, y) = \alpha_0 x^n + \alpha_1 x^{n-1} y + \cdots + \alpha_n y^n,$$

to which we associate a point $(\alpha_0, \alpha_1, \dots, \alpha_n) \in P\mathbb{C}^n$.

There is a natural bijective correspondence between points of the associated projective space $P\mathbb{C}^n$ and the sets of n roots (in $P\mathbb{C}^1$) of $f(x, y) = 0$. Consider the subgroup, $Sl(2, \mathbb{C})$, of $Gl(2, \mathbb{C})$ consisting of matrices with determinant 1 and the projective linear group, $PGl(2, \mathbb{C})$, given by

$$Gl(2, \mathbb{C})/cI_2,$$

where $c \in \mathbb{C}$. The action of $Sl(2, \mathbb{C})$, (or $PGl(2, \mathbb{C})$) on $P\mathbb{C}^1$ determines an action on V_{n+1} $(= P\mathbb{C}^n)$.

Definition 4.7.9 *The* **linear group** *action of $Sl(2, \mathbb{C})$ (or $PGl(2, \mathbb{C})$) on the space, V_{n+1} $(= P\mathbb{C}^n)$, of binary forms is given by*

$$g.f(x, y) = f\Big(g^{-1}(x, y)\Big),$$

for any $f \in V_{n+1}$, $g \in Sl(2, \mathbb{C})$ (or $PGl(2, \mathbb{C})$).

As already mentioned we are principally concerned with binary *quartics.*

Definition 4.7.10 *The space, V_5, of* **binary quartics** *consists of polynomials of the form*

$$f(x,y) = a_0x^4 + 4a_1x^3y + 6a_2x^2y^2 + 4a_3xy^3 + a_4y^4,$$

to which, for convenience, we associate a point $(a_0, a_1, a_2, a_3, a_4) \in P\mathbb{C}^4$.

Proposition 4.7.11 *Under the $Sl(2,\mathbb{C})$ action, quartics have two basic invariants*

$$I \equiv a_0a_4 - 4a_1a_3 + 3a_2^2,$$

$$J \equiv a_0a_2a_4 + 2a_1a_2a_3 - a_0a_3^2 - a_1^2a_4 - a_2^3.$$

In other words f and $g.f$ yield the same values of I and J for $g \in Sl(2,\mathbb{C})$.

□

Definition 4.7.12 *We define $\Delta : V_5 \to \mathbb{C}$ $(\Delta : P\mathbb{C}^4 \to \mathbb{C})$ by*

$$\Delta(q) = I^3 - 27J^2,$$

where $q \in V_5$.

Since *both* I and J are invariants of $Sl(2,\mathbb{C})$-equivalence, it follows immediately that Δ is also an $Sl(2,\mathbb{C})$-invariant. The following result is deduced in [Newstd].

Proposition 4.7.13 *A quartic $q \in V_5$ has a repeated root if and only if $\Delta(q) = 0$.* □

□

From now on we think of quartics as points of $P\mathbb{C}^4$. We denote the set of quartics with repeated roots by $\bar{\Delta}$, i.e.

$$\bar{\Delta} = \{q \in P\mathbb{C}^4 : \Delta(q) = 0\}.$$

Excluding this set from $P\mathbb{C}^4$, we consider the set, $P\mathbb{C}^4_\Delta$, of quartics with *no* repeated roots, in other words

$$P\mathbb{C}^4_\Delta = P\mathbb{C}^4 \backslash \bar{\Delta}.$$

It follows that each point of $P\mathbb{C}^4_\Delta$ is naturally represented by a set of four distinct points of $P\mathbb{C}^1$. Arranging four such points in some particular order, x_1, x_2, x_3, x_4, by the *Three Point Lemma* (Lemma 1.3.4) there is a *unique* element $g \in PGl(2,\mathbb{C})$ for which

$$gx_1 = (1,0), \qquad gx_2 = (0,1), \qquad gx_3 = (1,1);$$

with

$$gx_4 = (\lambda, 1),$$

for some $\lambda \neq 0$, 1. The value of λ is the *cross-ratio* of x_1, x_2, x_3, x_4 in this *order*. The 24 possible orders in which these four points can be arranged give, in general, six different values of cross-ratio; if any one of these is λ, the full set is

$$\lambda, \ \ , 1-\lambda, \ \ , 1/\lambda, \ \ , (\lambda-1)/\lambda, \ \ \lambda/(\lambda-1), \ \ 1/(1-\lambda).$$

It can be verified that the function $\mu : \mathbb{C} \to \mathbb{C}$, given by

$$\mu = \left[\frac{(2\lambda-1)(\lambda-2)(\lambda+1)}{\lambda(\lambda-1)}\right]^2,$$

has the *same* value for all six cross-ratios of points $\{x_1, x_2, x_3, x_4\}$.

By direct computation it can be shown that, for quartics with no repeated roots,

$$\mu = 3^6 J^2/\Delta.$$

This leads to the map

$$J^2/\Delta : P\mathbb{C}^4_\Delta \to \mathbb{C}. \tag{4.32}$$

Consider the action of $PGl(2,\mathbb{C})$ on $P\mathbb{C}^1$, which as described in Definition 4.7.9 determines an action on $P\mathbb{C}^4$.

Lemma 4.7.14 *The map $J^2/\Delta : P\mathbb{C}^4_\Delta \to \mathbb{C}$ is well-defined and* **invariant** *under $PGl(2,\mathbb{C})$-equivalence. Furthermore, up to this equivalence, the space of binary quartics, $P\mathbb{C}^4$, has uncountably many orbits.*

Proof Since J^2 and Δ are both homogeneous of degree 6 in the variables a_i clearly the quotient, J^2/Δ, is well-defined on $P\mathbb{C}^4_\Delta$.

We next show this map is a $PGl(2,\mathbb{C})$-invariant. Consider two elements, f_1, f_2 of $P\mathbb{C}^4_\Delta$ represented by points $\{x_1, x_2, x_3, x_4\}$ and $\{y_1, y_2, y_3, y_4\}$ respectively.

The system $\{y_1, y_2, y_3, y_4\}$ is $PGl(2,\mathbb{C})$-equivalent to $\{x_1, x_2, x_3, x_4\}$ if there is an element, P_1, of $PGl(2,\mathbb{C})$ taking one set to the other. There is also a unique element, $P_u \in PGl(2,\mathbb{C})$, fixing three points of $\{x_1, x_2, x_3, x_4\}$ and giving a cross-ratio, λ. It follows that the element $P_u P_1$, of $PGl(2,\mathbb{C})$ gives the same cross-ratio of $\{y_1, y_2, y_3, y_4\}$. It follows that f_1, f_2 have the same value of J^2/Δ as required.

It follows that to any value of J^2/Δ there corresponds at least one (distinct) $PGl(2,\mathbb{C})$-orbit in $P\mathbb{C}^4_\Delta$. We refer to each value of J^2/Δ as the j-invariant of elements in the corresponding orbit(s) of $P\mathbb{C}^4_\Delta$ and hence, up to $PGl(2,\mathbb{C})$-equivalence, $P\mathbb{C}^4_\Delta$ has uncountably many orbits. Since $\bar{\Delta}$ is a proper algebraic subset of $P\mathbb{C}^4$, $P\mathbb{C}^4_\Delta$ is dense, and (up to $PGl(2,\mathbb{C})$-equivalence) any point of $P\mathbb{C}^4$ has any neighbourhood meeting uncountably many orbits. □

Chapter 5

Using Transversal

As mentioned in the previous chapter, our method of classification involves using the specialist computer package, `Transversal`, (developed by N.P.Kirk at Liverpool), which runs under the (symbolic algebra system) `Maple`, see [MapleV]. This package is principally used for calculating and manipulating tangent spaces of orbits of jet-group actions on jet-spaces, $J^k(r, N)$, for numeric values of k, r and N. It is invaluable for calculating the tangent spaces to (members of) parametrised families of jets and hence, as previously discussed, is particularly useful for moduli detection and simplification using Mather's Lemma.

Although `Transversal` proves to be a valuable companion throughout a given set of calculations, providing a general idea of how the classification progresses, for example by suggesting the presence of series of distinct determined jets, there are points where it is more efficient or indeed necessary to use hand calculations. This is particularly the case when proving the existence of a series of k-determined germs for general values of k.

In the following we discuss, in some detail, the various procedures and functions of the package used for our calculations. In the final section we illustrate the techniques used by providing some worked examples taken from these calculations. For further details on `Transversal` we refer the reader to [KirkTr].

Adopting notation from [KirkTr], throughout this chapter we denote the Lie algebra of a jet-group by L and the action of this Lie algebra on a jet A by $L.A$. (In terms of the notation used in Sections 4.3 and 4.4 $L.A$ represents an action

$LG.A$, such as $J^k\mathcal{G}.A$.)

5.1 Overview of Package

Before any calculation can be performed by the package, it is necessary to define the Lie algebra, L, we wish to work with. Then, for a given k, and jet A the tangent space, $J^k(L.A)$ [1], to the orbit of A is calculated using the main routine `jetcalc`. This routine also finds a basis for this tangent space, and stores, globally, several useful by-products of the calculation which can be subsequently accessed by other functions of the package.

5.2 Lie algebra and Initialisation

We first discuss how to specify the Lie algebra. This requires several different items of data which, within a typical class of problems, remain fixed. Therefore L is specified using a set of global variables, which are accessed internally by the package's routines. Before any calculation is attempted the user must first assign these global variables accordingly.

In general, the Lie algebra L is decomposed as the direct sum of two components consisting of source and target vector fields where the source component can be user specified as an $\mathcal{O}_r$-module of vector fields. This Lie algebra is made more specific by determining which of these components appear in its *defining equation*. This is done by setting the 'type' of the Lie algebra.

There are five broad 'types' which we denote $\mathcal{R}$, $\mathcal{L}$, $\mathcal{C}$, $\mathcal{A}$ and $\mathcal{K}$. The required 'type' is set by the global variable `equiv` which takes the string constant value R, L, C, A or K accordingly. (Note, although these 'types' can be used to give Lie algebras to standard groups used in singularity theory, they need not be associated with just these groups, for example possibilities for Lie algebras of 'type' $\mathcal{R}$ extend beyond those associated with the standard $\mathcal{R}$-group.)

For example, since the tangent spaces we wish to consider are just $\mathcal{O}_r$-submodules of $\mathcal{O}_r^N$ we can define the Lie algebra by a source component only.

[1] $J^k(L.A)$ represents the tangent space $L(J^k\mathcal{G}).A = TJ^k\mathcal{G}.A$ discussed in Section 4.4

So for our situation L is chosen to be of 'type' $\mathcal{R}$ (the variable `equiv` takes the value R) and its action on a jet A is given by the *defining equation*

$$L.A = \mathcal{M}_r^{t_1}\langle \xi_1.A, \ldots, \xi_s.A\rangle, \tag{5.1}$$

where the exponent t_1 is a user-defined integer global variable, `source_power`, and ξ_i are user-defined vector fields. These vector fields take the form

$$\xi_i = g_1\frac{\partial}{\partial x_1} + \cdots + g_r\frac{\partial}{\partial x_r} \qquad \text{where} \qquad g_j \in \mathcal{O}_r, \tag{5.2}$$

and are defined for the package via a procedure which, on being called from within the main routine `jetcalc`, takes a jet A as parameter and returns the set $\{\xi_i.A : i = 1, \ldots, s\}$. The global variable `liealg` holds the name of this procedure. Before proceeding further we need to clarify the following technicality. Although this `liealg` procedure defines the Lie algebra, it is actually the main routine `jetcalc` which, using it, generates the actual tangent space, to a given jet A, from the defining equation (5.1).

In particular, for our calculations we have written the `liealg` procedure

`skewmatrix` := proc(A, N, `tgtspace`),

where A, a list, stores a jet passed to it, and N is a positive integer, the target dimension, deduced from the number of components of A. Both these parameters are pre-determined in `jetcalc` before it calls `skewmatrix`. The third parameter, `tgtspace`, is of Maple type 'table' and is assigned within the procedure. We briefly summarise the function of this procedure here; for more detail we refer the reader to Appendix D which provides the full (annotated) source code, together with a short explanation of how it works.

The `skewmatrix` procedure is structured along the same lines as the general `liealg` procedure discussed in Section 4.2.2 of [KirkTr]. It has, essentially, three tasks.

First it assigns Maple names to the source coordinates. The code used for this requires the source dimension, r, to be specified. This is done using the (user-defined) global variable `source_dim` which controls the number of source coordinates for ensuing calculations. Then a list of names for these source coordinates are found and stored in a global list `coords`. Although we use $x_1, \ldots, x_r$ for source coordinates their actual Maple names are defined to be

$x1, \ldots, xr$, are stored as such in **coords** and then used to describe any jet A passed to `jetcalc`.

Secondly, a set of generators for the (source) Lie algebra are specified and stored in the parameter **tgtspace**. Each entry of **tgtspace** is *itself* of type 'table' with N components and corresponds to a generator, ξ_i, specifying how ξ_i acts on A. The precise syntax is as follows. Suppose A is given, in Maple, by a list of N entries, $A = [a_1, \ldots, a_N]$, and the ith vector, ξ_i, in the generating set is of the form given in (5.2). Then the ith entry entry of **tgtspace** specifies the N components of $\xi_i.A$ and is defined in Maple by

$$\begin{aligned}
\mathbf{tgtspace}[i][1] &:= g_1 * \mathrm{diff}(A[1], \mathbf{coords}[1]) + \cdots + g_r * \mathrm{diff}(A[1], \mathbf{coords}[r]);\\
&\quad\ldots\\
\mathbf{tgtspace}[i][N] &:= g_1 * \mathrm{diff}(A[N], \mathbf{coords}[1]) + \cdots + g_r * \mathrm{diff}(A[N], \mathbf{coords}[r]);
\end{aligned}$$

The tangent spaces we require `jetcalc` to generate are of the form

$$L.A = \mathcal{M}_r^{p_1} \langle \xi_1.A, \ldots, \xi_r.A \rangle + \mathcal{M}_r^{p_2} \langle \xi_{r+1}.A, \ldots, \xi_{r+n^2}.A \rangle, \tag{5.3}$$

where $\{\xi_1.A, \ldots, \xi_r.A\}$ are the standard generators of the Jacobian ideal J_A, $\{\xi_{r+1}.A, \ldots, \xi_{r+n^2}.A\}$ are the generators

$$(A_{ij} = C_{ij}(A) + R_{ij}(A), 1 \leq i, j \leq n),$$

described in Proposition 4.3.2 and $p_1 \geq p_2$ (the powers of the maximal ideal $\mathcal{M}_r$) differ depending on whether we require the $\mathcal{G}_1$, $\mathcal{G}$ or $\mathcal{G}_e$-tangent spaces to A. Comparing (5.3) with the defining equation for Lie algebras of 'type' $\mathcal{R}$, (see (5.1)), in each case we obtain most of the required tangent space by setting the global variable **source_power** to $p_1 = \max\{p_1, p_2\}$. However, the space subsequently generated by `jetcalc` using

$$L.A = \mathcal{M}_r^{p_1} \langle \xi_1.A, \ldots, \xi_{r+n^2}.A \rangle, \tag{5.4}$$

omits the tangent vectors of (5.3) given by

$$(\mathcal{M}_r^{p_2} / \mathcal{M}_r^{p_1}) \langle \xi_{r+1}.A, \ldots, \xi_{r+n^2}.A \rangle.$$

This problem is resolved by defining a global variable to add these 'extra' vectors to those obtained from (5.4). The variable used is **R_nilp** and is of

Maple type 'list'. Each entry in this 'list' specifies an 'extra' vector which is to be added to (the source component of) L. These vectors take the form $g\xi_i$ where $g \in \mathcal{O}_r$ and ξ_i $(r+1 \leq i \leq r+n^2)$ is an existing vector from (5.4). The precise syntax is as follows. For $g \in \mathcal{O}_r$ and ξ_i the ith vector specified by table `tgtspace`, the entry $[g, i]$ in the list `R_nilp` indicates that `jetcalc` will add the vector $g\xi_i$ to the tangent space (generated by (5.4)).

Clearly, these 'extra' vectors depend on which tangent space we are considering. In particular, we generate the $\mathcal{G}_1$ (or CT-group) tangent space by setting `source_power` to 2 and `R_nilp` to be the set of vectors

$$\{x_j\xi_i : r+1 \leq i \leq r+n^2, 1 \leq j \leq r\}, \tag{5.5}$$

and the $\mathcal{G}$ (or full group) tangent space is given by `source_power` 1 with `R_nilp` the set of vectors

$$\{\xi_i : r+1 \leq i \leq r+n^2\}. \tag{5.6}$$

It is convenient to define both these sets (5.5) and (5.6) in the procedure `skewmatrix`, storing them in global variables `RCT_nilp` and `RG_nilp` respectively. Then depending on which tangent space we wish `jetcalc` to generate we can assign the appropriate set to the global variable `R_nilp`, discussed above.

These 'extra' vectors will be included in the tangent space generated by `jetcalc` provided another global variable `nilp` is set to `true`. If however this variable is set to `false`, then, when called, `jetcalc` will ignore any 'extra' vectors in `R_nilp`. This is what is required when generating the 'extended' tangent space,

$$T\mathcal{G}_e.A = \langle \xi_1.A, \ldots, \xi_{r+n^2}.A \rangle,$$

obtained by setting `source_power` 0 and `nilp` to `false`.

5.3 The Algorithm : jetcalc

Briefly, we discuss the algorithm carried out by the main routine `jetcalc`. Having specified the Lie algebra L, by assigning the global variables `equiv`, `liealg`, `source_power`, `R_nilp` and `nilp`, then for a given k and jet A `jetcalc` first calculates the tangent space $J^k(L.A)$ to the orbit of A. Specifically, the algorithm calculates a spanning set, for this tangent space, as a vector subspace of $J^k(r,N)$, where $J^k(r,N)$ is identified with N-tuples of *all* polynomials, in r indeterminates, (over $\mathbb{C}$) of degree $\leq k$, including those which don't vanish at the origin.

By calling the procedure `skewmatrix`, `jetcalc` finds the vectors which generate $L.A$ as the $\mathcal{O}_r$-module $\langle \xi_1.A, \dots, \xi_{r+n^2}.A\rangle$. Then it finds all non-zero k-jets in

$$L.A = \mathcal{M}_r^{p_1}\langle \xi_1.A, \dots, \xi_{r+n^2}.A\rangle$$

by multiplying each of these generators $(\xi_i.A)$ by all monomials of degree p_1 and higher until jets, whose components all have initial degree greater then k, are obtained. By adding to this set of k-jets any 'extra' vectors, from `R_nilp`, `jetcalc` obtains a spanning set for the required tangent space $J^k(L.A)$.

Having done this the next and main computational task of `jetcalc` is to reduce this spanning set, by Gaussian elimination, to a basis for the tangent space. Although the algorithm is more sophisticated than the proceeding description, it is sufficient to interprete it by the following linear algebra problem. For more details on the actual algorithm see Chapter 3 in [KirkTr].

We have a basis of $J^k(r,N)$ consisting of an ordering of the monomial vectors, $x^I e_j$, (see Remark 4.4.6[2]) and with respect to this basis each jet in $J^k(r,N)$ corresponds to an element of a vector space $\mathbb{C}^M$ of large dimension. The spanning set, just discussed, corresponds to a matrix whose rows consist of these vectors. The ordering `jetcalc` chooses, for the basis, is such that the columns of this matrix corresponding to monomial vectors of degree k appear, together, in a block at the right hand end of the matrix. Reducing this matrix to row echelon form using Gaussian elimination gives a basis for the tangent space.

The advantage of writing the package in a symbolic system such as Maple is that we can allow parameters to be present in the jets we work with, thus enabling us to perform calculations over whole *families* of jets. On passing such a family to `jetcalc` the matrix representing the spanning set (of the tangent space to jets in this family) will contain non-constant (polynomial) terms. This must be taken into account by the reduction algorithm.

[2]with regards to this Remark, note that $|I| = 0$ is also considered here.

Where possible the algorithm chooses numerical pivotal elements. If forced to choose a non-constant pivotal element division is *not* performed on the row, in which it occurs, to reduce it to unity. However, division is still performed (working in the field of rational functions) when using the pivot to reduce the rest of the column to zero. In other words the leading entry of this row remains the non-constant pivot rather than unity. Clearly, for values of the parameters (of the family passed to `jetcalc`) for which this pivot vanishes the row operations used in the elimination are not valid, but, by preserving the pivot, conditions when this occurs are retained. We shall discuss this further later on.

Despite the size of this matrix being large it is also highly sparse and can, in most cases, be reduced relatively quickly. We take this opportunity to mention a further global variable which needs to be set before calling `jetcalc`, namely `jetcalc_verbosity`. During a calculation `jetcalc` relays information relating to how the calculation is progressing and this variable controls how much of that information is displayed to the user. Taking an integer value from 0 (forcing `jetcalc` to work in silence) through to 3 (for which `jetcalc` displays all aspects of the calculation) the `jetcalc_verbosity` setting enables the user not only to check that the calculation being performed is the one required but can also give an idea of the size of the matrix involved. For our calculations we set `jetcalc_verbosity` to 2 and in the worked examples at the end of this chapter we reproduce the information typical to that displayed by `jetcalc` in response to this setting.

The result of the reduction performed by `jetcalc` is therefore a row echelon matrix of the form (a_{ij}) with pivotal elements $a_{1j_1}, a_{2j_2}, \ldots, a_{uj_u}$ (these are non-zero elements with $1 \leq j_1 < j_2 < \cdots < j_u \leq q = \dim J^k(r, N)$ and $u = \dim J^k(L.A))$. The rows of this matrix then represent a basis for the tangent space, the principal objective of the algorithm.

The reduction process, used to achieve this objective, yields various by-products which are of considerable use as regards the singularity theory discussed in the previous chapter. These are stored in global variables for access by other routines of the package. We proceed to discuss these routines, which are used to manipulate and extrapolate the results of the reduction.

The routine `jetcalc` also calculates a basis, C, for the complementary (or normal) space to the tangent space it has found. This is the set of monomial vectors required to extend the basis for the tangent space to one of full rank in $J^k(r,N)$. Having already calculated a basis for the tangent space, represented by the u rows of the echelon matrix (a_{ij}) above, the set of canonical vectors

$$\{f_1,\ldots,\hat{f_{j_1}},\ldots,\hat{f_{j_2}},\ldots,\hat{f_{j_u}},\ldots,f_q\}$$

(where $\{f_i \in \mathbb{C}^M : 1 \leq i \leq M\}$ represents the ordered set of monomial vectors, $x^I e_j$, chosen as a basis for $J^k(r,N)$ and $\hat{f_j}$ denotes the exclusion of f_j from this set of vectors) correspond to the basis C. Once this set, C, of monomial vectors has been found they are then stored globally and can be accessed by the function `pcomp()` (i.e. `pcomp()` displays the complementary basis). The dimension of this complementary space is stored as the global variable `codim`.

It is important to highlight the following point, when interpreting calculations. As previously mentioned the routine `jetcalc` regards $J^k(r,N)$ as the vector space of N-tuples of *all* polynomials of degree $\leq k$ thereby including those which do not vanish at the origin. This enables the package to be used for unfolding calculations and working with 'extended' tangent spaces. So any basis for a complementary space in $J^k(r,N)$ found by `jetcalc` usually contains constant vectors.

5.4 Interpreting `jetcalc` Calculations in terms of Singularity Theory

We discuss how the results obtained by `jetcalc` can be interpreted as regards the singularity theory covered in the previous chapter.

5.4.1 Complete Transversals and Determinacy

Due to the ordering, chosen by `jetcalc`, of the basis of $J^k(r,N)$, the columns corresponding to monomials of degree k appear in a block at the right hand end of the echelon matrix (a_{ij}). It follows that the set of tangent vectors from this matrix, with leading entries in this final block, can be extended to a basis for $H^k(r,N)$ by adding to it all elements of the complementary basis, C, of degree k.

In particular, given a $(k-1)$-jet A, if we specify the Lie algebra so that `jetcalc` generates the $J^k\mathcal{G}_1$-tangent space to A then by Theorem 4.4.13 the terms of the complementary basis, calculated by `jetcalc`, of degree k provide a *complete transversal* for A.

This complete transversal can be extracted from the complementary basis by, before calling `jetcalc`, setting a further global variable, `compltrans`, to `true`. Then after running `jetcalc` this ensures that the function `pcomp()` displays *only* the terms of the complementary basis of degree k.

If there are no terms of degree k in the complementary basis then the function `pcomp()` returns a message indicating that the k-transversal of the $(k-1)$-jet is empty and, since the tangent space is an $\mathcal{O}_r$-module, by Corollary 4.4.19, A is $(k-1)$-$\mathcal{G}_1$-determined.

5.4.2 Using Mather's Lemma

Having discussed how, by setting the Lie algebra to the *CT-group*, we can use `jetcalc` to find complete transversals and detect determined jets we consider what useful information can be gleaned when working with the full jet-group, $J^k\mathcal{G}$.

So given a k-jet A we define the Lie algebra so that `jetcalc` generates the $J^k\mathcal{G}$-tangent space to A. This setup is used, principally, to simplify families of k-jets into a finite number of $J^k\mathcal{G}$-orbits by applying Mather's Lemma (in the form of Lemma 4.5.1). Whereas the two conditions for Mather's Lemma are extremely difficult to check by hand they are easily dealt with by this package. For this work, we require the *full* complementary basis, calculated by `jetcalc`, to be displayed by the function `pcomp()`. To ensure this, before any calculation is performed by `jetcalc`, it is necessary to set the global variable `compltrans` to `false`. It is worth pointing out that the value of `compltrans` has no effect on the actual calculations carried out within `jetcalc`, it is just a user-interface device.

A useful invariant for distinguishing $J^k\mathcal{G}$-orbits is the $J^k\mathcal{G}$-codimension of jets lying in such orbits. Naturally, this codimension is found from the complementary basis calculated by `jetcalc`. However, when looking for the $J^k\mathcal{G}$-

codimension of k-jets which *vanish at* 0 we disregard, from this complementary basis, any constant jets. The $J^k\mathcal{G}$-codimension is therefore the number of non-constant monomial vectors in the basis displayed by `pcomp()`.

Typically, given a complete transversal of a $(k-1)$-jet A_0, represented by a t-parameter family, $\{A_a \in J^k(r, N) : a \in \mathbb{C}^t\}$, of k-jets, we use `jetcalc` to simplify, where possible, its members into a finite number of $J^k\mathcal{G}$-orbits. Since, by Theorem 4.4.13, this transversal meets *each* $J^k\mathcal{G}$-orbit of a jet with $(k-1)$-jet A_0, this means that, at the k-jet level, a neighbourhood of any such jet consists of finitely many $J^k\mathcal{G}$-orbits.

We have previously mentioned how `jetcalc` approaches the reduction when passed a parametrised family of jets. The resulting row echelon matrix (a_{ij}) may well contain pivotal elements which are rational functions in the parameters. The algorithm `jetcalc` used to derive the complementary basis, C, from (a_{ij}) also collects all of these non-numeric pivots, and stores them in a 'checklist' for global access once `jetcalc` has terminated. The function `plist()` can be used to display this checklist. Recall that for values of the parameters for which these non-numeric pivots vanish the reduction process used by `jetcalc` is not valid. Hence the numerators of each element of this list define a *finite* set of proper algebraic varieties, within the parameter space, where this occurs.

So the reduction performed by `jetcalc` applies to members of the family, passed to it, corresponding to values of the parameters *not* lying on any of these varieties and therefore determines the generic behaviour by default. For the following discussion assume that a t-parameter family, A_a, of k-jets has been passed to `jetcalc` and it has calculated the tangent space, $J^k(L.A_g)$, to the orbit of A_g for generic values, g, of a. We refer to the sub-manifold of $J^k(r, N)$ consisting of these generic jets by $Z \subset J^k(r, N)$.

We first use the package to check whether both conditions of Mather's Lemma are satisfied for Z. Clearly, condition (ii) is satisfied since, by default, all k-jets in this sub-manifold have the same $J^k\mathcal{G}$-codimension, given by the number of non-constant jets in the complementary basis found by `jetcalc`.

Condition (i) involves determining, for each k-jet $A_g \in Z$, whether the $J^k(L.A_g)$-tangent space contains the tangent space to Z (at A_g). Since `jetcalc` has calculated a basis for $J^k(L.A_g)$, represented by the row echelon matrix (a_{ij}), it is relatively easy to check this. The tangent space to Z, at each point A_g, is the subspace $\mathrm{sp}\{w_1, \ldots, w_t\}$, where each vector w_i is obtained by differentiating the family of k-jets, A_a, with respect to the ith parameter. The routine `intangent` can then be used to test whether each one of these vectors is contained in the default tangent space, $J^k(L.A_g)$, calculated by `jetcalc`.

In general, this routine is passed a list of jets $\{v_1, \ldots, v_p\}$, considered as vectors of $\mathbb{C}^M$, and determines whether this set together with the basis for the tangent space, $J^k(L.A_g)$, form a dependent set of vectors. It returns `true` when a dependent set results and `false` when the set is independent. We refer the reader to Section 4.4 of [KirkTr] for the technical details of this routine. It follows that if a *single* vector v is passed to `intangent` it will return `true` if v is in the tangent space but `false` if not.

Given a spanning set $\{w_1, \ldots, w_t\}$ for the tangent space to the submanifold, $Z \subset J^k(r, N)$, for our purposes it is sufficient to use `intangent` individually on *each* vector in this set. If `intangent` returns `true` for each one of these vectors it follows that the tangent space to Z is contained in the tangent space $J^k(L.A_g)$ and condition (ii) of Mather's Lemma is satisfied.

Once both conditions have been shown to hold for the sub-manifold Z of $J^k(r, N)$, (consisting of jets A_g) by Mather's Lemma we can deduce this sub-manifold to be contained in a single $J^k\mathcal{G}$-orbit. By choosing a particular value of g, g_0, the corresponding jet A_{g_0} is a representative for this orbit.

Alternatively, if for any one of these vectors, `intangent` were to return `false` *or* it is observed that one of them is present in the complementary basis, C, calculated by `jetcalc`, then, the tangent space to the sub-manifold Z cannot be contained in $J^k(L.A_g)$. Consequently, the criterion for simplicity, in Lemma 4.5.3, is not met and the family of jets A_a has moduli. We have just described how the package can be used to detect moduli. It is also possible to use the package to determine the number of moduli present in a particular family. However since we are only concerned with $\mathcal{G}$-simples this is not required.

Having found a (representative of) a $J^k\mathcal{G}$-orbit for generic members, A_g, of the family of jets A_a, we must also investigate the *exceptional behaviour*, i.e.

the members of the family having parameter values for which the reduction performed by `jetcalc` is not valid. To do this we need to inspect the list of non-numeric pivots, displayed by the function `plist()`, obtaining from each entry in this list a condition on the parameters for which it vanishes. From each condition, we find a variety (of the parameter space) on which the pivot vanishes. If this variety can itself be parametrised, then we can substitute the parameter values on this variety back into the original family. The resulting family is then passed to `jetcalc`. (Note in this way we have reduced the number of parameters of the family (passed to `jetcalc`) by 1.) Denote *generic* members of this family by, A_s. Hence `jetcalc` calculates the tangent space, $J^k(L.A_s)$, to the orbits of k-jets A_s. As before we use Mather's Lemma to determine whether the corresponding sub-manifold of $J^k(r, N)$ consisting of these k-jets are contained in a single $J^k\mathcal{G}$-orbit. Of course we must also investigate any exceptional behaviour for this modified family by inspecting the list of non-numeric pivots used for the reduction and so on.

This process is repeated until either we find the original family to be contained in a finite number of $J^k\mathcal{G}$-orbits, we identify distinct $J^k\mathcal{G}$-orbits by comparing the $J^k\mathcal{G}$-codimension of jets in each using `pcomp()`, or moduli are detected. Although this process appears horrendous, for our purposes we use it when dealing with families of only one or two parameters and the subsequent calculations don't prove to be too arduous.

We conclude this section by discussing how the package can be used to identify a family of k-jets, $Z_a = \{A_a \in J^k(r, N) : a \in \mathbb{C}^t\}$, which is $J^k\mathcal{G}$-trivial.

Typically if, on passing a family Z_a to `jetcalc`, the subsequent calculation of the tangent space, $J^k(L.A_a)$, is valid for *all* values of its parameters, a, (i.e. the 'checklist' displayed by `plist()` is empty) *and* both conditions of Mather's Lemma are satisfied (for this tangent space) then the family is $J^k\mathcal{G}$-trivial.

5.5 Examples of Calculations using `Transversal`

In this section we demonstrate how some standard calculations, needed for our classification, can be carried out using `Transversal`. This we do by means of a sample of worked examples taken from calculations discussed in the following chapter. These examples are specifically chosen to illustrate the various techniques, described in the previous sections, and implemented during the proceeding calculations.

In particular providing details at this stage, such as actual commands used and some of the corresponding Maple responses, allows us to gloss over (or omit completely) these technicalities later on so that we can concentrate, instead, on the calculations themselves. We first describe how, in practice, the various Lie algebra setups are defined before dealing with the calculations themselves.

5.5.1 Initialising Calculations

Before going any further note that calculations are carried out on Maple V Release 4 via the graphical interface "xmaple" on Unix. Also after initiating such a Maple session we first load `Transversal` by the command,

```
> with(transversal);
[canonical_vector, classify, coeff_table, determined, gausselim,

    get_coeff, get_deg, get_monomial, get_ref_tables, get_wt,

    increment, intangent, jetcalc, ldegree_vector, pcomp, pdetterms,

    plist, pmons, ptangent, pvars, scalar_multn, setup_Aclassn,

    setup_Agroup, setup_Aunf, stdjacobian]
```

to which, as shown, Maple responds by listing all subroutines, functions that are available within the package.

All calculations involve working in parallel with three different jet groups, the $J^k\mathcal{G}_1$ or CT-group for complete transversal and determinacy results, the group, $J^k\mathcal{G}$, for using Mather's Lemma and moduli detection, and the extended group, $J^k\mathcal{G}_e$ for finding 'extended' codimensions (and unfolding calculations).

As mentioned in Section 5.2 each of these groups require different Lie algebra setups and the most efficient way to do this is to create three entirely different Maple sessions, one for each type of calculation. We proceed to define these three setups. Unless otherwise stated it is assumed we are classifying germs $\mathbb{C}^2, 0 \to Sk(4, \mathbb{C})$ up to the $\mathcal{G}$-equivalence defined above, which amounts to considering germs $\mathbb{C}^2, 0 \to \mathbb{C}^6, 0$ up to this equivalence (see Lemma 4.2.5).

The global variable setup defining the Lie algebra of the *CT-group*, used in all complete transversal and determinacy calculations, is stored for convenience in the file `skewsetup`. Reading in this file by the command :

```
read skewsetup;
                    liealg := skewmatrix
                        equiv := R
                    compltrans := true
                     source_dim := 2
                    source_power := 2
                    target_power := 0
                      nilp := true
                    R_nilp := RCT_nilp
                      L_nilp := []
                 jetcalc_verbosity := 2
```

Maple responds by printing the values assigned by this file to the global variables (described in the previous sections).

Several of these (user-defined) global variables are assigned the same values for all three setups, i.e. the globals : `equiv`, `liealg`, `jetcalc_verbosity` take the values prescribed in Sections 5.2 and 5.3.

Another constant for this set of germs is the source dimension, specified by setting `source_dim` to be 2, and as a result the Lie algebra procedure, `skewmatrix` denotes the source coordinates by `x1`, `x2`. However, when not reproducing Maple commands or output we shall represent them by x, y respectively.

The remaining (user-defined) global variables are assigned values intrinsic to defining tangent spaces, to $J^k\mathcal{G}_1$-orbits of germs, A, given by

$$T\left(J^k\mathcal{G}_1.A\right) = j^k\left[\mathcal{M}_2^2\{A_x, A_y\} + \mathcal{M}_2\{C_{ij}(A) + R_{ij}(A) : 1 \leq i, j \leq 4\}\right].$$

In particular, we refer the reader to Section 5.2 to remind them of the values of `source_power` and `R_nilp` required for the CT-group.

We perhaps should say a little more concerning our use of the variables `nilp/R_nilp`. The more usual use of the variables `nilp/R_nilp/L_nilp` is for determinacy calculations using a *nilpotent* Lie algebra — hence their name. For complete transversal calculations (using a nilpotent space) the homogeneous jets must be ordered as dictated by the nilpotent filtration. To achieve this `nilp` is set equal to `true_order` and two further variables, `nilp_source_wt` and `nilp_target_wt`, are brought into play for defining the ordering.

However, for our case we use the variable `R_nilp` for a different purpose, i.e. to add vectors missing from the tangent space generated by `jetcalc`, as discussed in Section 5.2. Notice we do not use `L_nilp` since, by setting `equiv` to be 'type' R, we are only considering a source component Lie algebra. (The same applies to the variable `target_power`.)

By setting `nilp` to `true`, thereby enabling us to use `R_nilp`, a default lexicographical ordering is used which ensures the basis of $J^k(2,6)$ is ordered such that all monomial vectors of degree k appear last. This facilitates the extraction of the complete transversal, of a jet, from the complementary basis found by `jetcalc` (see Section 5.4.1) provided `compltrans` is set to `true`, as above.

For the *group* Lie algebra setup we create a different session and, after loading `Transversal` and reading in the previous setup, we *reassign* the three (user-defined) global variables `source_power`, `R_nilp` and `compltrans` as follows :

```
source_power:=1; R_nilp:=RG_nilp; compltrans:=false;
                                  source_power := 1

                                  R_nilp := RG_nilp

                                 compltrans := false
```

This modification to the previous setup defines tangent spaces, to $J^k\mathcal{G}$-orbits of germs, A, given by

$$T\left(J^k\mathcal{G}.A\right) = j^k\left[\mathcal{M}_2\{A_x, A_y\} + \mathcal{O}_2\{C_{ij}(A) + R_{ij}(A) : 1 \le i, j \le 4\}\right].$$

We again refer to Section 5.2 for the justification for the first two reassignments. Recall from Section 5.4.2 that setting **compltrans** to **false** enables us to display the full complementary basis (to the tangent space to a jet) calculated by `jetcalc`.

When setting up a Lie algebra in this way, i.e. by reassigning several global variables, it is advisable to inspect the current values of all the user-defined global variables to check this has been done correctly. For this purpose we use the function **pvars()**.

Finally, for finding $\mathcal{G}_e$-codimensions (of determined jets) we need the *extended group* setup obtained by reassigning the following two variables of the *group* setup :

```
> source_power:=0; nilp:=false;

                    source_power := 0

                      nilp := false
```

These modifications define $J^k\mathcal{G}_e$-tangent spaces to germs A, given by

$$T\left(J^k\mathcal{G}_e.A\right) = j^k\left[\mathcal{O}_2\{A_x, A_y,\ C_{ij}(A) + R_{ij}(A) : 1 \le i, j \le 4\}\right],$$

and are mentioned, briefly, at the end of Section 5.2. Clearly, on setting **source_power** to 0 means that `jetcalc` generates all vectors of the 'extended' tangent space and so we can ignore any 'extra' vectors present in the variable **R_nilp** by setting **nilp** to **false**. Also note the value of **compltrans** is kept **false**, since for codimension calculations we require the full complementary basis.

Having thus defined each of the three Lie algebra setups, used for the following examples, from this point on we refer to them as the CT-group, group and extended group setups without any further explanation. We also remind the reader that each such setup and the calculations performed using it occurs on a different Maple session.

5.5.2 Worked Examples

The following worked examples are taken from calculations used to give Theorem 6.1.10, in the following chapter, and concern the classification of $\mathcal{G}$-simple determined germs with 2-jet $[0, x, 0, y, x^2, 0]$.

We start by considering the $J^3\mathcal{G}_1$-orbits over the 2-jet $[0, x, 0, y, x^2, 0]$ using the CT-group setup. Specifying the jet $A = [0, x, 0, y, x^2, 0]$ by entering

```
> A:=[0,x1,0,x2,x1^2,0];
                                              2
                         A := [0, x1, 0, x2, x1 , 0]
```

we calculate the $J^3\mathcal{G}_1$-tangent space to A in $J^3(2, 6)$ by the command

```
> jetcalc(A,3);
                              defined map:
                                            2
                           [0, x1, 0, x2, x1 , 0]

                    working in 3-jet space with R-equivalence

                          defined coordinates:

                                [x1, x2]

               ***    calculating right tangent space    ***

               ***    performing Gaussian elimination    ***

                         using default ordering

                        calculating tangent space

                        matrix dimensions: 76, 60

                                 Ready.
```

where the second parameter (passed to `jetcalc`) indicates the degree of the jet space in which the calculation is performed.

As promised in Section 5.3, this output is typical of the response from `jetcalc`, when the `jetcalc_verbosity` is set to 2. In the remainder of this section, although always providing the commands entered, we will not always show the corresponding Maple response. For example, in future, the response to an assignment, such as `A` above, is usually omitted — as is output from `jetcalc` of the type shown above.

As described in Section 5.3, this command calculates a basis for both the $J^3\mathcal{G}_1$-tangent space to A and its complementary space in $J^3(2,6)$, storing all the results. Recalling the discussion of Section 5.4.1, the basis for a complete transversal of A is displayed by typing

```
> pcomp();
                                  3
                        [0, 0, x2 , 0, 0, 0]

                                 2
                       [0, 0, x2  x1, 0, 0, 0]
```

Consequently, a complete 3-transversal to A is the 2-parameter family of 3-jets :

$$A_{ab} = [0, x, axy^2 + by^3, y, x^2, 0],$$

where a, $b \in \mathbb{C}$.

As discussed in Section 5.4.2 we can use the group setup for applying Mather's Lemma (Lemma 4.5.1) to simplify such a family into a finite number of $J^3\mathcal{G}$-orbits.

We start by calculating the $J^3\mathcal{G}$-tangent space to the family of 3-jets, A_{ab}. (Note for the purposes of `Transversal` we assign this family to the variable `fam1`.)

```
> fam1 := [0, x1, a* x1 * x2^2 + b* x2^3, x2, x1^2, 0];
> jetcalc(fam1,3);

          WARNING: global variable 'checklist' is non-empty !!!

                               Ready.
```

Here in addition to the standard output displayed, `jetcalc` adds a warning concerning the 'checklist'. As discussed earlier, in Sections 5.3 and 5.4.2, if `jetcalc` is passed a parametrised family of jets it is quite possible that the row echelon matrix obtained (by the reduction of the spanning set for the tangent space being calculated) contains a number of non-numeric pivots. Hence the calculation performed by `jetcalc` is only valid for those parameter values for which *none* of these non-numeric pivots vanish. The 'checklist' mentioned in the warning is a list of all these non-numeric pivots and is displayed by typing

```
> plist();
                              #1, b

                              #2, a
```

The output consists of two columns, the first indicating the index number of the pivotal element, as an entry in the table where the checklist is stored, and the second column contains the actual pivotal element. By default `plist()` factorises the pivots before displaying them thereby enabling the exceptional values (values of the parameters where a pivot vanishes) to be more easily identified. In future we demonstrate this situation by giving the non-empty 'checklist' and omit the associated warning message.

So, in the present case the reduction performed by `jetcalc` applies to members of A_{ab} for which $a \neq 0$ and $b \neq 0$. This set of (generic) 3-jets of A_{ab} is a connected submanifold of the 3-jet space and, for convenience, we denote it as follows :

$$Z = \{z \in A_{ab} : a \neq 0, b \neq 0\} \subset J^3(2,6).$$

All the results of the reduction, carried out by `jetcalc`, therefore hold for each element $z \in Z$. With this in mind we check whether both conditions of Lemma 4.5.1 are satisfied by these results.

Firstly, a basis for the complement to the $J^3\mathcal{G}$-tangent space, for any element $z \in Z$, is displayed by the command

```
> pcomp();
                         [1, 0, 0, 0, 0, 0]

                         [0, 1, 0, 0, 0, 0]
```

```
                [0, 0, 1, 0, 0, 0]

                [0, 0, 0, 1, 0, 0]

                [0, 0, 0, 0, 1, 0]

                [0, 0, 0, 0, 0, 1]

               [0, 0, x2, 0, 0, 0]

               [0, 0, x1, 0, 0, 0]

               [0, 0, 0, 0, x1, 0]

                         2
               [0, 0, x2 , 0, 0, 0]

              [0, 0, x2 x1, 0, 0, 0]
```

The variable codim stores the number of these basis vectors

```
> codim;
                                  11
```

Recall, from Section 5.4.2, that since the jets we are considering all vanish at 0 the $J^k\mathcal{G}$-codimension (of such jets) is given by the number of non-constant monomial vectors in the complementary basis. So, clearly each element $z \in Z$ has the same $J^3\mathcal{G}$-codimension, namely 5, and condition (ii) of Lemma 4.5.1 is satisfied.

The tangent space, T_zZ, to each element z of Z is given by the span of the two vectors

$$\{[0,0,xy^2,0,0,0],[0,0,y^3,0,0,0]\}$$

and, as described in Section 5.4.2, showing condition (i) of Lemma 4.5.1 to be satisfied involves verifying whether *both* these vectors lie in the $J^3\mathcal{G}$-tangent space calculated by jetcalc. This is done by typing the following

```
> intangent([0,0,x1*x2^2,0,0,0]);

      WARNING: original matrix contains non-numeric elements,
```

```
                    check checklist !!!

                              true

> intangent([0,0,x2^3,0,0,0]);

        WARNING: original matrix contains non-numeric elements,

                    check checklist !!!

                              true
```

The warning message in each case is carried over from the previous calculation (by `jetcalc`) and reminds us that we still need to investigate the 'exceptional' values $a = 0$ and $b = 0$.

So condition (i) of Lemma 4.5.1 is also satisfied and it follows that all elements of A_{ab}, for which $a \neq 0$ and $b \neq 0$, are contained in a single $J^3\mathcal{G}$-orbit of $J^3(2,6)$ with $J^3\mathcal{G}$-codimension 5. By choosing the parameter values $(a,b) = (1,1)$ (i.e. satisfying $a \neq 0$, $b \neq 0$) the corresponding element of A_{ab},

$$[0, x, xy^2 + y^3, y, x^2, 0], \tag{5.7}$$

is a representative for this orbit.

It remains to check the 'exceptional' values $a = 0$ and $b = 0$. We start by considering parameter values for which the first pivot, a, vanishes. Substituting $a = 0$ into A_{ab} we calculate the $J^3\mathcal{G}$-tangent space of the resulting family of 3-jets, $A_{0b} = [0, x, by^3, y, x^2, 0]$, (which for **Transversal** is denoted **fam21**) :

```
fam21:=subs(a=0,fam1);
                                       3          2
                  fam21 := [0, x1, b x2 , x2, x1 , 0]

> jetcalc(fam21,3);

> plist();

                             #1, b

                            #2, -4 b
```

(Note : subs is a standard Maple function for substitution) Here the reduction performed by `jetcalc` is only valid for the 3-jets of A_{0b} with parameter values $b \neq 0$. We denote the connected submanifold, of Z, consisting of these 3-jets by

$$W = \{w \in A_{0b} : b \neq 0\}.$$

By the techniques described above, we find that the $J^3\mathcal{G}$-codimension of each 3-jet $w \in W$ is 5 and the vector, $[0,0,y^3,0,0,0]$, spanning T_wW (the tangent space to W at w) is contained in the $J^3\mathcal{G}$-tangent space calculated by `jetcalc`. It therefore follows, by Lemma 4.5.1, that all elements of A_{0b}, for which $b \neq 0$, are contained in a single $J^3\mathcal{G}$-orbit of $J^3(2,6)$ with $J^3\mathcal{G}$-codimension 5 and representative

$$[0,x,y^3,y,x^2,0]. \tag{5.8}$$

Notice that the 3-jets (5.7) and (5.8) have the same $J^3\mathcal{G}$-codimension 5 and it is possible that they both represent the same $J^3\mathcal{G}$-orbit. To investigate this we construct the 1-parameter family, of $J^3(2,6)$, connecting them :

$$A_t = [0,x,y^3 + txy^2,y,x^2,0].$$

Denoting A_t by `fam22` we use the group setup to find its $J^3\mathcal{G}$-tangent space.

```
> fam22:=[0,x1,x2^3+t*x1*x2^2,x2,x1^2,0];
> jetcalc(fam22,3);
```

Although `jetcalc` is passed a 1-parameter family of jets, there is no warning message concerning the 'checklist', as in the previous cases. In fact on inspecting the checklist

```
> plist();
                  ***    CHECKLIST EMPTY    ***
```

we find it is empty. It follows that the calculation performed by `jetcalc` is valid for every value of the parameter t, i.e. for every element of the family A_t.

By the previous techniques we find the $J^3\mathcal{G}$-codimension of each 3-jet of A_t to be 5 and that the vector, $[0,0,xy^2,0,0,0]$, spanning the tangent space to A_t is contained in the $J^3\mathcal{G}$-tangent space calculated by `jetcalc`. It therefore follows, by Lemma 4.5.1, that the entire family, A_t, is contained in a single $J^3\mathcal{G}$-orbit of $J^3(2,6)$. In other words, A_t is a $J^3\mathcal{G}$-trivial family and therefore the 3-jets (5.7) and (5.8) do, in fact, lie in the same $J^3\mathcal{G}$-orbit.

We have just demonstrated that provided the parameter $b \neq 0$ the corresponding elements of A_{ab} all lie in a single $J^3\mathcal{G}$-orbit, of $J^3\mathcal{G}$-codimension 5, which has a representative

$$[0, x, y^3, y, x^2, 0]. \tag{5.9}$$

So, with a little thought it follows that the 3-jets of A_{ab} which remain to be considered are all given by substituting $b = 0$ into A_{ab} giving the family

$$A_{a0} = [0, x, axy^2, y, x^2, 0].$$

However, to illustrate the technique used for more complicated siuations we proceed with a more exhaustive approach.

Backtracking slightly, we need to consider the 'exceptional' value, $b = 0$, for the `jetcalc` calculation for `fam21`. Substituting $b = 0$ into A_{0b} and calculating the $J^3\mathcal{G}$-tangent space to the resulting 3-jet :

```
> A3:=subs(b=0,fam21);
> jetcalc(A3,3);
```

we find this jet,

$$A_3 = [0, x, 0, y, x^2, 0], \tag{5.10}$$

to have $J^3\mathcal{G}$-codimension 7. This 3-jet lies in a distinct $J^3\mathcal{G}$-orbit to that represented by (5.9) since their $J^3\mathcal{G}$-codimensions differ.

Having thus considered the parameter values for which the first pivot (of the `jetcalc` calculation for `fam1`) vanishes it is also necessary to investigate the 3-jets of A_{ab} for which $b = 0$. Again we proceed with the formal technique by substituting $b = 0$ into A_{ab} and calculating the $J^3\mathcal{G}$-tangent space to the family

$$A_{a0} = [0, x, axy^2, y, x^2, 0].$$

We find the subsequent reduction performed by `jetcalc` to be valid for 3-jets of A_{a0} for which $a \neq 0$. By a similar method to that used previously we find these elements of A_{a0} to be contained in the single $J^3\mathcal{G}$-orbit with $J^3\mathcal{G}$-codimension 6 and representative

$$A_2 = [0, x, xy^2, y, x^2, 0]. \tag{5.11}$$

It follows that this represents a further (distinct) $J^3\mathcal{G}$-orbit. Note that the exceptional value from the previous calculation by `jetcalc` i.e. $(b = 0)$, $a = 0$ has already been considered.

5.5.3 Further Illustrative Examples

In the previous subsection we have established that the complete transversal, $A_{ab} = [0, x, axy^2 + by^3, y, x^2, 0]$, (to the 2-jet $[0, x, 0, y, x^2, 0]$) is contained in three distinct $J^3\mathcal{G}$-orbits with representatives :

$$A_1 = [0, x, y^3, y, x^2, 0], \qquad A_2 = [0, x, xy^2, y, x^2, 0]$$

and

$$A_3 = [0, x, 0, y, x^2, 0].$$

The remaining examples in this section consider the continuation of the classification of each one of these representatives.

Example 1

Taking the 3-jet $A_1 = [0, x, y^3, y, x^2, 0]$ we seek a complete transversal by investigating its $J^4\mathcal{G}_1$-tangent space. In the CT-group setup we enter the commands :

```
> A1:= [0, x1, x2^3, x2, x1^2, 0];
> jetcalc(A1, 4);
```

Then, as before, a basis for a complete 4-transversal for A_1 is obtained by typing

```
> pcomp();
                    ***   THE NORMAL SPACE IS EMPTY   ***
```

This response indicates that the 4-transversal to A_1 is empty and (since the tangent space to the germ is an $\mathcal{O}_2$-module) we can apply Nakayama's Lemma (see Corollary 4.4.19) and deduce that A_1 is 3-$\mathcal{G}$-determined.

So, having found a finitely determined germ it is desirable to calculate its $\mathcal{G}_e$-codimension which, here, is given by the $J^3\mathcal{G}_e$-codimension of $A_1 = [0, x, y^3, y, x^2, 0]$.

As remarked above we need to use the extended group setup for this purpose. In this setup, by the commands

```
> A1 := [0, x1, x2^3, x2, x1^2, 0];
> jetcalc(A1,3);
```

`jetcalc` calculates the $J^3\mathcal{G}_e$-tangent space to A_1.

We take this opportunity to check that the (user-defined) global variables for this setup have been correctly defined :

```
> pvars();
```

```
liealg = skewmatrix

equiv = R

compltrans = false

source_dim = 2

source_power = 0

target_power = 0

nilp = false

R_nilp:

[[1, 13], [1, 14], [1, 15], [1, 16], [1, 17], [1, 18], [1, 3], [1, 4],
    [1, 5], [1, 6], [1, 7], [1, 8], [1, 9], [1, 10], [1, 11], [1, 12]]

L_nilp:

[]

nilp_source_wt:
```

```
nilp_source_wt

nilp_target_wt:

nilp_target_wt
```

Observe that, as a result of running `jetcalc`, the variable `R_nilp` holds a list of 'extra' vectors (in this case those needed for the group setup) defined and stored, originally, in the global variable `RG_nilp` within the Lie algebra procedure, `skewmatrix`. A similar list would appear for the other two setups (although for the CT-group case the 'extra' vectors would differ, being those defined and stored, originally, in the global variable `RCT_nilp`). The interpretation of the syntax for these vectors is given in Section 5.2. However, for the extended group setup we do not require these vectors to be added (to the tangent space) and to ensure this the value of `nilp` is set to `false`.

Another point worth noticing is the appearance of (user-defined) global variables, `nilp_source_wt` and `nilp_target_wt`. As discussed in Section 5.5.1, these are only used for nilpotent Lie algebra setups, when the value of `nilp` is set equal to `true_order`, and so for all our calculations are irrelevant.

The basis of the complement to the $J^3\mathcal{G}_e$-tangent space to A_1 is displayed in the usual way

```
> pcomp();

                    [1, 0, 0, 0, 0, 0]

                    [0, 0, 1, 0, 0, 0]

                    [0, 0, 0, 0, 1, 0]

                    [0, 0, 0, 0, 0, 1]

                   [0, 0, x2, 0, 0, 0]

                   [0, 0, x1, 0, 0, 0]

                   [0, 0, 0, 0, x1, 0]

                            2
```

```
[0, 0, x2 , 0, 0, 0]

[0, 0, x2 x1, 0, 0, 0]
```

Since when considering $J^3\mathcal{G}_e$-tangent spaces we work over the complete jet-space, $J^3(2,6)$, of 6-tuples of *all* polynomials truncated to degree 3, i.e. including those which do not vanish at the origin, the $J^3\mathcal{G}_e$-codimension is given by the number of all the monomial vectors in this basis. So the variable

```
> codim;

                                    9
```

gives the $\mathcal{G}_e$-codimension of $A_1 = [0, x, y^3, y, x^2, 0]$. Furthermore this basis yields a versal unfolding for A_1.

Example 2

Taking the 3-jet, $A_2 = [0, x, xy^2, y, x^2, 0]$, we find using the CT-group setup that it has a complete transversal

$$A_c = [0, x, xy^2 + cy^4, y, x^2, 0],$$

in $J^4(2,6)$. By definition (of complete transversals) any 4-jet with 3-jet A_2 is $J^4\mathcal{G}$-equivalent to something in this transversal. Denoting A_c by `fam3` we use the group setup to investigate its $J^4\mathcal{G}$-tangent space.

```
> fam3:=[0,x1,x1*x2^2+c*x2^4,x2,x1^2,0];
                                   2          4         2
            fam3 := [0, x1, x2  x1 + c x2 , x2, x1 , 0]

> jetcalc(fam3,4);
> plist();

                       ***    CHECKLIST EMPTY    ***
```

Here, we find the 'checklist' to be empty i.e. the calculation performed by `jetcalc` is valid for every value of the parameter c. By previously described techniques we find this entire family to be contained in a single $J^4\mathcal{G}$-orbit of

$J^4(2,6)$ of $J^4\mathcal{G}$-codimension 6. In other words, A_c is a $J^4\mathcal{G}$-trivial family and we can represent it by the 4-jet

$$[0, x, xy^2, y, x^2, 0].$$

So any 4-jet, with 3-jet $[0, x, xy^2, y, x^2, 0]$ is $J^4\mathcal{G}$-equivalent to $[0, x, xy^2, y, x^2, 0]$. Furthermore using the CT-group setup we find that this 4-jet has an empty 5-transversal in $J^5(2,6)$. It follows that the 3-jet,

$$[0, x, xy^2, y, x^2, 0],$$

is 3-$\mathcal{G}$-determined and using the extended group setup we calculate its $\mathcal{G}_e$-codimension ($J^3\mathcal{G}_e$-codimension) to be 10.

Example 3

Taking the 3-jet $A_3 = [0, x, 0, y, x^2, 0]$, we find, using the CT-group setup, a complete 4-transversal

$$A_{cd} = [0, x, cxy^3 + dy^4, y, x^2, 0].$$

On, using the group setup, to calculate the $J^4\mathcal{G}$-tangent space to this family of 4-jets we find that both pivots don't vanish provided $d \neq 0$. In other words, initially, we only have to check one exceptional condition in this case. Compare this with the more complicated situation encountered when considering the family A_{ab} above. We find that elements of A_{cd} for which $d \neq 0$ are contained in a single $J^4\mathcal{G}$-orbit of $J^4\mathcal{G}$-codimension 7 with representative,

$$[0, x, y^4, y, x^2, 0].$$

This representative is 4-$\mathcal{G}$-determined and has $\mathcal{G}_e$-codimension ($J^4\mathcal{G}_e$-codimension) 11.

Investigating the exceptional case $d = 0$, i.e. the family of 4-jets

$$A_{c0} = [0, x, cxy^3, y, x^2, 0],$$

we find that elements of this family for which $c \neq 0$ are contained in the single $J^4\mathcal{G}$-orbit with representative

$$A_4 = [0, x, xy^3, y, x^2, 0],$$

which has $J^4\mathcal{G}$-codimension 8. (Note we could also deduce this, by hand, using the scaling changes described in Section 4.5 and demonstrated in the following chapter.)

Using the CT-group setup we find a complete transversal to A_4 in $J^5(2,6)$:

$$A_e = [0, x, xy^3 + ey^5, y, x^2, 0].$$

We investigate the $J^5\mathcal{G}$-tangent space to this family using the group setup (denoting A_e by `fam4`)

```
> fam4:=[0,x1,x1*x2^3+e*x2^5,x2,x1^2,0];
> jetcalc(fam4,5);
> plist();
                    ***    CHECKLIST EMPTY    ***
```

We first note that the 'checklist' is empty, so the calculation is valid for all values of the parameter, e. Inspecting the basis for the complementary space to the $J^5\mathcal{G}$-tangent space

```
> pcomp();
                        [1, 0, 0, 0, 0, 0]

                        [0, 1, 0, 0, 0, 0]

                        [0, 0, 1, 0, 0, 0]

                        [0, 0, 0, 1, 0, 0]

                        [0, 0, 0, 0, 1, 0]

                        [0, 0, 0, 0, 0, 1]

                       [0, 0, x2, 0, 0, 0]

                       [0, 0, x1, 0, 0, 0]

                       [0, 0, 0, 0, x1, 0]
                                 2
                       [0, 0, x2 , 0, 0, 0]
```

```
         [0, 0, x2 x1, 0, 0, 0]
                   3
          [0, 0, x2 , 0, 0, 0]
                 2
        [0, 0, x2  x1, 0, 0, 0]
                   4
          [0, 0, x2 , 0, 0, 0]
                   5
          [0, 0, x2 , 0, 0, 0]
```

we observe that the vector, $[0,0,y^5,0,0,0]$, spanning the tangent space to the family, A_e, is present in the complementary basis and therefore is *not* contained in the $J^5\mathcal{G}$-tangent space (for any e). Consequently, the criterion for simplicity, given in Lemma 4.5.3 is not met for $k=5$ and we deduce that any germ with 5-jet $J^5\mathcal{G}$-equivalent to something in A_e cannot be simple. It follows by, the definition of a complete transversal, that any germ with 4-jet lying in the $J^4\mathcal{G}$-orbit represented by

$$A_4 = [0,x,xy^3,y,x^2,0]$$

also cannot be simple and, since we are only classifying simples, this case needn't be considered any further.

Normally, we would need to inspect the exceptional behaviour of the family A_{c0} when $c=0$. However, the jet, obtained by substituting $c=0$ into A_{c0},

$$A_5 = [0,x,0,y,x^2,0]$$

needn't be considered any further since in *any* neighbourhood of it there is a 4-jet,

$$[0,x,\epsilon xy^3,y,x^2,0]$$

where ϵ is small, of a germ for which, by the above, any neighbourhood meets uncountably many $J^5\mathcal{G}$-orbits.

In conclusion, these worked examples have demonstrated, computationally, that any $\mathcal{G}$-simple map $A : \mathbb{C}^2,0 \to \mathbb{C}^6,0$ with 2-jet, $[0,x,0,y,x^2,0]$, is $\mathcal{G}$-equivalent to one of the three finitely determined germs :

$$[0,x,y^3,y,x^2,0],$$

$$[0,x,xy^2,y,x^2,0],$$

$$[0, x, y^4, y, x^2, 0],$$

with $\mathcal{G}_e$-codimensions 9, 10 and 11 respectively.

Chapter 6

Calculations

In this chapter we apply the theory and techniques of the previous two chapters to classify simple families of skew-symmetric matrices, represented by smooth germs
$$\mathbb{C}^r, 0 \longrightarrow \mathbb{C}^N, 0,$$
under the action $\mathcal{G} = \mathcal{R} \times \mathcal{H}$. As discussed in Section 4.6 we have already covered the cases when $r = 1$ and when $r > 1$, $n = 2, 3$. Here we are concerned with the case $r \geq 2$, $n = 4$ i.e. two or more parameter families of 4×4 skew-symmetric matrices.

We start with some general results useful for these classifications.

Lemma 6.0.1 *Any map $\hat{A} : \mathbb{C}^r, 0 \to Sk(4, \mathbb{C})$ of the form,*
$$\hat{A} = \begin{bmatrix} 0 & a_1 & a_2 & a_3 \\ -a_1 & 0 & a_4 & a_5 \\ -a_2 & -a_4 & 0 & a_6 \\ -a_3 & -a_5 & -a_6 & 0 \end{bmatrix},$$
where $a_i \in \mathcal{M}_r$, is $\mathcal{H}$-equivalent to both,
$$\tilde{A} = \begin{bmatrix} 0 & -a_3 & a_1 & a_2 \\ a_3 & 0 & a_5 & a_6 \\ -a_1 & -a_5 & 0 & a_4 \\ -a_2 & -a_6 & -a_4 & 0 \end{bmatrix}$$

and

$$\bar{A}_t = \begin{bmatrix} 0 & a_4 & a_1 & a_5 \\ -a_4 & 0 & a_2 & a_6 \\ -a_1 & -a_2 & 0 & -a_3 \\ -a_5 & -a_6 & a_3 & 0 \end{bmatrix}.$$

In particular, $\bar{A}$ *and* $\bar{A}_t$ *are* $\mathcal{H}$*-equivalent.*

Proof Recall, from Corollary 4.2.1, that parametrised simultaneous row and column operations on such matrices correspond to the action of elements of $\mathcal{H}$. Here we rely on simultaneous row and column interchanges, discussed in Remark 2.1.15 in Chapter 2.

Starting with

$$\hat{A} = \begin{bmatrix} 0 & a_1 & a_2 & a_3 \\ -a_1 & 0 & a_4 & a_5 \\ -a_2 & -a_4 & 0 & a_6 \\ -a_3 & -a_5 & -a_6 & 0 \end{bmatrix},$$

by interchanging C_2 and C_4 (and R_2 and R_4) we obtain the matrix

$$\begin{bmatrix} 0 & a_3 & a_2 & a_1 \\ -a_3 & 0 & -a_6 & -a_5 \\ -a_2 & a_6 & 0 & -a_4 \\ -a_1 & a_5 & a_4 & 0 \end{bmatrix}.$$

Interchanging C_3 and C_4 (and R_3 and R_4) gives

$$\begin{bmatrix} 0 & a_3 & a_1 & a_2 \\ -a_3 & 0 & -a_5 & -a_6 \\ -a_1 & a_5 & 0 & a_4 \\ -a_2 & a_6 & -a_4 & 0 \end{bmatrix},$$

which on multiplying R_2 (and C_2) by -1 gives the required matrix

$$\bar{A} = \begin{bmatrix} 0 & -a_3 & a_1 & a_2 \\ a_3 & 0 & a_5 & a_6 \\ -a_1 & -a_5 & 0 & a_4 \\ -a_2 & -a_6 & -a_4 & 0 \end{bmatrix}.$$

Similarly, returning to $\hat{A}$ and applying the sequence of simultaneous row and column operations : $C_1 \leftrightarrow C_3$ $(R_1 \leftrightarrow R_3)$, $C_1 \leftrightarrow C_2$ $(R_1 \leftrightarrow R_2)$ and $-1R_3$

$(-1C_3)$, we obtain the matrix

$$\bar{A}_t = \begin{bmatrix} 0 & a_4 & a_1 & a_5 \\ -a_4 & 0 & a_2 & a_6 \\ -a_1 & -a_2 & 0 & -a_3 \\ -a_5 & -a_6 & a_3 & 0 \end{bmatrix}.$$

□

In fact we can represent any smooth map $\bar{A} : \mathbb{C}^r, 0 \to Sk(4, \mathbb{C})$ by the block matrix

$$\bar{A} = \begin{bmatrix} A_1 & A \\ -A^T & A_2 \end{bmatrix}, \tag{6.1}$$

where A is a smooth map germ $\mathbb{C}^r, 0 \to M(2, \mathbb{C})$ and A_1, A_2 are smooth map germs $\mathbb{C}^r, 0 \to Sk(2, \mathbb{C})$.

Consider the subgroup $\mathcal{H}_q \subset \mathcal{H}$ consisting of germs, $\mathbb{C}^r, 0 \to Gl(4, \mathbb{C})$,

$$X_q = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}$$

where both X_1 and X_2 are smooth germs, $\mathbb{C}^r, 0 \to Gl(2, \mathbb{C})$. The action of elements of this subgroup on matrices of type (6.1) gives the ($\mathcal{H}$-)equivalent matrix

$$\begin{bmatrix} X_1^T A_1 X_1 & X_1^T A X_2 \\ -(X_1^T A X_2)^T & X_2^T A_2 X_2 \end{bmatrix}. \tag{6.2}$$

Setting $Y = X_1^T$, $X^{-1} = X_2$ we can therefore use a natural equivalence of families of square matrices, A, to aid the classification of $\mathcal{G}$-equivalent families of skew-symmetric matrices of the type in (6.1). We point out that, mostly, this equivalence is used when considering matrices, of type (6.1), for which $A_1 = A_2 = 0$ in which case the action of $(\mathcal{R} \times \mathcal{H}_q)$ amounts to a natural equivalence of the family of square matrices, A.

6.0.4 Families of Square Matrices

In the following section we briefly discuss the classification of families of square matrices under this equivalence. For further details we refer the reader to [BrTarSq].

As usual $M(n,\mathbb{C})$ denotes the space of $n \times n$ matrices over $\mathbb{C}$. The set of singular matrices is a hypersurface D in $M(n,\mathbb{C})$ given by the vanishing of the determinant.

We are classifying the set, $\mathcal{N}$, of smooth germs

$$A : \mathbb{C}^r, 0 \longrightarrow M(n,\mathbb{C}),$$

subject to the following equivalences.

Definition 6.0.2 *Let $\mathcal{Q}$ be the set of germs $\mathbb{C}^r, 0 \to Gl(n,\mathbb{C}) \times Gl(n,\mathbb{C})$ with the group structure inherited from that in the target. Then if A, $B : \mathbb{C}^r, 0 \to M(n,\mathbb{C})$ are smooth map germs we say they are $\mathcal{Q}$-equivalent if and only if for some $(X,Y) \in \mathcal{Q}$ we have*

$$B = YAX^{-1}.$$

Notice that $\mathcal{Q}$-equivalence is the parametrised analogue of the standard action of $Gl(n,\mathbb{C}) \times Gl(n,\mathbb{C})$ on the square matrices, $M(n,\mathbb{C})$, corresponding to basis change. Since $\mathcal{N}$ is the set of $n \times n$ matrices defined over $\mathcal{O}_r$ and $\mathcal{Q}$ is the group of pairs of invertible matrices over this integral domain we can deduce the following result from Theorem 2.1.11 in Chapter 2.

Corollary 6.0.3 *Two germs A, $B \in \mathcal{N}$ are $\mathcal{Q}$-equivalent if it is possible to pass from one to the other by a series of elementary row/column operations.*

Proof The proof follows from Theorem 2.1.11, with $R = \mathcal{O}_r$. $\square$

This result is an analogue of Corollary 4.2.1 in Section 4.1 although note here, since we are working with $\mathcal{N}$, as opposed to Sk, all parametrised row/column operations are independent of each other.

As before, the action of $\mathcal{Q}$ is combined with an $\mathcal{R}$ change of source coordinates to give the following equivalence on $\mathcal{N}$.

Definition 6.0.4 *If A, $B : \mathbb{C}^r, 0 \to M(n,\mathbb{C})$ are smooth map germs we say they are $(\mathcal{R} \times \mathcal{Q})$-equivalent if and only if for some $(\phi, (X,Y)) \in \mathcal{R} \times \mathcal{Q}$ we have*

$$B = Y(A \circ \phi)X^{-1}.$$

An element of $\mathcal{N}$ can also be thought of as map $\mathbb{C}^r, 0 \to \mathbb{C}^P$, where $P = n^2$. We state the following result which is an analogue of Lemma 4.2.5.

Lemma 6.0.5 *The group $\mathcal{R} \times \mathcal{Q}$ acts on the space of mappings $\mathbb{C}^r, 0 \to \mathbb{C}^P$ as a subgroup of the corresponding contact group $\mathcal{K}$.*

Proof The proof is similar to that of Lemma 4.2.5. In particular if $\mathcal{C}$ is the group of mappings

$$\mathbb{C}^r, 0 \to Gl(P, \mathbb{C}),$$

there is group homomorphism from $\mathcal{Q}$ to a subgroup of $\mathcal{C}$. We represent the image of an element $(X, Y) \in \mathcal{Q}$ under this homomorphism by $\tilde{X} \in \mathcal{C}$. □

In fact it can be shown, by a similar procedure to that outlined in Appendix B, that $\mathcal{R} \times \mathcal{Q}$ is one of Damon's geometric subgroups of $\mathcal{K}$. As a consequence we can apply all the techniques of singularity theory, discussed in Sections 4.4 and 4.5 of Chapter 4, to these germs. However in order to do this we need to find a set of generators for the $(\mathcal{R} \times \mathcal{Q})$-tangent space to germs $A \in \mathcal{N}$.

Proposition 6.0.6 *(i) The $\mathcal{R}$-tangent space to the orbit of the element $A \in \mathcal{N}$ is the $\mathcal{O}_r$-module spanned by the $x_j A_{x(i)} = x_j \partial A/\partial x_i$, where $1 \leq i, j \leq r$.*

(ii) Let $R_j(A)$ (respectively $C_i(A)$) denote the jth row (respectively ith column) of A. Then the tangent space to the orbit of A under the subgroup $\mathcal{Q}$ of $\mathcal{C}$ is the $\mathcal{O}_r$-module spanned by the set of matrices $R_{jl}(A)$ (respectively $C_{im}(A)$), with lth row (respectively mth column) $R_j(A)$ (respectively $C_i(A)$) and zeros elsewhere, for $1 \leq l, m \leq n$ and $1 \leq i, j \leq n$. So the tangent space to the $(\mathcal{R} \times \mathcal{Q})$-orbit of A is

$$T(\mathcal{R} \times \mathcal{Q}).A = \mathcal{M}_r\{A_{x(i)}\} + \mathcal{O}_r\{R_{jl}(A), C_{im}(A)\}.$$

Proof The vectors emerging from the action of the $\mathcal{R}$-group are found in the usual way. For the $\mathcal{Q}$-group we consider the action (on the left) of the path $(I_n + taE_{ij}, I_n)$ in $\mathcal{Q}$ on matrix A for t small, where I_n is the $n \times n$ identity matrix, $(E_{ij} : 1 \leq i, j \leq n)$ give the basis vectors for $M(n, \mathbb{C})$, as described in the Proof of Proposition 4.3.2, and $a \in \mathcal{O}_r$. The tangent vector of the resulting path in $\mathcal{N}$, at $t = 0$, is given by

$$\lim_{t \to 0} \left\{ \frac{(I_n + taE_{ij})AI_n - A}{t} \right\}$$

$$= a(E_{ij}A),$$

yielding the tangent vectors

$$\mathcal{O}_r \{R_{ji} : 1 \le i, j \le n\}. \tag{6.3}$$

Similarly by considering the action (on the right) of the path $(I_n, I_n + taE_{ij}) \in \mathcal{Q}$ on matrix A, for small t, we find that the tangent vector of the resulting path in $\mathcal{N}$, at $t = 0$, is

$$a(AE_{ij}),$$

thereby yielding the tangent vectors

$$\mathcal{O}_r \{C_{ij} : 1 \le i, j \le n\}. \tag{6.4}$$

The sum of both sets of vectors, (6.3) and (6.4), gives the $\mathcal{Q}$-tangent space to A. So the $(\mathcal{R} \times \mathcal{Q})$-tangent space to A is

$$\mathcal{M}_r \left\{A_{x(i)} : 1 \le i \le r\right\} + \mathcal{O}_r \{R_{ji}, C_{ij} : 1 \le i, j \le n\}.$$

□

The subgroup of $(\mathcal{R} \times \mathcal{Q})$ required for the complete transversal theory of Section 4.4 is given here by the semi-direct product $(\mathcal{R}_1 \times \mathcal{Q}_0)$ where $\mathcal{R}_1$ is as described in Definition 4.3.3 and $\mathcal{Q}_0$ is the subgroup of $\mathcal{Q}$ consisting of germs $\mathbb{C}^r, 0 \to Gl(n, \mathbb{C}) \times Gl(n, \mathbb{C})$ with constant part the identity (I_n, I_n). The following corollary gives the tangent space to the action of this group on a germ $A \in \mathcal{N}$.

Corollary 6.0.7 *The $(\mathcal{R}_1 \times \mathcal{Q}_0)$-tangent space to a germ, $A \in \mathcal{N}$, is given by*

$$\mathcal{M}_r^2 \{A_{x(i)} : 1 \le i \le r\} + \mathcal{M}_r \{R_{ji}(A), C_{ij}(A) : 1 \le i, j \le n\}.$$

Proof The first set of vectors are given by the standard $\mathcal{R}_1$-tangent space to a germ $\mathbb{C}^r, 0 \to \mathbb{C}^P$. The second set are given by considering paths in $\mathcal{Q}_0$. For example in the case of the left action we have paths such as $(I_n + taE_{ij}, I_n)$ where $a \in \mathcal{M}_r$. The result then follows from the proof of Proposition 6.0.6. □

We continue this summary of $(\mathcal{R} \times \mathcal{Q})$-equivalence with the analogues of Definition 4.2.8 and Proposition 4.2.10.

Definition 6.0.8 *The* **discriminant** *of an element $A \in \mathcal{N}$ is the set $\mathcal{D}(A) = \{x \in \mathbb{C}^r, 0 : \det A(x) = 0\}$.*

Proposition 6.0.9 *If A, $B : \mathbb{C}^r, 0 \to M(n, \mathbb{C})$ are $(\mathcal{R} \times \mathcal{Q})$-equivalent then their determinants are $\mathcal{K}$-equivalent. Geometrically, this means there is a germ of a diffeomorphism $\mathbb{C}^r, 0 \to \mathbb{C}^r, 0$ taking $\mathcal{D}(A)$ to $\mathcal{D}(B)$.*

Proof This is similar to that of Proposition 4.2.10. □

To find versal unfoldings of germs $A : \mathbb{C}^r, 0 \to M(n, \mathbb{C})$ we need to consider the 'extended tangent space' (see Section 4.4.3).

Definition 6.0.10 *Consider the mapping $A : \mathbb{C}^r, 0 \to M(n, \mathbb{C})$, which can also be thought of as an element $A \in \mathcal{O}_r^P$. The 'extended tangent space' or $(\mathcal{R} \times \mathcal{Q})_e$-tangent space of A, denoted $T(\mathcal{R} \times \mathcal{Q})_e.A$, is defined to be*

$$T(\mathcal{R} \times \mathcal{Q})_e.A = \mathcal{O}_r\{A_{x(i)} : 1 \leq i \leq r,\ R_{jl}(A), C_{im}(A) : 1 \leq i, j \leq n\}.$$

The $(\mathcal{R} \times \mathcal{Q})_e$-codimension of A is the codimension of $T(\mathcal{R} \times \mathcal{Q})_e.A$ in $\mathcal{O}_r^P$, that is

$$(\mathcal{R} \times \mathcal{Q})_e\text{-}codim A = \dim_{\mathbb{C}} \left(\mathcal{O}_r^P / T(\mathcal{R} \times \mathcal{Q})_e.A\right).$$

We use results analogous to those discussed in Section 4.4.3 to calculate these codimensions and to find a basis for $\mathcal{O}_r^P / T(\mathcal{R} \times \mathcal{Q})_e.A$.

The determinacy results, corresponding to those discussed in Section 4.4.2, have the following geometric interpretation, relating to the natural stratification of $M(n, \mathbb{C})$.

We need to say a few words here. Recall that we have an action of $Gl(n, \mathbb{C}) \times Gl(n, \mathbb{C})$ on $M(n, \mathbb{C})$. There are finitely many orbits (simply determined by the rank of the matrices). The orbits are smooth manifolds, and the union of these is the *stratification* referred to. When we say that a map $A : \mathbb{C}^r \to M(n, \mathbb{C})$ is transverse to the set of singular matrices, D, we mean that it is transverse to each of the orbits (or *strata*) in the stratification.

Lemma 6.0.11 *Consider the germ of a smooth mapping $A : \mathbb{C}^r, 0 \to M(n, \mathbb{C})$, vanishing at the origin. If A is finitely $(\mathcal{R} \times \mathcal{Q})$-determined then off $0 \in \mathbb{C}^r$ A is transverse to the set of singular matrices in $M(n, \mathbb{C})$ and in particular $A^{-1}(0)\backslash\{0\}$ is a smooth manifold of dimension $r - n^2$.*

Proof Consider the extended tangent space $T(\mathcal{R} \times \mathcal{Q})_e.A$. Fixing an ordering for the elements of the set, $\{R_{jl}(A), C_{im}(A)\}$, described in Proposition 6.0.6, we denote the sth vector in this set $Q_s A$, $(1 \leq s \leq 2n^2)$, and write

$$T(\mathcal{R} \times \mathcal{Q})_e.A = \mathcal{O}_r\{A_{x(i)},\ Q_s A : 1 \leq i \leq r,\ 1 \leq s \leq 2n^2\}.$$

If A is finitely determined it has finite $(\mathcal{R}\times\mathcal{Q})_e$-codimension which, by Lemma 4.4.23 in Section 4.4.3, implies, for some integer $N \geq 0$, that $\mathcal{M}_r^N \cdot \mathcal{O}_r^P \subset T(\mathcal{R}\times\mathcal{Q})_e.A$. Let e_k $(1 \leq k \leq P)$ denote the standard basis vectors for $\mathbb{C}^P$. Then, in particular,

$$x_j^N e_k = \underbrace{\sum_{i=1}^{r} \alpha_i^{jk}(x) A_{x(i)}(x)}_{\in dA_x(\mathbb{C}^r)} + \underbrace{\sum_{s=1}^{2P} \beta_s^{jk}(x) Q_s A(x)}_{\text{tangent to } D}, \tag{6.5}$$

for some α_i^{jk}, $\beta_s^{jk} \in \mathcal{O}_r$, (where $1 \leq j \leq r$, $1 \leq k \leq P$ and D is the hypersurface defined at the beginning of this section). Choose any non-zero point $\mathbf{x} \in \mathbb{C}^r$, for which $A(\mathbf{x}) \in D$, that is $A(\mathbf{x})$ is a singular matrix. It follows that some coordinate of $\mathbf{x}$, $x_j \neq 0$, for the sake of argument let this be x_1. Substituting this value of $\mathbf{x}$ into (6.5) gives

$$x_1^N e_k = \sum_{i=1}^{r} \alpha_i^{1k}(\mathbf{x}) A_{x(i)}(\mathbf{x}) + \sum_{s=1}^{2P} \beta_s^{1k}(\mathbf{x}) Q_s A(\mathbf{x}),$$

for each $1 \leq k \leq P$, and since $x_1^N \neq 0$ it follows that each $e_k \in dA_{\mathbf{x}}(\mathbb{C}^r) + D_{A(\mathbf{x})}$, where $D_{A(\mathbf{x})}$ is the tangent space to D at $A(\mathbf{x})$. Since this argument applies to any choice of non-zero $x \in \mathbb{C}^r$, then it follows for all $x \neq 0$ with $A(x)$ singular, that

$$\mathrm{im}\,(dA_x) + D_{A(x)} = M(n, \mathbb{C}).$$

In other words A is transverse to D off $0 \in \mathbb{C}^r$. In particular if we consider the (codimension P) submanifold of D consisting of just the null matrix, i.e.

$$Q = \{0 \in M(n, \mathbb{C})\},$$

then clearly, off $0 \in \mathbb{C}^r$, A is also transverse to Q and applying Proposition 4.1.3 $(A^{-1}(0)\backslash\{0\})$ is a smooth submanifold of $\mathbb{C}^r$ of dimension $r - P$. □

We also have a similar result for mappings into the space of $n \times n$ skew-symmetric matrices. (A map, $A : \mathbb{C}^r \to Sk(n, \mathbb{C})$, is transverse to the set of singular skew matrices if it is transverse to each of the strata in the stratification of $Sk(n, \mathbb{C})$.)

Lemma 6.0.12 *Consider the germ of a smooth mapping $A : \mathbb{C}^r, 0 \to Sk(n, \mathbb{C})$, vanishing at the origin. If A is finitely $\mathcal{G}$-determined then off $0 \in \mathbb{C}^r$ A is transverse to the set of singular matrices in $Sk(n, \mathbb{C})$ and in particular $A^{-1}(0)\backslash\{0\}$ is a smooth manifold of dimension $r - n(n-1)/2$.*

Proof This is similar to that for Lemma 6.0.11. □

6.0.5 Comparing Equivalences

In this section we link the $\mathcal{G}$-classification of smooth families of 4×4 skew-symmetric matrices, $\tilde{A} : \mathbb{C}^r, 0 \to Sk(4, \mathbb{C})$, of the form

$$\tilde{A} = \begin{bmatrix} 0 & 0 & a & b \\ 0 & 0 & c & d \\ -a & -c & 0 & 0 \\ -b & -d & 0 & 0 \end{bmatrix}, \tag{6.6}$$

with the $(\mathcal{R} \times \mathcal{Q})$-classification of the corresponding 2×2 matrices, $A : \mathbb{C}^r, 0 \to M(2, \mathbb{C})$,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}. \tag{6.7}$$

Starting with several results we finish with a brief discussion of how they can be used to aid the ensuing calculations.

Lemma 6.0.13 *Given two smooth germs A, $B : \mathbb{C}^r, 0 \to M(2, \mathbb{C})$ which are $(\mathcal{R} \times \mathcal{Q})$-equivalent then the corresponding skew-symmetric germs, $\bar{A}$, $\bar{B} : \mathbb{C}^r, 0 \to Sk(4, \mathbb{C})$, given by block matrices :*

$$\bar{A} = \begin{bmatrix} 0 & A \\ -A^T & 0 \end{bmatrix}, \qquad \bar{B} = \begin{bmatrix} 0 & B \\ -B^T & 0 \end{bmatrix}, \tag{6.8}$$

(where 0 is the 2×2 null matrix) are $(\mathcal{R} \times \mathcal{H}_q)$-equivalent (and therefore $\mathcal{G}$-equivalent).

Proof If A and B are $(\mathcal{R} \times \mathcal{Q})$-equivalent then for some germ of a diffeomorphism $\phi : \mathbb{C}^r, 0 \to \mathbb{C}^r, 0$ and some pair $(X, Y) \in \mathcal{Q}$

$$B = Y(A \circ \phi)X^{-1}.$$

Then by choosing $X_q \in \mathcal{H}_q$, such that

$$X_q = \begin{bmatrix} Y^T & 0 \\ 0 & X^{-1} \end{bmatrix},$$

it follows from (6.2) that

$$\bar{B} = X_q^T(\bar{A} \circ \phi)X_q,$$

as required. □

Lemma 6.0.14 *Consider a smooth germ, $\bar{A} : \mathbb{C}^r, 0 \to Sk(4, \mathbb{C})$, of the form*

$$\bar{A} = \begin{bmatrix} 0 & 0 & a & b \\ 0 & 0 & c & d \\ -a & -c & 0 & 0 \\ -b & -d & 0 & 0 \end{bmatrix},$$

where a, b, c, $d \in \mathcal{M}_r$ and define $A : \mathbb{C}^r, 0 \to M(2, \mathbb{C})$ to be

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Then it follows that :

(i) the ($\mathcal{G}$-)discriminant of $\bar{A}$ is just the (($\mathcal{R} \times \mathcal{Q}$)-) discriminant of A, and

(ii) we have the following relation

$$\mathcal{G}_e\text{-}codim_{\mathcal{S}k}\tilde{A} = (\mathcal{R} \times \mathcal{Q})_e\text{-}codim_{\mathcal{N}} A + 2 \dim_{\mathbb{C}} \mathcal{O}_r / \langle a, b, c, d \rangle. \tag{6.9}$$

Proof (i) Since $\det \tilde{A} = (\det A)^2$ this follows immediately from Corollary 4.2.9 and Definition 6.0.8.

(ii) Associating any skew-symmetric matrix

$$\tilde{A} = \begin{bmatrix} 0 & 0 & a & b \\ 0 & 0 & c & d \\ -a & -c & 0 & 0 \\ -b & -d & 0 & 0 \end{bmatrix},$$

with the 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

it follows by an explicit calculation of the generators of the $\mathcal{G}$-tangent space to $\tilde{A}$, as described in Proposition 4.3.2, that

$$T\mathcal{G}.\tilde{A} \cong T(\mathcal{R} \times \mathcal{Q}).A + \mathcal{O}_r \left\{ \langle a, b, c, d \rangle e_1, \langle a, b, c, d \rangle e_6 \right\},$$

where $T(\mathcal{R} \times \mathcal{Q}).A$ is given by Proposition 6.0.6.

In other words, we can consider slots e_2, e_3, e_4, e_5 of $\tilde{A}$ together as a 2×2 matrix and each of slots e_1 and e_6 separately, in both the latter cases the tangent vectors, arising from the action of the $\mathcal{H}$-group only, being given by the span of the ideal $\mathcal{O}_r \langle a, b, c, d \rangle$.

In particular, the extended tangent space is given by

$$T\mathcal{G}_e.\tilde{A} \cong T(\mathcal{R} \times \mathcal{Q})_e.A + \mathcal{O}_r \left\{ \langle a, b, c, d \rangle e_1, \langle a, b, c, d \rangle e_6 \right\},$$

and the $\mathcal{G}_e$-codimension of $\tilde{A}$ is

$$\mathcal{G}_e\text{-codim}\tilde{A} = (\mathcal{R} \times \mathcal{Q})_e\text{-codim}A + 2 \dim_{\mathbb{C}} \mathcal{O}_r / \langle a, b, c, d \rangle.$$

□

Theorem 6.0.15 *Consider the map germs $\tilde{A} : \mathbb{C}^r, 0 \to Sk(4, \mathbb{C})$,*

$$\tilde{A} = \begin{bmatrix} 0 & 0 & a & b \\ 0 & 0 & c & d \\ -a & -c & 0 & 0 \\ -b & -d & 0 & 0 \end{bmatrix},$$

and $A : \mathbb{C}^r, 0 \to M(2,\mathbb{C})$,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

(a, b, c, $d \in \mathcal{M}_r$) described in Lemma 6.0.14, where r is either 1, 2, 3 *or* 4. *Then $\bar{A}$ is finitely $\mathcal{G}$-determined if and only if A is finitely $(\mathcal{R} \times \mathcal{Q})$-determined.*

Proof We use the relation, (6.9), between codimensions of $\bar{A}$ and A, found in Lemma 6.0.14 :

$$\mathcal{G}_e\text{-codim}_{\mathcal{S}k}\bar{A} = (\mathcal{R} \times \mathcal{Q})_e\text{-codim}_{\mathcal{N}}A + 2\dim_{\mathbb{C}} \mathcal{O}_r/\langle a,b,c,d\rangle. \tag{6.10}$$

Clearly, if $\mathcal{G}_e$-codim $\bar{A} < \infty$, it follows from this relation that the $(\mathcal{R}\times\mathcal{Q})_e$-codim of A must also be finite and, by applying Lemma 4.4.23 and Theorem 4.4.16, A is finitely $(\mathcal{R} \times \mathcal{Q})$-determined.

The converse is not as simple. If A is $(\mathcal{R}\times\mathcal{Q})$-determined, then it is required to prove that $\dim_{\mathbb{C}} (\mathcal{O}_r/\mathcal{O}_r\langle a,b,c,d\rangle)$ is finite.

However, using Lemma 6.0.11 $A^{-1}(0)\backslash\{0\}$ is a smooth manifold of dimension $(r-4)$. If $1 \leq r \leq 3$ this submanifold is empty and $A^{-1}(0) = \{0\}$. If $r = 4$, $A^{-1}(0)\backslash\{0\}$ is a smooth 0 manifold, consisting of isolated points. So by the *Curve Selection Lemma*, see [Milnor] Chapter 3, $A^{-1}(0)\backslash\{0\}$ cannot have 0 in its closure and since we are considering $A^{-1}(0)$ as a germ it follows that $A^{-1}(0) = \{0\}$ for this case as well.

Having therefore established that $A^{-1}(0) = \{0\}$ it follows automatically that

$$\mathbf{V}(a,b,c,d) = \{0\},$$

and from Lemma 4.7.7 this implies that

$$\dim_{\mathbb{C}} (\mathcal{O}_r/\mathcal{O}_r\langle a,b,c,d\rangle) < \infty.$$

This enables us to deduce from (6.10) that the $\mathcal{G}_e$-codim of $\bar{A}$ is finite, as required. □

Remarks 6.0.16 Given a skew-symmetric matrix of the form $\bar{A}$ (6.6) we can use the result of Lemma 6.0.13 by considering $(\mathcal{R} \times \mathcal{Q})$-equivalence on its corresponding 2×2 block A (6.7). This is particularly useful when proving a family of skew-symmetric matrices is $J^k\mathcal{G}$-trivial, as the number of tangent space generators is much smaller for the 2×2 matrix, therefore making the calculations more managable.

Recall that $(\mathcal{R} \times \mathcal{Q})$-equivalence on A amounts to $(\mathcal{R} \times \mathcal{H}_q)$-equivalence on $\bar{A}$. However since $(\mathcal{R} \times \mathcal{H}_q)$ is a proper subgroup of $\mathcal{G}$ then clearly $(\mathcal{R} \times \mathcal{Q})$-equivalence on A and $\mathcal{G}$-equivalence on $\bar{A}$ are not in direct correspondence. As a result two matrices which are not in the same $(\mathcal{R} \times \mathcal{Q})$-orbit could still correspond to $\mathcal{G}$-equivalent skew-symmetric matrices of type $\bar{A}$.

For example, in [BrTarSq], it is found that a germ, $A : \mathbb{C}^r, 0 \to M(2, \mathbb{C})$,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

is, up to $(\mathcal{R} \times \mathcal{Q})$-equivalence, usually *distinct* from its transpose, A^T,

$$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

However, if we consider the skew-symmetric matrix of type, $\bar{A}$, corresponding to A :

$$\bar{A} = \begin{bmatrix} 0 & 0 & a & b \\ 0 & 0 & c & d \\ -a & -c & 0 & 0 \\ -b & -d & 0 & 0 \end{bmatrix},$$

then applying the results of Lemma 6.0.1 we find that $\bar{A}$ is $\mathcal{H}$-equivalent to $\overline{A^T}$,

$$\overline{A^T} = \begin{bmatrix} 0 & 0 & a & c \\ 0 & 0 & b & d \\ -a & -c & 0 & 0 \\ -b & -d & 0 & 0 \end{bmatrix}.$$

Consequently, the two $(\mathcal{R} \times \mathcal{Q})$-inequivalent germs A and A^T correspond to the same $\mathcal{G}$-orbit with representative $\bar{A}$ (or $\overline{A^T}$).

So when using $(\mathcal{R} \times \mathcal{Q})$-classification, to simplify matrices of type $\bar{A}$ it is important to use $\mathcal{G}$-invariants for identifying distinct orbits. In fact Lemma 6.0.14 is a useful tool for finding the $\mathcal{G}_e$-codimensions and discriminants of matrices of type $\bar{A}$ by considering those of the corresponding matrix A.

Representing skew-symmetric matrices of type $\bar{A}$ with the 2×2 matrix A is also more convenient, notationally, and since applying a row/column operation to the matrix A corresponds to applying a corresponding simultaneous row and column operation to $\bar{A}$ this makes the description of explicit row and column operations neater (avoiding references to 'counterpart' operations etc).

As a consequence of this result, from here on, we adhere to the following convention. Unless specified otherwise any 2×2 matrix of the form in (6.7) is interpreted as representing a skew-symmetric matrix of the form in (6.6). Furthermore, to match the notation used for 4×4 skew-symmetric matrices we denote the basis vectors for $M(2, \mathbb{C})$ by $\{e_2, e_3, e_4, e_5\}$, the corresponding slots, e_i, being given by :

$$\begin{bmatrix} e_2 & e_3 \\ e_4 & e_5 \end{bmatrix}.$$

Hence a basis for $J^k(r, 4)$ (the space of k-jets of germs $A \in \mathcal{N}$) is given by the set of *monomial vectors* $x^I e_i$, $1 \leq |I| \leq k$, $2 \leq i \leq 5$ (see Remark 4.4.6 of Section 4.4). Occasionally for notational ease, i.e. when manipulating $(\mathcal{R} \times \mathcal{Q})$-tangent spaces, we may also express such germs/jets as 4-tuples $[a, b, c, d]$ in *row major order*, by which we mean the rows of the jet are, in descending order, listed end to end in a single vector.

It is however worth mentioning that care must be taken when finding complete transversals for a jet of type $\bar{A}$ using the jet A. This is due to having to account for the two slots e_1 and e_6 neglected when considering the 2×2 matrix. So although quite a few of the classification lists obtained in the following are replicated in lists given in [BrTarSq] the approach we adopt is different. In particular as a backup resource any calculations performed by `Transversal` are on matrices of type $\bar{A}$ up to $\mathcal{G}$-equivalence, we only consider $(\mathcal{R} \times \mathcal{Q})$-equivalence on the corresponding square matrix to simplify 'hand' calculations.

We start with the first of these cases, $r = 2$.

6.1 Case $r = 2$, $n = 4$

Consider smooth germs $A : \mathbb{C}^2, 0 \to Sk(4, \mathbb{C})$ which we also represent as maps

$$A : \mathbb{C}^2, 0 \to \mathbb{C}^6, 0.$$

As discussed at the end of Section 4.4.2 we start classification by considering the 1-jets. Here these 1-jets are of the form

$$A(x, y) = xA_1 + yA_2,$$

where A_1, $A_2 \in Sk(4, \mathbb{C})$. The corresponding jet-group of $\mathcal{R} \times \mathcal{H}$ acting on this space is the product $J^1\mathcal{R} \times J^1\mathcal{H}$, where $J^1\mathcal{R}$ is the set of invertible linear changes of coordinates in $\mathbb{C}^2, 0$ and $J^1\mathcal{H}$ the set of 1-jets of maps $\mathbb{C}^2, 0 \longrightarrow Gl(4, \mathbb{C})$:

$$X_1 : \mathbb{C}^2, 0 \longrightarrow Gl(4, \mathbb{C}).$$

Furthermore the action of this jet-group on $J^1(2, 6)$, as given in Definition 4.4.9, amounts to the group action $Gl(2, \mathbb{C}) \times Gl(4, \mathbb{C})$ on pairs of 4×4 skew-symmetric matrices, discussed in the first three chapters. Recall, from Example 3.4.1 in Section 3.4, that under this action we have the following six normal forms (expressed in upper triangular notation):

$$[0, 0, 0, 0, 0, 0],$$

$$[x, 0, 0, 0, 0, 0],$$

$$[x, 0, 0, 0, 0, x],$$

$$[x, 0, 0, 0, 0, y],$$

$$[x, 0, 0, y, 0, x],$$

$$[0, x, 0, y, 0, 0].$$

Having established these 1-jets we start the classification using techniques described in Chapter 4. We are assisted in this task by **Transversal**, as discussed in the previous chapter. The result of our calculations is the following list of simple finitely $\mathcal{G}$-determined germs.

Theorem 6.1.1 *Any $\mathcal{G}$-simple germ $A : \mathbb{C}^2, 0 \to Sk(4, \mathbb{C})$, where $A(0) = 0$, lies in one of the following (distinct) finitely $\mathcal{G}$-determined orbits. (Note germs written in 2×2 form.)*

Normal form	*Discriminant*	*$\mathcal{G}_e$-codimension*	*Label*
$\begin{bmatrix} x & y^k \\ y^l & x \end{bmatrix}, (1 \le k \le l)$	A_{k+l-1}	$4k+l-1$	B_{kl}
$\begin{bmatrix} x & xy \\ y & x^k \end{bmatrix}, (k \ge 2)$	D_{k+2}	$k+5$	S_k
$\begin{bmatrix} x & y^3 \\ y & x^2 \end{bmatrix}$	E_6	9	M_9
$\begin{bmatrix} x & xy^2 \\ y & x^2 \end{bmatrix}$	E_7	10	M_{10}
$\begin{bmatrix} x & y^4 \\ y & x^2 \end{bmatrix}$	E_8	11	M_{11}
$\begin{bmatrix} x & 0 \\ 0 & y^2 + x^k \end{bmatrix}, (k \ge 2)$	D_{k+2}	$k+8$	F_k
$\begin{bmatrix} x & 0 \\ 0 & xy + y^k \end{bmatrix}, (k \ge 3)$	D_{2k}	$5k$	G_k
$\begin{bmatrix} x & y^k \\ y^l & xy \end{bmatrix}, (2 \le k \le l)$	D_{k+l+1}	$4k+l+1$	H_{kl}
$\begin{bmatrix} x & y^2 \\ y^2 & x^2 \end{bmatrix}$	E_6	12	T_{12}
$\begin{bmatrix} x & y^2 \\ 0 & x^2 + y^3 \end{bmatrix}$	E_7	13	T_{13}
$\begin{bmatrix} x & y^2 \\ y^3 & x^2 \end{bmatrix}$	E_8	14	T_{14}
$\begin{bmatrix} x & 0 \\ 0 & x^2 + y^3 \end{bmatrix}$	E_7	16	T_{16}

(6.11)

We proceed by describing how this list is obtained.

6.1.1 1-jets : pencils

We first consider the three pencils.

Consider first the 1-jet, $[x,0,0,0,0,y]$, i.e. the family of skew-symmetric matrices:

$$\begin{bmatrix} 0 & x & 0 & 0 \\ -x & 0 & 0 & 0 \\ 0 & 0 & 0 & y \\ 0 & 0 & -y & 0 \end{bmatrix}.$$

This has an empty 2-transversal and by Corollary 4.4.19 is 1-$\mathcal{G}$-determined. Its $\mathcal{G}$-codimension (given by its $J^1\mathcal{G}$-codimension) is 0 and we deduce $[x,0,0,0,0,y]$ to be a finitely-determined representative for the open $\mathcal{G}$-orbit. By considering its Pfaffian this germ has discriminant

$$xy = 0,$$

which has an A_1 singularity and we calculate its $\mathcal{G}_e$-codimension to be 4.

The 1-jet, $[x,0,0,y,0,x]$, is the family of skew-symmetric matrices

$$\begin{bmatrix} 0 & x & 0 & 0 \\ -x & 0 & y & 0 \\ 0 & -y & 0 & x \\ 0 & 0 & -x & 0 \end{bmatrix}.$$

Finding a 2-transversal to this jet to be given by the family

$$A_a = [x, 0, ay^2, y, 0, x],$$

$a \in \mathbb{C}$, we use scale changes, mentioned in Section 4.5, to show this family is contained in a finite number of $J^2\mathcal{G}$-orbits of the 2-jet space.

Lemma 6.1.2 *Each member of the family of 2-jets (of germs $A : \mathbb{C}^2, 0 \to Sk(4,\mathbb{C})$) given by*

$$A_a = [x, 0, ay^2, y, 0, x],$$

is $J^2\mathcal{G}$-equivalent to either A_1 or A_0 in $J^2(2,6)$.

Proof Consider the $J^2\mathcal{G}$-action, on the family

$$A_a = \begin{bmatrix} 0 & x & 0 & ay^2 \\ -x & 0 & y & 0 \\ 0 & -y & 0 & x \\ -ay^2 & 0 & -x & 0 \end{bmatrix},$$

given by the scale $\mathcal{R}$-change of coordinates $\phi_s : \mathbb{C}^2, 0 \to \mathbb{C}^2, 0$

$$(x, y) \mapsto (\lambda x, \mu y), \qquad \lambda\mu \neq 0,$$

followed by the action of a constant matrix $X_c \in \mathcal{H}$ of the form

$$X_c = \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \delta \end{bmatrix}, \qquad \alpha\beta\gamma\delta \neq 0.$$

(Recall the action of this matrix on A_a corresponds to multiplying each row and column of A_a by a non-zero constant, see the discussion which concludes Section 4.5.) The resulting matrix, $X_c^T(A_a \circ \phi_s)X_c$, written in upper triangular form is then

$$[\alpha\beta\lambda x, 0, \alpha\delta a\mu^2 y^2, \beta\gamma\mu y, 0, \gamma\delta\lambda x].$$

To preserve the 1-jet of A_a we require these constants to satisfy the following equations:

$$\alpha\beta\lambda = 1, \tag{6.12}$$

$$\beta\gamma\mu = 1, \tag{6.13}$$

$$\gamma\delta\lambda = 1. \tag{6.14}$$

Equations 6.12 and 6.13 are satisfied by *setting*

$$\lambda = 1/\alpha\beta \tag{6.15}$$

and

$$\mu = 1/\beta\gamma, \tag{6.16}$$

respectively. It then follows that Equation 6.14 is always satisfied if we set

$$\delta = \alpha\beta/\gamma. \tag{6.17}$$

Hence the set of scale changes defined above, for which conditions 6.15, 6.16 and 6.17 are satisfied for arbitrary non-zero constants α, β and γ, preserve the 1-jet of A_a and change the coefficient of the y^2 term in slot e_3 to

$$\frac{\alpha^2 a}{\gamma^3\beta}.$$

So for *any* $a \neq 0$ we can choose the constants α, β, γ so that this coefficient is 1 and hence by such a scale change any element of the family A_a, for which $a \neq 0$, is $\mathcal{G}$-equivalent to the 2-jet

$$[x, 0, y^2, y, 0, x].$$

By calculation this 2-jet has $J^2\mathcal{G}$-codimension 1. Alternatively for $a = 0$ we have the 2-jet $[x, 0, 0, y, 0, x]$ which lies in a distinct orbit by virtue of its differing $J^2\mathcal{G}$-codimension (2). □

This result is included, primarily, to demonstrate the use of scale changes for the reduction of families of k-jets to a finite number of representatives. This technique is used throughout our classification and from here on we apply it without much further explanation.

Taking each of the 2-jets in turn, we continue by finding a complete transversal at the 3-jet level.

The 2-jet $[x, 0, y^2, y, 0, x]$, has an empty 3-transversal in $J^3(2, 6)$ and we deduce it to be 2-$\mathcal{G}$-determined.

We find the 2-jet $[x, 0, 0, y, 0, x]$ to have a 3-transversal

$$[x, 0, ay^3, y, 0, x],$$

$a \in \mathbb{C}$ which suggests there is a *series* of (distinct) finitely determined germs occuring here. Having used `Transversal` to indicate the presence of a series we can only prove that there is one by hand calculations. Compared to the mechanistic approach adopted by `Transversal` these calculations are more ad-hoc and as a sample of those to come, we do the present calculation in great detail. As we proceed and the approach becomes more familiar less explanation is required.

Theorem 6.1.3 *Let* $A : \mathbb{C}^2, 0 \to Sk(4, \mathbb{C})$ *be a smooth germ with* 1*-jet*

$$\begin{bmatrix} 0 & x & 0 & 0 \\ -x & 0 & y & 0 \\ 0 & -y & 0 & x \\ 0 & 0 & -x & 0 \end{bmatrix}, \tag{6.18}$$

then A is $\mathcal{G}$-equivalent to a l-$\mathcal{G}$-determined germ of the form

$$\begin{bmatrix} 0 & x & 0 & y^l \\ -x & 0 & y & 0 \\ 0 & -y & 0 & x \\ -y^l & 0 & -x & 0 \end{bmatrix},$$

where $l \geq 2$, or A is $\mathcal{G}$-equivalent to a germ whose l-jet, for any $l \geq 1$, is (6.18). So we have the series *of finitely determined germs (written in upper-triangular form):*

$$[x, 0, y^l, y, 0, x], \qquad (l \geq 2).$$

Proof We work, as before, at the jet-level and use the results of Corollary 4.4.15 and Theorem 4.4.18 in Section 4.4 to interpret this at the germ level.

Assume, for any $l \geq 2$, A has a $(l-1)$-jet,

$$j^{l-1}A = \begin{bmatrix} 0 & x & 0 & 0 \\ -x & 0 & y & 0 \\ 0 & -y & 0 & x \\ 0 & 0 & -x & 0 \end{bmatrix}. \tag{6.19}$$

We calculate the $J^l\mathcal{G}_1$-tangent space to this jet in the l-jet space. This is given by

$$j^l\left[\mathcal{M}_2^2\{A_x, A_y\} + \mathcal{M}_2\{g_{ij} = C_{ij}(A) + R_{ij}(A) : 1 \leq i, j \leq 4\}\right],$$

where $g_{ij} = C_{ij}(A) + R_{ij}(A)$ are the generators of the $\mathcal{H}$-tangent space to A defined in Proposition 4.3.2, in Section 4.3. Recall that each generator, g_{ij}, is a skew-symmetric matrix whose entries are all zero with the exception of its jth row and column which are found by superimposing the ith row and column of matrix (6.19). The $\mathcal{G}_1$-tangent space is given by suitable $\mathcal{O}_2$-linear combinations of these generators. Having found the set, $\{(A_x, A_y); g_{ij} : 1 \leq i, j \leq 4\}$, of these 18 generators we list, in upper-triangular form, the subset of distinct ones. (The labels (i.e. $\mathcal{M}_2^2$, $\mathcal{M}_2$) preceeding both types of generators indicate the appropriate ideals of $\mathcal{O}_2$ to be used as coefficients for obtaining tangent vectors.)

$$\mathcal{M}_2^2$$

$$[1,0,0,0,0,1], \quad [0,0,0,1,0,0];$$

$$\mathcal{M}_2$$

$$\begin{array}{lll} [x,0,0,0,0,0], & [0,0,0,x,0,0], & [0,0,0,0,x,0] \\ [0,y,0,0,0,0], & [x,0,0,y,0,0], & [0,x,0,0,0,0] \\ [0,0,x,0,0,-y], & [-y,0,x,0,0,0], & [0,0,0,y,0,x] \\ [0,0,0,0,y,0], & [0,0,0,0,0,x]. & \end{array} \tag{6.20}$$

Recall that $J^l(2,6)$ is a finite dimensional vector space, with a basis consisting of 6-tuples with a monomial of degree $\leq l$ in one slot and zeros elsewhere, and that finding a complete transversal of our $(l-1)$-jet amounts to finding a subspace, T of $H^l(2,6)$ such that

$$TJ^l\mathcal{G}_1.j^{l-1}A + T \supset H^l(2,6),$$

where $H^l(2,6)$ is the subspace of $J^l(2,6)$ of all 6-tuples of homogeneous polynomials of degree l. (See Theorem 4.4.13 in Section 4.4.) Furthermore, a basis for $H^l(2,6)$ consists of the subset of the basis vectors of $J^l(2,6)$, which are homogeneous of degree l.

The approach we adopt to find a complete l-transversal of $j^{l-1}A$ is as follows. We first use the spanning set, (6.20), to find as many as possible of the basis vectors of $H^l(2,6)$ which are contained in the $J^l\mathcal{G}_1$-tangent space. This is achieved by taking each slot e_i $1 \leq i \leq 6$ of the 6-tuple and determining how many degree l monomials we can get in this slot by suitable $\mathcal{O}_2$-linear combinations of the generators modulo $(k+1)$. Having done this we look for a complement to the tangent space giving all the remaining basis vectors of $H^l(2,6)$. This complement then provides a complete transversal for $j^{l-1}A$.

For future ease of notation, we refer to a general basis vector of $H^l(2,6)$ (or $J^l(2,6)$) in a given slot e_i as me_i, where m is a monomial of degree l (degree $\leq l$). Furthermore, we use the shorthand $\langle x\rangle e_i$ and $\langle y\rangle e_i$ for the set of all basis vectors consisting of a monomial in slot e_i which are divisible by x and y, respectively. Clearly, if we can find both sets of monomials $\langle x\rangle e_i$ and $\langle y\rangle e_i$ in a slot, e_i, this means that every basis vector with a monomial (of degree l) in that slot is contained in the tangent space.

We illustrate this technique by solving the current problem. Consider the set of generators (6.20).

Multiplying each of $[x,0,0,0,0,0]$, $[0,x,0,0,0,0]$, $[0,0,0,x,0,0]$, $[0,0,0,0,x,0]$ and $[0,0,0,0,0,x]$, by monomials of degree $(l-1)$ we get, respectively,

$$\langle x\rangle e_1,\ \langle x\rangle e_2,\ \langle x\rangle e_4,\ \langle x\rangle e_5,\ \langle x\rangle e_6.$$

Furthermore, multiplying $[0,y,0,0,0,0]$ and $[0,0,0,0,y,0]$ by y^{l-1} gives the missing y^l terms in slots e_2 and e_5. Also

$$y^{l-1}\left([x,0,0,y,0,0]-[x,0,0,0,0,0]\right),$$

gives a y^l term in e_4. It remains to find $\langle x\rangle e_3$, $\langle y\rangle e_3$ and y^l terms in e_1, e_6. A quick look at the generators confirms that it is not possible to get a y^l term in e_3. However we can get $\langle x\rangle e_3$ by multiplying

$$[0,0,x,0,0,-y]+[-y,0,x,0,0,0]+y[1,0,0,0,0,1]=[0,0,2x,0,0,0]$$

by monomials of degree $(l-1)$. Finally

$$y^{l-1}\left([0,0,x,0,0,-y]-[-y,0,x,0,0,0]\right)\pm y^l[1,0,0,0,0,1],$$

gives a y^l term in e_1 and e_6 respectively.

We deduce that the $J^l\mathcal{G}_1$- tangent space to $j^{l-1}A$ contains every basis vector of $H^l(2,6)$ *except* $[0,0,y^l,0,0,0]$ and so a complete transversal is the family of l-jets

$$A_a=[x,0,ay^l,y,0,x].$$

By scale changes, of the type described in the proof of Lemma 6.1.2, any element of this family for which $a\neq 0$ is $J^l\mathcal{G}$-equivalent to the l-jet

$$A_1=[x,0,y^l,y,0,x].$$

Notice if $a=0$ we have the l-jet $[x,0,0,y,0,x]$, for which we can repeat the same procedure as above, replacing $(l-1)$ with l and so on.

To show that the l-jet

$$A_l = \begin{bmatrix} 0 & x & 0 & y^l \\ -x & 0 & y & 0 \\ 0 & -y & 0 & x \\ -y^l & 0 & -x & 0 \end{bmatrix}, \tag{6.21}$$

is l-determined we need to show that it has an empty complete transversal in the $(l+1)$-jetspace (see Corollary 4.4.19). We therefore calculate the $J^{l+1}\mathcal{G}_1$-tangent space to this jet in the $(l+1)$-jetspace. This tangent space is determined by the following list of distinct generators (where, compared with those obtained from the $(l-1)$-jet $[x,0,0,y,0,x]$, new or changed generators are denoted by an asterisk),

$$\begin{array}{lll}
\mathcal{M}_2^2 & & \\
[1,0,0,0,0,1], & [0,0,ly^{l-1},1,0,0]_*; & \\
\mathcal{M}_2 & & \\
[x,0,y^l,0,0,0]_*, & [0,0,0,-x,0,y^l]_*, & [0,0,0,0,x,0], \\
[0,y,0,0,0,0], & [x,0,0,y,0,0], & [0,x,0,0,0,0], \\
[0,0,x,0,0,-y], & [-y,0,x,0,0,0], & [0,0,0,y,0,x], \\
[0,0,0,0,y,0], & [0,0,y^l,0,0,x]_*, & [y^l,0,0,-x,0,0]_*, \\
[0,0,0,0,y^l,0]_*, & [0,y^l,0,0,0,0]_*. &
\end{array} \tag{6.22}$$

Notice that we can ignore the last two generators since these already lie in

$$\mathcal{M}_2\{[0,y,0,0,0,0],[0,0,0,0,y,0]\}. \tag{6.23}$$

We apply the same method as before, this time looking to show that all basis vectors of $H^{l+1}(2,6)$ are contained in this tangent space. Here the notation $\langle x\rangle e_i$ will denote the set of all monomials of degree $l+1$ in the slot e_i. We follow the argument used to find the complete transversal of $[x,0,0,y,0,x]$ indicating modifications where necessary.

To get $\langle x\rangle e_2$, $\langle x\rangle e_5$, $\langle x\rangle e_1$, $\langle x\rangle e_4$, $\langle x\rangle e_6$, respectively, we multiply generators $[0,x,0,0,0,0]$, $[0,0,0,0,x,0]$, $[x,0,y^l,0,0,0]$, $[0,0,0,-x,0,y^l]$, $[0,0,y^l,0,0,x]$ by monomials of degree l, noting that the y^l term in each of the last three contributes monomials of degree $2l=(l+1)+(l-1)$ which drop out in $(l+1)$ jetspace since $l \geq 2$.

To get the y^{l+1} monomials in e_2 and e_5, as before, we use the generators in (6.23), only here we multiply each by y^l. The y^{l+1} term in e_4 is given, in a similar way to that above, by combining $[x,0,0,y,0,0]$ with $[x,0,y^l,0,0,0]$ and noting that in $(l+1)$-jetspace the subsequent y^{2l} term in e_3 drops out.

We can also use $[0,0,x,0,0,-y]$, $[-y,0,x,0,0,0]$ and $y[1,0,0,0,0,1]$ as above to obtain $\langle x\rangle e_3$ and y^{l+1} monomials in slots e_1 and e_6. Observe here, that having previously demonstrated a basis vector me_i (of $H^l(2,6)$) to be contained in the $J^l\mathcal{G}_1$-tangent space, to $j^{l-1}A$, then it follows that the basis vectors xme_i, yme_i of $H^{l+1}(2,6)$ are *also* contained in the $J^{l+1}\mathcal{G}_1$-tangent space to A_1.

It therefore remains to find a y^{l+1} term in slot e_3. Compared with the previous case we have the new generators $[x,0,y^l,0,0,0]$, $[0,0,y^l,0,0,x]$ available, which combined with $x[1,0,0,0,0,1]$ give us a y^{l+1} in this slot as follows:

$$y\left([x,0,y^l,0,0,0]+[0,0,y^l,0,0,x]-x[1,0,0,0,0,1]\right).$$

So it follows that the $J^{l+1}\mathcal{G}_1$-tangent space to $[x,0,y^l,y,0,x]$ contains all the basis vectors of $H^{l+1}(2,6)$, therefore it has an empty complete transversal and is l-determined as required. □

Lemma 6.1.4 *The l-$\mathcal{G}$-determined germ $A=[x,0,y^l,y,0,x]$, $(l\geq 2)$, has $\mathcal{G}_e$-codimension $l+3$ and a discriminant of type A_l.*

Proof We use the result of Corollary 4.4.34, that is,

$$\mathcal{G}_e\text{-codim}A=\sum_{s=0}^{l}\text{cod}_sA, \tag{6.24}$$

where, for $s\geq 0$,

$$\text{cod}_sA=\dim\left(\frac{T\mathcal{G}_e.A+\mathcal{M}_2^s\mathcal{O}_2^6}{T\mathcal{G}_e.A+\mathcal{M}_2^{s+1}\mathcal{O}_2^6}\right).$$

First we calculate the 'extended' tangent space to A, $T\mathcal{G}_e.A$. This is the $\mathcal{O}_2$-module consisting of *all* $\mathcal{O}_2$-linear combinations of the 16 vectors in (6.22), above. Instead of reproducing this set we refer to (6.22).

We proceed to calculate cod_sA for each $s\geq 0$, using the method outlined in Remark 4.4.35.

Firstly, we consider $s = 0$, i.e. we find $\mathrm{cod}_0 A$. Clearly, the tangent vector $[0,0,ly^{l-1},1,0,0]$ means that $[0,0,0,1,0,0] \in T\mathcal{G}_e.A + \mathcal{M}_2\mathcal{O}_2^6$. However, by observation, neither of $[1,0,0,0,0,0]$, $[0,1,0,0,0,0]$, $[0,0,1,0,0,0]$, $[0,0,0,0,1,0]$ or $[0,0,0,0,0,1]$ are present in $T\mathcal{G}_e.A + \mathcal{M}_2.\mathcal{O}_2^6$. Choosing a basis for a supplement consisting of the first four of these missing vectors (we can omit $[0,0,0,0,0,1]$ since the vector $[1,0,0,0,0,1] \in T\mathcal{G}_e.A$) gives $\mathrm{cod}_0 A = 4$.

Next we consider values of $s \geq 1$. Here $\langle x\rangle e_i$ and $\langle y\rangle e_i$ represent sets of basis vectors of degree s which are, respectively, divisible by x and y. Using $[0,0,ly^{l-1},1,0,0]$ we see that $\langle 1\rangle e_4 \in T\mathcal{G}_e.A + \mathcal{M}_2^{s+1}\mathcal{O}_2^6$ (where $\langle 1\rangle e_4$ represents any basis vector of degree s in slot e_4). Clearly using the tangent vectors $[x,0,y^l,0,0,0]$, $[0,x,0,0,0,0]$, $[0,0,0,0,x,0]$ and $[0,0,y^l,0,0,x]$ we find $\langle x\rangle e_1$, $\langle x\rangle e_2$, $\langle x\rangle e_5$ and $\langle x\rangle e_6$, respectively, in $T\mathcal{G}_e.A + \mathcal{M}_2^{s+1}\mathcal{O}_2^6$.

Using $[0,0,x,0,0,-y]$, $[-y,0,x,0,0,0]$ and $y[1,0,0,0,0,1]$ gives $\langle x\rangle e_3$, $\langle y\rangle e_1$ and $\langle y\rangle e_6$. The two tangent vectors $[0,y,0,0,0,0]$ and $[0,0,0,0,y,0]$ give $\langle y\rangle e_2$ and $\langle y\rangle e_5$ which leaves us to check for y^s in slot e_3.

We observe that for $1 \leq s \leq l-1$ the vector $[0,0,y^s,0,0,0]$ is not present in $T\mathcal{G}_e.A + \mathcal{M}_2^{s+1}\mathcal{O}_2^6$ and clearly forms a basis for a supplement, for this space, in $T\mathcal{G}_e.A + \mathcal{M}_2^s\mathcal{O}_2^6$. However, using vectors $[x,0,y^l,0,0,0]$, $[0,0,y^l,0,0,x]$ and $x[1,0,0,0,0,1]$ we find the vector $[0,0,y^l,0,0,0]$ to be in $T\mathcal{G}_e.A$. So, for $s \geq l$, $[0,0,y^s,0,0,0] \in T\mathcal{G}_e.A + \mathcal{M}_2^{s+1}\mathcal{O}_2^6$.

In summary, for values $s \geq 1$,

$$\mathrm{cod}_s = \begin{cases} 1 & \text{for } 1 \leq s \leq l-1 \\ 0 & \text{for } s \geq l \end{cases} .$$

Therefore, from (6.24), $\mathcal{G}_e$-codim $A = 4 + (l-1) = (l+3)$.

Finally, by calculating the determinant of

$$A = \begin{bmatrix} 0 & x & 0 & y^l \\ -x & 0 & y & 0 \\ 0 & -y & 0 & x \\ -y^l & 0 & -x & 0 \end{bmatrix}$$

we find its discriminant to be given by the vanishing of the Pfaffian, i.e.

$$x^2 + y^{l+1} = 0,$$

which has an A_l singularity. □

The 1-jet $[0, x, 0, y, 0, 0]$ is the family of skew-symmetric matrices

$$g = \begin{bmatrix} 0 & 0 & x & 0 \\ 0 & 0 & y & 0 \\ -x & -y & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Again for illustrative purposes we consider the classification of the germs with this 1-jet in some detail. Firstly we find this 1-jet to have a complete 2-transversal

$$A_{ct} = [0, x, axy + by^2, y, cxy + dx^2, 0],$$

where $(a, b, c, d) \in \mathbb{C}^4$. In this case applying scale changes of the type described in the proof of Lemma 6.1.2 is not sufficient to reduce this family to a finite number of representatives. For example by such scale changes we obtain the $J^2\mathcal{G}$-equivalent family

$$[0, \alpha\gamma\lambda x, \alpha\delta(a\lambda\mu xy + b\mu^2 y^2), \beta\gamma\mu y, \beta\delta(c\lambda\mu xy + d\lambda^2 x^2), 0].$$

To preserve the 1-jet we set

$$\lambda = 1/\alpha\gamma,$$

$$\mu = 1/\beta\gamma.$$

and denoting the coefficients of the degree 2 monomials, after this scaling, by uppercase letters they become:

$$A = \frac{a\delta}{\beta\gamma^2}, \qquad B = \frac{b\alpha\delta}{\beta^2\gamma^2},$$

$$C = \frac{c\delta}{\alpha\gamma^2}, \qquad D = \frac{d\delta\beta}{\alpha^2\gamma^2}.$$

It can be shown that, in general, it is only possible to scale two of these coefficients to unity. In fact we observe, for any choice of α, β, γ and δ, the two ratios :

$$\frac{AC}{BD}, \qquad \frac{A^3 D}{C^3 B},$$

are *invariant* under scaling. So, returning to A_{ct}, provided $ac \neq 0$ we can, by scaling, obtain the family

$$A_s = [0, x, xy + By^2, y, xy + Dx^2, 0],$$

for which, as regards scale changes, any neighbourhood of each element meets uncountably many orbits. Using `Transversal` we find this family to be a union of a finite number of $J^2\mathcal{G}$-orbits. However the calculations required to find representatives for these orbits are untidy, involving the parametrisation of certain varieties of the parameter space where the calculation, for finding the $J^2\mathcal{G}$-tangent space to generic members of A_s, no longer applies (see Section 5.4.2). Furthermore we must also consider the possiblities when $a = 0$ or $c = 0$. Instead we gain more insight by reducing A_{ct} in the 2-jet space. We start with a result which applies to any germ with 1-jet $[0, x, 0, y, 0, 0]$.

Proposition 6.1.5 *Any germ $A : \mathbb{C}^2, 0 \to Sk(4, \mathbb{C})$ with 1-jet $[0, x, 0, y, 0, 0]$ is $J^k\mathcal{G}_1$-equivalent to a k-jet of the form*

$$\begin{bmatrix} 0 & 0 & x & f(x,y) \\ 0 & 0 & y & g(x,y) \\ -x & -y & 0 & 0 \\ -f(x,y) & -g(x,y) & 0 & 0 \end{bmatrix}, \tag{6.25}$$

where $2 \leq \deg f, \deg g \leq k$. We refer to (6.25) as a **prenormal form** *for A.*

Proof The proof is by induction at the jet level. Firstly the base case $k = 1$ is trivial. For the inductive step we assume the result holds for $(k-1)$, with $k \geq 2$. Consider the $(k-1)$-jet

$$A_{k-1} = \begin{bmatrix} 0 & 0 & x & \bar{f}(x,y) \\ 0 & 0 & y & \bar{g}(x,y) \\ -x & -y & 0 & 0 \\ -\bar{f}(x,y) & -\bar{g}(x,y) & 0 & 0 \end{bmatrix},$$

where $2 \leq \deg \bar{f}, \deg \bar{g} \leq k-1$. The $\mathcal{G}_1$-tangent space is generated by the following vectors (again the labels refer to the appropriate ideals of $\mathcal{O}_r$ to be used as coefficients for obtaining tangent vectors).

$$\mathcal{M}_2^2$$

$$[0, 1, \bar{f}_x, 0, \bar{g}_x, 0], \quad [0, 0, \bar{f}_y, 1, \bar{g}_y, 0]; \tag{6.26}$$

$$\mathcal{M}_2$$

$$[x, 0, 0, 0, 0, 0], \quad [y, 0, 0, 0, 0, 0], \tag{6.27}$$

$$[0,0,0,0,0,x], \quad [0,0,0,0,0,y], \tag{6.28}$$

$$[0,x,\bar{f},0,0,0], \quad [0,0,0,x,\bar{f},0], \tag{6.29}$$

$$[0,y,\bar{g},0,0,0], \quad [0,0,0,y,\bar{g},0], \tag{6.30}$$

$$[0,x,0,y,0,0], \quad [0,0,x,0,y,0], \tag{6.31}$$

$$[0,\bar{f},0,\bar{g},0,0], \quad [0,0,\bar{f},0,\bar{g},0]. \tag{6.32}$$

As usual we look for a complete k-transversal (an affine subspace which complements the $J^k\mathcal{G}_1$-tangent space in $H^k(2,6)$).

We first observe that the vectors in (6.27) and (6.28) give every monomial of degree k in slots e_1 and e_6 respectively. Furthermore multiplying the $\mathcal{R}$-tangent vectors (6.26) by monomials of degree k gives everything in slots e_2 and e_4. The key here being that the terms of $\bar{f}_x$, $\bar{g}_x$ all drop out in the k-jet space. It follows that a complete k-transversal is something of the form

$$[0,x,\bar{f}+f_k,y,\bar{g}+g_k,0],$$

where f_k, g_k are homogeneous polynomials of degree k. The result then follows by induction. □

Remark 6.1.6 We observe from the $\mathcal{G}$-tangent space generators given above, that the vectors (6.26) and (6.29) – (6.32) can also be thought of as generators of the $(\mathcal{R}\times\mathcal{Q})$-tangent space to the square germ $\mathbb{C}^2,0\to M(2,\mathbb{C})$

$$\begin{bmatrix} x & \bar{f} \\ y & \bar{g} \end{bmatrix}. \tag{6.33}$$

Therefore we can continue the $\mathcal{G}$-classification of a k-jet of the form (6.25) by representing it in this form and considering the action of the $(\mathcal{R}\times\mathcal{Q})$-group (see Section 6.0.4).

Having thus given an idea of where this classification is headed we return to the simplification of a 2-transversal of $[0,x,0,y,0,0]$ of the form $[0,x,Q_1,y,Q_2,0]$, where $Q_1(x,y) = ax^2+bxy+cy^2$, $Q_2(x,y) = Ax^2+Bxy+Cy^2$. As just discussed we can represent this by the 2-jet

$$\begin{bmatrix} x & Q_1 \\ y & Q_2 \end{bmatrix}, \tag{6.34}$$

and simplify by using elements of the $J^2(\mathcal{R} \times \mathcal{Q})$-jet group not present in the $J^2(\mathcal{R}_1 \times \mathcal{Q}_0)$-group. In particular the scale changes already discussed are used for this purpose. In addition we also have available the set of transformations consisting of linear coordinate changes of the form

$$(x,y) \mapsto (x+\alpha y, y),$$

followed by a suitable row operation such as

$$R_1 - \alpha R_2,$$

to preserve the 1-jet. In particular these can be used to eliminate the y^2 term in Q_1 and the x^2 term in Q_2. Then we can further simplify the quadratic terms in the second column by parametrised column operations of the form

$$C_2 - LC_1,$$

where $L = rx + sy$ is a suitable linear combination of x and y. In this way we can kill off the x^2 term in e_3 and the y^2 term in e_5 leaving only xy terms in both slots. Finally we can scale the coefficients of both these xy terms to 1. The resulting representative is therefore given by the 2-jet

$$\begin{bmatrix} x & xy \\ y & xy \end{bmatrix}.$$

It is instructive to give the details of this method since it yields representatives for the more degenerate orbits covering this family.

Lemma 6.1.7 *Any square 2-jet of the form*

$$\begin{bmatrix} x & Q_1(x,y) \\ y & Q_2(x,y) \end{bmatrix},$$

where $Q_1(x,y) = ax^2 + bxy + cy^2$, $Q_2(x,y) = Ax^2 + Bxy + Cy^2$ are arbitrary quadratics, is $J^2(\mathcal{R} \times \mathcal{Q})$-equivalent to one of the following four 2-jets:

$$\begin{bmatrix} x & xy \\ y & xy \end{bmatrix},$$

$$\begin{bmatrix} x & xy \\ y & 0 \end{bmatrix}, \qquad \begin{bmatrix} x & 0 \\ y & x^2 \end{bmatrix}, \qquad \begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix},$$

the first of which is a representative of the generic orbit.

Proof The proof follows the method of simplification sketched out above. Consider the 2-jet in question,

$$\begin{bmatrix} x & Q_1(x,y) \\ y & Q_2(x,y) \end{bmatrix}. \tag{6.35}$$

The first step involves combining a linear change of coordinates of the source with the action of an element of $\mathcal{Q}$ which not only preserves the first column but also eliminates the y^2 term from the $(1,2)$ entry and the x^2 term from the $(2,2)$ entry. We start with an arbitrary linear $\mathcal{R}$-change

$$(x,y) \longmapsto (\alpha x + \beta y, \gamma x + \delta y), \tag{6.36}$$

where α, β, γ, $\delta \in \mathbb{C}$ and $\alpha\delta - \beta\gamma \neq 0$. The matrix, A_r, representing this map is

$$A_r = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

and

$$\begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = (\alpha\delta - \beta\gamma) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So applying this $\mathcal{R}$-change of coordinates, in (6.36), to the matrix (6.35) followed by premultiplying the result by

$$\mathrm{adj}(A_r) = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix},$$

we obtain the $(\mathcal{R} \times \mathcal{Q})$-equivalent matrix

$$\begin{bmatrix} (\alpha\delta - \beta\gamma)x & \delta Q_1(\alpha x + \beta y, \gamma x + \delta y) - \beta Q_2(\alpha x + \beta y, \gamma x + \delta y) \\ (\alpha\delta - \beta\gamma)y & -\gamma Q_1(\alpha x + \beta y, \gamma x + \delta y) + \alpha Q_2(\alpha x + \beta y, \gamma x + \delta y) \end{bmatrix}. \tag{6.37}$$

Consider the matrix in (6.37). It is a requirement of our choice of $\mathcal{R}$-change of coordinates that $\alpha\delta - \beta\gamma \neq 0$ so we can scale the coefficients of x and y in column 1 to unity by a column operation. It remains to find which of these $\mathcal{R}$-changes will eliminate the required terms from the second column. By expanding the terms in the second column of (6.37) we find that the coefficients of y^2 terms in the first entry and of x^2 terms in the second entry are, respectively:

$$y^2: \qquad \delta\left(a\beta^2 + b\beta\delta + c\delta^2\right) - \beta\left(A\beta^2 + B\beta\delta + C\delta^2\right),$$

$$x^2: \qquad -\gamma\left(a\alpha^2 + b\alpha\gamma + c\gamma^2\right) + \alpha\left(A\alpha^2 + B\alpha\gamma + C\gamma^2\right).$$

We therefore need to find values of α, β, γ and δ so that both of these coefficients vanish. This amounts to solving both of the following equations,

$$y^2: \quad A\beta^3 + (B-a)\beta^2\delta + (C-b)\beta\delta^2 - c\delta^3 = 0 \tag{6.38}$$

and

$$x^2: \quad A\alpha^3 + (B-a)\alpha^2\gamma + (C-b)\alpha\gamma^2 - c\gamma^3 = 0, \tag{6.39}$$

subject to the condition $\alpha\delta - \beta\gamma \neq 0$. Dividing through Equations 6.38 and 6.39 by δ^3 and γ^3 respectively, we find that this problem can be solved provided the cubic

$$f(w) = Aw^3 + (B-a)w^2 + (C-b)w - c,$$

has at least 2 *distinct* roots, w_1, w_2, in which case we can choose α, β, γ and δ such that

$$\frac{\beta}{\delta} = w_1, \qquad \frac{\alpha}{\gamma} = w_2.$$

Suppose that $f(w) = 0$ has at least 1 root, w_1. Then we can always choose β and δ such that $\beta/\delta = w_1$. Furthermore, we then choose α, γ such that $\alpha\delta - \beta\gamma \neq 0$. With these choices the corresponding $\mathcal{R}$-change (followed by the action of an appropriate element of $\mathcal{Q}$) guarantees the elimination of the y^2 term in the $(1,2)$ entry. The resulting $(\mathcal{R} \times \mathcal{Q})$-equivalent matrix (to matrix (6.35)) is

$$\begin{bmatrix} x & a'x^2 + b'xy \\ y & A'x^2 + B'xy + C'y^2 \end{bmatrix}. \tag{6.40}$$

This matrix is also of the form (6.35), and by applying the previous argument to it we obtain a cubic

$$f(w) = A'w^3 + (B'-a')w^2 + (C'-b')w,$$

which has two distinct roots unless $B' = a'$ and $C' = b'$.

So, unless $B' = a'$ and $C' = b'$ we can reduce (6.40) to the $(\mathcal{R} \times \mathcal{Q})$-equivalent matrix

$$\begin{bmatrix} x & a''x^2 + b''xy \\ y & B''xy + C''y^2 \end{bmatrix},$$

which by the column operation $C_2 - (a''x + C''y)C_1$ is $\mathcal{Q}$-equivalent to

$$\begin{bmatrix} x & Rxy \\ y & Sxy \end{bmatrix}, \tag{6.41}$$

where R, $S \in \mathbb{C}$. There are three possibilities.

(i) Both R, $S \neq 0$. In this case we can scale both to unity thence obtaining the representative

$$\begin{bmatrix} x & xy \\ y & xy \end{bmatrix},$$

of the least degenerate $J^2(\mathcal{R} \times \mathcal{Q})$-orbit of (6.35).

(ii) Only one of R, S is non-zero. By scaling we have either

$$\begin{bmatrix} x & xy \\ y & 0 \end{bmatrix}, \tag{6.42}$$

or

$$\begin{bmatrix} x & 0 \\ y & xy \end{bmatrix}. \tag{6.43}$$

However they both represent the same $J^2(\mathcal{R} \times \mathcal{Q})$-orbit since we can pass from (6.42) to (6.43) by switching rows followed by the $\mathcal{R}$-coordinate switch

$$(x, y) \longmapsto (y, x).$$

(iii) Finally $R = S = 0$ and we have the representative,

$$\begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix}.$$

If however $B' = a'$ and $C' = b'$, then (6.40) becomes

$$\begin{bmatrix} x & a'x^2 + b'xy \\ y & A'x^2 + a'xy + b'y^2 \end{bmatrix}$$

and the column operation $C_2 - (a'x + b'y)C_1$ gives the $\mathcal{Q}$-equivalent matrix

$$\begin{bmatrix} x & 0 \\ y & A'x^2 \end{bmatrix}.$$

If $A' \neq 0$ we can scale it to unity, giving the representative,

$$\begin{bmatrix} x & 0 \\ y & x^2 \end{bmatrix},$$

otherwise we have $A' = 0$ and the representative,

$$\begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix},$$

found above.

Finally if $f(w) \equiv 0$ then $A = B - a = C - b = c = 0$ which is a special case of that above.

Having exhausted all the possibilities we have found the four representatives given in the statement of the lemma. □

Taking each of these representatives in turn, we continue our calculations by finding complete transversals at the 3-jet level.

We find that the 2-jet $[0, x, xy, y, xy, 0]$, i.e.

$$\begin{bmatrix} x & xy \\ y & xy \end{bmatrix},$$

is 2-$\mathcal{G}$-determined. Furthermore it has $\mathcal{G}_e$-codimension 7 and its discriminant is given by $xy(x-y) = 0$ which is of type D_4.

The 2-jet $[0, x, xy, y, 0, 0]$, i.e.

$$\begin{bmatrix} x & xy \\ y & 0 \end{bmatrix},$$

has $J^2\mathcal{G}$-codimension 4. It has a complete 3-transversal

$$\begin{bmatrix} x & xy \\ y & ax^3 \end{bmatrix}, \qquad a \in \mathbb{C}.$$

If $a \neq 0$ by scaling, we obtain the $J^3\mathcal{G}$-equivalent jet,

$$\begin{bmatrix} x & xy \\ y & x^3 \end{bmatrix},$$

which is 3-$\mathcal{G}$-determined with $\mathcal{G}_e$-codimension 8. Furthermore, if $a = 0$ the corresponding 3-jet,

$$\begin{bmatrix} x & xy \\ y & 0 \end{bmatrix},$$

has a 4-transversal

$$\begin{bmatrix} x & xy \\ y & ax^4 \end{bmatrix},$$

which suggests the presence of a series.

Theorem 6.1.8 *Let $A : \mathbb{C}^2, 0 \to Sk(4, \mathbb{C})$ be a smooth germ with 2-jet*

$$\begin{bmatrix} 0 & 0 & x & xy \\ 0 & 0 & y & 0 \\ -x & -y & 0 & 0 \\ -xy & 0 & 0 & 0 \end{bmatrix},$$

then A is $\mathcal{G}$-equivalent to a k-$\mathcal{G}$-determined germ of the form

$$\begin{bmatrix} 0 & 0 & x & xy \\ 0 & 0 & y & x^k \\ -x & -y & 0 & 0 \\ -xy & -x^k & 0 & 0 \end{bmatrix},$$

where $k \geq 3$, or for each $k \geq 2$ A is $\mathcal{G}$-equivalent to a germ with k-jet j^2A.

Proof By Proposition 6.1.5 it suffices to consider the $(k-1)$-jet $(k \geq 3)$

$$j^{k-1}A = \begin{bmatrix} x & xy \\ y & 0 \end{bmatrix},$$

up to $(\mathcal{R} \times \mathcal{Q})$-equivalence.

Using Corollary 6.0.7 we find that the $J^k(\mathcal{R}_1 \times \mathcal{Q}_0)$-tangent space to this jet is generated by the following vectors : (the labels preceding the generators refer to the ideals of $\mathcal{O}_r$ from which we can select coefficients for $\mathcal{O}_r$-linear combinations to yield valid tangent vectors)

$$\mathcal{M}_2^2 : \quad \begin{bmatrix} 1 & y \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & x \\ 1 & 0 \end{bmatrix},$$

$$\mathcal{M}_2 : \quad \begin{bmatrix} x & xy \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ x & xy \end{bmatrix}, \quad \begin{bmatrix} y & 0 \\ 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 \\ y & 0 \end{bmatrix}, \quad \begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & x \\ 0 & y \end{bmatrix},$$

$$\begin{bmatrix} xy & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & xy \\ 0 & 0 \end{bmatrix}.$$

As before we use this spanning set to find the basis vectors of $H^k(2,4)$ contained in the tangent space, refer to the proof of Theorem 6.1.3 for further details.

Multiplying

$$\begin{bmatrix} y & 0 \\ 0 & 0 \end{bmatrix},$$

by terms of degree $k-1$ gives $\langle y \rangle e_2$. The pure power of x term in e_2 is given by

$$x^{k-1} \begin{bmatrix} x & xy \\ 0 & 0 \end{bmatrix},$$

the resulting $x^k y$ term in e_3 dropping out in $J^k(2,4)$. Similarly the generators,

$$\begin{bmatrix} 0 & 0 \\ y & 0 \end{bmatrix}, \qquad \text{and} \qquad \begin{bmatrix} 0 & 0 \\ x & xy \end{bmatrix},$$

give everthing in e_4.

Multiplying the generator,

$$\begin{bmatrix} 0 & xy \\ 0 & 0 \end{bmatrix},$$

by terms of degree, $k-2$, (which is valid since $k \geq 3$) gives everything in e_3 divisible by xy. To find the pure powers of both x and y in this slot we need to use the $\mathcal{R}_1$-tangent vectors, e.g.

$$y^{k-1} \begin{bmatrix} 1 & y \\ 0 & 0 \end{bmatrix} - y^{k-2} \begin{bmatrix} y & 0 \\ 0 & 0 \end{bmatrix},$$

gives the y^k term in e_3. The x^k term in this slot is given by combining the vector

$$x^{k-1} \begin{bmatrix} 0 & x \\ 1 & 0 \end{bmatrix},$$

with the two vectors

$$x^{k-2} \begin{bmatrix} 0 & 0 \\ x & xy \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ x^{k-1} & x^{k-1}y \end{bmatrix}$$

and

$$x^{k-1}\begin{bmatrix} 0 & x \\ 0 & y \end{bmatrix} = \begin{bmatrix} 0 & x^k \\ 0 & x^{k-1}y \end{bmatrix}.$$

Looking at the generators it is clear that we cannot get a x^k term in slot e_5. However,

$$x\begin{bmatrix} 0 & x \\ 0 & y \end{bmatrix} - x\begin{bmatrix} 0 & x \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -x & xy \end{bmatrix},$$

combined with

$$\begin{bmatrix} 0 & 0 \\ x & xy \end{bmatrix}$$

gives a tangent vector

$$\begin{bmatrix} 0 & 0 \\ 0 & xy \end{bmatrix},$$

and we have everything in e_5 divisible by xy. Finally

$$y^{k-1}\begin{bmatrix} 0 & x \\ 0 & y \end{bmatrix}$$

gives the y^k term in this slot, since we have already shown that we have everything divisible by xy in e_3. So a k-transversal is

$$\begin{bmatrix} x & xy \\ y & ax^k \end{bmatrix},$$

where $a \in \mathbb{C}$. If $a \neq 0$ it can be shown, by scaling this is $(\mathcal{R} \times \mathcal{Q})$-equivalent to the k-jet

$$\begin{bmatrix} x & xy \\ y & x^k \end{bmatrix}. \tag{6.44}$$

Notice if $a = 0$ we have a k-jet

$$\begin{bmatrix} x & xy \\ y & 0 \end{bmatrix},$$

and we repeat the above procedure, replacing $k-1$ with k, and so on.

We return to the k-jet in (6.44). For determinacy we need to show that a $(k+1)$-complete transversal to this jet is empty. We find the $J^{k+1}(\mathcal{R}_1 \times \mathcal{Q}_0)$-tangent space to this jet to be generated by the vectors (where, compared with

those obtained from the $(k-1)$-jet, changed vectors are denoted by an asterisk)
:

$$\mathcal{M}_2^2: \quad \begin{bmatrix} 1 & y \\ 0 & kx^{k-1} \end{bmatrix}_*, \quad \begin{bmatrix} 0 & x \\ 1 & 0 \end{bmatrix},$$

$$\mathcal{M}_2: \quad \begin{bmatrix} x & xy \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ x & xy \end{bmatrix}, \quad \begin{bmatrix} y & x^k \\ 0 & 0 \end{bmatrix}_*,$$

$$\begin{bmatrix} 0 & 0 \\ y & x^k \end{bmatrix}_*, \quad \begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & x \\ 0 & y \end{bmatrix},$$

$$\begin{bmatrix} xy & 0 \\ x^k & 0 \end{bmatrix}_*, \quad \begin{bmatrix} 0 & xy \\ 0 & x^k \end{bmatrix}.$$

By following a method similar to that used to find the complete transversal of the $(k-1)$-jet,

$$\begin{bmatrix} x & xy \\ y & 0 \end{bmatrix},$$

it follows that these generators give all the basis vectors of $H^{k+1}(2,4)$, corresponding to those found above.

It remains to find the x^{k+1} term in e_5. This can be obtained by combining

$$x\begin{bmatrix} 0 & 0 \\ y & x^k \end{bmatrix} - xy\begin{bmatrix} 0 & x \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -x^2y \\ 0 & x^{k+1} \end{bmatrix},$$

with

$$x\begin{bmatrix} 0 & xy \\ 0 & x^k \end{bmatrix}.$$

So we have shown that the $(k+1)$-transversal of

$$\begin{bmatrix} x & xy \\ y & x^k \end{bmatrix},$$

is empty and it is therefore k-$(\mathcal{R} \times \mathcal{Q})$-determined. The required result follows from Proposition 6.1.5 and Corollary 4.4.19. □

Corollary 6.1.9 *We have the series of finitely k-$\mathcal{G}$-determined germs*

$$[0, x, xy, y, x^k, 0], \qquad (k \geq 2),$$

each of which has $\mathcal{G}_e$-codimension $(k+5)$ and discriminant of type D_{k+2}.

Proof By using a method similar to that used in the proof of Theorem 6.1.7 it can be shown that the two squares

$$\begin{bmatrix} x & xy \\ y & xy \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} x & xy \\ y & x^2 \end{bmatrix}$$

are $(\mathcal{R} \times \mathcal{Q})$-equivalent. So it follows that

$$[0, x, xy, y, x^2, 0],\cdot$$

is (like $[0, x, xy, y, xy, 0]$) also 2-$\mathcal{G}$-determined and we can add it to the series found in Theorem 6.1.8.

The $\mathcal{G}_e$-codimension of $[0, x, xy, y, x^2, 0]$ is therefore just the $\mathcal{G}_e$-codimension of $[0, x, xy, y, xy, 0]$ found, using `Transversal`, to be 7. We proceed by finding the $\mathcal{G}_e$-codimension of the germ $[0, x, xy, y, x^k, 0]$ for $k \geq 3$. For this task it is more convenient to find the $(\mathcal{R} \times \mathcal{Q})_e$-codimension of the k-jet

$$A = \begin{bmatrix} x & xy \\ y & x^k \end{bmatrix}, \tag{6.45}$$

in $\mathcal{O}_2^4$ and then use the formula given in part (ii) of Lemma 6.0.14 to obtain the $\mathcal{G}_e$-codimension for $[0, x, xy, y, x^k, 0]$. Since this is a slightly different approach to that described in the proof of Lemma 6.1.4 we give some details. Note that as regards finding the $(\mathcal{R} \times \mathcal{Q})_e$-codimension of (6.45) the method is essentially the same as before.

Firstly, the 'extended tangent space' to A is the $\mathcal{O}_2$-module generated by the vectors given in the latter part of the proof of Lemma 6.1.8. Rewriting them as 4-tuples (in row major order) these vectors are :

$$\begin{array}{cccc} [1, y, 0, kx^{k-1}], & [0, x, 1, 0] & & \\ [x, xy, 0, 0], & [0, 0, x, xy], & [y, x^k, 0, 0] & [0, 0, y, x^k] \\ [x, 0, y, 0], & [0, x, 0, y], & [xy, 0, x^k, 0] & [0, xy, 0, x^k] \end{array} . \tag{6.46}$$

Using a similar result to Corollary 4.4.34 (in Section 4.4.3) the codimension of A is given by

$$(\mathcal{R} \times \mathcal{Q})_e\text{-codim}A = \sum_{s=0}^{k} \dim \left(\frac{T(\mathcal{R} \times \mathcal{Q})_e.A + \mathcal{M}_2^s \mathcal{O}_2^4}{T(\mathcal{R} \times \mathcal{Q})_e.A + \mathcal{M}_2^{s+1} \mathcal{O}_2^4} \right), \tag{6.47}$$

where for convenience we denote the summands $\mathrm{cod}_s A$, see Definition 4.4.33.

We start by considering $s = 0$. Using vectors $[1, y, 0, kx^{k-1}]$ and $[0, x, 1, 0]$ it can be seen that $[1, 0, 0, 0]$ and $[0, 0, 1, 0]$ are both in $T(\mathcal{R} \times \mathcal{Q})_e.A + \mathcal{M}_2\mathcal{O}_2^4$. However $[0, 1, 0, 0]$, $[0, 0, 0, 1]$ are both missing from $T(\mathcal{R} \times \mathcal{Q})_e.A + \mathcal{M}_2\mathcal{O}_2^4$ and form a basis for a complement to this space in $\mathcal{O}_2^4$. Hence $\mathrm{cod}_0 A = 2$.

Next we look at the case when $s = 1$. Again using $[1, y, 0, kx^{k-1}]$ and $[0, x, 1, 0]$ all linear terms in both slots e_2 and e_4 are in $T(\mathcal{R} \times \mathcal{Q})_e.A + \mathcal{M}_2^2\mathcal{O}_2^4$. However the four vectors $[0, x|y, 0, 0]$, $[0, 0, 0, x|y]$ are not present in $T(\mathcal{R} \times \mathcal{Q})_e.A + \mathcal{M}_2^2\mathcal{O}_2^4$. Since $[0, x, 0, y]$ is however in this space, a basis for its complement in $T(\mathcal{R} \times \mathcal{Q})_e.A + \mathcal{M}_2\mathcal{O}_2^4$ consists of the three vectors

$$\{[0, x, 0, 0],\ [0, 0, 0, x],\ [0, y, 0, 0]\}.$$

Hence $\mathrm{cod}_0 A = 3$.

Finally we consider values of $s \geq 2$. Again using $[1, y, 0, kx^{k-1}]$ and $[0, x, 1, 0]$ all monomials of degree s in both slots e_2 and e_4 are in $T(\mathcal{R} \times \mathcal{Q})_e.A + \mathcal{M}_2^{s+1}\mathcal{O}_2^4$. It remains to consider monomial vectors for slots e_3 and e_5. Using similar arguments to those used in the proof of Lemma 6.1.8 it can be shown that for $2 \leq s \leq k-1$ the only monomial vector missing from $T(\mathcal{R} \times \mathcal{Q})_e.A + \mathcal{M}_2^{s+1}\mathcal{O}_2^4$ is $[0, 0, 0, x^s]$ which clearly forms a basis for a complement of $T(\mathcal{R} \times \mathcal{Q})_e.A + \mathcal{M}_2^s\mathcal{O}_2^4$.

However by using vectors $[1, y, 0, kx^{k-1}]$, $[x, xy, 0, 0]$ it is clear that $[0, 0, 0, x^k]$ is present in $T(\mathcal{R} \times \mathcal{Q})_e.A$ and so for values $s \geq 2$

$$\mathrm{cod}_s = \begin{cases} 1 & \text{for } 2 \leq s \leq k-1 \\ 0 & \text{for } s \geq k \end{cases}.$$

Hence from (6.47) A has $(\mathcal{R} \times \mathcal{Q})_e$-codimension $2 + 3 + (k - 2) = k + 3$.

Returning to the skew-symmetric germ $[0, x, xy, y, x^k, 0]$, since

$$\mathcal{O}_2/\langle x, xy, y, x^k\rangle = \{1\},$$

by using the relation, (6.9), in Lemma 6.0.14 this germ has $\mathcal{G}_e$-codimension $(k + 5)$.

Using part (i) of the same Lemma the discriminant of $[0, x, xy, y, x^k, 0]$ is given by the determinant of

$$\begin{bmatrix} x & xy \\ y & x^k \end{bmatrix}$$

and is $x^{k+1} - xy^2 = 0$ which has a D_{k+2} singularity. □

The 2-jet $[0, x, 0, y, x^2, 0]$, i.e.

$$\begin{bmatrix} 0 & 0 & x & 0 \\ 0 & 0 & y & x^2 \\ -x & -y & 0 & 0 \\ 0 & -x^2 & 0 & 0 \end{bmatrix},$$

has $J^2\mathcal{G}$-codimension 5. The following result, obtained computationally as described in Section 5.5, gives all $\mathcal{G}$-simple germs arising from this 2-jet.

Theorem 6.1.10 *Any $\mathcal{G}$-simple map $A : \mathbb{C}^2, 0 \to \mathbb{C}^6, 0$, with 2-jet $J^2\mathcal{G}$-equivalent to $[0, x, 0, y, x^2, 0]$, is $\mathcal{G}$-equivalent to one of the following finitely determined germs :*

$$[0, x, y^3, y, x^2, 0],$$

$$[0, x, xy^2, y, x^2, 0],$$

$$[0, x, y^4, y, x^2, 0],$$

with $\mathcal{G}_e$-codimensions 9, 10, 11 *and discriminants* E_6, E_7 *and* E_8 *respectively.*

Proof The following is an outline of the classification and we refer the reader to the worked examples in Section 5.5 for a full description.

The 2-jet $[0, x, 0, y, x^2, 0]$ has a complete 3-transversal

$$A_{ab} = [0, x, axy^2 + by^3, y, x^2, 0],$$

where a, $b \in \mathbb{C}$.

This transversal is contained in three distinct $J^3\mathcal{G}$-orbits of $J^3(2, 6)$ and we consider each in turn.

1. The elements of A_{ab} for which $b \neq 0$ are contained in a single $J^3\mathcal{G}$-orbit with $J^3\mathcal{G}$-codimension 5 and a representative

$$A_1 = [0, x, y^3, y, x^2, 0].$$

Furthermore this 3-jet, A_1, is 3-$\mathcal{G}$-determined and has $\mathcal{G}_e$-codimension 9. Representing it by its corresponding 2×2 submatrix,

$$\begin{bmatrix} x & y^3 \\ y & x^2 \end{bmatrix},$$

we see that it has discriminant

$$x^3 - y^4 = 0,$$

which has an E_6 singularity.

2. The elements of A_{ab} for which $b = 0$ and $a \neq 0$ are contained in a single $J^3\mathcal{G}$-orbit with $J^3\mathcal{G}$-codimension 6 and a representative

$$A_2 = [0, x, xy^2, y, x^2, 0].$$

This 3-jet has a complete 4-transversal

$$A_c = [0, x, xy^2 + cy^4, y, x^2, 0],$$

which is found to be a $J^4\mathcal{G}$-trivial family contained in the single $J^4\mathcal{G}$-orbit with $J^4\mathcal{G}$-codimension 6 and representative

$$[0, x, xy^2, y, x^2, 0].$$

This 4-jet is 4-$\mathcal{G}_1$-determined. So, since any germ with 3-jet A_2 is $J^4\mathcal{G}$-equivalent to the 4-jet $[0, x, xy^2, y, x^2, 0]$ (by the definition of a complete transversal), it follows that A_2 is in fact 3-$\mathcal{G}$-determined. We find the $\mathcal{G}_e$-codimension of A_2 to be 10 and representing it by its corresponding 2×2 submatrix,

$$\begin{bmatrix} x & xy^2 \\ y & x^2 \end{bmatrix},$$

we see it has discriminant

$$x^3 - xy^3 = 0,$$

which has an E_7 singularity.

3. The element of A_{ab} for which $b = a = 0$ is given by

$$A_3 = [0, x, 0, y, x^2, 0]$$

and has $J^3\mathcal{G}$-codimension 7. This 3-jet has a complete 4-transversal

$$A_{cd} = [0, x, cxy^3 + dy^4, y, x^2, 0].$$

The elements of A_{cd} for which $d \neq 0$ are contained in a single $J^4\mathcal{G}$-orbit, of $J^4(2,6)$, with $J^4\mathcal{G}$-codimension 7 and representative

$$[0, x, y^4, y, x^2, 0].$$

This 4-jet is 4-$\mathcal{G}$-determined and has $\mathcal{G}_e$-codimension 11. Representing it by its corresponding 2×2 submatrix,

$$\begin{bmatrix} x & y^4 \\ y & x^2 \end{bmatrix},$$

we see it has discriminant

$$x^3 - y^5 = 0,$$

which has an E_8 singularity.

It remains to show there are no more $\mathcal{G}$-simple germs. The elements of A_{cd} for which $d = 0$ and $c \neq 0$ are contained in a single $J^4\mathcal{G}$-orbit, of $J^4(2,6)$, with representative

$$A_4 = [0, x, xy^3, y, x^2, 0]$$

and $J^4\mathcal{G}$-codimension 8. This 4-jet has a complete 5-transversal

$$A_e = [0, x, xy^3 + ey^5, y, x^2, 0].$$

All elements of this family have $J^5\mathcal{G}$-codimension 9. Associating each $e \in \mathbb{C}$ to the corresponding element of this family, their Pfaffians are given by the family of function germs $x^3 - xy^4 - ey^6$. However for any such $e \in \mathbb{C}$ we claim that any neighbourhood of e consists of uncountably many (corresponding) $\mathcal{K}$-inequivalent germs.

Provided $27e^2 - 4 \neq 0$ each germ $x^3 - xy^4 - ey^6$ is 6-$\mathcal{K}$-determined. Working in $J^6(2,1)$, by a similar argument to that given in the proof of Lemma 4.4.10, all $J^6\mathcal{K}$-orbits are constructible and so meet the 1-dimensional affine space

$$X = \{x^3 - xy^4 - ey^6 : e \in \mathbb{C}\}$$

in constructible sets. These are either a finite set of isolated points or the complement of a finite set of points.

If one of the $J^6\mathcal{K}$-orbits meets X in the complement of a finite set of points then for all except a finite number of values of e the $J^6\mathcal{K}$-tangent space to a germ $x^3 - xy^4 - ey^6$ would contain the tangent vector to X, y^6. A calculation shows this is not the case. So every $J^6\mathcal{K}$-orbit of $J^6(2,1)$ meets X in a finite set of points thus proving our claim.

Hence applying Proposition 4.2.10, in any neighbourhood of each element of A_e there are uncountably many $J^5\mathcal{G}$-orbits (in $J^5(2,6)$). Since any germ with a 4-jet, lying in the $J^4\mathcal{G}$-orbit represented by A_4, is $J^5\mathcal{G}$-equivalent to something in A_e (by the definition of a complete transversal) it follows that such germs cannot be simple.

Finally, the element of A_{cd} for which $d = c = 0$,

$$A_5 = [0, x, 0, y, x^2, 0]$$

cannot be the 4-jet of a $\mathcal{G}$-simple germ, since in any neighbourhood of it (in $J^4(2,6)$) there is a 4-jet

$$[0, x, \epsilon xy^3, y, x^2, 0],$$

where ϵ is small, of a germ for which, as demonstrated above, any neighbourhood meets uncountably many $J^5\mathcal{G}$-orbits of $J^5(2,6)$. $\square$

Investigating germs with 2-jet $[0, x, 0, y, 0, 0]$ we establish the following result.

Lemma 6.1.11 *There are no $\mathcal{G}$-simple germs $A : \mathbb{C}^2, 0 \to \mathbb{C}^6, 0$ with 2-jet* $[0, x, 0, y, 0, 0]$.

Proof The proof relies on a calculation carried out by `Transversal`. The 2-jet $[0, x, 0, y, 0, 0]$ has $J^2\mathcal{G}$-codimension 7 and a complete 3-transversal

$$A_{cd} = [0, x, c(x,y), y, dx^3, 0],$$

where $c(x,y) = c_1x^3 + c_2x^2y + c_3xy^2 + c_4y^3$ and $(c_1, c_2, c_3, c_4, d) \in \mathbb{C}^5$. Observe, with regard to using Lemma 4.5.3, that A_{cd} is a smooth constructible subset of $J^3(2,6)$ which, obviously, passes through each of its constituent 3-jets.

We find, using `Transversal`, that for almost all values of the parameters (c_1, c_2, c_3, c_4, d), (the possible exceptions being values of these parameters which occur in the union of a *finite* number of affine varieties of the parameter space) the corresponding elements of A_{cd} have $J^3\mathcal{G}$-codimension 8 and the $J^3\mathcal{G}$-tangent space to each does not contain the vector, $[0,0,0,0,x^3,0]$, (which lies in the tangent space to A_{cd}) and hence cannot contain the tangent space to A_{cd}.

Consider any 3-jet of the transversal A_{cd}. It follows, by the above calculation, that on any neighbourhood U of this 3-jet (in $J^3(2,6)$) there is no Zariski open set $V \subset A_{cd}$ satisfying the criterion for simplicity given in Lemma 4.5.3. So, no germ with a 3-jet in the complete transversal, A_{cd}, can be $\mathcal{G}$-simple and, by the definition of a complete transversal, this is also the case for all germs with 2-jet $[0,x,0,y,0,0]$. □

6.1.2 1-jets : jetrank ≤ 1

In this section we consider the remaining 1-jets, $[x,0,0,0,0,x]$, $[x,0,0,0,0,0]$ and $[0,0,0,0,0,0]$. Initially we manipulate the first two into more convenient forms. Using Lemma 6.0.1 we find the 1-jets $[x,0,0,0,0,x]$ and $[x,0,0,0,0,0]$ are $\mathcal{H}$-equivalent to the 1-jets

$$\begin{bmatrix} 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \\ -x & 0 & 0 & 0 \\ 0 & -x & 0 & 0 \end{bmatrix} \tag{6.48}$$

and

$$\begin{bmatrix} 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 \\ -x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

respectively.

First consider the 1-jet $[0,x,0,0,x,0]$. Exploratory techniques, already familiar to the reader from the previous section, suggest that any germ with this 1-jet occurs in a series of (distinct) finitely $\mathcal{G}$-determined germs. An initial step to proving this is the following result.

Lemma 6.1.12 *Any germ $\mathbb{C}^2,0 \to Sk(4,\mathbb{C})$ with $(k-1)$-jet ($k \geq 2$),*

$$\begin{bmatrix} 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \\ -x & 0 & 0 & 0 \\ 0 & -x & 0 & 0 \end{bmatrix},$$

is $J^k\mathcal{G}$-equivalent to one of the following three k-jets:

$$[0, x, y^k, y^k, x, 0],$$

$$[0, x, y^k, 0, x, 0],$$

or

$$[0, x, 0, 0, x, 0].$$

Proof It is fairly easy to show that the $(k-1)$-jet,

$$\begin{bmatrix} 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \\ -x & 0 & 0 & 0 \\ 0 & -x & 0 & 0 \end{bmatrix},$$

has a complete k-transversal

$$\begin{bmatrix} 0 & a_1y^k & x & a_3y^k \\ -a_1y^k & 0 & a_4y^k & x + a_5y^k \\ -x & -a_4y^k & 0 & a_6y^k \\ -a_3y^k & -x - a_5y^k & -a_6y^k & 0 \end{bmatrix}, \qquad (6.49)$$

where, as usual, $a_i \in \mathbb{C}$.

We use the $J^k\mathcal{G}$-group to simplify this family to a finite number of representatives. There are two cases to consider

(i) $a_3 \neq 0$,

(ii) $a_3 = 0$.

(i) If $a_3 \neq 0$ we can assume $a_1 = a_6 = 0$, since by using the simultaneous row and column operation involving the column operation,

$$C_2 - \frac{a_1}{a_3}C_4,$$

followed by another involving the row operation,

$$R_3 - \frac{a_6}{a_3}R_1,$$

eliminates the y^k terms in slots e_1 and e_6 respectively. Assuming $a_5 \neq 0$ then by scaling we obtain the $J^k\mathcal{G}$-equivalent matrix

$$\begin{bmatrix} 0 & 0 & x & y^k \\ 0 & 0 & A_4 y^k & x + y^k \\ -x & -A_4 y^k & 0 & 0 \\ -y^k & -x - y^k & 0 & 0 \end{bmatrix}, \tag{6.50}$$

where A_4 is invariant with respect to scaling.

Representing this k-jet by the square

$$\begin{bmatrix} x & y^k \\ A_4 y^k & x + y^k \end{bmatrix},$$

by explicit row/column operations followed by an $\mathcal{R}$-change of coordinates, we obtain the $J^k(\mathcal{R} \times \mathcal{Q})$-equivalent matrix

$$\begin{bmatrix} x & y^k \\ a y^k & x \end{bmatrix}.$$

(We also obtain a k-jet of this form if $a_5 = 0$ above.) If $a \neq 0$ it can be scaled to unity and we conclude that all elements of the k-transversal (6.49) for which $a_3 \neq 0$ are $J^k\mathcal{G}$-equivalent to one of two k-jets :

$$[0, x, y^k, y^k, x, 0] \qquad \text{or} \qquad [0, x, y^k, 0, x, 0].$$

(ii) Alternatively, if $a_3 = 0$, we can also assume $a_1 = a_4 = a_5 = a_6 = 0$, since otherwise by a series of simultaneous row and column operations we can move a non-zero multiple of y^k into slot e_3 thereby obtaining a k-jet considered in case (i). This leaves the third and final k-jet

$$[0, x, 0, 0, x, 0].$$

□

We can use this lemma to give us the following result.

Theorem 6.1.13 *Let $A : \mathbb{C}^2, 0 \to Sk(4, \mathbb{C})$ be a smooth germ with 1-jet*

$$\begin{bmatrix} 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \\ -x & 0 & 0 & 0 \\ 0 & -x & 0 & 0 \end{bmatrix}, \tag{6.51}$$

then A is $\mathcal{G}$-equivalent to an l-$\mathcal{G}$-determined germ of the form

$$\begin{bmatrix} 0 & 0 & x & y^k \\ 0 & 0 & y^l & x \\ -x & -y^l & 0 & 0 \\ -y^k & -x & 0 & 0 \end{bmatrix}, \qquad 2 \le k \le l, \tag{6.52}$$

or for any $k \ge 1$, A is $\mathcal{G}$-equivalent to a germ whose k-jet is (6.51) or for any $2 \le k \le l$, A is $\mathcal{G}$-equivalent to a germ whose l-jet is

$$\begin{bmatrix} 0 & 0 & x & y^k \\ 0 & 0 & 0 & x \\ -x & 0 & 0 & 0 \\ -y^k & -x & 0 & 0 \end{bmatrix}.$$

Each germ of the series, (6.52), has $\mathcal{G}_e$-codimension $(4k+l-1)$ and a discriminant of type A_{k+l-1}.

Proof Assume for any $k \ge 2$ that A has a $(k-1)$-jet

$$\begin{bmatrix} 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \\ -x & 0 & 0 & 0 \\ 0 & -x & 0 & 0 \end{bmatrix}.$$

By Lemma 6.1.12 we can represent this jet by the matrix

$$\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix}$$

for which there are two possible k-jets :

$$\begin{bmatrix} x & y^k \\ y^k & x \end{bmatrix} \tag{6.53}$$

or

$$\begin{bmatrix} x & y^k \\ 0 & x \end{bmatrix}. \tag{6.54}$$

The third possible k-jet

$$\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix},$$

corresponds to replacing $(k-1)$ in the original assumption by k etc. The first of these, (6.53), is k-$(\mathcal{R} \times \mathcal{Q})$-determined. Hence since $\mathcal{M}_2^{k+1} \subset \mathcal{M}_2\langle x, y^k \rangle$, this implies that the corresponding skew k-jet is k-$\mathcal{G}$-determined.

Assume any germ with the second k-jet has (for $l > k$) an $(l-1)$-jet

$$\begin{bmatrix} x & y^k \\ 0 & x \end{bmatrix}.$$

This has a complete l-transversal

$$\begin{bmatrix} x & y^k \\ ay^l & x \end{bmatrix},$$

where $a \in \mathbb{C}$. Note if $a = 0$ we replace $(l-1)$, in the previous assumption, by l etc. Considering $a \neq 0$, by scaling we obtain the $J^l(\mathcal{R} \times \mathcal{Q})$-equivalent l-jet

$$\begin{bmatrix} x & y^k \\ y^l & x \end{bmatrix},$$

which is l-$(\mathcal{R} \times \mathcal{Q})$-determined.

Noting that the k-determined jets

$$\begin{bmatrix} x & y^k \\ y^k & x \end{bmatrix}$$

are included in this series by allowing $l = k$, we have found the required series.

By finding the $(\mathcal{R} \times \mathcal{Q})_e$-codimension of the l-determined germ

$$\begin{bmatrix} x & y^k \\ y^l & x \end{bmatrix},$$

and using the relation (6.9) in Lemma 6.0.14 we find the corresponding germ $[0, x, y^k, y^l, x, 0]$ has $\mathcal{G}_e$-codimension $(4k + l - 1)$. Furthermore its discriminant is given by $x^2 - y^{k+l} = 0$ which has an A_{k+l-1} singularity. □

We can extend this series to include all previously found finitely determined germs whose discriminants have A-type singularities.

Corollary 6.1.14 *We have a series, B_{kl}, of finitely l-$\mathcal{G}$-determined germs*

$$[0, x, y^k, y^l, x, 0], \qquad 1 \geq k \geq l, \tag{6.55}$$

each of which has $\mathcal{G}_e$-codimension $(4k + l - 1)$, a discriminant of type A_{k+l-1} and represents a distinct $\mathcal{G}$-orbit.

Proof First consider the series of germs obtained in Theorem 6.1.3 :

$$\begin{bmatrix} 0 & x & 0 & y^l \\ -x & 0 & y & 0 \\ 0 & -y & 0 & x \\ -y^l & 0 & -x & 0 \end{bmatrix}, \qquad l \geq 2. \tag{6.56}$$

Interchanging C_2 and C_3 (and R_2 and R_3) we obtain the matrix

$$\begin{bmatrix} 0 & 0 & x & y^l \\ 0 & 0 & -y & x \\ -x & y & 0 & x \\ -y^l & -x & 0 & 0 \end{bmatrix}.$$

Representing this by the matrix

$$\begin{bmatrix} x & y^l \\ -y & x \end{bmatrix},$$

interchanging R_1 and R_2 followed by interchanging C_1 and C_2 we obtain the $(\mathcal{R} \times \mathcal{Q})$-equivalent matrix

$$\begin{bmatrix} x & -y \\ y^l & x \end{bmatrix},$$

which by a scale change is equivalent to

$$\begin{bmatrix} x & y \\ y^l & x \end{bmatrix}. \tag{6.57}$$

Hence, using Lemma 6.0.13, we deduce the two germs $[x, 0, y^l, y, 0, x]$ and $[0, x, y, y^l, x, 0]$, $l \geq 2$, are $\mathcal{G}$-equivalent and represent them by the 2×2 square in (6.57). So the series of l-determined germs found in Theorem 6.1.3 can be added to those found above, in Theorem 6.1.13, by allowing $k = 1$. It is easily verified that when $k = 1$ the invariants of these two series match.

We can also include the open $\mathcal{G}$-orbit of germs in this enlarged series. Recall, a representative for such a germ is given by the 1-determined jet

$$\begin{bmatrix} 0 & x & 0 & 0 \\ -x & 0 & 0 & 0 \\ 0 & 0 & 0 & y \\ 0 & 0 & -y & 0 \end{bmatrix}. \tag{6.58}$$

As discussed in Section 3.4.1 any non-singular pencil of 4×4 skew-symmetric matrices with two distinct eigenvalues (a non-degenerate pencil) is skew-equivalent to this 1-jet. Clearly the 1-jet

$$\begin{bmatrix} 0 & 0 & x & y \\ 0 & 0 & y & x \\ -x & -y & 0 & 0 \\ -y & -x & 0 & 0 \end{bmatrix}$$

is such a pencil as the roots of its Pfaffian are given by $x^2 - y^2 = 0$. Hence we can add 1-determined jets represented by (6.58) to our series by allowing $k = l = 1$.

It remains to distinguish each of the germs of this series. By Lemma 4.2.7, in Section 4.2, the germ $[0, x, y^k, y^l, x, 0]$ is $\mathcal{K}$-equivalent to $[0, x, y^k, 0, 0, 0]$ which has $\mathcal{K}_e$-codimension $(5k - 1)$. Since, by Lemma 4.2.6, the $\mathcal{K}_e$-codimension of a germ is a $\mathcal{G}$-invariant it follows that the value of k in the germ $[0, x, y^k, y^l, x, 0]$ is an invariant. Furthermore since both the discriminant and $\mathcal{G}_e$-codimension of a germ are $\mathcal{G}$-invariants this implies that l is also an invariant. It follows each germ in this series represents a distinct $\mathcal{G}$-orbit. □

Consider the 1-jet

$$\begin{bmatrix} 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 \\ -x & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{6.59}$$

This 1-jet has a complete 2-transversal

$$\begin{bmatrix} 0 & a_1y^2 & x & a_3y^2 \\ -a_1y^2 & 0 & a_4y^2 & p(x,y) \\ -x & -a_4y^2 & 0 & a_6y^2 \\ -a_3y^2 & -p(x,y) & -a_6y^2 & 0 \end{bmatrix}, \tag{6.60}$$

where $p(x, y) = ax^2 + bxy + cy^2$ is a arbitrary quadratic. Next we look to identify, from this 2-transversal, a finite number of distinct $J^2\mathcal{G}$-orbits.

Assume first that $a_3 \neq 0$. By a pair of simultaneous row and column operations, the first involving

$$C_2 - \frac{a_1}{a_3}C_4,$$

and the second involving

$$R_3 - \frac{a_6}{a_3}R_1,$$

we obtain the $J^2\mathcal{G}$-equivalent matrix

$$\begin{bmatrix} 0 & 0 & x & a_3y^2 \\ 0 & 0 & A_4y^2 & p(x,y) \\ -x & -A_4y^2 & 0 & 0 \\ -a_3y^2 & -p(x,y) & 0 & 0 \end{bmatrix}.$$

There are two possiblities, each obtained by scaling.

If $A_4 \neq 0$ then we have the $J^2\mathcal{G}$-equivalent matrix

$$\begin{bmatrix} 0 & 0 & x & y^2 \\ 0 & 0 & y^2 & P(x,y) \\ -x & -y^2 & 0 & 0 \\ -y^2 & -P(x,y) & 0 & 0 \end{bmatrix}. \tag{6.61}$$

where the quadratic $P(x,y)$ is different to $p(x,y)$.

If $A_4 = 0$ then we have the $J^2\mathcal{G}$-equivalent matrix

$$\begin{bmatrix} 0 & 0 & x & y^2 \\ 0 & 0 & 0 & P(x,y) \\ -x & 0 & 0 & 0 \\ -y^2 & -P(x,y) & 0 & 0 \end{bmatrix}. \tag{6.62}$$

If in (6.60) $a_3 = 0$, provided one of a_1, a_4 or a_6 is non-zero we can, by simultaneous row and column operations, move a non-zero multiple of y^2 into slot e_3 — thereby reducing to one of the previous cases, (6.61) or (6.62). There remain those 2-jets for which $a_3 = a_1 = a_4 = a_6 = 0$ i.e.

$$\begin{bmatrix} 0 & 0 & x & 0 \\ 0 & 0 & 0 & P(x,y) \\ -x & 0 & 0 & 0 \\ 0 & -P(x,y) & 0 & 0 \end{bmatrix}. \tag{6.63}$$

For convenience we represent each of these families of 2-jets, (6.61), (6.62) and (6.63), by the matrices

$$\begin{bmatrix} x & y^2 \\ y^2 & P(x,y) \end{bmatrix}, \tag{6.64}$$

$$\begin{bmatrix} x & y^2 \\ 0 & P(x,y) \end{bmatrix}, \tag{6.65}$$

and

$$\begin{bmatrix} x & 0 \\ 0 & P(x,y) \end{bmatrix}, \tag{6.66}$$

respectively, with a view to obtaining, from them, all distinct $J^2(\mathcal{R} \times \mathcal{Q})$-orbits.

Lemma 6.1.15 *Any 2-jet of the form (6.64), (6.65) or (6.66) lies in one of eleven (distinct) $J^2(\mathcal{R} \times \mathcal{Q})$-orbits with representatives, written in row major order (see Remarks 6.0.16), given by*

$$\begin{gathered}
[x, 0, 0, x^2 + y^2], \\
[x, 0, 0, y^2], \\
[x, y^2, y^2, xy], \\
[x, y^2, y^2, x^2], \\
[x, y^2, y^2, 0], \\
[x, y^2, 0, xy], \\
[x, y^2, 0, x^2], \\
[x, y^2, 0, 0], \\
[x, 0, 0, xy], \\
[x, 0, 0, x^2], \\
[x, 0, 0, 0].
\end{gathered}$$

Proof Here we will show that, given a 2-jet in (6.64), (6.65) or (6.66), then by $J^2(\mathcal{R} \times \mathcal{Q})$-equivalence we obtain one of the above cases. We need to check the corresponding $J^2\mathcal{G}$-orbits are distinct and we defer a proof of this to a later corollary. Clearly, if two $J^2\mathcal{G}$-orbits represented by $[0, a_1, b_1, c_1, d_1, 0]$ and $[0, a_2, b_2, c_2, d_2, 0]$ are distinct then so are the corresponding $J^2(\mathcal{R} \times \mathcal{Q})$-orbits represented by $[a_1, b_1, c_1, d_1]$ and $[a_2, b_2, c_2, d_2]$.

For illustrative purposes we consider 2-jets of the form (6.64)

$$\begin{bmatrix} x & y^2 \\ y^2 & P(x,y) \end{bmatrix}. \tag{6.67}$$

The argument used for this case can also be applied to the simpler case of the 2-jets of type (6.65) and (6.66).

First we classify the quadratic, $P(x,y) = p_1x^2 + p_2xy + p_3y^2$, up to the subgroup of linear coordinate changes consisting of elements of the form

$$(x,y) \mapsto (\alpha x, \beta x + \gamma y), \qquad \alpha\gamma \neq 0. \tag{6.68}$$

These coordinate changes preserve $x = 0$ and hence, after scaling, the 1-jet of (6.67).

In fact the action of this subgroup yields a classification of *binary cubics*, $xP(x,y) \in V_4$, up to a linear equivalence preserving $x = 0$. Recall from Definition 4.7.8 and the succeeding remarks that we associate a binary cubic with the set of its 3 roots, in $P\mathbb{C}^1$. Consequently, we expect the action (given in Definition 4.7.9) on the set,

$$V = \{q \in P\mathbb{C}^3 : q = \lambda xP(x,y),\ \lambda \neq 0, \text{ with } P(x,y) \in V_2\},$$

of the subgroup of $PGl(2,\mathbb{C})$ which *fixes* the point $(0:1) \in P\mathbb{C}^1$, to give 4 distinct types. Representing these by their set of roots in $P\mathbb{C}^1$, they are :

(1) a simple point $(0:1)$ and two further simple points,

(2) a simple point $(0:1)$ and a further double point,

(3) a double point $(0:1)$ and a further simple point,

(4) a triple point $(0:1)$.

Adding to these the further possibility of $0 \in V_4$ gives five possible types under this classification. In the following we find the normal forms for these types.

The effect of such a coordinate change, (6.68), on the quadratic $P(x,y)$ is

$$\begin{aligned} P(\alpha x, \beta x + \gamma y) &= p_1\alpha^2x^2 + p_2\alpha x(\beta x + \gamma y) + p_3(\beta x + \gamma y)^2 \\ &= (p_1\alpha^2 + p_2\alpha\beta + p_3\beta^2)x^2 \\ &\quad + (p_2\alpha\gamma + 2p_3\beta\gamma)xy + p_3\gamma^2y^2. \end{aligned} \tag{6.69}$$

If $p_3 \neq 0$ and $P(x,y)$ has distinct roots, by choosing suitable values for (α,β,γ), we can reduce it to x^2+y^2. For example, from (6.69), it can be seen, by choosing $\gamma = 1/\sqrt{p_3}$ and $(\alpha,\beta) = \lambda(2p_3, -p_2)$ for some $\lambda \neq 0$, that the coefficient of y^2 it scaled to unity and the coefficient of xy is zero. Furthermore the coefficient of the x^2 term is then given by $p_3\lambda^2(4p_1p_3 - p_2^2)$. So denoting the discriminant, $4p_1p_3 - p_2^2$, of $P(x,y)$ by δ it follows that if $\delta \neq 0$ then by choosing $\lambda = 1/\sqrt{p_3\delta}$ we can scale this coefficient to unity and get $x^2 + y^2$.

If however, $p_3 \neq 0$ but $P(x,y)$ has repeated roots ($\delta = 0$) then choosing the same values, $(\alpha, \beta, \gamma) = (\lambda 2p_3, -\lambda p_2, 1/\sqrt{p_3})$ for any $\lambda \neq 0$, we reduce $P(x,y)$ to y^2.

If $p_3 = 0$ then (6.69) becomes

$$\alpha x\left((p_1\alpha + p_2\beta)x + p_2\gamma y\right),$$

and provided $p_2 \neq 0$ choosing $(\alpha, \beta, \gamma) = (1, -p_1/p_2, 1/p_2)$ reduces $P(x,y)$ to xy.

If $p_3 = p_2 = 0$ but $p_1 \neq 0$ then by choosing $\alpha = 1/\sqrt{p_1}$ we obtain x^2. Finally we are left with the possibility, $P(x,y) = 0$.

So by the action of elements of the form (6.68) we can reduce $P(x,y)$ to a quadratic, $Q(x,y)$, taken from the set :

$$\{x^2 + y^2, y^2, xy, x^2, 0\}. \tag{6.70}$$

Premultiplying each of these by some non-zero constant multiple of x gives each of the five normal forms of cubics, $xP(x,y)$, (up to a linear equivalence preserving $x = 0$) discussed overleaf.

The effect of this action on a 2-jet of type (6.67) results in an equivalent matrix of the form

$$\begin{bmatrix} ax & b_1x^2 + b_2xy + b_3y^2 \\ b_1x^2 + b_2xy + b_3y^2 & Q(x,y) \end{bmatrix}, \tag{6.71}$$

where $a \neq 0$ and $b_3 \neq 0$. Similarly, the action of this group on 2-jets of type (6.65) or (6.66) result in equivalent matrices of type

$$\begin{bmatrix} ax & b_1x^2 + b_2xy + b_3y^2 \\ 0 & Q(x,y) \end{bmatrix}, \tag{6.72}$$

and

$$\begin{bmatrix} ax & 0 \\ 0 & Q(x,y) \end{bmatrix}, \tag{6.73}$$

respectively. We continue by considering (6.71). Since we are working in the 2-jet space we can use the x-term in slot e_2 to kill off both the terms in slot e_3 involving x. For example by the column operation,

$$C_2 - \frac{1}{a}(b_1 x + b_2 y)C_1,$$

we obtain the $J^2(\mathcal{R} \times \mathcal{Q})$-equivalent jet

$$\begin{bmatrix} ax & b_3 y^2 \\ b_1 x^2 + b_2 xy + b_3 y^2 & Q(x,y) \end{bmatrix}.$$

The same operation applied to the rows of this matrix gives

$$\begin{bmatrix} ax & b_3 y^2 \\ b_3 y^2 & Q(x,y) \end{bmatrix}. \tag{6.74}$$

Applying the same argument to (6.72) we obtain the $J^2(\mathcal{R} \times \mathcal{Q})$-equivalent jets

$$\begin{bmatrix} ax & b_3 y^2 \\ 0 & Q(x,y) \end{bmatrix}. \tag{6.75}$$

We also have jets of type (6.73).

The next step is to scale the non-zero constants a, b_3 in (6.74) to unity while preserving the quadratic $Q(x,y)$. Since $Q(x,y)$ is one of the five homogeneous quadratic types in (6.70) this amounts to combining a scaling coordinate change, $(x,y) \mapsto (\lambda x, \lambda y)$ with the $\mathcal{Q}$-action of a pair $(X,Y) \in Gl(2,\mathbb{C}) \times Gl(2,\mathbb{C})$, of the form

$$X = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}, \qquad Y = \begin{bmatrix} \rho & 0 \\ 0 & \sigma \end{bmatrix}, \quad \alpha\beta\rho\sigma \neq 0.$$

By such an action we obtain, from (6.74), the $J^2(\mathcal{R} \times \mathcal{Q})$-equivalent matrix

$$\begin{bmatrix} a\alpha\rho\lambda x & b_3\alpha\sigma\lambda^2 y^2 \\ b_3\beta\rho\lambda^2 y^2 & \beta\sigma\lambda^2 Q(x,y) \end{bmatrix}.$$

It can be verified that by choosing $(\lambda, \alpha, \beta, \rho, \sigma) = (a/b_3^2, b_3^3/a^2, b_3^4/a^2, 1/b_3, 1)$, we obtain the required form

$$\begin{bmatrix} x & y^2 \\ y^2 & Q(x,y) \end{bmatrix}. \tag{6.76}$$

By a similar method we can scale 2-jets of type (6.75) and (6.73) to

$$\begin{bmatrix} x & y^2 \\ 0 & Q(x,y) \end{bmatrix}, \tag{6.77}$$

and

$$\begin{bmatrix} x & 0 \\ 0 & Q(x,y) \end{bmatrix}. \tag{6.78}$$

Finally we observe for types (6.76) and (6.77), that if the quadratic $Q(x,y)$ contains a non-zero y^2 term then it can be used to kill off any y^2 terms present in slots e_3 or e_4, thereby reducing to a 2-jet of type (6.78). For example, consider 2-jet of type (6.76) where $Q(x,y) = x^2 + y^2$,

$$\begin{bmatrix} x & y^2 \\ y^2 & x^2+y^2 \end{bmatrix}.$$

By the row operation $R_1 - R_2$ we get the 2-jet

$$\begin{bmatrix} x-y^2 & -x^2 \\ y^2 & x^2+y^2 \end{bmatrix}.$$

Then the column operation $C_2 + xC_1$ gives the $J^2(\mathcal{R} \times \mathcal{Q})$-equivalent jet

$$\begin{bmatrix} x-y^2 & 0 \\ y^2 & x^2+y^2 \end{bmatrix},$$

since degree 3 terms drop out in 2-jet space. By the analogous operations $C_1 - C_2$ followed by $R_2 + xR_1$ we obtain the $J^2(\mathcal{R} \times \mathcal{Q})$-equivalent jet

$$\begin{bmatrix} x-y^2 & 0 \\ 0 & x^2+y^2 \end{bmatrix},$$

which by the $J^2\mathcal{R}$-change $(x,y) \mapsto (x+y^2, y)$ is $J^2(\mathcal{R} \times \mathcal{Q})$-equivalent to

$$\begin{bmatrix} x & 0 \\ 0 & x^2+y^2 \end{bmatrix}.$$

By similar arguments any 2-jet of type (6.76) or (6.77) with $Q(x,y)$ either $x^2 + y^2$ or y^2 is $J^2(\mathcal{R} \times \mathcal{Q})$-equivalent to the 2-jet of type (6.78) with this $Q(x,y)$. Bearing this in mind we obtain from, (6.76), (6.77) and (6.78), the eleven 2-jets in the statement of the lemma. □

We wish to determine, from this list of 2-jets $\{[x, b_i, c_i, d_i] \in J^2(2,4)\}$, whether the corresponding 2-jets $\{[0, x, b_i, c_i, d_i, 0] \in J^2(2,6)\}$ represent *distinct* $J^2\mathcal{G}$-orbits of $J^2(2,6)$. To this end we identify an invariant of such 2-jets, using the following results. Recall, from Corollary 4.2.4, that if two germs A, $B : \mathbb{C}^r, 0 \to Sk(n, \mathbb{C})$ are $\mathcal{G}$-equivalent then for some ring isomorphism, $\phi^* : \mathcal{O}_r \to \mathcal{O}_r$,

$$\phi^* \left(I_1(A)\right) = I_1(B).$$

The following gives a related invariant for germs (of 4×4 skew-symmetric matrices) which are equivalent as 1-jets.

Lemma 6.1.16 *Consider two germs A, $B : \mathbb{C}^r, 0 \to Sk(4, \mathbb{C})$, both vanishing at the origin, with 1-jets A_1, B_1 respectively.*

(i) If A and B are $J^1\mathcal{H}$-equivalent then

$$\left(I_1(A_1) + \mathcal{M}_r^2\right) / \mathcal{M}_r^2 = \left(I_1(B_1) + \mathcal{M}_r^2\right) / \mathcal{M}_r^2. \tag{6.79}$$

(ii) If A and B are $J^1\mathcal{G}$-equivalent then, for some germ of a diffeomorphism $\phi : \mathbb{C}^r, 0 \to \mathbb{C}^r, 0$ and the corresponding ring isomorphism $\phi^* : \mathcal{O}_r \to \mathcal{O}_r$,

$$\left(\phi_1^* \left(I_1(A_1)\right) + \mathcal{M}_r^2\right) / \mathcal{M}_r^2 = \left(I_1(B_1) + \mathcal{M}_r^2\right) / \mathcal{M}_r^2, \tag{6.80}$$

where ϕ_1 is the 1-jet of ϕ.

Proof (i) Write $A = A_1 + A_2$, $B = B_1 + B_2$, where A_1, B_1 are skew-symmetric matrices with each entry linear in the variables $\{x_i : 1 \leq i \leq r\}$ and A_2, $B_2 \in \mathcal{M}_r^2.\mathcal{O}_r^6$. If A and B are $J^1\mathcal{H}$-equivalent then, for some $X \in J^1\mathcal{H}$,

$$B_1 + B_2 \equiv X^T(A_1 + A_2)X \bmod \mathcal{M}_r^2,$$

In particular if $X(0) = X_1$ then

$$B_1 = X_1^T A_1 X_1.$$

Clearly,

$$\left(I(B_1) + \mathcal{M}_r^2\right) / \mathcal{M}_r^2 \subset \left(I(A_1) + \mathcal{M}_r^2\right) / \mathcal{M}_r^2.$$

Writing $A \equiv (X^T)^{-1}BX^{-1} \bmod \mathcal{M}_r^2$ and following the same argument gives the reverse inclusion and hence the result in (6.79).

(ii) If A and B are $\mathcal{G}$-equivalent as 1-jets then, for some germ of a diffeomorphism $\phi : \mathbb{C}^r, 0 \to \mathbb{C}^r, 0$, $A \circ \phi$ and B are $\mathcal{H}$-equivalent as 1-jets. Since we are working modulo $\mathcal{M}_r^2$ the only relevant part of ϕ is its 1-jet which we denote ϕ_1. From (6.79) it follows that

$$\left(I_1(A_1 \circ \phi_1) + \mathcal{M}_r^2\right) / \mathcal{M}_r^2 = \left(I_1(B_1) + \mathcal{M}_r^2\right) / \mathcal{M}_r^2,$$

and denoting the ring isomorphism, $\mathcal{O}_r \to \mathcal{O}_r$, corresponding to ϕ_1 by ϕ_1^* we have (6.80). □

Lemma 6.1.17 *If the 2-jets A_2, $B_2 : \mathbb{C}^r, 0 \to Sk(4, \mathbb{C})$ of the form, $A_2 = [0, x_1 + a_1, b_1, c_1, d_1, 0]$, $B_2 = [0, x_1 + a_2, b_2, c_2, d_2, 0]$ (with each a_i, b_i, c_i, d_i homogeneous of degree 2), are $J^2\mathcal{H}$-equivalent then*

$$(Pf(B_2) - \lambda Pf(A_2)) \in \langle x_1 \rangle \mathcal{M}_r^3 + \mathcal{M}_r^5, \tag{6.81}$$

where λ is some non-zero constant. In particular,

$$j^3(x_1 d_2) = \lambda j^3(x_1 d_1). \tag{6.82}$$

The 3-jet $j^3(x_1 d_1)$ is therefore, up to non-zero multiples, a $J^2\mathcal{H}$-invariant of 2-jets of this form.

Proof Write

$$A = \begin{bmatrix} 0 & 0 & x_1 + a_1 & b_1 \\ 0 & 0 & c_1 & d_1 \\ -x_1 - a_1 & -c_1 & 0 & 0 \\ -b_1 & -d_1 & 0 & 0 \end{bmatrix},$$

and

$$B = \begin{bmatrix} 0 & \nu & x_1 + a_2 + \alpha & b_2 + \beta \\ -\nu & 0 & c_2 + \gamma & d_2 + \delta \\ -x_1 - a_2 - \alpha & -c_2 - \gamma & 0 & \eta \\ -b_2 - \beta & -d_2 - \delta & -\eta & 0 \end{bmatrix},$$

where ν, α, β, γ, δ, $\eta \in \mathcal{M}_r^3$. Then supposing, $A_2 = [0, x_1 + a_1, b_1, c_1, d_1, 0]$, and $B_2 = [0, x_1 + a_2, b_2, c_2, d_2, 0]$ are $J^2\mathcal{H}$-equivalent we know, for some $X : \mathbb{C}^r, 0 \to Gl(4, \mathbb{C})$, that

$$X^T A X = B.$$

Taking determinants of both sides we have

$$(\det X)^2 (x_1 d_1 + a_1 d_1 - b_1 c_1)^2 = (\nu\eta - [(x_1 + a_2 + \alpha)(d_2 + \delta) - (c_2 + \gamma)(b_2 + \beta)])^2.$$

Taking square roots and denoting $\det X$ by Λ ($\Lambda(0) \neq 0$), this gives

$$\begin{aligned} \pm\Lambda(x_1 d_1 + a_1 d_1 - b_1 c_1) &= \nu\eta - [(x_1 + a_2 + \alpha)(d_2 + \delta) - (c_2 + \gamma)(b_2 + \beta)] \\ &= -x_1 d_2 - a_2 d_2 + c_2 b_2 - x_1 \delta + \phi, \end{aligned}$$

where $\phi \in \mathcal{M}_r^5$. Writing $\Lambda = -\lambda + \Lambda_1$, where $\lambda \neq 0$ is a constant and $\Lambda_1 \in \mathcal{M}_r$, we obtain the following expression

$$\begin{aligned} \pm\lambda(x_1 d_1 + a_1 d_1 - b_1 c_1) - (x_1 d_2 + a_2 d_2 - c_2 b_2) &= x_1 \delta \pm \Lambda_1(x_1 d_1 + a_1 d_1 - b_1 c_1) + \phi \\ &= x_1 \delta_1 + \phi_1, \end{aligned}$$

where $\delta_1 \in \mathcal{M}_r^3$, $\phi_1 \in \mathcal{M}_r^5$. Recalling that the Pfaffian of a skew-symmetric matrix is given by a square root of its determininant, the LHS of this expression is $\mp\lambda Pf(A_2) + Pf(B_2)$ and we have the result in (6.81). Clearly, taking the 3-jets of both sides gives

$$j^3(x_1 d_2) = \pm\lambda j^3(x_1 d_1).$$

In both cases, since λ is arbitrary, the sign is irrelevant. □

We can use these two lemmas to identify an invariant of two $J^2\mathcal{G}$-equivalent jets of the form $[0, x_1, b_i, c_i, d_i, 0]$.

Corollary 6.1.18 *If the 2-jets A_2, $B_2 : \mathbb{C}^r, 0 \to Sk(4, \mathbb{C})$ of the form, $A_2 = [0, x_1, b_1, c_1, d_1, 0]$, $B_2 = [0, x_1, b_2, c_2, d_2, 0]$ (with each b_i, c_i, d_i homogeneous of degree 2), are $J^2\mathcal{G}$-equivalent then, there is a (invertible) linear change of coordinates $\phi_1 : \mathbb{C}^r, 0 \to \mathbb{C}^r, 0$ of the form*

$$(x_1, \ldots, x_r) \mapsto (\alpha x_1, L(x_1, \ldots x_r)),$$

which takes d_1 to a non-zero scalar multiple of d_2, i.e.

$$d_2 = \lambda(d_1 \circ \phi_1),$$

where $\lambda \neq 0$.

Proof If $A_2 = [0, x_1, b_1, c_1, d_1, 0]$, $B_2 = [0, x_1, b_2, c_2, d_2, 0]$ are equivalent as 2-jets then for some germ of a diffeomorphism, $\phi : \mathbb{C}^r, 0 \to \mathbb{C}^r, 0$, $A_2 \circ \phi$ and B_2 are $\mathcal{H}$-equivalent as 2-jets. Clearly, $A_2 \circ \phi$ and B_2 must also be $\mathcal{H}$-equivalent as

1-jets and denoting the 1-jet of ϕ by ϕ_1, it follows from (6.80) in Lemma 6.1.16 that

$$\left(\mathcal{O}_r\langle x_1 \circ \phi_1\rangle + \mathcal{M}_r^2\right)/\mathcal{M}_r^2 = \left(\mathcal{O}_r\langle x_1\rangle + \mathcal{M}_r^2\right)/\mathcal{M}_r^2.$$

Any such diffeomorphism, ϕ, must therefore have a 1-jet of the form

$$\phi_1(x_1, \ldots, x_r) = (\alpha x_1, L(x_1, \ldots, x_r)),$$

above.

For such a diffeomorphism, ϕ,

$$j^2(A_2 \circ \phi) = \begin{bmatrix} 0 & 0 & \alpha x_1 + q & b_1 \circ \phi \\ 0 & 0 & c_1 \circ \phi & d_1 \circ \phi \\ -\alpha x_1 - q & -c_1 \circ \phi & 0 & 0 \\ -b_1 \circ \phi & -d_1 \circ \phi & 0 & 0 \end{bmatrix},$$

where q is some homogeneous polynomial of degree 2 introduced by the 2-jet of ϕ. It follows, by a simultaneous row and column operation, that if $A_2 \circ \phi$ is $J^2\mathcal{H}$-equivalent to B_2 then so is

$$\begin{bmatrix} 0 & 0 & x_1 + q_1 & b_1 \circ \phi \\ 0 & 0 & C_1 \circ \phi & d_1 \circ \phi \\ -x_1 - q_1 & -C_1 \circ \phi & 0 & 0 \\ -b_1 \circ \phi & -d_1 \circ \phi & 0 & 0 \end{bmatrix},$$

where $q_1 = (1/\alpha)q$ and $C_1 = (1/\alpha)c_1$. Applying Lemma 6.1.17, this implies that

$$j^3(x_1 d_2) = \lambda j^3\left(x_1(d_1 \circ \phi)\right),$$

for some non-zero constant λ. Since we are taking 3-jets, we need only consider the effect of the 1-jet, ϕ_1, of ϕ on the LHS of this expression. So,

$$x_1 d_2 = \lambda x_1(d_1 \circ \phi_1),$$

where ϕ_1 preserves $x_1 = 0$.

We therefore deduce that if A_2 and B_2 are $J^2\mathcal{G}$-equivalent then there is a germ of a diffeomorphism $\phi : \mathbb{C}^r, 0 \to \mathbb{C}^r, 0$ whose linear part, ϕ_1, fixes $x_1 = 0$ and takes $d_1 = 0$ to $d_2 = 0$, as required. □

We return to considering the case $r = 2$ i.e. germs $\mathbb{C}^2, 0 \to Sk(4, \mathbb{C})$, in particular all possible $J^2\mathcal{G}$-orbits of germs with 1-jet $[0, x, 0, 0, 0, 0]$.

We can use the invariant, identified by Corollary 6.1.18, to distinguish the $J^2\mathcal{G}$-orbits which correspond to the $(\mathcal{R} \times \mathcal{Q})$-orbits listed in Lemma 6.1.15.

Corollary 6.1.19 *Any germ* $A : \mathbb{C}^2, 0 \to Sk(4, \mathbb{C})$ *with* 1*-jet,* $[0, x, 0, 0, 0, 0]$*, is* $J^2\mathcal{G}$*-equivalent to one of the eleven distinct* 2*-jets :*

	$J^2\mathcal{G} - codim$	$xd(x,y)$
$[0, x, 0, 0, x^2 + y^2, 0]$	6	$x(x^2 + y^2)$
$[0, x, 0, 0, y^2, 0]$	7	xy^2
$[0, x, y^2, y^2, xy, 0]$	7	x^2y
$[0, x, y^2, y^2, x^2, 0]$	8	x^3
$[0, x, y^2, y^2, 0, 0]$	9	0
$[0, x, y^2, 0, xy, 0]$	8	x^2y
$[0, x, y^2, 0, x^2, 0]$	9	x^3
$[0, x, y^2, 0, 0, 0]$	10	0
$[0, x, 0, 0, xy, 0]$	11	x^2y
$[0, x, 0, 0, x^2, 0]$	12	x^3
$[0, x, 0, 0, 0, 0]$	13	0

Proof We have demonstrated above that any element in a 2-transversal of $[0, x, 0, 0, 0, 0]$ is $J^2\mathcal{G}$-equivalent to one of three broad types : $[0, x, y^2, y^2, P(x, y), 0]$, $[0, x, y^2, 0, P(x, y), 0]$ or $[0, x, 0, 0, P(x, y), 0]$. Furthermore, using Lemma 6.1.15, we can further simplify this 2-transversal to one of the eleven possibilities listed in the statement. Finally by comparing both the $J^2\mathcal{G}$-codimensions and the zero sets of $xd(x, y)$ ($d(x, y)$ being the quadratic in slot e_5 of each 2-jet) of these representatives, it is evident that they are all distinct. □

Taking each of these 2-jets in turn we can, with the aid of transversal, complete a classification of all $\mathcal{G}$-simples with 1-jet $[0, x, 0, 0, 0, 0]$.

Firstly, $[0, x, 0, 0, x^2 + y^2, 0]$ is 2-$\mathcal{G}$-determined and has $\mathcal{G}_e$-codimension 10.

Investigating jets with 2-jet, $[0, x, 0, 0, y^2, 0]$, we detect the presence of a series.

Lemma 6.1.20 *Let $A : \mathbb{C}^2, 0 \to Sk(4, \mathbb{C})$ be any smooth germ with 2-jet*

$$[0, x, 0, 0, y^2, 0]. \tag{6.83}$$

Then A is $\mathcal{G}$-equivalent to a k-determined germ of the form

$$[0, x, 0, 0, y^2 + x^k, 0],$$

where $k \geq 3$, or for any $k \geq 2$ A is $\mathcal{G}$-equivalent to a germ whose k-jet is (6.83). Thus, including the 2-$\mathcal{G}$-determined jet $[0, x, 0, 0, x^2 + y^2, 0]$, we have a series of (distinct) $\mathcal{G}$-finitely determined germs

$$[0, x, 0, 0, y^2 + x^k, 0], \qquad (k \geq 2),$$

each with $\mathcal{G}_e$-codimension $(k + 8)$ and a discriminant of type D_{k+2}

Proof Assume, for any $k \geq 3$, A has a $(k - 1)$-jet

$$j^{k-1} A = [0, x, 0, 0, y^2, 0].$$

It is not difficult to show that the only terms of $H^k(2, 6)$ missing from the $J^k\mathcal{G}_1$-tangent space to $j^{k-1}A$ are scalar multiples of $[0, 0, 0, 0, x^k, 0]$. In other words the k-transversal of $j^{k-1}A$ is

$$[0, x, 0, 0, y^2 + ax^k, 0], \tag{6.84}$$

where $a \in \mathbb{C}$. If $a \neq 0$ then by scaling we obtain the $J^k\mathcal{G}$-equivalent jet

$$[0, x, 0, 0, y^2 + x^k, 0], \tag{6.85}$$

which is k-$\mathcal{G}$-determined.

If, in (6.84), $a = 0$ we would have the k-jet $[0, x, 0, 0, y^2, 0]$ and can repeat the previous argument, replacing $k - 1$ by k, and so on. The $\mathcal{G}_e$-codimension and the discriminant of the germ (6.85) are found, in familiar fashion, by considering the corresponding k-$(\mathcal{R} \times \mathcal{Q})$-determined jet

$$\begin{bmatrix} x & 0 \\ 0 & y^2 + x^k \end{bmatrix}.$$

□

The 2-jet $[0, x, y^2, y^2, xy, 0]$ has a 3-transversal given by the 1-parameter family

$$g_a = [0, x, y^2, y^2, xy + ay^3, 0].$$

This transversal is a connected submanifold $X \subset J^3(2,6)$, where

$$X = \{g_a : a \in \mathbb{C}\},$$

and any germ with 2-jet $[0, x, y^2, y^2, xy, 0]$ is $J^3\mathcal{G}$-equivalent to some element of this space. Using **Transversal** we find that for *all* values of a, the jet g_a has $J^3\mathcal{G}$-codimension 7 *and* the vector, $[0, 0, 0, 0, y^3, 0]$, which spans the tangent space $T_{g_a}X$, is contained in the $J^3\mathcal{G}$-tangent space to g_a. It follows, by Lemma 4.5.1, that g_a is a $J^3\mathcal{G}$-trivial family and is contained in a single $J^3\mathcal{G}$-orbit of $J^3(2,6)$ with representative

$$[0, x, y^2, y^2, xy, 0].$$

Furthermore, since $[0, x, y^2, y^2, xy, 0]$ is 3-$\mathcal{G}_1$-determined, we deduce that the 2-jet, $[0, x, y^2, y^2, xy, 0]$, is in fact 2-$\mathcal{G}$-determined.

Lemma 6.1.21 *Any germ $A : \mathbb{C}^2, 0 \to Sk(4, \mathbb{C})$ with 2-jet lying in the $J^2\mathcal{G}$-orbit represented by $[0, x, y^2, 0, xy, 0]$ is $J^3\mathcal{G}$-equivalent to one of two 3-jets:*

$$[0, x, y^2, y^3, xy, 0] \quad \text{or} \quad [0, x, y^2, 0, xy, 0],$$

the first of which, $[0, x, y^2, y^3, xy, 0]$, is 3-$\mathcal{G}$-determined.

Proof The 2-jet $[0, x, y^2, 0, xy, 0]$ has a 3-transversal

$$[0, x, y^2, ay^3, xy + by^3, 0], \tag{6.86}$$

which for convenience we represent by the matrix

$$\begin{bmatrix} x & y^2 \\ ay^3 & xy + by^3 \end{bmatrix}.$$

By an $\mathcal{R}$-change of coordinates, $(x, y) \mapsto (x - (b/2)y^2, y)$, followed the column/row operations, $C_1 + (b/2)C_2$ and $R_2 - (b/2)yR_1$, we obtain the $(\mathcal{R} \times \mathcal{Q})$-equivalent matrix

$$\begin{bmatrix} x & y^2 \\ (a + b^2/4)\, y^3 & xy \end{bmatrix}.$$

If $a + b^2/4 \neq 0$, scaling gives the 2×2 matrix

$$\begin{bmatrix} x & y^2 \\ y^3 & xy \end{bmatrix},$$

which is 3-$\mathcal{G}$-determined. Whereas if $a + b^2/4 = 0$ we have

$$\begin{bmatrix} x & y^2 \\ 0 & xy \end{bmatrix}.$$

Therefore any 3-jet in (6.86) lies in one of two $J^3\mathcal{G}$-orbits. All those for which $a+b^2/4 \neq 0$ lie in the $J^3\mathcal{G}$-orbit with representative $[0, x, y^2, y^3, xy, 0]$, have $J^3\mathcal{G}$-codimension 8 and are 3-$\mathcal{G}$-determined. However all 3-jets for which $a+b^2/4 = 0$ lie in the $J^3\mathcal{G}$-orbit with representative $[0, x, y^2, 0, xy, 0]$ and $J^3\mathcal{G}$-codimension 9. □

The following result deals with germs with 3-jets in the second orbit, $[0, x, y^2, 0, xy, 0]$.

Lemma 6.1.22 *Let $A : \mathbb{C}^2, 0 \to Sk(4, \mathbb{C})$ be any smooth germ with 3-jet*

$$[0, x, y^2, 0, xy, 0]. \tag{6.87}$$

Then A is $\mathcal{G}$-equivalent to a k-determined germ of the form

$$[0, x, y^2, y^k, xy, 0],$$

where $k \geq 4$, or for any $k \geq 3$, A is $\mathcal{G}$-equivalent to a germ whose k-jet is (6.87).

Thus including the, previously found, $\mathcal{G}$-determined jets $[0, x, y^2, y^2, xy, 0]$ and $[0, x, y^2, y^3, xy, 0]$ we have a series of (distinct) $\mathcal{G}$-finitely determined germs

$$[0, x, y^2, y^l, xy, 0], \qquad (l \geq 2),$$

which have $\mathcal{G}_e$-codimension $(l + 9)$ and a discriminant of type D_{l+3}.

Proof Assume, for any $k \geq 4$, A has a $(k-1)$-jet

$$j^{k-1}A = [0, x, y^2, 0, xy, 0].$$

This jet has a complete k-transversal

$$[0, x, y^2, ay^k, xy, 0], \tag{6.88}$$

where $a \in \mathbb{C}$. If $a \neq 0$ then by scaling we obtain the $J^k\mathcal{G}$-equivalent jet

$$[0, x, y^2, y^k, xy, 0],$$

which is k-$\mathcal{G}$-determined.

If, in (6.88), $a = 0$ we would have the k-jet $[0, x, y^2, 0, xy, 0]$ and can repeat the previous argument, replacing $k-1$ by k, and so on.

By adding to these k-determined jets the $\mathcal{G}$-determined jets $[0, x, y^2, y^2, xy, 0]$ and $[0, x, y^2, y^3, xy, 0]$ we obtain the series of finitely determined germs given in the statement. The $\mathcal{G}_e$-codimensions and discriminants of such germs are found in the usual manner by considering the l-$(\mathcal{R} \times \mathcal{Q})$ determined jet

$$\begin{bmatrix} x & y^2 \\ y^l & xy \end{bmatrix}, \qquad l \geq 2.$$

□

In fact we can subsume this series into one which is even more extensive, by considering those germs with 2-jet $[0, x, 0, 0, xy, 0]$.

Lemma 6.1.23 *Let $A : \mathbb{C}^2, 0 \to Sk(4, \mathbb{C})$ be any smooth germ with 2-jet*

$$[0, x, 0, 0, xy, 0]. \tag{6.89}$$

Then A is $\mathcal{G}$-equivalent to, either, a k-determined germ of the form

$$[0, x, 0, 0, xy + y^k, 0], \tag{6.90}$$

where $k \geq 3$, an l-determined germ of the form

$$[0, x, y^k, y^l, xy, 0], \tag{6.91}$$

$3 \leq k \leq l$, or for any $k \geq 2$, A is $\mathcal{G}$-equivalent to a germ with k-jet $[0, x, 0, 0, xy, 0]$, or for any $3 \leq k \leq l$, A is $\mathcal{G}$-equivalent to a germ whose l-jet is $[0, x, y^k, 0, xy, 0]$. Germs in (6.90) have $\mathcal{G}_e$-codimension $5k$ and discriminants of type D_{2k}. Germs in (6.91) have $\mathcal{G}_e$-codimension $(4k + l + 1)$ and discriminants of type D_{k+l+1}.

Proof If, for any $k \geq 3$, A has a $(k-1)$-jet

$$j^{k-1}A = [0, x, 0, 0, xy, 0],$$

it is fairly easy to show that a k-transversal is of the form

$$\begin{bmatrix} 0 & a_1y^k & x & a_3y^k \\ -a_1y^k & 0 & a_4y^k & xy + a_5y^k \\ -x & -a_4y^k & 0 & a_6y^k \\ -a_3y^k & -x - a_5y^k & -a_6y^k & 0 \end{bmatrix}. \tag{6.92}$$

If $a_3 \neq 0$ then in the usual manner we can use a couple of simultaneous row and column operations, followed by a scaling change to obtain the $J^k\mathcal{G}$-equivalent matrix

$$\begin{bmatrix} 0 & 0 & x & y^k \\ 0 & 0 & A_4y^k & xy + A_5y^k \\ -x & -A_4y^k & 0 & 0 \\ -y^k & -x - A_5y^k & 0 & 0 \end{bmatrix}. \tag{6.93}$$

There are three possibilities here.

(i) If $A_5 \neq 0$ by a further scaling change we have the family of k-jets

$$[0, x, y^k, ay^k, xy + y^k, 0]. \tag{6.94}$$

(ii) If $A_5 = 0$ and $A_4 \neq 0$ by scaling we obtain the $J^k\mathcal{G}$-equivalent jet

$$[0, x, y^k, y^k, xy, 0]. \tag{6.95}$$

(iii) If $A_5 = A_4 = 0$ then we have the k-jet

$$[0, x, y^k, 0, xy, 0]. \tag{6.96}$$

Alternatively if, in (6.93) $a_3 = 0$ we can also assume that $a_1 = a_4 = a_5 = a_6 = 0$, since otherwise we could, by various simultaneous row and column operations, move an a_iy^k $(i = 1, 4, 5, 6)$ into slot e_3 and thereby obtain one of the $k$jets just considered. We are then left with the k-jet

$$[0, x, 0, 0, xy, 0],$$

which by replacing $(k-1)$ with k in the original assumption is covered by the following arguments.

We consider each of the three possibilities (i) – (iii) in turn.

(i) The family of k-jets $[0, x, y^k, ay^k, xy + y^k, 0]$.

Calculations performed by `Transversal` for *specific* values of k suggest such a family to be $J^k\mathcal{G}$-trivial. To prove this for *all* $k \geq 3$ we need to demonstrate, by hand, that the family satisfies both conditions of Lemma 4.5.1. It is instructive to give the details of the technique used.

By Lemma 6.0.13 it is sufficient to show the family of k-jets,

$$\begin{bmatrix} x & y^k \\ ay^k & xy + y^k \end{bmatrix},$$

to be $J^k(\mathcal{R} \times \mathcal{Q})$-trivial. For convenience we represent elements of the jet space $J^k(2,4)$ by 4-tuples (in row major order).

By Corollary 6.0.6, the $J^k(\mathcal{R} \times \mathcal{Q})$-tangent space to the family of k-jets $g_a = [x, y^k, ay^k, xy + y^k]$ is generated by the vectors

$$\begin{array}{lll}
\mathcal{M}_2 : & & \\
[1,0,0,y], & [0, ky^{k-1}, kay^{k-1}, x + ky^{k-1}]; & \\
\mathcal{O}_2 : & & \\
[x, y^k, 0, 0], & [0, 0, x, y^k], & [ay^k, xy + y^k, 0, 0], \\
[0, 0, ay^k, xy + y^k], & [x, 0, ay^k, 0], & [0, x, 0, ay^k], \\
[y^k, 0, xy + y^k, 0] & [0, y^k, 0, xy + y^k]. &
\end{array} \tag{6.97}$$

To satisfy the first condition of Lemma 4.5.1, we need to show that the tangent vector to the family g_a (at each of its elements), $[0, 0, y^k, 0]$, is contained in this $J^k(\mathcal{R} \times \mathcal{Q})$-tangent space for *all* values of the parameter a. This is seen to be the case, by combining the three $J^k(\mathcal{R} \times \mathcal{Q})$-tangent vectors,

$$y^k[1, 0, 0, y] = [y^k, 0, 0, 0], \tag{6.98}$$

$$y[0, 0, x, y^k] = [0, 0, xy, 0]$$

and

$$[y^k, 0, xy + y^k, 0].$$

The second condition of Lemma 4.5.1 requires us to demonstrate that the codimension of the $J^k(\mathcal{R} \times \mathcal{Q})$-tangent space doesn't vary with the parameter a. We achieve this by showing that this space, spanned by the generators in (6.97), is also spanned by a set of generators independent of a. We do this by looking to use existing tangent vectors, independent of a, to eliminate terms involving a from the remaining generators.

Consider the generators, $[x, 0, ay^k, 0]$, $[0, 0, ay^k, xy + y^k]$. We have already shown for all values of a, that the tangent vector $[0, 0, y^k, 0]$ lies in the $J^k(\mathcal{R} \times \mathcal{Q})$-tangent space. We can amend the set of generators in (6.97) by replacing $[x, 0, ay^k, 0]$ and $[0, 0, ay^k, xy + y^k]$ by $[x, 0, 0, 0]$ and $[0, 0, 0, xy + y^k]$ respectively, the resulting set of generators spanning the same space as before. Using the tangent vector, (6.98), we can replace the generator, $[ay^k, xy + y^k, 0, 0]$ by $[0, xy + y^k, 0, 0]$, again preserving the space spanned. It remains to consider the remaining generators involving a:

$$\mathcal{M}_2\,[0, ky^{k-1}, kay^{k-1}, x + ky^{k-1}] \qquad \text{and} \qquad [0, x, 0, ay^k],$$

where the $\mathcal{M}_2$ before the first vector denotes it is always multiplied by terms in the maximal ideal $\mathcal{M}_2$. We observe that, regarding the first of these, the kay^{k-1} term in e_4 drops out automatically except when multiplying it by linear terms. Consider each of these possibilities. The tangent vector

$$y[0, ky^{k-1}, kay^{k-1}, x + ky^{k-1}] = [0, ky^k, kay^k, xy + ky^k],$$

and the ay^k term in e_4 can be eliminated by the tangent vector $[0, 0, y^k, 0]$. Alternatively the tangent vector obtained by multiplying by x is

$$[0, kxy^{k-1}, kaxy^{k-1}, x^2 + kxy^{k-1}],$$

but using the tangent vector $y^{k-1}[0, 0, x, y^k] = [0, 0, xy^{k-1}, 0]$ we can eliminate the axy^{k-1} term in e_4. In summary, we can further amend our generating set by replacing the generator $[0, ky^{k-1}, kay^{k-1}, x + ky^{k-1}]$ with a generator $[0, ky^{k-1}, 0, x + ky^{k-1}]$, while leaving the space spanned unchanged.

Finally, we need to consider the generator $[0, x, 0, ay^k]$. Using the previous generator, we find the tangent vector

$$y[0, ky^{k-1}, 0, x + ky^{k-1}] = [0, ky^k, 0, xy + ky^k],$$

which, combined with generator $[0, y^k, 0, xy + y^k]$, gives a tangent vector

$$[0, (k-1)y^k, 0, (k-1)y^k]. \tag{6.99}$$

Furthermore the generator $[x, 0, 0, 0]$, discussed earlier, combined with $[x, y^k, 0, 0]$ gives a tangent vector

$$[0, y^k, 0, 0]. \tag{6.100}$$

Combining vectors (6.99) and (6.100) gives the tangent vector $[0, 0, 0, y^k]$, which justifies replacing the final generator dependent on a, $[0, x, 0, ay^k]$, by $[0, x, 0, 0]$. We have therefore replaced the generator set (6.97) by a set of generators, each of which is independent of the parameter a, and which generate the *same* space. Hence, for all values of a, the $J^k(\mathcal{R} \times \mathcal{Q})$-tangent space to the k-jet $[x, y^k, ay^k, xy + y^k]$ is the same and therefore, for each element of the family this tangent space has constant dimension. So the second condition of Lemma 4.5.1 is satisfied and this family is $J^k(\mathcal{R} \times \mathcal{Q})$-trivial.

It follows that the corresponding family of k-jets, $[0, x, y^k, ay^k, xy + y^k, 0]$ is $J^k\mathcal{G}$-trivial and the $J^k\mathcal{G}$-orbit in which it lies has representative

$$[0, x, y^k, 0, xy + y^k, 0].$$

Writing this in matrix form,

$$\begin{bmatrix} 0 & 0 & x & y^k \\ 0 & 0 & 0 & xy + y^k \\ -x & 0 & 0 & 0 \\ -y^k & -xy - y^k & 0 & 0 \end{bmatrix},$$

it can be seen by the simultaneous row and column operation involving $R_1 - R_2$ followed by the simultaneous row and column operation involving $C_4 + yC_3$ that this is equivalent to the k-jet $[0, x, 0, 0, xy + y^k, 0]$, which is k-$\mathcal{G}$-determined. Hence we have found a series of determined germs of the form (6.90).

(ii) The k-jet $[0, x, y^k, y^k, xy, 0]$.

This k-jet has a complete $(k+1)$-transversal

$$[0, x, y^k, y^k, xy + ay^{k+1}, 0]. \tag{6.101}$$

Calculations performed by Transversal suggest this family to be $J^{k+1}\mathcal{G}$-trivial. Using a method similar to that introduced above we can prove this (for all values of $k \geq 3$).

As before, this amounts to showing both conditions of Mather's Lemma (Lemma 4.5.1) are satisfied for the $J^{k+1}(\mathcal{R} \times \mathcal{Q})$-action on the 2×2 matrix,

$$\begin{bmatrix} x & y^k \\ y^k & xy + ay^{k+1} \end{bmatrix}.$$

Representing this family by the 4-tuple $[x, y^k, y^k, xy+ay^{k+1}]$, the $J^{k+1}(\mathcal{R} \times \mathcal{Q})$-tangent space to it is generated by

$$\begin{array}{lll}
\mathcal{M}_2 : & & \\
[1, 0, 0, y], & [0, ky^{k-1}, ky^{k-1}, x + (k+1)ay^k]; & \\
\mathcal{O}_2 : & & \\
[x, y^k, 0, 0], & [0, 0, x, y^k], & [y^k, xy + ay^{k+1}, 0, 0] \\
[0, 0, y^k, xy + ay^{k+1}] & [x, 0, y^k, 0], & [0, x, 0, y^k], \\
[y^k, 0, xy + ay^{k+1}, 0] & [0, y^k, 0, xy + ay^{k+1}]. &
\end{array} \tag{6.102}$$

To satisfy the first condition of Lemma 4.5.1 we need to show that the tangent vector, $[0, 0, 0, y^{k+1}]$, to the family, $[x, y^k, y^k, xy+ay^{k+1}]$, is contained in this $J^{k+1}(\mathcal{R} \times \mathcal{Q})$-tangent space. This is significantly more difficult to achieve, compared with the previous example.

We wish to show that the monomial vector $y^{k+1}e_5$ is contained in the tangent space. Using the list of generators, (6.102), we compile a set of tangent vectors each with y^{k+1} in slot e_5 but with the number of other monomial vectors involved kept to a minimum. To isolate the required monomial vector it is more than likely that this set needs to be augmented with further tangent vectors, although this is done with a view to keeping the total number of distinct monomial vectors involved as small as possible.

Denoting the number of tangent vectors in this set by t and the total number of distinct monomial vectors involved in it by s, we first require $t \geq s$. If we can show the span of this set to consist of s linearly independent vectors, then by various linear combinations of them, we obtain all s monomial vectors and in particular the required one (here $y^{k+1}e_5$).

Equivalently, we could represent this set by a $t \times s$ matrix over $\mathbb{C}$, where each row corresponds to a tangent vector, which we then require to have maximal rank. This will become clearer as we proceed with the present example.

The vectors $y^k[1,0,0,y] = [y^k,0,0,y^{k+1}]$, $y[0,0,x,y^k] = [0,0,xy,y^{k+1}]$, each contain a y^{k+1} term in slot e_5 and introduce two further monomial vectors $y^k e_2$ and xye_4. The tangent vector

$$[y^k, 0, xy + ay^{k+1}, 0], \tag{6.103}$$

contains both these monomial vectors, but also introduces a y^{k+1} term into slot e_4. The two tangent vectors $y[x,0,y^k,0] = [xy,0,y^{k+1},0]$, $y[0,0,y^k,xy+ay^{k+1}] = [0,0,y^{k+1},xy^2]$, although both containing a y^{k+1} term in e_4, introduce two more monomial vectors xye_2 and xy^2e_5 respectively. At present we have found five tangent vectors which are combinations of six distinct monomial vectors and so by adding to these the tangent vector, $xy[1,0,0,y] = [xy,0,0,xy^2]$ which introduces no new monomial vectors, we have a set of tangent vectors satisfying the first requirement, i.e. $t \geq s$. By showing these six vectors to be linearly independent it would then follow that each of the six monomial vectors are contained in the $J^k(\mathcal{R} \times \mathcal{Q})$-tangent space.

It can be easily seen, from the spanning set of the three tangent vectors,

$$\{[xy,0,0,xy^2],\ \ [0,0,y^{k+1},xy^2],\ \ [xy,0,y^{k+1},0]\}, \tag{6.104}$$

that

$$[0,0,y^{k+1},xy^2] - [xy,0,0,xy^2] + [xy,0,y^{k+1},0] = [0,0,2y^{k+1},0],$$

gives the monomial vector $y^{k+1}e_4$ and consequently we can also find the monomial vectors $[xy,0,0,0]$ and $[0,0,0,xy^2]$ from this spanning set.

With this in mind we consider the remaining three tangent vectors :

$$\{[y^k,0,xy+ay^{k+1},0],\ \ [0,0,xy,y^{k+1}],\ \ [y^k,0,0,y^{k+1}]\}. \tag{6.105}$$

Since we have just shown that $[0,0,y^{k+1},0]$ is in the tangent space we can replace the first of these vectors by $[y^k,0,xy,0]$. Then we see that

$$[0,0,xy,y^{k+1}] - [y^k,0,xy,0] + [y^k,0,0,y^{k+1}] = [0,0,0,2y^{k+1}],$$

and therefore, for all values of a, the monomial vector $y^{k+1}e_5$ lies in the $J^{k+1}(\mathcal{R}\times\mathcal{Q})$-tangent space to $[x, y^k, y^k, xy+ay^{k+1}]$, i.e. the first condition of Lemma 4.5.1 is satisfied.

However by this process we have also found five further monomial vectors which are contained in this tangent space. (The final two $y^k e_2$ and xye_4 are found from (6.105) using the fact that $y^{k+1}e_5$ is in this spanning set.) This makes the task of showing the second condition of Lemma 4.5.1 is satisfied considerably easier. Since the method adopted for this is similar to that described above, when proving part (i), we omit the details and give the result. We therefore find, by applying Mather's Lemma, that $[x, y^k, y^k, xy + ay^{k+1}]$ is a $J^{k+1}(\mathcal{R}\times\mathcal{Q})$-trivial family with representative, the $(k+1)$-jet

$$[x, y^k, y^k, xy].$$

We have therefore shown that any germ with k-jet $[x, y^k, y^k, xy]$ is $J^{k+1}(\mathcal{R}\times\mathcal{Q})$-equivalent to a germ with $(k+1)$-jet $[x, y^k, y^k, xy]$ and to show this k-jet to be k-determined it is enough to show it to be $(k+1)$-determined. This follows easily from the above work.

Recall that the k-jet $[x, y^k, y^k, xy]$ has the $(k+1)$-transversal $[x, y^k, y^k, xy+ay^{k+1}]$. It follows that the $(k+2)$-transversal of $[x, y^k, y^k, xy]$ can only be missing the term $[0, 0, 0, y^{k+2}]$. By the triviality result we demonstrated that the $J^{k+1}(\mathcal{R}\times\mathcal{Q})$-tangent space to $[x, y^k, y^k, xy]$, contains $[0, 0, 0, y^{k+1}]$. Furthermore the (six) tangent vectors used to obtain this are, with the exception of (6.103), in the $J^{k+1}(\mathcal{R}_1\times\mathcal{Q}_0)$-tangent space to $[x, y^k, y^k, xy]$. By multiplying these tangent vectors by y (in particular obtaining from the $J^{k+1}(\mathcal{R}\times\mathcal{Q})$-tangent vector, (6.103), the $J^{k+2}(\mathcal{R}_1\times\mathcal{Q}_0)$-tangent vector $y[y^k, 0, xy + ay^{k+1}, 0]$) we hence find that $[0, 0, 0, y^{k+2}]$ lies in the $J^{k+2}(\mathcal{R}_1\times\mathcal{Q}_0)$-tangent space to $[x, y^k, y^k, xy]$. It therefore follows that $[x, y^k, y^k, xy]$ is $(k+1)$-determined as required.

We must interprete these calculations for the corresponding skew-symmetric k-jet

$$[0, x, y^k, y^k, xy, 0].$$

It follows by the previous triviality result that any germ with this k-jet is $J^{k+1}\mathcal{G}$-equivalent to the $(k+1)$-jet $[0, x, y^k, y^k, xy, 0]$. So to show it to be k-$\mathcal{G}$-determined it is sufficient to show it to be $(k+1)$-$\mathcal{G}$-determined.

It is easily verified that the $J^{k+2}\mathcal{G}_1$-tangent space to $[0, x, y^k, y^k, xy, 0]$ contains all monomials of degree $(k+2)$ in both slots e_1 and e_6. So since the corresponding 2×2 matrix

$$\begin{bmatrix} x & y^k \\ y^k & xy \end{bmatrix},$$

has been proved to be $(k+1)$-$(\mathcal{R} \times \mathcal{Q})$-determined it follows that

$$[0, x, y^k, y^k, xy, 0], \tag{6.106}$$

is k-$\mathcal{G}$-determined.

(iii) The k-jet $[0, x, y^k, 0, xy, 0]$.

This jet has a complete $(k+1)$-transversal

$$[0, x, y^k, ay^{k+1}, xy + by^{k+1}, 0], \tag{6.107}$$

which we represent by the 2×2 matrix

$$\begin{bmatrix} x & y^k \\ ay^{k+1} & xy + by^{k+1} \end{bmatrix}.$$

Generalising the argument of the proof of Lemma 6.1.21, we can by an $\mathcal{R}$-change, $(x, y) \mapsto (x - (b/2)y^k, y)$, followed by appropriate row and column operations, reduce this 2×2 matrix to the $(\mathcal{R} \times \mathcal{Q})$-equivalent matrix

$$\begin{bmatrix} x & y^k \\ (a + b^2/4)\, y^{k+1} & xy \end{bmatrix}.$$

By the same reasoning used in this proof, it follows that any $(k+1)$-jet in (6.107) lies in one of the two $J^{k+1}\mathcal{G}$-orbits represented by

$$[0, x, y^k, y^{k+1}, xy, 0], \qquad \text{and} \qquad [0, x, y^k, 0, xy, 0]. \tag{6.108}$$

The first of these representatives is $(k+1)$-$\mathcal{G}$-determined. and it remains to consider germs with $(k+1)$-jet $[0, x, y^k, 0, xy, 0]$.

By a similar approach to that adopted above when proving Lemma 6.1.22 (concerning germs with 2-jet $[0, x, y^2, 0, xy, 0]$) we can prove that any germ with $(k+1)$-jet $[0, x, y^k, 0, xy, 0]$ is either $\mathcal{G}$-equivalent to the l-$\mathcal{G}$-determined germ

$$[0, x, y^k, y^l, xy, 0], \tag{6.109}$$

where $l \geq k+2$, or for any $l \geq k+1$ is $\mathcal{G}$-equivalent to a germ whose l-jet is $[0, x, y^k, 0, xy, 0]$.

Adding the finitely determined germs $[0, x, y^k, y^k, xy, 0]$, $[0, x, y^k, y^{k+1}, xy, 0]$ ($k \geq 3$), found seperately from cases (ii) and (iii) respectively, to the series given in (6.109), we have the second series of finitely determined germs, (6.91), in the statement. The codimensions and discriminants of all the finitely determined germs in the statement are found in the usual manner. □

Remark 6.1.24 Clearly, the series referred to prior to this result is given by adding the series of finitely determined germs from Lemma 6.1.22 to those of (6.91) in Lemma 6.1.23, and hence consists of germs

$$[0, x, y^k, y^l, xy, 0], \qquad (2 \leq k \leq l)$$

with $\mathcal{G}_e$-codimensions $(4k + l + 1)$ and discriminants with D_{k+l+1} singularities.

From the set of $J^2\mathcal{G}$-orbits not yet considered we look for further $\mathcal{G}$-simple finitely determined germs.

Lemma 6.1.25 *From the six $J^2\mathcal{G}$-orbits : $[0, x, y^2, y^2, x^2, 0]$, $[0, x, y^2, y^2, 0, 0]$, $[0, x, y^2, 0, x^2, 0]$, $[0, x, y^2, 0, 0, 0]$, $[0, x, 0, 0, x^2, 0]$ and $[0, x, 0, 0, 0, 0]$, we can find four $\mathcal{G}$-simple finitely determined germs :*

$$[0, x, y^2, y^2, x^2, 0],$$

$$[0, x, y^2, 0, x^2 + y^3, 0],$$

$$[0, x, y^2, y^3, x^2, 0],$$

$$[0, x, 0, 0, x^2 + y^3, 0],$$

with $\mathcal{G}_e$-codimensions 12, 13, 14, 16 and discriminants E_6, E_7, E_8, E_7 respectively.

Proof The remainder of this classification is done using `Transversal` and we give an outline of the results.

(i) The 2-jet $[0, x, y^2, y^2, x^2, 0]$.

This 2-jet has a complete 3-transversal

$$g_{ab} = [0, x, y^2, y^2, x^2 + axy^2 + by^3, 0].$$

This family is found to be $J^3\mathcal{G}$-trivial and contained in the $J^3\mathcal{G}$-orbit with representative

$$[0, x, y^2, y^2, x^2, 0].$$

Since this jet is 3-$\mathcal{G}_1$-determined we conclude

$$[0, x, y^2, y^2, x^2, 0]$$

is 2-$\mathcal{G}$-determined. A straightforward calculation using the extended tangent space verifies that this germ has $\mathcal{G}_e$-codimension 12. Representing it by the 2×2 matrix

$$\begin{bmatrix} x & y^2 \\ y^2 & x^2 \end{bmatrix},$$

it has a discriminant given by $x^3 - y^4 = 0$ which has an E_6 singularity.

(ii) The 2-jet $[0, x, y^2, 0, x^2, 0]$.

This 2-jet has a complete 3-transversal

$$[0, x, y^2, ay^3, x^2 + bxy^2 + cy^3, 0]. \tag{6.110}$$

Provided $c \neq 0$, by scaling we obtain the $J^3\mathcal{G}$-equivalent family

$$[0, x, y^2, Ay^3, x^2 + Bxy^2 + y^3, 0].$$

which is $J^3\mathcal{G}$-trivial and lies the single $J^3\mathcal{G}$-orbit with representative

$$[0, x, y^2, 0, x^2 + y^3, 0],$$

and $J^3\mathcal{G}$-codimension 9. Furthermore, this 3-jet is 3-$\mathcal{G}$-determined, has $\mathcal{G}_e$-codimension 13 and its discriminant is an E_7 singularity.

If, in (6.110), $c = 0$, but $a \neq 0$ we can scale to the $J^3\mathcal{G}$-equivalent family

$$[0, x, y^2, y^3, x^2 + bxy^2, 0],$$

which is $J^3\mathcal{G}$-trivial and lies in the $J^3\mathcal{G}$-codimension 10-orbit with representative

$$[0, x, y^2, y^3, x^2, 0].$$

This 3-jet has a complete a 4-transversal

$$[0, x, y^2, y^3, x^2 + dy^4, 0],$$

which is a $J^4\mathcal{G}$-trivial family and is represented by 4-$\mathcal{G}_1$-determined jet

$$[0, x, y^2, y^3, x^2, 0].$$

We can therefore deduce that, from the case $c = 0$, $a \neq 0$ in (6.110), we obtain the 3-$\mathcal{G}$-determined germ

$$[0, x, y^2, y^3, x^2, 0],$$

which has $\mathcal{G}_e$-codimension 14 and discriminant with an E_8-singularity.

The final possiblity arising from the 3-transversal, (6.110), is $c = a = 0$:

$$[0, x, y^2, 0, x^2 + bxy^2, 0],$$

which is a $J^3\mathcal{G}$-trivial family, contained in a $J^3\mathcal{G}$-codimension 11-orbit represented by the 3-jet

$$[0, x, y^2, 0, x^2, 0].$$

This 3-jet has a complete 4-transversal

$$g_{de} = [0, x, y^2, dy^4, x^2 + ey^4, 0],$$

but for values of $d \neq 0$ the $J^4\mathcal{G}$-tangent space to g_{de} *does not* contain the tangent vector $[0, 0, 0, 0, y^4, 0]$ (to the submanifold of $J^4(2, 6)$ corresponding to g_{de}). Consequently, the criterion for simplicity in Lemma 4.5.3 is not met for any germ with a 4-jet in the complete transversal g_{de}. Therefore the neighbourhood of any germ with 3-jet, $[0, x, y^2, 0, x^2, 0]$, meets uncountably many $\mathcal{G}$-orbits and the case ($c = a = 0$) cannot give us any $\mathcal{G}$-simple germs.

(iii) The 2-jet $[0, x, 0, 0, x^2, 0]$.

This 2-jet has a 3-transversal, written in matrix form,

$$\begin{bmatrix} 0 & a_1y^3 & x & a_3y^3 \\ -a_1y^3 & 0 & a_4y^3 & x^2 + b_5xy^2 + c_5y^3 \\ -x & -a_4y^3 & 0 & a_6y^3 \\ -a_3y^3 & -x^2 - b_5xy^2 - a_5y^3 & -a_6y^3 & 0 \end{bmatrix}. \quad (6.111)$$

Assuming $a_3 \neq 0$, we have a $J^3\mathcal{G}$-equivalent family,

$$\left[\begin{array}{cccc} 0 & 0 & x & y^3 \\ 0 & 0 & ay^3 & x^2+bxy^2+cy^3 \\ -x & -ay^3 & 0 & 0 \\ -y^3 & -x^2-bxy^2-cy^3 & 0 & 0 \end{array}\right]. \tag{6.112}$$

Provided $c \neq 0$ we can scale to obtain

$$[0, x, y^3, Ay^3, x^2 + Bxy^2 + y^3, 0],$$

which is $J^3\mathcal{G}$-trivial and is contained in a $J^3\mathcal{G}$-codimension 12-orbit with representative

$$[0, x, y^3, 0, x^2 + y^3, 0], \tag{6.113}$$

which is 3-$\mathcal{G}$-determined. Representing this 3-jet by the 2×2 matrix

$$\left[\begin{array}{cc} x & y^3 \\ 0 & x^2+y^3 \end{array}\right],$$

it can be seen, by a couple of explicit row/column operations, to be $J^3\mathcal{G}$-equivalent to $[0, x, 0, 0, x^2 + y^3, 0]$. Furthermore this 3-jet has $\mathcal{G}_e$-codimension 16 and its discriminant is an E_7-singularity.

If, in (6.112), $c = 0$, we have a family of 3-jets

$$g_{ab} = [0, x, y^3, ay^3, x^2 + bxy^2, 0].$$

For values of $a \neq 0$ the vector, $[0, 0, 0, 0, xy^2, 0]$, tangent to this submanifold, is *not* contained in the $J^3\mathcal{G}$-tangent space to g_{ab}. Consequently, any 3-jet in g_{ab} fails the criterion for simplicity in Lemma 4.5.3 and therefore any neighbourhood of a germ with such a 3-jet meets uncountably many $\mathcal{G}$-orbits. It follows that, in (6.112), the case $c = 0$ cannot give any $\mathcal{G}$-simples.

Finally, if in (6.111) $a_3 = 0$, unless $a_1 = a_4 = a_6 = c_5 = 0$ we can, by simultaneous row and column operations, move an y^3 term into slot e_3 and thereby obtain one of the 3-jets just considered. So, it remains to consider 3-jets originating from the family

$$[0, x, 0, 0, x^2 + b_5xy^2, 0].$$

However, in any neighbourhood of a 3-jet in this family there is a 3-jet of the form

$$[0, x, \epsilon y^3, 0, x^2 + b_5xy^2, 0]. \tag{6.114}$$

for some small $\epsilon \neq 0$. Clearly, (6.114), is in g_{ab} and by the above argument any neighbourhood of it meets uncountably many $\mathcal{G}$-orbits. This in turn means any neighbourhood of the 3-jet, $[0, x, 0, 0, x^2 + b_5 xy^2, 0]$, meets uncountably many $\mathcal{G}$-orbits and therefore any 3-jet in this family cannot give $\mathcal{G}$-simple germs.

Having exhausted all possibilities we deduce that we have all $\mathcal{G}$-simple germs arising from the set of 2-jets in the statement of the lemma. □

We conclude this discussion of germs with 1-jet $[0, x, 0, 0, 0, 0]$ by collecting together our results.

Theorem 6.1.26 *Any $\mathcal{G}$-simple map $A : \mathbb{C}^2, 0 \to \mathbb{C}^6, 0$ with 1-jet $[0, x, 0, 0, 0, 0]$ lies in one of the following finitely $\mathcal{G}$-determined orbits (where it is convenient to provide each representative in 2×2 form) :*

	Discriminant	*$\mathcal{G}_e$-codimension*	*Label*
$\begin{bmatrix} x & 0 \\ 0 & y^2 + x^k \end{bmatrix}, (k \geq 2)$	D_{k+2}	$k+8$	F_k
$\begin{bmatrix} x & 0 \\ 0 & xy + y^k \end{bmatrix}, (k \geq 3)$	D_{2k}	$5k$	G_k
$\begin{bmatrix} x & y^k \\ y^l & xy \end{bmatrix}, (2 \leq k \leq l)$	D_{k+l+1}	$4k + l + 1$	H_{kl}
$\begin{bmatrix} x & y^2 \\ y^2 & x^2 \end{bmatrix}$	E_6	12	T_{12}
$\begin{bmatrix} x & y^2 \\ 0 & x^2 + y^3 \end{bmatrix}$	E_7	13	T_{13}
$\begin{bmatrix} x & y^2 \\ y^3 & x^2 \end{bmatrix}$	E_8	14	T_{14}
$\begin{bmatrix} x & 0 \\ 0 & x^2 + y^3 \end{bmatrix}$	E_7	16	T_{16}

(6.115)

Furthermore, each of these orbits is distinct.

Proof We show that each of these orbits are distinct.

Since the discriminant of a germ is a $\mathcal{G}$-invariant we can immediately divide the above list of germs into those with type E-discriminants and those with type D-discriminants. Considering the four germs, with E-discriminants, it can be seen that their $\mathcal{G}_e$-codimensions are all different and, since codimension is another invariant, each of these germs are distinct.

Consider the list of germs with D-discriminants. By looking at either its discriminant or $\mathcal{G}_e$-codimension it is clear that for each germ in the series, F_k, the value of $k \geq 2$, is a $\mathcal{G}$-invariant and they are all distinct. Furthermore, using Corollary 6.1.19, we can distinguish any germ in this series from the germs in the two remaining series, G_k and H_{kl}, by virtue of its 2-jet.

We are left with these two remaining series. Clearly, the germs in the series, G_k, are distinct from each other, since by looking at either their discriminant or codimension the value of k is an invariant.

By Lemma 4.2.7, in Section 4.1, the germ $[0, x, y^k, y^l, xy, 0]$, $(2 \leq k \leq l)$, is $\mathcal{K}$-equivalent to to $[0, x, y^k, 0, 0, 0]$ which has $\mathcal{K}_e$-codimension $(5k-1)$. Since the $\mathcal{K}_e$-codimension of a germ is a $\mathcal{G}$-invariant it follows that the value of k in the germ $[0, x, y^k, y^l, xy, 0]$ is an invariant. Furthermore, since each element of this series has a discriminant with a D_{k+l+1}-singularity, $(k+l+1)$ and therefore l is also a $\mathcal{G}$-invariant. It follows that each germ of the H_{kl} series represents a distinct $\mathcal{G}$-orbit.

Finally, comparing the G_k and H_{kl} series,

$$[0, x, 0, 0, xy + y^k, 0] \quad k \geq 3, \tag{6.116}$$

and

$$[0, x, y^a, y^b, xy, 0] \quad 3 \leq a \leq b, \tag{6.117}$$

it is possible that for some values of $k \geq 3$ and $3 \leq a \leq b$ a germ in series, (6.116), lies in the same $\mathcal{G}$-orbit as a germ in (6.117). Note that we needn't worry about germs $[0, x, y^k, y^l, xy, 0]$ with $k = 2$, occuring in the $\mathcal{G}$-orbits of (6.116), by virtue of Corollary 6.1.19.

If a germ in (6.116) is $\mathcal{G}$-equivalent to a germ in (6.117), then they must both have the same discriminant and be $\mathcal{K}$-equivalent. In other words for some $k \geq 3$ and $3 \leq a \leq b$ it is necessary that

$$2k = a + b + 1, \tag{6.118}$$

and

$$k = a. \tag{6.119}$$

Substituting (6.119) into (6.118) gives

$$b = a - 1,$$

which contradicts the initial assumption $a \leq b$. It follows that we can distinguish between germs in (6.116) and (6.117).

We conclude that each germ in the list, (6.115), represents a distinct $\mathcal{G}$-orbit.
□

Finally, we consider germs with 1-jet $[0,0,0,0,0,0]$. It follows that any such germ has a 2-jet of the form

$$A = \begin{bmatrix} 0 & Q_1 & Q_2 & Q_3 \\ -Q_1 & 0 & Q_4 & Q_5 \\ -Q_2 & -Q_4 & 0 & Q_6 \\ -Q_3 & -Q_5 & -Q_6 & 0 \end{bmatrix}, \tag{6.120}$$

where $Q_i(x,y) = a_{i1}x^2 + a_{i2}xy + a_{i3}y^2$, $1 \leq i \leq 6$. The space of these jets can be thought of as an 18-dimensional subspace of $J^2(2,6)$:

$$V_A = \{A \in J^2(2,6) : j^1A = 0\},$$

which we represent by $\mathbb{C}^{18}$. The action of the jet-group $J^2\mathcal{G}$ on V_A, as given in Definition 4.4.9, amounts to the obvious action of $Gl(2,\mathbb{C}) \times Gl(4,\mathbb{C})$ on these matrices. It follows, by the same argument used in the proof of Proposition 4.2.10, that if two 2-jets A_1, $A_2 \in V_A$ are $J^2\mathcal{G}$-equivalent then there is a linear coordinate change (corresponding to an element of $Gl(2,\mathbb{C})$) taking the discriminant of A_1 to that of A_2. We have already discussed how the Pfaffian of finitely determined germs is an important invariant, we also find it useful for considering the 2-jets in V_A.

Since $\det A$ has degree 8, its Pfaffian is a quartic of the form

$$f(x,y) = b_0x^4 + 4b_1x^3y + 6b_2x^2y^2 + 4b_3xy^3 + b_4y^4, \tag{6.121}$$

where each coefficient, b_j, is a polynomial in the coefficients of the quadratics Q_i (or the coordinates, in $\mathbb{C}^{18}$, of $A \in V_A$). Identifying the binary quartic, (6.121), with the point $(b_0,b_1,b_2,b_3,b_4) \in P\mathbb{C}^4$ (see Section 4.7.2), we can therefore represent the Pfaffian of elements $A \in V_A$ as images of a polynomial map $P : \mathbb{C}^{18} \to P\mathbb{C}^4$ determined by

$$(Q_1, \ldots, Q_6) \mapsto \sqrt{\det A}.$$

Furthermore if two elements A_1, A_2 in V_A lie in the same $Gl(2, \mathbb{C}) \times Gl(4, \mathbb{C})$-orbit their Pfaffians are $PGl(2, \mathbb{C})$-equivalent (up to the action given by Definition 4.7.9).

We will show, in the following theorem, that all 2-jets of type (6.120) have moduli. By Lemma 4.7.14 their Pfaffians have (up to linear equivalence) a modulus. To show that the corresponding 2-jets in V_A also have moduli it is enough to show, near any $A \in V_A$, that the modulus of the resulting Pfaffian varies.

Theorem 6.1.27 *There are no $\mathcal{G}$-simple map germs with zero 1-jet.*

Proof Assuming all of the above preparatory work we also use notation from Section 4.7.2 without further explanation.

We consider the composite map

$$\mathbb{C}^{18} \backslash P^{-1}(\bar{\Delta}) \xrightarrow{P} P\mathbb{C}^4 \backslash \bar{\Delta} \xrightarrow{j} \mathbb{C},$$

where $\bar{\Delta}$ is the set of elements of $P\mathbb{C}^4$ with repeated roots, P the polynomial map described above and j the rational map

$$q \mapsto J^2(q)/\Delta(q).$$

It follows, using Lemma 4.7.14, that distinct values of $j \circ P$ correspond to distinct $Gl(2, \mathbb{C}) \times Gl(4, \mathbb{C})$-orbits of V_A. Alternatively the *fibres*,

$$(j \circ P)^{-1}(\mu) = \{A \in \mathbb{C}^{18} \backslash P^{-1}(\bar{\Delta}) : j\,(P(A)) = \mu\},$$

of $j \circ P$, lie in distinct $Gl(2, \mathbb{C}) \times Gl(4, \mathbb{C})$-orbits of V_A.

Each fibre (of some element $\mu \in \mathbb{C}$) consists of points $A \in \mathbb{C}^{18} \backslash P^{-1}(\bar{\Delta})$ satisfying

$$\frac{J^2\,(P(A))}{\Delta\,(P(A))} = \mu,$$

or

$$J^2\left(P(A)\right) - \mu\Delta\left(P(A)\right) = 0.$$

The latter defines an algebraic set in $\mathbb{C}^{18}$. To reclaim the *fibre*, $(j \circ P)^{-1}(\mu)$, we must remove, from this set, its *intersection* with the algebraic set $P^{-1}(\bar{\Delta})$ and thus the fibres of $j \circ P$ are *differences of algebraic sets.*

By definition, these fibres fill the 18-dimensional space $\mathbb{C}^{18}\backslash P^{-1}(\bar{\Delta})$. The proof therefore hinges on showing that these fibres have dimension ≤ 17, in which case any neighbourhood of any point of the space would meet uncountably many fibres. It then follows, from previous remarks, that any neighbourhood of any point of this space would meet uncountably many $Gl(2,\mathbb{C}) \times Gl(4,\mathbb{C})$-orbits. Furthermore, provided $P^{-1}(\bar{\Delta})$ is a proper algebraic subset of $\mathbb{C}^{18}$, $\mathbb{C}^{18}\backslash P^{-1}(\bar{\Delta})$ is dense in $\mathbb{C}^{18}$ which also implies that any neighbourhood of any point of V_A meets uncountably many $Gl(2,\mathbb{C}) \times Gl(4,\mathbb{C})$-orbits, meaning there can be no simple germs with one of these 2-jets.

Assume the contrary, i.e. that some fibre of $j \circ P$ has dimension 18. As already stated, this fibre is a difference, $(S\backslash V)$, of algebraic sets, S and V in $\mathbb{C}^{18}$. If $S\backslash V$ has dimension 18 then so does S, but the only 18 dimensional subvariety of $\mathbb{C}^{18}$ is $\mathbb{C}^{18}$ itself. It follows that any 18-dimensional fibre of $j \circ P$ is $\mathbb{C}^{18}\backslash V$, for some algebraic set V. By the definition of a fibre, $j \circ P$ is *constant* on $\mathbb{C}^{18}\backslash V$. Furthermore since $\mathbb{C}^{18}\backslash V$ is dense in $\mathbb{C}^{18}$ and $j \circ P$ is continuous $j \circ P$ is constant on $\mathbb{C}^{18}$.

For this assumption to be false we must therefore demonstrate two things. Firstly, we need to verify that $P^{-1}(\bar{\Delta})$ is a proper algebraic subset of $\mathbb{C}^{18}$ and secondly show that $j \circ P$ varies near some point $A_0 \in \mathbb{C}^{18}$.

For example, consider the following 1-parameter family :

$$A_a = \begin{bmatrix} 0 & 0 & x^2 - y^2 & x^2 \\ 0 & 0 & ay^2 & x^2 + y^2 \\ -(x^2 - y^2) & -ay^2 & 0 & 0 \\ -x^2 & -(x^2 + y^2) & 0 & 0 \end{bmatrix} \qquad a \in \mathbb{C}.$$

Here $P(A_a) = x^4 - y^4 - ax^2y^2$, and by calculation $\Delta\left(P(A_a)\right) = -(1 + a^2/4)^2$ and

$$\frac{J^2}{\Delta}\left(P(A_a)\right) = -\left(\frac{a}{6}\right)^2 \frac{\left(1 + (\frac{a}{6})^2\right)^2}{\left(1 + \frac{a^2}{4}\right)^2}, \qquad (a^2 + 4 \neq 0). \tag{6.122}$$

So elements of this family, for which $a^2 + 4 \neq 0$, have discriminants which don't lie on $\bar{\Delta}$. Hence, $\Delta\,(P(A))$ is not identically zero and $P^{-1}(\bar{\Delta})$ is a proper algebraic subset of $\mathbb{C}^{18}$. Furthermore, from (6.122), $j \circ P(A_a) = (J^2/\Delta)\,(P(A_a))$ varies with a and we have contradicted the assumption that $j \circ P$ has a fibre of dimension 18. It follows that V_A cannot be $\mathcal{G}$-simple. $\square$

6.2 Case $r = 3,\ n = 4$

Having classified all $\mathcal{G}$-simple 2-parameter families of 4×4 skew-symmetric matrices, we can use these calculations as a foundation for exploring higher parameter families of 4×4 skew-symmetric matrices. In this section we consider 3-parameter families, that is the space of smooth germs $A : \mathbb{C}^3, 0 \to Sk(4, \mathbb{C})$.

As before, the first step of any classification involves considering the 1-jets, which here are of the form

$$A(x, y, z) = xA_1 + yA_2 + zA_3, \tag{6.123}$$

with $A_1,\ A_2,\ A_3 \in Sk(4, \mathbb{C})$. The corresponding jet-group $J^1\mathcal{G}$ action on this subspace of $J^1(3, 6)$, as defined in Definition 4.4.9, amounts to the group action of $Gl(3, \mathbb{C}) \times Gl(4, \mathbb{C})$ on *triples* of 4×4 skew-symmetric matrices.

These 1-jets, (6.123), can be thought of as linear maps $\mathbb{C}^3, 0 \to Sk(4, \mathbb{C})$, of rank ≤ 3, this rank being determined by the number of independent matrices in the set $\{A_1, A_2, A_3\}$ (see Definition 3.4.2 of *jetrank*, given in Section 3.4.1). Before going any further we give a couple of useful definitions which are analogous to Definitions 1.2.1 and 1.2.2 in Section 1.2.

Definition 6.2.1 *Let (A_1, A_2, A_3), (B_1, B_2, B_3) be ordered triples of $n \times n$ skew-symmetric matrices over $\mathbb{C}$. We say that they are* **skew-equivalent** *if for some matrix $X \in Gl(n, \mathbb{C})$ we have*

$$B_j = X^T A_j X, \quad j = 1, 2, 3.$$

Definition 6.2.2 *Given a triple (A_1, A_2, A_3), then the points $(x_i : y_i : z_i)$ in the projective plane, $P\mathbb{C}^2$, which satisfy*

$$\det(xA_1 + yA_2 + zA_3) = 0,$$

are refered to as **eigenvalues** *of the triple. Associated with each (distinct) eigenvalue, $(x_i : y_i : z_i)$, there is an* **eigenvector** $\mathbf{u_i} \neq \mathbf{0}$ *satisfying*

$$(x_iA_1 + y_iA_2 + z_iA_3)\mathbf{u_i} = \mathbf{0}.$$

Corollary 6.2.3 *The set of eigenvalues is an invariant of skew-equivalent triples.*

Proof By Definition 6.2.1, any triple skew-equivalent to (A_1, A_2, A_3) is of the form $(X^T A_1 X, X^T A_2 X, X^T A_3 X)$ where $X \in Gl(n, \mathbb{C})$. Such 3-tuples have eigenvalues given by the zeros of

$$\begin{aligned} \det\left(xX^T A_1 X + yX^T A_2 X + zX^T A_3 X\right) &= \det\left[X^T\left(xA_1 + yA_2 + zA_3\right)X\right] \\ &= (\det X)^2 \det(xA_1 + yA_2 + zA_3), \end{aligned}$$

which since $\det X \neq 0$ are the eigenvalues of (A_1, A_2, A_3). □

Consider the following definition, which extends the arguments in Section 1.3, concerning pairs, to triples of skew-symmetric matrices.

Definition 6.2.4 *A* **net** *of (skew-symmetric) matrices is a plane in the projective space, $P(n, \mathbb{C})$, of such matrices. It is determined by any* **three non-collinear** *points in that plane. In other words a net, A, is determined by any triple, (A_1, A_2, A_3), of* **linearly independent** *matrices lying on it and is represented by*

$$A(x, y, z) = xA_1 + yA_2 + zA_3,$$

with points $(x : y : z)$ in the projective plane, $P\mathbb{C}^2$, giving its members. The set $\{A_1, A_2, A_3\}$ is a **basis** *for the net.*

Lemma 6.2.5 *Two triples (A_1, A_2, A_3) and (D_1, D_2, D_3) determine the same net if and only if there is some $P \in Gl(3, \mathbb{C})$ taking the basis $\{A_1, A_2, A_3\}$ to $\{D_1, D_2, D_3\}$.*

Proof Denoting the net, determined by (A_1, A_2, A_3), by A we write

$$A(\lambda, \mu, \nu) = \lambda A_1 + \mu A_2 + \nu A_3. \tag{6.124}$$

If (D_1, D_2, D_3) determines the same net then

$$(D_1, D_2, D_3) = \Big(\sum_{i=1}^{3} \alpha_i A_i, \sum_{i=1}^{3} \beta_i A_i, \sum_{i=1}^{3} \gamma_i A_i\Big),$$

for some α_i, β_i, $\gamma_i \in \mathbb{C}$ with $\{D_1, D_2, D_3\}$ linearly independent. This implies that the three vectors $(\alpha_1, \alpha_2, \alpha_3)$, $(\beta_1, \beta_2, \beta_3)$ and $(\gamma_1, \gamma_2, \gamma_3)$ are linearly independent, that is the matrix

$$P = \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix} \in Gl(3, \mathbb{C}). \tag{6.125}$$

This is a change of basis matrix from $\{A_1, A_2, A_3\}$ to $\{D_1, D_2, D_3\}$, as required.

(In particular,

$$\begin{aligned} A &= x\left(\sum_{i=1}^{3}\alpha_i A_i\right) + y\left(\sum_{i=1}^{3}\beta_i A_i\right) + z\left(\sum_{i=1}^{3}\gamma_i A_i\right) \\ &= (x\alpha_1 + y\beta_1 + z\gamma_1)A_1 + (x\alpha_2 + y\beta_2 + z\gamma_2)A_2 + (x\alpha_3 + y\beta_3 + z\gamma_3)A_3. \end{aligned}$$

Comparing this with (6.124) we see that the coordinates, $Z = (\lambda : \mu : \nu)$, of A, with respect to the old basis $\{A_1, A_2, A_3\}$, are expressed in terms of the coordinates $\hat{Z} = (x : y : z)$, with respect to the new basis $\{D_1, D_2, D_3\}$, by

$$Z = P\hat{Z}.)$$

The converse is clear from the above. □

Remark 6.2.6 It follows from Lemma 6.2.5 that nets are representatives for orbits, under this action of $Gl(3, \mathbb{C})$, of the set of linearly independent triples.

A triple of skew-symmetric matrices, (A_1, A_2, A_3), determines a net if they are independent (have rank 3), a pencil if they span a line in $P(n, \mathbb{C})$ (have rank 2), or a point if they are all multiples of some (non-zero) matrix (have rank 1).

It is easy to see that if the rank is 2 (respectively 1) a $Gl(3, \mathbb{C})$ change of coordinates on the triple reduces the family $xA_1 + yA_2 + zA_3$ to a pencil $x'B_1 + y'B_2$ (respectively $x'C_1$). We then have the normal forms (3.58), (3.56) and (3.53) (respectively (3.55) and (3.57)) of Example 3.4.1 in Section 3.4.

We can thus interpret the action of $Gl(3, \mathbb{C}) \times Gl(n, \mathbb{C})$ on 1-jets, (6.123), as the standard action of $Gl(n, \mathbb{C})$ on their images in $Sk(n, \mathbb{C})$.

Clearly, a net contains pencils, for example by considering the members of $A = \lambda A_1 + \mu A_2 + \nu A_3$ with coordinates $(\lambda : \mu : 0)$ we see that they constitute a pencil determined by the pair (A_1, A_2). Assume that A contains a pencil determined by a pair, (D_1, D_2), of its members and let D_3 be a further member of A which does not lie on this pencil. By a $Gl(3, \mathbb{C})$ change of basis

$$(A_1, A_2, A_3) \mapsto (D_1, D_2, D_3),$$

we can represent elements of A by

$$A = xD_1 + yD_2 + zD_3,$$

where $(x : y : z)$ are coordinates with respect to the basis $\{D_1, D_2, D_3\}$. So when considering a pencil contained in a net we can always, by a change of coordinates, suppose the pencil is given by $z = 0$.

The following definition concerns equivalent nets under the action of $Gl(n, \mathbb{C})$ on $Sk(n, \mathbb{C})$ (Compare with Definition 1.3.2 and Lemma 1.3.3 of Section 1.3.)

Definition 6.2.7 *Two nets, A, B are said to be* **skew-equivalent** *if the action of $Gl(n, \mathbb{C})$ on $Sk(n, \mathbb{C})$ (inducing an action on the projective space $P(n, \mathbb{C})$) takes one net to the other. Equivalently if the action of $Gl(3, \mathbb{C}) \times Gl(4, \mathbb{C})$ takes any independent triple of A to any independent triple of B.*

We describe a classification of skew-equivalent nets using results from the classification of pencils, discussed earlier in Chapters 2 and 3. Although the following can be applied to nets of *any even* order of skew-symmetric matrices, here we restrict ourselves to considering nets of 4×4 skew-symmetric matrices where we can use the list of normal forms of 4×4 pencils given by Example 3.4.1 of Section 3.4.

6.2.1 Nets Containing Non-degenerate Pencils

Consider, first, all nets containing a non-degenerate pencil (see Section 3.4.1). (Recall for a pencil of 4×4 skew-symmetric matrices the Pfaffian is a quadratic, so a non-degenerate pencil has two distinct eigenvalues.) We may suppose this pencil is given by $z = 0$. We therefore represent our net by

$$A = xA_1 + yA_2 + zA_3, \tag{6.126}$$

where $xA_1 + yA_2$ is a non-degenerate pencil. From the classification of pencils we found that any non-degenerate pencil, $xA_1 + yA_2$, is skew-equivalent (or the pair (A_1, A_2) is $Gl(2, \mathbb{C}) \times Gl(4, \mathbb{C})$-equivalent) to the normal form :

$$\begin{bmatrix} 0 & x & 0 & 0 \\ -x & 0 & 0 & 0 \\ 0 & 0 & 0 & y \\ 0 & 0 & -y & 0 \end{bmatrix}.$$

Hence by the action of $Gl(4,\mathbb{C})$ on nets, (6.126), for which $xA_1 + yA_2$ is a non-degenerate pencil we obtain a skew-equivalent net of the form

$$\begin{bmatrix} 0 & x+a_1z & a_2z & a_3z \\ -x-a_1z & 0 & a_4z & a_5z \\ -a_2z & -a_4z & 0 & y+a_6z \\ -a_3z & -a_5z & -y-a_6z & 0 \end{bmatrix},$$

where $a_i \in \mathbb{C}$, $1 \leq i \leq 6$. Furthermore by a change of basis this net is equivalent to

$$\begin{bmatrix} 0 & x & a_2z & a_3z \\ -x & 0 & a_4z & a_5z \\ -a_2z & -a_4z & 0 & y \\ -a_3z & -a_5z & -y & 0 \end{bmatrix}. \tag{6.127}$$

Assuming $a_2 \neq 0$ we can, by a pair of simultaneous row and column operations, use the z term in the second slot to kill off the a_3z and a_4z terms in the third and fourth slots. Furthermore we can then scale the coefficient of this z term to unity, obtaining the skew-equivalent net

$$\begin{bmatrix} 0 & x & z & 0 \\ -x & 0 & 0 & az \\ z & 0 & 0 & y \\ 0 & -az & -y & 0 \end{bmatrix}.$$

This then gives rise to two inequivalent nets.

(i) If $a \neq 0$

$$\begin{bmatrix} 0 & x & z & 0 \\ -x & 0 & 0 & z \\ z & 0 & 0 & y \\ 0 & -z & -y & 0 \end{bmatrix},$$

which has $J^1\mathcal{G}$-codimension 0.

(ii) If $a = 0$

$$\begin{bmatrix} 0 & x & z & 0 \\ -x & 0 & 0 & 0 \\ z & 0 & 0 & y \\ 0 & 0 & -y & 0 \end{bmatrix},$$

which has $J^1\mathcal{G}$-codimension 1.

Returning to the net (6.127), if $a_2 = 0$ we can also assume $a_3 = a_4 = a_5 = 0$ since otherwise by simultaneous row and column operations we can move a non-zero multiple of z into the second slot giving the previous case. But setting $a_2 = a_3 = a_4 = a_5 = 0$ gives the normal form for a non-degenerate pencil, i.e. we are no longer considering a net. We have therefore deduced that any net which contains a non-degenerate pencil has one of the two normal forms (i) and (ii) above.

6.2.2 Nets Containing No Non-degenerate Pencils

The next type of net to consider is one which doesn't contain any non-degenerate pencils but which still contains a non-singular pencil. In the present case this amounts to nets containing a pencil with repeated eigenvalues. As before suppose this pencil, with repeated eigenvalues, is given by $z = 0$ in (6.126). Recall from the classification of pencils that any 4×4 pencil $xA_1 + yA_2$ (pair (A_1, A_2)) with repeated eigenvalues (that is those whose Pfaffian is a non-zero perfect square) has the skew-equivalent ($Gl(2,\mathbb{C}) \times Gl(4,\mathbb{C})$-equivalent) normal form :

$$\begin{bmatrix} 0 & x & 0 & 0 \\ -x & 0 & y & 0 \\ 0 & -y & 0 & x \\ 0 & 0 & -x & 0 \end{bmatrix}.$$

Therefore by an action of $Gl(4,\mathbb{C})$ on these nets we have skew-equivalent nets of the form

$$\begin{bmatrix} 0 & x + a_1 z & a_2 z & a_3 z \\ -x - a_1 z & 0 & y + a_4 z & a_5 z \\ -a_2 z & -y - a_4 z & 0 & x + a_6 z \\ -a_3 z & -a_5 z & -x - a_6 z & 0 \end{bmatrix},$$

which, by a change of basis, are equivalent to

$$\begin{bmatrix} 0 & x & a_2 z & a_3 z \\ -x & 0 & y & a_5 z \\ -a_2 z & -y & 0 & x + a_6 z \\ -a_3 z & -a_5 z & -x - a_6 z & 0 \end{bmatrix}, \tag{6.128}$$

where a_6 is different to that above. This gives us a representation for all nets containing a pencil with repeated eigenvalues. However we may suppose these nets contain *no* non-degenerate pencils. To identify such nets, we need an expression for the set of *all pencils* contained in nets of type (6.128).

Consider the set of linear maps $\Psi_{\alpha,\beta} : \mathbb{C}^2, 0 \to \mathbb{C}^3, 0$, defined by

$$(x, y) \mapsto (x, y, \alpha x + \beta y).$$

For each pair of values $(\alpha, \beta) \in \mathbb{C}^2$ the map $\Psi_{\alpha,\beta}$ is a parametrisation (of points $(x : y : z)$) of the line $\alpha x + \beta y - z = 0$ in $P\mathbb{C}^2$. Hence varying (α, β) over the whole complex plane gives every line, $\Psi_{\alpha,\beta}(x, y)$, in $P\mathbb{C}^2$, other than those through $(0 : 0 : 1)$. It follows that the composition of linear maps, $A \circ \Psi_{\alpha,\beta} : \mathbb{C}^2 \to Sk(4, \mathbb{C})$ represents a pencil contained in net A.

Substituting $z = \alpha x + \beta y$ into (6.128) gives :

$$A_{\alpha,\beta} = \begin{bmatrix} 0 & x & a_2(\alpha x + \beta y) & a_3(\alpha x + \beta y) \\ -x & 0 & y & a_5(\alpha x + \beta y) \\ -a_2(\alpha x + \beta y) & -y & 0 & x + a_6(\alpha x + \beta y) \\ -a_3(\alpha x + \beta y) & -a_5(\alpha x + \beta y) & -x - a_6(\alpha x + \beta y) & 0 \end{bmatrix} . \tag{6.129}$$

So

$$\det(A_{\alpha,\beta}) = Pf(A_{\alpha,\beta})^2,$$

and the Pfaffian, $Pf(A_{\alpha,\beta})$, of $A_{\alpha,\beta}$ is a quadratic.

As already stated, we are only concerned with nets of type, (6.128), whose complete set of pencils (6.129) contains *none* which are non-degenerate. Since a non-degenerate pencil of 4×4 skew-symmetric matrices has two distinct eigenvalues, or equivalently its Pfaffian has two distinct roots, we are looking for conditions on the a_i's for which $Pf(A_{\alpha,\beta})$ has repeated roots, for every $(\alpha, \beta) \in \mathbb{C}^2$.

By calculation we find that

$$\begin{aligned} Pf(A_{\alpha,\beta}) &= \left(-1 + a_2 a_5 \alpha^2 - a_6 \alpha\right) x^2 + \left(2 a_2 a_5 \alpha\beta - a_3 \alpha - a_6 \beta\right) xy \\ &\quad + \left(-a_3 \beta + a_2 a_5 \beta^2\right) y^2, \end{aligned}$$

which has repeated roots provided its discriminant

$$a_3^2 \alpha^2 + \left(a_6^2 + 4 a_2 a_5\right) \beta^2 - 2 a_3 a_6 \alpha\beta - 4 a_3 \beta$$

vanishes. So we seek conditions on a_2, a_3, a_5 and a_6 so that this expression vanishes for all α, β. This results in the following conditions on a_2, a_3, a_5, a_6 in (6.128) :

$$a_3 = 0, \tag{6.130}$$

$$a_6^2 + 4a_2a_5 = 0. \tag{6.131}$$

We can parametrise these conditions by

$$(a_2, a_3, a_5, a_6) = (u^2, 0, -v^2, 2uv),$$

which is regular provided $(u, v) \neq (0, 0)$. Therefore all nets containing no non-degenerate pencils but containing at least one (non-singular) pencil with repeated eigenvalues are skew-equivalent to a net of the form

$$\begin{bmatrix} 0 & x & u^2z & 0 \\ -x & 0 & y & -v^2z \\ -u^2z & -y & 0 & x + 2uvz \\ 0 & v^2z & -x - 2uvz & 0 \end{bmatrix}, \tag{6.132}$$

where u, $v \in \mathbb{C}$.

This net has Pfaffian $(x+uvz)^2$, which determines a repeated line in $P\mathbb{C}^2$. It therefore follows that the net is tangent to the singular set P, (see Section 3.4.1) of $Sk(4, \mathbb{C})$ along a pencil. This makes sense since any non-singular pencil of this net is also tangent to P, at some point on the aforementioned (singular) pencil, and hence cannot be non-degenerate.

By using skew-equivalence we can reduce the nets in (6.132) to a finite list of representatives. Assume $u \neq 0$. By the basis change given by $(x, y, z) \mapsto (x - uvz, y, z)$ we can represent such nets by

$$\begin{bmatrix} 0 & x - uvz & u^2z & 0 \\ -x + uvz & 0 & y & -v^2z \\ -u^2z & -y & 0 & x + uvz \\ 0 & v^2z & -x - uvz & 0 \end{bmatrix}.$$

By the simultaneous row and column operations $C_2 + (v/u)C_3$ $(R_2 + (v/u)R_3)$, followed by $R_4 + (v/u)R_1$ $(C_4 + (v/u)C_1)$ we obtain the skew-equivalent nets

$$\begin{bmatrix} 0 & x & u^2z & 0 \\ -x & 0 & y & 0 \\ -u^2z & -y & 0 & x \\ 0 & 0 & -x & 0 \end{bmatrix}.$$

Scaling gives their representative,

$$\begin{bmatrix} 0 & x & z & 0 \\ -x & 0 & y & 0 \\ -z & -y & 0 & x \\ 0 & 0 & -x & 0 \end{bmatrix}, \tag{6.133}$$

which has $J^1\mathcal{G}$-codimension 3.

Alternatively, if in (6.132) $u = 0$ but $v \neq 0$, by scaling we have the representative

$$\begin{bmatrix} 0 & x & 0 & 0 \\ -x & 0 & y & -z \\ 0 & -y & 0 & x \\ 0 & z & -x & 0 \end{bmatrix},$$

which we can obtain from (6.133) by the sequence of simultaneous row and column operations : $C_4 + C_1$ $(R_4 + R_1)$, $C_2 + C_3$ $(R_2 + R_3)$, $R_1 - R_4$ $(C_1 - C_4)$, $C_3 - C_2$ $(R_3 - R_2)$. Therefore nets in this family lie in the same orbit as (6.133).

Finally the case $u = v = 0$ yields a pencil and not a net.

The final type of net (of 4×4 skew-symmetric matrices) are those which contain *only* singular pencils. All nets containing a non-singular pencil are covered by one of the previous two types.

As before, we represent the net by

$$A = xA_1 + yA_2 + zA_3,$$

where here the pencil $xA_1 + yA_2$ is *singular*. Recall that all singular pencils of 4×4 skew-symmetric matrices are skew-equivalent to the normal form :

$$\begin{bmatrix} 0 & 0 & x & 0 \\ 0 & 0 & y & 0 \\ -x & -y & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, by an action of $Gl(4, \mathbb{C})$ on A we obtain the skew-equivalent nets

$$\begin{bmatrix} 0 & a_1 z & x + a_2 z & a_3 z \\ -a_1 z & 0 & y + a_4 z & a_5 z \\ -x - a_2 z & -y - a_4 z & 0 & a_6 z \\ -a_3 z & -a_5 z & -a_6 z & 0 \end{bmatrix},$$

which, by a change of basis, are equivalent to

$$\begin{bmatrix} 0 & a_1 z & x & a_3 z \\ -a_1 z & 0 & y & a_5 z \\ -x & -y & 0 & a_6 z \\ -a_3 z & -a_5 z & -a_6 z & 0 \end{bmatrix}. \tag{6.134}$$

This is therefore a representation for any net containing a singular pencil, however again we may choose to consider only those nets which contain no non-singular pencils. Again replacing z by $\alpha x + \beta y$ we obtain a pencil given by

$$A_{\alpha,\beta} = \begin{bmatrix} 0 & a_1(\alpha x+\beta y) & x & a_3(\alpha x+\beta y) \\ -a_1(\alpha x+\beta y) & 0 & y & a_5(\alpha x+\beta y) \\ -x & -y & 0 & a_6(\alpha x+\beta y) \\ -a_3(\alpha x+\beta y) & -a_5(\alpha x+\beta y) & -a_6(\alpha x+\beta y) & 0 \end{bmatrix}.$$

It follows from Definition 1.4.1 (of singular pencils), that nets in (6.134) yield only singular pencils provided $Pf(A_{\alpha,\beta}) \equiv 0$ for *all* $(\alpha, \beta) \in \mathbb{C}^2$. By calculation we find that

$$\begin{aligned} Pf(A_{\alpha,\beta}) &= \left(a_1a_6\alpha^2 - a_5\alpha\right)x^2 + \left(2a_1a_6\alpha\beta + a_3\alpha - a_5\beta\right)xy \\ &\quad + \left(a_1a_6\beta^2 + a_3\beta\right)y^2, \end{aligned}$$

which vanishes identically for all α, β if and only

$$a_5 = 0,$$

$$a_1a_6 = 0,$$

and

$$a_3 = 0.$$

Substituting the conditions on a_3 and a_5, into (6.134) gives the family of nets

$$\begin{bmatrix} 0 & a_1z & x & 0 \\ -a_1z & 0 & y & 0 \\ -x & -y & 0 & a_6z \\ 0 & 0 & -a_6z & 0 \end{bmatrix},$$

where $a_1a_6 = 0$. It can be seen, by scaling, that those elements of this family for which $a_1 \neq 0$, are skew-equivalent to the representative

$$\begin{bmatrix} 0 & z & x & 0 \\ z & 0 & y & 0 \\ -x & -y & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{6.135}$$

Alternatively if $a_6 \neq 0$ we have the representative

$$\begin{bmatrix} 0 & 0 & x & 0 \\ 0 & 0 & y & 0 \\ -x & -y & 0 & z \\ 0 & 0 & -z & 0 \end{bmatrix},$$

which, using Lemma 6.0.1, is skew-equivalent to the net

$$\begin{bmatrix} 0 & 0 & 0 & x \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & y \\ -x & -z & -y & 0 \end{bmatrix}. \tag{6.136}$$

We observe here that these two singular nets, (6.135) and (6.136), both have the same $J^1\mathcal{G}$-codimension, 6, and at first sight we might expect them to be skew-equivalent. This however is not the case, as we will show in Chapter 7.

If $a_1 = a_6 = 0$ then we have a pencil. In summary we have just established the following result.

Theorem 6.2.8 *Any net of* 4×4 *skew-symmetric matrices over* $\mathbb{C}$ *is skew-equivalent to one of five possible normal forms (written in upper triangular form)* :

$$[x, z, 0, 0, z, y],$$

$$[x, z, 0, 0, 0, y],$$

$$[x, z, 0, y, 0, x],$$

$$[z, x, 0, y, 0, 0],$$

$$[0, 0, x, 0, z, y].$$

It follows that a complete list of orbits of 1-jets of maps, $A : \mathbb{C}^3, 0 \to Sk(4, \mathbb{C})$, under $J^1\mathcal{G}$-equivalence (or alternatively $Gl(3, \mathbb{C}) \times Gl(4, \mathbb{C})$-equivalence) is obtained by appending the five normal forms in Theorem 6.2.8, above, to the list of normal forms of 1-jets of two parameter families found in Example 3.4.1 of Section 3.4.

6.2.3 Selective Classification ($r = 3$, $n = 4$)

Using this list of 1-jets as a starting point we can proceed to classify all $\mathcal{G}$-simple germs $A : \mathbb{C}^3, 0 \to Sk(4, \mathbb{C})$ using the methods employed for the two parameter case. However, instead we limit our consideration to the $\mathcal{G}$-simples arising from the 1-jets found in Theorem 6.2.8, leaving the complete classification to a later date.

Here for all calculations using `Transversal` the only alteration to the Lie algebra setups (see Section 5.5 of Chapter 5) is to change the value of the global variable `source_dim` to 3, thereby introducing an extra source coordinate, **x3**, to represent z.

Firstly, $[0, x, z, z, y, 0]$ is 1-$\mathcal{G}$-determined and is a representative of the open $\mathcal{G}$-orbit (it has $J^1\mathcal{G}$-codimension 0). Furthermore its $\mathcal{G}_e$-codimension is 3 and its discriminant is given by $xy - z^2 = 0$, which has an A_1 singularity.

Investigating jets with 1-jet $[0, x, z, 0, y, 0]$ we detect the presence of the following series.

Lemma 6.2.9 *Let $A : \mathbb{C}^3, 0 \to Sk(4, \mathbb{C})$ be any smooth germ with 1-jet*

$$[0, x, z, 0, y, 0]. \tag{6.137}$$

Then A is $\mathcal{G}$-equivalent to a k-determined germ of the form

$$[0, x, z, z^k, y, 0],$$

where $k \geq 2$, or for any $k \geq 1$, A is $\mathcal{G}$-equivalent to a germ whose k-jet is (6.137).

Each germ $[0, x, z, z^k, y, 0]$, $k \geq 2$, has $\mathcal{G}_e$-codimension $(k+2)$ and a discriminant of type A_k.

Proof Assume, for any $k \geq 2$, that A has a $(k-1)$-jet

$$j^{k-1}A = \begin{bmatrix} 0 & 0 & x & z \\ 0 & 0 & 0 & y \\ -x & 0 & 0 & 0 \\ -z & -y & 0 & 0 \end{bmatrix}.$$

Firstly, we observe that any monomial $x^r y^s z^t$, $r+s+t = k$, in either slot e_1 or e_6 is in the $J^k\mathcal{H}_0$-tangent space to $j^{k-1}A$ due to the $\mathcal{H}_0$-tangent space generators

$$[x|y|z, 0, 0, 0, 0, 0], \qquad [0, 0, 0, 0, 0, x|y|z].$$

Hence when looking for a complete k-transversal it is sufficient to represent $j^{k-1}A$ by the corresponding $(k-1)$-jet,

$$A_q = \begin{bmatrix} x & z \\ 0 & y \end{bmatrix},$$

in $J^k(3,4)$ and calculate its $J^k(\mathcal{R}_1 \times \mathcal{Q}_0)$-tangent space. (See Sections 6.0.4 and 6.0.5, in particular Corollary 6.0.7 and Remarks 6.0.16.) For ease we represent elements of $J^k(3,4)$ by 4-tuples. So the $J^k(\mathcal{R}_1 \times \mathcal{Q}_0)$-tangent space to A_q is generated by the following vectors :

$$\mathcal{M}_3^2 :$$

$$[1,0,0,0], \quad [0,1,0,0] \quad [0,0,0,1]; \tag{6.138}$$

$$\mathcal{M}_3 :$$

$$[x,z,0,0], \quad [0,0,x,z], \quad [0,y,0,0]$$

$$[0,0,0,y], \quad [x,0,0,0], \quad [0,x,0,0]$$

$$[z,0,y,0], \quad [0,z,0,y].$$

We are looking for all elements of the set of vectors,

$$\{x^r y^s z^t e_i : r+s+t=k, \ \ 2 \le i \le 5\},$$

which are contained in this tangent space.

Clearly, the three generators in (6.138) give everything in slots e_2, e_3 and e_5 respectively. The only generators which have terms in e_4 are :

$$[0,0,x,z], \qquad [z,0,y,0].$$

Provided $r \ge 1$ or $s \ge 1$ we obtain terms $x^r y^s z^t e_4$ by combining one of these vectors with an appropriate $\mathcal{R}_1$-tangent vector from (6.138). For example if $r_1 \ge 1$ the vector $[0,0,x^{r_1}y^{s_1}z^{t_1},0]$ is given by

$$x^{r_1-1}y^{s_1}z^{t_1}[0,0,x,z] - x^{r_1-1}y^{s_1}z^{t_1+1}[0,0,0,1].$$

So, the only elements of $H^k(3,4)$ missing from this tangent space are scalar multiples of $z^k e_4$ and we have a k-transversal

$$[x,z,az^k,y], \tag{6.139}$$

where $a \in \mathbb{C}$.

If $a \neq 0$ by scaling we obtain the $J^k(\mathcal{R} \times \mathcal{Q})$-equivalent jet

$$[x, z, z^k, y].$$

It is clear, from what has gone before, that the $J^{k+1}(\mathcal{R}_1 \times \mathcal{Q}_0)$-tangent space to this jet contains all elements of $H^{k+1}(3,4)$ except, possibly, scalar multiples of $[0,0,z^k,0]$. The significant vectors of the $J^{k+1}(\mathcal{R}_1 \times \mathcal{Q}_0)$-tangent space are :

$$\mathcal{M}_3^2 : \quad [0,1,kz^{k-1},0] \quad ([1,0,0,0])$$

$$\mathcal{M}_3 : \quad [0,0,z^k,y] \quad [x,0,z^k,0].$$

By combining $z[x,0,z^k,0]$ and $xz[1,0,0,0]$ it follows that this tangent space also contains all scalar multiples of $z^{k+1}e_4$. Therefore $[x,z,z^k,y]$ is k-$(\mathcal{R} \times \mathcal{Q})$-determined, which, since every monomial of degree $k+1$ in slots e_1 and e_6 is present in the $J^{k+1}\mathcal{G}_1$-tangent space, means $[0,x,z,z^k,y,0]$ is k-$\mathcal{G}$-determined.

If, in (6.139), $a = 0$ then we would have the k-jet $[0,x,z,0,y,0]$ (or $[x,z,0,y]$) and can repeat the previous argument replacing $k-1$ by k and so on.

The $\mathcal{G}_e$-codimension and discriminant of each germ $[0,x,z,z^k,y,0]$, $k \geq 2$, are found by considering the corresponding invariants of the k-$(\mathcal{R} \times \mathcal{Q})$-determined germ

$$\begin{bmatrix} x & z \\ z^k & y \end{bmatrix},$$

and using the results of Lemma 6.0.14 of Section 6.0.5. □

Consider the 1-jet $[x,z,0,y,0,x]$. By the $\mathcal{R}$-change $(x,y,z) \mapsto (z,y,x)$ this 1-jet is equivalent to (in matrix form) :

$$\begin{bmatrix} 0 & z & x & 0 \\ -z & 0 & y & 0 \\ -x & -y & 0 & z \\ 0 & 0 & -z & 0 \end{bmatrix},$$

and has a 2-transversal of the form

$$\begin{bmatrix} 0 & z & x & Q_1(x,y) \\ -z & 0 & y & Q_2(x,y) \\ -x & -y & 0 & z \\ -Q_1(x,y) & -Q_2(x,y) & -z & 0 \end{bmatrix},$$

where $Q_1(x,y) = ax^2 + bxy + cy^2$, $Q_2(x,y) = Ax^2 + Bxy + Cy^2$ are arbitrary quadratics. The following result uses Lemma 6.1.7 to simplify the 2-jets in this transversal into a finite number of $J^2\mathcal{G}$-orbits.

Lemma 6.2.10 *Any 2-jet of the form*

$$\begin{bmatrix} 0 & z & x & Q_1(x,y) \\ -z & 0 & y & Q_2(x,y) \\ -x & -y & 0 & z \\ -Q_1(x,y) & -Q_2(x,y) & -z & 0 \end{bmatrix},$$

where $Q_1(x,y) = ax^2 + bxy + cy^2$, $Q_2(x,y) = Ax^2 + Bxy + Cy^2$ are arbitrary quadratics, is $J^2\mathcal{G}$-equivalent to one of the following four 2-jets:

$$\begin{bmatrix} 0 & z & x & xy \\ -z & 0 & y & xy \\ -x & -y & 0 & z \\ -xy & -xy & -z & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & z & x & xy \\ -z & 0 & y & 0 \\ -x & -y & 0 & z \\ -xy & 0 & -z & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & z & x & 0 \\ -z & 0 & y & x^2 \\ -x & -y & 0 & z \\ 0 & -x^2 & -z & 0 \end{bmatrix}, \qquad \begin{bmatrix} 0 & z & x & 0 \\ -z & 0 & y & 0 \\ -x & -y & 0 & z \\ 0 & 0 & -z & 0 \end{bmatrix},$$

the first of which is a representative of the generic orbit.

Proof Consider the action of the subgroup of $\mathcal{G}$, $(\mathcal{R} \times \mathcal{H}_q)$, on the skew-symmetric matrix,

$$\begin{bmatrix} 0 & z & x & Q_1(x,y) \\ -z & 0 & y & Q_2(x,y) \\ -x & -y & 0 & z \\ -Q_1(x,y) & -Q_2(x,y) & -z & 0 \end{bmatrix}, \tag{6.140}$$

(see the discussion following Lemma 6.0.1). Clearly, any $\mathcal{R}$-change of coordinates preserving the z-coordinate only affects the 2×2 sub-matrix

$$\begin{bmatrix} x & Q_1(x,y) \\ y & Q_2(x,y) \end{bmatrix}. \tag{6.141}$$

Furthermore the action of elements of $\mathcal{H}_q$ on (6.140) have the effect, apart from that on (6.141) itself, of multiplying the z terms in slots e_1 and e_6 by non-zero function germs. We can then use the results of Lemma 6.1.7 to deduce that a matrix of the form (6.140) is $J^2\mathcal{G}$-equivalent to something of the form

$$\begin{bmatrix} 0 & cz & x & q_1(x,y) \\ -cz & 0 & y & q_2(x,y) \\ -x & -y & 0 & dz \\ -q_1(x,y) & -q_2(x,y) & -dz & 0 \end{bmatrix}, \tag{6.142}$$

where

$$\begin{bmatrix} x & q_1(x,y) \\ y & q_2(x,y) \end{bmatrix}$$

is one of the four 2-jets given in the statement of Lemma 6.1.7 and c, d are non-zero function germs. It remains to 'scale' away the function germs c and d.

We can first assume $c = 1$ by the coordinate change $(x, y, z) \mapsto (x, y, z/c)$. By the further coordinate change given by $(x, y, z) \mapsto (\lambda x, \lambda y, z)$, where λ is a non-zero function germ, combined with multiplying R_3, C_3 by a non-zero function germ α and multiplying R_4, C_4 by a non-zero function germ β we obtain the $J^2\mathcal{G}$-equivalent matrix

$$\begin{bmatrix} 0 & z & \alpha\lambda x & \beta\lambda^2 q_1(x,y) \\ -z & 0 & \alpha\lambda y & \beta\lambda^2 q_2(x,y) \\ -\alpha\lambda x & -\alpha\lambda y & 0 & \alpha\beta dz \\ -\beta\lambda^2 q_1(x,y) & -\beta\lambda^2 q_2(x,y) & -\alpha\beta dz & 0 \end{bmatrix}.$$

By choosing $\alpha = 1/\lambda$, $\beta = 1/\lambda^2$, with $\lambda = d^{1/3}$ we obtain the required result. □

We consider each of these representatives in turn.

The 2-jet $[z, x, xy, y, xy, z]$ is 2-$\mathcal{G}$-determined. It has $\mathcal{G}_e$-codimension 6 and its discriminant is given by $z^2 - xy(x - y) = 0$, which is of type D_4.

Investigating jets with 2-jet $[z, x, xy, y, 0, z]$ we detect the presence of the following series.

Lemma 6.2.11 *Let $A : \mathbb{C}^3, 0 \to Sk(4, \mathbb{C})$ be any smooth germ with 2-jet*

$$[z, x, xy, y, 0, z]. \tag{6.143}$$

Then A is $\mathcal{G}$-equivalent to a k-determined germ of the form

$$[z, x, xy, y, x^k, z],$$

where $k \geq 3$, or for any $k \geq 2$ A is $\mathcal{G}$-equivalent to a germ whose k-jet is (6.143).

Proof The main idea of this argument is to use the proof of Lemma 6.1.8 to simplify the calculations.

Assume, for any $k \geq 3$, that A has a $(k-1)$-jet

$$j^{k-1}A = \begin{bmatrix} 0 & z & x & xy \\ -z & 0 & y & 0 \\ -x & -y & 0 & z \\ -xy & 0 & -z & 0 \end{bmatrix}. \tag{6.144}$$

The $J^k\mathcal{G}_1$-tangent space to this jet is generated by the following vectors :

$\mathcal{M}_3^2$:

$$[1,0,0,0,0,1], \quad [0,1,y,0,0,0] \quad [0,0,x,1,0,0];$$

$\mathcal{M}_3$:

$$[x,0,0,0,z,0], \quad [0,0,0,0,z,x], \tag{6.145}$$

$$[-y,0,z,0,0,0], \quad [0,0,z,0,0,-y], \tag{6.146}$$

$$[z,x,xy,0,0,0], \quad [0,0,xy,0,0,z], \tag{6.147}$$

$$[0,0,0,x,xy,0], \quad [0,y,0,0,0,0], \tag{6.148}$$

$$[0,0,x,0,y,0], \quad [0,xy,0,0,0,0], \tag{6.149}$$

$$[z,0,0,y,0,0], \quad [0,x,0,y,0,z], \tag{6.150}$$

$$[0,z,0,0,0,0], \quad [0,0,0,-z,0,xy] \quad [xy,0,0,-z,0,0]. \tag{6.151}$$

First, it is clear that scalar multiples of $x^k e_5$ are not contained in this tangent space, but we will show that all other elements of $H^k(3,4)$ arè present.

We next demonstrate that this generating set gives every monomial $x^r y^s z^t$ $(r+s+t=k)$ in both slots e_1 and e_6. Specifically the spanning set consisting of :

(i) $x[1,0,0,0,0,1]$ and both vectors in (6.145) give xe_1, xe_6, ze_5;

(ii) $y[1,0,0,0,0,1]$ and both vectors in (6.146) give ye_1, ye_6, ze_3;

(iii) $z[1,0,0,0,0,1]$, $x[0,1,y,0,0,0]$ and both vectors in (6.147) give ze_1, ze_6 and xye_3.

Hence we have everything in slots e_1 and e_6.

Next we show that every monomial $x^r y^s z^t e_i$ ($r+s+t=k$, $2 \leq i \leq 5$) for which $t \geq 1$ is contained in the $J^k\mathcal{G}_1$-tangent space. This is clear since from (i) and (ii) we have $\langle z\rangle e_5$, $\langle z\rangle e_3$ and the vectors in (6.151) give $\langle z\rangle e_2$ and $\langle z\rangle e_4$.

From (iii) we have the vector $[0,0,xy,0,0,0]$ and using ze_1 and ze_6 we obtain, from the vectors in (6.150), the vectors $[0,0,0,y,0,0]$ and $[0,x,0,y,0,0]$ respectively. These three vectors along with the vectors $[0,1,y,0,0,0]$, $[0,0,x,1,0,0]$ and all the vectors in (6.148) and (6.149) give a spanning set corresponding to the $J^k(\mathcal{R}_1 \times \mathcal{Q}_0)$-tangent space to

$$\begin{bmatrix} x & xy \\ y & 0 \end{bmatrix},$$

given in the proof of Lemma 6.1.8. It then follows, using the calculations in this proof, that the $J^k\mathcal{G}_1$-tangent space contains every monomial vector $x^r y^s e_i$ ($r+s=k$, $2 \leq i \leq 5$) — with the exception of $x^k e_5$.

So a k-transversal to (6.144) is

$$[z, x, xy, y, ax^k, z], \qquad a \in \mathbf{C}. \tag{6.152}$$

Provided $a \neq 0$, by scaling, we have the $\mathcal{G}$-equivalent k-jet

$$[z, x, xy, y, x^k, z].$$

From the above it is clear that the $J^{k+1}\mathcal{G}_1$-tangent space to this jet contains all elements of $H^{k+1}(3,4)$ except possibly scalar multiples of $[0,0,0,0,x^{k+1},0]$.

Relevant tangent vectors of the $J^{k+1}\mathcal{G}_1$-tangent space are,

$$\mathcal{M}_3^2: \quad [1,0,0,0,0,1], \quad [0,0,x,1,0,0],$$

$$\mathcal{M}_3: \quad [z,0,0,y,x^k,0], \quad [0,0,xy,0,x^k,z],$$

which give the required $[0,0,0,0,x^{k+1},0]$. It follows that $[z,x,xy,y,x^k,z]$ is k-$\mathcal{G}$-determined.

As usual if, in (6.152), $a=0$ we have the k-jet $[z,x,xy,y,0,z]$ and we can repeat the previous argument, replacing $(k-1)$ by k, and so on. □

Corollary 6.2.12 *We have the series of finitely $\mathcal{G}$-determined germs*

$$[z,x,xy,y,x^k,z], \qquad (k \geq 2), \tag{6.153}$$

each of which has $\mathcal{G}_e$-codimension $(k+4)$ and a discriminant of type D_{k+2}.

Proof By a similar approach to that used to prove Lemma 6.2.10 it can be shown that the two 2-jets

$$\begin{bmatrix} 0 & z & x & xy \\ -z & 0 & y & xy \\ -x & -y & 0 & z \\ -xy & -xy & -z & 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 0 & z & x & xy \\ -z & 0 & y & x^2 \\ -x & -y & 0 & z \\ -xy & -x^2 & -z & 0 \end{bmatrix},$$

are $J^2\mathcal{G}$-equivalent. So it follows that

$$[z,x,xy,y,x^2,z],$$

(like $[z,x,xy,y,xy,z]$) is also 2-$\mathcal{G}$-determined and has $\mathcal{G}_e$-codimension 6. We add it to the series found in Lemma 6.2.11.

Using a method similar to that described in Remark 4.4.35 of Section 4.4.3 we find, for each $k \geq 3$, that the corresponding germ of the series, (6.153), has $\mathcal{G}_e$-codimension $(k+4)$. (Setting $k=2$ this agrees with the $\mathcal{G}_e$-codimension of $[z,x,xy,y,x^2,z]$.) Finally each germ of the series has a discriminant given by $z^2-(x^{k+1}-xy^2)=0$, i.e. of type D_{k+2}. □

Investigating the 2-jet $[z,x,0,y,x^2,z]$ we obtain the following result, analogous to that given by Theorem 6.1.10.

Theorem 6.2.13 *Any $\mathcal{G}$-simple map $A:\mathbb{C}^3,0 \to \mathbb{C}^6,0$, with 2-jet $J^2\mathcal{G}$-equivalent to $[z,x,0,y,x^2,z]$, is $\mathcal{G}$-equivalent to one of the following finitely determined germs :*

$$[z,x,y^3,y,x^2,z],$$

$$[z, x, xy^2, y, x^2, z],$$

$$[z, x, y^4, y, x^2, z],$$

with $\mathcal{G}_e$-codimensions 8, 9, 10 *and discriminants E_6, E_7 and E_8 respectively.*

Proof The classification is carried out using `Transversal` in a similar fashion to that used to obtain the results of Theorem 6.1.10 (see Section 5.5). Starting the classification, by finding a complete 3-transversal to

$$[z, x, 0, y, x^2, z],$$

the calculations involved and the results obtained are analogous to those for Theorem 6.1.10.

The $\mathcal{G}_e$-codimensions of the resulting finitely determined germs, listed in the statement, are also found using `Transversal`.

By finding their determinants, the discriminants of $[z, x, y^3, y, x^2, z]$, $[z, x, xy^2, y, x^2, z]$ and $[z, x, y^4, y, x^2, z]$ are given by

$$z^2 - (x^3 - y^4) = 0,$$

$$z^2 - (x^3 - xy^3) = 0,$$

$$z^2 - (x^3 - y^5) = 0,$$

which have E_6, E_7, and E_8 singularities, respectively. □

The investigation of germs with 2-jet $[z, x, 0, y, 0, z]$ is found to be analogous to that discussed in the proof of Lemma 6.1.11.

Lemma 6.2.14 *There are no $\mathcal{G}$-simple germs $A : \mathbb{C}^3, 0 \to \mathbb{C}^6, 0$ with 2-jet $[z, x, 0, y, 0, z]$.*

Proof This result relies on a similar calculation and argument to that used for Lemma 6.1.11. □

Chapter 7

Geometry

7.1 Duality of Classification

Setting $V = K^n$, we have seen that we can think of $n \times n$ skew-symmetric matrices as the set of skew forms, $Alt(V)$, on V. We have also seen that the space of skew forms is canonically isomorphic to $(\bigwedge^2 V)^*$, i.e. linear maps $\bigwedge^2 V \longrightarrow K$. We have the following result.

Lemma 7.1.1 *(see [Lang].) Let V_1 and V_2 be a pair of vector spaces. Then any linear map*

$$\alpha : V_1 \longrightarrow V_2$$

induces a linear map

$$\hat{\alpha} : Alt(V_2) \to Alt(V_1)$$

and a linear map

$$\bigwedge^2 \alpha : \bigwedge^2 V_1 \longrightarrow \bigwedge^2 V_2.$$

Moreover if ϕ, $\psi \in Alt(V)$ and $\bar{\phi}$, $\bar{\psi}$ are the corresponding elements of $(\bigwedge^2 V)^*$ then the diagram on the left below commutes if and only if that on the right commutes (if and only if $\hat{\alpha}(\psi) = \phi$).

$$\begin{tikzcd} V_1 \times V_1 \arrow[dd, "(\alpha,\alpha)"'] \arrow[dr, "\phi"] & \\ & K \\ V_2 \times V_2 \arrow[ur, "\psi"'] & \end{tikzcd} \longleftrightarrow \begin{tikzcd} \bigwedge^2 V_1 \arrow[dd, "\bigwedge^2 \alpha"'] \arrow[dr, "\bar{\phi}"] & \\ & K . \\ \bigwedge^2 V_2 \arrow[ur, "\bar{\psi}"'] & \end{tikzcd}$$

So we can think of our skew-symmetric matrices as elements of $(\bigwedge^2 V)^*$ and, by setting $W = K^r$, of our 1-jets as elements $f \in \hom(W, (\bigwedge^2 V)^*)$. The classification of these 1-jets corresponds to the natural action of $Gl(W) \times Gl(V)$ on this space. It is not hard to see that the type of f is determined by the image of f; see below. Before considering this we need a few results from linear algebra.

Lemma 7.1.2 *There is a canonical isomorphism (identification) between the two vector spaces $Alt(V)^*$ and $Alt(V^*)$. Alternatively we have a canonical isomorphism between*

$$(\bigwedge^2 V)^* \qquad \text{and} \qquad \bigwedge^2 V^*.$$

If we choose a basis $\{e_1, \ldots, e_n\}$ for V, the dual basis $\{e_1^*, \ldots, e_n^*\}$ for V^* and corresponding bases for $(\bigwedge^2 V)^*$ and $\bigwedge^2(V^*)$ then the isomorphism takes $(e_i \wedge e_j)^*$ to $e_i^* \wedge e_j^*$.

Proof We give two (equivalent) proofs. Consider the map $\theta : Alt(V) \times Alt(V^*) \to K$ defined by $(\phi, \psi) \mapsto \text{trace}(\psi^* \circ \phi^*)$, where $\phi^* : V \to V^*$ and $\psi^* : V^* \to (V^*)^* \equiv V$. We claim that θ is a natural non-degenerate bilinear form. Bilinearity and naturality is clear. So we need to establish non-degeneracy.

First as in the statement the dual basis for V^* is $\{e_1^*, \ldots, e_n^*\}$ where

$$e_i^*(e_j) = \delta_{ij}.$$

Now we have seen that there is a natural map $Alt(V) \to Hom(V, V^*)$. With respect to the given basis for V and the dual basis for V^* we can think of this as a homomorphism $Sk(n, K) \to M(n, K)$, and it is easy to check that this is the transpose map. So the bilinear form becomes a map $Sk(n, K) \times Sk(n, K) \to K$ given by $(A, B) \mapsto \text{trace}(BA) = -\sum_i \sum_j a_{ij} b_{ij}$. This is clearly nondegenerate. The map θ now gives an isomorphism $\theta^* : Alt(V^*) \to Alt(V)^*$.

For the second proof consider the pairing $\bigwedge^2 V \times \bigwedge^2 V^* \longrightarrow K$ determined by

$$(v_1 \wedge v_2, v_1^* \wedge v_2^*) \mapsto (v_1^*(v_1)v_2^*(v_2) - v_1^*(v_2)v_2^*(v_1)),$$

which we denote by $\langle v_1 \wedge v_2, v_1^* \wedge v_2^* \rangle$ and extend linearly. This is a canonical well defined bilinear form which we will demonstrate is also non-degenerate.

Consider for $1 \leq i < j \leq n$, $1 \leq r < s \leq n$ the expression $\langle e_i \wedge e_j, e_r^* \wedge e_s^* \rangle$. It is easy to see that this is 1 if $i = r, j = s$ and 0 otherwise. (It is not possible that $i = s, j = r$.) So the bilinear form is therefore non-degenerate.

Consequently this bilinear form yields a natural isomorphism $\bigwedge^2(V^*) \longrightarrow (\bigwedge^2 V)^*$ determined by

$$e_r^* \wedge e_s^* \mapsto \langle -, e_r^* \wedge e_s^* \rangle,$$

as required, which has the required effect on $(e_i \wedge e_j)^*$. □

A consequence of this lemma is that we can identify $Alt(V)^*$ with $Alt(V^*)$ (resp. $(\bigwedge^2 V)^*$ with $\bigwedge^2(V^*)$). In the latter case we denote either by $\bigwedge^2 V^*$ from now on.

The following definitions and results will prove useful.

Definition 7.1.3 *Given a subspace U of a vector space, V, there is an associated subspace $U^\perp$ of the dual space, V^*,*

$$U^\perp = \{v^* \in V^* : v^*(u) = 0 \qquad \textit{for all } u \in U \subset V\}.$$

So $U^\perp$ consists of those elements of V^* which kill off every element of U.

Lemma 7.1.4 *If V is finite dimensional and U is a subspace of V then* $\dim U^\perp = \dim V - \dim U$ *and* $(U^\perp)^\perp = U$.

Proof Let $\{e_1, \ldots, e_r\}$ be any basis for U. We extend this basis to the basis $\{e_1, \ldots, e_N\}$ of V, where N is its dimension. Consider the corresponding basis for the 'dual space' V^*, $\{e_1^*, \ldots, e_N^*\}$ where

$$e_i^*(e_j) = \delta_{ij}. \tag{7.1}$$

A general element $v^* \in V^*$ can be written

$$v^* = \sum_{i=1}^{N} \lambda_i e_i^*, \qquad \lambda_i \in K.$$

For $v^* \in U^\perp$, then Definition 7.1.3 requires that

$$v^*(u) = 0,$$

for all $u \in Sp\{e_1, \dots, e_r\}$. This holds provided

$$\sum_{i=1}^{N} \lambda_i e_i^*(e_j) = 0,$$

for each $1 \le j \le r$. From the identity (7.1) it follows that $\lambda_j = 0$, $1 \le j \le r$ and we can express any element $u^* \in U^\perp$ by

$$u^* = \sum_{i=r+1}^{N} \lambda_i e_i^*,$$

for some $(\lambda_{r+1}, \dots, \lambda_N) \in \mathbf{C}^{N-r}$. Clearly, $\{e_{r+1}^*, \dots, e_N^*\}$ is a basis for $U^\perp$ and $\dim U^\perp = N - r$ as required. It is not hard to see that $(U^\perp)^\perp \supset U$ and since both spaces have the same dimension the result follows. □

Lemma 7.1.5 *If V_1, V_2 are vector spaces, U_1, U_2 respectively subspaces with $\dim V_1 = \dim V_2$ and $\dim U_1 = \dim U_2$ then an isomorphism $\alpha : V_1 \to V_2$ takes U_1 to U_2 if and only if the dual isomorphism $\alpha^* : V_2^* \to V_1^*$ takes $U_2^\perp$ to $U_1^\perp$.*

Proof The map α^* is defined by $\alpha^*(v_2^*)(v_1) = v_2^*(\alpha(v_1))$. The result is now easy. □

We tailor these results to suit the case at hand, where we consider subspaces of the vector space $Alt(V)$ and its dual $Alt(V)^* \equiv Alt(V^*)$ (or equivalently $\bigwedge^2 V^*$ and its dual, $(\bigwedge^2(V^*))^* \equiv \bigwedge^2 V$). We have a canonical isomorphism $\gamma : Alt(V)^* \to Alt(V^*)$ (denote the isomorphism $(\bigwedge^2(V^*))^* \equiv \bigwedge^2 V$ by the same symbol) .

Lemma 7.1.6 *So given any subspace, $U \subset Alt(V)$ (resp. $U \subset \bigwedge^2 V^*$), there is an associated subspace, $\gamma(U^\perp)$, of $Alt(V^*)$ (resp. $(\bigwedge^2 V^*)^* \equiv \bigwedge^2 V$), given by*

$$\gamma(U^\perp) = \left\{\hat{\phi} \in Alt(V^*) : \text{trace}\left(\hat{\phi}^* \circ \psi^*\right) = 0 \qquad \text{for all } \psi \in U \subset Alt(V)\right\},$$

(respectively

$$\gamma(U^\perp) = \left\{\hat{v} \in \bigwedge^2 V : \langle \hat{v}, \hat{u}^* \rangle = 0 \qquad \text{for all } \hat{u}^* \in U \subset \bigwedge^2 V^*\right\},$$

where $\langle \hat{v}, \hat{u}^* \rangle$ is the bilinear form described in the proof of Lemma 7.1.2.)

Proof Underlying this result is the following. Let $\theta : V \times W \to K$ be a non-degenerate bilinear form. Then there is a natural isomorphism $\theta^* : W \to V^*$. Let U be a subspace of V, so $U^\perp \subset V^*$. Then $(\theta^*)^{-1}(U^\perp) = \{w \in W : \theta(u, w) = 0 \text{ for all } u \in U\}$. To see this note that $(\theta^*)^{-1}(v^*) = w$ where $v^*(v) = \theta(v, w)$ for all $v \in V$. The result follows simply by writing down the definitions of the two sets. The stated results are immediate consequences. □

Definition 7.1.7 *Let V_1, V_2 be vector spaces and U_i subspaces of $\bigwedge^2 V_i$, $i = 1, 2$, respectively. Then the pairs (V_1, U_1) and (V_2, U_2) are defined to be* **equivalent**, *written*

$$(V_1, U_1) \sim (V_2, U_2),$$

if and only if there exists an isomorphism $\alpha : V_1 \to V_2$ with

$$\bigwedge^2 \alpha(U_1) = U_2.$$

With this definition in mind we give the following lemma.

Lemma 7.1.8 *Let V_i, W_i, $i = 1, 2$ be finite dimensional vector spaces with* $\dim W_1 = \dim W_2$. *Given linear maps*

$$f_j : W_j \longrightarrow \bigwedge^2 V_j, \qquad j = 1, 2,$$

the following are equivalent :

(1) There is a commutative diagram

$$\begin{array}{cccc} & W_1 & \xrightarrow{f_1} & \bigwedge^2 V_1 & \\ \alpha & \downarrow & & \downarrow & \bigwedge^2 \beta \, , \\ & W_2 & \xrightarrow{f_2} & \bigwedge^2 V_2 & \end{array}$$

where $\alpha : W_1 \to W_2$, $\beta : V_1 \to V_2$ are both isomorphisms. (In which case we can write $f_1 \sim f_2$.)

(2)

$$(V_1, im f_1) \sim (V_2, im f_2).$$

Proof The equivalence is clear. □

Note that the result shows that the only invariant of these maps $f : W \to \bigwedge^2 V$ are the pair $(V, \operatorname{im} f)$ and the dimension of W.

The result we require now follows.

Theorem 7.1.9 *Two subspaces U_1 and U_2 of $(\bigwedge^2 V)^*$ are equivalent if and only if the corresponding subspaces $\gamma^{-1}(U_1^\perp)$ and $\gamma^1(U_2^\perp)$ of $(\bigwedge^2 V^*)^*$ are equivalent.*

Proof Since γ is natural $\gamma^{-1}(U_1^\perp)$ and $\gamma^{-1}(U_2^\perp)$ are equivalent in $(\bigwedge^2 V^*)^*$ if and only if $U_1^\perp$ and $U_2^\perp$ are equivalent in $(\bigwedge^2 V)^{**} = (\bigwedge^2 V)$. But this is true if and only if U_1 and U_2 are equivalent in $(\bigwedge^2 V)^*$. This is because if $\bigwedge^2 \alpha$ takes $U_1^\perp$ to $U_2^\perp$ then $(\bigwedge^2 \alpha)^*$ takes U_2 to U_1. □

We deduce that if we have a classification of elements $A \in \hom(X, \bigwedge^2 V^*)$ when $\dim(\operatorname{im} A = r)$ then we have a similar classification of elements of $B \in \hom(Y, (\bigwedge^2 V^*)^*)$ where $\dim(\operatorname{im} B) = (N - r)$ where $N = n(n-1)/2$, where X, Y are finite dimensional vector spaces of equal dimension. Working with co-ordinates we see that once we have classified r-dimensional subspaces of $Sk(n, K)$ we have a classification of $(N - r)$-dimensional subspaces of $Sk(n, K)$.

We illustrate this with the following example where $K = \mathbb{C}$, $V = \mathbb{C}^4$.

Example 7.1.10 Consider the 1-dimensional subspace, U, of the 6-dimensional space $Sk(4, \mathbb{C})$, consisting of non-singular skew-symmetric matrices and represented by the normal form

$$A(x) = \begin{bmatrix} 0 & x & 0 & 0 \\ -x & 0 & 0 & 0 \\ 0 & 0 & 0 & x \\ 0 & 0 & -x & 0 \end{bmatrix},$$

with respect to some basis of $\mathbb{C}^4$. The corresponding dual 5-dimensional subspace of $Sk(4, \mathbb{C})$ is given by

$$\gamma(U^\perp) = \{B \in Sk(4, \mathbb{C}) : \operatorname{trace}(BA) = 0\}. \tag{7.2}$$

With respect to the dual basis of $(\mathbb{C}^4)^*$, a representative for an element of this space is given by

$$B = \begin{bmatrix} 0 & b_{12} & b_{13} & b_{14} \\ -b_{12} & 0 & b_{23} & b_{24} \\ -b_{13} & -b_{23} & 0 & b_{34} \\ -b_{14} & -b_{24} & -b_{34} & 0 \end{bmatrix},$$

where $b_{ij} \in \mathbb{C}$ satisfy $2x(b_{12} + b_{34}) = 0$, for all $x \in \mathbb{C}$. Hence the corresponding 'dual' space, for $A(x)$, is the 5-dimensional subspace of $Sk(4, \mathbb{C})$ consisting of skew-symmetric matrices of the form

$$\begin{bmatrix} 0 & -b_{34} & b_{13} & b_{14} \\ b_{34} & 0 & b_{23} & b_{24} \\ -b_{13} & -b_{23} & 0 & b_{34} \\ -b_{14} & -b_{24} & -b_{34} & 0 \end{bmatrix}.$$

For $s \geq 5$ this gives a 1-jet of an s-parameter family of 4×4 skew-symmetric matrices which has cojetrank 1.

7.2 Some Geometry

Consider the two normal forms for singular nets, found in Section 6.2.2,

$$A = \begin{bmatrix} 0 & z & x & 0 \\ z & 0 & y & 0 \\ -x & -y & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & 0 & x \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & z \\ -x & -y & -z & 0 \end{bmatrix}. \tag{7.3}$$

These represent 3-dimensional subspaces of the 6-dimensional space $Sk(4, \mathbb{C})$, and are clearly dual to each other. Both these 1-jets have the same $J^1\mathcal{G}$-codimension (namely 6) and a natural question is whether they are $J^1\mathcal{G}$-equivalent (or skew-equivalent). In fact we can distinguish between them as follows.

Following the above, we can think of A, B as 3-parameter families of skew-symmetric forms $V \times V \to \mathbb{C}$, where $\dim V = 4$. Consider some non-zero vector $v \in V$ such that

$$v^T A(\mathbf{x}) \equiv 0, \tag{7.4}$$

for all $\mathbf{x} \in \mathbb{C}^3$. (In other words, if $A(\mathbf{x})$ is a matrix representative for a family of skew-symmetric forms $\alpha_{\mathbf{x}} : V \times V \to \mathbb{C}$, then $\alpha_{\mathbf{x}}(v, w) = 0$ for all $w \in V$, and

$\mathbf{x} \in \mathbb{C}^3$.) Any net skew-equivalent to A is of the form

$$X^T(A \circ \phi)X,$$

for some $\phi \in Gl(3, \mathbb{C})$, $X \in Gl(4, \mathbb{C})$. Consider,

$$(X^{-1}v)^T X^T (A \circ \phi)X$$

noting that $X^{-1}v$ is a non-zero vector of V. It follows that

$$\begin{aligned} (X^{-1}v)^T X^T \left(A \circ \phi(x)\right) X &= v^T \left(A \circ \phi(x)\right) \\ &\equiv 0, \end{aligned}$$

from (7.4). So if there exists a $v \in V$ $(v \neq 0)$ such that $v^T A(\mathbf{x}) \equiv 0$ for all $\mathbf{x} \in \mathbb{C}^3$ then for some non-zero vector, $v_1 \in V$,

$$(v_1)^T X^T (A \circ \phi) X \equiv 0.$$

Returning to the two nets above, (7.3), by choosing

$$v = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

$v^T A \equiv 0$ for all $(x, y, z) \in \mathbb{C}^3$. For B to be skew-equivalent to A there must be a non-zero vector

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix},$$

such that

$$(v_1, v_2, v_3, v_4) \begin{bmatrix} 0 & 0 & 0 & x \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & z \\ -x & -y & -z & 0 \end{bmatrix} [-v_4x, -v_4y, -v_4z, v_1x + v_2y + v_3z],$$

is identically zero for all $(x, y, z) \in \mathbb{C}^3$. This is not the case and so A and B cannot be skew-equivalent, even though they are dual.

Consider the space $Sk(4, \mathbb{C})$ consisting of elements of the form

$$A = \begin{bmatrix} 0 & a_1 & a_2 & a_3 \\ -a_1 & 0 & a_4 & a_5 \\ -a_2 & -a_4 & 0 & a_6 \\ -a_3 & -a_5 & -a_6 & 0 \end{bmatrix},$$

where $a_i \in \mathbb{C}$. The set of singular skew-symmetric matrices, P, is given by the vanishing of Pfaffian $Pf(A) = 0$. Since $Pf(A)$ has degree 2 this set is a *quadric* of dimension 5 in $Sk(4, \mathbb{C})$. Alternatively, considering the associated projective space $P(4, \mathbb{C})$ of $Sk(4, \mathbb{C})$, denoted by $P\mathbb{C}^5$, the singular set is a quadric of dimension 4, which we shall denote by Q^4.

Since a net of skew-symmetric matrices is a plane in $P\mathbb{C}^5$ it follows that singular nets are 2-dimensional subspaces of $P\mathbb{C}^5$ contained in a quadric of dimension 4. Griffiths and Harris in their book, [GrifHar], Pg. 735 show that there are two 3-parameter families of planes in Q^4. We show how one family is parameterised in an natural way by $P\mathbb{C}^3$; the other family is obtained as their duals.

Lemma 7.2.1 *Let V be a finite dimensional vector space of dimension n. Consider any $v \in V \backslash \{0\}$. Define a linear map $\psi_v : Alt(V) \to V^*$ by*

$$\phi \mapsto \phi(v, -).$$

Then $im\psi_v = v^\perp$, where

$$v^\perp = \{v^* \in V^* : v^*(v) = 0\}.$$

Proof Since ϕ is skew-symmetric $\phi(v, v) = 0$, and therefore $\mathrm{im}\psi_v \subset v^\perp$.

We show the reverse inclusion by considering bases. Choose a basis, $\{e_1, \ldots, e_n\}$, of V with $v = e_n$. Clearly, $\{e_1^*, \ldots, e_{n-1}^*\}$ is a basis for $v^\perp = e_n^\perp$. Therefore to show $v^\perp \subset \mathrm{im}\psi_v$ it is sufficient to show

$$\{e_1^*, \ldots, e_{n-1}^*\} \subset \mathrm{im}\ \psi_v.$$

Define, for $1 \leq j \leq n-1$, $\phi_j \in Alt(V)$ such that, for $r < s$,

$$\phi_j(e_r, e_s) = \begin{cases} 1 & \text{if } (r, s) = (j, n) \\ 0 & \text{otherwise} \end{cases}.$$

So under ψ_v,

$$\phi_j \mapsto \phi_j(e_n, -) = -e_j^*, \qquad 1 \leq j \leq n-1,$$

which implies $\{e_1^*, \ldots, e_{n-1}^*\} \subset \mathrm{im}\psi_v$, as required. $\square$

Consider, $\ker \psi_v$. If $\phi \in \ker \psi_v$ then $\phi(v, w) = 0$ for all $w \in V$, i.e. if A is a matrix of ϕ, with respect to some basis of V,

$$v^T A w = 0,$$

for all $w \in V$. Choosing a basis for V with v as one of its elements, a row and column of the corresponding skew-symmetric matrix, A, is null and $\det A = 0$. So, for any $v \in V \backslash \{0\}$ there is a linear subspace, $\ker \psi_v$, of $Sk(n, \mathbb{C})$ contained in the set of singular skew-symmetric matrices (given by the vanishing of the Pfaffian). From the proof of Lemma 7.2.1, $\dim(\text{im } \psi_v) = \dim V - 1$ and so by *rank-nullity* this subspace has dimension,

$$\dim \ker \psi_v = \dim(Alt(V)) - (\dim V - 1). \tag{7.5}$$

Corresponding to the action of $Gl(n, \mathbb{C})$ on $A \in Sk(n, \mathbb{C})$ there is an automorphism of the vector space V. Hence given any $v \in V \backslash \{0\}$, by this automorphism we can suppose $v = e_n$ and

$$\ker \psi_{e_n} = \{\phi \in Alt(V) : \phi(e_n, -) = 0\}$$

gives a skew-equivalent normal form for all such subspaces $\ker \psi_v$ of $Sk(n, \mathbb{C})$.

This has a projective interpretation. It is easy to see that given two vectors v, $w \in V \backslash \{0\}$, that $\ker \psi_v = \ker \psi_w$ if and only if $v = \lambda w$, $\lambda \neq 0$. Considering the projective space $P(V)$, associated with V by identifying each $v \in V \backslash \{0\}$ with its scalar multiples, there is a $P(v)$-family of subspaces $\ker \psi_v$ contained in the singular set.

We finish by applying these results to the example introduced at the begining of this section.

Example 7.2.2 Consider any non-trivial vector $v \in V$, where $\dim V = 4$. The kernel of the map ψ_v is a 3-dimensional subspace of $Sk(4, \mathbb{C})$ (using (7.5)). Furthermore, this subspace is contained in the singular set of $Sk(4, \mathbb{C})$. We therefore have a 4-parameter family of singular nets. Corresponding to the action of $Gl(4, \mathbb{C})$ on $Sk(4, \mathbb{C})$ there is an automorphism of V and we can choose the vector v to be e_4. It follows that up to skew-equivalence this family of nets has the single normal form

$$A = \begin{bmatrix} 0 & z & x & 0 \\ z & 0 & y & 0 \\ -x & -y & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Projectively, this is a skew-equivalent normal form for all members of a 3-parameter family of nets contained in the quadric Q^4 (the singular set of $P(4,\mathbb{C})$).

Appendix A

An Alternative Approach

Although the following is of interest it is not necessary for the main purpose of the thesis.

In Section 2.4, of Chapter 2, we considered a pair of skew-symmetric matrices (A_1, A_2), for which A_2 is non-singular, as a λ-matrix $A = A_1 + \lambda A_2$. In so doing we used work on λ-matrices along with the result of Lemma 2.3.10 to derive a skew-equivalent normal form for this pair. Naturally we could also represent this pair by the λ-matrix $A_2 + \lambda A_1$ and we discuss this presently.

Consider the λ-matrix

$$\bar{A} = (A_2 + \lambda A_1),$$

and so (again using Lemma 1.1.8)

$$\det \bar{A} = \left(\bar{f}(\lambda)\right)^2,$$

for some $\bar{f}(\lambda)$. Here $\deg \bar{f} = \rho \leq r$ and $\bar{f}(0) \neq 0$. So over $\mathbb{C}$ we can write

$$\bar{f} = \prod_{i=1}^{q} (c_i\lambda + 1)^{s_{ir}},$$

there possibly being a c_i which is zero (if $\rho < r$). Hence using Theorem 2.2.4, the invariant factors of $\bar{A}$ are given by

$$\begin{array}{rcl} d_1 & = & \prod_{i=1}^{q}(c_i\lambda + 1)^{s_{i1}} \\ d_1 d_2 & = & \prod_{i=1}^{q}(c_i\lambda + 1)^{s_{i2}} \\ \vdots & & \vdots \\ d_1 d_2 \cdots d_r & = & \prod_{i=1}^{q}(c_i\lambda + 1)^{s_{ir}}, \end{array} \tag{A.1}$$

where for each $1 \le i \le q$, $s_{i1} \le s_{i2} \le \cdots \le s_{ir}$. By following a similar argument to that in Section 2.4 we can construct, for $A_2 + \lambda A_1$, the skew λ-equivalent normal form $\bar{N} = \bar{N}_1 + \lambda \bar{N}_2$ consisting, exclusively, of blocks of the type in (2.38) and represented by the shorthand :

$$\text{distinct}\left\{\begin{array}{lllll} c_1 & r_{11} & r_{12} & \cdots & r_{1n(1)} \\ \vdots & & & & \\ \vdots & & & & \\ c_q & r_{q1} & r_{q2} & \cdots & r_{qn(q)} \end{array}\right. ,$$

where for each $1 \le i \le q$, $r_{i1} \le r_{i2} \le \cdots \le r_{in(i)}$. Note, if $\deg \bar{f} < r$ there will be $n(\hat{\imath})$ blocks for which $c_{\hat{\imath}} = 0$ that are of the form

$$\underbrace{\left[\begin{array}{cc|cc|c|cc} 0 & 1 & & & & & \\ -1 & 0 & 1 & & & & \\ \hline & -1 & 0 & 1 & & & \\ & & -1 & 0 & 1 & & \\ \hline & & & -1 & & & \\ & & & & \ddots & & \\ & & & & & 1 & \\ \hline & & & & -1 & 0 & 1 \\ & & & & & -1 & 0 \end{array}\right]}_{2r_{\hat{\imath}j}} .$$

This normal form is more elegant than the one obtained for $A_1 + \lambda A_2$, but since A_1 is not necessarily non-singular we can no longer use Lemma 2.3.10 to equate the skew λ-equivalence of $A_2 + \lambda A_1$ and $\bar{N}_1 + \lambda \bar{N}_2$ with the skew-equivalence of the pairs they represent.

From this arises the question of whether the normal forms $N_1 + \lambda N_2$ and $\bar{N}_1 + \lambda \bar{N}_2$, of $A_1 + \lambda A_2$ and $A_2 + \lambda A_1$ respectively, are related. Since skew λ-equivalent matrices have the same invariants this can be investigated, to some extent, by comparing the invariants of $A_1 + \lambda A_2$ and $A_2 + \lambda A_1$. It seems sensible here to broaden our consideration of these invariants to those under λ-equivalence (see Section 2.3), rather than just skew λ-equivalence. This is discussed in the following section. However before doing so we give the following corollary.

Corollary A.0.3 *Consider the λ-matrices $A_1 + \lambda A_2$, $B_1 + \lambda B_2$. If A_2 is non-singular and $A_1 + \lambda A_2$ is λ-equivalent to $B_1 + \lambda B_2$ then $A_2 + \lambda A_1$ and $B_2 + \lambda B_1$ are λ-equivalent.*

Proof If $A_1+\lambda A_2$ and $B_1+\lambda B_2$ are λ-equivalent then by following a similar argument to that used in the proof of Lemma 2.3.10, here applying Theorem 2.3.9, it follows that the pairs (A_1, A_2) and (B_1, B_2) are equivalent. So we can write

$$B_2+\lambda B_1 = y(A_2+\lambda A_1)x,$$

for some invertible constant matrices x, y, and hence $B_2+\lambda B_1$ and $A_2+\lambda A_1$ are λ-equivalent. □

A.1 Comparing Invariants

From Lemma 2.2.3 and using the result of Lemma 2.1.16 invariants of λ-equivalent matrices are the *principal generators* of the ideals generated by their $r \times r$ minors. With this in mind, we need the following result.

Lemma A.1.1 *Denote an $r \times r$ minor of $(A_2+\lambda A_1)$ by*

$$P_r(\lambda) = \det(A_2+\lambda A_1)_r,$$

where $(A_2+\lambda A_1)_r$ is the associated sub-matrix of $A_2+\lambda A_1$. If the sub-matrix $(A_1+\lambda A_2)_r$ is obtained from $A_1+\lambda A_2$ by the same row and column deletions then the corresponding $r \times r$ minor of $A_1+\lambda A_2$ is given by

$$Q_r(\lambda) = \det(A_1+\lambda A_2)_r.$$

Then

$$Q_r(\lambda) = \lambda^r P_r(\lambda^{-1}). \tag{A.2}$$

Proof So

$$\begin{aligned}
Q_r(\lambda) &= \det(A_1+\lambda A_2)_r \\
&= \det(A_{1_r}+\lambda A_{2_r}) \\
&= \det \lambda(\lambda^{-1}A_{1_r}+A_{2_r}) \\
&= \lambda^r \det(\lambda^{-1}A_{1_r}+A_{2_r}) \\
&= \lambda^r \det(\lambda^{-1}A_1+A_2)_r \\
&= \lambda^r P_r(\lambda^{-1}),
\end{aligned}$$

as required. □

Note, from (A.2), by replacing λ with λ^{-1} we also find that

$$P_r(\lambda) = \lambda^r Q_r(\lambda^{-1}).$$

As previously mentioned our aim is to find a relationship between the invariants of $A_2 + \lambda A_1$ and those of $A_1 + \lambda A_2$. We can immediately use the result of Lemma A.1.1 to this effect for the last invariants.

The last invariant of $A_2 + \lambda A_1$ is $\det(A_2 + \lambda A_1) = P_n(\lambda)$. So, from (A.2), the relation between this and the last invariant, $\det(A_1 + \lambda A_2) = Q_n(\lambda)$, of $A_1 + \lambda A_2$ is given by

$$Q_n(\lambda) = \lambda^n P_n(\lambda^{-1}). \tag{A.3}$$

If A_2 is non-singular, $P_n(0) \neq 0$, and it follows from (A.3) that $Q_n(\lambda)$ has degree n as was shown in the proof of Lemma 2.3.10.

Given the list $f_1, \ldots, f_N$ of all $r \times r$ minors of $A_2 + \lambda A_1$, the associated invariant, g_r, of this matrix is given by

$$\langle g_r \rangle = \langle f_1, \ldots, f_N \rangle. \tag{A.4}$$

It follows from Lemma A.1.1 that we can write the ideal generated by the corresponding $r \times r$ minors of $A_1 + \lambda A_2$ as

$$\langle h_r \rangle = \langle \lambda^r f_1(\lambda^{-1}), \ldots, \lambda^r f_N(\lambda^{-1}) \rangle$$

with h_r the associated invariant. We look for some relation between g_r and h_r.

Definition A.1.2 *The function $\sigma : K[\lambda] \longrightarrow K[\lambda]$ is defined as follows. Given any polynomial $p \in K[\lambda]$, of degree d, $\sigma(p)$ is the polynomial*

$$\sigma(p) = \lambda^d p(\lambda^{-1}).$$

Lemma A.1.3 *Given a polynomial $p \in K[\lambda]$, the polynomial, $\sigma(p)$, defined above, has no zero roots.*

If $p \in \mathbb{C}[\lambda]$ the function, σ, in addition to killing off the zero root of p, sends each of its (distinct) non-zero roots, α_i, to the non-zero root $1/\alpha_i$ of $\sigma(p)$, of the same multiplicity. As a result, the number of roots of $\sigma(p)$ is equal to the number of non-zero roots of p.

Proof Writing

$$p(\lambda) = p_0 + p_1\lambda + \cdots + p_{d-1}\lambda^{d-1} + p_d\lambda^d,$$

where each $p_i \in K$, if p has degree d then $p_d \neq 0$. So since

$$\begin{aligned}\sigma(p) &= \lambda^d(p_0 + p_1\lambda^{-1} + \cdots + p_{d-1}\lambda^{1-d} + p_d\lambda^{-d}) \\ &= p_d + p_{d-1}\lambda + \cdots + p_1\lambda^{d-1} + p_0\lambda^d\end{aligned}$$

it can be seen that $\sigma(p)$ has no zero roots.

If $p \in \mathbb{C}[\lambda]$ it can be expressed in the form

$$p(\lambda) = c\lambda^k \prod_{i=1}^{q}(\lambda - \alpha_i)^{m_i} \qquad (k \geq 0),$$

where q is the number of non-zero roots. So

$$\begin{aligned}\sigma(p) &= \lambda^{(k+\sum_{i=1}^{q} m_i)} c\lambda^{-k} \prod_{i=1}^{q}(\lambda^{-1} - \alpha_i)^{m_i} \\ &= c\lambda^{\sum_{i=1}^{q} m_i} \prod_{i=1}^{q}(\lambda^{-1} - \alpha_i)^{m_i} \\ &= c\prod_{i=1}^{q}(1 - \alpha_i\lambda)^{m_i}. \qquad \text{(A.5)}\end{aligned}$$

Note that, irrespective of its multiplicity, any zero root of p is killed off by σ. Furthermore it can be seen, from (A.5), that for each non-zero root, α_i, of p there is a non-zero root, $1/\alpha_i$ in $\sigma(p)$, of the same multiplicity. Hence the number of roots of $\sigma(p)$ is the same as the number of non-zero roots of p. □

Lemma A.1.4 *Let g_r and h_r be the principal generators of the ideals generated by the $r \times r$ minors of the λ-matrices $A_2 + \lambda A_1$ and $A_1 + \lambda A_2$, respectively. Then it follows that*

$$\sigma(g_r) | h_r$$

and

$$\sigma(h_r) | g_r.$$

Furthermore, over the polynomial ring $\mathbb{C}[\lambda]$, we can write

$$g_r = \lambda^{u_r}\sigma(h_r) \qquad \text{(A.6)}$$

and

$$h_r = \lambda^{v_r} \sigma(g_r), \tag{A.7}$$

there being a one to one correspondence between the non-zero roots, α_i*, of* g_r *and the non-zero roots,* $1/\alpha_i$*, of* h_r.

Proof Adopting the notation in (A.4), above,

$$I_r(A_2 + \lambda A_1) = \langle g_r \rangle = \langle f_1, \ldots, f_N \rangle, \tag{A.8}$$

from which we recall, using Lemma A.1.1, that

$$I_r(A_1 + \lambda A_2) = \langle h_r \rangle = \langle \lambda^r f_1(\lambda^{-1}), \ldots, \lambda^r f_N(\lambda^{-1}) \rangle. \tag{A.9}$$

From (A.8), for each $1 \leq i \leq N$,

$$f_i = g_r q_i,$$

for some $q_i \in K[\lambda]$. Let $\deg g_r = s$. Then

$$\deg q_i = \deg f_i - \deg g_r = \deg f_i - s,$$

and since $\deg f_i \leq r$ it follows that

$$\deg q_i \leq r - s.$$

So

$$\begin{aligned} \lambda^r f_i(\lambda^{-1}) &= \lambda^r g_r(\lambda^{-1}) q_i(\lambda^{-1}) \\ &= \lambda^s g_r(\lambda^{-1}) \lambda^{r-s} q_i(\lambda^{-1}). \end{aligned} \tag{A.10}$$

Since $\deg q_i \leq r - s$ then $\lambda^{r-s} q_i(\lambda^{-1})$ is a polynomial and so, for each $1 \leq i \leq N$, $\lambda^s g_r(\lambda^{-1}) = \sigma(g_r)$ divides $\lambda^r f_i(\lambda^{-1})$. From (A.9) this implies that $\sigma(g_r)$ divides h_r.

Alternatively, writing

$$I_r(A_1 + \lambda A_2) = \langle h_r \rangle = \langle H_1, \ldots, H_N \rangle, \tag{A.11}$$

then by the note after Lemma A.1.1

$$I_r(A_2 + \lambda A_1) = \langle g_r \rangle = \langle \lambda^r H_1(\lambda^{-1}), \ldots, \lambda^r H_N(\lambda^{-1}) \rangle. \tag{A.12}$$

Letting $\deg h_r = t$, then by applying to (A.11) the argument used above on (A.8) it can be shown, for each $1 \leq i \leq N$, that $\sigma(h_r) = \lambda^t h_r(\lambda^{-1})$ divides $\lambda^r H_i(\lambda^{-1})$. Then from (A.12) it follows that $\sigma(h_r)$ divides g_r.

We consider the case when $K = \mathbb{C}$ and write

$$g_r = \sigma(h_r)K_1(\lambda), \tag{A.13}$$

$$h_r = \sigma(g_r)K_2(\lambda), \tag{A.14}$$

where g_r, h_r, $\sigma(g_r)$, $\sigma(h_r)$, K_1 and K_2 are polynomials in $\mathbb{C}[\lambda]$. Let m_1, m_2 be the number of non-zero roots of g_r and h_r respectively and n_1, n_2 be the number of non-zero roots of $K_1(\lambda)$ and $K_2(\lambda)$ respectively. Then, by Lemma A.1.3 the number of (non-zero) roots of $\sigma(h_r)$ is equal to the number, m_2, of non-zero roots of h_r. So from (A.13) we find

$$n_1 = m_1 - m_2. \tag{A.15}$$

Similarly, since the number of non-zero roots of $\sigma(g_r)$ and g_r are the same it follows from (A.14) that

$$n_2 = m_2 - m_1. \tag{A.16}$$

It follows from (A.15) and (A.16) that the only non-negative solution for n_1 and n_2 is $n_1 = n_2 = 0$. So both $K_1(\lambda)$ and $K_2(\lambda)$ have no non-zero roots. So

$$g_r = \lambda^{u_r}\sigma(h_r), \tag{A.17}$$

$$h_r = \lambda^{v_r}\sigma(g_r), \tag{A.18}$$

Note that since we are dealing with principal generators we can neglect any constant multiples.

1). From Lemma A.1.3 $\sigma(h_r)$, has no zero roots and so, over $\mathbb{C}$, we can write

$$\sigma(h_r) = \prod_{i=1}^{q}(\lambda - \alpha_i)^{m_i}.$$

Hence from (A.17) we can write

$$g_r = \lambda^{u_r}\prod_{i=1}^{q}(\lambda - \alpha_i)^{m_i},$$

where α_i are the distinct non-zero roots of g_r. Substituting this expression into (A.18) and using Lemma A.1.3 we get

$$h_r = \lambda^{v_r} \prod_{i=1}^{q} (1 - \alpha_i \lambda)^{m_i},$$

and the one to one correspondence between the non-zero roots, α_i, of g_r and the non-zero roots, $1/\alpha_i$, of h_r is clear.

It is the second of these equations, (A.7), which is needed for our purposes. Particularly we would like to determine the unknown power, v_r. However before considering this, we show that for some values of r the powers u_r and v_r vanish.

Corollary A.1.5 *Over the polynomial ring* $\mathbb{C}[\lambda]$ *let*

$$I_r(A_2 + \lambda A_1) = \langle g_r \rangle \qquad \textit{and} \qquad I_r(A_1 + \lambda A_2) = \langle h_r \rangle,$$

for some $1 \leq r \leq n$. *By Lemma A.1.4*

$$g_r = \lambda^{u_r} \sigma(h_r),$$

$$h_r = \lambda^{v_r} \sigma(g_r).$$

If $\mathrm{rank} A_2 \geq r$ *then* $u_r = 0$ *and*

$$g_r = \sigma(h_r).$$

Similarly, provided $\mathrm{rank} A_1 \geq r$, *it can be shown that* $v_r = 0$ *and*

$$h_r = \sigma(g_r).$$

Proof Recall from (A.8), in the proof of Lemma A.1.4, that

$$I_r(A_2 + \lambda A_1) = \langle f_1(\lambda), \ldots, f_N(\lambda) \rangle = \langle g_r \rangle,$$

for some $1 \leq r \leq n$. If $\mathrm{rank} A_2 \geq r$ then some $r \times r$ minor of A_2 is non-zero. If the minor of $A_2 + \lambda A_1$, corresponding to the same row and column deletions, is $f_\gamma(\lambda) = \det(A_2 + \lambda A_1)_r$, it follows that $f_\gamma(0) \neq 0$. Since $g_r = \gcd(f_1, \ldots f_N)$, see Lemma 2.1.16, this implies that $g_r(0) \neq 0$, hence $u_r = 0$ in (A.6) and

$$g_r(\lambda) = \sigma(h_r).$$

From (A.11), in the proof of Lemma A.1.4,

$$I_r(A_1 + \lambda A_2) = \langle h_r \rangle = \langle H_1, \ldots, H_N \rangle.$$

If rank$A_1 \geq r$ then by applying, to this, the same argument we can find some $r \times r$ minor $H_\delta(\lambda)$ of $A_1 + \lambda A_2$ for which $H_\delta \neq 0$. This implies that $h_r(0) \neq 0$, so $v_r = 0$ in (A.7) and

$$h_r = \sigma(g_r).$$

□

For the remaining values of r we would like to determine u_r and v_r, particularly the latter since for the purpose discussed in the preamble we need relation (A.7).

However before considering this we can use Corollary A.1.5 to establish the result of Corollary A.0.3 directly, without having to use Theorem 2.3.9.

Lemma A.1.6 *If A_2 is non-singular and the matrices $A_1 + \lambda A_2$ and $B_1 + \lambda B_2$ are λ-equivalent then $A_2 + \lambda A_1$ is λ-equivalent to $B_2 + \lambda B_1$.*

Proof Let the sequence of invariants of $A_1 + \lambda A_2$ be $\{h_1, \ldots, h_n\}$. Since A_2 is non-singular rank$A_2 = n$ and so for each invariant, h_r, where $1 \leq r \leq n$, $r \leq$ rankA_2. By Corollary A.1.5 we deduce that the corresponding invariants, g_r, of $A_2 + \lambda A_1$ are given by

$$g_r = \sigma(h_r).$$

If $B_1 + \lambda B_2$ is λ-equivalent to $A_1 + \lambda A_2$ then by Lemma 2.2.3 it has, up to constant multiples, the same sequence of invariants. In particular, the last invariants of each, $\det(B_1 + \lambda B_2)$ and $\det(A_1 + \lambda A_2)$, differ by a constant and as A_2 is non-singular it follows, by the argument used in the proof of Lemma 2.3.10, that B_2 is also non-singular. So rank$B_2 = n$ and for each invariant, h_r, of $B_1 + \lambda B_2$, $r \leq$ rankB_2. Applying Corollary A.1.5 we deduce that, for each $1 \leq r \leq n$, the corresponding invariants, g'_r, of $B_2 + \lambda B_1$ are given by

$$g'_r = \sigma(h_r).$$

Therefore the sequence of invariants of $A_2 + \lambda A_1$ and $B_2 + \lambda B_1$ are equal and by Lemma 2.2.3 it follows that $A_2 + \lambda A_1$ and $B_2 + \lambda B_1$ are λ-equivalent as required. □

We return to the consideration of

$$h_r = \lambda^{v_r}\sigma(g_r),$$

with a view to determining the power v_r.

Lemma A.1.7 *Given the λ-matrix $A_1 + \lambda A_2$, the rank, k, of A_1 is invariant under λ-equivalence. Furthermore reducing $A_1 + \lambda A_2$ to the λ-equivalent matrix*

$$\tilde{A}_1 + \lambda\tilde{A}_2 = \left[\begin{array}{c|c} I_k & 0 \\ \hline 0 & 0 \end{array}\right] + \lambda\left[\begin{array}{c|c} A_{21} & A_{22} \\ \hline A_{23} & A_{24} \end{array}\right],$$

we denote the rank of the $(n-k)\times(n-k)$ sub-matrix A_{24} by s.

Then for $1 \le r \le k+s$ the invariant, h_r, of $A_1 + \lambda A_2$ can be expressed, in terms of the invariant, g_r, of $A_2 + \lambda A_1$, as follows

$$h_r = \begin{cases} \sigma(g_r) & \text{for } 1 \le r \le k \\ \lambda^{r-k}\sigma(g_r) & \text{for } k < r \le k+s \end{cases}.$$

However, as yet, for $k + s < r \le n$, it can only be determined that $v_r > r - k$.

Proof Firstly, given any matrix $B_1 + \lambda B_2$ which is λ-equivalent to $A_1 + \lambda A_2$ then for the invertible matrices $X(\lambda)$, $Y(\lambda)$

$$B_1 + \lambda B_2 = Y(\lambda)(A_1 + \lambda A_2)X(\lambda).$$

Setting $\lambda = 0$ it follows that $B_1 = Y(0)A_1X(0)$ and hence $\text{rank}B_1 = \text{rank}A_1$. So $\text{rank}A_1 = k$ is invariant under λ-equivalence.

By a series of elementary row and column operations (over K) on $A_1 + \lambda A_2$ we obtain the equivalent matrix

$$\tilde{A}_1 + \lambda\tilde{A}_2 = \left[\begin{array}{c|c} I_k & 0 \\ \hline 0 & 0 \end{array}\right] + \lambda\left[\begin{array}{c|c} A_{21} & A_{22} \\ \hline A_{23} & A_{24} \end{array}\right]. \tag{A.19}$$

For $r \le k$ it follows from Corollary A.1.5 that

$$h_r(\lambda) = \sigma(g_r).$$

If $r > k$ then the gcd, h_r, of all the $r \times r$ minors of (A.19) is given by

$$h_r = \lambda^{v_r}\sigma(g_r),$$

where $v_r > 0$ and $\sigma(g_r)(0) \neq 0$. So the factor, λ^{v_r}, is the highest power of λ which divides all these $r \times r$ minors. Hence from these minors, those with the minimal power of λ as a factor will yield v_r.

Minors, with the *lowest possible* multiple of λ as a factor, are those obtained from sub-matrices, of (A.19), containing the $k \times k$ sub-matrix $I_k + \lambda A_{21}$. The remaining $r - k$ rows and columns of each such sub-matrix are obtained by deleting $n - r$ rows and columns from the last $n - k$ rows and columns of $\tilde{A}_1 + \lambda \tilde{A}_2$. These $r \times r$ sub-matrices are of the following form,

$$\left[\begin{array}{cccc|c} 1+\lambda\alpha_{11} & & & & \\ & 1+\lambda\alpha_{22} & & & \\ & & \ddots & & \\ & & & 1+\lambda\alpha_{kk} & \\ \hline & & & & M \end{array}\right], \tag{A.20}$$

where all non-specified entries are some constant multiple of λ (possibly zero) and M is some $(r-k) \times (r-k)$ sub-matrix of A_{24}. It follows that, provided $\det M \neq 0$, the corresponding minor of $\tilde{A}_1 + \lambda \tilde{A}_2$ is

$$\lambda^{r-k}(1 + O(1)),$$

where $O(1)$ is a λ-polynomial consisting of linear and higher order terms. So as long as one $(r-k)$-minor of A_{24} is non-zero we have a minor of $\tilde{A}_1 + \lambda \tilde{A}_2$ whose multiple of λ is the lowest possible, i.e. $r - k$. Since this power is therefore the minimal power of λ of any minor of $\tilde{A}_1 + \lambda \tilde{A}_2$, we deduce that if $\operatorname{rank} A_{24} \geq r - k$, then $v = r - k$ and

$$h_r = \lambda^{r-k}\sigma(g_r).$$

If however $\operatorname{rank} A_{24} < r - k$ then all $(r-k) \times (r-k)$ minors of A_{24} are zero, and no minor of $\tilde{A}_1 + \lambda \tilde{A}_2$ can have, as a factor, a multiple of λ with degree $r - k$. So $v_r > r - k$. □

Having established this, we ask whether the reverse of Lemma A.1.6 is also true.

Conjecture A.1.8 *Given that A_2 is non-singular and the matrices $A_2 + \lambda A_1$ and $B_2 + \lambda B_1$ are λ-equivalent then does this imply that $A_1 + \lambda A_2$ is λ-equivalent to $B_1 + \lambda B_2$?*

If $A_2 + \lambda A_1$ is λ-equivalent to $B_2 + \lambda B_1$ then by Lemma 2.2.3 they both have the same sequence of invariants, $\{g_1, \ldots, g_n\}$. We would like to compare the sequence of invariants,

$$\{h_1, \ldots, h_n\},$$

of $A_1 + \lambda A_2$ and the sequence of invariants,

$$\{h'_1, \ldots, h'_n\},$$

of $B_1 + \lambda B_2$. To do this we use the results of the previous lemma. First note, since A_2 is non-singular the λ-equivalence of $A_2 + \lambda A_1$ and $B_2 + \lambda B_1$ implies that B_2 is also non-singular. However, from this equivalence there is no reason why the ranks of A_1 and B_1 should be equal. Let rank$A_1 = k_1$ and rank$B_1 = k_2$. As described in Lemma A.1.7 above, we can reduce $A_1 + \lambda A_2$ and $B_1 + \lambda B_2$ to $\tilde{A}_1 + \lambda \tilde{A}_2$ and $\tilde{B}_1 + \lambda \tilde{B}_2$ respectively, where

$$\tilde{A}_1 + \lambda \tilde{A}_2 = \left[\begin{array}{c|c} I_{k_1} & 0 \\ \hline 0 & 0 \end{array}\right] + \lambda \left[\begin{array}{c|c} A_{21} & A_{22} \\ \hline A_{23} & A_{24} \end{array}\right]$$

and

$$\tilde{B}_1 + \lambda \tilde{B}_2 = \left[\begin{array}{c|c} I_{k_2} & 0 \\ \hline 0 & 0 \end{array}\right] + \lambda \left[\begin{array}{c|c} B_{21} & B_{22} \\ \hline B_{23} & B_{24} \end{array}\right].$$

If rank$A_{24} = \alpha$ and rank$B_{24} = \beta$, then, by this lemma, the invariants, $\{h_1, \ldots, h_n\}$, of $A_1 + \lambda A_2$ are

$$\{\sigma(g_1), \ldots, \sigma(g_{k_1}), \lambda\sigma(g_{k_1+1}), \ldots, \lambda^{\alpha}\sigma(g_{k_1+\alpha}), \lambda^{v_{k_1+\alpha+1}}\sigma(g_{k_1+\alpha+1}), \ldots, \lambda^{v_n}\sigma(g_n)\}$$

and the invariants, $\{h'_1, \ldots, h'_n\}$, of $B_1 + \lambda B_2$ are

$$\{\sigma(g_1), \ldots, \sigma(g_{k_2}), \lambda\sigma(g_{k_2+1}), \ldots, \lambda^{\beta}\sigma(g_{k_2+\beta}), \lambda^{w_{k_2+\beta+1}}\sigma(g_{k_2+\beta+1}), \ldots, \lambda^{w_n}\sigma(g_n)\}.$$

So the sequence of invariants for $A_1 + \lambda A_2$ and $B_1 + \lambda B_2$ are equal only if all of the following three conditions are satisfied.

(i) $k_1 = k_2$,

(ii) $\alpha = \beta$,

(iii) $v_i = w_i$ for $k_1 + \alpha + 1 \leq i \leq n$.

Hence, by Lemma 2.2.3, the conjecture that $A_1 + \lambda A_2$ is λ-equivalent to $B_1 + \lambda B_2$ also depends on these conditions. Clearly this conjecture generally fails as shown by the following counter example.

Consider the linear λ-matrix

$$A_2 + \lambda A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \lambda \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Premultiplying this by the invertible λ-matrix,

$$\begin{bmatrix} 1+\lambda & 1 \\ \lambda & 1 \end{bmatrix},$$

it follows that

$$\begin{aligned} \begin{bmatrix} 1+\lambda & 1 \\ \lambda & 1 \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \lambda \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \\ &= B_2 + \lambda B_1 \end{aligned}$$

is λ-equivalent to $A_2 + \lambda A_1$.

However,

$$A_1 + \lambda A_2 = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

has the sequence of invariants $\{\lambda, \lambda^2\}$, whereas the sequence of invariants of

$$B_1 + \lambda B_2 = \begin{bmatrix} 1+\lambda & \lambda \\ 1 & \lambda \end{bmatrix}$$

is $\{1, \lambda^2\}$. Since the first invariants, of each, are different $A_1 + \lambda A_2$ and $B_1 + \lambda B_2$ are not λ-equivalent. Note that rank$A_1 = 0$, whereas rank$B_1 = 1$ and hence this is an example of $k_1 \neq k_2$.

It therefore appears that the method described in Section 2.4 is the most suitable for classifying pairs of skew-symmetric matrices, where at least one is non-singular.

Appendix B

Justification for $\mathcal{G}$ being one of Damon's geometric subgroups of $\mathcal{K}$

It is required to establish that our group $\mathcal{G}$ is one of Damon's geometric subgroups of $\mathcal{K}$. Having shown that it is a subgroup of $\mathcal{K}$ (see Lemma 4.2.5) we need to show that it satisfies the four conditions given in [Damon] Pgs. 40–42, labelled by him : naturality, tangent space structure, exponential map and filtration condition. It is easy to see that if we establish the relevant conditions for the action of $\mathcal{H}$ then they will also hold for $\mathcal{G} = \mathcal{R} \times \mathcal{H}$. So to simplify matters we consider the subgroup $\mathcal{H}$ of $\mathcal{C}$ consisting of smooth germs $\mathbb{C}^r, 0 \to Gl(n, \mathbb{C})$ acting on the space of germs

$$\mathcal{S}k_e = \{A : \mathbb{C}^r, 0 \to Sk(n, \mathbb{C})\},$$

with the action $X(x).A(x) = X^T(x)A(x)X(x)$.

In fact using Proposition 4.3.5 we can assume all our germs vanish at $0 \in \mathbb{C}^r$, i.e. we consider the action of $\mathcal{H}$ on the space of germs

$$\mathcal{S}k = \{A : \mathbb{C}^r, 0 \to Sk(n, \mathbb{C}), 0\}.$$

The subgroup $\mathcal{C}$ of $\mathcal{K}$ just consists of pairs of germs (id, ϕ) with $id : \mathbb{C}^r, 0 \to \mathbb{C}^r, 0$ the identity, $\phi : \mathbb{C}^r \times Sk(n, \mathbb{C}), (0,0) \to Sk(n, \mathbb{C}), 0$, $\phi(x, 0) = 0$ for all x and $\phi(0, -) : Sk(n, \mathbb{C}), 0 \to Sk(n, \mathbb{C}), 0$ the germ of a diffeomorphism. This ϕ can roughly be thought of as a family $\tilde{\phi} : \mathbb{C}^r, 0 \to Diff(Sk(n, \mathbb{C}))$ (where

$Diff\,(Sk(n,\mathbb{C}))$ is the group of diffeomorphisms $Sk(n,\mathbb{C}),0 \to Sk(n,\mathbb{C}),0)$. Our group $\mathcal{H}$ merely corresponds to the subgroup of C where the target is $Gl(n,\mathbb{C})$, a subgroup of $Diff\,(Sk(n,\mathbb{C}))$.

In the following we demonstrate that $\mathcal{H}$ satisfies each of Damon's requirements of a geometric subgroup.

1. **Naturality** An unfolding of a germ $A : \mathbb{C}^r,0 \to Sk(n,\mathbb{C}),0$ is a germ $\overline{\bar{A}} : \mathbb{C}^r \times \mathbb{C}^q,(0,0) \to Sk(n,\mathbb{C}) \times \mathbb{C}^q,(0,0)$ of the form

$$\overline{\bar{A}}(x,u) = \left(\bar{A}(x,u),u\right),$$

with $\bar{A}(x,0) = A(x)$ (see Definition 4.4.25). The set of unfoldings of germs in Sk is the translate of the linear space Sk_{un},

$$Sk_{un} = \{\bar{A} : \mathbb{C}^r \times \mathbb{C}^q,(0,0) \to Sk(n,\mathbb{C}),0\},$$

by the addition of the projection $\pi(x,u) = u$.

Given a smooth mapping $\lambda : \mathbb{C}^p,0 \to \mathbb{C}^q,0$ then the *pull-back* of $\overline{\bar{A}}$, $\lambda^*\overline{\bar{A}}$, is defined by $\lambda^*\overline{\bar{A}}(x,v) = \left(\bar{A}(x,\lambda(v)),v\right)$, for $x \in \mathbb{C}^r$, $v \in \mathbb{C}^p$ (compare with Definition 4.4.27).

Using notation corresponding to that used in [Damon] Pgs. 4–5,

$$\begin{aligned}\mathcal{H}_{un}(q) \quad = \quad & \Big\{\overline{\bar{X}} : \mathbb{C}^r \times \mathbb{C}^q,(0,0) \to Gl(n,\mathbb{C}) \times \mathbb{C}^q : \overline{\bar{X}} \text{ is} \\ & \quad \text{an unfolding of } X \in \mathcal{H}\Big\},\end{aligned}$$

with $\overline{\bar{X}}(x,u) = \left(\bar{X}(x,u),u\right) \in Gl(n,\mathbb{C}) \times \mathbb{C}^q$.

If $\overline{\bar{X}} \in \mathcal{H}_{un}(q)$ for naturality under pull-back (see [Damon] Pgs. 5 and 40) we need $\lambda^*\overline{\bar{X}}$, given by $\lambda^*\overline{\bar{X}}(x,v) = \left(\bar{X}(x,\lambda(v)),v\right)$, to lie in $\mathcal{H}_{un}(p)$. This is obviously true.

2. **Algebraic Structure** (of the tangent spaces) These conditions are relatively trivial since all tangent spaces are $\mathcal{O}_r$-modules (i.e. modules over the ring of function germs in the source).

Here we have our group $\mathcal{H}$ acting on the space Sk, defined above.

Proposition B.0.9 *Consider the spaces*

$$\mathcal{S}k_{un,e}(q) = \{\bar{A} : \mathbb{C}^r \times \mathbb{C}^q, (0,0) \to Sk(n,\mathbb{C})\},$$

$$\left(\mathcal{S}k_{un}(q) = \{\bar{A} : \mathbb{C}^r \times \mathbb{C}^q, (0,0) \to Sk(n,\mathbb{C}), 0\}\right)$$

and

$$\begin{aligned}\mathcal{H}_{un,e}(q) &= \mathcal{H}_{un}(q) \\ &= \left\{\overline{X} : \mathbb{C}^r \times \mathbb{C}^q, (0,0) \to Gl(n,\mathbb{C}) \times \mathbb{C}^q\right\}.\end{aligned}$$

Using Damon's notation (see [Damon] Pgs. 2–3),

$$T\mathcal{S}k_{un,e}(q) = \{\bar{A} : \mathbb{C}^r \times \mathbb{C}^q, (0,0) \to Sk(n,\mathbb{C})\},$$

$$\left(T\mathcal{S}k_{un}(q) = \{\bar{A} : \mathbb{C}^r \times \mathbb{C}^q, (0,0) \to Sk(n,\mathbb{C}), 0\}\right)$$

and

$$T\mathcal{H}_{un,e}(q) = T\mathcal{H}_{un}(q) = \mathcal{N}_{un}(q),$$

where

$$\mathcal{N}_{un}(q) = \{\bar{Y} : \mathbb{C}^r \times \mathbb{C}^q, (0,0) \to M(n,\mathbb{C})\}.$$

Proof Firstly, consider a 1-parameter family in $\mathcal{S}k_{un,e}(q)$:

$$\tilde{A} : \mathbb{C}^r \times \mathbb{C}^q \times \mathbb{C}, (0,0,0) \to Sk(n,\mathbb{C}),$$

with members $\tilde{A}(x,u,t)$.

(The corresponding 1-parameter family in $\mathcal{S}k_{un}(q)$ is given by germs $\tilde{A}$: $\mathbb{C}^r \times \mathbb{C}^q \times \mathbb{C}, (0,0,0) \to Sk(n,\mathbb{C}), 0$ with $\tilde{A}(0,0,t) \equiv 0$.)

Now the tangent vector to $\tilde{A}(x,u,t) \in \mathcal{S}k_{un,e}(q)$ at $\tilde{A}(x,u,0)$,

$$\frac{\partial \tilde{A}}{\partial t}(x,u,0) : \mathbb{C}^r \times \mathbb{C}^q, (0,0) \to Sk(n,\mathbb{C}),$$

is clearly in $\mathcal{S}k_{un,e}(q)$ which implies that $T\mathcal{S}k_{un,e}(q) \subset \mathcal{S}k_{un,e}(q)$.

(The tangent vector to $\tilde{A}(x,u,t) \in \mathcal{S}k_{un}(q)$ at $\tilde{A}(x,u,0)$, given by

$$\frac{\partial \tilde{A}}{\partial t}(x,u,0) : \mathbb{C}^r \times \mathbb{C}^q, (0,0) \to Sk(n,\mathbb{C}),$$

vanishes at $(0,0)$, since $\tilde{A}(0,0,t) \equiv 0$, and hence $\partial\tilde{A}/\partial t(x,u,0) \in \mathcal{S}k_{un}(q)$.)

Conversely, let $\bar{B}(x,u) : \mathbb{C}^r \times \mathbb{C}^q, (0,0) \to Sk(n,\mathbb{C})$ be any element of $\mathcal{S}k_{un,e}(q)$. Considering the path $\bar{A}(x,u) + t\bar{B}(x,u) \in \mathcal{S}k_{un,e}(q)$, and differentiating with respect to t, it follows that $\bar{B}(x,u) \in T\mathcal{S}k_{un,e}(q)$. Hence $\mathcal{S}k_{un,e}(q) = T\mathcal{S}k_{un,e}(q)$ as required.

(By a similar argument if $\bar{B}(x,u) : \mathbb{C}^r \times \mathbb{C}^q, (0,0) \to Sk(n,\mathbb{C}), 0$ is a general element of $\mathcal{S}k_{un}(q)$ we can also show that $\mathcal{S}k_{un}(q) \subset T\mathcal{S}k_{un}(q)$.)

Secondly, for $\mathcal{H}_{un,e}(q)$ consider the 1-parameter family

$$\tilde{X} : \mathbb{C}^r \times \mathbb{C}^q \times \mathbb{C}, (0,0,0) \to Gl(n,\mathbb{C}),$$

i.e. $\tilde{X}(x,u,t) \in Gl(n,\mathbb{C})$ for x, u, t in a neighbourhood of $(0,0,0)$.

Then the tangent vector to this path, at $\tilde{X}(x,u,0)$, is

$$\frac{\partial\tilde{X}}{\partial t}(x,u,0) : \mathbb{C}^r \times \mathbb{C}^q, (0,0) \to M(n,\mathbb{C})$$

and $T\mathcal{H}_{un,e}(q) \subset \mathcal{N}_{un}(q)$.

Conversely if $\bar{Y} : \mathbb{C}^r \times \mathbb{C}^q, (0,0) \to M(n,\mathbb{C})$ is a general element of $\mathcal{N}_{un}(q)$ then by considering the 1-parameter family

$$\bar{X}(x,u) + t\bar{Y}(x,u),$$

the elements of which, for x, u, t in a neighbourhood of $(0,0,0)$, are in $Gl(n,\mathbb{C})$, (since $\bar{X}(x,u) : \mathbb{C}^r \times \mathbb{C}^q, (0,0) \to Gl(n,\mathbb{C})$ then for t small, and x, u in a neighbourhood of $(0,0)$, $\bar{X}(x,u) + t\bar{Y}(x,u) \in Gl(n,\mathbb{C})$) it follows that $\mathcal{N}_{un}(q) \subset T\mathcal{H}_{un,e}(q)$. $\square$

We now demonstrate these tangent spaces satisfy the algebraic structure given in [Damon], Pg. 41.

Firstly, clearly $T\mathcal{S}k_{un,e}(q)$ is a finitely generated $\mathcal{O}_{r,q}$-module, $\mathcal{O}_{r,q}\langle E^{ij} : 1 \leq i < j \leq n\rangle$, (where E^{ij} are the standard basis vectors for $Sk(n,\mathbb{C})$, see Section 1.1) which clearly contains $T\mathcal{S}k_e$ as an $\mathcal{O}_r$-module, by setting $q = 0$.

Note that TSk_{un} is also a finitely generated $\mathcal{O}_{r,q}$-module generated by the set

$$\{x_k E^{ij}, u_l E^{ij} : 1 \le i < j \le n, 1 \le k \le r, 1 \le l \le q\}.$$

The space $T\mathcal{H}_{un,e}(q)$ is the finitely generated $\mathcal{O}_{r,q}$-module $\mathcal{O}_{r,q}\langle E_{ij} : 1 \le i,j \le n\rangle$, (where E_{ij} are the standard basis vectors for $M(n,\mathbb{C})$, see proof of Proposition 4.3.2) which contains $T\mathcal{H}_e$ as an $\mathcal{O}_r$-module.

Given $\bar{A} \in \mathcal{S}k_{un}$ we define the orbit map

$$\sigma_{\bar{A}} : \mathcal{H}_{un}(q) \to \mathcal{S}k_{un}(q)$$

$$\bar{X}(x,u) \mapsto \bar{X}(x,u).\bar{A} = \bar{X}^T(x,u)\bar{A}(x,u)\bar{X}(x,u),$$

with derivative

$$d\sigma_{\bar{A}} : \mathcal{N}_{un}(q) \to \mathcal{S}k_{un,e}(q)$$

$$\bar{Y}(x,u) \mapsto \bar{Y}^T\bar{A}(x,u) + \bar{A}(x,u)\bar{Y}(x,u).$$

(This derivative is found by considering the action of paths $I + t\bar{Y}(x,u) \in \mathcal{H}_{un,e}(q)$ on $\bar{A}(x,u) \in \mathcal{S}k_{un,e}(q)$ in the same fashion used when proving Proposition 4.3.1.)

It can be easily verified that $d\sigma_{\bar{A}}$ is a homomorphism of $\mathcal{O}_{r,q}$-modules.

Secondly, the natural map

$$T\mathcal{S}k_{un,e}/\mathcal{M}_u.T\mathcal{S}k_{un,e} \to T\mathcal{S}k_e$$

is clearly an isomorphism of $\mathcal{O}_r$-modules (since the basis of $T\mathcal{S}k_{un,e}$ doesn't depend on the unfolding parameters u_l). Similarly the natural map

$$T\mathcal{H}_{un,e}/\mathcal{M}_u.T\mathcal{H}_{un,e} \to T\mathcal{H}_e$$

is also an isomorphism of $\mathcal{O}_r$-modules.

Finally the inclusions

$$\mathcal{M}_r.T\mathcal{S}k_e \subset T\mathcal{S}k \qquad \text{and} \qquad \mathcal{M}_r.T\mathcal{H}_e \subset T\mathcal{H}$$

are obvious.

3. **Exponential Map** Here we check the exponential condition. We first review this map for the group $\mathcal{C}$ (acting on the space of smooth map germs $f : \mathbb{C}^r, 0 \to$

$\mathbb{C}^p, 0)$. This is the set of diffeomorphisms $H_0 : \mathbb{C}^r \times \mathbb{C}^p, 0 \to \mathbb{C}^r, 0 \times \mathbb{C}^p, 0$ of the form $H_0(x, y) = (x, h_0(x, y))$ with $h_0(x, 0) \equiv 0$.

Definition B.0.10 *We define*

$$\begin{aligned} \mathcal{C}_{un}(q) &= \{H : \mathbb{C}^r \times \mathbb{C}^p \times \mathbb{C}^q, 0 \to \mathbb{C}^r \times \mathbb{C}^p \times \mathbb{C}^q, 0 : \\ & \qquad H \text{ is an unfolding of } H_0 \in \mathcal{C}\}, \end{aligned}$$

and $H(x, y, u) = (x, h(x, y, u), u)$ *with* $h(x, 0, u) \equiv 0$ *(see [Damon] Pg. 4).*

Then two unfoldings $\bar{f}(x, u)$, $\bar{g}(x, u)$, $\bar{f}, \bar{g} : \mathbb{C}^r \times \mathbb{C}^q, 0 \to \mathbb{C}^p, 0$ *are isomorphic if for a diffeomorphism* $\lambda : \mathbb{C}^q, 0 \to \mathbb{C}^q, 0$ *and a map* $H \in \mathcal{C}_{un}(q)$

$$\bar{g}(x, v) = h\left(x, \bar{f}(x, \lambda(v)), v\right),$$

where $v \in \mathbb{C}^q$ *(see [Damon] Pgs. 6–7).*

The group of such equivalences is labelled $\mathcal{C}_{eq}(q)$ and

$$T\mathcal{C}_{eq,e}(q) = \mathcal{M}_p.\mathcal{O}^p_{r,p,q} \oplus \mathcal{O}^q_q,$$

(see [Damon] Pg. 7, l -1) i.e. elements of $T\mathcal{C}_{eq,e}(q)$ can be identified with pairs of smooth germs (α, β) where

$$\alpha : \mathbb{C}^r \times \mathbb{C}^p \times \mathbb{C}^q, 0 \to \mathbb{C}^p, 0 \quad \text{with} \quad \alpha(x, 0, u) \equiv 0, \tag{B.1}$$

$$\beta : \mathbb{C}^q, 0 \to \mathbb{C}^q.$$

We now define the exponential map for $\mathcal{C}$.

Definition B.0.11 *The exponential map is a map*

$$\exp : T\mathcal{C}_{eq,e}(q) \to \mathcal{C}_{eq}(q+1),$$

where $\exp(\alpha, \beta)$ *is identified with a germ* $\phi_t(x, y, u, t) = (x, h(x, y, u, t), \lambda(u, t))$ *with the property that*

$$\left.\begin{array}{c} h(x, y, u, 0) = y \\ \lambda(u, 0) = u \end{array}\right\} \text{ i.e. } \phi_0 = id$$

and

$$\frac{\partial h}{\partial t}(x, y, u, t) = \alpha\left(x, h(x, y, u, t), \lambda(u, t)\right),$$

$$\frac{\partial \lambda}{\partial t}(u,t) = \beta\left(\lambda(u,t)\right),$$

(see [Damon] Pg. 8).

Remark B.0.12 Such maps h, λ exist by the fundamental existence (and uniqueness) theorem for the solution of systems of ordinary differential equations (see [ArnMec], Pg. 56 and preceding pages). Note that it follows from the above definition that $h(x,0,u,t) \equiv 0$ (it is easy to check this is a solution, and the solution is unique) which is required for ϕ_t to be in $\mathcal{C}_{eq}(q+1)$.

Returning to our subgroup $\mathcal{H}$ of $\mathcal{C}$, two unfoldings $\bar{A}(x,u)$, $\bar{B}(x,u)$, $\bar{A}$, $\bar{B} : \mathbb{C}^r \times \mathbb{C}^q, (0,0) \to Sk(n,\mathbb{C}), 0$ are ***isomorphic*** if, for a diffeomorphism $\lambda : \mathbb{C}^q, 0 \to \mathbb{C}^q, 0$ and a map $\bar{X} : \mathbb{C}^r \times \mathbb{C}^q, (0,0) \to Gl(n,\mathbb{C})$ we have

$$\bar{B}(x,v) = \bar{X}^T(x,v)\bar{A}(x,\lambda(v))\bar{X}(x,v),$$

where $v \in \mathbb{C}^q$ (see Definition 4.4.26). This follows from Definition B.0.10 by replacing $\mathcal{C}$ by $\mathcal{H}$.

The group of such equivalences is clearly $\mathcal{H}_{un}(q) \times \mathcal{D}(q)$ (where $\mathcal{D}(q)$ is the group of diffeomorphisms $\mathbb{C}^q, 0 \to \mathbb{C}^q, 0$). This group, $\mathcal{H}_{un}(q) \times \mathcal{D}(q)$, is labelled as $\mathcal{H}_{eq}(q)$ in [Damon], Pg. 41. It follows that

$$T\mathcal{H}_{eq,e}(q) = T\mathcal{H}_{un,e}(q) \oplus T_I\mathcal{D}(q),$$

(see [Damon], Pg. 41, l -5).

For $\mathcal{H}$ to satisfy Damon's condition on the exponential map we need to prove the following result.

Proposition B.0.13 *The restriction of the exponential map for $\mathcal{C}$ (see Definition B.0.11) induces a map*

$$\exp : T\mathcal{H}_{eq,e}(q) \to \mathcal{H}_{eq}(q+1).$$

(See [Damon], Pg. 41.)

Proof We need to check that when we restrict the exponential map to $T\mathcal{H}_{eq,e}(q)$ we obtain a map in $\mathcal{H}_{eq}(q+1)$. As we have just remarked the space $T\mathcal{H}_{eq,e}(q)$

consists of elements $(\bar{\alpha}(x,u), \beta(u))$ where $\bar{\alpha} : \mathbb{C}^r \times \mathbb{C}^q, (0,0) \to M(n,\mathbb{C})$ and $\beta : \mathbb{C}^q, 0 \to \mathbb{C}^q$. This $\bar{\alpha}$ corresponds to the α associated with $T\mathcal{C}_{eq,e}(q)$ above (see (B.1)) via

$$\alpha : \mathbb{C}^r \times Sk(n,\mathbb{C}) \times \mathbb{C}^q, 0 \to Sk(n,\mathbb{C}), 0$$

$$\alpha(x,A,u) = \bar{\alpha}(x,u)^T A + A\,\bar{\alpha}(x,u).$$

(Note that $\beta : \mathbb{C}^q, 0 \to \mathbb{C}^q$ coincides with that associated with $T\mathcal{C}_{eq,e}(q)$.)

Applying the exponential map for $\mathcal{C}$ to (α, β) we obtain the equations

$$h(x,A,u,0) = A, \qquad \lambda(u,0) = u,$$

$$\frac{\partial h}{\partial t}(x,A,u,t) = \bar{\alpha}\,(x,\lambda(u,t))^T\, h(x,A,u,t) + h(x,A,u,t)\,\bar{\alpha}\,(x,\lambda(u,t)), \quad \text{(B.2)}$$

$$\frac{\partial \lambda}{\partial t}(u,t) = \beta\,(\lambda(u,t)),$$

which, as stated in Remark B.0.12, has a unique solution $A \mapsto h(x,A,u,t)$, λ (for each (α,β)). We need to show that the maps $A \mapsto h(x,A,u,t)$ are linear and of the form $A \mapsto \bar{X}(x,u,t)^T A \bar{X}(x,u,t)$ for some $\bar{X} : \mathbb{C}^r \times \mathbb{C}^q \times \mathbb{C}, 0 \to Gl(n,\mathbb{C})$.

Suppose this was the case. Then the above equation (B.2) reduces to

$$\begin{aligned}\frac{\partial \bar{X}}{\partial t}(x,u,t)^T A\bar{X}(x,u,t) + \bar{X}(x,u,t)^T A \frac{\partial \bar{X}}{\partial t}(x,u,t) =&\\ \bar{\alpha}\,(x,\lambda(u,t))^T\, \bar{X}(x,u,t)^T A\bar{X}(x,u,t) +&\\ \bar{X}(x,u,t)^T A\bar{X}(x,u,t)\,\bar{\alpha}\,(x,\lambda(u,t)).&\end{aligned}$$

This is satisfied if

$$\frac{\partial \bar{X}}{\partial t}(x,u,t) = \bar{X}(x,u,t)\,\bar{\alpha}\,(x,\lambda(u,t)).$$

Setting $\bar{X}(x,u,t) = exp\,(t\bar{\alpha}\,(x,\lambda(u,t)))$, where $exp : M(n,\mathbb{C}) \to Gl(n,\mathbb{C})$ is the familiar exponential map given by $exp(A) = \sum_{k=0}^{\infty} A^k/k!$; then

$$\frac{\partial \bar{X}}{\partial t}(x,u,t) = \bar{X}(x,u,t)\,\bar{\alpha}\,(x,\lambda(u,t))$$

as required, and $\bar{X}(x,u,0) = I_n$. Since the solution for h above was unique this proves the result. □

4. **Filtration Condition** Finally, by considering the action of $\bar{X}(x,u) \in \mathcal{H}_{un}$ on $\bar{A}(x,u) \in \mathcal{M}_r^l.\mathcal{S}k_{un}$, $l \geq 0$,

$$\bar{X}(x,u).\bar{A}(x,u) = \bar{X}^T(x,u)\bar{A}(x,u)\bar{X}(x,u),$$

it is obvious that $\mathcal{H}_{un}$ preserves the filtration $\mathcal{M}_r^l.\mathcal{S}k_{un}$ on $\mathcal{S}k_{un}$.

Appendix C

Vector Fields on Varieties

Our group of equivalences $\mathcal{G}$ has two parts, $\mathcal{R}$ and $\mathcal{H}$, the latter corresponding to a subgroup of Mather's group $\mathcal{C}$. This arises naturally from the group action of $Gl(n, \mathbb{C})$ on $Sk(n, \mathbb{C})$. We show here that the corresponding vector fields can also be interpreted as coming from those tangent to the set of singular skew matrices.

Let X be an analytic variety in $\mathbb{C}^n$, and let $Sing(X)$ denote the singular part of X.

Definition C.0.14 *A germ of a vector field ξ is* **tangent** *to X if it is tangent in the usual sense to $X \setminus Sing(X)$. The set of smooth vector fields tangent to X is denoted $\Theta(X)$. These clearly form an $\mathcal{O}_n$-module.*

We shall largely be concerned with the case when X is a hypersurface given by the reduced equation $f = 0$. One can then prove the following.

Proposition C.0.15 *The vector field ξ is tangent to $X = \{f = 0\}$ if and only if $\xi(f) = \alpha.f$ for some smooth α.*

Now suppose that the Lie group G acts smoothly on the germ $\mathbb{C}^n, 0$; in particular G fixes 0. Then given an element l of the Lie algebra LG we can define the germ of a vector field $l\xi$ on $\mathbb{C}^n, 0$ as follows. Since $l \in T_eG$ there is a smooth curve $\gamma : \mathbb{C}, 0 \to G, e$ with $\gamma'(0) = l$. For each $x \in \mathbb{C}^n$ define $l\xi(x)$ to be $d(\gamma(t).x)/dt|_{t=0}$.

Proposition C.0.16 *The vector fields $l\xi$ are well defined.*

Proof Clearly if $\phi : G \times X \to X$ is the map yielding the group action then $l\xi(x) = d\phi(e,x)(l,0)$. □

This set of vector fields is, of course, a finite dimensional vector space. We can consider the $\mathcal{O}_n$-module generated by these fields; this is denoted by $\Theta(G)$.

Proposition C.0.17 *The module $\Theta(G)$ is finitely generated and given any $\xi \in \Theta(G)$ we find that $\xi(x)$ is tangent to the orbit $G.x$ through x.*

Proof The module is clearly finitely generated. We have only to show that the final result holds true for the vector fields of the form $l\xi$. However $l\xi(x) = d\phi(e,x)(l,0)$ which is clearly tangent to $\phi(G \times \{x\})$ the orbit of x. □

Example C.0.18 As usual we work over the complex numbers. Let $M(n,p)$ denote the set of $n \times p$ matrices. Clearly there is an action of the Lie group $G = Gl(n) \times Gl(p)$ on this space given by

$$(Gl(n) \times Gl(p)) \times M(n,p) \to M(n,p); ((X,Y),A) \mapsto X^{-1}AY.$$

Let E_{ij} denote the matrix with a 1 in the $(i,j)th$ place and 0's elsewhere. Consider the path γ_{ij} in $G = Gl(n) \times Gl(p)$ given by $t \mapsto (I, I + tE_{ij})$ for t small. Clearly $\gamma_{ij}(t).A = A(I + tE_{ij}) = A + tAE_{ij}$ and the corresponding tangent vector is AE_{ij}. Similarly from the obvious paths in $Gl(n)$ we obtain the tangent vector field $E_{ij}A$. (The key here is that when computing this one can ignore the inverse. Take the paths to be $t \mapsto (I + tE_{ij}, I)^{-1}$.) From this we deduce the following.

The vector fields in $\Theta(G)$ are generated by the A_{ij}, $1 \leq i,j \leq p$ and ${}_{ij}A$, $1 \leq i,j \leq n$, where A_{ij} (respectively ${}_{ij}A$) is the matrix whose jth column (respectively row) is the ith column (respectively row) of A, and whose remaining columns (respectively rows) are zero.

Example C.0.19 Now let $Sk(n)$ denote the set of skew $n \times n$ matrices. There is an action of the Lie group $G = Gl(n)$ on this space given by

$$Gl(n) \times Sk(n) \to Sk(n); (X,A) \mapsto X^T AX.$$

Let E_{ij} denote the matrix with a 1 in the $(i,j)th$ place and 0's elsewhere. Consider the path γ_{ij} in $Gl(n)$ given by $t \mapsto I + tE_{ij}$ for t small. Clearly $\gamma_{ij}(t).A = (I + tE_{ji})A(I + tE_{ij}) = A + t(E_{ji}A + AE_{ij}) + t^2 E_{ji}AE_{ij}$ and the corresponding tangent vector is $E_{ji}A + AE_{ij}$. From this we deduce the following.

The vector fields in $\Theta(G)$ are generated by the matrices obtained by placing the ith row of A in the jth row and the ith column in the jth column, with zeros elswhere, and adding. Note that the result is skew-symmetric, and in particular the diagonal entries are all zero.

Now we know that the elements of $\Theta(G)$ are tangent to the orbits. In the first case above and the second case when n is even there is one open orbit (those matrices with maximal rank) and the remaining orbits form a hypersurface D given by the vanishing of the determinant. The main result we wish to establish is the following.

Theorem C.0.20 *For the square matrices ($n = p$), and the even skew matrices* $\Theta(G) = \Theta(D)$.

We first consider the case of (arbitrary) square matrices, and start to try to compute $\Theta(D)$. We do this because the basic algebra is more familiar.

Lemma C.0.21 *The defining equation of the set of singular matrices D is homogeneous of degree n, and consequently there is an euler vector field $e = \sum_{1 \leq i,j \leq n} a_{ij}\partial/\partial a_{ij}$ with $e(\det(A)) = n\det(A)$.*

This is just Euler's theorem.

Now to check if a vector field ξ is in $\Theta(D)$ we need to find all solutions to the equation

$$\xi(\det(A)) = \alpha(A)\det(A)$$

for some smooth α. However note that given such a ξ the vector field $\xi' = \xi - \alpha e/n$ has the property $\xi'(\det) = \xi(\det) - \alpha e(\det)/n = \alpha.\det - n\alpha.\det/n = 0$. So we are reduced to finding solutions to the equation $\xi(\det) = 0$. Now any ξ can be written as a linear combination of $\partial/\partial a_{ij}$, so we are looking for relations between the $\partial \det / \partial a_{ij}$. On the other hand if we expand the determinant by its ith row we obtain

$$\det(A) = \sum_{k=1}^{n} -1^{i+k} a_{ik} A_{ik}$$

where the A_{ik} are the $(n-1)\times(n-1)$ minors of the matrix A, and do not contain any entry in the ith row. So $\partial \det / \partial a_{ij} = (-1)^{i+j} A_{ij}$ and we are reduced to finding relations between the (maximal) minors of the matrix A. One obvious set of relations is provided by the equations $A.adj(A) = adj(A).A = \det(A)I$. The next result shows that the vector fields one can construct from these are those we obtained from the group action.

Proposition C.0.22 *The equations $A.adj(A) = adj(A).A = \det(A)I$ yield $2n^2 - 1$ relations between the minors of the generic matrix A, namely*

$$\sum_k a_{ik}(-1)^{k+j} A_{jk} = 0,\ i \neq j;\ \sum_k a_{ik}(-1)^{k+i} A_{ik} = \det A,\ 1 \leq i \leq n$$

$$\sum_k (-1)^{k+j} A_{kj} a_{ki} = 0,\ i \neq j;\ \sum_k (-1)^{k+i} A_{ki} a_{ki} = \det A,\ 1 \leq i \leq n.$$

Moreover these relations immediately show that the corresponding vector fields, $\sum_k a_{ik}\partial/\partial a_{jk}$ and $\sum_k a_{ki}\partial/\partial a_{kj}$, are tangent to D. These are the vector fields produced geometrically from the group action.

We are consequently reduced to showing that these are the only relations between the minors of a generic matrix. Note that we are working here over the ring of smooth/analytic functions, and not the polynomial functions, but this is also a Noetherian ring. This result is provided by the exactness of a complex due to Gulliksen and Negard, see [BrunV]. In what follows M_n is the space of $n \times n$ matrices with entries in $\mathcal{O}_r$, and let $A \in M_n$. We are interested in the case when A is the generic matrix $A = (a_{ij})$, $r = n^2$ and $\mathcal{O}_r$ is the ring of smooth functions in the a_{ij}. So consider the sequence

$$\mathcal{GN}(A) : 0 \longrightarrow \mathcal{O}_r \xrightarrow{d_4} M_n \xrightarrow{d_3} G_2 \xrightarrow{d_2} M_n \xrightarrow{d_1} \mathcal{O}_r \longrightarrow 0.$$

To explain all of the symbols here we need the sequence

$$\mathcal{O}_r \xrightarrow{i} M_n \oplus M_n \xrightarrow{\pi} \mathcal{O}_r$$

where $i(\lambda) = (\lambda I, \lambda I)$ and $\pi(V, W) = tr(V - W)$. With this noted $G_2 = \ker \pi / \operatorname{im} i$, and then $d_1(V) = tr(\tilde{A}V)$ where $\tilde{A}$ is the adjoint of A, $d_4(\lambda) = \lambda\tilde{A}$, $d_2(V, W) = VA - AW$, $d_3(V) = (AV, VA)$. It is not difficult to prove that this is a complex. The following is more difficult.

Theorem C.0.23 *Let $I(A)$ denote the ideal generated by the $(n-1) \times (n-1)$ minors of A. Then if grade $I(A) \geq 4$ then the above sequence is exact. Moreover in the case when A is the generic $n \times n$ matrix then the grade of $I(A)$ is 4, and the sequence is exact.*

Completion of the proof of Theorem C.0.20 in square matrix case. Since the sequence is exact it follows that the image of d_2 coincides with the kernel of d_1. Now $d_1(V) = \sum_j \sum_i (-1)^{i+j} v_{ij} A_{ij}$ so the kernel of d_1 coincides precisely with the set of relations between the maximal minors of A. We have now to find the image of d_2; we can clearly extend d_2 to the whole of $\ker \pi$. But $\ker \pi$ is spanned by $(E_{ij}, 0)$, $(0, E_{ij})$ $i \neq j$, and (E_{ii}, E_{jj}), $1 \leq i, j \leq n$. However $d_2(E_{ij}, 0) = E_{ij}A$ and $d_2(0, E_{ij}) = AE_{ij}$ and these correspond precisely to the first occuring in the two sets of relations in Proposition C.0.22. On the other hand $d_2(E_{ii}, E_{jj}) = E_{ii}A - AE_{jj}$ and the corresponding relations here are simply those from the second two sets. □

We now need to carry out the same process for the skew-symmetric matrices. What is different here is that if A is an $n \times n$ skew-symmetric matrix with entries in a ring R, then $\det(A)$ is the square of an element in R, the Pfaffian $Pf(A)$. Now for a sequence of integers $I = 1 \leq i_1 < i_2 < \ldots < i_k \leq n$ the matrices obtained by deleting the rows and columns with these indices is also skew. We write Pf^I or $Pf^{i_1 i_2 \ldots i_k}$ for its Pfaffian, said to be a Pfaffian of order $n - k$. We next show that our vector field problem reduces to one concerning Pfaffians of order $n - 2$. Let us suppose that $n = 2m$ is even, and let D be the set of skew matrices of rank$< n$. Here the equation $\det A = 0$ does not now give a reduced equation for D, but $Pf(A) = 0$ does; this is a homogenous equation of degree $m = n/2$.

So to check if a vector field ξ is in $\Theta(D)$ we need to find all solutions to the equation

$$\xi(Pf(A)) = \alpha(A)Pf(A)$$

for some smooth α. As above note that given such a ξ the vector field $\xi' = \xi - \alpha e/m$ has the property $\xi'(Pf) = \xi(Pf) - \alpha e(Pf)/m = \alpha.Pf - m\alpha.Pf/m = 0$. So we are reduced to finding solutions to the equation $\xi(Pf) = 0$. Now any ξ can be written as a linear combination of $\partial/\partial a_{ij}$, so we are looking for relations between the $\partial Pf/\partial a_{ij}$.

The next step involves a not so familiar result.

Lemma C.0.24 *If $A = (a_{ij})$ is a generic skew matrix let $A^* = (b_{ij})$ denote the skew matrix with entries*

$$b_{ij} = \begin{cases} (-1)^{i+j-1} Pf^{ij}(A) & \text{if } j < i \\ 0 & \text{if } i = j \\ (-1)^{i+j} Pf^{ij}(A) & \text{if } i < j \end{cases} .$$

Then $AA^* = A^*A = Pf(A).I$.

Proof The fact that the identity holds for the diagonal elements follows from Proposition 1.1.10 in Chapter 1. For the off-diagonal entries we obtain a sum $\sum_{k=1}^{n}(-1)^{j+k-1}a_{ik}Pf^{jk}(A)$. However by the aforementioned result this is just the Pfaffian of the skew-symmetric matrix obtained by replacing the j^{th} row and column by the i^{th} row and column. This matrix is clearly singular if $j \neq i$ and the result follows. □

Note that the Pf^{ij} do not contain any entry in the ith or jth rows, or columns. So $\partial Pf/\partial a_{ij} = (-1)^{i+j-1}Pf^{ij}$ and we are reduced to finding relations between the Pfaffians of order $n-2$ of the matrix A. One set of relations is provided by the above equations $A.A^* = A^*.A = Pf(A)I$. The next result shows that the vector fields one can construct from these are those we obtained from the group action.

Proposition C.0.25 *The equations $A.A^* = A^*.A = Pf(A)I$ yield the following relations between the Pfaffians of order $n-2$ of the generic skew matrix A.*

$$\sum_k a_{ik}(-1)^{j+k-1}Pf^{jk} = 0,\ i \neq j;\ \sum_k a_{ik}(-1)^{i+k-1}Pf^{ik} = Pf(A),\ 1 \leq i \leq n.$$

Moreover these relations immediately show that vector fields $\sum_k a_{ik}\partial/\partial a_{jk}$ are *tangent to D. These are the vector fields produced geometrically from the group action.*

Proof These relations immediately show that the indicated vector fields are tangent to D. (Note that there is only one set of relations because the matrix is skew.) Moreover these are the vector fields produced geometrically from the group action. □

We are consequently reduced to showing that these are the only relations between these Pfaffians of order $n-2$ of a generic matrix. Note that we are working here over the ring of smooth/analytic functions, and not the polynomial functions, but this is also a Noetherian ring. This result is provided by the exactness of a complex due to Jozefiak and Pragaz, see [JozPra]. In what follows M_n (respectively Sym_n, Sk_n) is the space of $n \times n$ matrices (respectively symmetric matrices, skew symmetric matrices) with entries in $\mathcal{O}_r$. Let $A \in Sk_n$ be the generic skew-symmetric matrix $A = (a_{ij})$ with $a_{ij} = -a_{ji}$, $r = n(n-1)/2$ and $\mathcal{O}_r$ is the ring of smooth functions in the a_{ij}, $1 \leq i < j \leq n$. So consider the sequence

$$\mathcal{JP}(A) : 0 \longrightarrow \mathcal{O}_r \xrightarrow{d_6} Sk_n \xrightarrow{d_5} L_4 \xrightarrow{d_4} (M_n/Sk_n) \oplus Sym_n \xrightarrow{d_3}$$

$$L_2 \xrightarrow{d_2} M_n/Sym_n \xrightarrow{d_1} \mathcal{O}_r \longrightarrow 0.$$

Here $L_4 = \operatorname{coker}(\mathcal{O}_r \to M_n,\ r \mapsto rI)$ and $L_2 = \ker(M_n \to R,\ X \mapsto tr(X))$. With this noted $d_1(V) = tr(A^*V)$, $d_2(V) = AV \bmod Sym_n$, $d_3(V,W) = A^*(V+V^T) - WA$, $d_4(V) = (AV \bmod Sk_n, VA^* + (VA^*)^T)$, $d_5(V) = VA \bmod \mathcal{O}_r I$, $d_6(r) = rI$. It is not difficult to prove that this is a complex. The following is more difficult.

Theorem C.0.26 *Let $I(A)$ denote the ideal generated by the Pfaffians of order $n-2$ in A. Then if grade $I(A) \geq 6$ then the above sequence is exact. Moreover in the case when (as usual $n = 2m$ is even) and A is the generic skew $n \times n$ matrix then the grade of $I(A)$ is 6, and the sequence is exact.*

Completion of the proof of Theorem C.0.20 for the even skew case. Since the sequence is exact it follows that the image of d_2 coincides with the kernel of d_1. Now the rest goes as before. The point is that $tr(A^*V) = 0$ gives the relations between the Pfaffian minors of A. We can work modulo Sym_n because A^* is skew and if W is symmetric then $tr(A^*W)$ is identically 0. Exactness shows that these relations all arise as the image of d_2. But elements in the image of d_2 we may suppose are of the form $(AV - V^TA^T)/2$. Choosing V to be the basis matrix E_{ij} where $i \neq j$ or $E_{ii} - E_{nn}$ for $1 \leq i < n$ yields exactly the relations given above. (Taking $V = E_{ii}$ we obtain the relation giving the Pfaffian, and considering only those matrices with trace 0 eliminates the Pfaffian.) □

Appendix D

The procedure `skewmatrix`

Here we provide the full Maple source code for the Lie algebra procedure `skewmatrix` mentioned in Chapter 5. First we make a few remarks.

Principally, the objective of this procedure is to define the $\mathcal{G}$-tangent space, to a family of skew-symmetric matrices, described in Proposition 4.3.2. As usual this involves finding a finite set of generators for this space. It is convenient and more efficient (for carrying out calculations in the jetspaces) to represent families of skew-symmetric matrices by N-tuples in upper triangular form. However for finding the generators for the tangent space we require the matrix form.

Therefore the main problem tackled by this procedure is that of converting upper triangular N-tuples into skew-symmetric matrices, to find the generators of a tangent space, and then converting the resulting skew-symmetric matrices back into N-tuples for the purposes of calculations using `jetcalc`.

The equivalence for families of *symmetric* matrices (used in [BrTarSy]) is the same as the $\mathcal{G}$-equivalence defined here for the skew-symmetric case. Hence the method for finding generators of the tangent space to the orbit of a family of symmetric matrices, under this action, is the same as for our skew-symmetric case. Consequently, to write this procedure we have made use of a procedure drafted by N.P.Kirk for the symmetric case. In particular the code used for finding the $\mathcal{H}$-tangent space generators is suggested by his routine.

We finish with a few further points on the following code.

Defining the source coordinates $x1, \ldots, xr$ and the $\mathcal{R}$-tangent space generating set are standard and we refer to [Kirk]. Regarding the $\mathcal{H}$-tangent space, the code for converting an N-tuple into a skew-symmetric matrix is fairly simple and uses the Maple array indexing function **antisymmetric**. The code for converting a skew-symmetric generator into an upper triangular N-tuple uses the relation between the index, rs, of an upper triangular entry a_{rs} of a skew-symmetric matrix and the index, k, of its corresponding entry b_k in the representative N-tuple. That is, if $a_{rs} = b_k$ then

$$k = (r-1)n + s - r(r+1)/2.$$

The code used for finding the 'extra' vectors, described in Section 5.2, is also taken from the aforementioned symmetric procedure.

D.1 The code

In the following, all annotations are prefixed by the symbol '#'.

```
# Procedure to define Lie algebra corresponding to group R x H
# used for classifying Families of Skew-symmetric Matrices (is used
# within the routine initialised by the call jetcalc(A,k);)

# NOTES:
# -----

# Represent (n x n) skew-symmetric matrices as
# n(n-1)/2 (upper triangular) vectors, e.g
# [ 0, a1, a2, a3 ]
# [-a1, 0, a4, a5 ]  <-->  [ a1, a2, a3, a4, a5, a6 ]
# [-a2,-a4, 0, a6 ]
# [-a3,-a5,-a6, 0 ]

# Dimension of source manifold given by global variable
#  'source_dim'.
# Source coordinates defined as (x1, x2, ..., xr)
#  where r = source_dim.
# Target dimension passed in as the argument target_dim; must be a
#  triangular number.
# Nilpotent vectors: need two different sets - one for CT calculations,
```

```
#  one for full group tangent space. (Note these are the extra
# tangent vectors which have been missed by setting
# source_power = 2/1 for R - part. Not 'true' Nilpotents.)
# Therefore define two global lists, RCT_nilp and RG_nilp which
# define these. One then assigns the appropriate one to R_nilp
# (and sets L_nilp := [];).
# NB: CT:  source_power := 2; R_nilp := RCT_nilp;
#     group  source_power := 1; R_nilp := RG_nilp;



skewmatrix:=proc(A,target_dim,tgtspace)

local coords_temp,n,i,j,F,k,count,r,s,v,b_rs,temp_vect,F_t,K,
temp,last_tgt;
global source_dim,coords,RCT_nilp,RG_nilp;

# DEFINE COORDINATES (global variable, data type 'list')
# first check all source coordinates are formal indeterminates
# i.e are unassigned as Maple expressions

if not type(source_dim,posint) then
ERROR(`global variable 'source_dim' must be a
+ve integer`);
fi;
for i from 1 to source_dim do
if assigned(`x`.i) then
ERROR(`not all source coordinates are unassigned
Maple names`);
fi;
od;
# CREATE SOURCE COORDINATES
coords_temp := array(1..source_dim);
for i from 1 to source_dim do
coords_temp[i] := `x`.i;  # concatenates "string" name :
# `x`.3 = "x3"
od;
# now convert vector coords_temp to type list to form coords
coords := convert(coords_temp,list);
```

```
# CHECK TARGET DIMENSION VALID
# (The target_dim holds the number of components
# of the jet, A, and is
# determined within jetcalc.)

n:= (1 + sqrt(1 + 8*target_dim))/2;
if n <> trunc(n) then
ERROR ('the target dimension must be a triangular number');
fi;

# DEFINE LIE ALGEBRA GENERATING SET (data type 'table')
tgtspace := table();

# 1) R-TANGENT SPACE

for i from 1 to source_dim do
for j from 1 to target_dim do
tgtspace[i][j] := diff(A[j],coords[i]);
od;
od;

# 2) H-TANGENT SPACE
#
#
#

# first need to convert vector A into skew-symmetric matrix
F:=array(antisymmetric,1..n,1..n);
k:=1;
for r from 1 to n-1 do
for s from r+1 to n do
F[r,s]:=A[k];
k:=k+1;
od;
od;
# need to convert this to n^2-vector using row major order
v:=convert(F,vector);

# following code calculates the H tangent space generators

count:= source_dim+1;
for i from 1 to n do
for j from 1 to n do
```

```
temp_vect:=array(1..n^2); # stores the generator
# whose ith row and column is obtained
# by superimposing the jth row and
# column of A

# build B(i,j) and assign to temp_vect
for r from 1 to n do
for s from 1 to n do
# finds C_{ji}(A)
if i = s then
b_rs := v[n*(r-1)+j];
else
b_rs := 0;
fi;
# finds R_{ji}(A)
if i = r then
b_rs := b_rs + v[n*(j-1)+s];
fi;
temp_vect[n*(r-1)+s] := b_rs;
  od;
od;
# converts 'temp_vect' back to its corresponding
# skew-symmetric matrix
F_t := linalg[matrix](n,n,temp_vect);
# Finally need to convert this tangent matrix
# into an upper triangular vector
# and assigns it to tgtspace[count]
for r from 1 to n-1 do
for s from r+1 to n do
K:= (r-1)*n + s -r*(r+1)/2;
tgtspace[count][K]:= F_t[r,s];
od;
od;

count:= count+1;
od;
od;

# store last element
last_tgt := count-1;
```

```
# Define "NILPOTENT" VECTORS (global variables,
# data type 'list')
# (see above notes)

# RCT_nilp: extra vectors: x.k * B(i,j)
# NB: the relevant B(i,j) vectors are stored as
#  tgtspace[source_dim +1], ..., tgtspace[last_tgt]
# (also see manual)

count:=1;
temp:=table();
for i from source_dim+1 to last_tgt do
for j from 1 to source_dim do
temp[count]:= [`x`.j,i];
count:=count+1;
od;
od;
RCT_nilp := convert(temp,list);

# RG_nilp: extra vectors: B(i,j)   (Not needed for CT)

count:=1;
temp:=table();
for i from source_dim+1 to last_tgt do
temp[count] := [1,i];
count := count+1;
od;
RG_nilp := convert(temp,list);

# RETURN NULL
NULL;

end:
```

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